06. For the system shown in the given figure the moment of inertia of the weight W and the ball about the pivot point is 10 ($I_0$). The natural frequency of the system is given by

$f_n = (1/2\pi)\sqrt{(Ka^2 - Wb)/I_0}$

The system will vibrate when

$$f_n = \frac{1}{2\pi}\sqrt{\frac{Ka^2 - Wb}{I_0}} \qquad f_n > 0$$

$Ka^2 - Wb > 0$

only then system will oscillate.

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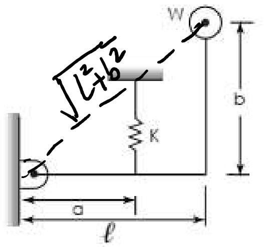



$$\boxed{b < \frac{Ka^2}{W}}$$

W is responsible for $\delta_{static}$

given position itself can be assumed as equilibrium position.

$$I = \frac{W}{g}(L^2 + b^2)$$

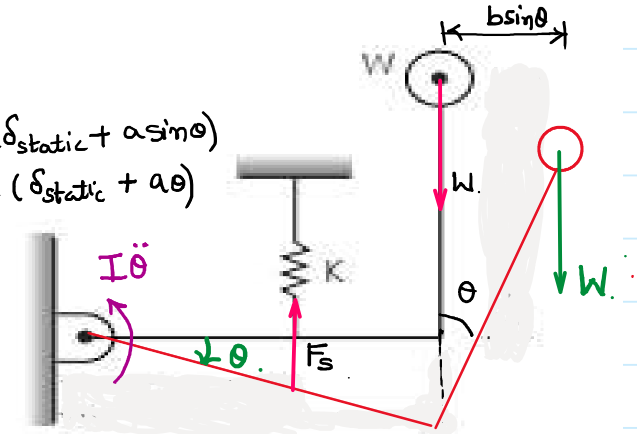
(a) b <(Ka²/W) (b) b = (Ka²/W)

(c) b >(Ka²/W) (d) a = 0

$$F_s = K.(\delta_{static} + a\sin\theta)$$
$$= K.(\delta_{static} + a\theta)$$

$\sin\theta \approx \theta$

$\cos\theta \approx 1$




$\Sigma M_{Hinge} = 0$

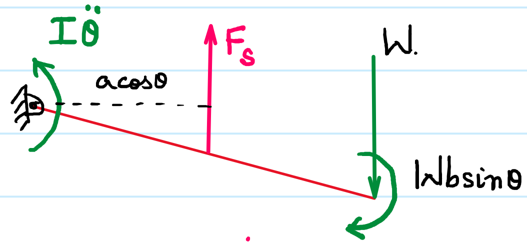
$$I\ddot{\theta} + F_s a\cos\theta - WL\cos\theta - Wb\sin\theta = 0$$

$$I\ddot{\theta} + K.(\delta_{static} + a\theta)a - WL - Wb\theta = 0$$

constant terms

$$\frac{W}{g}(L^2+b^2)\ddot{\theta} + \boxed{K\delta_{static}\,a} + Ka^2\theta - \boxed{WL} - Wb\theta = 0$$

0 0



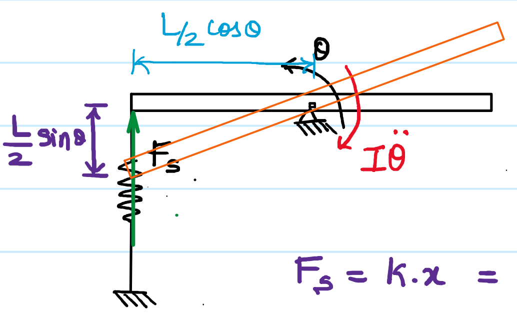

$$\frac{W}{g}(L^2+b^2)\ddot{\theta} + (Ka^2 - Wb)\theta = 0$$

$$\ddot{\theta} + \frac{Ka^2 - Wb}{\frac{W}{g}(L^2+b^2)}\theta = 0 \qquad \omega_n = \sqrt{\frac{Ka^2 - Wb^2}{\frac{W}{g}(L^2+b^2)}}$$

05. A uniform rigid rod of mass m = 1kg and length L = 1 m is hinged at its centre & laterally supported at one end by a spring of spring constant k = 300N/m. The natural frequency $\omega_n$ in rad/s is (GATE-08)

(a) 10 (b) 20

(c) 30 (d) 40




Disp. variable – θ

DOF = 1.

$\sin\theta \approx \theta$

$\cos\theta \approx 1$

$$F_s = K.x = K.\frac{L}{2}\sin\theta$$
$$= (KL/2\,\theta)$$

$\Sigma M_{Hinge} = 0 \qquad I\ddot{\theta} + F_s.L/2.\cos\theta = 0$

$$\frac{ML^2}{12}\ddot{\theta} + (K.L/2.\theta).L/2 = 0 \Rightarrow \frac{ML^2}{12}\ddot{\theta} + \frac{KL^2}{4}\theta = 0 \Rightarrow \ddot{\theta} + \frac{3K}{M}\theta = 0$$

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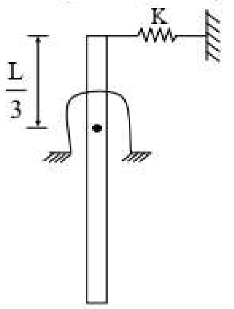
09. A slender link of mass 'm' and length L is pivoted at a length of L/3 from the top and is connected as shown in figure and it oscillates in the vertical plane. The stiffness of the spring is 'K'. The natural frequency of the system is given by




(a) $\sqrt{\frac{3g}{2L}-\frac{K}{m}}$ (b) $\sqrt{\frac{3g}{2L}+\frac{K}{m}}$

(c) $\sqrt{\frac{g}{L}+\frac{K}{m}}$ (d) $\sqrt{\frac{g}{L}-\frac{K}{m}}$

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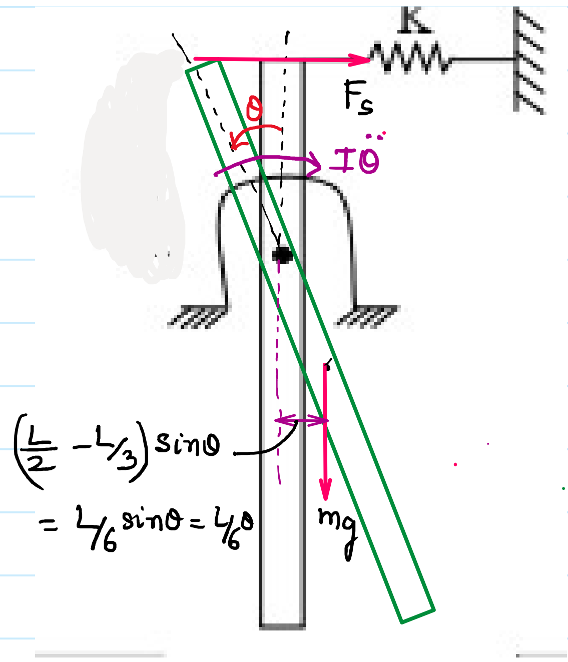


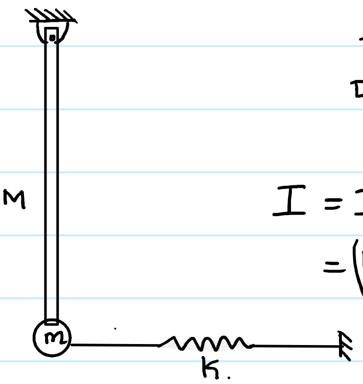

$$I = \bar{I} + md^2$$

$$= \frac{ML^2}{12} + M\left(\frac{L}{6}\right)^2$$

$$= \frac{ML^2}{9}.$$

$$I\ddot{\theta} + \left(F_s \times \frac{L}{3}\right) + mg \cdot \frac{L}{6} \cdot \theta = 0$$

$$\frac{ML^2}{9}\ddot{\theta} + \frac{K.L^2}{9}\theta + mg \cdot \frac{L}{6}\,\theta = 0 \Rightarrow \ddot{\theta} + \left(\frac{K}{M} + \frac{3g}{2L}\right)\theta = 0$$

