06.12.2021

YZV 231E

Probability Theory & Stats

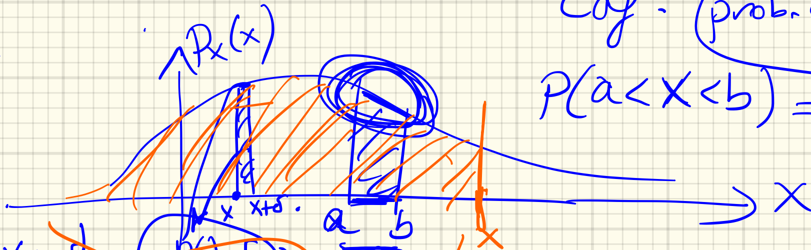
Gü.

Continuous R.V.'s (r.v. X) → prob. density function (pdf) $p_X(x)$

1) $p_X(x) \geq 0$
2) $\int_{-\infty}^{\infty} p_X(x)\,dx = 1$

cdf. prob. densities.

$$P(a<X<b) = \int_a^b p_X(x)\,dx$$

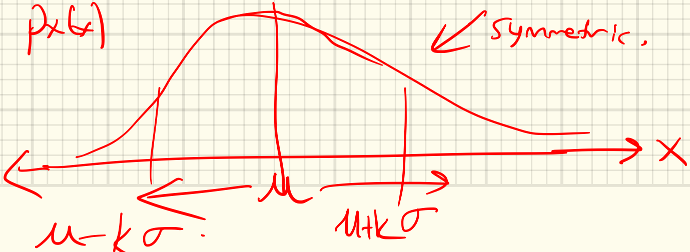


$$P(x \leq X < x+\delta) = p_X(x)\cdot\delta$$

CDF: $F_X(x) = P(X \leq x)$

pdf: Gaussian.

$$\mathcal{N}(\mu, \sigma^2)$$



symmetric.

Standard Normal

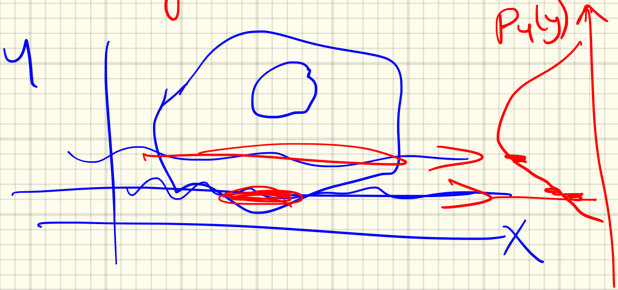
$$\mathcal{N}(0, 1)$$

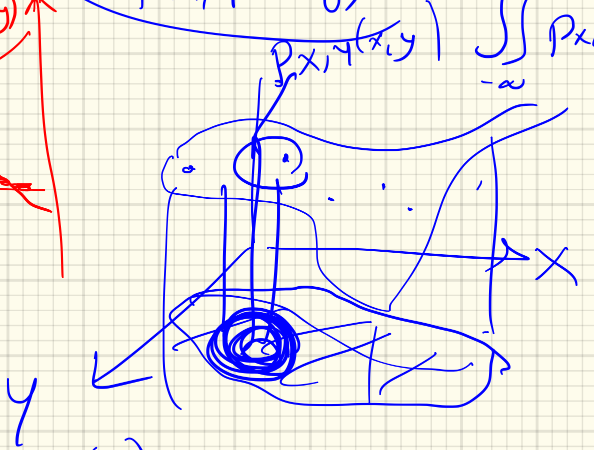
$E[X]$

$Var(X)$

Joint Pdfs : X, Y r.v.s $p_{X,Y}(x,y) \rightarrow p_{X,Y} \geq 0$

$$\iint_{-\infty}^{\infty} p_{X,Y}(x,y)\,dx\,dy = 1$$





Independence $p_{X,Y}(x,y) = p_X(x) \cdot p_Y(y)$

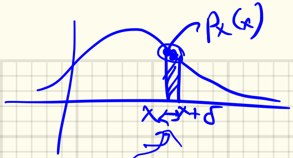
X, Y indep.

getting marginal pdf from the joint

$$p_Y(y) = \int_{-\infty}^{\infty} p_{X,Y}(x,y)\,dx.$$

Conditional Pdfs:

Recall: $P(x < X < x+\delta) \approx \underbrace{p_X(x)}\cdot\delta$



y

$P(x < X < x+\delta \mid Y \approx y) \approx p_{X|Y}(x|y)\cdot\delta.$

in light new knowledge

Given y

$$\to p_{X|Y}(x|y) = \frac{p_{X,Y}(x,y)}{p_Y(y) + \varepsilon}, \quad \text{if } \underline{p_Y(y) > 0}.$$

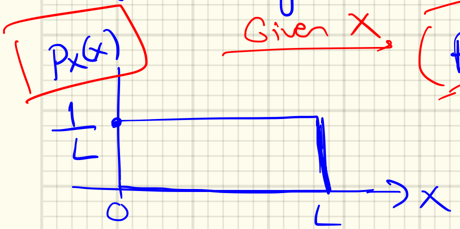
If X & Y are independent: Given Y = y

$$p_{X|Y}(x|y) = p_X(x) \quad \text{: marginal pdf.}$$

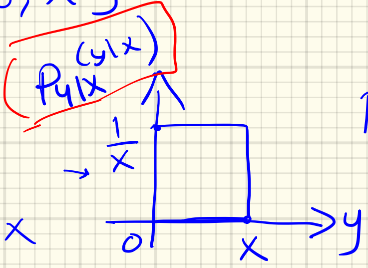
$\boxed{\text{Ex}}$: Start w/ a stick of certain length L; break it at a random location X;

$X \sim \text{Uniform}[0, L]$; break again at a random location; → Y: location

$Y \sim \text{Uniform}[0, X]$;



Given X →



Q. What's the joint pdf of X & Y:

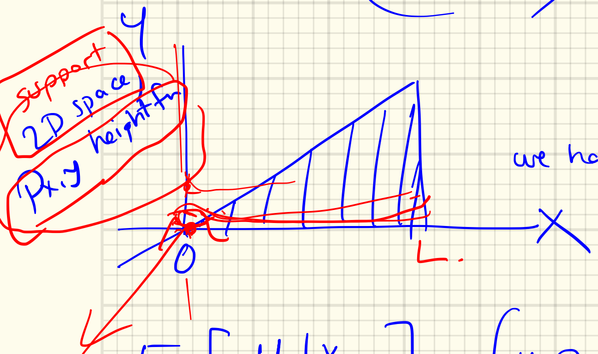
$$p_{X,Y}(x,y) = p_{Y|X}(y|x)\, p_X(x)$$

$$p_{X,Y}(x,y) = \frac{1}{L}\cdot\frac{1}{x} \; ; \quad x \in [0, L], \; y \in [0, x]$$

imagine in 3D.

$x = 0 \rightarrow p_{X,Y} \rightarrow \infty$

$x = L \quad p_{X,Y} = \frac{1}{L^2}$



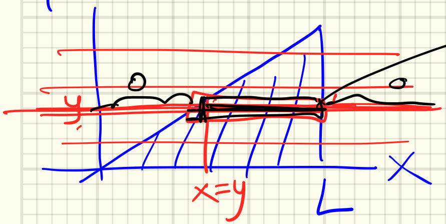
we have: $p_{X,Y}$ ✓ we have everything we need for our calculations.

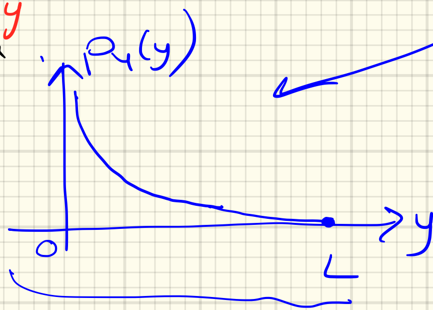
$$E[Y \mid X=x] = \int y \cdot p_{Y|X}(y \mid X=x)\, dy = \int_0^x y \cdot \frac{1}{x}\, dy = \frac{1}{x}\, \frac{y^2}{2}\Big|_0^x = \frac{x}{2}$$ ✓

Q. What is the density of Y : $p_Y(y)$?

$$p_Y(y) = \int \underbrace{p_{X,Y}(x,y)}_{\frac{1}{L.x}}\, dx = \int_y^L \frac{1}{L.x}\, dx = \frac{1}{L.}\ln x \Big|_y^L = \boxed{\frac{1}{L}\log\frac{L}{y}}$$

$0 \leq y \leq L.$





$$E[Y] = ? \int y\, p_Y(y)\, dy = \int_0^L \frac{1}{L}\, \underbrace{y \cdot \log\frac{L}{y}}\, dy = \boxed{\frac{L}{4}}$$

I.B.P $\rightarrow$ $\boxed{-\int y.\log y\, dy}$

Continuous Bayes' Rule: To make Inference:

unknown signal / information X → Random phenomenon → Y (observation)

we observe r.v. Y.

