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Solutions to Problem 1 of Homework 2

Name: GOWTHAM GOLI (N17656180) *Due: Wednesday, October 5*

(a) **Solution:**

Let $A = \begin{pmatrix} a_{11} & a_0 \\ a_0 & a_{22} \end{pmatrix} B = \begin{pmatrix} b_{11} & b_0 \\ b_0 & b_{22} \end{pmatrix}$ so that $A = A^T$ and $B = B^T$

If AB is not symmetric $\implies (AB) \neq (AB)^T \implies AB \neq B^T A^T \implies AB \neq BA$

$AB = \begin{pmatrix} a_{11}b_{11} + a_0 b_0 & a_{11}b_0 + a_0 b_{22} \\ a_0 b_{11} + a_{22}b_0 & a_0 b_0 + a_{22}b_{22} \end{pmatrix} BA = \begin{pmatrix} a_{11}b_{11} + a_0 b_0 & a_0 b_{11} + b_0 a_{22} \\ b_0 a_{11} + b_{22}a_0 & a_0 b_0 + a_{22}b_{22} \end{pmatrix}$

If $AB = BA$

$$a_{11}b_0 + a_0 b_{22} = a_0 b_{11} + b_0 a_{22}$$
$$a_0 b_{11} + a_{22}b_0 = b_0 a_{11} + b_{22}a_0$$

From the above two equations, we get $b_0(a_{11} - a_{22}) = a_0(b_{11} - b_{22})$. Let $a_0 = 1, b_0 = 1$ then $a_{11} - a_{22} = b_{11} - b_{22}$. It is easy to choose any values so that this equation is violated.
$\therefore A = \begin{pmatrix} 4 & 1 \\ 1 & 3 \end{pmatrix} B = \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix} \implies AB = \begin{pmatrix} 13 & 4 \\ 6 & 1 \end{pmatrix}$

Clearly AB is not symmetric $\square$

(b) **Solution:**

From part (a), if $AB = BA$, we have $b_0(a_{11} - a_{22}) = a_0(b_{11} - b_{22})$

Let $a_0 = 1, b_0 = 1$ and $a_{11} - a_{22} = b_{11} - b_{22} = 2$

$\therefore A = \begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix} B = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} \implies AB = \begin{pmatrix} 13 & 5 \\ 5 & 3 \end{pmatrix}$

Clearly AB is symmetric $\square$

(c) **Solution:**

From part(a), if $A = \begin{pmatrix} a_{11} & a_0 \\ a_0 & a_{22} \end{pmatrix} B = \begin{pmatrix} b_{11} & b_0 \\ b_0 & b_{22} \end{pmatrix}$

then we have $b_0(a_{11} - a_{22}) = a_0(b_{11} - b_{22})$ (this equation has already derived in part (a)) $\square$

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Solutions to Problem 2 of Homework 2

Name: GOWTHAM GOLI (N17656180) *Due: Wednesday, October 5*

(a) **Solution:** Assume that all the diagonal elements of R are non-zero i.e $r_{ii} \neq 0 \; \forall i$ and R is a singular a matrix $\therefore \exists$ a non zero vector y such that $Ry = 0$

$$\begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ & \ddots & & \\ & & r_{n-1,n-1} & r_{n-1,n} \\ & & & r_{nn} \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_{n-1} \\ y_n \end{pmatrix} = 0$$

$$\implies \begin{pmatrix} r_{11}y_1 + r_{12}y_2 + \cdots + r_{1n}y_n \\ \vdots \\ r_{n-1,n-1}y_{n-1} + r_{n-1,n}y_n \\ r_{nn}y_n \end{pmatrix} = 0$$

$$\implies r_{nn}y_n = 0 \implies y_n = 0 \because r_{nn} \neq 0$$

$$\implies r_{n-1,n-1}y_{n-1} = 0 \implies y_{n-1} = 0 \because r_{n-1,n-1} \neq 0$$

$$\vdots$$

$$\implies r_{11}y_1 = 0 \implies y_1 = 0 \because r_{11} \neq 0$$

$$\implies y = 0$$

But we know that y is a non zero matrix. This a contradiction. Hence our assumption was wrong. Therefore R is singular only if atleast one of the diagonal elements is zero □