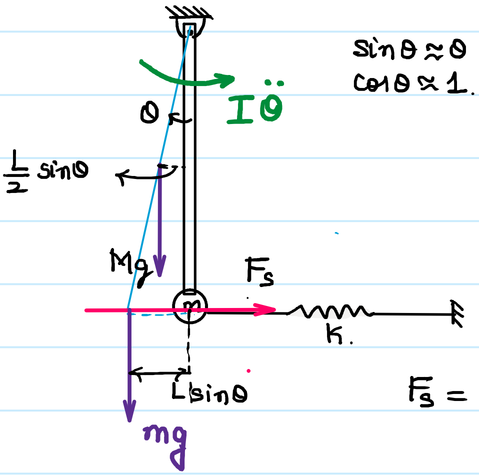



DOF = 1

Disp. variable — θ.

$$I = I_{Rod} + I_{Point\ mass}.$$

$$= \left(\frac{ML^2}{3} + m.L^2\right)$$




$\sin\theta \approx \theta$

$\cos\theta \approx 1.$

$F_s = K.(L\theta)$

$$\Sigma M_{Hinge.} = 0$$

$$\left(\frac{ML^2}{3} + mL^2\right)\ddot{\theta} + Mg.\frac{L}{2}.\theta + mg.L.\theta + F_s.L = 0$$

$$\left(\frac{ML^2}{3} + mL^2\right)\ddot{\theta} + K.L^2\theta + \left(\frac{M}{2} + m\right)g.L\theta = 0 \Rightarrow \ddot{\theta} + \frac{KL^2 + \left(\frac{M}{2}+m\right)g.L}{\left(\frac{ML^2}{3} + mL^2\right)}\theta = 0$$

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Wt of Rod will produced Static

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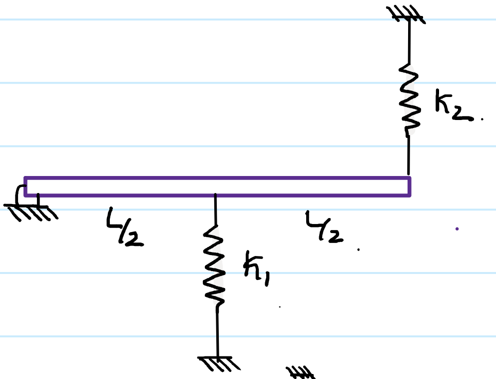
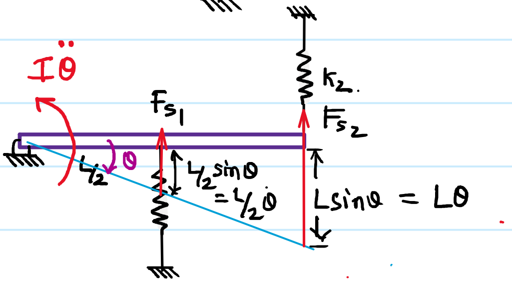



$\sin\theta = \theta$

$\cos\theta = 1$




$$\Sigma M_{Hinge} = 0$$

$$I\ddot{\theta} + F_{s_1} \times L/2 \cdot \cos\theta + F_{s_2} \cdot L\cos\theta = 0$$

$$\frac{ML^2}{3}\ddot{\theta} + (k_1 \cdot L/2\,\theta)\,L/2 + (k_2 L\theta) \cdot L = 0$$

$$\frac{ML^2}{3}\ddot{\theta} + \left(k_1 \frac{L^2}{4} + k_2 L^2\right)\theta = 0$$

$$\ddot{\theta} + \left(\frac{3k_1}{4M} + \frac{3k_2}{M}\right)\theta = 0$$

$$\Sigma M_{Ref\ point} = 0$$

$$I_{Ref\cdot point}\,\ddot{\theta} + k(\perp^{\text{lar}} \text{distance from ref}\cdot\text{point})^2\theta \pm mg.(\perp^{\text{lar}} \text{distance from ref-point})\theta = 0$$

$$\frac{ML^2}{3}\ddot{\theta} + k_1 (L/2)^2\theta + k_2 (L)^2\theta = 0$$

10. A spring mass system with a natural frequency of 5 rad/sec is set in to free oscillations by giving an initial displacement of 10 cm. The amplitude of vibration is

(a) 50 cm (b) 2 cm

(c) 15 cm (d) 10 cm

$\omega_n = 5$ rad/s.

intial displacement $X_0 = 10$ cm.

Amplitude = ?

$$x(t) = X_0 \cdot \sin(\omega t + \phi)$$

@ $t = 0$ $\quad x(t=0) = 10\text{cm} = X_0 \cdot \sin(0 + \phi)$

$$10\text{cm} = X_0 \cdot \sin\phi$$

$\underbrace{@\ t = 0 \quad \dot{x}(t=0)}_{\text{intial velocity}} = 0$ $\quad \dot{x}(t) = X_0\,\omega \cdot \cos(\omega t + \phi)$

$$0 = X_0 \cdot \times 5 \cos(0 + \phi)$$

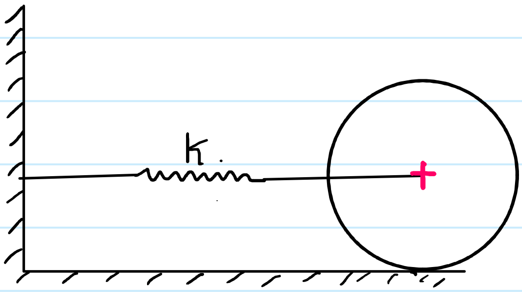
$X_0 \cdot \cos\phi = 0$ $\quad \cos\phi = 0$

$\phi = 90^\circ$

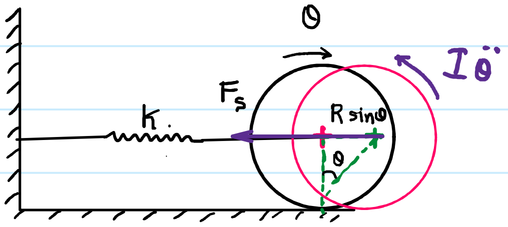
Amplitude

$$10 = X_0 \cdot \sin 90^\circ \Rightarrow X_0 = 10\text{ cm}$$

Cylinder Disc. Higher Pair – Pure Rolling.




$\sin\theta \simeq \theta$
$\cos\theta \simeq 1$




$$\Sigma M_{p.o.c} = 0$$

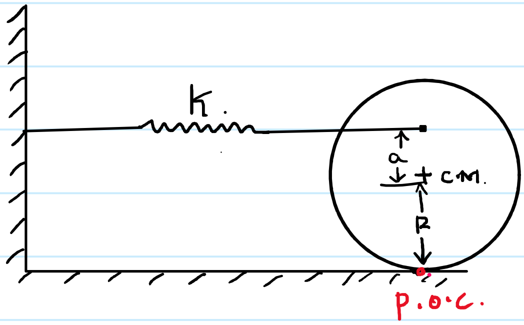
$$I_{p.o.c}\cdot\ddot{\theta} + F_s \times R\cdot\cos\theta = 0$$

