# 20: Maximum Likelihood Estimation

Jerry Cain
May 12, 2021

# Quick slide reference

# Intro to parameter estimation

# Story so far

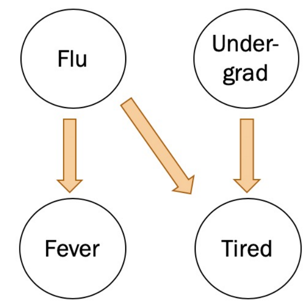
At this point:

If you are given a **model** with all the necessary probabilities, you can make predictions.

$Y \sim \text{Poi}(5)$

$X_1, \ldots, X_n$ i.i.d. $X_i \sim \text{Ber}(0.2)$, $X = \sum_{i=1}^{n} X_i$

$P(F_{lu} = 1) = 0.1$ $P(U = 1) = 0.8$




$P(F_{ev} = 1 | F_{lu} = 1) = 0.9$
$P(F_{ev} = 1 | F_{lu} = 0) = 0.05$

$P(T = 1 | F_{lu} = 0, U = 0) = 0.1$
$P(T = 1 | F_{lu} = 0, U = 1) = 0.8$
$P(T = 1 | F_{lu} = 1, U = 0) = 0.9$
$P(T = 1 | F_{lu} = 1, U = 1) = 1.0$

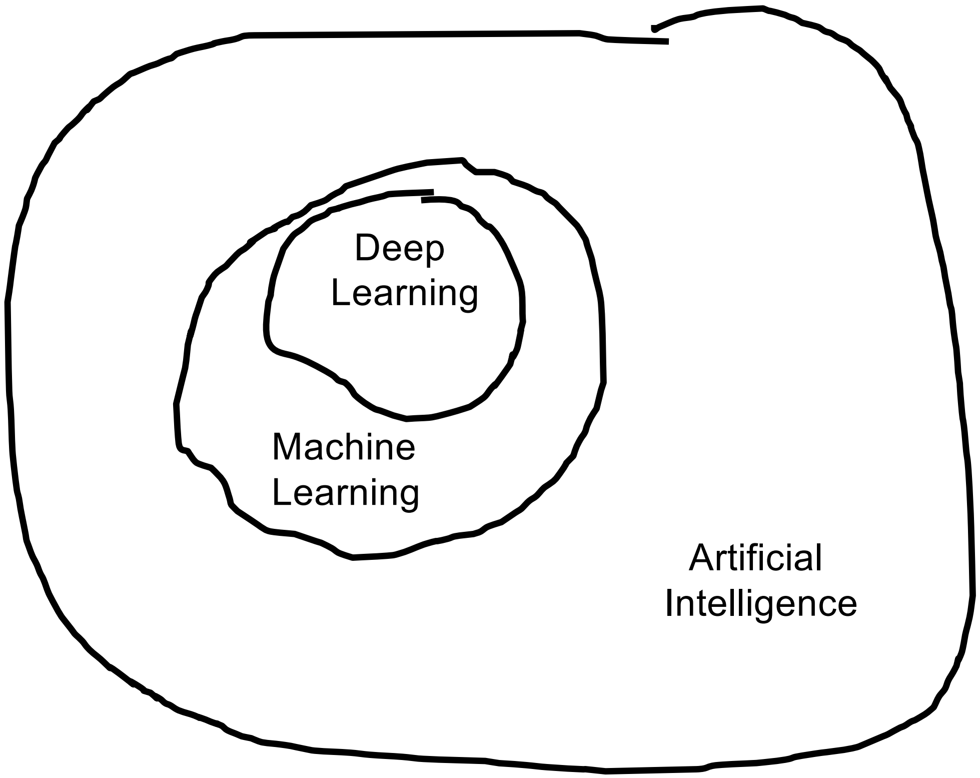
But what if you want to **learn** the probabilities in the model?

~~What if you want to learn the structure of the model, too?~~

(I wish… another day)

