Intro to Inequalities!

Part 1 : AM - GM

AM-GM Inequality

For any reals $x_1, x_2, \ldots, x_n \geq 0$, we have:

$$\underbrace{\frac{x_1 + x_2 + \ldots + x_n}{n}}_{\text{AM}} \geq \underbrace{\sqrt[n]{x_1 x_2 \ldots x_n}}_{\text{GM}}$$

Idea (Strange Induct) $n=2$, we want $\frac{a+b}{2} \geqslant \sqrt{ab}$ or

$$(a+b)^2 \geqslant 4ab$$

i.e. $$(a-b)^2 \geqslant 0. \quad \blacksquare$$

Now, $n=3$

$\frac{a+b+c}{3} \geqslant \sqrt[3]{abc}$, already no direct way to induct

$$(a+b+c)^3 \geq 27abc$$

$$\underbrace{a^3+b^3+c^3} + 3(\underbrace{a^2b+b^2a+c^2b}+\underbrace{c^2a+a^2c+b^2c}) \geq 21abc$$

$$\downarrow$$

$$3(3abc + 3abc)$$

$$\geq 3abc \qquad + \qquad \geq 18abc$$

$n=4$

$$\frac{a+b+c+d}{4} \geqslant \sqrt[4]{abcd}$$

$$\Updownarrow$$

$$\frac{\frac{a+b}{2} + \frac{c+d}{2}}{2} \geqslant \sqrt{\sqrt{ab}\cdot\sqrt{cd}}$$

Looks so much better!

Step 1

$$\frac{a+b}{2} \geqslant \sqrt{ab}$$

Step 2

$$\frac{c+d}{2} \geqslant \sqrt{cd}$$

Step 3

$$\frac{\sqrt{ab}+\sqrt{cd}}{2} \geqslant \sqrt[4]{abcd}$$

Write the argument for n power of 2

Status: We can prove the result for $n = 2, 4, 8, 16, \ldots$

Question: Shouldn't $n = 4$ be harder than $n = 3$??

$$\underline{n=4} \implies \frac{a+b+c+\frac{a+b+c}{3}}{4} \geq \sqrt[4]{abc \cdot \left(\frac{a+b+c}{3}\right)}$$

$$\implies \frac{a+b+c}{3} \geq \sqrt[4]{abc\left(\frac{a+b+c}{3}\right)} \implies \frac{a+b+c}{3} \geq \sqrt[3]{abc}$$

Backward induct!!

IDEA 2

(I really really wanna induct)

For convenience, let's say $a+b+c=3$

↳ Why can we assume this?

Now, we want

$$1 \geq abc$$

Now, let's say $a \geq 1$ & $b \leq 1$ then

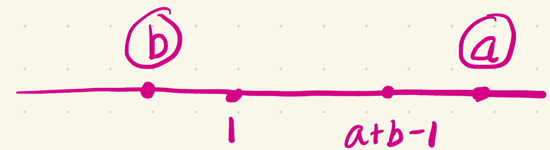
$$a-1 \geq 0, \quad 1-b \geq 0 \Rightarrow (a-1)(1-b) \geq 0$$

$$\Rightarrow a+b-1 \geq ab$$

Now, we know $c + a + b - 1 = 2$

$$\Rightarrow 1 \geq c(a+b-1) \Rightarrow 1 \geq abc$$

Thus, we are done!



The same proof idea can also be done without

$$\frac{a+b+c}{3} = 1$$

Idea 3
(Smoothing)

We know that if we fix $a+b$ then ab is maximized when $a=b$!

$x_1+x_2+\dots+x_n$ is fixed & $x_1x_2\dots x_n$ is maximized then $x_1=x_2=\dots=x_n$?

Proved? Why does maximum need to exist?