$$\left(\frac{MR^2}{2} + MR^2\right)\ddot{\theta} + (K.R\theta)\cdot R = 0$$

$R\sin\theta = R\theta$

$$\frac{3}{2}MR^2\ddot{\theta} + KR^2\theta = 0$$

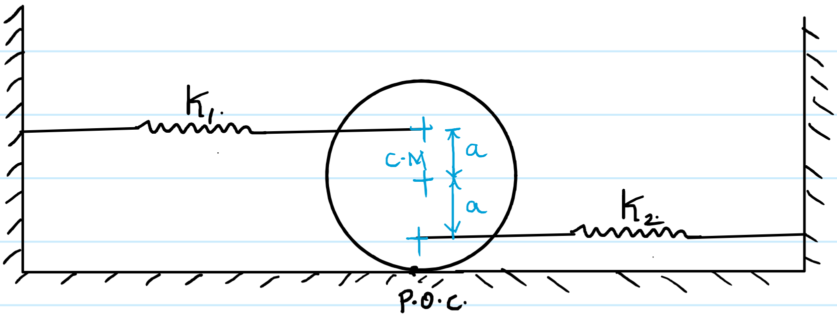
$$\ddot{\theta} + \frac{2K}{3M}\theta = 0$$




$$\Sigma M_{p.o.c} = 0$$

$$I_{p.o.c}\ddot{\theta} + K\cdot(\perp^{lar} \text{ from } p.o.c)^2\theta = 0 \Rightarrow \tfrac{3}{2}MR^2\ddot{\theta} + K\cdot(R+a)^2\theta = 0$$

$$\ddot{\theta} + \frac{2K\cdot(R+a)^2}{3MR^2}\theta = 0$$



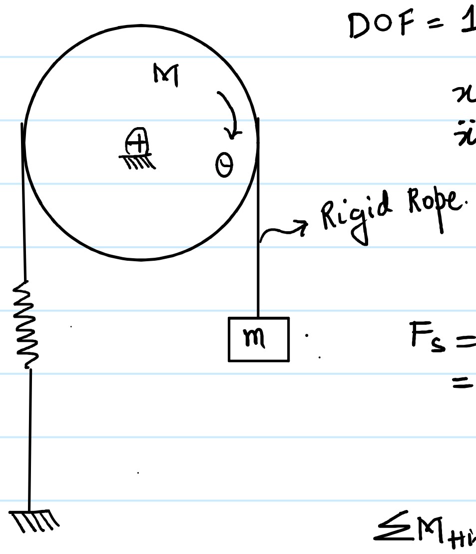

$$\Sigma M_{P.O.C} = 0$$

$$I_{P.O.C}\cdot\ddot{\theta} + K_1(R+a)^2\theta + K_2(R-a)^2\theta = 0$$

$$\frac{3}{2}MR^2\ddot{\theta} + \left[K_1(R+a)^2 + K_2(R-a)^2\right]\theta = 0$$

$$\ddot{\theta} + \frac{2\cdot\left[K_1(R+a)^2 + K_2(R-a)^2\right]}{3MR^2}\theta = 0$$

DOF = 1.

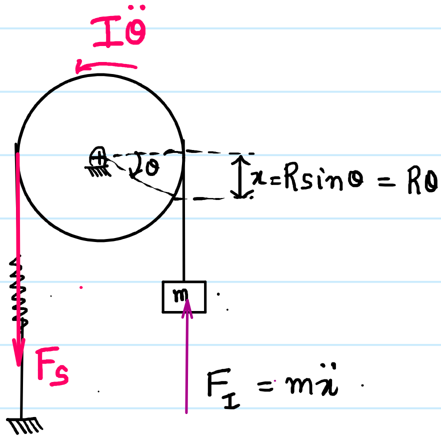



$x = R\theta$

$\ddot{x} = R\cdot\ddot{\theta}$

$F_s = kx$

$= k\cdot R\theta$



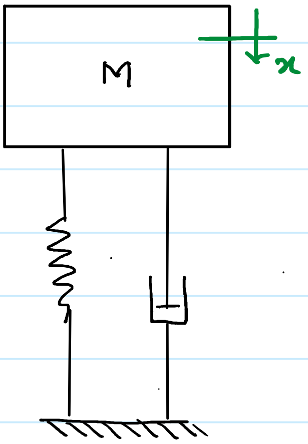

$$\Sigma M_{Hinge} = 0$$

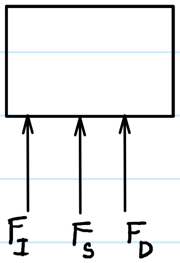
$$I_{Hinge}\,\ddot{\theta} + F_I\times R + F_s\times R = 0$$

$$\frac{MR^2}{2}\ddot{\theta} + (mR\ddot{\theta}\times R) + K\cdot R^2\theta = 0$$

$$\ddot{\theta} + \frac{K}{\left(m+\frac{M}{2}\right)}\theta = 0$$

## Free Damped Vibrations — Loss in energy after each cycle.




$\Sigma F = 0$

$F_I + F_D + F_S = 0$

$M\ddot{x} + C\dot{x} + Kx = 0$

$$\ddot{x} + \frac{C}{M}\dot{x} + \frac{K}{M}x = 0 \qquad \frac{K}{M} \to \omega_n^2$$

$$\ddot{x} + \frac{C}{2\sqrt{MK}} \cdot \frac{\sqrt{K}}{\sqrt{M}} \cdot 2\dot{x} + \omega_n^2 \cdot x = 0 \qquad \frac{C}{2\sqrt{MK}} \to \xi, \quad \frac{\sqrt{K}}{\sqrt{M}} \to \omega_n$$

$M\ddot{x} + c\dot{x} + kx = 0$

$\ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2 x = 0$ $\qquad a = 1 \quad b = 2\xi\omega_n \quad c = \omega_n^2$

$$\frac{d^2x}{dt^2} + 2\xi\omega_n \cdot \frac{dx}{dt} + \omega_n^2 \cdot x = 0 \Rightarrow (D^2 + 2\xi\omega_n D + \omega_n^2)x = 0$$

$$D = \frac{-2\xi\omega_n \pm \sqrt{(2\xi\omega_n)^2 - 4(1)\cdot\omega_n^2}}{2(1)}$$

$$D = -\xi\omega_n \pm \sqrt{(\xi\omega_n)^2 - \omega_n^2}$$

$$D = \omega_n\left[-\xi \pm \sqrt{\xi^2 - 1}\right]$$

$\xi = 1 \qquad D = -\omega_n, -\omega_n$ $\qquad$ complimentary function

$$x(t) = (A + Bt)e^{-\omega_n t}$$

$\xi > 1 \qquad D = \underbrace{\omega_n\left[-\xi + \sqrt{\xi^2 - 1}\right]}_{-\alpha_1}, \ \underbrace{\omega_n\left[-\xi - \sqrt{\xi^2 - 1}\right]}_{-\alpha_2}$

C.F. $\quad x(t) = Ae^{-\alpha_1 t} + Be^{-\alpha_2 t}$

$\boxed{\xi = 1,\ \xi > 1 \text{ system is coming to rest without executing oscillations.}}$

$\xi < 1 \qquad D = \omega_n\left[-\xi + i\sqrt{1-\xi^2}\right], \ \omega_n\left[-\xi - i\sqrt{1-\xi^2}\right]$

$D = -\xi\omega_n \pm i\omega_n\sqrt{1-\xi^2} \qquad \omega_n\sqrt{1-\xi^2} \to \omega_d \qquad D = -\xi\omega_n \pm \omega_d$

C.F. $\quad x(t) = \left[A\sin(\omega_d t) + B\cos(\omega_d t)\right]e^{-\xi\omega_n t}$

$x(t) = X_0 \sin(\omega_d t + \phi)\, e^{-\xi\omega_n t}$

$\boxed{\xi < 1 \text{ system is coming to rest by executing decaying oscillations}}$

Loss in energy.