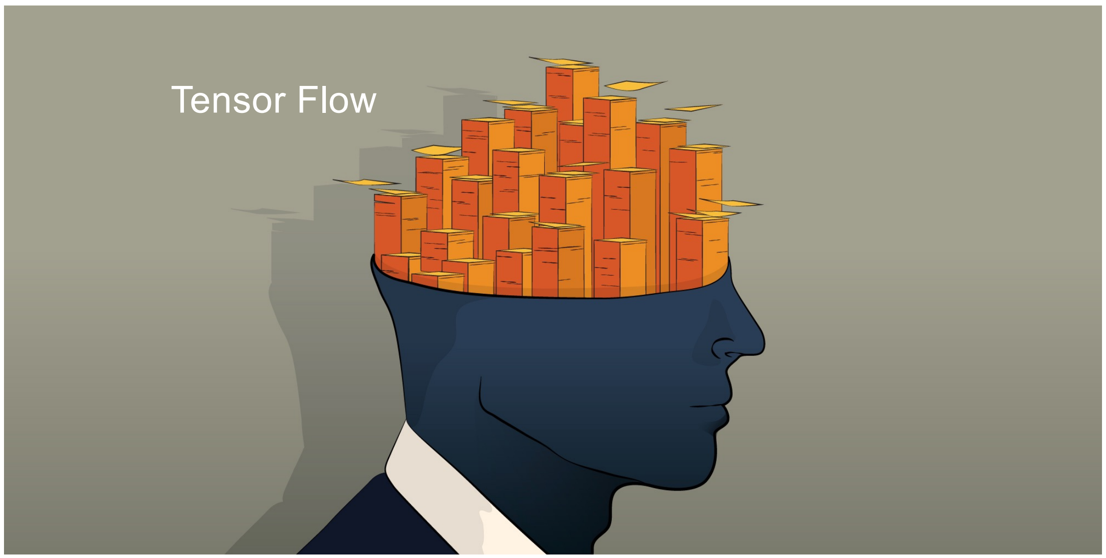
Machine Learning

# AI and Machine Learning




ML: Rooted in probability theory

# Alright, so Deep Learning now?




Not so fast…


Walt Disney
PICTURES

# Once upon a time…

…there was parameter estimation.

# Recall some estimators

$X_1, X_2, \ldots, X_n$ are $n$ i.i.d. random variables,
where $X_i$ drawn from distribution $F$ with $E[X_i] = \mu, \mathrm{Var}(X_i) = \sigma^2$.

Sample mean:

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

unbiased estimate of $\mu$

Sample variance:

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

unbiased estimate of $\sigma^2$

# What are parameters?

def Many random variables we have learned so far are parametric models:

$$\text{Distribution} = \text{model} + \text{parameter } \theta$$

ex The distribution Ber$(0.2)$ = Bernoulli model, parameter $\theta = 0.2$.

For each of the distributions below, what is the parameter $\theta$?

1. Ber$(p)$ $\theta = p$
2. Poi$(\lambda)$
3. Uni$(\alpha, \beta)$
4. $\mathcal{N}(\mu, \sigma^2)$
5. $Y = mX + b$

# What are parameters?

def Many random variables we have learned so far are parametric models:

Distribution = model + parameter $\theta$

ex The distribution $\text{Ber}(0.2)$ = Bernoulli model, parameter $\theta = 0.2$.

For each of the distributions below, what is the parameter $\theta$?

1. $\text{Ber}(p)$ $\quad\theta = p$
2. $\text{Poi}(\lambda)$ $\quad\theta = \lambda$
3. $\text{Uni}(\alpha, \beta)$ $\quad\theta = (\alpha, \beta)$
4. $\mathcal{N}(\mu, \sigma^2)$ $\quad\theta = (\mu, \sigma^2)$
5. $Y = mX + b$ $\quad\theta = (m, b)$

$\theta$ is the parameter of a distribution.
$\theta$ can be a vector of parameters!

# Why do we care?

In the real world, we don't know the "true" parameters.

- But we do get to observe data: (# times coin comes up heads, lifetimes of disk drives produced, # visitors to website per day, etc.)

def estimator $\hat{\theta}$: random variable estimating parameter $\theta$ from data.

In parameter estimation,

We use the point estimate of parameter estimate (best single value):

- Better understanding of the process producing data
- Future predictions based on model
- Simulation of future processes

# Maximum Likelihood Estimator

# Defining the likelihood of data: Bernoulli

Consider a sample of $n$ i.i.d. random variables $X_1, X_2, \ldots, X_n$.

- $X_i$ was drawn from distribution $F = \text{Ber}(\theta)$ with unknown parameter $\theta$.
- Observed data:

$$[0, 0, 1, 1, 1, 1, 1, 1, 1, 1] \qquad (n = 10)$$

How likely was the observed data if $\theta = 0.4$?

$0.6^2 \quad 0.4^8$

$$P(\text{sample}|\theta = 0.4) = \underbrace{(0.4)^8(0.6)^2}_{\text{Likelihood of data given parameter } \theta = 0.4} = 0.000236$$

Likelihood of data
given parameter $\theta = 0.4$

Is there a better parameter $\theta$?

# Defining the likelihood of data

Consider a sample of $n$ i.i.d. random variables $X_1, X_2, \dots, X_n$.

- $X_i$ was drawn from a distribution with density function $f(X_i|\theta)$.
  or mass
- Observed data: $(X_1, X_2, \dots, X_n)$

Likelihood question:

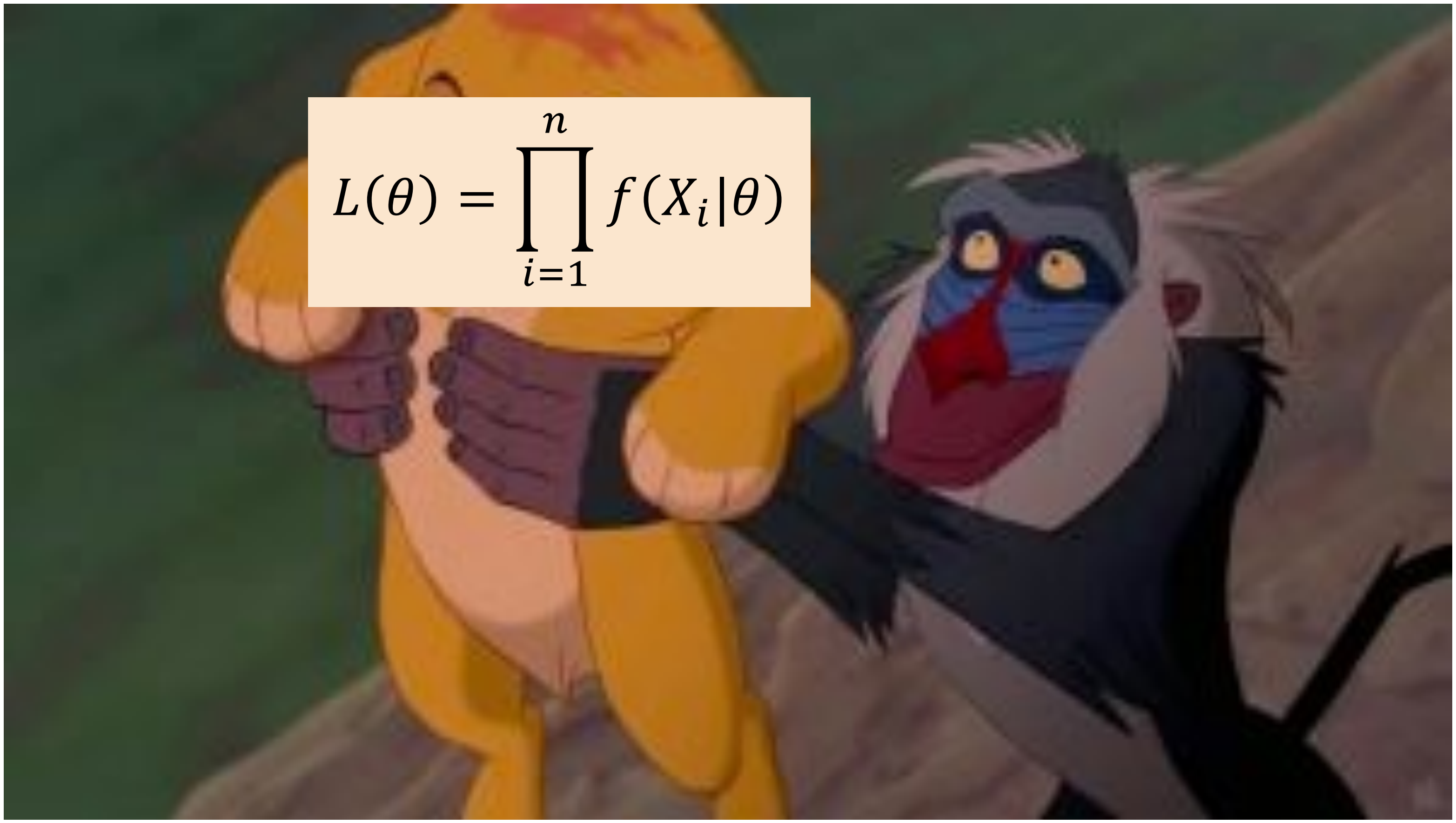
How likely is the observed data $(X_1, X_2, \dots, X_n)$ given parameter $\theta$?

**Likelihood function, $L(\theta)$:**

$$L(\theta) = f(X_1, X_2, \dots, X_n|\theta) = \boxed{\prod_{i=1}^{n} f(X_i|\theta)}$$

This is just a product, since $X_i$ are i.i.d.

# Defining the likelihood of data



$$L(\theta) = \prod_{i=1}^{n} f(X_i|\theta)$$

# Maximum Likelihood Estimator

Consider a sample of $n$ i.i.d. random variables $X_1, X_2, \ldots, X_n$, drawn from a distribution $f(X_i|\theta)$.

def The **Maximum Likelihood Estimator (MLE)** of $\theta$ is the value of $\theta$ that maximizes $L(\theta)$.

$$\theta_{MLE} = \arg\max_{\theta} \; L(\theta)$$

# Maximum Likelihood Estimator

Consider a sample of $n$ i.i.d. random variables $X_1, X_2, \ldots, X_n$, drawn from a distribution $f(X_i|\theta)$.

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$$\theta_{MLE} = \arg\max_{\theta} \boxed{L(\theta)}$$

Likelihood of your sample

$$L(\theta) = \prod_{i=1}^{n} f(X_i|\theta)$$

For continuous $X_i$, $f(X_i|\theta)$ is PDF; for discrete $X_i$, $f(X_i|\theta)$ is PMF

# Maximum Likelihood Estimator

Consider a sample of $n$ i.i.d. random variables $X_1, X_2, \ldots, X_n$, drawn from a distribution $f(X_i|\theta)$.

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$$\theta_{MLE} = \arg\max_{\theta} L(\theta)$$

The argument $\theta$
that maximizes $L(\theta)$

Stay tuned!

