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Solutions to Problem 1 of Homework 5

Name: GOWTHAM GOLI (N17656180) *Due: Tuesday, November 29*

(1) **Solution:**

$$20 = 5 \times 4$$

This is possible under the circumstances that

- the order of paths matter
- Any of the 5 paths can be chosen on the trip from bottom to top.
- On the trip from top to bottom, we can't choose the same path as we chose on the trip to top, so we have only 4 choices.
- Therefore, the total number of choices = 5×4

Acceptable round trips are $\{(1,2),(1,3),(1,4),(1,5),(2,1),(2,3),(2,4),(2,5),(3,1),(3,2),(3,4),(3,5),(4,1),(4,2),(4,3),(4,5),(5,1),(5,2),(5,3),(5,4)\}$ □

(2) **Solution:**

$$25 = 5 \times 5$$

This is possible under the circumstances that

- the order of paths matter
- Any of the 5 paths can be chosen on the trip from bottom to top.
- Any of the 5 paths can be chosen on the trip from top to bottom.
- Therefore, the total number of choices = 5×5

Acceptable round trips are $\{(1,1),(1,2),(1,3),(1,4),(1,5),(2,1),(2,2),(2,3),(2,4),(2,5),(3,1),(3,2),(3,3),(3,4),(3,5),(4,1),(4,2),(4,3),(4,4),(4,5),(5,1),(5,2),(5,3),(5,4),(5,5)\}$ □

(3) **Solution:**

$$\binom{5}{2} = 10$$

It is clear that, we are any 2 paths from the given 5 paths. Therefore, this is possible under the circumstances that

- the order of the paths don't matter.
- the path from bottom to top is different from the path from bottom to top i.e., there are no repetitions.

Acceptable round trips are $\{(1,2),(1,3),(1,4),(1,5),(2,3),(2,4),(2,5),(3,4),(3,5),(4,5)\}$ □

(4) **Solution:**

$$\binom{6}{2} = 15$$

This is in the form of $\binom{n+k-1}{k}$ where $n = 5, k = 2$. Therefore, this is possible under the circumstances that

- the order of the paths don't matter.
- the path from bottom to top and top to bottom could be equal i.e., repetitions are allowed

Acceptable round trips are $\{(1,2),(1,2),(1,3),(1,4),(1,5),(2,2),(2,3),(2,4),(2,5),(3,3),(3,4),(3,5),(4,4),(4,5),(5,5)\}$ $\square$

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Solutions to Problem 2 of Homework 5

Name: GOWTHAM GOLI (N17656180) *Due: Tuesday, November 29*

(1) **Solution:**

Let b, g represent boy and girl, then we have 4 possibilities as follows: $\{(b,b),(b,g),(g,g),(g,b)\}$ where the first element in the tuple is the younger child and the second element in the older child.

Let C be the event that the both children are girls. Therefore, we need to compute $P[C|A]$.

$$P[C|A] = \frac{P[C \cap A]}{P[A]}$$

- $P[A]$ = Probability the older child is a girl. In the above set we can see that g occurs at the second position in the tuple twice $\implies P[A] = 2/4 = 1/2$
- $C \cap A$ = Both the children are girls and the older child is a girl. If C happens then A happens automatically $\implies C \cap A = C$ $\therefore$ From the above set we can see that, $P[C] = 1/4$

$$P[C|A] = \frac{1/4}{1/2} = 1/2$$

□

(2) **Solution:** Let C be the event that both the children are girls and D be the event that atleast one of the children is a girl. Therefore, we need to compute $P[C|D]$.

$$P[C|D] = \frac{P[C \cap D]}{P[D]}$$

- $P[D]$ = Probability the atleast one of the children is a girl. In the above set we can see that g belongs to a tuple 3 times. $\implies P[D] = 3/4$
- $C \cap D$ = Both the children are girls and atleast one of the children is a girl. If C happens then D happens automatically $\implies C \cap D = C$ $\therefore$ From the above set we can see that, $P[C] = 1/4$

$$P[C|D] = \frac{1/4}{3/4} = 1/3$$

□

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Solutions to Problem 3 of Homework 5

Name: GOWTHAM GOLI (N17656180) *Due: Tuesday, November 29*

(a) **Solution:** Using the formula in the lecture notes, we know that, the probability of k successes in n trials where the success probability is p, failure probability is $q = 1 - p$ is

$$\binom{n}{k} p^k q^{n-k}$$

In this case, we have $n = 11$ (total number of carrots), $k = 11$ (number of acceptable carrots), $q = 0.14$ (unacceptable carrot probability), $p = 1 - 0.14 = 0.86$. Let A be the event that all the carrots in the sample will be sweet (or no carrot in the sample is sour), then

$$P[A] = \binom{11}{11} \times 0.86^{11} \times 0.14^{(11-11)} = 0.1903$$

□

(b) **Solution:** Let B be the event that 1 or more carrots in the sample is sour, then $B = A^c$

$$P[B] = P[A^c] = 1 - P[A] = 1 - 0.1903 = 0.8097$$

□

(c) **Solution:** Let C be the event that at most 1 carrot in the sample is sour $\implies$ 0 carrots in the sample is sour or exactly 1 carrot in the sample is sour. We know from part(a), $P[A] =$ the probability of 0 carrots in the sample is sour is 0.1905. Let D be the event that exactly 1 carrot in the sample is sour (or 10 carrots are sweet), then similar to part (a), we get

$$P[D] = \binom{11}{1} \times 0.86^{10} \times 0.14^{(11-10)} = 0.3408$$
$$\therefore P[C] = P[A] + P[D] = 0.1905 + 0.3408 = 0.5313$$