$A = X_0 \cos\phi$

$B = X_0 \sin\phi$

$\zeta = 0$ $\quad x(t) = X_0 \cdot \sin(\omega_n t + \phi)$ $\quad$ undamped system. (No loss in energy)

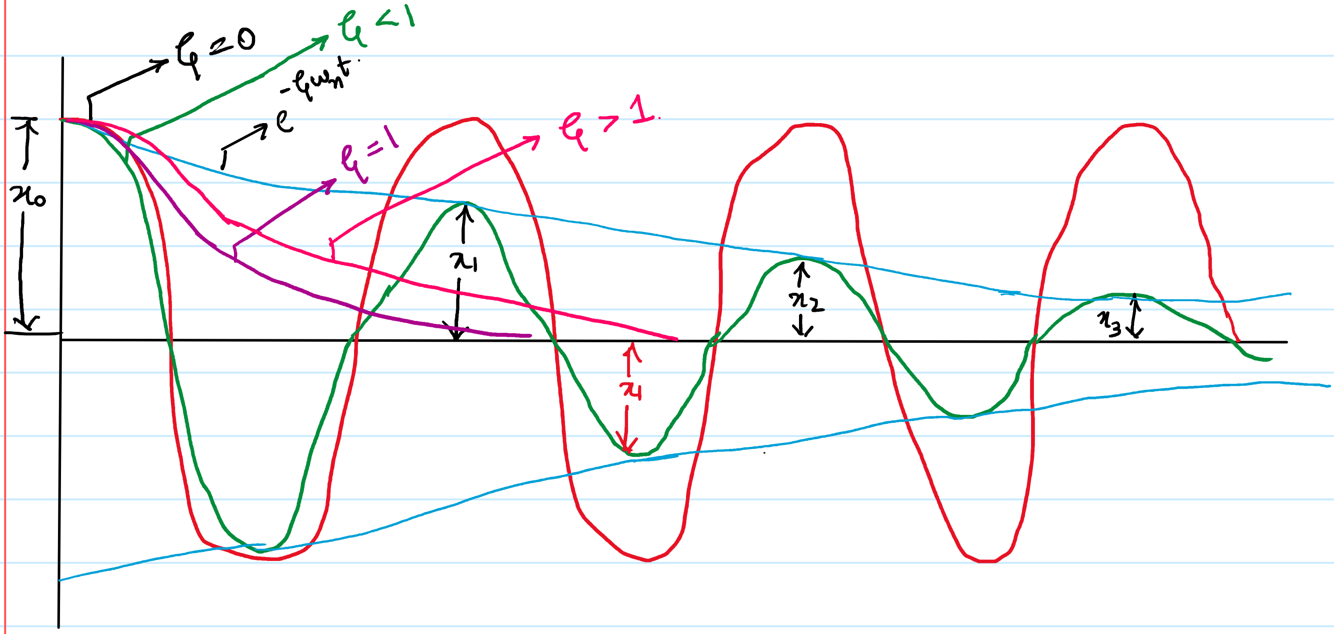
$\zeta < 1$ $\quad x(t) = X_0 e^{-\zeta \omega_n t} \sin(\omega_d t + \phi)$

Loss in energy. $\quad$ underdamped system.

$\zeta = 1.$ $\quad \boxed{x(t) = (A + Bt) e^{-\omega_n t}}$ $\quad$ critically damped system.

$\zeta > 1$ $\quad \boxed{x(t) = A e^{-\alpha_1 t} + B e^{-\alpha_2 t}}$ $\quad$ Overdamped system.

↳ aperiodic/non-periodic response.




$\zeta$ – Damping Ratio/Damping factor. $\quad \zeta = \dfrac{c}{2\sqrt{MK}} = \dfrac{\text{Actual damping coeff.}}{\text{Critical damping coeff.}}$

Damping factor is defined as Ratio of Actual damping to critical damping. Damping factor also represents the % of critical damping.

$$\zeta = \frac{c}{c_c} = \frac{c}{2\sqrt{MK}}.$$

$$c_c = 2\sqrt{\frac{M^2 \cdot K}{M}} = 2 \cdot M \cdot \omega_n.$$

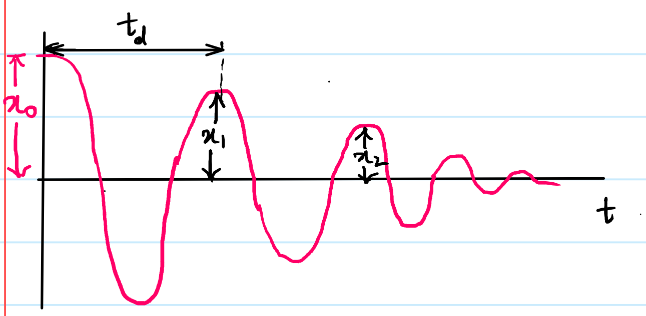
$\zeta < 1$. $\quad x(t) = X_0 \cdot e^{-\zeta\omega_n t} \sin(\omega_d t + \phi)$

initial condition @ $t = 0$

$$x(t=0) = X_0 e^{-\zeta\omega_n \cdot (0)} \sin(\omega_d (0) + \phi)$$

Amplitude $\rightarrow X_0$

$$x_0 = X_0.$$




$\omega_d$ – frequency of damped oscillation

$t_d$ – time period of damped oscillation.

$$t_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n\sqrt{1-\zeta^2}}$$

$$x_1 = x(t=t_d) = X_0 . e^{-\zeta\omega_n t_d} \sin(\omega_d t_d + \phi)$$

Amplitude

$$x_1 = X_0 . e^{-\zeta\omega_n t_d}$$

displacement · Amplitudes

$$x_0 = X_0 . e^{-\zeta\omega_n \cdot (0)}$$
$$x_1 = X_0 . e^{-\zeta\omega_n . t_d}$$
$$x_2 = X_0 . e^{-\zeta\omega_n . 2t_d}$$
$$x_n = X_0 . e^{-\zeta\omega_n . n . t_d}$$

Ratio of consecutive displacements.

$$\frac{x_0}{x_1} = \frac{X_0}{X_0 . e^{-\zeta\omega_n t_d}} \Rightarrow \frac{x_0}{x_1} = e^{\zeta\omega_n t_d}.$$

$$\frac{x_0}{x_1} = e^{\zeta\omega_n t_d}$$

$$\frac{x_0}{x_1} = \frac{x_1}{x_2} = \frac{x_2}{x_3} = \frac{x_3}{x_4} = \dots = \frac{x_{n-1}}{x_n} = e^{\zeta\omega_n t_d}.$$

$x_0, x_1, x_2 \dots x_n$ are in Geometric Progression.

$$\frac{x_0}{x_n} = \underbrace{\frac{x_0}{x_1}}_{\zeta\omega_n t_d} . \frac{x_1}{x_2} . \frac{x_2}{x_3} . \frac{x_3}{x_4} \dots \frac{x_{n-1}}{x_n}$$

$$\frac{x_0}{x_n} = e^{\zeta\omega_n t_d . n}.$$

$n$ – no. of cycles.

Logarithamic decrement $\quad \delta = \ln\left(\frac{x_0}{x_1}\right) = \zeta\omega_n t_d.$

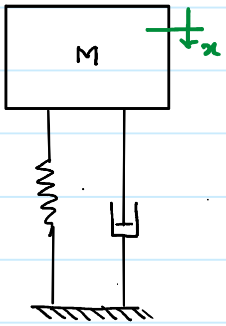
$$\ln\left(\frac{x_0}{x_1}\right) = \zeta \cdot \omega_n \cdot \frac{2\pi}{\omega_n\sqrt{1-\zeta^2}}$$

$$\delta = \ln\left(\frac{x_0}{x_1}\right) = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

$$\ln\left(\frac{x_0}{x_n}\right) = \zeta\omega_n t_d . n.$$

$$\ln\left(\frac{x_0}{x_n}\right) = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} . n$$

$$\frac{1}{n}\ln\left(\frac{x_0}{x_n}\right) = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = \delta$$




Equation of motion

$$M\ddot{x} + C\dot{x} + Kx = 0$$

$\hookrightarrow$ Translatory motion



For Rotatory motion

$$I_{eq}\ddot{\theta} + C_{eq}\dot{\theta} + K_{eq}\theta = 0$$

Natural frequency.

$$\omega_n = \sqrt{\frac{K}{M}} \rightsquigarrow \text{Translatory motion.}$$

$$\omega_n = \sqrt{\frac{K_{eq}}{I_{eq}}} \rightsquigarrow \text{Rotatory motion.}$$

