11.10.2021

YZV 231E

Probability Theory & Stats

Week 2

Gü.

Recap : Essential Elements in a probability model : Random experiment is performed.

1) Sample space defined Ω → All outcomes of the experiment.
2) Define a probability law
3) Define an event(s)
4) Calculate probability of events.

experiment: verbal description : ⟶ 1 coin toss

eg. Consider random experiment of 4 coin tosses.

$\Omega_1 = \{H, T\}$

1 or (5)

$\Omega = \{\{HHHH\}, \{HHHT\}, \{HHTT\}, \ldots, \{TTTT\}\}$

$\Omega = \{H,T\} \times \{H,T\} \times \{H,T\} \times \{H,T\}$

Size of $\Omega = 2 \times 2 \times 2 \times 2$.

$|\Omega| = 16$

2 All outcomes are equally likely = Discrete Uniform Probability Law

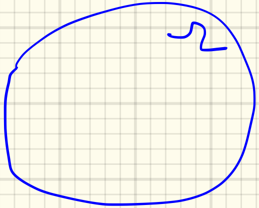
$P(\{THTH\}) = \frac{1}{16}$ → Any / each outcome has prob. $\frac{1}{16}$

3 $E = \{\text{getting exactly 3 heads}\} = \{\{HHHT\}, \{HHTH\}, \{HTHH\}, \{THHH\}\}$

4 $P(E) = ?\ \frac{|E|}{|\Omega|} = \frac{4}{16} = \frac{1}{4}$

Recap: Our probability model should obey 3 axioms of Prob.

Def: A collection of subsets of Ω is called a σ-field (σ-algebra) or Borel field $\mathcal{B}$, if it satisfies:



(i) $\emptyset \in \mathcal{B}$

(ii) If $A \in \mathcal{B}$ then $A^c \in \mathcal{B}$ ($\mathcal{B}$ is closed under complementation)

(iii) If $A_1, A_2 \ldots \in \mathcal{B}$ then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{B}$ ($\mathcal{B}$ is closed under countable unions)

By De Morgan's law: $\mathcal{B}$ is also closed under countable intersections

* If Ω is finite or countable, then $\mathcal{B} = \{$all subsets of Ω, including Ω itself$\}$

ex countably ∞ set. $A_i = \frac{1}{2^i}$, $i = 1, \ldots, \infty$.

can be put to 1-1 correspondence w/ set of Natural Numbers

Let $\Omega = \{1, 2, 3\} \rightarrow \mathcal{B} = 2^3 = 8$.

* Let $\Omega = (-\infty, \infty)$ the real line

choose $\mathcal{B}$ to contain all sets of the form

A_i: $[a,b]$, $(a,b]$, $[a,b)$, (a,b) $\forall a,b \in \mathbb{R}$ $\hookrightarrow \mathcal{B}$

$\mathcal{B}$ contains all sets that can be formed by taking (countably ∞) unions & complementations of such sets

Axioms of Probability:

Given a sample space Ω, and an associated σ-field $\mathcal{B}$, a probability fn. P w/ domain $\mathcal{B}$: that satisfies:

(1) $P(A) \geq 0$, for all $A \in \mathcal{B}$. $\quad (P : \mathcal{B} \to [0,1])$

(2) $P(\Omega) = 1$.

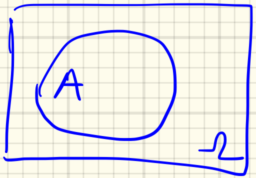
(3) If $\underbrace{A_1, A_2, \dots \in \mathcal{B}}$ are pairwise disjoint, then $\quad (A_i \cap A_j = \emptyset \ \forall i,j)$

$$P\Big(\underbrace{\bigcup_i A_i}_{\in \mathcal{B}}\Big) = \sum_{i=1}^{\infty} P(A_i)$$

Only 3 axioms!! We can derive many others based on these:

Thm: If P is a probability fn. (i.e. it satisfies the 3 axioms of prob) & $A \in \mathcal{B}$ (any set in $\mathcal{B}$), then

* $P(\emptyset) = 0$
* $P(A) \leq 1$. $\quad$: $A \subset \Omega$



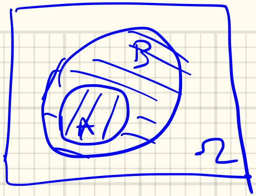
A & A^c are disjoint

$$\underbrace{P(\underline{A \cup A^c})} = P(\Omega)$$

$$P(A) + \underbrace{P(A^c)}_{\geq 0} = 1 \implies P(A) \leq 1$$

* $P(A^c) = 1 - P(A)$

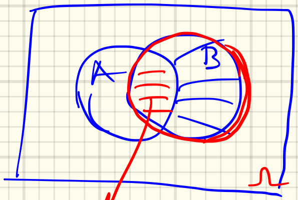
$\Rightarrow$ * If $A \subset B$ then $P(A) \leq P(B)$.



