CSCI-GA.1180:
Mathematical Techniques for Computer Science Applications
New York University, Fall 2015
Instructor: Margaret Wright
Solutions to Homework Assignment 5
Assigned 3 December 2015; due 11:59pm, 14 December 2015

HW5–1. *This question involves independent coin tosses. For parts (a) and (b), give the answer in two ways: as an algebraic expression in terms of quantities p, n, and k, and as a number. (Use Matlab to do the calculations.)*

(a) *A fair two-sided coin is tossed* 5 *times. What is the probability that exactly three of these tosses produce heads? Explain your answer.*

Consider n Bernoulli trials in which the probability of success in a single trial is p. The probability of exactly k successes in the n trials is given by the binomial probability distribution:

$$B(n,p,k) = \binom{n}{k} p^k (1-p)^{n-k}.$$

In this problem, $p = \frac{1}{2}$, so that $p^k(1-p)^{n-k} = p^n$. The number of tosses is the number of trials, so that $n = 5$, and the number of heads is the number of successes, i.e., $k = 3$:

$$B(n,p,k) = \binom{5}{3} \times (\tfrac{1}{2})^5 = 0.3125.$$

The Matlab calculation is

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nchoosek(5,3)* (half^3) * (half^2) = 0.3125.
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(b) *A fair two-sided coin is tossed* 40 *times. What is the probability that exactly* 20 *of these tosses will produce heads? Explain your answer.*

Using the same reasoning as in part (a), we let $n = 40$ and $k = 20$, so that the answer is $\binom{40}{20} p^{40} \approx 0.12537$, calculated as follows by Matlab:

```
n = 40;  k = 20;  p = 1/2
pfact = p^(k)*q^(n-k); nchoice = nchoosek(n,k)
pfact = 9.0949e-13; nchoice = 1.3785e+11
probk = nchoice*pfact = 1.2537e-01
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(c) *Under the same circumstances as in part (b), explain how you could justify saying "The most likely outcome of the* 40 *tosses is* 20 *heads".*

Although the probability of obtaining exactly 20 heads in 40 tosses is not very large (0.125), the value $k = 20$ is the most likely in the sense that the probability of exactly 20 heads is larger than the probability for any other value of k. Here are the relevant calculations for values of k on either side of 20, where the probabilities are, as expected, symmetric around $k = 20$, since $\binom{n}{k} = \binom{n}{n-k}$:

```
k = 18;   probk = 1.0312e-01

k = 19;   probk = 1.1940e-01

k = 21;   probk = 1.1940e-01

k = 22;   probk = 1.0312e-01
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HW5–2. *This problem refers to the days when people wore hats and checked them when going to a restaurant.*

Suppose that n people come to the restaurant in a group and check their hats. Unfortunately, the person in charge of the hat check is totally disorganized and does not keep track of which hat belongs to each person. Thus, when the people leave the restaurant, each person's probability of getting his/her own hat is $1/n$. Let A_i be the event that person i gets his or her own hat when leaving the restaurant; let the random variable X_i be equal to 1 if person i gets his/her own hat back, and $X_i = 0$ otherwise. Assume that $n = 4$.

1. *What is the probability of A_1? What is the expected value of X_1? Explain your answer.*

The final arrangement of the 4 hats is a permutation of the four integers 1, 2, 3, and 4, and there are $4! = 24$ such arrangements. The probability of A_1, that person 1 receives his or her own hat, is the fraction of the permutations of the 4 integers in which the first digit is 1. There are 6 such permutations, so that the probability of A_1 is $6/24 = 1/4$.

When event A_1 occurs, the random variable X has the value 1, and is zero otherwise. Hence the expected value of X_1 is

$$P[A_1][\text{value of } X_1] = \frac{1}{4}(1) = \frac{1}{4}.$$

2. *What is the probability of the event $A_1 \cap A_2$, i.e., that both persons 1 and 2 will receive their own hats? What is the expected value of the random variable $X_1 \times X_2$? Explain your answer.*

The event $A_1 \cap A_2$ happens when persons 1 and 2 both receive their own hats, i.e., when first two digits of the 4-digit permutation are 12. This happens for only two events, the permutations 1234 and 1243, out of the 24 possible events. Hence the probability of $A_1 \cap A_2$ is $2/24 = 1/12$.

