SUBLINEAR BILIPSCHITZ EQUIVALENCE AND THE QUASIISOMETRIC CLASSIFICATION OF SOLVABLE LIE GROUPS

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Abstract. We prove a product theorem for sublinear bilipschitz equivalences which generalizes the classical work of Kapovich, Kleiner and Leeb on quasiisometries between product spaces. Independently we develop a new tool, based on a theorem of Cornulier on sublinear bilipschitz equivalences between solvable Lie groups, to evaluate the distortion of certain subgroups in central extensions. This is useful to provide lower bounds on Dehn functions. Building on work Cornulier and Tessera, we compute the Dehn functions of all simply connected solvable Lie groups of exponential growth up to dimension 5. We employ our product theorem to distinguish up to quasiisometry certain families among these groups which share the same dimension, cone-dimension and Dehn function. Finally, using a theorem of Peng together with our computations of cone-dimensions and Dehn functions, we establish the quasiisometric rigidity of the rank-two, five-dimensional solvable Lie group acting simply transitively by isometries on the horosphere in general position inside the product of three real hyperbolic planes.

Contents

1. Introduction

This study is motivated by the quasiisometric classification of connected Lie groups. Such a classification would be complete if one could classify the completely solvable Lie groups up to quasiisometry, as every connected Lie group G is quasiisometric to a completely solvable[1] group $\rho_0(G)$, called the trigshadow of G [Cor08].

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[1]The completely solvable Lie groups are the closed subgroups of the upper triangular real matrix group; they are also called real-triangulable, or split-solvable.

Cornulier conjectured that two quasiisometric completely solvable Lie groups should be isomorphic [Cor18, Conjecture 19.113]. This is currently open even within the smaller class of simply connected nilpotent groups.

The process of going from G to $\rho_0(G)$ is a reduction procedure. Cornulier went further in the reduction procedure and defined two subclasses of completely solvable Lie groups, $(\mathcal{C}_1)$ and $(\mathcal{C}_\infty)$, such that every connected Lie group G is $O(\log)$-bilipschitz equivalent to some group $\rho_1(G)$ in $(\mathcal{C}_1)$, and $O(u)$-bilipschitz equivalent to some group $\rho_\infty(G)$ in $(\mathcal{C}_\infty)$ for some explicit sublinear function u depending on G (one has $(\mathcal{C}_\infty) \subset (\mathcal{C}_1)$, $\rho_\infty \circ \rho_1 = \rho_\infty$ and $\rho_1 \circ \rho_0 = \rho_1$) [Cor11]. We will recall the definition of $O(u)$-bilipschitz equivalence and Cornulier's reductions farther; let us only specify here that in this language, quasiisometry is $O(1)$-bilipschitz equivalence, that $O(\log)$ and $O(u)$-bilipschitz equivalence are weaker than quasiisometry, and that when G is nilpotent $\rho_\infty(G)$ is the graded nilpotent group associated to the lower central filtration of G, which is known to be a quasiisometry invariant by the work of Pansu.

Theorem 1.1 ([Pan83, Pan89])**.** *Let G and G' be quasiisometric simply connected nilpotent Lie groups. Then $\rho_\infty(G)$ and $\rho_\infty(G')$ are isomorphic.*

Our purpose here is to demonstrate that the reduction procedure, in the present case at the level of ρ_1, has applications to the quasiisometric classification in the class of Lie groups of exponential growth, which is disjoint from that of nilpotent groups. This stems from two facts, both relying on [Cor11]:

- Since the group $\rho_1(G)$ in the class $(\mathcal{C}_1)$ have less complicated structure than G, it can happen that it splits in a direct product while G did not. Under certain assumptions on the factors, we can then apply a suitable generalization of the Kapovich-Kleiner-Leeb theorems on quasiisometries between products ([KKL98]), in order to rule out the existence of a $O(u)$-equivalence (hence of a quasiisometry) between G and G' when $\rho_1(G)$ and $\rho_1(G')$ split as non-isomorphic direct products. This is the strategy of Theorem A and Theorem C below.
- The distortion of certain subgroups in central extensions $\widetilde{G}$ of G can be estimated by comparing $\widetilde{G}$ and $\rho_1(\widetilde{G})$. In return, this gives estimates on the Dehn function of G, which allows one to progress in the quasiisometry classification. We develop our second tool, Proposition D, for that purpose.

1.A. **Sublinear bilipschitz equivalence and products.** For a pointed metric space (X, x_0) we denote by $|x| = d(x, x_0)$ the *size of* $x \in X$. A *sublinear function* is any real function $u\colon \mathbf{R}_{\geqslant 0} \to \mathbf{R}_{\geqslant 0}$ such that $\lim_{r \to \infty} \frac{u(r)}{r} = 0$. We often write $u(x)$ instead of $u(|x|)$; this should not cause any confusion. Given two maps f and g between pointed metric spaces and a sublinear function u, we say that f and g are $O(u)$-close if $d\big(f(x), g(x)\big) = O\big(u(x)\big)$.

Definition 1.2 (Commuting up to sublinear error)**.** Let n be a positive integer. Let (X_i) and (Y_i) be families of pointed metric spaces, $i = 1, \ldots, n$. We say that the diagram

$$\begin{tikzcd} \prod_{i=1}^n X_i \arrow[r, "\phi"] \arrow[d, "\pi_i"] & \prod_{i=1}^n Y_i \arrow[d, "\pi_i"] \\ X_i \arrow[r, "\phi_i"] & Y_i \end{tikzcd}$$

commutes up to sublinear error if there exists a sublinear function u such that $\phi_i \circ \pi_i$ and $\pi_i \circ \phi$ are $O(u)$-close for all i.

The precise definitions for the following theorem will be given in Section 2. Key examples of spaces of coarse type I are simply connected Riemannian manifolds with sectional curvature bounded above by a negative constant; examples of spaces of coarse type II are irreducible Riemannian symmetric spaces of noncompact type and of higher rank.

Theorem A. *Let $X = M \times \prod_{i=1}^n X_i$ and $Y = N \times \prod_{i=1}^m Y_j$ be product metric spaces. Assume the metric spaces X_i and Y_j are of coarse type I or II in the sense of Kapovich, Kleiner and Leeb [KKL98], and M and N are geodesic metric spaces with asymptotic cones homeomorphic to $\mathbf{R}^p$ and $\mathbf{R}^q$ respectively. Let u be a subadditive sublinear function, and let $\phi\colon X \to Y$ be an $O(u)$-bilipschitz equivalence. Then $p = q$, $n = m$, and there exists a bijection $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$ and, for every $i \in \{1, \dots, n\}$, an $O(u)$-bilipschitz equivalence $\phi_i : X_i \to Y_{\sigma(i)}$ such that the diagram*

$$\begin{tikzcd} M \times \prod_{i=1}^n X_i \arrow[r, "\phi"] \arrow[d, "\pi_i"] & N \times \prod_{i=1}^n Y_i \arrow[d, "\pi_{\sigma(i)}"] \\ X_i \arrow[r, "\phi_i"] & Y_{\sigma(i)} \end{tikzcd}$$

commutes up to sublinear error. Moreover, this error is in the class $O(u)$.

Remark 1.3. In the statement above, we do not explicitly specify how the distance on X and Y is built from that of the factors. In the proof we work with the ℓ^2 (or Pythagorean product) metric, which is more natural when the factors are Riemannian. However, the theorem applies if one considers any distances on X and Y quasiisometric to the ℓ^2 distances, e.g. the ℓ^1 distance.

Remark 1.4. When p or q vanish, the conclusion of Theorem A can be stated in the following way: ϕ is $O(u)$-close to the composition of a map of the form

$$(x_1, \dots, x_n) \mapsto (\phi_1(x_1), \dots, \phi_n(x_n)),$$

where the ϕ_i are $O(u)$-bilipschitz equivalences, and a map permuting the factors.

Theorem A generalizes [KKL98, Theorem B], which is the case $u = 1$. We state a first application below, which uses some of the results of [Pal20a, Gra23]. By pluriisometry between symmetric spaces we mean that we allow a rescaling in each factor of the de Rham decomposition.

Corollary B. *Let X and Y be two Riemannian globally symmetric spaces with no compact factors. Let u be a subadditive sublinear function. If X and Y are $O(u)$-bilipschitz equivalent, then X and Y are pluriisometric.*

Before stating the second application below, we need to recall Cornulier's ρ_1 reduction (see §3.A for a more comprehensive account). Given a completely solvable group S we denote by $\mathrm{R}_{\exp} S$ the smallest normal subgroup such that $S/\mathrm{R}_{\exp} S$ is nilpotent; this is also the intersection of the descending central series of S. The homomorphism $\alpha\colon S \to \mathrm{Aut}(\mathrm{R}_{\exp} S) = \mathrm{Aut}(\mathrm{Lie}(\mathrm{R}_{\exp} S))$ determined by the Adjoint action of S is algebraic over $\mathbf{R}$, and can be decomposed into $\alpha = \alpha_\sigma \alpha_\nu$, where α_σ is valued in a diagonal $\mathbf{R}$-torus of $\mathrm{Aut}(\mathrm{Lie}(\mathrm{R}_{\exp} S))$, and α_ν is valued in the unipotent radical of $\mathrm{Aut}(\mathrm{Lie}(\mathrm{R}_{\exp} S))$. Since $\mathrm{R}_{\exp} S$ is nilpotent (it is contained in $[S, S]$), it sits in $\ker \alpha_\sigma$, so that α_σ defines a homomorphism $S/\mathrm{R}_{\exp} S \to \mathrm{Aut}(\mathrm{R}_{\exp} S)$, that we still denote α_σ.

Definition 1.5. Let S, α_σ and α_ν be as above. One defines $\rho_1(S)$ as the group $\mathrm{R}_{\exp} S \rtimes_{\alpha_\sigma} S/\mathrm{R}_{\exp} S$. We say that S is in the class $(\mathcal{C}_1)$ if $S \simeq \rho_1(S)$, that is, if $\mathrm{R}_{\exp} S$ is split and $\alpha_\nu = 1$. (In [Cor11], $\rho_1(S)$ is denoted S_1.)

Theorem 1.6 (Cornulier, [Cor11]). *Let S be a completely solvable group. Then S and $\rho_1(S)$ are $O(\log)$-bilipschitz equivalent.*

Combining this theorem with our product theorem, and using some further previous results on sublinear bilipschitz equivalences, we obtain the following.

Theorem C. *Let S and S' be two quasiisometric completely solvable groups. Assume that*

$$\rho_1(S) \simeq \mathbf{R}^n \times P \times H_1 \times \cdots \times H_m \quad \text{and} \quad \rho_1(S') \simeq \mathbf{R}^{n'} \times P' \times H'_1 \times \cdots \times H'_{m'}$$

for some $n, m, n', m' \geqslant 0$, where

(1) *$P = AN$ and $P' = A'N'$ are maximal completely solvable subgroups in semisimple groups $G = KAN$ and $G' = K'A'N'$ respectively.*
(2) *For $i = 1, \ldots, m$, H_i has a left-invariant Riemannian metric that is negatively curved, and an abelian derived subgroup; same assumption for H'_j, $j = 1, \ldots, m'$.*

Then, $\rho_1(S)$ and $\rho_1(S')$ are isomorphic.

The case $S = P$ is equivalent to the former Corollary B while the case where S is equal to a single factor H_1 is the main theorem of [Pal20b], used in the proof.

A more sophisticated (and slightly more general) version of Theorem C will be given in Theorem 3.17; in particular, the geometric assumption on curvature in (2) can be reformulated in terms of the structure of H with the help of Heintze's theorem [Hei74].

Theorems A and C allow us to distinguish between several families of completely solvable groups up to quasiisometry.

We will summarize this contribution for groups of dimension 5 in Corollary 5.22. We now give an example of this strategy in dimension 4.

Example 1.7. Let $\alpha \in (0, 1)$. The groups $S = G^0_{4,9}$ and $S' = \mathbf{R} \times G^\alpha_{3,5}$ (the names are from the classification in [Mub63]) are the four dimensional, completely solvable Lie groups whose respective Lie algebras $\mathfrak{g}^0_{4,9}$ and $\mathbf{R} \times \mathfrak{g}^\alpha_{3,5}$ are spanned by $e_1, \ldots, e_4$, subject to the following nonzero brackets:

$$\begin{aligned} \mathfrak{g}^0_{4,9} : \quad & [e_4, e_1] = e_1, [e_4, e_2] = e_2, [e_2, e_3] = e_1 \\ \mathbf{R} \times \mathfrak{g}^\alpha_{3,5} : \quad & [e_4, e_1] = e_1, [e_4, e_2] = \alpha e_2. \end{aligned}$$

It follows from Theorem C that S and S' are not quasiisometric. Here, $\rho_1(S)$ splits as a direct product, that is $\mathbf{R} \times P$ where P is the maximal completely solvable subgroup of $\mathrm{SO}(3, 1)$. See section 3.B for a detailed proof of this example.

Remark 1.8. The parameter α in the definition of $G^\alpha_{3,5}$ can actually be taken in $(-1, 1) \setminus \{0\}$; the two limit cases $\alpha = -1$ and $\alpha = 1$ give the groups $\mathbf{R} \times \mathrm{Sol}_3$ and $\mathbf{R} \times P$ respectively, where P is as above. A direct inspection of the asymptotic cone suffice to tell $G^0_{4,9}$ apart from the groups $\mathbf{R} \times G^\alpha_{3,5}$ when $\alpha < 0$ and from $\mathbf{R} \times \mathrm{Sol}_3$.

Keeping the goal of quasiisometric classification in mind, results like Theorem C should be complemented by other ones establishing that among the class of groups considered, the subset of those in the class $(\mathcal{C}_1)$ should be quasiisometrically closed. Peng notably achieved this for some groups [Pen11b, Corollary 5.3.8] by measuring the divergence of vertical geodesics; in her setting the groups of class $(\mathcal{C}_1)$ are characterized by pure exponential divergence (and Peng is able to prove that the quasiisometries respect vertical geodesics). However Peng's technique and results usually cannot be applied to the groups we consider here, since they are not nondegenerate in Peng's sense.

1.B. **Distortion, Dehn function, and applications.** The next results are about distortion of one-parameter subgroups and the Dehn functions of completely solvable Lie groups. The Dehn function is a notorious quasiisometry invariant of compactly presented (and especially connected Lie) groups.

Definition 1.9. Let G be a simply connected Lie group, and let X be a nonzero element of its Lie algebra $\mathfrak{g}$. The distortion function of a one-parameter subgroup $L = \{\exp(tX)\}$ in G is the growth type of the function

$$\Delta_L^G(r) = \sup\{t : \exp tX \in B_G(r)\}.$$

The proposition below interpolates between [Osi01] (which corresponds to the case when the group S in the statement is nilpotent) and [Osi02]; we use both theorems of Osin in the proof. Although it is logically independent from the aforementioned results, it rests on a more elaborate version of Theorem 1.6, also due to Cornulier (See Theorem 3.7).

Proposition D (Evaluating distortion in completely solvable groups)**.** *Let S be a completely solvable group with Lie algebra $\mathfrak{s}$. Let $X \in \mathfrak{s}$ be nonzero, and let $c_X = \sup\{j \in \mathbf{Z}_{\geqslant 1} \cup \{\infty\} \colon X \in C^j\mathfrak{s}\}$, where $C^1\mathfrak{s} = \mathfrak{s}$ and $C^{j+1}\mathfrak{s} = [\mathfrak{s}, C^j\mathfrak{s}]$ for all $j \geqslant 1$. Let L be the one-parameter subgroup generated by X in S.*

(1) If $c_X < \infty$, and if X is in a Cartan subalgebra (e.g. if X is a regular or central element) then there exists $\alpha > 0$ such that

$$\Delta_L^S(r) \asymp r^{c_X}.$$

(2) [Osi02] If $c_X = \infty$ then Δ_L^S is exponential.

As an illustration of Proposition D, using the relation between the Dehn function and the distortion in a central extension (Proposition 4.2), we can distinguish up to quasiisometry the two completely solvable groups of dimension 4 and whose asymptotic cones have dimension 3.

Example 1.10. The groups named $S = G_{4,3}$ in [Mub63] and $S' = \mathbf{R}^2 \times A_2$ are the only four dimensional, completely solvable Lie groups of cone dimension 3. Their Lie algebras $\mathfrak{g}_{4,3}$ and $\mathbf{R}^2 \times \mathfrak{a}_2$ are spanned by $e_1, \dots, e_4$, subject to the following nonzero brackets:

$$\begin{aligned} \mathfrak{g}_{4,3}: &\quad [e_4, e_1] = e_1, [e_4, e_3] = e_2. \\ \mathbf{R}^2 \times \mathfrak{a}_2: &\quad [e_4, e_1] = e_1. \end{aligned}$$

Let $(\omega_1, \dots, \omega_4)$ denote the basis of $Z^1(\mathfrak{g}_{4,3}, \mathbf{R})$ dual to $(e_1, \dots e_4)$ in $\mathfrak{g}_{4,3}$. Then $[\omega_2 \wedge \omega_4] \in H^2(\mathfrak{g}_{4,3}, \mathbf{R})$ generates a central extension (isomorphic to the Lie algebra $\mathfrak{g}_{5,10}$ in Table 2) whose generator is cubically distorted in the corresponding Lie group by Proposition D. It follows that the Dehn function of $G_{4,3}$ is at least cubic. On the other hand, S' has a symmetric, hence a CAT(0) left invariant Riemannian metric, so that the Dehn function of S' is quadratic. So S and S' are not quasiisometric.

Remark 1.11. It turns out that in order to compute the Dehn functions of completely solvable groups of dimensions 4 and 5, it is enough to use the versions of Proposition D and Proposition 4.2 for nilpotent groups, which are well known (See e.g. [BW97] and also [Pit97, Theorem 3.1] for a different formulation). In Appendix B we provide an example (of dimension 13) for which our version for solvable groups is actually necessary and improves on previously known lower bounds techniques. See discussion in Section 4.D.

Proposition E. *The simply connected solvable Lie groups of dimension less or equal five and exponential growth have Dehn function of growth type n, n^2, n^3, n^4 or $\exp(n)$ exactly, all given for the groups of dimension less than 5 in Table 4 and for the groups of dimension 5 in Table 5.*

Remark 1.12. The Dehn functions for all solvable Lie groups of dimension up to 5 and polynomial growth follow from those of nilpotent groups of these dimensions, which were completely determined by Pittet [Pit97, Proposition 7.1]. We restrict our computations to the groups of exponential growth in this paper.

Finally, we use theorems of Peng [Pen11a, Pen11b] that we combine with our work on the quasiisometric classification to reach a rigidity result for the group Sol_5 (also named $G_{5,33}^{-1,-1}$ in [Mub63]), which is the semidirect product $\mathbf{R}^2 \ltimes \mathbf{R}^3$ with diagonal action of the $\mathbf{R}^2$ torus, such that the three weights $\varpi_1, \varpi_2, \varpi_3 \in \mathrm{Hom}(\mathbf{R}^2, \mathbf{R})$ are linearly independent and sum to zero.

Among all the left-invariant metrics on this group, one makes it isometric to a hypersurface

$$\mathcal{H} = \{(z_1, z_2, z_3) \in \mathbb{H}^2 \times \mathbb{H}^2 \times \mathbb{H}^2 \colon b_1(z_1) + b_2(z_2) + b_3(z_3) = 0\}$$

where for $i = 1, 2, 3$, b_i is a horofunction only depending on the projection to the i^{th} factor. As such, it is a higher-rank generalization of the three-dimensional group Sol_3 (or SOL) which admits the same description as a horosphere in $\mathbb{H}^2 \times \mathbb{H}^2$.

Proposition F (Propositions 6.5–6.8)**.** *The following hold:*

(1) Let G be a group of class $(\mathcal{C}_0)$, quasiisometric to Sol_5. Then G is isomorphic to Sol_5.
(2) Let Γ be a finitely generated group quasiisometric to Sol_5. Then there is a finite-index subgroup Γ_0 in Γ and a homomorphism $\Gamma_0 \to \mathrm{Sol}_5$ with finite kernel and whose image is a lattice in Sol_5.

The distinctive property of Sol_5 used in the proof of Proposition F is that, unlike all the other completely solvable Lie groups of the same dimension and cone-dimension, it has an exactly quadratic Dehn function. The fact that its Dehn function is quadratic was proved by Druţu [Dru04, Theorem 1.1] and Leuzinger-Pittet [LP04, Corollary 2.1].

Remark 1.13. Passing to the finite index subgroup Γ_0 of Γ is necessary in the quasiisometric rigidity statement (2) of Proposition F above, as the following example shows. Let K be a number field of degree 3 with Galois group $\Sigma = \mathrm{Sym}_3$ and consider the group $\Lambda = \mathrm{PSL}(2, \mathcal{O}_K)$, which embeds as a non-uniform $\mathbf{Q}$-rank one lattice in $X = \mathbb{H}^2 \times \mathbb{H}^2 \times \mathbb{H}^2$. The group Σ operates by automorphisms on Λ; these extend as conjugations in $G = \mathrm{Isom}(X)$ by maps of $\mathbb{H}^2 \times \mathbb{H}^2 \times \mathbb{H}^2$ permuting the three factors. If $\mathcal{H}$ is a horosphere bounding a cusp in the quotient $\Gamma \backslash X$, then by [Pra73, Proposition 2.1(3)], $\Lambda \rtimes \Sigma$ intersects $\mathrm{Isom}(\mathcal{H})$ in a uniform lattice of the latter group; let us denote this lattice by Γ. One may write $\Gamma = \Gamma_0 \rtimes \Sigma$, where Γ_0 is a lattice in $\mathrm{Sol}_5 < \mathrm{Isom}(\mathcal{H})$. Thus Γ is quasiisometric to Sol_5. However, if there was $\phi \colon \Gamma \to \mathrm{Sol}_5$ a group homomorphism with finite kernel, then $\ker \phi$ would contain Σ (which is torsion), but intersect Γ_0 trivially, since $\Gamma_0 < \mathrm{Sol}_5$ has no non-trivial finite subgroup; it would follow that $\ker \phi = \Sigma$ and Γ would split as a direct product, which is not the case.

1.C. **Organization of the paper.** In Section 2 we prove Theorem A. The sublinear adaptation is mainly in Section 2.C, resulting in Proposition 2.11. The conclusion of Theorem A is in Section 2.D, and done exactly as in [KKL98].

Section 3 introduces some of the theory of completely solvable Lie groups and the significance of SBE to this theory. Corollary B and Theorem C are proved in Section 3.C. We actually deduce both of these statements from Theorem 3.17, although Corollary B could be derived directly. Theorem 3.17 is stated in terms of diagonal Heintze groups, which are the Gromov-hyperbolic groups of class $(\mathcal{C}_1)$ and thus are of coarse type I.

Section 4 is dedicated to the proof of the lower bounds estimates of Dehn functions. In Section 4.A we prove the distortion estimates stated in Proposition D, and in Section 4.B we apply them to obtain the lower bounds. Section 4.D compares our results with other lower bound methods. In Appendix B we give an example in which our tool improves on previously known techniques.

Sections 5 and 6 are devoted to mapping the contribution of our work towards the quasiisometric classification of low dimensional completely solvable Lie groups. We list these groups up to dimension 5 and compute their image by ρ_1 (Tables 1 and 2) and their Dehn functions (Tables 4 and 5). Our computations are based on a list of criteria, mostly following [CT17], which are explained in Section 5.A. We explain our computations in Section 5.C. In Corollary 5.22 we show which new groups we are able to distinguish up to quasiisometry using Theorem A. Section 5.D elaborates on particular families of completely solvable Lie groups which exhibit interesting behavior. Finally in Section 6 we prove the quasiisometric rigidity of the group $G_{5,33}^{-1,-1}$ (Proposition F), building on the work of Peng [Pen11a], [Pen11b] for the part concerning finitely generated groups.

1.D. **Acknowledgements.** We thank Yves Cornulier for a useful discussion and pointing us to Theorems 6.E.2 and 10.H.1 in [CT17].

2. The product Theorem A

In this section we prove Theorem A. The argument is exactly that of Theorem B of Kapovich, Kleiner and Leeb in [KKL98]. Most of their proof works *verbatim* also for the case of SBE. In particular, this is true for sections 3, 4, 5 and 6 of their work. In these sections Kapovich, Kleiner and Leeb deal with properties of homeomorphisms between asymptotic cones, and to this end it makes no difference whether these homeomorphisms arise as the cone maps of quasiisometries or SBE.

The only new ingredient we need is a generalization of Section 2 of [KKL98], proving that whenever the cone maps preserve the product structure, the original maps coarsely preserve the product structure. Our Proposition 2.11 generalizes [KKL98, Proposition 2.6], and our Lemma 2.17 generalizes [KKL98, Lemma 2.7].

The key assumption in this section is that the cone maps decompose as products. A sufficient condition for this is that the original factors are of *coarse type* I and II [KKL98, Theorem 5.1]. This is the reason for the hypothesis of Theorem A. We only briefly sketch the definitions because in essence all we need to know about these spaces is the property that their cone maps preserve the product structure. We refer to [KKL98, Section 3] for more details.

