SUBLINEAR BILIPSCHITZ EQUIVALENCE AND THE QUASIISOMETRIC CLASSIFICATION OF SOLVABLE LIE GROUPS

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Abstract. We prove a product theorem for sublinear bilipschitz equivalences which generalizes the classical work of Kapovich, Kleiner and Leeb on quasiisometries between product spaces. We employ our product theorem to distinguish up to quasiisometry certain families among these groups which share the same dimension, conedimension and Dehn function; actually we do this by distinguishing them up to sublinear bilipschitz equivalence, which is slightly stronger. Especially, we recover the fact, recently obtained by Bourdon and Rémy with different groups, that there exists uncountably many quasiisometry classes of indecomposable, non-unimodular, higher rank solvable Lie groups.

Contents

1. Introduction

This study is motivated by the quasiisometric classification of connected Lie groups. Such a classification would be complete if one could classify the completely solvable Lie groups up to quasiisometry, as every connected Lie group G is quasiisometric to a completely solvable[1] group $\rho_0(G)$, called the trigshadow of G [Cor08].

Cornulier conjectured that two quasiisometric completely solvable Lie groups should be isomorphic [Cor18, Conjecture 19.113]. This is currently open even within the smaller class of simply connected nilpotent groups.

The process of going from G to $\rho_0(G)$ is a reduction procedure. Cornulier went further in the reduction procedure and defined two subclasses of completely solvable Lie groups, $(\mathcal{C}_1)$ and $(\mathcal{C}_\infty)$, such that every connected Lie group G is $O(\log)$-bilipschitz equivalent to some group $\rho_1(G)$ in $(\mathcal{C}_1)$, and $O(u)$-bilipschitz equivalent to some group $\rho_\infty(G)$ in $(\mathcal{C}_\infty)$ for some explicit sublinear function u depending on G (one has $(\mathcal{C}_\infty) \subset (\mathcal{C}_1)$,

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[1]The completely solvable Lie groups are the closed subgroups of the upper triangular real matrix group; they are also called real-triangulable, or split-solvable.

$\rho_\infty \circ \rho_1 = \rho_\infty$ and $\rho_1 \circ \rho_0 = \rho_1$) [Cor11]. We will recall the definition of $O(u)$-bilipschitz equivalence and Cornulier's reductions farther; let us only specify here that in this language, quasiisometry is $O(1)$-bilipschitz equivalence, that $O(\log)$ and $O(u)$-bilipschitz equivalence are weaker than quasiisometry, and that when G is nilpotent $\rho_\infty(G)$ is the graded nilpotent group associated to the lower central filtration of G, which is known to be a quasiisometry invariant by the work of Pansu.

Theorem 1.1 ([Pan83, Pan89])**.** *Let G and G' be quasiisometric simply connected nilpotent Lie groups. Then $\rho_\infty(G)$ and $\rho_\infty(G')$ are isomorphic.*

Our purpose here is to demonstrate that the reduction procedure, in the present case at the level of ρ_1, has applications to the quasiisometric classification in the class of Lie groups of exponential growth, which is disjoint from that of nilpotent groups. This stems from the following fact, relying on [Cor11]. Since the group $\rho_1(G)$ in the class $(\mathcal{C}_1)$ have less complicated structure than G, it can happen that it splits in a direct product while G did not. Under certain assumptions on the factors, we can then apply a suitable generalization of the Kapovich-Kleiner-Leeb theorems on quasiisometries between products [KKL98], in order to rule out the existence of an $O(u)$-equivalence (hence of a quasiisometry) between G and G' when $\rho_1(G)$ and $\rho_1(G')$ split as non-isomorphic direct products. This is the strategy of Theorem A and Theorem C below.

1.A. **Sublinear bilipschitz equivalence and products.** For a pointed metric space (X, x_0) we denote by $|x| = d(x, x_0)$ the *size of* $x \in X$. A *sublinear function* is any real function $u\colon \mathbf{R}_{\geqslant 0} \to \mathbf{R}_{\geqslant 0}$ such that $\lim_{r\to\infty} \frac{u(r)}{r} = 0$. We often write $u(x)$ instead of $u(|x|)$; this should not cause any confusion. Given two maps f and g between pointed metric spaces and a sublinear function u, we say that f and g are $O(u)$-close if $d\big(f(x), g(x)\big) = O\big(u(x)\big)$.

Definition 1.2 (Commuting up to sublinear error)**.** Let n be a positive integer. Let (X_i) and (Y_i) be families of pointed metric spaces, $i = 1, \ldots, n$. We say that the diagram

$$\begin{tikzcd} \prod_{i=1}^n X_i \arrow[r, "\phi"] \arrow[d, "\pi_i"] & \prod_{i=1}^n Y_i \arrow[d, "\pi_i"] \\ X_i \arrow[r, "\phi_i"] & Y_i \end{tikzcd}$$

commutes up to sublinear error if there exists a sublinear function u such that $\phi_i \circ \pi_i$ and $\pi_i \circ \phi$ are $O(u)$-close for all i.

The precise definitions for the following theorem will be given in Section 2. Key examples of spaces of coarse type I are simply connected Riemannian manifolds with sectional curvature bounded above by a negative constant; examples of spaces of coarse type II are irreducible Riemannian symmetric spaces of noncompact type and of higher rank.

Theorem A. *Let $X = M \times \prod_{i=1}^n X_i$ and $Y = N \times \prod_{i=1}^m Y_j$ be product metric spaces. Assume the metric spaces X_i and Y_j are of coarse type I or II in the sense of Kapovich, Kleiner and Leeb [KKL98], and M and N are geodesic metric spaces with asymptotic cones homeomorphic to $\mathbf{R}^p$ and $\mathbf{R}^q$ respectively. Let u be a subadditive sublinear function, and let $\phi\colon X \to Y$ be an $O(u)$-bilipschitz equivalence. Then $p = q$, $n = m$, and there exists a bijection $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ and, for every $i \in \{1, \ldots, n\}$, an*

$O(u)$-bilipschitz equivalence $\phi_i : X_i \to Y_{\sigma(i)}$ such that the diagram

$$\begin{tikzcd} M \times \prod_{i=1}^n X_i \arrow[r, "\phi"] \arrow[d, "\pi_i"] & N \times \prod_{i=1}^n Y_i \arrow[d, "\pi_{\sigma(i)}"] \\ X_i \arrow[r, "\phi_i"] & Y_{\sigma(i)} \end{tikzcd}$$

commutes up to sublinear error. Moreover, this error is in the class $O(u)$.

Remark 1.3. In the statement above, we do not explicitly specify how the distance on X and Y is built from that of the factors. In the proof we work with the ℓ^2 (or Pythagorean product) metric, which is more natural when the factors are Riemannian. However, the theorem applies if one considers any distances on X and Y quasiisometric to the ℓ^2 distances, e.g. the ℓ^1 distance.

Remark 1.4. When p or q vanish, the conclusion of Theorem A can be stated in the following way: ϕ is $O(u)$-close to the composition of a map of the form

$$(x_1, \ldots, x_n) \mapsto (\phi_1(x_1), \ldots, \phi_n(x_n)),$$

where the ϕ_i are $O(u)$-bilipschitz equivalences, and a map permuting the factors.

Theorem A generalizes [KKL98, Theorem B], which is the case $u = 1$. We state a first application below, which uses some of the results of [Pal20a, Gra23]. By pluriisometry between symmetric spaces we mean that we allow a rescaling in each factor of the de Rham decomposition.

Corollary B. *Let X and Y be two Riemannian globally symmetric spaces with no compact factors. Let u be a subadditive sublinear function. If X and Y are $O(u)$-bilipschitz equivalent, then X and Y are pluriisometric.*

Before stating the second application below, we need to recall Cornulier's ρ_1 reduction (see §3.A for a more comprehensive account). Given a completely solvable group S we denote by $\mathrm{R}_{\exp} S$ the smallest normal subgroup such that $S/\mathrm{R}_{\exp} S$ is nilpotent; this is also the intersection of the descending central series of S. The homomorphism $\alpha\colon S \to \mathrm{Aut}(\mathrm{R}_{\exp} S) = \mathrm{Aut}(\mathrm{Lie}(\mathrm{R}_{\exp} S))$ determined by the Adjoint action of S is algebraic over $\mathbf{R}$, and can be decomposed into $\alpha = \alpha_\sigma \alpha_\nu$, where α_σ is valued in a diagonal $\mathbf{R}$-torus of $\mathrm{Aut}(\mathrm{Lie}(\mathrm{R}_{\exp} S))$, and α_ν is valued in the unipotent radical of $\mathrm{Aut}(\mathrm{Lie}(\mathrm{R}_{\exp} S))$. Since $\mathrm{R}_{\exp} S$ is nilpotent (it is contained in $[S, S]$), it sits in $\ker \alpha_\sigma$, so that α_σ defines a homomorphism $S/\mathrm{R}_{\exp} S \to \mathrm{Aut}(\mathrm{R}_{\exp} S)$, that we still denote α_σ.

