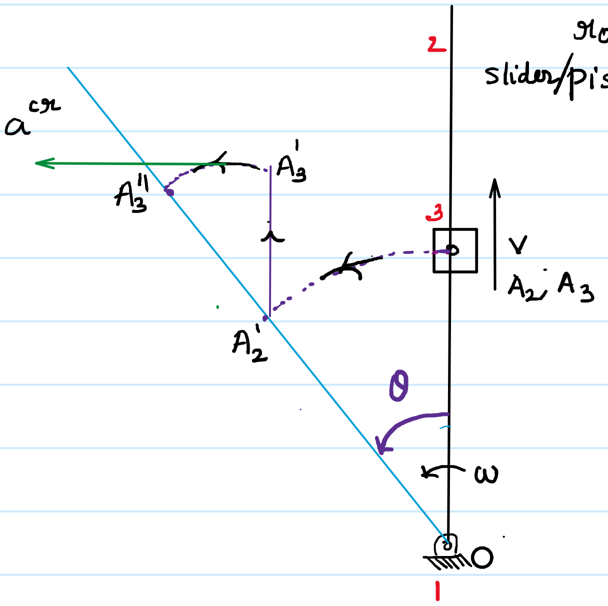
rod – link 2
slider/piston – link 3.



A_2, A_3 are points on link 2 and link 3 instantly co-incident.

Displacement of link 3 / Slider $= \overset{\frown}{A_2A_2'} + \overline{A_2'A_3'} + \overset{\frown}{A_3'A_3''}$

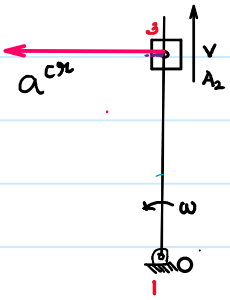
$\overset{\frown}{A_2A_2'} \rightarrow$ Angular displacement due to velocity ω.

$\overline{A_2'A_3'} \rightarrow$ linear displacement due to velocity V.

$\overset{\frown}{A_3'A_3''} \rightarrow$ Addition displacement of slider due to coriolis acceleration

$$\vec{a} = \vec{a}^{r}_{rod} + \vec{a}^{t}_{rod} + \vec{a}^{cr} + \vec{a}^{slider}$$

$$\vec{a}^{cr} = 2.\dot{r}\dot{\theta}(\hat{e}_t) = 2.(\vec{\omega} \times \vec{V}).$$



$$a^{cr}\,\hat{a}^{cr} = 2.V_{slider}.\omega_{Rod}.(\hat{\omega}_{Rod} \times \hat{V}_{slider})$$

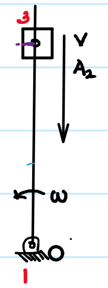
$$\hat{\omega}_{Rod} = +\hat{k}$$

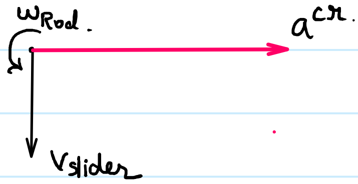
$$\hat{V}_{slider} = +\hat{j}$$

$$\hat{a}^{cr} = \hat{\omega}_{Rod} \times \hat{V}_{slider} = \hat{k} \times \hat{j} = -\hat{i}$$

Rotate V_{slider} in the direction of ω by 90°.

a^{cr}





$$\hat{a}^{cr} = \hat{\omega} \times \hat{V}$$
$$= \hat{k} \times -\hat{j} = \hat{i}$$

Newton's laws
D'Alembert Principle
FBD
Second order D.E.

Coriolis acceleration exists in
1.Crank Slotted Lever Mechanism 2.Whitworth Mechanism 3.Oscillating Cylinder Mechanism.

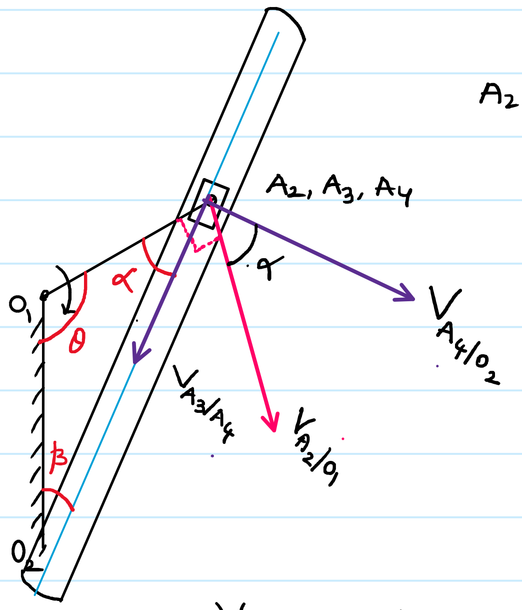
Analysis of Crank Slotted Lever Mechanism

$\alpha = 180 - (\theta + \beta)$

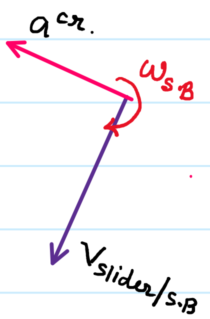
V_{A_2/O_1} – V_{crank}.

V_{A_3/A_4} – $V_{slider/slotted\ bar}$.

V_{A_4/O_2} – $V_{slotted\ bar/O_2}$



A_2, A_3, A_4 are points on link 2 (crank), link 3 (slider) and link 4 (slotted bar)



$$V_{A_3/A_4} = V_{A_2/O_1} \cdot \sin\alpha.$$

$$V_{A_4/O_2} = O_2A_4 \cdot \omega_{S.B.}$$

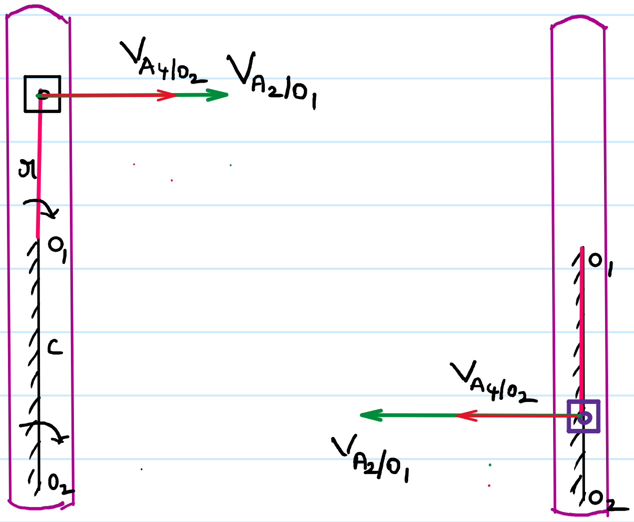
$$V_{A_4/O_2} = V_{A_2/O_1} \cdot \cos\alpha = O_2A_4 \cdot \omega_{S.B.}$$

Coriolis acceleration

$$\vec{a}^{cr} = 2 \cdot (\vec{\omega}_{S.B} \times \vec{V}_{slider/S.B})$$

Special Cases of Crank Slotted Lever Mechanism

Case I When the Slotted bar is at the Mean position



$V_{A_3/A_4} = 0$ because link 3 is @ IDC / ODC.

$\alpha = 0, 180$

$V_{A_3/A_4} = 0$

O_1O_2 – fixed link.
C – distance b/w fixed centers.
O_1A_1 – crank.
r – lenght of crank.

$$V_{A_2/O_1} = V_{A_4/O_2}$$

$$O_1A_2 \cdot \omega_{O_1A_2} = O_2 \cdot A_4 \cdot \omega_{O_2 A_4}$$

$$(r) \cdot \omega = (C+r) \cdot (\omega_{S.B})_{max.1} \longrightarrow A$$

$$V_{A_2/O_1} = V_{A_4/O_2}$$

$$r \cdot \omega = (C-r) \cdot (\omega_{S.B})_{max\ 1} \rightarrow (B)$$

$$\frac{A}{B} = 1 = \frac{(C+r) \cdot \omega_{max\ 1}}{(C-r) \cdot \omega_{max\ 2}}$$

$$\frac{(\omega_{max-2})_{\text{Return}}}{(\omega_{max-1})_{\text{Cutting}}} = \frac{C+r}{C-r} > 1$$

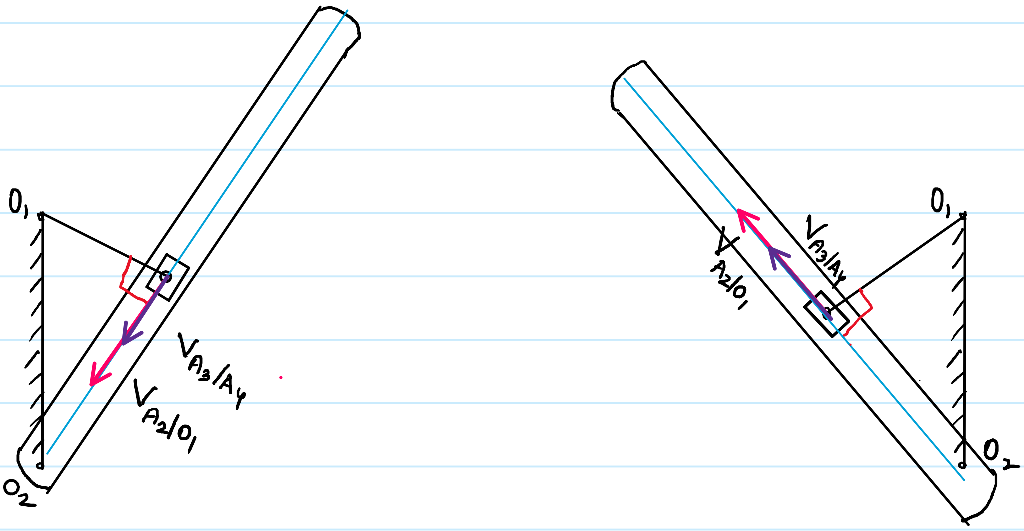
Coriolis acceleration

$$\vec{a}^{cr} = 2 \cdot (\vec{\omega}_{S.B} \times \vec{V}_{slider/S.B})$$

$\vec{V}_{slider/S.B} = 0$

$$\vec{a}^{cr} = 0$$

Case II **When the slotted bar is at the extreme positions**



$\alpha = 90^\circ$ $\quad V_{A_4/O_2} = V_{A_2/O_1} \cdot \cos\alpha = 0 \Rightarrow \omega_{S.B} = 0$

$\vec{a}^{cr} = 0$

$$V_{A_3/A_4} = V_{A_2/O_1} \cdot \sin\alpha$$

$$= r\omega.$$

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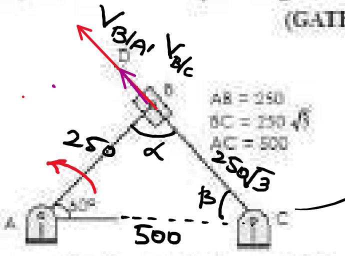
16. For the configuration shown, the angular velocity of link AB is 10 rad/s counterclockwise. The magnitude of the relative sliding velocity (in ms⁻¹) of slider B with respect to rigid link CD is (GATE – 10)



(a) 0 (b) 0.86 (c) 1.25 (d) ~~2.50~~

Toggle position.

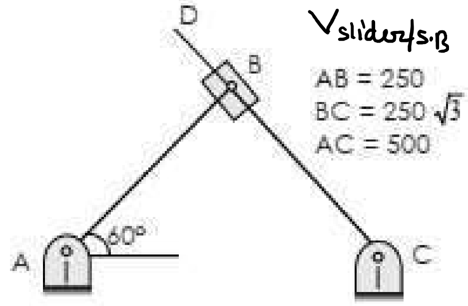
slotted is @ extreme position.

$\omega_{SB} = 0$

$V_{S \cdot B/O_2} = V_{crank} \cdot \cos\alpha$.

$V_{slider/S.B.} = V_{crank} \cdot \sin\alpha$.

$V_{slider/S.B} = V_{crank}$.



$V_{slider/S.B} = r\omega$

$= 0.25 \times 10$

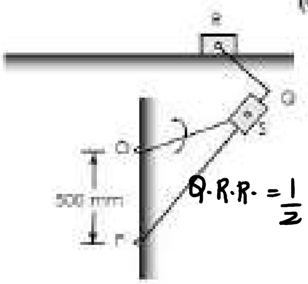
$= 2.5\,m/s$

Sine Rule.

$$\frac{250}{\sin 60} = \frac{500}{\sin\alpha} \qquad \alpha = 90^\circ \qquad \beta = 30^\circ$$

Statement for Common data Q. 17 & Q.18

A quick return mechanism is shown below. The crank OS is driven at 2 rev/s in counterclockwise direction. (GATE-07)



$$Q.R.R. = \frac{1}{2} = \frac{\theta_{Return}}{\theta_{cutting}}$$

17. If the quick return ratio is 1:2 then the length of the crank in mm is

(a) 250 (b) $250\sqrt{3}$

(c) 500 (d) $500\sqrt{3}$

slotted bar.

18. The angular speed PQ in rev / s when the block R attains maximum speed during forward stroke (stroke with slower speed) is (ω_{max}) cutting

(a) $\frac{1}{3}$ (b) $\frac{2}{3}$

(c) 2 (d) 3

$\omega_{crank} = 2\,rev/s$. C.C.W.

lenght of fixed link = 500.

Q.R.R. = 1:2 lenght of crank = ?

Whitworth. Mechanism.

$$\frac{\text{lenght of fixed link}/L_{min}}{\text{lenght of crank}} = \frac{1}{Q.R.R.}$$

lenght of crank = 1000mm.

Crank. Slotted Lever

$$\frac{\text{lenght of crank}/L_{min.}}{\text{lenght of fixed link}} = \frac{1}{Q.R.R.}$$

$$\text{lenght of crank} = \frac{500}{2} = 250mm.$$

$r \cdot \omega = (c + r) \cdot \omega_{max}$,

$250 \times 2 = (500 + 250)(\omega_{max})_{cutting}$

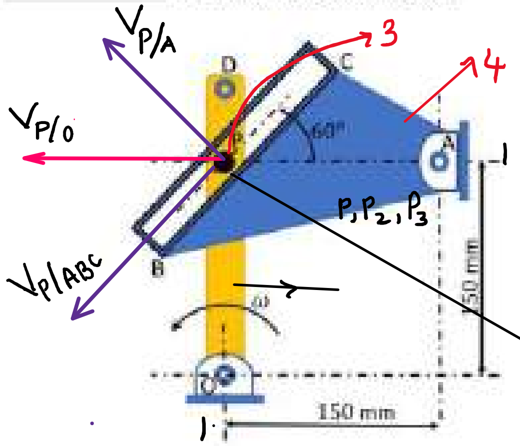
$(\omega_{max})_{cutting} = \frac{2}{3}\,rev/s$.

$r\omega = (c - r) \cdot (\omega_{max-2})_{Return}$.

$$(\omega_{max})_{Return} = \frac{250 \times 2}{(500 - 250)}$$

$= 2\,rev/s$.

Q.48 At the instant when OP is vertical and AP is horizontal, the link OD is rotating counter clockwise at a constant rate ω = 7 rad/s. Pin P on link OD slides in the slot BC of link ABC which is hinged at A, and causes a clockwise rotation of the link ABC. The magnitude of angular velocity of link ABC for this instant is ________ rad/s (rounded off to 2 decimal places).



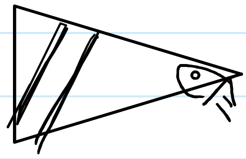
Pin is moving along slot

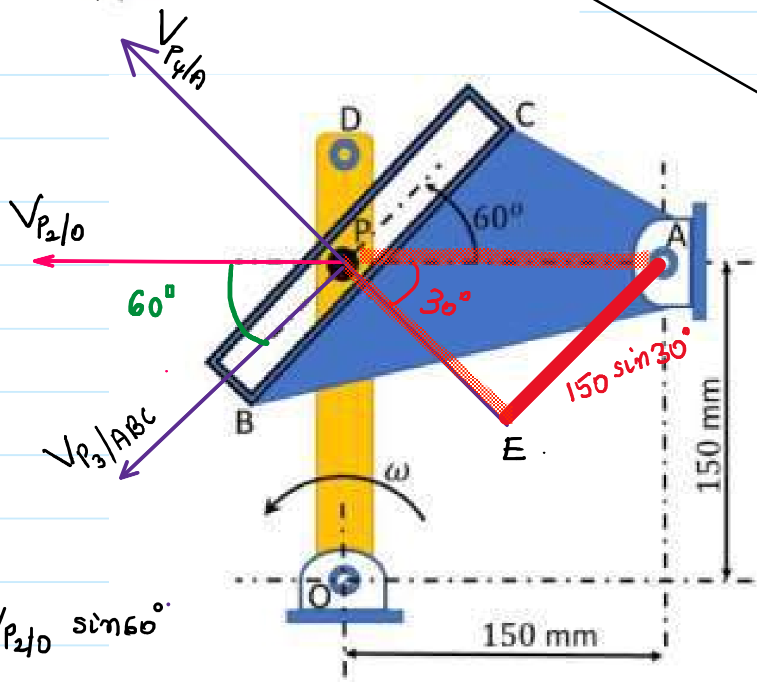
disp. of pin wrt slot of link ABC.

$\omega_{OD} = 7 \text{ rad/s}$ C.C.W.

$OP = 150 mm$

$AP = 150 mm$.





P_2, P_3, P_4 is the point on link 2, link 3, link 4.

$V_{P_2/O}$ — velocity of link 2

$V_{P_3/ABC}$ — velocity of link 3 wrt link 4

$V_{P_4/A}$ — velocity of link 4.

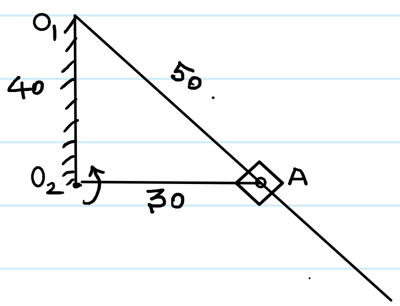
$V_{P_4/A} = V_{P_2/O} \sin 60°$

$AE \cdot \omega_{ABC} = OP \cdot \omega_{OP} \cdot \sin 60°$

$150 \sin 30° \times \omega_{ABC} = 150 \times 7 \times \sin 60°$

$$\omega_{ABC} = \frac{7 \sin 60}{\sin 30°} = 12.124 \text{ rad/s}$$

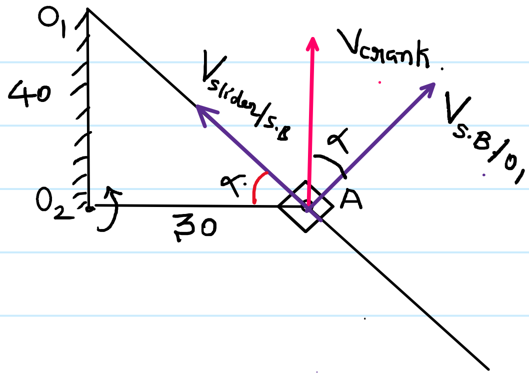
C.R.P.Q. - volume - 2

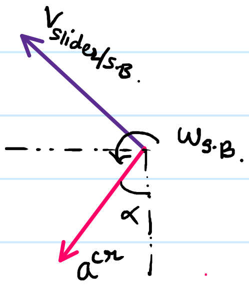


$O_2A = 4 \text{ rad/s}$.

$V_{slider} = ?$ $\omega_{S.B.} = ?$

$\dfrac{\omega_{cutting}}{\omega_{Return}} = ?$ coriolis acc'n. $a^{cr} = ?$





$$\tan\alpha = \frac{40}{30}$$

$$\alpha = 53.13^\circ$$

$$V_{slider/s.B} = V_{crank} \cdot \sin\alpha.$$

$$= 30 \times 4 \sin 53.13 = 120 \times 0.8 = 96 \text{ cm/s}.$$

V_{slider} – $V_{s.B/O_1} = V_{crank} \cdot \cos\alpha$

$$= 30 \times 4 \times \cos 53.13 = 120 \times 0.6 = 72 \text{ cm/s}.$$

$V_{s.B.}$ – $V_{s.B/O_1} = 72 = O_1A \cdot \times \cdot \omega_{s.B.} \Rightarrow \omega_{s.B.} = 72/50 = 1.44 \text{ rad/s}.$

Ratio of cutting velocity to Return Velocity.

$$\frac{(\omega_{max})_{cutting}}{(\omega_{max})_{Return}} = \frac{c-r}{c+r} = \frac{40-30}{40+30} = \frac{1}{7}.$$

Coriolis acceleration

$$a^{cr} = 2 \cdot \times V_{slider/s.B.} \times \omega_{s.B.}$$

$$= 2 \times 96 \times 1.44$$

$$= 276.48\,(-\hat{i}\sin 53.13 - \hat{j}\cos 53.13)$$

$$= -276.48 \times 0.8\,\hat{i} - 276.48 \times 0.6\,\hat{j} \quad \text{cm/s.}$$

$$= -221.184\,\hat{i} - 168.44\,\hat{j} \quad \text{cm/s.}$$

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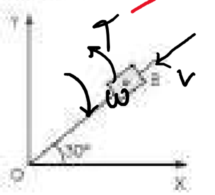
In the mechanism shown at the instant OB = 40 cm and the block B slides inward with a velocity of 0.2 m/sec and decelerates at the rate of 0.5 m/sec². The link rotates at 1 rad/sec in CW direction and decelerates at the rate of 0.5 rad/sec². Determine



$OB = 40\,cm.$

$V_{slider} = 0.2\,m/s.$

$a_{slider} = -0.5\,m/s^2$

$\omega_{OB} = 1\,rad/s.$

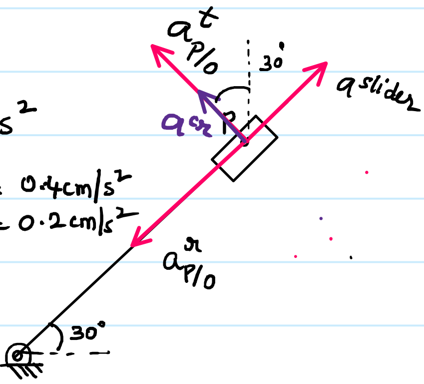
$\alpha_{OB} = 0.5\,rad/s^2$

$a^r = OP.\omega^2 = 0.4\times1^2 = 0.4\,cm/s^2$

$a^t = OP.\alpha = 0.4\times0.5 = 0.2\,cm/s^2$

$a^{slider} = 0.5\,m/s^2$

$a^{cr} = 2.V.\omega = 2\times0.2\times1 = 0.4\,m/s^2$



$\vec{a}_{Resultant} = \vec{a}^r + \vec{a}^t + \vec{a}^{cr} + \vec{a}^{slider}$

$\vec{a}_{Resultant} = 0.4(-i\cos30^\circ - j\sin30) + 0.2(-i\sin30^\circ + j\cos30^\circ) + 0.4(-i\sin30^\circ + j\cos30) + 0.5(i\cos30 + j\sin30)$

$a_{Resultant} = (-0.4\cos30 - 0.2\sin30 - 0.4\sin30^\circ + 0.5\cos30)\hat{i} + \hat{j}(-0.4\sin30^\circ + 0.2\cos30^\circ + 0.4\cos30^\circ + 0.5\sin30)$

$a_{Resultant} = 0.608\,m/s^2$

Direction $\alpha = \tan^{-1}\left(\frac{j-coff}{i-coff}\right) = 110^\circ$

$\alpha = 180 - \tan^{-1}\left(\frac{j-coff}{i-coff}\right)$ → −ve

31. The resultant acceleration of the block centre B with respect to the centre of rotation of the link 'O'.

(a) 0.608 m/sec² (b) 1.64 m/sec²
(c) 0.88 m/sec² (d) 0.94 m/sec²

32. The direction of the above acceleration with respect to OX

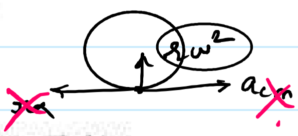
(a) 63.7° (b) 110.53°
(c) 93.7° (d) 176.3°

Pure Rolling

33. A solid disc of radius r rolls without slipping on a horizontal floor with angular velocity ω and angular acceleration α. The magnitude of the acceleration of the point of contact on the disc is (GATE-12)

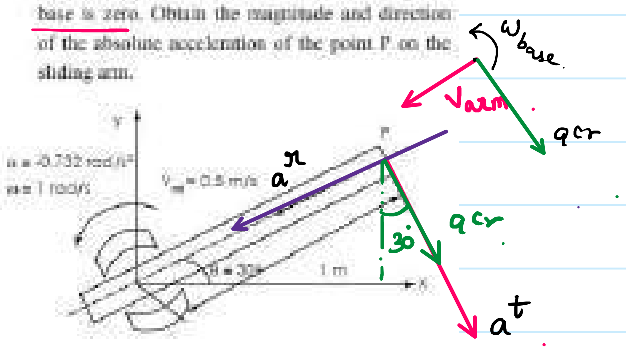
(a) zero (b) rα
(c) $\sqrt{(r\alpha)^2+(r\omega^2)^2}$ (d) $r\omega^2$

$a_{cm} = r\alpha$.



38. Fig. shows a 2 degree of freedom manipulator consisting of a rotary base and a sliding arm which slides radially with respect to the base. The instantaneous position, angular velocity, angular acceleration and relative velocity of arm with respect to the base are as shown in fig. The radial relative acceleration of the arm with respect to the base is zero. Obtain the magnitude and direction of the absolute acceleration of the point P on the sliding arm.

$a^{slider} = 0$



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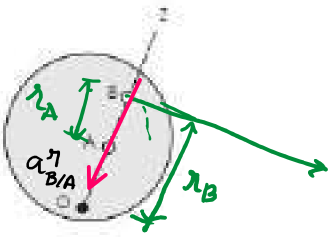
Common Data for Question 34 & 35 :

The circular disc shown in its plan view in the figure rotates in a plane parallel to the horizontal plane about the point O at a uniform angular velocity ω. Two other points A and B are located on the line OZ at distances r_A and r_B from O respectively. (GATE-03)



V_B. $\perp^{lar}$ to line joining AB.

$$V_{B/A} = V_{B/O} - V_{A/O}$$
$$= (OB\cdot\omega - OA\,\omega)$$
$$= (OB - OA)\cdot\omega$$
$$= (r_B - r_A)\cdot\omega$$

34. The velocity of point B with respect to point A is a vector of magnitude

(a) 0 ✗

(b) $\omega(r_B - r_A)$ and direction opposite to the direction of motion of point B ✗

(c) $\omega(r_B - r_A)$ and direction same as the direction of motion of point B ✓

(d) $\omega(r_B - r_A)$ and direction being from O to Z

35. The acceleration of point B with respect to point A is a vector of magnitude

(a) 0

(b) $\omega(r_B^2 - r_A^2)$ and direction same as the direction of motion of point B

(c) $\omega^2(r_B - r_A)$ and direction opposite to the direction of motion of point B ✗

(d) $\omega^2(r_B - r_A)$ and direction being from Z to 'O' ✓

$$a^r_{B/A} = a^r_{B/O} - a^r_{A/O}$$
$$= (OB\cdot\omega^2 - OA\cdot\omega^2)$$
$$= (r_B - r_A)\,\omega^2$$

39. In a reciprocating engine mechanism, the crank and connecting rod of same length r meters. At a given instant, when the crank makes an angle of 45° with TDC and the crank rotates with a uniform velocity of ω rad/s, the angular acceleration of the connecting rod will be

(a) $2\omega^2 r$ (b) $\omega^2 r$

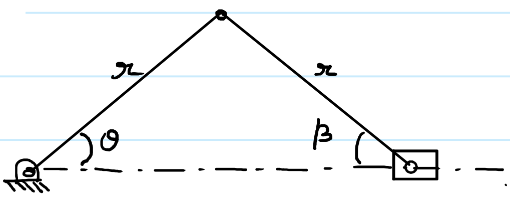
(c) ω^2/r (d) zero

$\theta = 45^\circ$

ω = constant

$\alpha_{conn.Rod} = ?$

$r = l$.



$$r\sin\theta = l.\sin\beta.$$
$$\sin\theta = \sin\beta.$$
$$\cos\theta\cdot\frac{d\theta}{dt} = \cos\beta\cdot d\beta/dt$$
$$\cos\theta\cdot\omega_{crank.} = \cos\beta\cdot\omega_{conn.Rod.}$$
$$\cos 45^\circ \times \omega = \cos 45^\circ\ \omega_{conn.Rod.}$$

@ $\theta = 45^\circ$ $\boxed{\omega_{crank.} = \omega_{conn.Rod.}}$

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$$(-\sin\theta)\cdot\frac{d\theta}{dt}\cdot\omega_{crank} = -\sin\beta\cdot\frac{d\beta}{dt}\cdot\omega_{C.R.} + \cos\beta\cdot\frac{d^2\beta}{dt^2}$$

$$-\sin\theta\cdot\omega^2_{crank} = -\sin\beta\cdot\omega^2_{C.R} + \cos\beta\cdot\alpha_{C.R}$$

$$-\sin 45^\circ\cdot\cancel{\omega^2_{crank}} = -\sin 45^\circ\times\cancel{\omega^2_{C.R}} + \cos 45^\circ\,\alpha_{C.R}$$

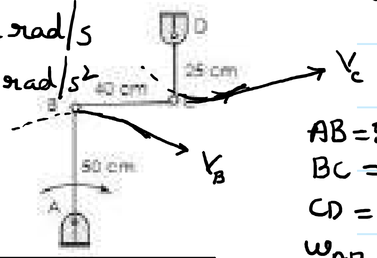
$$\alpha_{C.R.} = 0$$

36. A four bar mechanism along with the dimensions is shown in figure. The links AB and CD are vertical while the link BC is horizontal in the given configuration. The input link has an instantaneous angular velocity of 0.2 rad/sec and acceleration of 0.1 rad/sec². The angular velocity and acceleration of the output link DC are

$\alpha_{DC} = ?$

$\omega_{AB} = 0.2\ rad/s$

$\alpha_{AB} = 0.1\ rad/s^2$



$AB = 50cm$

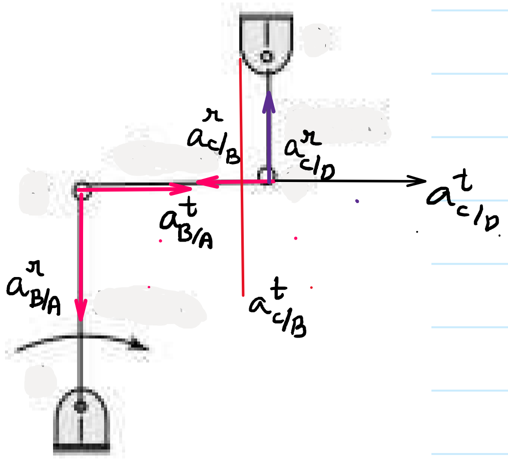
$BC = 40cm$

$CD = 25cm$

$\omega_{AB} = 0.2\ rad/s$

$\omega_{CD} = 0.4\ rad/s.$

$\alpha_{AB} = 0.1\ rad/s^2$



$\boxed{\text{(a) 0.4 rad/sec and 0.2 rad/sec}^2}$

(b) 0.2 rad/sec and 0.2 rad/sec²

(c) 0.1 rad/sec and 0.4 rad/sec²

(d) 0.4 rad/sec and 0.1 rad/sec²

Velocity Analysis.

AB is $\parallel^{le}$ to CD

$$V_{B/A} = V_{c/D} \qquad \boxed{\omega_{BC} = 0}$$

$$AB\cdot\omega_{AB} = CD\cdot\omega_{CD}.$$

$$50\times 0.2 = 25\times\omega_{CD}$$

$$\omega_{CD} = 0.4\ rad/s.$$

$$\vec{a}_c = \vec{a}_B + \vec{a}_{c/B}$$

$$\vec{a}^r_{c/D} + \vec{a}^t_{c/D} = \vec{a}^r_{B/A} + \vec{a}^t_{B/A} + \vec{a}^r_{c/B} + \vec{a}^t_{c/B}$$

$$CD\cdot\omega^2_{CD}(+\hat{j}) + CD\cdot\alpha_{CD}(\hat{i}) = AB\cdot\omega^2_{AB}(-\hat{j}) + AB\,\alpha_{AB}(\hat{i}) + CB\cdot\cancel{\omega^2_{CB}}(-\hat{i}) + CB\,\alpha_{CB}(\hat{j})$$

$$25(0.4)^2(\hat{j}) + 25\times\alpha_{CD}(\hat{i}) = 50\times 0.2^2(-\hat{j}) + 50\times 0.1(\hat{i}) + 0 + 40\,\alpha_{CB}(\hat{j})$$

$$4\hat{j} + 25\,\alpha_{CD}(\hat{i}) = -2\hat{j} + 5\hat{i} + 40\,\alpha_{CB}(\hat{j})$$

$\hat{j}$-coff $\quad 4 = -2 + 40\,\alpha_{CB} \Rightarrow \alpha_{CB} = 6/40 = 0.15\ rad/s^2$

$\hat{i}$-coff $\quad 25\,\alpha_{CD} = 5 \Rightarrow \alpha_{CD} = 0.2\ rad/s^2$

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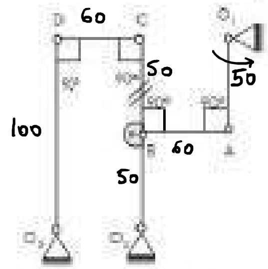
37. In the linkage shown in figure, $O_1A = 50$ mm, $AB = 60$ mm, $O_2B = 50$ mm, $BC = 50$ mm, $CD = 60$ mm, $O_3D = 100$ mm. In the position shown in the figure, if O_1A has momentarily an angular velocity of 2 rad/s without any angular acceleration, then the velocity and acceleration of D will be:

(GATE-04)



(a) 100 mm/s, 100 mm/s² (b) 200 mm/s, 100 mm/s²
(c) 200 mm/s, 300 mm/s² (d) 200 mm/s, 400 mm/s²

$\omega_{O_1A} = 2$ rad/s.

$\alpha_{O_1A} = 0$ rad/s²

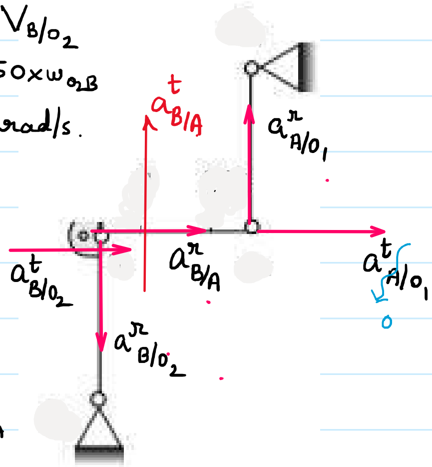
$V_D,\ a_D = ?$

O_1A and O_2B are $\parallel^{le}$ $\Rightarrow$ $\boxed{\omega_{AB} = 0}$

$V_{A/O_1} = V_{B/O_2}$

$50 \times 2 = 50 \times \omega_{O_2B}$

$\omega_{O_2B} = 2$ rad/s.



$\vec{a}_B = \vec{a}_A + \vec{a}_{B/A}$

$\vec{a}^{\,r}_{B/O_2} + \vec{a}^{\,t}_{B/O_2} = \vec{a}^{\,r}_{A/O_1} + \underbrace{\vec{a}^{\,t}_{A/O_1}}_{0} + \underbrace{\vec{a}^{\,r}_{B/A}}_{0} + \vec{a}^{\,t}_{B/A}$

$O_2B \cdot \omega^2_{O_2B}(-\hat{j}) + O_2B \cdot \alpha_{O_2B} \cdot (\hat{i}) = O_1A \cdot \omega^2_{O_1A}(\hat{j}) + \underbrace{O_1A \cdot \alpha_{O_1A}}_{0}(\hat{i}) + \underbrace{AB \cdot \omega^2_{AB}}_{0}(-\hat{i}) + AB \cdot \alpha_{AB}(\hat{j})$

$50 \times 2^2(-\hat{j}) + 50\alpha_{O_2B} \cdot (\hat{i}) = 50 \times 2^2(\hat{j}) + 0 + 0 + 60\alpha_{AB}$

j-coff $\quad -200 = 200 + 60\alpha_{AB} \Rightarrow \alpha_{AB} = -400/60$ rad/s²

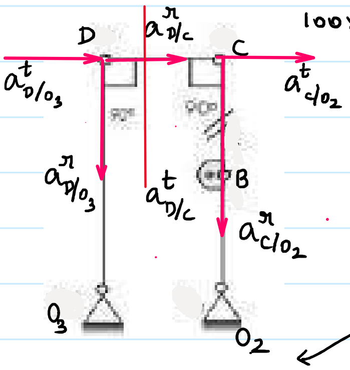
i-coff $\quad 50\alpha_{O_2B} = 0 \Rightarrow \boxed{\alpha_{O_2B} = 0}$

$\omega_{O_2C} = \omega_{O_2D} = 2$ rad/s. $\quad V_{C/O_2} = V_{D/O_3} \Rightarrow \omega_{CD} = 0,\ \alpha_{O_2C} = 0$

O_2C is $\parallel^{le}$ to O_3D $\quad O_2C \times \omega_{O_2C} = O_3D \times \omega_{O_3D}$

$100 \times 2 = 100 \times \omega_{O_3D}$

$\omega_{O_3D} = 2$ rad/s.