$B = A \cup (A^c \cap B)$ disjoint

$P(B) = P(A) + \underbrace{P(A^c \cap B)}_{\geq 0} \Rightarrow P(A) \leq P(B)$.

* $P(B \cap A^c) = P(B) - P(A \cap B)$



$(A \cap B) \cup (B \cap A^c) = B$

disjoint

* $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\Rightarrow$ show this.

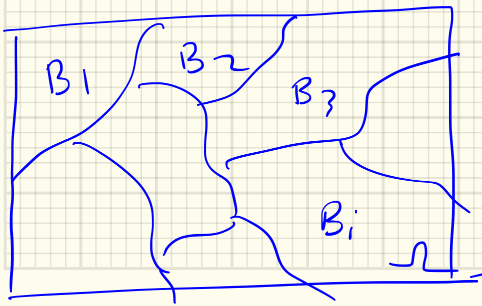
Show: * $\{B_i\}$ is any partition of the Ω

$\downarrow$

$P(A) = \sum_{i=1}^{\infty} P(A \cap B_i)$.

$\left\{ \begin{array}{l} (B_i \cap B_j = \emptyset) \ \forall i,j \ \& \\ \bigcup_i B_i = \Omega \end{array} \right.$

Def: Partition of Ω.



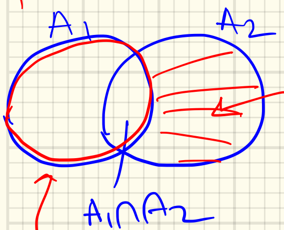
Pf: $\Omega = \bigcup_{i=1}^{\infty} B_i$; $A = A \cap \Omega = A \cap (\bigcup_i B_i) = \bigcup_i (A \cap B_i)$

$P(A) = P(\bigcup_i (A \cap B_i)) = \sum_i P(A \cap B_i)$

$\ast$ **Union Bound** : $\boxed{P\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} P(A_i)}$

Attention: A_i's are not necessarily disjoint. (finite or countable (countably ∞.)

Show for 2 sets.



$$A_1 \cup A_2 = A_1 \cup (A_2 \cap A_1^c)$$

$$P(A_1 \cup A_2) = P(A_1) + P(A_2 \cap A_1^c)$$

$$(A_2 \cap A_1^c) \subset A_2 .$$

$$\leq P(A_1) + P(A_2)$$

We've shown for 2 sets

This generalizes to countably ∞. sequences A_i.

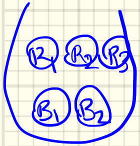
$\rightarrow$ Show these derived axioms.

→ We'll study counting → important in probabilistic modeling, particularly discrete uniform probability models.

A probability model we'll use (binomial)

e.g. QUALITY CONTROL (Later)

Ex: An Urns (Sampling)
(torba)



→ 3 R
2 B

we sample a ball → we replace it back

E = { getting first a "Red" then a "Black" ball }

$P("R, B") = ?$

{R, B}

List of Balls

R_1, R_2, R_3, B_1, B_2.

Experiment: Sample 2 balls sequentially, independently.

sampling a ball

$\Omega_1 = \{R_1, R_2, R_3, B_1, B_2\}$

$\Omega_2 = \{R_1, R_2, R_3, B_1, B_2\}$

$\Omega = \Omega_1 \times \Omega_2$) Think of the sample space as the Cartesian product

1st ball Draw X 2nd ball draw

$|\Omega| = 25$

$E = \{ "R_1, B_1", "R_1, B_2", "R_2, B_1", "R_2, B_2", "R_3, B_1", "R_3, B_2" \}$

$$P(E) = \frac{|E|}{|S|} = \frac{6}{25}$$ ← Here, we manually counted the cardinalities of events ✓

Now

→ Ex: Dealing a 52-card deck.
What is the prob. that 5 cards you draw contain 4 aces?

Sample space size? Huge space!