Definition 1.5. Let S, α_σ and α_ν be as above. One defines $\rho_1(S)$ as the group $\mathrm{R}_{\exp} S \rtimes_{\alpha_\sigma} S/\mathrm{R}_{\exp} S$. We say that S is in the class $(\mathcal{C}_1)$ if $S \simeq \rho_1(S)$, that is, if $\mathrm{R}_{\exp} S$ is split and $\alpha_\nu = 1$. (In [Cor11], $\rho_1(S)$ is denoted S_1.)

Theorem 1.6 (Cornulier, [Cor11])**.** *Let S be a completely solvable group. Then S and $\rho_1(S)$ are $O(\log)$-bilipschitz equivalent.*

Combining this theorem with our product theorem, and using some further previous results on sublinear bilipschitz equivalences, we obtain the following.

Theorem C. *Let S and S' be two quasiisometric completely solvable groups. Assume that*

$$\rho_1(S) \simeq \mathbf{R}^n \times P \times H_1 \times \cdots \times H_m \quad \textit{and} \quad \rho_1(S') \simeq \mathbf{R}^{n'} \times P' \times H_1' \times \cdots \times H_{m'}'$$

for some $n, m, n', m' \geqslant 0$, where

(1) *$P = AN$ and $P' = A'N'$ are maximal completely solvable subgroups in semisimple groups $G = KAN$ and $G' = K'A'N'$ respectively.*
(2) *For $i = 1, \ldots, m$, H_i has a left-invariant Riemannian metric that is negatively curved, and an abelian derived subgroup; same assumption for H'_j, $j = 1, \ldots, m'$.*

Then, $\rho_1(S)$ and $\rho_1(S')$ are isomorphic.

The case $S = P$ is equivalent to the former Corollary B while the case where S is equal to a single factor H_1 is the main theorem of [Pal20b], used in the proof.

A more sophisticated (and slightly more general) version of Theorem C will be given in Theorem 3.16; in particular, the geometric assumption on curvature in (2) can be reformulated in terms of the structure of H with the help of Heintze's theorem [Hei74].

Theorems A and C allow us to distinguish between several families of completely solvable groups up to quasiisometry. The following is an example of this strategy in dimension 4.

Example 1.7. Let $\alpha \in (0, 1)$. The groups $S = G^0_{4,9}$ and $S' = \mathbf{R} \times G^\alpha_{3,5}$ (the names are from the classification in [Mub63]) are the four dimensional, completely solvable Lie groups whose respective Lie algebras $\mathfrak{g}^0_{4,9}$ and $\mathbf{R} \times \mathfrak{g}^\alpha_{3,5}$ are spanned by $e_1, \ldots, e_4$, subject to the following nonzero brackets:

$$\begin{aligned} \mathfrak{g}^0_{4,9}: &\quad [e_4, e_1] = e_1, [e_4, e_2] = e_2, [e_2, e_3] = e_1 \\ \mathbf{R} \times \mathfrak{g}^\alpha_{3,5}: &\quad [e_4, e_1] = e_1, [e_4, e_2] = \alpha e_2. \end{aligned}$$

It follows from Theorem C that S and S' are not quasiisometric. Here, $\rho_1(S)$ splits as a direct product, that is $\mathbf{R} \times P$ where P is the maximal completely solvable subgroup of $\mathrm{SO}(3, 1)$. See section 3.B for a detailed proof of this example.

Remark 1.8. The parameter α in the definition of $G^\alpha_{3,5}$ can actually be taken in $(-1, 1) \setminus \{0\}$; the two limit cases $\alpha = -1$ and $\alpha = 1$ give the groups $\mathbf{R} \times \mathrm{Sol}_3$ and $\mathbf{R} \times P$ respectively, where P is as above. A direct inspection of the asymptotic cone suffice to tell $G^0_{4,9}$ apart from the groups $\mathbf{R} \times G^\alpha_{3,5}$ when $\alpha < 0$ and from $\mathbf{R} \times \mathrm{Sol}_3$. As the group $\mathbf{R} \times P$ bears a symmetric metric with a Euclidean factor, one can describe the finitely generated groups quasiisometric to $\mathbf{R} \times P$ [KL01]; however, due to the presence of the Euclidean factor we do not know whether similar methods can describe the Lie groups quasiisometric to $\mathbf{R} \times P$; we raise this question in Appendix A.

For completely solvable groups, primary quasiisometry invariants are the cone dimension (that is, the covering dimension of the asymptotic cone) and the Dehn function. In a subsequent paper [GP], we use the wide range of tools developed in [CT17] by Cornulier and Tessera in order to determine the Dehn function of all completely solvable groups of exponential growth and dimension up to 5. In Corollary 4.2 below we will summarize the contribution of our strategy to the quasiisometric classification of these groups. More precisely, Corollary 4.2 lists the 5-dimensional groups which share the same cone dimension and Dehn function, but for which Theorem C implies that they are nonetheless quasiisometrically distinct. We recover a fact already obtained recently (in a more specific form that we explain below) by Bourdon and Rémy [BR23].

Corollary D. *There exists uncountably many quasiisometry classes of indecomposable, non-unimodular, completely solvable Lie groups with quadratic Dehn function.*

The family of pairwise non-isomorphic groups we consider in order to deduce this corollary are named $G^{1,\beta}_{5,19}$ with $\beta \in (0, +\infty)$ in [Mub63] while Bourdon and Rémy's family is $G^{\alpha,1-\alpha}_{5,33}$ with $\alpha \in [1/2, +\infty)$ (it is indecomposable for $\alpha \neq 1$). All these groups have dimension 5 and cone dimension 2, but the groups considered by Bourdon and Rémy have left-invariant proper CAT(0) metrics (a condition known to imply a quadratic isoperimetric inequality, see e.g. [Wen05]), while the groups in the $G_{5,19}^{1,\beta}$ family do not have such metrics, as can be checked using Azencott and Wilson's criterion [AW76]. Note that Peng obtained uncountably many quasiisometry classes among the completely solvable groups [Pen11a, Pen11b]. The groups in the $G_{5,19}^{1,\beta}$ family fall outside of the CAT(0) groups and those considered by Peng, which are unimodular.

In order to formulate our final question, let us call a group of type NPC a completely solvable group with a CAT(0) left-invariant proper Riemannian metric, as characterized in [AW76]. Examples of groups of type NPC are provided by the AN subgroups of (higher rank) semisimple Lie groups $G = KAN$, but this family is considerably more vast; in dimension 5 already, it includes the uncountable family studied by Bourdon and Rémy and mentionned above.

Question 1.9. In Theorem A, can one allow the factors Y_j to be taken among non-abelian, indecomposable groups of type NPC?

The question can equally be raised in the weaker form, for the original version of the Kapovich-Kleiner-Leeb product theorem instead of Theorem A. Among all the examples in dimension 5 that we reviewed in Section 4, none of Cornulier's ρ_1-reductions (which are $O(\log)$-bilipschitz equivalences) could rule out a positive answer to Question 1.9.

1.B. **Organization of the paper.** In Section 2 we prove Theorem A. The sublinear adaptation is mainly in Section 2.C, resulting in Proposition 2.11. The conclusion of Theorem A is in Section 2.D, and done exactly as in [KKL98].

Section 3 introduces some of the theory of completely solvable Lie groups and the significance of sublinear bilipschitz equivalence (SBE) to this theory. Corollary B and Theorem C are proved in Section 3.C. We actually deduce both of these statements from Theorem 3.16, although Corollary B could be derived directly. Theorem 3.16 is stated in terms of diagonal Heintze groups, which are the Gromov-hyperbolic groups of class $(\mathcal{C}_1)$ and thus are of coarse type I. Finally, in Section 4 we formulate and prove Corollary 4.2 about the families of 5-dimensional Lie groups that can be distinguished up to quasiisometry using our product theorem; corollary D appears as a byproduct of this study.

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2. The product Theorem A

In this section we prove Theorem A. The argument is exactly that of Theorem B of Kapovich, Kleiner and Leeb in [KKL98]. Most of their proof works *verbatim* also for the case of SBE. In particular, this is true for sections 3, 4, 5 and 6 of their work. In these sections Kapovich, Kleiner and Leeb deal with properties of homeomorphisms between asymptotic cones, and to this end it makes no difference whether these homeomorphisms arise as the cone maps of quasiisometries or SBE.

The only new ingredient we need is a generalization of Section 2 of [KKL98], proving that whenever the cone maps preserve the product structure, the original maps coarsely preserve the product structure. Our Proposition 2.11 generalizes [KKL98, Proposition 2.6], and our Lemma 2.17 generalizes [KKL98, Lemma 2.7].

