TST Mock 2

EGMOTC 2023 - Rohan

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Problems

Problem 1. There are three cities each of which has exactly the same number of citizens, say n. Every citizen in each city has exactly a total of $(n+1)$ friends in the other two cities. Show that there exist three people, one from each city, such that they are friends. We assume that friendship is mutual (that is, a symmetric relation).

Problem 2. Find all real numbers $x_1, \ldots, x_{2024}$ that satisfy the following equation for each $1 \leq i \leq 2024$. (Here $x_{2025} = x_1$.)

$$x_i^2 + x_i - 1 = x_{i+1}$$

Problem 3. Let m be the product of the first 100 primes, and let S denote the set of divisors of m greater than 1 (hence S has exactly $2^{100} - 1$ elements). We wish to color each element of S with one of k colors such that

- every color is used at least once; and
- any three elements of S whose product is a perfect square have exactly two different colors used among them.

Find, with proof, all values of k for which this coloring is possible.