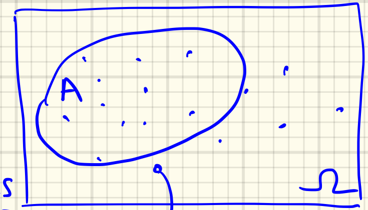
That's why we study

⇒ COUNTING: part of COMBINATORICS → huge field.

↳ probabilistic modeling.

$$P(A) = \frac{|A|}{|\Omega|}$$

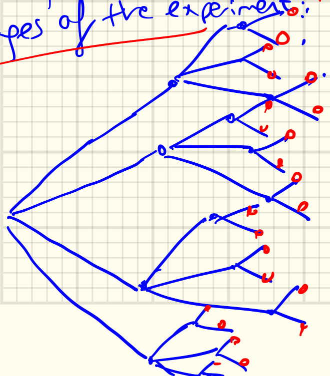
finite sample space → N outcomes equally likely



$\frac{1}{N}$

→ Sequential experiments

3 stages of the experiments:



1st stage: $n_1 = 4$
2nd stage $n_2 = 3$
3rd stage $n_2 = 2$.

24

How many outcomes (choices) in total?

total # possible outcomes of the experiment $= n_1 \times n_2 \times n_3$

Ex: # License plates w/ 3 letters 4 numbers

$\underbrace{\cdot \quad \cdot \quad \cdot}$ $\cdot \quad \cdot \quad \cdot \quad \cdot$ →

$22 \times 22 \times 22 \times 10 \times 10 \times 10 \times 10 = (22)^3 (10)^4 \Rightarrow |\Omega|$

from 22 letters (of the Turkish alphabet)

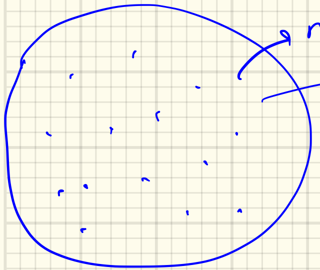
(w/ replacement = repetition allowed)

→ If repetition is <u>not</u> allowed:

$22 \times 21 \times 20 \times 10 \times 9 \times 8 \times 7 = |\Omega|$ (w/o replacement)

* n-element set

choose k elements in an ordered fashion



$n \cdot (n-1)(n-2) \cdot \cdot \cdot \cdot \cdot (n-(k-1))$

$= n-k+1$

k-permutation.

Permutation: # of possible arrangements of n objects:

$n \cdot (n-1) \cdot (n-2) \cdot \cdot \cdot 1 = \underbrace{n!}$ n-factorial ways to order n objects

→ k-permutation : $\dfrac{n!}{(n-k)!} = n(n-1)\cdots(n-k+1) \triangleq n_k$.

Selecting k objects from a collection of n objects.

* Say we have a set $A = \{1, 2, 3, \dots, n\}$

Create subsets → how many subsets does A have? 2^n?

$$\underbrace{2 \cdot 2 \cdot 2 \cdots 2}_{n} = 2^n.$$

$\underbrace{A = \{1, 2\}}_{n=2}$

00

01

10

11

$\{1\}$

$\{2\}$

$\{1, 2\}$

$\emptyset$.

$= 2^2 = 4$

∴

Def: Number of subsets of a n-element set $= 2^n$.

Ex: We are given a fair die (w/ six sides) : roll it 5 times

(independent

- $P(5,4,2,2,1) \doteq \; ? \quad \frac{1}{6^5}$

$$|\Omega| = 6^5 \qquad 6 \times 6 \times 6 \times 6 \times 6 = 6^5$$

w/ replacement.

- $A = \{$ all rolls give different numbers $\}$

$$P(A) = \frac{6 \times 5 \times 4 \times 3 \times 2}{6^5}$$

$$6 \times 5 \times 4 \times 3 \times 2 . = n_k = 6_5 = \frac{n!}{(n-k)!} = 6!$$

if $k = n$

$$\frac{6!}{0!} = 6! \qquad 0! = 1.$$

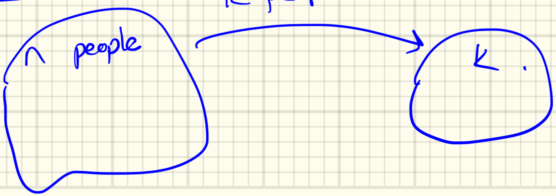
$1! = 1$.