Definition 2.1. Let X be a geodesic metric space. We say that $p, q \in X$ are in the same *leaf* if there is a continuous path $\gamma : I \to X$ joining p and q such that every other continuous path which joins p and q contains γ. Being in the same leaf is an equivalence relation, and every leaf of X is a closed convex subset. The space X is called *type* I if all its leaves are geodesically complete trees which branch everywhere.

Definition 2.2. A geodesic metric space is of *type* II if it is a thick irreducible Euclidean building with transitive affine Weyl group of rank $r \geq 2$.

Definition 2.3. A space X is of *coarse type* I (resp. II) if every asymptotic cone of X is of type I (resp. II).

Symmetric spaces of noncompact type and higher rank are coarse type II [KL97], and so are Euclidean buildings of higher rank and cocompact affine Weyl group. Examples of spaces of coarse type I are Gromov hyperbolic groups whose Gromov boundary contains at least 3 points. These examples are enough for all our purposes in this paper, which arise in Section 3 in the proofs of Corollary B and Theorem 3.17.

We start with some preliminaries on SBE and asymptotic cones, then prove Proposition 2.11 and its variant Proposition 2.18. In Section 2.D we deduce Theorem A from these propositions and from some further results of [KKL98].

2.A. **Preliminaries.** Let X and Y be metric spaces. After fixing $x_0 \in X$ and $y_0 \in Y$, we denote by $|\cdot|$ the distance to the respective basepoints in X and Y. We denote $|x_1| \vee |x_2| := \max\{|x_1|, |x_2|\}$.

Let $u \colon \mathbf{R}_{\geqslant 0} \to \mathbf{R}_{\geqslant 1}$ be a sublinear function, that is,

$$\lim_{r \to +\infty} \frac{u(r)}{r} = 0.$$

We say that u is admissible if it is nondecreasing and $\limsup u(2r)/u(r) < +\infty$. Often, we will also assume that u is subadditive, that is, $u(r_1 + r_2) \leqslant u(r_1) + u(r_2)$ for all $r_1, r_2 \geqslant 0$. This implies admissibility.

Definition 2.4. Let X, Y, x_0, y_0 and u be as above. Let $L \geqslant 1$. We say that $f \colon X \to Y$ is

- (L, u, x_0)-Lipschitz if for every $x, x' \in X$, $d\big(f(x), f(x')\big) \leqslant Ld(x,x') + u(|x| \vee |x'|)$
- (L, u, x_0)-expansive if $L^{-1}d(x,x') - u(|x| \vee |x'|) \leqslant d\big(f(x), f(x')\big)$
- (u, y_0)-surjective if for every y in Y, there is $x \in X$ such that $d\big(y, f(x)\big) \leqslant u(|y|)$.

We say that f is a (L, u)-bilipschitz embedding if it is (L, cu, x_0)-Lipschitz and (L, cu, x_0)-expansive for some $c \geqslant 0$. If f is additionally (u, y_0)-surjective for some y_0, then for all $y_0' \in Y$ there is $c' > 0$ such that it is $(c'u, y_0')$-surjective; in this case, we say that f realizes a $\big(L, O(u)\big)$-bilipschitz equivalence, or for short, a $O(u)$-bilipschitz equivalence between X and Y. When no reference is made to L and u we will call a (L, u)-bilipschitz embedding a sublinear bilipschitz embedding.

2.B. **Going through cones.**

Definition 2.5. Let X be a metric space. Let $(\sigma_n)_{n \in \mathbf{N}}$ be a sequence of positive real numbers. Let $(x_n)_n \in X^{\mathbf{N}}$. We call precone of X with data $\big((x_n)_n, \sigma\big)$ the set of sequences

$$\mathrm{Precone}(X, (x_n)_n, \sigma) = \{(x_n')_n \in X^{\mathbf{N}} : \exists M \in [0, +\infty), \forall n \in \mathbf{N}, d(x_n, x_n') \leqslant M\sigma_n\}.$$

Given a nonprincipal ultrafilter ω over $\mathbf{N}$ we denote by $\mathrm{Cone}\big(X, (x_n)_n, (\sigma_n)_n, \omega\big)$ the quotient of the set $\mathrm{Precone}(X, (x_n)_n, \sigma)$ by the relation

$$(x_n')_n \sim (x_n'')_n \iff \lim_{n \to \omega} \frac{d(x_n', x_n'')}{\sigma_n} = 0$$

equipped with the distance

$$d_\omega([x'], [x'']) = \lim_{n \to \omega} \frac{d(x_n', x_n'')}{\sigma_n}.$$

If σ goes to $+\infty$ we call this an asymptotic cone. When it is not relevant to mention all the remaining data, we will denote an asymptotic cone of X by X_ω.

Definition 2.6. Given a metric space X, a triple $\big(X, (x_n)_n, (\sigma_n)_n\big)$ is called u-admissible if for some (any) $v \in O(u)$

$$\lim_{i \to +\infty} \frac{v(x_i)}{\sigma_i} = 0.$$

Note that we did not specify a basepoint when writing $v(x_i)$ in the above definition; such a choice turns out to have no influence on the notion of u-admissibility.

Lemma 2.7 below is a basic fact, usually applied to cones with fixed base point. We will need asymptotic cones with moving base points, which are slightly less common. In the statement below, given a map $f : X \to Y$ we still denote f the map between the power sets $f : X^{\mathbf{N}} \to Y^{\mathbf{N}}$.

Lemma 2.7. *Let $L \geqslant 1$. Let u be an admissible function, let X and Y be metric spaces and let $f : X \to Y$ an (L,u)-bilipschitz equivalence. Assume $\big(X, (x_n)_n, (\sigma_n)_n\big)$ is u-admissible. Then*

$$f\big(\operatorname{Precone}(X, x_n, \sigma_n)\big) \subseteq \operatorname{Precone}\big(Y, f(x_n), \sigma_n\big),$$

and for any nonprincipal ultrafilter ω, a quotient map

$$\operatorname{Cone}\big(f, (x_n)_n, (\sigma_n)_n, \omega\big) \colon \operatorname{Cone}\big(X, (x_n)_n, (\sigma_n)_n, \omega\big) \to \operatorname{Cone}\Big(Y, \big(f(x_n)\big)_n, (\sigma_n)_n, \omega\Big)$$

is well-defined and an L-bilipschitz embedding. Moreover, if f is additionally $O(u)$-surjective, then $\operatorname{Cone}\big(f, (x_n)_n, (\sigma_n)_n, \omega\big)$ is a bilipschitz homeomorphism.

Proof. This follows from [Pal22, Proposition 6]. □

Remark 2.8. Lemma 2.7 means that when the asymptotic cones are u-admissible, the cone maps of $O(u)$-bilipschitz equivalences have the same property as cone maps of quasiisometries. We therefore feel free to use results from [KKL98] whose hypothesis require some property to hold for *all* asymptotic cones, while we can only guarantee it for u-admissible cones. We do not reformulate or reprove these statements in our work, and the reader should only note that whenever we use cones, they are u-admissible.

Definition 2.9. Let X, Y be two metric spaces, $v : \mathbf{R}_{\geqslant 0} \to \mathbf{R}_{\geqslant 1}$. Two maps $f_1, f_2 : X \to Y$ are *at distance* v if for all $x \in X$ one has $d_Y\big(f_1(x), f_2(x)\big) \leq v(|x|)$. The maps are called *sublinearly close* if f_1, f_2 are at distance v for some sublinear function v.

2.C. **SBE Preserve Product Structure.**

Definition 2.10. A bilipschitz homeomorphism $F : \prod_{i=1}^n X_i \to \prod_{i=1}^m Y_i$ is said to *preserve the product structure* if up to reindexing the factors and ignoring factors not in the image of F, there are bilipschitz homeomorphisms F_i such that for each i, the following diagram commutes:

$$\begin{tikzcd}
\prod_{i=1}^n X_i \arrow[r, "F"] \arrow[d, "\pi_i"] & \prod_{i=1}^n Y_i \arrow[d, "\pi_i"] \\
X_i \arrow[r, "F_i"] & Y_i
\end{tikzcd}$$

Throughout this section, the standing assumptions on the map ϕ and spaces X, Y are those of Proposition 2.11.

Proposition 2.11 (SBE version of Proposition 2.6 in [KKL98])**.** *Suppose $\phi : X = \prod_i X_i \to Y = \prod Y_j$ is an (L,u)-bilipschitz equivalence. Assume that for all u-admissible cones X_ω and Y_ω of X and Y, the ultralimit $F = \operatorname{Cone}(\phi) : X_\omega \to Y_\omega$ preserves the product structure. Then:*

(1) Up to reindexing the factors, there are (L', u_i)-bilipschitz equivalences $\phi_i : X_i \to Y_i$ with $u_i \in O(u)$.
(2) The diagram

$$\begin{array}{ccc} \prod_{i=1}^n X_i & \xrightarrow{\phi} & \prod_{i=1}^n Y_i \\ \downarrow{\scriptstyle \pi_i} & & \downarrow{\scriptstyle \pi_i} \\ X_i & \xrightarrow{\phi_i} & Y_i \end{array}$$

commutes up to $O(u)$ sublinear error. In particular, ϕ is sublinearly close to a product of $O(u)$-bilipschitz equivalences.

Remark 2.12. One notable difference between our proofs and those of the product theorem for quasiisometries concerns the 'nontranslatability' condition [KKL98, Definitions 2.2, 2.3]. In Section 2 of [KKL98] the nontranslatability of the factors was used both implicitly in the assumption that cone maps preserve product structure, and then again explicitly in order to prove that certain pairs of quasi-isometries are at uniform bounded distance. In contrast, we use the nontranslatability only through its implicit use as part of the product decomposition of cone maps between spaces of types I and II. We use this result as a black box from [KKL98] and therefore do not need to explicitly define or discuss nontranslatability. In particular our hypothesis in Proposition 2.11 is formally (and perhaps essentially) weaker than its quasiisometry predecessor [KKL98, Proposition 2.6].

The proof requires some sublinear adaptations to the notions that appear in [KKL98].

Definition 2.13. Let $d > 0$ be a positive constant and $d_0 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 1}$ a function. A pair of points $x, x' \in X$ is called *d-separated* if $d_X(x, x') \geqslant d$, and *d_0-separated* if $d(x, x') \geqslant d_0(|x| \vee |x'|)$.

Definition 2.14. A pair of points $x, x' \in X = \prod_i X_i$ is called *i-horizontal* if x and x' agree on all $m \neq i$ coordinates.

Definition 2.15. Fix $L_1 > L$, $\epsilon < L_1^{-1}$, $i, j \in \{1, \dots, n\}$. A pair of i-horizontal points $x, x' \in X$ is called:

(1) *j-compressed* if

$$\frac{d_{Y_j}\Big(\pi_{Y_j}\big(\phi(x)\big), \pi_{Y_j}\big(\phi(x')\big)\Big)}{d_X(x, x')} \leq \epsilon$$

(2) *j-uncompressed* if

$$L_1^{-1} \leq \frac{d_{Y_j}\Big(\pi_{Y_j}\big(\phi(x)\big), \pi_{Y_j}\big(\phi(x')\big)\Big)}{d_X(x, x')} \leq L_1$$

(3) *j-semi-compressed* if

$$\epsilon \leq \frac{d_{Y_j}\Big(\pi_{Y_j}\big(\phi(x)\big), \pi_{Y_j}\big(\phi(x')\big)\Big)}{d_X(x, x')} \leq {L_1}^{-1}$$

Two pairs are *j-compatible* if they are simultaneously j-semi/un/compressed, and *j-incompatible* otherwise.

When it is clear from the context, we occasionally do not specify the space in which the distance is computed. Also, we sometimes write π_j instead of π_{X_j} or π_{Y_j}.

Let x, x' be i-horizontal. Notice two trivial facts: a) $d_{X_i}\big(\pi_i(x), \pi_i(x')\big) = d_X\big(x, x'\big)$, and b) the geodesic $[x, x']$ in X joining x and x' is contained in a leaf of X_i. In particular, for each $x'' \in [x, x']$, (x, x'') and (x', x'') are i-horizontal pairs.

Lemma 2.16. *Being j-compressed is transitive along a geodesic, i.e. if x'' lies on a geodesic joining a pair (x, x') of i-horizontal points, and both (x, x'') and (x'', x') are j-compressed, then also (x, x') is j-compressed.*

Proof.

$$\begin{aligned} d_{Y_j}\Big(\pi_j\big(\phi(x)\big), \pi_j\big(\phi(x')\big)\Big) &\leq d_{Y_j}\Big(\pi_j\big(\phi(x)\big), \pi_j\big(\phi(x'')\big)\Big) + d_{Y_j}\Big(\pi_j\big(\phi(x'')\big), \pi_j\big(\phi(x')\big)\Big) \\ &\leq \varepsilon d(x, x'') + \varepsilon d(x'', x') = \varepsilon d(x, x') \end{aligned}$$

The last equality follows from $x'' \in [x, x']$. $\square$

Lemma 2.17 (SBE version of Lemma 2.7 in [KKL98])**.** *There exists a sublinear subadditive function $d_0 \in O(u)$ such that for every fixed $i, j \in \{1, \dots, n\}$, either all d_0-separated i-horizontal pairs are j-compressed or all such pairs are j-uncompressed.*

The existence of d_0 as in Lemma 2.17 in particular implies that d_0-separated i-horizontal pairs are never j-semi-compressed. Our proof follows that of Lemma 2.7 in [KKL98], with the required sublinear adaptations. The meaning of the lemma is that ϕ coarsely preserve the product structure of X and Y. The idea of the proof is that if ϕ fails to do so, this failure is manifested by a sequence of points and constants that give rise to u-admissible asymptotic cones of X and Y in which $\mathrm{Cone}(\phi)$ also fails to preserve the product structure, contradicting the hypothesis.

Proof. Fix $i, j \in \{1, \dots, n\}$.
Step 1. Let $x \in X$. We claim that there is a constant $d_0(x)$ such that for all $d \geq d_0(x)$, all d-separated i-horizontal pairs in the ball $B := B\big(x, 99d\big)$ are simultaneously j-compressed or simultaneously j-uncompressed.

First we show there is $d_0(x)$ such that for $d > d_0(x)$ there are no d-separated i-horizontal pairs in $B(x, 99d)$ that are j-semi-compressed. Assume towards contradiction there were such pairs for all $d > 0$. They give rise to a sequence of radii $d_k \to \infty$ and points $x^k, z^k \in B(x, 99d_k)$ that are i-horizontal, $d(x^k, z^k) = d_k$, with (x^k, z^k) being j-semi-compressed. Consider the asymptotic cone $X_\omega := \mathrm{Cone}_\omega(X, x, d_k)$, and the points $x_\omega := (x^k)_{k \in \mathbf{N}}, z_\omega := (z^k)_{k \in \mathbf{N}} \in X_\omega$. These points agree on the $(X_\omega)_m$-coordinate for all $m \neq i$. The hypothesis on X and Y assures that the cone map $F = \mathrm{Cone}(\phi)$ respects the product structure of X_ω and Y_ω, therefore if $m \neq i$ we have

$$d_{(Y_\omega)_m}\Big(\pi_{(Y_\omega)_m}\big(F(x_\omega)\big), \pi_{(Y_\omega)_m}\big(F(z_\omega)\big)\Big) = 0 \tag{2.1}$$

and so

$$d_{(Y_\omega)}\big(F(x_\omega), F(z_\omega)\big) = d_{(Y_\omega)_i}\Big(\pi_{(Y_\omega)_i}\big(F(x_\omega)\big), \pi_{(Y_\omega)_i}\big(F(z_\omega)\big)\Big). \tag{2.2}$$

Finally, for every $m \in \{1, \dots, n\}$ it holds that $\pi_{(Y_\omega)_m}\big(F(x_\omega)\big) = \lim_\omega \pi_{Y_m}\big(\phi(x^k)\big)$ and similarly for z_ω, yielding

$$\lim_\omega \frac{1}{d_k} d_{Y_m}\Big(\pi_{Y_m}\big(\phi(x^k)\big), \pi_{Y_m}\big(\phi(z^k)\big)\Big) = d_{(Y_\omega)_m}\Big(\pi_{(Y_\omega)_m}\big(F(x_\omega)\big), \pi_{(Y_\omega)_m}\big(F(z_\omega)\big)\Big).$$

Plugging this formula into Equations 2.1 and 2.2 above, we get:

(1) For $j \neq i$ $\lim_\omega \frac{1}{d_k} d_{Y_j}\Big(\pi_{Y_j}\big(\phi(x^k)\big), \pi_{Y_j}\big(\phi(z^k)\big)\Big) = 0$.
(2) For $j = i$, the fact that F is L-bilipschitz and preserves the product structure, together with $d_{X_\omega}(x_\omega, z_\omega) = 1$, implies

$$\lim_\omega \frac{1}{d_k} d_{Y_i}\Big(\pi_{Y_i}\big(\phi(x^k)\big), \pi_{Y_i}\big(\phi(z^k)\big)\Big) \in [L^{-1}, L].$$

The assumption that $0 < \epsilon < L_1^{-1} < L^{-1}$ contradicts the assumption that (x^k, z^k) are j-semi-compressed.

Next, we show that there is $d_0(x)$ such that for every $d > 0$, every two pairs of d-separated i-horizontal points inside $B(x, 99d)$ are j-compatible. This is done exactly as above. If there is no such $d_0(x)$, we obtain sequences of radii $d_k \to \infty$ and points $x^{1,k}, z^{1,k}, x^{2,k}, z^{2,k} \in B(x, 99d_k)$ such that for each $k \in \mathbf{N}$ and $l \in \{1,2\}$, the pair $(x^{l,k}, z^{l,k})$ is i-horizontal with $d_X(x^{l,k}, z^{l,k}) \in [d_k, 99d_k)$ and $(x^{1,k}, z^{1,k})$ is j-compressed while $(x^{2,k}, z^{2,k})$ is j-uncompressed. We consider the cones $X_\omega := \mathrm{Cone}(X, x, d_k), Y_\omega := \mathrm{Cone}\big(Y, \phi(x), d_k\big)$ and the cone map $F := \mathrm{Cone}(\phi)$. For $l \in \{1,2\}$, the points $x^l_\omega = (x^{l,k})_{k\in\mathbf{N}}, z^l_\omega = (z^{l,k})_{k\in\mathbf{N}} \in X_\omega$ both lie within the ball of radius 99 around the base point of the cone, and agree on the $m \neq i$ coordinates. Since F respects products there are L-bilipschitz homeomorphisms $F_m : (X_\omega)_m \to (Y_\omega)_m$ for which $F_i \circ \pi_{(X_\omega)_m} = \pi_{(Y_\omega)_m} \circ F$, hence for $m \neq i$ we must have for $l \in \{1,2\}$

$$d_{(Y_\omega)_m}\Big(\pi_{(Y_\omega)_m}\big(F(x^l_\omega)\big), \pi_{(Y_\omega)_m}\big(F(z^l_\omega)\big)\Big) = 0$$

Since this does not depend on l, we see as in step 1 that for $j \neq i$ for all large enough k, both $(x^{1,k}, z^{1,k})$ and $(x^{2,k}, z^{2,k})$ are j-compressed and in particular compatible. A similar argument shows that these pairs are both j-uncompressed for $j = i$. We conclude that the pairs are j-compatible for all j.

Step 2. We now show that we can choose d_0 to be a sublinear function in $O(u)$. We start by taking, for each $x \in X$, the infimal $\tilde{d}_0(x)$ such that for all $d \geq \tilde{d}_0(x)$, all i-horizontal d-separated pairs in $B(x, 99d)$ are j-compatible. Next define the radial function $d_0(|x|) := \sup_{y:|y|=|x|} \tilde{d}_0(y)$.

A-priori $d_0 : \mathbf{R}_{\geq 0} \to \mathbf{R}_{\geq 0} \cup \{\infty\}$, and we need to show it is in fact real valued. Assume towards contradiction that d_0 is not bounded along some sphere S around x_0, and get points $x^k \in S$ and real numbers $r_k \to \infty$ such that the balls $B(x^k, 10r_k)$ contain two i-horizontal r_k-separated pairs $(x^{1,k}, z^{1,k}), (x^{2,k}, z^{2,k})$ that are j-incompatible. We consider the cone $X_\omega := (X, x_0, r_k)$, based at $x_\omega = (x_0)$. Since $x^k \in S$ we have $d(x^k, x_0) = R$ for some constant R, so $(x^{l,k}), (z^{l,k}) \in X_\omega$. Moreover, the assumption that $(x^{1,k}, z^{1,k})$ are r_k-separated and lie in $B(x^k, 99r_k)$ means that $d_\omega(x^l_\omega, z^l_\omega) \in [1, 198]$. Again, for every k $x^{l,k}$ and $z^{l,k}$ agree on the $m \neq i$ coordinates, hence so do x^l_ω and z^l_ω. As in previous steps, we get a contradiction to the assumption that $(x^{1,k}, z^{1,k}), (x^{2,k}, z^{2,k})$ are j-incompatible.

The function d_0 is therefore a radial real valued function. We show it is $O(u)$. Assume towards contradiction that there is a sequence $x^k \to \infty$ so that $\lim_\omega \frac{u(x^k)}{d_0(x^k)} = 0$. By the construction of d_0 we may take x^k such that each ball $B\big(x^k, 99d_0(x^k)\big)$ contains two $(d_0(x^k) - 1)$-separated i-horizontal pairs that are j-incompatible. The assumption $\lim_\omega \frac{u(x^k)}{d_0(x^k)} = 0$ implies that the cone with moving base points $X_\omega := \mathrm{Cone}_\omega\big(X, x^k, d_0(x^k)\big)$ is u-admissible. By Lemma 2.7 the cone map $F = \mathrm{Cone}(\phi)$ is L-bilipschitz, and so respects products. One gets a contradiction as in step 2.

Step 3. We have established the following situation: there is $d_0 \in O(u)$ such that for every $x \in X$ and every $d \geq d_0(x)$, the property of whether a d-separated i-horizontal pair inside $B = B(x, 99d)$ is j-uncompressed or j-compressed depends only on x, i and j, and not on the specific pair (in particular such a pair cannot be j-semi-compressed). We call it the j-type of B. We assume, as we may, that d_0 is subadditive. We can right away conclude that there are no d_0-separated i-horizontal j-semi-compressed pairs at all in X: such a pair (x, x') must admit $d(x, x') \geq d_0(x)$ and therefore for $d = d(x, x') \geq d_0(x)$, x, x' are d-separated and $x' \in B(x, 99d)$. Step 1 shows that (x, x') is not j-semi-compressed.

Our next goal is to lose any restriction on the radius of the balls $B(x, 99d)$. Fix $x \in X$. Our first observation is that the j-type of d_0-separated i-horizontal pairs which involve x depend only on x: they are the type of the ball $B := B\big(x, 99d_0(x)\big)$. Indeed if (x, x') is an i-horizontal pair, one can consider the geodesic $\gamma := [x, x']$. If $x' \in B$ the claim is immediate. In particular the pair $\Big(x, \gamma\big(98d_0(x)\big)\Big)$ has the j-type of B. Therefore for $R_1 := 98d_0(x)$, the ball $B_1 := B(x, 99 \cdot R_1)$ has the j-type of B. We can continue to enlarge the radius of the balls until we find one which contains x', so indeed (x, x') has the j-type of B, which is what we were set out to prove.

We push the above argument a bit further in order to show that any two d_0-separated i-horizontal pairs are j-compatible, regardless of which fiber of X_i they are in. Let (x, x') and (z, z') be two d_0-separated i-horizontal pairs. Assume first that $x, x', z, z' \in X$ are pairwise i-horizontal, so they all lie in the same fiber. Since each X_i can be assumed to have infinite diameter, we can find a point p in the same X_i fiber which is far enough do that (p, w) is d_0-separated for all $w \in \{x, x', z, z'\}$. It is clear that the j-type of all points must be equal to that of $B\big(p, 99d_0(p)\big)$.

This allows us to define the j-type of each X_i-fiber, defined by a choice of $\hat{x} \in \hat{X} := \prod_{m \neq i} X_m$. To finish the proof we need to show that the (i,j)-type of $\hat{x} \in \hat{X}$ depends only on (i, j) and not on the choice of $\hat{x}$. Indeed, fix $\hat{x}, \hat{z} \in \prod_{m \neq i} X_m$, and $\gamma : \mathbf{R}_{\geq 0} \to X_i$ an infinite ray in X_i (we assume X_i is of infinite diameter and geodesic and so by an Arzela-Ascoli argument there exists an infinite geodesic ray).

Consider the points $x^k = \big(\gamma(k), \hat{x}\big), z^k = \big(\gamma(k), \hat{z}\big) \in X$. For all large enough k, (x^0, x^k) and (z^0, z^k) are d_0-separated i-horizontal. For all large enough $k, m \in \mathbb{N}$, all pairs $(x^0, x^k), (x^0, x^m)$ are j-compatible and all $(z^0, z^k), (z^0, z^m)$ are j-compatible. Consider the cone $X_\omega := (X, x^0, k)$. Let x^0_ω be the base point of X_ω, $x_\omega := (x^k)_k, z_\omega := (z^k)_k$. Clearly $x_\omega = z_\omega$ and $d(x^0_\omega, x_\omega) = 1$. We conclude that for all large enough $k, m \in \mathbb{N}$, it holds that (x^0, x^k) and (z^0, z^m) are j-compatible, so $\hat{x}, \hat{z}$ have the same (i, j)-type. The lemma stands proven. □

Proof of Proposition 2.11. Together with Lemma 2.17, the assumption on the product decomposition of the relevant cone maps implies that for every i there is a unique j for which the projection on Y_j is uncompressed for all i-horizontal pairs. We may reindex and assume Y_i is that factor. To ease the notation, we fix from now $i = 1$. The proof is identical for all other i.