Unknown.

$p_X(x)$: prior belief

Measurement

$p_{Y|X}(y|x)$: make a model conditional pdf (likelihood)

Target: $p_{X|Y}(x|y)$: posterior density

unknown.

Recall: Discrete case, we worked on it

$X = 1/0$: airplane Present / Absent

$Y = 1/0$: radar fired / or not.

$p_{X|Y}(x|y)$ unobserved.

$p_{Y|X}$ observed side

Now, in continous case

$$\underbrace{p_{X|Y}(x|y)}_{\text{posterior density}} = \frac{p_{X,Y}(x,y)}{p_Y(y)} = \frac{\overbrace{p_{Y|X}(y|x)}^{\text{likelihood}} \cdot \overbrace{p_X(x)}^{\text{prior}}}{p_Y(y)}$$

Evidence: $p_Y(y) = \int_x \underbrace{p_X(x)\, p_{Y|X}(y|x)}_{p_{X,Y}(x,y)}\, dx$

CT Inference:

X some signal : "prior" $p_X(x)$

Y : noisy version of the signal X

$p_{Y|X}(y|x)$: model of the noise.

$p_{X|Y}(x|y)$

std example in communications / signal processing:

Ex: Discrete : $X = 0, 1 \longrightarrow$ xmit

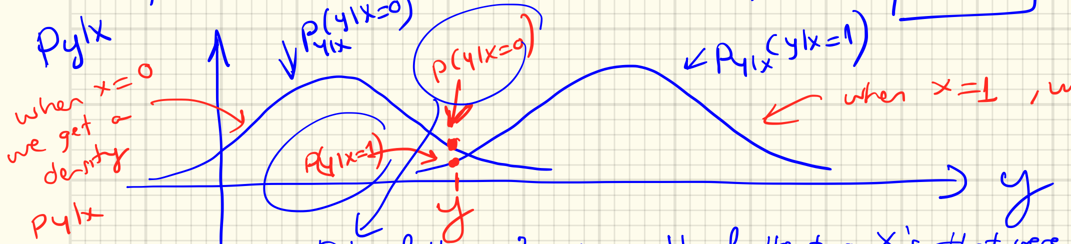
we measure

Continuous : $Y = \underbrace{X + W}$ say Gaussian distributed noise

model of the measurement device

measurement model.

$X \longrightarrow \boxed{Y|X} \longrightarrow Y$



ratio of those 2 gives odds of the two X's that were sent.

$y \leq Y \leq y+\delta$

$$P(\overbrace{X=x}^{\text{discrete r.v.}}, \overbrace{y \leq Y \leq y+\delta}^{\text{cont. r.v.}}) = \underbrace{P(X=x)}_{\text{discret}} \underbrace{P(y \leq Y \leq y+\delta \mid X=x)}_{\text{cont.}} = \underbrace{P(y \leq Y \leq y+\delta)}_{\text{cont}} \cdot \underbrace{P(X=x \mid \frown)}_{\text{discrete}}$$

$$\underbrace{p_X(x)}_{\text{pmf}} \cdot \underbrace{p_{Y|X}(y|x)}_{\text{cond. pdf}} \cdot \delta \approx \underbrace{p_Y(y)}_{\text{cont. pdf}} \cdot \delta \cdot \underbrace{p_{X|Y}(x|y)}_{\text{conditional pmf}}$$

posterior:

$$p_{X|Y}(x|y) = \frac{p_{Y|X}(y|x) \cdot p_X(x)}{p_Y(y)}$$

Ex: If Continuous X, Discrete Y:

X (light intensity) $\rightarrow$ Measuring Device $\rightarrow$ Y: photon count: measurement.

