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Solutions to Problem 1 of Homework 1

Name: GOWTHAM GOLI (N17656180) *Due: Tuesday, September 26*

(a) **Solution:**

Let, $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \implies A^2 = \begin{pmatrix} a_{11}^2 + a_{12}a_{21} & a_{11}a_{12} + a_{12}a_{22} \\ a_{21}a_{11} + a_{22}a_{21} & a_{21}a_{12} + a_{22}^2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

Therefore,

$$a_{11}^2 + a_{12}a_{21} = -1 \tag{1}$$
$$a_{11}a_{12} = -a_{22}a_{12} \tag{2}$$
$$a_{11}a_{21} = -a_{22}a_{21} \tag{3}$$
$$a_{22}^2 + a_{21}a_{12} = -1 \tag{4}$$

From equations (1) and (4), it is clear that $a_{12} \neq 0$ and $a_{21} \neq 0$. Therefore from equations (2) and (4), we can conclude that $a_{11} = -a_{22}$. Let $a_{11} = 1, a_{12} = 1 \implies a_{22} = -1$ and $a_{21} = -2$

Therefore $A = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}$ □

(b) **Solution:** Let, $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \implies B^2 = \begin{pmatrix} b_{11}^2 + b_{12}b_{21} & b_{11}b_{12} + b_{12}b_{22} \\ b_{21}b_{11} + b_{22}b_{21} & b_{21}b_{12} + b_{22}^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Therefore,

$$b_{11}^2 = -b_{12}b_{21} \tag{5}$$
$$b_{11}b_{12} = -b_{22}b_{12} \tag{6}$$
$$b_{11}b_{21} = -b_{22}b_{21} \tag{7}$$
$$b_{22}^2 = -b_{21}b_{12} \tag{8}$$

From the above equations, it can be easily observed that if $b_{11} = 0$, B becomes zero matrix. So assume, $b_{11} \neq 0$. From equations (5) and (8), we can conclude that, $b_{11}^2 = b_{22}^2$ and $b_{12}b_{21} < 0 \implies (b_{12} - b_{21}) \neq 0$. From equations (6) and (7) we get, $(b_{12} - b_{21})(b_{11} + b_{22}) = 0$. Therefore, $b_{11} = -b22$. Let $b_{11} = 1, b_{12} = 1$ then $b_{22} = -1$ and $b_{21} = -1$.

Therefore, $B = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$ □

(c) **Solution:** Let $C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}, D = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}$ Given that $CD = -DC \implies$

$$2c_{11}d_{11} = -c_{12}d_{21} - d_{12}c_{21} \tag{9}$$
$$c_{11}d_{12} + c_{12}d_{22} = -d_{11}c_{12} - d_{12}c_{22} \tag{10}$$
$$c_{21}d_{11} + c_{22}d_{21} = -d_{21}c_{11} - d_{22}c_{21} \tag{11}$$
$$2c_{22}d_{22} = -d_{21}c_{12} - c_{21}d_{12} \tag{12}$$

Equations (9), (10) $\implies c_{11}d_{11} = c_{22}d_{22}$
Equations (10), (11) $\implies c_{12}(d_{11}+d_{22}) = -d_{12}(c_{11}+c_{22})$ and $d_{21}(c_{11}+c_{22}) = -c_{21}(d_{11}+d_{22})$

Let $c_{11} = 1, c_{22} = -1, d_{11} = 1$ then $d_{22} = -1$. Now pick $c_{12} = 1$ and $c_{21} = 1$, from equation (9) we get, $-d_{21} - d_{12} = -2$. Pick $d_{21} = -3$ then $d_{12} = 1$.
Therefore $C = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, D = \begin{pmatrix} 1 & 1 \\ -3 & -1 \end{pmatrix}$ where $CD = \begin{pmatrix} -2 & 0 \\ 4 & 2 \end{pmatrix}$ and $DC = \begin{pmatrix} 2 & 0 \\ -4 & -2 \end{pmatrix}$

□

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Solutions to Problem 2 of Homework 1

Name: GOWTHAM GOLI (N17656180) *Due: Tuesday, September 26*

Solution:

- If A is a singular matrix, then for any non-zero vector x we know that $Ax \neq 0$
- Since B is non singular, there exists a non-zero vector y such that $By = x$.
- Therefore, $Ax \neq 0 \implies A(By) \neq 0 \implies (AB)y \neq 0$ where y is a non-zero matrix.