$\vec{a}_D = \vec{a}_C + \vec{a}_{D/C}$

$\vec{a}^{\,r}_{D/O_3} + \vec{a}^{\,t}_{D/O_3} = \vec{a}^{\,r}_{C/O_2} + \vec{a}^{\,t}_{C/O_2} + \vec{a}^{\,r}_{D/C} + \vec{a}^{\,t}_{D/C}$

$O_3D \cdot \omega^2_{O_3D} \cdot (-\hat{j}) + O_3D \cdot \alpha_{O_3D}(\hat{i}) = O_2C \cdot \omega^2_{O_2C}(-\hat{j}) + \underbrace{O_2C\,\alpha_{O_2C}}_{0}(\hat{i}) + \underbrace{DC \cdot \omega^2_{CD}}_{0}(+\hat{i}) + DC\,\alpha_{CD}(\hat{j})$

$100 \times 2^2(-\hat{j}) + 100\alpha_{O_3D}(\hat{i}) = 100 \times 2^2(-\hat{j}) + 0 + 0 + 60\alpha_{CD}(\hat{j})$

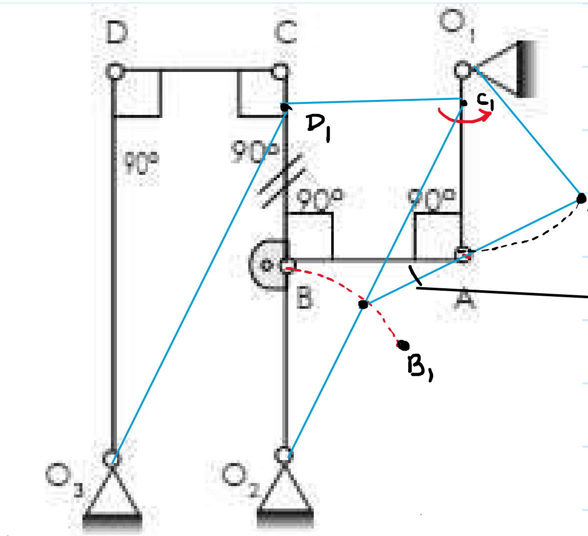
j-coff $\quad -400 = -400 + 60\alpha_{CD} \Rightarrow \alpha_{CD} = 0$

i-coff $\quad 100\alpha_{O_3D} = 0 \Rightarrow \alpha_{O_3D} = 0$.

CD and C_1D_1 are $\parallel^{le}$.

$\omega_{CD} = 0$

$\alpha_{CD} = 0$



$\omega_{AB} = 0$

$\alpha_{AB} = -400/60$

Link AB having angular retardation.

Link CD is in Translation throughout the transmission of motion.

$a_D = ?$

$V_D = ?$

$V_{D/O_3} = O_3D \cdot \omega_{O_3D} = 100 \times 2 = 200\text{mm/s}$.

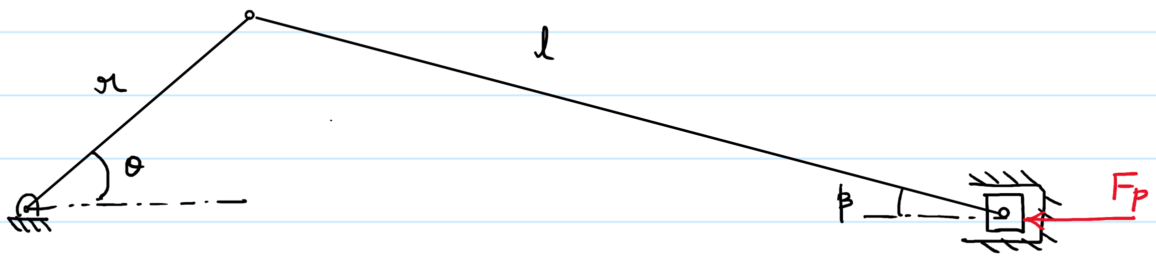
$a^{r}_{D/O_3} = O_3D \cdot \omega^2_{O_3D} = 100 \times 2^2 = 400\text{mm/s}$.

$\alpha_{O_3D} = 0$.

Dynamic Analysis of Crank Slider Mechanism

Assumptions

1. Mass of Crank and Connecting Rod is neglected.
2. Crank is rotating with constant angular velocity.



F_p – Piston Effort.

Piston Effort:

$$F_p = F_G \pm F_I \pm F_W \pm f$$

F_G - Gas force.

$$F_G = p_{max} \cdot \frac{\pi}{4} D^2 - p_{min} \cdot \frac{\pi}{4} \cdot (D^2 - d^2)$$

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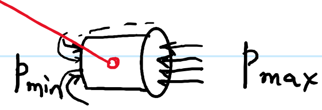


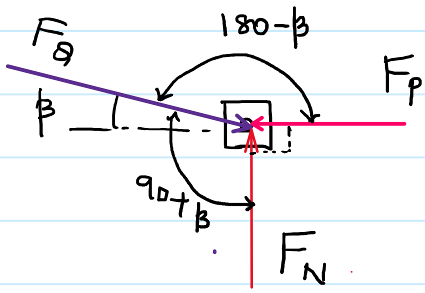
D - Dia of piston , d - dia of connecting

F_I - Inertia force $\quad F_I = m_{piston} \cdot a_{piston}$

$$a_p = r\omega^2 \left[\cos\theta + \frac{\cos 2\theta}{n}\right]$$

F_W - Weight of piston - (Considered only for vertical cylinder engine)

f - frictional force



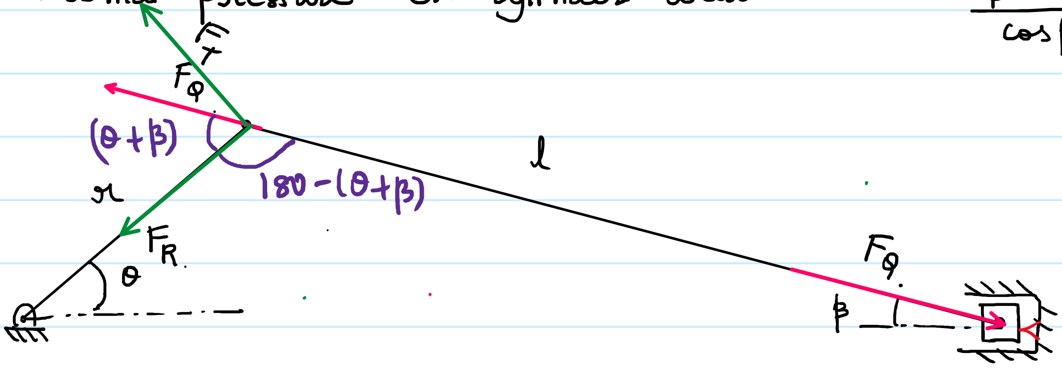
$$\frac{F_P}{\sin(90+\beta)} = \frac{F_Q}{\sin 90^\circ} = \frac{F_N}{\sin(180-\beta)}$$

$$F_Q = \frac{F_P}{\cos\beta}$$

F_Q - Thrust on connecting Rod.

F_N - Normal pressure on cylinder wall.

$$F_N = \frac{F_P \cdot \sin\beta}{\cos\beta} = F_P \tan\beta$$



F_T - Tangential force / Crank Effort.

F_R - Radial load on crank Bearings.

$$F_T = F_Q \cdot \sin(\theta+\beta)$$

$$F_T = \frac{F_P}{\cos\beta} \sin(\theta+\beta)$$

$$F_R = F_Q \cdot \cos(\theta+\beta) = \frac{F_P}{\cos\beta} \cos(\theta+\beta)$$

$$\text{Torque on crank shaft} = F_T \times r = \frac{F_P}{\cos\beta} \sin(\theta+\beta) \times r.$$

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Dynamic Analysis

Statement for Common data Q.40 & Q. 41

In a single acting steam engine when the piston is at the middle of the expansion stroke the net gas force on the Piston is 2 kN. The crank length is 20 cm and the connecting rod length is 80 cm

→ Piston effort

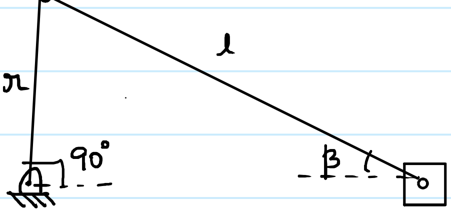
40. The thrust in the connecting rod is
    (a) 2 kN (b) 2.065 kN
    (c) 1.935 kN (d) 1.5 kN

41. The turning moment is
    (a) 0.4 kN-m (b) 4 kN-m
    (c) 0.3 kN-m (d) None

$F_p = 2$ KN.

$r = 20$cm. $l = 80$cm.

$\theta = 90°$



$r\sin\theta = l\sin\beta$

$20 \sin 90° = 80 \sin\beta$

$\beta = \sin^{-1}\left(\frac{20}{80}\right)$

$= 14.47°$

Thrust in connecting Rod.

$F_Q = \frac{F_p}{\cos\beta} = \frac{2}{\cos 14.47°} = 2.06$ KN.

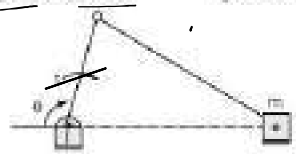
Turning Moment.

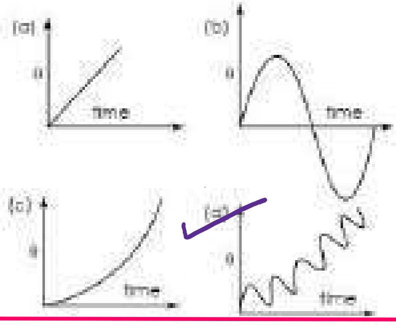
$T = \frac{F_p}{\cos\beta} \cdot \sin(\theta+\beta) \times r$

$T = \frac{2}{\cos 14.47} \sin(90 + 14.47) \times 0.2$

$T = 0.4$ KN-m.

45. Consider a slider crank mechanism with nonzero masses and inertia. A constant torque τ is applied on the crank as shown in the figure. Which of the following plots best resembles variation of crank angle, θ versus time (GATE –15 –Set I)





I - inertia of crank and connecting Rod.

$$\tau = I\alpha \pm F_p \times r$$

$$F_I = m r \omega^2 \left[\cos\theta + \frac{\cos 2\theta}{n}\right]$$

$$\tau = I\alpha \pm F_I \times r$$

$$\alpha = \frac{\tau \pm F_I \times r}{I}$$

$f(\theta)$

$$\alpha = \frac{d^2\theta}{dt^2}$$

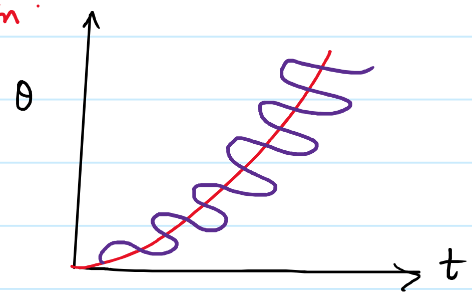
τ = constant.

I = constant.

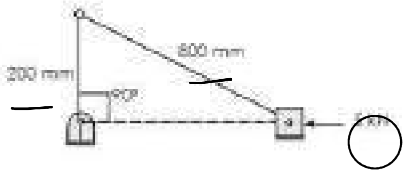
$$\theta = \frac{\tau}{I} t^2 \pm \iint \frac{F_I \times r}{I}$$

Parabolic variation.

cyclic / harmonic.



46. A slider crank mechanism with crank radius 200 mm and connecting rod length 800 mm is shown. The crank is rotating at 600 rpm in the counterclockwise direction. In the configuration shown, the crank makes an angle of 90° with the sliding direction of the slider, and a force of 5 kN is acting on the slider. Neglecting the inertia forces, the turning moment on the crank (in kN-m) is_______ (GATE –16)



$F_p = 5$ kN. , $\theta = 90°$

$$T = \frac{F_p}{\cos\beta} \sin(\theta+\beta) \times r$$

$$T = \frac{F_p \cdot \sin(90+\beta)}{\cos\beta} \times r = F_p \times r$$

$$T = 5 \times 0.2 = 1\ \text{kN-m}$$

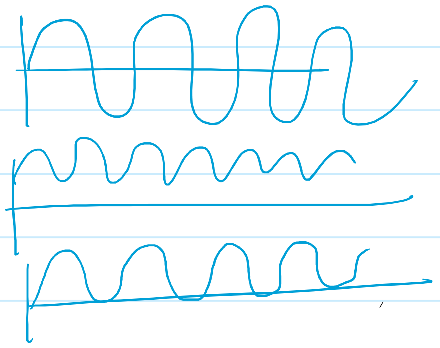
Vibrations
↓
Pythagorus.
↳ sound waves have repeated pattern.

Repeated pattern → oscillations.

Oscillations produced under the influence of Mechanical energy are called Mechanical Vibrations.

Mechanical Energy = (P.E) + (K.E)

Vibrations → caused due to presence of unbalanced element.
↓
Dynamic Loads / Unsteady loads / Transient loading.



↓
Variable stresses are setup in the m/c component.
↓
Design criteria → Endurance strength
Fatigue strength.

Theories of failure.

$\underbrace{\text{static}}$ Load = const. $\dfrac{d(\text{Load})}{dt} = 0$

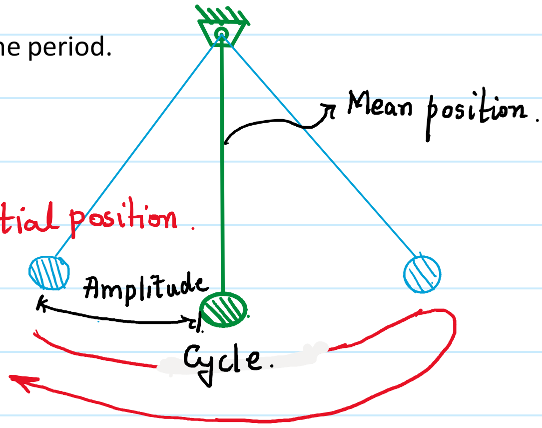
↳ Yield strength
ultimate strength.

Vibration - A motion which repeats after a fixed interval of time period is called as **vibratory motion.**

Cycle - Motion of body to reach the same starting point when is displaced is called as cycle.

Time Period - The time required to complete one cycle is called as time period.

Frequency/Speed - Number of Cycles completed per unit time.



circular frequency

ω — rad/sec.

linear frequency f - cycle/sec./Hz.

$$\omega = 2\pi f$$

$$f = \frac{1}{T} \qquad T\text{ - Time period}$$

$$\boxed{\omega = \frac{2\pi}{T}}$$

Amplitude - Maximum Deviation from the mean position is called Amplitude.

Mean Position - About this position the body oscillates and at mean position the system will be in equilibrium.

Vibratory motion will never start on its own, for the vibratory to be started some disturbance/external source is required.

For the vibratory motion to occur the system must have
1. Elasticity/Stiffness
2. Inertia.

Vibrations will continue until the imparted energy is fully lost.

If the system continues to oscillate without loss in energy, then system is executing undamped vibrations.**(Conservation of Energy is valid)**.

If there is loss in energy while executing oscillations then the damping is present.

Damping represent the loss in energy.

Classification of Vibration on the basis of excitation.

1. Free Vibrations - System continues to oscillate under the influence of initial condition (disturbance/input/Displacement/Velocity).
2. Forced Vibration - System continues to oscillate under the influence of some external source.

Free Vibration
- Free undamped Vibration – under the influence of intial condition only. No loss in energy after each cycle. (conservation of Energy is valid)
- Free damped Vibrations – under the influence of intial condition only. Loss in energy occurs after each cycle.

Free undamped Vibrations – No. loss in energy / System will never return to rest; Hypothetical situation.

Free damped Vibration – Loss in energy after each cycle. System will come to rest finally.

Types of External Excitation/Forced Vibration

1.Excitation due to Unbalance masses
  a. Unbalance Rotating Mass
  b. Unbalance Reciprocating Mass

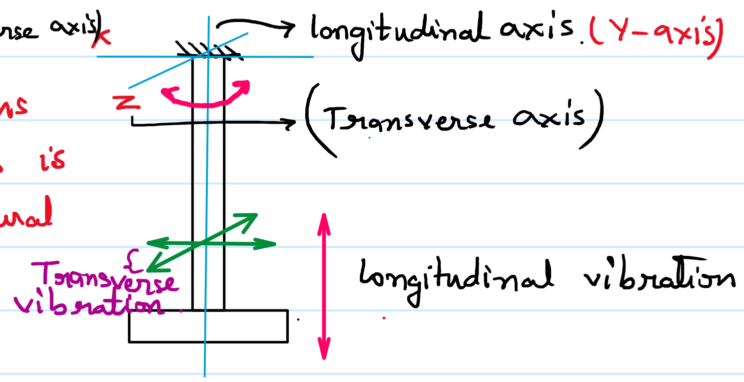
2. Base Excitation.

Classification of Vibrations on the basis of direction.

1.Longitudinal Vibrations
2.Transverse Vibrations
3.Torsional Vibrations

Longitudinal Vibrations - If the particles tend to move along the longitudinal axis thereby causing the elongation and shortening of the structural member , these oscillations are called Longitudinal Vibrations

Due to these vibrations variable axial stress is setup in the structural member (shaft).



Transverse Vibrations - If the particles tend to move laterally along the Transverse axis thereby causing the bending of the structural member , these oscillations are called Transverse Vibrations.

Due to these the structural member will be subjected to variable bending stress.

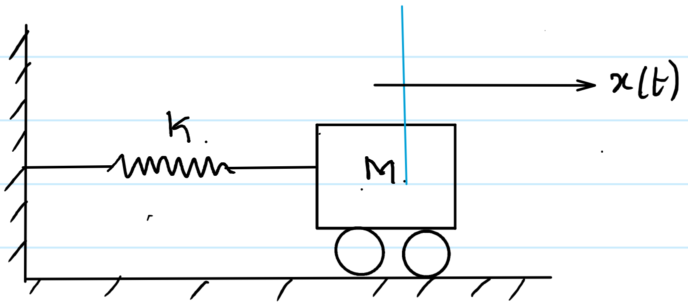
Torsional Vibrations - If the particles tend to move about the longitudinal axis thereby causing the twisting and untwisting of the structural member , these oscillations are called Torsional Vibrations.
The structural member is subjected to variable torsional shear stress.

Classification of Vibrations on the basis of DOF.

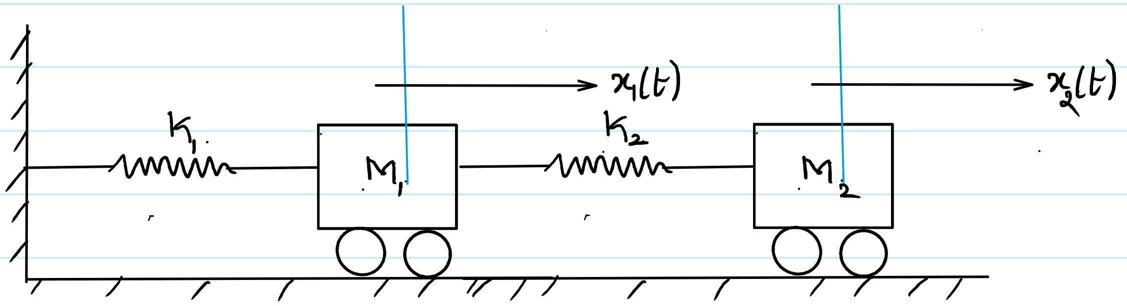
No. of DOF = No. of equation of motion.

1. Single DOF system (SDOF) — U.G. level.
2. Multiple DOF system (MDOF) – ✓
3. Infinite DOF system. – [elementary level].

DoF = 1.

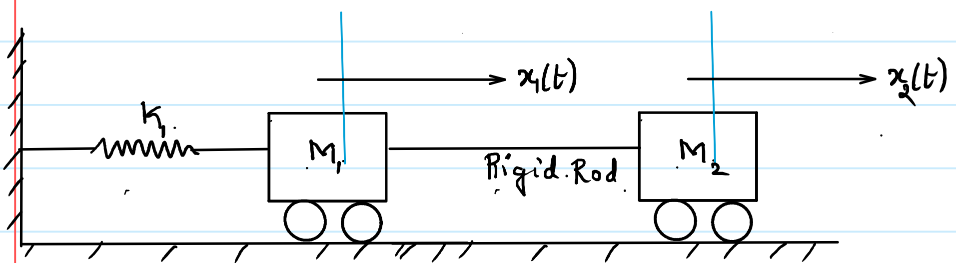


x_1, x_2 – independent variable.



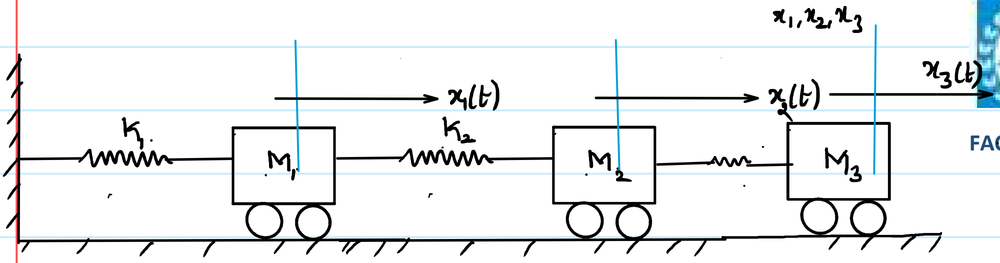
DoF = 2.

x_1, x_2



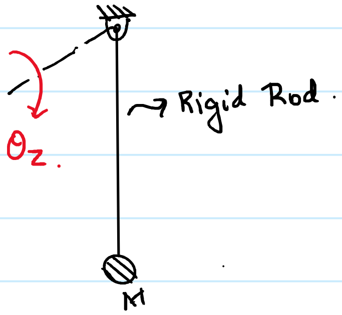
$x_1 = x_2$

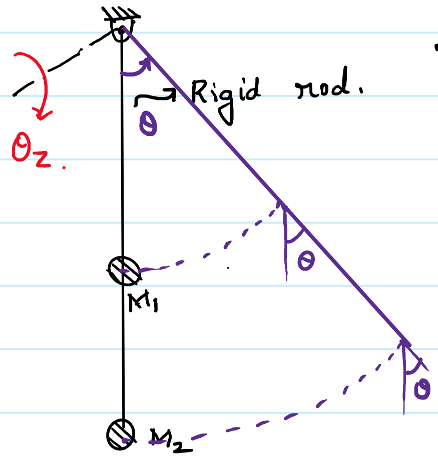
DoF = 1.



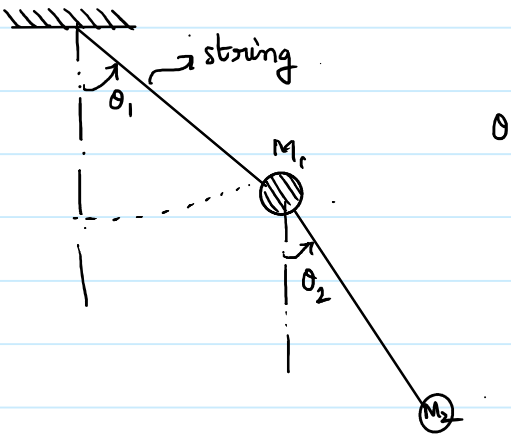
x_1, x_2, x_3 — independent variables

DoF = 3



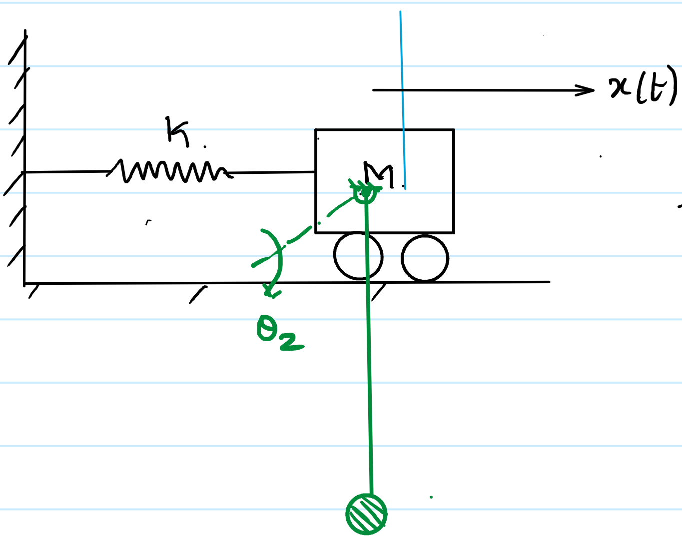


DoF = 1



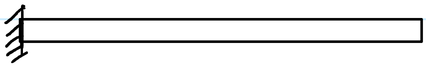
θ_1, θ_2 are independent variable

DoF = 2.



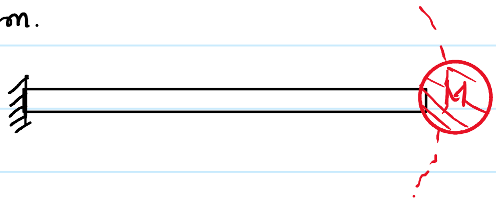
DoF = 2

Rigid Beam.



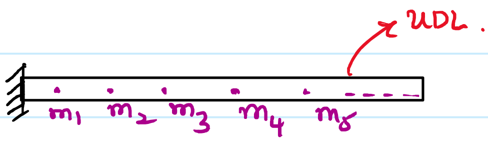
$\boxed{DoF = 0}$

Flexible Massless beam.



DoF = 1.

Flexible beam of mass "m"



DoF $= \infty$

System Parameters

1. Inertia
2. Restoration
3. Damping

Inertia – It is the tendency of a object to be rest. Resistance against the change in state. } Newton 1st law.

Motion
- → Translation – Mass of object is the measure of inertia
- → Rotation/Angular motion – Second moment of mass is the measure of inertia.

Mass Moment of Inertia $= \int r^2 \cdot dm.$

r → ⊥ lar distance from axis of rotation.

$I = \bar{I} + md^2$

$I = mk^2$

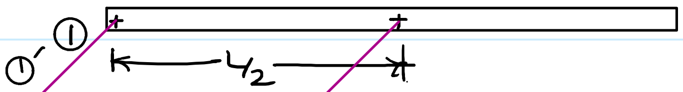
$\bar{I}$ – Moment of Inertia about the centriodal axis.

m – mass.

d – ⊥ lar distance b/w. (C.A) and other axis.

K – Radius of gyration.

Slender Rod.

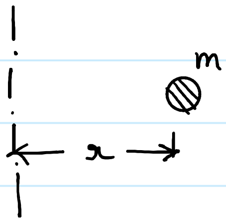


$I = \frac{ML^2}{3}$ $\qquad$ $\bar{I} = \frac{ML^2}{12}$

$$I = \bar{I} + Md^2$$

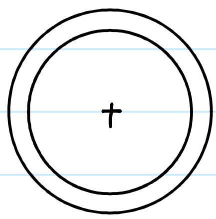
$$I_{1-1} = \frac{ML^2}{12} + M\cdot\left(\frac{L}{2}\right)^2 = \frac{ML^2}{3}.$$

Point Mass.



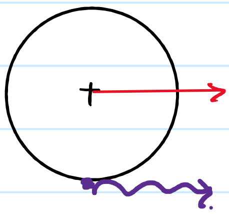
$I = mr^2$

Rim.



$I = mR^2$

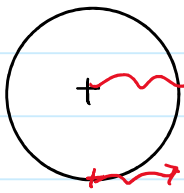
Cylinderical Disc.



$\frac{MR^2}{2}$

$\frac{MR^2}{2} + MR^2 = \frac{3MR^2}{2}.$

Spherical Roller

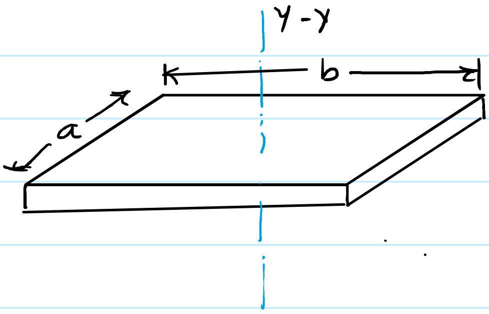


$\frac{2}{5}mR^2$

$\frac{2}{5}mR^2 + mR^2$

$= \frac{7}{5}mR^2.$

Rectangular plate.

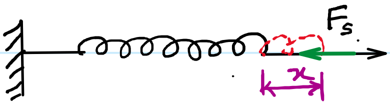


$$\bar{I}_{y-y} = \frac{1}{12}\cdot(a^2+b^2)$$

2. Restoration – It is the ability of system to return back to the initial position.

1. Springs.
2. Gravity
3. Fluid.

Hooke's law of spring.



F – Applied force / External force

F_s – Spring force / Internal force.

$F \propto x$.

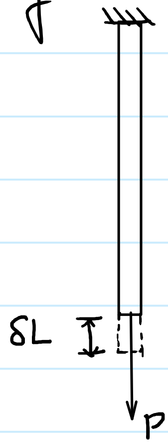
$\boxed{F_s \propto -x}$

$F_s = -K \cdot x$

↳ spring constant.

F_s will be resisting the displacement or elongation.

Axial loading



elongation $\delta L = \frac{PL}{AE}$

stiffness. $- \frac{\text{load}}{\text{deflection}} = \frac{P}{\delta L} = \frac{AE}{L}$

axial stiffness.

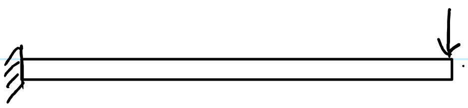
stiffness – Rigidity per unit lenght.

AE – Axial Rigidity. A – c/s area. E – Young Modulus.

Rigidity = f (c/s area, Material constant)

Rigidity = f (c/s property, Material property)

Bending.



$\text{stiffness} = \frac{\text{load}}{\text{Deflection}} = \frac{W}{\frac{WL^3}{EI \times \text{const}}} = \frac{EI}{L^3} \times \text{constant}$

EI – flexural Rigidity. Rigidity = f (c/s. property, Material property)

I – Area moment of inertia.

Twisting / Torsion.

Torsional stiffness.

$$\frac{T}{J} = \frac{\tau}{R} = \frac{G\theta}{L} \quad \Rightarrow \quad \frac{T}{\theta} = \frac{GJ}{L}$$

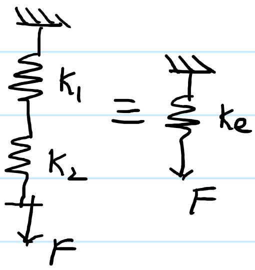
GJ – Torsional Rigidity.

J – Area Polar moment of inertia.

Spring in series and Parallel combinations.

$F_{s_1} = F_{s_2} = F$.

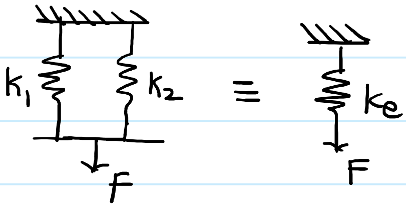
$F = k_1\delta_1 = k_2\delta_2 = k_e\delta_e$



$\delta_e = \delta_1 + \delta_2$

$$\frac{F}{k_e} = \frac{F}{k_1} + \frac{F}{k_2}$$

$$k_e = \frac{k_1 k_2}{k_1 + k_2}$$

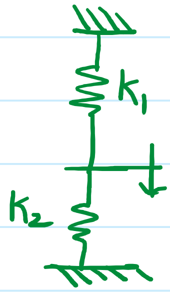


$\delta_1 = \delta_2 = \delta_e$

$F_{s_1} + F_{s_2} = F$

$k_1\delta_1 + k_2\delta_2 = k_e \cdot \delta_e$

$k_e = k_1 + k_2$

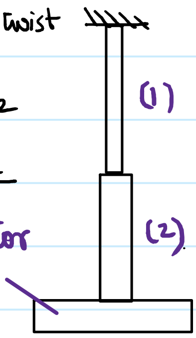


Shafts in series.

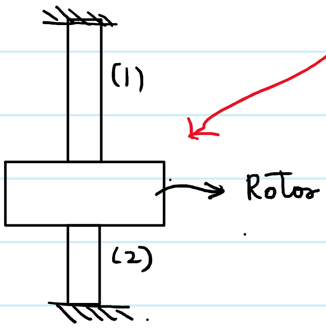
θ – Angle of twist

$\theta_e = \theta_1 + \theta_2$

$$k_e = \frac{k_{T_1} \cdot k_{T_2}}{k_{T_1} + k_{T_2}}$$



Shafts in parallel.



$\theta_e = \theta_1 = \theta_2$

$k_{T_e} = k_{T_1} + k_{T_2}$

K_T – Torsional stiffness.

$$k_T = \frac{T}{\theta} = \frac{GJ}{L}$$

$$K_{axial} = \frac{AE}{L}$$

$$K_{bending} = \frac{EI}{L^3}$$

$$K_{Torsional} = \frac{GJ}{L}$$

$$\boxed{\text{spring stiffness} \quad K = \frac{Gd^4}{8D^3 n}}$$

$$K \propto \frac{1}{\text{Lenght of member}}$$

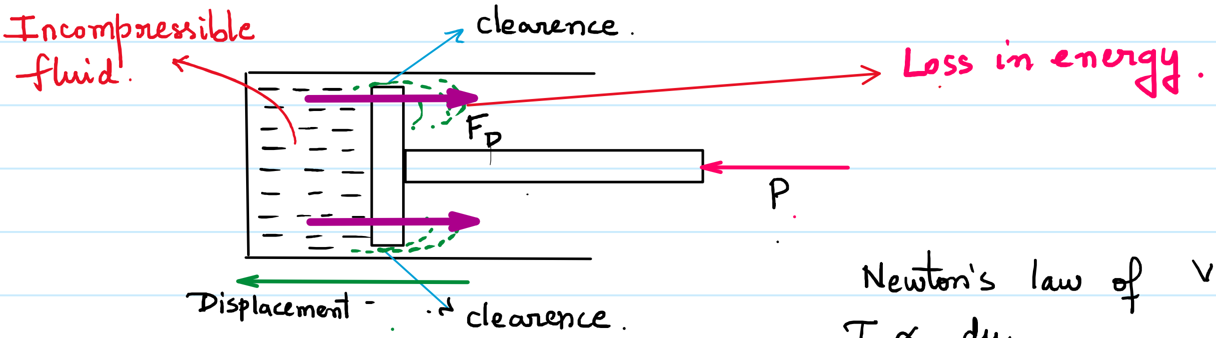
$$K \propto \frac{1}{\text{no. of turns}}$$

If the Spring is cut into 'n' no. of pieces then the stiffness of each Spring will be increased by n times.

Damping – Ability of system to resistance the relative motion.

1. Coloumb's Damping / Frictional Damping → frictional force $\propto$ –velocity.
2. Hysteresis Damping.
3. $\boxed{\text{Viscous Damping}}$ ✓

Cylinder – Piston



F_D – Damping force.

Newton's law of Viscosity

$$\tau \propto \frac{du}{dy}$$

$$\tau = \frac{\mu \cdot V}{h}$$

$$P = \tau \times A = \frac{\mu \cdot V}{h} \times A$$

μ – constant
h – constant / clearence
A – Area - constant.

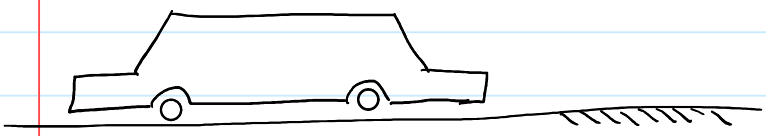
$$F_D = \frac{\mu \cdot V}{h} \times A$$

$F_D \propto$ – velocity

$$F_D = \underbrace{\left(\frac{\mu \cdot A}{h}\right)}_{\text{Damping coefficient}} \cdot V = CV = C\dot{x}$$

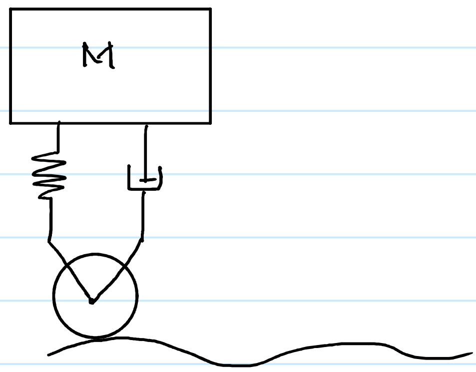
$$\boxed{F_D \propto -\dot{x}}$$

The process of deriving the mathematical equation for a system which is having motion of masses is **mathematical modelling.**



Actual.

Lumped model of car.



Methods for deriving equation of motion

1. D'Alembert's Principle (SDOF)
2. Conservation of Energy (SDOF | MDOF)
3. Rayleigh Method
4. Deflection Method
5. Galerkin Method
6. Lagrangian Method.

D'Alembert Principle - method of converting a **kinematic behaviour to static** by applying a **virtual force in direction opposite to the direction** of motion.

Inertia force is in equilibrium with the net forces acting on the system.

Newton's 2nd law. $\Sigma F = ma$.

Newton's 1st law. Inertia $\propto$ – change in state.

Translation $F_I = -ma$.

Rotation $T_I = -I\alpha$.

D'Alembert Principle

$$\Sigma F - ma = 0$$

$$\Sigma F - (-ma) = 0$$

$$\Sigma F + ma = 0$$

By the reversing the direction of force causing motion and adding with net forces acting on system the dynamic problem is converted into a equivalent static equilibrium problem.

