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ECE498 :- 4 credit hours

HW0 :- Basic Probability

Problem 1

(a) **Axioms of Probability**

$S \equiv$ Sample Space

$P(A) \equiv$ Probability of event A.

$\phi \equiv$ Null set

A1 :- For any event A, $P(A) \geq 0$

A2 :- $P(S) = 1$

i.e Probability of sample space events as a whole is 1.

that implies, if an action occurs some event in S must occur

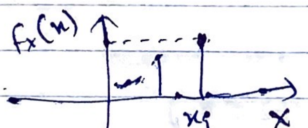
A3 :- $P(A \cup B) = P(A) + P(B)$, if A & B events are mutually exclusive i.e $P(A \cap B) = 0$ or $A \cap B = \phi$

A3 (General) :- for a countable sequence of events, $A_1, \ldots A_n$ if they are pairwise disjoint i.e $A_i \cap A_j = \phi$, $i \neq j$ $= A_i$, $i = j$

then $P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n)$

(b) Pmf = Probability mass function = $f(x)$

Discrete distributions of 'x'

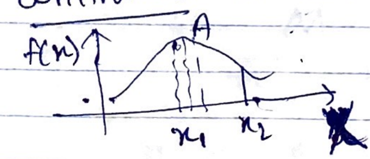


$P(a) = P\{x = a\}$ & $\sum_{i=1}^{\infty} P(x_i) = 1$

It is the probability that $x = x_i$ is true

Pdf = Probability density function = $f(x)$

Continuous 'x' random variable



$P(X \in A) = \int_A f(x)\,dx$; $P\{x = x_1\} = \int_{x_1}^{x_1} f(x)\,dx = 0$

Here, Pdf is a non-negative function defined for all real $x \in$ domain (X).

(c) $P(A) = 0.8$; $P(B) = 0.5$ also, since A, B are independent $P(A \cap B) = P(A) * P(B) = 0.8 \times 0.5 = 0.4$

$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.8 + 0.5 - 0.4 = \underline{\underline{0.9}}$

$\Rightarrow \boxed{P(A \cup B) = 0.9}$

To prove,

(d) $P(A, B \mid C) = P(A \mid B, C) \times P(B \mid C)$

applying conditional joint probabilities successively

Proof:- $P(A, B \mid C) = \dfrac{P(A, B, C)}{P(C)}$

$= \dfrac{P(A \mid B, C) P(B, C)}{P(C)}$

$= P(A \mid B, C) * \dfrac{P(B \mid C) \cancel{P(C)}}{\cancel{P(C)}}$

$\Rightarrow \boxed{P(A, B \mid C) = P(A \mid B, C) * P(B \mid C)}$

Proved

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Problem 4

Exponential distribution

(a) $f(x) = \lambda e^{-\lambda x}, \quad x > 0$

$= 0, \quad$ otherwise

(i) cdf = cumulative distribution function = $F(x)$

$$F(x) = \int_{-\infty}^{x} f(x)\,dx = \int_{0}^{x} \lambda e^{-\lambda x}\,dx$$

$$= \left[\cancel{\lambda}\frac{e^{-\lambda x}}{-\cancel{\lambda}}\right]_0^x$$

$$= \left[e^{-\lambda x}\right]_x^0$$

$$\Rightarrow \boxed{F(x) = 1 - e^{-\lambda x}} \text{ for } x > 0$$

(ii) Exponential distribution is said to be memoryless due to the nature of the function definition. The past history of the random variable does not play any role in the current state. (Conditional probability)

also called the Markov property.

ex: Machine running for 't' hours. Does it run for 's+t' hours.

$$P(x > s+t \mid x > t) = \frac{P(x > s+t \text{ and } x > t)}{P(x > t)} = \frac{P(x > s+t)}{P(x > t)}$$

checking if it runs for 's' hr's; history

$$= \frac{\lambda e^{-\lambda(s+t)}}{\lambda e^{-\lambda(t)}} = e^{-\lambda s} \rightarrow \text{Not a function of 't'.}$$