$0! = 1$

Next →

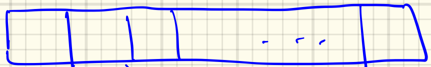
form

k people committees



choose k among n

Start w/ ordered subsets



$$\underbrace{n \cdot (n-1)(n-2) \cdots n-k+1}_{} = \frac{n!}{(n-k)!} = n_k$$

How many ways can I permute these k-list subsets : $k!$

$$\rightarrow \binom{n}{k} = \frac{n!}{(n-k)!} \cdot \frac{1}{k!} \leftarrow (\text{unordered})$$

"n choose k"

$\binom{n}{k}$: binomial coefficient $\rightarrow$ how to choose k-element subsets out of n-elements.

(Unordered)

Combination :

If $k=n$ $\binom{n}{n} = \frac{n!}{0!} \cdot \frac{1}{n!} = 1$. ✓

Note: $\underbrace{\sum_{k=0}^{n}}_{\text{sums of all possible \# element subsets of an n-element set}} \binom{n}{k} = 2^n$ = Total # subsets

(Cont'd) 52-cards: Prob of having 4 "1"s (aces)? Choose 5-cards.

Ex: $P\left(\boxed{1 \mid 1 \mid 1 \mid 1 \mid X}\right) = ?$ Sample space $\Rightarrow |\Omega| = \binom{52}{5}$

$\underbrace{1 \cdot 1 \cdot 1 \cdot 1}$ 48 ways to choose the 5th slot.

$$= \frac{52!}{47!} \cdot \frac{1}{5!} = \frac{52 \times 51 \times 50 \times 49 \times 48}{5!}$$

$$P(4 \text{ aces}) = \frac{|E|}{|\Omega|} = \frac{48}{\binom{52}{5}}$$

$\binom{52}{5}$ ways to choose 5 cards from 52 cards.

$$\frac{48}{2,598,960}$$ very low!

→ $P(\text{having 4 cards of the same kind}) = ?$

exercise.

$$13 \times \frac{48}{\binom{52}{5}}$$

$\boxed{\cdot \mid \cdot \mid \cdot \mid \cdot \mid X}$

1 1 1 1 →
2 2 2 2 →
3 3 3 3 →

13

10 10 10 10

11
12
13

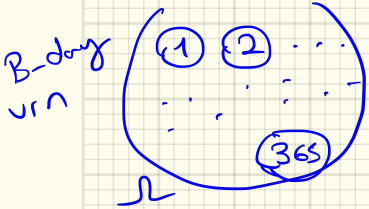
Ex 3.12 [Kay] Birthday (b'day) Problem: A class has N students.

What is the prob. that

$A = \{$ at least 2 students have the same bday $\}$?

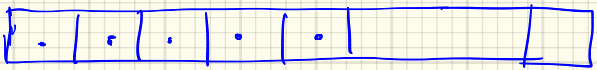
see Fig 3.9

B-day urn



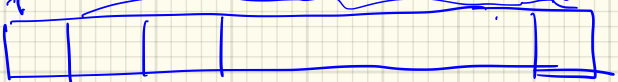
(Sampling w/ replacement)

1 ... N



$365 \cdot 365 \cdot \ldots = (365)^N = |\Omega|$

Consider $A^c \doteq \{$ no two students have the same bday $\}$



$365 \times 364 \times 363 \; \ldots \times 365-N+1$

$365-(N-1)$

$= (365)_N = \dfrac{365!}{(365-N)!} = |A^c|$

$$P(A^c) = \frac{|A^c|}{|\Omega|} = \frac{(365)_N}{(365)^N}$$

$N=4$: $\dfrac{365 \times 364 \times 363 \times 362}{}$ $\dfrac{365!}{(365-4)!} = \dfrac{365!}{(365-N)!}$

Recall: k-permutation out of n elements: $n_k = \dfrac{n!}{(n-k)!}$

$$P(A) = 1 - P(A^c)$$
$$= 1 - \frac{(365)_N}{(365)^N}$$

Summarize: # possible arrangements of size k from N objects:

	w/o Replacement	w/ Replacement
Ordered	$N_r = \frac{N!}{(N-r)!}$	N^r
Unordered	$\binom{N}{r}$	$\binom{N+r-1}{r}$ ← given w/o show/proof.

