04.10.2021

YZV 231E

Probability Theory & Stats

Week 1

Gü.

Today, we learn what it takes to set up a probabilistic model:

1) $\boxed{\text{Sample Space:}}$ contains all outcomes of a random experiment.

eg. Experiment: flipping a coin: Sample Space:

Outcomes: → Heads
→ Tails

cardinality of a set

$|S| = 2$

2) $\boxed{\text{Probability Law}}$: describe our beliefs which outcomes are more likely to occur compared to other outcomes.

Probability Laws should obey certain properties

→ AXIOMS OF PROBABILITY. :

3) Define $\boxed{\text{Events}}$ → sets of outcomes / subsets of the sample space.

4) Calculate Probabilites of Events

Sample space: we execute a particular experiment & we list all the outcomes

(experiment) eg. rolling a die

$\Omega = \{1, 2, 3, 4, 5, 6\}$

$|\Omega| = 6$

$|A| \triangleq$ cardinality of a set A
$= \#$ elements in A

Sample Space: List of outcomes:
should be:

1) Mutually exclusive: $\equiv$
from the experiment, we get only one of the outcomes.

2) Collectively exhaustive:
all the outcomes that may happen in the experiment are in S.

experiment: rolling two dice.

(1,1) (2,1) (3,1) (4,1) (5,1) (6,1)
(1,2) (2,2)
(1,3)
(1,4)
(1,5)
(1,6) (2,6) ... (6,6)

outcome $\leftarrow$ (1,1)
outcome $\leftarrow$ (1,4)

$S = \{(1,1), \ldots, (1,4), \ldots, (6,6)\}$

$|S| = 36$

$A \times B$ $\leftarrow$ cartesian product.

$A = \{\{1\},\{2\},\{3\},\{4\},\{5\},\{6\}\}$
$\times$
$B = \{\{1\},\{2\},\{3\},\{4\},\{5\},\{6\}\}$
$= \{(1,1), (1,2), \ldots\}$

(1,1) . . . . .

Ω:

- • H
- • T — you're wearing a white shirt
- • T — you're not wearing a white shirt

Ω:

- • H
- • T

$\hat{\Omega}$:

- H •
- • T

3) How much "granularity" you need in defining Ω

picking the "right" granularity is an art of engineering

→ Simplified models vs Complicated models.

Occam's Razor.

needed granularity

Einstein: "Everything should be made as simple as possible, but no simpler".

Ex: Rolling the die twice : single experiment → sequentially executed experiment.

hypothetical die w/ 4 faces → Single roll outcomes : $\{1\}, \{2\}, \{3\}, \{4\}$

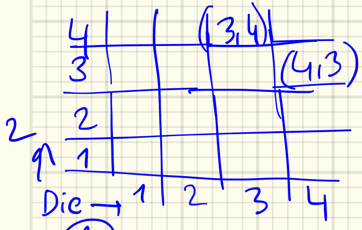
Q - What is the sample space of this experiment?

possible outcomes :

$$\Omega = \begin{matrix} (1,1) & (2,1) & (3,1) & (4,1) \\ (1,2) & (2,2) & (3,2) & (4,2) \\ (1,3) & (2,3) & (3,3) & (4,3) \\ (1,4) & (2,4) & (3,4) & (4,4) \end{matrix}$$

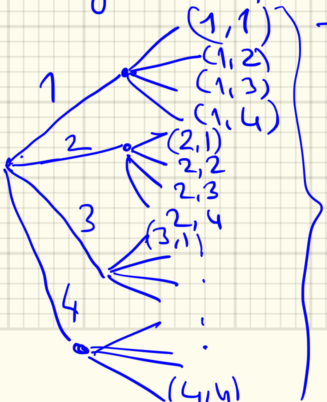
$(3,4)$ is different from $(4,3)$

$|\Omega| = 16$.



← visualization.

For a sequential description of the experiment, we can use a **TREE DIAGRAM**



→ leaves of the tree

Each leave ≡ outcome

16 leaves (outcomes).

Finite Sample space.

