Size and Guesstimation: Olympiad

EGMOTC 2023 - Rohan

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Problems

Remark. ** marked problems are considered harder.*
*** marked problems are strictly optional and harder than the rest.*
Remark. Try to do the first three problems atleast.

Problem 1. Let n, m be integers greater than 1, and let $a_1, a_2, \dots, a_m$ be positive integers not greater than n^m. Prove that there exist positive integers $b_1, b_2, \dots, b_m$ not greater than n, such that

$$\gcd(a_1 + b_1, a_2 + b_2, \dots, a_m + b_m) < n,$$

where $\gcd(x_1, x_2, \dots, x_m)$ denotes the greatest common divisor of $x_1, x_2, \dots, x_m$.
Problem 2. Suppose $q_0,\ q_1,\ q_2, \ldots$ is an infinite sequence of integers satisfying the following two conditions:

- $m - n$ divides $q_m - q_n$ for $m > n \geq 0$,
- there is a polynomial P such that $|q_n| < P(n)$ for all n

Prove that there is a polynomial Q such that $q_n = Q(n)$ for all n.
Problem 3. Does there exist a nonnegative integer a for which the equation

$$\left\lfloor \frac{m}{1} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{m}{3} \right\rfloor + \cdots + \left\lfloor \frac{m}{m} \right\rfloor = n^2 + a$$

has more than one million different solutions (m, n) where m and n are positive integers?
Problem 4.* Let $m \geq 2$ be an integer, A a finite set of integers (not necessarily positive) and $B_1, B_2, ..., B_m$ subsets of A. Suppose that, for every $k = 1, 2, ..., m$, the sum of the elements of B_k is m^k. Prove that A contains at least $\dfrac{m}{2}$ elements.
Problem 5.** For a positive integer n we denote by $s(n)$ the sum of the digits of n. Let $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ be a polynomial, where $n \geqslant 2$ and a_i

is a positive integer for all $0 \leqslant i \leqslant n-1$. Could it be the case that, for all positive integers k, $s(k)$ and $s(P(k))$ have the same parity?

Problem 6.** Let p be an odd prime, and put $N = \frac{1}{4}(p^3 - p) - 1$. The numbers $1, 2, \ldots, N$ are painted arbitrarily in two colors, red and blue. For any positive integer $n \leqslant N$, denote $r(n)$ the fraction of integers $\{1, 2, \ldots, n\}$ that are red. Prove that there exists a positive integer $a \in \{1, 2, \ldots, p-1\}$ such that $r(n) \neq a/p$ for all $n = 1, 2, \ldots, N$.