CSCI-GA.1180: Mathematical Techniques for Computer Science Applications New York University, Fall 2016

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Homework Assignment 1

Assigned Monday, 19 September 2016; due Monday, 26 September 2016

Unless stated otherwise, A is assumed to be a real $m \times n$ matrix, where m may be different from n. If a matrix-vector product or matrix-matrix product is mentioned, assume that the dimensions are compatible.

Whenever calculations are needed to solve a problem, those calculations must be submitted as part of the homework assignment.

Homework must be submitted electronically, by 11:59pm on the due date. Unless express permission has been given in advance by the instructor for a late homework submission, a 30% percent penalty will be deducted for each late day (or part of a late day).

Homework grades are based on the quality and clarity of your explanations and proofs.

Exercise 1.1. In each part, give a specific numerical example of a 2×2 matrix such that every one of its elements is a nonzero integer, and such that the matrix satisfies the given conditions. Explain how you found each answer.

(a) $A^2 = -I$, where I is the identity matrix;

(b) $B^2 = 0$, where 0 is the zero matrix;

(c) $CD = -DC$, with $CD \neq 0$.

Exercise 1.2. If A and B are nonsingular, prove that their product AB is nonsingular. (Do not use determinants.)

Exercise 1.3. Let

$$A = \begin{pmatrix} 1 & 8 & 7 \\ 2 & 10 & 8 \\ 3 & 12 & 9 \end{pmatrix}.$$

Confirm that the three columns of A (a_1, a_2, and a_3) are linearly dependent by expressing a_3 as a linear combination of a_1 and a_2, i.e.,

$$a_3 = \lambda_1 a_1 + \lambda_2 a_2, \quad \text{so that} \quad a_3 - \lambda_1 a_1 - \lambda_2 a_2 = 0,$$

where λ_1 and λ_2 are scalars. Give the numerical values of λ_1 and λ_2 and explain how you found them.

Exercise 1.4. Let A be an $n \times n$ real matrix. Prove that there is a unique n-vector x satisfying $Ax = b$ for any nonzero n-vector b if and only if the only solution of $Ay = 0$ is $y = 0$. (Prove both the "if" and "only if" results.)

Exercise 1.5.

(a) If A has linearly independent columns and $Ax = Ay$ for vectors x and y, show that $x = y$. (This result implies that we can "cancel" a matrix with linearly independent columns appearing on the left of both sides of an equation.)

(b) Give a specific numerical example where A has linearly independent *rows* and $Ax = Ay$, but $x \neq y$. (The contrast between parts (a) and (b) emphasizes the differing roles of rows and columns in matrix multiplication.)

Exercise 1.6. If A is $m \times n$ and has rank m, what does this imply about the relative sizes of m and n? Explain.

Exercise 1.7. Fredholm's alternative[1] is a famous result that can be expressed in the form of a *theorem of the alternative* as follows: given any matrix A and vector b of appropriate dimensions, precisely one of the following two relations is true:

(1) there exists a vector x such that $Ax = b$, or

(2) there exists a vector y such that $A^T y = 0$ and $y^T b \neq 0$.

Show that condition (1) and condition (2) are contradictory, i.e., they cannot both be true.

Exercise 1.8. For a given nonzero $m \times n$ matrix A and nonzero m-vector x, assume that x may be written as $x = x_R + x_N$, where x_R is a linear combination of the columns of A, i.e., x lies in the range of A, and $A^T x_N = 0$, i.e., x_N is in the null space of A^T. (The vector x_R is called the *range-space portion* of x [with respect to A], and x_N is called the *null-space portion* of x.)

(a) Show that $x_R^T x_N = 0$.

(b) Show that x_R and x_N are unique.

(c) If x_R and x_N are both nonzero, show that they are linearly independent.

Exercise 1.9.

(a) Let C be a given $m \times n$ matrix with full column rank, and let d be a given m-vector. If there is a solution x to the linear system $Cx = d$ (i.e., if the system is compatible), show that x is unique.

(b) Use part (a) and the uniqueness of the decomposition of b into its range- and null-space portions to show that if A is an $m \times n$ matrix with rank m, then the system $Ax = b$ is compatible for *every* $b \in \Re^m$. (Which means that every $b \in \Re^m$ lies in the range of A.)

(c) Construct a 2×4 matrix A and a 2×1 right-hand side vector b to show that the result of part (b) may not be true if the columns of A are linearly dependent.

Exercise 1.10.

(a) Give a 2×5 matrix A such that (i) $\text{rank}(A) = 2$ and (ii) the 2×2 submatrix consisting of the first 2 columns of A has rank 1. Explain how you constructed A, and confirm numerically that properties (i) and (ii) hold.

(b) Give a nonzero 2-vector b such that b lies in the range of A (i.e., b can be written as a linear combination of the columns of A), and explain how you constructed b.

(c) Is it possible to find a 2-vector b that does not lie in the range of A? Explain your answer.

(d) Find a vector c that does *not* lie in the range of A^T. Explain how you chose c.

[1] Erik Ivar Fredholm, 1866–1927, from Sweden.