Extra: 4th case:

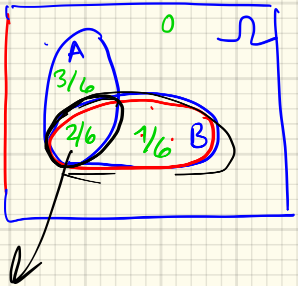
Sampling w/ replacement unordered: eg. $\{1, 2, 3\}$ sample 2 balls

$(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)$ $\binom{3+2-1}{2} = \binom{4}{2}$

CONDITIONAL PROBABILITY :

$P(A)$ ✓

Now, we have new information that event B occured



$$P(A|B) = \frac{P(A\cap B)}{\underbrace{P(B)}_{\text{reduced (revised) sample space.}}} = \frac{2}{3} = \frac{2/6}{3/6}$$ ✓

$$P(B) = \frac{1}{2}$$

$$P(A|B) \triangleq \frac{P(A\cap B)}{P(B)}$$ ✓, undefined if $P(B)=0$

$$\Rightarrow P(A\cap B) = \underbrace{P(A|B)}\,.\,\underbrace{P(B)}$$ ✓

★ → Conditional probabilities are ordinary probabilities:

$P(A|C) \geq 0$

$P(\Omega|C) = 1.$

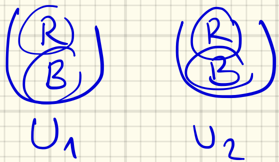
$$P(A\cup B|C) = P(A|C) + P(B|C)$$

A & B disjoint

we get new information.

Axioms of prob. satisfied.

Ex: Compound experiment → Two urns : say we select one of them randomly



U_1 : p_1 : proportion of Red balls ; $(1-p_1)$ prob. blacks ($=$ prob.)

U_2 : p_2 : " " " ; $(1-p_2)$.

P(Redball is selected) : $P(A) \overset{?}{=} P(A\cap B_1) + P(A\cap B_2)$

A =

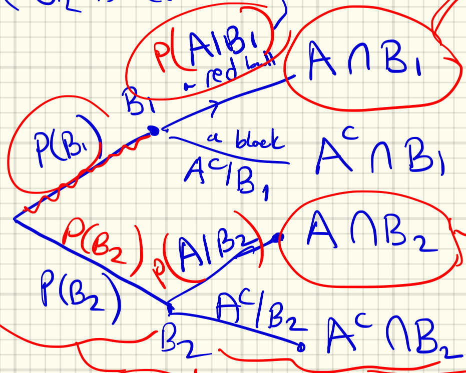
$$= P(A|B_1)P(B_1) + P(A|B_2)P(B_2)$$

$$= p_1 \cdot \frac{1}{2} + p_2 \cdot \frac{1}{2}$$

$$= \frac{1}{2}(p_1+p_2)$$

$B_1 = \{U_1 \text{ is selected}\}$

$B_2 = \{U_2 \text{ is selected}\}$

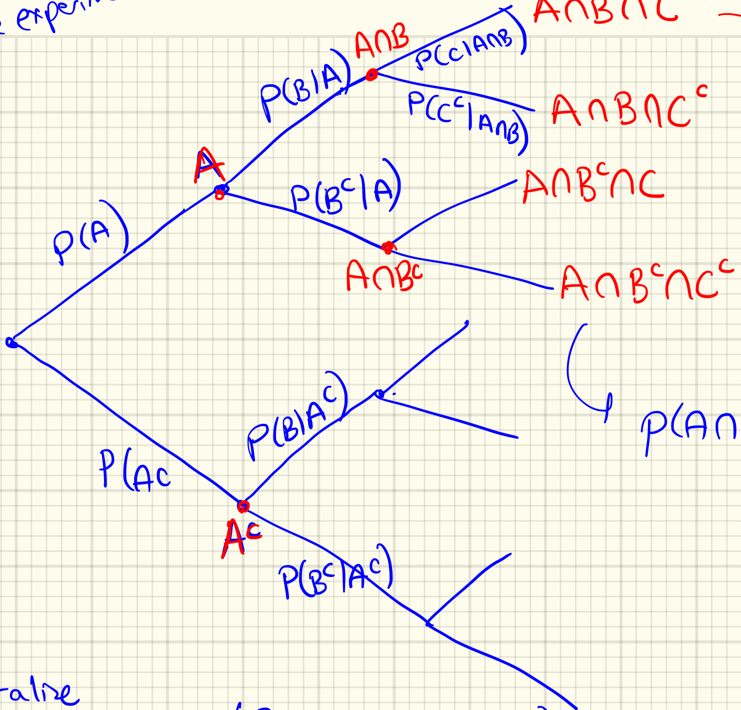


2 stage experiment : 1st select an urn w/ equally likely probabilities ; 2nd select a ball.

exercise: $P(C =$ (selecting a black ball from the 1st urn)).

