EGMO TST Day 1

Date: 29 December 2023

Instructions:

i) You have 4 hours and 30 minutes for three problems.

ii) Each problem is worth 10 points. Attempt all three.

iii) Any claim you make must be accompanied by a proper justification.

Rubric P2
Problem and Solution

Problem 2.

Given that $a_1, a_2, \dots, a_{10}$ are positive real numbers, determine the smallest possible value of

$$\sum_{i=1}^{10} \left\lfloor \frac{7a_i}{a_i + a_{i+1}} \right\rfloor$$

where we define $a_{11} = a_1$.

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Solution 1. We claim the minimum is 6. This is attained when $a_i = 7^i$ for $i \in \{1, \dots, 10\}$.

Now to prove this is the minimum, make the substitution $a_{i+1}/a_i = x_i$. The problem becomes: for positive reals $x_1, \dots, x_{10}$ with $x_1 x_2 \cdots x_{10} = 1$. Prove that

$$\sum_{i=0}^{10} \left\lfloor \frac{7}{1 + x_i} \right\rfloor \geq 6.$$

Let $z_i = \left\lfloor \frac{7}{1+x_i} \right\rfloor$. If at most 4 of the z_i's are zero, then at least 6 of them are ≥ 1, and thus the sum is at least 6.
If not, then at least five of them are 0. Suppose $z_1 = \cdots = z_5 = 0$ without loss of generality. Note that $z_i = 0$ implies

$$\frac{7}{1 + x_i} < 1 \implies x_i > 6,$$

so that $x_1, \dots, x_5 > 6$. Thus $x_1 x_2 x_3 x_4 x_5 > 6^5$, which means $x_6 x_7 x_8 x_9 x_{10} < 1/6^5$. Assuming WLOG $x_6 = \min\{x_6, x_7, x_8, x_9, x_{10}\}$, this implies $x_6 < 1/6$. However, then we have

$$z_6 \geq \left\lfloor \frac{7}{1 + \frac{1}{6}} \right\rfloor = 6,$$

and the conclusion follows. □

Solution 2. We claim the minimum is 6. This is attained when $a_i = 7^i$ for $i \in \{1, \dots, 10\}$.

We first observe that if $a_i \geq a_{i+1}$, then

$$\left\lfloor \frac{7a_i}{a_i + a_{i+1}} \right\rfloor \geq \left\lfloor \frac{7a_i}{2a_i} \right\rfloor \geq 3.$$

Thus the sequence (taken cyclically) decreases at most once. WLOG $a_1 < a_2 < \cdots < a_{10}$.

Make the substitution $a_{i+1}/a_i = x_i$. We have $x_i > 1$ for $1 \leq i \leq 9$, and $x_{10} = 1/x_1 x_2 \dots x_9$. The expression becomes

$$\sum_{i=1}^{9} \left\lfloor \frac{7}{1 + x_i} \right\rfloor + \left\lfloor \frac{7x_1 x_2 \dots x_9}{x_1 x_2 \dots x_9 + 1} \right\rfloor.$$

If we have $x_i \geq 6$ for any $1 \leq i \leq 9$, then we get

$$\begin{aligned} x_1 x_2 \dots x_9 &\geq 6 \\ 7x_1 x_2 \dots x_9 &\geq 6x_1 x_2 \dots x_9 + 6 \\ \left\lfloor \frac{7x_1 x_2 \dots x_9}{x_1 x_2 \dots x_9 + 1} \right\rfloor &\geq 6, \end{aligned}$$

and we are done.
If not, $x_i < 6$ for all $1 \leq i \leq 9$. But then, for $1 \leq i \leq 9$,

$$\left\lfloor \frac{7}{1 + x_i} \right\rfloor \geq 1,$$

which means the expression is at least 9, which is greater than our bound!