Either $X_1 = 0$ or $X_2 = 0$ except when event $A_1 \cap A_2$ occurs, and the probability of $A_1 \cap A_2$ was just shown to be $1/12$. Hence the numerical value of the product $X_1 \times X_2$ is nonzero (and equal to 1) only when this event occurs, so that the expected value of $X_1 \times X_2$ is $\frac{1}{12}$.

3. *In general, are events A_i and A_j, where $i \neq j$, independent? Explain your answer.*

For any value of i, the probability of A_i is $1/4$. If $i \neq j$, the product $P[A_i]\,P[A_j]$ of the probabilities of events A_i and A_j is $1/(16)$. However, as just shown in part (b), the probability of $A_i \cap A_j$ for $i \neq j$ is $1/12$. Thus $P[A_i \cap A_j]$ is not equal to $P[A_i]P[A_j]$, which shows that the two events are not independent.

HW5–3. *Assume that it rains in a big city on half of the days. The weather forecaster is reasonably reliable.*

- *The probability is* $2/3$ *that it will rain given a forecast of rain.*
- *The probability is* $2/3$ *that it will not rain given a forecast that it will not rain.*

A professor of probability relies to some extent on the forecasts, but is exceptionally cautious about rain: when rain is forecast, the professor brings an umbrella to the office; when the forecast is that it will not rain, the professor brings an umbrella with probability $1/3$.

1. *The probability that the professor does not bring an umbrella to the office given that it rains that day, is* $2/9$. *Explain why this answer is correct, giving all relevant formulas used to calculate probabilities.*

Given any events A and B, the following general rules apply:

$$P[A] = P[A|B]\,P[B] + P[A|{\sim}B]\,P[{\sim}B]; \tag{5.1}$$

$$P[A|B] = 1 - P[{\sim}A|B]; \tag{5.2}$$

$$P[A] = 1 - P[{\sim}A].$$

Let R denote the event that it rains (so that ${\sim}R$ is the event that it does not rain); let F denote the event that the forecast is for rain (so that ${\sim}F$ is the forecast that it will not rain); and let U denote the event that the professor takes an umbrella.

Here are the probabilities that are given:

(i) $P[R] = \frac{1}{2}$, so that $P[{\sim}R] = \frac{1}{2}$;

(ii) $P[R|F] = \frac{2}{3}$, so that $P[{\sim}R|F] = \frac{1}{3}$, using the general rule (5.2);

(iii) $P[{\sim}R|{\sim}F] = \frac{2}{3}$, so that $P[R|{\sim}F] = \frac{1}{3}$;

(iv) $P[U|F] = 1$, so that $P[{\sim}U|F] = 0$;

(v) $P[U|{\sim}F] = \frac{1}{3}$, so that $P[{\sim}U|{\sim}F] = \frac{2}{3}$.

The probability wanted in this part of the problem, that the professor does not bring an umbrella to the office given that it rains, is $P[{\sim}U|R]$.

The only factor affecting whether the professor brings an umbrella is the forecast, not whether it actually rains. So we try to find the probability that rain is forecast. Using the general rule (5.1), we can write

$$P[R] = P[R|F]\,P[F] + P[R|{\sim}F]\,P[{\sim}F]. \tag{5.3}$$

Substituting the known values for $P[R]$, $P[R|F]$, and $P[R|{\sim}F]$, and using the fact that $P[{\sim}F] = 1 - P[F]$, this gives an equation that must be satisfied by $P[F]$:

$$\frac{1}{2} = \frac{2}{3}P[F] + \frac{1}{3}(1 - P[F]), \quad \text{so that} \quad \left(\frac{2}{3} - \frac{1}{3}\right) P[F] = \frac{1}{6},$$

which means that

$$P[F] = \frac{1}{2} \quad \text{and} \quad P[\sim F] = \frac{1}{2}.$$

In words, the unconditional probability of a forecast of rain is $P[F] = \frac{1}{2}$, which implies that $P[\sim F] = \frac{1}{2}$.