We're trying to infer $X|Y$.

$$P_{X|Y}(x|y) = \frac{P_{Y|X}(y|x)\, P_X(x)}{P_Y(y)}$$

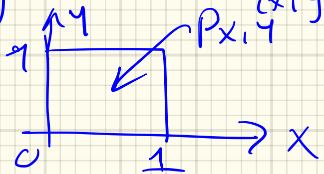
: The same formula Bayes,

How you model these probability densities or pmfs.] matter.

priors, conditionals $\rightarrow$

Distribution of Transformed R.V.'s $\equiv$ Derived Distributions.

e.g. Given $p_{X,Y}(x,y)$



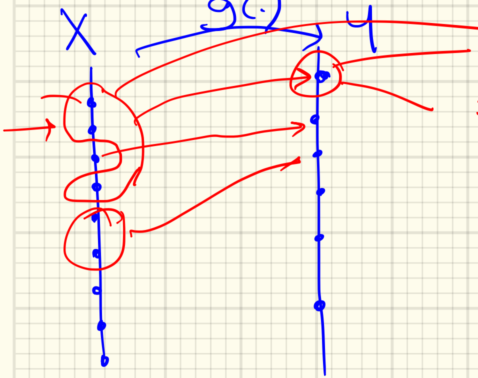
$$g(X,Y) = \underbrace{\frac{Y}{X}}_{\text{ratio of } X \,\&\, Y \text{ r.v.s.}} = Z$$

$\Rightarrow Z$: a new r.v.

$$\rightarrow E[g(X,Y)] = \iint g(x,y)\, p_{X,Y}(x,y)\, dx\, dy$$

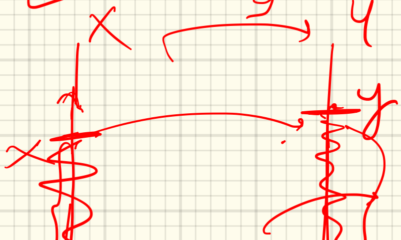
Now But, we want distribution of $g(X,Y)$:

Recall: Discrete case (very easy)



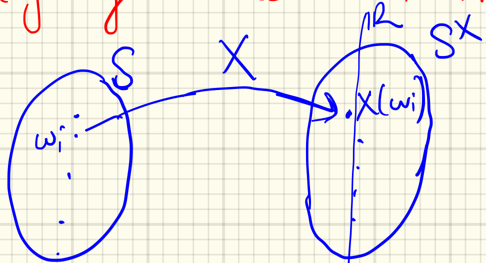
these are equivalent events :

$$p_Y(y_i) = \sum_{x_i \in g^{-1}(y_i)} p_X(x_i)$$

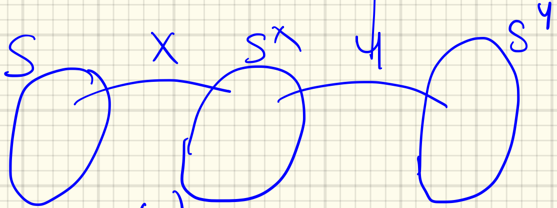


$$P(Y \le y) \ne P(X \le x)$$

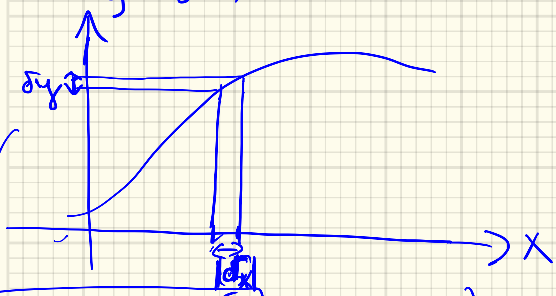
1) Use Change of Variables Formula to Transform R.V.s.



$X : S \longrightarrow S^X \subseteq \mathbb{R}$



$y = g(x)$



$p_Y(y)\,|\delta y| = p_X(x)\,|\delta x|$

prob. prob.

$y = g(x)$

Given r.v. X $p_X(x)$, $y = g(x)$ $\Longrightarrow$ $p_Y(y) = ?$

$$p_Y(y) = p_X(x)\left|\frac{dx}{dy}\right|$$

$$p_X(x) = p_Y(y)\left|\frac{dy}{dx}\right|$$

$(x_1 \dots x_n) \to (y_1 \dots y_m)$

Jacobian matrix

$$J = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \cdot & \\ \vdots & & & \\ \frac{\partial y_m}{\partial x_1} & \cdots & & \end{bmatrix}$$

$$p_Y(y) = p_X(\underbrace{g^{-1}(y)}_{x})\left|\frac{d\, g^{-1}(y)}{dy}\right|$$

$g(.)$ is one-to-one (invertible)

eg. ~~Linear~~ (Affine) Transformation of r.v.'s : $Y = aX + b$, $a \in \mathbb{R}$, $a \neq 0$, $b \in \mathbb{R}$,

