Part 3

Example Problems

Problem 1

Let $a_1 < a_2 < a_3 < \dots < a_n$ be positive integers

Then

$$\sum_{i=1}^{n-1} \frac{1}{\operatorname{lcm}(a_i, a_{i+1})} < 1$$

- $$\sum_{i=1}^{n-1} \frac{1}{\operatorname{lcm}(a_i, a_{i+1})} < 1 \quad \Bigg| \quad \sum_{i=1}^{n-1} \frac{\gcd(a_i, a_{i+1})}{a_i\, a_{i+1}} < 1$$

$$\hookrightarrow \operatorname{lcm}(a_i, a_{i+1}) = \frac{a_i \cdot a_{i+1}}{\gcd(a_i, a_{i+1})}$$

$$\Rightarrow \frac{1}{\operatorname{lcm}(a_i, a_{i+1})} = \frac{\gcd(a_i, a_{i+1})}{a_i \cdot a_{i+1}}$$

$$\underline{\gcd(a_i, a_{i+1})} \leq \underline{a_{i+1} - a_i} \, !$$

$$\Rightarrow \frac{1}{\operatorname{lcm}(a_i, a_{i+1})} \leq \frac{\underline{a_{i+1} - a_i}}{a_i a_{i+1}} = \boxed{\frac{1}{a_i} - \frac{1}{a_{i+1}}}$$

$$\Rightarrow \sum_{i=1}^{n} \frac{1}{\operatorname{lcm}(a_i, a_{i+1})} \leq \sum_{i=1}^{n} \frac{1}{a_i} - \frac{1}{a_{i+1}}$$

$$= \boxed{\frac{1}{a_1}} - \frac{1}{a_n} < 1 \, .$$

Problem 2

Let $\underset{\in \mathbb{R}^+}{\underline{a+b+c}} = 1$. Prove that:

$$\sqrt{a + \frac{(b-c)^2}{4}} + \sqrt{b} + \sqrt{c} \leq \sqrt{3}$$

$$\Downarrow$$

$$\sqrt{a} + \sqrt{b} + \sqrt{c} \leq \sqrt{3}$$

Can't directly apply AM-GM type ineq.

We ignore the $\frac{(b-c)^2}{4}$ term for now.

then

$$\sqrt{a}+\sqrt{b}+\sqrt{c} \leqslant \sqrt{3}$$

but close to equality when $b \sim c$.

So we make them close!

$$\sqrt{a+\frac{(b-c)^2}{4}}+\sqrt{b}+\sqrt{c}=\sqrt{a+\frac{(b-c)^2}{4}}+\left(\frac{\sqrt{b}+\sqrt{c}}{2}\right)+\left(\frac{\sqrt{b}+\sqrt{c}}{2}\right)$$

So,

$$\frac{\sqrt{a+\frac{(b-c)^2}{4}}+\sqrt{b}+\sqrt{c}}{3}\leqslant\sqrt{\frac{a+\frac{(b-c)^2}{4}+\frac{b+c+2\sqrt{bc}}{2}}{3}}$$

$$\boxed{\sqrt{\frac{a^2+b^2+c^2}{3}}\geqslant\frac{a+b+c}{3}}$$ ←

Just need $\underbrace{a + \frac{b+c}{2} + \sqrt{bc} + \frac{(b-c)^2}{4} \leq 1}$

i.e $\quad a + \frac{b+c}{2} + \sqrt{bc} + \frac{(b-c)^2}{4} \leq \cancel{a}+b+c$

$$\iff \sqrt{bc} + \frac{(b-c)^2}{4} \leq \frac{b+c}{2}$$

$$\iff \boxed{\frac{(b-c)^2}{2}} \leq \boxed{\left(\sqrt{b}-\sqrt{c}\right)^2}$$

$$b+c+2\sqrt{bc} \leq 2b+2c \leq 2$$

$$\Downarrow$$

$$\iff \left(\sqrt{b}+\sqrt{c}\right)^2 \leq 2 \iff \sqrt{b}+\sqrt{c} \leq \sqrt{2} \quad \blacksquare$$

Thank You

→ Try the problem set (I will post solutions by December 15)

→ See you at TC