(live)

# 20: Maximum Likelihood Estimation

Jerry Cain

May 21, 2020

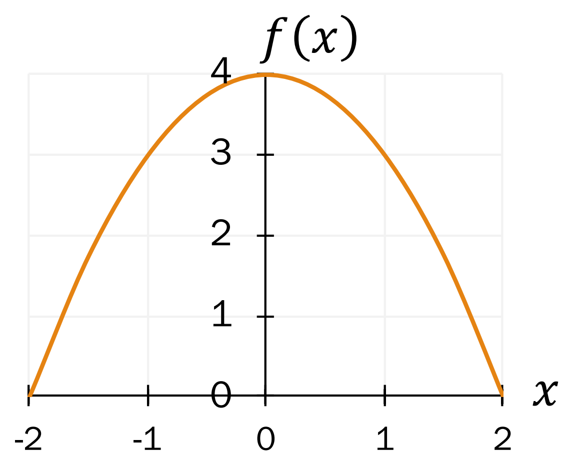
# argmax and log likelihood

# New function: arg max

$$\arg\max_x f(x)$$

The argument $x$ that maximizes the function $f(x)$.

Let $f(x) = -x^2 + 4$, where $-2 < x < 2$.




1. $\max_x f(x)$ ?

$$= 4$$

2. $\arg\max_x f(x)$ ?

$$= 0$$

# Argmax properties

$$\arg\max_x f(x)$$

The argument $x$ that maximizes the function $f(x)$.

$$= \arg\max_x \log f(x)$$

(log is an increasing function: $x < y \Leftrightarrow \log x < \log y$)

$$= \arg\max_x (c \log f(x))$$

for any positive constant $c$

$(x < y \Leftrightarrow c \log x < c \log y)$

# Finding the argmax with calculus

$$\hat{x} = \arg\max_x f(x)$$

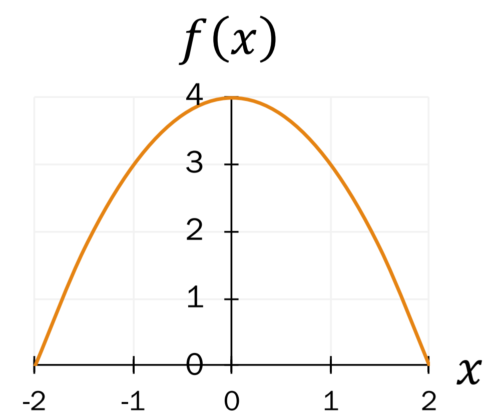
Let $f(x) = -x^2 + 4$,
where $-2 < x < 2$.

Differentiate w.r.t. argmax's argument

$$\frac{d}{dx} f(x) = \frac{d}{dx}(x^2 + 4) \quad = 2x$$

Set to 0 and solve

$$2x = 0 \qquad \Rightarrow \qquad \hat{x} = 0$$




Make sure $\hat{x}$ is a maximum

- Check $f(\hat{x} \pm \epsilon) < f(\hat{x})$
- Often ignored in expository derivations
- We'll ignore it here too (and won't require it in class)

# Maximum Likelihood Estimator

Consider a sample of $n$ i.i.d. random variables $X_1, X_2, \ldots, X_n$, drawn from a distribution $f(X_i|\theta)$.

$$L(\theta) = \prod_{i=1}^{n} f(X_i|\theta)$$

$\theta_{MLE}$ maximizes the likelihood of our sample, $L(\theta)$:

$$\theta_{MLE} = \arg\max_{\theta} \ L(\theta)$$

$\theta_{MLE}$ also maximizes the **log-likelihood function, $LL(\theta)$**:

$$\theta_{MLE} = \arg\max_{\theta} \ LL(\theta)$$

$$LL(\theta) = \log L(\theta) = \log\left(\prod_{i=1}^{n} f(X_i|\theta)\right) = \sum_{i=1}^{n} \log f(X_i|\theta)$$

$LL(\theta)$ is often easier to differentiate than $L(\theta)$.

# MLE: Bernoulli

# Computing the MLE

$$\theta_{MLE} = \arg\max_{\theta} LL(\theta)$$

General approach for finding $\theta_{MLE}$ , the MLE of $\theta$:

1. Determine formula for $LL(\theta)$

$$LL(\theta) = \sum_{i=1}^{n} \log f(X_i|\theta)$$

2. Differentiate $LL(\theta)$ w.r.t. (each) $\theta$

$$\frac{\partial LL(\theta)}{\partial \theta}$$

To maximize:
$$\frac{\partial LL(\theta)}{\partial \theta} = 0$$

3. Solve resulting equations

(algebra or computer)

4. Make sure derived $\hat{\theta}_{MLE}$ is a maximum
   - Check $LL(\theta_{MLE} \pm \epsilon) < LL(\theta_{MLE})$
   - Often ignored in expository derivations
   - We'll ignore it here too (and won't require it in class)

$LL(\theta)$ is often easier to differentiate than $L(\theta)$.

# Maximum Likelihood with Bernoulli

Consider a sample of $n$ i.i.d. RVs $X_1, X_2, \dots, X_n$.

What is $\theta_{MLE} = p_{MLE}$?