□

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Solutions to Problem 4 of Homework 5

Name: GOWTHAM GOLI (N17656180) *Due: Tuesday, November 29*

Solution:

```
format short e;
p = 0.75;
n = 12;
z = zeros(1,n);
x = zeros(1,n);
for i=1:n
    z(i) = rand
    if (z(i) <= p)
        x(i) = 1;
    else
        x(i) = 0;
    end
end
```

Trial 1:

```
 z =
Columns 1 through 8
1.1921e-01 9.3983e-01 6.4555e-01 4.7946e-01 6.3932e-01 5.4472e-01 6.4731e-01
5.4389e-01
Columns 9 through 12
7.2105e-01 5.2250e-01 9.9370e-01 2.1868e-01
x =
1 0 1 1 1 1 1 1 1 1 0 1 = 10 heads = 0.83
```

Trial 2:

```
 z =
Columns 1 through 7
1.0580e-01 1.0970e-01 6.3591e-02 4.0458e-01 4.4837e-01 3.6582e-01 7.6350e-01
Columns 8 through 12
6.2790e-01 7.7198e-01 9.3285e-01 9.7274e-01 1.9203e-01
x =
1 1 1 1 1 1 0 1 0 0 0 1 = 8 heads = 0.66
```

Trial 3:

```
 z =
Columns 1 through 7
7.0405e-01 4.4231e-01 1.9578e-02 3.3086e-01 4.2431e-01 2.7027e-01 1.9705e-01
Columns 8 through 12
8.2172e-01 4.2992e-01 8.8777e-01 3.9118e-01 7.6911e-01
x =
1 1 1 1 1 1 1 0 1 0 1 0 = 9 heads = 0.75
```

Trial 4:

```
 z =
Columns 1 through 7
4.7952e-01 8.0135e-01 2.2784e-01 4.9809e-01 9.0085e-01 5.7466e-01 8.4518e-01
Columns 8 through 12
7.3864e-01 5.8599e-01 2.4673e-01 6.6642e-01 8.3483e-02
x =
1 0 1 1 0 1 0 1 1 1 1 1 = 9 heads = 0.75
```

Trial 5:

```
 z =
Columns 1 through 7
6.2596e-01 6.6094e-01 7.2975e-01 8.9075e-01 9.8230e-01 7.6903e-01 5.8145e-01
Columns 8 through 12
9.2831e-01 5.8009e-01 1.6983e-02 1.2086e-01 8.6271e-01
x =
1 1 1 0 0 0 1 0 1 1 1 0 = 7 heads = 0.58
```

Trial 6:

```
 z =
Columns 1 through 8
4.8430e-01 8.4486e-01 2.0941e-01 5.5229e-01 6.2988e-01 3.1991e-02 6.1471e-01
3.6241e-01
Columns 9 through 12
4.9533e-02 4.8957e-01 1.9251e-01 1.2308e-01
x =
1 0 1 1 1 1 1 1 1 1 1 1 = 11 heads = 0.91
```

Trial 7:

```
 z =
Columns 1 through 8
4.9036e-01 8.5300e-01 8.7393e-01 2.7029e-01 2.0846e-01 5.6498e-01 6.4031e-01
4.1703e-01
Columns 9 through 12
2.0598e-01 9.4793e-01 8.2071e-02 1.0571e-01
x =
1 0 0 1 1 1 1 1 1 0 1 1 = 9 heads = 0.75
```

Trial 8:

```
 z =
Columns 1 through 8
1.4204e-01 1.6646e-01 6.2096e-01 5.7371e-01 5.2078e-02 9.3120e-01 7.2866e-01
7.3784e-01
Columns 9 through 12
6.3405e-02 8.6044e-01 9.3441e-01 9.8440e-01
x =
1 1 1 1 1 0 1 1 1 0 0 0 = 9 heads = 0.75
```

We can see that in trials 3,4,7,8 the ratio of success is exactly equal to 0.75 and in the rest of the trails, it is very close to 0.75 □

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Solutions to Problem 5 of Homework 5

Name: GOWTHAM GOLI (N17656180) *Due: Tuesday, November 29*

(a) **Solution:** This is similar to linear least squares problem of minimizing $\|Ax - b\|_2^2$ where the m vector e corresponds to $m \times n$ matrix A and the scalar β corresponds to the n vector x. $\therefore e^T e\beta = e^T b \implies \beta = \frac{e^T b}{e^T e}$. $e^T e = m, e^T b = b_1 + \cdots + b_m \implies \beta = \frac{b_1 + \cdots + b_m}{m}$ □

(b) **Solution:** The maximum absolute value of an element is the infinity norm of a vector. Therefore, we need to choose β such that $|b_i - \beta|$ is overall minimized. Intuitively, this will happen when β is mid-way between the maximum (b_{max}) and minimum (b_{min}) elements of b. This can be heuristically justified as follows

- If we choose β closer to b_{min} then $|b_i - \beta|$ will become too big $\forall b_i$ that are closer to b_{max}.
- If we choose β closer to b_{max} then $|b_i - \beta|$ will become too big $\forall b_i$ that are closer to b_{min}.