The key assumption in this section is that the cone maps decompose as products. A sufficient condition for this is that the original factors are of *coarse type* I and II [KKL98, Theorem 5.1]. This is the reason for the hypothesis of Theorem A. We only briefly sketch the definitions because in essence all we need to know about these spaces is the property that their cone maps preserve the product structure. We refer to [KKL98, Section 3] for more details.

Definition 2.1. Let X be a geodesic metric space. We say that $p, q \in X$ are in the same *leaf* if there is a continuous path $\gamma : I \to X$ joining p and q such that every other continuous path which joins p and q contains γ. Being in the same leaf is an equivalence relation, and every leaf of X is a closed convex subset. The space X is called *type* I if all its leaves are geodesically complete trees which branch everywhere.

Definition 2.2. A geodesic metric space is of *type* II if it is a thick irreducible Euclidean building with transitive affine Weyl group of rank $r \geq 2$.

Definition 2.3. A space X is of *coarse type* I (resp. II) if every asymptotic cone of X is of type I (resp. II).

Symmetric spaces of noncompact type and higher rank are coarse type II [KL97], and so are Euclidean buildings of higher rank and cocompact affine Weyl group. Examples of spaces of coarse type I are Gromov hyperbolic groups whose Gromov boundary contains at least 3 points. These examples are enough for all our purposes in this paper, which arise in Section 3 in the proofs of Corollary B and Theorem 3.16.

We start with some preliminaries on SBE and asymptotic cones, then prove Proposition 2.11 and its variant Proposition 2.18. In Section 2.D we deduce Theorem A from these propositions and from some further results of [KKL98].

2.A. **Preliminaries.** Let X and Y be metric spaces. After fixing $x_0 \in X$ and $y_0 \in Y$, we denote by $|\cdot|$ the distance to the respective basepoints in X and Y. We denote $|x_1| \vee |x_2| := \max\{|x_1|, |x_2|\}$.

Let $u \colon \mathbf{R}_{\geqslant 0} \to \mathbf{R}_{\geqslant 1}$ be a sublinear function, that is,

$$\lim_{r \to +\infty} \frac{u(r)}{r} = 0.$$

We say that u is admissible if it is nondecreasing and $\limsup u(2r)/u(r) < +\infty$. Often, we will also assume that u is subadditive, that is, $u(r_1 + r_2) \leqslant u(r_1) + u(r_2)$ for all $r_1, r_2 \geqslant 0$. This implies admissibility.

Definition 2.4. Let X, Y, x_0, y_0 and u be as above. Let $L \geqslant 1$. We say that $f \colon X \to Y$ is

- (L, u, x_0)-Lipschitz if for every $x, x' \in X$, $d\big(f(x), f(x')\big) \leqslant L d(x, x') + u(|x| \vee |x'|)$
- (L, u, x_0)-expansive if $L^{-1} d(x, x') - u(|x| \vee |x'|) \leqslant d\big(f(x), f(x')\big)$
- (u, y_0)-surjective if for every y in Y, there is $x \in X$ such that $d\big(y, f(x)\big) \leqslant u(|y|)$.

We say that f is a (L, u)-bilipschitz embedding if it is (L, cu, x_0)-Lipschitz and (L, cu, x_0)-expansive for some $c \geqslant 0$. If f is additionally (u, y_0)-surjective for some y_0, then for all $y_0' \in Y$ there is $c' > 0$ such that it is $(c'u, y_0')$-surjective; in this case, we say that f realizes a $\big(L, O(u)\big)$-bilipschitz equivalence, or for short, a $O(u)$-bilipschitz equivalence between X and Y. When no reference is made to L and u we will call a (L, u)-bilipschitz embedding a sublinear bilipschitz embedding.

2.B. Going through cones.

Definition 2.5. Let X be a metric space. Let $(\sigma_n)_{n\in\mathbf{N}}$ be a sequence of positive real numbers. Let $(x_n)_n \in X^{\mathbf{N}}$. We call precone of X with data $\big((x_n)_n, \sigma\big)$ the set of sequences

$$\mathrm{Precone}(X, (x_n)_n, \sigma) = \{(x'_n)_n \in X^{\mathbf{N}} : \exists M \in [0, +\infty), \forall n \in \mathbf{N}, d(x_n, x'_n) \leqslant M\sigma_n\}.$$

Given a nonprincipal ultrafilter ω over $\mathbf{N}$ we denote by $\mathrm{Cone}\big(X, (x_n)_n, (\sigma_n)_n, \omega\big)$ the quotient of the set $\mathrm{Precone}(X, (x_n)_n, \sigma)$ by the relation

$$(x'_n)_n \sim (x''_n)_n \iff \lim_{n\to\omega} \frac{d(x'_n, x''_n)}{\sigma_n} = 0$$

equipped with the distance

$$d_\omega([x'], [x'']) = \lim_{n\to\omega} \frac{d(x'_n, x''_n)}{\sigma_n}.$$

If σ goes to $+\infty$ we call this an asymptotic cone. When it is not relevant to mention all the remaining data, we will denote an asymptotic cone of X by X_ω.

Definition 2.6. Given a metric space X, a triple $\big(X, (x_n)_n, (\sigma_n)_n\big)$ is called u-admissible if for some (any) $v \in O(u)$

$$\lim_{i\to+\infty} \frac{v(x_i)}{\sigma_i} = 0.$$

Note that we did not specify a basepoint when writing $v(x_i)$ in the above definition; such a choice turns out to have no influence on the notion of u-admissibility.

Lemma 2.7 below is a basic fact, usually applied to cones with fixed base point. We will need asymptotic cones with moving base points, which are slightly less common. In the statement below, given a map $f : X \to Y$ we still denote f the map between the power sets $f : X^{\mathbf{N}} \to Y^{\mathbf{N}}$.

Lemma 2.7. *Let $L \geqslant 1$. Let u be an admissible function, let X and Y be metric spaces and let $f : X \to Y$ an (L, u)-bilipschitz equivalence. Assume $\big(X, (x_n)_n, (\sigma_n)_n\big)$ is u-admissible. Then*

$$f\big(\mathrm{Precone}(X, x_n, \sigma_n)\big) \subseteq \mathrm{Precone}\left(Y, f(x_n), \sigma_n\right),$$

and for any nonprincipal ultrafilter ω, a quotient map

$$\mathrm{Cone}\left(f, (x_n)_n, (\sigma_n)_n, \omega\right) \colon \mathrm{Cone}\left(X, (x_n)_n, (\sigma_n)_n, \omega\right) \to \mathrm{Cone}\left(Y, \big(f(x_n)\big)_n, (\sigma_n)_n, \omega\right)$$

is well-defined and an L-bilipschitz embedding. Moreover, if f is additionally $O(u)$-surjective, then $\mathrm{Cone}\left(f, (x_n)_n, (\sigma_n)_n, \omega\right)$ is a bilipschitz homeomorphism.

Proof. This follows from [Pal22, Proposition 6]. □

Remark 2.8. Lemma 2.7 means that when the asymptotic cones are u-admissible, the cone maps of $O(u)$-bilipschitz equivalences have the same property as cone maps of quasiisometries. We therefore feel free to use results from [KKL98] whose hypothesis require some property to hold for *all* asymptotic cones, while we can only guarantee it for u-admissible cones. We do not reformulate or reprove these statements in our work, and the reader should only note that whenever we use cones, they are u-admissible.

Definition 2.9. Let X, Y be two metric spaces, $v : \mathbf{R}_{\geqslant 0} \to \mathbf{R}_{\geqslant 1}$. Two maps $f_1, f_2 : X \to Y$ are *at distance* v if for all $x \in X$ one has $d_Y\big(f_1(x), f_2(x)\big) \leq v(|x|)$. The maps are called *sublinearly close* if f_1, f_2 are at distance v for some sublinear function v.

2.C. SBE Preserve Product Structure.

Definition 2.10. A bilipschitz homeomorphism $F : \prod_{i=1}^n X_i \to \prod_{i=1}^m Y_i$ is said to *preserve the product structure* if up to reindexing the factors and ignoring factors not in the image of F, there are bilipschitz homeomorphisms F_i such that for each i, the following diagram commutes:

$$\begin{tikzcd} \prod_{i=1}^n X_i \arrow[r, "F"] \arrow[d, "\pi_i"] & \prod_{i=1}^n Y_i \arrow[d, "\pi_i"] \\ X_i \arrow[r, "F_i"] & Y_i \end{tikzcd}$$

Throughout this section, the standing assumptions on the map ϕ and spaces X, Y are those of Proposition 2.11.

Proposition 2.11 (SBE version of Proposition 2.6 in [KKL98])**.** *Suppose $\phi : X = \prod_i X_i \to Y = \prod Y_j$ is an (L, u)-bilipschitz equivalence. Assume that for all u-admissible cones X_ω and Y_ω of X and Y, the ultralimit $F = \mathrm{Cone}(\phi) : X_\omega \to Y_\omega$ preserves the product structure. Then:*

(1) Up to reindexing the factors, there are (L', u_i)-bilipschitz equivalences $\phi_i : X_i \to Y_i$ with $u_i \in O(u)$.

