CSCI-GA.1180-001	October 21, 2016

Solutions to Problem 1 of Homework 3

Name: GOWTHAM GOLI (N17656180) *Due: Wednesday, October 21*

(a) **Solution:**

$$\begin{aligned}||A||_1 &= \max_{1\le j\le n}\sum_{i=1}^{n}|a_{ij}| \implies \text{(maximum of the absolute column sums)}\\ &= \max\{|1|+|1|, |-2|+|-1|\}\\ &= 3\\ ||A||_\infty &= \max_{1\le i\le n}\sum_{j=1}^{n}|a_{ij}| \implies \text{(maximum of the absolute row sums)}\\ &= \max\{|1|+|-2|, |1|+|-1|\}\\ &= 3\end{aligned}$$

$\square$

(b) **Solution:**

Let $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, ||v||_1 = 1 \implies |u_1| + |u_2| = 1$

$Au = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_1 - 2u_2 \\ u_1 - u_2 \end{pmatrix}, ||Au||_1 = ||A||_1 \implies |u_1 - 2u_2| + |u_1 - u_2| = 3$ $\square$

Assume, $u_1 \ge 0$ and $u_2 < 0 \implies u_1 - 2u_2 > 0$ and $u_1 - u_2 > 0$

$\therefore u_1 - u_2 = 1$ and $(u_1 - 2u_2) + (u_1 - u_2) = 3 \implies 2u_1 - 3u_2 = 3$. Solving these two equations we get $u_1 = 0, u_2 = -1 \implies u = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

(c) **Solution:** Let $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, ||v||_\infty = 1 \implies \max\{|v_1|, |v_2|\} = 1$

$Av = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 - 2v_2 \\ v_1 - v_2 \end{pmatrix}, ||Av||_\infty = ||A||_\infty \implies \max\{|v_1 - 2v_2|, |v_1 - v_2|\} = 3$

$\max\{|v_1|, |v_2|\} = 1 \implies$ one of v_1, v_2 has to be either 1 or -1. Let $v_1 = 1$

$\therefore \max\{|1 - 2v_2|, |1 - v_2|\} = 3$. We easily see that $v_2 = -1$ satisfies this equation

$\therefore v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\square$

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Solutions to Problem 2 of Homework 3

Name: GOWTHAM GOLI (N17656180) *Due: Wednesday, October 21*

Solution:

Since A is a non-singular matrix, A^{-1} will exist

$$\begin{aligned} Ax &= b \\ \implies x &= A^{-1}b \\ \implies ||x|| &\leq ||A^{-1}||\ ||b|| \\ \implies ||A^{-1}|| &\geq \frac{||x||}{||b||} \\ \implies ||A^{-1}|| &\geq \frac{1}{10^{-6}} \\ \implies ||A^{-1}|| &\geq 10^6 \end{aligned}$$

$\therefore cond(A) = ||A||\ ||A^{-1}|| \geq 10^6 \implies cond(A)$ is large therefore A is ill-conditioned

$\square$

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Solutions to Problem 3 of Homework 3

Name: GOWTHAM GOLI (N17656180) *Due: Wednesday, October 21*

(a) **Solution:**

Using GEPP, $M_1 = \begin{pmatrix} 1 & 0 \\ -m_{21} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -a_{21}/a_{11} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -0.8736 & 1 \end{pmatrix}$

$$\begin{aligned} M_1 A x &= M_1 b \\ \implies \begin{pmatrix} 1 & 0 \\ -0.8736 & 1 \end{pmatrix} \begin{pmatrix} 0.625 & 0.4376 \\ 0.546 & 0.3823 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ -0.8736 & 1 \end{pmatrix} \begin{pmatrix} 1.0626 \\ 0.9283 \end{pmatrix} \\ \implies \begin{pmatrix} 0.625 & 0.4376 \\ 0 & 0.00001264 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} 1.0626 \\ 0.00001264 \end{pmatrix} \\ \implies x_2 &= 1 \\ \implies 0.625 x_1 + 0.4376 &= 1.0626 \\ \implies x_1 &= 1 \\ \therefore x &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{aligned}$$

