ON THE LARGE-SCALE SUBLINEAR GEOMETRY OF HEINTZE SPACES

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Analysis and Geometry Seminar, Bristol

Outline

Heintze spaces

Large-scale geometry and the boundary

Invariants

HEINTZE SPACES

Heintze space: Definition

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Let Y be a connected Riemannian manifold of **negative sectional curvature**. If the isometry group $\mathrm{Isom}(Y)$ acts transitively on Y, then Y is called a Heintze space.

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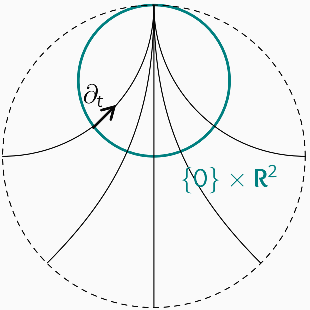
Example: Y is $\mathbf{R} \times \mathbf{R}^2$ with coordinates (t, x, y) and the Riemannian metric $ds^2 = dt^2 + e^{-2t}(dx^2 + dy^2)$.

Y is the hyperbolic 3-space (constant curvature -1) and (x, y, t) are **horospherical coordinates**.

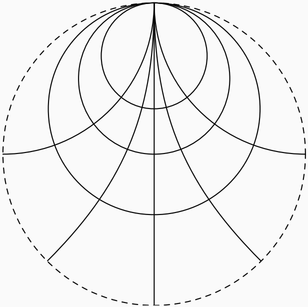


Isometries of $\mathbb{H}^3$ seen as a Heintze space

$$ds^2 = dt^2 + e^{-t}(dx^2 + dy^2).$$

Let us check that $\mathrm{Isom}(Y)$ is transitive.

1. For all $(u, v) \in \mathbf{R}^2$, $(t, x, y) \mapsto (t, x + u, y + v)$ is an isometry.



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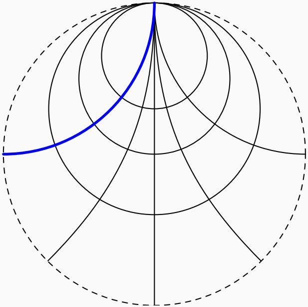

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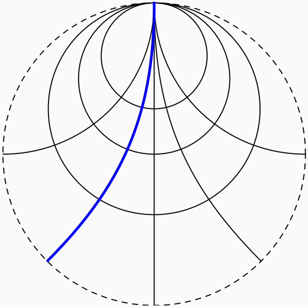

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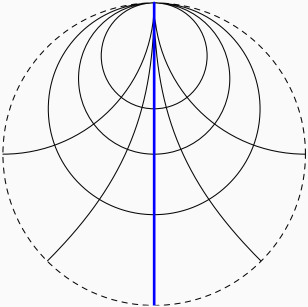

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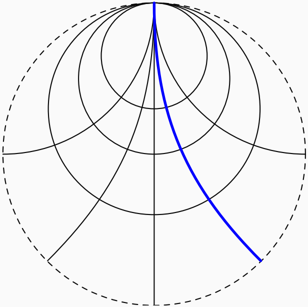

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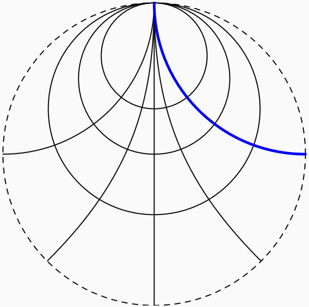

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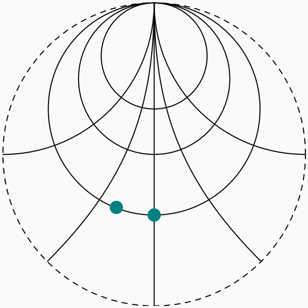

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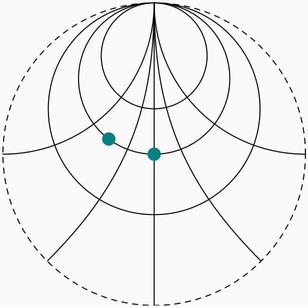

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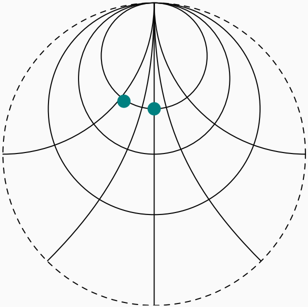

$G = \mathbf{R} \ltimes \mathbf{R}^2$ acts **simply transitively** by isometries on Y.