The $O(u)$-embeddings. We denote $\hat{X} := \prod_{m=2}^n X_m$, and write points $x \in X$ as a pair $(z, \hat{x}) \in X_1 \times \hat{X}$. Fixing $\hat{x} \in \hat{X}$ we obtain a map $\phi_{\hat{x}} : X_1 \to Y_1$ given by $z \mapsto \pi_{Y_1}\big(\phi(z, \hat{x})\big)$. We prove that these are all $(L_1, u_{\hat{x}})$-bilipschitz embeddings, for $u_{\hat{x}} = u + d_0 + (u + d_0)(|\hat{x}|_{\hat{X}}) \in O(u)$, where $L_1 \geqslant 1$ and $d_0 \in O(u)$ are given by Lemma 2.17. Indeed, let $z, z' \in X_1$ be a two points such that the pair $(z, \hat{x}), (z', \hat{x})$ is

d_0-separated. By assumption on X_1 this pair is 1-uncompressed so

$$L_1^{-1}d(z,z') \leqslant d\Big(\pi_{Y_1}\big(\phi(z,\hat{x})\big), \pi_{Y_1}\big(\phi(z',\hat{x})\big)\Big) \leqslant L_1 d(z,z')$$

If the pair $(z,\hat{x}),(z',\hat{x})$ is not d_0-separated, then subadditivity of d_0 gives $d(z,z') = d_{X_1}(z,z') \leqslant d_0(z\vee z') + d_0(\hat{x})$. Recalling that $L_1^{-1} < 1$, this gives a lower bound:

$$L_1^{-1}d(z,z') - d_0(z\vee z') - d_0(\hat{x}) \leqslant 0 \leqslant d\Big(\pi_{Y_1}\big(\phi(z,\hat{x})\big), \pi_{Y_1}\big(\phi(z',\hat{x})\big)\Big).$$

The Pythagorean formula yields the upper bound:

$$d\Big(\pi_{Y_1}\big(\phi(z,\hat{x})\big), \pi_{Y_1}\big(\phi(z',\hat{x})\big)\Big) \le d\big(\phi(z,\hat{x}), \phi(z',\hat{x})\big) \le Ld(z,z') + u(z\vee z') + u(\hat{x}).$$

We conclude that $\phi_{\hat{x}} : X_i \to Y_i$ is an $(L, u_{\hat{x}})$-bilipschitz embedding.

$O(u)$**-surjectivity.** We now show that $\phi_{\hat{x}}$ is $v_{\hat{x}}$-quasi-surjective with $v_{\hat{x}} \in O(u)$. For convenience we show it in the case $\hat{x} = \hat{0}$, the base point of $\hat{X} = \prod_{m=2}^n X_m$. Assume towards contradiction that there is a sequence $y_1^n \in Y_1$ for which $\lim_\omega \frac{u(y_1^n)}{d\big(\operatorname{Im}(\phi_{\hat{0}}), y_1^n\big)} = 0$. Consider the points $y^n := (y_1^n, \hat{0}_Y)$, where $\hat{0}_Y$ is the base point of $\hat{Y} := \prod_{m=2}^n Y_m$. The quasi-surjectivity of ϕ implies that there are corresponding $x^n \in X$ for which $d\big(\phi(x^n), y^n\big) \le u(y^n) = u(y_1^n)$. It is a general fact of sublinear bilipschitz embeddings that x^n and y_1^n admit $\frac{1}{L'}|y_1^n| \leqslant |x^n| \leqslant L'|y_1^n|$ for some $L' > L$ and all large enough y_1^n.

Denote $\sigma_n := d\big(\operatorname{Im}(\phi_{\hat{0}}), y_1^n\big)$, and consider the cones $X_\omega := \operatorname{Cone}(X, x^n, \sigma_n)$, $Y_\omega := \operatorname{Cone}\big(Y, \phi(x^n), \sigma_n\big)$. The previous paragraph implies $\lim_\omega \frac{u(x^n)}{\sigma_n} = 0$ and therefore by Lemma 2.7 the map $F = \operatorname{Cone}(\phi)$ is a bilipschitz homeomorphism. The hypothesis on the factors of X and Y implies that the map F respects the product structure of X_ω and Y_ω, i.e. there are bilipschitz maps F_i for $1 \leqslant i \leqslant n$ such that if $x_\omega, x'_\omega \in X_\omega$ agree on the j-coordinate, then

$$\pi_{(Y_\omega)_j}\big(F(x_\omega)\big) = F_j\big(\pi_{(X_\omega)_j}(x_\omega)\big) = F_j\big(\pi_{(X_\omega)_j}(x'_\omega)\big) = \pi_{(Y_\omega)_j}\big(F(x'_\omega)\big) \tag{2.3}$$

Set $x_\omega := (x^n)_{n\in\mathbf{N}}$, $x_\omega^0 := (x_1^n, \hat{0}_{\hat{X}})_{n\in\mathbf{N}}$ (where x_1^n is the first coordinate of x^n). These two points in X_ω agree on the first coordinate, thus

$$\Big(\pi_{Y_1}\big(\phi(x_1^n, \hat{0})\big)\Big)_{n\in\mathbf{N}} = \pi_{(Y_\omega)_1}\big(F(x_\omega^0)\big) = \pi_{(Y_\omega)_1}\big(F(x_\omega)\big) = \pi_{(Y_\omega)_1}\Big(\big(\phi(x^n)\big)_{n\in\mathbf{N}}\Big)$$

(Equalities are in $(Y_\omega)_1$). On the other hand $d\big(\phi(x^n), (y_1^n, \hat{0}_Y)\big) \leqslant u(y_1^n)$ and $\lim_\omega \frac{y_1^n}{\sigma_n} = 0$, so $\big(\phi(x^n)\big)_{n\in\mathbf{N}} = (y_1^n, \hat{0}_Y)_{n\in\mathbf{N}}$ (equality in Y_ω). By definition $\phi_{\hat{0}}(x_1^n) = \pi_{Y_1}\big(\phi(x_1^n, \hat{0})\big)$. We conclude

$$\big(\phi_{\hat{0}}(x_1^n)\big)_{n\in\mathbf{N}} = \pi_{(Y_\omega)_1}\big(y_1^n, \hat{0}_Y)_{n\in\mathbf{N}}\big) = (y_1^n)_{n\in\mathbf{N}},$$

and from the Pythagorean formula we get $\lim_\omega \frac{d_{Y_1}\big(\phi_{\hat{0}}(x_1^n), y_1^n\big)}{\sigma_n} = 0$, contradicting the definition of σ_n. We conclude there is a function $v_{\hat{0}} \in O(u)$ such that $\phi_{\hat{0}}$ is $v_{\hat{0}}$-quasi-surjective.

Sublinear Control. Finally, we show there is $v \in O(u)$ such that for $\hat{x}, \hat{w} \in \hat{X}$ and $x_1 \in X_1$, we have $d_{Y_1}\big(\phi_{\hat{x}}(x_1), \phi_{\hat{w}}(x_1)\big) \le v(|(\hat{x}, x_1)|, |(\hat{w}, x_1|) \le v(\hat{x}\vee\hat{w}) + v(x_1)$. For convenience, we show it in the case where $\hat{w} = \hat{0}$, which clearly implies the general case. Assume towards contradiction that there are sequences $x_1^n \in X_1$ and $\hat{x^n} \in \hat{X}$ with $\lim_\omega \frac{u(\hat{x^n}, x_1^n)}{d_{Y_1}\big(\phi_{\hat{x^n}}(x_1^n), \phi_{\hat{0}}(x_1^n)\big)} = 0$. Denote $x^n = (x_1^n, \hat{x^n})$ and $\sigma_n := d_{Y_1}\big(\phi_{\hat{x^n}}(x_1^n), \phi_{\hat{0}}(x_1^n)\big)$.

Consider the cones $X_\omega = \mathrm{Cone}(X, x^n, \sigma_n), Y_\omega := \mathrm{Cone}\big(Y, \phi(x^n), \sigma_n\big)$. From Lemma 2.7 we know that the map $F = \mathrm{Cone}(\phi)$ is an L-bilipschitz homeomorphism hence respects the product structure of X_ω and Y_ω. We thus have $F_1 : (X_\omega)_1 \to (Y_\omega)_1$ an L-bilipschitz map such that for $x_\omega := (x^n)_{n\in\mathbf{N}}, x_\omega^0 = (x_1^n, \hat{0})_{n\in\mathbf{N}}$ we have:

$$\big(\phi_{\hat{x^n}}(x_1^n)\big)_{n\in\mathbf{N}} = \pi_{(Y_\omega)_1}\big(F(x_\omega)\big) = \pi_{(Y_\omega)_1}\big(F(x_\omega^0)\big) = \big(\phi_{\hat{0}}(x_1^n)\big)_{n\in\mathbf{N}}$$

And so $\lim_\omega \frac{d_{Y_1}\left(\phi_{\hat{x^n}}(x_1^n), \phi_{\hat{0}}(x_1^n)\right)}{\sigma_n} = 0$, contradicting the definition of σ_n. We conclude that indeed there is $v \in O(u)$ such that for all $\hat{x} \in \hat{X}$ and $x_1 \in X_1$ we have

$$d_{Y_1}\big(\phi_{\hat{x}}(x_1), \phi_{\hat{0}}(x_1)\big) \leq v(|(\hat{x}, x_1)|).$$

This inequality concludes the proof. □

As in [KKL98], in order to include Euclidean factors (or, more generally, factors with asymptotic cones homeomorphic to Euclidean space) we will also need a slight variation of the above result.

Proposition 2.18. *Let $X := \bar{X} \times Z$, $Y := \bar{Y} \times W$ be two geodesic metric spaces. Assume that $f : X \to Y$ is an (L, u)-bilipschitz equivalence, and that for every pair of u-admissible cones X_ω, Y_ω the induced map $F = \mathrm{Cone}(f) : X_\omega \to Y_\omega$ preserves the product $\bar{X}_\omega \times Z_\omega$ and $\bar{Y}_\omega \times W_\omega$ by the Z_ω, W_ω factors. Equivalently, there exists a homeomorphism $\bar{F} : \bar{X}_\omega \to \bar{Y}_\omega$ such that the following diagram commutes:*

$$\begin{array}{ccc} X_\omega & \xrightarrow{F} & Y_\omega \\ \downarrow{\scriptstyle \bar{\pi}_\omega} & & \downarrow{\scriptstyle \bar{\pi}_\omega} \\ \bar{X}_\omega & \xrightarrow{\bar{F}} & \bar{Y}_\omega \end{array}$$

Then there is an (L', v)-bilispchitz equivalence $\bar{f} : \bar{X} \to \bar{Y}$, with $v \in O(u)$ and L' depending only on L and u such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle \bar{\pi}} & & \downarrow{\scriptstyle \bar{\pi}} \\ \bar{X} & \xrightarrow{\bar{f}} & \bar{Y} \end{array}$$

commutes up to sublinear $O(u)$-error.

Proof. We define for each $z \in Z$ the map $\phi_z : \bar{X} \to \bar{Y}$ by $\phi_z(\bar{x}) = \bar{\pi}(f(x, z))$. A similar computation to that done in the proof of Proposition 2.11 shows that these are (L, u_z)-bilipschitz equivalences with $u_z \in O(u)$. To be a bit more explicit (but without repeating the entire argument): the quantity $d_{\bar{Y}}\big(\phi_z(\bar{x_1})\big), \phi_z(\bar{x_2})\big)$ is clearly bounded above by the upper bound on $d\big(f(\bar{x_1}, z), f(\bar{x_2}, z)\big)$ given by f. For the lower bound, the reverse triangle inequality implies that one has to find a sublinear function $v_z \in O(u)$ such that $d_W\Big(\pi_W\big(f(\bar{x_1}, z)\big), \pi_W\big(f(\bar{x_2}, z)\big)\Big) < v_z(\bar{x_1} \vee \bar{x_2})$. This is done in the usual way: a sequence of pairs $(\bar{x_1^n}, z)$ and $(\bar{x_2^n}, z)$ defying this property yields two points in an u-admissible cone which defy the fact that the cone map $F = \mathrm{Cone}(f)$ respects the product structure $X_\omega = \bar{X}_\omega \times Z_\omega$.

The quasi-surjectivity and sublinear control when varying z are done exactly like in Proposition 2.11, concluding the proof. □

2.D. **Concluding Theorem A.** By assumption, the asymptotic cones of X and Y decompose into a product of a Euclidean factor and a non-Euclidean factor, where the non-Euclidean factor is a product of factors of types I and II. From [KKL98, Theorem 5.1] we see the cone maps between u-admissible cones preserve this product structure, and that the dimension of the Euclidean factors equal, i.e. $p = q$ in the notation of the statement. Proposition 2.18 gives rise to an $O(u)$-bilipschitz equivalence $\bar{\phi} : \prod_{i=1}^{n} X_i \to \prod_{j=1}^{m} Y_j$. Using [KKL98, Theorem 5.1] once more we see that $n = m$ and that in u-admissible cones the cone maps of $\bar{\phi}$ preserve the product structure. Theorem A now follows immediately from Proposition 2.11.

3. Completely solvable groups, Corollary B and Theorem C

3.A. **Some preliminaries on distortion and Cornulier's class.** As announced in the beginning of the introduction, the quasiisometry classification of connected Lie groups amounts to that of the completely solvable ones. Given a simply connected solvable Lie group G there are several possible, equivalent definitions for $\rho_0(G)$. We give below that of Jablonski, building on Gordon and Wilson.

Proposition 3.1 ([Jab19, §4.1 and §4.2], after [GW88])**.** *Let G be a simply connected solvable Lie group. There exists a (possibly non-unique) left-invariant metric $g_{\max}$ on G whose isometry group contains a transitive completely solvable group G_0. Moreover, the group G_0 obtained in this way is unique up to isomorphism and it does not depend on $g_{\max}$.*

Definition 3.2. Let G be a simply connected solvable Lie group. We define $\rho_0(G)$ as G_0. We say that a group is in the class $(\mathcal{C}_0)$ if $G = \rho_0(G)$, that is, if G is completely solvable.

It is clear that the groups G and G_0 are quasiisometric, being closed co-compact subgroups of the isometry group $\widehat{G}$ of $g_{\max}$. They are commable in the terminology of [Cor18]. The role of the group $\widehat{G}$ is played by the group denoted G_3 in Cornulier's treatment ([Cor20, Lemme 1.3], summarizing [Cor08]).

Definition 3.3. Let G be a group in the class $(\mathcal{C}_0)$. The exponential radical $\mathrm{R}_{\exp} G$ of G is the smallest normal subgroup N of G such that G/N is nilpotent.

The exponential radical was named by Osin [Osi02] as it is the subgroup of exponentially distorted elements in G (together with 1). We call $\dim G/\mathrm{R}_{\exp} G$ the rank of G. If $\widehat{G}$ is a real semisimple Lie group with trivial center, writing an Iwasawa decomposition $\widehat{G} = KAN$ and setting $G = AN$, we recover that the real rank of $\widehat{G}$ is the rank of G. More generally, the rank as defined here is still the dimension of one (or any) Cartan subgroup of G.

Definition 3.4. Let G be a completely solvable Lie group with exponential radical N. Say that G is in $(\mathcal{C}_1)$ if the extension $1 \to N \to G \to G/N \to 1$ splits and the action of G/N on N is $\mathbf{R}$-diagonalizable.

Definition 3.5. Let G be a completely solvable group with $N = \mathrm{R}_{\exp} G$, and set $H = G/N$. Decompose $\phi = \mathrm{ad}\colon \mathfrak{g} \to \mathrm{Der}(\mathfrak{n})$ into

$$\phi = \phi_\delta + \phi_\nu$$

where ϕ_δ is $\mathbf{R}$-diagonalisable and ϕ_ν is nilpotent [Bou75]. Note that ϕ_δ is zero when restricted to $\mathfrak{n}$, so that it is well-defined on $\mathfrak{h}$. Let $\rho_1(G)$ be $N \rtimes H$, where $\mathfrak{h}$ acts on $\mathfrak{n}$ through ϕ_δ. We also write $\rho_1(\mathfrak{g})$ for $\mathrm{Lie}(\rho_1(G))$.

Although in this paper the focus is on ρ_1, we recall below the definition of ρ_∞, a further reduction that we mentioned in the introduction.

Definition 3.6 (Cornulier, [Cor11])**.** Let G be a completely solvable group, and let $H = G/\operatorname{R}_{\exp G}$. The Lie algebra of H has a filtration by its the derived central series. Define

$$\mathfrak{h}_\infty = \bigoplus_{i>0} C^i\mathfrak{h}/C^{i+1}\mathfrak{h}$$

with the brackets induced from those of $\mathfrak{h}$. The action of H on $\operatorname{R}_{\exp} G$, after being factored through $H/C^2H \simeq H_\infty/C^2H_\infty$, lifts a new action of H_∞ on $\operatorname{R}_{\exp} G$; define $\rho_\infty(G)$ as the corresponding semidirect product $\operatorname{R}_{\exp G} \rtimes H_\infty$.

Theorem 3.7 (Cornulier, [Cor11])**.** *Let G be a completely solvable group, and let $H = G/\operatorname{R}_{\exp} G$. Then*

(1) *G and $\rho_1(G)$ are $O(\log)$-bilipschitz equivalent.*
(2) *H is a $O(\log)$-Lipschitz retract of G, more precisely:*
  (a) $\pi : G \to H$ is $O(\log)$-Lipschitz;
  (b) Let X be a nonzero vector in a Cartan subalgebra of $\mathfrak{g}$. Then there exists $f : H \to G$ (depending on X) which is $O(\log)$-Lipschitz and such that
    (i) $\pi \circ f$ is $O(\log)$-close to the identity of H.
    (ii) $f \circ \pi(\exp(tX)) = \exp(tX)$ for all t in $\mathbf{R}$.
(3) *G and $\rho_\infty(G)$ are $O(u)$-bilipschitz equivalent, where the function u depends on G.*

Proof. Parts (1) and (3) are stated by Cornulier [Cor11]. Parts (2a) and (2bi) express that $\pi\colon G \to H$ is a retract in the category of $O(\log)$-Lipschitz maps, which is also stated in [Cor19, Example 2.6] and the content of the proof can be found [Cor11, Theorem 4.4], where in the notation of [Cor11], f is the map $\psi_{|V}^{-1}$ (before the statement of Theorem 4.4). See also the few lines before [Cor08, Lemma 5.2] where the map f that we need is named ψ. To check part (2bii) we have to specify the construction of f; for this we refer to some parts of Cornulier's proof in [Cor11]. In Cornulier's construction, $\pi(X)$ is identified with an element ξ in a subspace of V, the complement of the Lie algebra of $\mathfrak{h} \cap \operatorname{R}_{\exp} \mathfrak{g}$ in $\mathfrak{h}$ where $\mathfrak{h}$ can be taken to be any Cartan subalgebra of $\mathfrak{g}$, and then, $f \circ \pi(\exp(tX)) = \exp_G(t\xi)$. Since in our assumption, X lies in a Cartan subalgebra $\mathfrak{h}$ of X, we can take the Cartan subalgebra in the construction of f to be $\mathfrak{h}$, and then, with this choice, take ξ to be equal to X. In this way, $f \circ \pi(\exp(tX)) = \exp(tX)$ for all $t \in \mathbf{R}$. ☐

The following notion will be useful for us, as it allows to use the work of Cornulier and Tessera for upper bounds on Dehn functions.

Definition 3.8 (After Cornulier and Tessera, [CT17, Definition 1.2])**.** Let G be a group in $(\mathcal{C}_0)$, $\mathfrak{n} = \operatorname{R}_{\exp} \mathfrak{g}$. We say that G is standard solvable if its exponential radical splits, the quotient $A = G/\operatorname{R}_{\exp} G$ is abelian, and the action of $\mathfrak{a}$ on $\mathfrak{n}/[\mathfrak{n},\mathfrak{n}]$ has a trivial kernel.

It is not completely obvious why the definition above is equivalent to that of Cornulier and Tessera (which applies in a wider setting including non-Archimedean Lie groups as well). Especially, in [CT17], it is asked that the action of $\mathfrak{a}$ on every proper quotient of $\mathfrak{n}/[\mathfrak{n},\mathfrak{n}]$ does not admit zero as a nontrivial weight, and $\mathfrak{n}$ is not a-priori required to be the exponential radical. The connection is made by the following proposition.

Proposition 3.9. *Let G be a completely solvable group. Assume that G splits as $U \rtimes A$, where A is abelian, U is nilpotent, and the action of A on $U/[U,U]$ has no fixed point. Then*

$$U = [G,G] = \mathrm{R}_{\exp} G.$$

Moreover, the action of A on any non-trivial quotient of $U/[U,U]$ has no fixed point, hence G is standard solvable in the sense of [CT17].

Proof. With notation as before, consider

$$\mathfrak{g} = \mathfrak{u} \rtimes \mathfrak{a},$$

and denote $\rho : \mathfrak{a} \to \mathfrak{gl}(\mathfrak{u}/[\mathfrak{u},\mathfrak{u}])$. In order to prove that $\mathfrak{u} \subset [\mathfrak{g},\mathfrak{g}]$, it is enough to prove that $\rho(\mathfrak{a})(\mathfrak{u}/[\mathfrak{u},\mathfrak{u}]) = \mathfrak{u}/[\mathfrak{u},\mathfrak{u}]$. Indeed, if for $u \in \mathfrak{u}$, we can write

$$u = [a, u'] + w$$

where $u' \in \mathfrak{u}$ and $w \in [\mathfrak{u},\mathfrak{u}] \subseteq [\mathfrak{g},\mathfrak{g}]$, then u is the linear combination of Lie brackets in $\mathfrak{g}$, so $u \in [\mathfrak{g},\mathfrak{g}]$. Let us now prove the claim. Let $\mathfrak{v} = \mathfrak{u}/[\mathfrak{u},\mathfrak{u}]$ and let $v \in \mathfrak{v}$. Write $v = v_1 + \cdots + v_s$, where $v_i \in V^{\lambda_i}$, $\lambda_i \in \mathrm{Hom}(\mathfrak{a}, \mathbf{R})$ nonzero. It is sufficient to prove that v_i is in the image of $\rho(\mathfrak{a})$ for every $i = 1, \ldots, s$. Choosing $a_i \in \mathfrak{a}$ such that $\lambda_i(a_i) \neq 0$, we find that $\rho(a_i)_{|V^{\lambda_i}}$ has nonzero diagonal entries, hence it is surjective, so that $v_i \in \rho(a_i)(\mathfrak{u}/[\mathfrak{u},\mathfrak{u}])$. The fact that $\rho(\mathfrak{a})(\mathfrak{u}/[\mathfrak{u},\mathfrak{u}]) = \mathfrak{u}/[\mathfrak{u},\mathfrak{u}]$ also implies that in any non-trivial quotients of $\mathfrak{u}/[\mathfrak{u},\mathfrak{u}]$, the zero weight of the $\mathfrak{a}$-action is trivial.

We proved that $\mathfrak{u} \subseteq [\mathfrak{g},\mathfrak{g}]$. The converse containment follows from the fact that $\mathfrak{g}/\mathfrak{u} = \mathfrak{a}$ is abelian. Now, $C^3\mathfrak{g} = [\mathfrak{g},\mathfrak{g}]$ because of the following series of equalities

$$[\mathfrak{g},\mathfrak{u}] = [\mathfrak{a} + \mathfrak{u}, \mathfrak{u}] = [\mathfrak{a},\mathfrak{u}] + [\mathfrak{u},\mathfrak{u}] = \mathfrak{u}.$$

and in view of the fact that $\mathrm{R}_{\exp}\,\mathfrak{g}$ is the limit of the central series, $\mathfrak{u} = \mathrm{R}_{\exp}\,\mathfrak{g}$. ☐

We can check that all the groups of class $(\mathcal{C}_0)$ of dimension less than 5, save for one, are standard solvable (See Table 4). The only exception is the group $G_{4,3}$, since $G_{4,3}/\mathrm{R}_{\exp} G_{4,3}$ is non-abelian. There are many non-standard solvable groups of dimension 5 and class $(\mathcal{C}_0)$ - see Section 5.C.2.

3.B. **Warm-up: the groups in Example 1.7 are not quasiisometric.** We prove below that the groups in Example 1.7 are not quasiisometric. The proof is less involved than that of Theorem C but the main idea already intervenes, so we give it before.

