CSCI-GA.1180-001 December 14, 2016

Solutions to Problem 1 of Homework 6

Name: GOWTHAM GOLI (N17656180) *Due: Wednesday, December 14*

(a) **Solution:**

We can see this as $n = 5$ Bernoulli trials in which a success is when the toss produces a head. Since the coin is fair, the probability of success, $p = 1/2$. Therefore, the probability of getting $k = 3$ successes (producing a head) in $n = 5$ trails with success probability $p = 1/2$ and failure probability $q = 1 - p = 1/2$ is

$$\begin{aligned}\binom{n}{k}p^k q^{n-k} &= \binom{5}{3}(\frac{1}{2})^3(\frac{1}{2})^{5-3} \\ &= 0.3125\end{aligned}$$

```
n = 5; k = 3; p = 0.5 ;
nchoosek(n,k)*p^k*(1-p)^(n-k)
```

$\square$

(b) **Solution:**

We can see this as $n = 40$ Bernoulli trials in which a success is when the toss produces a head. Since the coin is fair, the probability of success, $p = 1/2$. Therefore, the probability of getting $k = 20$ successes (producing a head) in $n = 40$ trails with success probability $p = 1/2$ and failure probability $q = 1 - p = 1/2$ is

$$\begin{aligned}\binom{n}{k}p^k q^{n-k} &= \binom{40}{20}(\frac{1}{2})^{20}(\frac{1}{2})^{40-20} \\ &= 0.1254\end{aligned}$$

```
n = 40; k = 20; p = 0.5 ;
nchoosek(n,k)*p^k*(1-p)^(n-k)
```

$\square$

(c) **Solution:** $\binom{n}{k}$ will be maximum for $k = \lceil n/2 \rceil, \lfloor n/2 \rfloor$

We know that probability of k heads out of 40 tosses is $p = \binom{40}{k}(\frac{1}{2})^{40} \implies p$ will be maximum when $\binom{40}{k}$ will be maximum $\implies \binom{40}{k}$ will be maximum for $k = 40/2 = 20$ $\square$

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Solutions to Problem 2 of Homework 6

Name: GOWTHAM GOLI (N17656180) *Due: Wednesday, December 14*

(1) **Solution:**

Let h_1, h_2, h_3, h_4 are the hats. Now, we have $4! = 24$ arrangements of the hats. In one of the 24 arrangements, each person gets his/her own hat correctly. Let (h_1, h_2, h_3, h_4) be that arrangement i.e., h_i goes to person i.

A_1 is the event that person 1 correctly gets h_1. This means this arrangement will have the form $(h_1, -, -, -)$. Now, we can arrange the remaining three hats in the three blanks in 3! ways. Therefore, the probability of A_1 is $\frac{3!}{4!} = 1/4$

To calculate $E[X_1]$, $X_1 = 1$ when A_1 occurs and $X_1 = 0$ otherwise.

$\therefore E[X_1] = 1 \times P[A_1] = 1/4$ $\square$

(2) **Solution:**

$A_1 \cap A_2$ is the event that person 1 correctly gets h_1 and person 2 correctly gets h_2. This means this arrangement will have the form $(h_1, h_2, -, -)$. Now, we can arrange the remaining two hats in the two blanks in 2! ways. Therefore, the probability of $A_1 \cap A_2$ is $\frac{2!}{4!} = 1/12$

To calculate $E[X_1 \times X_2]$, $X_1 \times X_2 = 1$ only when $X_1 = 1, X_2 = 1$ i.e. when A_1, A_2 both happen and $X_1 \times X_2 = 0$ otherwise.

$\therefore E[X_1 \times X_2] = 1 \times P[A_1 \cap A_2] = 1/12$ $\square$

(3) **Solution:** Consider the events A_1, A_2. It is easy to see that $P[A_2] = P[A_1] = 1/4 \implies P[A_1] \times P[A_2] = 1/4 \times 1/4 = 1/16$.