'Compactness so max exists'

Avoiding using maximas

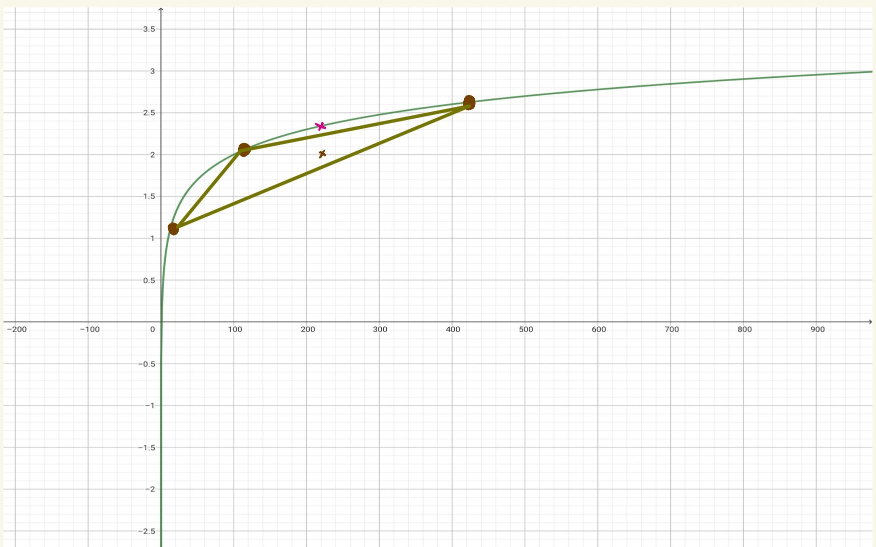
↳ Replace a number with the average!

Idea 4

(Let's graph it!)

We take log on both sides for cleaner picture:

$$\log\left(\frac{x_1+x_2+\dots+x_n}{n}\right) \geq \frac{\log(x_1)+\log(x_2)+\dots+\log(x_n)}{n}$$



$\log(x)$ is a concave function!

↳ also gives induct idea!

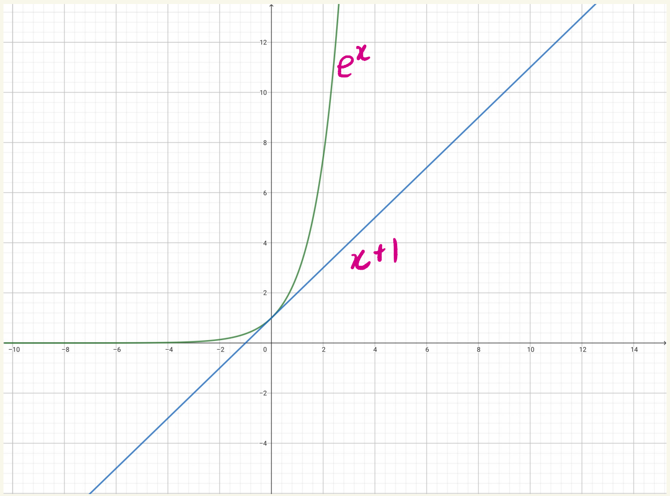
Idea 5

(Magic)

Let $A = \frac{x_1 + x_2 + \dots + x_n}{n}$, $G = \sqrt[n]{x_1 x_2 \cdots x_n}$

$$e^{x-1} \geq x$$

$$e^x \geq x+1 \quad \forall x \in \mathbb{R}.$$



$$\Rightarrow \left.\begin{aligned} e^{\frac{x_1}{A}-1} &\geq \frac{x_1}{A} \\ &\times \\ e^{x_2/A-1} &\geq \frac{x_2}{A} \\ &\times \\ &\vdots \\ &\times \\ e^{x_n/A-1} &\geq \frac{x_n}{A} \end{aligned}\right\} \Rightarrow e^{\frac{x_1}{A}+\frac{x_2}{A}+\dots+\frac{x_n}{A}-n} \geq \frac{x_1 x_2 \cdots x_n}{A^n}$$

$$\Rightarrow e^{\frac{nA}{A}-n} \geq \frac{G^n}{A^n}$$

$$\Rightarrow A^n \geq G^n$$