$\therefore$ If A and B are non-singular matrices then their product AB is also non-singular. $\square$

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Solutions to Problem 3 of Homework 1

Name: GOWTHAM GOLI (N17656180) *Due: Tuesday, September 26*

Solution:

Given $A = \begin{pmatrix} 1 & 8 & 7 \\ 2 & 10 & 8 \\ 3 & 12 & 9 \end{pmatrix}$

$$\begin{aligned} & a_3 = \lambda_1 a_1 + \lambda_2 a_2 \\ & \lambda_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda_2 \begin{pmatrix} 8 \\ 10 \\ 12 \end{pmatrix} = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} \\ \implies & \lambda_1 + 8\lambda_2 = 7 && (1) \\ & 2\lambda_1 + 10\lambda_2 = 10 && (2) \\ & 3\lambda_1 + 12\lambda_2 = 9 && (3) \end{aligned}$$

Solving equations (1) and (2), we get $\lambda_1 = -1$ and $\lambda_2 = 1$. Substitute these values in (3), we get $-3 + 12 = 9$. Therefore the columns of the matrix are linearly dependent. $\square$

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Solutions to Problem 4 of Homework 1

Name: GOWTHAM GOLI (N17656180) *Due: Tuesday, September 26*

Solution:

Theorem 1. *Let A be an $n \times n$ real matrix. If the only solution of $Ay = 0$ is $y = 0$ then prove that there is a unique n-vector x satisfying $Ax = b$ for any nonzero n-vector b*

Proof. Let us assume that there are two vectors x_1 and x_2 where $x_1 \neq x_2$ such that $Ax_1 = b$ and $Ax_2 = b \implies A(x_1 - x_2) = 0$. But we know that if $Ay = 0 \implies y$ has to be zero. Therefore, $(x_1 - x_2) = 0 \implies x_1 = x_2$.

This is a contradiction to the given hypothesis. Thus our assumption was wrong i.e. x_1 and x_2 should be equal. Hence, there is a unique n-vector x satisfying $Ax = b$ for any nonzero n-vector b if the only solution of $Ay = 0$ is $y = 0$ □

Theorem 2. *Let A be an $n \times n$ real matrix. If there is a unique n-vector x satisfying $Ax = b$ for any nonzero n-vector b then prove that the only solution of $Ay = 0$ is $y = 0$.*

Proof. Let us assume that there exists a non-zero vector y such that $Ay = 0$. We know that $Ax = b$ where x is a unique n-vector and b is a non-zero vector. Adding these two equations, we get $A(x + y) = b$ where $x + y \neq x$ since y is a non-zero vector.

This is a contradiction to the given hypothesis. Thus our assumption was wrong i.e. y can not be a non-zero vector. Hence, the only solution of $Ay = 0$ is $y = 0$ if there is a unique n-vector x satisfying $Ax = b$ for any nonzero n-vector b. □

□

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Solutions to Problem 5 of Homework 1

Name: GOWTHAM GOLI (N17656180) *Due: Tuesday, September 26*

(a) **Solution:**

If columns of A are linearly independent then $Az = 0$ only if $z = 0$.

If $Ax = Ay \implies A(x - y) = 0$. Given that the columns of A are linearly independent, $\therefore x - y = 0 \implies x = y$ $\square$

(b) **Solution:**

$$A = \begin{pmatrix} -1 & 2 & 3 \\ 0 & -2 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \implies Ax = \begin{pmatrix} 12 \\ -1 \end{pmatrix}$$

Let $y = \begin{pmatrix} 5 \\ \lambda_2 \\ 4 \end{pmatrix} \implies Ay = \begin{pmatrix} 12 \\ -1 \end{pmatrix}$

Solve for λ_2, we get $\lambda_2 = 5/2 \therefore y = y = \begin{pmatrix} 5 \\ 5/2 \\ 4 \end{pmatrix}$ $\square$

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Solutions to Problem 6 of Homework 1

Name: GOWTHAM GOLI (N17656180) *Due: Tuesday, September 26*

Solution:

Rank of A is equal to the maximum number of linearly independent rows and columns of A (which are equal). Thus $r \leq m$ and $r \leq n$. But it is given that $r = m$. Therefore $m \leq n$ $\square$

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Solutions to Problem 7 of Homework 1

Name: GOWTHAM GOLI (N17656180) *Due: Tuesday, September 26*

Solution:

Assume that both (1) and (2) are true

$$\begin{aligned}
y^T b &= y^T(Ax) && \textit{Using (1)} \\
&= (y^T A)x && \textit{Associative property of Matrix Multiplication} \\
&= ((y^T A)^T)^T x \\
&= (A^T y)^T x \\
&= 0^T x && \textit{Using (2)} \\
&= 0
\end{aligned}$$