Critical damping

$$C_c = 2\cdot\sqrt{MK}. \rightsquigarrow \text{Translatory motion}$$

$$C_c = 2\cdot\sqrt{I_{eq}\cdot K_{eq}} \rightsquigarrow \text{Rotatory motion.}$$

Damping factor

$$\zeta = \frac{C}{2\sqrt{MK}}. \rightsquigarrow \text{Translatory motion.}$$

$$\zeta = \frac{C_{eq}}{2\cdot\sqrt{I_{eq}\cdot K_{eq}}} \rightsquigarrow \text{Rotatory motion.}$$

| SNo. | System Parameter | Motion. | | | |
|---|---|---|---|---|---|
| | | Translation. | | Rotation | |
| | | Force. | Energy. | Couple. | Energy. |
| 1. | Inertia | $M\ddot{x}$ | $\frac{1}{2}\cdot M\dot{x}^2$ | $I_{eq}\ddot{\theta}$ | $\frac{1}{2} I_{eq}\dot{\theta}^2$ |
| 2. | Restoration | $Kx.$ | $\frac{1}{2}Kx^2$ | $K_{eq}\theta$ | $\frac{1}{2} K_{eq}\cdot\theta^2$ |
| 3. | Damping | $C\dot{x}$ | $\int F_D\cdot dx = \int C\dot{x}\,dx$ | $C_{eq}\dot{\theta}$ | $\int T_D\cdot d\theta = \int C_{eq}\dot{\theta}\,d\theta$ |

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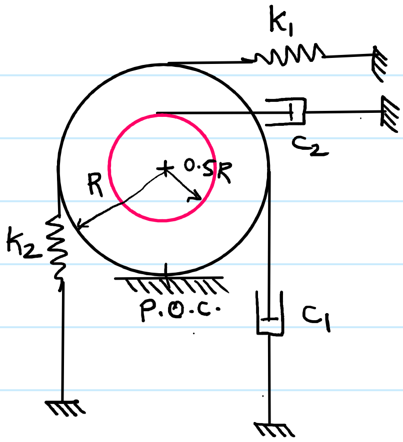



$E.O.M = ?$

$\omega_n = ?$

$C_c = ?$

$\zeta = ?$

$\omega_d = ?$



$$\Sigma M_{P.O.C} = 0$$

$$I_{P.O.C}\,\ddot{\theta} + c_1(\perp^{lar}\text{ distance from P.O.C})^2\dot{\theta} + c_2(\perp^{lar}\text{ distance from P.O.C})^2\dot{\theta} + k_1(\perp^{lar}\text{ distance from P.O.C})^2\theta + k_2(\perp^{lar}\text{ distance from P.O.C})^2\theta = 0$$

$$\frac{3}{2}MR^2\ddot{\theta} + c_1(R)^2\dot{\theta} + c_2(1.5R)^2\dot{\theta} + k_1(2R)^2\theta + k_2(R)^2\theta = 0$$

$$I_{eq} = \frac{3}{2}MR^2 \qquad C_{eq} = c_1R^2 + 2.25c_2R^2 \qquad K_{eq} = 4k_1R^2 + k_2R^2$$

Natural frequency $\omega_n = \sqrt{\dfrac{K_{eq}}{I_{eq}}} \Rightarrow \omega_n = \sqrt{\dfrac{(4k_1+k_2)R^2}{3/2\,MR^2}} = \sqrt{\dfrac{8k_1+2k_2}{3M}}$

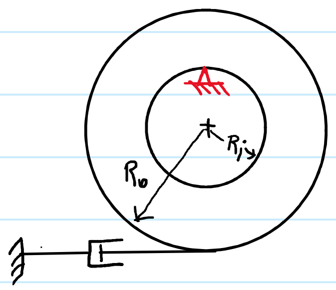
Critical Damping $C_c = 2\sqrt{I_{eq}\cdot K_{eq}} \qquad C_c = 2\sqrt{3/2\,MR^2\cdot(4k_1+k_2)R^2}$

Damping factor $\zeta = \dfrac{C_{eq}}{2\sqrt{I_{eq}\cdot K_{eq}}} = \dfrac{c_1R^2 + 2.25c_2R^2}{2\sqrt{3/2\,MR^2\cdot(4k_1+k_2)R^2}}$

Frequency of Damped oscillation

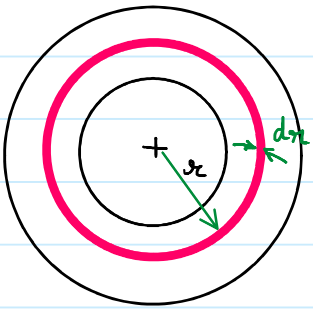
$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

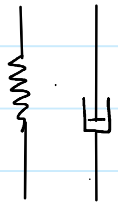
$$\omega_d = \sqrt{\frac{8k_1+2k_2}{3M}}\sqrt{1-\left(\frac{c_1R^2+2.25c_2R^2}{2\sqrt{3/2\,MR^2\cdot(4k_1+k_2)R^2}}\right)^2}$$




E.O.M.

$R_o = R$.
$R_i = 0.5R$.



Area of elemental ring
$= 2\pi r . dr$.

Volume of elemental ring
$= 2\pi r . dr . t$

Mass of elemental Ring
$dm = \rho . 2\pi r . dr . t$

Moment of Inertia about centroidal axis $= \int r^2 . dm = \int_{R_i}^{R_o} r^2 . (\rho . 2\pi r . dr . t)$

$$I = \rho . 2\pi . t \int_{R_i}^{R_o} r^3 . dr = \rho . \cancel{2}\pi . t \cdot \frac{(R_o^4 - R_i^4)}{\cancel{4}\,2}$$

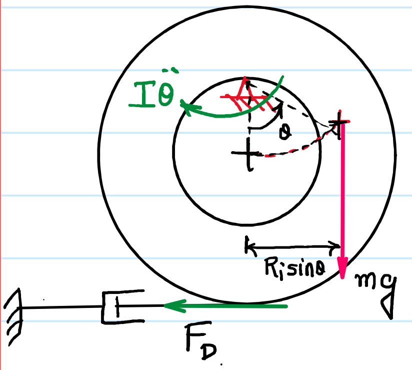
$$I = \boxed{\rho . \pi . (R_o^2 - R_i^2) . t} \cdot \frac{(R_o^2 + R_i^2)}{2} = \frac{m . (R_o^2 + R_i^2)}{2}$$

Mass of Annular Ring

$I_{Hinge} = \bar{I} + m . d^2$

$$I_{Hinge} = \frac{m . (R^2 + (0.5R)^2)}{2} + m (0.5R)^2$$

$I_{Hinge} = 0.875\, mR^2$




$\sin\theta \approx \theta$.

$$I_{Hinge}\, \ddot{\theta} + mg (R_i \sin\theta) + C . (R + 0.5R)^2\, \dot{\theta} = 0$$

$$0.875\, mR^2 \ddot{\theta} + 2.25\, CR^2 \dot{\theta} + 0.5\, mg R . \theta = 0$$

$$I_{eq} \ddot{\theta} + C_{eq} \dot{\theta} + k_{eq} \theta = 0$$

**Damped free Vibration**

18. Critical damping is the (GATE-14)
(a) ~~largest~~ amount of damping for which no oscillation occurs in free vibration
(b) $\boxed{\text{smallest amount of damping for which no oscillation occurs in free vibration}}$
(c) largest amount of damping for which the motion is ~~simple harmonic in free vibration~~
(d) smallest amount of damping for which the motion is simple harmonic in free vibration

Critical damping is least amount of damping for there will be no oscillations.

19. A mass attached at the end of a spring has a natural frequency of 50 rad/sec. When the mass is increased to four times the original mass and a damper is attached, the frequency of oscillations is found to be 20 rad/sec. What is the damping ratio of the resulting system?
(a) 60 % (b) 40 % (c) 80 % (d) 16 %

$\omega_n = 50 \text{ rad/s.} \rightarrow$ for mass $- m.$

$\hookrightarrow \omega_n = \sqrt{\frac{K}{M}}$

mass is increased to 4m. and damper is added.