Other Heintze spaces

To form G, we used $\mathbf{R} \curvearrowright \mathbf{R}^2$ by $s.(x,y) = \delta^s(x,y) = (e^s x, e^s y)$. More general:

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1. **Diagonal** action: Let $\mu \geqslant 1$ be a parameter

$$\delta^s(x,y) = (e^s x, e^{\mu s} y) = \exp\left[s\begin{pmatrix}1 & 0\\ 0 & \mu\end{pmatrix}\right]\begin{pmatrix}x\\ y\end{pmatrix}$$

   a left inv. metric is $ds^2 = dt^2 + e^{-2t}(dx^2 + e^{2(1-\mu)t}dy^2)$.

2. **Unipotent** action:

$$\delta^s(x,y) = (e^s x + se^s y, e^s y) = \exp\left[s\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\right]\begin{pmatrix}x\\ y\end{pmatrix}$$

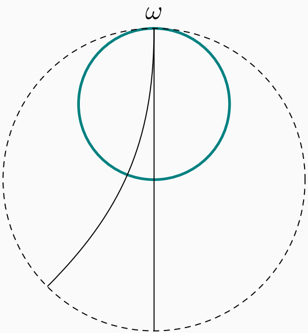
   a left inv. metric is $ds^2 = dt^2 + e^{-2t}(dx^2 + (1+t^2)dy^2 - 2t dx dy)$.

Fact (Consequence of Heintze's theorem)

The corresponding groups G, G_μ, G' with all their left invariant metrics are **all the** 3**-dim Heintze spaces**.

The boundary of a 3-dimensional Heintze space

All the isometries described fix a **special boudary point** of Y that we denote ω. Horospheres centered at ω are **Euclidean planes** (if $\dim Y = 3$).



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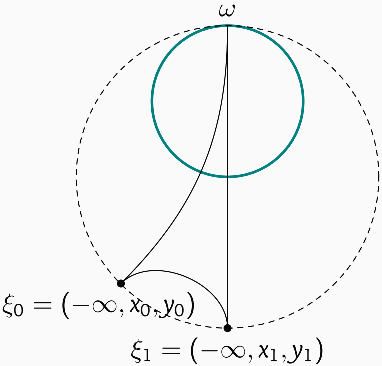

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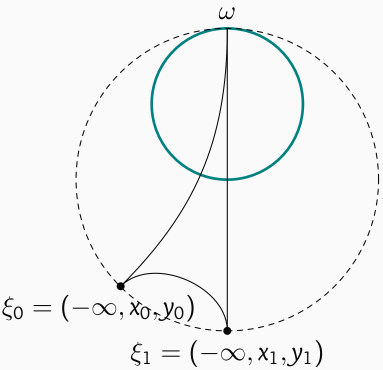

A quasidistance on the boundary minus ω

$\rho((-\infty,x,y),(-\infty,x',y')) :=$
$\exp\left(-\frac{1}{2}\lim_{t\to-\infty} d_Y((-t,x_0,y_0),(-t,(x_1,y_1)) + 2t\right)$

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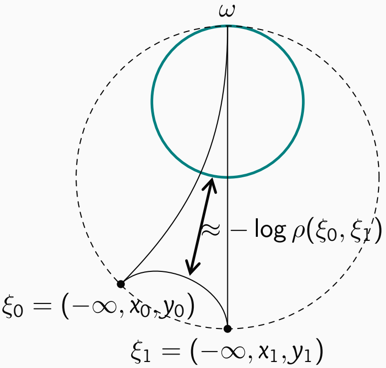

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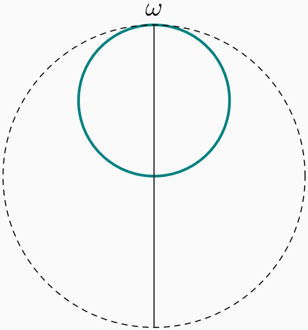
Why such a quasidistance on the boundary?

Isometries of Y extend to ∂_∞^*, by homeomorphisms.



