EGMO TST Day 2

Date: 30 December 2023

Instructions:

i) You have 4 hours and 30 minutes for three problems.

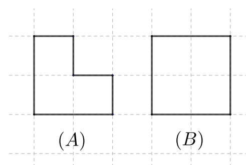
ii) Each problem is worth 10 points. Attempt all three.

iii) Any claim you make must be accompanied by a proper justification.

Rubric P2

Problem 2.

1. Can a 7×7 square be tiled with the two types of tiles shown in the figure. (Tiles can be rotated and reflected but cannot overlap or be broken)
2. Find the least number N of tiles of type A that must be used in the tiling of a 1011×1011 square. Give an example of a tiling that contains exactly N tiles of type A.



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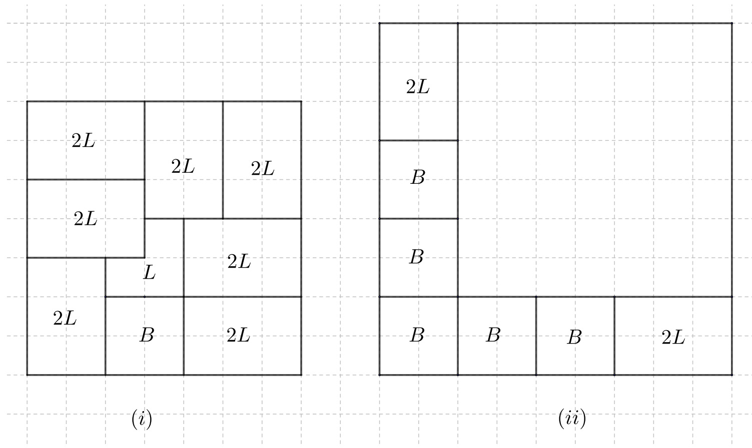
Solution. We prove a more general fact: the number of $L-$tiles in any tiling of a $(2n-1)\times(2n-1)$ square with tiles of the two given types is not less than $4n-1$ for any $n > 4$. In particular, for $n = 506$ the minimum number of $L-$tiles is 2023. Color the big square in four colors $1, 2, 3, 4$ as shown below.

1	2	1	2	1	2
3	4	3	4	3	4
1	2	1	2	1	2
3	4	3	4	3	4
1	2	1	2	1	2
3	4	3	4	3	4

C1early, no matter how we fill this square with our ti1es, the tiles of type B will always cover four unit squares of different colors. So all these tiles will cover an equal number of unit squares of each color. However, the total numbers of unit squares of different colors is different. Suppose we have x tiles of type A and y tiles of types B. Then $3x + 4y = (2n-1)^2$. On the other hand each tile covers no more than one square of color 1 and the total number of 1-colored tiles is n^2. Hence $x + y \geq n^2$. Hence

$$4x \geq 4n^2 - 4y = 4n^2 - (2n-1)^2 + 3x = 4n - 1 + 3x$$

Hence $x \geq 4n - 1$. When $n = 25$, the minimum number of $L-$tiles required is 99.
In the figure, (i) exhibits a tiling of 7×7 square with 15 $L-$tiles. Figure (ii) shows how to extend a tiling of $(2n-1)\times(2n-1)$ containing $4n-1$ $L-$tiles to a tiling of $(2n+1)\times(2n+1)$ with additional 4 $L-$tiles obtaining a tiling with $4n-1+4 = 4n+3 = 4(n+1)-1$ $L-$ tiles. Thus starting with a tiling of 7×7 square with 15 $L-$tiles, we obtain a tiling of 1011×1011 square with $15 + 4 \times 502 = 2023$ $L-$tiles. $\square$



Rubric

The first part is worth 3 points and the second part is worth 7 points.

Part 1)

+3: Correct construction, should be clearly made. Incomplete attempts at a construction will not be awarded points.

Part 2), 0+

(A) **+1:** Extending to a valid construction of 1011×1011 or larger $2n-1 \times 2n-1$ grids.

(B) **+1:** Extending to a construction of $2n-1 \times 2n-1$ grid using exactly $4n-1$ $L-$tiles and claiming that this is the optimal.

(C) **+2:** Using a correct colouring even if conclusion is not drawn. 1 point can be awarded if the colouring is only for small cases ≥ 7 and not in general.

(D) **+2:** Arguing using a correct colouring for small cases, for example 7 or 9.

All of the above points are additive but incomplete solutions will not be awarded more than 7 points. If the student has only argued small cases, they will not be awarded more than 6 points.

Other points in case none of the other 7 are awarded.

(A) **+1:** Claiming that the answer is 2023. This is only awarded if none of the other points are awarded.

(B) **+0:** Showing that it is not possible to tile a 3×3 or 5×5 grid.

(C) **+1:** Having some other colouring which lets you prove a weaker result that tells you that more than 3 $L-$ tiles are required.

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(A) **-3:** If 7×7 colouring is wrong but easy to fix and the rest of the solution extends

(B) **-2:** If the extension from 7×7 to 1011×1011 does not have correct number of $L-$ tiles but is easily fixed and the rest of the argument is fine. If the extension is completely wrong or too many $L-$ tiles are used then grade as $0+$.

(C) **-1:** If there are minor ± 1 errors in the final bounding after the coloring. Should be easy to fix

If there are any major errors, grade as $0+$.