(iii) Mean of Exponential Distribution $= \mu = \dfrac{\int_{-\infty}^{\infty} x f(x)\,dx}{\int_{-\infty}^{\infty} f(x)\,dx} = E(x)$

$$= \frac{\int_0^{\infty} x \cancel{\lambda} e^{-\lambda x}\,dx}{\int_0^{\infty} \cancel{\lambda} e^{-\lambda x}\,dx}$$

$$= \frac{\left[\frac{x e^{-\lambda x}}{-\lambda} + \frac{1}{\lambda}\int e^{-\lambda x}\,dx\right]_0^{\infty}}{\left[\frac{e^{-\lambda x}}{-\lambda}\right]_0^{\infty}}$$

$$= \left[x e^{-\lambda x} + \frac{e^{-\lambda x}}{\lambda}\right]_{\infty}^{0}$$

$$\Rightarrow \boxed{\mu = \frac{1}{\lambda}}$$

Variance of Exponential distribution $= \sigma^2 = E(x^2) - [\underbrace{E(x)}_{\frac{1}{\lambda^2}}]^2$

$$E(x^2) = \frac{\int_{-\infty}^{\infty} x^2 f(x)\,dx}{\int_{-\infty}^{\infty} f(x)\,dx} = \int_0^{\infty} x^2 \lambda e^{-\lambda x}\,dx$$

$$= \lambda\left[x^2 \frac{e^{-\lambda x}}{-\lambda} + \frac{1}{\lambda}\int 2x e^{-\lambda x}\,dx\right]_0^{\infty}$$

$$= \lambda\left[\frac{-x^2 e^{-\lambda x}}{\lambda} + \frac{2}{\lambda}\left[\frac{x e^{-\lambda x}}{-\lambda} + \frac{(-1)}{\lambda^2} e^{-\lambda x}\right]\right]_0^{\infty}$$

$$= \left[0 + \frac{2}{\lambda}[0+0] - \left[0 + \frac{2}{\lambda}\left[0 + \frac{-1}{\lambda^2}\right]\right]\right]$$

$$= \frac{2}{\lambda^2}$$

$$\Rightarrow \sigma^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} \Rightarrow \boxed{\sigma^2 = \frac{1}{\lambda^2}} \text{ Variance}$$

Problem 4 (continued)

Poisson distribution

(from Binomial distribution)

(b)

general binomial dist. $\equiv$ $P(X=r) = {}^nC_r \, p^r (1-p)^{n-r}$ — (1)

where 'p' is the 0/1 probability of unary event.
↳ Bernoulli random variable.

& ${}^nC_r = \frac{n!}{(n-r)!\,r!}$, n = number of trails

general Poisson dist. $\equiv$ $P(X=i) = \frac{e^{-\lambda}\lambda^i}{i!}$ — (2)

derivation :- If we close observe (1), we can expand it as

$$P(X=r) = \frac{n!}{(n-r)!\,r!} p^r (1-p)^{n-r}$$

Here, assumptions are

$$\boxed{\begin{array}{ll} n \to \infty & — (3) \\ \lambda = np \ \& \ p \to 0 & — (4) \end{array}}$$

"infinite trials"

"constant number of successes"

incorporating (3), (4) in (1), we get

$$P(X=r) = \lim_{n\to\infty} \frac{n!}{(n-r)!\,r!} \left(\frac{\lambda}{n}\right)^r \left(1-\frac{\lambda}{n}\right)^{n-r}$$

$$= \frac{\lambda^r}{r!} \lim_{n\to\infty} \frac{n!}{(n-r)!\,n^r} \left(1-\frac{\lambda}{n}\right)^{n-r}$$

$[n\to\infty \Rightarrow n-r\to\infty]$

$\Rightarrow$

$$= \frac{\lambda^r}{r!} \lim_{n\to\infty} \frac{n(n-1)(n-2)\cdots(n-r+1)\cancel{(n-r)!}}{\cancel{(n-r)!}\, n^r} \times \left(1-\frac{\lambda}{n}\right)^{n-r}$$

$$= \frac{\lambda^r}{r!} \lim_{n\to\infty} \left[1 \times \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{r-1}{n}\right)\right] \times \left(1-\frac{\lambda}{n}\right)^{n} \Big/ \left(1-\frac{\lambda}{n}\right)^{r}$$

$$= \frac{\lambda^r}{r!} \lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^{n}$$

$$\boxed{P(x=r) = \frac{\lambda^r}{r!} e^{-\lambda}}$$

because

$$e^{x} = \lim_{n\to\infty} \left(1+\frac{x}{n}\right)^{n}$$

$\therefore$ product rule of limits is applied.