P(selecting a black ball)

3-stage experiment



$P(A\cap B\cap C)$: move along the tree, multiply the probabilities along the path.

$$p(A\cap B\cap C) = p(A)\,p(B|A)\,p(C|A\cap B)$$

$$p(A\cap B^c\cap C^c) = P(A)\,P(B^c|A)\,P(C^c|A\cap B^c)$$

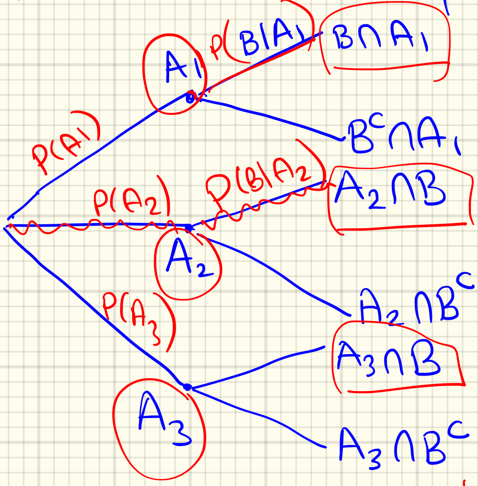
Generalize

→ Multiplication (Probability Chain) Rule :

$$p\left(\bigcap_i A_i\right) = p(A_1)\,p(A_2|A_1)\,p(A_3|A_1\cap A_2)\cdots p\left(A_n\,\middle|\,\bigcap_{i=1}^{n-1}A_i\right)$$

Total Probability Theorem

Partition the sample space into $\bigcup_i A_i$'s, A_i's disjoint.

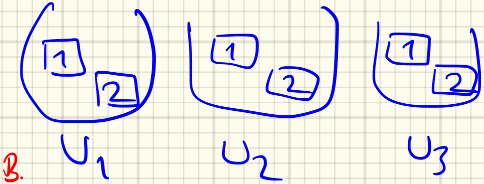


(B happened & A_1 happened)

(B not happened & ")

again like sampling from 3 urns,

Urns have 2 types of products $\boxed{1}$ & $\boxed{2}$



; B = select type $\boxed{1}$;

A_1 : select urn 1

A_2 : " " 2

A_3 : " " 3

total prob of B.

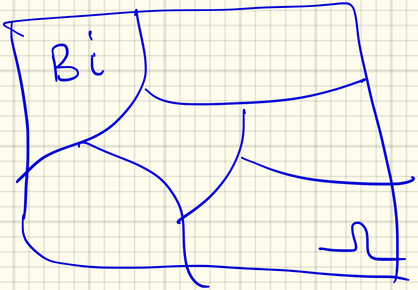
$$P(B) = P(B|A_1)P(A_1) + P(B|A_2) \cdot P(A_2) + P(B|A_3)P(A_3)$$

$$\sum_i P(A_i) = 1$$

Total Probability weighted addition of conditional probabilities.

Total Probability Law :

Let B_i be a $\underbrace{\text{partition}}$ of $\underbrace{\Omega = \bigcup_{i=1}^{N} B_i}$; $\boxed{B_i \cap B_j = \emptyset}$



$$P(A) = \sum_{i=1}^{N} P(A \cap B_i) = \sum_{i=1}^{N} P(A|B_i) \cdot P(B_i)$$

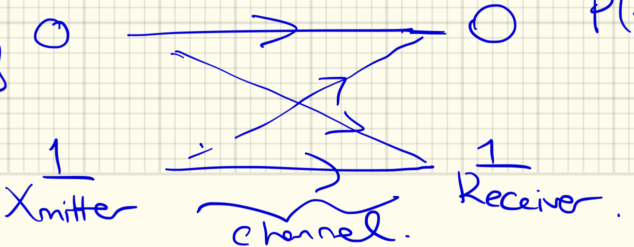
Example 4.3 (Kay) Prob. of error in a digital communication system.

$B_1 = \{0 \text{ xmitted}\}$

$B_2 = \{1 \text{ xmitted}\}$

$A = \{\text{error at the rcvr.}\}$

$P(A) = ?$



$$P(A) = P(B_1)\, P(A|B_1) + P(B_2)\, \underbrace{P(A|B_2)}$$

prob of receiving 1

prob. of receiving a 0.

$\boxed{P(A|B_i) = \varepsilon}$