$\omega_d = 20 \text{ rad/s.}$

$\rightarrow \omega_n = \sqrt{\frac{K}{4m}} = \frac{1}{2}\sqrt{\frac{K}{M}} = \frac{1}{2} \times 50 = 25 \text{ rad/s}$

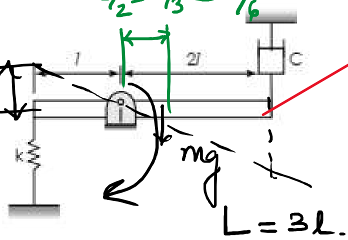
$\omega_d = \omega_n \cdot \sqrt{1-\xi^2}$

$20 = 25 \cdot \sqrt{1-\xi^2}$

$(0.8)^2 = 1-\xi^2 \Rightarrow \xi = \sqrt{1-(0.8)^2} = 0.6 = 60\%$

**Common data for 21 & 22**

A bar of mass m assumed to be distributed uniformly is constrained to under go small oscillations as shown in figure




Mg will cause. $\delta_{static}$

This can be assumed as equilibrium position.



21. The equivalent Inertia and stiffness of the system are
    (a) $(ml^2)$ & $kl^2$ (b) $(3ml^2)$ & $kl^2$
    (c) $(7ml^2/3)$ & $kl^2$ (d) $(7ml^2/3)$ & $(kl^2)/2$

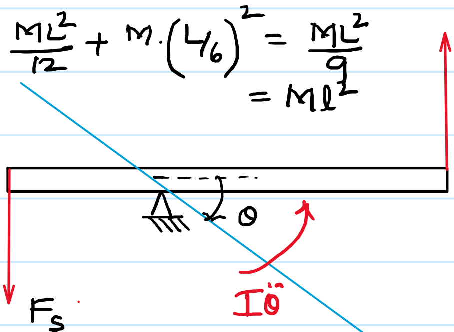
22. The damping ratio of the system is
    (a) $\dfrac{2C}{\sqrt{\dfrac{7km}{3}}}$ (b) $\dfrac{2C}{\sqrt{km}}$
    (c) $\dfrac{2C}{2\sqrt{km\,\ell^2}}$ (d) $\dfrac{2C}{\sqrt{3km}}$

L = 3l

$$I = \bar{I} + md^2$$

$$= \frac{ML^2}{12} + M\left(\frac{L}{6}\right)^2 = \frac{ML^2}{9}$$

$$= Ml^2$$




$\Sigma M_{Hinge} = 0$

$I_{Hinge}\ddot{\theta} + C(\perp^{lar}\text{ distance from hinge})^2\dot{\theta} + K.(\perp^{lar}\text{ distance from hinge})^2\theta = 0$

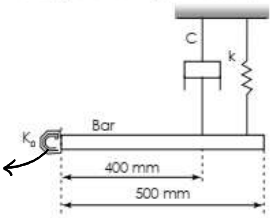
$$Ml^2\ddot{\theta} + c(2l)^2\dot{\theta} + k.(l)^2\theta = 0$$

$$I_{eq} = Ml^2, \quad k_{eq} = kl^2$$

$$\zeta = \frac{C_{eq}}{2\sqrt{I_{eq}\cdot k_{eq}}} = \frac{4cl^2}{2\sqrt{Ml^2\cdot kl^2}} = \frac{2c}{\sqrt{MK}}$$

**Common data questions 25 & 26**

A uniform rigid slender bar of mass 10 kg is hinged at the left end is suspended with the help of spring and damper arrangement as shown in the figure where K = 2kN/m, $\boxed{C = 500\text{ Ns/m}}$ and the stiffness of the torsional spring $K_\theta$ is 1 kN-m/rad. Ignore the hinge dimensions.




Torsional spring

L = 500mm. a = 400mm.

25. The un-damped natural frequency of oscillations of the bar about the hinge point is → $\omega_n = ?$ (GATE-03)
    (a) 42.43 rad/s (b) 30 rad/s
    (c) 17.32 rad/s (d) 14.14 rad/s

26. The damping coefficient in the vibration equation is given by (GATE-03) $C_{eq}$
    (a) 500 Nms/rad (b) 500 N/(m/s)
    $\boxed{\text{(c) 80 Nms/rad}}$ $\boxed{\text{(d) 80 N/(m/s)}}$

Rotational system. Translatory system.

$\Sigma M_{Hinge} = 0$

$I_{Hinge}\ddot{\theta} + c(\perp^{lar}\text{distance from hinge})^2\dot{\theta} + K.(\perp^{lar}\text{ distance from hinge})^2\theta + k_T\theta = 0$

$$\frac{ML^2}{3}\ddot{\theta} + ca^2\dot{\theta} + KL^2\theta + k_T\theta = 0$$

$$I_{eq} = ML^2/3 \quad C_{eq} = Ca^2 \quad K_{eq} = (KL^2 + K_T)$$

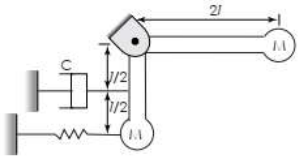
$$\frac{10(0.5)^2}{3}\ddot{\theta} + 500(0.4)^2\dot{\theta} + 2000(0.5)^2\theta + 1000\theta = 0$$

$$\frac{2.5}{3}\ddot{\theta} + 80\dot{\theta} + 1500\theta = 0$$

$$\omega_n = \sqrt{\frac{K_{eq}}{I_{eq}}} \qquad \omega_n = \sqrt{\frac{1500}{(2.5/3)}} = 42.43\text{ rad/s.}$$

**Common data for 23 & 24**

In the single degree freedom system shown in figure is in free vibration $l$ = 1 m, M = 10 kg at each end k = 400 N/m, C = 400 N-sec/m. The L shaped bar is mass less and rigid and hinged at O.

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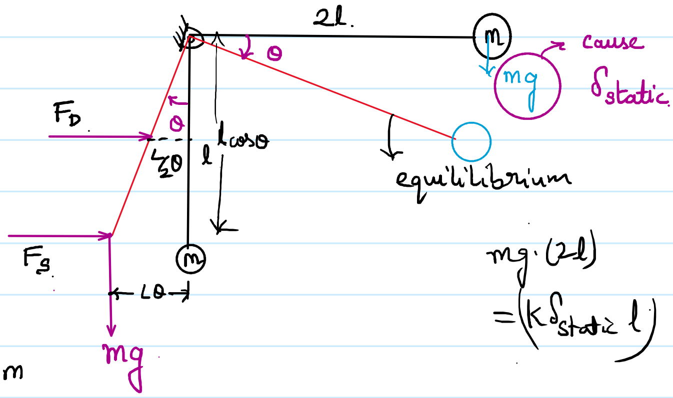


23. The undamped natural frequency of the system is

(a) 3.162 rad/sec (b) 6.32 rad/sec

(c) 4.62 rad/sec (d) 2.83rad/sec

24. The damping ratio of the system is

(a) 0.316 (b) 0.79

(c) 0.46 (d) 0.54

$M = 10\,kg.\ L = 1m \quad K = 400N/m$

$C = \frac{400N\text{-}sec}{m.}$

$\Sigma M_{Hinge} = 0$

$$I_{Hinge}\ddot{\theta} + F_s \times L\cos\theta + F_D \times L/_2 \cos\theta + mg\,L\sin\theta - mg\,(2l)\cos\theta = 0$$

$$[m\cdot(2l)^2 + m(l)^2]\ddot{\theta} + C(L/_2\cdot\dot{\theta})\cdot L/_2 + k\cdot(\delta_{static} + L\theta)\cdot L + mgL\theta - 2mgL = 0$$

$$5ml^2\ddot{\theta} + \frac{CL^2}{4}\dot{\theta} + (kL^2\theta + mg\,L\theta) + k\,\delta_{static}L - 2mgL = 0$$

$$5ml^2\ddot{\theta} + \frac{cL^2}{4}\dot{\theta} + (kL^2 + mgL)\theta = 0$$

$$I_{eq}\ddot{\theta} + C_{eq}\dot{\theta} + k_{eq}\theta = 0$$

$$\omega_n = \sqrt{\frac{kL^2 + mgL}{5ml^2}}$$

$$\zeta = \frac{C_{eq}}{2\cdot\sqrt{I_{eq}\cdot k_{eq}}} = \frac{(CL^2/4)}{2\cdot\sqrt{5ml^2\cdot(kl^2 + mgL)}}$$

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28. Which of the following statements are TRUE for damped vibrations?