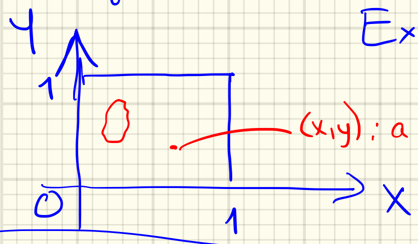
$$P((2,1)) = \frac{|E|}{|\Omega|} = \frac{1}{16}$$

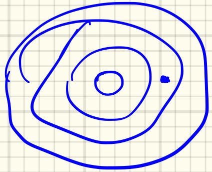
$E = \{(2,1)\}$

→ "uniform prob. law"

→ Infinite Sample Space : / continuous example.

Experiment : throwing a dart into the square $[0,1]\times[0,1]$





(x,y) : a possible outcome $(x,y)\in[0,1]\times[0,1]$

$P((x,y)=(0.5,0.1)) = ?\ 0$

From prev.

$$P(\underbrace{\text{sum of the numbers on the 2 dice being } = 5}) = \frac{4}{16} = \frac{1}{4}$$

$$E = \{(1,4),(4,1),(2,3),(3,2)\} \to |E| = 4$$

$\exists\ \infty$ many outcomes → ∞ many real numbers

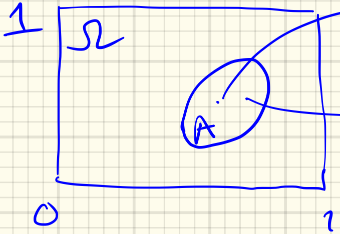
→ How do we calculate probabilities ?

↳ How do we assign probabilities to outcomes ?

In the continuous ex: any individual point has zero probability !

→ How do we work w/ this ? →

→ We assign probabilities to SUBSETS of the sample space; called an EVENT



Event A: We assign prob. to event A to find how likely that event is to occur.

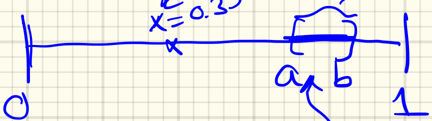
$P(A) = \text{Area of } A.$

Another ex

1D Dart

$p(x) = 0!$

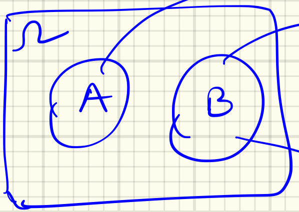
$x = 0.3$



$P(\text{hitting a certain interval}) = \text{Length of that interval.}$

$P([a,b]) = ?$

ex



$P(A \cup B) \Rightarrow$ total prob. is sum of individual probabilities of the two event.

Physical Analogy:

total mass := sum of 2 masses.

To study axioms of probability, let's refresh SETS;

sets: collections of object

$A = \{ \xi_1, \xi_2, \dots, \xi_n \}$:

← elements of A

$\xi_i \in A$

$\xi_i \notin A$

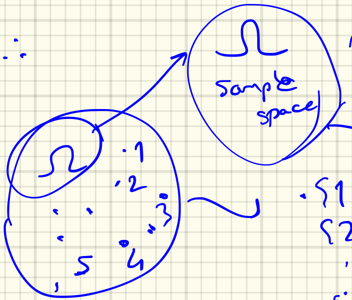
Empty set: $\emptyset$: contains no elements

(null set / <u>impossible event</u>).

$\emptyset$: has to be an event: due to axioms

— If A is an event, its complement A^c has to be an event.

∴ Ω is an event : $\Omega^c = \emptyset$ is an event.



eg $n=5$ → 32 subsets

$\{1\}$ $\{1,2\}$ $\{1,2,3\}$

$\{2\}$ $\{1,3\}$ ⋮

⋮ ⋮

$\{5\}$

* If <u>a set has n elements</u> → $\exists$ <u>2^n subsets</u> including $\emptyset$ & itself

there exists

Set Operations:

Ω: Universal set (Sample Space)

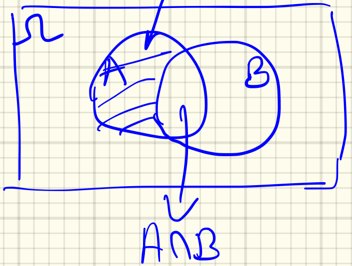
Union: $A \cup B = \{ x : x \in A \ \underline{\text{or}} \ x \in B \}$