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Marginal/Joint Distribution

Problem 5

$$f(x, y)_{X,Y} = \begin{cases} 10\, e^{-(2x+5y)} & x \geq 0, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

(a) Marginal distribution of $X = f_X(x) = \int_Y f_{X,Y}(x,y)\, dy$

$$\Rightarrow f_X(x) = \int_{y=0}^{\infty} 10 e^{-(2x+5y)}\, dy$$

$$= 10 e^{-2x} \int_{y=0}^{\infty} e^{-5y}\, dy$$

$$\boxed{f_X(x) = 2e^{-2x}} \quad , y \geq 0$$

$$= 0 \quad , y < 0$$

(b) Marginal distribution of $Y = f_Y(y) = \int_X f_{X,Y}(x,y)\, dx$

$$\Rightarrow f_Y(y) = \int_{x=0}^{\infty} 10\, e^{-(2x+5y)}\, dy$$

$$\Rightarrow \boxed{f_Y(y) = 5 e^{-5y}} \quad , x \geq 0$$

$$= 0 \quad , x < 0$$

(c) Here $F(x,y) = F_X(x) \times f_Y(y)$ for $x, y \geq 0$ and otherwise

$$10\, e^{-(2x+5y)} = 2e^{-2x} \times 5e^{-5y}$$

Yes, they are independent!

$$f_{Y/X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{10e^{-(2x+5y)}}{2e^{-2x}} = 5e^{-5y}, \quad x \leq 0$$

5(d) They are independent hence $f_{Y/X}(y|x) = f_Y(y)$

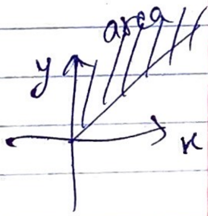
(i) It is well defined for all $x \geq 0$

(ii) It is zero when $y \to +\infty$

$$f_Y(y) = \begin{cases} 5e^{-5y}, & x \geq 0,\ y \geq 0 \\ 0, & y < 0 \end{cases}$$

& $x < 0$, it is undefined

5(e)



$$P(y > x) = \int_0^\infty \left(\int_x^\infty 10 e^{-2x-5y}\, dy \right) dx$$

$$= \int_0^\infty 10 e^{-2x} \left[\frac{e^{-5y}}{-5} \right]_x^\infty dx$$

$$= 2 \int_0^\infty e^{-2x} \left[0 + e^{-5x} \right] dx$$

$$= -\frac{2}{7} \left[e^{-7x} \right]_0^\infty$$

$$= \frac{2}{7}$$

$$= \underline{\underline{0.285}}$$

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Co-Variance and correlation Coefficient

Problem 7

(a) $Var(X+Y) = 7$, $Var(2X-2Y) = 12 \Rightarrow Var[X-Y] = 3$

we Know $Cov(X,Y) = E[XY] - E[X]E[Y]$

& $Var(X) = E[X^2] - [E[X]]^2$

$Var(x+y) = E[(x+y)^2] - (E(x+y))^2$

$= E[x^2+y^2+2xy] - [E(x)+E(y)]^2$

$= E[x^2] + E[y^2] + 2E[xy] - (E(x))^2 - (E(y))^2 - 2E(x)E(y)$

$= E(x^2) - (E(x))^2 + E(y^2) - (E(y))^2 + 2[E(x,y) - E(x)E(y)]$

$\Rightarrow \boxed{Var(X+Y) = Var(X) + Var(Y) + 2\,Cov(X,Y)}$ — (1)

Similarly $\boxed{Var(X-Y) = Var(x) + Var(y) - 2\,Cov(X,Y)}$ (2)

(i) Covariance of X, Y

(1) - (2) $= 4\,Cov(X,Y)$

$\Rightarrow 4\,Cov(X,Y) = Var(X+Y) - Var(X-Y)$

$\Rightarrow Cov(X,Y) = \frac{1}{4}(7-3) = 1$

$\Rightarrow \boxed{Cov(X,Y) = 1}$

(ii) Given $Var(X) = 1$ and $Cov(X,Y) = 1$

we can find $Var(Y)$ from ①

$$\Rightarrow Var(Y) = Var(X+Y) - Var(X) - 2\,Cov(X,Y)$$

$$= 7 - 1 - 2$$

$$\Rightarrow \boxed{Var(Y) = 4}$$

Correlation coefficient $\rho_{X,Y} = \dfrac{Cov(X,Y)}{\sqrt{Var(X)\,Var(Y)}}$

$$= \frac{1}{\sqrt{4 \times 1}}$$

$$= \frac{1}{2}$$

$$\Rightarrow \boxed{\rho_{X,Y} = 1/2}$$

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see Part (b) in next Page

Problem 7

Part (b)

① $X_1, X_2, \ldots X_{10}$ are uncorrelated.