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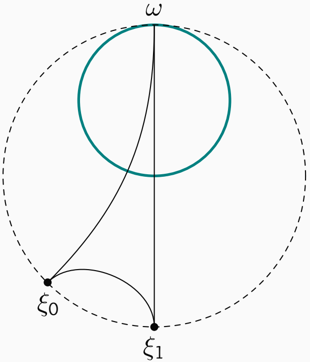

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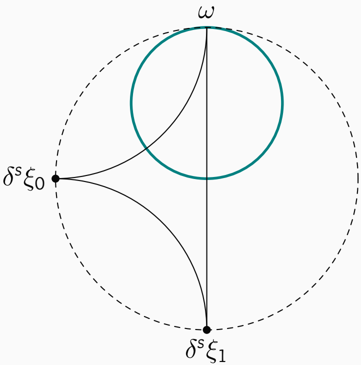
$$\forall \xi_0, \xi_1 \in \partial^*_\infty Y, \rho(\delta^s \xi_0, \delta^s \xi_1) = e^s \rho(\xi_0, \xi_1).$$

Equipped with ρ, ∂^*_∞ is a self-similar space. Identified with $\mathbf{R}^2$, self-similarities are the δ^s.



How do the small balls of ρ look like?

Group	G	G'	G_μ $(\mu > 1)$
self-similarities	$\left\{ \begin{pmatrix} e^s & 0 \\ 0 & e^s \end{pmatrix} \right\}$	$\left\{ \begin{pmatrix} e^s & se^s \\ 0 & e^s \end{pmatrix} \right\}$	$\left\{ \begin{pmatrix} e^s & 0 \\ 0 & e^{\mu s} \end{pmatrix} \right\}$

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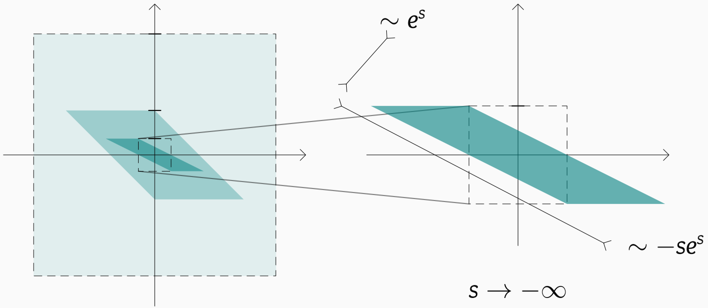

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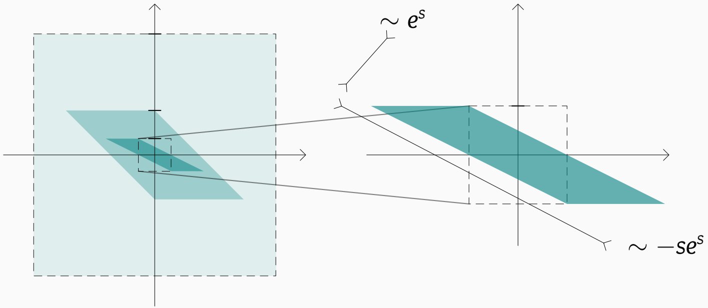

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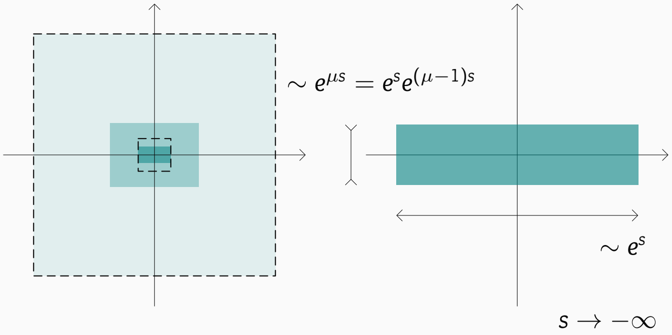

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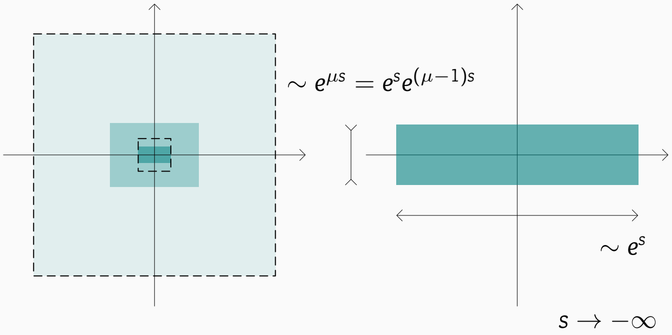

LARGE-SCALE GEOMETRY AND THE BOUNDARY

Quasiisometry and Sublinearly biLipschitz Equivalence

Y, Y' are pointed metric spaces, $\lambda \geqslant 1$.