(2) The diagram

$$\begin{tikzcd} \prod_{i=1}^n X_i \arrow[r, "\phi"] \arrow[d, "\pi_i"] & \prod_{i=1}^n Y_i \arrow[d, "\pi_i"] \\ X_i \arrow[r, "\phi_i"] & Y_i \end{tikzcd}$$

commutes up to $O(u)$ sublinear error. In particular, ϕ is sublinearly close to a product of $O(u)$-bilipschitz equivalences.

Remark 2.12. One notable difference between our proofs and those of the product theorem for quasiisometries concerns the 'nontranslatability' condition [KKL98, Definitions 2.2, 2.3]. In Section 2 of [KKL98] the nontranslatability of the factors was used both implicitly in the assumption that cone maps preserve product structure, and then again explicitly in order to prove that certain pairs of quasi-isometries are at uniform bounded distance. In contrast, we use the nontranslatability only through its implicit use as part of the product decomposition of cone maps between spaces of types I and II. We use this result as a black box from [KKL98] and therefore do not need to explicitly define or discuss nontranslatability. In particular our hypothesis in Proposition 2.11 is formally (and perhaps essentially) weaker than its quasiisometry predecessor [KKL98, Proposition 2.6].

The proof requires some sublinear adaptations to the notions that appear in [KKL98].

Definition 2.13. Let $d > 0$ be a positive constant and $d_0 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 1}$ a function. A pair of points $x, x' \in X$ is called *d-separated* if $d_X(x, x') \geqslant d$, and *d_0-separated* if $d(x, x') \geqslant d_0(|x| \vee |x'|)$.

Definition 2.14. A pair of points $x, x' \in X = \prod_i X_i$ is called *i-horizontal* if x and x' agree on all $m \neq i$ coordinates.

Definition 2.15. Fix $L_1 > L$, $\epsilon < L_1^{-1}$, $i, j \in \{1, \dots, n\}$. A pair of i-horizontal points $x, x' \in X$ is called:

(1) *j-compressed* if

$$\frac{d_{Y_j}\Big(\pi_{Y_j}\big(\phi(x)\big), \pi_{Y_j}\big(\phi(x')\big)\Big)}{d_X(x,x')} \leq \epsilon$$

(2) *j-uncompressed* if

$$L_1^{-1} \leq \frac{d_{Y_j}\Big(\pi_{Y_j}\big(\phi(x)\big), \pi_{Y_j}\big(\phi(x')\big)\Big)}{d_X(x,x')} \leq L_1$$

(3) *j-semi-compressed* if

$$\epsilon \leq \frac{d_{Y_j}\Big(\pi_{Y_j}\big(\phi(x)\big), \pi_{Y_j}\big(\phi(x')\big)\Big)}{d_X(x,x')} \leq {L_1}^{-1}$$

Two pairs are *j-compatible* if they are simultaneously j-semi/un/compressed, and *j-incompatible* otherwise.

When it is clear from the context, we occasionally do not specify the space in which the distance is computed. Also, we sometimes write π_j instead of π_{X_j} or π_{Y_j}.

Let x, x' be i-horizontal. Notice two trivial facts: a) $d_{X_i}\big(\pi_i(x), \pi_i(x')\big) = d_X\big(x, x'\big)$, and b) the geodesic $[x, x']$ in X joining x and x' is contained in a leaf of X_i. In particular, for each $x'' \in [x, x']$, (x, x'') and (x', x'') are i-horizontal pairs.

Lemma 2.16. *Being j-compressed is transitive along a geodesic, i.e. if x'' lies on a geodesic joining a pair (x, x') of i-horizontal points, and both (x, x'') and (x'', x') are j-compressed, then also (x, x') is j-compressed.*

Proof.

$$\begin{aligned} d_{Y_j}\Big(\pi_j\big(\phi(x)\big), \pi_j\big(\phi(x')\big)\Big) &\leq d_{Y_j}\Big(\pi_j\big(\phi(x)\big), \pi_j\big(\phi(x'')\big)\Big) + d_{Y_j}\Big(\pi_j\big(\phi(x'')\big), \pi_j\big(\phi(x')\big)\Big) \\ &\leq \varepsilon d(x, x'') + \varepsilon d(x'', x') = \varepsilon d(x, x') \end{aligned}$$

The last equality follows from $x'' \in [x, x']$. □

Lemma 2.17 (SBE version of Lemma 2.7 in [KKL98])**.** *There exists a sublinear subadditive function $d_0 \in O(u)$ such that for every fixed $i, j \in \{1, \dots, n\}$, either all d_0-separated i-horizontal pairs are j-compressed or all such pairs are j-uncompressed.*

The existence of d_0 as in Lemma 2.17 in particular implies that d_0-separated i-horizontal pairs are never j-semi-compressed. Our proof follows that of Lemma 2.7 in [KKL98], with the required sublinear adaptations. The meaning of the lemma is that ϕ coarsely preserve the product structure of X and Y. The idea of the proof is that if ϕ fails to do so, this failure is manifested by a sequence of points and constants that give rise to u-admissible asymptotic cones of X and Y in which $\mathrm{Cone}(\phi)$ also fails to preserve the product structure, contradicting the hypothesis.

Proof. Fix $i, j \in \{1, \dots, n\}$.
Step 1. Let $x \in X$. We claim that there is a constant $d_0(x)$ such that for all $d \geq d_0(x)$, all d-separated i-horizontal pairs in the ball $B := B\big(x, 99d\big)$ are simultaneously j-compressed or simultaneously j-uncompressed.

First we show there is $d_0(x)$ such that for $d > d_0(x)$ there are no d-separated i-horizontal pairs in $B(x, 99d)$ that are j-semi-compressed. Assume towards contradiction there were such pairs for all $d > 0$. They give rise to a sequence of radii $d_k \to \infty$ and points $x^k, z^k \in B(x, 99d_k)$ that are i-horizontal, $d(x^k, z^k) = d_k$, with (x^k, z^k) being j-semi-compressed. Consider the asymptotic cone $X_\omega := \mathrm{Cone}_\omega(X, x, d_k)$, and the points $x_\omega := (x^k)_{k\in\mathbf{N}}, z_\omega := (z^k)_{k\in\mathbf{N}} \in X_\omega$. These points agree on the $(X_\omega)_m$-coordinate for all $m \neq i$. The hypothesis on X and Y assures that the cone map $F = \mathrm{Cone}(\phi)$ respects the product structure of X_ω and Y_ω, therefore if $m \neq i$ we have

$$d_{(Y_\omega)_m}\Big(\pi_{(Y_\omega)_m}\big(F(x_\omega)\big), \pi_{(Y_\omega)_m}\big(F(z_\omega)\big)\Big) = 0 \tag{2.1}$$

and so

$$d_{(Y_\omega)}\big(F(x_\omega), F(z_\omega)\big) = d_{(Y_\omega)_i}\Big(\pi_{(Y_\omega)_i}\big(F(x_\omega)\big), \pi_{(Y_\omega)_i}\big(F(z_\omega)\big)\Big). \tag{2.2}$$

Finally, for every $m \in \{1, \dots, n\}$ it holds that $\pi_{(Y_\omega)_m}\big(F(x_\omega)\big) = \lim_\omega \pi_{Y_m}\big(\phi(x^k)\big)$ and similarly for z_ω, yielding

$$\lim_\omega \frac{1}{d_k} d_{Y_m}\Big(\pi_{Y_m}\big(\phi(x^k)\big), \pi_{Y_m}\big(\phi(z^k)\big)\Big) = d_{(Y_\omega)_m}\Big(\pi_{(Y_\omega)_m}\big(F(x_\omega)\big), \pi_{(Y_\omega)_m}\big(F(z_\omega)\big)\Big).$$

Plugging this formula into Equations 2.1 and 2.2 above, we get:

(1) For $j \neq i$ $\lim_\omega \frac{1}{d_k} d_{Y_j}\Big(\pi_{Y_j}\big(\phi(x^k)\big), \pi_{Y_j}\big(\phi(z^k)\big)\Big) = 0$.

(2) For $j = i$, the fact that F is L-bilipschitz and preserves the product structure, together with $d_{X_\omega}(x_\omega, z_\omega) = 1$, implies

$$\lim_\omega \frac{1}{d_k} d_{Y_i}\Big(\pi_{Y_i}\big(\phi(x^k)\big), \pi_{Y_i}\big(\phi(z^k)\big)\Big) \in [L^{-1}, L].$$

The assumption that $0 < \epsilon < L_1^{-1} < L^{-1}$ contradicts the assumption that (x^k, z^k) are j-semi-compressed.