Next we need to find the probability of a forecast of rain given that it rains. For this we use Bayes' law:

$$P[F|R] = \frac{P[F]\ P[R|F]}{P[R]} = \frac{(\frac{1}{2})(\frac{2}{3})}{\frac{1}{2}} = \frac{2}{3}.$$

Using the general rule (5.2), the prediction will be for no rain 1/3 of the time that it actually rains, i.e.,

$$P[\sim F|R] = \frac{1}{3}.$$

So we know that, when it rains, the probability is $\frac{1}{3}$ that the forecast will say "no rain". Relation (v) in the initial data says that $P[\sim U|\sim F] = \frac{2}{3}$. Thus we also know that, when the forecast says "no rain", the probability is $\frac{2}{3}$ that the professor will not bring an umbrella. Putting these pieces of information together, it follows that

$$P[\sim U|R] = \frac{2}{3} \times P[\sim F|R] = \left(\frac{2}{3}\right) \times \left(\frac{1}{3}\right) = \frac{2}{9},$$

as needed to be shown.

2. *The probability that the professor brings an umbrella to the office given that it does not rain that day is* 5/9. *Explain why this answer is correct, giving all relevant formulas used to calculate probabilities.*

In the second part, we want to show why $P[U|\sim R] = \frac{5}{9}$. The professor brings an umbrella with probability 1 if the forecast is for rain, and with probability $\frac{1}{3}$ if the forecast is for no rain. Hence the probability that the forecast is for no rain given that there is no rain is

$$P[\sim F|\sim R] = \frac{P[\sim R]P[\sim R|\sim F]}{P[\sim R]} = \frac{(\frac{1}{2})(\frac{2}{3})}{\frac{1}{2}} = \frac{2}{3}.$$

Thus the probability of the professor's bringing an umbrella when there is no rain is

$$\left(\frac{1}{3}\right)\times 1 + \left(\frac{1}{3}\right) \times \left(\frac{2}{3}\right) = \frac{1}{3} + \frac{2}{9} = \frac{5}{9},$$

as was to be shown.

HW5–4. *A restaurant offers two kinds of pie (strawberry and cherry), and always begins each day with an equal number of pies of these two kinds. Every day exactly* 10 *customers each request a pie, and the probability that a given customer will choose one kind or the other is* 1/2.

(a) *If the restaurant stocks* 5 *strawberry pies and* 5 *cherry pies every day, what is the probability that every customer will receive his/her requested kind of pie? Explain your answer.*

Note first that, since $p = \frac{1}{2}$, $p^k(1-p)^{n-k} = p^n$. (The value of $(\frac{1}{2})^{10}$ is approximately 9.7656×10^{-4}.) By assumption, the restaurant always stocks an equal number of pies for each of the two kinds. Let m denote the number of each kind of pie, and let k denote the number of customers who want cherry pie. (This means that $10-k$ want strawberry.) In order for every one of the 10 customers to receive his/her requested kind of pie, it must be true that $m \geq 5$, since 10 pies are always needed; if $m \geq 10$, it is a certainty that every customer will receive his/her preferred kind of pie.

If $m = 5$, every customer will receive his/her preferred only if exactly 5 people want cherry and exactly 5 people want strawberry. The probability that this will happen is the probability of exactly 5 successes in 10 trials, namely

$$B(10, \tfrac{1}{2}, 5) = \binom{10}{5} p^{10} = 0.24609.$$

(b) *Answer the same question as in part (a), but assuming that the restaurant stocks* 8 *strawberry pies and* 8 *cherry pies each day. Explain your answer.*

When $m = 8$, let N_8 denote the event that every customer will *not* receive his/her preferred kind of pie. Event N_8 will happen if $k = 0$, $k = 1$, $k = 9$, or $k = 10$. (Otherwise, there will be enough pies.) Letting A_k denote the event that exactly k customers want cherry pie, we have

$$P[A_0] = \binom{10}{0}(\tfrac{1}{2})^{10} = (\tfrac{1}{2})^{10}; \qquad P[A_1] = \binom{10}{1}(\tfrac{1}{2})^{10} = 10(\tfrac{1}{2})^{10};$$
$$P[A_9] = \binom{10}{9}(\tfrac{1}{2})^{10} = 10(\tfrac{1}{2})^{10}; \qquad P[A_{10}] = \binom{10}{10}(\tfrac{1}{2})^{10} = (\tfrac{1}{2})^{10}.$$

