Probabilistic Method Pset 2

EGMOTC 2023 - Rohan

December 13, 2023

Problems

Remark. ** marked problems are considered harder.*
Remark. *Try to do as much as possible and submit whatever progress you have. You can then look at the solutions after submitting. Try to spend atleast somewhere around 30-40 minutes on this set.*

1. (MP4G 2022) Across the face of a rectangular post-it note, you idly draw lines that are parallel to its edges. Each time you draw a line, there is a 50% chance it'll be in each direction and you never draw over an existing line or the edge of the post-it note. After a few minutes, you notice that you've drawn 20 lines. What is the expected number of rectangles that the post-it note will be partitioned into?

2. (Folklore) Let $v_1, v_2, \ldots v_n \in \mathbb{R}^n$, all $|v_i| = 1$ where $|x|$ refers to the Euclidean distance of x from the origin. Then there exist $\epsilon_1, \epsilon_2, \ldots \epsilon_n = \pm 1$ such that

$$|\epsilon_1 v_1 + \epsilon_2 v_2 + \cdots \epsilon_n v_n| \leq \sqrt{n}$$

and also there exist $\epsilon_1, \epsilon_2, \ldots \epsilon_n = \pm 1$ such that

$$|\epsilon_1 v_1 + \epsilon_2 v_2 + \cdots \epsilon_n v_n| \geq \sqrt{n}$$

3. (*) Suppose $p > n > 10m^2$, with p prime, and let $0 < a_1 < a_2 < \cdots a_m < p$ be integers. Prove that there is an integer $0 < x < p$ for which the m numbers

$$(xa_i \pmod{p}) \pmod{n}$$

are pairwise distinct.

Solutions

Problem 1

Problem 1. (MPFG 2022) Across the face of a rectangular post-it note, you idly draw lines that are parallel to its edges. Each time you draw a line, there is a 50% chance it'll be in each direction and you never draw over an existing line or the edge of the post-it note. After a few minutes, you notice that you've drawn 20 lines. What is the expected number of rectangles that the post-it note will be partitioned into?

Solution. Initially the paper is exactly one rectangle. Let X_i be the RV corresponding to the number of more rectangles introduced by the ith line. Now, the final number of rectangles created is precisely $1 + X_1 + \cdots X_{20}$.

Now, we have by linearity of expectation that

$$\mathbb{E}[1 + X_1 + \cdots X_{20}] = 1 + \mathbb{E}[X_1] + \mathbb{E}[X_2] + \cdots \mathbb{E}[X_{20}]$$

Now, X_i just depends on the number of lines already drawn not parallel to it i.e. if j lines are already drawn not parallel to it, it divides the rectangle into $j+1$ regions.

Thus, $\mathbb{E}[X_i] = 1 + \mathbb{E}[\text{number of lines drawn not parallel to the } i\text{th line}] = 1 + \frac{i-1}{2}$ (each line is parallel to the i th line with probability $\frac{1}{2}$).

Thus, the final answer is $1 + \frac{2}{2} + \frac{3}{2} + \cdots \frac{21}{2} = \frac{1+231}{2} = \boxed{116}$. □ □

Problem 2

Problem 2. (Folklore) Let $v_1, v_2, \dots v_n \in \mathbb{R}^n$, all $|v_i| = 1$ where $|x|$ refers to the Euclidean distance of x from the origin. Then there exist $\epsilon_1, \epsilon_2, \dots \epsilon_n = \pm 1$ such that

$$|\epsilon_1 v_1 + \epsilon_2 v_2 + \cdots \epsilon_n v_n| \leq \sqrt{n}$$

and also there exist $\epsilon_1, \epsilon_2, \dots \epsilon_n = \pm 1$ such that

$$|\epsilon_1 v_1 + \epsilon_2 v_2 + \cdots \epsilon_n v_n| \geq \sqrt{n}$$

Solution. We let $v = \epsilon_1 v_1 + \epsilon_2 v_2 + \cdots \epsilon_n v_n$ where v is a random variable and ϵ_i are picked as ± 1 with probability $\frac{1}{2}$ each.

Thus, finally let X be the RV corresponding to $v \cdot v = |v|^2$ where $(\cdot)$ is used for the dot product.[1].
We will show that $\mathbb{E}[X] = n$ which will immediately imply both the results!

$$\mathbb{E}[X] = \left(\sum_i \mathbb{E}[\epsilon_i^2] v_i \cdot v_i \right) + \left(\sum_{i \neq j} \mathbb{E}[\epsilon_i \epsilon_j] v_i \cdot v_j \right)$$

$$\mathbb{E}[\epsilon_i^2] = 1$$

since $\epsilon_i^2 = 1$ always and

$$\mathbb{E}[\epsilon_i \epsilon_j] = \mathbb{E}[\epsilon_i]\mathbb{E}[\epsilon_j] = 0 \cdot 0$$

as ϵ_i and ϵ_j are independent. Finally, $v_i \cdot v_i = 1$ for all i as we know $\sqrt{v_i \cdot v_i} = |v_i| = 1$ for all i. Thus,

$$\mathbb{E}[X] = \left(\sum_i \mathbf{1} \right) + \mathbf{0} = n$$

Thus, we are done! □

[1] check inequalities material if you don't remember definition of dot product.

Problem 3

Problem 3. (Probabilistic Method by Alon and Spencer Exercise. 2.7.4) Suppose $p > n > 10m^2$, with p prime, and let $0 < a_1 < a_2 < \cdots a_m < p$ be integers. Prove that there is an integer $0 < x < p$ for which the m numbers

$$(xa_i \pmod{p}) \pmod{n}$$

are pairwise distinct.

Solution. Remind me to put up a solution. Writing it is more convoluted than I initially expected. $\square$