Next, we show that there is $d_0(x)$ such that for every $d > 0$, every two pairs of d-separated i-horizontal points inside $B(x, 99d)$ are j-compatible. This is done exactly as above. If there is no such $d_0(x)$, we obtain sequences of radii $d_k \to \infty$ and points $x^{1,k}, z^{1,k}, x^{2,k}, z^{2,k} \in B(x, 99d_k)$ such that for each $k \in \mathbf{N}$ and $l \in \{1, 2\}$, the pair $(x^{l,k}, z^{l,k})$ is i-horizontal with $d_X(x^{l,k}, z^{l,k}) \in [d_k, 99d_k)$ and $(x^{1,k}, z^{1,k})$ is j-compressed while $(x^{2,k}, z^{2,k})$ is j-uncompressed. We consider the cones $X_\omega := \mathrm{Cone}(X, x, d_k), Y_\omega := \mathrm{Cone}\big(Y, \phi(x), d_k\big)$ and the cone map $F := \mathrm{Cone}(\phi)$. For $l \in \{1, 2\}$, the points $x^l_\omega = (x^{l,k})_{k\in\mathbf{N}}, z^l_\omega = (z^{l,k})_{k\in\mathbf{N}} \in X_\omega$ both lie within the ball of radius 99 around the base point of the cone, and agree on the $m \neq i$ coordinates. Since F respects products there are L-bilipschitz homeomorphisms $F_m : (X_\omega)_m \to (Y_\omega)_m$ for which $F_i \circ \pi_{(X_\omega)_m} = \pi_{(Y_\omega)_m} \circ F$, hence for $m \neq i$ we must have for $l \in \{1, 2\}$

$$d_{(Y_\omega)_m}\Big(\pi_{(Y_\omega)_m}\big(F(x^l_\omega)\big), \pi_{(Y_\omega)_m}\big(F(z^l_\omega)\big)\Big) = 0$$

Since this does not depend on l, we see as in step 1 that for $j \neq i$ for all large enough k, both $(x^{1,k}, z^{1,k})$ and $(x^{2,k}, z^{2,k})$ are j-compressed and in particular compatible. A similar argument shows that these pairs are both j-uncompressed for $j = i$. We conclude that the pairs are j-compatible for all j.

Step 2. We now show that we can choose d_0 to be a sublinear function in $O(u)$. We start by taking, for each $x \in X$, the infimal $\tilde{d}_0(x)$ such that for all $d \geq \tilde{d}_0(x)$, all i-horizontal d-separated pairs in $B(x, 99d)$ are j-compatible. Next define the radial function $d_0(|x|) := \sup_{y:|y|=|x|} \tilde{d}_0(y)$.

A-priori $d_0 : \mathbf{R}_{\geq 0} \to \mathbf{R}_{\geq 0} \cup \{\infty\}$, and we need to show it is in fact real valued. Assume towards contradiction that d_0 is not bounded along some sphere S around x_0, and get points $x^k \in S$ and real numbers $r_k \to \infty$ such that the balls $B(x^k, 10r_k)$ contain two i-horizontal r_k-separated pairs $(x^{1,k}, z^{1,k}), (x^{2,k}, z^{2,k})$ that are j-incompatible. We consider the cone $X_\omega := (X, x_0, r_k)$, based at $x_\omega = (x_0)$. Since $x^k \in S$ we have $d(x^k, x_0) = R$ for some constant R, so $(x^{l,k}), (z^{l,k}) \in X_\omega$. Moreover, the assumption that $(x^{1,k}, z^{1,k})$ are r_k-separated and lie in $B(x^k, 99r_k)$ means that $d_\omega(x^l_\omega, z^l_\omega) \in [1, 198]$. Again, for every k $x^{l,k}$ and $z^{l,k}$ agree on the $m \neq i$ coordinates, hence so do x^l_ω and z^l_ω. As in previous steps, we get a contradiction to the assumption that $(x^{1,k}, z^{1,k}), (x^{2,k}, z^{2,k})$ are j-incompatible.

The function d_0 is therefore a radial real valued function. We show it is $O(u)$. Assume towards contradiction that there is a sequence $x^k \to \infty$ so that $\lim_\omega \frac{u(x^k)}{d_0(x^k)} = 0$. By the construction of d_0 we may take x^k such that each ball $B\big(x^k, 99d_0(x^k)\big)$ contains two $(d_0(x^k) - 1)$-separated i-horizontal pairs that are j-incompatible. The assumption $\lim_\omega \frac{u(x^k)}{d_0(x^k)} = 0$ implies that the cone with moving base points $X_\omega := \mathrm{Cone}_\omega\big(X, x^k, d_0(x^k)\big)$ is u-admissible. By Lemma 2.7 the cone map $F = \mathrm{Cone}(\phi)$ is L-bilipschitz, and so respects products. One gets a contradiction as in step 2.

Step 3. We have established the following situation: there is $d_0 \in O(u)$ such that for every $x \in X$ and every $d \geq d_0(x)$, the property of whether a d-separated i-horizontal pair inside $B = B(x, 99d)$ is j-uncompressed or j-compressed depends only on x, i, j and d, and not on the specific pair (in particular such a pair cannot be j-semi-compressed). We call it the j-type of B. We assume, as we may, that d_0 is subadditive. We can right away conclude that there are no d_0-separated i-horizontal j-semi-compressed pairs at all in X: such a pair (x, x') must admit $d(x, x') \geq d_0(x)$ and therefore for $d = d(x, x') \geq d_0(x)$, x, x' are d-separated and $x' \in B(x, 99d)$. Step 1 shows that (x, x') is not j-semi-compressed.

Our next goal is to lose any restriction on the radius of the balls $B(x, 99d)$. Fix $x \in X$. Our first observation is that the j-type of d_0-separated i-horizontal pairs which involve x depend only on x: they are the type of the ball $B := B\big(x, 99d_0(x)\big)$. Indeed if (x, x') is an i-horizontal pair, one can consider the geodesic $\gamma := [x, x']$. If $x' \in B$ the claim is immediate. In particular the pair $\Big(x, \gamma\big(98d_0(x)\big)\Big)$ has the j-type of B. Therefore for $R_1 := 98d_0(x)$, the ball $B_1 := B(x, 99 \cdot R_1)$ has the j-type of B. We can continue to enlarge the radius of the balls until we find one which contains x', so indeed (x, x') has the j-type of B, which is what we were set out to prove.

We push the above argument a bit further in order to show that any two d_0-separated i-horizontal pairs are j-compatible, regardless of which leaf of X_i they are in. Let (x, x') and (z, z') be two d_0-separated i-horizontal pairs. Assume first that $x, x', z, z' \in X$ are pairwise i-horizontal, so they all lie in the same leaf. Since each X_i can be assumed to have infinite diameter, we can find a point p in the same X_i leaf which is far enough so that (p, w) is d_0-separated for all $w \in \{x, x', z, z'\}$. It is clear that the j-type of all points must be equal to that of $B\big(p, 99d_0(p)\big)$.

This allows us to define the j-type of each X_i-leaf, defined by a choice of $\hat{x} \in \hat{X} := \prod_{m \neq i} X_m$. To finish the proof we need to show that the (i, j)-type of $\hat{x} \in \hat{X}$ depends only on (i, j) and not on the choice of $\hat{x}$. Indeed, fix $\hat{x}, \hat{z} \in \prod_{m \neq i} X_m$, and $\gamma : \mathbf{R}_{\geq 0} \to X_i$ an infinite ray in X_i (we assume X_i is of infinite diameter and geodesic and so by an Arzela-Ascoli argument there exists an infinite geodesic ray).

Consider the points $x^k = \big(\gamma(k), \hat{x}\big), z^k = \big(\gamma(k), \hat{z}\big) \in X$. For all large enough k, (x^0, x^k) and (z^0, z^k) are d_0-separated i-horizontal. For all large enough $k, m \in \mathbb{N}$, all pairs $(x^0, x^k), (x^0, x^m)$ are j-compatible and all $(z^0, z^k), (z^0, z^m)$ are j-compatible.

Consider the cone $X_\omega := (X, x^0, k)$. Let x^0_ω be the base point of X_ω, $x_\omega := (x^k)_k$, $z_\omega := (z^k)_k$. Clearly $x_\omega = z_\omega$ and $d(x^0_\omega, x_\omega) = 1$. We conclude that for all large enough $k, m \in \mathbb{N}$, it holds that (x^0, x^k) and (z^0, z^m) are j-compatible, so $\hat{x}, \hat{z}$ have the same (i,j)-type. The lemma stands proven. $\square$

Proof of Proposition 2.11. Together with Lemma 2.17, the assumption on the product decomposition of the relevant cone maps implies that for every i there is a unique j for which the projection on Y_j is uncompressed for all i-horizontal pairs. We may reindex and assume Y_i is that factor. To ease the notation, we fix from now $i = 1$. The proof is identical for all other i.