□

(b) **Solution:**

```
format long e;
A = [0.625 0.4376;
     0.546 0.3823];
b = [1.0626; 0.9283];
x_star = [1;1];
x_tilda = A\b
```

$\widetilde{x} = \begin{pmatrix} 9.999999999975465e-01 \\ 1.000000000003504e+00 \end{pmatrix}$

i.

```
d = x_tilda - x_star
norm_d = norm(d)
```

$$d = \begin{pmatrix} -2.453481862119133e-12 \\ 3.504085910321919e-12 \end{pmatrix}$$
$$||d|| = 4.277638520803758e-12$$

Since $||d||$ is almost zero, it means that $\widetilde{x}$ and x are almost equal

ii.
```
r_star = b - A*x_star
norm_r_star = norm(r_star)
```

$$r^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$||r^*|| = 0$$

Since $||r^*||$ is zero, it means that b and Ax^* are exactly equal

iii.
```
r_tilda = b - A*x_tilda
norm_r_tilda = norm(r_tilda)
```

$$\widetilde{r} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$||\widetilde{r}|| = 0$$

Since $||\widetilde{r}||$ is zero, it means that b and $A\widetilde{x}$ are exactly equal. (Although they aren't exactly equal the difference is so low that matlab rounded it off to 0)

□

(c) **Solution:**

```
x_hat = [-27.678;41.958];
r_hat = b - A*x_hat;
E = (1/((transpose(x_hat))*x_hat))*(r_hat*transpose(x_hat))
b1 = (A+E)*x_hat
```

$$E = \begin{pmatrix} -5.797322035777278e-06 & 8.788353131625949e-06 \\ 6.069003037814685e-07 & -9.200203391163686e-07 \end{pmatrix}$$
$$b1 = \begin{pmatrix} 1.062600000000000e+00 \\ 9.282999999999986e-01 \end{pmatrix} \approx b$$

□

(d) **Solution:**

```
norm_x_hat_x_star = norm(x_hat - x_star)
norm_x_hat_x_tilda = norm(x_hat - x_tilda)
```

$$||\hat{x} - x^*|| = 4.999985447978824e+01$$
$$||\hat{x} - \widetilde{x}|| = 4.999985447978396e+01$$

We can see that $||\hat{x} - \widetilde{x}|| \approx ||\hat{x} - x^*||$ and $||\hat{x} - \widetilde{x}|| < ||\hat{x} - x^*||$ this means that $\hat{x}$ is more closer to $\widetilde{x}$ than $\hat{x}$ is closer to x^*. □

(e) **Solution:**

- $\hat{r} = \begin{pmatrix} 5.292000000030050e-04 \\ -5.539999999670808e-05 \end{pmatrix}$
- $E = \begin{pmatrix} -5.797322035777278e-06 & 8.788353131625949e-06 \\ 6.069003037814685e-07 & -9.200203391163686e-07 \end{pmatrix}$
- $||E|| = 1.058578570328090e-05$
- $||A|| = 1.013108113647895e+00$

$||E|| << ||A||$. Therefore we can imply that $||E||$ is small $\square$

(f) **Solution:**

```
x_bar = (A+E)\b
```

$$\bar{x} = \begin{pmatrix} -2.767800001642768e+01 \\ 4.195800002346206e+01 \end{pmatrix}$$

$\square$

(g) **Solution:**

```
norm_x_bar_x_hat = norm(x_bar - x_hat)
```

$$||\bar{x} - \hat{x}|| = 2.864152054095992e-08$$

Yes the norm is small i.e $\bar{x}$ is close to $\hat{x}$ $\square$

(h) **Solution:**

Yes it can be said that $\hat{x}$ is close to the exact solution of a system that is close to the original system because $||\hat{x} - x^*|| = 4.999985447978824e+01 \approx ||\hat{x} - \widetilde{x}|| = 4.999985447978396e+01$