P. For a system having critical damping, the value of damping ratio is unity and system does not undergo a vibratory motion. → $\xi = 1$ no oscillation

$x(t) = (A + Bt)e^{-\xi\omega_n t}$ → exponential decay

Q. Logarithmic decrement method is used to determine the amount of damping in a physical system.

R. In case of damping due to dry friction between moving surfaces resisting force of constant magnitude acts opposite to the relative motion.

friction force $\propto$ − velocity

$\boxed{f \propto -\dot{x}}$

S. For the case of viscous damping, drag force is directly proportional to the square of relative velocity. (GATE –15)

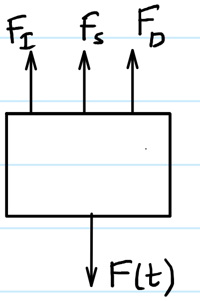
→ $F_D \propto -\dot{x}^1$

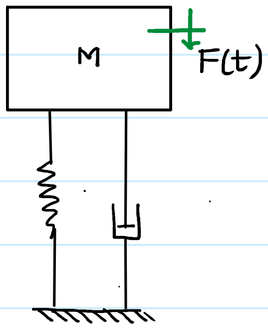
(a) P and Q only (b) P and S only

(c) P,Q and R only (d) Q and Sonly

## Forced Vibrations

$F(t) = F\sin(\omega t)$




$F_I + F_s + F_D = F(t)$

$M\ddot{x} + C\dot{x} + Kx = F(t)$

$x(t) = C.F + P.I.$

$(M.D^2 + CD + K)x = F\sin(\omega t)$

$$x = \frac{F\sin(\omega t)}{(MD^2 + CD + K)} \qquad \boxed{D^2 = -\omega^2} \qquad D = \frac{d}{dt}$$

$$x = \frac{F\sin(\omega t)}{(CD) + (K - M\omega^2)} \times \frac{(K - M\omega^2) - (CD)}{(K - M\omega^2) - (CD)}$$

$$x = \frac{F\sin(\omega t).(K - M\omega^2) - C.\left(\frac{d}{dt}.(F\sin(\omega t))\right)}{(K - M\omega^2)^2 - (CD)^2}$$

$$x = \frac{F\sin(\omega t)\cdot(K-M\omega^2) - C\cdot\left(\frac{d}{dt}\cdot(F\sin(\omega t))\right)}{(K-M\omega^2)^2 - (CD)^2}$$

$$D^2 = -\omega^2$$

$$x = \frac{F\sin(\omega t)\,(K-M\omega^2) - C\cdot F\cdot\cos(\omega t)\cdot\omega}{(K-M\omega^2)^2 - (C^2(-\omega^2))}$$

$$x = \frac{F\cdot\left[(K-M\omega^2)\cdot\sin(\omega t) - (C\omega)\cdot\cos(\omega t)\right]}{(K-M\omega^2)^2 + (C\omega)^2}$$

$$K-M\omega^2 = R\cdot\cos\phi.$$
$$C\omega = R\sin\phi$$
$$R = \left[(K-M\omega^2)^2 + (C\omega)^2\right]^{1/2}$$

$$x = \frac{F\cdot\left[R\cos\phi.\sin(\omega t) - R\sin\phi.\cos(\omega t)\right]}{R^2}$$

$$x = \frac{F\cancel{R}\cdot\left[\sin(\omega t-\phi)\right]}{R^{\cancel{2}}} \qquad x = \frac{F\sin\cdot(\omega t-\phi)}{\sqrt{(K-M\omega^2)^2 + (C\omega)^2}}$$

$$x(t) = C.F + P.I.$$

$$= \underbrace{X_0\cdot e^{-\xi\omega_n t}\sin\cdot(\omega_d t+\phi)}_{\text{Transient Response}} + \underbrace{\frac{F\sin(\omega t-\phi)}{\sqrt{(K-M\omega^2)^2 + (C\omega)^2}}}_{\text{steady state Response.}}$$

As $t\to\infty$ $\quad e^{-\xi\omega_n t} \longrightarrow 0$ $\quad$ Transient Response $\longrightarrow 0$

$$x(t) = \frac{F\sin(\omega t-\phi)}{\sqrt{(K-M\omega^2)^2 + (C\omega)^2}}$$

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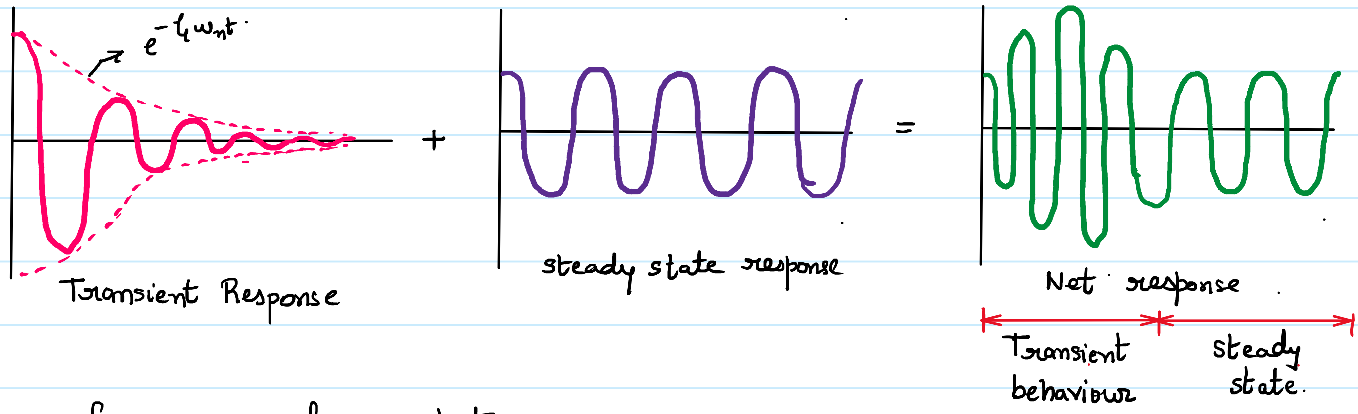
$$x(t) = C.F + P.I.$$

$$= X_0 \cdot e^{-\zeta \omega_n t} \sin(\omega_d t + \phi) + \frac{F \sin(\omega t - \phi)}{\sqrt{(K - M\omega^2)^2 + (c\omega)^2}}$$




$\omega$ – frequency of excitation.

for static loading $\quad \omega = 0 \qquad F(t) = F$

$F$ = constant
$K$ = constant

$$x = \frac{F}{\sqrt{(K - M\omega^2)^2 + (c\omega)^2}} \qquad x_{static} = \frac{F}{K}$$

(with $\omega^2 \to 0$ and $c\omega \to 0$)

Response of forced vibration

$$x(t) = \frac{F \sin(\omega t - \phi)}{\sqrt{(K - M\omega^2)^2 + (c\omega)^2}} \longrightarrow x(t) = A \cdot \sin(\omega t - \phi)$$

$$A = \frac{F}{\sqrt{(K - M\omega^2)^2 + (c\omega)^2}}$$

$A$ – Amplitude of forced vibration.