Proposition 3.10. *Let $\alpha \in (0,1)$. The groups $G_{4,9}^0$ and $\mathbf{R} \times G_{3,5}^\alpha$ are not quasiisometric, for any $\mu \neq 1$.*

Proof. Let's check that $\mathbf{R} \times G_{3,3}$ is the group in the class $(\mathcal{C}_1)$ associated to $G_{4,9}^0$ by [Cor11], and so there is a $O(\log)$-sublinear bilipschitz equivalence $\phi \colon G_{4,9}^0 \to \mathbf{R} \times G_{3,3}$. The Lie algebra $\mathfrak{g} = \mathfrak{g}_{4,9}^0$ of $G_{4,9}^0$ is a semidirect product $\mathfrak{heis} \rtimes \delta$ where $\mathfrak{heis}$ has a basis (e_1, e_2, e_3) with $[e_1, e_2] = e_3$ and $\delta e_i = e_i$ if $i = 1, 3$ or 0 if $i = 2$. The derived subalgebra is $\mathfrak{u} = \langle e_1, e_3 \rangle$ and $[\mathfrak{g},\mathfrak{u}] = \mathfrak{u}$, so that $\mathfrak{u}$ is the Lie algebra of the exponential radical. The matrices of ad_{e_3} and ad_{e_4} in the basis (e_1, e_2) are, respectively,

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

From the definition of ρ_1 it follows that, using the same basis and changing the brackets, $\rho_1(\mathfrak{g}_{4,9})^0$ is the Lie algebra $\mathfrak{a} \rtimes \langle e_4 \rangle \times \mathbf{R} \simeq \mathfrak{g}_{3,3} \times \mathbf{R}$.

Assume now towards contradiction that $G_{4,9}^0$ and $\mathbf{R} \times G_{3,5}^\alpha$ are quasiisometric; then by Cornulier's Theorem 3.7 there is a $O(\log)$-bilispchitz equivalence $\psi \colon \mathbf{R} \times G_{3,3} \to \mathbf{R} \times G_{3,5}^\alpha$. By Theorem A there is a sublinear bilipschitz equivalence $\underline{\psi} \colon G_{3,3} \to G_{3,5}^\alpha$. However, by

[Pal20b], $O(u)$-sublinear bilipschitz equivalences between Heintze groups preserve the conformal dimensions of their Gromov boundaries. $G_{3,3}$ and $G_{3,5}^\alpha$ are Heintze groups, and the conformal dimensions are

$$\operatorname{Cdim} \partial_\infty G_{3,3} = 2, \quad \text{while} \quad \operatorname{Cdim} \partial_\infty G_{3,5}^\alpha = 1 + 1/\alpha > 2.$$

So we reach a contradiction. □

3.C. **Proof of Corollary B and Theorem C.** We will achieve both corollaries by means a technical result, Theorem 3.17. Before that we need some preparation.

Definition 3.11. A Heintze group of diagonal type is a completely solvable Lie group $M \rtimes \mathbf{R}$, where M is simply connected nilpotent and $t \in \mathbf{R}$ acts by $\exp(tD)$, where $D \in \operatorname{Der}(\mathfrak{m})$ is diagonalizable and has strictly positive eigenvalues.

If H is a Heintze group of diagonal type, then M as above is its nilradical.

Theorem 3.12 (Heintze, [Hei74])**.** *A group in $(\mathcal{C}_1)$ is a Heintze group of diagonal type if and only if it carries a left-invariant metric of strictly negative curvature.*

Given a metric space X (or a group) and an admissible function u, we denote by $\operatorname{SBE}(X)^{O(u)}$ the group of self $O(u)$-bilipschitz equivalences of X.

Definition 3.13. Let H be a Heintze group of diagonal type. We say that H satisfies the strong pointed sphere property if for every admissible u the group $\operatorname{SBE}(H)^{O(u)}$ is not transitive on $\partial_\infty H$.

Some Heintze groups of diagonal type are parabolic subgroups of isometries of rank one symmetric spaces; they do not have the strong pointed sphere property, since the group of isometries of a rank one symmetric space acts transitively on its Gromov boundary. Among the Heintze groups with an abelian nilradical, the ones of the latter kind turn out to be the only exceptions:

Lemma 3.14 ([Pal22, Lemma 4.1])**.** *Let H be a completely solvable Heintze group with an abelian nilradical. If H is not isomorphic to a maximal completely solvable subgroup of $\mathrm{SO}(n,1)$ for any $n \geqslant 2$, then H has the strong pointed sphere property.*

We will also need the following Lemma:

Lemma 3.15 (Consequence of [Gra23, Theorem 1.4])**.** *Let G_{II} and G'_{II} be two semisimple Lie groups with trivial centers, no factors of rank one and no compact factors. If G_{II} and G'_{II} are $O(u)$-bilipschitz equivalent, then they are isomorphic.*

Proof. Let X be the Riemannian symmetric space associated to G_{II}. It follows from [Gra23] (with $X = X_0$) that the group $\operatorname{SBE}(G_{\mathrm{II}})^{O(u)} = \operatorname{SBE}(X)^{O(u)}$ is isomorphic to G_{II}. Similarly, $\operatorname{SBE}(G'_{\mathrm{II}})^{O(u)}$ is isomorphic to G'_{II}. Assuming that G_{II} and G'_{II} are $O(u)$-bilipschitz equivalent, the groups $\operatorname{SBE}(G_{\mathrm{II}})^{O(u)}$ and $\operatorname{SBE}(G'_{\mathrm{II}})^{O(u)}$ are isomorphic, concluding the proof. □

Remark 3.16. The way Lemma 3.15 is deduced from [Gra23] is the same the way the QI classification was deduced in [KL97]. One may nevertheless long for a direct way of proving this.

Theorem 3.17. *Let m, n, m' and n' be nonnegative integers. Let G, and G' be two real semisimple Lie groups with trivial center and no compact factors. Let $\{H_i\}_{1 \leqslant i \leqslant m}$ and $\{H'_j\}_{1 \leqslant j \leqslant m'}$ be families of diagonal Heintze groups satisfying the strong pointed sphere* *property; let M_i, resp. M'_j be the nilradical of H_i, resp. of H'_j. Write $G = KAN$, and $G' = K'A'N'$. Form the following solvable groups P, P', $R \lhd P$ and $R' \lhd P'$*

$$P = \mathbf{R}^n \times AN \times \prod_{i=1}^{m} H_i \quad R = N \times \textstyle\prod_{i=1}^{m} M_i;$$

$$P' = \mathbf{R}^{n'} \times A'N' \times \prod_{j=1}^{m'} H'_j \quad R' = N' \times \textstyle\prod_{j=1}^{m'} M'_j.$$

Let S and S' be completely solvable groups. Assume that S and S' are quasiisometric, that $\rho_1(S) = P$ and that $\rho_1(S') = P'$. Then $m = m'$, $n = n'$, G and G' are isomorphic, and there is a bijection σ of $\{1, \ldots, m\}$ such that H_i is $O(\log)$-equivalent to $H_{\sigma(i)}$ for all i.

Remark 3.18. The strong pointed sphere property is known to hold for some Heintze groups with nonabelian nilradicals [Pal22, Remark 9]. So the assumptions of Theorem 3.17 are indeed weaker than those of Theorem C.

Proof of Corollary B assuming Theorem 3.17. Let L and L' be semisimple Lie group with no compact factors such that $X = L/K$ and $Y = L'/K'$. Set $S = AN$, $S' = A'N'$, $n = n' = m = m' = 0$ and apply the theorem to S and S'. It follows that $L = G$ and $L' = G'$ are isomorphic, so that X and Y are pluriisometric. □

Proof of Theorem C assuming Theorem 3.17. Let S, S', P and P' be as in the assumptions of the Theorem C. Then S, S', P and P' also satisfy the assumptions of Theorem 3.17, since the groups H_i and H'_j satisfy the strong pointed sphere property as recalled above. By Theorem 3.17, the groups $\mathbf{R}^n \times AN$ and $\mathbf{R}^{n'} \times A'N'$ are isomorphic, while, after possibly reindexing the groups H'_j, H_i is $O(\log)$-equivalent to H'_i for all i. Now by the main theorem of [Pal20b], H_i and H'_i are isomorphic for all i, and so P and P' are isomorphic. □

Proof of Theorem 3.17. Let n, m, n', m', S, S', P, P', G and G' be as in the assumptions of Theorem 3.17. Decomposing G and G' into products of simple factors, we find that there exists p, p', q and q' so that

$$G = \prod_{i=1}^{p+q} G_i \quad \text{and} \quad G' = \prod_{j=1}^{p'+q'} G_j,$$

where G_i has $\mathbf{R}$-rank one for $1 \leqslant i \leqslant p$ and $\mathbf{R}$-rank at least two for $p+1 \leqslant i \leqslant p+q$, similarly for G'_j. Let $\ell = p + m$ and $\ell' = p' + m'$. For $i \in \{m+1, \ell\}$ define $M_i := N_{i-m}$ and $H_i := A_{i-m}N_{i-m}$, where $G_k = K_kA_kN_k$ for $1 \leqslant k \leqslant p$; similarly, define M'_j and H'_j for j in $\{m'+1, \ell'\}$. In this way, $\{H_i\}_{i=1}^{m}$, resp. $\{H_j\}_{j=1}^{m'}$ are the Heintze groups that were originally present in the statement, and $\{H_i\}_{i=m+1}^{m+p}$, resp. $\{H_j\}_{j=m'+1}^{m'+p'}$ are the Borel subgroups of the rank 1 factors of G, resp. G' (which are also Heintze groups).

It follows from Theorem 3.7(1) that the groups S and P on the one hand, S' and P' on the other hand, are $O(\log)$-bilipschitz equivalent. By assumption, S and S' are quasiisometric, especially they are $O(\log)$-bilipschitz equivalent. So P and P' are $O(\log)$-bilipschitz equivalent.

Applying Theorem A to P and P' we find that $n = n'$, $q = q'$, $\ell = \ell'$, and there are two bijections σ_1 of $\{1, \ldots, \ell\}$ and σ_2 of $\{p+1, \ldots p+q\}$ so that

(1) G_i is $O(\log)$-bilipschitz equivalent to $G_{\sigma_2(i)}$ for all $p < i \leqslant p+q$;
(2) H_i is $O(\log)$-bilipschitz equivalent to $H_{\sigma_1(i)}$ for all $1 \leqslant i \leqslant \ell$;

From (1), by Lemma 3.15 we deduce that the groups

$$G_{\mathrm{II}} = \prod_{i=p+1}^{p+q} G_i \quad \text{and} \quad G'_{\mathrm{II}} = \prod_{j=p'+1}^{p'+q'} G'_j,$$

are isomorphic. We now claim that $\sigma_1(\{1,\dots,p\}) = \{1,\dots,p'\}$, especially $p = p'$. If it was not the case, then some H_i, say H_{i_0}, would be $O(u)$-equivalent to a rank one symmetric space. But then the group of self-SBEs of H_i would be transitive on its sphere at infinity. This cannot be, as the pointed sphere conjecture holds for H_i by assumption. It now follows from (2) and [Pal20a] that the groups

$$G_{\mathrm{I}} = \prod_{i=1}^{p} G_i \quad \text{and} \quad G'_{\mathrm{I}} = \prod_{j=1}^{p} G'_j,$$

are isomorphic. Finally, by the claim above $\sigma_1(\{p+1,\dots,\ell\}) = \{p+1,\dots,\ell\}$ so that H_i is $O(\log)$-bilipschitz equivalent to $H'_{\sigma(i)}$ for all $i \in \{p+1,\dots,\ell\}$. ☐

4. Distortion in completely solvable groups and Dehn function estimates

The theory of Dehn function and filling pairs is well known for finitely generated groups, and generalizes naturally for compactly presented groups. We refer to [CT17] for the basic definitions and a detailed exposition of this subject in the context of compactly presented groups.

Given two functions f, g from $\mathbf{R}_{\geqslant 0}$ or $\mathbf{Z}_{\geqslant 0}$ to itself, we will write $f(r) \preccurlyeq g(r)$ if there is some constant $C > 0$ such that $f(r) \preccurlyeq Cg(Cr+C)+Cr+C$. We also write $f(r) \asymp g(r)$ if $f(r) \preccurlyeq g(r)$ and $g(r) \preccurlyeq f(r)$.

4.A. **Proof of Proposition D.** We now proceed with the proof of Proposition D. We first recall the setting. Let G be a completely solvable group, let $H = G/\operatorname{R_{exp}} G$ and let $\pi : G \to H$ be the projection. Let $X \in \mathfrak{g} \setminus \{0\}$, let L be the one-parameter subgroup generated by X, and let c_X be as in the statement of Theorem D, namely,

$$c_X = \sup\{j \in \mathbf{Z}_{\geqslant 1} \cup \{\infty\} \colon X \in C^j\mathfrak{g}\};$$

we recall that we use the convention $C^1\mathfrak{g} = \mathfrak{g}$ so that, for instance, $C^2\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$.

Our goal is to evaluate the distortion of the subgroup generated by X.

In the case where $c_X = \infty$, X is in $\operatorname{R_{exp}} G$, and the conclusion follows directly from [Osi02].

In the case $c_X \neq \infty$, $\pi(L)$ is polynomially distorted with degree c_X in H by [Osi01]. This means that there exists a constant $M \geqslant 1$ so that for t large enough,

$$\frac{1}{M} t^{1/c_X} \leqslant d_H(\exp(t\pi(X)), 1) \leqslant M t^{1/c_X}$$

(here we still denote π the map $\mathrm{Lie}(\pi)$ for convenience). Now, by Cornulier's theorem 3.7, the map $\pi \colon G \to H$ is a $O(\log)$-retract. This implies, on the one hand, that π is $O(\log)$-Lipschitz. So, for some $\lambda > 0$ and $c \geqslant 0$, $d_H(\exp(t\pi(X), 1) \leqslant \lambda d_G(\exp(tX), 1) + c \log t$. So $d_G(\exp(tX), 1) \geqslant \frac{1}{\lambda M} t^{1/c_X} - \frac{c}{\lambda} \log t$, and then for t large enough,

$$d_G(\exp(tX), 1) \geqslant \frac{t^{1/c_X}}{2\lambda M} \tag{4.1}$$

Then, for $r > 0$ large enough,

$$\sup\{t : \exp tX \in B_G(r)\} \leqslant (2\lambda M)^{c_X} r^{c_X}.$$

Thus $\Delta_L^G(r) \preccurlyeq r^{c_X}$. On the other hand, using part (2b) in Theorem 3.7, there exists a $O(\log)$-lipschitz map $f\colon H \to G$ such that $f(\exp(t\pi(X))) = \exp(tX)$. Taking larger constants λ and c if needed, we have that

$$\begin{aligned} d_G(\exp(tX), 1) &\leqslant \lambda d_H(\exp(t\pi(X)), 1) + c \log d_H(\exp(t\pi(X)), 1) \\ &\leqslant \lambda M t^{1/c_X} + \frac{c}{c_X} \log t, \end{aligned}$$

so that $\lambda M \Delta_L^G(r)^{1/c_X} + \frac{c}{c_X} \log \Delta_L^G(r) \geqslant r$, and then, $\Delta_L^G(r) \succcurlyeq r^{c_X}$.

Remark 4.1. For our use of Proposition D, namely in Proposition 4.2 below, the element X will always lie in the centre of $\mathfrak{g}$, therefore will always lie in a Cartan subalgebra. In particular the assumption that X lies in a Cartan subalgebra of $\mathfrak{g}$ does not impose any restrictions to us later on. We do not know whether this assumption is necessary in the statement of Proposition D.

4.B. **From the distortion in an extension to the Dehn function.**

Proposition 4.2. *Let G be a simply connected solvable Lie group. Let $\omega \in Z^2(G, \mathbf{R})$; assume that in the central extension*

$$1 \to \mathbf{R} \xrightarrow{\iota} \widetilde{G} \xrightarrow{\pi} G \to 1$$

associated to ω, the subgroup $L = \iota(\mathbf{R})$ is distorted, and $\Delta_L^{\widetilde{G}}(n) \succcurlyeq n^k$. Then the Dehn function of G has growth type at least $n \mapsto n^k$.

The main ingredient is the following lemma.

Lemma 4.3. *Let G be a simply connected solvable Lie group. Equip G with a left-invariant Riemannian metric. Let $\alpha \in \Omega^1(G, \mathbf{R})$ be a smooth one-form; assume that*

- *$d\alpha$ is left-invariant, and*
- *There exists a family $(\gamma_n)_{n \geqslant 1}$ of piecewise smooth loops, with $\operatorname{length}(\gamma_n) \leqslant cn$ and $\int_{\gamma_n} \alpha \geqslant c' n^k$ for some positive constants c, c'.*

Then the Dehn function of G has growth type at least $n \mapsto n^k$.

Proof. We will prove that the filling area $\operatorname{Fill}(\gamma)$ of γ_n is larger or equal than a constant times n^k. For every $n \geqslant 1$ let Δ_n be a Lipschitz disk in G such that $\partial \Delta_n = \gamma_n$, by which we mean that $\Delta\colon B^2 \to G$ is a Lipschitz embedding of the Euclidean 2-disk B^2 into G such that $\Delta_{n|S^1}$ is a reparametrization of γ_n. Δ defines an integral Lipschitz chain in G, as defined in [Fed74, 2.11]. On the other hand, since it is smooth and has bounded exterior derivative, α represents a flat cochain as defined in [Fed74, 4.6]. By Federer's version of Stokes' theorem [Fed74, 6.2],

$$\int_{\Delta_n} d\alpha = \int_{\gamma_n} \alpha \geqslant c' n^k. \tag{4.2}$$

Now $d\alpha$ is left-invariant, so there is a constant $L > 0$ (namely, the point-wise comass norm of $d\alpha$ with respect to the Riemannian metric) such that

$$\left| \int_{\Delta_n} d\alpha \right| \leqslant L \operatorname{Area}(\Delta_n) =: L \int_{B^2} |\Lambda^2 d\Delta(x)| dx \tag{4.3}$$

for all n. Combining (4.2) and (4.3) yields

$$\operatorname{Area}(\Delta_n) \geqslant \frac{c'}{L} n^k \tag{4.4}$$

for all n. Since $\mathrm{length}(\gamma_n) \leqslant cn$, this finishes the proof that the filling area of G, defined by

$$\mathrm{Fill}_G(r) = \sup_{\gamma\colon S^1 \to G,\ \mathrm{length}(\gamma) \leqslant r} \inf\{\mathrm{Area}(\Delta) \colon \partial\Delta = \gamma\}$$

is at least of growth type $n \mapsto n^k$. Now $\delta_G(n) \succeq \mathrm{Fill}(n)$ by [CT17, Proposition 2.C.1].[2] ☐

Proof of Proposition 4.2 using Lemma 4.3. Let G and $\omega \in Z^2(G, \mathbf{R})$ be as in the statement of Proposition 4.2, and let α be a one-form on G such that $d\alpha = \omega$. For all n, let $\widetilde{\gamma}_n$ be a piecewise C^1 loop in $\widetilde{G}$ from 1 to $\iota(n^k)$. By the assumption on the distortion, we can assume that there is a constant $c > 0$ such that $\mathrm{length}(\widetilde{\gamma}_n) \leqslant cn$. We now let γ_n be the projection of $\widetilde{\gamma}_n$ in G. This is a piecewise C^1-loop, with length $\leqslant cn$. We now claim that $\int_{\gamma_n} \alpha = n^k$. This follows from [GMIP23, Lemma 3.1]; there, the Lemma is stated for simply connected nilpotent Lie groups, but the nilpotency assumption is actually not used; the lemma holds for simply connected Lie groups. ☐

Remark 4.4. Exponentially distorted central extensions also yield lower bounds on the Dehn function. For a group G of class $(\mathcal{C}_1)$, the existence of an exponentially distorted central extension is equivalent to the 2-homological obstruction of Cornulier and Tessera that we will discuss in the next section; see [CT17, 11.E] on this equivalence.

Remark 4.5. Proposition 4.2 is very close to [GMIP23, Proposition 3.7]. It is slightly stronger, even when restricted to nilpotent groups, since in [GMIP23, Proposition 3.7] there is the additional assumption that the nilpotent group G should be of nilpotency class $k-1$. The difference comes from the different versions of Stokes theorem used. When G is nilpotent and under the additional assumption that it has a lattice Γ, the result of Proposition 4.2 can be obtained by combinatorial arguments considering central extensions of Γ instead of G; this method does not require any assumption on the nilpotency class of G and Γ. It is described already in [BW97].

4.C. **Example.** There are two completely solvable groups of dimension 4 and cone dimension 3. These are $G_{4,3}$ and $A_2 \times \mathbf{R}^2$. We will prove that the Dehn function of $G_{4,3}$ is cubic, while the Dehn function of $A_2 \times \mathbf{R}^2$ is quadratic.

The Lie algebra of $G_{4,3}$ has a basis (e_1, e_2, e_3, e_4) in which the nonzero Lie brackets are $[e_4, e_1] = e_1$ and $[e_4, e_3] = e_2$. (Our e_4 is the opposite of the one in [PSWZ76], for convenience; the others are the same.) The derived subalgebra is the abelian ideal generated by e_1 and e_2. The next term in the central series (and exponential radical) is $\mathbf{R}e_1$, and the center is $\mathbf{R}e_2$. Consider the dual basis $(\omega_1, \ldots, \omega_4)$ to $(e_1, \ldots e_4)$. The 2-form $\omega_4 \wedge \omega_2$ is closed, since

$$d(\omega_4 \wedge \omega_2) = d\omega_4 \wedge \omega_2 - \omega_4 \wedge d\omega_2 = -\omega_4 \wedge \omega_3 \wedge \omega_4 = 0.$$

It is not exact, since $Z^2(\mathfrak{g}_{4,3}, \mathbf{R})$ is spanned by $\omega_1 \wedge \omega_4$ and $\omega_3 \wedge \omega_4$. Hence, there is a nontrivial central extension

$$1 \to \mathbf{R} \to G \to G_{4,3} \to 1.$$

where G is a 5-dimensional, completely solvable group (this is $G_{5,10}$ on Table 2). Moreover, the kernel of this extension is cubically distorted, as we explain now. Let us write $\widetilde{e}_i$ such that $\widetilde{e}_5$ generates the kernel of the central extension, and $\widetilde{e}_i$ projects to e_i in $\mathfrak{g}_{4,3}$ for $1 \leqslant i \leqslant 4$. $\widetilde{e}_5$ lies in the third term of the descending central series of $\mathfrak{g}$, so that we can apply Proposition D.

[2] Actually [CT17] gives the stronger result that $\delta_G(n) \asymp \mathrm{Fill}_G(n)$, however, the converse inequality is much more involved.

4.D. **Comparison with other known bounds on the Dehn functions.** The following theorem is essentially proven by Cornulier and Tessera. Since they do not state it in this way, we will provide some explanations on how to deduce it from [CT17] using their tools. The main ingredient is [CT17, Theorem 10.H.1]. The theorem essentially states that for a Lie group G, either the Dehn function is exponential, or it is well estimated (with error terms) by the Dehn function of the largest nilpotent quotient of a completely solvable group quasiisometric to G.

In the statement below, we say that a function $f\colon \mathbf{R}_{\geqslant 0} \to \mathbf{R}_{\geqslant 0}$ is *regular* if there exists $\alpha > 1$ such that $f(r)/r^\alpha$ is non-decreasing.

Theorem 4.6 (After Cornulier and Tessera)**.** *Let G be a completely solvable group, U its exponential radical, and $N = G/U$. Then*

(1) Either, G has an exponential Dehn function
(2) Or, there exists e (depending on G) such that

$$\delta_N(r)/\log^e(r) \preceq \delta_G(r) \preceq r \cdot \widehat{\delta_N}(r) \cdot \log^e(r), \tag{4.5}$$

where in the right inequality, $\widehat{\delta_N}$ is any regular function larger than δ_N.

Moreover, if c is the nilpotency class of N, then one can take $e = 2(c+1)$ in (4.5).

Remark 4.7. As much as Cornulier and Tessera in [CT17], we do not know a single example of a nilpotent group with a Dehn function that would not be equivalent (in the $\asymp$ sense) to a larger regular function, i.e. in all examples we know one can take $\widehat{\delta_N} = \delta_N$ in (4.5).

We present the ingredients and then assemble the proof.

4.D.1. *Dehn functions and $O(\log)$-bilipschitz equivalences.* The Dehn functions of two groups that are $O(log)$-bilipschitz equivalent are equal up to a factor of log. We will use it for the pair G and $\rho_1(G)$. The following statements are essentially [CT17, Corollary 3.C.2], formulated in a slightly more general way. We omit the proof, which is identical.

Lemma 4.8. *Let G and H be two locally compact compactly presented groups, with filling pairs (f_G, g_G) and (f_H, g_H) respectively. Assume there is an (L,u)-bilipschitz equivalence $\phi : H \to G$. Then we have the following:*

(1) $f_H(n) \preccurlyeq f_G\big(nu(n)\big) \cdot f_H \circ u \circ g_G\big(nu(n)\big)$.
(2) $g_H(n) \preccurlyeq g_H \circ u \circ g_G\big(nu(n)\big) + g_G\big(nu(n)\big)$.