$f : Y \to Y'$ is a **quasiisometry** (QI) if $\exists c \geqslant 0$ s.t. $\forall y_1, y_2 \in Y, \forall y' \in Y'$,

$$\begin{cases} \lambda^{-1} d(y_1, y_2) - c \leqslant d(f(y_1), f(y_2)) \leqslant \lambda d(y_1, y_2) + c \\ d(y', f(Y)) \leqslant c. \end{cases}$$

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$f : Y \to Y'$ is a **sublinearly biLipschitz equivalence** (SBE) if there exists a sublinear $v : \mathbf{R}_{\geqslant 0} \to \mathbf{R}_{\geqslant 1}$ s.t. $\forall y_1, y_2 \in Y$ and $\forall y' \in Y'$,

$$\begin{cases} \lambda^{-1} d(y_1, y_2) - v(|y_1| + |y_2|) & \leqslant d(f(y_1), f(y_2)) \\ & \leqslant \lambda d(y_1, y_2) + v(|y_1| + |y_2|) \\ d(y', f(Y)) \leqslant v(|y'|), & \end{cases}$$

where $|\cdot|$ denotes the distance to base-point.

Large-scale geometries of Heintze spaces

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$$Y, Y' \text{ isometric}$$
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Question

Can (2), or even (2) and (3) be reversed? If not, how? Positive partial answers can be obtained through the search of QI or SBE **invariants.**

Quasiisometries and the boundary

Let $t \geqslant 1$. A pair of subsets (a, a^+) of a quasimetric space is a t-**ring** if there is a ball B such that $B \subseteq a \subseteq a^+ \subseteq tB$. radius$(B)$ is an **inner radius** and $\tau = \log t$ is called an **asphericity** for (a, a^+). If $a = a^+$, **round set.**

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Theorem (known under some form since the 70s)

Let Y and Y' be Heintze spaces. Assume that there exists a quasiisometry $f : Y \to Y'$. Then $\partial_\infty f$ extends to a homeomorphism $\partial_\infty f : \partial_\infty Y \to \partial_\infty Y'$. Further one can assume that f preserves maps the distinguished points one to another, and $\partial_\infty f : \partial_\infty^* Y \to \partial_\infty^* Y'$ **is quasisymmetric.**

SBEs and the boundary

Let $s_n \to -\infty$. A family of rings (a_n, a_n^+) on $(\partial_\infty^*, \rho)$ with inner radii e^{s_n} and asphericities τ_n is said to have **sublinear asphericity** if $\tau_n \ll |s_n|$.

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Theorem

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2. (P. 2018) SBEs induce sublinearly quasisymmetric homeomorphisms between the boundaries.

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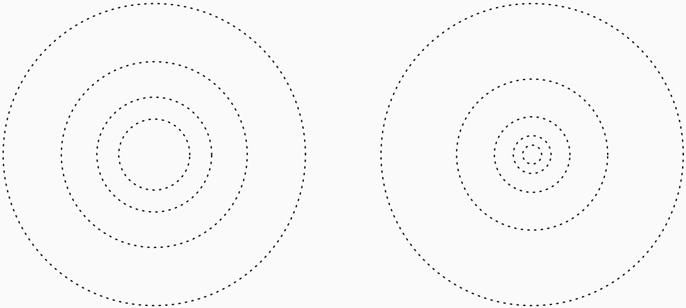

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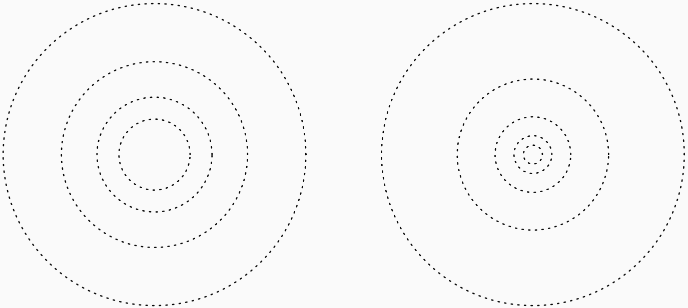

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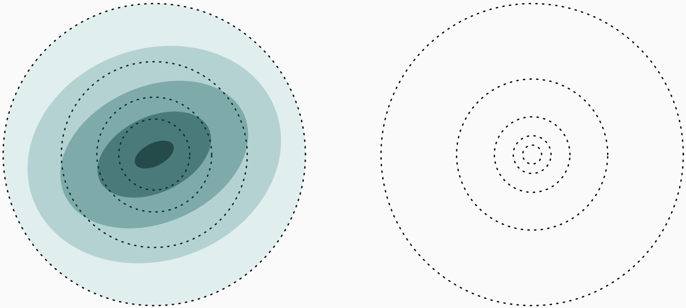

