CSCI-GA.1180:
Mathematical Techniques for Computer Science Applications
New York University, Fall 2016

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Homework Assignment 2

Assigned Monday, 26 September 2016;
due 11:59pm, Wednesday, 5 October 2016

Unless stated otherwise, A is assumed to be a real $m \times n$ matrix, where m may be different from n. If a matrix-vector product or matrix-matrix product is mentioned, assume that the dimensions are compatible.

Whenever calculations are needed to solve a problem, those calculations must be submitted as part of the homework assignment.

Homework must be submitted electronically, by 11:59pm on the due date. Unless express permission has been given in advance by the instructor for a late homework submission, a 30% percent penalty will be deducted for each late day (or part of a late day).

Homework grades are based on the quality and clarity of your explanations and proofs.

Exercise 2.1. A square matrix A is *symmetric* if $a_{ij} = a_{ji}$.

(a) Find two specific symmetric square matrices A and B of dimension at least 2×2, with nonzero integer entries, such that the product AB is not symmetric. Give A, B, and AB.

(b) Find two specific symmetric square matrices A and B of dimension at least 2×2, with nonzero integer entries, whose product AB is symmetric. Give A, B, and AB.

(c) What condition is necessary in the 2×2 case for AB to be symmetric when A and B are symmetric? Explain how you derived it.

Exercise 2.2. [Rank of a square triangular matrix.]

(a) Prove that a square upper-triangular matrix R is singular only if at least one diagonal element is zero. (The same result is true for a lower-triangular matrix.)

(b) Assume that R is an $n \times n$ triangular matrix, and that k of its diagonal elements are zero. Must the rank of R be exactly $n - k$? Explain your answer. If your answer is "yes", give a proof. If your answer is "no", give a counterexample.

Exercise 2.3. Consider the $n \times n$ elementary matrix $E = I - \alpha x y^T$.

(a) Show that E is singular if and only if $\alpha x^T y = 1$.

(b) If $\alpha x^T y \neq 1$, show that $E^{-1} = I - \beta x y^T$, where $\beta = \alpha/(\alpha x^T y - 1)$.

Exercise 2.4. Let x and y be n-vectors, and let Z be the rank-one matrix $Z = xy^T$. Describe an efficient way to compute Z^k (the k-th power of Z) for $k > 1$.

Exercise 2.5. Consider the following A and b:

$$A = \begin{pmatrix} 3 & 3 & 6 \\ 2 & -3 & 6 \\ 1 & 6 & 0 \\ 2 & 0 & 1 \\ 6 & 3 & 0 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 4 \\ 3 \\ 1 \\ -1 \\ -5 \end{pmatrix}.$$

(a) What is the rank of A? How did you determine this value? What conclusion do you draw about whether the columns of A are linearly independent (or not)? Explain.

(b) Is b in the range of A? If your answer is "yes", explain how you decided this. In this case, find the coefficients in the linear combination such that b is a linear combination of the columns of A, and explain how you found them. If your answer is "no", explain your reasoning.

(c) Let $r = \text{rank}(A)$. Find two subsets of r linearly independent rows of A (called "basic sets"), and explain how you determined that the designated rows were linearly independent.

(d) Use these two basic sets to solve the system $Ax = b$ for x, using a different basic set in each calculation of x.

(e) Is x the same in both cases? Explain why or why not.

Exercise 2.6. [Properties of $A^T A$.]

(a) If the columns of A are linearly independent, show that the matrix $A^T A$ is nonsingular.

(b) If A has linearly dependent columns, show that $A^T A$ is singular.

Exercise 2.7. Here are the four standard properties of the norm $\|A\|$ of a real matrix A: (1) $\|A\| > 0$ if $A \neq 0$, and $\|0\| = 0$; (2) $\|\gamma A\| = |\gamma| \, \|A\|$ for any scalar γ; (3) $\|A + B\| \leq \|A\| + \|B\|$ for any B of compatible dimensions; (4) $\|AB\| \leq \|A\| \, \|B\|$ for any B of compatible dimensions.

(a) Show that the quantity $M = \max_{i,j} |a_{ij}|$, i.e., the largest absolute value of any element in the matrix A, satisfies properties (1), (2), and (3) of a matrix norm.

(b) Show that property (4) does not hold by giving a counterexample, i.e., specific matrices A and B such that property (4) is not satisfied.

Exercise 2.8. [Properties of triangular matrices.] Show that the product of two square upper-triangular matrices is upper triangular.

Exercise 2.9.

(a) Consider an $n \times n$ upper-triangular matrix U such that

$$u_{11} u_{22} \cdots u_{n-1,n-1} \neq 0, \quad \text{but} \quad u_{nn} = 0,$$

i.e., U is singular. Give a general algorithm for computing a *nonzero* vector x such that $Ux = 0$.

(b) Verify your algorithm by finding a solution x when U is given by

$$U = \begin{pmatrix} 1 & -2 & \frac{1}{2} \\ & 3 & 2 \\ & & 0 \end{pmatrix}.$$

(c) What is the general form of x satisfying $Ux = 0$ for this particular U?