Hence β must lie in the mid way between b_{min} and b_{max} i.e., $\beta = \frac{b_{min} + b_{max}}{2}$ □

(d) Using the code below for each part in (i), (ii), (iii) we get,

```
b = %Insert value of b here for each part
e = [1 1 1 1 1 1 1];
beta_one_norm = median(b)
beta_two_norm = sum(b)/7
beta_inf_norm = (min(b) + max(b))/2

r1 = b-beta_one_norm*e
norm_r1 = norm(r1,1)

r2 = b-beta_two_norm*e
norm_r2 = norm(r2,2)

rinf = b-beta_inf_norm*e
norm_rinf = norm(rinf,inf)
```

(i) **Solution:**

$$\begin{aligned}
\beta_1 = \beta_2 = \beta_\infty &= 0 \\
r_1 &= \begin{pmatrix}-150 & 25 & 0 & -70 & 70 & 150 & -25\end{pmatrix}^T \\
\|r_1\|_1 &= 490 \\
r_2 &= \begin{pmatrix}-150 & 25 & 0 & -70 & 70 & 150 & -25\end{pmatrix}^T \\
\|r_2\|_2 &= 236.7488 \\
r_\infty &= \begin{pmatrix}-150 & 25 & 0 & -70 & 70 & 150 & -25\end{pmatrix}^T \\
\|r_\infty\| &= 150
\end{aligned}$$

$\beta_1 = 0 \implies$ The median of b is 0. Therefore the minimum value of b must be less than or equal to 0 and $\beta_\infty = 0 \implies$ that $b_{\max} = -b_{\min}$ and $\beta_2 = 0 \implies$ the sum of all elements of b is 0. This could be possible when the elements of b are symmetric around 0. □

(ii) **Solution:**

$$\begin{aligned}
\beta_1 &= 1 \\
\beta_2 &= 150.5714 \\
\beta_\infty &= 250.5000 \\
r_1 &= \begin{pmatrix}0 & 499 & 0 & 249 & 0 & 299 & 0\end{pmatrix}^T \\
\|r_1\| &= 1047 \\
r_2 &= \begin{pmatrix}-149.5714 & 349.4286 & -149.5714 & 99.4286 & -149.5714 & 149.4286 & -149.5714\end{pmatrix}^T \\
\|r_2\|_2 &= 493.7628 \\
r_\infty &= \begin{pmatrix}-249.5000 & 249.5000 & -249.5000 & -0.5000 & -249.5000 & 49.5000 & -249.5000\end{pmatrix}^T \\
\|r_\infty\| &= 249.5000
\end{aligned}$$

$\beta_1 = 1 \implies$ The median of b is 1. Therefore the minimum value of b must be less than or equal to 1 and $\beta_\infty \approx 250 \implies$ the maximum value of b must be greater than or equal to 499 and $\beta_2 \approx 150 \implies$ the sum of all elements of b is $150 \times 7 = 750$ □

(iii) **Solution:**

$$\begin{aligned}
\beta_1 &= 1 \\
\beta_2 &= 14285 \\
\beta_\infty &= 49995 \\
r_1 &= \begin{pmatrix}0 & -3 & -11 & 3 & 6 & -6 & 99999\end{pmatrix}^T \\
\|r_1\| &= 100028 \\
r_2 &= \begin{pmatrix}-14284 & -14287 - 14295 & -14281 & 14278 & -14290 & 85715\end{pmatrix}^T \\
\|r_2\|_2 &= 9.2583e + 04 \\
r_\infty &= \begin{pmatrix}-49994 & -49997 & -50005 & -49991 & -49988 & -50000 & 59005\end{pmatrix}^T \\
\|r_\infty\| &= 59005
\end{aligned}$$

$\beta_1 = 1 \implies$ The median of b is 1. Therefore the minimum value of b must be less than or equal to 1 and $\beta_\infty = 49995 \implies b_{max}$ must be large and $\beta_2 = 14285 \implies b_{max}$ must have been contributed to the overall increase of the sum of the elements of b. $\square$

(e) **Solution:** From part (iii) in the above question, we can see that β_2 and β_∞ will be affected the most due to outliers and β_1 has no significant affect because

- β_1 depends only on the median.
- β_2 depends on the average sum of the elements of b. So it can be significantly increased or decreased with large or small outliers.
- β_∞ depends on the maximum and minimum elements of b.

$\square$