- Let $X_i \sim \text{Ber}(p)$.

1. Determine formula for $LL(\theta)$

$$LL(\theta) = \sum_{i=1}^{n} \log f(X_i|p)$$

?

$$f(X_i|p) = \begin{cases} p & \text{if } X_i = 1 \\ 1 - p & \text{if } X_i = 0 \end{cases}$$


2. Differentiate $LL(\theta)$ w.r.t. (each) $\theta$, set to 0

3. Solve resulting equations

# Maximum Likelihood with Bernoulli

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What is $\theta_{MLE} = p_{MLE}$?

- Let $X_i \sim \text{Ber}(p)$.
- $f(X_i|p) = p^{X_i}(1-p)^{1-X_i}$

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$$LL(\theta) = \sum_{i=1}^{n} \log f(X_i|p)$$

$$f(X_i|p) = \begin{cases} p & \text{if } X_i = 1 \\ 1-p & \text{if } X_i = 0 \end{cases}$$

$$f(X_i|p) = p^{X_i}(1-p)^{1-X_i} \text{ where } X_i \in \{0,1\}$$

2. Differentiate $LL(\theta)$ w.r.t. (each) $\theta$, set to 0

3. Solve resulting equations

✅
- Is differentiable with respect to $p$
- Valid PMF over discrete domain

# Maximum Likelihood with Bernoulli

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1. Determine formula for $LL(\theta)$
2. Differentiate $LL(\theta)$ w.r.t. (each) $\theta$, set to 0
3. Solve resulting equations

$$LL(\theta) = \sum_{i=1}^{n} \log f(X_i|p) = \sum_{i=1}^{n} \log\left(p^{X_i}(1-p)^{1-X_i}\right)$$

$$= \sum_{i=1}^{n} \left[X_i \log p + (1 - X_i)\log(1-p)\right]$$

$$= Y(\log p) + (n - Y)\log(1-p), \text{ where } Y = \sum_{i=1}^{n} X_i$$

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$$LL(\theta) = \sum_{i=1}^{n} [X_i \log p + (1 - X_i) \log(1-p)]$$

$$= Y(\log p) + (n - Y) \log(1-p), \text{ where } \; Y = \sum_{i=1}^{n} X_i$$

2. Differentiate $LL(\theta)$ w.r.t. (each) $\theta$, set to 0

$$\frac{\partial LL(\theta)}{\partial p} = Y\frac{1}{p} + (n - Y)\frac{-1}{1-p} = 0$$

3. Solve resulting equations

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2. Differentiate $LL(\theta)$ w.r.t. (each) $\theta$, set to 0

$$\frac{\partial LL(\theta)}{\partial p} = Y\frac{1}{p} + (n-Y)\frac{-1}{1-p} = 0$$

3. Solve resulting equations

$$p_{MLE} = \frac{1}{n}Y = \frac{1}{n}\sum_{i=1}^{n} X_i$$

MLE of the Bernoulli parameter, $p_{MLE}$, is the unbiased estimate of the mean, $\bar{X}$ (sample mean)

# Quick check

- You draw $n$ i.i.d. random variables $X_1, X_2, \ldots, X_n$ from the distribution $F$, yielding the following sample:

$$[0, 0, 1, 1, 1, 1, 1, 1, 1, 1] \qquad (n = 10)$$

- Suppose distribution $F = \text{Ber}(p)$ with unknown parameter $p$.

1. What is $p_{MLE}$, the MLE of the parameter $p$?

   A. 1.0
   B. 0.5
   C. 0.8
   D. 0.2
   E. None/other

$$p_{MLE} = \bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

# Quick check

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$$[0, 0, 1, 1, 1, 1, 1, 1, 1, 1] \qquad (n = 10)$$

- Suppose distribution $F = \text{Ber}(p)$ with unknown parameter $p$.