(b) **Solution:**

The answer is No. Consider a simple 2×2 triangular matrix with all it's diagonal elements to be 0. Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. In this case, $n = 2, k = 2$ but $r(A) = 1 \neq n - k$ □

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Solutions to Problem 3 of Homework 2

Name: GOWTHAM GOLI (N17656180) *Due: Wednesday, October 5*

Given $E = I - \alpha x y^T$

(a) **Solution:**

Theorem 1. *If E is singular then $\alpha x^T y = 1$*

Proof. If $= 0$ or $y = 0$ or $\alpha = 0$ then $E = I$ which is non-singular. Hence assume that $x \neq 0$, $y \neq 0$, $\alpha \neq 0$. If E is singular then there will be a non zero vector z such that $Ez = 0$

$$\begin{aligned}
&Ez = 0 \\
\implies &(I - \alpha x y^T)z = 0 \\
\implies &z - \alpha x(y^T z) = 0 \qquad (1)\\
\implies &z - \alpha\beta x = 0 \;\; (\text{let } y^T z = \beta \text{ where } \beta \text{ is some scalar}) \\
\implies &z = \alpha\beta x
\end{aligned}$$

We know that z is non zero and $\alpha \neq 0, x \neq 0 \implies \beta \neq 0$. Let $\gamma = \alpha\beta \neq 0 \implies z = \gamma x$. Substitute this value in (1), we get

$$\begin{aligned}
\gamma x - \alpha x(y^T \gamma x) = 0 \\
\gamma x(1 - \alpha y^T x) = 0
\end{aligned}$$

We know that $\gamma \neq 0, x \neq 0 \implies 1 - \alpha y^T x = 0 \implies \alpha y^T x = 1 \implies \alpha x^T y = 1$ ☐

Theorem 2. *If $\alpha x^T y = 1$ then E is singular*

Proof. If $= 0$ or $y = 0$ or $\alpha = 0$ then $\alpha x^T y \neq 1$. Hence assume that $x \neq 0$, $y \neq 0$, $\alpha \neq 0$. Let us suppose that E is non singular. $\therefore Ez = 0 \implies z = 0$ ☐

☐

(b) **Solution:**

Let $E^{-1} = I - \beta x y^T$

$$\begin{aligned}
&EE^{-1} = I \\
\implies &(I - \alpha x y^T)(I - \beta x y^T) = I \\
\implies &I - \beta x y^T - \alpha x y^T + \alpha\beta(x y^T)(x y^T) = I \\
\implies &\alpha x y^T + \beta x y^T - \alpha\beta(x y^T)(x y^T) = 0 \qquad (1)
\end{aligned}$$

Consider $xy^Txy^T = x(y^Tx)y^T$, y^Tx is a scalar $\implies xy^Txy^T = (y^Tx)(xy^T)$. Substitute this value in the above equation

$$\begin{aligned}
&\implies \alpha xy^T + \beta xy^T - \alpha\beta(y^Tx)(xy^T) = 0 \\
&\implies xy^T(\alpha + \beta - \alpha\beta y^Tx) = 0 \\
&\implies \alpha + \beta - \alpha\beta y^Tx = 0 \\
&\implies \beta(\alpha y^Tx - 1) = \alpha \\
&\implies \beta(\alpha x^Ty - 1) = \alpha \\
&\implies \beta = \frac{\alpha}{\alpha x^Ty - 1}
\end{aligned}$$

$\square$

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Solutions to Problem 4 of Homework 2