Intersection: $A \cap B = \{ x : x \in A \ \underline{\text{and}} \ x \in B \}$

Complement: $A^c = \{ x \ x \in \Omega \text{ but } x \notin A \}$

Difference: $A - B = \{ x : x \in A \text{ and } x \notin B \}$

$P(A \cap B^c)$



* Sets A & B are disjoint $\underline{\text{if}}$ $A \cap B = \emptyset$

* A_i are called a <u>PARTITION of Ω</u> iff
$$\begin{cases} 1) \ A_i\text{'s are disjoint} \\ 2) \ \bigcup_{i=1}^{\infty} A_i = \Omega \end{cases}$$

For any subsets of Ω:

Commutative Law: $A \cap B = B \cap A$; $A \cup B = B \cup A$

Associative Law: $A \cup (B \cup C) = (A \cup B) \cup C$

$A \cap (B \cap C) = (A \cap B) \cap C$

Distributive Law: $(A \cup B) \cap C = (A \cap C) \cup (A \cap C)$

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

De Morgan Law: $\left(\bigcup_i A_i \right)^c = \bigcap_i A_i^c$

$\left(\bigcap_i A_i \right)^c = \bigcup_i A_i^c$

$A \subseteq B$: A is a subset of B

$\Leftrightarrow B \supseteq A$

ex: $[A \cap (B \cup C)]^c = ?\ A^c \cup (B^c \cap C^c)$

– If $B \subseteq A$ and $A \subseteq B$ then $A = B$.

Note: Duality: exchange intersection & union

complement any set

exchange Ω w/ $\emptyset$

Reverse inclusion $\subseteq \rightarrow \supseteq$

AXIOMS of PROBABILITY: They make our probability models work properly.

1) Nonnegativity : $P(A) \geq 0$

2) Normalization $P(\Omega) = 1$ — we're certain that outcome is an element of the sample space.

3) Additivity: If $A \cap B = \emptyset$ (A & B are disjoint events) $\rightarrow P(A \cup B) = P(A) + P(B)$

b/c (because)

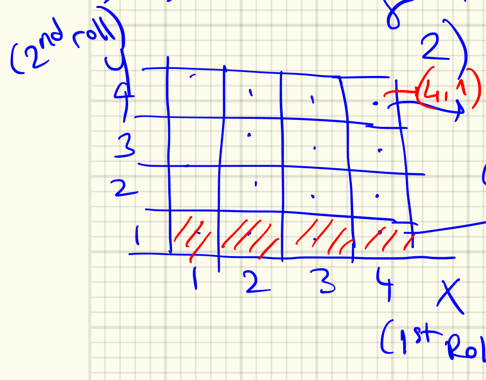
(2) b/c: Ω is collectively exhaustive

Probability Modeling

Continue

→ Ex: Rolling two dice (4-face die): 1) Defined the Sample Space Ω

2) Assign. probability Law: Discrete Uniform Probability Law



→ each outcome has a probability: $1/16$

Q: Probability that the 2nd roll gives "1" on the die?

→ Sum of all these "singleton" events (outcomes)

$(1,1), (2,1), (3,1), (4,1)$

$A_1 \quad A_2 \quad A_3 \quad A_4$

$$\bigcap_i A_i = \emptyset \rightarrow P\left(\bigcup_i A_i\right) = \sum_i P(A_i) = 4 \cdot \frac{1}{16} = \frac{1}{4}$$

some examples of games.

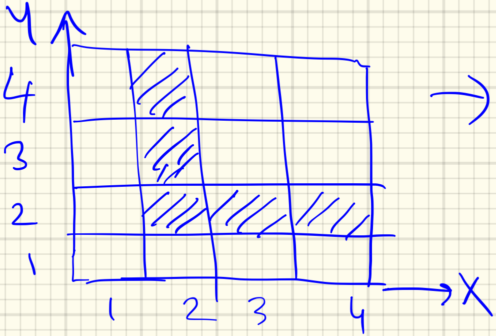
→ Discrete Uniform Probability Law :

→ – fair coins
– dice
– well-shuffled decks

- All outcomes are equally likely
- Computing probability → Counting

$$P(A) = \frac{|A|}{|\Omega|}$$

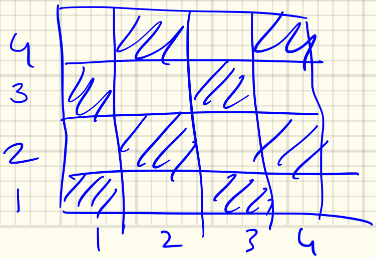
Q: → $P(\min(X, Y) = 2) = ?$ Event : min of the 2 rolls = 2