$P[A_1 \cap A_2] = 1/12 \neq P[A_1] \times P[A_2] \implies A_1, A_2$ are not independent events. $\square$

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Solutions to Problem 3 of Homework 6

Name: GOWTHAM GOLI (N17656180) *Due: Wednesday, December 14*

(1) **Solution:**

Let

- R be the event that it rains.
- F be the event that there is a forecast of rain.
- U be the event that the professor brings an umbrella.

We need to find $P[U|R^c]$ and we have ,

$$\begin{aligned} P[R] &= 1/2 \\ P[R|F] &= 2/3 \implies P[R^c|F] = 1/3 \\ P[R^c|F^c] &= 2/3 \implies P[R|F^c] = 1/3 \\ P[U|F] &= 1 \implies P[U^c|F] = 0 \\ P[U|F^c] &= 1/3 \implies P[U^c|F^c] = 2/3 \\ \therefore P[R] &= P[R|F] \cdot P[F] + P[R|F^c] \cdot P[F^c] \end{aligned} \tag{1}$$

Solve for $P[F]$ using $P[F^c] = 1 - P[F]$ and substituting the above values in (1), we get, $P[F] = 1/2$.

The professor does not bring an umbrella to the office, given that it rains that day is equivalent to

- Given that it rains that day, the forecast is that there will be no rain and given that forecast is that there will be no rain, the professor does not bring an umbrella (or)
- Given that it rains that day, the forecast is that there will be rain and given that forecast is that there will be rain, the professor does not bring an umbrella.

Mathematically we can express the same as follows

$$\begin{aligned} P[U^c|R] &= P[U^c|F^c] \cdot P[F^c|R] + P[U^c|F] \cdot P[F|R] \\ \implies P[U^c|R] &= 2/3 \times P[F^c|R] + 0 \times P[F|R] \end{aligned} \tag{2}$$

Now we need to calculate $P[F^c|R]$

$$P[F^c|R] = \frac{P[R|F^c] \cdot P[F^c]}{P[R]} = \frac{\frac{1}{3} \times \frac{1}{2}}{\frac{1}{2}} = \frac{1}{3}$$

Substituting $P[F^c|R]$ in (2) we get, $P[U^c|R] = 2/3 \times 1/3 = 2/9$ □

(2) **Solution:**

The professor brings an umbrella to the office, given that it does not rain that day is equivalent to

- Given that it does not rain that day, the forecast is that there will be no rain and given that forecast is that there will be no rain, the professor brings an umbrella (or)
- Given that it does not rain that day, the forecast is that there will be rain and given that forecast is that there will be rain, the professor brings an umbrella.

Mathematically we can express the same as follows

$$\begin{aligned} & P[U|R^c] = P[U|F^c] \cdot P[F^c|R^c] + P[U|F] \cdot P[F|R^c] \\ \implies & P[U|R^c] = 1/3 \times P[F^c|R^c] + 1 \cdot P[F|R^c] \end{aligned} \tag{3}$$

Now we need to calculate $P[F|R^c]$

$$P[F|R^c] = \frac{P[R^c|F] \cdot P[F]}{P[R^c]} = \frac{\frac{1}{3} \times \frac{1}{2}}{\frac{1}{2}} = \frac{1}{3} \implies P[F^c|R^c] = \frac{2}{3}$$

Substituting $P[F|R^c], P[F^c|R^c$ in (2) we get, $P[U|R^c] = 1/3 \times 2/3 + 1 \times 1/3 = 5/9$

$\square$

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Solutions to Problem 4 of Homework 6