so, $\text{cov}(X_i, X_j) = 0 \quad \forall\ i,j = \{1,2,\ldots 10\}$
$\quad \quad \& \ i \neq j$

② also $E[X_i] = i$ and $\text{Var}(X_i) = 5 \quad \forall\ i \in \{1,2,\ldots 10\}$

To find $\text{Var}\left(\frac{S_{10}}{\sqrt{10}}\right)$ where $S_{10} = X_1 + X_2 \cdots + X_{10}$

Let's try to generalize the result: $\text{var}\left(\frac{S_n}{\sqrt{n}}\right) = \frac{f_n}{n}$

$\Rightarrow \text{Var}\left(\frac{S_n}{\sqrt{n}}\right) = \text{Var}\left(\frac{X_n + X_{n-1} + X_{n-2} + \cdots + X_1}{\sqrt{n}}\right) = \frac{1}{n}\text{Var}(X_n + \cdots X_1)$

$$= \frac{1}{n}\left[\text{Var}(X_n) + \text{Var}(X_{n-1}, \ldots X_1) + 2\,\text{cov}(X_n, X_{n-1} + X_{n-2} + \cdots X_1)\right]$$

$$= \frac{1}{n}\left[\text{var}(X_n) + f_{n-1} + \underbrace{2\,\text{cov}(X_n, X_{n-1} + X_{n-2} + \cdots X_1)}_{0}\right]$$

$\Rightarrow \frac{f_n}{\cancel{n}} = \frac{1}{\cancel{n}}\left[\text{var}(X_n) + f_{n-1}\right]$:- recurrence relation (rather straigh forward)

$f_1 = \text{Var}(X_1) = 5$

$f_2 = \cancel{1}\left[\text{Var}(X_2) + f_1\right] = 10$

$\Rightarrow \quad f_n/n = \frac{1}{n}\left[\sum_{i=1}^{10} \text{Var}(X_i)\right] \Rightarrow \boxed{\text{Var}\left(\frac{S_{10}}{\sqrt{10}}\right) = 5}$

Problem 9 — Central Limit Theorem

$n = 400$ independent components

$P(Y_i = 1) = 0.98 = p$; $Y_i = 1$ indicated i^{th} component is working properly } Bernoulli random Variable

1. X = number of properly functioning components

Here we can choose 'i' working components out of 'n' Components

each component works independently of others.

so, $$\boxed{P(X=i) = {}^{n}C_i \; p^i \, (1-p)^{n-i}}$$

${}^{n}C_i = \binom{n}{i}$ Combinations; p^i: properly functioning; $(1-p)^{n-i}$: not properly functioning; n, p: Parameters

It is a Binomial distribution with $n = 400$, $p = 0.98$

2. Next page

Problem 9

(2)

It actually is a binomial distribution,
→ This is to approximate & make things easy (computation)

To use central limit theorem, let's define

$$X = \sum_{i=1}^{400} X_i$$

where X_i represents component i.

$$X_i = \begin{cases} 1 & \text{, component 'i' works properly} \\ 0 & \text{, component 'i' doesn't work} \end{cases}$$

each X_i follows a Bernoulli distribution with mean, $\mu = 0.98$ and $\sigma^2 = P(1-P) = 0.0196$

Assuming $N = 400$, is sufficiently large, we can approximate modified(X) to be following a Normal distribution $N(0,1)$.

$$X' = \text{modified}(X) = \frac{X_1 + X_2 + \cdots + X_{400} - \mu * 400}{\sigma\sqrt{400}}$$

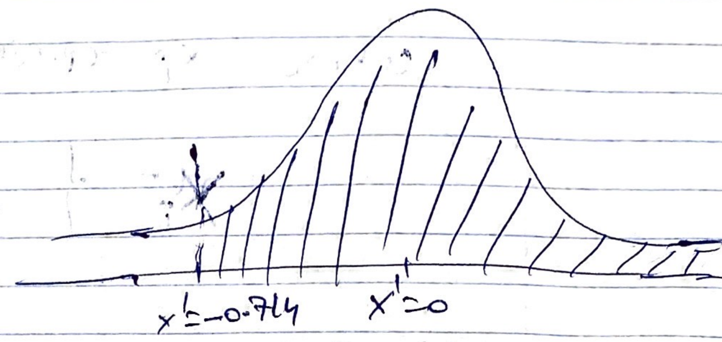
$$= \frac{\dfrac{\sum_{i=1}^{400} X_i}{400} - \mu}{\dfrac{\sigma}{\sqrt{400}}}$$

and $P(X' \leq a) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a} e^{-x^2/2}\,dx$ → use z-tables to get values

