Probabilistic Method Pset 2

EGMOTC 2023 - Rohan

November 5, 2023

Problems

Remark. ** marked problems are considered harder.*
Remark. *Try to do as much as possible and submit whatever progress you have. You can then look at the solutions after submitting. Try to spend atleast somewhere around 30-40 minutes on this set.*

1. (MP4G 2022) Across the face of a rectangular post-it note, you idly draw lines that are parallel to its edges. Each time you draw a line, there is a 50% chance it'll be in each direction and you never draw over an existing line or the edge of the post-it note. After a few minutes, you notice that you've drawn 20 lines. What is the expected number of rectangles that the post-it note will be partitioned into?

2. (Folklore) Let $v_1, v_2, \ldots v_n \in \mathbb{R}^n$, all $|v_i| = 1$ where $|x|$ refers to the Euclidean distance of x from the origin. Then there exist $\epsilon_1, \epsilon_2, \ldots \epsilon_n = \pm 1$ such that

$$|\epsilon_1 v_1 + \epsilon_2 v_2 + \cdots \epsilon_n v_n| \leq \sqrt{n}$$

and also there exist $\epsilon_1, \epsilon_2, \ldots \epsilon_n = \pm 1$ such that

$$|\epsilon_1 v_1 + \epsilon_2 v_2 + \cdots \epsilon_n v_n| \geq \sqrt{n}$$

3. (*) Suppose $p > n > 10m^2$, with p prime, and let $0 < a_1 < a_2 < \cdots a_m < p$ be integers. Prove that there is an integer $0 < x < p$ for which the m numbers

$$(xa_i \pmod{p}) \pmod{n}$$

are pairwise distinct.