The $O(u)$-embeddings. We denote $\hat{X} := \prod_{m=2}^n X_m$, and write points $x \in X$ as a pair $(z, \hat{x}) \in X_1 \times \hat{X}$. Fixing $\hat{x} \in \hat{X}$ we obtain a map $\phi_{\hat{x}} : X_1 \to Y_1$ given by $z \mapsto \pi_{Y_1}\big(\phi(z, \hat{x})\big)$. We prove that these are all $(L_1, u_{\hat{x}})$-bilipschitz embeddings, for $u_{\hat{x}} = u + d_0 + (u + d_0)(|\hat{x}|_{\hat{X}}) \in O(u)$, where $L_1 \geqslant 1$ and $d_0 \in O(u)$ are given by Lemma 2.17. Indeed, let $z, z' \in X_1$ be a two points such that the pair $(z, \hat{x}), (z', \hat{x})$ is d_0-separated. By assumption on X_1 this pair is 1-uncompressed so

$$L_1^{-1} d(z, z') \leqslant d\Big(\pi_{Y_1}\big(\phi(z, \hat{x})\big), \pi_{Y_1}\big(\phi(z', \hat{x})\big)\Big) \leqslant L_1 d(z, z')$$

If the pair $(z, \hat{x}), (z', \hat{x})$ is not d_0-separated, then subadditivity of d_0 gives $d(z, z') = d_{X_1}(z, z') \leqslant d_0(z \vee z') + d_0(\hat{x})$. Recalling that $L_1^{-1} < 1$, this gives a lower bound:

$$L_1^{-1} d(z, z') - d_0(z \vee z') - d_0(\hat{x}) \leqslant 0 \leqslant d\Big(\pi_{Y_1}\big(\phi(z, \hat{x})\big), \pi_{Y_1}\big(\phi(z', \hat{x})\big)\Big).$$

The Pythagorean formula yields the upper bound:

$$d\Big(\pi_{Y_1}\big(\phi(z, \hat{x})\big), \pi_{Y_1}\big(\phi(z', \hat{x})\big)\Big) \leq d\big(\phi(z, \hat{x}), \phi(z', \hat{x})\big) \leq L d(z, z') + u(z \vee z') + u(\hat{x}).$$

We conclude that $\phi_{\hat{x}} : X_i \to Y_i$ is an $(L, u_{\hat{x}})$-bilipschitz embedding.

$O(u)$-surjectivity. We now show that $\phi_{\hat{x}}$ is $v_{\hat{x}}$-quasi-surjective with $v_{\hat{x}} \in O(u)$. For convenience we show it in the case $\hat{x} = \hat{0}$, the base point of $\hat{X} = \prod_{m=2}^n X_m$. Assume towards contradiction that there is a sequence $y_1^n \in Y_1$ for which $\lim_\omega \frac{u(y_1^n)}{d\big(\operatorname{Im}(\phi_{\hat{0}}), y_1^n\big)} = 0$. Consider the points $y^n := (y_1^n, \hat{0}_Y)$, where $\hat{0}_Y$ is the base point of $\hat{Y} := \prod_{m=2}^n Y_m$. The quasi-surjectivity of ϕ implies that there are corresponding $x^n \in X$ for which $d\big(\phi(x^n), y^n\big) \leq u(y^n) = u(y_1^n)$. It is a general fact of sublinear bilipschitz embeddings that x^n and y_1^n admit $\frac{1}{L'}|y_1^n| \leqslant |x^n| \leqslant L'|y_1^n|$ for some $L' > L$ and all large enough y_1^n.

Denote $\sigma_n := d\big(\operatorname{Im}(\phi_{\hat{0}}), y_1^n\big)$, and consider the cones $X_\omega := \operatorname{Cone}(X, x^n, \sigma_n)$, $Y_\omega := \operatorname{Cone}\big(Y, \phi(x^n), \sigma_n\big)$. The previous paragraph implies $\lim_\omega \frac{u(x^n)}{\sigma_n} = 0$ and therefore by Lemma 2.7 the map $F = \operatorname{Cone}(\phi)$ is a bilipschitz homeomorphism. The hypothesis on the factors of X and Y implies that the map F respects the product structure of X_ω and Y_ω, i.e. there are bilipschitz maps F_i for $1 \leqslant i \leqslant n$ such that if $x_\omega, x'_\omega \in X_\omega$ agree on the j-coordinate, then

$$\pi_{(Y_\omega)_j}\big(F(x_\omega)\big) = F_j\big(\pi_{(X_\omega)_j}(x_\omega)\big) = F_j\big(\pi_{(X_\omega)_j}(x'_\omega)\big) = \pi_{(Y_\omega)_j}\big(F(x'_\omega)\big) \tag{2.3}$$

Set $x_\omega := (x^n)_{n \in \mathbf{N}}$, $x^0_\omega := (x_1^n, \hat{0}_{\hat{X}})_{n \in \mathbf{N}}$ (where x_1^n is the first coordinate of x^n). These two points in X_ω agree on the first coordinate, thus

$$\Big(\pi_{Y_1}\big(\phi(x_1^n, \hat{0})\big)\Big)_{n \in \mathbf{N}} = \pi_{(Y_\omega)_1}\big(F(x^0_\omega)\big) = \pi_{(Y_\omega)_1}\big(F(x_\omega)\big) = \pi_{(Y_\omega)_1}\Big(\big(\phi(x^n)\big)_{n \in \mathbf{N}}\Big)$$

(Equalities are in $(Y_\omega)_1$). On the other hand $d\big(\phi(x^n),(y_1^n,\hat{0}_Y)\big)\leqslant u(y_1^n)$ and $\lim_\omega \frac{y_1^n}{\sigma_n}=0$, so $\big(\phi(x^n)\big)_{n\in\mathbf{N}}=(y_1^n,\hat{0}_Y)_{n\in\mathbf{N}}$ (equality in Y_ω). By definition $\phi_{\hat{0}}(x_1^n)=\pi_{Y_1}\big(\phi(x_1^n,\hat{0})\big)$. We conclude

$$\big(\phi_{\hat{0}}(x_1^n)\big)_{n\in\mathbf{N}}=\pi_{(Y_\omega)_1}\big(y_1^n,\hat{0}_Y)_{n\in\mathbf{N}}\big)=(y_1^n)_{n\in\mathbf{N}},$$

and from the Pythagorean formula we get $\lim_\omega \frac{d_{Y_1}\left(\phi_{\hat{0}}(x_1^n),y_1^n\right)}{\sigma_n}=0$, contradicting the definition of σ_n. We conclude there is a function $v_{\hat{0}}\in O(u)$ such that $\phi_{\hat{0}}$ is $v_{\hat{0}}$-quasi-surjective.

Sublinear Control. Finally, we show there is $v\in O(u)$ such that for $\hat{x},\hat{w}\in\hat{X}$ and $x_1\in X_1$, we have $d_{Y_1}\big(\phi_{\hat{x}}(x_1),\phi_{\hat{w}}(x_1)\big)\leq v(|(\hat{x},x_1)|,|(\hat{w},x_1|)\leq v(\hat{x}\vee\hat{w})+v(x_1)$. For convenience, we show it in the case where $\hat{w}=\hat{0}$, which clearly implies the general case. Assume towards contradiction that there are sequences $x_1^n\in X_1$ and $\hat{x^n}\in\hat{X}$ with $\lim_\omega \frac{u(\hat{x^n},x_1^n)}{d_{Y_1}\left(\phi_{\hat{x^n}}(x_1^n),\phi_{\hat{0}}(x_1^n)\right)}=0$. Denote $x^n=(x_1^n,\hat{x^n})$ and $\sigma_n:=d_{Y_1}\big(\phi_{\hat{x^n}}(x_1^n),\phi_{\hat{0}}(x_1^n)\big)$. Consider the cones $X_\omega=\mathrm{Cone}(X,x^n,\sigma_n),Y_\omega:=\mathrm{Cone}\big(Y,\phi(x^n),\sigma_n\big)$. From Lemma 2.7 we know that the map $F=\mathrm{Cone}(\phi)$ is an L-bilipschitz homeomorphism hence respects the product structure of X_ω and Y_ω. We thus have $F_1:(X_\omega)_1\to(Y_\omega)_1$ an L-bilipschitz map such that for $x_\omega:=(x^n)_{n\in\mathbf{N}},x_\omega^0=(x_1^n,\hat{0})_{n\in\mathbf{N}}$ we have:

$$\big(\phi_{\hat{x^n}}(x_1^n)\big)_{n\in\mathbf{N}}=\pi_{(Y_\omega)_1}\big(F(x_\omega)\big)=\pi_{(Y_\omega)_1}\big(F(x_\omega^0)\big)=\big(\phi_{\hat{0}}(x_1^n)\big)_{n\in\mathbf{N}}$$

And so $\lim_\omega \frac{d_{Y_1}\left(\phi_{\hat{x^n}}(x_1^n),\phi_{\hat{0}}(x_1^n)\right)}{\sigma_n}=0$, contradicting the definition of σ_n. We conclude that indeed there is $v\in O(u)$ such that for all $\hat{x}\in\hat{X}$ and $x_1\in X_1$ we have

$$d_{Y_1}\big(\phi_{\hat{x}}(x_1),\phi_{\hat{0}}(x_1)\big)\leq v(|(\hat{x},x_1)|).$$

This inequality concludes the proof. □

As in [KKL98], in order to include Euclidean factors (or, more generally, factors with asymptotic cones homeomorphic to Euclidean space) we will also need a slight variation of the above result.

