EGMO TST Day 1

Date: 29 December 2023

Instructions:

i) You have 4 hours and 30 minutes for three problems.

ii) Each problem is worth 10 points. Attempt all three.

iii) Any claim you make must be accompanied by a proper justification.

Rubric P1
Problem and Solution

Problem 1.

Let ABC be a triangle with circumcentre O and centroid G. Let M be the midpoint of BC and N be the reflection of M across O. Prove that $NO = NA$ iff $\angle AOG = 90°$. *Pranjal Srivastava*

Solution 1. Let H be the orthocenter of ΔABC and let X be the midpoint of AH. Then we know that $AXON$ is a parallelogram.

Now, observe that $\angle AOH = \angle AOG$. Now,

$$NO = NA \iff XA = XO \iff \angle AOH = 90°$$

Thus, we are done. □

Solution 2. Let H be the orthocenter of ΔABC and let A' be the antipode of A in (ABC). Then we know that $ANA'M$ is a parallelogram as the common midpoint of AA' and NM is O. We also know that $MH = MA'$.

Now, observe that $\angle AOH = \angle AOG$. Now,

$$NO = NA \iff MA' = MO \iff MA' = MO' = MH \iff \angle A'OH = 90° \iff \angle AOH = 90°$$

Thus, we are done. □

Solution 3. Say $\angle AOG = 90°$. Let the perpendicular from N to AO and AG be D and E respectively. Note that $ND||OG$. But we have O as the midpoint of MN. So G is the midpoint of EM. But we know the centroid divides the median in a ratio $2 : 1$. Hence E is the midpoint of AG. Since $E{-}D{-}N$ is parallel to OG, we get that D is the midpoint of AO and hence $NO = AN$.

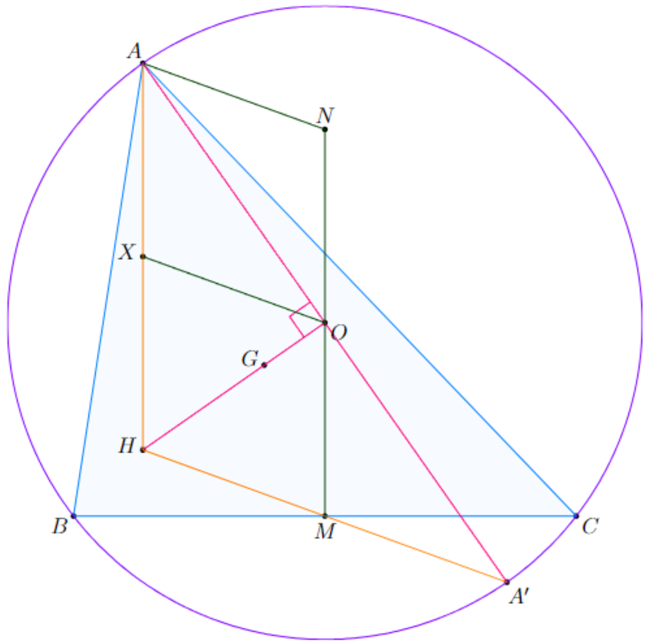
For the other direction, say $AN = NO$, define E as the midpoint of AG and D as the midpoint of AO. So

$$AE = EG = GM.$$

Then note that $DE||OG$ and $NE||OG$ and hence we get $N - D - E$ collinear and hence

$$90° = \angle ADE = \angle AOG.$$

□



Rubric

Incomplete solutions will be given almost 7 marks

Solution 1.

A **+1:** Introducing H (check diagram, angle chases, rough and see if H is introduced.)

B **+1:** Noticing G is not important, that is, $\angle AOG = \angle AOH$.

C **+1:** Introducing midpoint of AH (say X)

D **+2:** showing $AXON$ is a parallelogram (this can have various ways of showing, but if unable to show it is parallelogram but made significant progress which can lead to a way then give +1)

E **+2:** Proving $NA = NO$ iff $XA = XO$

F **+2:** Proving $XA = XO$ iff $XH = XO = XA$ iff $\angle XOH = 90°$

G **+1:** Using the above two to conclude the proof

All points are additive.

Solution 2.

A **+1:** Introducing H and A' (check diagram, angle chases, rough and see if H is introduced.)

B **+1:** Noticing G is not important, that is, $\angle AOG = \angle AOH$.

C **+2:** Showing $ANA'M$ is a parallelogram. (this can have various ways of showing, but if unable to show it is parallelogram but made significant progress which can lead to a way then give $+1$)

D **+2:** Proving $NA = NO$ iff $MA' = MO$

E **+2:** Proving $MA' = MO$ iff $MA' = MO = MH$ iff $\angle A'OH = 90°$

F **+1:** Proving that $\angle A'OH = 90°$ iff $\angle AOH = 90°$

G **+1:** Using the above three to conclude the problem.

All points are additive.

Solution 3. **+1:** Introducing D and E (check diagram, angle chases, rough and see if D or E is introduced.)

First direction: $\angle AOG = 90^\circ$

1. **+1:** Proving $ND||OG$
2. **+2:** Proving G is midpoint of EM
3. **+1:** Proving D is midpoint of AO

Second direction: $NA = NO$

1. **+1:** Proving $AE = EG = GM$
2. **+2:** Proving $NE||OG$ and Proving $DE||OG$
3. **+2:** Proving $N - D - E$ collinear and hence we get $90^\circ = \angle ADE = \angle AOG$.