CSCI-GA.1180:
Mathematical Techniques for Computer Science Applications
New York University, Fall 2016
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Homework Assignment 6
Assigned 30 November 2016; due 11:59pm, 14 December 2016

Exercise 6.1. This question involves independent coin tosses. For parts (a) and (b), give the answer in two ways: as an algebraic expression in terms of quantities p, n, and k, and as a number. (Use Matlab to do the calculations.)

(a) A fair two-sided coin is tossed 5 times. What is the probability that exactly three of these tosses produce heads? Explain your answer.

(b) A fair two-sided coin is tossed 40 times. What is the probability that exactly 20 of these tosses will produce heads? Explain your answer and specify how you calculated the associated number.

(c) Under the same circumstances as in part (b), explain how you could justify saying "The most likely outcome of the 40 tosses is 20 heads", including a clear definition of "most likely".

Exercise 6.2. This problem refers to the days when people wore hats and checked them when going to a restaurant.

Suppose that n people come to the restaurant in a group and check their hats. Unfortunately, the person in charge of the hat check is totally disorganized and does not keep track of which hat belongs to each person. Thus, when the people leave the restaurant, each person's probability of getting his/her own hat is $1/n$. Let A_i be the event that person i gets his or her own hat when leaving the restaurant; let the random variable X_i be equal to 1 if person i gets his/her own hat back, and $X_i = 0$ otherwise. Assume that $n = 4$.

(1) What is the probability of A_1? What is the expected value of X_1? Explain your answer.

(2) What is the probability of the event $A_1 \cap A_2$, i.e., that both persons 1 and 2 will receive their own hats? What is the expected value of the random variable $X_1 \times X_2$? Explain your answer.

(3) In general, are events A_i and A_j, where $i \neq j$, independent? Explain your answer.

Exercise 6.3. Assume that it rains in a big city on half of the days. The weather forecaster is reasonably reliable.

- Given a forecast of rain, the probability is 2/3 that it will rain.

- Given a forecast that it will not rain, the probability is 2/3 that it will not rain.

A professor of probability relies to some extent on the forecasts, but is exceptionally cautious about rain: when rain is forecast, the professor brings an umbrella to the office; when the forecast is that it will not rain, the professor brings an umbrella with probability 1/3.

(1) The probability that the professor does not bring an umbrella to the office, given that it rains that day, is 2/9. Show how this correct answer is obtained, giving all relevant formulas used to calculate probabilities and explaining each step in your calculation.

(2) The probability that the professor brings an umbrella to the office, given that it does not rain that day, is 5/9. Show how this correct answer is obtained, giving all relevant formulas used and explaining each step of your calculation.

Exercise 6.4. A restaurant offers two kinds of pie (strawberry and cherry), and always begins each day with an equal number of pies of these two kinds. Every day exactly 10 customers each request a pie, and the probability that a given customer will choose one kind or the other is 1/2.

(a) If the restaurant stocks 5 strawberry pies and 5 cherry pies every day, what is the probability that every customer will receive his/her requested kind of pie? Explain your answer.

(b) Answer the same question as in part (a), but assuming that the restaurant stocks 8 strawberry pies and 8 cherry pies each day. Explain your answer.

Exercise 6.5. Given a fair coin, a gambler will win a large amount of money if the gambler manages to toss "heads", and the gambler has three chances to do so. Once the coin comes up "heads", the tosses will stop. Otherwise, the gambler may make three tosses in total. Let X be the number of times that heads is tossed, and Y be the number of times that tails is tossed. For each case, explain how you obtained your results.

(1) For X, give (i) the expected value and (ii) the variance.

(2) For Y, give (i) the expected value and (ii) the variance.