In our case we want atleast 390 of the components to work so,

$$X \geq 390$$

$$\Rightarrow X' = \frac{\frac{390}{400} - 0.98}{\frac{\sqrt{0.0196}}{20}} = -0.714$$



$$P(X' \geq -0.714) = 1 - P(X < -0.714)$$

$$= 1 - 0.2389$$

Probability that car functions (when ≥ 390 components work) $= \boxed{0.7611}$ answer

Aside:

wrote a simple program to calculate it { If we use binomial distribution, answer is 0.818

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Bayes Theorem and Conditional Probabilities

Problem 10:-

M ≡ Malfunctions did occur ; M=0 & M=1
(M=0 → false ; M=1 → True)

D ≡ Disengagement event

given,

$P(D \mid M=1) = 0.85$; $P(M=1) = 0.0002$

$P(D \mid M=0) = 0.002$; $P(M=0) = 1 - P(M=1) = 0.9998$

to find,

$P(M=1 \mid D)$: i.e when a disengagement occured, it is due to malfunction (happening = True)

apply Bayes' theorem

$$P(M=1 \mid D) = \frac{P(D/M=1)\,P(M=1)}{P(D/M=1)\,P(M=1) + P(D/M=0) + P(M=0)}$$

$$= \frac{0.85 \times 0.0002}{0.85 \times 0.0002 + 0.002 \times 0.9998}$$

$$= \frac{1}{1 + \frac{0.002}{0.85} \times \frac{9998}{2}}$$

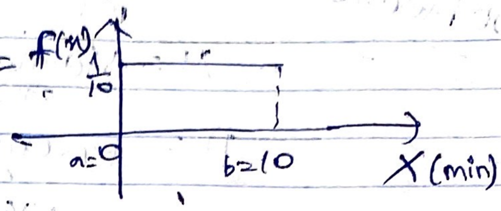
$$= \frac{1}{1 + \frac{9.998}{0.85}}$$

$$\boxed{P(M=1 \mid D) = 0.078}$$

Problem 12 — Uniform Distribution

X, Y f uniform distribution $[0, 10]$ & continuous

X, Y are independent

pdf = 

$Cov(X, Y) = 0$

$\Rightarrow E(XY) = E(X)E(Y)$

$P(x \in A) = \int_A \frac{1}{10} dx$

similarly Y exists

(a) $P(X = Y) = 0$ because this is a continuous distribution

$Mean(X) = \frac{b-a}{2} = \frac{10}{2} = 5$; $Var(X) = \frac{(b-a)^2}{12} = \frac{10^2}{12} = \frac{25}{3}$

$Mean(Y) = 5$; $Var(Y) = \frac{25}{3}$

$E(X-Y) = E(X) - E(Y) = 5 - 5 = 0$

$Var(X-Y) = Var(X) + Var(Y) - 2\cancel{Cov(X,Y)}^{\,0} = \frac{50}{3}$

(b) ~~Box A arrives earlier than box B~~

$f(X, Y) = f(x) * f(y)$ because X, Y are independent

Joint Probability distribution,

$$P(X = Y) = \int_0^{10} \left(\int_y^y f(x, y) \, dx \right) dy$$

$$= \int_0^{10} 0 \, dy$$

$\boxed{P(X = Y) = 0}$ (answer)

$P(\text{A arrives earlier than B}) = P(X \leq Y)$

(b) $P(X \leq Y) = \int_0^{10} \left(\int_0^{y} f(x,y)\, dx \right) dy$

$= \int_0^{10} \left(\int_0^{y} f(x) f(y)\, dx \right) dy$

$= \int_0^{10} \left(\int_0^{y} \frac{1}{100}\, dx \right) dy$

$= \frac{1}{100} \int_0^{10} y\, dy$

$= \frac{1}{\cancel{100}} \times \frac{\cancel{100}}{2}$

$\boxed{P(X \leq Y) = \frac{1}{2}}$ $\boxed{\text{answer}}$

$= 0.5$

(c) $Z = \max(X, Y)$

$\Rightarrow \quad Z = \begin{cases} X, & \text{if } X \geq Y \\ Y, & \text{if } Y \geq X \end{cases}$

CDF of each bus is given by $F(X)$, $F(Y)$

$F(x) = \frac{x}{10}$, $F(Y) = \frac{y}{10}$

$x \in [0, 10]$ $\quad y \in [0, 10]$

To get pdf get CDF and perform a derivative.

CDF of 'Z' is as follows.