Given X & $P_X(x)$, $P_Y(y) = ?$

$$Y = g(X)$$

$$g^{-1}(y) = \frac{y-b}{a}$$

$$P_Y(y) = P_X\left(\frac{y-b}{a}\right) \cdot \left|\frac{dx}{dy}\right|$$

$$\left|\frac{d(g^{-1}y)}{dy}\right| = \frac{1}{a}.$$

$$P_Y(y) = \frac{1}{|a|} P_X\left(\frac{y-b}{a}\right)$$

$\boxed{\text{Ex:}}$ $Y = aX + b$ Affine transform, $X \sim \mathcal{N}(0,1) \to Y = ?$

a, b scalars , $a \neq 0$

$\underline{S_X = (-\infty, \infty)} \to S_Y = (-\infty, \infty)$

$$P_Y(y) = \frac{1}{|a|} P_X\left(\frac{y-b}{a}\right) = \frac{1}{|a|} \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-b)^2}{2a^2}} \to Y \sim \mathcal{N}(b, a^2)$$

affine transformed Gaussian r.v.'s are still Gaussians w/ a new mean & variance.

Ex: $\underbrace{X \sim \mathcal{N}(0,1)}_{S_X = (-\infty, \infty)}$; $\underbrace{Y = e^X}_{S_Y = (0, \infty)}$;

$y = g(x) = e^x$

$\hookrightarrow x = \ln y = g^{-1}(y)$

$$\frac{d\, g^{-1}(y)}{dy} = \frac{1}{y}$$

$$P_Y(y) = \begin{cases} P_X(\ln y) \cdot \left|\frac{1}{y}\right|, & y > 0 \\ 0, & y \leq 0 \end{cases}$$

$$P_Y(y) = \begin{cases} \frac{1}{\sqrt{2\pi}\, y} e^{-\frac{1}{2}(\ln y)^2}, & y > 0 \\ 0, & y \leq 0 \end{cases}$$

called log-normal pdf

Note: Always determine the support of pdfs $P_X(x) \to P_Y(y)$

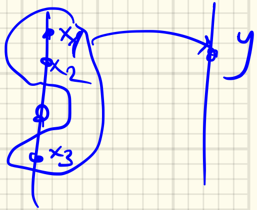
* If the transformation $g(\cdot)$ is $\boxed{\text{many-to-one}}$ mapping

$y = g(x)$ $\quad$ (eg $\underbrace{g(x) = x^2 = y}_{y=x^2}$) $\Rightarrow$ $g^{-1}(y) = \mp\sqrt{y}$

$\rightarrow x = \mp\sqrt{y}$

$= \{x : g(x) = y\}$

$$P_y(y) = \sum_{x \in g^{-1}(y)} p_x(x) \left| \frac{1}{\frac{dg(x)}{dx}} \right|$$



re-write

or $\quad x_i = g_i^{-1}(y) \quad$ for $\quad i = 1, \ldots, M$; M elements in $g^{-1}(y)$.

$$\boxed{P_y(y) = \sum_{i=1}^{M} p_x\left(g_i^{-1}(y)\right) \left| \frac{dg_i^{-1}(y)}{dy} \right|}$$

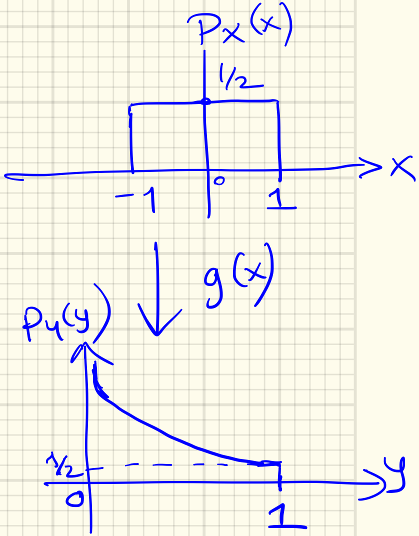
Ex: $Y = X^2$, $X \sim U[-1, 1]$

$Y = g(X)$, $S_X = [-1, 1]$

$S_Y = [0, 1]$ $\Longleftrightarrow$

$\boxed{x_1 = \sqrt{y}} \longrightarrow \dfrac{dg^{-1}}{dy} = \dfrac{1}{2\sqrt{y}}$

