HW 3-1:

a) $||x||_\infty$ is given as,

$$||x||_\infty = \max_{1 \le i < n} |x_i|$$

So, basically, $||x||_\infty$ is single maximum absolute value.

$||x||_2$ is given as,

$$||x||_2 = \sqrt{x_1^2 + \ldots + x_n^2}$$

$||x||_1$ is given as -

$$||x||_1 = |x_1| + |x_2| + \ldots + |x_n|$$

So, $||x||_1$ encompasses $||x||_\infty$.

$$\Rightarrow ||x||_1 \geq ||x||_\infty$$

Also, we know that $(a+b) \geq \sqrt{a^2+b^2}$

Thus,

$$||x||_1 \geq ||x||_2 \geq ||x||_\infty$$

b) The 3-vector z satisfying $||z||_1 = ||z||_2 = ||z||_\infty$ will have following property:

$$\max_{1 \le i < 3} |z_i| = \sqrt{z_1^2 + z_2^2 + z_3^2} = |z_1| + |z_2| + |z_3|$$

This is possible only when one of the element is non-zero and other element are zero.

$\therefore$ One of the example is -

$\therefore \quad z = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}$

c) The n-vector x that satisfy $\|x\|_1 = \|x\|_2 = \|x\|_\infty$ can be generalized as $x_i \neq 0$ and all other x_j's $= 0$.

HW 3-2:

$$A = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}$$

a) $\|A\|_1$ is maximum column sum.

$$\therefore \|A\|_1 = |-2| + |-1|$$
$$\therefore \|A\|_1 = 3$$

$\|A\|_\infty$ is maximum row sum.

$$\therefore \|A\|_\infty = |-2| + |1|$$
$$\therefore \|A\|_\infty = 3$$

b) Let $u = \begin{pmatrix} x \\ y \end{pmatrix}$

$$\therefore \|u\|_1 = |x| + |y|$$

and $$Au = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$
$$= \begin{pmatrix} x - 2y \\ x - y \end{pmatrix}$$

$$\therefore \|Au\|_1 = |x - 2y| + |x - y|$$

and $\|A\|_1 = 3$

$$\Rightarrow |x - 2y| + |x - y| = 3$$

Let us consider $x = 0$ & $y = 1$

$\therefore \ \|u\|_1 = |0| + |1|$

$\therefore \ \|u\|_1 = 1$

and,

$\|Au\|_1 = |x - 2y| + |x - y|$

$= |0 - 2| + |0 - 1|$

$\therefore \ \|Au\|_1 = 3$

and,

$\|A\|_1 = 3$

$\therefore$ for $u = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$; $\|u\|_1 = 1$ & $\|Au\|_1 = \|A\|_1$

c) Let $v = \begin{pmatrix} x \\ y \end{pmatrix}$

Consider, $x = 1$ & $y = 0$

$\therefore \ \|v\|_\infty =$ maximum row sum

$= 1$

Now,

Let $x = 4$ & $y = 1$

$\therefore \ Av = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$\therefore \ \|Av\|_\infty = 3$

& $\|v\|_\infty = 3 \Rightarrow \|Av\|_\infty = \|v\|_\infty$

HW 3-3:

a) Consider $I+E$ is a singular matrix.
So, As $(I+E)$ is a singular matrix, there exist $x \neq 0$ for which $(I+E)x = 0$

$$\therefore (I+E)x = 0$$

$$\therefore Ix + Ex = 0$$

$$\therefore Ix = -Ex$$

Taking Norms -

$$\|Ix\| = \|-Ex\|$$

$$\therefore \|I\| \, \|x\| \leq \|E\| \, \|x\|$$

$$\because \|\alpha E\| = |\alpha| \, \|E\|$$

As $x \neq 0$, $\|x\| \neq 0$

$$\therefore \|I\| \leq \|E\|$$

But, $\|I\| = 1$

$$\therefore \|E\| \geq 1$$

b) Consider I-F is a non-singular matrix.
So, as (I-F) is a non-singular matrix, ~~there~~ exists ~~no~~ $(I-F)X=0$ only when $X=0$.

Now,

$$(I-F)X = 0$$

$$\therefore IX - FX = 0$$

$$IX = FX$$

Taking Norm,

$$\|IX\| = \|FX\|$$

$$\|I\| \, \|X\| \geq \|F\| \, \|X\|$$

But $\|x\| = 0$ as $x = 0$

$$\Rightarrow \|I\| > \|F\|$$

But $\|I\| = 1$

$$\therefore \underline{\|F\| < 1}$$

HW 3-4:

We are given:

$$Ax = b$$

$$r = b - A\tilde{x}$$

$$E = r\tilde{x}^T \frac{1}{\tilde{x}^T\tilde{x}}$$

Substituting value of r -

$$\therefore E = (b - A\tilde{x})\tilde{x}^T \frac{1}{\tilde{x}^T\tilde{x}}$$

$$\therefore \tilde{x}^T\tilde{x} E = (b - A\tilde{x})\tilde{x}^T$$

$$\therefore \tilde{x} E = (b - A\tilde{x})$$

$$\cancel{\therefore \tilde{x}(A+E) = b}$$

$$\therefore \cancel{\tilde{x}}(A+E)\tilde{x} = b$$

$$\Rightarrow E \text{ satisfies } (A+E)\tilde{x} = b$$

HW 3-5

a) As third row is zero, we are left with two equations and three variables. So, it leads to infinite solution. Thus, it is necessary to make some assumptions regarding the x_n element.

So, the algorithm will work as assuming arbitrary value for x_n element and then solving the solutions for that arbitrary value & getting $x_1 \ldots x_{n-1}$

b) $$U = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$x = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Let us assume c as arbitrary and consider $c = 2$.

$\therefore$ $a - b - c = 0$ and

$2b + c = 0$

$\Rightarrow$ $b = -1$ & $a = 3$

$\therefore$ $x = (3 \ -1 \ 2)^T$

HW 3-6:

$$Ax = b$$

$\therefore$ ~~$x = A b$~~

$$\therefore x = A^{-1} b$$

$$\therefore \|x\| \le \|A^{-1}\| \, \|b\|$$

$$\therefore \|A^{-1}\| \geq \frac{1}{10^{-6}}$$

$$\therefore \|A^{-1}\| \geq 10^{6}$$

$$\therefore \text{Cond}(A) = \|A\| \, \|A^{-1}\|$$

$$\therefore \text{Cond}(A) = 1 \times 10^{6}$$

$$\text{Cond}(A) = 10^{6}$$

So, it can be said that A is ill-conditioned due to large magnitude of Cond (A) and small changes in the problem data can lead to large changes in exact solution.