Corollary 4.9. *If $(f_G, g_G) = (n^d, n^e)$, then $\frac{f_H(n)}{f_H\left(u(n^{2e})\right)} \leq n^{2d}$. If moreover $u = \log$, then f_H is polynomial. In particular, for two connected Lie groups G and H that are $(L, \log)$-bilipschitz, if $(f_G, g_G) = (n^d, n^d)$ then $f_H \preccurlyeq n^d \log^{2d}(n)$.*

4.D.2. *Lower bounds and Lipschitz Retracts.* Among finitely presented groups, going to a group-theoretic retract makes the Dehn function go down, as can be seen by choosing an adequate pair of presentations for which the retract corresponds to an enlargement of the set of relators; see e.g. [BMS93, Lemma 1]. This is still valid for Lie groups; this fact is used several times in [CT17] but we provide a proof below for completeness:

Proposition 4.10. *Let G be a simply connected Lie group, and let H be a retract of G in the Lie group category. Then $\delta_H \preccurlyeq \delta_G$.*

Proof. Consider the epimorphism $\pi : G \to H$ and let $\sigma\colon H \to G$ be a section of π. Let d_G be a left-invariant Riemannian distance on G, and let d_H be a left-invariant

Riemannian distance on H such that π is a Riemannian submersion. Then for every $h, h' \in H$,

$$d_H(h,h') = \mathrm{dist}_G(\sigma(h)N, \sigma(h')N) \tag{4.6}$$

where $N = \ker \pi$ (see e.g. [HP13b, Lemma 4.6]). Let $\gamma : S^1 \to H$ be a Lipschitz loop of d_H-length exactly n, and consider the loop $\widehat{\gamma} = \sigma \circ \gamma$. The d_G-length of $\widehat{\gamma}$ is less or equal to n thanks to (4.6); it is also greater or equal than n, since π is 1-Lipschitz and sends $\widehat{\gamma}$ onto γ. Using the equivalence of the Dehn function and the filling function in G [CT17, Proposition 2.C], there is a filling of $\widehat{\gamma}$ in G by a Lipschitz disk Δ of area at most the order of $\delta_G(n)$. Since $\pi\colon (G, d_G) \to (H, d_H)$ is 1-Lipschitz, $\pi \circ \Delta$ has area less than Δ. Using again the equivalence of the Dehn function and the filling function, in H and in the reverse direction, we conclude that $\delta_H(n) \preccurlyeq \delta_G(n)$. □

4.D.3. *Generalized Standard Solvable Groups.* A special case of interest where the group G retracts to a subgroup is when the short exact sequence determined by the exponential radical of G splits. This case is captured by Cornulier and Tessera's definition of *generalized standard solvable* groups:

Definition 4.11 ([CT17], Section 10.H.1)**.** Let G be a completely solvable group. We call G *generalized standard solvable* if $G = V \rtimes N$ where N is nilpotent and such that the following condition on the action of N on V is met: there is no nontrivial quotient of $V/[V,V]$ on which N acts as the identity.

If $G \in \mathcal{C}_0$ admits a splitting $G = V \rtimes N$ as a generalized standard solvable group, then if N is abelian then $V = \mathrm{R}_{\exp}(G)$ (Proposition 3.9) and G is standard solvable. In general, any such splitting with N nilpotent forces V to contain the exponential radical. On the other hand, if $G \in \mathcal{C}_0$ is moreover in $(\mathcal{C}_1)$, it is automatically generalized standard solvable with $V = \mathrm{R}_{\exp}(G)$:

Lemma 4.12. *If G is of class $(\mathcal{C}_1)$, then it is generalized standard solvable via the splitting $G = \mathrm{R}_{\exp} G \rtimes N$.*

Proof. Let U be the exponential radical of G and assume towards contradiction that $H_1(\mathfrak{u})$ has zero as a nontrivial weight. Let X be a nonzero vector in the corresponding kernel, and let $\widehat{X} \in \mathfrak{u}$ be such that $X = \widehat{X} + [\mathfrak{u}, \mathfrak{u}]$. Then $[\mathfrak{n}, \widehat{X}] \subseteq [\mathfrak{u}, \mathfrak{u}]$. But since $\mathfrak{u}$ is the exponential radical of $\mathfrak{g}$, one has $[\mathfrak{g}, \mathfrak{u}] = \mathfrak{u}$, especially $[\mathfrak{g}, \mathfrak{u}]$ should contain $\widehat{X}$. However the map $\mathfrak{g} \times \mathfrak{u} \to \mathfrak{u}$ which to (Y, U) associates $[Y, U]$ is not surjective, since its image cannot contain $\widehat{X}$ (remember that since G is in $(\mathcal{C}_1)$, the action of $\mathfrak{n}$ on $\mathfrak{u}/[\mathfrak{u}, \mathfrak{u}]$ is diagonalizable. So a vector in the kernel cannot be in the image). This is a contradiction. □

The following is the main result of Cornulier and Tessera on generalized standard solvable groups. See Section 5.C for the definition of $\mathrm{Kill}(\mathfrak{v})$.

Theorem 4.13 ([CT17], Theorem 10.H.1)**.** *Let $G = V \rtimes N$ be a generalized standard solvable group whose Dehn function is non-exponential (i.e. strictly smaller than exponential). Then $\delta_G(n) \preccurlyeq n \cdot \widehat{\delta_N}(n)$, where $\widehat{\delta_N}$ denotes any regular function larger than δ_N. If, moreover, $\mathrm{Kill}(\mathfrak{v})_0 = 0$ then $\delta_G(n) \preccurlyeq \widehat{\delta_N}(n)$.*

We are now ready to complete the proof of Theorem 4.6 assuming the results we took from [CT17].

Proof of Theorem 4.6. Let G be a Lie group. It is quasiisometric to a completely solvable group G_0 and $\delta_G \asymp \delta_{G_0}$. In turn, G_0 is $O(\log)$ bilipschitz equivalent to $G_1 := \rho_1(G_0)$

(Theorem 3.7), hence Corollary 4.9 gives $\delta_{G_1}(n)/\log^e(n) \preccurlyeq \delta_{G_0}(n) \preccurlyeq \delta_{G_1}(n)\cdot\log^e(n)$. By definition, $G = \mathrm{R}_{\exp} G \rtimes N$ hence $\delta_N(n) \preccurlyeq \delta_{G_1}(n)$ by Proposition 4.10. By Lemma 4.12 G_1 is generalized standard solvable, and Theorem 4.13 gives $\delta_{G_1}(n) \preccurlyeq n \cdot \widehat{\delta_N}(n)$. Combining all inequalities completes the proof. □

A careful read of the proof of Theorem 4.6 sheds light on the theoretical contribution of Proposition D and Proposition 4.2 over their well known nilpotent groups analogues. If G is in $(\mathcal{C}_1)$ and $N = G/\mathrm{R}_{\exp} G$, then any lower bound on δ_N (in particular those coming from distorted central extensions) is automatically a lower bound on δ_G. However in general in order to retract to a nilpotent group one might have to pass to $\rho_1(G)$, which comes at a cost of a power of log factor on the lower bound. Our version allows using distorted central extensions without passing to the nilpotent quotient, therefore removing this factor. Moreover, it is also theoretically possible that G would admit distorted central extensions of degree higher than those of N (though only by at most 1, due to the upper bound of Theorem 4.6; in this context one should also keep in mind [CT17, Theorem 11.C.1]).

In practice, after establishing the list of Dehn functions for completely solvable groups of dimensions 4 and 5 we found that in the few cases where the distortion in central extensions tool was helpful, the groups were in fact in $(\mathcal{C}_1)$. In Appendix B we present an example where our theoretical contribution is practical, i.e. where Proposition 4.2 improves on the known lower bound obtained by Theorem 4.6.

5. The solvable Lie groups of low dimensions and their QI-invariants

The goal of this section is to apply our results for the quasiisometric classification of completely solvable groups of dimensions 4 and 5. The solvable Lie groups (or more precisely the real solvable Lie algebras) of dimensions 4 and 5 were completely classified by Mubarakzyanov [Mub63]; the list is also available in the more accessible [PSWZ76]. In view of the diversity observed, when it comes to the quasiisometric classification and rigidity we cannot expect more than a vast array of specific techniques to hope to cover all cases in the present day.

We will say that a given completely solvable Lie group G is QI-rigid within $(\mathcal{C}_0)$ if any group in $(\mathcal{C}_0)$ quasiisometric to it is isomorphic to G, and that a class $\mathcal{G}$ of groups is QI-complete within $(\mathcal{C}_0)$ if any group in $(\mathcal{C}_0)$ quasiisometric to a group in $\mathcal{G}$ is isomorphic to a group in $\mathcal{G}$. By dimension we mean the dimension of the Lie algebra over $\mathbf{R}$; for the simply connected solvable Lie groups this is also the asymptotic dimension, or the asymptotic Assouad-Nagata dimension ([HP13a]) so that, for instance, the family of completely solvable groups of dimension 5 is QI-complete within $(\mathcal{C}_0)$.

By cone dimension, we mean the covering dimension of any asymptotic cone; it is given by Cornulier's formula [Cor08], and for simply connected solvable Lie groups, it is exactly the codimension of the exponential radical. The cone dimension is obviously a quasiisometry invariant, so that the first refinement of the simply connected solvable groups in QI-complete families is the ordered pair of positive integers

$$(\mathrm{conedim}, \dim).$$

The cone dimensions of Lie groups up to dimension 4 was computed and tabulated by Kivioja, Le Donne and Nicolussi Golo [KLDNG22, Table 1]. The same authors also listed the simply connected solvable Lie groups G of polynomial growth of dimension 5, and their associated $\rho_0(G)$ (that they call the real shadow of G).

We list the real solvable Lie groups of exponential growth up to dimension 5, compute their image by ρ_1 (Tables 1 and 2) and their Dehn functions (Tables 4 and 5). The tools for our computations are presented in Section 5.B, and elaborated examples with detailed explanations on the computations are given in Section 5.C. In Section 5.D we discuss two families of groups which exhibit interesting behavior in the context of the quasiisometric classification. Then in Section 5.E we summarize the contribution of our product theorem in the light of the Dehn functions tables; this is Corollary 5.22.

5.A. **Structure, dimension, and cone dimension.** We list in Table 1 the groups G in $(\mathcal{C}_0)$ and of dimension 2 to 4 and of exponential growth that do not split in direct product and their associated $\rho_1(G)$ in $(\mathcal{C}_1)$ (the cone dimensions can be found in [KLDNG22]). In Tables 2–3 we continue the list in dimension 5 to all indecomposable simply connected solvable groups of exponential growth G, and computing their associated $\rho_0(G)$; we also compute $\rho_1(G)$ and list the cone dimension. There are 39 families of indecomposable real five-dimensional solvable Lie algebras, including 18 with parameters. In [Mub63] they are named $g_{5,i}$ for $1 \leqslant i \leqslant 39$. For $1 \leqslant i \leqslant 7$, $g_{5,i}$ is nilpotent and for $i \in \{14, 17, 18, 26\}$ and certain particular values of the parameters, the corresponding group has polynomial growth; we deliberately exclude them from our tables, since the computation of ρ_0 was done for them in [KLDNG22, Table 3].

In order to ease the determination of whether a given irreducible simply connected solvable group belongs to $(\mathcal{C}_0)$ or $(\mathcal{C}_1)$, we always list the group in the rightmost possible column. For instance for some group G in $(\mathcal{C}_1)$, the column below G and (in Tables 2–3) the column $\rho_0(G)$ will be left blank, only the column $\rho_1(G)$ will be filled with G.

The structure of the Lie algebra is given in [Mub63] and [PSWZ76] as a list of nonzero brackets; however this is not quite convenient when it comes to computing $\rho_1(G)$, and checking our computations. It turns out that in all cases but two, namely $G_{5,38}$ and $G_{5,39}$, the nilradical is split, and the Lie algebra decomposes as $\mathfrak{n} \rtimes \mathbf{R}$ or $\mathfrak{n} \rtimes \mathbf{R}^2$, where the Lie algebra $\mathfrak{n}$ of the nilradical can be either $\mathbf{R}^d$ for $d \in \{1, \dots, 4\}$, the Lie algebra $\mathfrak{heis}$ of the Heisenberg group, the Lie algebra $\mathfrak{fil}$ of the 4-dimensional filiform group, or a product of $\mathfrak{heis}$ with an abelian ideal of dimension 1. We fix bases $(e_1, \dots, e_d)$ for all the Lie algebras among the former, in the following way: (e_1, e_2, e_3) is a basis of $\mathfrak{heis}$ in which $[e_1, e_2] = e_3$ and e_3 is central, (e_1, e_2, e_3, e_4) is the basis of $\mathfrak{fil}$ in which $[e_1, e_2] = e_3$, $[e_1, e_3] = e_4$ and e_4 is central; when we write the product $\mathbf{R} \times \mathfrak{heis}$ the nonzero bracket is $(e_2, e_3) = e_4$ while when $\mathfrak{heis} \times \mathbf{R}$ the nonzero bracket is $(e_1, e_2) = e_3$. To denote the torus of derivations, we write $\Delta(\mathbf{b}_1, \dots, \mathbf{b}_r)$ for a derivation of $\mathfrak{n}$ which has diagonal blocks $\mathbf{b}_1, \dots, \mathbf{b}_r$. By block we mean one of the following:

- a scalar block corresponding to an eigenspace of eigenvalue $\lambda \in \mathbf{R}$ which we write $\mathbf{b}_i = (\lambda)$.
- a complex scalar block corresponding to
$$\begin{pmatrix} \sigma & \tau \\ -\tau & \sigma \end{pmatrix}$$
which we write $(\sigma \pm i\tau)$.
- a non-scalar irreducible Jordan block of a generalized eigenspace of eigenvalue λ, which we write (λ^s) where s is the dimension, e.g. (2^2) corresponds to the block
$$\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

Finally,
$$\mathfrak{g} = \mathfrak{n} \rtimes \{\Delta(\mathbf{b}_1, \dots, \mathbf{b}_{r_i})\}_i$$
denotes the semidirect product of $\mathfrak{n}$ by the derivations listed. For a few real Lie algebras of dimension 5, the description above is not possible and we provide it separately.

The parameters in our table are in the same range[3] and order as in [PSWZ76], who took the list in [Mub63], but denoted $A_{5,i}^{a,b,c\cdots}$ the corresponding algebras with parameters; however we used greek letters for the parameters, and sometimes used different letters; this is in order to "type" the parameters, for instance τ always denotes an imaginary part and ϵ is a sign. We normalize as much as possible to reduce the number of parameters when applicable, but we did not reparametrize.

Remark 5.1. The real Lie algebra named M^8 (for more general ground fields) in [dG07] is missing[4] in [PSWZ76], and we did not find the Lie algebra named $\mathfrak{s}_{5,26}$ in [vW14, p.237], for the value of the parameter $a = 1$, in [Mub63] nor in [PSWZ76]. We named the corresponding groups M^8 and $S^1_{5,26}$ in our table; that is our only departures from the taxonomy of Mubarakzyanov and followers. In addition, there is an entry in [PSWZ76], named $A_{5,40}$ and marked there as solvable, however it turns out that it has a nontrivial Levi decomposition; this is $\mathfrak{sl}(2, \mathbf{R}) \ltimes \mathbf{R}^2$ with the tautological representation of $\mathfrak{sl}(2, \mathbf{R})$. There are two quasiisometry classes of Lie groups with this Lie algebra:

(1) The simply connected Lie group
$$\widetilde{\mathrm{SL}(2, \mathbf{R})} \ltimes \mathbf{R}^2$$
whose QI type is that of $\mathbf{R} \times G_{4,8}$;

(2) The connected Lie group $\mathrm{SL}(2, \mathbf{R}) \ltimes \mathbf{R}^2$, whose QI type is that of $G_{4,9}^{-1/2}$; we discuss this further in subsection 5.D.

Since they are not solvable, we do not list these groups. Otherwise, we found a few less serious inconsistencies: in [PSWZ76] one should not have a parameter c in the definition of $G_{5,8}$; in $G_{5,9}^{b,c}$ we found the condition $bc \neq 0$ to be missing in both [Mub63] and [PSWZ76]; in the definition of $G_{5,26}^{p,\epsilon}$, resp. of $G_{5,33}^{a,b}$ one should assume $p > 0$, resp. $a \leqslant b$ to avoid redundancy.

Remark 5.2. The exponential radical of $G^0_{5,20}$ is not split. This group $G^0_{5,20}$ is the group named G in [Cor08, Example 4.2]. There are no completely solvable group for which the exponential radical does not split in dimension four, so that Cornulier's example is minimal for the dimension. It follows from our study that $G^0_{5,20}$ is the only such group in dimension 5, so the six-dimensional $\mathbf{Q}$-algebraic group with no $\mathbf{Q}$-split torus that has a non-split exponential radical given in [Cor08, Example 4.1] in $(\mathcal{C}_0)$ is minimal for the dimension among groups with all these properties.

5.B. **Dehn Functions.** In tables 4–5 we compute, as accurately as we can, the Dehn functions of the groups listed in Tables 1-3. To this end we used the following list of criteria:

- Gromov-hyperbolicity [Hei74];
- Azencott-Wilson criterion [AW76];
- Standard solvability [CT17];
- SOL obstruction [CT17];
- 2-homological obstruction [CT17];
- Vanishing of the zero weight subspace in the Killing module [CT17];
- Bound for generalized tame groups [CT17];

[3] In a few cases, a relevant range for the parameters is not clearly indicated in [PSWZ76], but the invariants given there can be used to determine one. We provide explicit ranges here.

[4] It is also missing in [vW14], and in [KLDNG22, Table 1]; the latter relied on [PSWZ76]. However no other group G in the class $(\mathcal{C}_0)$ has $\rho_0(G) = M_8$, so that this does not create an issue in [KLDNG22, Table 2].

- The distortion of all central extensions as Lie group (Section 4.B).

To the best of our knowledge, this list exhausts the general, or algorithmic, criteria for estimating Dehn functions. For each group the table states how we obtain this Dehn function, so our computations could be easily verified. We now explain how we obtain our estimates based on the above criteria.

- The Azencott-Wilson criterion checks whether a Lie group acts simply transitively on a non-positively curved Riemannian manifold [AW76]. In particular such groups have Dehn function at most quadratic.
- The definitions of *standard solvable*, SOL *obstruction* and 2-*homological obstruction* are given by Cornulier and Tessera. They prove the following result for a completely solvable group G ([CT17], Theorem E):
  - G has exponential Dehn function if and only if it admits either the SOL obstruction or the 2-homological obstruction. Otherwise its Dehn function is polynomially bounded.
  - If G does not satisfy the SOL or 2-homological obstruction and is standard solvable, then it has at most cubic Dehn function.

  The group G is *generalized tame* if it can be written as $G = U \rtimes N$, N nilpotent compactly generated with some element $c \in N$ acting as a compaction on U (see [CT17, Section 6.E]). Theorem 6.E.2 of [CT17] states that in this case, the Dehn functions of G and N are almost equivalent: if f is a function for which $f(r)/r^\alpha$ is non-decreasing, and the Dehn function of N admits $\delta_N < f$, then f also upper bounds δ_G. Moreover, $\delta_N \preccurlyeq \delta_G$ (Proposition 4.10). In all cases of completely solvable groups of dimensions 4 and 5, f can be taken to equal δ_N so $\delta_G \asymp \delta_N$.
- For standard solvable groups there are two ways of concluding they have at most quadratic Dehn function. The first is using Theorem D in [CT17] stating that for some strong version of standard solvable groups, not having the SOL obstruction implies a quadratic upper bound on the Dehn function. The second is using Theorem 10.E.1 in [CT17], which involves computing the zero weight subspace in the *Killing module*. Finally, if a standard solvable group has a co-dimension 1 exponential radical and the group does not satisfy the SOL obstruction, it is hyperbolic ([CT17], Corollary E.3.a).
- Section 4.B allows to derive Dehn function estimates for a group G using its possible central extensions. The 2-cohomology of a group classifies its central extensions. We compute it, and for each possible extension check its distortion. In the tables we list the names of the cohomology classes that give the maximal degree distortions. The name of a cohomology class is given with respect to the ordered basis in which the group is presented in tables 1–2.

For readability we only write down the Dehn function and the Reason column, that depicts which criteria were used to concluded the Dehn function. The reason is enough in order to completely determine which criteria led us to this decision. For example in $G_{5,10}$ we write n^4 D.Ex $\{\omega_{15}, \omega_{23}\}$. This means that the distorted central extensions corresponding to the cohomology classes of ω_{15} and ω_{23} are n^4 distorted and that there are no higher degree distorted extension; here ω_{ij} is dual to $e_i \wedge e_j$ in the given basis of the Lie algebra. The fact that the group is not standard solvable and does not admit the Azencott-Wilson criterion follows from these facts.

In the Reason column we use the following abbreviations for our reasoning:

- SOL: G admits the SOL obstruction.

G	$\rho_1(G)$	structure of $\mathfrak{g} = \mathrm{Lie}(G)$	conedim
-	A_2	$\mathbf{R} \rtimes \Delta(1)$	1
$G_{3,2}$	$G_{3,3}$	$\mathbf{R}^2 \rtimes \Delta(1^2)$	1
-	$G_{3,3}$	$\mathbf{R}^2 \rtimes \Delta(1,1)$	1
-	$G_{3,4}$	$\mathbf{R}^2 \rtimes \Delta(1,-1)$	1
-	$G_{3,5}^{\alpha}$	$\mathbf{R}^2 \rtimes \Delta(1,\alpha)$, $-1 < \alpha < 1$, $\alpha \neq 0$.	1
$G_{4,2}^{\alpha}$	$G_{4,5}^{1,\alpha'}$ (†)	$\mathbf{R}^3 \rtimes \Delta(1^2,\alpha)$, $\alpha \neq 0$.	1
-	$G_{4,3}$	$\mathbf{R}^3 \rtimes \Delta(1,0^2)$	3
$G_{4,4}$	$G_{4,5}^{1,1}$	$\mathbf{R}^3 \rtimes \Delta(1^3)$	1
-	$G_{4,5}^{\alpha,\beta}$	$\mathbf{R}^3 \rtimes \Delta(1,\alpha,\beta)$, $-1 \leqslant \alpha \leqslant \beta \leqslant 1$, $\alpha\beta \neq 0$.	1
$G_{4,7}$	$G_{4,9}^{1}$	$\mathfrak{heis} \rtimes \Delta(1^2,2)$	1
-	$G_{4,8}$	$\mathfrak{heis} \rtimes \Delta(1,-1,0)$	1
$G_{4,9}^{0}$	$\mathbf{R} \times G_{3,3}$	$\mathfrak{heis} \rtimes \Delta(1,0,1)$.	2
-	$G_{4,9}^{\beta}$	$\mathfrak{heis} \rtimes \Delta(1,\beta,1+\beta)$, $-1 < \beta \leqslant 1$, $\beta \neq 0$	1
-	M_8	$\mathbf{R}^2 \rtimes \{\Delta(1,0),\Delta(0,1)\}$	2

(†) α' may differ from α.

TABLE 1. The completely solvable, indecomposable Lie groups of exponential growth and dimension d, $2 \leqslant d \leqslant 4$.

- Hyp: G is standard solvable with co-dimension 1 exponential radical and does not admit the SOL obstruction.
- not Hyp: a sufficient condition to check non-hyperbolicity is when the cone dimension is larger than 1.
- ρ_1: The Dehn functions of G and $\rho_1(G)$ are related by Cornulier's Theorem 3.7 and Proposition 4.9.
- $\rho_1 = \rho_0$: When $\rho_1 = \rho_0$ then G is quasiisometric to $\rho_1(G)$ and all Dehn functions equal.
- A-W: G admits Azencott-Wilson criterion, and so its Dehn function is either linear (if and only if G is hyperbolic), or quadratic.
- n^k D.Ex.: The group admits a n^k-distorted central extension, and does not admit extensions of higher degree distortions.
- C-T: A group that is standard solvable and does not admit SOL or 2-homological obstructions has at most cubic Dehn function.
- K: A standard solvable group without SOL or 2-homological obstruction can admit the Killing module criterion, by which its Dehn function is at most quadratic.
- GT: If $G = U \ltimes N$ is generalized tame and the Dehn function of N is at least r^{α} for some $\alpha > 1$, then the Dehn function of G is equivalent to that of N.