Name: GOWTHAM GOLI (N17656180) *Due: Wednesday, October 5*

Solution: Given $Z = xy^T$

$$
\begin{aligned}
Z^k &= (xy^T)^k \\
\implies Z^k &= \underbrace{(xy^T)(xy^T)(xy^T)\ldots(xy^T)}_{\text{k times}}
\end{aligned}
$$

Consider $(xy^T)(xy^T) = x(y^Tx)y^T$, y^Tx is a scalar $\implies (xy^T)(xy^T) = (y^Tx)(xy^T)$. Substitute this in the above equation

$$
\begin{aligned}
\implies Z^k &= (y^Tx)\underbrace{(xy^T)(xy^T)\ldots(xy^T)}_{\text{k-1 times}} \\
&= (y^Tx)(y^Tx)\underbrace{(xy^T)(xy^T)\ldots(xy^T)}_{\text{k-2 times}} \\
&= \vdots \\
&= (y^Tx)^{k-1}(xy^T) \\
\implies Z^k &= (y^Tx)^{k-1}z
\end{aligned}
$$

□

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Solutions to Problem 5 of Homework 2

Name: GOWTHAM GOLI (N17656180) *Due: Wednesday, October 5*

(a) **Solution:**

Since A is 5×3 matrix, the rank has to be ≤ 3. Check if the three columns of A are linearly independent or not. If not, check if any two columns are linearly independent. If not, then the rank has to be one.

1. Check if $r(A) = 3$

$$x_1 \begin{pmatrix} 3 \\ 2 \\ 1 \\ 2 \\ 6 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ -3 \\ 6 \\ 0 \\ 3 \end{pmatrix} + x_3 \begin{pmatrix} 6 \\ 6 \\ 0 \\ 1 \\ 0 \end{pmatrix} = 0$$

$$\implies x_1 + x_2 + 2x_3 = 0 \tag{1}$$
$$2x_1 - 3x_2 + 6x_3 = 0 \tag{2}$$
$$x_1 + 6x_2 = 0 \tag{3}$$
$$2x_1 + x_3 = 0 \tag{4}$$
$$2x_1 + x_2 = 0 \tag{5}$$

Solving (3) and (5), we get $x_1 = x_2 = 0$, substitute these values in (1), we get $x_3 = 0$ $\therefore r(A) = 3$

□

(b) **Solution:**

$$x_1 \begin{pmatrix} 3 \\ 2 \\ 1 \\ 2 \\ 6 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ -3 \\ 6 \\ 0 \\ 3 \end{pmatrix} + x_3 \begin{pmatrix} 6 \\ 6 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 1 \\ -1 \\ -5 \end{pmatrix}$$

$$\implies 3x_1 + 3x_2 + 6x_3 = 4 \tag{1}$$
$$2x_1 - 3x_2 + 6x_3 = 3 \tag{2}$$
$$x_1 + 6x_2 = 1 \tag{3}$$
$$2x_1 + x_3 = -1 \tag{4}$$
$$6x_1 + 3x_2 = -5 \tag{5}$$

Solving (3) and (5), we get $x_1 = -1, x_2 = 1/3$, substitute these values in (4), we get $x_3 = 1$. Now substitute these values in (1) and (2) to check if they satisfy those equations.

Substitute in (1) $\implies$ 3(-1) + 3(1/3) + + 6(1) = 4

Substitute in (2) $\implies$ 2(-1) - 3(1/3) + + 6(1) = 3

$\therefore B$ lies in the range of A $\square$

(c) **Solution:**

Consider rows 3,4,5 of $A \implies y_1(1\ 6\ 0) + y_2(2\ 0\ 1) + y_3(6\ 3\ 0) = 0$

$$\begin{aligned}
&y_1 + 2y_2 + 6y_3 = 0\\
&6y_1 + 3y_3 = 0\\
&y_2 = 0\\
&\text{Substitue } y_2 \text{ in the above equations and solving we get}\\
\implies &y_1 = y_3 = 0
\end{aligned}$$

