Inequalities PSet

EGMOTC 2023 - Rohan

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Problems

Remark. ** marked problems are considered harder.*
*** marked problems are strictly optional for the ones feeling extremely curious about this particular setup.*

Problem 1. Watch the first video about the AM-GM inequality. Based on the video, write two different proofs of AM-GM inequality in your own words. (the video alludes to 6-7 different proofs)

Problem 2. (Rearrangement Inequality) Prove the rearrangement inequality: Let $a_1 < a_2 < \ldots < a_n$ and $b_1 < b_2 < \ldots < b_n$ be real numbers. Prove that for any permutation σ of $\{1, 2, \ldots n\}$, we have:

$$a_1b_1 + a_2b_2 + \ldots a_nb_n \geq a_1b_{\sigma(1)} + a_2b_{\sigma(2)} + \ldots a_nb_{\sigma(n)}$$

Problem 3. (INMO 2020) Let $n \geqslant 2$ be an integer and let $1 < a_1 \leq a_2 \leq \cdots \leq a_n$ be n real numbers such that $a_1 + a_2 + \cdots + a_n = 2n$. Prove that

$$a_1a_2 \ldots a_{n-1} + a_1a_2 \ldots a_{n-2} + \cdots + a_1a_2 + a_1 + 2 \leq a_1a_2 \ldots a_n$$

Problem 4. (ISL 2001 A3) Let $x_1, x_2, \ldots, x_n$ be arbitrary real numbers. Prove the inequality

$$\frac{x_1}{1 + x_1^2} + \frac{x_2}{1 + x_1^2 + x_2^2} + \cdots + \frac{x_n}{1 + x_1^2 + \cdots + x_n^2} < \sqrt{n}.$$