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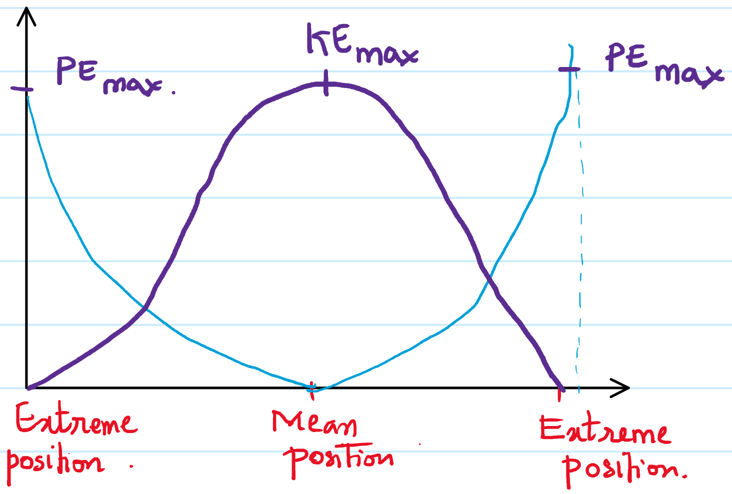
(2) Conservation of Energy.

$$E = P.E + KE = \text{constant}$$

(3) Rayliegh Method.

$$(KE)_{max} = (PE)_{max}$$

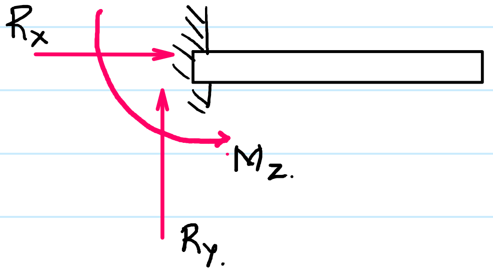


Rules for Drawing FBD

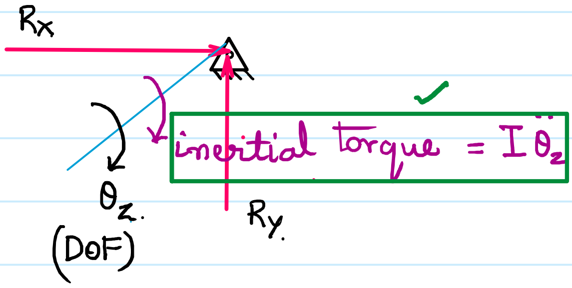
1. Identify the type of system (**SDOF**/MDOF).
2. Identify the displacement variable (x - translation / θ - Rotation(or) Angular Motion)
3. Identify the mean position or equilibrium position.
4. Provide the initial displacement to the body.
5. Isolate the body from the system.
6. Replace the connections with equivalent Reaction forces.
7. Write the equation of motion.

F.B.D.

Fixed end.

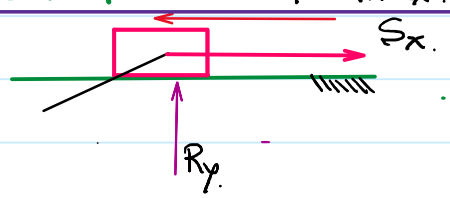


Hinge support.



Block sliding on horizontal surface.

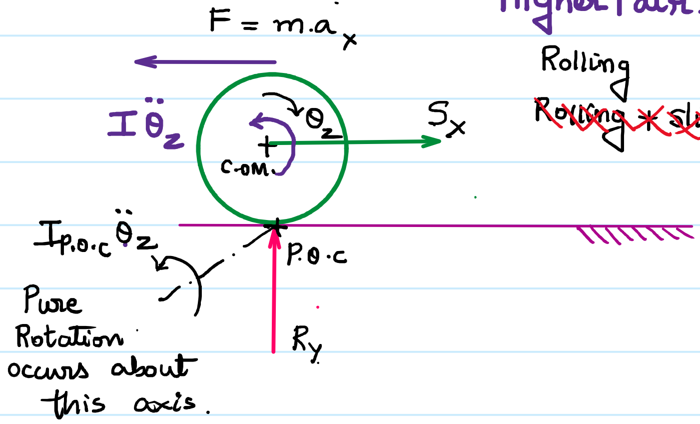
$\boxed{\text{Inertia force.} \quad F = ma_x}$



Higher Pair.

Rolling

~~Rolling + slipping~~



P.O.C – point of contact.

Rolling = Translation + Rotation @ C.O.M.

Rolling = Rotation about P.O.C.

Spring force / Restoring force.

$F_s \propto -x$.

F_s will resist the displacement x

Restoring Torque.

$T_s \propto -\theta$

T_s will resist the displacement θ.

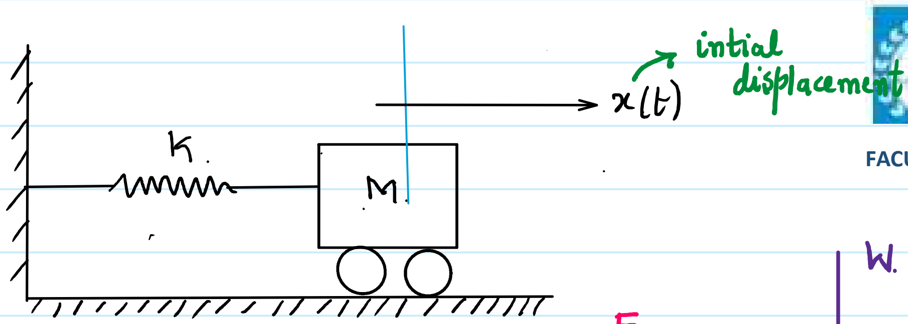
Damping force. – $F_D \propto -\dot{x}$ F_D will resist the relative motion.

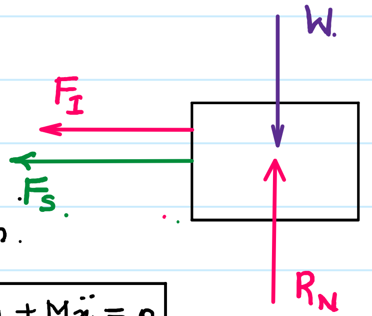
Damping Torque. – $T_D \propto -\dot{\theta}$ T_D will resist the relative motion

Inertia force $F_I \propto -\ddot{x}$ F_I will resist the acceleration.

Inertial torque $T_I \propto -\ddot{\theta}$ T_I will resist the acceleration.

1.





$\ddot{x} = \frac{d^2x}{dt}$

$\underbrace{x, \dot{x}, \ddot{x}}_{\text{variables}}$

$\Sigma F_y = 0 \qquad R_N - W = 0$

$R_N = W \rightsquigarrow$ constant.

In Y-direction it is static equilibrium.

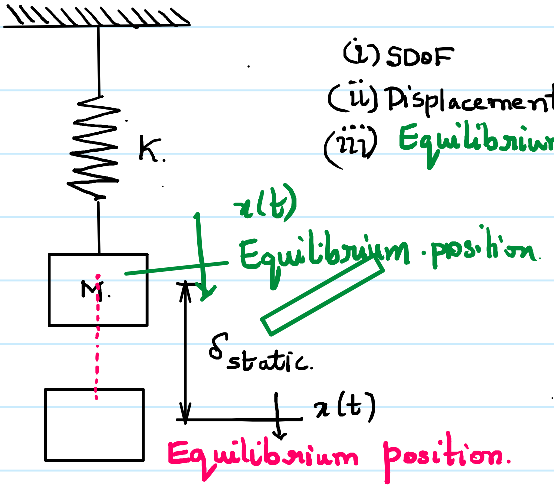
$\Sigma F_x = 0 \qquad F_S + F_I = 0 \rightsquigarrow \boxed{kx + M\ddot{x} = 0}$

$-Kx + (-M\ddot{x}) = 0$

$\boxed{\underbrace{M\ddot{x}}_{\text{Variable}} + \underbrace{kx}_{\text{Variable}} = 0} \rightsquigarrow$ Equation of motion.

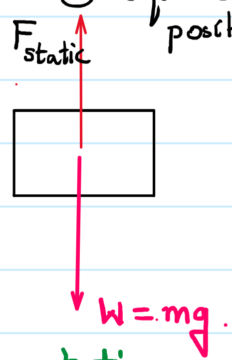
$M\ddot{x}(t) + K.x(t) = 0$

2.



(i) SDoF
(ii) Displacement variable – x.
(iii) Equilibrium posn – identified.

F.B.D. @ Equilibrium position.



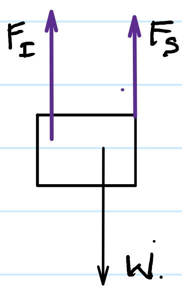
$\Sigma F_y = 0$

$F_{static} = W$

$\boxed{K.\delta_{static} = W.}$

↳ System is in static equilibrium.

F.B.D. after giving intial displacement.



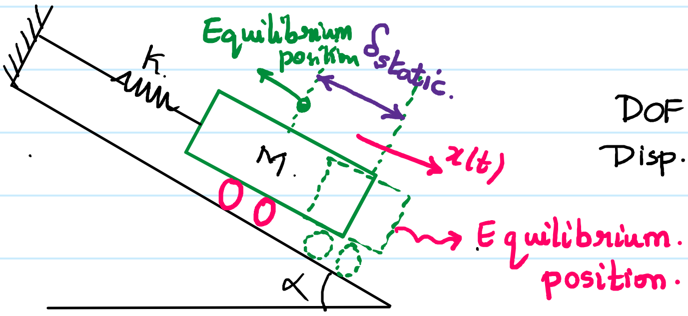
$\Sigma F_y = 0$

$F_I + F_S - W = 0$

$M\ddot{x} + K.(\delta_{static} + x) - W = 0$

$M\ddot{x} + Kx. + \boxed{K\delta_{static} - W} = 0$ → static terms

$M\ddot{x} + kx = 0$



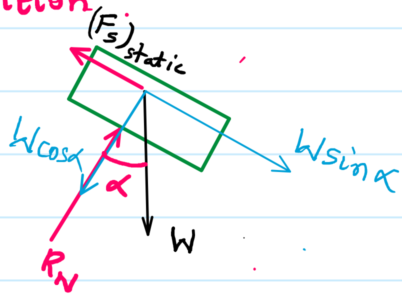
DOF = 1

Disp. variable – x

W – constant

α – constant

@ Equilibrium position



both in x and y-direction it is in static equilibrium.

$$\Sigma F_x = 0$$

$$(F_s)_{static} = W\sin\alpha$$

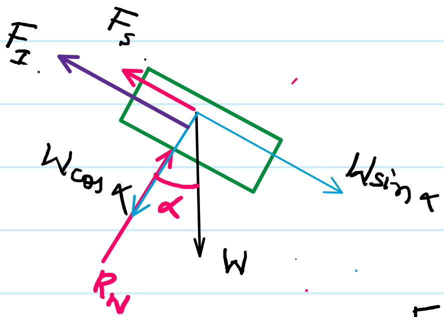
$$K\cdot\delta_{static} = W\sin\alpha$$

$$\Sigma F_y = 0$$

$$R_N - W\cos\alpha = 0$$

$$R_N = W\cos\alpha$$

F.B.D. after giving intial displacement.



$$\Sigma F_y = 0 \Rightarrow R_N = W\cos\alpha.$$

$$\Sigma F_x = 0$$

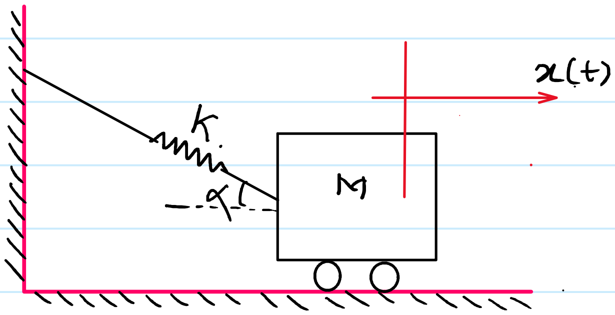
$$F_I + F_s - W\sin\alpha = 0$$

$$M\ddot{x} + K(\delta_{static} + x) - W\sin\alpha = 0$$

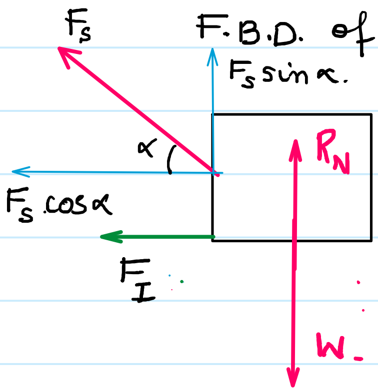
$$M\ddot{x} + Kx = 0.$$

★ If Mg is responsible for δ_{static} then the given position itself can be assumed as equilibrium position, by giving intial displacement we can write the equation of motion.

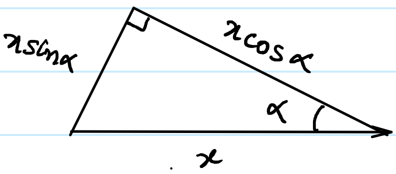
★ If Mg is responsible for δ_{static} then in the final equation of motion Mg will not appear.



F.B.D. of block.



Resolving displacement along spring.



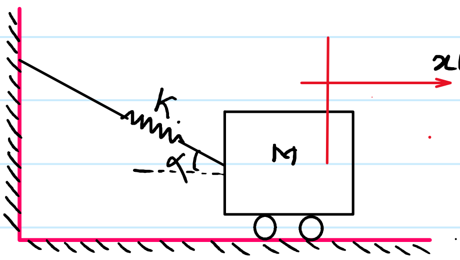
$$F_s = k \cdot x_s = k \cdot x \cdot \cos\alpha.$$

$$\Sigma F_y = 0 \qquad R_N + F_s \sin\alpha = W.$$

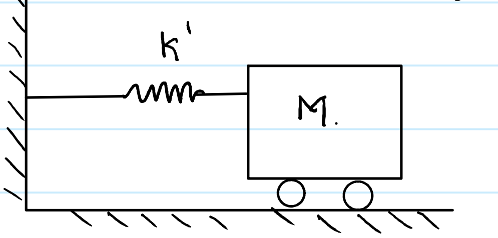
$$-F_s \cdot \cos\alpha - F_I = 0$$

$$F_s \cos\alpha + F_I = 0 \Rightarrow k \cdot x \cdot \cos^2\alpha + M\ddot{x} = 0$$

$$\boxed{M\ddot{x} + k x \cdot \cos^2\alpha = 0}$$



$$k' = k \cdot \cos^2\alpha$$



Equation of motion

$$M\ddot{x} + Kx = 0$$

$$M \cdot \frac{d^2x}{dt^2} + Kx = 0$$

$$\frac{d}{dt} = D$$

$$\frac{d^2x}{dt^2} + \frac{K}{M}x = 0$$

$$\left(D^2 + \frac{K}{M}\right)x = 0$$

$$D = \pm i\sqrt{\frac{K}{M}}$$

$\sqrt{\frac{K}{M}} = \omega_n \longrightarrow$ natural frequency.

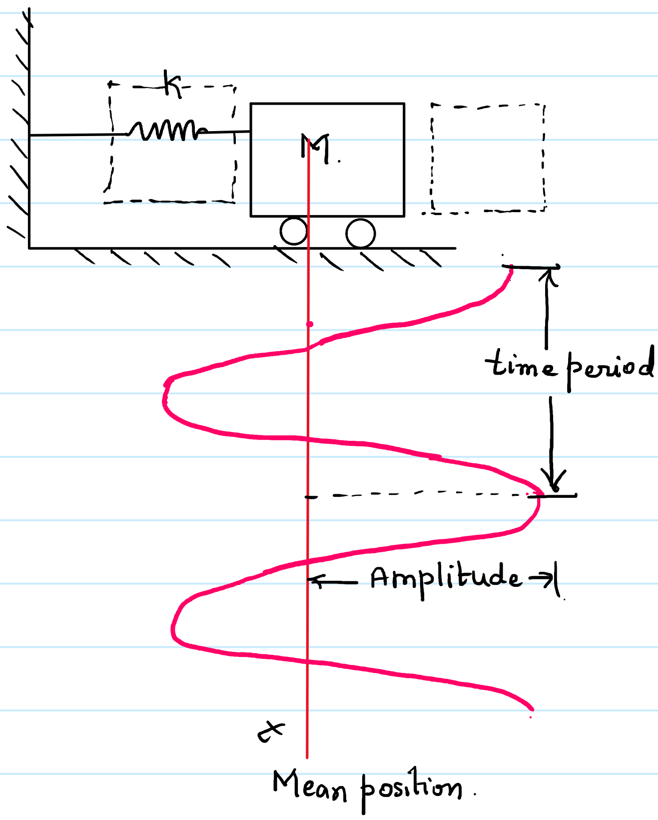
Complimentary function $\quad x(t) = A \cdot \sin\left(\sqrt{\frac{K}{M}} \cdot t\right) + B\cos\left(\sqrt{\frac{K}{M}} \cdot t\right)$

$A = X_0 \cos\phi \qquad B = X_0 \cdot \sin\phi$

$$x(t) = X_0 \cos\phi \cdot \sin\left(\sqrt{\frac{K}{M}}\,t\right) + X_0 \sin\phi \cdot \cos\left(\sqrt{\frac{K}{M}}\,t\right)$$

$$x(t) = X_0 \cdot \sin(\omega_n t + \phi)$$

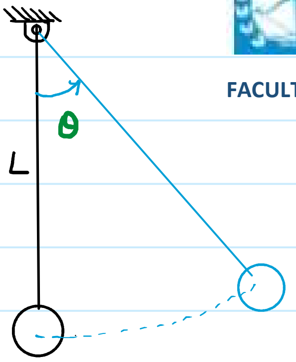
Displacement, Velocity, Acceleration are harmonic in nature.



Simple Pendulum.

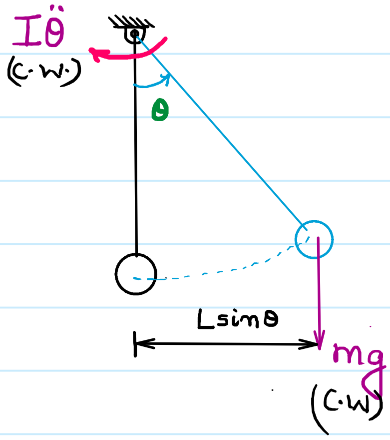
DoF = 1.

Displacement Variable = θ_z.



$\theta \to \sin\theta \approx \theta,\ \cos\theta = 1.$

$\tan\theta \approx \theta.$



$$I_{Hinge} = ML^2$$

$$\Sigma M_{Hinge} = 0$$

$$I_{Hinge}\ddot{\theta} + MgL\sin\theta = 0$$

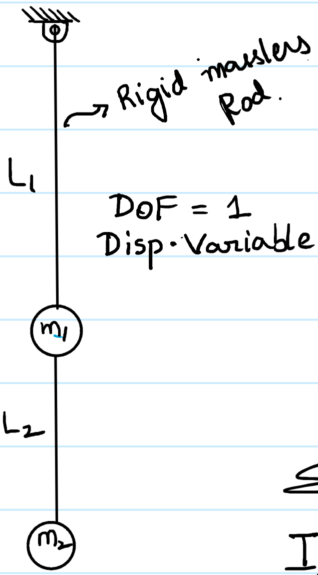
$$I_{Hinge}\ddot{\theta} + MgL\theta = 0$$

$$ML^2\ddot{\theta} + MgL\theta = 0$$

$$\ddot{\theta} + \frac{g}{L}\theta = 0$$

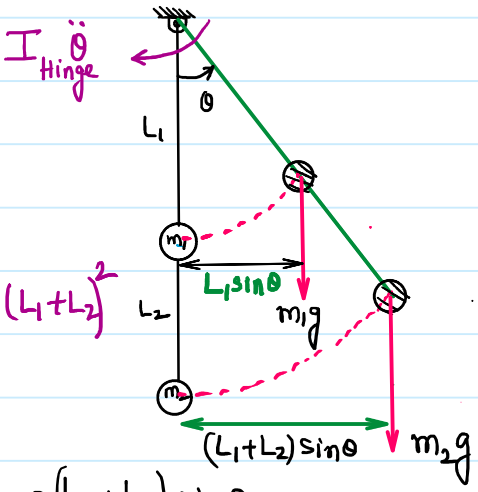
$$\omega_n = \sqrt{\frac{g}{L}}$$

time period. $t_n = \dfrac{2\pi}{\omega_n} = 2\pi\sqrt{L/g}$



DoF = 1

Disp. Variable = θ.



$$I_{Hinge} = m_1L_1^2 + m_2.(L_1+L_2)^2$$

$$\Sigma M_{Hinge} = 0$$

$$I_{Hinge}\ddot{\theta} + m_1gL_1\sin\theta + m_2g(L_1+L_2)\sin\theta = 0$$

$\sin\theta \approx \theta.$

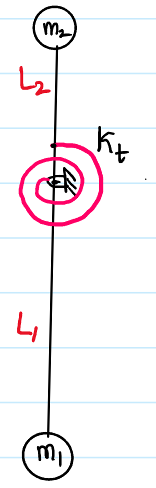
$$\left[m_1L_1^2 + m_2.(L_1+L_2)^2\right]\ddot{\theta} + \left[m_1L_1 + m_2.(L_1+L_2)\right]g.\theta = 0$$

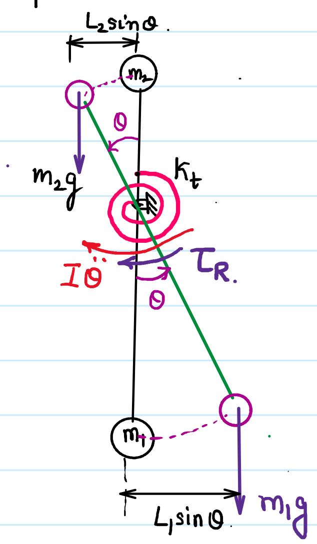
$$\omega_n = \sqrt{\frac{\left[m_1L_1 + m_2.(L_1+L_2)\right]g}{m_1L_1^2 + m_2.(L_1+L_2)^2}}$$

k_t - Torsional stiffness.

DoF = 1.

Disp. variable = θ.





$$\boxed{\tau_R = K_T \theta}$$

$$I_{Hinge} = m_1 L_1^2 + m_2 L_2^2$$

$\sin\theta = \theta$

$$\Sigma M_{Hinge} = 0$$

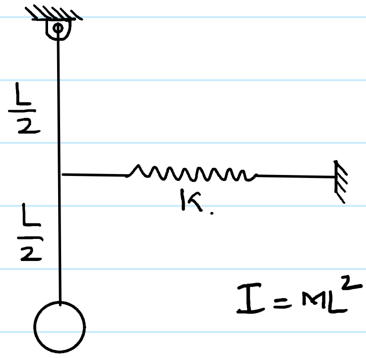
$$I_{Hinge}\ddot{\theta} + \tau_R + m_1 g L_1 \sin\theta - m_2 g L_2 \sin\theta = 0$$

$$(m_1 L_1^2 + m_2 L_2^2)\ddot{\theta} + k_T \theta + m_1 g L_1 \theta - m_2 g L_2 \theta = 0$$

$$(m_1 L_1^2 + m_2 L_2^2)\ddot{\theta} + (k_T + m_1 g L_1 - m_2 g L_2)\theta = 0$$

$$\ddot{\theta} + \frac{(k_T + m_1 g L_1 - m_2 g L_2)\theta}{m_1 L_1^2 + m_2 L_2^2} = 0$$

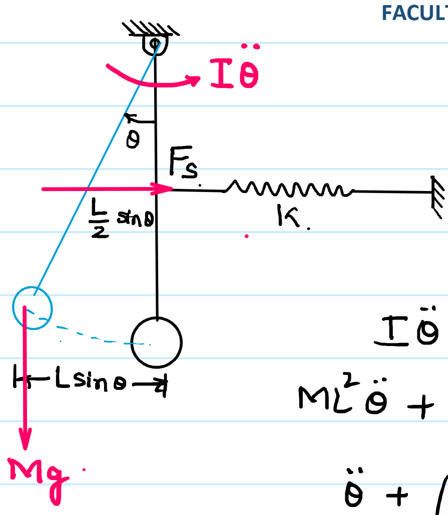
$$\omega_n = \sqrt{\frac{(k_T + m_1 g L_1 - m_2 g L_2)}{m_1 L_1^2 + m_2 L_2^2}}$$



$I = ML^2$

$F_s = k \cdot L/2 \, \sin\theta$

$= \frac{k \cdot L}{2} \cdot \theta$

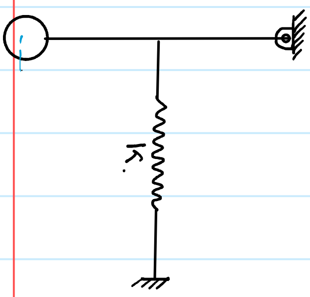


$\Sigma M_{Hinge} = 0$

$I\ddot{\theta} + F_s \times L/2 + Mg \cdot L \sin\theta = 0$

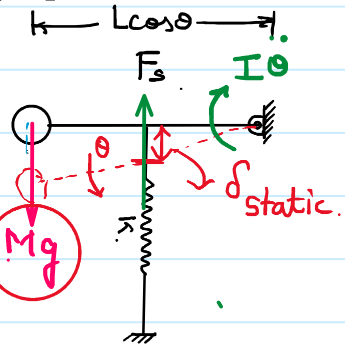
$ML^2\ddot{\theta} + \frac{k \cdot L^2}{4}\theta + Mg L\theta = 0$

$\ddot{\theta} + \left(\frac{k}{4M} + \frac{g}{L}\right)\theta = 0$



$\sin\theta = \theta$

$\cos\theta = 1$ ✓



It will cause δ_{static}.

Mg is not responsible for restoring.

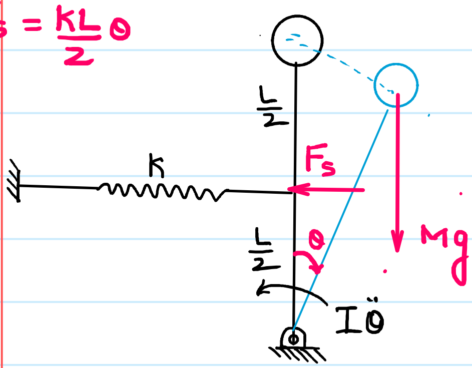
$\Sigma M_{Hinge} = 0$

$I\ddot{\theta} + F_s \times L/2 - MgL\cos\theta = 0$

$ML^2\ddot{\theta} + \frac{kL^2}{4}\theta - MgL + k\delta_s \cdot L/2 = 0$

$$\boxed{\ddot{\theta} + \frac{k}{4M}\theta = 0}$$

$F_s = \frac{kL}{2}\theta$



$\Sigma M_{Hinge} = 0$

$I\ddot{\theta} - Mg \cdot L\sin\theta + F_s \times L/2 = 0$

$ML^2\ddot{\theta} + \frac{kL^2}{4}\theta - Mg \cdot L\theta = 0$

$\ddot{\theta} + \left(\frac{k}{4M} - g/L\right)\theta = 0$

01. In a cuckoo clock the pendulum has a mass of 50gms and has a time period of 0.5 sec. Its length is approximately
(a) 52 mm (b) 62 mm
(c) 48 mm (d) 24 mm

→ simple pendulum.

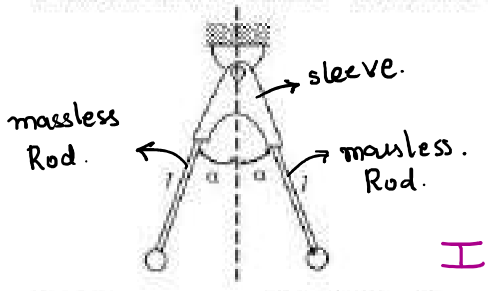
$\omega_n = \sqrt{g/l}$

$t_n = 2\pi\sqrt{L/g} \Rightarrow 0.5 = 2\pi\sqrt{\frac{L}{9.81}}$

$L = \frac{0.5^2}{4\pi^2} \times 9.81$ m.

$L = 62$mm.

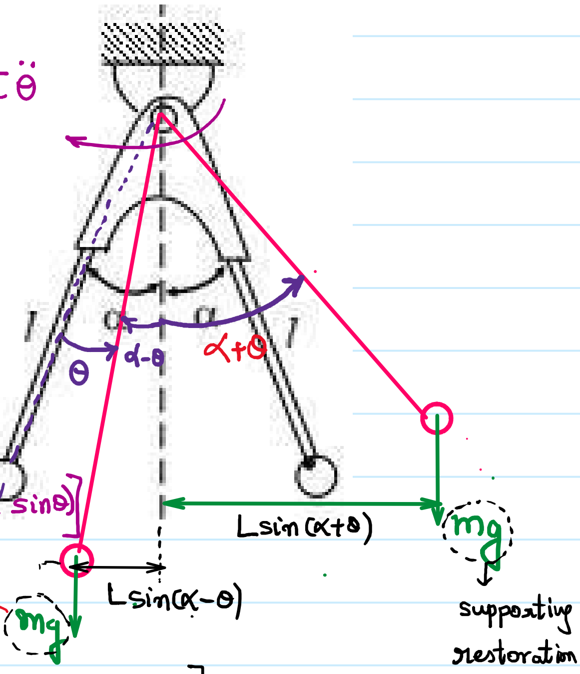
02. The assembly shown in the figure is composed of two mass less rods of length l with two particles, each of mass m. The natural frequency of this assembly for small oscillations is (GATE-2001)



(a) $\sqrt{g/l}$ (b) $\sqrt{2g/(l\cos\alpha)}$
(c) $\sqrt{g/(l\cos\alpha)}$ (d) $\sqrt{(g\cos\alpha)/l}$

DoF = 1 Disp. variable = θ.

$I = mL^2 + mL^2$
$= 2mL^2$



$\Sigma M_{Hinge} = 0$

$I\ddot{\theta} + mgL\sin(\alpha+\theta) - mgL\sin(\alpha-\theta) = 0$

$2mL^2\ddot{\theta} + mgL[(\sin\alpha\cos\theta + \cos\alpha\sin\theta) - (\sin\alpha\cos\theta - \cos\alpha\sin\theta)]$

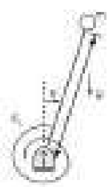
$\sin\theta \approx \theta$

$2mL^2\ddot{\theta} + mgL[\cancel{\sin\alpha\cos\theta} + \cos\alpha\sin\theta - \cancel{\sin\alpha\cos\theta} + \cos\alpha\sin\theta]$

$2mL^2\ddot{\theta} + 2mgL\cos\alpha\cdot\theta = 0 \Rightarrow \boxed{\ddot{\theta} + \frac{g\cos\alpha}{L}\theta = 0}$

Common data for Q. 11 & 12

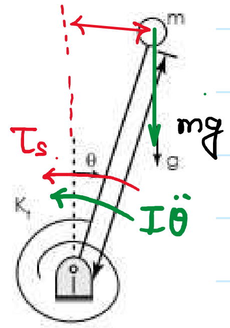
For the system shown in figure θ = 0 is the static equilibrium position and the torsional spring of stiffness K_t is uncompressed at this position. Assume small oscillations about this position and consider θ as the co-ordinate.



11. The Equivalent Inertia is
(a) m kg (b) m l kgm
(c) ml^2 kgm² (d) $ml^2/2$ kgm²

12. The Equivalent torsional stiffness is
(a) K_t (b) $K_t + gl$
(c) $K_t - mgl$ (d) $K_t + mgl$

$L\sin\theta \approx L\theta$



$T_s = K_T\theta$

ΣM_{Hinge}.

$I\ddot{\theta} + T_s \boxed{-mgL\theta} = 0 \Rightarrow mL^2\ddot{\theta} + K_T\theta - mgL\theta = 0$ → against Restoration.

$\underbrace{mL^2}_{I_{eq}}\ddot{\theta} + \underbrace{(K_T - mgL)}_{K_{eq}}\theta = 0$

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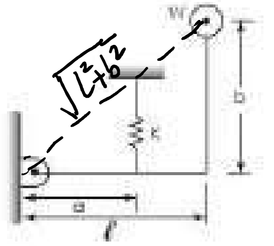
06. For the system shown in the given figure the moment of inertia of the weight W and the ball about the pivot point is I_0. The natural frequency of the system is given by

$f_n = (1/2\pi)\sqrt{(Ka^2 - Wb)/I_0}$

The system will vibrate when



(a) $b < (Ka^2/W)$ (b) $b = (Ka^2/W)$
(c) $b > (Ka^2/W)$ (d) $a = 0$

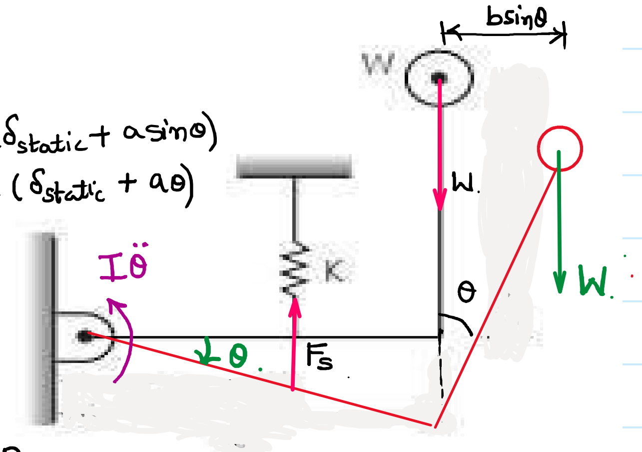
$$f_n = \frac{1}{2\pi}\sqrt{\frac{Ka^2 - Wb}{I_0}} \qquad f_n > 0$$

$Ka^2 - Wb > 0$ only then system will oscillate.

$$\boxed{b < \frac{Ka^2}{W}}$$

$$I = \frac{W}{g}(L^2+b^2)$$

W is responsible for δ_{static}. given position itself can be assumed as equilibrium position.



$$F_s = K(\delta_{static} + a\sin\theta)$$
$$= K(\delta_{static} + a\theta)$$

$\sin\theta \approx \theta$
$\cos\theta \approx 1$

$$\Sigma M_{Hinge} = 0$$

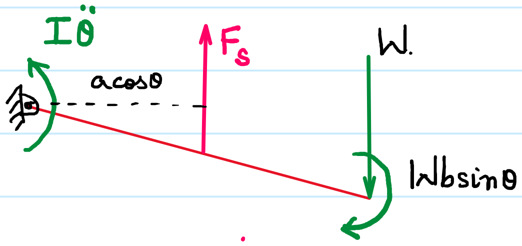
$$I\ddot{\theta} + F_s a\cos\theta - WL\cos\theta - Wb\sin\theta = 0$$

$$I\ddot{\theta} + K(\delta_{static} + a\theta)a - WL - Wb\theta = 0$$

constant terms

$$\frac{W}{g}(L^2+b^2)\ddot{\theta} + \boxed{K\delta_{static}a} + Ka^2\theta - \boxed{WL} - Wb\theta = 0$$

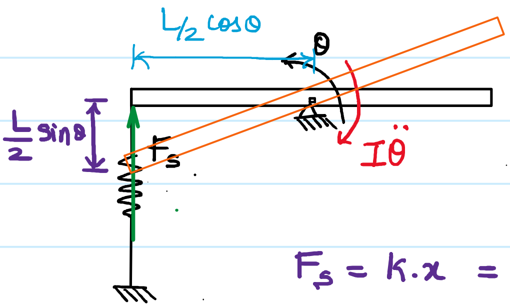
0 0



$$\frac{W}{g}(L^2+b^2)\ddot{\theta} + (Ka^2 - Wb)\theta = 0$$

$$\ddot{\theta} + \frac{Ka^2 - Wb}{\frac{W}{g}(L^2+b^2)}\theta = 0 \qquad \omega_n = \sqrt{\frac{Ka^2 - Wb^2}{\frac{W}{g}(L^2+b^2)}}$$

05. A uniform rigid rod of mass m = 1kg and length L = 1 m is hinged at its centre & laterally supported at one end by a spring of spring constant k = 300N/m. The natural frequency ω_n in rad/s is (GATE-08)
(a) 10 (b) 20
(c) 30 (d) 40



Disp. variable – θ
DOF = 1.