Consider rows 1,2,4 of $A \implies y_1(3\ 3\ 6) + y_2(2\ \ -3\ 6) + y_3(2\ 0\ 1) = 0$

$$\begin{aligned}
&3y_1 + 2y_2 + 2y_3 = 0\\
&3y_1 - 3y_2 = 0\\
&6y_1 + 6y_2 + y_3 = 0\\
\implies &y_1 = y_2\\
&\text{Substitue } y_1 = y_2 \text{ in the above equations and solving we get}\\
&y_1 = y_2 = y_3 = 0
\end{aligned}$$

Therefore rows 1,2,4 and rows 3,4,5 are linearly independent rows $\square$

(d) **Solution:**

$\square$

(e) **Solution:**

Since the columns of A are linearly independent, if there is a solution to $Ax = b$ it has to be unique.

Assume that there exists more than one solution then $Ax_1 = b$ and $Ax_2 = b \implies A(x_1 - x_2) = 0 \implies x_1 - x_2 = 0$ ($\because$ the columns of A are linearly independent) $\implies x_1 = x_2$ $\square$

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Solutions to Problem 6 of Homework 2

Name: GOWTHAM GOLI (N17656180) *Due: Wednesday, October 5*

(a) **Solution:**

Suppose that A^TA is singular $\implies \exists$ a non zero vector z such that

$(A^TA)z = 0 \implies z^T(A^TA)z = 0 \implies (Z^TA^T)(AZ) = 0 \implies (AZ)^T(AZ) = 0$

$\therefore ||Az||_2^2 = 0 \implies Az = 0$ but since the columns of A are linearly independent $\implies$ if $Az = 0$ then $z = 0$. This is a contradiction. Hence our assumption is wrong. Therefore (A^TA) is non singular. $\square$

(b) **Solution:**

If A has linearly independent columns then $\exists$ a non-zero vector z such that $Az = 0$

$Az = 0 \implies A^T(Az) = 0 \implies (A^TA)z = 0$ where z is non-zero $\implies (A^TA)$ is singular

$\square$

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Solutions to Problem 7 of Homework 2

Name: GOWTHAM GOLI (N17656180) *Due: Wednesday, October 5*

(a) **Solution:**

1. $||A|| > 0$ if $A \neq 0$ and $||0|| = 0$
   If $A \neq 0$ then $max_{i,j}|a_{ij}| > 0$ as $\exists$ atleast one $a_{ij} \neq 0$ and $|a_{ij}| > 0 \implies ||A|| > 0$
   If $A = 0$ then $a_{ij} = 0 \ \forall i, j \implies |a_{ij}| = 0 \ \forall i, j \implies max_{i,j}|a_{ij}| = 0 \implies ||A|| = 0$
2. $||\gamma A|| = |\gamma| \, ||A||$ for any scalar γ
   In γA, each element of A is multiplied by γ. Therefore i, j element now becomes γa_{ij}. If a_{ij} is the largest absolute value of any element in A then γa_{ij} is the largest absolute value of any element in γA and $|\gamma a_{ij}| = |\gamma| \, |a_{ij}|$
   $\therefore ||\gamma A|| = max_{i,j}|\gamma a_{ij}| = max_{i,j}|\gamma| \, |a_{ij}| = |\gamma| max_{i,j}|a_{ij}| = |\gamma| \, ||A||$
3. $||A + B|| \leq ||A|| + ||B||$
   Let $C = A + B$ then $c_{ij} = a_{ij} + b_{ij} \implies |c_{ij}| \leq |a_{ij}| + |b_{ij}|$
   $\implies max_{i,j}|c_{ij}| \leq max_{i,j}|a_{ij}| + max_{i,j}|b_{ij}| \implies ||C|| \leq ||A|| + ||B||$

$\square$

(b) **Solution:**

Let $A = \begin{pmatrix} 2 & 9 \\ -3 & 4 \end{pmatrix} \implies ||A|| = 9$ and $B = \begin{pmatrix} 5 & 2 \\ 4 & 1 \end{pmatrix} \implies ||B|| = 5$