Name: GOWTHAM GOLI (N17656180) *Due: Wednesday, December 14*

(a) **Solution:**

Since there are stocks exactly 5 strawberry pies and 5 cherry pies, every customer will receive his/her requested kind of pie only if 5 people want strawberry pie and the rest of the 5 people want cherry pies. This can be seen as $n = 10$ Bernoulli trials in which a success is when the 5 persons that requests the strawberry pie gets strawberry pie (which automatically means the rest of the 5 persons get cherry pie as requested). Thus, the probability of success, $p = 1/2$. Therefore, the probability of getting $k = 5$ successes in $n = 10$ trails with success probability $p = 1/2$ and failure probability $q = 1 - p = 1/2$ is

$$\begin{aligned}\binom{n}{k} p^k q^{n-k} &= \binom{10}{5} (\frac{1}{2})^5 (\frac{1}{2})^{10-5} \\ &= 0.2461\end{aligned}$$

□

(b) **Solution:**

Since there are stocks 8 strawberry pies and 8 cherry pies, every customer will receive his/her requested kind of pie only if

- 2 people want strawberry pie and the rest of the 8 people want cherry pies $\implies k = 2$.
- 3 people want strawberry pie and the rest of the 7 people want cherry pies $\implies k = 3$.
- $\vdots$
- 8 people want strawberry pie and the rest of the 2 people want cherry pies $\implies k = 8$.

Summing up we get, probability that every customer will receive his/her requested kind of pie is

$$\begin{aligned}&\binom{n}{2} p^2 q^{n-2} + \binom{n}{3} p^3 q^{n-3} + \cdots + \binom{n}{8} p^8 q^{n-8} \\ &= \binom{10}{2} (1/2)^{10} + \binom{10}{3} (1/2)^{10} + \cdots + \binom{10}{8} (1/2)^{10} \\ &= (1/2)^{10} (2\binom{10}{2} + 2\binom{10}{3} + 2\binom{10}{4} + \binom{10}{5}) \\ &= 0.9785\end{aligned}$$

```
n = 10; p = 0.5;
(p^10)*(2*nchoosek(n,2)+2*nchoosek(n,3)
+2*nchoosek(n,4)+nchoosek(n,5))
```

□

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Solutions to Problem 5 of Homework 6

Name: GOWTHAM GOLI (N17656180) *Due: Wednesday, December 14*

(1) **Solution:** Let $\{X = k\}$ denote that heads has been tossed k times in 3 chances. The gambler will win when $\{X = 1\}$ i.e., if he tosses either $H/TH/TTH$ and the gambler will lose when $\{X = 0\}$ i.e., if he tosses TTT.

$$\begin{aligned}
&P(H) = 1/2, P(TH) = 1/4, P(TTH) = 1/8 \\
\implies &P_X(1) = 1/2 + 1/4 + 1/8 = 7/8 \\
&P(TTT) = 1/8 \\
\implies &P_X(0) = 1/8 \\
\therefore\ &E[X] = (0 \times P_X(0)) + (1 \times P_X(1)) = 7/8
\end{aligned}$$

$$\begin{aligned}
E[X^2] &= (0^2 \times P_X(0)) + (1^2 \times P_X(1)) = 7/8 \\
\therefore var[X] &= E[X^2] - (E[X])^2 = 7/8 - (7/8)^2 = 7/64
\end{aligned}$$

□

(2) **Solution:** Let $\{Y = k\}$ denote that tails has been tossed k times in 3 chances.

$$\begin{aligned}
P_Y(1) &= P(TH) = 1/4 \\
P_Y(2) &= P(TTH) = 1/8 \\
P_Y(3) &= P(TTT) = 1/8 \\
\therefore E[Y] &= (0 \times P_Y(0)) + (1 \times P_Y(1)) + (2 \times P_Y(2)) + (3 \times P_Y(3)) \\
&= (1 \times 1/4) + (2 \times 1/8) + (3 \times 1/8) = 7/8
\end{aligned}$$

$$\begin{aligned}
E[Y^2] &= (0^2 \times P_Y(0)) + (1^2 \times P_Y(1)) + (2^2 \times P_Y(2)) + (3^2 \times P_Y(3)) = 15/8 \\
\therefore var[X] &= E[X^2] - (E[X])^2 = 15/8 - (7/8)^2 = 71/64
\end{aligned}$$

□