But from (2), we know that $y^T b \neq 0$. This is a contradiction. Thus our assumption was wrong, (1) and (2) cannot both be true and thus they are contradictory. □

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Solutions to Problem 8 of Homework 1

Name: GOWTHAM GOLI (N17656180) *Due: Tuesday, September 26*

(a) **Solution:**

Given that x_R is the linear combination of columns of A. Therefore, there is a n vector y such that $Ay = x_R$.

$$\begin{aligned} x_R^T x_N &= (Ay)^T x_N \\ &= (y^T A^T) x_N \\ &= y^T (A^T x_N) && \textit{Associative property of Matrix Multiplication} \\ &= y^T 0 \\ &= 0 \end{aligned}$$

Hence proved $\square$

(b) **Solution:**

Let us assume that x_R and x_N are not unique. Let $\exists\ x_{R_1}, x_{R_2} \in \mathbb{R}(A)$ and $x_{N_1}, x_{N_2} \in \mathbb{N}(A^T)$ where $x_{R_1} \neq x_{R_2}$ and $x_{N_1} \neq x_{N1_2}$ such that

$$\begin{aligned} &x = x_{R_1} + x_{N_1} \\ &x = x_{R_2} + x_{N_2} \\ &\text{Subtracting these two equations we get} \\ &0 = (x_{R_1} - x_{R_2}) + (x_{N_1} - x_{N_2}) \\ &x_{R_1} - x_{R_2} = -(x_{N_1} - x_{N_2}) \qquad (1) \end{aligned}$$

But we know that Range space and Null space are closed under addition and subtraction. $\therefore (x_{R_1} - x_{R_2}) \in \mathbb{R}(A)$ and $(x_{N_1} - x_{N_2}) \in \mathbb{N}(A^T) \implies (x_{R_1} - x_{R_2}) \perp (x_{N_1} - x_{N_2})$ but from (1) we have, $(x_{R_1} - x_{R_2}) \parallel (x_{N_1} - x_{N_2}) \implies (x_{R_1} - x_{R_2}) = 0$ and $(x_{N_1} - x_{N_2}) = 0 \implies$ $x_{R_1} = x_{R_2}$ and $x_{N_1} = x_{N_2}$

This is a contradiction. Hence our assumption was wrong. Therefore x_R and x_N are unique. $\square$

(c) **Solution:**

It is given that $x_R \neq 0$ and $x_N \neq 0$

Assume that $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$ such that $\lambda_1 x_R + \lambda_2 x_N = 0 \implies x_R$ and x_N are parallel but we also know that x_R and x_N are perpendicular to each other $\because x_R \in \mathbb{R}(A)$ and $x_n \in \mathbb{N}(A^T) \implies$ $x_R = x_N = 0$. But this a contradiction to the given hypothesis. Hence our assumption is wrong. Therefore if x_R and x_N are both nonzero then they are linearly independent. $\square$

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Solutions to Problem 9 of Homework 1

Name: GOWTHAM GOLI (N17656180) *Due: Tuesday, September 26*

(a) **Solution:**

Given that C is an $m \times n$ matrix with full column rank. Hence all the columns of C are linearly independent. Therefore $Cy = 0 \implies y = 0$. Assume there are two vectors x_1 and x_2 such that $Cx_1 = d$ and $Cx_2 = d$. $\implies C(x_1 - x_2) = 0$. But we know that if $cy = 0$, $y = 0$. $\therefore x_1 - x_2 = 0 \implies x_1 = x_2$. Therefore x is unique □

(b) **Solution:**

We know $\exists$ unique x_R, x_N such that $b = x_R + x_N$ and $\exists$ y such that $x_R = Ay \because x_R \in \mathbb{R}(A)$

$\implies b = Ax = Ay + x_N \implies Ax - Ay = X_N \implies Ax - Ay \parallel x_N$.

$Ax - Ay \in \mathbb{R}(A)$ using closure property of Range space $\implies Ax - Ay \perp X_N$

$\therefore Ax - Ay = 0 \implies x = y$. We are sure that $\exists$ y for the unique $x_R \implies x$ exists too.

$\therefore$ The system $Ax = b$ is compatible $\forall b \in \mathbb{R}^m$ □

(c) **Solution:**

The rank of the 2×4 matrix cannot be two because if it is 2 then from part (b) the system $Ax = b$ is compatible for every $b \in \mathbb{R}^m$. So choose a matrix whose rank is 1 and columns are linearly independent. It easy to choose such a matrix. Choose the columns such that they are multiples of column 1, so they are linearly dependent.

