EGMO TST Day 1

Date: 29 December 2023

Instructions:

i) You have 4 hours and 30 minutes for three problems.

ii) Each problem is worth 10 points. Attempt all three.

iii) Any claim you make must be accompanied by a proper justification.

Rubric P3
Problem and Solution

Problem 3.

Find all functions $f : \mathbb{N} \to \mathbb{N}$ so that for any positive integer n and finite sequence of positive integers $a_0, \dots, a_n$, whenever the polynomial $a_0 + a_1 x + \cdots + a_n x^n$ has at least one integer root, so does $f(a_0) + f(a_1)x + \cdots + f(a_n)x^n$.

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Solution. The answer is all functions of the form $f(x) = kx$ for some $k \in \mathbb{N}$. These clearly work: now let us prove they are the only possibilities.

Since $x+a$ has an integer root (namely, $-a$) for any $a \in \mathbb{N}$, so does $f(1)x + f(a)$, implying $f(1)|f(a)$ for all $a \in \mathbb{N}$. Now the function $g : \mathbb{N} \to \mathbb{N}$ defined by $g(x) = f(x)/f(1)$ satisfies the same conditions as f, so WLOG, we may assume $f(1) = 1$.
Now the polynomial for any $n \in \mathbb{N}$, $nx^2+(n+1)x+1$ has a the root -1, so $f(n)x^2+f(n+1)x+f(1) = f(n)x^2+f(n+1)x+1$ has an integer root, which must be negative. Call this number $-k$. Therefore

$$f(n+1)k = f(n)k^2 + 1 > f(n) \cdot k \implies f(n+1) > f(n),$$

so that $f(n+1) \geq f(n) + 1$.
Now we prove by induction that $f(n) = n$ for every $n \in \mathbb{N}$. The base case is clear, now consider some $m = n+1 > 1$ and assume $f(n) = n$. The polynomial $x^2 + (n+1)x + n$ has the root -1, so $f(1)x^2+f(n+1)x+f(n) = x^2+f(n+1)x+n$ has an integer. Since the sum of its roots is $-f(n+1)$, an integer, both roots are in fact integers. Further, the roots must be negative and their product is n, so they are $-d$ and $-n/d$ for some positive divisor d of n. Therefore $f(n+1) = d+\frac{n}{d}$. However, the inequality

$$d + \frac{n}{d} \leq n+1 \iff (n-d)\left(1 - \frac{1}{d}\right) \geq 0$$

holds, so we must have

$$n+1 \geq d + \frac{n}{d} = f(n+1) \geq f(n) + 1 = n+1,$$

forcing $f(n+1) = n+1$ as desired. □

Remark There are several ways to finish the induction step. One can argue that looking at $(x+1)(x+n)$, $x^2 + f(n+1) + f(n)$ has an integer root and thus $f(n+1)^2 - 4f(n)$ is a square. But observe that $f(n+1)^2 > f(n+1)^2 - 4f(n) \geq f(n+1)^2 - 4(f(n+1)-1) = (f(n+1)-2)^2$ and thus equality must hold (note that $f(n+1)^2 - 4f(n) \neq (f(n+1)-1)^2$ for parity reasons), so $f(n) = f(n+1) - 1$.
Alternatively, one notes that $q(x) = (n+1)x^{2k+1} + x^{2k} + x^{2k-1} + \cdots x^3 + 2x^2 + x + n$ has -1 as a root, and therefore if x is an integer root of $f(n+1)x^{2k+1} + x^{2k} + x^{2k-1} + \cdots x^3 + 2x^2 + x + f(n)$,

$$f(n+1)|x^{2k+1}| \leq \sum_{i=0}^{2k} |x^i| + |x^2| + n - 1.$$

But if $|x| \geq 2$, we have $\sum |x^i| < 2^{2k+1} \implies (f(n+1)-1)|x^{2k}| < f(n)$ which is absurd. Thus, $|x| = 1$ and since all coefficients are positive, we have -1 as the root. By the inductive hypothesis, we get $f(n+1) = n+1$.

Rubric

0+

(A) **+0:** Guessing the answer is kx for any $k \in \mathbb{N}$.

(B) **+1:** Showing that $f(1)|f(n)$ for any n.

(C) **+3:** Showing that $f(n+1) > f(n)$.

- **+2:** Can be awarded if the conclusion is not drawn but correct type of equation is considered which gives the result. For example, $nx^2 + (n+1)x + 1$.
- **+1:** Can be awarded if it's shown that $f(1) < f(2)$, $f(3) > f(2)$ and 2 can be awarded if these are shown in a way that generalizes.

(D) **+1:** Trying to induct on the claim that $f(n) = kn$ for all n.

(E) **+3:** Considering a good equation or family of equations that lets one induct to show that $f(n+1) \leq f(n) + f(1)$ or $f(n) = kn$. (even if they could not complete the argument). A few such families are mentioned in the remark.

(F) **+1:** Showing that $f(n) = nf(1)$

All points are considered additive. If all points are there, mark using $10-$ scheme.

10-

(A) **-0:** Clear typos that do not influence calculation or bounds.

(B) **-1:** Claimed answer is not verified.

(C) **-1:** Minor errors that lead to calculation error but easy to fix.

(D) Mark using $0+$ scheme if any major bounding or other error or multiple minor errors.