Constructions

EGMOTC 2023 - Rohan

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Problems

Problem 1

ISL 1995: Let a and b be non-negative integers such that $ab \geq c^2$, where c is an integer. Prove that there is a number n and integers $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ such that

$$\sum_{i=1}^{n} x_i^2 = a, \sum_{i=1}^{n} y_i^2 = b, \text{ and } \sum_{i=1}^{n} x_i y_i = c$$

Problem 2

ELMO: Sahil chooses a functional expression* E which is a finite nonempty string formed from a set $x_1, x_2, \ldots$ of variables and applications of a function f, together with addition, subtraction, multiplication (but not division), and fixed real constants. He then considers the equation $E = 0$, and lets S denote the set of functions $f \colon \mathbb{R} \to \mathbb{R}$ such that the equation holds for any choices of real numbers $x_1, x_2, \ldots$. (For example, if Sahil chooses the functional equation

$$f(2f(x_1) + x_2) - 2f(x_1) - x_2 = 0,$$

then S consists of one function, the identity function.

1. Let X denote the set of functions with domain $\mathbb{R}$ and image exactly $\mathbb{Z}$. Show that Sahil can choose his functional equation such that S is nonempty but $S \subseteq X$.

2. Can Sahil choose his functional equation such that $|S| = 1$ and $S \subseteq X$?

*These can be defined formally in the following way: the set of functional expressions is the minimal one (by inclusion) such that (i) any fixed real constant is a functional expression, (ii) for any positive integer i, the variable x_i is a functional expression, and (iii) if V and W are functional expressions, then so are $f(V)$, $V + W$, $V - W$, and $V \cdot W$.