G	$\rho_0(G)$	$\rho_1(G)$	structure of $\mathfrak{g} = \mathrm{Lie}(G)$	$\mathrm{conedim}(G)$
-	-	$G_{5,7}^{\alpha,\beta,\gamma}$	$\mathbf{R}^4 \rtimes \mathrm{diag}(1,\alpha,\beta,\gamma)$, $-1 \leqslant \gamma \leqslant \beta \leqslant \alpha \leqslant 1, \alpha\beta\gamma \neq 0$.	1
-	-	$G_{5,8}^{\gamma}$	$\mathbf{R}^4 \rtimes \Delta(0^2,1,\gamma)$, $-1 \leqslant \gamma \leqslant 1, \gamma \neq 0$.	3
-	$G_{5,9}^{\beta,\gamma}$	$G_{5,7}^{1,\beta,\gamma}$	$\mathbf{R}^4 \rtimes \Delta(1^2,\beta,\gamma)$, $\beta\gamma \neq 0$, $\beta \leqslant \gamma$.	1
-	-	$G_{5,10}$	$\mathbf{R}^4 \rtimes \Delta(0^3,1)$	4
-	$G_{5,11}^{\gamma}$	$G_{5,7}^{1,1,\gamma}$	$\mathbf{R}^4 \rtimes \Delta(1^3,\gamma)$, $\gamma \neq 0$.	1
-	$G_{5,12}$	$G_{5,7}^{1,1,1}$	$\mathbf{R}^4 \rtimes \Delta(1^4)$	1
$G_{5,13}^{\alpha,0,1}$	-	$\mathbf{R}^2 \times G_{3,5}^{\alpha}$	$\mathbf{R}^4 \rtimes \Delta(1,\alpha,\pm i)$, $\alpha \neq 0$.	3
$G_{5,13}^{\alpha,\beta,\tau}$	-	$G_{5,7}^{\alpha,\beta,\beta}$	$\mathbf{R}^4 \rtimes \Delta(1,\alpha,\beta \pm i\tau)$, $-1 \leqslant \alpha \leqslant 1$, $\alpha\beta\tau \neq 0$.	1
$G_{5,14}^{\alpha}$	-	$G_{5,8}^{1}$	$\mathbf{R}^4 \rtimes \Delta(0^2,\alpha \pm i)$, $\alpha \neq 0$.	3
-	$G_{5,15}^{0}$	$G_{5,8}^{1}$	$\mathbf{R}^4 \rtimes \Delta(0^2,1^2)$	3
-	$G_{5,15}^{\beta}$	$G_{5,7}^{1,\beta,\beta}$	$\mathbf{R}^4 \rtimes \Delta(1^2,\beta^2)$, $\beta \neq 0$.	1
$G_{5,16}^{0,\tau}$	-	$\mathbf{R}^2 \times G_{3,3}$	$\mathbf{R}^4 \rtimes \Delta(\pm i\tau,1^2)$, $\tau \neq 0$.	3
$G_{5,16}^{\beta,1}$	-	$G_{5,7}^{1,\beta,\beta}$	$\mathbf{R}^4 \rtimes \Delta(1^2,\beta \pm i)$, $\beta \neq 0$.	1
$G_{5,17}^{\tau,0,1}$	-	$\mathbf{R}^2 \times G_{3,3}$	$\mathbf{R}^4 \rtimes \Delta(\pm i, 1 \pm i\tau)$, $\tau \neq 0$.	3
$G_{5,17}^{\tau,\alpha,\beta}$	-	$G_{5,7}^{1,\beta/\alpha,\beta/\alpha}$	$\mathbf{R}^4 \rtimes (\alpha \pm i, \beta \pm i\tau)$	1
$G_{5,18}^{\alpha}$	$G_{5,11}^{1}$	$G_{5,7}^{1,1,1}$	$\mathbf{R}^4 \rtimes \Delta((\alpha \pm i)^2)$, $\alpha \neq 0$	1
-	-	$G_{5,19}^{0,\beta}$	$(\mathfrak{heis} \times \mathbf{R}) \rtimes \Delta(1,-1,0,\beta)$, $\beta \neq 0$	2
-	$G_{5,19}^{1,\beta}$	$\mathbf{R} \times G_{4,5}^{1,\beta}$	$(\mathfrak{heis} \times \mathbf{R}) \rtimes \Delta(1,0,1,\beta)$, $\beta \neq 0$	2
-	-	$G_{5,19}^{\alpha,\beta}$	$(\mathfrak{heis} \times \mathbf{R}) \rtimes \Delta(1,\alpha-1,\alpha,\beta)$, $(\alpha-1)\beta \neq 0$.	1
-	$G_{5,20}^{0}$	$\mathbf{R} \times G_{4,8}$	$(\mathfrak{heis} \times \mathbf{R}) \rtimes \Delta(1,-1,0^2)$	2
-	$G_{5,20}^{1}$	$G_{5,19}^{1,1}$	$(\mathfrak{heis} \times \mathbf{R}) \rtimes \Delta(1,0,1^2)$	2
-	$G_{5,20}^{\alpha}$	$G_{5,19}^{\alpha,\alpha}$	$(\mathfrak{heis} \times \mathbf{R}) \rtimes \Delta(1,\alpha-1,\alpha^2)$, $\alpha \neq 1$.	1

Table 2. The simply connected, real, indecomposable, solvable Lie groups of exponential growth and dimension five (to be continued on Table 3).

G	$\rho_0(G)$	$\rho_1(G)$	structure of $\mathfrak{g}$	conedim(G)
-	$G_{5,21}$	$G_{5,19}^{2,1}$	$(\mathbf{R} \times \mathfrak{heis}) \rtimes \Delta(1^3, 2)$	1
-	-	$G_{5,22}$	$(\mathbf{R} \times \mathfrak{heis}) \rtimes \Delta(1, 0^2, 0)$	4
-	$G_{5,23}^{\beta}$	$G_{5,19}^{2,\beta}$	$(\mathfrak{heis} \times \mathbf{R}) \rtimes \Delta(1^2, 2, \beta)$, $\beta \neq 0$	1
-	$G_{5,24}^{\epsilon}$	$G_{5,19}^{2,2}$	$(\mathfrak{heis} \times \mathbf{R}) \rtimes \phi_{5,24}^{\epsilon}$, see Example 5.5.	1
$G_{5,25}^{1,0}$	-	$\mathbf{Heis} \times A_2$	$(\mathfrak{heis} \times \mathbf{R}) \rtimes \Delta(\pm i, 0, 1)$	4
$G_{5,25}^{\beta,\alpha}$	-	$G_{5,19}^{2,\beta/\alpha}$	$(\mathfrak{heis} \times \mathbf{R}) \rtimes \Delta(\alpha \pm i, 2\alpha, \beta), \alpha \neq 0$	1
-	$S_{5,26}^{1}$	$G_{5,19}^{2,2}$	$(\mathfrak{heis} \times \mathbf{R}) \rtimes \Delta(1, 1, 2^2)$	1
$G_{5,26}^{\alpha,\epsilon}$	$S_{5,26}^{1}$	$G_{5,19}^{2,2}$	$(\mathfrak{heis} \times \mathbf{R}) \rtimes \phi_{5,26}^{\alpha,\epsilon}$, $\alpha > 0$, $\epsilon = \pm 1$; see Example 5.6.	1
-	$G_{5,27}$	$\mathbf{R} \times G_{4,5}^{1,1}$	$(\mathfrak{heis} \times \mathbf{R}) \rtimes \phi_{5,27}$; see Example 5.7	2
-	$G_{5,28}^{1}$	$\mathbf{R} \times G_{4,5}^{1,1}$	$(\mathbf{R} \times \mathfrak{heis}) \rtimes \Delta(1^2, 0, 1)$	2
-	$G_{5,28}^{\alpha}$	$G_{5,19}^{\alpha,1}$	$(\mathbf{R} \times \mathfrak{heis}) \rtimes \Delta(1^2, \alpha - 1, \alpha)$, $\alpha > 1$	1
-	$G_{5,29}$	$G_{5,8}^{1}$	$(\mathbf{R} \times \mathfrak{heis}) \rtimes \Delta(0^2, 1, 1)$	3
-	-	$G_{5,30}^{-1}$	$\mathfrak{fil} \rtimes \Delta(1, -2, -1, 0)$	1
-	-	$G_{5,30}^{0}$	$\mathfrak{fil} \rtimes \Delta(1, -1, 0, 1)$	1
-	$G_{5,30}^{1}$	$\mathbf{R} \times G_{4,9}^{1}$	$\mathfrak{fil} \rtimes \Delta(1, 0, 1, 2)$	2
-	-	$G_{5,30}^{\alpha}$	$\mathfrak{fil} \rtimes \Delta(1, \alpha - 1, \alpha, \alpha + 1)$	1
-	$G_{5,31}$	$G_{5,30}^{2}$	$\mathfrak{fil} \rtimes \Delta(1^2, 2, 3)$	1
-	$G_{5,32}^{0}$	$\mathbf{R} \times G_{4,5}^{1,1}$	$\mathfrak{fil} \rtimes \Delta(0, 1, 1, 1)$	2
-	$G_{5,32}^{\alpha}$	$\mathbf{R} \times G_{4,5}^{1,1}$	$\mathfrak{fil} \rtimes \phi_{5,32}^{\alpha}$; see Example 5.8	2
-	-	$G_{5,33}^{0,\beta}$	$\mathbf{R}^3 \rtimes \{\Delta(0, 1, 0), \Delta(1, 0, \beta)\}$, $\beta \neq 0$.	2
-	-	$G_{5,33}^{\alpha,\beta}$	$\mathbf{R}^3 \rtimes \{\Delta(0, 1, \alpha), \Delta(1, 0, \beta)\}$, $\alpha \leqslant \beta$, $\alpha \neq 0$	2
-	-	$G_{5,34}^{\alpha}$	$\mathbf{R}^3 \rtimes \{\Delta(\alpha, 1, 1), \Delta(1, 0, 1)\}$	2
$G_{5,35}^{\alpha,\beta}$	-	$G_{5,33}^{\alpha,\beta}$	$\mathbf{R}^3 \rtimes \{\Delta(\alpha, \pm i), \Delta(\beta, 1, 1)\}$, $\alpha \neq 0$.	2
$G_{5,35}^{0,\beta}$	-	$\mathbf{R} \times G_{4,5}^{1,\beta}$	$\mathbf{R}^3 \rtimes \{\Delta(0, \pm i), \Delta(\beta, 1, 1)\}$, $\beta \neq 0$	2
-	-	$G_{5,36}{}^{(\ddagger)}$	$\mathfrak{heis} \rtimes \{\Delta(1, 0, 1), \Delta(-1, 1, 0)\}$	2
$G_{5,37}$		$\mathbf{R} \times G_{4,9}^{1}$	$\mathfrak{heis} \rtimes \{\Delta(1, 1, 2), \Delta(\pm i, 0)\}$	2
		$G_{5,38}$	$\mathbf{R}^2 \rtimes \mathfrak{heis}$; see Example 5.9	3
$G_{5,39}$	-	$G_{5,8}^{1}$	see Example 5.9	3

$^{(\ddagger)}$ $G_{5,36}$ is the maximal completely solvable subgroup of the rank two simple group $\mathrm{SL}(3, \mathbf{R})$.

TABLE 3. The simply connected, real, indecomposable, solvable Lie groups of exponential growth and dimension five (started on Table 2).

Group	ρ_1	S.S.	SOL	HOM	AW	D. Ex.	$\delta(n)$	Reason
$G_{3,2}$	$G_{3,3}$	✓	×				n	Hyp
$G_{3,3}$	$G_{3,3}$						n	ρ_1
$G_{3,4}$	$G_{3,4}$	✓	✓				$\exp(n)$	SOL
$G_{3,5}^{\alpha>0}$	$G_{3,5}^{\alpha}$	✓					n	Hyp
$G_{3,5}^{\alpha<0}$	$G_{3,5}^{\alpha}$	✓	✓				$\exp(n)$	SOL
$G_{4,2}^{\alpha>0}$	$G_{4,5}^{1,\alpha}$	✓	×				n	Hyp
$G_{4,2}^{\alpha<0}$	$G_{4,5}^{1,\alpha}$	✓	✓				$\exp(n)$	SOL
$G_{4,3}$	$G_{4,3}$	×	×	×	×	$\{\omega_{2,3}, \omega_{2,4}\}$	n^3	n^3 D.Ex., GT
$G_{4,4}$	$G_{4,5}^{1,1}$	✓	×				n	Hyp
$G_{4,5}^{0<\alpha\leq\beta}$	$G_{4,5}^{\alpha,\beta}$	✓	×				n	Hyp
$G_{4,5}^{\alpha<0,\beta\neq 0}$	$G_{4,5}^{\alpha,\beta}$	✓	✓				$\exp(n)$	SOL
$G_{4,7}$	$G_{4,9}^{1}$	✓	×				n	Hyp
$G_{4,8}$	$G_{4,8}$	✓	✓				$\exp(n)$	SOL
$G_{4,9}^{0}$	$\mathbf{R} \times G_{3,3}$	✓			×		n^2	ρ_1, K
$G_{4,9}^{0<\beta\leq 1}$	$G_{4,9}^{\beta}$	✓	×				n	HYP
$G_{4,9}^{-1\leq\beta<0}$	$G_{4,9}^{\beta}$	✓	✓				$\exp(n)$	SOL
M_8	M_8	✓	×	×	✓		n^2	AW

TABLE 4. Indecomposable completely solvable groups of exponential growth and dimension less than 5 and their Dehn functions.

5.C. **Computations.** In this section we explain how to compute the data and criteria described above. We do not give the complete computations, but rather provide full details on a few selected examples in each category.

5.C.1. *Computing $\rho_1(G)$ for simply connected solvable G.* Beware that we do not take the same bases as in [Mub63, PSWZ76] when working in the Lie algebras: often, the order and the sign of the vectors is changed according to our needs.

Example 5.3. The Lie algebra of the group $G_{4,3}$ is $\mathbf{R}^3 \rtimes \Delta(1, 0^2)$. Let $(e_1, \dots e_4)$ be its basis. The derived subalgebra is $C^2\mathfrak{g} = \langle e_1, e_2 \rangle$, and the exponential radical is $C^3\mathfrak{g} = \langle e_1 \rangle$. The quotient $\mathfrak{g}_{4,3} / \mathrm{R}_{\exp} \mathfrak{g}_{4,3}$ is isomorphic to $\mathfrak{heis}$, the exponential radical is split, and the action on it is $\mathbf{R}$-diagonalizable. So $G_{4,3}$ is in $(\mathcal{C}_1)$.

Example 5.4. The Lie algebra of the group $G_{4,7}$ is $\mathfrak{heis} \rtimes \Delta(1^2, 2)$, which means that it has a basis (e_1, e_2, e_3, e_4) where e_1, e_2 and e_3 generate a Heisenberg ideal, with $[e_1, e_2] = e_3$, and that

$$\mathrm{ad}_{e_4} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

in the basis (e_1, e_2, e_3). We have $C^3 G_{4,7} = C^2 G_{4,7}$, so the exponential radical is the nilradical. To compute $\rho_1(G_{4,7})$ we remove the nilpotent part in the Jordan decomposition of ad_{e_4}, this gives a new Lie algebra law which is that of $\mathfrak{heis} \rtimes \Delta(1, 1, 2)$. We find that it is $G_{4,9}^1$. ($G_{4,7}$ is QI rigid within $(\mathcal{C}_0)$ by the combination of [CT11] and [CPS17]).

Group	$\delta(n)$	Reason	Group	$\delta(n)$	Reason
$G_{5,7}^{\alpha\geq\beta\geq\gamma>0}$	n	Hyp	$G_{5,23}^{\beta>0}$	n	Hyp
$G_{5,7}^{0\neq\alpha,0\neq\beta,\gamma<0}$	$\exp(n)$	SOL	$G_{5,23}^{\beta<0}$	$\exp(n)$	SOL
$G_{5,8}^{-1\leq\gamma<0}$	$\exp(n)$	SOL	$G_{5,24}^{\epsilon}$	n	ρ_1
$G_{5,8}^{0<\gamma\leq 1}$	n^3	n^3 D.Ex. $\{\omega_{51}\}$, GT	$G_{5,25}^{1,0}$	n^3	$\rho_1=\rho_0$
$G_{5,9}^{0<\beta\leq\gamma}$	n	Hyp	$G_{5,25}^{\beta,\alpha,\beta\alpha<0}$	$\exp(n)$	ρ_1
$G_{5,9}^{0>\beta,\gamma\neq 0}$	$\exp(n)$	SOL	$G_{5,25}^{\beta,\alpha,\beta\alpha>0}$	n	$\rho_1=\rho_0$
$G_{5,10}$	n^4	n^4 D.Ex. $\{\omega_{15},\omega_{23}\}$, GT	$S_{5,26}^{1}$	n	ρ_1
$G_{5,11}^{\gamma>0}$	n	Hyp	$S_{5,26}^{\alpha,\epsilon}$	n	ρ_1
$G_{5,11}^{\gamma<0}$	$\exp(n)$	SOL	$G_{5,27}$	n^2	ρ_1, K
$G_{5,12}$	n	ρ_1	$G_{5,28}^{1}$	n^2	ρ_1, K
$G_{5,13}^{1>\alpha,0,1}$	$\exp(n)$	ρ_1	$G_{5,28}^{\alpha>1}$	n	ρ_1
$G_{5,13}^{1<\alpha,0,1}$	n^2	$\rho_1=\rho_0$	$G_{5,28}^{\alpha<1}$	$\exp(n)$	ρ_1
$G_{5,13}^{\alpha\leq\beta,\tau:\alpha<0}$	$\exp(n)$	ρ_1	$G_{5,29}$	n^3	n^3 D.Ex. $\{\omega_{12},\omega_{15}\}$, GT
$G_{5,13}^{0<\alpha\leq\beta,\tau}$	n	ρ_1	$G_{5,30}^{-1}$	$\exp(n)$	SOL
$G_{5,14}^{\alpha\neq 0}$	n^3	$\rho_1=\rho_0$	$G_{5,30}^{0}$	$\exp(n)$	SOL
$G_{5,15}^{0}$	n^3	$\rho_1=\rho_0$	$G_{5,30}^{1}$	n^2	ρ_1, K
$G_{5,15}^{\beta<0}$	$\exp(n)$	ρ_1	$G_{5,30}^{1>\alpha\notin\{-1,0\}}$	$\exp(n)$	SOL
$G_{5,15}^{\beta>0}$	n	ρ_1	$G_{5,30}^{1<\alpha}$	n	Hyp
$G_{5,16}^{0,\tau\neq 0}$	n^2	$\rho_1=\rho_0$	$G_{5,31}$	n	ρ_1
$G_{5,16}^{0>\beta,1}$	$\exp(n)$	ρ_1	$G_{5,32}^{\alpha}$	n^2	ρ_1, K
$G_{5,16}^{0<\beta,1}$	n	ρ_1	$G_{5,33}^{0,\beta<0}$	$\exp(n)$	SOL
$G_{5,17}^{0\neq\tau,0,1}$	n^2	$\rho_1=\rho_0$	$G_{5,33}^{0,\beta>0}$	n^2	A-W, not Hyp
$G_{5,17}^{0\neq\tau,\alpha,\beta:\alpha\beta>0}$	n	ρ_1	$G_{5,33}^{\alpha<\beta=0}$	$\exp(n)$	SOL
$G_{5,17}^{0\neq\tau,\alpha,\beta:\alpha\beta<0}$	$\exp(n)$	ρ_1	$G_{5,33}^{0<\alpha,\beta=0}$	n^2	C-T, not Hyp, K
$G_{5,18}^{\alpha\neq 0}$	n	ρ_1	$G_{5,33}^{0<\alpha\leq\beta}$	n^2	A-W, not Hyp
$G_{5,19}^{0,\beta\neq 0}$	$\exp(n)$	SOL	$G_{5,33}^{\alpha\leq\beta<0}$	n^2	C-T, not Hyp, K
$G_{5,19}^{1,\beta<0}$	$\exp(n)$	SOL	$G_{5,34}^{\alpha}$	n^2	A-W, not Hyp
$G_{5,19}^{1,\beta>0}$	n^2	ρ_1, K	$G_{5,35}^{0,\beta<0}$	$\exp(n)$	$\rho_1=\rho_0$
$G_{5,19}^{\alpha,\beta:(\alpha-1)\beta<0}$	$\exp(n)$	SOL	$G_{5,35}^{0,\beta>0}$	n^2	$\rho_1=\rho_0$
$G_{5,19}^{\alpha,\beta:(\alpha-1)\beta>0}$	n	Hyp	$G_{5,35}^{\alpha<\beta=0}$	$\exp(n)$	$\rho_1=\rho_0$
$G_{5,20}^{0}$	$\exp(n)$	SOL	$G_{5,35}^{0<\alpha,\beta=0}$	n^2	$\rho_1=\rho_0$
$G_{5,20}^{1}$	n^2	ρ_1, K	$G_{5,35}^{0<\alpha\leq\beta}$	n^2	$\rho_1=\rho_0$
$G_{5,20}^{\alpha:(\alpha-1)\alpha<0}$	$\exp(n)$	ρ_1	$G_{5,35}^{\alpha\leq\beta<0}$	n^2	$\rho_1=\rho_0$
$G_{5,20}^{\alpha:(\alpha-1)\alpha>0}$	n	ρ_1	$G_{5,36}$	n^2	A-W, not Hyp
$G_{5,21}$	n	ρ_1	$G_{5,37}$	n^2	$\rho_1=\rho_0$
$G_{5,22}$	n^4	n^4 D.Ex. $\{\omega_{25},\omega_{34}\}$, GT	$G_{5,38}$	n^3	n^3 D. Ex. $\{\omega_{35};\omega_{34}\}$, GT
			$G_{5,39}$	n^3	$\rho_1=\rho_0$

TABLE 5. Dehn functions of 5-dimensional simply connected indecomposable solvable Lie groups.

Example 5.5. The Lie algebra of the group $G_{5,24}^{\epsilon}$ has a nilradical $\mathfrak{n}=\mathfrak{heis}\times\mathbf{R}$, with basis (e_1,e_2,e_3,e_4) such that $[e_1,e_2]=e_3$, and

$$\mathrm{ad}_{e_5}=\phi_{5,24}^{\epsilon}:=\begin{pmatrix}1&1&0&0\\0&1&0&0\\0&0&2&\epsilon\\0&0&0&2\end{pmatrix},\quad \epsilon=\pm 1.$$

(Beware the basis in [Mub63] would rather be $(e_2, e_1, -e_3, e_4, -e_5)$). Since ad_{e_5} is non-degenerate on $\mathfrak{n}$, the exponential radical is again equal to the nilradical. If $\epsilon = 1$ then $\phi^{\epsilon}_{5,24}$ has Jordan normal form in the given basis, and its diagonal part is that of type $\Delta(1,1,2,2)$. If $\epsilon = -1$, then $\phi^{\epsilon}_{5,24}$ has its standard Jordan form in the basis $(f_1, f_2, f_3, f_4) := (e_1, e_2, -e_3, e_4)$ and the diagonal part the same. (Note that $[f_1, f_2] = -f_3$, which is why we could not express the structure of $\mathfrak{g}^{\epsilon}_{5,24}$ in the table.) In both cases, $\rho_1(\mathfrak{g}^{\epsilon}_{5,24}) = (\mathfrak{heis} \times \mathbf{R}) \rtimes \Delta(1,1,2,2) = \mathfrak{g}^{2,2}_{5,19}$.

Example 5.6. The Lie algebra of the group $G^{\alpha,\epsilon}_{5,26}$ has a basis $(e_1, \ldots, e_5)$ with $[e_1, e_2] = e_3$, $[e_5, e_1] = \alpha e_1 + e_2$, $[e_5, e_2] = \alpha e_2 - e_1$, $[e_5, e_3] = 2\alpha e_3$ and $[e_5, e_4] = 2\alpha e_4 + \epsilon e_3$. Without loss of generality the parameter α is positive[5]; set

$$(f_1, f_2, f_3, f_4, f_5) = \left(\frac{\epsilon}{\sqrt{\alpha}} e_1, \frac{1}{\sqrt{\alpha}} e_2, \frac{\epsilon}{\alpha} e_3, e_4, \frac{1}{\alpha} e_5 \right).$$

Then, with $\tau = \alpha^{-1/2}$,

$$[\mathrm{ad}_{f_5}]_{(f_1,f_2,f_3,f_4)} = \phi^{\alpha,\epsilon}_{5,26} = \begin{pmatrix} 1 & -\tau & 0 & 0 \\ \tau & 1 & 0 & 0 \\ 0 & 0 & 2 & \epsilon \\ 0 & 0 & 0 & 2 \end{pmatrix}, \quad \epsilon = \pm 1.$$

The derivation $\phi^{\alpha,\epsilon}_{5,26}$ has Jordan type $\Delta(1 + i\tau, 2^2)$, so $\rho_0(G^{\alpha,\epsilon}_{5,26})$ is $S^1_{5,26}$.

Example 5.7. The Lie algebra of $G_{5,27}$ has nonzero brackets $[e_1, e_2] = e_3$, $[e_5, e_3] = e_3$, $[e_5, e_2] = e_2 + e_4$, $[e_5, e_4] = e_3 + e_4$. Thus $C^2 G_{5,27} = C^3 G_{5,27} = \langle e_2, e_3, e_4 \rangle$ and this is the exponential radical. The cone dimension is therefore 2. Moreover, the exponential radical splits, and in the basis (e_3, e_4, e_2), ad_{e_5} has type $\Delta(1^3)$ while ad_{e_1} is nilpotent. From this we deduce that $\rho_1(G_{5,27})$ is $\mathbf{R} \times G^{1,1}_{4,5}$.