Solution 3. This is a third approach for showing the lower bound. Let a_{10} be the maximum across all a_i. Then, we have that $\left\lfloor \frac{7a_1}{a_1+a_2} \right\rfloor \geq \left\lfloor \frac{a_1}{a_1+a_{10}} \right\rfloor$. Thus,

$$\left\lfloor \frac{7a_1}{a_1 + a_2} \right\rfloor + \left\lfloor \frac{7a_{10}}{a_{10} + a_1} \right\rfloor \geq \left\lfloor \frac{7a_1}{a_1 + a_{10}} \right\rfloor + \left\lfloor \frac{7a_{10}}{a_{10} + a_1} \right\rfloor \geq \left\lfloor \frac{7(a_1 + a_{10})}{a_1 + a_{10}} \right\rfloor - 1 \geq 6$$

Rubric

Solution 1.

0+

Upper Bound. (3)

(A) **+1:** Guessing the answer is 6

- The conjecture should be explicitly stated. Something like being boxed as a number is rough work is not considered sufficient.

(B) **+2:** Correct construction achieving 6

- **+1:** If construction is not explicitly written saying that it achieves 6 but the sequence $1, 7, 7^2, \cdots$ is mentioned upto atleast the square.

Lower Bound. (7)

(A) **+2:** Shifting to working with $\frac{a_{i+1}}{a_i}$ or x_i

- The full two points can be awarded if only the expression is rewritten as $\sum_{i=0}^{10} \left\lfloor \frac{7}{1+x_i} \right\rfloor$ or it is clear that only the x_i are being considered.
- 1 point can still be awarded if things like $\frac{a_{i+1}}{a_i} > 6 \implies z_i = 0$ and equivalent things are written. It is not considered to be the same as writing $a_{i+1} > 6a_i$. It should be clear that the ratio is being considered.

(B) **+1:** Concluding if at most 4 z_i are 0

(C) **+1:** Showing that $z_i = 0 \iff x_i > 6$

(D) **+1:** Showing that atleast 1 x_i is less than $\frac{1}{6}$ if 5 z_is are 0

(E) **+2:** Concluding that the total sum is atleast 6 then

All points are considered additive.

10-

A solution with both the upper and lower bound is considered complete. 1 mark can be deducted for minor errors which are easily fixable. Typos and such should not lead to deductions.

Solution 2.

0+

Upper Bound. (3)

(A) **+1:** Guessing the answer is 6

- The conjecture should be explicitly stated. Something like being boxed as a number is rough work is not considered sufficient.

(B) **+2:** Correct construction achieving 6

- **+1:** If construction is not explicitly written saying that it achieves 6 but the sequence $1, 7, 7^2, \cdots$ is mentioned upto atleast the square.

Lower Bound. (7)

(A) **+2:** Shifting to working with $\frac{a_{i+1}}{a_i}$ or x_i

- The full two points can be awarded if only the expression is rewritten as $\sum_{i=0}^{10} \left\lfloor \frac{7}{1+x_i} \right\rfloor$ or it is clear that only the x_i are being considered.
- 1 point can still be awarded if things like $\frac{a_{i+1}}{a_i} > 6 \implies z_i = 0$ and equivalent things are written. It is not considered to be the same as writing $a_{i+1} > 6a_i$. It should be clear that the ratio is being considered.

(B) **+2:** Concluding that in the sequence if $a_i > a_{i+1}$ for more than 1 i's ($1 \leq i \leq 10$) then sum will be greater than 6. In other words, sequence is atmost once decreasing.

- This is also equivalent to concluding 9 of the x_i's are > 1, and the product $x_1 x_2 .... x_{10}$ = 1.

(C) **+2:** Showing that if $\left\lfloor \frac{7}{1+x_{10}} \right\rfloor < 6$ then all the x_i's for $1 \leq i \leq 9$ are < 6 and hence the sum is atleast 9

(D) **+2:** Showing that if $\left\lfloor \frac{7}{1+x_{10}} \right\rfloor > 6$, then also we are done

Getting A and B both results gets 3 marks.
Incomplete solution gets atmost 7 marks.
Rest points are additive.

10-

A solution with both the upper and lower bound is considered complete. 1 mark can be deducted for minor errors which are easily fixable. Typos and such should not lead to deductions.