1. What is $p_{MLE}$, the MLE of the parameter $p$?
2. What is the likelihood $L(\theta)$ of this particular sample?

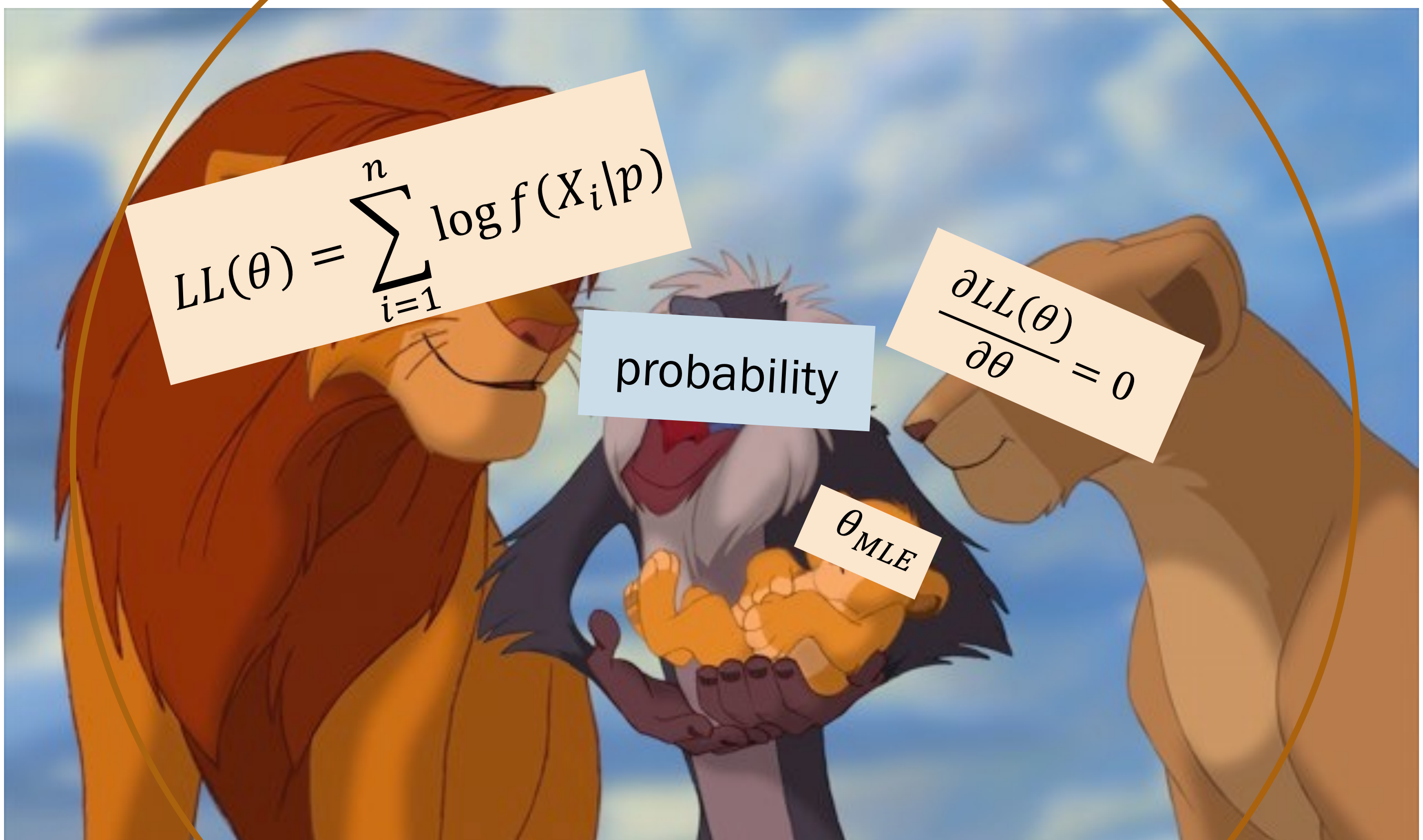
C. 0.8

$f(X_i|p) = p^{X_i}(1-p)^{1-X_i}$ where $X_i \in \{0,1\}$

$$L(\theta) = \prod_{i=1}^{n} f(X_i|p) \quad \text{where } \theta = p$$

$$= p^8(1-p)^2$$

Life Life$^C$



$LL(\theta) = \sum_{i=1}^{n} \log f(X_i|p)$
probability
$\frac{\partial LL(\theta)}{\partial \theta} = 0$
$\theta_{MLE}$

# Maximum Likelihood with Poisson

Consider a sample of $n$ i.i.d. RVs $X_1, X_2, \ldots, X_n$.

What is $\theta_{MLE} = \lambda_{MLE}$?

- Let $X_i \sim \text{Poi}(\lambda)$.
- PMF: $f(X_i|\lambda) = \dfrac{e^{-\lambda}\lambda^{X_i}}{X_i!}$

1. Determine formula for $LL(\theta)$

$$LL(\theta) = \sum_{i=1}^{n} \log\left(\frac{e^{-\lambda}\lambda^{X_i}}{X_i!}\right) = \sum_{i=1}^{n}(-\lambda \log e + X_i \log \lambda - \log X_i!)$$

$$= -n\lambda + \log(\lambda)\sum_{i=1}^{n} X_i - \sum_{i=1}^{n} \log(X_i!)$$

(using natural log, $\ln e = 1$)

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1. Determine formula for $LL(\theta)$

$$LL(\theta) = \sum_{i=1}^{n} \log\left(\frac{e^{-\lambda}\lambda^{X_i}}{X_i!}\right) = \sum_{i=1}^{n} (-\lambda \log e + X_i \log \lambda - \log X_i!)$$

$$= -n\lambda + \log(\lambda)\sum_{i=1}^{n} X_i - \sum_{i=1}^{n} \log(X_i!)$$

(using natural log, $\ln e = 1$)

2. Differentiate $LL(\theta)$ w.r.t. (each) $\theta$, set to 0

$$\frac{\partial LL(\theta)}{\partial \lambda} = ?$$

A. $-n + \dfrac{1}{\lambda}\sum_{i=1}^{n} X_i + n \log \lambda - \sum_{i=1}^{n} \dfrac{1}{X_i!} \cdot \dfrac{\partial X_i!}{\partial X_i}$