Example 5.8. The Lie algebra $\mathfrak{g}^{\alpha}_{5,32}$ has a basis $(e_1, \ldots, e_5)$ where $[e_1, e_2] = e_3$, $[e_1, e_3] = e_4$ and the matrix of ad_{e_5} in this basis is

$$[\mathrm{ad}_{e_5}] = \phi^{\alpha}_{5,32} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \alpha & 0 & 1 \end{pmatrix}$$

The derived subgroup is $\langle e_2, \ldots, e_4 \rangle$ and this is the exponential radical. e_1 acts nilpotently on the exponential radical. Hence the Lie algebra of $\rho_1(G^{\alpha}_{5,32})$ is, in the same basis, $\langle e_2, e_3, e_4 \rangle \rtimes \langle e_1, e_5 \rangle$, where $\mathrm{ad}_{e_1} = 0$ and $\mathrm{ad}_{e_5} = 1$. This further splits as $(\langle e_2, e_3, e_4 \rangle \rtimes \langle e_5 \rangle) \times \langle e_1 \rangle$, with the isomorphism type of $G^{1,1}_{4,5} \times \mathbf{R}$.

Example 5.9 ($G_{5,38}$ and $G_{5,39}$)**.** The Lie algebra $\mathfrak{g}_{5,38}$ has a two-dimensional abelian ideal $\mathfrak{r}$, generated by the basis element (e_1, e_2), and splits as a semidirect product $\mathfrak{r} \rtimes \mathfrak{heis}$, where a section of $\mathfrak{heis}$ is generated by (e_3, e_4, e_5), where $[e_5, e_4] = e_3$,

$$\mathrm{ad}_{e_4} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathrm{ad}_{e_5} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

in the basis (e_1, e_2) of $\mathfrak{r}$. The nilradical is $\mathfrak{n} = \mathfrak{r} + \mathbf{R} e_5$, it is not split. We compute that $C^2 \mathfrak{g}_{5,38} = \mathfrak{n}$ and $C^3 \mathfrak{g}_{5,38} = C^4 \mathfrak{g}_{5,38} = \mathfrak{r}$, so that $\mathfrak{r}$ is the exponential radical. The action of $\mathrm{span}(e_3, e_4, e_5)$ being diagonal, $\mathfrak{g}_{5,38}$ is in $(\mathcal{C}_1)$.

[5] We might as well define the group for $\alpha < 0$. However, exchanging e_1 and e_2 while turning e_3 and e_4 into their opposite we see that $G^{-\alpha,\epsilon}_{5,26} \simeq G^{\alpha,-\epsilon}_{5,26}$; there are no further isomorphism in this family thanks to the invariant given in [PSWZ76], where α is denoted by p.

Let us now turn to $\mathfrak{g}_{5,39}$ The structure of the Lie algebra $\mathfrak{g}_{5,39}$ is also $\mathfrak{r} \rtimes \mathfrak{heis}$ but this time the adjoint action of e_4 and e_5 on $\mathfrak{r}$ are, instead,

$$\mathrm{ad}_{e_4} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathrm{ad}_{e_5} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

in the basis (e_1, e_2) of $\mathfrak{r}$. Again, the nilradical $\mathfrak{n} = \mathfrak{r} + \mathbf{R}e_3$ is not split, and $\mathfrak{r}$ is the exponential radical. However the action of e_5 on the nilradical has a purely imaginary type, so that, $\rho_0(\mathfrak{g}_{5,39}) = \mathfrak{r} \rtimes \mathfrak{heis}$ with e_5 centralizing $\mathfrak{r}$. Further, $\rho_0(\mathfrak{g}_{5,39})$ admits the following description: $\mathrm{span}(e_1, e_2, e_3, e_5)$ is an abelian ideal and, in the basis $(-e_3, e_5, e_1, e_2)$, ad_{e_4} has matrix of type $\Delta(0^2, 1, 1)$. Consequently, $\rho_0(\mathfrak{g}_{5,39}) = \mathfrak{g}_{5,8}^1$.

5.C.2. *Standard Solvability.* We need to check whether G splits as $G = U \rtimes A$, where U is the exponential radical of G, A is abelian and the action of A on $U/[U,U]$ has no fixed points (Proposition 3.9). When A is 1-dimensional this becomes a very easy task. In the general case, in the presence of an abelian complement A to U in G, it is quite straight forward to check the eigenvalues of the action of A on $U/[U,U]$. When G/U is non-abelian, this rules out the possibility of U having an abelian complement in G.

Example 5.10. Let us check that $G_{5,8}^{\gamma}$ is not standard solvable. The nonzero Lie brackets are:

$$[e_5, e_2] = e_1, [e_5, e_3] = e_3, [e_5, e_4] = \gamma e_4;$$

The exponential radical $\mathfrak{u}$ is generated by $\{e_3, e_4\}$ and is isomorphic to $\mathbf{R}^2$. Any complement of this must be isomorphic to the quotient of $\mathfrak{g}$ by $\mathfrak{u}$, which is the Heisenberg algebra. In particular there is no abelian complement, and $G_{5,8}^{\gamma}$ is not standard solvable.

5.C.3. *Azencott-Wilson criterion.* The Azencott-Wilson criterion from [AW76] for a real Lie algebra $\mathfrak{g}$ states that G admits a left-invariant nonpositively curved Riemannian metric if the following conditions are met:

(1) $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a}$, where $\mathfrak{n}$ is its nilradical and $\mathfrak{a}$ is abelian.
(2) For every root α and $H \in \mathfrak{a}$ with $\alpha(H) = 0$, ad_H is semisimple.
(3) The set of roots that are different from 0 lie in an open half-space.
(4) The zero weight space is central in $\mathfrak{n}$.
(5) For each root α, let

$$\mathfrak{n}_{\alpha}^0 = \{X \in \mathfrak{n}_{\alpha}\} \mid [X, \mathfrak{n}_{\beta}] = 0 \text{ whenever } \gamma \text{ is a root linearly independent of } \alpha\}.$$

Then for all α, the space $\mathfrak{n}_{\alpha}^0$ is $\mathfrak{a}$-invariant and admits an $\mathfrak{a}$-invariant complement on which $\mathfrak{a}$ acts semisimply.

Example 5.11. We show that $G_{5,36}$ admits the Azencott-Wilson criterion. We remark that this group is the Borel subgroup of $\mathrm{SL}_3(\mathbf{R})$ so the fact that it acts on a nonpositively curved space is well known. We give this example in detail only to familiarize the reader with the Azencott-Wilson criterion.

The Lie algebra $\mathfrak{g}_{5,36}$ is given by the brackets:

$$[e_1, e_2] = e_3, [e_4, e_1] = e_1, [e_4, e_3] = e_3, [e_5, e_1] = -e_1, [e_5, e_2] = e_2$$

The nilradical of $\mathfrak{g}_{5,36}$ is $\mathfrak{n} = \langle e_1, e_2, e_3 \rangle$, isomorphic to $\mathfrak{heis}$. It has a natural abelian complement in $\mathfrak{a} := \langle e_4, e_5 \rangle$. The action of all elements of $\mathfrak{a}$ is semisimple, so condition 2 is met. The set of roots of $\mathfrak{a}$ is $\{(1, -1), (0, 1)(1, 0)\}$, all lying in an open half-space of $\mathbf{R}^2$. The 0-root space is trivial and in particular central in $\mathfrak{n}$. Condition (5) is the most involved: For the root $\alpha = (1, -1)$, $\mathfrak{n}_{\alpha} = \langle e_1 \rangle$. Since $[e_1, e_2] \neq 0$, the space $\mathfrak{n}_{\alpha}^0 = \{0\}$. Obviously this space is $\mathfrak{a}$-invariant, its complement in $\mathfrak{n}$ is the whole $\mathfrak{n}$ which is $\mathfrak{a}$-invariant. Finally, $\mathfrak{a}$ acts semisimply on $\mathfrak{n}$, so condition (5) is met for the root

$(-1,1)$. The same argument works for the root $(0,1)$, as $\mathfrak{n}^0_{(0,1)} = \{0\}$. For the root $(1,0)$ we have $\mathfrak{n}^0_{(1,0)} = \langle e_3 \rangle$. But also this space is easily seen to satisfy the requirements of condition (5). We conclude that $\mathfrak{g}_{5,36}$ admits the Azencott-Wilson criterion.

Example 5.12. We show that the group $G^1_{5,30}$ does not admit the Azencott-Wilson criterion. Notice that this group does have Dehn function n^2, and could possibly act on a nonpositively curved space.

The Lie algebra is defined by the filiform brackets $[e_1, e_2] = e_3, [e_1, e_3] = e_4$, and non-zero action of e_5 given by $[e_5, e_1] = e_1, [e_5, e_3] = e_3, [e_5, e_4] = 2e_4$. We see that the nilradical is spanned by $\{e_1, e_3, e_4\}$ and that the complement spanned by $\{e_2, e_5\}$ is abelian. However, e_2 acts on the nilradical with ordered basis (e_3, e_1, e_4) via the following matrix:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus the corresponding root is $(0,0,0)$, but e_2 does not act semisimply. This violates condition (2) of the criterion.

5.C.4. SOL *obstruction.* Propositions 4.C.3 and 4.9.D of [CT17] state that a group admits the SOL obstruction if and only if its exponential radical admits two quasi-opposite principal weights. By weight we mean elements of $\mathrm{Hom}(G/U, \mathbf{R})$ with non-zero eigenspaces; weights are principal if they are weights of the action on $U/[U,U]$.

Example 5.13. We show that $G^{\beta<0}_{5,23}$ admits the SOL obstruction. The defining Lie brackets are

$$[e_1, e_2] = e_3, [e_5, e_1] = e_1, [e_5, e_2] = e_1 + e_2, [e_5, e_3] = 2e_3, [e_5, e_4] = \beta e_4$$

The Lie algebra of the exponential radical is $\mathfrak{u} = \langle e_1, e_2, e_3, e_4 \rangle$, a direct product of $\mathfrak{heis}$ and $\mathbf{R}$. Therefore $\mathfrak{u}/[\mathfrak{u},\mathfrak{u}] = \langle e_1, e_2, e_4 \rangle$, which is $\mathbf{R}^3$. The action of $\mathfrak{a} = \langle e_5 \rangle$ on this space is given by the roots $(1), (1), (\beta)$. Since $\beta < 0$, we see that 0 is in the convex hull of the principal roots, and conclude $G^{\beta<0}_{5,23}$ admits the SOL obstruction.

5.C.5. 2-*homological obstruction.* Let $\mathfrak{u}$ be the Lie algebra of the exponential radical U, and consider the action of G/U on U. This action extends to an action on $H_2(\mathfrak{u})$, the second homology of $\mathfrak{u}$, defined by linearly extending the action on 2-vectors given by $t.(v_1 \wedge v_2) = t.v_1 \wedge v_2 + v_1 \wedge t.v_2$. The 2-homological obstruction states that if this action on $\mathrm{H}_2(\mathfrak{u})$ has a non-trivial zero eigenspace, then G has exponential Dehn function. The computation of this condition is algorithmic. We give one example to manifest it.

Example 5.14. We show the group $G_{5,29}$ does not have the 2-homological obstruction. The defining brackets are:

$$[e_2, e_3] = e_4, [e_5, e_2] = e_1, [e_5, e_3] = e_3, [e_5, e_4] = e_4$$

The Lie algebra of the exponential radical is $\mathfrak{u} := \langle e_3, e_4 \rangle$ which is isomorphic to $\mathbf{R}^2$. The second homology group of $\mathbf{R}^2$ is one dimensional generated by $e_3 \wedge e_4$. Since it is 1-dimensional, it is enough to find one element of $\mathfrak{a}$ which acts non-trivially on it. Indeed, $e_5.(e_3 \wedge e_4) = 2e_3 \wedge e_4$. We conclude that $G_{5,29}$ does not admit the 2-homological obstruction.

We remark that in [CT17, Section 1.5.3], Cornulier and Tessera give an example of a group that admits the 2-homological obstruction but not the SOL obstruction. We note that in all groups we checked (i.e. up to dimension 5) there is no such group.

5.C.6. *Generalized tame groups.* Assume $G = U \rtimes N$. An element $c \in N$ acts as a *compaction* on U if there is a compact set $\Omega \subset U$ such that for every compact subset $K \subset U$ there is $n \geq 0$ such that $c^n(K) \subset \Omega$.

Definition 5.15 ([CT17], Definition 6.E.1)**.** A locally compact group G is generalized tame if it has a semi-direct product decomposition $G = U \rtimes N$ where some element c of N acts on U as a compaction, and N is nilpotent and compactly generated.

Example 5.16. Consider $G_{5,8}^{\gamma}$ for $\gamma \in (0,1]$. The defining brackets are:

$$[e_5, e_1] = 0, [e_5, e_2] = e_1, [e_5, e_3] = e_3, [e_5, e_4] = \gamma e_4$$

The exponential radical is $\langle e_3, e_4 \rangle$, and the quotient by it is $\langle e_1, e_2, e_5 \rangle$, isomorphic to the Heisenberg group where e_1 is central. The element e_5 acts as a compaction. The Dehn function of the Heisenberg group is cubic, therefore G has cubic Dehn function.

5.C.7. *Central Extensions.* See example in Section 4.C.

5.C.8. *Computing* $\mathrm{Kill}(\mathrm{R}_{\exp}\,\mathfrak{g})_0$. When a non-hyperbolic standard solvable group G does not have the SOL nor the 2-homological obstruction, [CT17, Theorem F] gives a sufficient condition for the Dehn function of G to be quadratic. This condition is given by the vanishing of the zero weight submodule in the Killing module of $\mathrm{R}_{\exp}\,\mathfrak{g}$, that is, in the quotient of the symmetric square of $\mathrm{R}_{\exp}\,\mathfrak{g}$ by the submodule spanned by elements of the form $[x, y] \odot z - x \odot [y, z]$ for $x, y, z \in \mathrm{R}_{\exp}\,\mathfrak{g}$. When the exponential radical is abelian, this is not a proper quotient and we will still denote the elements of the Killing module by their representatives in the symmetric square.

Example 5.17. Let us prove that $\mathrm{Kill}(\mathrm{R}_{\exp}\,\mathfrak{g}_{4,9}^0)_0 = 0$. The nonzero Lie brackets in $\mathfrak{g}_{4,9}^0$ are $[e_1, e_2] = e_3$, $[e_4, e_1] = e_1$. The exponential radical is spanned by e_1 and e_3; it is abelian, so that $\mathrm{Kill}(\mathrm{R}_{\exp}\,\mathfrak{g}_{4,9}^0)$ is three dimensional, spanned by $e_1 \odot e_1$, $e_1 \odot e_3 = e_3 \odot e_1$, and $e_3 \odot e_3$. Using $e_i \cdot (e_j \odot e_k) = [e_i, e_j] \odot e_k + e_j \odot [e_i, e_k]$ for $i \in \{2, 4\}$ and $j, k \in \{1, 3\}$, we obtain that

$$\begin{aligned} e_2 \cdot (e_1 \odot e_1) &= -2e_1 \odot e_3 & \qquad e_4 \cdot (e_1 \odot e_1) &= 2e_1 \odot e_1 \\ e_2 \cdot (e_1 \odot e_3) &= -e_3 \odot e_3 & e_4 \cdot (e_1 \odot e_3) &= 2e_1 \odot e_3 \\ e_2 \cdot (e_3 \odot e_3) &= 0 & e_4 \cdot (e_3 \odot e_3) &= 2e_3 \odot e_3, \end{aligned}$$

so that $\mathrm{Kill}(\mathrm{R}_{\exp}\,\mathfrak{g}_{4,9}^0)_0 = 0$.

Example 5.18 (An example with non-abelian exponential radical)**.** Let us check that $\mathrm{Kill}(\mathrm{R}_{\exp}\,\mathfrak{g}_{5,30}^1)_0 = 0$. The nonzero Lie brackets are $[e_1, e_2] = e_3$, $[e_1, e_3] = e_4$, $[e_5, e_1] = e_1$, $[e_5, e_3] = e_3$, $[e_5, e_4] = 2e_4$. The exponential radical is spanned by e_1, e_3, e_4; it is non abelian, and the Killing module is a quotient of its symmetric square by the submodule spanned by all the symmetric tensors involving e_4; indeed,

$$\begin{aligned} &[e_1, e_3] \odot e_3 - e_1 \odot [e_3, e_4] = e_4 \odot e_4; \\ &[e_1, e_3] \odot e_3 - e_1 \odot [e_3, e_3] = e_3 \odot e_4; \\ &[e_1, e_3] \odot e_1 - e_1 \odot [e_3, e_1] = e_1 \odot e_4 - (e_1 \odot -e_4) = 2e_1 \odot e_4. \end{aligned}$$

Now

$$\begin{aligned} e_2 \cdot [e_1 \odot e_1] &= -2[e_1 \odot e_3] & \qquad e_5 \cdot [e_1 \odot e_1] &= 2[e_1 \odot e_1] \\ e_2 \cdot [e_1 \odot e_3] &= -[e_1 \odot e_3] & e_5 \cdot [e_1 \odot e_3] &= 2[e_1 \odot e_3] \\ e_2 \cdot [e_3 \odot e_3] &= 0 & e_5 \cdot [e_3 \odot e_3] &= 2[e_3 \odot e_3], \end{aligned}$$

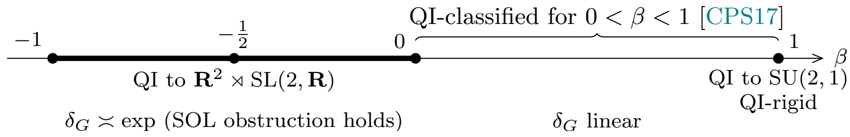

FIGURE 1. Together with $G_{4,8}$, the groups in the $G_{4,9}^{\beta}$ family are determined by the parameter $\beta \in (-1, 1]$. The three classes of groups defined by $\beta > 0$, $\beta = 0$ and $\beta < 0$ are quasiisometrically distinct. The groups with $\beta < 0$ (with exponential Dehn function) have not been classified up to quasiisometry so far.

finishing the proof that $\mathrm{Kill}(\mathrm{R}_{\exp}\, \mathfrak{g}_{5,30}^{1})_0 = 0$. The group $G_{5,30}^{1}$ is standard solvable, and sublinear bilipschitz equivalent to $G_{4,9}^{1} \times \mathbf{R}$, so that its Dehn function was a priori between n^2 and $n^2 \log^4 n$ by Corollary 4.9; the computation of the zero weight subspace in the Killing module above raises this indetermination, and the Dehn function of $G_{5,30}^{1}$ is quadratic.

Example 5.19 (An example with parameters)**.** Let us compute $\mathrm{Kill}(\mathrm{R}_{\exp}\, \mathfrak{g}_{5,33}^{\alpha,\beta})_0$ depending on α and β. The nonzero Lie brackets are as follows:

$$[e_4, e_1] = 0,\ [e_5, e_1] = e_1,\ [e_4, e_2] = e_2,\ [e_5, e_2] = 0,\ [e_4, e_3] = \alpha e_3,\ [e_5, e_3] = \beta e_3.$$

We can assume that $(\alpha, \beta) \neq (0, 0)$, otherwise $\langle e_3 \rangle$ becomes a direct factor, and $\alpha \leqslant \beta$ without loss of generality. Since $\mathfrak{u} = \mathrm{R}_{\exp}\, \mathfrak{g}_{5,33}^{\alpha,\beta} = \mathrm{span}(e_1, e_2, e_3)$ is abelian, $\mathrm{Kill}(\mathfrak{u}) = \mathfrak{u} \odot \mathfrak{u}$. We compute that

$$\begin{aligned} e_4 \cdot (e_1 \odot e_1) &= 0 & e_4 \cdot (e_2 \odot e_2) &= 2 e_2 \odot e_2 \\ e_4 \cdot (e_1 \odot e_2) &= e_1 \odot e_2 & e_4 \cdot (e_2 \odot e_3) &= (1+\alpha) e_2 \odot e_3 \\ e_4 \cdot (e_1 \odot e_3) &= \alpha e_1 \odot e_3 & e_4 \cdot (e_3 \odot e_3) &= 2\alpha e_3 \odot e_3 \end{aligned}$$

and similarly (by exchanging e_1 and e_2, α and β)

$$\begin{aligned} e_5 \cdot (e_1 \odot e_1) &= 2 e_1 \odot e_1 & e_5 \cdot (e_2 \odot e_2) &= 0 \\ e_5 \cdot (e_1 \odot e_2) &= e_1 \odot e_2 & e_5 \cdot (e_2 \odot e_3) &= \beta e_2 \odot e_3 \\ e_5 \cdot (e_1 \odot e_3) &= (1+\beta) e_1 \odot e_3 & e_5 \cdot (e_3 \odot e_3) &= 2\beta e_3 \odot e_3. \end{aligned}$$

Thus we get that

$$\mathrm{Kill}(\mathrm{R}_{\exp}\, \mathfrak{g}_{5,33}^{\alpha,\beta})_0 = \begin{cases} \langle e_2 \odot e_3 \rangle & (\alpha, \beta) = (-1, 0) \\ 0 & \text{otherwise.} \end{cases}$$

(The case $(\alpha, \beta) = (-1, -1)$ is treated in [CT17, 1.5.2].) Note that when $\alpha > 0$, the fact that $\delta_G(n) \asymp n^2$ follows already from the Azencott-Wilson criterion.

5.D. **Some Particular Families.** We discuss in details two particular families, to indicate where some progress would be needed to complete the quasiisometry classification.

5.D.1. *$G_{4,8}$ and the $G_{4,9}^{\beta}$ family.* Together with $G_{4,8}$, the groups of the form $G_{4,9}^{\beta}$ may be represented on a line segment, so that the eigenvalues of e_4 acting on the abelianization of $\mathrm{R}_{\exp}\, \mathfrak{g}_{4,9}$ are 1 and $\beta \in (-1, 1]$ (see Figure 1.) Note that $G_{4,8}$ is the limit case $\beta = -1$. The group $G_{4,9}^{1}$ is the maximal completely solvable subgroup of $\mathrm{SU}(2, 1)$; as such, it is QI-rigid within $(\mathcal{C}_0)$; See Theorem A.1 in Appendix A. When $\beta > 0$ the group $G_{4,9}^{\beta}$ is

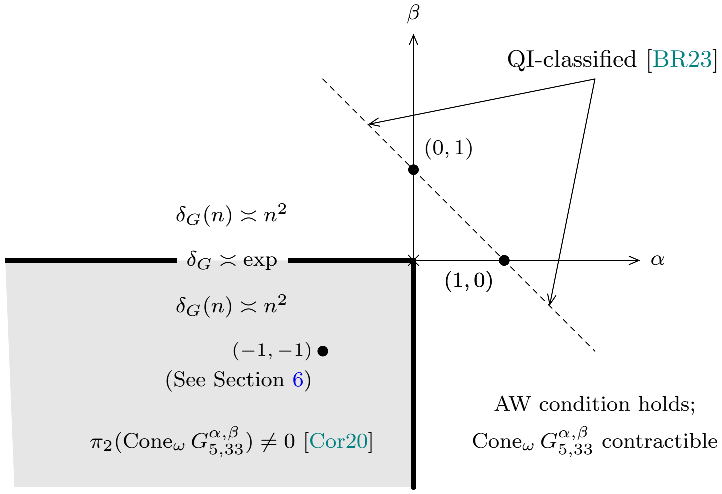

FIGURE 2. The groups in the $G_{5,33}^{\alpha,\beta}$ family are determined by the coordinates (α,β) of the third principal weight in the above weight diagram, so that one point in the plane represents a group; $G_{5,33}^{\alpha,\beta}$ and $G_{5,33}^{\alpha',\beta'}$ are isomorphic if and only if $\{\alpha,\beta\} = \{\alpha',\beta'\}$. The groups in the three areas (grey, black, and white) are quasiisometrically distinct. $G_{5,33}^{-1,-1}$ is the group Sol_5 discussed in Section 6.

hyperbolic; the fact that the $\{G_{4,9}^{\beta} \colon \beta > 0\}$ is QI-complete within $(\mathcal{C}_0)$ can be deduced from [KLDNG22], and the internal QI-classification is done by [CPS17]. The group $G_{4,9}^0$ is not in $(\mathcal{C}_1)$ and its cone dimension is 2. This is the only group in the family with this cone dimension. When $\beta < 0$ the cone dimension is again 1 and the SOL obstruction holds, so that the Dehn function is exponential.

The group $G_{4,9}^{-1/2}$ is of particular interest, since it is quasiisometric to $G = \mathrm{SL}(2,\mathbf{R}) \ltimes \mathbf{R}^2$; precisely it is isomorphic to its subgroup ANR, where R denotes the radical $\mathbf{R}^2$ and $L = KAN$ denotes the Levi factor $\mathrm{SL}(2,\mathbf{R})$ in the Levi decomposition LR of G. De la Harpe [dlH00, IV.25.(viii)] observed that the lattices in G are nonuniform and asked whether there are finitely generated groups quasiisometric to G.

Question 5.20. Can one describe the space $\mathrm{QI}(G_{4,9}^{\beta}, G_{4,9}^{\beta'})$ for $\beta, \beta' < 0$?

In the special case $\beta = \beta' = -1/2$, this amounts to the knowledge of the group of self-quasiisometries of $G_{4,9}^{-1/2}$ and would likely shed light on de la Harpe's question mentionned above.