$\square$

Solutions to Problem 4 of Homework 3

Name: GOWTHAM GOLI (N17656180) *Due: Wednesday, October 21*

(a) **Solution:**

$$\begin{aligned} cond(BC) &= ||BC|| \cdot ||(BC)^{-1}|| \\ &= ||BC|| \cdot ||C^{-1}B^{-1}|| \\ &\leq ||B|| \cdot ||C|| \cdot ||C^{-1}|| \cdot ||B^{-1}|| \\ &\leq ||B|| \cdot ||B^{-1}|| \cdot ||C|| \cdot ||C^{-1}|| \\ \implies cond(BC) &\leq cond(B)cond(C) \end{aligned}$$

□

(b) **Solution:**

According to SVD, we know that if $A = USV^T$ then $||A||_2 = \sigma_1$ and $||A^{-1}||_2 = 1/\sigma_n$

$$\begin{aligned} A &= USV^T \\ \implies A^T &= VS^TU^T \\ \implies A^T &= VSU^T \end{aligned}$$

This looks very much like the SVD of A except that the orthogonal matrices are interchanged but S remains the same while holding the crucial property that $\sigma_1 \geq \cdots \geq \sigma_n$. Therefore the the maximum in $||A^T||_2 = \max\limits_{||x||_2 \neq 0} \frac{||A^Tx||_2}{||x||_2} = \max\limits_{||y||_2=1} ||A^Ty||_2$ is achieved when x is a multiple of u_1. As a result, the unit-norm of vector y for which $||A^T||_2$ is maximized is u_1
$\therefore \max\limits_{||y||_2=1} ||A^Ty||_2 = ||A^Tu_1||_2 = \sigma_1 = ||A^T||_2$. Similar to the arguments given to $||A^{-1}||_2$ in the lecture notes, we can prove that $||(A^T)^{-1}||_2 = 1/\sigma_n$ as A and A^T are structurally similar in SVD.

$$\begin{aligned} cond(A^T) &= ||A^T||_2 \cdot ||(A^T)^{-1}||_2 \\ &= \sigma_1/\sigma_n \\ &= cond(A) \end{aligned}$$

□

(c) **Solution:**

$$cond(A) = ||A||_1 \cdot ||A^{-1}||_1 \tag{1}$$
$$cond(A^T) = ||A^T||_1 \cdot ||(A^T)^{-1}||_1$$
$$\implies cond(A^T) = ||A^T||_1 \cdot ||(A^{-1})^T||_1 \tag{2}$$

If we compare (1) and (2), the first term in both the expressions are $||A||_1$ and $||A^T||_1$. We know that the one norm is the maximum column sum of A. Since the columns of A^T are the rows of A, these two terms will be equal only if the maximum column sum of A = maximum row sum of A i.e. $||A||_1 = ||A^T||_1 \Leftrightarrow ||A||_1 = ||A||_\infty$.

The similar argument holds for the second term in (1) and (2) i.e.
$||A^{-1}||_1 = |(A^{-1})^T||_1 \Leftrightarrow ||A^{-1}||_1 = ||A^{-1}||_\infty$

$\therefore$ $cond(A) = cond(A^T)$ only when both the above conditions hold or if the conditions fail but the products become equal. It easy to choose any A such that these conditions are violated.

Let $A = \begin{pmatrix} 1 & 3 & 4 \\ 0 & 5 & -8 \\ 6 & 3 & 0 \end{pmatrix} \implies ||A||_1 = 12, ||A||_\infty = 13 \implies ||A||_1 \neq ||A||_\infty$.

```
A = [1 3 4; 0 5 -8; 6 3 0];
x = cond(A,1)
y = cond(transpose(A),1)
```

$$cond(A) = x = 5.100000000000001e + 00$$
$$cond(A^T) = y = 4.333333333333334e + 00$$
$$\implies cond(A) \neq cond(A^T)$$

$\square$

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Solutions to Problem 5 of Homework 3

Name: GOWTHAM GOLI (N17656180) *Due: Wednesday, October 21*

(a) **Solution:**

$\because A$ is 3 $\times 2$ matrix, $r(A) \leq 2$. But clearly $a_2 = 2a_1$. $\therefore r(A) \neq 2 \implies r(A) = 1$ □

(b) **Solution:**

$\mathbb{R}$(A) includes all the vectors of the form

$$A \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 2\lambda_1 + 4\lambda_2 \\ \lambda_1 + 2\lambda_2 \\ \lambda_1 + 2\lambda_2 \end{pmatrix} \in \mathbb{R}(\mathrm{A})$$

