EGMO TST Day 1

Date: 29 December 2023

Instructions:

i) You have 4 hours and 30 minutes for three problems.

ii) Each problem is worth 10 points. Attempt all three.

iii) Any claim you make must be accompanied by a proper justification.

Rubric P2

Problem and Solution

Problem 2.

Given that $a_1, a_2, \ldots, a_{10}$ are positive real numbers, determine the smallest possible value of

$$\sum_{i=1}^{10} \left\lfloor \frac{7a_i}{a_i + a_{i+1}} \right\rfloor$$

where we define $a_{11} = a_1$.

Sutanay Bhattacharya

Solution. We claim the minimum is 6. This is attained when $a_i = 7^i$ for $i \in \{1, \ldots, 10\}$.

Now to prove this is the minimum, make the substitution $a_{i+1}/a_i = x_i$. The problem becomes: for positive reals $x_1, \ldots, x_{10}$ with $x_1 x_2 \cdots x_{10} = 1$. Prove that

$$\sum_{i=0}^{10} \left\lfloor \frac{7}{1 + x_i} \right\rfloor \geq 6.$$

Let $z_i = \left\lfloor \frac{7}{1+x_i} \right\rfloor$. If at most 4 of the z_i's are zero, then at least 6 of them are ≥ 1, and thus the sum is at least 6.
If not, then at least five of them are 0. Suppose $z_1 = \cdots = z_5 = 0$ without loss of generality. Note that $z_i = 0$ implies

$$\frac{7}{1 + x_i} < 1 \implies x_i > 6,$$

so that $x_1, \ldots, x_5 > 6$. Thus $x_1 x_2 x_3 x_4 x_5 > 6^5$, which means $x_6 x_7 x_8 x_9 x_{10} < 1/6^5$. Assuming WLOG $x_6 = \min\{x_6, x_7, x_8, x_9, x_{10}\}$, this implies $x_6 < 1/6$. However, then we have

$$z_6 \geq \left\lfloor \frac{7}{1 + \frac{1}{6}} \right\rfloor = 6,$$

and the conclusion follows. $\square$

Rubric

0+

Upper Bound. (3)

(A) **+1:** Guessing the answer is 6

- The conjecture should be explicitly stated. Something like being boxed as a number is rough work is not considered sufficient.

(B) **+2:** Correct construction achieving 6

- **+1:** If construction is not explicitly written saying that it achieves 6 but the sequence $1, 7, 7^2, \cdots$ is mentioned upto atleast the square.

Lower Bound. (7)

(A) **+2:** Shifting to working with $\frac{a_{i+1}}{a_i}$ or x_i.

- The full two points can be awarded if only the expression is rewritten as $\sum_{i=0}^{10} \left\lfloor \frac{7}{1+x_i} \right\rfloor$ or it is clear that only the x_i are being considered.
- 1 point can still be awarded if things like $\frac{a_{i+1}}{a_i} > 6 \implies z_i = 0$ and equivalent things are written. It is not considered to be the same as writing $a_{i+1} > 6a_i$. It should be clear that the ratio is being considered.

(B) **+1:** Concluding if at most 4 z_i are 0.

(C) **+1:** Showing that $z_i = 0 \iff x_i > 6$.

(D) **+1:** Showing that atleast 1 x_i is less than $\frac{1}{6}$ if 5 z_is are 0.

(E) **+2:** Concluding that the total sum is atleast 6 then.

All points are considered additive.

10-

A solution with both the upper and lower bound is considered complete. 1 mark can be deducted for minor errors which are easily fixable. Typos and such should not lead to deductions.