B. $-n + \dfrac{1}{\lambda}\sum_{i=1}^{n} X_i$

C. None/other/don't know

# Maximum Likelihood with Poisson

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(using natural log, $\ln e = 1$)

2. Differentiate $LL(\theta)$ w.r.t. (each) $\theta$, set to 0

$$\frac{\partial LL(\theta)}{\partial \lambda} = -n + \frac{1}{\lambda}\sum_{i=1}^{n} X_i = 0$$

3. Solve resulting equations

$$\lambda_{MLE} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

MLE of the Poisson parameter, $\lambda_{MLE}$, is the unbiased estimate of the mean, $\bar{X}$ (sample mean)

# Quick check

1. A particular experiment can be modeled as a Poisson RV with parameter $\lambda$, in terms of events/minute.
   Collect data: observe 53 events over the next 10 minutes. What is $\lambda_{MLE}$?

$$\lambda_{MLE} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

2. Is the Bernoulli MLE an unbiased estimator of the Bernoulli parameter $p$? ✅

3. Is the Poisson MLE an unbiased estimator of the Poisson variance? ✅

4. What does unbiased mean?

$E[\text{estimator}] = \text{true\_thing}$

Unbiased: If you could repeat your experiment, on average you would get what you are looking for.

# Maximum Likelihood with Uniform

Consider a sample of $n$ i.i.d. random variables $X_1, X_2, \dots, X_n$.

Let $X_i \sim \text{Uni}(\alpha, \beta)$.

$$f(X_i|\alpha,\beta) = \begin{cases} \dfrac{1}{\beta - \alpha} & \text{if } \alpha \le x_i \le \beta \\ 0 & \text{otherwise} \end{cases}$$

1. Determine formula for $L(\theta)$

$$L(\theta) = \begin{cases} \left(\dfrac{1}{\beta - \alpha}\right)^n & \text{if } \alpha \le x_1, x_2, \dots, x_n \le \beta \\ 0 & \text{otherwise} \end{cases}$$

2. Differentiate $LL(\theta)$ w.r.t. (each) $\theta$, set to 0

A. Great, let's do it
B. Differentiation is hard
C. Constraint $\alpha \le x_1, x_2, \dots, x_n \le \beta$ makes differentiation hard

# Example sample from a Uniform

Consider a sample of $n$ i.i.d. random variables $X_1, X_2, \ldots, X_n$.

Let $X_i \sim \text{Uni}(\alpha, \beta)$.

$$L(\theta) = \begin{cases} \left(\dfrac{1}{\beta - \alpha}\right)^n & \text{if } \alpha \le x_1, x_2, \ldots, x_n \le \beta \\ 0 & \text{otherwise} \end{cases}$$

Suppose $X_i \sim \text{Uni}(0,1)$.
You observe data:

$[0.15, 0.20, 0.30, 0.40, 0.65, 0.70, 0.75]$

Which parameters would give you maximum $L(\theta)$?

A. $\text{Uni}(\alpha = 0, \beta = 1)$

B. $\text{Uni}(\alpha = 0.15, \beta = 0.75)$

C. $\text{Uni}(\alpha = 0.15, \beta = 0.70)$

# Example sample from a Uniform

Consider a sample of $n$ i.i.d. random variables $X_1, X_2, \dots, X_n$.

Let $X_i \sim \text{Uni}(\alpha, \beta)$.

$$L(\theta) = \begin{cases} \left(\frac{1}{\beta - \alpha}\right)^n & \text{if } \alpha \le x_1, x_2, \dots, x_n \le \beta \\ 0 & \text{otherwise} \end{cases}$$

Suppose $X_i \sim \text{Uni}(0,1)$.
You observe data:

$$[0.15, 0.20, 0.30, 0.40, 0.65, 0.70, 0.75]$$

Which parameters would give you maximum $L(\theta)$?

A. $\text{Uni}(\alpha = 0 \quad, \beta = 1 \quad)$ $\quad (1)^7 = 1$

B. $\text{Uni}(\alpha = 0.15, \beta = 0.75)$ $\quad \left(\frac{1}{0.6}\right)^7 = 59.5$

C. $\text{Uni}(\alpha = 0.15, \beta = 0.70)$ $\quad \left(\frac{1}{0.55}\right)^6 \cdot 0 = 0$

⚠ Original parameters may not yield maximum likelihood.

# Maximum Likelihood with Uniform

Consider a sample of $n$ i.i.d. random variables $X_1, X_2, \ldots, X_n$.

Let $X_i \sim \text{Uni}(\alpha, \beta)$.

$$L(\theta) = \begin{cases} \left(\dfrac{1}{\beta - \alpha}\right)^n & \text{if } \alpha \le x_1, x_2, \ldots, x_n \le \beta \\ 0 & \text{otherwise} \end{cases}$$

$\theta_{MLE}$: $\alpha_{MLE} = \min(x_1, x_2, \ldots, x_n)$ $\qquad \beta_{MLE} = \max(x_1, x_2, \ldots, x_n)$

Intuition:

- Want interval size $(\beta - \alpha)$ to be as small as possible to maximize likelihood function per datapoint
- Need to make sure all observed data is in interval (if not, then $L(\theta) = 0$)

(demo)

# Small samples = problems with MLE

Maximum Likelihood Estimator $\theta_{MLE}$ :

$$\theta_{MLE} = \arg\max_{\theta} L(\theta)$$

- Best explains data we have seen
- Does not attempt to generalize to unseen data.

