EGMO TST Day 1

Date: 29 December 2023

Instructions:

i) You have 4 hours and 30 minutes for three problems.

ii) Each problem is worth 10 points. Attempt all three.

iii) Any claim you make must be accompanied by a proper justification.

Rubric P1

Problem and Solution

Problem 1.

Let $N \geq 3$ be an integer, and $a_0, \ldots, a_{N-1}$ be pairwise distinct reals so that $a_i \geq a_{2i}$ for all i (indices are taken mod N). Find all possible N for which this is possible.

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Solution. The only such N are powers of 2.

If N is not a power of 2, let's say $p|N$ where p is an odd prime. Now, observe that $p \nmid 1$ but $p \mid 2^k - 1$ for some $k > 1$. Then, let $\alpha = \frac{N}{p}$. Now,

$$a_\alpha > a_{2\alpha} \geq a_{4\alpha} \ldots \geq a_{2^k\alpha} \implies a_\alpha > a_{2^k\alpha}$$

But now, $a_{2^k\alpha} = a_\alpha$ as $N|(2^k-1)\alpha$ but then this is a contradiction! We have that $a_\alpha > a_{2\alpha}$ since all the given reals are distinct and $N \nmid \alpha$.

Now, if $N = 2^m$ is a power of 2, let $a_j = -\nu_2(j) + \frac{j}{2N}$ for $j \neq 0$ and let $a_0 = -v_2(N)$. This is clearly okay if $j = 0$ and if $j \neq 0$ then $\nu_2(2j) = \nu_2(j) + 1$ and thus $a_j \geq a_{2j}$. $\square$

Rubric

Incomplete solutions will be given almost 7 marks.

Solution 1. There are 2 parts to a complete solution. Showing that powers of 2 work and the second part is to show that other N don't. The first part is worth 3 marks and the other direction 7.

Construction for $N = 2^n$

(A) **+1:** Claiming that the answer is only powers of 2.

(B) **+3:** For a complete valid construction for 2^n. In case the full $+3$ are not awarded, one of the following can be awarded:

- **+2:** These can be awarded if there is no general construction but constructions for $n = 8$ or some other power of $2 \geq 8$ is found. Proof is required that construction works, otherwise awarded at most 1.
- **+2:** Can be awarded if the construction has some minor issue like all numbers are not distinct but the idea is correct. Proof is required that construction works, otherwise awarded at most 1.
- **+1:** Can be awarded if construction is there for $N = 4$. Could be in rough work and no proof is required.
- **+2:** Correct general construction but no proof is given.

A and B are not additive.

Other N don't work.

(A) **+1:** Writing $N = 2^\alpha \cdot \beta$ where β is odd.

(B) **+3:** If p is an odd factor of N, then looking at a chain of the form $a_{\frac{N}{p}}, a_{\frac{2N}{p}}, \cdots$

(C) **+3:** Finding a contradiction and completing the argument that numbers with odd part fail.

If 3 marks are not awarded from the above part. The following non-additive points can still be given:

(a) **+1:** Showing that some small odd N do not work. At least 2 different odd numbers ≥ 3 need to be shown.

(b) **+2:** Odd numbers do not work.

An incomplete argument for showing that non primes of 2 do not work cannot fetch more than 3 points. Thus, an incomplete solution cannot fetch a total of more than $3 + 3 = 6$ points.

10-

(A) Any major errors should result in the solution being treated as a $0+$.

(B) **-1:** Construction not having all numbers distinct.

(C) **-1:** Typos and errors which are easy to fix.

Any error bigger than this should be treated as a $0+$ and not be awarded more than 6.