Inequalities PSet

EGMOTC 2023 - Rohan

November 29, 2023

Problems

Remark. ** marked problems are considered harder.*
** marked problems are strictly optional for the ones feeling extremely curious about this particular setup.
Remark. Try to do the first two parts atleast and submit whatever progress you get on the last two parts.

Problem 1. Watch the first video about the AM-GM inequality. Based on the video, write two different proofs of AM-GM inequality in your own words. (the video alludes to 6-7 different proofs)

Problem 2. (Rearrangement Inequality) Prove the rearrangement inequality: Let $a_1 < a_2 < \ldots < a_n$ and $b_1 < b_2 < \ldots < b_n$ be real numbers. Prove that for any permutation σ of $\{1, 2, \ldots n\}$, we have:

$$a_1 b_1 + a_2 b_2 + \ldots a_n b_n \geq a_1 b_{\sigma(1)} + a_2 b_{\sigma(2)} + \ldots a_n b_{\sigma(n)}$$

Problem 3. (INMO 2020) Let $n \geqslant 2$ be an integer and let $1 < a_1 \leq a_2 \leq \cdots \leq a_n$ be n real numbers such that $a_1 + a_2 + \cdots + a_n = 2n$. Prove that

$$a_1 a_2 \ldots a_{n-1} + a_1 a_2 \ldots a_{n-2} + \cdots + a_1 a_2 + a_1 + 2 \leq a_1 a_2 \ldots a_n$$

Problem 4. (ISL 2001 A3) Let $x_1, x_2, \ldots, x_n$ be arbitrary real numbers. Prove the inequality

$$\frac{x_1}{1 + x_1^2} + \frac{x_2}{1 + x_1^2 + x_2^2} + \cdots + \frac{x_n}{1 + x_1^2 + \cdots + x_n^2} < \sqrt{n}.$$