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✅ In many cases, $\mu_{MLE} = \frac{1}{n}\sum_{i=1}^{n} X_i$ Sample mean (MLE for Bernoulli $p$, Poisson $\lambda$, Normal $\mu$)

- Unbiased ($E[\mu_{MLE}] = \mu$ regardless of size of sample, $n$)

⚠️ For some cases, like Uniform: $\alpha_{MLE} \geq \alpha, \qquad \beta_{MLE} \leq \beta$

- Biased. Problematic for small sample size
- Example: If $n = 1$ then $\alpha = \beta$, yielding an invalid distribution

# Properties of MLE

Maximum Likelihood Estimator:

$$\theta_{MLE} = \arg\max_{\theta} L(\theta)$$

- Best explains data we have seen
- Does not attempt to generalize to unseen data.

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- Often used when sample size $n$ is large relative to parameter space
- Potentially biased (though asymptotically less so, as $n \to \infty$)
- Consistent: $\lim_{n\to\infty} P\left(\left|\hat{\theta} - \theta\right| < \varepsilon\right) = 1$ where $\varepsilon > 0$

  As $n \to \infty$ (i.e., more data), probability that $\hat{\theta}$ significantly differs from $\theta$ is zero

# Extra: Gaussian

# Maximum Likelihood with Normal

Consider a sample of $n$ i.i.d. random variables $X_1, X_2, \dots, X_n$.

- Let $X_i \sim \mathcal{N}(\mu, \sigma^2)$.

$$f(X_i|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(X_i-\mu)^2/(2\sigma^2)}$$

What is $\theta_{MLE} = (\mu_{MLE}, \sigma^2_{MLE})$?

1. Determine formula for $LL(\theta)$
2. Differentiate $LL(\theta)$ w.r.t. (each) $\theta$, set to 0
3. Solve resulting equations

$$LL(\theta) = \sum_{i=1}^{n} \log\left(\frac{1}{\sqrt{2\pi}\sigma} e^{-(X_i-\mu)^2/(2\sigma^2)}\right) = \sum_{i=1}^{n} \left[-\log\left(\sqrt{2\pi}\sigma\right) - (X_i-\mu)^2/(2\sigma^2)\right]$$

(using natural log)

$$= -\sum_{i=1}^{n} \log\left(\sqrt{2\pi}\sigma\right) - \sum_{i=1}^{n} \left[(X_i-\mu)^2/(2\sigma^2)\right]$$

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with respect to $\mu$

$$LL(\theta) = -\sum_{i=1}^{n} \log\left(\sqrt{2\pi}\sigma\right) - \sum_{i=1}^{n} \left[(X_i - \mu)^2/(2\sigma^2)\right]$$

$$\frac{\partial LL(\theta)}{\partial \mu} = \sum_{i=1}^{n} \left[2(X_i - \mu)/(2\sigma^2)\right]$$

$$= \frac{1}{\sigma^2} \sum_{i=1}^{n} (X_i - \mu) = 0$$

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with respect to $\mu$

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with respect to $\sigma$

$$\frac{\partial LL(\theta)}{\partial \sigma} = -\sum_{i=1}^{n}\frac{1}{\sigma} + \sum_{i=1}^{n} 2(X_i-\mu)^2/(2\sigma^3)$$

$$= -\frac{n}{\sigma} + \frac{1}{\sigma^3}\sum_{i=1}^{n}(X_i-\mu)^2 = 0$$

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3. Solve resulting equations

Two equations, two unknowns:

$$\frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i - \mu) = 0 \qquad -\frac{n}{\sigma} + \frac{1}{\sigma^3}\sum_{i=1}^{n}(X_i - \mu)^2 = 0$$

First, solve for $\mu_{MLE}$:

$$\frac{1}{\sigma^2}\sum_{i=1}^{n} X_i - \frac{1}{\sigma^2}\sum_{i=1}^{n}\mu = 0 \quad \Rightarrow \quad \sum_{i=1}^{n} X_i = n\mu \quad \Rightarrow \quad \mu_{MLE} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

unbiased

# Maximum Likelihood with Normal

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Two equations, two unknowns: $\frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i - \mu) = 0$ $\quad -\frac{n}{\sigma} + \frac{1}{\sigma^3}\sum_{i=1}^{n}(X_i - \mu)^2 = 0$

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unbiased

Next, solve for $\sigma_{MLE}$: $\frac{1}{\sigma^3}\sum_{i=1}^{n}(X_i - \mu)^2 = \frac{n}{\sigma} \quad \Rightarrow \sum_{i=1}^{n}(X_i - \mu)^2 = \sigma^2 n \quad \Rightarrow \sigma^2_{MLE} = \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu_{MLE})^2$

biased