Probabilistic Method Lecture 1

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Markov's inequality and beyond!

± 1 on the number line For this section, we will restrict our attention to a *random walk* on the integers. We start at $x = 0$ and in each move, we move forward or backward each with probability $\frac{1}{2}$. Now, we expect that it should be unlikely that after n steps that we are too far from 0 (the expected value). In this section, we will try to find some bounds for the same.

We will now write the setup a little more formally: let $X_1 = X_2 = \cdots$ be independent identically distributed random variables taking value ± 1 with probability $1/2$ each. Now, $S_n = X_1 + X_2 + \cdots X_n$ and we would like to understand the values of S_n.

Definition 1. *The $Var(X)$ or the variance of a random variable X is defined as* $\mathbb{E}[X^2] - \mathbb{E}[X]^2 = \mathbb{E}[(X - \mathbb{E}[X])^2]$.

Now, prove the following:

- **(Markov's Inequality)** Prove that for any non-negative r.v. X, we have $\mathbb{P}[X \geq a] \leq \frac{\mathbb{E}[X]}{a}$.
- **(Chebyshev's Inequality)** Prove that for any $a > 0$, we have $\mathbb{P}[|X - \mathbb{E}[X]| \geq a] \leq \frac{Var(X)}{a^2}$
- **(Random Walk** 1**)**: Prove that for the random variable S_n described before, we have that $\mathbb{P}[|S_n| > 2\sqrt{n}] < \frac{1}{2}$
- **(Chernoff Bound)**: Prove that for $0 \leq k \leq \sqrt{n}$, we have:

$$\mathbb{P}[|S_n| \geq k\sqrt{n}] \leq 2e^{-\frac{k^2}{2}}$$

- **(Binomial Coefficients)**: As a consequence of the above, prove that

$$\frac{\sum\limits_{i=0}^{\frac{n}{2}-\frac{k\sqrt{n}}{2}} \binom{n}{i}}{2^n} \leq e^{\frac{-k^2}{2}}$$

To provide context on these bounds: Chebyshev's inequality will tell you that $\mathbb{P}[|S_n| \geq 10\sqrt{n}] \leq 0.01$ but Chernoff will tell you, it is $\leq 2 \cdot e^{-50} \sim 3.9 \cdot 10^{-22}$.

An Olympiad problem:

A problem from the USAMO: For integer $n \geq 2$, let $x_1, x_2, \ldots, x_n$ be real numbers satisfying

$$x_1 + x_2 + \ldots + x_n = 0, \qquad \text{and} \qquad x_1^2 + x_2^2 + \ldots + x_n^2 = 1.$$

For each subset $A \subseteq \{1, 2, \ldots, n\}$, define

$$S_A = \sum_{i \in A} x_i.$$

(If A is the empty set, then $S_A = 0$.)
Prove that for any positive number λ, the number of sets A satisfying $S_A \geq \lambda$ is at most $2^{n-3}/\lambda^2$. For which choices of $x_1, x_2, \ldots, x_n, \lambda$ does equality hold?

Hints:

For more details, check the solution document.

- Markov: Write down both the sides and compare.
- Chebyshev: Apply Markov to $Y = |X - \mathbb{E}[X]|^2$.
- Apply Chebyshev
- Discuss in class
- Apply Chernoff
- **(USAMO)**: Apply Chebyshev