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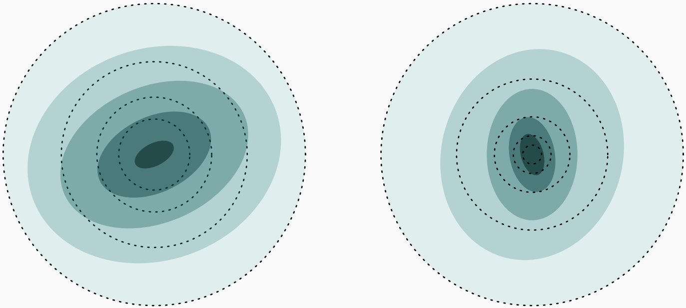

$\tau_n = O(\sqrt{n})$ $\qquad$ $\tau'_n = O(\sqrt{n})$

Figure: Round sets and their images in Euclidean $\mathbf{R}^2$.

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Negative answer to the second part of the Question (even restricted to 3-dim Heintze groups).

A method to produce sublin-q.s. homeos (I)

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An infinite rooted binary tree,

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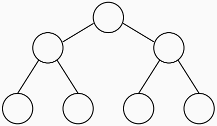

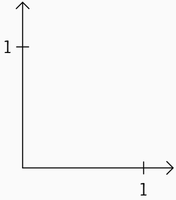

A method to produce sublin-q.s. homeos (I)

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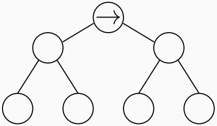

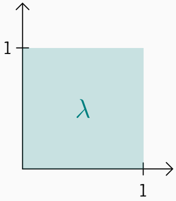

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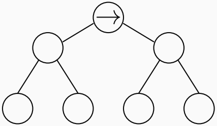

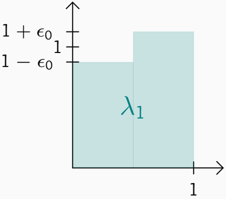

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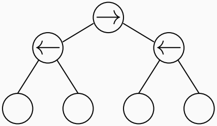

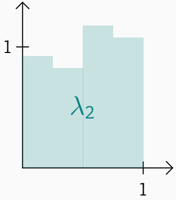

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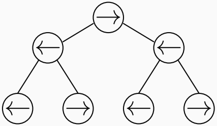

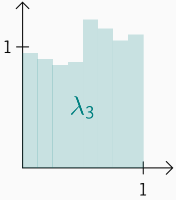

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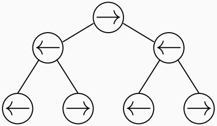
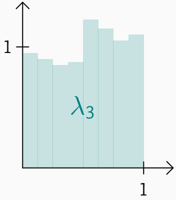


$$M = \lim_n \lambda_n$$



A method to produce sublin-q.s. homeos (II)

2nd step: Take the primitive $\phi : [0,1] \to [0,1]$ in the distributional sense.

ϕ is not absolutely continuous. The derivative is λ-a.e. 0. The modulus of continuity deviates sublinearly from that of a Lipschitz function:
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Proposition

ϕ and Φ are sublinearly quasisymmetric. The asphericity distorsions at scale s for ϕ and Φ are bounded by $(\sum_{n<\log_2 s} \epsilon_n)$ (in fact they are a.e. much lower).

INVARIANTS

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Self-similar space	**Sublinear conformal dimension**
$\mathbb{R}^2$ with scalar or unipotent δ	2
$\mathbb{R}^2$ with $\delta = \mathrm{diag}(1, \mu)$	$1 + \mu$
General (nilpotent)	**trace** of the generator of dilations δ

Large-scale classifications of 3-dim Heintze spaces

Csq of Xie 2011 and of Carrasco Piaggio 2014

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With the sublinear conformal dimension

The 3-dimensional Heintze spaces Y and Y' are SBE if and only if

1. Either $\mathrm{Isom}(Y)$ and $\mathrm{Isom}(Y')$ are isomorphic.
2. Or Y and Y' are isometric to left-invariant Riemannian metrics on G and G'.