$\rightarrow \frac{5}{16}$

Q. $P(X+Y \text{ is even}) = ?$

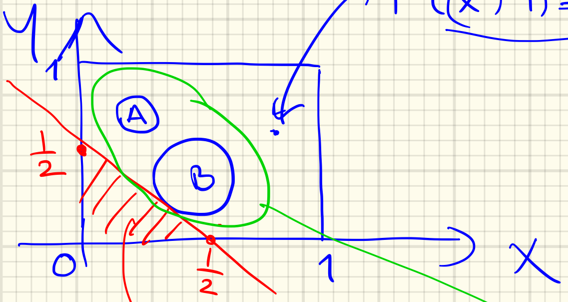
$= \frac{1}{2}$



→ Continuous Uniform Probability Law: → we measure areas

Recall the dart problem → now assign probabilities

$P((x,y)=(0.2,0.6)) = 0$ → This event has zero prob.



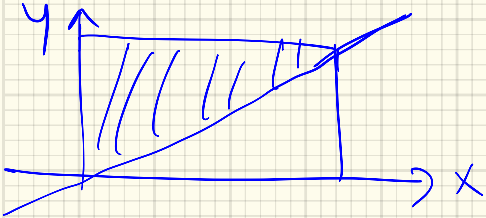
$P(x+y \leq \frac{1}{2}) = ?$

→ find prob (here area) of this event.

$= \frac{1}{8}$.

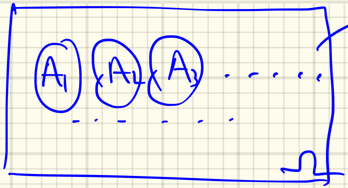
If B has double the area of A then $P(B) = 2 \times P(A)$

$P(X - Y \leq 0)$



$P(X \leq Y) = \frac{1}{2}$

→ Go to Axiom 3

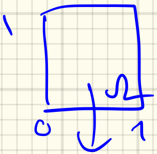


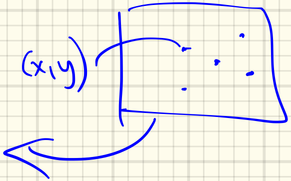
A_i are ∞- sequence of disjoint events

$\therefore$: therefore

$$P(A_1 \cup A_2 \cup A_3 \cup \ldots) = P(A_1) + P(A_2) + P(A_3) + \ldots$$

Consider again our continuous sample space





$$\Omega = \bigcup_{\forall (x,y)} \{(x,y)\}$$

$$P(\Omega) = 1 = P\left(\bigcup_{\forall (x,y)} \{(x,y)\}\right) \cancel{=} \underbrace{\sum P(\{(x,y)\})} = \sum_{\forall (x,y)} 0 = 0 \,!$$

Additivity axiom $\longrightarrow$ Countable additive ! $\therefore$ 3rd axiom does not apply to uncountable sets !

This union $\underbrace{\bigcup_{\forall (x,y)} \{(x,y)\}}_{\text{Uncountable.}} \neq \bigcup_{i=1}^{\infty} A_i \longrightarrow$ countable seq. of events.

(x,y) in $[0,1] \times [0,1]$

3) Countable Additivity.

3rd Axiom

∞. countable seq. of events $\{A_i\}_{i=1}^{\infty}$ $\rightarrow$ $\{1, 2, 3, 4, \ldots \ldots\}_{\infty}$

$\{A_1, A_2, \ldots \ldots\}_{\infty}$.

1–1 correspondence w/ the integers $\underline{\mathbb{Z}^+}$

$\rightarrow$ Note: Some weird things happening w/ continuous models

First, we will start w/ discrete models the go to continuous models.

Summarize: Probabilistic Modeling: Random experiment

1) Specify Sample Space: mutually exclusive, Collectively exhaustive, "Right" amount of granularity
2) Define a Probability Law;
3) Identify an Event of Interest:
4) Calculate probability of the Event.

$\rightarrow$ Axioms of Probability + Kolmogorov

$P(\Omega) = 1$

$P(A) \geq 0$

$\{A_i\}_{i=1}^{\infty}$ disjoint $P(\bigcup_i A_i) = \sum_i P(A_i)$