5.D.2. *The $G_{5,33}^{\alpha,\beta}$ family.* Groups of the form $G_{5,33}^{\alpha,\beta}$ may be represented on a plane, so that the three principal weights in the basis of $\mathrm{Hom}(\mathfrak{a},\mathbf{R})$ dual to (e_4, e_5) are $(1,0)$, $(0,1)$, and (α,β). See Figure 2. This family is interesting because it exhibit various behaviours.

If α or β is strictly positive, then $G_{5,33}^{\alpha,\beta}$ has the Azencott-Wilson criterion; as such, it has a quadratic Dehn function. If moreover $\alpha + \beta = 1$, then $G_{5,33}^{\alpha,\beta}$ contains $G_{4,5}^{1,1}$ (namely the Borel subgroup of $\mathrm{SO}(4,1)$) as a codimension 1 subgroup. Bourdon and

Rémy recently used this fact to compute critical exponents in L^p-cohomology for the groups within the line $\alpha+\beta=1$; they obtain that two such groups are quasiisometric if and only if they are isomorphic [BR23, Theorem 1].

If α and β are nonpositive, on the other hand, then 0 lies in the convex hull of the set of principal weights, which changes drastically the geometry. If α or β is zero, then $G_{5,33}^{\alpha,\beta}$ has the SOL obstruction (as 0 lies in the segment between two principal weights), therefore its Dehn function is exponential. If α and β are both negative, then the Dehn function is again quadratic, as we compute in Example 5.19. However, we can still distinguish the groups $G_{5,33}^{\alpha,\beta}$ with $\alpha,\beta<0$ from the other ones using the asymptotic cone: $\pi_2(\mathrm{Cone}_\omega G_{5,33}^{\alpha,\beta})$ is nontrivial for $\alpha,\beta<0$ (See [Cor14, p.9]; the asymptotic cone is $\mathbf{T}^3_{D(3)}$) while when α or β are positive the asymptotic cone is contractible (since it is bilipschitz to a CAT(0) space). The group $G_{5,33}^{-1,-1}$ is the only unimodular group in the $G_{5,33}$ family; it is in Peng's class $(\mathcal{P})$ (See Definition 6.1) and the description of its self-quasiisometries is given by [Pen11a, Pen11b]. See Section 6 for more on Peng's class and the group $G_{5,33}^{-1,-1}$.

5.E. **Contribution to the Quasi-Isometric Classification.** We recall our motivation for the product theorem and the strategy of our work. Let G and H be two completely solvable Lie groups which we want to determine whether they are quasiisometric or not. If they were quasiisometric, we would have obtained an $O(\log)$-bilipschitz equivalence between $\rho_1(G)$ and $\rho_1(H)$. If it so happens that $\rho_1(G)$ or $\rho_1(H)$ decompose as products, and if the factors of these products admit the conditions of Theorem A, we obtain an $O(\log)$-bilipschitz equivalence between each of the respective factors. If we are lucky, we may be able to rule out these factor maps by various reasons which were not applicable to the product groups. This would rule out the existence of the original quasiisometry between G and H.

To conclude, our strategy is beneficial whenever:

(1) A quasiisometry between G and H cannot be ruled out with known tools.
(2) $\rho_1(G)$ or $\rho_1(H)$ decompose as direct products, whereas G and H did not.
(3) There is some obstruction for $O(\log)$-bilipschitz equivalence between the factors of $\rho_1(G)$ and $\rho_1(H)$.

Using the Dehn function estimates, we can now summarize to which groups this applies. First, we extract the groups whose ρ_1 decompose as products from tables 1–2. We then divide them according to their Dehn function estimates. The result is Table 6.

$\rho_1(G)$	Dehn function		
	$\exp(n)$	n^2	n^3
$\mathbf{R}^2\times G_{3,3}$		$G_{5,16}^{0,\tau}$, $G_{5,17}^{\tau,0,1}$	
$\mathbf{R}^2\times G_{3,5}^{\alpha}$	$G_{5,13}^{\alpha<1,0,1}$	$G_{5,13}^{\alpha>1,0,1}$	
$\mathbf{R}\times G_{4,5}^{1,\beta}$	$G_{5,19}^{1,\beta<0}$, $G_{5,35}^{0,\beta<0}$	$G_{5,19}^{1,\beta>0}$, $G_{5,35}^{0,\beta>0}$ $G_{5,27}$, $G_{5,28}^{1}$, $G_{5,32}^{\alpha}$	
$\mathbf{R}\times G_{4,8}$	$G_{5,20}^{0}$		
$\mathbf{Heis}\times A_2$			$G_{5,25}^{1,0}$
$\mathbf{R}\times G_{4,9}^{1}$		$G_{5,30}^{1}$, $G_{5,37}$	

TABLE 6. Indecomposable, completely solvable groups G such that $\rho_1(G)$ is decomposable, and their possible Dehn functions.

The next step is to determine which of the factors that appear in these decompositions admit the assumptions of Theorem A. Since hyperbolic groups are coarse type I, Table 4 recovers the following fact:

Lemma 5.21. *The groups $G_{3,3}, G_{3,5}^{\alpha\geq 0}, G_{4,5}^{1,\beta>0}, G_{4,9}^{\beta>0}$ are of coarse type I.*

We conclude that if a group G appearing in Table 6 has quadratic Dehn function then $\rho_1(G)$ admits the hypothesis of Theorem A.

Corollary 5.22. *Denote:*

- $\mathcal{G}^2_{3,3} := \{G_{5,16}^{0,\tau}, G_{5,17}^{\tau,0,1}\}$
- $\mathcal{G}^2_{3,5} := \{G_{5,13}^{\alpha>1,0,1}\}$
- $\mathcal{G}^2_{4,5} := \{G_{5,19}^{1,\beta>0}, G_{5,35}^{0,\beta>0}, G_{5,27}, G_{5,28}^{1}, G_{5,32}^{\alpha}\}$
- $\mathcal{G}^2_{4,9} = \{G_{5,30}^{1}, G_{5,37}\}$

If $G \in \mathcal{G}^2_{i,j}, H \in \mathcal{G}^2_{k,l}$ are quasi-isometric, then $(i,j) = (k,l)$.

Remark 5.23. The corollary could be deduced mostly from Theorem C, or entirely from Theorem 3.17. However for simplicity we give here a direct and short proof, using just Theorem A.

Proof. Let $G$and H be as in the statement of the Corollary. We need to rule out the possibility of an SBE between the factors of $\rho_1(G)$ and $\rho_1(H)$. Dimension considerations imply right away that $i = k$ (See [Pal22, Theorem B] whose substantial part is done by [HP13a]). Next, $G_{4,5}^{1,\beta}$ is not SBE to $G_{4,9}^{1}$: if $\beta \neq 2$, this follows from Proposition 3.8 in [Pal20b]. If $\beta = 2$, this follows from [Pal22, Theorem E]. Finally, $G_{3,3}$ and $G_{3,5}^{\alpha<1}$ are not SBE by the main theorem in [Pal20b]. □

Remark 5.24. A basic consequence of Theorem A is that the number of direct factors of $\rho_1(G)$ is a SBE invariant of the group G satisfying the assumptions of the theorem. Looking at Table 5 hints at groups for which the knowledge of whether or not they can be SBE to a direct product would be beneficial in the context of quasiisometric distinction. An example for such a family of groups is $G_{5,33}^{\alpha,\beta}$, specifically with $\alpha > 0$ and $\beta > 0$. We pose it as a question, and refer to 5.D for a short discussion on this family of groups.

Question 5.25. Can $G_{5,33}^{\alpha>0,\beta>0}$ be SBE to a direct product of two infinite groups?

6. Groups quasiisometric to Sol_5 proof of Proposition F

Eskin, Fisher and Whyte have conjectured that the class of virtually polycyclic groups should be quasiisometrically complete within finitely generated groups [EF10, Conjecture 1.2]. The first evidence for this came from the work of Shalom, who proved that any finitely generated group quasiisometric to a polycyclic group has a finite index subgroup with nonvanishing first Betti number [Sha04]. Since every polycyclic group contains a finite-index subgroup which is a uniform lattice in a simply connected solvable Lie group, the quasiisometry classification conjecture for completely solvable Lie groups [Cor18, Conjecture 19.25] would complement [EF10, Conjecture 1.2] in that it would complete the internal QI-classification of polycyclic group. (Note that when we restrict attention to the smaller class of virtually nilpotent groups, we face a similar picture, but while the quasiisometric classification of simply connected nilpotent Lie groups is still open, Gromov's polynomial growth theorem [Gro81] can be taken as a replacement of the above conjecture of Eskin, Fisher and Whyte.)

Peng [Pen11a, Pen11b] made significant progress towards both conjectures. To state her theorems we make the following definition.

Definition 6.1. Let G be a standard solvable group. We say that G is of class $(\mathcal{P})$ if the following holds:

(1) $\mathrm{R}_{\exp}G$ is equal to the nilradical of G
(2) $\mathrm{R}_{\exp}G$ is abelian
(3) $G/\mathrm{R}_{\exp}G$ is abelian
(4) G is unimodular

Theorem 6.2 ([Pen11b, Corollary 5.3.7])**.** *If two groups of class $(\mathcal{P})$ and $(\mathcal{C}_0)$ are quasiisometric, then they are isomorphic.*

Theorem 6.3 ([Pen11b, Corollary 5.3.9])**.** *If a finitely generated group is quasiisometric to a group of class $(\mathcal{P})$, then it is virtually polycyclic.*

Remark 6.4. In [Pen11b], the statement of Theorem 6.2 is different and involves real parts of Jordan form of the adjoint action. This is because the groups are not supposed to be in $(\mathcal{C}_0)$ a priori. It translates as follows: if two groups G and G' of class $(\mathcal{P})$ are quasiisometric, then $\rho_0(G)$ and $\rho_0(G')$ are isomorphic. With the current formulation it is easier to see that [Pen11b, Corollary 5.3.7] implies [Pen11b, Corollary 5.3.8].

The completely solvable groups of class $(\mathcal{P})$ and dimension less or equal 5 are $G_{3,4}$, $G_{4,2}^{-2}$, $G_{4,5}^{-(1+\delta)/2,-(1-\delta)/2}$ for $0 \leqslant \delta < 1$, $G_{5,7}^{\alpha,\beta,\gamma}$ with $\alpha+\beta+\gamma=1$, $G_{5,9}^{-1-\delta,-1+\delta}$ with $0 \leqslant \delta < 1$, $G_{5,11}^{-3}$, $G_{5,15}^{-1}$, and $G_{5,33}^{-1,-1}$, which is the group Sol_5.

Using Peng's rigidity theorem and our work in Section 5, we can prove that the finitely generated groups quasiisometric to Sol_5 are almost lattices in this group.

Proposition 6.5. *Let Γ be a finitely generated group quasiisometric to* Sol_5*. Then there is a finite-index subgroup Γ_0 in Γ and a homomorphism $\Gamma_0 \to \mathrm{Sol}_5$ with finite kernel and closed co-compact image.*

Remark 6.6. The group Sol_5 does have lattices, see Remark 1.13.

Proof. The proof is done by exhaustion. The group Sol_5 is of class $(\mathcal{P})$, so that by Peng's rigidity theorem 6.3 we know that Γ is virtually polycyclic; let G be a simply connected solvable group such that there exists a finite-index subgroup of Γ that surjects with finite kernel onto a uniform lattice in G [Rag72, Theorem 4.28]. We know that G is quasiisometric to Sol_5, so that $\dim G = 5$ and $\operatorname{conedim} G = 2$. Therefore G is among the following list of groups:

$$G_{5,19}^{0,\beta},\ G_{5,19}^{1,\beta},\ G_{5,20}^{0},\ G_{5,20}^{1},\ G_{5,27},\ G_{5,28}^{1},\ G_{5,30}^{1},\ G_{5,32}^{0},\ G_{5,32}^{\alpha},\ G_{5,33}^{\alpha,\beta}, G_{5,34},\ G_{5,35},$$

$$G_{5,36},\ G_{5,37},\ G_{4,2}^{\alpha}\times\mathbf{R}, G_{4,4}\times\mathbf{R}, G_{4,5}^{\alpha,\beta}\times\mathbf{R},\ G_{4,7}\times\mathbf{R},\ G_{4,8}\times\mathbf{R}, G_{4,9}^{0}\times\mathbf{R},\ G_{4,9}^{\beta}\times\mathbf{R}.$$

We can rule out most of the groups in this list since they are not unimodular. The unimodular ones are

$$G_{5,19}^{1,-2},\ G_{5,20}^{0},\ G_{5,33}^{-1,-1},\ G_{5,35}^{0,-2},\ G_{4,2}^{-2}\times\mathbf{R},\ G_{4,5}^{\alpha,\beta\colon\,\alpha+\beta=-1}\times\mathbf{R},\ G_{4,8}\times\mathbf{R}.$$

By our work in Tables 4-5 all these except $G_{5,33}^{-1,-1} \simeq \mathrm{Sol}_5$ have exponential Dehn function; the Dehn function of G is quadratic, since it is quasiisometric to Sol_5 (by [Dru04, Theorem 1.1], Leuzinger-Pittet [LP04, Theorem 2.1], or the computation in Example 5.19, using Cornulier and Tessera's [CT17, Theorem F]). So G must be isomorphic to Sol_5. □

Remark 6.7. According to Peng [Pen11b, Corollary 5.3.11], a completely solvable group quasiisometric to Sol_5 must be a semidirect product of the form $\mathbf{R}^2 \ltimes \mathbf{R}^3$. This would allow to rule out some of the groups above without estimating Dehn functions, but it does not rule out $G_{4,5}^{\alpha,1-\alpha} \times \mathbf{R}$.

Proposition 6.8. *Let G be a completely solvable group, quasiisometric to* Sol_5*. Then G is isomorphic to* Sol_5*.*

Proof. The group Sol_5 is amenable and unimodular, hence geometrically amenable, and geometric amenability is invariant under quasiisometry, so that G must be geometrically amenable (See §11 in [Tes08] and especially Corollary 11.13 there). Moreover, since G is completely solvable, it is geometrically amenable if and only if it is unimodular. So G is unimodular as well, and belongs to the list of groups already considered in the proof of Proposition 6.5. The end of the proof is the same as that of Proposition 6.5. □

Appendix A. Completely solvable groups quasiisometric to symmetric spaces

The theorem below follows from the combined works of many authors on the quasi-isometric rigidity of symmetric spaces in the 1990s, complemented by an improvement of Kleiner-Leeb [KL09] and synthetized in [Cor18, Theorem 19.25].

Theorem A.1. *Let G and H be two completely solvable groups. Assume that*

(1) G and H are quasiisometric, and
(2) G or H admits a symmetric left-invariant Riemannian metric with no Euclidean factor.

Then G and H are isomorphic.

Note that Theorem A.1 without assumption (2) would be [Cor18, Conjecture 19.113].

Proof. Without loss of generality, we can assume that H admits a left-invariant symmetric metric g, so that (H, g) is isometric to the symmetric space X. Since H is completely solvable, X must be of non-compact type. H acts simply transitively by isometries on X, so any larger connected Lie group H' of isometries of X will contain non-trivial point stabilizers; the latter are compact, and a completely solvable group does not have non-trivial compact subgroups. So H is maximal among the completely solvable groups of isometries of X. Now, G is quasiisometric to X, therefore by [Cor18, Theorem 19.25] it has a continuous, proper, cocompact action by isometries on X. The kernel of this action is a compact, therefore trivial, subgroup of G so that we may consider G as a subgroup of $\mathrm{Isom}(X)$; it is a closed subgroup by properness of the action. Let $\widehat{G}$ be a maximal completely solvable subgroup of $\mathrm{Isom}(X)$ containing G. By the combination of [GW88, Theorem 1.11] and [GW88, Theorem 4.3], H and $\widehat{G}$ are isomorphic. It remains to show that $G = \widehat{G}$. By [HP13a] we know that

$$\dim G = \mathrm{asdim}_{\mathrm{AN}}\, G = \mathrm{asdim}_{\mathrm{AN}}\, H = \dim H = \dim \widehat{G},$$

so G and $\widehat{G}$ have the same dimension. Since $G \subseteq \widehat{G}$ and G and $\widehat{G}$ are both completely solvable, $G = \widehat{G}$. □

Question A.2. In the theorem above, can one allow Euclidean factors in assumption (2)?

The answer to this question is not obviously yes since in [Cor18, Lemma 19.29], the assumption that there is no Euclidean factors is necessary.

Appendix B. A completely solvable group with non-split exponential radical and polynomial Dehn function

We give an example of a group for which the tool we developed in Proposition D and Proposition 4.2, namely a lower bound on the Dehn function using distortion in central extensions of solvable Lie groups, improves on previously known techniques, in particular on the bound given by distortion in central extensions of nilpotent groups and on Theorem 4.6.

Our example is a variation on [Cor08, Examples 4.1, 4.2], using [CT17, Example 1.5.4] as a building block. It is of dimension 13, and we do not claim this dimension is minimal for the properties we need. We first explain the logic of our construction, which is extracted from the discussion in the end of Section 4.D. We are looking for a group with quite a few properties: it must be completely solvable (i.e. in class $(\mathcal{C}_0)$), it must not split with a nilpotent quotient (in particular it should not be generalized standard solvable, hence not in $(\mathcal{C}_1)$), its Dehn function must be polynomial, and its distorted central extensions must be necessary for giving a lower bound on its Dehn function (so for example it must not admit a left-invariant nonpositively curved Riemannian metric). In low dimensions and for the groups of class $(\mathcal{C}_0)$, the property of not splitting over the exponential radical is the hardest to come by. Cornulier's construction [Cor08, Examples 4.1, 4.2] give such groups, and hints at how to obtain them in general. We vary his building blocks in order to assure that the group admits the other desired properties; the challenge of escaping the SOL obstruction is the main reason we chose [CT17, Example 1.5.4] as a building block for our example.

Let $\mathfrak{g}_2$ be the Lie algebra corresponding to the group $U \rtimes A$ presented in [CT17, Example 1.5.4]. The Lie algebra $\mathfrak{u} = \mathrm{Lie}(U)$ is given by:

$$\mathfrak{u} := \langle X_1, X_2, X_3, X_4, X_5, X_6, X_9, X_{12} \rangle$$

with the nonzero brackets

$$[X_1, X_2] = X_4, [X_1, X_3] = X_5, [X_2, X_3] = X_6,$$
$$[X_1, X_6] = X_9, [X_3, X_4] = X_{12}, [X_2, X_5] = X_9 + X_{12}.$$

(The indices of the generators indeed skip 7, 8, 10 and 11: this is reminiscent to the fact that $\mathfrak{u}$ is the quotient of the free 3-step nilpotent Lie algebra on 3 generators $\{X_1, X_2, X_3\}$ by the ideal generated by $[X_i, [X_i, X_j]]$ for $i \neq j \in \{1, 2, 3\}$).

Let $\mathfrak{a} := \mathrm{Lie}(A)$ be the abelian Lie algebra on two generators $\langle T_1, T_2 \rangle$, and $\mathfrak{g}_2 := \mathfrak{u} \rtimes \mathfrak{a}$ with the following non-zero brakcets:

$$[T_1, X_1] = -X_1, [T_1, X_3] = X_3, [T_1, X_4] = -X_4, [T_1, X_6] = X_6,$$
$$[T_2, X_1] = -X_1, [T_2, X_2] = 2X_2, [T_2, X_3] = -X_3, [T_2, X_4] = X_4, [T_2, X_5] = -2X_5,$$
$$[T_2, X_6] = X_6$$

Let $\mathfrak{fil} = \langle E_1, E_2, E_3, E_4 \rangle$ be the 4-dimensional filiform algebra, with non-zero brackets $[E_1, E_2] = E_3, [E_1, E_3] = E_4$. Define $\mathfrak{g}_3 := \mathfrak{g}_2 \times \mathfrak{fil}$. The centre of $\mathfrak{g}_3$ is the product of the centres of the factors, which is $\langle X_9, X_{12}, E_4 \rangle$. Let $\mathfrak{z}$ be the 1-dimensional subspace generated by the diagonal of the centre $Z := X_9 + X_{12} + E_4$.

Our example is the group G whose Lie algebra is $\mathfrak{g} := \mathfrak{g}_3/\mathfrak{z}$. We can write it in the basis

$$\langle X_1, X_2, X_3, X_4, X_5, X_6, X_9, X_{12}, T_1, T_2, E_1, E_2, E_3 \rangle,$$

with all non-zero brackets exactly as in $\mathfrak{g}_3$, except for $[E_1, E_3] = -X_9 - X_{12}$.

The group G admits the following properties:

- $\rho_0(G) = G$, i.e. G is in $(\mathcal{C}_0)$.
- $U = \mathrm{R}_{\exp} G$ and the short exact sequence
$$1 \to U \to G \to G/U \to 1$$
does not split. In particular G is not in $(\mathcal{C}_1)$.
- G is not generalized standard solvable.
- G does not have a nonabelian nilpotent retract.
- G has a polynomially bounded Dehn function.
- G admits a cubically distorted central extension.

We supply short reasoning for the above claims. Due to the high dimension of this group, we do not give the details of the computations. The reader may consult Section 5.C for the relevant techniques.

The nilradical of $\mathfrak{g}$ is $\mathfrak{u} + \mathfrak{fil}/\langle Z \rangle$, it splits with complement $\mathfrak{a}$ acting with only real eigenvalues. Therefore G is in $(\mathcal{C}_0)$.

The exponential radical is $\mathfrak{u}$. The quotient $\mathfrak{g}/\mathfrak{u}$ is $\mathbf{R}^2 \times \mathfrak{heis}$, where in the above basis $\mathfrak{heis} = \langle E_1, E_2, E_3 \rangle / \mathfrak{u}$ is the Lie algebra of the 3-dimensional real Heisenberg group with central element $E_3 \cdot \mathfrak{u}$. If $\mathfrak{u}$ did split, we would have $\mathfrak{g} = \mathfrak{u} \rtimes \mathfrak{m}$ with $\mathfrak{m}$ isomorphic to $\mathbf{R}^2 \times \mathfrak{heis}$ and its action on $\mathfrak{u}$ would factor through the quotient. In particular, the central element of $\mathfrak{heis}$ in $\mathfrak{m}$, which is central in $\mathfrak{m}$, would act trivially on $\mathfrak{u}$. Therefore the centre of $\mathfrak{g}$ would intersect $\mathfrak{m}$ nontrivially. It can be verified however that the centre of $\mathfrak{g}$ is exactly $\langle X_9, X_{12} \rangle \subset \mathfrak{u}$.

If, towards contradiction, G was generalized standard solvable with nilpotent quotient N, then $\mathfrak{n} := \mathrm{Lie}(N)$ would have to be a quotient of $\mathfrak{m} = \mathbf{R}^2 \times \mathfrak{heis}$ (recall $G/\mathrm{R}_{\exp}(\mathrm{G})$ is the largest nilpotent quotient of G). Therefore if $\mathfrak{n}$ is nonabelian, it must contain $\mathfrak{heis}$ and the same argument as above yields a contradiction. This moreover proves that G does not retract to a nonabelian nilpotent group. If on the other hand N was abelian, then by Proposition 3.9 G would have to split over the exponential radical, which is not the case. So G is not generalized standard solvable.

One may check that G does not admit the SOL or 2-homological obstructions, and therefore has a polynomial Dehn function [CT17, Theorem E].

It is easily observed that G_3 (the Lie group corresponding to $\mathfrak{g}_3$) is a central extension of G. The generator of the extension is $Z = X_9 + X_{12} + E_4$, which is in $C^3\mathfrak{g}_3$ but not in $C^4\mathfrak{g}_3$, i.e. $c_Z = 3$. By Proposition D, $\langle Z \rangle$ is n^3-distorted in $G_1 := \rho_1(G)$, hence by Proposition 4.2, the Dehn function of G is bounded from below by n^3. A direct computation of the second cohomology group proves that we cannot improve this lower bound using distortion in other central extensions.

To the best of our knowledge, the most accurate evaluation of the Dehn function of G prior to our work is given Theorem 4.6. Concretely, it is easy to check that $\rho_1(\mathfrak{g}) = \mathfrak{u} \rtimes (\mathfrak{a} \times \mathfrak{heis})$, where the only difference in brackets from $\mathfrak{g}$ is that $[E_1, E_3] = 0$. In particular, the action of $\mathfrak{a} \times \mathfrak{heis}$ on the exponential radical $\mathfrak{u}$ is the same as in $\mathfrak{g}$ and so the corresponding Lie group $G_1 = U \rtimes (A \times \mathrm{Heis})$ is generalized standard solvable in the sense of [CT17, Section 10.H]. Theorem 4.6 yields:
$$n^3/\log^e(n) \preceq \delta_G(n) \preceq n^4 \cdot \log^e(n).$$
where $e \leq 8$ is twice the bound on the exponent of δ_G.

Finally, we remark that $\mathrm{Kill}(\mathfrak{u})_0 \neq 0$, so the upper bound of n^4 cannot be improved to n^3 using only [CT17, Theorem 10.H.1]. Moreover, $\rho_1(G)$ is not generalized tame as can be seen by drawing the weight diagram, and using [CT17, Proposition 4.B.5].

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