$x_2 = -\sqrt{y} \longrightarrow \dfrac{dg^{-1}}{dy} = -\dfrac{1}{2\sqrt{y}}$



$$p_Y(y) = p_X(\sqrt{y}) \cdot \frac{1}{|2\sqrt{y}|} + p_X(-\sqrt{y}) \left| \frac{1}{-2\sqrt{y}} \right|$$

$$= \frac{1}{2} \frac{1}{2\sqrt{y}} + \frac{1}{2} \cdot \frac{1}{2\sqrt{y}} = \frac{1}{2\sqrt{y}}, \quad y \in [0, 1]$$

Exercise: $Y = X^2$, $X \sim N(0, 1)$

Derive $p_Y(y)$.

[S. Kay Example 10.7]

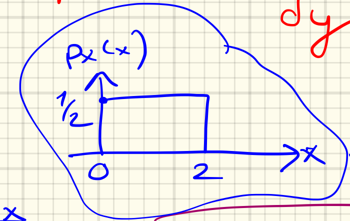
② nd Way: CDF Approach to determine Transformed R.V.'s pdf:

2-step procedure: $X \xrightarrow{g} Y$.

1) Get CDF of Y : $F_Y(y) = P(Y \le y)$
2) Differentiate to get $p_Y(y) = \dfrac{dF_Y(y)}{dy}$

Ex: X : uniform on $[0,2]$

$Y = X^3$, Find pdf of Y.



$S_X = [0,2]$
$S_Y = [0,8]$

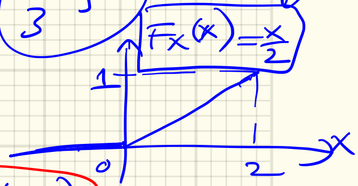
① st way: $p_Y(y) = p_X(\underset{\uparrow\, g^{-1}(y)}{x}) \left| \dfrac{d\, \overbrace{g^{-1}(y)}^{x}}{dy} \right| = \underbrace{p_X(y^{1/3})}_{\frac{1}{2}} \cdot \dfrac{1}{3y^{2/3}}$

$\dfrac{1}{\left|\dfrac{dy}{dx}\right|}$

$\left|\dfrac{dx}{dy}\right|$; $x = (y)^{1/3} \Rightarrow \dfrac{1}{3} y^{-2/3}$

$\to 3y^{2/3}$

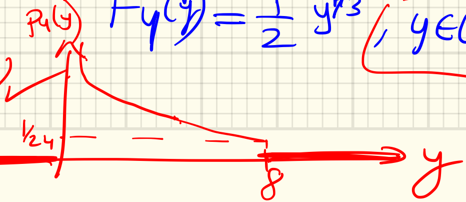
2nd way: i) $F_Y(y) = P(Y \le y)$
$= P(X^3 \le y) = P(X \le y^{1/3}) = F_X(y^{1/3})$



$F_Y(y) = \dfrac{1}{2} y^{1/3}$, $y \in (0,8)$

(ii) $p_Y(y) = \dfrac{d}{dy}\left(\dfrac{1}{2} y^{1/3}\right) = \boxed{\dfrac{1}{6y^{2/3}}}$

$y = 8$: $\dfrac{1}{6(2^3)^{2/3}}$



Ex: Ece driving from istanbul to Edirne. Her speed uniformly distrib. btw (30, 60) km/h. What is the distribution of the duration of the trip?

distance = 200km.

$$V \sim U[30, 60] \leftarrow p_V(v)$$

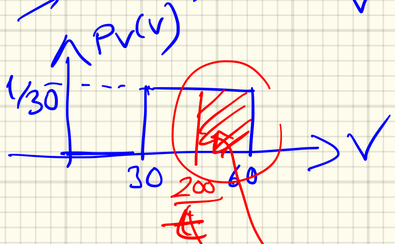
r.v.