$\sin\theta \approx \theta$
$\cos\theta \approx 1$

$$F_s = K.x = K.\frac{L}{2}\sin\theta = (KL/2\,\theta)$$

$$\Sigma M_{Hinge} = 0 \qquad I\ddot{\theta} + F_s \cdot L/2 \cdot \cos\theta = 0$$

$$\frac{ML^2}{12}\ddot{\theta} + (K \cdot L/2 \cdot \theta)\cdot L/2 = 0 \Rightarrow \frac{ML^2}{12}\ddot{\theta} + \frac{KL^2}{4}\theta = 0 \Rightarrow \ddot{\theta} + \frac{3K}{M}\theta = 0$$

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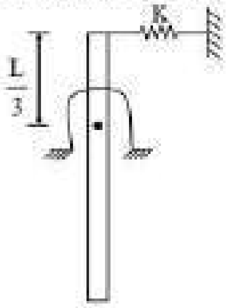
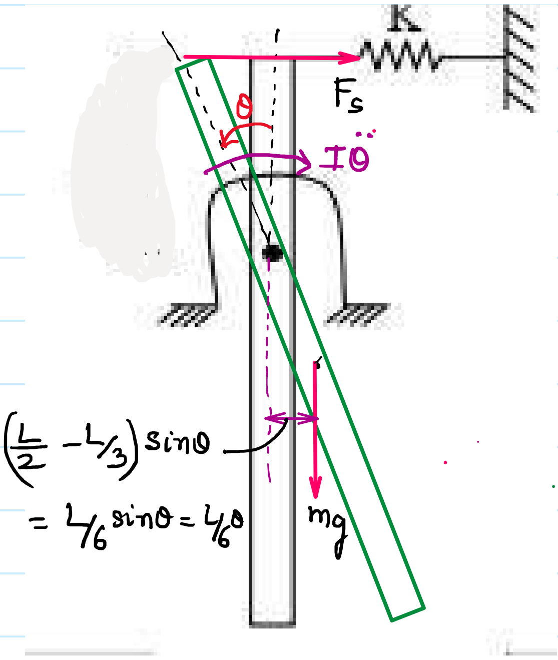
09. A slender link of mass 'm' and length L is pivoted at a length of L/3 from the top and is connected as shown in figure and it oscillates in the vertical plane. The stiffness of the spring is 'K'. The natural frequency of the system is given by



(a) $\sqrt{\frac{3g}{2L} - \frac{K}{m}}$ (b) $\sqrt{\frac{3g}{2L} + \frac{K}{m}}$

(c) $\sqrt{\frac{g}{L} + \frac{K}{m}}$ (d) $\sqrt{\frac{g}{L} - \frac{K}{m}}$



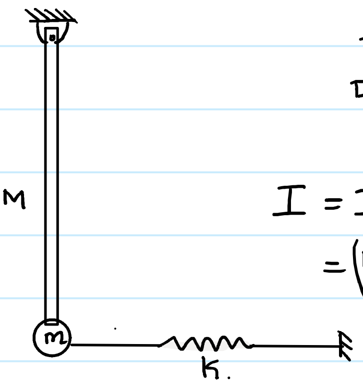
$$I = \bar{I} + md^2$$

$$= \frac{ML^2}{12} + M\left(\frac{L}{6}\right)^2$$

$$= \frac{ML^2}{9}.$$

$$I\ddot{\theta} + \left(F_s \times \frac{L}{3}\right) + mg \cdot \frac{L}{6} \cdot \theta = 0$$

$$\frac{ML^2}{9}\ddot{\theta} + \frac{K \cdot L^2}{9}\theta + mg \cdot \frac{L}{6}\theta = 0 \Rightarrow \ddot{\theta} + \left(\frac{K}{M} + \frac{3g}{2L}\right)\theta = 0$$

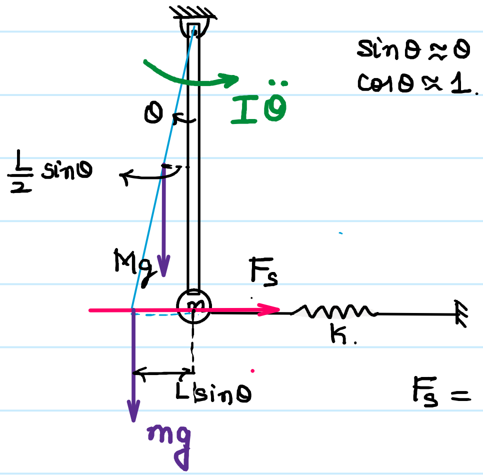


DOF = 1

Disp. variable — θ.

$$I = I_{Rod} + I_{Point\ mass}.$$

$$= \left(\frac{ML^2}{3} + m \cdot L^2\right)$$



$\sin\theta \approx \theta$

$\cos\theta \approx 1.$

$F_s = K.(L\theta)$

$$\Sigma M_{Hinge.} = 0$$

$$\left(\frac{ML^2}{3} + mL^2\right)\ddot{\theta} + Mg \cdot \frac{L}{2} \cdot \theta + mg \cdot L \cdot \theta + F_s \cdot L = 0$$

$$\left(\frac{ML^2}{3} + mL^2\right)\ddot{\theta} + K \cdot L^2\theta + \left(\frac{M}{2} + m\right) g \cdot L\theta = 0 \Rightarrow \ddot{\theta} + \frac{KL^2 + \left(\frac{M}{2} + m\right)g \cdot L}{\left(\frac{ML^2}{3} + mL^2\right)}\theta = 0$$

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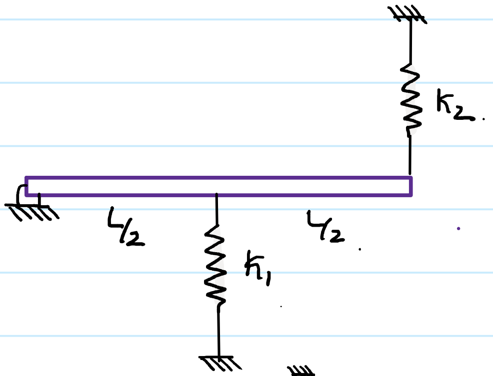

Wt of Rod will produced δ_{static}

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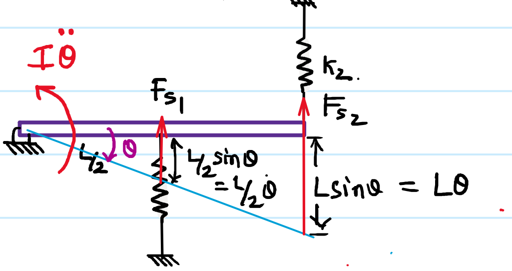
$\sin\theta = \theta$
$\cos\theta = 1$



$$\Sigma M_{Hinge} = 0$$

$$I\ddot{\theta} + F_{s_1} \times \frac{L}{2}\cdot\cos\theta + F_{s_2}\cdot L\cos\theta = 0$$

$$\frac{ML^2}{3}\ddot{\theta} + \left(k_1 \cdot \frac{L}{2}\theta\right)\frac{L}{2} + (k_2 L\theta)\cdot L = 0$$

$$\frac{ML^2}{3}\ddot{\theta} + \left(\frac{k_1 L^2}{4} + k_2 L^2\right)\theta = 0$$

$$\ddot{\theta} + \left(\frac{3k_1}{4M} + \frac{3k_2}{M}\right)\theta = 0$$

$$\Sigma M_{Ref\ point} = 0$$

$$I_{Ref\cdot point}\,\ddot{\theta} + k(\perp^{lar}\text{ distance from ref}\cdot\text{point})^2\theta \pm mg\cdot(\perp^{lar}\text{ distance from ref-point})\theta = 0$$

$$\frac{ML^2}{3}\ddot{\theta} + k_1\left(\frac{L}{2}\right)^2\theta + k_2(L)^2\theta = 0$$

10. A spring mass system with a natural frequency of 5 rad/sec is set in to free oscillations by giving an initial displacement of 10 cm. The amplitude of vibration is
(a) 50 cm (b) 2 cm
(c) 15 cm (d) 10 cm

$\omega_n = 5\,rad/s.$

intial displacement $X_o = 10\,cm.$

Amplitude $= ?$

$$x(t) = X_o\cdot\sin(\omega t + \phi)$$

$$@\ t=0 \qquad x(t=0) = 10\,cm = X_o\cdot\sin(0+\phi)$$

$$10\,cm = X_o\cdot\sin\phi$$

$$@\ \underbrace{t=0 \qquad \dot{x}(t=0)}_{\text{intial velocity}} = 0$$

$$\dot{x}(t) = X_o\,\omega\cdot\cos(\omega t+\phi)$$

$$0 = X_o \cdot \times 5\cos(0+\phi)$$

$$X_o\cdot\cos\phi = 0 \qquad \cos\phi = 0$$

$$\phi = 90^\circ$$

Amplitude

$$10 = X_o\cdot\sin 90^\circ \Rightarrow X_o = 10\,cm$$

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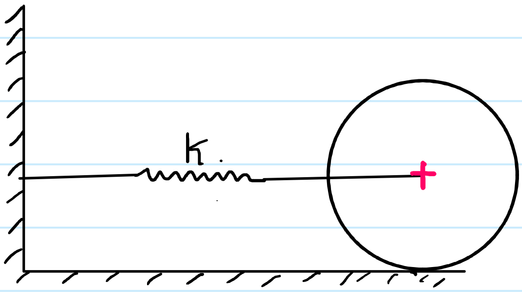
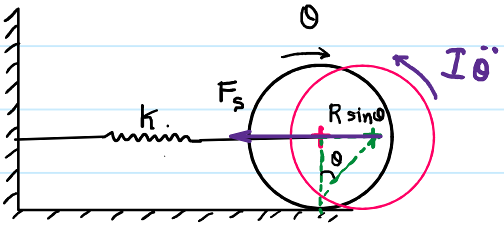
Cylinder Disc. Higher Pair – Pure Rolling.



$\sin\theta \simeq \theta$
$\cos\theta \simeq 1$



$R\sin\theta = R\theta$

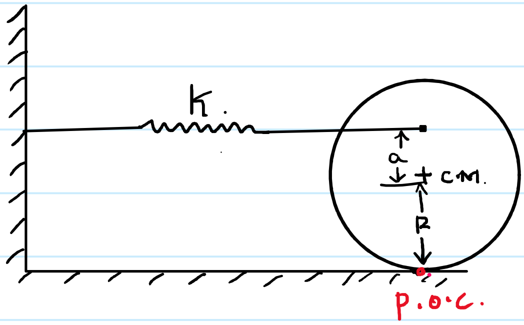
$$\Sigma M_{p.o.c} = 0$$

$$I_{p.o.c}\cdot\ddot{\theta} + F_s \times R\cdot\cos\theta = 0$$

$$\left(\frac{MR^2}{2} + MR^2\right)\ddot{\theta} + (K\cdot R\theta)\cdot R = 0$$

$$\frac{3}{2}MR^2\ddot{\theta} + KR^2\theta = 0$$

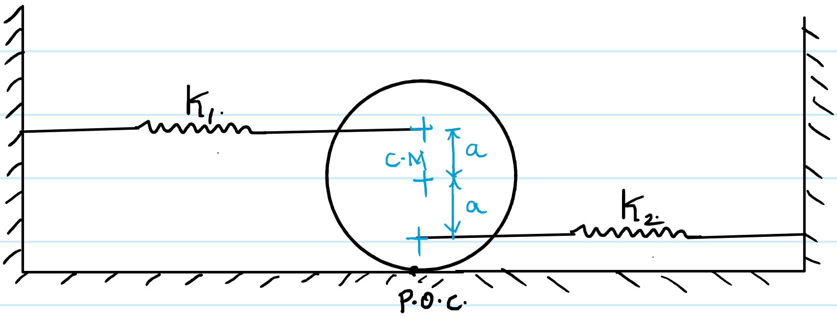
$$\ddot{\theta} + \frac{2K}{3M}\theta = 0$$



$$\Sigma M_{p.o.c} = 0$$

$$I_{p.o.c}\ddot{\theta} + K\cdot(\perp^{lar} \text{ from } p.o.c)^2\theta = 0 \Rightarrow \frac{3}{2}MR^2\ddot{\theta} + K\cdot(R+a)^2\theta = 0$$

$$\ddot{\theta} + \frac{2K\cdot(R+a)^2}{3MR^2}\theta = 0$$

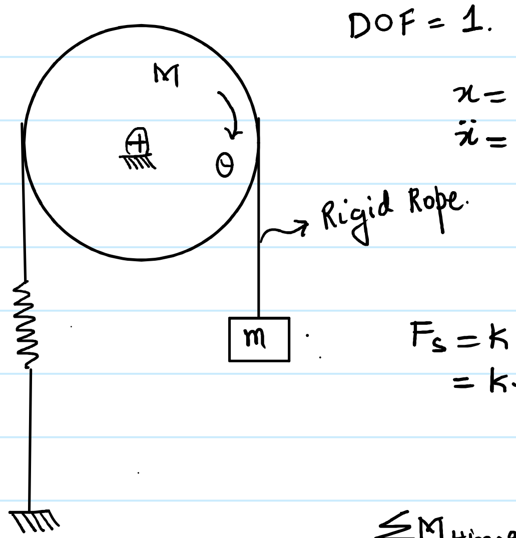


$$\Sigma M_{p.o.c} = 0$$

$$I_{p.o.c}\,\ddot{\theta} + k_1 (R+a)^2 \theta + k_2 (R-a)^2 \theta = 0$$

$$\frac{3}{2} MR^2 \ddot{\theta} + \left[k_1 (R+a)^2 + k_2 (R-a)^2\right] \theta = 0$$

$$\ddot{\theta} + \frac{2\left[k_1 (R+a)^2 + k_2 (R-a)^2\right]}{3MR^2}\,\theta = 0$$



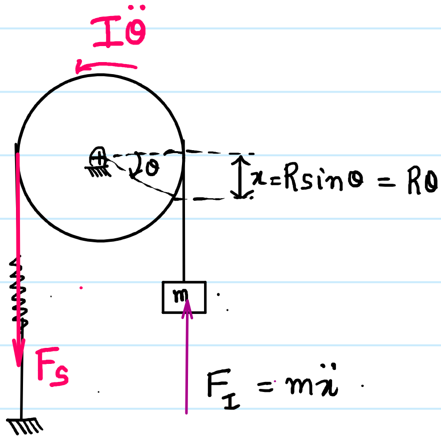
DOF = 1.

$x = R\theta$

$\ddot{x} = R\cdot\ddot{\theta}$

$F_s = kx$

$\quad = k\cdot R\theta$



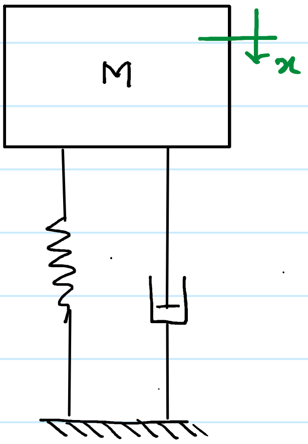
$$\Sigma M_{Hinge} = 0$$

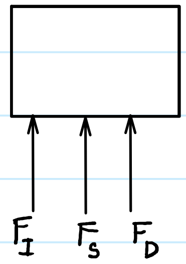
$$I_{Hinge}\,\ddot{\theta} + F_I \times R + F_s \times R = 0$$

$$\frac{MR^2}{2}\ddot{\theta} + (mR\ddot{\theta} \times R) + k\cdot R^2\theta = 0$$

$$\ddot{\theta} + \frac{k}{\left(m + \frac{M}{2}\right)}\,\theta = 0$$

Free Damped Vibrations — Loss in energy after each cycle.





$$\Sigma F = 0$$

$$F_I + F_D + F_S = 0$$

$$M\ddot{x} + C\dot{x} + Kx = 0$$

$$\ddot{x} + \frac{C}{M}\dot{x} + \frac{K}{M}x = 0 \qquad \frac{K}{M} \rightarrow \omega_n^2$$

$$\ddot{x} + \frac{C}{2\sqrt{MK}} \cdot \frac{\sqrt{K}}{\sqrt{M}} \cdot 2\dot{x} + \omega_n^2 \cdot x = 0 \qquad \frac{C}{2\sqrt{MK}} \rightarrow \zeta, \quad \frac{\sqrt{K}}{\sqrt{M}} \rightarrow \omega_n$$

$$M\ddot{x} + C\dot{x} + Kx = 0$$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$$

$$a = 1 \quad b = 2\zeta\omega_n \quad c = \omega_n^2$$

$$\frac{d^2x}{dt^2} + 2\zeta\omega_n \cdot \frac{dx}{dt} + \omega_n^2 \cdot x = 0 \Rightarrow (D^2 + 2\zeta\omega_n D + \omega_n^2)x = 0$$

$$D = \frac{-2\zeta\omega_n \pm \sqrt{(2\zeta\omega_n)^2 - 4(1)\cdot\omega_n^2}}{2(1)}$$

$$D = -\zeta\omega_n \pm \sqrt{(\zeta\omega_n)^2 - \omega_n^2}$$

$$D = \omega_n\left[-\zeta \pm \sqrt{\zeta^2 - 1}\right]$$

$\zeta = 1$ $\quad D = -\omega_n, -\omega_n$ $\quad$ complimentary function

$$x(t) = (A + Bt)e^{-\omega_n t}$$

$\zeta > 1$

$$D = \underbrace{\omega_n\left[-\zeta + \sqrt{\zeta^2 - 1}\right]}_{-\alpha_1}, \quad \underbrace{\omega_n\left[-\zeta - \sqrt{\zeta^2 - 1}\right]}_{-\alpha_2}$$

C.F. $\quad x(t) = Ae^{-\alpha_1 t} + Be^{-\alpha_2 t}$

$\boxed{\zeta = 1,\ \zeta > 1 \text{ system is coming to rest without executing oscillations.}}$

$\zeta < 1$

$$D = \omega_n\left[-\zeta + i\sqrt{1 - \zeta^2}\right], \quad \omega_n\left[-\zeta - i\sqrt{1 - \zeta^2}\right]$$

$$D = -\zeta\omega_n \pm i\,\omega_n\sqrt{1 - \zeta^2} \qquad \omega_n\sqrt{1-\zeta^2} \rightarrow \omega_d \qquad D = -\zeta\omega_n \pm \omega_d$$

C.F. $\quad x(t) = \left[A\sin(\omega_d t) + B\cos(\omega_d t)\right]e^{-\zeta\omega_n t}$

$$x(t) = X_0 \cdot \sin(\omega_d t + \phi)\, e^{-\zeta\omega_n t}$$

$\boxed{\zeta < 1 \text{ system is coming to rest by executing decaying oscillations}}$

Loss in energy

$A = X_0 \cos\phi$
$B = X_0 \sin\phi$

(No loss in energy)

$\zeta = 0$ $\quad x(t) = X_0 \cdot \sin(\omega_n t + \phi)$ undamped system.

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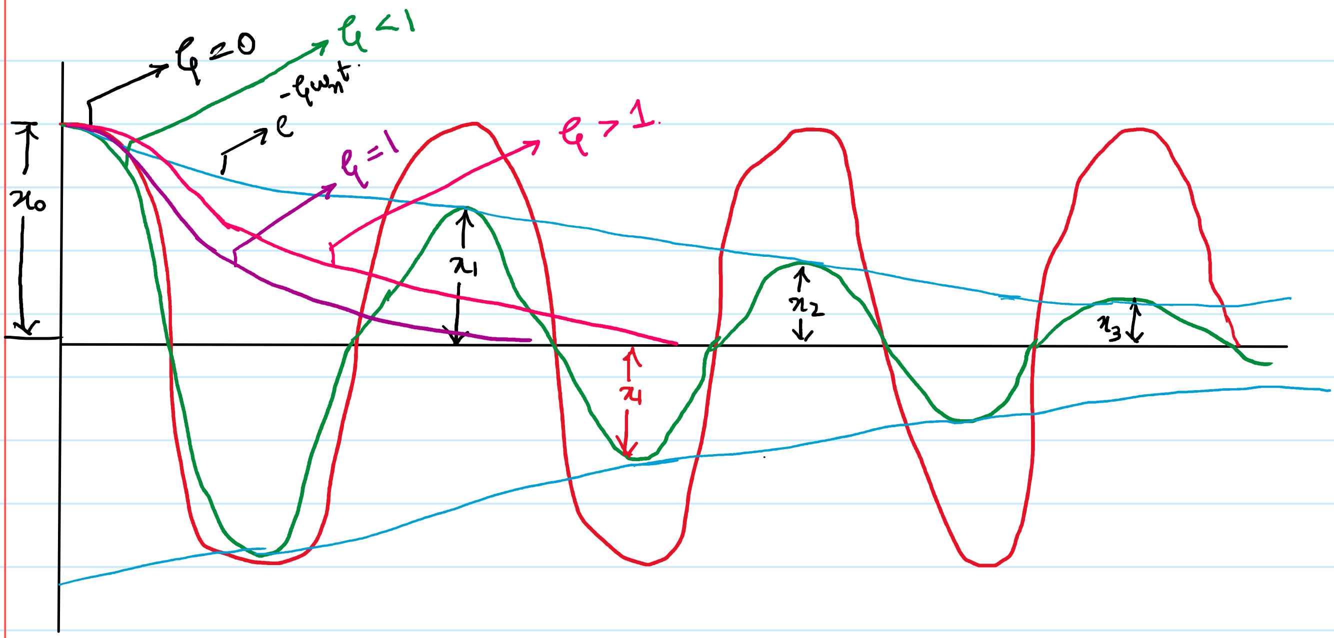
$\zeta < 1$ $\quad x(t) = X_0 e^{-\zeta\omega_n t} \sin(\omega_d t + \phi)$ underdamped system.

Loss in energy.

$\zeta = 1$. $\quad \boxed{x(t) = (A + Bt)e^{-\omega_n t}}$ critically damped system.

$\zeta > 1$ $\quad \boxed{x(t) = Ae^{-\alpha_1 t} + Be^{-\alpha_2 t}}$ Overdamped system.

$\hookrightarrow$ aperiodic/non-periodic response.



ζ – Damping Ratio/Damping factor. $\quad \zeta = \dfrac{c}{2\sqrt{MK}} = \dfrac{\text{Actual damping coeff.}}{\text{Critical damping coeff.}}$

Damping factor is defined as Ratio of Actual damping to critical damping. Damping factor also represents the % of critical damping.

$$\zeta = \frac{c}{c_c} = \frac{c}{2\sqrt{MK}}.$$

$$c_c = 2\sqrt{\frac{M^2 \cdot K}{M}} = 2 \cdot M \cdot \omega_n.$$

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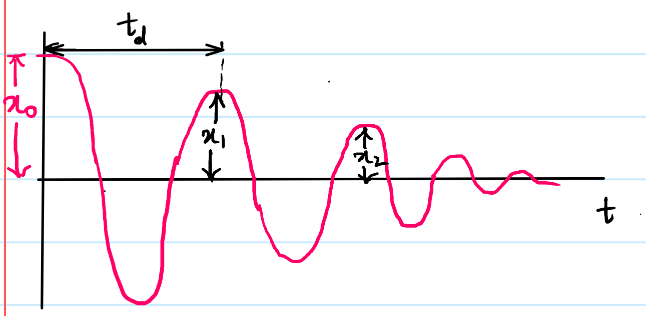
$\zeta < 1$. $\quad x(t) = X_0 \cdot e^{-\zeta\omega_n t} \sin(\omega_d t + \phi)$

intial condition @ $t = 0$

$$x(t=0) = X_0 e^{-\zeta\omega_n \cdot (0)} \sin(\omega_d (0) + \phi)$$

Amplitude $\rightarrow$ X_0

$$x_0 = X_0.$$



ω_d – frequency of damped oscillation

t_d – time period of damped oscillation.

$$t_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n\sqrt{1-\zeta^2}}$$

$$x_1 = x(t=t_d) = X_0 \cdot e^{-\zeta\omega_n t_d} \sin(\omega_d t_d + \phi)$$

Amplitude

$$x_1 = X_0 \cdot e^{-\zeta\omega_n t_d}$$

displacement $\quad$ Amplitudes

$$x_0 = X_0 \cdot e^{-\zeta\omega_n \cdot (0)}$$
$$x_1 = X_0 \cdot e^{-\zeta\omega_n \cdot t_d}$$
$$x_2 = X_0 \cdot e^{-\zeta\omega_n \cdot 2t_d}$$
$$x_n = X_0 \cdot e^{-\zeta\omega_n \cdot n \cdot t_d}$$

Ratio of consecutive displacements.

$$\frac{x_0}{x_1} = \frac{X_0}{X_0 \cdot e^{-\zeta\omega_n t_d}} \Rightarrow \frac{x_0}{x_1} = e^{\zeta\omega_n t_d}$$

$$\frac{x_0}{x_1} = e^{\zeta\omega_n t_d}$$

$$\frac{x_0}{x_1} = \frac{x_1}{x_2} = \frac{x_2}{x_3} = \frac{x_3}{x_4} = \cdots = \frac{x_{n-1}}{x_n} = e^{\zeta\omega_n t_d}$$

$x_0, x_1, x_2 \cdots x_n$ are in Geometric Progression.

$$\frac{x_0}{x_n} = \overbrace{\frac{x_0}{x_1}}^{\zeta\omega_n t_d} \cdot \frac{x_1}{x_2} \cdot \frac{x_2}{x_3} \cdot \frac{x_3}{x_4} \cdots\cdots \frac{x_{n-1}}{x_n}$$

$$\frac{x_0}{x_n} = e^{\zeta\omega_n t_d \cdot n}$$

n – no. of cycles.

Logarithamic decrement. $\quad \delta = \ln\left(\frac{x_0}{x_1}\right) = \zeta\omega_n t_d$

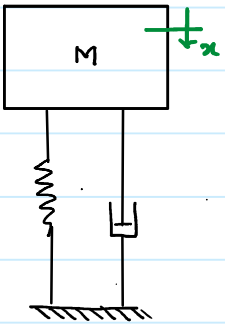
$$\ln\left(\frac{x_0}{x_1}\right) = \zeta \cdot \omega_n \cdot \frac{2\pi}{\omega_n\sqrt{1-\zeta^2}}$$

$$\delta = \ln\left(\frac{x_0}{x_1}\right) = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

$$\ln\left(\frac{x_0}{x_n}\right) = \zeta\omega_n t_d \cdot n$$

$$\ln\left(\frac{x_0}{x_n}\right) = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \cdot n$$

$$\frac{1}{n}\ln\left(\frac{x_0}{x_n}\right) = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = \delta$$



Equation of motion

$$M\ddot{x} + c\dot{x} + kx = 0$$

$\hookrightarrow$ Translatory motion

For Rotatory motion

$$I_{eq}\ddot{\theta} + C_{eq}\dot{\theta} + K_{eq}\theta = 0$$

Natural frequency.

$$\omega_n = \sqrt{\frac{K}{M}} \rightsquigarrow \text{Translatory motion.}$$

$$\omega_n = \sqrt{\frac{K_{eq}}{I_{eq}}} \rightsquigarrow \text{Rotatory motion.}$$

Critical damping

$$C_c = 2 \cdot \sqrt{MK}. \rightsquigarrow \text{Translatory motion}$$

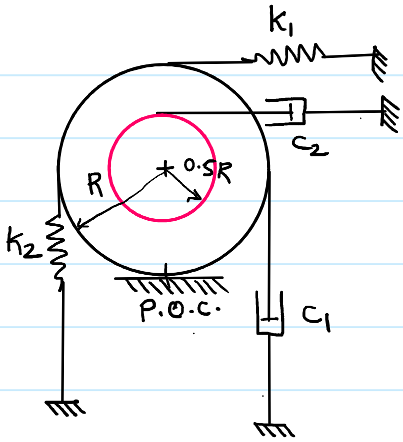
$$C_c = 2 \cdot \sqrt{I_{eq} \cdot K_{eq}} \rightsquigarrow \text{Rotatory motion.}$$

Damping factor

$$\zeta = \frac{c}{2\sqrt{MK}} \rightsquigarrow \text{Translatory motion.}$$

$$\zeta = \frac{C_{eq}}{2 \cdot \sqrt{I_{eq} \cdot K_{eq}}} \rightsquigarrow \text{Rotatory motion.}$$

SNo.	System Parameter	Motion.			
		Translation.		Rotation	
		Force.	Energy.	Couple.	Energy.
1.	Inertia	$M\ddot{x}$	$\frac{1}{2} \cdot M\dot{x}^2$	$I_{eq}\ddot{\theta}$	$\frac{1}{2} I_{eq}\dot{\theta}^2$
2.	Restoration	$kx.$	$\frac{1}{2}kx^2$	$K_{eq}\theta$	$\frac{1}{2} K_{eq} \cdot \theta^2$
3.	Damping	$C\dot{x}$	$\int F_D \cdot dx = \int C\dot{x}\, dx$	$C_{eq}\dot{\theta}$	$\int T_D \cdot d\theta = \int C_{eq}\dot{\theta}\, d\theta$



$E.O.M = ?$
$\omega_n = ?$
$c_c = ?$
$\xi = ?$
$\omega_d = ?$

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$\sum M_{P.O.C} = 0$

$I_{P.O.C}\,\ddot{\theta} + c_1(\perp^{lar}\text{ distance from P.O.C})^2\dot{\theta} + c_2(\perp^{lar}\text{ distance from P.O.C})^2\dot{\theta} + k_1(\perp^{lar}\text{ distance from P.O.C})^2\theta + k_2(\perp^{lar}\text{ distance from P.O.C})^2\theta = 0$

$\frac{3}{2}MR^2\ddot{\theta} + c_1(R)^2\dot{\theta} + c_2(1.5R)^2\dot{\theta} + k_1(2R)^2\theta + k_2(R)^2\theta = 0$

$I_{eq} = \frac{3}{2}MR^2 \qquad C_{eq} = c_1R^2 + 2.25c_2R^2 \qquad K_{eq} = 4k_1R^2 + k_2R^2$

Natural frequency $\quad \omega_n = \sqrt{\frac{K_{eq}}{I_{eq}}} \Rightarrow \omega_n = \sqrt{\frac{(4k_1+k_2)R^2}{3/2\,MR^2}} = \sqrt{\frac{8k_1+2k_2}{3M}}$

Critical Damping $\quad C_c = 2\cdot\sqrt{I_{eq}\cdot K_{eq}} \qquad C_c = 2\cdot\sqrt{3/2\,MR^2\cdot(4k_1+k_2)R^2}$

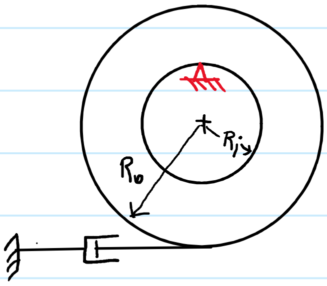
Damping factor $\quad \xi = \frac{C_{eq}}{2\sqrt{I_{eq}\cdot K_{eq}}} = \frac{c_1R^2 + 2.25c_2R^2}{2\cdot\sqrt{3/2\,MR^2\cdot(4k_1+k_2)R^2}}$

Frequency of Damped oscillation

$$\omega_d = \omega_n\cdot\sqrt{1-\xi^2}$$

$$\omega_d = \sqrt{\frac{8k_1+2k_2}{3M}}\sqrt{1-\left(\frac{c_1R^2+2.25c_2R^2}{2\cdot\sqrt{3/2\,MR^2\cdot(4k_1+k_2)R^2}}\right)^2}$$

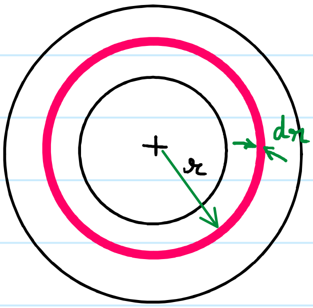
→

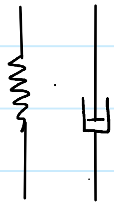


E.O.M.

$R_o = R.$

$R_i = 0.5R.$





Area of elemental ring

$= 2\pi r . dr.$

Volume of elemental ring

$= 2\pi r . dr . t$

Mass of elemental Ring

$dm = \rho . 2\pi r . dr . t$

Moment of Inertia about centroidal axis. $= \int r^2 . dm . = \int_{R_i}^{R_o} r^2 . (\rho . 2\pi r . dr . t)$

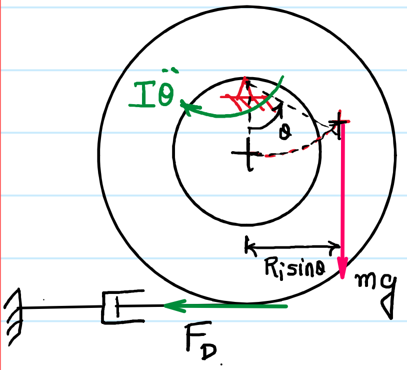
$$I = \rho . 2\pi . t \int_{R_i}^{R_o} r^3 . dr = \rho . \cancel{2}\pi . t . \frac{(R_o^4 - R_i^4)}{\cancel{4}\,2}.$$

$$I = \underbrace{\boxed{\rho . \pi . (R_o^2 - R_i^2) . t}}_{\text{Mass of Annular Ring}} . \frac{(R_o^2 + R_i^2)}{2} = \frac{m . (R_o^2 + R_i^2)}{2}$$

$$I_{Hinge} = \bar{I} + m . d^2$$

$$I_{Hinge} = \frac{m . (R^2 + (0.5R)^2)}{2} + m(0.5R)^2$$

$$I_{Hinge} = 0.875\, mR^2$$



$\sin\theta \approx \theta.$

$$I_{Hinge}\, \ddot{\theta} + mg(R_i \sin\theta) + C . (R + 0.5R)^2 \dot{\theta} = 0$$

$$0.875 mR^2 \ddot{\theta} + 2.25 . CR^2 \dot{\theta} + 0.5 mg . R . \theta = 0$$

$$I_{eq} \ddot{\theta} + C_{eq} \dot{\theta} + k_{eq} \theta = 0$$

Damped free Vibration

18. Critical damping is the **(GATE-14)**
 (a) largest amount of damping for which no oscillation occurs in free vibration
 (b) $\boxed{\text{smallest amount of damping for which no oscillation occurs in free vibration}}$
 (c) largest amount of damping for which the motion is ~~simple harmonic in free vibration~~
 (d) smallest amount of damping for which the motion is simple harmonic in free vibration

Critical damping is least amount of damping for there will be no oscillations.

19. A mass attached at the end of a spring has a natural frequency of 50 rad/sec. When the mass is increased to four times the original mass and a damper is attached, the frequency of oscillations is found to be 20 rad/sec. What is the damping ratio of the resulting system?
 (a) 60 % (b) 40 % (c) 80 % (d) 16 %

$\omega_n = 50$ rad/s. $\rightarrow$ for mass $- m$.

$\hookrightarrow \omega_n = \sqrt{\frac{K}{M}}$

mass is increased to $4m$. and damper is added.

$\omega_d = 20$ rad/s.

$\omega_n = \sqrt{\frac{K}{4m}} = \frac{1}{2}\sqrt{\frac{K}{M}} = \frac{1}{2} \times 50 = 25$ rad/s

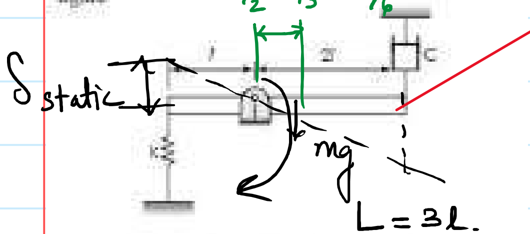
$\omega_d = \omega_n \cdot \sqrt{1-\xi^2}$

$20 = 25 \cdot \sqrt{1-\xi^2}$

$(0.8)^2 = 1-\xi^2 \Rightarrow \xi = \sqrt{1-(0.8)^2} = 0.6 = 60\%$

Common data for 21 & 22

A bar of mass m assumed to be distributed uniformly is constrained to under go small oscillations as shown in figure



Mg will cause δ_{static}

This can be assumed as equilibrium position.

21. The equivalent Inertia and stiffness of the system are

(a) (ml^2) & kl^2 ✓ (b) ($3ml^2$) & kl^2

(c) ($7ml^2/3$) & kl^2 (d) ($7ml^2/3$) & (kl^2)/2

22. The damping ratio of the system is

(a) $\dfrac{2C}{\sqrt{\frac{7km}{3}}}$ (b) $\dfrac{2C}{\sqrt{km}}$ ✓

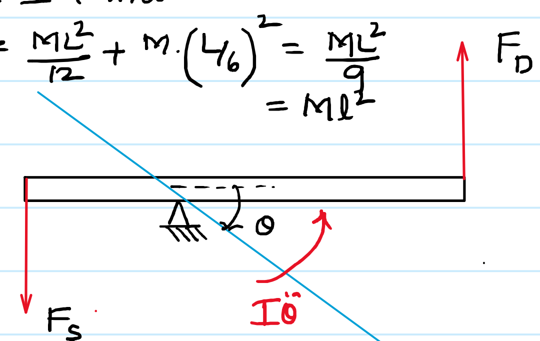
(c) $\dfrac{2C}{2\sqrt{km\,l^2}}$ (d) $\dfrac{2C}{\sqrt{3km}}$

$L = 3l$

$$I = \bar{I} + md^2$$

$$= \frac{ML^2}{12} + M\cdot\left(\frac{L}{6}\right)^2 = \frac{ML^2}{9}$$

$$= Ml^2$$



$$\Sigma M_{Hinge} = 0$$

$$I_{Hinge}\ddot{\theta} + C(\perp^{lwr}\text{ distance from hinge})^2\dot{\theta} + K.(\perp^{lwr}\text{ distance from hinge})^2\theta = 0$$

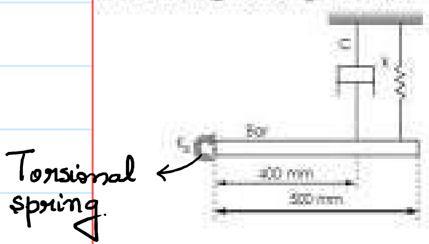
$$Ml^2\ddot{\theta} + c(2l)^2\dot{\theta} + K\cdot(l)^2\theta = 0$$

$$I_{eq} = Ml^2, \quad K_{eq} = Kl^2$$

$$\zeta = \frac{C_{eq}}{2\sqrt{I_{eq}\cdot K_{eq}}} = \frac{4Cl^2}{2\sqrt{Ml^2\cdot Kl^2}} = \frac{2c}{\sqrt{MK}}$$

Common data questions 25 & 26

A uniform rigid slender bar of mass 10 kg is hinged at the left end is suspended with the help of spring and damper arrangement as shown in the figure where K = 2kN/m, $\boxed{C = 500\ \text{Ns/m}}$ and the stiffness of the torsional spring K_t is 1 kN-m/rad. Ignore the hinge dimensions.



$L = 500$ mm. $a = 400$ mm.

25. The un-damped natural frequency of oscillations of the bar about the hinge point is → $\omega_n = ?$ (GATE-03)

(a) 42.43 rad/s (b) 30 rad/s

(c) 17.32 rad/s (d) 14.14 rad/s

$$\Sigma M_{Hinge} = 0$$

$$I_{Hinge}\ddot{\theta} + c(\perp^{lwr}\text{ distance from hinge})^2\dot{\theta} + K.(\perp^{lwr}\text{ distance from hinge})^2\theta + K_T\theta = 0$$

$$\frac{ML^2}{3}\ddot{\theta} + ca^2\dot{\theta} + KL^2\theta + K_T\theta = 0$$

$$I_{eq} = ML^2/3 \quad C_{eq} = Ca^2 \quad K_{eq} = (KL^2 + K_T)$$

$$\frac{10(0.5)^2}{3}\ddot{\theta} + 500(0.4)^2\dot{\theta} + 2000(0.5)^2\theta + 1000\theta = 0$$

$$\frac{2.5}{3}\ddot{\theta} + 80\dot{\theta} + 1500\theta = 0$$

26. The damping coefficient in the vibration equation is given by (GATE-03)

(a) 500 Nms/rad (b) 500 N/(m/s)

(c) $\boxed{\text{80 Nms/rad}}$ ✓ (d) $\boxed{\text{80 N/(m/s)}}$

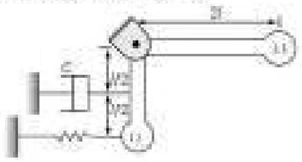
↓ Rotational system. ↓ Translatory system.

