Mathematical Techniques for Computer Science Applications

CSCI-GA.1180-001

New York University, Fall 2016
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Homework Assignment 3

Assigned Monday 10 October 2016; due 11:59pm, Friday 21 October 2016

Whenever calculations are needed to solve a problem, those calculations must be submitted as part of the homework assignment.

Homework must be submitted electronically, by 11:59pm on the due date. Unless express permission has been given in advance by the instructor for a late homework submission, a 30% percent penalty will be deducted for each late day (or part of a late day).

Homework grades are based on the quality and clarity of your explanations and proofs.

Exercise 3.1. [Matrix norms.] Consider the matrix

$$A = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix}.$$

(a) Give the value of $\|A\|_1$ and the value of $\|A\|_\infty$.

(b) Find a 2-vector u such that $\|u\|_1 = 1$ and $\|Au\|_1 = \|A\|_1$.

(c) Find a 2-vector v such that $\|v\|_\infty = 1$ and $\|Av\|_\infty = \|A\|_\infty$.

Exercise 3.2. Given a nonsingular matrix A and a vector b, let x denote the *exact* solution of $Ax = b$, and let $\|\cdot\|$ denote any subordinate norm. Consider a matrix A such that $\|A\| = 1$ and a vector b such that $\|b\| = 10^{-6}$. If $\|x\| = 1$, what do we know about $\mathrm{cond}(A)$ measured in the given norm? Explain.

Exercise 3.3. Note: numerical calculations (using the two-norm) will be needed to solve this problem.

Let

$$A = \begin{pmatrix} 0.625 & 0.4376 \\ 0.546 & 0.3823 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1.0626 \\ 0.9283 \end{pmatrix}.$$

(a) Show that the exact solution x^* to $Ax = b$ is $x^* = (1, 1)^T$.

(b) Give the solution $\widetilde{x}$ computed by executing the Matlab "backslash" command `A\b` (or the equivalent in octave or SciPy). Compute and print the vectors (i) $d = \widetilde{x} - x^*$, (ii) $r^* = b - Ax^*$, and (iii) $\widetilde{r} = b - A\widetilde{x}$. Comment on the relative size of their norms.

(c) Let $\widehat{x}$ be a potential solution of $Ax = b$, with residual $\widehat{r} = b - A\widehat{x}$, and define E as the rank-one matrix

$$E = \frac{1}{\widehat{x}^T \widehat{x}} \, \widehat{r} \, \widehat{x}^T. \tag{1}$$

Show mathematically that the *exact* matrix E defined in (1) satisfies the relation

$$(A + E)\widehat{x} = b.$$

(d) Consider the specific vector $\widehat{x}$

$$\widehat{x} = \begin{pmatrix} -27.678 \\ 41.958 \end{pmatrix}.$$

Would you say that $\widehat{x}$ is close to x^* from (a)? Would you say that $\widehat{x}$ is close to $\widetilde{x}$ from (b)? Explain.

(e) Given the vector $\widehat{x}$ from (d), compute and print (i) its residual $\widehat{r} = b - A\widehat{x}$, (ii) the matrix E from (1), and (iii) $\|E\|_2$. Would you describe $\|E\|_2$ as "small" or "large"? Explain.

(f) Compute the solution $\bar{x}$ to $(A + E)\bar{x} = b$, using the Matlab backslash command, and print $\bar{x}$.

(g) Print $\|\bar{x} - \widehat{x}\|$. Based on this norm, is $\bar{x}$ "close to" $\widehat{x}$ from part (d)? Explain why or why not.

(h) Do you agree with the statement that $\widehat{x}$ is close to the exact solution of a system that is close to the original system? Explain why or why not.

Exercise 3.4.

(a) Show that, if the square matrices B and C are nonsingular, then $\mathrm{cond}(BC) \le \mathrm{cond}(B)\ \mathrm{cond}(C)$, where the condition number is measured in any of the "standard" matrix norms (one, two, or infinity).

(b) Show that, measured in the matrix two-norm, $\mathrm{cond}(A^T) = \mathrm{cond}(A)$.

(c) If cond is measured in the matrix one-norm (the maximum absolute column sum), is $\mathrm{cond}(A^T)$ necessarily equal to $\mathrm{cond}(A)$? Explain your answer, giving a specific example for illustration if your answer is "no".

Exercise 3.5. Consider the following matrix A and vector b:

$$A = \begin{pmatrix} 2 & 4 \\ 1 & 2 \\ 1 & 2 \end{pmatrix}, \quad b = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}.$$

(a) What is the rank of A?

(b) Give a general form, expressed in terms of a single scalar, for any vector in the range of A.

(c) Explain why the dimension of the null space of A^T is two, and display two linearly independent vectors z_1 and z_2 in the null space of A^T. Give a general form, expressed in terms of two scalars, for every vector z in $\mathrm{null}(A^T)$, i.e., such that $A^T z = 0$.

(d) Given the particular vector b, find the specific vectors $b_R \in \mathrm{range}(A)$ (i.e., the range-space component of b) and $b_N \in \mathrm{null}(A^T)$ such that $b = b_R + b_N$.

(e) What is the dimension of the null space of A? (Note: of A, not A^T.) What is the general form of any vector $q = (q_1, q_2)^T$ such that $Aq = 0$?

(f) Find a *specific* two-vector $v = (\alpha_1, \alpha_2)^T$ such that $Av = b_R$, where b_R is the vector found in part (d). (In this part, give numbers for the components of vb.)

(g) Consider a two-vector b_A defined as $b_A = v + q$, where $Av = b_R$ from part (d) and $Aq = 0$, and use this form to obtain two specific (different) instances of b_A (i.e., with numbers).