Let $\alpha = \lambda_1 + 2\lambda_2$ where α is a scalar then $\lambda = \begin{pmatrix} 2\alpha \\ \alpha \\ \alpha \end{pmatrix} \in \mathbb{R}(\mathrm{A})$ □

(c) **Solution:**

$\mathbb{N}(\mathrm{A}^\mathrm{T})$ has the dimension $m - r(A) = 3 - 1 = 2$. $\mathbb{R}(\mathrm{A}^\mathrm{T})$ contains all the vectors of the form z such that

$$A^T z = 0$$

$$\implies \begin{pmatrix} 2 & 1 & 1 \\ 4 & 2 & 2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = 0 \implies 2z_1 + z_2 + z_3 = 0$$

$$\implies z = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ -2\gamma_1 - \gamma_2 \end{pmatrix} \in \mathbb{N}(\mathrm{A}^\mathrm{T}) \text{ where } \gamma_1, \gamma_2 \text{ are any scalars}$$

Let $\gamma_1 = 1, \gamma_2 = 1 \implies z_1 = \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix}$

Let $\gamma_1 = 0, \gamma_2 = 1 \implies z_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

$$\lambda_1 z_1 + \lambda_2 z_2 = 0$$
$$\implies \lambda_1 = 0, \lambda_1 + \lambda_2 = 0, 3\lambda_1 + \lambda_2 = 0$$
$$\implies \lambda_1 = \lambda_2 = 0$$

$\therefore z_1, z_2$ are linearly independent □

(d) **Solution:** Let $b_R = \begin{pmatrix} 2\alpha \\ \alpha \\ \alpha \end{pmatrix}$, $b_N = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ -2\gamma_1 - \gamma_2 \end{pmatrix}$

$$\begin{aligned} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} &= \begin{pmatrix} 2\alpha \\ \alpha \\ \alpha \end{pmatrix} + \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ -2\gamma_1 - \gamma_2 \end{pmatrix} \\ \implies 2\alpha + \gamma_1 &= 3 \\ \alpha + \gamma_2 &= 2 \\ \alpha - 2\gamma_1 - \gamma_2 &= 1 \end{aligned}$$

Solving these three equations, we get $b_R = \begin{pmatrix} 3 \\ 3/2 \\ 3/2 \end{pmatrix}$ and $b_N = \begin{pmatrix} 0 \\ 1/2 \\ -1/2 \end{pmatrix}$ □

(e) **Solution:** $\mathbb{N}$(A) has dimension $n - r(A) = 1$. $\mathbb{N}$(A) contains all the vectors of the form q such that

$$\begin{aligned} Aq &= 0 \\ \implies \begin{pmatrix} 2 & 4 \\ 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} &= 0 \implies q_1 + 2q_2 = 0 \\ \implies q = \begin{pmatrix} -2q_2 \\ q_2 \end{pmatrix} &\in \mathbb{N}(\text{A}) \end{aligned}$$

□

(f) **Solution:**

$$\begin{aligned} Av &= b_R \\ \implies \begin{pmatrix} 2 & 4 \\ 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 3 \\ 3/2 \\ 3/2 \end{pmatrix} \implies v_1 + 2v_2 = 3/2 \\ \implies v &= \begin{pmatrix} 3/2 - 2v_2 \\ v_2 \end{pmatrix} \end{aligned}$$

Let $v_2 = 1/2 \implies v_1 = 1/2 \implies v = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$ □

(g) **Solution:**

- Let $v_2 = 1/2 \implies v_1 = 1/2 \implies v = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$

  Let $q_2 = 1 \implies q = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

  $\implies b_A = \begin{pmatrix} -3/2 \\ 3/2 \end{pmatrix}$

- Let $v_2 = 1 \implies v_1 = -1/2 \implies v = \begin{pmatrix} -1/2 \\ 1 \end{pmatrix}$

  Let $q_2 = -1 \implies q = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

  $\implies b_A = \begin{pmatrix} 3/2 \\ 0 \end{pmatrix}$

$\square$