C_{eq}

$$\omega_n = \sqrt{\frac{K_{eq}}{I_{eq}}} \qquad \omega_n = \sqrt{\frac{1500}{(2.5/3)}} = 42.43\ \text{rad/s}.$$

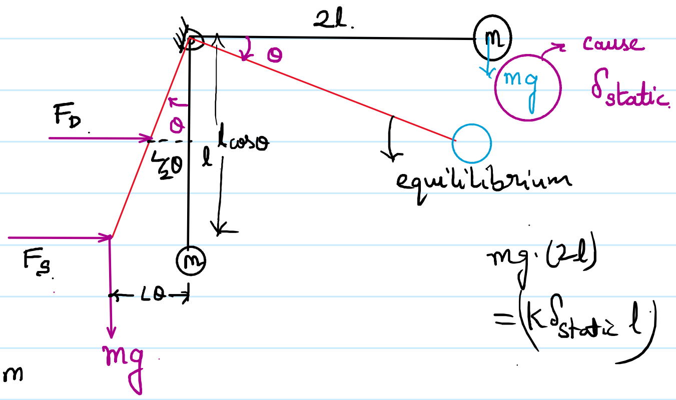
Common data for 23 & 24

In the single degree freedom system shown in figure is in free vibration l = 1 m, M = 10 kg at each end k = 400 N/m, C = 400 N-sec/m. The L shaped bar is mass less and rigid and hinged at O.



23. The undamped natural frequency of the system is
(a) 3.162 rad/sec (b) 6.32 rad/sec
(c) 4.62 rad/sec (d) 2.83rad/sec

24. The damping ratio of the system is
(a) 0.316 (b) 0.79
(c) 0.46 (d) 0.54



cause δ_{static}

equilibrium

$mg\cdot(2l) = (k\delta_{static}\, l)$

$M = 10\,kg$, $L = 1\,m$, $K = 400\,N/m$, $C = \dfrac{400\,N\text{-}sec}{m}$

$\Sigma M_{Hinge} = 0$

$$I_{Hinge}\ddot{\theta} + F_s \times L\cos\theta + F_D \times L/2\cos\theta + mg\,L\sin\theta - mg\,(2l)\cos\theta = 0$$

$$[m\cdot(2l)^2 + m(l)^2]\ddot{\theta} + C(L/2\cdot\dot{\theta})\cdot L/2 + k\cdot(\delta_{static} + L\theta)\cdot L + mgL\theta - 2mgL = 0$$

$$5ml^2\ddot{\theta} + \frac{CL^2}{4}\dot{\theta} + (kL^2\theta + mg\,L\theta) + k\delta_{static}L - 2mgL = 0$$

$$5ml^2\ddot{\theta} + \frac{cl^2}{4}\dot{\theta} + (kl^2 + mgL)\theta = 0$$

$$I_{eq}\ddot{\theta} + C_{eq}\dot{\theta} + k_{eq}\theta = 0$$

$$\omega_n = \sqrt{\frac{kL^2 + mgL}{5ml^2}}$$

$$\zeta = \frac{C_{eq}}{2\cdot\sqrt{I_{eq}\cdot k_{eq}}} = \frac{(cL^2/4)}{2\sqrt{5ml^2\cdot(kl^2 + mgL)}}$$

28. Which of the following statements are TRUE for damped vibrations?

P. For a system having critical damping, the value of damping ratio is unity and system does not undergo a vibratory motion. → $\xi = 1$ no oscillation
$x(t) = (A + Bt)e^{-\xi\omega_n t}$ → exponential decay

Q. Logarithmic decrement method is used to determine the amount of damping in a physical system.

R. In case of damping due to dry friction between moving surfaces resisting force of constant magnitude acts opposite to the relative motion.

friction force $\propto$ − velocity

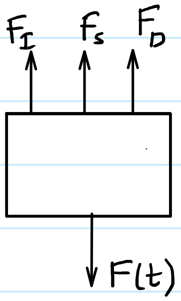
$\boxed{f \propto -\dot{x}}$

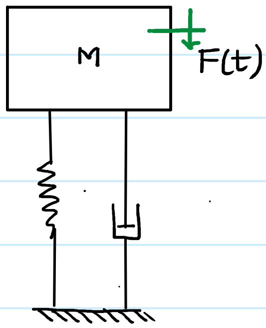
S. For the case of viscous damping, drag force is directly proportional to the square of relative velocity. (GATE –15) → $F_D \propto -\dot{x}^{1}$

(a) P and Q only
(b) P and S only
(c) $\boxed{\text{P,Q and R only}}$
(d) Q and S only

Forced Vibrations

$F(t) = F\sin(\omega t)$





$$F_I + F_s + F_D = F(t)$$

$$M\ddot{x} + C\dot{x} + Kx = F(t)$$

$$x(t) = C.F + P.I.$$

$$(M.D^2 + CD + K)x = F\sin(\omega t)$$

$$x = \frac{F\sin(\omega t)}{(MD^2 + CD + K)} \qquad \boxed{D^2 = -\omega^2} \qquad D = \frac{d}{dt}$$

$$x = \frac{F\sin(\omega t)}{(CD) + (K - M\omega^2)} \times \frac{(K - M\omega^2) - (CD)}{(K - M\omega^2) - (CD)}$$

$$x = \frac{F\sin(\omega t).(K - M\omega^2) - C.\left(\frac{d}{dt}.(F\sin(\omega t))\right)}{(K - M\omega^2)^2 - (CD)^2}$$

$$x = \frac{F\sin(\omega t)\cdot(K-M\omega^2) - C\cdot\left(\frac{d}{dt}\cdot(F\sin(\omega t))\right)}{(K-M\omega^2)^2 - (CD)^2}$$

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$$D^2 = -\omega^2$$

$$x = \frac{F\sin(\omega t)\,(K-M\omega^2) - C\cdot F\cdot\cos(\omega t)\cdot\omega}{(K-M\omega^2)^2 - (C^2(-\omega^2))}$$

$$x = \frac{F\cdot[(K-M\omega^2)\cdot\sin(\omega t) - (C\omega)\cdot\cos(\omega t)]}{(K-M\omega^2)^2 + (C\omega)^2}$$

$$K-M\omega^2 = R\cdot\cos\phi$$
$$C\omega = R\sin\phi$$
$$R = \left[(K-M\omega^2)^2 + (C\omega)^2\right]^{1/2}$$

$$x = \frac{F\cdot[R\cos\phi\cdot\sin(\omega t) - R\sin\phi\cdot\cos(\omega t)]}{R^2}$$

$$x = \frac{F\cancel{R}\cdot[\sin(\omega t-\phi)]}{R^{\cancel{2}}} \qquad x = \frac{F\sin\cdot(\omega t-\phi)}{\sqrt{(K-M\omega^2)^2 + (C\omega)^2}}$$

$$x(t) = C.F + P.I.$$
$$= \underbrace{X_o\cdot e^{-\xi\omega_n\cdot t}\sin\cdot(\omega_d t+\phi)}_{\text{Transient Response}} + \underbrace{\frac{F\sin(\omega t-\phi)}{\sqrt{(K-M\omega^2)^2+(C\omega)^2}}}_{\text{steady state Response}}$$

As $t\to\infty$ $\quad e^{-\xi\omega_n t} \longrightarrow 0$ $\quad$ Transient Response $\longrightarrow 0$

$$x(t) = \frac{F\sin(\omega t-\phi)}{\sqrt{(K-M\omega^2)^2+(C\omega)^2}}$$

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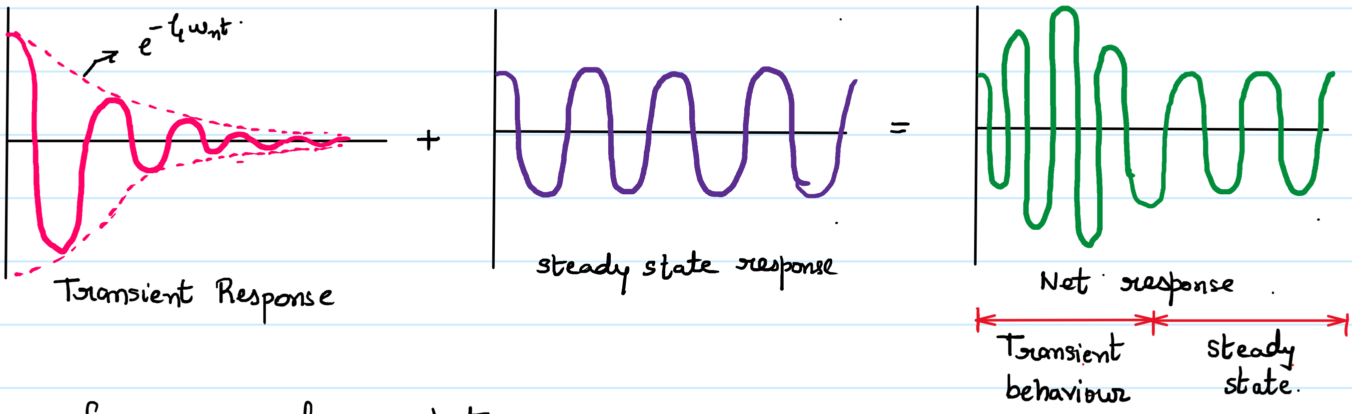
$$x(t) = C.F + P.I.$$

$$= X_0 \, e^{-\zeta\omega_n t} \sin(\omega_d t + \phi) + \frac{F\sin(\omega t - \phi)}{\sqrt{(K - M\omega^2)^2 + (c\omega)^2}}$$



ω – frequency of excitation.

for static loading $\omega = 0$ $F(t) = F$

F = constant
K = constant

$$x = \frac{F}{\sqrt{(K - M\omega^2)^2 + (c\omega)^2}} \qquad x_{static} = \frac{F}{K}$$

(with $\omega^2 \to 0$ and $(c\omega)^2 \to 0$)

Response of forced vibration

$$x(t) = \frac{F\sin(\omega t - \phi)}{\sqrt{(K - M\omega^2)^2 + (c\omega)^2}} \longrightarrow x(t) = A \cdot \sin(\omega t - \phi)$$

$$A = \frac{F}{\sqrt{(K - M\omega^2)^2 + (c\omega)^2}}$$

A – Amplitude of forced vibration.

$$x(t) = \frac{\left(\frac{F}{K}\right) \cdot \sin(\omega t - \phi)}{\sqrt{\left(\frac{K}{K} - \frac{M}{K}\omega^2\right)^2 + \left(\frac{c\omega}{K}\right)^2}}$$

$$\frac{M}{K} = \frac{1}{\omega_n^2}$$

$$x(t) = \frac{x_{static} \; \sin(\omega t - \phi)}{\sqrt{\left(1 - \frac{M}{K}\omega^2\right)^2 + \left(\frac{c\omega}{K}\right)^2}}$$

$$\frac{c\omega}{K} = \frac{2c \cdot \omega}{2\sqrt{KM}} \cdot \frac{\sqrt{M}}{\sqrt{K}}$$

$$= 2 \cdot \frac{c}{2\sqrt{MK}} \cdot \sqrt{\frac{M}{K}} \cdot \omega$$

$$= 2\zeta \cdot \frac{\omega}{\omega_n} = 2\zeta r$$

$$x(t) = \frac{x_{static} \; \sin(\omega t - \phi)}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}$$

r – ratio of frequencies – $\dfrac{\text{excitation frequency}}{\text{Natural frequency}}$ $\qquad r = \dfrac{\omega}{\omega_n}$

ζ – Damping Ratio

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Dynamic Magnification factor – It is the ratio of Amplitude of forced vibration to static deflection.

$$D.M.F = \frac{A}{x_{static}} = \frac{\dfrac{\cancel{F}}{\sqrt{(k-M\omega^2)^2+(c\omega)^2}}}{\left(\cancel{F}/k\right)} = \frac{1}{\sqrt{\left[\left(\frac{k}{k}-\frac{M\omega^2}{k}\right)^2+\left(\frac{c\omega}{k}\right)^2\right]}}$$

$$D.M.F = \frac{1}{\sqrt{(1-r^2)^2+(2\xi r)^2}}$$

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$$D.M.F = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

$$D.M.F = f(\zeta, r) = f(r)$$

$\zeta < 1$ (under damped)

$$r = \frac{\omega}{\omega_n} = \frac{\text{forcing frequency}}{\text{Natural frequency}}.$$

So D.M.F = Max. then $\frac{d}{dr}(D.M.F) = 0$

$$\frac{d}{dr}\left(\frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}\right) = 0$$

$$-\frac{1}{2}\left[(1-r^2)^2 + (2\zeta r)^2\right]^{-\frac{1}{2}-1} \times \left[2.(1-r^2).(0-2r) + 2.(2\zeta r).(2\zeta)\right] = 0$$

$$-4r(1-r^2) + 8\zeta^2 . r = 0$$

$$-1 + r^2 + 2\zeta^2 = 0$$

$$r^2 = 1 - 2\zeta^2$$

$$\boxed{r = \sqrt{1-2\zeta^2}}$$

$$\frac{\omega}{\omega_n} = \sqrt{1-2\zeta^2} \rightsquigarrow \boxed{\omega_p = \omega_n . \sqrt{1-2\zeta^2}}$$

ω_p — frequency at which peak value of D.M.F/Amplitude of forced vibration occurs.

$$D.M.F = \frac{A \rightsquigarrow \text{Variable.}}{x_{static} \rightsquigarrow \text{constant}}$$

D.M.F vs r Graph

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$$D.M.F = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

1. $r = 0$ $\quad \zeta = 0$ (undamped system) $\quad D.M.F = 1$

   $\hookrightarrow \omega = 0$ static loading.

2. $r = 0$ $\quad \zeta \neq 0$ $\quad D.M.F = 1$. ✓

3. $r = 1$ $\quad \zeta = 0$ $\quad D.M.F = \infty$

4. $r = 1$ $\quad \zeta \neq 0$ $\quad D.M.F = (1/2\zeta) \longrightarrow \zeta = 1 \quad r = 1 \quad D.M.F = 0.5$

5. $r = \sqrt{2}$ $\quad \zeta = 0$ $\quad D.M.F = 1$.

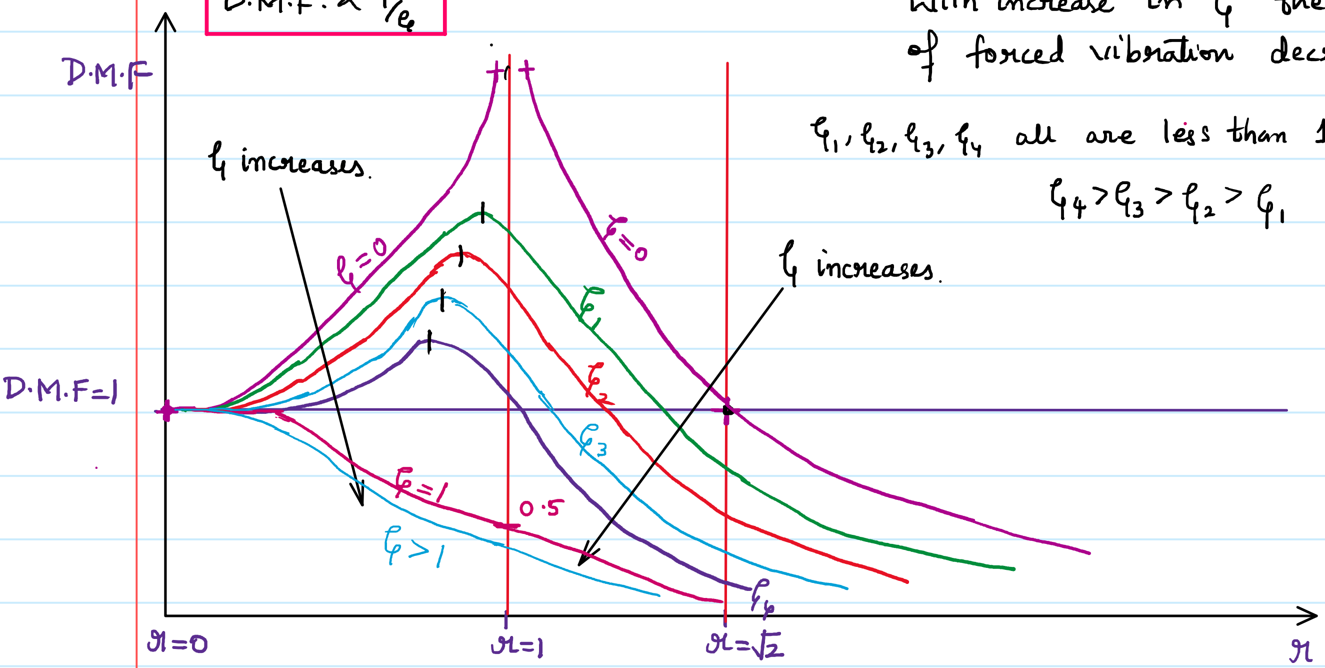
6. $r = \sqrt{2}$ $\quad \zeta \neq 0$ $\quad D.M.F = \dfrac{1}{\sqrt{1+8\zeta^2}}$

$\boxed{A \propto 1/\zeta.}$

$D.M.F \propto \dfrac{1}{\zeta}$ $\quad \dfrac{A}{x_{static}} \propto \dfrac{1}{\zeta}$

Constant

$\boxed{D.M.F. \propto 1/\zeta}$



With increase in ζ the amplitude of forced vibration decreases.

$\zeta_1, \zeta_2, \zeta_3, \zeta_4$ all are less than 1.

$\zeta_4 > \zeta_3 > \zeta_2 > \zeta_1$

$r = \sqrt{1-2\zeta^2} \rightarrow D.M.F = Max.$

As $\quad \zeta \uparrow \quad 2\zeta^2 \uparrow\uparrow \quad 1-2\zeta^2 \downarrow. \quad r \downarrow. \quad \omega_p \downarrow$

As the value of ζ increases the peak of D.M.F vs r graph tends to shift towards Leftside.

For damped systems the peak value of D.M.F occurs at r slightly less than 1.

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Response of forced Vibration.

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$$x(t) = X_0 \cdot e^{-\zeta\omega_n t} \sin(\omega_d t + \phi) + \frac{F}{[(K-M\omega^2)^2 + (C\omega)^2]^{0.5}} \sin(\omega t - \phi)$$

0

as $t \to \infty$

Phase Angle.

$R\cos\phi = K - M\omega^2, \quad R \cdot \sin\phi = C\omega.$

$$\frac{C\omega}{K} = \frac{C\omega}{2\sqrt{MK}} \cdot \frac{\sqrt{M}}{\sqrt{K}} \times 2$$

$$\operatorname{Tan}\phi = \frac{C\omega}{K - M\omega^2}$$

$$= 2 \cdot \frac{C}{2\sqrt{MK}} \cdot \frac{\omega}{\omega_n}$$

$$\operatorname{Tan}\phi = \frac{\frac{C\omega}{K}}{1 - \frac{M}{K}\cdot\omega^2} = \frac{2\zeta\left(\frac{\omega}{\omega_n}\right)}{1 - \left(\frac{\omega}{\omega_n}\right)^2} = \frac{2\zeta r}{1 - r^2}$$

$$= 2\zeta r.$$

Phase Angle vs. r Graph.

$0 < r < 1 \rightsquigarrow \operatorname{Tan}\phi = \frac{2\zeta r}{1 - r^2}$ $\quad \boxed{\operatorname{Tan}\phi \propto \zeta}$

$1 < r < \sqrt{2}$

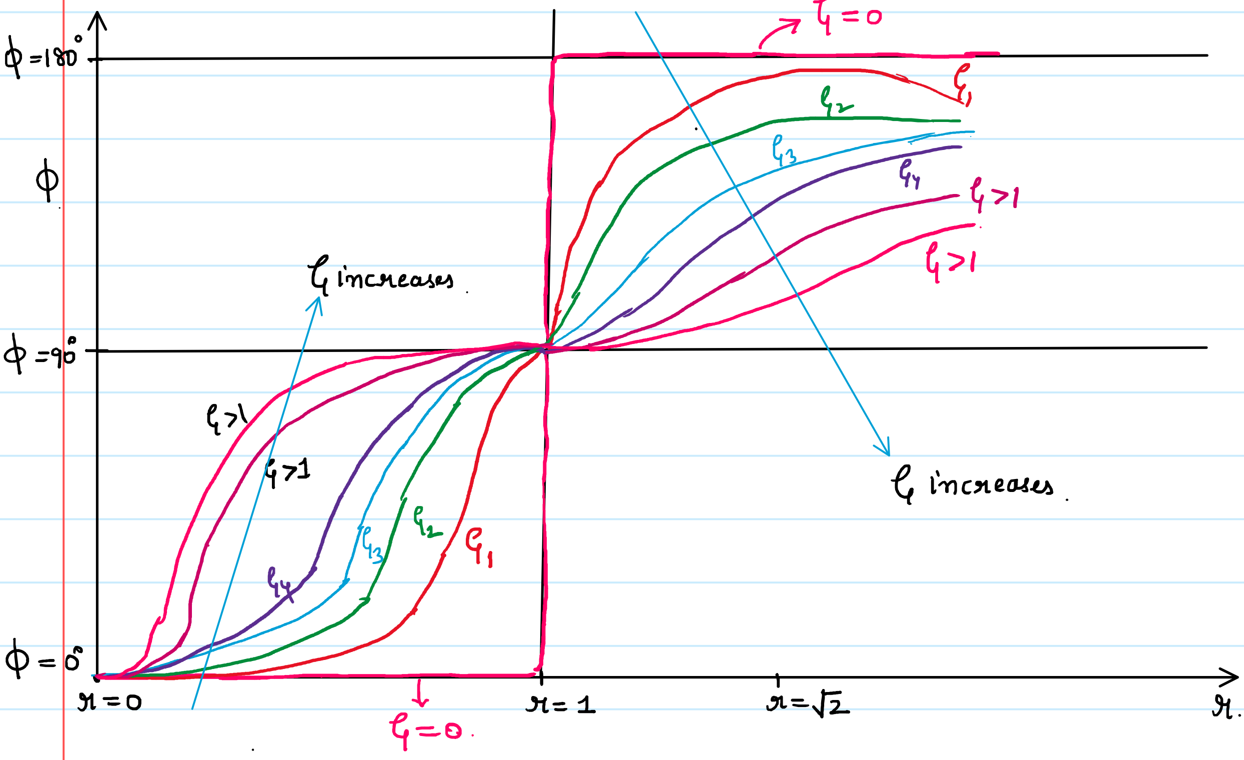
$\hookrightarrow \phi = 180 - \operatorname{Tan}^{-1}\left(\frac{2\zeta r}{1 - r^2}\right)$

As. $\zeta \uparrow \quad 180 - \operatorname{Tan}^{-1}\left(\frac{2\zeta r}{1 - r^2}\right) \downarrow \quad \phi \downarrow$

1. $r = 0 \quad \zeta = 0 \quad \phi = 0^\circ$
2. $r = 0 \quad \zeta \neq 0 \quad \phi = 0^\circ$
3. $r = 1 \quad \zeta = 0 / \zeta \neq 0 \quad \phi = 90^\circ$
4. $r = \sqrt{2} \quad \zeta = 0 \quad \operatorname{Tan}\phi = \frac{2 \times 0 \times \sqrt{2}}{1 - (\sqrt{2})^2} \Rightarrow \operatorname{Tan}\phi = 0^- \quad \phi = \operatorname{Tan}^{-1}(0^-) = 180^\circ$
5. $r = \sqrt{2} \quad \zeta \neq 0 \quad \operatorname{Tan}\phi = \frac{2\zeta \cdot \sqrt{2}}{1 - (\sqrt{2})^2} \Rightarrow \operatorname{Tan}\phi = -2\sqrt{2}\zeta \Rightarrow \phi = \operatorname{Tan}^{-1}(-2\sqrt{2}\zeta)$

$$\downarrow$$

$$\phi = 180 - \operatorname{Tan}^{-1}(2\sqrt{2}\zeta)$$



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Amplitude of Forced Vibration

Displacement $x(t) = A \cdot \sin(\omega t - \phi)$

$$A = \frac{F}{\left[(k - M\omega^2)^2 + (c\omega)^2\right]^{0.5}}$$

$$A = \frac{x_{static}}{\left[(1 - r^2)^2 + (2\xi r)^2\right]^{0.5}}$$

Velocity $\dot{x}(t) = A \cdot \omega \cdot \cos(\omega t - \phi) = A\omega \cdot \sin\left(\frac{\pi}{2} + (\omega t - \phi)\right)$

Acceleration $\ddot{x}(t) = -A\omega^2 \sin(\omega t - \phi) = A\omega^2 \sin(\pi + \omega t - \phi)$

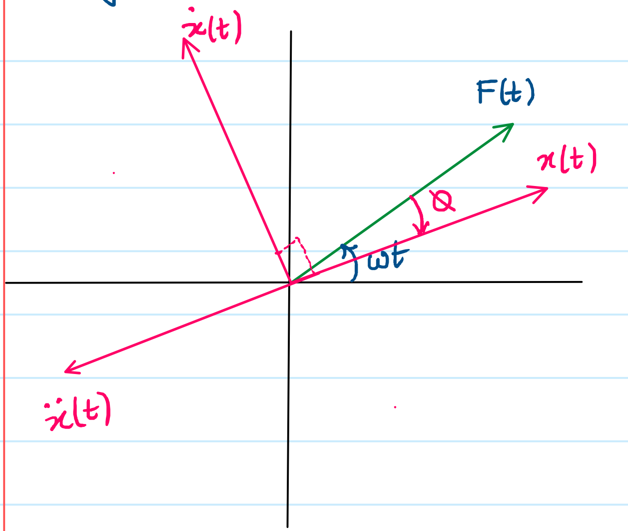
$$M\ddot{x} + c\dot{x} + kx = F(t) \qquad F(t) = F\sin(\omega t)$$

$$M\cdot(-A\omega^2 \sin(\omega t - \phi)) + c\cdot(A\omega \cdot \cos(\omega t - \phi)) + k \cdot A\sin(\omega t - \phi) = F\sin(\omega t)$$

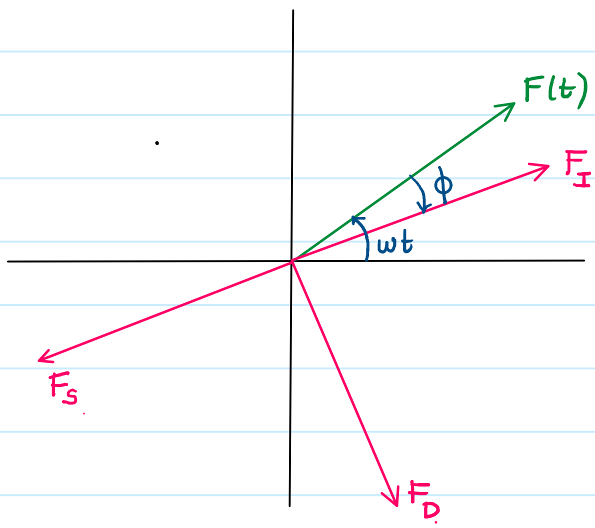
$$F\sin(\omega t) + MA\omega^2 \sin(\omega t - \phi) - cA\omega\cos(\omega t - \phi) - kA\sin(\omega t - \phi) = 0$$

$$\underbrace{F\sin(\omega t)}_{\vec{F}_{ext}} + \underbrace{MA\omega^2 \sin(\omega t - \phi)}_{\vec{F}_I} + \underbrace{cA\omega \cdot \sin\left(\frac{3\pi}{2} + \omega t - \phi\right)}_{\vec{F}_D} + \underbrace{kA \cdot \sin(\pi + \omega t - \phi)}_{\vec{F}_S} = 0$$

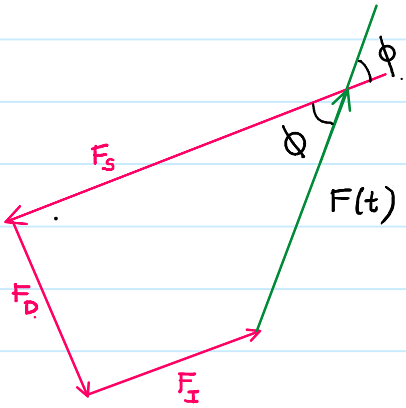
Phase Angle b/w $\vec{F}_{ext}$, displacement, velocity & Acceleration



Phase Angle b/w $\vec{F}_{ext}$, F_I, $\vec{F}_D$, $\vec{F}_S$



ϕ – Angle subtended b/w $x(t)$ [Displacement of vibration] (o/p variable) and $F_{external}$ (i/p variable). [External force / Excitation]

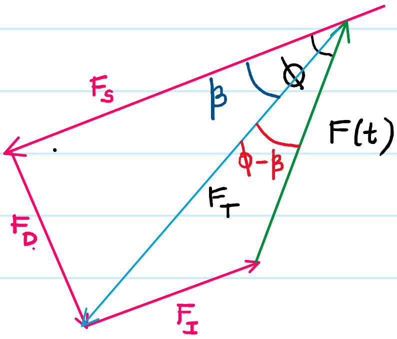


$$\tan\phi = \frac{F_D}{F_S - F_I}$$

$$\tan\phi = \frac{CA\omega}{KA - MA\omega^2}$$

$$\tan\phi = \frac{C\omega}{K - M\omega^2} = \frac{2\xi r}{1 - r^2}$$

F_T – Force transmitted to the foundation.



$$F_T = \sqrt{F_S^2 + F_D^2}$$

$$F_T = \sqrt{(KA)^2 + (CA\omega)^2}$$

$$F_T = \sqrt{K^2 + (C\omega)^2} \cdot A$$

$$\tan\beta = \frac{F_D}{F_S} = \frac{C\omega}{K} = 2\xi r$$

$(\phi - \beta)$ = Angle b/w. force transmitted to foundation and applied force.

$$\phi - \beta = \tan^{-1}\left(\frac{2\xi r}{1 - r^2}\right) - \tan^{-1}(2\xi r)$$

$$A = \frac{F}{\sqrt{(K - M\omega^2)^2 + (C\omega)^2}}$$

$$F_T = \sqrt{K^2 + (C\omega)^2} \cdot \frac{F}{\sqrt{(K - M\omega^2)^2 + (C\omega)^2}}$$

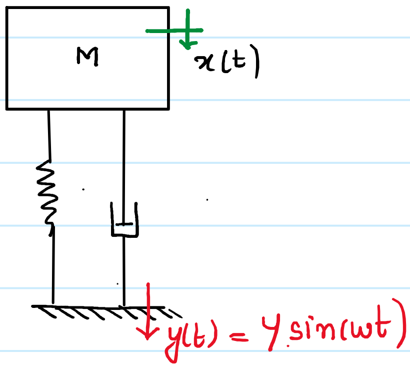
$$\boxed{\epsilon = \frac{F_T}{F} = \sqrt{\frac{K^2 + (C\omega)^2}{(K - M\omega^2)^2 + (C\omega)^2}}}$$

ϵ – Transmissibility Ratio $\quad \epsilon = \dfrac{F_T}{F} = \dfrac{\text{force transmitted to foundation}}{\text{force applied}}$

$$\epsilon = \frac{F_T}{F} = \sqrt{\frac{K^2 + (C\omega)^2}{(K - M\omega^2)^2 + (C\omega)^2}} = \sqrt{\frac{\frac{K^2 + (C\omega)^2}{K^2}}{\frac{(K - M\omega^2)^2 + (C\omega)^2}{K^2}}} = \sqrt{\frac{1 + \left(\frac{C\omega}{K}\right)^2}{\left(1 - \frac{M}{K}\cdot\omega^2\right)^2 + \left(\frac{C\omega}{K}\right)^2}}$$

$$\boxed{\epsilon = \sqrt{\frac{1 + (2\xi r)^2}{(1 - r^2)^2 + (2\xi r)^2}}}$$

Base Excitation



$$M\ddot{x} + c(\dot{x} - \dot{y}) + k.(x - y) = 0$$

$$M\ddot{x} + c\dot{x} + kx = c\dot{y} + ky$$

$$M\ddot{x} + c\dot{x} + kx = c.Y.\omega\cos(\omega t) + k.Y.\sin(\omega t)$$

$$kY = R\cos\alpha \qquad cY\omega = R\sin\alpha$$

$$M\ddot{x} + c\dot{x} + kx = R\sin(\omega t + \alpha)$$

Displacement

$$x(t) = \frac{R}{\sqrt{(k - M\omega^2)^2 + (c\omega)^2}}\sin(\omega t + \alpha - \phi_1) \qquad R = \sqrt{k^2 + (c\omega)^2}\,.\,Y$$

$$x(t) = X.\sin(\omega t + \alpha - \phi_1) \qquad Y = \frac{R}{\sqrt{k^2 + (c\omega)^2}}$$

$$\frac{X}{Y} = \frac{\dfrac{R}{\sqrt{(k - M\omega^2)^2 + (c\omega)^2}}}{\dfrac{R}{\sqrt{k^2 + (c\omega)^2}}} = \sqrt{\frac{k^2 + (c\omega)^2}{(k - M\omega^2)^2 + (c\omega)^2}} = \sqrt{\frac{1 + (2\xi r)^2}{(1 - r^2)^2 + (2\xi r)^2}}$$

X – Amplitude of vibration for mass 'm'

Y – Amplitude of displacement given at the base

Transmissibility Ratio vs r Graph

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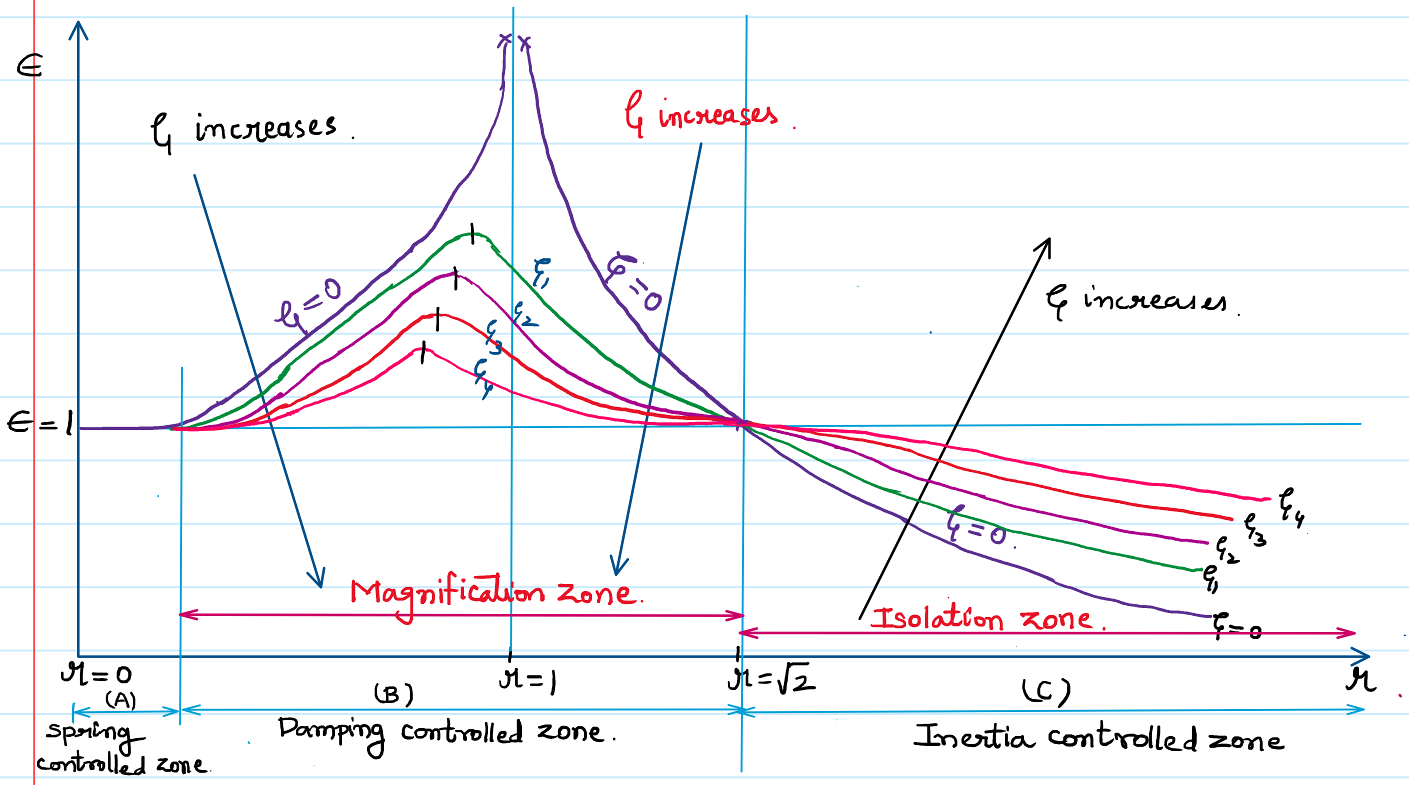
$$\epsilon = \frac{X}{Y} = \frac{F_T}{F} = \sqrt{\frac{1+(2\zeta r)^2}{(1-r^2)^2+(2\zeta r)^2}}$$

$\zeta < 1 \longrightarrow$ underdamped system.

1. $r=0 \quad \zeta = 0 \quad \epsilon = 1$
2. $r=0 \quad \zeta \neq 0 \quad \epsilon = 1.$
3. $r=1 \quad \zeta = 0 \quad \epsilon = \infty$
4. $r=1 \quad \zeta \neq 0 \quad \epsilon = \frac{\sqrt{1+4\zeta^2}}{2\zeta}$

$\boxed{\epsilon \propto \frac{1}{\zeta}}$

5. $r=\sqrt{2} \quad \zeta = 0 \quad \epsilon = 1$
6. $r=\sqrt{2} \quad \zeta \neq 0 \quad \epsilon = 1.$



$r <<< 1 \quad . \quad r^2 \to 0 \qquad (1-r^2)^2 + (2\zeta r)^2 \longrightarrow 1, \qquad 1+(2\zeta r)^2 \longrightarrow 1.$

$\epsilon \to 1. \Rightarrow X = Y, \quad F_T = F$

$r <<< 1 \qquad \frac{\omega}{\omega_n} <<< 1 \qquad \omega_n >>> \omega \qquad \sqrt{\frac{K}{M}} >>> \omega. \to \quad K \uparrow\uparrow\uparrow \checkmark$

lower excitation frequency. $\quad M \downarrow\downarrow\downarrow \times$

(A) – Spring controlled zone. $\quad X \approx Y \quad F_T \approx F \quad$ Sesimograph.