Adding up these probabilities, the probability of N_8 is

$$P[N_8] = (1 + 10 + 10 + 1)(9.7656 \times 10^{-4}) = 2.1484 \times 10^{-2}.$$

The complement of N_8 is the happy event that every customer will receive his/her choice of pie, namely

$$P[\sim N_8] = 1 - P[N_8] = 0.97852.$$

One also obtains this probability by adding up the probabilities that exactly 2, 3, ..., 7, 8 customers will want cherry:

$$(\binom{10}{2} + \binom{10}{3} + \binom{10}{4} + \binom{10}{5} + \binom{10}{6} + \binom{10}{7} + \binom{10}{8}) * (\tfrac{1}{2})^{10} = 0.97852.$$

HW5–5. *[For clarity, the wording has been changed from the original assignment.] Given a fair coin, a gambler will win a large amount of money if the gambler manages to toss* *"heads", and the gambler has three chances to do so. Once the coin comes up "heads", the tosses will stop. Otherwise, the gambler may make three tosses in total. Let X be the number of times that heads is tossed, and Y be the number of times that tails is tossed. For each case, explain how you obtained your results.*

1. *For X, give (i) the expected value and (ii) the variance.*

There are four possible events, A_i, $i = 1$,..., 4, where H denotes heads and T denotes tails:

$$A_1 = H, \quad A_2 = TH, \quad A_3 = TTH, \quad A_4 = TTT.$$

The probability of A_1 is $\frac{1}{2}$, since A_1 involves one flip of a fair coin. Similarly, A_2 involves two independent tosses of a fair coin, with $P[A_2] = \frac{1}{4}$, and $P[A_3] = P[A_4] = \frac{1}{8}$.

The random variable X is equal to 1 in events A_1, A_2, and A_3, and equal to 0 in event A_4. By definition of the expected value,

$$\begin{aligned} E[X] &= \sum_{i=1}^{4} P[A_i]\times(\text{value of } X \text{ in event } A_i) \\ &= P[A_1]\times 1 + P[A_2]\times 1 + P[A_3]\times 1 + P[A_4]\times 0 = \frac{7}{8}. \end{aligned}$$

The variance of X is given by

$$\begin{aligned} \text{var}[X] &= E[X^2] - (E[X])^2 \\ &= \sum_{i=1}^{4} P[A_i] \times (\text{value of } X^2 \text{ in event } A_i) - (E[X])^2 \\ &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} - \left(\frac{7}{8}\right)^2 = \frac{7}{8} - \frac{49}{64} = \frac{7}{64}. \end{aligned}$$

2. *For Y, give (i) the expected value and (ii) the variance.*

Analyzing the values of Y in each of the four events, we have $Y = 0$ in event A_1, $Y = 1$ in event A_2, $Y = 2$ in event A_3, and $Y = 3$ in event A_4. By definition of the expected value,

$$\begin{aligned} E[Y] &= \sum_{i=1}^{4} P[A_i] \times (\text{value of } Y \text{ in event } A_i) \\ &= P[A_1]\times 0 + P[A_2]\times 1 + P[A_3]\times 2 + P[A_4]\times 3 \\ &= 0 + \left(\frac{1}{4}\right)\times 1 + \left(\frac{1}{8}\right)\times 2 + \left(\frac{1}{8}\right)\times 3 = \frac{7}{8}. \end{aligned}$$

The variance of Y is given by

$$\begin{aligned}
\text{var}[Y] &= E[Y^2] - (E[Y])^2 \\
&= \sum_{i=1}^{4} P[A_i] \times (\text{value of } Y^2 \text{ in event } A_i) - (E[Y])^2 \\
&= \left(\frac{1}{2}\right)\times 0 + \left(\frac{1}{4}\right)\times 1 + \left(\frac{1}{8}\right)\times 4 + \left(\frac{1}{8}\right)\times 9 - \left(\frac{7}{8}\right)^2 \\
&= \frac{15}{8} - \frac{49}{64} = \frac{71}{64}.
\end{aligned}$$