CSCI-GA.1170-001/002 Fundamental Algorithms December 3, 2015

Solutions to Problem 3 of Homework 10 (12 points)

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The diameter of an undirected tree $T = (V, E)$ on n vertices V (and $(n-1)$ edges E) is the largest of all shortest paths distances in the tree: $D = \max_{x,y \in V} \delta(x, y)$. You will design an $O(n)$ algorithm to compute D and will prove its correctness as follow.

(a) (7 pts) Let r be the root of T. Let b is the furthest node from r in T. Show that the diameter path in T either ends or starts at b.

Solution:

Let $\delta(a, b)$ be the diameter of T and b is the farthest node from r. Note that the root r may or may not lie in the path between a and b.

Consider that r lies in the path between a and b i.e r is the lowest common ancestor of a and b. Therefore $\delta(a, b) = \delta(a, r) + \delta(r, b)$. Suppose that $\delta(a, b)$ is not the diameter of T and let $\delta(a, c)$ be the diameter of T. Therefore $\delta(a, c) = \delta(a, r) + \delta(r, c)$. We know that $\delta(r, c) < \delta(r, b)$ as b is the farthest node from r. Hence $\delta(a, c) < \delta(a, b)$. Hence our assumption is wrong i.e c has to be the farthest node from r. Hence $c = b$

Now consider that r doesn't lie in the path between a and b and let p be the lower common ancestor of a and b. Therefore $\delta(a, b) = \delta(a, p) + \delta(p, b)$. Suppose that $\delta(a, b)$ is not the diameter of T and let $\delta(a, c)$ be the diameter of T. Therefore $\delta(a, c) = \delta(a, p) + \delta(p, c)$. Since b is the farthest node from r we have, $\delta(r, c) < \delta(r, b) \implies \delta(r, p) + \delta(p, c) < \delta(r, p) + \delta(p, b) \implies \delta(p, c) < \delta(p, b)$. Hence $\delta(a, c) < \delta(a, b)$. Hence our assumption is wrong i.e c has to be the farthest node from r. Hence $c = b$

Note that in the above two cases we considered that the diameter ends at b. We can similarly argue when the diameter starts at b. Therefore we can conclude that the diameter path in T starts or ends at b where b is the farthest node from the root of the tree □

(b) (5 pts) Assuming part (a), irrespective of whether or not you solved it, design an $O(n)$ algorithm to compute D. For partial credit, give a slower algorithm.

Solution:

- Pick the root node r and perform BFS on it
- Let b be the farthest node from r. Therefore from part (a), the diameter of T either starts or ends at b
- Perform BFS on b. Let a be the the farthest node from b
- $\delta(a, b)$ is the the diameter of T

In the above algorithm, BFS is called twice so the running time is $O(m + n)$ but we know that $m = n - 1$. Therefore the running time is $O(n)$ □