$(r<1, \zeta<1)$, $1+(2\zeta r)^2 \longrightarrow 1$, $(1-r^2)^2+(2\zeta r)^2$ = Decimal value.

$\epsilon > 1 \longrightarrow \quad \epsilon \propto \frac{1}{\zeta}$

$X > Y, \; F_T > F$

$(0<r<1) \longrightarrow \quad \epsilon \propto \frac{1}{\zeta}$

$(1<r<\sqrt{2}) \quad \epsilon = \sqrt{\dfrac{1+(2\zeta r)^2}{(1-r^2)^2+(2\zeta r)^2}} \rightarrow \epsilon \propto \frac{1}{\zeta}$

Damper is supressing the vibration Amplitude or Transmissibility Ratio.

$\sqrt{2} < r <$ large finite value

$r\uparrow \quad (1-r^2)^2 \uparrow\uparrow\uparrow \quad (1-r^2)^2+(2\zeta r)^2 \longrightarrow (1-r^2)^2 \quad \epsilon \rightarrow \dfrac{\sqrt{1+(2\zeta r)^2}}{(1-r^2)}$

$\boxed{\epsilon \propto \zeta}$

$\dfrac{\omega}{\omega_n} >> 1 \qquad \omega_n << \omega. \qquad \dfrac{K}{M} << \omega.$

K is decreasing ✗
M is increasing ✓

As $r\uparrow\uparrow \quad (1-r^2)^2 \uparrow\uparrow\uparrow\uparrow \quad \epsilon \rightarrow 0$

$r=1 \quad A = Max. \quad \epsilon = Max. \longrightarrow$ dangerous situation.

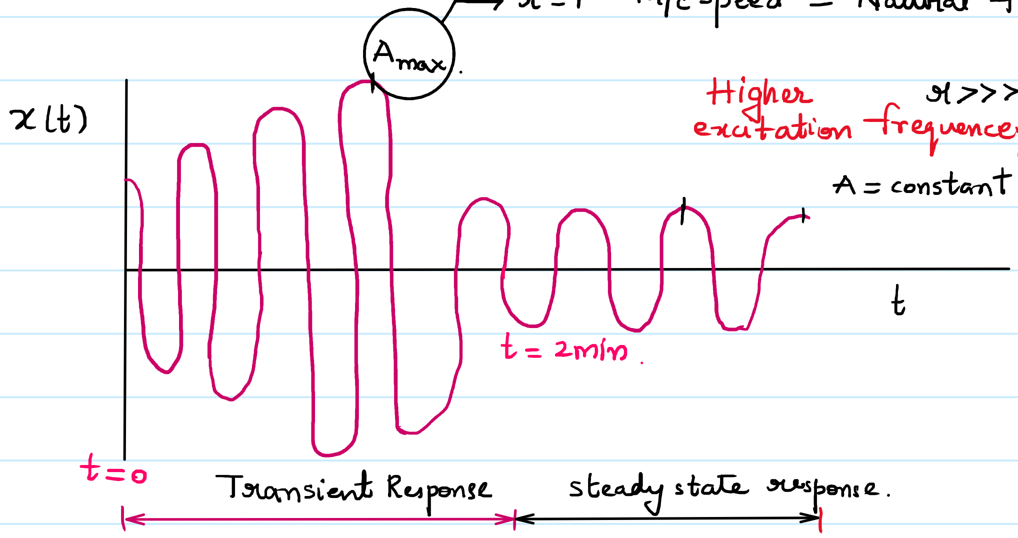
$\longrightarrow$ Resonance.

$\longrightarrow r=1$ M/c speed = Natural frequency of system.

Higher excitation frequency.

$r>>>1 \rightsquigarrow \epsilon \rightarrow 0 \quad F_T \rightarrow 0$

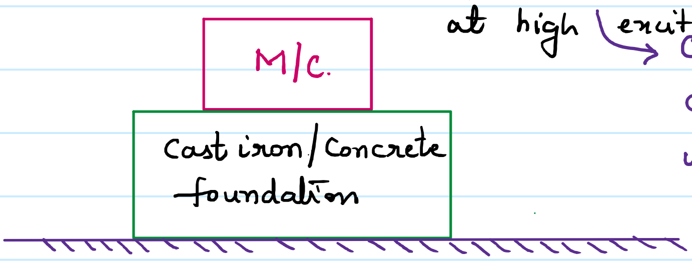
A = constant.



At $r >>> 1$ (Higher excitation frequency)

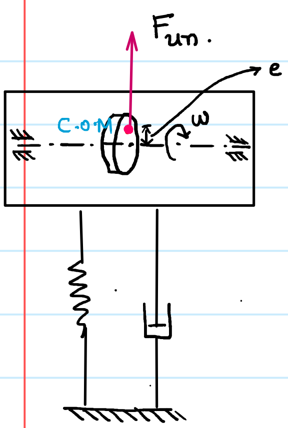
$M \rightarrow$ inertia controlled zone.

M/c can be operated at high excitation frequency



Combined centre of mass of machine and foundation will shift towards the ground. Isolation occurs.

Excitation of System due to unbalanced Rotating Mass



$$F_{un} = m e \omega^2 \sin(\omega t)$$

m – Mass of rotor

M – Mass of system.

e – eccentric b/w C.O.M. and axis of rotation

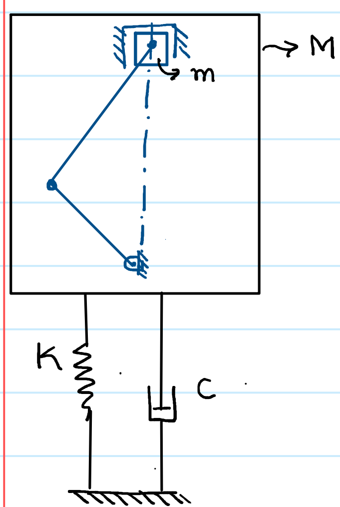
ω – excitation frequency / speed of rotor.

E.O.M.

$$M\ddot{x} + c\dot{x} + kx = m e \omega^2 \sin(\omega t)$$

$$\text{Amplitude of forced vibration} = \frac{m e \omega^2}{\sqrt{(k - M\omega^2)^2 + (c\omega)^2}}$$

Excitation due to reciprocating mass.



M – mass of system, m – mass of piston,
ω – Excitation frequency / crank speed. r – crank lenght.

$$F_{piston} = m_p \cdot a_p = m r \omega^2 \left[\cos\theta + \frac{\cos 2\theta}{n}\right]$$

$$M\ddot{x} + c\dot{x} + kx = m r \omega^2 \left[\cos\theta + \frac{\cos 2\theta}{n}\right]$$

0

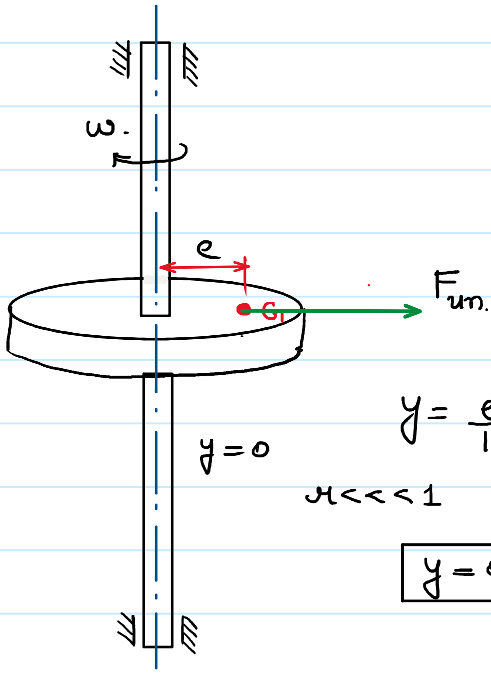
$n > 1$

$$M\ddot{x} + c\dot{x} + kx = m r \omega^2 \cos(\omega t)$$

Amplitude of forced vibration

$$A = \frac{m r \omega^2}{\sqrt{(k - M\omega^2)^2 + (c\omega)^2}}$$

Whirling of Shafts



$$F_{un} = m e \omega^2$$

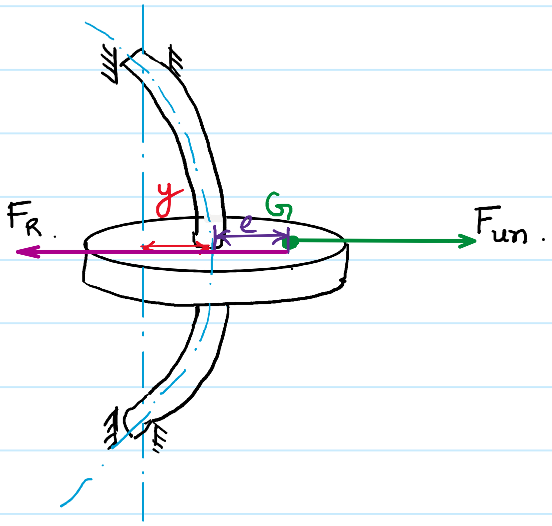
m – mass of rotor
e – eccentricity b/w C.O.M. of rotor and axis of rotation.
ω – speed of rotation.
k – stiffness of shaft.

$$y = \frac{e r^2}{1-r^2}$$

$r << 1 \quad r^2 \to 0$

$$\boxed{y = 0}$$

y – Deflection in shaft due to F_{un} / Radial of whirl / Amplitude of vibration.



Direction y and F_{un} is in same sense / same phase.

$$F_{un} = m\cdot(y+e)\,\omega^2$$

$$F_R = k y$$

$$F_R = F_{un}$$

$$k y = m\cdot(y+e)\,\omega^2$$

$$\frac{k}{m}\cdot y = \omega^2\cdot y + \omega^2\cdot e$$

$$\omega_n^2\cdot y = \omega^2\cdot y + \omega^2\cdot e$$

$$(\omega_n^2-\omega^2)\cdot y = \omega^2\cdot e$$

$r = \frac{\omega}{\omega_n}$

$$y = \frac{e\cdot\omega^2}{\omega_n^2-\omega^2} = \frac{e}{\left(\frac{\omega_n}{\omega}\right)^2-1} = \frac{e}{\frac{1}{r^2}-1}$$

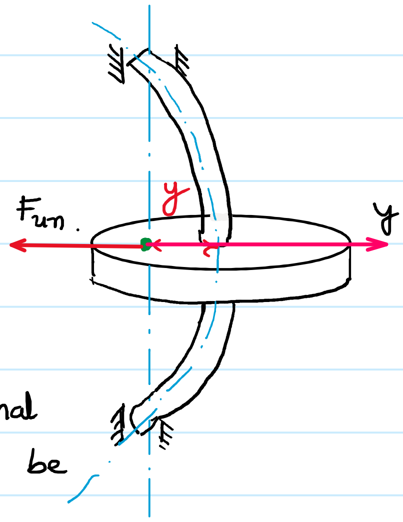
$$y = \frac{e r^2}{1-r^2}$$

$y = \frac{er^2}{1-r^2}$ if $r \to 1$. $y \to \infty$ resonance occurs.

$r = 1$ $\omega = \omega_n \longrightarrow$ Whirling speed – The speed at which the shaft will execute voilent oscillation is called whirling speed or critical speed.

if $r > 1$. $1 - r^2 = -ve$ value. $y = -ve$.

$$y = -\frac{er^2}{(1-r^2)}$$

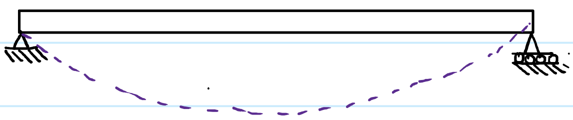


Fun and y are in opposite phase.

if ω is above resonant frequency then External force and displacement of forced vibration will be in opposite phase.

$r >>> 1$ $1 - r^2 \approx -r^2$ $y = \frac{er^2}{1-r^2} \rightarrow y = \frac{er^2}{-r^2}$ $y = -e$

At very high excitation frequencies the C.O.M. of rotor will coincide with shaft axis.



Deflection. $EI.\frac{d^2y}{dx^2} = M_{x-x}$.

$$EI.\frac{d^3y}{dx^3} = \frac{dM_{x-x}}{dx} = S.F_{x-x}.$$

$$EI.\frac{d^4y}{dx^4} = \frac{d(S.F)_{x-x}}{dx} = w(x) = \text{Rate of Loading}.$$

Boundary cond. $x=0 \quad y=0, \qquad x=0 \quad M_{x-x}=0 \Rightarrow \frac{d^2y}{dx^2}=0$

$x=L \quad y=0 \qquad x=L \quad M=0 \Rightarrow \frac{d^2y}{dx^2}=0$

$$EI.\frac{d^4y}{dx^4} - w(x) = 0 \qquad w(x) = m.y.\omega^2$$

m – mass per unit lenght of shaft.

y. – displacement due transverse vibration.

$$\frac{d^4y}{dx^4} - \frac{m.\omega^2}{EI}.y = 0 \qquad \frac{m.\omega^2}{EI} \rightarrow \lambda^4$$

$$(D^4-\lambda^4)y = 0$$

$$(D^2+\lambda^2)(D^2-\lambda^2).y = 0 \Rightarrow D = \pm i\lambda, \pm\lambda$$

C.F.

$$y = A\sin(\lambda x) + B\cos(\lambda x) + C\sinh(\lambda x) + D.\cosh(\lambda x)$$

(i) $x=0, \; y=0$ (ii) $x=0 \quad M=0 \quad \frac{d^2y}{dx^2}=0$

(iii) $x=L \quad y=0$ (iv) $x=L \quad M=0 \quad \frac{d^2y}{dx^2}=0$

After. putting B.C. in C.F.

$$A\sin(\lambda L) = 0 \Rightarrow \lambda L = 0, \pi, 2\pi, 3\pi, \ldots n\pi$$

$$\lambda^4 = \frac{m\omega^2}{EI} \Rightarrow \left[\frac{m\omega^2}{EI}\right]^{1/4}.L = \pi, 2\pi, 3\pi, \ldots n\pi.$$

$$\left[\frac{m\omega^2}{EI}\right]^{1/4}.L = \pi, 2\pi, 3\pi, \ldots n\pi$$

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$$\frac{m\omega^2}{EI}.L^4 = \pi^4, (2\pi)^4, (3\pi)^4 \ldots (n\pi)^4$$

$$\omega^2 = \pi^4.\frac{EI}{mL^4}, \quad (2\pi)^4.\frac{EI}{mL^4}, \quad \ldots\ldots \quad (n\pi)^4.\frac{EI}{mL^4}$$

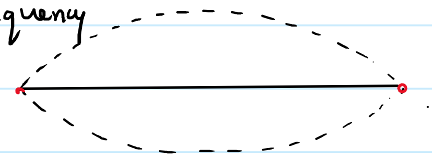
$$\omega = \underbrace{\pi^2\sqrt{\frac{EI}{mL^4}}}_{\omega_1}, \quad (2\pi)^2\sqrt{\frac{EI}{mL^4}}, \quad \ldots\ldots \quad (n\pi)^2\sqrt{\frac{EI}{mL^4}}$$

$$\omega = \omega_1, \quad 2^2.\omega_1, \quad 3^2\omega_1, \quad \ldots\ldots \quad n^2.\omega_1$$

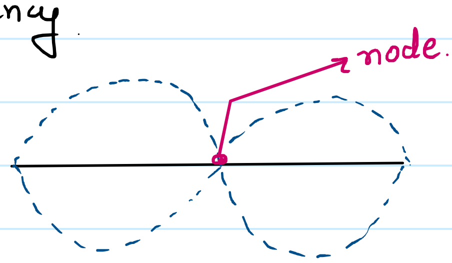
$$\omega_1, \quad \omega_2, \quad \omega_3, \quad \ldots\ldots$$

Mode shapes

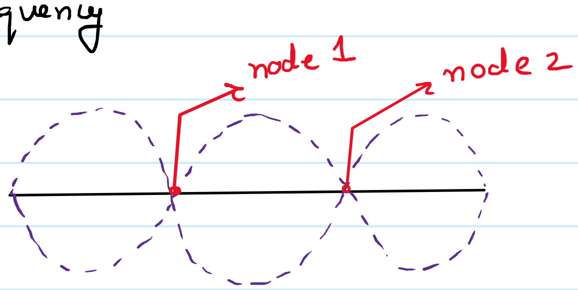
At ω_1 frequency



At ω_2. frequency



At ω_3 frequency



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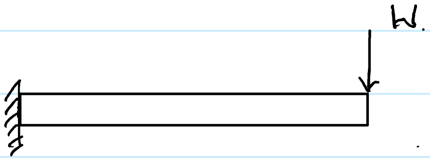
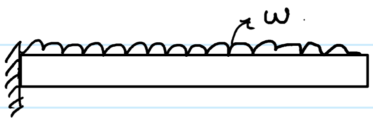
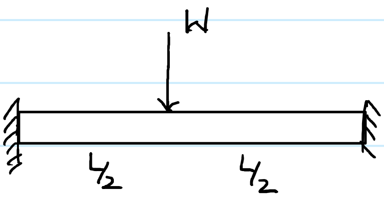
Deflection of beam.



$$\delta = \frac{WL^3}{3EI}$$



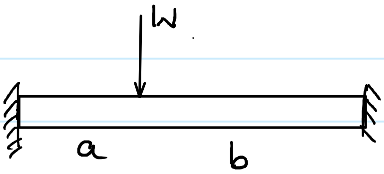
$$\delta = \frac{wL^4}{8EI}$$



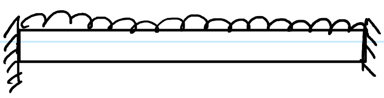
$a = L/2$
$b = L/2$

$$\delta = \frac{W \cdot (L/2)^3 \cdot (L/2)^3}{3EIL^3} = \frac{WL^3}{192EI} = \frac{1}{4}\left(\frac{WL^3}{48EI}\right)$$

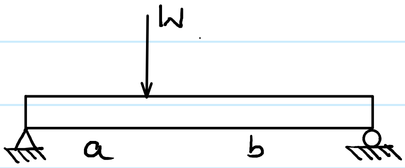
$$\delta_{f.B} = \frac{1}{4} \cdot \delta_{S.S.B}$$



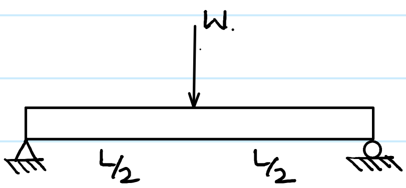
$$\delta = \frac{Wa^3 \cdot b^3}{3EIL^3}$$



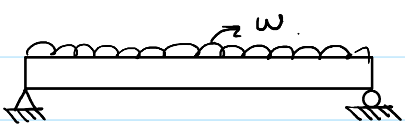
$$\delta = \frac{wL^4}{384EI}$$



$$\delta = \frac{Wa^2 \cdot b^2}{3EIL}$$



$$\delta = \frac{W \cdot (L/2)^2 \cdot (L/2)^2}{3EIL} = \frac{WL^3}{48EI}$$



$$\delta = \frac{5wL^4}{384EI}$$

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29. In a spring mass damper system excited by a harmonic force with a damping of 10%. The amplitude at resonance is found to be 10 cm. What is the amplitude at half the resonance frequency. (GATE-16)

(a) 9.81 cm (b) 4.9 cm
(c) 7.35 cm (d) $\boxed{2.64 \text{ cm}}$

29Q. $\xi = 10\% = 0.1$

$A = 10\text{cm}$ @ $r = 1$.

@ $r = 0.5$ $A = ?$

@ resonance. $\dfrac{A}{x_{static}} = \dfrac{1}{2\xi} \Rightarrow \dfrac{10}{x_{static}} = \dfrac{1}{2\times 0.1}$

$x_{static} = 2\text{cm}$.

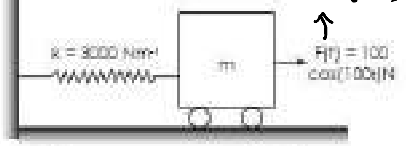
Amplitude @ $r = 0.5$.

$$A = \frac{x_{static}}{\left[(1-r^2)^2 + (2\xi r)^2\right]^{0.5}}$$

$$A = \frac{2}{\left[(1-0.5^2)^2 + (2\times 0.1\times 0.5)^2\right]^{0.5}} = 2.64.$$

30. A mass m attached to a spring is subjected to a harmonic force as shown in figure. The amplitude of the forced motion is observed to be 50 mm. The value of m (in kg) is (GATE-10)

$F(t) = F\sin(\omega t)$



(a) $\boxed{0.1}$ (b) 1.0
(c) 0.3 (d) $\boxed{0.5}$

30Q. $A = 50\text{mm}$.

$m = ?$ $k = 3000\text{N/m}$.

$F = 100\text{N}$ $\omega = 100 \text{ rad/s}$.

$C = 0$

$$A = \frac{F}{\left[(k-M\omega^2)^2 + (c\omega)^2\right]^{0.5}}$$

$$A = \left|\frac{F}{\left[(k-M\omega^2)^2 + (c\omega)^2\right]^{0.5}}\right| \qquad A = \frac{\pm F}{k - M\omega^2}$$

$$50\times 10^{-3} = \frac{\pm 100}{3000 - M(100)^2} \qquad M = 0.1\text{kg}, 0.5\text{kg}$$

Common data for 31 & 32

In a spring mass damper system k = 100 N/m and m = 1 kg under free vibrations the amplitude decays to half the initial value in every oscillation

31. The damping coefficient of the system is

(a) 2.19 N-sec /m (b) 3.29 N-sec/cm
(c) 5.15 N-sec/m (d) 3.45 N-sec/m

32. When a force F is applied statically it produces a deflection of 3 mm when the same force is applied with a frequency of 20 rad/sec the steady state amplitude is

(a) 1 mm along F (b) $\boxed{\text{1 mm opposite to F}}$
(c) 3 mm along F (d) 0.5 mm opposite to F

31, 32Q.

$k = 100\text{ N/m}$ $x_1 = 0.5 \cdot x_0$

$m = 1\text{kg}$. $C = ?$

Logarithmic decrement

$$\delta = \ln\left(\frac{x_0}{x_1}\right) = \frac{2\pi\xi}{\sqrt{1-\xi^2}}$$

$$\ln\left(\frac{1}{0.5}\right) = \frac{2\pi\xi}{\sqrt{1-\xi^2}} \qquad \xi = 0.1096$$

$$C = \xi \cdot 2\sqrt{MK} = 0.1096 \times 2 \times \sqrt{1\times 100} = 2.19 \frac{\text{N-s}}{\text{m}}.$$

$x_{static} = 3\text{mm}$.

$\omega = 20 \text{ rad/s}$.

$$\omega_n = \sqrt{\frac{K}{M}} = \sqrt{\frac{100}{1}} = 10 \text{ rad/s}.$$

$$r = \frac{\omega}{\omega_n} = \frac{20}{10} = 2$$

$$A = \frac{x_{static}}{\sqrt{(1-r^2)^2 + (2\xi r)^2}} \qquad A = \frac{3}{\left[(1-2^2)^2 + (2\times 0.1\times 2)^2\right]^{0.5}}$$

$A = 1\text{mm}$.

@ $r > 1$ Applied Force F and displacement $x(t)$ are in opposite phase.

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33. In a spring mass dampers system the magnification factor at resonance is found to be 20. The damping ratio is

(a) 0.2 (b) 0.25

$\boxed{\text{(c) 0.025}}$ (d) 0.05

339. @ $r=1$ resonance.

$$D.M.F = \frac{1}{2\xi} = 20.$$

$$\xi = \frac{1}{2\times 20} = 1/40$$

$$= 0.025$$

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34. A machine of 100 kg mass has a 20kg rotor with 0.5mm eccentrically. The mounting springs have stiffness 85kN/m and damping is negligible. If the operating speed is 20 π rad/s and the unit is constrained to move vertically, the dynamic amplitude of the machine will be

(a) 0.470×10^{-4} (b) 1.000×10^{-4} m

(c) 1.270×10^{-4} m (d) 2.540×10^{-4} m

$$A = \frac{me\omega^2}{\sqrt{(K-M\omega^2)^2+(C\omega)^2}}.$$

$M = 100kg$ $m = 20kg$ $e = 0.5mm.$

$K = 85000 N/m.$ $C = 0$ $\omega = 20\pi$ rad/s

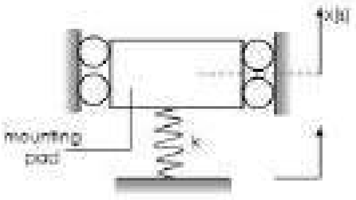
$A = ?$

$$A = \pm\frac{20\times0.5\times10^{-3}\times(20\pi)^2}{85000-100(20\pi)^2}$$

$$A = 1.270\times10^{-4} m.$$

35. A proper base isolation is to be designed for mounting a sensitive instrument as shown in Fig. At the point of mounting, the base vibration due to other disturbances is indicated in the figure. If the permitted absolute displacement amplitude on the rigid mounting pad is 0.01 mm, find the stiffness of the spring. Assume that the total mass of the mounting pad and the instrument is 50 kg.

(GATE-99)



$K = ?$ $M = 50kg.$ $C = 0$ $X = 0.01$

$Y = 0.2$

$$\frac{X}{Y} = \pm\left[\frac{K^2+(C\omega)^2}{(K-M\omega^2)^2+(C\omega)^2}\right]^{0.5}$$

$\omega = 200\pi$ rad/s

$$\frac{X}{Y} = \frac{K}{K-M\omega^2}$$

$$\frac{0.01}{0.2} = \frac{\pm K}{K-50(200\pi)^2}$$

$$K = 939.96 kN/m.$$

36. A single degree of freedom system is specified by the following differential equation

$5\ddot{x} + 20\dot{x} + 80x = 8\cos 4t,$

Match the following

Group A

P. Frequency of Oscillations – 4 rad/s.

Q. Amplitude of Oscillations – 0.1

R. Damping ratio – 0.5

S. Magnification factor – 1.

Group B

1. 0.8 2. 0.5

3. 4 4. 1 5. 0.1

Codes :

(a) A–4, B–3, C–2, D–1

(b) A–3, B–5, C–2, D–4

(c) A–3, B–1, C–5, D–2

(d) A–3, B–4, C–2, D–1

$M = 5kg$ $C = 20\,\frac{N\text{-}s}{m}.$ $K = 80 N/m.$

$F = 8N.$ $\omega = 4$ rad/s. $\xi = \frac{C}{2\sqrt{MK}} = \frac{20}{2\sqrt{5\times80}}$

$$\omega_n = \sqrt{\frac{K}{M}} = \sqrt{\frac{80}{5}} = 4 \text{ rad/s}.$$

$$@\ r=1\quad D.M.F = \frac{A}{x_{static}} = \frac{1}{2\xi} = \frac{1}{2\times0.5} = 1.$$

$$A = \frac{x_{static}}{2\xi} = \frac{F/K}{2\times0.5} = \frac{8/80}{1} = 0.1$$

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37. A machine of 250 kg mass is supported on springs of total stiffness 100 kN/m. Machine has an unbalanced rotating force of 350 N at speed of 3600 rpm. Assuming a damping factor of 0.15, the value of transmissibility ratio is (GATE-06)

(a) 0.0531 (b) 0.9922
(c) 0.0162 (d) 0.0028

37Q. $M = 250\,kg$ $\quad k = 100\,kN/m$

$F = 350\,N$ $\quad N = 3600\,rpm$

$\zeta = 0.15$ $\quad \epsilon = ?$

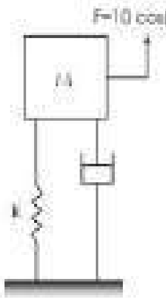
$$\epsilon = \sqrt{\frac{1+(2\zeta r)^2}{(1-r^2)^2+(2\zeta r)^2}}$$

$$r = \frac{\omega}{\omega_n} = \frac{\frac{2\pi N}{60}}{\sqrt{\frac{K}{M}}} = \frac{\frac{2\pi\times 3600}{60}}{\sqrt{\frac{100\times10^3}{250}}} = 6\pi >> 1. \qquad \epsilon << 1.$$

$$\epsilon = \sqrt{\frac{1+(2\times0.15\times6\pi)^2}{[1-(6\pi)^2]^2+(2\times0.15\times6\pi)^2}} = 0.0162$$

Efficiency of isolation $\eta = 1-\epsilon$
$= 1-0.0162$
$= 98.38\%$

38. A mass-spring-dashpot system with mass m=10 kg, spring constant k = 6250 N/m is excited by a harmonic excitation of 10 cos (25t) N. At the steady state, the vibration amplitude of the mass is 40mm. The damping coefficient (c, in N.s/m) of the dashpot is ______ (GATE-14)



38Q. $m = 10\,kg$

$K = 6250\,N/m.$

$F(t) = 10\cos(25t)$

$A = 40\,mm.$ $\quad C = ?$

$$A = \frac{F}{\sqrt{(K-M\omega^2)^2+(C\omega)^2}}$$

$$\Rightarrow 40\times10^{-3} = \frac{10}{[(6250-10(25)^2)^2+(C\times25)^2]^{0.5}} \Rightarrow C = 10\,\frac{N\text{-}s}{m}.$$

39. In vibration isolation, which one of the following statements is NOT correct regarding Transmissibility (T)? (GATE-14)

(a) T is nearly unity at small excitation frequencies → $\epsilon \approx 1$ @ $r <<< 1$
(b) T can be always reduced by using higher damping at ~~any~~ lower excitation frequency → $0 < r < \sqrt{2} \rightarrow \epsilon \propto 1/\zeta$ after $r > \sqrt{2}$ $\boxed{\epsilon \propto \zeta}$
(c) T is unity at the frequency ratio of √2 → $\epsilon = 1$ @ $r = \sqrt{2}$ $\quad \zeta = 0 / \zeta \neq 0$
(d) T is infinity at resonance for undamped systems → $\epsilon = \infty$ @ $r = 1$ $\quad \zeta = 0$

after $r > \sqrt{2}$ damping behaviour is determin-ental.

40. There are four samples P, Q, R and S with natural frequencies 64, 96, 128 and 256 Hz, respectively. They are mounted on test setups for conducting vibration experiments. If a loud pure note of frequency 144 Hz is produced by some instrument, which of the samples will show the most perceptible induced vibration? (GATE-05) → resonance

(a) P (b) Q $\boxed{\text{(c) R}}$ (d) S

128 Hz is closer to 144Hz.
so this sample will produced loud note.

41. A single degree freedom spring mass is subjected to a sinusoidal force of 10 N amplitude and frequency ω along the axis of the spring. The stiffness of the spring is 150N/m, damping factor is 0.2 and undamped natural frequency is 10ω. At steady state, the amplitude of vibration (in m) is approximately (GATE –15)

(a) 0.05 (b) 0.07
(c) 0.70 (d) 0.90

$F = 10N \quad K = 150 N/m.$

$\xi = 0.2 \quad \omega_n = 10\omega.$

$r = \frac{\omega}{\omega_n} = 0.1$

$x_{static} = F/K = 10/150.$

$$A = \frac{x_{static}}{\left[(1-r^2)^2 + (2\xi r)^2\right]^{0.5}} = \frac{10/150}{\left[(1-10^2)^2 + (2\times0.2\times10)^2\right]^{0.5}}$$

$A = 0.7 m.$

42. A precision instrument package (m=1kg) needs to be mounted on a surface vibrating at 60 Hz. It is desired that only 5% of the base surface vibration amplitude be transmitted to the instrument. Assume that the isolation is designed with its natural frequency significantly lesser than 60Hz, so that the effect of damping may be ignored. The stiffness (in N/m) of the required mounting pad is ______ (GATE-15)

Base excitation

$M = 1 kg$

$\omega = 2\pi f = 2\pi\times60 = 120\pi \ Hz.$

$$\epsilon = 5\% = 0.05 = \frac{\pm K}{K - M\omega^2} \Rightarrow 0.05 = \frac{\pm K}{K - 1(120\pi)^2}$$

$K =$

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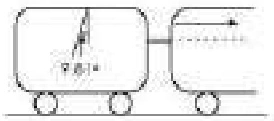
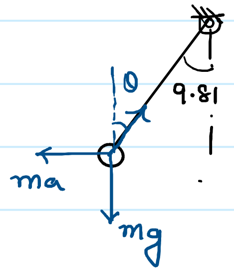
43. A pendulum of mass 50 gm and length 500 mm is suspended from the roof of a moving rail coach. If the pendulum is in equilibrium at the position shown, the acceleration of the train at that instant is



(a) 0.981 m/sec^2 (b) 9.81 m/sec^2
$\boxed{\text{(c) 1.69 m/sec}^2}$ (d) 1.86 m/sec^2

$m = 50\,gm \quad l = 500\,mm$



$\Sigma F_x = 0$

$T\sin\theta = ma.$

$T\cos\theta = mg$

$\tan\theta = a/g \Rightarrow a = g\cdot\tan\theta$

$a = 9.81\cdot\tan 9.81.$

$= 1.69\,m/s^2$

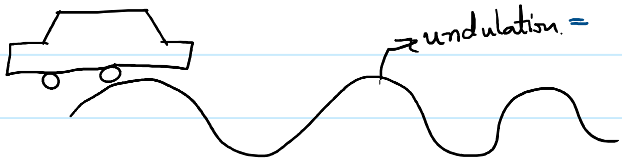
44. When a vehicle travels on a rough road whose undulations can be assumed to be sinusoidal, the resonant conditions of the base excited vibrations are determined by the

$\boxed{\text{(a) mass of the vehicle, stiffness of the suspension spring, speed of the vehicle, wavelength of the roughness curve}}$

(b) speed of the vehicle only

(c) speed of the vehicle and the stiffness of the suspension spring

(d) amplitude of the undulations



45. A shaft has an attached disc at the center of its length. The disc has its center of gravity located at a distance of 2 mm from the axis of the shaft. When the shaft is allowed to vibrate in its natural bow-shaped mode, it has a frequency of vibration of 10 rad/sec. When the shaft is rotated a 300 revolutions per minute, it will whirl with a radius of

whirling of shaft.

(a) 2 mm (b) 2.25 mm
(c) 2.50 mm (d) 3.00 mm

$e = 2\,mm.$

$\omega_n = 10\,rad/s.$

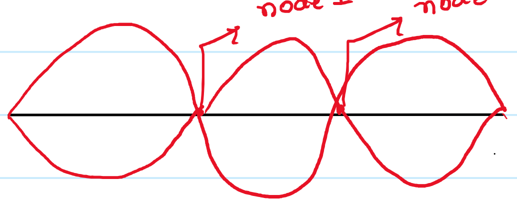
$\omega = \dfrac{2\pi\times 300}{60} = 10\pi\ rad/s.$

$r = \dfrac{\omega}{\omega_n} = \dfrac{10\pi}{10} = \pi$

$y = \dfrac{er^2}{1-r^2} = \dfrac{2\cdot(\pi)^2}{1-\pi^2} = -2.25\,mm.$

46. If two nodes are observed at a frequency of 1800 rpm during whirling of a simply supported long slender rotating shaft, the first critical speed of the shaft in rpm is (GATE-13)

(a) 200 (b) 450
(c) 600 (d) 900



$\omega_3 = 3^2\cdot\omega_1 = 1800$

$\omega_1 = \dfrac{1800}{9}$

$\omega_1 = 200\,rpm.$

47. The rotor shaft of a large electric motor supported between short bearings at both the ends shows a deflection of 1.8 mm in the middle of the rotor. Assuming the rotor to be perfectly balanced and supported at knife edges at both the ends, the likely critical speed (in rpm) of the shaft is (GATE-09)

(a) 350 (b) 705
(c) 2810 (d) 4430

$\delta = 1.8\,mm.$

$\omega_n = ?$

$\omega_n = \sqrt{\dfrac{K}{M}} = \sqrt{\dfrac{Mg/\delta_{static}}{M}}$

$\omega_n = \sqrt{\dfrac{g}{\delta_{static}}} \qquad \omega_n = \sqrt{\dfrac{9.81}{1.8\times10^{-3}}} = 73.82\,rad/s.$

$N_{cr} = \dfrac{\omega_n\times 60}{2\pi} = 705\,rpm.$

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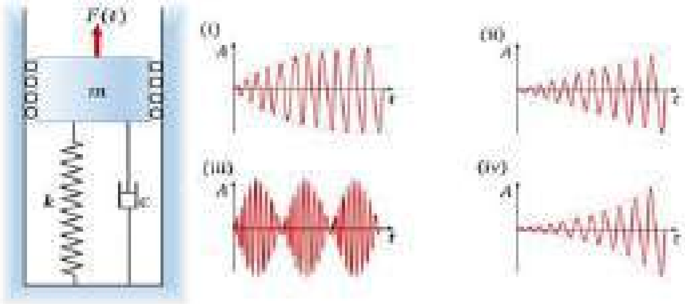
G-2022

Q.40 A spring mass damper system (mass m, stiffness k, and damping coefficient c) excited by a force $F(t) = B\sin\omega t$, where B, ω and t are the amplitude, frequency and time, respectively, is shown in the figure. Four different responses of the system (marked as (i) to (iv)) are shown just to the right of the system figure. In the figures of the responses, A is the amplitude of response shown in red color and the dashed lines indicate its envelope. The responses represent only the qualitative trend and those are not drawn to any specific scale.



Four different parameter and forcing conditions are mentioned below.

(P) $c > 0$ and $\omega = \sqrt{k/m}$ (Q) $c < 0$ and $\omega \neq 0$
(R) $c = 0$ and $\omega = \sqrt{k/m}$ (S) $c = 0$ and $\omega \cong \sqrt{k/m}$

Which one of the following options gives correct match (indicated by arrow →) of the parameter and forcing conditions to the responses?