$$\text{Let } T(V) = \frac{200}{V}, \quad p_T(t) = ?$$

$$v \in (30, 60)$$

$$30 \leq \frac{200}{t} \leq 60$$

$$\boxed{\frac{200}{60} \leq t \leq \frac{200}{30}}$$

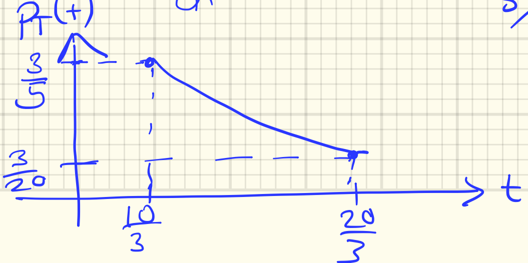


Use CDF way

i)
$$F_T(t) = P(T \leq t)$$
$$= P\left(\frac{200}{V} \leq t\right)$$
$$= P\left(V \geq \frac{200}{t}\right)$$

$$F_T(t) = \frac{1}{30} \cdot \left(60 - \frac{200}{t}\right),$$

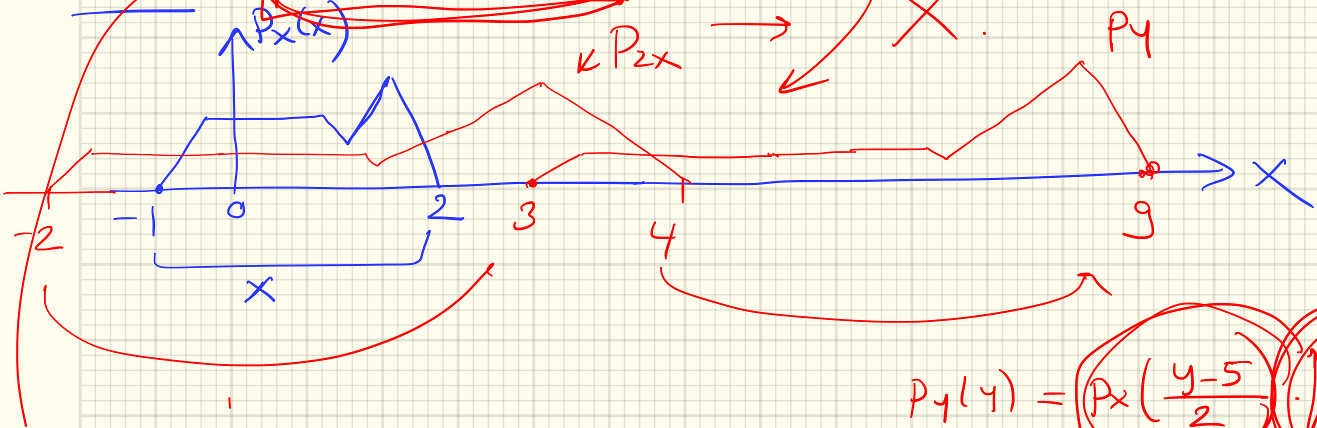
ii)
$$p_T(t) = \frac{d}{dt} F_T(t) = \frac{20\cancel{0}}{3\cancel{0}} \cdot \frac{1}{t^2}, \quad \frac{10}{3} \leq t \leq \frac{20}{3}$$



Ex:

$$\boxed{y = 2x + 5}$$

$$\cancel{y = \frac{x}{2} - 5}$$



$$P_y(y) = P_x\left(\frac{y-5}{2}\right) \cdot \left|\frac{dx}{dy}\right| \quad \frac{1}{2}$$

$$\left|\frac{dy}{dx}\right|$$

Distribution of $X+Y$:

$\boxed{W = X+Y}$, X & Y independent.

$\boxed{W = g(X,Y)}$ $\longrightarrow$ find $p_W(w)$

transformation of multiple r.v.s.

3 Use Conditioning to find $p_W(w)$:

1) Fix $X=x$, let $W|X=x = g(x,Y) = g_x(Y)$: function of Y (x is fixed)

2) Find $p_{W|X}(w|x)$ using transformation of $Y \to W$: $W = g_x(Y)$

3) Uncondition to find $p_W(w) = \displaystyle\int_{-\infty}^{\infty} \underbrace{p_{W|X}(w|x)\, p_X(x)}_{p_{W,X}}\, dx$

ep. $\boxed{W = \underset{\uparrow \text{const.}}{x} + Y} = g_x(Y)$

$p_W(w) = \displaystyle\int_{-\infty}^{\infty} p_{W|X}(w|x)\, p_X(x)\, dx = \boxed{\displaystyle\int_{-\infty}^{\infty} p_Y(w-x)\, p_X(x)\, dx}$

$= p_{Y|X}(w-x) = p_Y(w-x)$ (x const) (X & Y indep.)

CONVOLUTION. operation:

X & Y indep : $\boxed{W = X+Y}$

$\boxed{p_W(w) = p_X * p_Y}$

$w = x + x$

$w = x_i + x_i + x_i$