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One can define **functions of locally bounded** *p*-**energy** $\mathcal{W}^p_{\mathrm{loc}}$; if φ is a sublin-q.s. homeo then $\mathcal{W}^p_{\mathrm{loc}}(\Omega) \overset{\sim}{\to} \mathcal{W}^p_{\mathrm{loc}}(\varphi^{-1}\Omega)$ for Ω an open in the target. $\mathcal{W}^p_{\mathrm{loc}}(\Omega)$ is a Fréchet algebra whose **spectrum** is a quotient of Ω, the largest space of leaves that it separates.

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Work in progress

$\mathbf{R} \ltimes_{\delta_1} \mathbf{R}^n$ and $\mathbf{R} \ltimes_{\delta_2} \mathbf{R}^n$ with invariant metrics are SBE if and only if $\chi_{\delta_1} = \chi_{\delta_2}$.

General Heintze spaces

Theorem (Heintze 1974)

Every Heintze space is the left-invariant Riemannian metric of a $\mathbf{R} \ltimes_\delta N$ where N is a connected nilpotent Lie group and δ is a derivation of its Lie algebra with positive eigenvalues.

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Conjecture (Hamenstädt 1980s) — known for Carnot type (Pansu 1989)

If two Heintze spaces are quasiisometric then the underlying groups under purely real form are isomorphic.

Cornulier reduction

Theorem (Cornulier 2008)

Let N be a nilpotent Lie group. Let δ_1, δ_2 be Heintze derivations with semisimple parts σ_1, σ_2. If $G_1 = \mathbf{R} \ltimes_{\sigma_1} N$ and $G_2 = \mathbf{R} \ltimes_{\sigma_2} N$ are isomorphic then $G_1 = \mathbf{R} \ltimes_{\delta_1} N$ and $G_2 = \mathbf{R} \ltimes_{\delta_2} N$ with left-invariant metrics are SBE.

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1. Yes in the 3-dimensional case.
2. Ongoing work to a positive answer for abelian N.

Overview

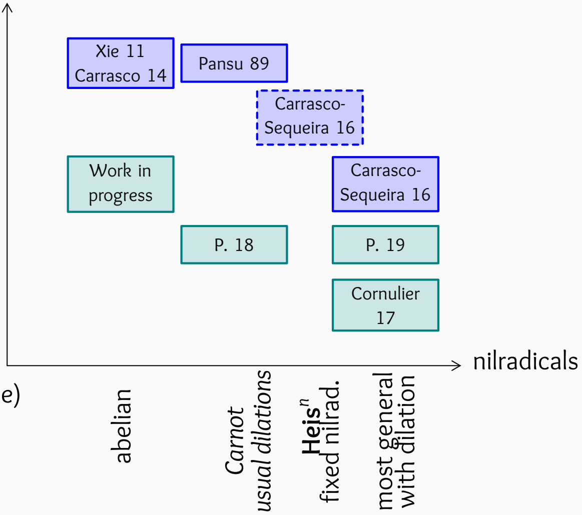
Isomorphic

same Jordan form of δ

same **char poly** of δ

same **trace** of δ

same dimension



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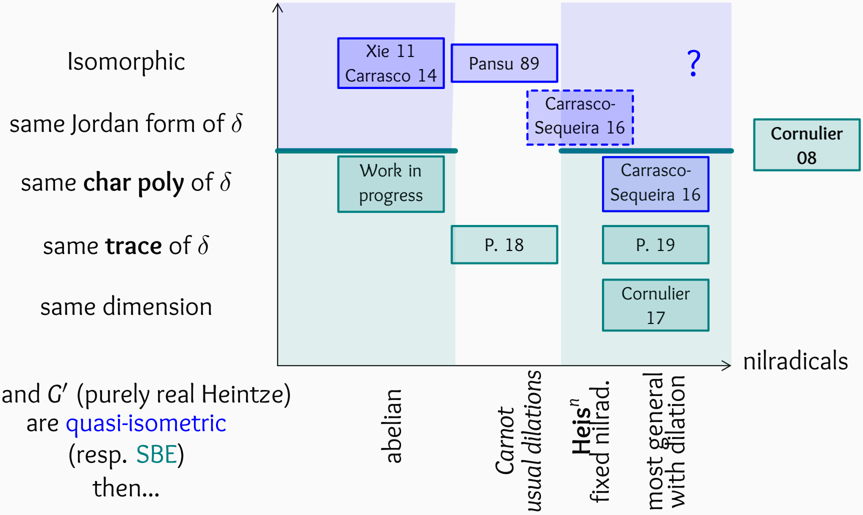

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Thank you for your attention!