(A)	$\boxed{(P) \to (i)}$,	(Q) → (iii),	(R) → (iv),	(S) → (ii)
(B)	(P) → (ii),	(Q) → (iii),	(R) → (iv),	(S) → (i)
(C)	$\boxed{(P) \to (i)}$,	$\boxed{(Q) \to (iv)}$,	(R) → (ii),	(S) → (iii)
(D)	(P) → (iii),	(Q) → (iv),	(R) → (ii),	(S) → (i)

Response of forced vibration

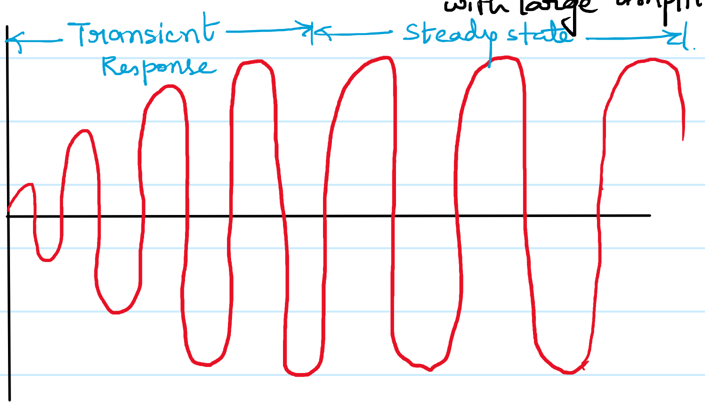
$$x(t) = X_0 e^{-\xi\omega_n t}\sin(\omega_d t + \phi) + \frac{x_{static}}{[(1-r^2)^2 + (2\xi r)^2]^{0.5}}\sin(\omega t - \phi)$$

Cond. (P) – $c > 0$ $\omega = \sqrt{\frac{k}{M}}$ (Resonance) $r = 1$

$$x(t) = \underbrace{X_0 \cdot e^{-\xi\omega_n t}\sin(\omega_d t + \phi)}_{} + \underbrace{\frac{x_{static}}{2\xi}\sin\cdot(\omega_n t - \phi)}_{}$$

as $t \to \infty$
$e^{-\xi\omega_n t} \to 0$

Steady state
with large amplitude



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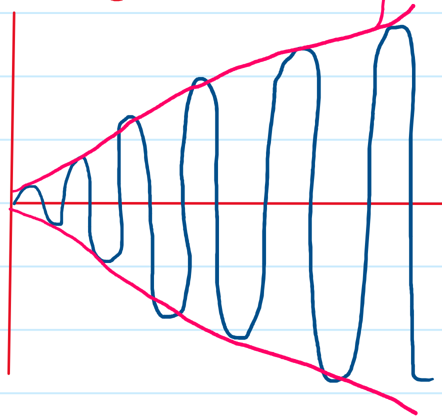
Q- $c<0 \quad \omega \neq 0 \quad \xi = -ve.$

$$x(t) = X_0 \cdot e^{+\xi\omega_n t} \sin(\omega_d t + \phi) + \frac{x_{static} \sin(\omega t - \phi)}{[(1-r^2)^2 + (2\xi r)^2]^{0.5}}$$

as $t \to \infty$

$e^{\xi\omega_n t} \to \infty$

$e^{\xi\omega_n t}$ exponential growth.



(R) $C = 0 \quad \omega = \sqrt{\frac{K}{M}} \quad r = 1. \quad \xi = 0$

$$x(t) = X_0 e^{-\xi\omega_n t} \sin(\omega_d t + \phi) + \frac{x_{static}}{\left[\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\xi\frac{\omega}{\omega_n}\right)^2\right]^{0.5}} \sin(\omega t - \phi)$$

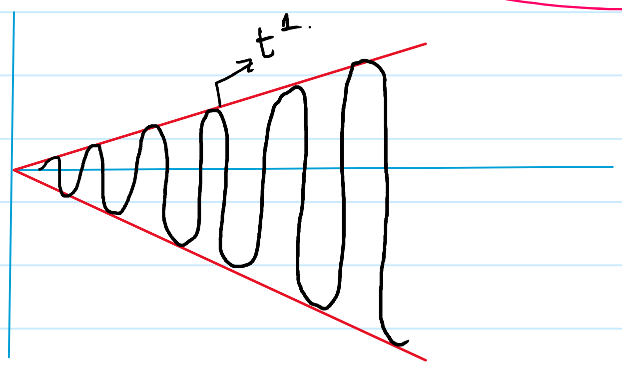
$\xi \to 1$, $\omega_d \to \omega_n$, $\frac{x_{static}}{0}$

$$x(t) = X_0 \cdot \sin(\omega_n t + \phi) + \frac{\frac{d}{d\omega}(x_{static} \sin(\omega t - \phi))}{\frac{d}{d\omega}\left[\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)\right]}$$

$$x(t) = X_0 \cdot \sin(\omega_n t + \phi) + \frac{x_{static} \cdot \cos(\omega t - \phi) \cdot t}{\left(0 - \frac{2\omega}{\omega_n}\right)}$$

$$x(t) = X_0 \cdot \sin(\omega_n t - \phi) - \frac{x_{static}\, \omega_n \cdot t^1}{2} \cos(\omega_n t - \phi)$$

linear variation wrt time.



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$C = 0 \qquad \omega \cong \sqrt{\frac{K}{M}}$

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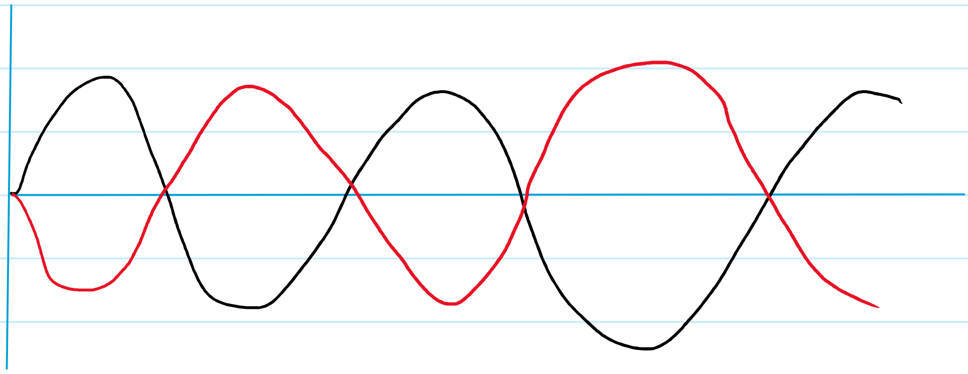
$$x(t) = X_0 e^{-\overset{1}{\xi}\omega_n t} \sin(\overset{\omega_n}{\omega_d} t + \phi) + \frac{x_{static}}{\left[(1-r^2)^2 + (2\xi r)^2\right]^{0.5}} \sin(\omega t - \phi)$$

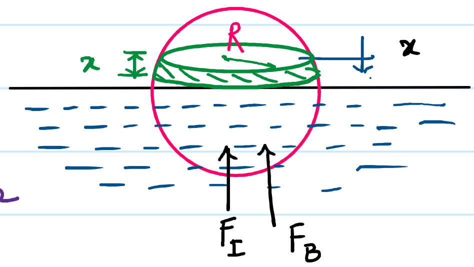
$$x(t) = X_0 \cdot \sin(\omega_n t + \phi) \pm \frac{x_{static}}{1 - \left(\frac{\omega}{\omega_n}\right)^2} \cdot \sin(\omega t - \phi)$$



Q-19;

A hollow sphere of radius R_o is 50% immersed in the water. Natural frequency of undamped vibration is ___________ for small displacement.

ρ is the density of water



Sol: Hollow sphere attain equilibrium after 50% of its volume is immersed.

Volume of water displaced = 50% volume of sphere $= 0.5 \times \frac{4}{3}\pi \cdot R_o^3 = \frac{2}{3}\pi R_o^3$

Mass of water displaced = Mass of 50% of sphere.

Mass of water displaced $= (\rho \times 50\% \text{ of } V_{sphere})$

Mass of sphere inside = Mass of water displaced.

$= \rho_w \cdot \frac{2}{3}\pi R_o^3$

Bouyancy force

Weight of sphere inside water after giving initial displacement (x)

$= \rho_w \cdot g \cdot \pi R_o^2 x.$

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$$F_I + F_B = 0$$

$$m_s \cdot \ddot{x}_s + F_B = 0$$

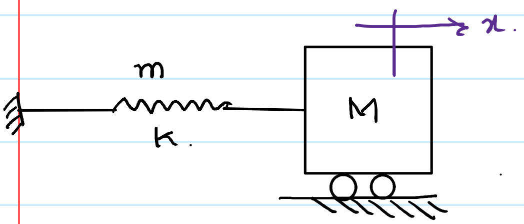
$$\rho_w \cdot \tfrac{2}{3}\pi R_0^3 \cdot \ddot{x} + \rho \cdot g \cdot \pi R_0^2 \cdot x = 0$$

$$\ddot{x} + \frac{3g}{2R_0} \cdot x = 0$$

$$\omega_n = \sqrt{\frac{3g}{2R_0}}.$$

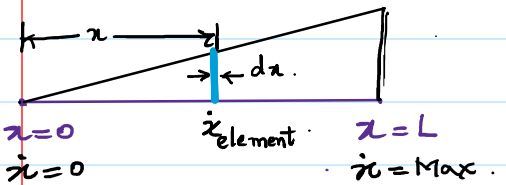
Considering mass of Spring.

M – mass of Block.
m – mass of spring.



Energy Approach.

$$(KE)_{block} + (KE)_{spring} + (SE) = \text{constant}$$



$$KE_{block} = \frac{1}{2} M \dot{x}^2$$

$$SE = \tfrac{1}{2} k x^2$$

$\dot{x}$ – velocity of block.

Velocity varies linearly along lenght of spring

$$(KE)_{spring} = \frac{1}{2} \cdot m_{element} \cdot \dot{x}_{element}^2$$

$$m_{element} = \frac{m}{L} \cdot dx.$$

$$\frac{\dot{x}_{element}}{x} = \frac{\dot{x}}{L} \Rightarrow \dot{x}_{element} = \frac{\dot{x}}{L} \cdot x.$$

$$(KE)_{spring} = \int \tfrac{1}{2} \cdot \left(\frac{m}{L}\right) \cdot \left(\frac{\dot{x}}{L} \cdot x\right)^2 \cdot dx = \frac{1}{2L^3} m \dot{x}^2 \int x^2 dx.$$

$$= \frac{1}{2L^3} m \dot{x}^2 \cdot \left[\frac{x^3}{3}\right]_0^L = \frac{1}{6} \cdot m \cdot \dot{x}^2$$

$$= \tfrac{1}{6} m \dot{x}^2$$

$$(KE)_{Block} + (KE)_{spring} + (SE) = \text{const}.$$

$$\frac{d}{dt}\left(\frac{1}{2} M \dot{x}^2 + \tfrac{1}{6} m \dot{x}^2 + \tfrac{1}{2} k x^2 = c.\right) \Rightarrow \cancel{\tfrac{1}{2}} \cdot M \cancel{2}\dot{x}\ddot{x} + \cancel{\tfrac{1}{6}}_3 m \cdot \cancel{2}\dot{x}\ddot{x} + \cancel{\tfrac{1}{2}} k \cancel{2} x \cdot \cancel{\dot{x}} = 0$$

$$M\ddot{x} + \frac{m}{3}\ddot{x} + kx = 0$$

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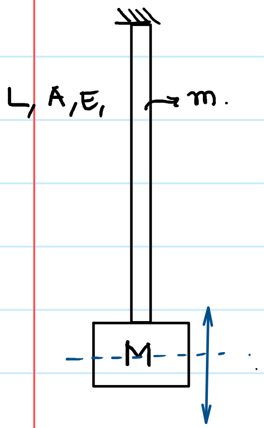
$$\left(M + \frac{m}{3}\right)\ddot{x} + kx = 0$$

Prismatic Rod.

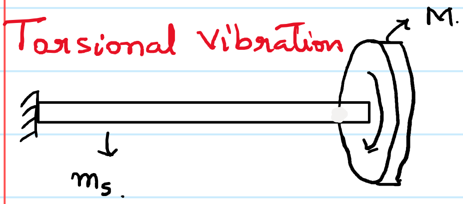
Longitudinal vibrations.



$$\left(M + \frac{m}{3}\right)\ddot{x} + K_{Axial} \cdot x = 0$$

$$\left(M + \frac{m}{3}\right)\ddot{x} + \left(\frac{AE}{L}\right)x = 0$$

Torsional Vibration



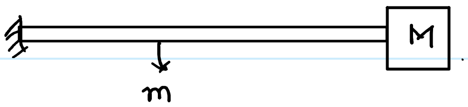
$$\left(I_{Rotor} + \frac{I_{shaft}}{3}\right)\ddot{\theta} + K_{Torsional} \cdot \theta = 0$$

$$\left(I_{Rotor} + \frac{I_{shaft}}{3}\right)\ddot{\theta} + \frac{GJ}{L}\theta = 0$$

M – Mass of Rotor.
m_s – Mass of shaft.

Transverse Vibration

$$\left(M + ml\right)\ddot{x} + K_{beam}x = 0$$



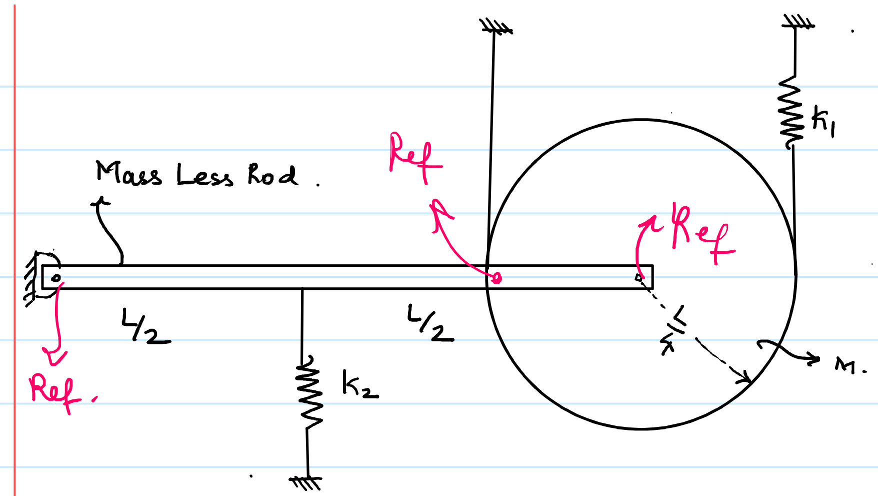
$$\omega_n = \sqrt{\frac{g}{\delta_{static}}} \qquad \delta_{Point\,load} = \frac{MgL^3}{3EI}$$

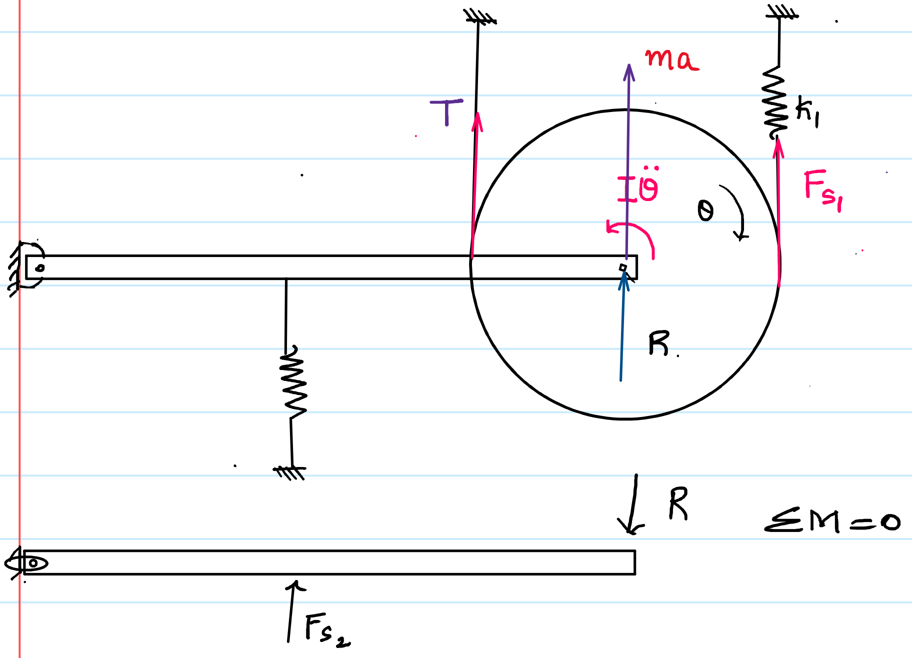
m – mass of beam per unit lenght.
M – mass of block.

$$\delta_{UDL} = \frac{mgL^4}{8EI}$$

$$\delta_{Total} = \delta_{P.L.} + \frac{\delta_{UDL}}{1.27}$$

$$f_n = \frac{1}{2\pi}\sqrt{\frac{g}{\delta_{static}}} = \frac{0.495}{\sqrt{\delta_{P.L.} + \frac{\delta_{UDL}}{1.27}}} \text{ Hz.}$$

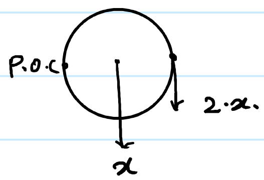


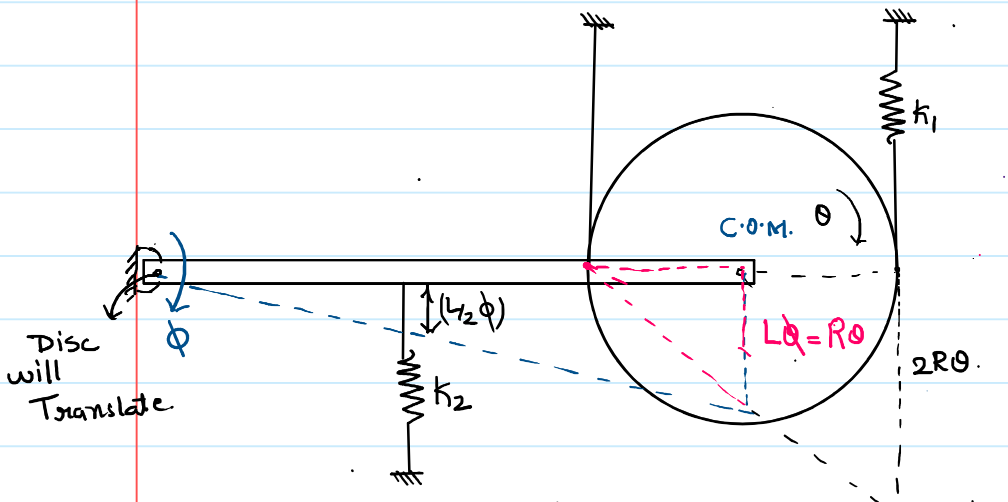


$\Sigma M_{c.o.m} = 0$

$\Sigma F_y = 0$

$\Sigma M = 0$





$R = L/4$

$L\phi = R\cdot\theta$

$\phi = \theta/4$

$\theta = 4\phi \Rightarrow \dot{\theta} = 4\dot{\phi}$

$I = \frac{MR^2}{2} = \frac{M(L/4)^2}{2} = \frac{ML^2}{32}$

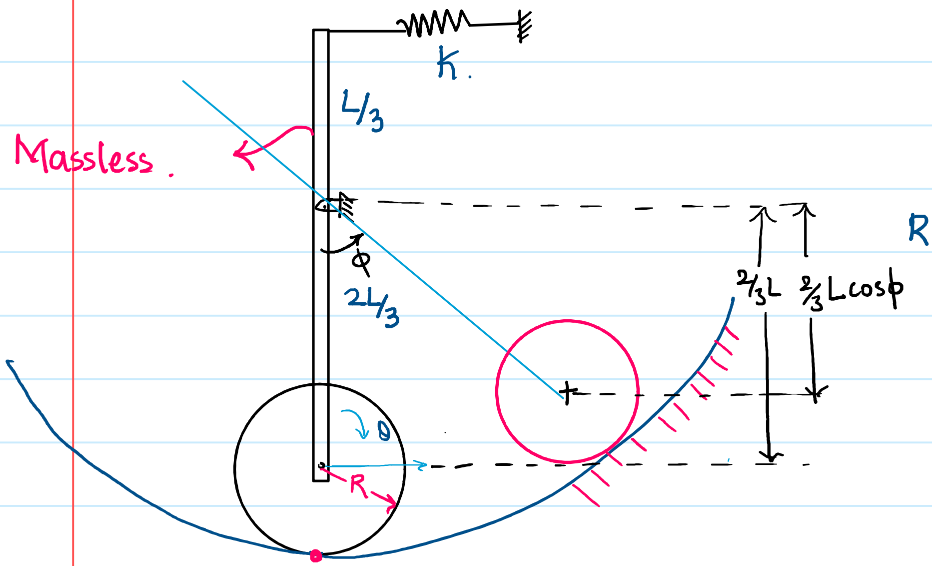
$(KE)_{Rot} + (KE)_{Trans} + (SE)_1 + (SE)_2 = \text{Constant}$

$$\frac{1}{2}\cdot I\dot{\theta}^2 + \frac{1}{2}\cdot M\cdot(L\dot{\phi})^2 + \frac{1}{2}\cdot k_1(2R\theta)^2 + \frac{1}{2}k_2(L/2\,\phi)^2 = C$$

$$\frac{d}{dt}\left[\frac{1}{2}\cdot\left(\frac{ML^2}{32}\right)(4\dot{\phi})^2 + \frac{1}{2}\cdot ML^2\dot{\phi}^2 + \frac{1}{2}\cdot k_1\left(L/4\cdot 4\phi\right)^2\cdot 4 + 1/2\cdot k_2\cdot(L/2\,\phi)^2 = C\right]$$

$$\frac{1}{\cancel{2}}\cdot\frac{ML^2}{\cancel{32}_{2}}\times\cancel{16}\cdot\cancel{2}\cdot\cancel{\dot{\phi}}\,\ddot{\phi} + \frac{1}{\cancel{2}}ML^2\cdot\cancel{2}\cancel{\dot{\phi}}\ddot{\phi} + \frac{1}{\cancel{2}}\cdot k_1L^2\cdot\cancel{2}\phi\cdot\cancel{\dot{\phi}}\times 4 + \frac{1}{\cancel{2}}\cdot k_2\cdot\frac{L^2}{4}\cdot\cancel{2}\phi\,\cancel{\dot{\phi}} = 0$$

$$\left(\frac{ML^2}{2} + ML^2\right)\ddot{\phi} + \left(4k_1 + k_2/4\right)L^2\phi = 0$$



$R\theta = \frac{2}{3} L \phi$

$R\dot{\theta} = \frac{2}{3} L \dot{\phi}$

$R\ddot{\theta} = \frac{2}{3} \cdot L \ddot{\phi}$

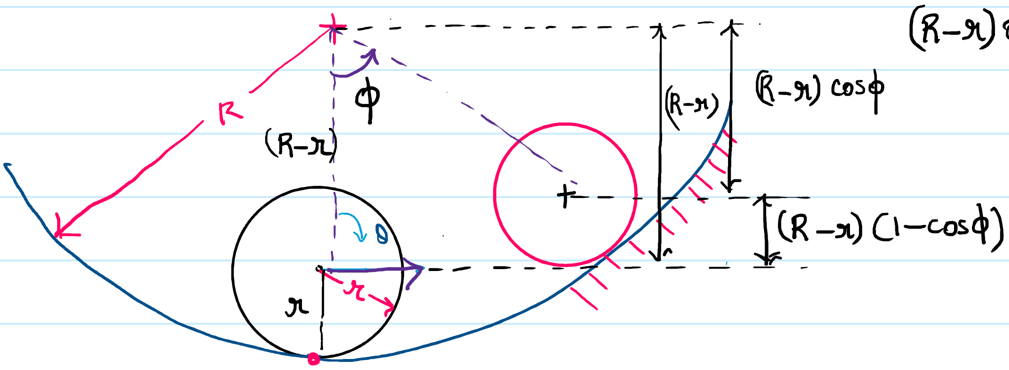
$(KE)_{Rot} + (KE)_{Trans} + (SE) + mg \frac{2}{3}L[1 - \cos\phi] = \text{const.}$

$\frac{1}{2}\left(\frac{MR^2}{2}\right)\dot{\theta}^2 + \frac{1}{2} \cdot M \cdot \left(\frac{2}{3} L \dot{\phi}\right)^2 + \frac{1}{2} \cdot k \cdot \left(\frac{L}{3}\phi\right)^2 + mg \frac{2}{3}L[1 - \cos\phi] = C.$

(2)

Pure Rolling

$(R - r)\phi = r\theta.$

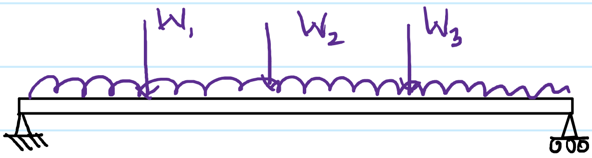


$(PE) + (KE)_{Rot} + (KE)_{Trans} = C$

$mg(R-r)[1 - \cos\phi] + \frac{1}{2}\left(\frac{Mr^2}{2}\right)\dot{\theta}^2 + \frac{1}{2} M \cdot (R - r)^2 \cdot \dot{\phi}^2 = C$

If $\omega_1, \omega_2, \omega_3, \ldots$ are natural frequencies of a Multi DOF system, then the lowest natural frequency is given by

$$\frac{1}{\omega^2} = \frac{1}{\omega_1^2} + \frac{1}{\omega_2^2} + \frac{1}{\omega_3^2} + \ldots\ldots$$



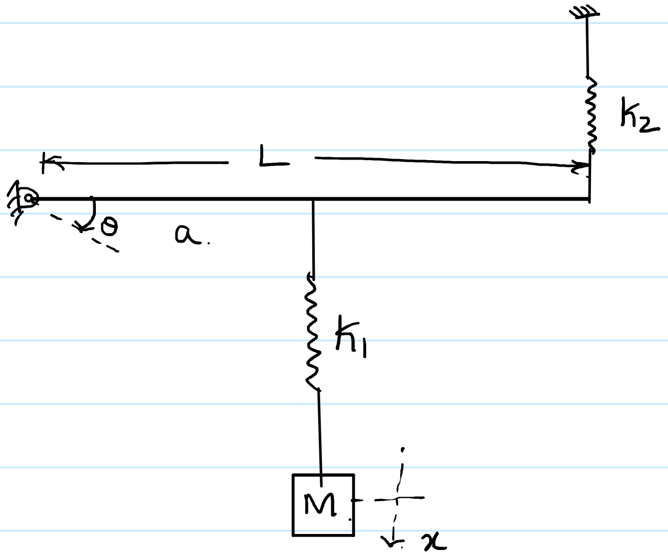
δ_1 – Deflection due to load W_1

δ_2 – " " " " W_2

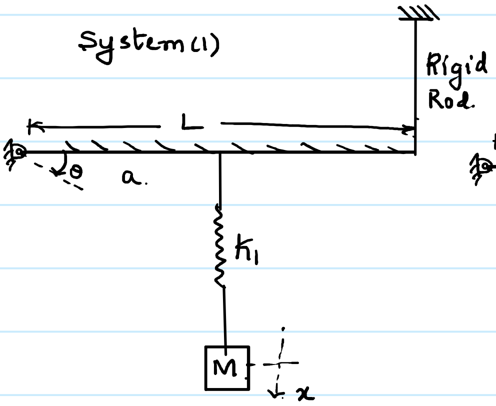
⋮

lowest natural frequency

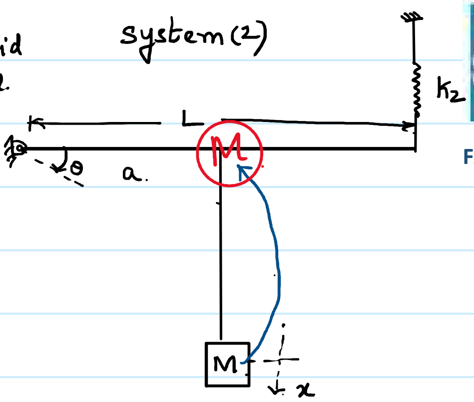
$$f_n = \frac{1}{2\pi}\sqrt{\frac{g}{\delta_{Total}}} \Rightarrow f_n = \frac{0.495}{\sqrt{\delta_1 + \delta_2 \ldots \frac{\delta_{UDL}}{1.27}}}$$



System (1)



System (2)



$$M\ddot{x} + k_1 x = 0$$

$$\ddot{x} + \frac{k_1}{M}\cdot x = 0$$

$$\omega_1 = \sqrt{\frac{k_1}{M}}$$

$$\sum M_{Hinge} = 0$$

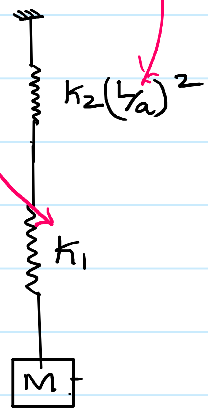
$$Ma^2\ddot{\theta} + k_2 . L^2\theta = 0$$

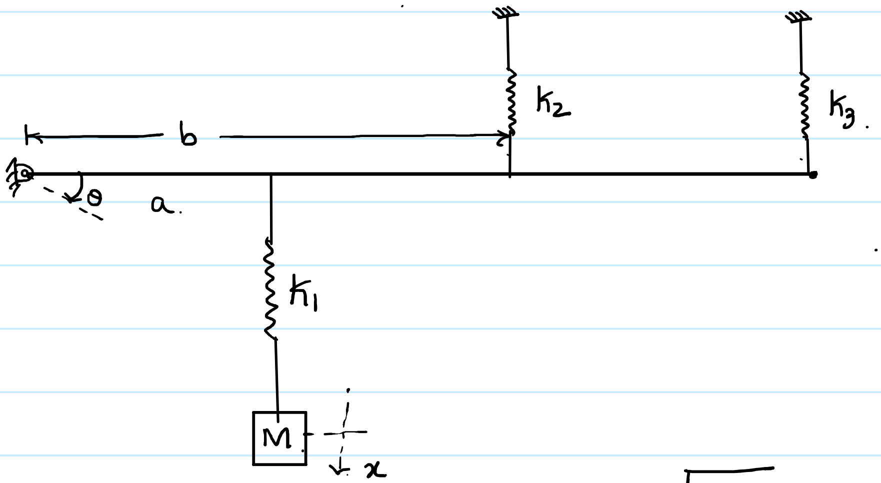
$$\ddot{\theta} + \frac{k_2 . L^2}{Ma^2}\theta = 0$$

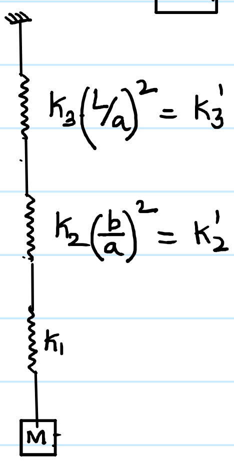
$$\omega_2 = \sqrt{\frac{k_2}{M}\left(\frac{L}{a}\right)^2}$$

$$\frac{1}{\omega^2} = \frac{1}{\omega_1^2} + \frac{1}{\omega_2^2}$$

$$\omega = \sqrt{\frac{\omega_1^2 \cdot \omega_2^2}{\omega_1^2 + \omega_2}} = \sqrt{\frac{\left(\frac{k_1}{M}\right)\cdot\left(\frac{k_2}{M}\right)\left(\frac{L}{a}\right)^2}{\left(\frac{k_1}{M}\right) + \left(\frac{k_2 . L^2}{Ma^2}\right)}} = \sqrt{\frac{k_1 \cdot k_2\left(\frac{L}{a}\right)^2}{M\left\{k_1 + k_2\left(\frac{L}{a}\right)^2\right\}}}$$

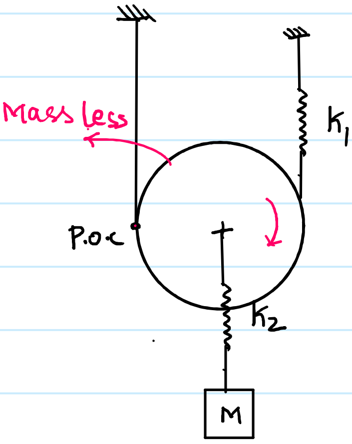






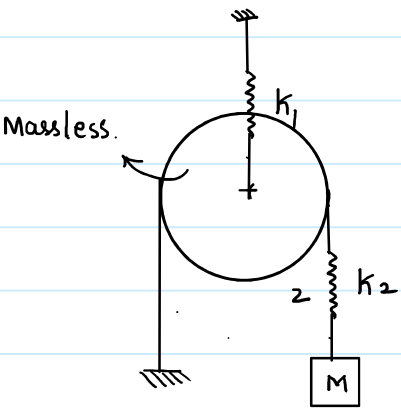
$$K_e = \frac{k_1 \cdot k_2' \cdot k_3'}{k_1 k_2' + k_2' k_3' + k_3' k_1}$$

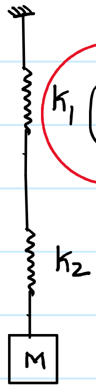
$$\omega_n = \sqrt{\frac{K_e}{M}}$$



$$k_1\left(\frac{2R}{R}\right)^2 = 4k_1$$

$$\omega_n = \sqrt{\frac{4k_1k_2}{(4k_1+k_2)M}}$$

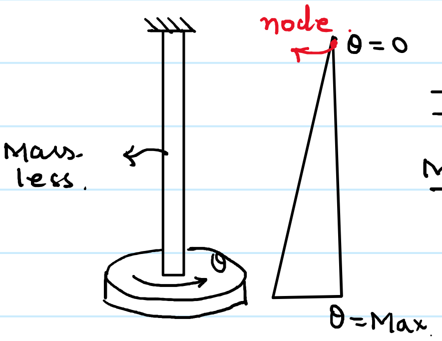




$$k_1\left(\frac{R}{2R}\right)^2 = \frac{k_1}{4}$$

$\rightarrow k_1 \times \dfrac{(\perp^{\text{lar}} \text{ distance of spring from P.o.c})^2}{(\perp^{\text{lar}} \text{ distance of mass from P.o.c})^2}$

Torsional Vibrations



S.D.OF

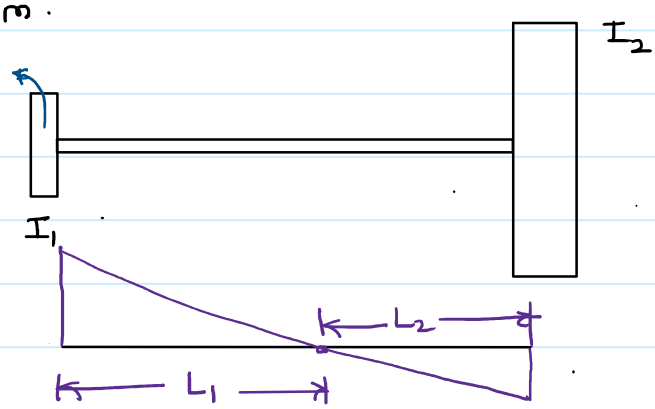
$$I\ddot{\theta} + k_T\theta = 0$$

$$\frac{MR^2}{2}\ddot{\theta} + \frac{GJ}{L}\theta = 0$$

$$\ddot{\theta} + \frac{GJ}{IL}\theta = 0 \qquad \omega_n = \sqrt{\frac{GJ}{IL}}$$

node – is a point having no displacement.

2. rotor system.



$$\omega_1 = \omega_2$$

$$\sqrt{\frac{GJ}{I_1L_1}} = \sqrt{\frac{GJ}{I_2 \cdot L_2}}$$

L_1, L_2 – distance of rotors 1, 2 from the node.

$$I \propto \frac{1}{\text{distance of node from rotor}}$$

$$\boxed{I_1L_1 = I_2L_2}$$

Number of nodes = No. of rotors – 1.

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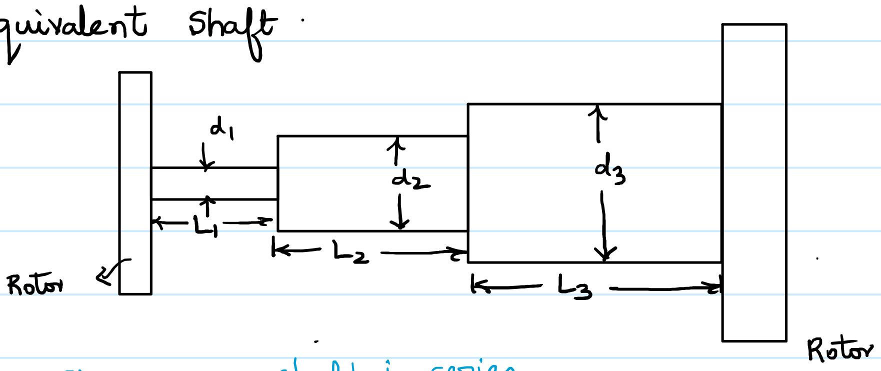
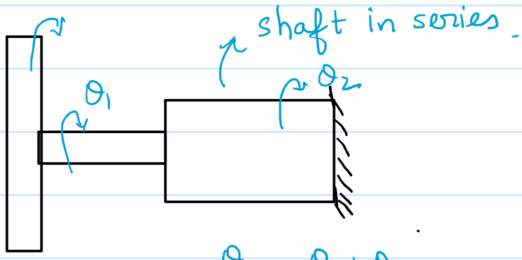
Equivalent Shaft.