Proposition 2.18. *Let $X:=\bar{X}\times Z$, $Y:=\bar{Y}\times W$ be two geodesic metric spaces. Assume that $f:X\to Y$ is an (L,u)-bilipschitz equivalence, and that for every pair of u-admissible cones X_ω,Y_ω the induced map $F=\mathrm{Cone}(f):X_\omega\to Y_\omega$ preserves the product $\bar{X}_\omega\times Z_\omega$ and $\bar{Y}_\omega\times W_\omega$ by the Z_ω,W_ω factors. Equivalently, there exists a homeomorphism $\bar{F}:\bar{X}_\omega\to\bar{Y}_\omega$ such that the following diagram commutes:*

$$\begin{tikzcd} X_\omega \arrow[r, "F"] \arrow[d, "\bar{\pi}_\omega"] & Y_\omega \arrow[d, "\bar{\pi}_\omega"] \\ \bar{X}_\omega \arrow[r, "\bar{F}"] & \bar{Y}_\omega \end{tikzcd}$$

Then there is an (L',v)-bilispchitz equivalence $\bar{f}:\bar{X}\to\bar{Y}$, with $v\in O(u)$ and L' depending only on L and u such that the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\bar{\pi}"] & Y \arrow[d, "\bar{\pi}"] \\ \bar{X} \arrow[r, "\bar{f}"] & \bar{Y} \end{tikzcd}$$

commutes up to sublinear $O(u)$-error.

Proof. We define for each $z \in Z$ the map $\phi_z : \bar{X} \to \bar{Y}$ by $\phi_z(\bar{x}) = \bar{\pi}(f(x,z))$. A similar computation to that done in the proof of Proposition 2.11 shows that these are (L, u_z)-bilipschitz equivalences with $u_z \in O(u)$. To be a bit more explicit (but without repeating the entire argument): the quantity $d_{\bar{Y}}\big(\phi_z(\bar{x_1})\big), \phi_z(\bar{x_2})\big)$ is clearly bounded above by the upper bound on $d\big(f(\bar{x_1}, z), f(\bar{x_2}, z)\big)$ given by f. For the lower bound, the reverse triangle inequality implies that one has to find a sublinear function $v_z \in O(u)$ such that $d_W\Big(\pi_W\big(f(\bar{x_1}, z)\big), \pi_W\big(f(\bar{x_2}, z)\big)\Big) < v_z(\bar{x_1} \vee \bar{x_2})$. This is done in the usual way: a sequence of pairs $(\bar{x_1^n}, z)$ and $(\bar{x_2^n}, z)$ defying this property yields two points in an u-admissible cone which defy the fact that the cone map $F = \mathrm{Cone}(f)$ respects the product structure $X_\omega = \bar{X}_\omega \times Z_\omega$.

The quasi-surjectivity and sublinear control when varying z are done exactly like in Proposition 2.11, concluding the proof. □

2.D. **Concluding Theorem A.** By assumption, the asymptotic cones of X and Y decompose into a product of a Euclidean factor and a non-Euclidean factor, where the non-Euclidean factor is a product of factors of types I and II. From [KKL98, Theorem 5.1] we see the cone maps between u-admissible cones preserve this product structure, and that the dimension of the Euclidean factors equal, i.e. $p = q$ in the notation of the statement. Proposition 2.18 gives rise to an $O(u)$-bilipschitz equivalence $\bar{\phi} : \prod_{i=1}^n X_i \to \prod_{j=1}^m Y_j$. Using [KKL98, Theorem 5.1] once more we see that $n = m$ and that in u-admissible cones the cone maps of $\bar{\phi}$ preserve the product structure. Theorem A now follows immediately from Proposition 2.11.

3. Completely solvable groups, Corollary B and Theorem C

3.A. **Some preliminaries on distortion and Cornulier's class.** As announced in the beginning of the introduction, the quasiisometry classification of connected Lie groups amounts to that of the completely solvable ones. Given a simply connected solvable Lie group G there are several possible, equivalent definitions for $\rho_0(G)$. We give below that of Jablonski, building on Gordon and Wilson.

Proposition 3.1 ([Jab19, §4.1 and §4.2], after [GW88])**.** *Let G be a simply connected solvable Lie group. There exists a (possibly non-unique) left-invariant metric $g_{\max}$ on G whose isometry group contains a transitive completely solvable group G_0. Moreover, the group G_0 obtained in this way is unique up to isomorphism and it does not depend on $g_{\max}$.*

Definition 3.2. Let G be a simply connected solvable Lie group. We define $\rho_0(G)$ as G_0. We say that a group is in the class $(\mathcal{C}_0)$ if $G = \rho_0(G)$, that is, if G is completely solvable.

It is clear that the groups G and G_0 are quasiisometric, being closed co-compact subgroups of the isometry group $\widehat{G}$ of $g_{\max}$. They are commable in the terminology of [Cor18]. The role of the group $\widehat{G}$ is played by the group denoted G_3 in Cornulier's treatment ([Cor20, Lemme 1.3], summarizing [Cor08]).

Definition 3.3. Let G be a group in the class $(\mathcal{C}_0)$. The exponential radical $\mathrm{R}_{\exp} G$ of G is the smallest normal subgroup N of G such that G/N is nilpotent.

The exponential radical was named by Osin [Osi02] as it is the subgroup of exponentially distorted elements in G (together with 1). We call $\dim G/\mathrm{R}_{\exp} G$ the rank of G. If $\widehat{G}$ is a real semisimple Lie group with trivial center, writing an Iwasawa decomposition

$\widehat{G} = KAN$ and setting $G = AN$, we recover that the real rank of $\widehat{G}$ is the rank of G. More generally, the rank as defined here is still the dimension of one (or any) Cartan subgroup of G.

Definition 3.4. Let G be a completely solvable Lie group with exponential radical N. Say that G is in $(\mathcal{C}_1)$ if the extension $1 \to N \to G \to G/N \to 1$ splits and the action of G/N on N is $\mathbf{R}$-diagonalizable.

Definition 3.5. Let G be a completely solvable group with $N = \mathrm{R}_{\exp} G$, and set $H = G/N$. Decompose $\phi = \mathrm{ad}\colon \mathfrak{g} \to \mathrm{Der}(\mathfrak{n})$ into

$$\phi = \phi_\delta + \phi_\nu$$

where ϕ_δ is $\mathbf{R}$-diagonalisable and ϕ_ν is nilpotent [Bou75]. Note that ϕ_δ is zero when restricted to $\mathfrak{n}$, so that it is well-defined on $\mathfrak{h}$. Let $\rho_1(G)$ be $N \rtimes H$, where $\mathfrak{h}$ acts on $\mathfrak{n}$ through ϕ_δ. We also write $\rho_1(\mathfrak{g})$ for $\mathrm{Lie}(\rho_1(G))$.

Although in this paper the focus is on ρ_1, we recall below the definition of ρ_∞, a further reduction that we mentioned in the introduction.

Definition 3.6 (Cornulier, [Cor11])**.** Let G be a completely solvable group, and let $H = G/\mathrm{R}_{\exp G}$. The Lie algebra of H has a filtration by its the derived central series. Define

$$\mathfrak{h}_\infty = \bigoplus_{i>0} C^i\mathfrak{h}/C^{i+1}\mathfrak{h}$$

with the brackets induced from those of $\mathfrak{h}$. The action of H on $\mathrm{R}_{\exp} G$, after being factored through $H/C^2H \simeq H_\infty/C^2 H_\infty$, lifts a new action of H_∞ on $\mathrm{R}_{\exp} G$; define $\rho_\infty(G)$ as the corresponding semidirect product $\mathrm{R}_{\exp G} \rtimes H_\infty$.

We will require the following theorem of Cornulier:

Theorem 3.7 (Cornulier, [Cor11])**.** *Let G be a completely solvable group, and let $H = G/\mathrm{R}_{\exp} G$. Then*

(1) G and $\rho_1(G)$ are $O(\log)$-bilipschitz equivalent.
(2) H is a $O(\log)$-Lipschitz retract of G, more precisely: $\pi : G \to H$ is $O(\log)$-Lipschitz.
(3) G and $\rho_\infty(G)$ are $O(u)$-bilipschitz equivalent, where the function u depends on G.