$G(Z) = P(Z \le z)$ where $z \in [0, 10]$ ← time variable

$= P(\max(X, Y) \le z)$

$= P(X \le z \text{ and } Y \le z)$

$= P(X \le z) \times P(Z \le z)$

$= \left(\frac{z}{10}\right) * \left(\frac{z}{10}\right)$

$= \frac{z^2}{100}$

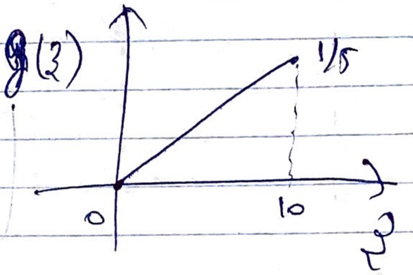
so $g(z) = \text{pdf of } Z = \frac{d}{dz} G(z) = \frac{d}{dz}\left(\frac{z^2}{100}\right) = \frac{z}{50}$ for $z \in [0, 10]$

$E(Z) = \int_0^{10} z \, g(z) \, dz$

$= \int_0^{10} \frac{z^2}{50} dz$

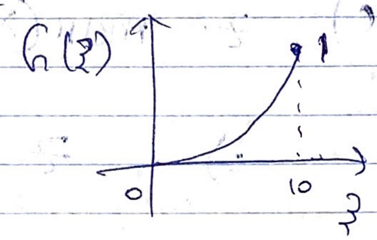
$= \frac{1}{\cancel{50}} \times \frac{1}{3} \times \overset{20}{\cancel{1000}}$

pdf



$\boxed{E(Z) = \frac{20}{3} \text{ min}}$

CDF



$\boxed{\text{where } Z = \max(X, Y)}$

(d) Now, $z = \min(x, y)$

following similar method as last time.

Define CDF of z and then get Pdf by derivating

$$CDF(z) = G(z) = G(z \le z) \; ; \; z \in [0, 10)$$

$$= 1 - G(z \ge z)$$

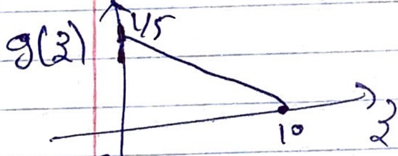
$$= 1 - G(\min(x, y) \ge z)$$

$$= 1 - [G(x \ge z) \times G(y \ge z)]$$

$$= 1 - \left[\left(1 - \frac{z}{10}\right) \times \left(1 - \frac{z}{10}\right)\right]$$

$$CDF = G(z) = 1 - \left(1 - \frac{z}{10}\right)^2$$

Now $Pdf(z) = g(z) = \frac{d}{dz} G(z) = \frac{d}{dz}\left[1 - \left(1 - \frac{z}{10}\right)^2\right]$



$$\boxed{g(z) = \frac{1}{5}\left(1 - \frac{z}{10}\right), z \in [0, 10]}$$

Expectation value

$$E(z) = \int_0^{10} z \, g(z) \, dz$$

$$= \int_0^{10} \frac{1}{5}\left(z - \frac{z^2}{10}\right) dz$$

$$= \frac{1}{5}\left[\frac{z^2}{2} - \frac{z^3}{30}\right]_0^{10} = \frac{100}{150}[15 - 10] = \frac{10}{3}$$

where $z = \min(x, y)$ $\boxed{E(z) = \frac{10}{3}, z \in [0, 10]}$

(e) Both bus A and B with waith for 5 minutes at the bus stop.

$$P(A, B \text{ are together}) = 1 - P(A, B \text{ are not together})$$

$$= 1 - \Big[P\big(X \in [0,5] \text{ and } Y \in [X+5, 10]\big) + P\big(X \in [Y+5, 10] \text{ and } Y \in [0,5]\big) \Big]$$

$$= 1 - 2 \times P\big(X \in [0,5] \text{ and } Y \in [X+5, 10]\big)$$

$$= 1 - 2 \int_0^5 \left(\int_{x+5}^{10} f(x,y)\, dy \right) dx$$

$$= 1 - 2 \times \int_0^5 \left[\frac{1}{100} \times (10 - x - 5) \right] dx$$

$$= 1 - 2 \times \frac{1}{100} \left[5x - \frac{x^2}{2} \right]_0^5$$

$$= 1 - \frac{1}{50} \times 5 \times \frac{5}{2}$$

$$= 1 - \frac{1}{4}$$

$$\Rightarrow \boxed{P(A, B \text{ are together}) = \frac{3}{4}}$$