Cauchy - Schwarz

Cauchy - Bunyakowski - Schwarz

For reals $a_1, a_2, \ldots a_n$ and $b_1, \ldots b_n$

$$\left(\sum a_i^2\right)\left(\sum b_i^2\right) \geq \left(\sum a_i b_i\right)^2$$

with equality iff

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n}$$

Inner Product (Think about dot product)

→ A map $\langle \cdot , \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$

is called an inner product iff :

↳ $\langle x, y \rangle = \langle y, x \rangle$ $\quad \forall x, y \in \mathbb{R}^n$

↳ $\langle x, x \rangle > 0$ $\quad \forall x \in \mathbb{R}^n \setminus \{\vec{0}\}$

↳ $\langle ax + by, z \rangle = a \langle x, y \rangle + b \langle y, z \rangle$

$\forall x, y, z \in \mathbb{R}^n$
$a, b \in \mathbb{R}.$

Dot Product :

Let $x = (x_1, x_2, \ldots x_n)$ and
$y = (y_1, \ldots y_n)$

Now define $\underline{x \cdot y} = \underline{x_1 y_1} + \underline{x_2 y_2} + \cdots + \underline{x_n y_n}$

Observe that the '$\cdot$' operator is an inner product!

In fact $\sqrt{x \cdot x}$ is the Euclidean distance of x from $\vec{0}$.

Back to CS

Version 0

$$\left(\sum a_i^2\right)\left(\sum b_i^2\right) \geq \left(\sum a_i b_i\right)^2$$

Version 1

Let

$a = (a_1, a_2, \ldots a_n) \in \mathbb{R}^n$ &

$b = (b_1, b_2, \ldots b_n) \in \mathbb{R}^n$

then

$$(a \cdot a)(b \cdot b) \geq (a \cdot b)^2$$

We prove this version

Version 2

Let $a, b \in \mathbb{R}^n$, $\langle \cdot, \cdot \rangle$ an inner product then we have:

$$\langle a, a \rangle \langle b, b \rangle \geq \langle a, b \rangle^2$$

Proving CS inequality

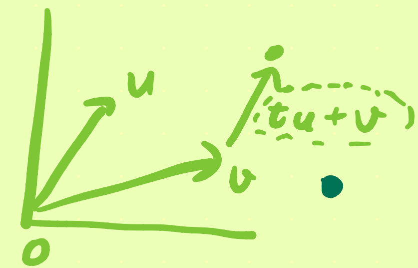
$\langle u,u\rangle \cdot \langle v,v\rangle \geq \langle u,v\rangle^2$

↳ let's consider $h(t) = \overset{\geq 0}{\langle tu+v, tu+v\rangle}$ for $t \in \mathbb{R}$.

Now, $h(t) = t^2 \langle u,u\rangle + 2t\langle u,v\rangle + \langle v,v\rangle$

$\boxed{\text{h is a quadratic polynomial}}$

But $h(t) \geq 0 \quad \forall t \in \mathbb{R}$



$\Rightarrow \left(2\langle u,v\rangle\right)^2 \leq 4\langle u,u\rangle\langle v,v\rangle$

$\Rightarrow \langle u,u\rangle\langle v,v\rangle \geq \langle u,v\rangle^2$

Yes, that's it!

Special Case (Titu's Lemma, Engel form, Sedrakyn's inequality, Bergström's inequality)

For any reals $(a_1, \dots a_n) \in \mathbb{R}^n$, $(b_1, \dots b_n) \in \mathbb{R}^n_{>0}$

$$\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \dots + \frac{a_n^2}{b_n} \geqslant \frac{(\Sigma a_i)^2}{(\Sigma b_i)}$$

Pf Let $u_i = a_i/\sqrt{b_i}$ | $u \cdot u \geqslant \dfrac{(u \cdot v)^2}{v \cdot v}$

$v_i = \sqrt{b_i}$

CS is very useful !!!

Thank You