$$x(t) = \frac{(F/K) \cdot \sin(\omega t - \phi)}{\sqrt{\left(\frac{K}{K} - \frac{M}{K}\cdot\omega^2\right)^2 + \left(\frac{c\omega}{K}\right)^2}}$$

$$\frac{M}{K} = \frac{1}{\omega_n^2}$$

$$x(t) = \frac{x_{static} \ \sin(\omega t - \phi)}{\sqrt{\left(1 - \frac{M}{K}\cdot\omega^2\right)^2 + \left(\frac{c\omega}{K}\right)^2}}$$

$$\frac{c\omega}{K} = \frac{2c \cdot \omega}{2\sqrt{KM}} \cdot \frac{\sqrt{M}}{\sqrt{K}}$$

$$= 2 \cdot \frac{c}{2\sqrt{MK}} \cdot \sqrt{\frac{M}{K}} \cdot \omega$$

$$= 2\zeta \cdot \frac{\omega}{\omega_n} = 2\zeta r$$

$$x(t) = \frac{x_{static} \ \sin(\omega t - \phi)}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}$$

$r$ – ratio of frequencies – $\dfrac{\text{excitation frequency}}{\text{Natural frequency}}$ $\qquad r = \dfrac{\omega}{\omega_n}$

$\zeta$ – Damping Ratio

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Dynamic Magnification factor - It is the ratio of Amplitude of forced vibration to static deflection.

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$$D.M.F = \frac{A}{x_{static}} = \frac{\dfrac{\cancel{F}}{\sqrt{(k - M\omega^2)^2 + (c\omega)^2}}}{\left(\cancel{F}/k\right)} = \frac{1}{\sqrt{\left[\left(\dfrac{k}{k} - \dfrac{M\omega^2}{k}\right)^2 + \left(\dfrac{c\omega}{k}\right)^2\right]}}$$

$$D.M.F = \frac{1}{\sqrt{(1-r^2)^2 + (2\xi r)^2}}$$

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$$D.M.F = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

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$$D.M.F = f(\zeta, r) = f(r)$$

$\zeta < 1$ (under damped).

$$r = \frac{\omega}{\omega_n} = \frac{\text{forcing frequency}}{\text{Natural frequency}}.$$

So D.M.F = Max. then. $\frac{d}{dr}(D.M.F) = 0$

$$\frac{d}{dr}\left(\frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}\right) = 0$$

$$-\frac{1}{2}\left[(1-r^2)^2 + (2\zeta r)^2\right]^{-\frac{1}{2}-1} \times \left[2.(1-r^2).(0-2r) + 2.(2\zeta r).(2\zeta)\right] = 0$$

$$-4r(1-r^2) + 8\zeta^2 . r = 0$$

$$-1 + r^2 + 2\zeta^2 = 0$$

$$r^2 = 1 - 2\zeta^2$$

$$\boxed{r = \sqrt{1-2\zeta^2}}$$

$$\frac{\omega}{\omega_n} = \sqrt{1-2\zeta^2} \rightsquigarrow \boxed{\omega_p = \omega_n . \sqrt{1-2\zeta^2}}$$

$\omega_p$ — frequency at which peak value of D.M.F/Amplitude of forced vibration occurs.

$$D.M.F = \frac{A \rightsquigarrow \text{Variable.}}{x_{static} \rightsquigarrow \text{constant}}$$

**D.M.F vs r Graph**

$$D.M.F = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

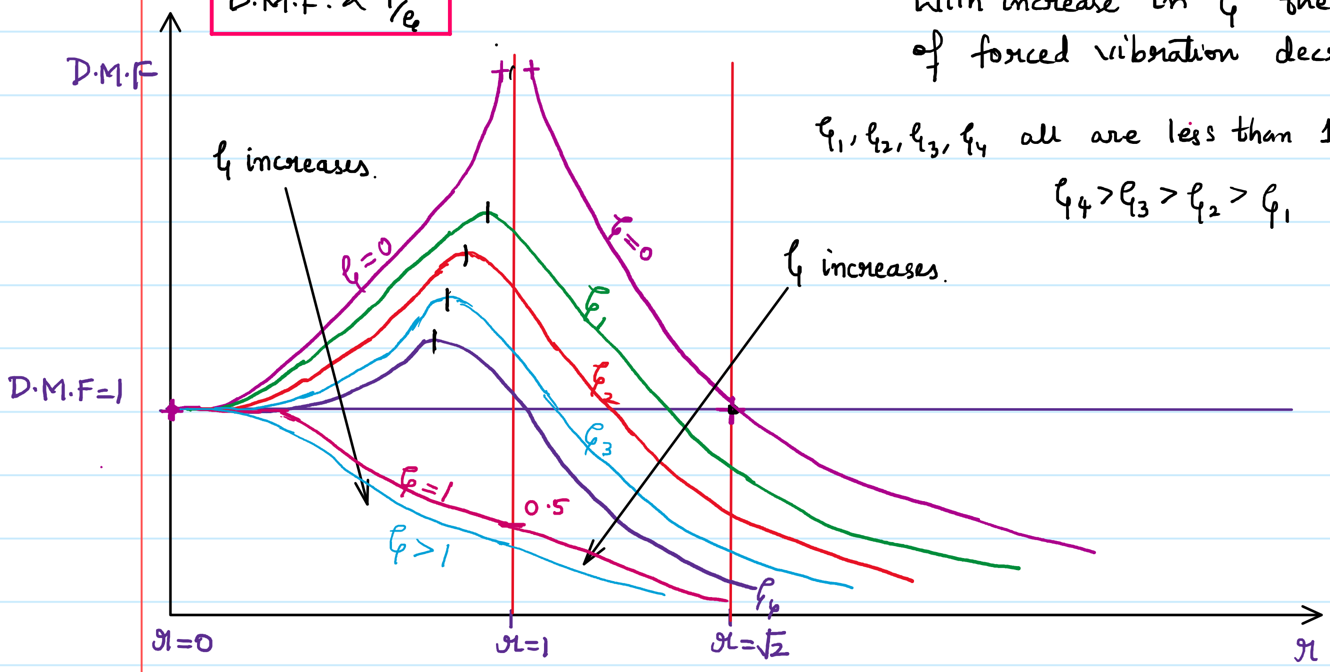
1. $r = 0$ $\zeta = 0$ (undamped system) $D.M.F = 1$
   $\hookrightarrow \omega = 0$ static loading.
2. $r = 0$ $\zeta \neq 0$ $D.M.F = 1$.
3. $r = 1$ $\zeta = 0$ $D.M.F = \infty$
4. $r = 1$ $\zeta \neq 0$ $D.M.F = (1/2\zeta)$ $\rightarrow$ $\zeta = 1$ $r = 1$ $D.M.F = 0.5$
5. $r = \sqrt{2}$ $\zeta = 0$ $D.M.F = 1$
6. $r = \sqrt{2}$ $\zeta \neq 0$ $D.M.F = \frac{1}{\sqrt{1+8\zeta^2}}$

$\boxed{A \propto 1/\zeta}$

Constant

$$D.M.F \propto \frac{1}{\zeta} \qquad \frac{A}{x_{static}} \propto \frac{1}{\zeta}$$

$\boxed{D.M.F. \propto 1/\zeta}$




With increase in $\zeta$ the amplitude of forced vibration decreases.

$\zeta_1, \zeta_2, \zeta_3, \zeta_4$ all are less than 1.

$\zeta_4 > \zeta_3 > \zeta_2 > \zeta_1$

$r = \sqrt{1-2\zeta^2} \rightarrow D.M.F = Max.$

As $\zeta \uparrow$ $2\zeta^2 \uparrow\uparrow$ $1-2\zeta^2 \downarrow$. $r \downarrow$. $\omega_p \downarrow$

As the value of $\zeta$ increases the peak of D.M.F vs $r$ graph tends to shift towards Leftside.

For damped systems the peak value of D.M.F occurs at $r$ slightly less than 1.