$AB = \begin{pmatrix} 46 & 13 \\ 1 & -2 \end{pmatrix} \implies ||AB|| = 46 > ||A|| \, ||B||$ $\square$

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Solutions to Problem 8 of Homework 2

Name: GOWTHAM GOLI (N17656180) *Due: Wednesday, October 5*

Solution:

Let A and B be two $n \times n$ square upper triangular matrices $\implies a_{ij} = 0, b_{ij} = 0$ when $i > j$
Let $C = AB$

$$\begin{aligned} c_{ij} &= \sum_{k=1}^{n} a_{ik}b_{kj} \\ &= \sum_{k=1}^{j-1} a_{ik}b_{kj} + \sum_{k=j}^{i-1} a_{ik}b_{kj} + \sum_{k=i}^{n} a_{ik}b_{kj} \end{aligned} \tag{1}$$

Now in the calculation of c_{ij} consider the case when $i > j$ and k varies from 1 to n

- When $k < j < i \implies a_{ik} = 0$, the first term in (1) becomes 0 i.e. $\sum_{k=1}^{j-1} a_{ik}b_{kj} = 0$
- When $j < k < i \implies a_{ik} = b_{kj} = 0$, the second term in (1) becomes 0 i.e. $\sum_{k=j}^{i-1} a_{ik}b_{kj} = 0$
- When $i < k < n \implies b_{kj}0$, the third term in (1) becomes 0 i.e. $\sum_{k=i}^{n} a_{ik}b_{kj} = 0$

$\therefore$ If $i > j$, $c_{ij} = 0 + 0 + 0 = 0 \implies C$ is an upper triangular matrix. $\square$

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Solutions to Problem 9 of Homework 2

Name: GOWTHAM GOLI (N17656180) *Due: Wednesday, October 5*

(a) **Solution:**

$$\begin{aligned}
&Ux = 0 \\
\implies &\begin{pmatrix} u_{11} & u_{12} & u_{13} & \dots & u_{1n} \\ & u_{22} & u_{23} & \dots & u_{2n} \\ & & \ddots & & \vdots \\ & & & & u_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = 0 \qquad (1) \\
\implies &u_{nn}x_n = 0 \\
&u_{n-1,n-1}x_{n-1} + u_{n-1,n}x_n = 0 \\
&u_{n-2,n-2}x_{n-2} + u_{n-2,n-1}x_{n-1} + u_{n-2,n}x_n = 0 \\
&\vdots \\
&u_{n-k,n-k}x_{n-k} + u_{n-k,n-k+1}x_{n-k+1} + \dots + u_{n-k,n}x_n = 0 \qquad (2)
\end{aligned}$$

From (1) $x_n = \alpha$ where α is some scalar

(2) is equivalent to

$$\begin{aligned}
&u_{k,k}x_k + u_{k,k+1}x_{k+1} + \dots + u_{k1,n}x_n = 0 \\
\implies &x_k = \frac{-\sum_{j=k+1}^{n} u_{kj}x_j}{u_{kk}} \quad \text{for } k = 1, \dots, n-1 \qquad (3)
\end{aligned}$$

□

(b) **Solution:**

Let $x_3 = \alpha$. Now use equation (3) from part (a) to calculate x_2 and x_1

$$\begin{aligned}
x_2 &= \frac{-u_{23}x_3}{u_2 2} = \frac{-2\alpha}{3} \\
x_1 &= \frac{-u_{12}x_2 - u_{13}x_3}{u_{11}} = \frac{2(-2\alpha/3) - \alpha/2}{1} = \frac{-11\alpha}{6} \\
\therefore x &= \begin{pmatrix} -11\alpha/6 \\ -2\alpha/3 \\ \alpha \end{pmatrix}
\end{aligned}$$

□

(c) **Solution:**

From part(b), $x = \begin{pmatrix} -11\alpha/6 \\ -2\alpha/3 \\ \alpha \end{pmatrix}$

$\square$