$T_1 = T_2 = T_3 = T_e$

$$\theta = \frac{TL}{GJ}$$



$\theta_e = \theta_1 + \theta_2$

$$\theta_e = \theta_1 + \theta_2 + \theta_3 + \cdots$$

$$\frac{T_e \cdot L_e}{G \cdot J_e} = \frac{T_1 L_1}{G \cdot J_1} + \frac{T_2 \cdot L_2}{G \cdot J_2} + \cdots$$

$$J_e = \frac{\pi}{32} d_e^4$$

$d_e = d_1$

$d_e = d_2$

$$L_e = J_e \left[\frac{L_1}{J_1} + \frac{L_2}{J_2} + \cdots \right]$$

$$L_e = d_e^4 \left[\frac{L_1}{d_1^4} + \frac{L_2}{d_2^4} + \cdots \right]$$

$$L_e = L_1 \left(\frac{d_e}{d_1}\right)^4 + L_2 \left(\frac{d_e}{d_2}\right)^4 + L_3 \left(\frac{d_e}{d_3}\right)^4 + \cdots$$

Gyroscope — Gyre → circular motion

Gyre → German word

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Translation.

Linear Moment $\quad \vec{P} = m\vec{v}$

$$P\cdot\hat{P} = m\cdot v\hat{v}$$

$$P = mv \quad , \quad \hat{P} = \hat{v}$$

Newton 2^{nd} law. $\quad \dfrac{d\vec{P}}{dt} = \vec{F}$

$$\vec{F} = m\vec{a}.$$

$$\hat{F} = \hat{a}$$

Angular Motion → cross-product.

Angular Moment $\quad L = mvr.$

$$\vec{L} = m\cdot(\vec{r}\times\vec{v}) \quad \Big\} \text{ point mass.}$$

$$\vec{L} = I\cdot\vec{\omega} \rightsquigarrow \text{complex body.}$$

$$L\cdot\hat{L} = I\cdot\omega\cdot\hat{\omega} \qquad \hat{L} = \hat{\omega}, \quad L = I\omega.$$

Newton 2^{nd}. $\quad \text{Torque} = \dfrac{d\vec{L}}{dt}$

$$\vec{\tau} = \frac{d}{dt}(I\cdot\omega\cdot\hat{\omega})$$

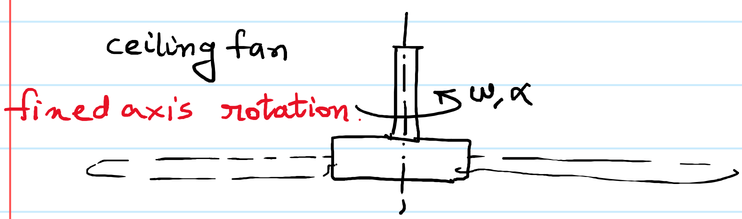
$I = \text{const}.$
ω – variable
$\hat{\omega}$ – variable.

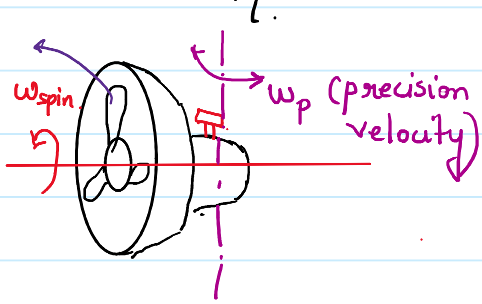
$$\vec{\tau} = I\cdot\frac{d\omega}{dt}\cdot\hat{\omega} + \underbrace{I\cdot\omega\cdot\frac{d\hat{\omega}}{dt}}$$

$$= I\alpha\cdot\hat{\alpha} + I\cdot\omega\cdot\omega_p. \qquad \text{Gyroscopic Torque.} \qquad (\hat{\omega} = \hat{\alpha})$$

$I\alpha\hat{\alpha}$ – Torque exerted due to rate of change in angular velocity.
↳ Fixed axis rotation.

$I\omega\cdot\omega_p$ → Torque exerted due to rate of change in direction of angular velocity.
↳ axis of rotation is moving with some velocity about another axis.





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Effect of Gyroscope on

1. Aeroplane
2. Naval Ship
3. 4 wheeler
4. 2 wheeler

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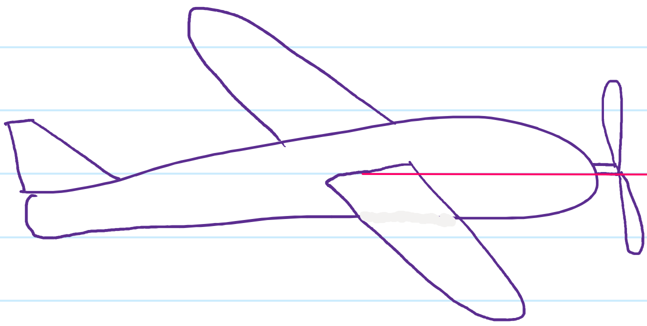
Effect of Gyroscope on the Aeroplane

1. Location of observer - Rear/Front
2. Direction of spin - CW/CCW
3. Direction of Precision (Steering) - Left/Right
   (Pitching) - Upwards/Downwards



Active Gyroscopic Torque $\vec{C} = I.(\vec{\omega_p} \times \vec{\omega_s})$

Reactive Gyroscopic Torque $\vec{C} = I.(\vec{\omega_s} \times \vec{\omega_p})$

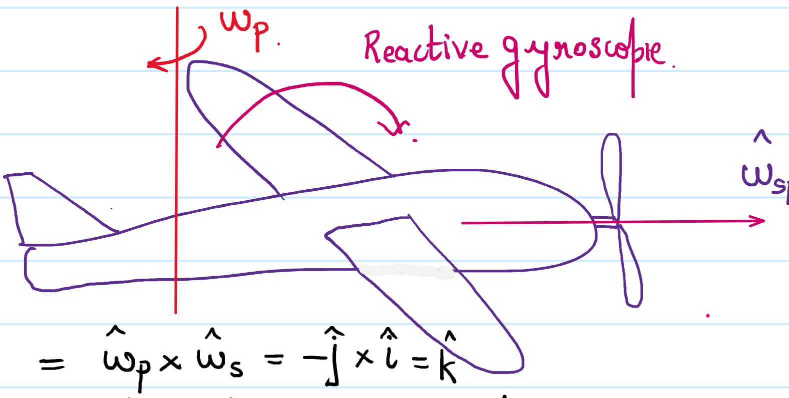
$\vec{C}, \vec{\omega_s}, \vec{\omega_p}$ are mutually $\perp^{lur}$ to each other.

case (i)

1. Location of observer - Rear
2. Direction of spin - CW
3. Direction of Precision (Steering) - Right

$\hat{\omega}_{spin} = \hat{i}$

$\hat{\omega}_p = -\hat{j}$



Active gyroscope $= \hat{\omega}_p \times \hat{\omega}_s = -\hat{j} \times \hat{i} = \hat{k}$

Reactive gyroscope $= \hat{\omega}_s \times \hat{\omega}_p = \hat{i} \times -\hat{j} = -\hat{k}$. (C.W)

Due to reactive gyroscope nose is dipped and tail is raised.

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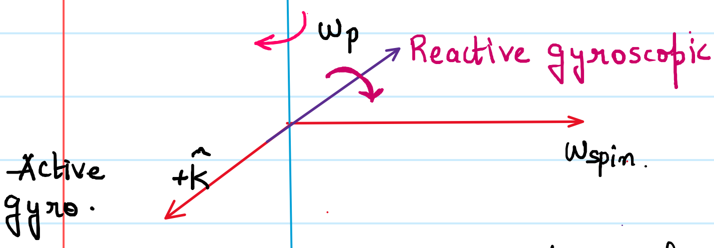
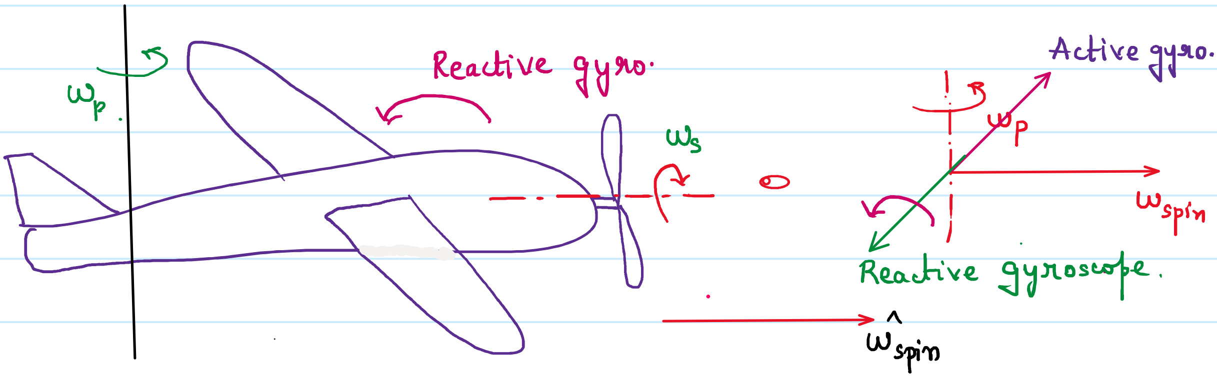


Rotate ω_{spin} in the direction of ω_p by 90°



1. Location of observer - Front
2. Direction of spin - CCW
3. Direction of Precision (Steering) - Left

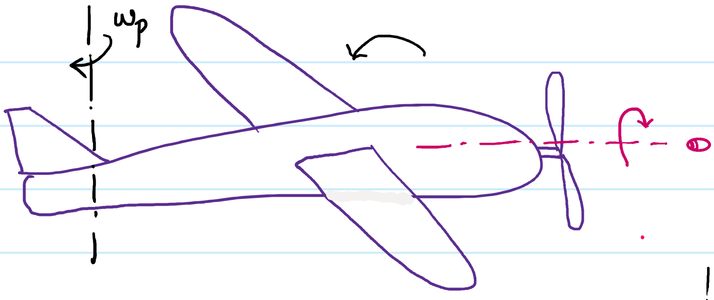
$\hat{\omega}_s = +\hat{i}$

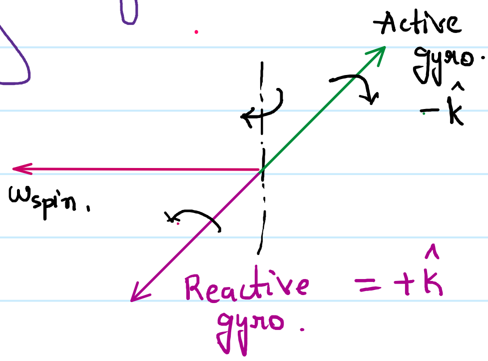
$\hat{\omega}_p = +\hat{j}$

Active gyro $-$ $\hat{\omega}_p \times \hat{\omega}_s = \hat{j} \times \hat{i} = -\hat{k}$

Reactive gyro. $-$ $\hat{\omega}_s \times \hat{\omega}_p = i \times j = +k$.

Effect of Reactive gyro. is to raise the nose and dip the tail.





1. Location of observer - Front
2. Direction of spin - CW
3. Direction of Precision (Steering) - Right

Active gyro. $= \hat{\omega}_p \times \hat{\omega}_s = -\hat{j} \times -\hat{i} = -\hat{k}$ $\quad$ $\hat{\omega}_s = -\hat{i}$

Reactive gyro. $= \hat{\omega}_s \times \hat{\omega}_p = -\hat{i} \times -\hat{j} = +\hat{k}$ $\quad$ $\hat{\omega}_p = -\hat{j}$

Due to Reactive gyro. nose is raised and tail is dipped.

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Effect of Gyroscope on Naval Ships

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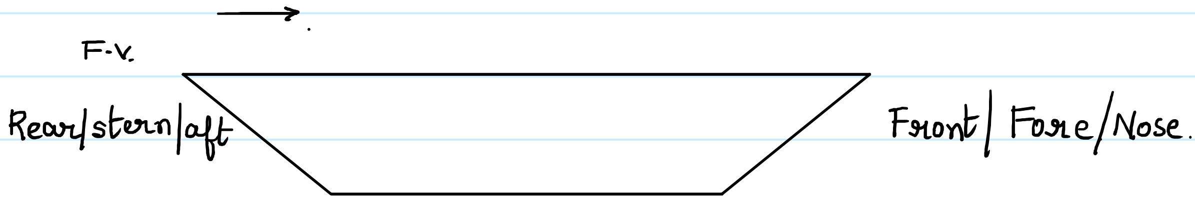
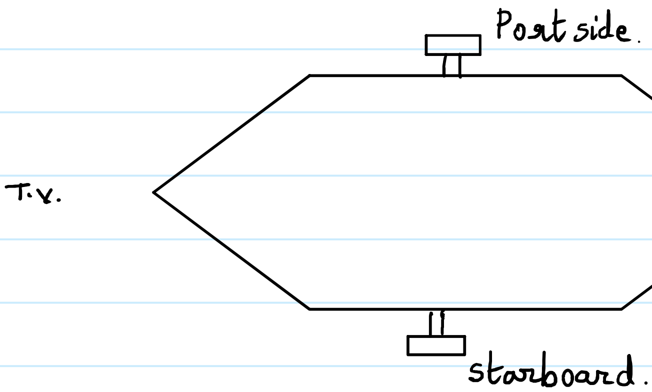
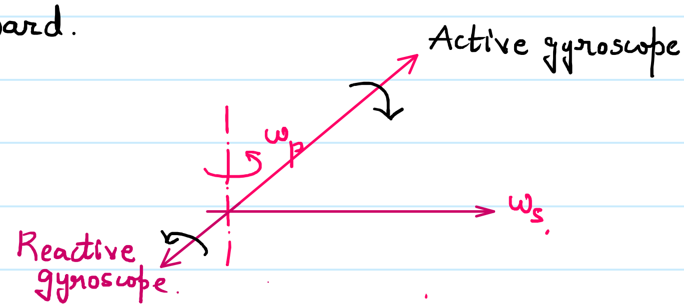
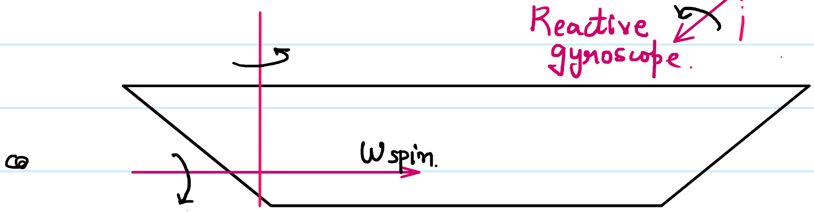






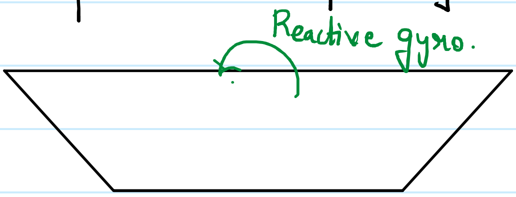
1. Location of observer - Rear
2. Direction of spin - CW
3. Direction of Precision (Steering) - Left



$$\hat{\omega}_s = \hat{i}$$
$$\hat{\omega}_p = +\hat{j}$$

Active gyroscope $= \hat{\omega}_p \times \hat{\omega}_s = \hat{j} \times \hat{i} = -\hat{k}$

Reactive gyroscope $= \hat{\omega}_s \times \hat{\omega}_p = \hat{i} \times \hat{j} = +\hat{k}$ (c.c.w)



Due to the effect of reactive gyro the stern is dipped and fore is raised.

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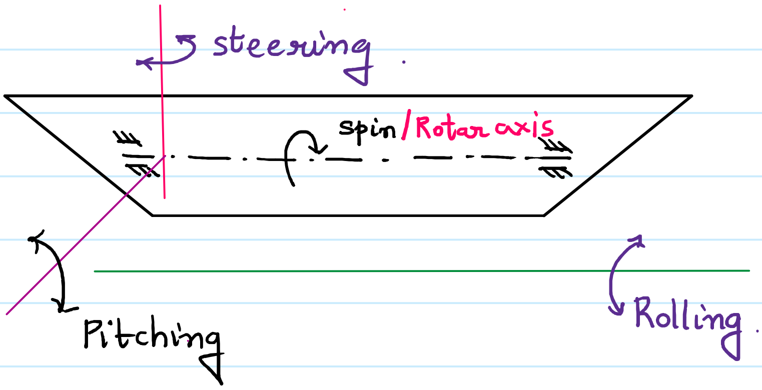
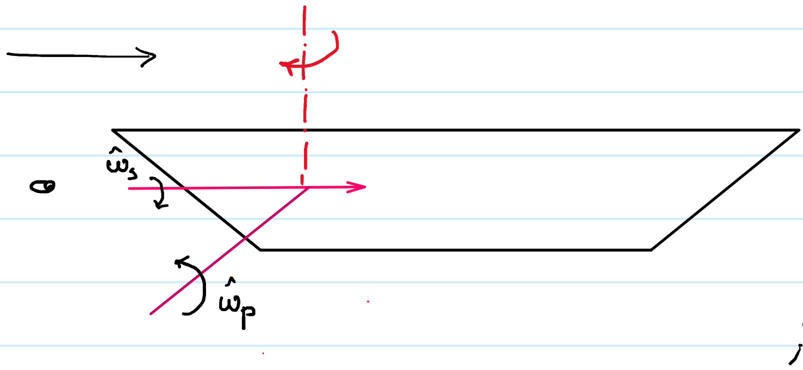
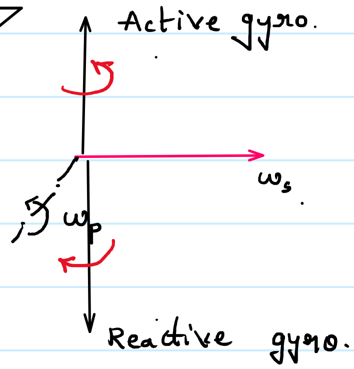


Pitching

1. Location of observer - Rear
2. Direction of spin - CW
3. Direction of Precision (Pitching) - Upwards





$\hat{\omega}_s = \hat{i}$

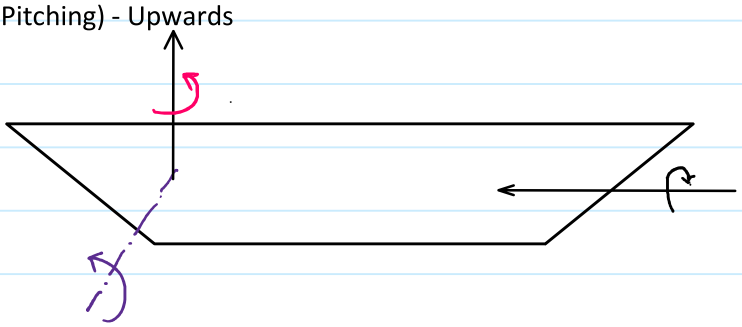
$\hat{\omega}_p = +\hat{k}$

Active gyro. $= \hat{\omega}_p \times \hat{\omega}_s = \hat{k} \times \hat{i} = \hat{j}$

Reactive gyro. $= \hat{\omega}_s \times \hat{\omega}_p = \hat{i} \times \hat{k} = -\hat{j}$

The ship will turn towards right side / starboard.

1. Location of observer - Front
2. Direction of spin - CW
3. Direction of Precision (Pitching) - Upwards



$\hat{\omega}_s = -\hat{i}$

$\hat{\omega}_p = +\hat{k}$

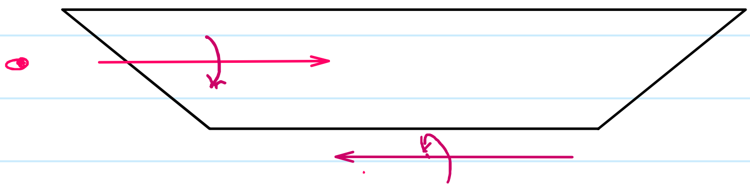
Active gyro. $= \hat{\omega}_p \times \hat{\omega}_s = \hat{k} \times -\hat{i} = -\hat{j}$

Reactive gyro $= \hat{\omega}_s \times \hat{\omega}_p = -\hat{i} \times k = +\hat{j}$

Ship will turn to left side / Portside.

Rolling Motion

1. Location of observer - Rear
2. Direction of spin - CW
3. Direction of Precision (Rolling) - CCW



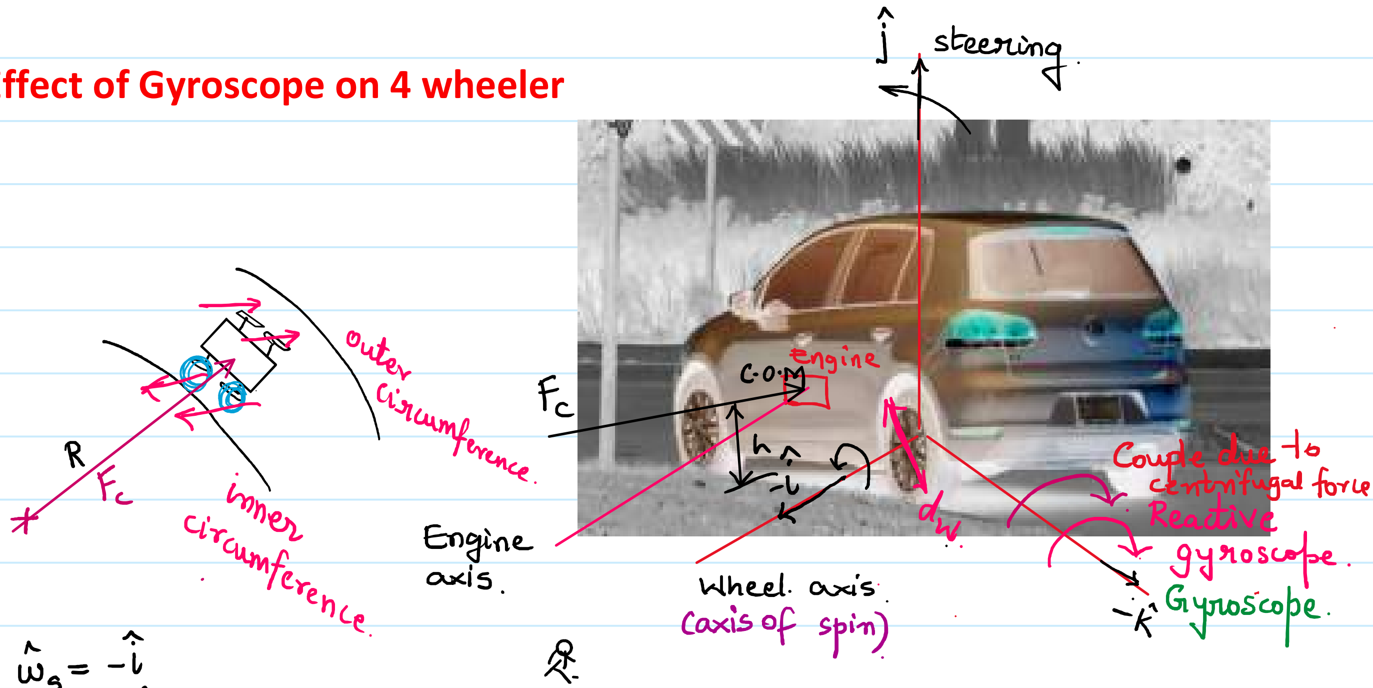
$\hat{\omega}_s = \hat{i}$

$\hat{\omega}_p = -\hat{i}$

Active gyro $= \hat{\omega}_p \times \hat{\omega}_s = -\hat{i} \times \hat{i} = 0$

Reactive gyro $= \hat{\omega}_s \times \hat{\omega}_p = \hat{i} \times -\hat{i} = 0$

Effect of Gyroscope on 4 wheeler



$\hat{\omega}_s = -\hat{i}$

$\hat{\omega}_p = +\hat{j}$

Active gyro $= \hat{\omega}_p \times \hat{\omega}_s = \hat{j} \times -\hat{i} = +\hat{k}$

Reactive gyro $= \hat{\omega}_s \times \hat{\omega}_p = -\hat{i} \times \hat{j} = -\hat{k}$

Due to the effect of Reactive gyro. the wheel on inner circumf-erence will be lifted from ground, wheels on outer circumference will be pressed against the ground.

Vehicle will tend to exprience overturning due to Reactive gyro & Couple due to centrifugal force.

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$$\omega_p = \frac{V}{R}$$

V – Velocity of vehicle
R – Turning Radius.

Gear Ratio $= \frac{\omega_E}{\omega_W} = \quad \omega_E = G \cdot \omega_W.$

Gyroscope on Engine $= I_E \cdot \omega_E \cdot \omega_p.$

$$= I_E \cdot G\, \omega_W \cdot \left(\frac{V}{R}\right)$$

Gyroscope on wheel $= 4 \cdot I_W \cdot \omega_W \cdot \omega_p$

$$= \left(4 \cdot I_W \cdot \frac{V}{R_W} \cdot \frac{V}{R}\right)$$

Angular velocity of wheel.

$$\omega_W = \frac{V}{R_W}.$$

R_W – Radius of wheel.

$$C_E \pm C_W = I_E \cdot G \cdot \frac{V}{R_W} \cdot \frac{V}{R} \pm 4 I_W \cdot \frac{V^2}{R \cdot R_W}.$$

Net Gyroscopic Torque.

$$C_1 = \underbrace{\left(G \cdot I_E \pm 4 I_W\right)} \cdot \frac{V^2}{R \cdot R_W}.$$

$$C_1 = I_{eq} \cdot \alpha_{eq}$$

I_{eq} – Equivalient Inertia
α_{eq} – Gyroscopic Acceleration.

Couple due to centrifugal force.

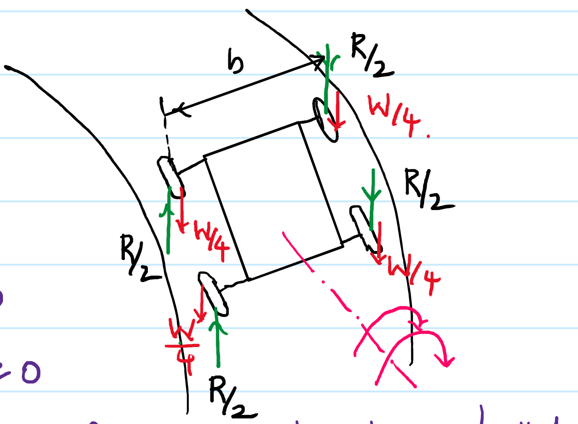
$$C_2 = F_c \times h = \frac{mV^2}{R} \times h.$$

Net Couple $= C_1 + C_2 = \left[\left(G \cdot I_E \pm 4 I_W\right) \cdot \frac{V^2}{R \cdot R_W} + \frac{mV^2}{R} \times h\right]$

Net couple on each axle.

$$= \frac{C_{Net}}{2} = \frac{R}{2} \times b.$$

$$\frac{R}{2} = \frac{C_{Net}}{2b}.$$



Reaction on outer wheel $= \frac{W}{4} + R/2$

Reaction on inner wheel $= W/4 - \frac{R}{2} \geq 0$

$$= \frac{W}{4} - \frac{C_{Net}}{2b} \geq 0$$

In order to maintain stability the value of V must be controlled.

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Condition of stability.

$$\frac{W}{4} - \frac{C_{Net}}{2b} \geq 0$$

$$\frac{W}{4} - \frac{1}{2b}\left[\frac{(G.I_E \pm 4I_W)}{R.R_W} + \frac{m.h}{R}\right].V^2 \geq 0$$

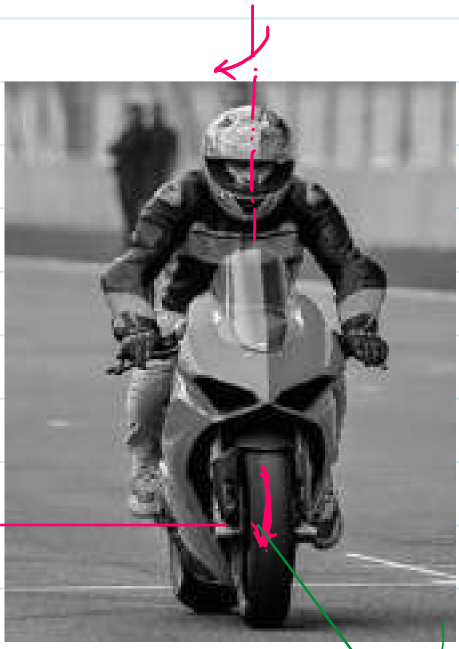
$$V^2 \leq \frac{W/4}{\frac{1}{2b}\left[(GI_E \pm 4I_W) + \frac{mh}{R}\right]}.$$

Effect of Gyroscope on 2 wheeler

$$\hat{\omega}_s = \hat{i}$$
$$\hat{\omega}_p = -\hat{j}$$

Active gyro. $= \hat{\omega}_p \times \hat{\omega}_s = -\hat{j} \times \hat{i} = \hat{k}$

Reactive gyro $= \hat{\omega}_s \times \hat{\omega}_p = \hat{i} \times -\hat{j} = -\hat{k}$



axis of spin.

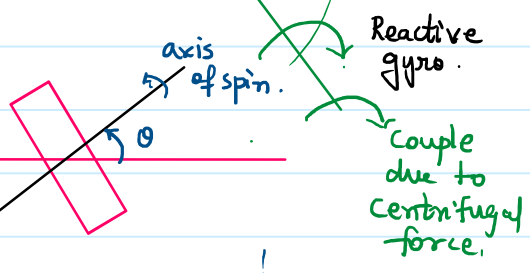
Gyroscope on 2 wheel.

$$C = C_W + C_E$$

$$C = 2.I_W.\omega_W.\omega_p + I_E.\omega_E.\omega_p$$

$$= 2.I_W.\frac{V}{R_W}\cos\theta.\frac{V}{R} + I_E.G.\frac{V}{R_w}\cos\theta.\frac{V}{R}$$

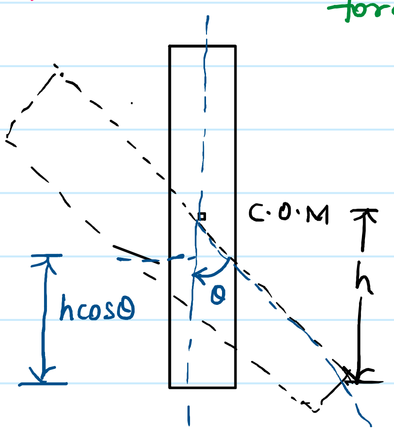
$$C_1 = (2I_W + GI_E).\frac{V^2}{R.R_W}\cos\theta \quad = I_{eq}.\alpha_{eq}$$



Couple due to centrifugal force.

$$C_2 = F_c \times h\cos\theta$$

$$= \frac{mv^2}{R}.\times h\cos\theta$$



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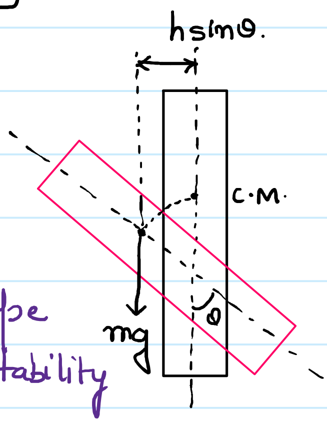
Net couple on vehicle

$$= C_1 + C_2$$

$$= \left[\left(2 \cdot I_W + G I_E\right) \cdot \frac{V^2}{R \cdot R_W} + \frac{mV^2}{R} \cdot h\right] \cos\theta \rightarrow A$$

Balancing couple $= mg \cdot h \sin\theta \rightarrow$ (B)

$$A = B$$



Centrifugal force and Reactive gyroscope together will cause distrubance / unstability

MSQ

01. When a four-wheeler moving forward at a speed just above critical speed for stability (Assume the engine rotates on a parallel axis to the wheels in the same direction) takes a turn to the right the wheel(s) that tends to leave the ground is
(a) outer front wheel
(b) outer rear wheel
(c) both the inner wheels → Due reactive gyro. + couple due centrifugal force wheel are lifted.
(d) none of the four wheels

02. Pitching motion of ship, carrying a rotor that rotates in clockwise sense as seen from stern produces couple

Rolling motion { (a) ~~in transverse vertical plane in clockwise sense~~ as seen from stern
(b) ~~in transverse vertical plane in counterclockwise~~ sense as seen from stern

(c) in longitudinal vertical plane → Pitching motion
(d) in horizontal plane → Gyroscope.

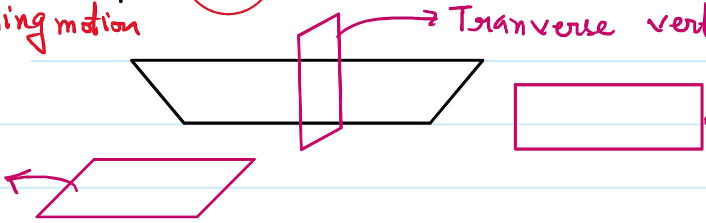
Horizontal plane.

07. Which of the following statement(s) is/are CORRECT?
(a) Spin, Precession and Gyroscope axes are mutually perpendicular to each other. ✓
(b) Gyroscope couple becomes zero, if spin and precession axes become collinear. ✓ → Rolling motion
(c) ~~Gyroscope effect and centrifugal force effects are opposing each other in case of automobiles.~~
(d) Rolling of ship causes no gyroscope effect. ✓

$\hat{\omega}_s = \pm\hat{i}$

$\hat{\omega}_p = \pm\hat{k}$

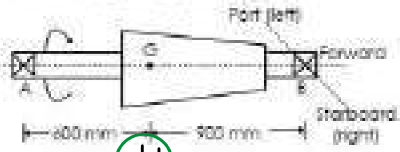
Gyroscope $= \pm\hat{j}$ ↗ plane normal to Y-axis



Tranverse vertical plane

longitudinal vertical plane

03. The turbine rotor in a ship's power plant has a mass of 1000 kg with center of mass at G and a radius of gyration of 200 mm. The rotor shaft is mounted in bearings A and B with its axis in the horizontal fore-and-aft direction and turns ccw at a speed of 5000 rpm when viewed from the stern. Determine the vertical components of the bearing reactions at A and B if the ship is making a turn to port (left) of 400 m radius at a speed of 25 knots (1 knot = 0.514 m/s). Does the bow of the ship tend to rise or fall because of gyroscopic action.



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$M = 1000\,kg$

$K = 200\,mm$.

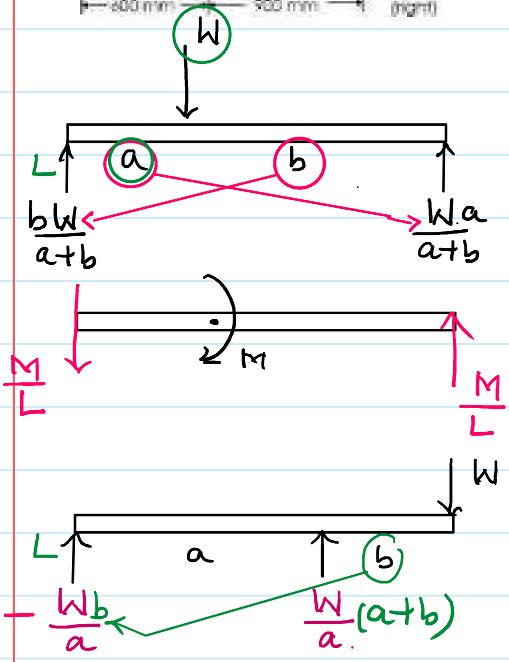
$N = 5000\,rpm$. (spin) c.c.w. from stern.

turning. left.

$R = 400\,m$ $\qquad V = 25$ Knots.

$V = 25 \times 0.514\ m/s$.

$V = 12.85\ m/s$.

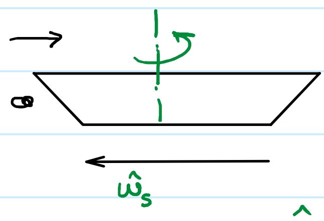


$C = I \cdot \omega_s \omega_p$.

$= mk^2 \cdot \left(\frac{2\pi N}{60}\right) \cdot \left(\frac{V}{R}\right)$

$= 1000 \times 0.2^2 \times \left(\frac{2\pi \times 5000}{60}\right) \times \left(\frac{12.85}{400}\right)$

$C = 672.824$. N-m.



$\hat{\omega}_s = -\hat{i}$

$\hat{\omega}_p = +\hat{j}$

Active gyro $= \hat{\omega}_p \times \hat{\omega}_s$

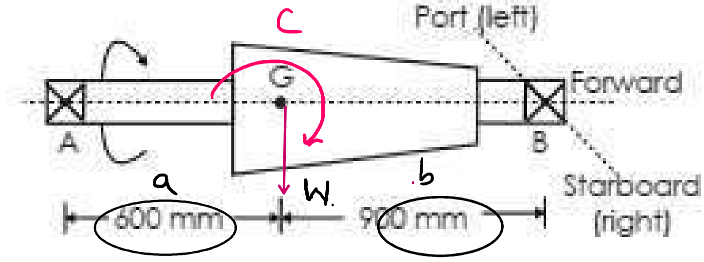
$= \hat{j} \times -\hat{i} = +\hat{k}$ c.c.w.

Reactive gyro. $= \hat{\omega}_s \times \hat{\omega}_p = -\hat{i} \times \hat{j} = -\hat{k}$ c.w.

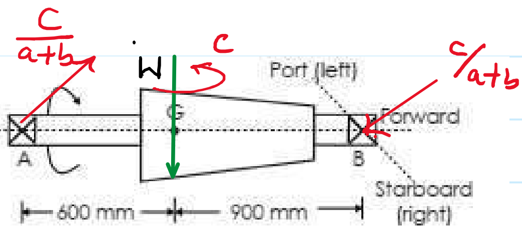
steering. Left.

$R_A = \frac{Wb}{a+b} - \frac{C}{a+b}$

$R_B = \frac{Wa}{a+b} + \frac{C}{a+b}$



Pitching Motion.



$\omega_s = \pm \hat{i}$ $\qquad C = \pm \hat{j}$

$\omega_p = \pm \hat{k}$

$R_A = \sqrt{\left(\frac{Wb}{a+b}\right)^2 + \left(\frac{C}{a+b}\right)^2}$

$R_B = \sqrt{\left(\frac{Wa}{a+b}\right)^2 + \left(\frac{C}{a+b}\right)^2}$

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