Let $A = \begin{pmatrix} a_1 \, a_2 \, a_3 \, a_4 \end{pmatrix}$ where $a_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $a_2 = 2a_1, a3 = 3a_1, a_4 = 4a_1$.

$\therefore A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \end{pmatrix} \implies b = Ax = \begin{pmatrix} x_1 + 2x_2 + 3x_3 + 4x_4 \\ 2x_1 + 4x_2 + 6x_3 + 8x_4 \end{pmatrix} \in \mathbb{R}(A)$

$\therefore \mathbb{R}(A)$ consists of only those vectors such that $b_2 = 2b_1$. Choose any vector b that violates this property. Let $b = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$. Therefore for the chosen A, b the system $Ax = b$ is not compatible.

□

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Solutions to Problem 10 of Homework 1

Name: GOWTHAM GOLI (N17656180) *Due: Tuesday, September 26*

(a) **Solution:**

Let $A = \begin{pmatrix} a_1\, a_2\, a_3\, a_4\, a_5 \end{pmatrix}$ where a_i is a 2 dimensional vector. Given that the $rank(A) = 2$ and the 2×2 submatrix consisting of the first two columns of A has rank 1. Therefore a_1 and a_2 should be linearly dependent on each other. Hence choose $a_2 = 2a_1$ where $a_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $a_2 = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$. Clearly a_1 and a_2 are linearly dependent. Thus $(a_1 a_2)$ has rank 1. Now we need to choose a_3, a_4, a_5 such that the maximum number of linear independent columns are only 2. Let $a_5 = \begin{pmatrix} 5 \\ 9 \end{pmatrix}$. Therefore a_1, a_5 and a_2, a_5 are linearly independent but a_1, a_2, a_5 are linearly dependent $\because a_2 = 2a_1 + 0a_5$. Thus the rank of the matrix constructed thus far is 2. Now choose a_3, a_4 such that the maximum number of linearly independent columns still remains to be 2. Simply choose a_3, a_4 to be multiples of a_1 so that a_i, a_5 for $i = 1, \ldots, 4$ are linearly independent. Therefore $A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 6 & 8 & 9 \end{pmatrix}$

a_1, a_2 are linearly dependent **(confirms (i))**

$a_1 + a_4 = a_2 + a_3 + 0a_5 \implies$ Rank cannot be 5

Randomly choose any 4 columns from A, if a_5 is one of the chosen columns, then simply choose the coefficient of a_5 to be 0 and the remaining columns will be linearly dependent on each other as they are all multiples of a_1. If a_5 is not chosen then it is straight forward that all the columns will be linearly dependent. A similar argument can be given for choosing any 3 columns from $A \implies$ Rank cannot be 3, 4

Choose a_1, a_5, $\lambda_1 a_1 + \lambda_2 a_5 = 0 \implies \lambda_1 = -5\lambda$ and $2\lambda_1 = -9\lambda_2 \implies \lambda_1 = \lambda 2 = 0 \therefore$ the rank of the matrix is 2 **(confirms (ii))** □

(b) **Solution:**

Let $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \implies b = Ax = \begin{pmatrix} x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 \\ 2(x_1 + 2x_2 + 3x_3 + 4x_4) + 9x_5 \end{pmatrix}$

For $x = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \implies b = \begin{pmatrix} 15 \\ 29 \end{pmatrix}$ □

(c) **Solution:**

Let $x_1 + 2x_2 + 3x_3 + 4x_4 = k$, then $b = Ax = \begin{pmatrix} k + 5x_5 \\ 2k + 9x_5 \end{pmatrix} \in range(A)$

For any 2-vector b the system of linear equations can be solved to get the values of k and x_5. Using k choose x_1, x_2, x_3, x_4 such that $x_1 + 2x_2 + 3x_3 + 4x_4 = k$ i.e. $\forall b\ \exists x$ such that $b = Ax$

$\therefore$ It is not possible to find any such $b \notin range(A)$ $\square$

(d) **Solution:**

$$A^T = \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \\ 4 & 8 \\ 5 & 9 \end{pmatrix} \implies b = A^T x = \begin{pmatrix} x_1 + 2x_2 \\ 2x_1 + 4x_2 \\ 3x_1 + 6x_2 \\ 4x_1 + 8x_2 \\ 5x_1 + 9x_2 \end{pmatrix} \in range(A^T)$$

Notice that rows 2 to 4 of b are multiples of row 1. Choose c such that c violates this property.

$$\therefore c = \begin{pmatrix} 3 \\ 7 \\ 10 \\ 11 \\ 14 \end{pmatrix} \notin range(A^T)$$

$\square$