3.B. **Warm-up: the groups in Example 1.7 are not quasiisometric.** We prove below that the groups in Example 1.7 are not quasiisometric. The proof is less involved than that of Theorem C but the main idea already intervenes, so we give it before.

Proposition 3.8. *Let $\alpha \in (0, 1)$. The groups $G^0_{4,9}$ and $\mathbf{R} \times G^\alpha_{3,5}$ are not quasiisometric, for any $\mu \neq 1$.*

Proof. Let's check that $\mathbf{R} \times G_{3,3}$ is the group in the class $(\mathcal{C}_1)$ associated to $G^0_{4,9}$ by [Cor11], and so there is a $O(\log)$-sublinear bilipschitz equivalence $\phi\colon G^0_{4,9} \to \mathbf{R} \times G_{3,3}$. The Lie algebra $\mathfrak{g} = \mathfrak{g}^0_{4,9}$ of $G^0_{4,9}$ is a semidirect product $\mathfrak{heis} \rtimes \delta$ where $\mathfrak{heis}$ has a basis (e_1, e_2, e_3) with $[e_1, e_2] = e_3$ and $\delta e_i = e_i$ if $i = 1, 3$ or 0 if $i = 2$. The derived subalgebra is $\mathfrak{u} = \langle e_1, e_3 \rangle$ and $[\mathfrak{g}, \mathfrak{u}] = \mathfrak{u}$, so that $\mathfrak{u}$ is the Lie algebra of the exponential radical. The matrices of ad_{e_3} and ad_{e_4} in the basis (e_1, e_2) are, respectively,

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

From the definition of ρ_1 it follows that, using the same basis and changing the brackets, $\rho_1(\mathfrak{g}_{4,9})^0$ is the Lie algebra $\mathfrak{a} \rtimes \langle e_4 \rangle \times \mathbf{R} \simeq \mathfrak{g}_{3,3} \times \mathbf{R}$.

Assume now towards contradiction that $G_{4,9}^0$ and $\mathbf{R} \times G_{3,5}^\alpha$ are quasiisometric; then by Cornulier's Theorem 3.7 there is a $O(\log)$-bilispchitz equivalence $\psi \colon \mathbf{R} \times G_{3,3} \to \mathbf{R} \times G_{3,5}^\alpha$. By Theorem A there is a sublinear bilipschitz equivalence $\underline{\psi} \colon G_{3,3} \to G_{3,5}^\alpha$. However, by [Pal20b], $O(u)$-sublinear bilipschitz equivalences between Heintze groups preserve the conformal dimensions of their Gromov boundaries. $G_{3,3}$ and $G_{3,5}^\alpha$ are Heintze groups, and the conformal dimensions are

$$\operatorname{Cdim} \partial_\infty G_{3,3} = 2, \quad \text{while} \quad \operatorname{Cdim} \partial_\infty G_{3,5}^\alpha = 1 + 1/\alpha > 2.$$

So we reach a contradiction. $\square$

3.C. **Proof of Corollary B and Theorem C.** We will achieve both corollaries by means a technical result, Theorem 3.16. Before that we need some preparation.

Definition 3.9. A Heintze group of diagonal type is a completely solvable Lie group $M \rtimes \mathbf{R}$, where M is simply connected nilpotent and $t \in \mathbf{R}$ acts by $\exp(tD)$, where $D \in \operatorname{Der}(\mathfrak{m})$ is diagonalizable and has strictly positive eigenvalues.

If H is a Heintze group of diagonal type, then M as above is its nilradical.

Theorem 3.10 (Heintze, [Hei74])**.** *A group in $(\mathcal{C}_1)$ is a Heintze group of diagonal type if and only if it carries a left-invariant metric of strictly negative curvature.*

Some examples of Heintze groups of diagonal type come from rank one symmetric spaces. More generally, we define Iwasawa subgroups as follows.

Definition 3.11. Let G be a simple Lie group with trivial center. Let $G = KAN$ be an Iwasawa decomposition of G. We call AN an Iwasawa subgroup of G; it is unique up to conjugation in G so that we may speak of "the Iwasawa subgroup of G". If G has real rank one, its Iwasawa subgroup is a Heintze group of diagonal type, see [Pan89, §9.3].

Given a metric space X (or a group) and an admissible function u, we denote by $\operatorname{SBE}(X)^{O(u)}$ the group of self $O(u)$-bilipschitz equivalences of X.

Definition 3.12. Let H be a Heintze group of diagonal type. We say that H satisfies the strong pointed sphere property if for every admissible u the group $\operatorname{SBE}(H)^{O(u)}$ is not transitive on $\partial_\infty H$.

The Iwasawa subgroups of simple Lie groups of rank one do not have the strong pointed sphere property, since the group of isometries of a rank one symmetric space acts transitively on its Gromov boundary. Among the Heintze groups with an abelian nilradical, the ones of the latter kind turn out to be the only exceptions:

Lemma 3.13 ([Pal22, Lemma 4.1])**.** *Let H be a completely solvable Heintze group with an abelian nilradical. If H is not isomorphic to a maximal completely solvable subgroup of $\mathrm{SO}(n,1)$ for any $n \geqslant 2$, then H has the strong pointed sphere property.*

We will also need the following Lemma:

Lemma 3.14 (Consequence of [Gra23, Theorem 1.4])**.** *Let G_{II} and G'_{II} be two semisimple Lie groups with trivial centers, no factors of rank one and no compact factors. If G_{II} and G'_{II} are $O(u)$-bilipschitz equivalent, then they are isomorphic.*

Proof. Let X be the Riemannian symmetric space associated to G_{II}. It follows from [Gra23] (with $X = X_0$) that the group $\mathrm{SBE}(G_{\mathrm{II}})^{O(u)} = \mathrm{SBE}(X)^{O(u)}$ is isomorphic to G_{II}. Similarly, $\mathrm{SBE}(G'_{\mathrm{II}})^{O(u)}$ is isomorphic to G'_{II}. Assuming that G_{II} and G'_{II} are $O(u)$-bilipschitz equivalent, the groups $\mathrm{SBE}(G_{\mathrm{II}})^{O(u)}$ and $\mathrm{SBE}(G'_{\mathrm{II}})^{O(u)}$ are isomorphic, concluding the proof. □

Remark 3.15. The way Lemma 3.14 is deduced from [Gra23] is the same the way the QI classification was deduced in [KL97]. One may nevertheless long for a direct way of proving this.

Theorem 3.16. *Let m, n, m' and n' be nonnegative integers. Let G, and G' be two real semisimple Lie groups with trivial center and no compact factors. Let $\{H_i\}_{1 \leqslant i \leqslant m}$ and $\{H'_j\}_{1 \leqslant j \leqslant m'}$ be families of diagonal Heintze groups satisfying the strong pointed sphere property. Write $G = KAN$, and $G' = K'A'N'$. Form the following solvable groups P and P' :*

$$P = \mathbf{R}^n \times AN \times \prod_{i=1}^{m} H_i \text{ and } P' = \mathbf{R}^{n'} \times A'N' \times \prod_{j=1}^{m'} H'_j.$$

Let S and S' be completely solvable groups. Assume that S and S' are quasiisometric, that $\rho_1(S) = P$ and that $\rho_1(S') = P'$. Then $m = m'$, $n = n'$, G and G' are isomorphic, and there is a bijection σ of $\{1, \ldots, m\}$ such that H_i is $O(\log)$-equivalent to $H_{\sigma(i)}$ for all i.

Remark 3.17. The strong pointed sphere property is known to hold for some Heintze groups with nonabelian nilradicals [Pal22, Remark 9]. So the assumptions of Theorem 3.16 are indeed weaker than those of Theorem C. (See also Remark 4.5 for concrete examples.)

Proof of Corollary B assuming Theorem 3.16. Let L and L' be semisimple Lie group with no compact factors such that $X = L/K$ and $Y = L'/K'$. Set $S = AN$, $S' = A'N'$, $n = n' = m = m' = 0$ and apply the theorem to S and S'. It follows that $L = G$ and $L' = G'$ are isomorphic, so that X and Y are pluriisometric. □

Proof of Theorem C assuming Theorem 3.16. Let S, S', P and P' be as in the assumptions of the Theorem C. Then S, S', P and P' also satisfy the assumptions of Theorem 3.16, since the groups H_i and H'_j satisfy the strong pointed sphere property as recalled above. By Theorem 3.16, the groups $\mathbf{R}^n \times AN$ and $\mathbf{R}^{n'} \times A'N'$ are isomorphic, while, after possibly reindexing the groups H'_j, H_i is $O(\log)$-equivalent to H'_i for all i. Now by the main theorem of [Pal20b], H_i and H'_i are isomorphic for all i, and so P and P' are isomorphic. □

We now come to the proof of Theorem 3.16. Applying Theorem A to P and P' we will get $O(\log)$-equivalences between the direct factors of P and P'. The proof consists in establishing which factors can be paired with one another. While it requires some notation, the reality is quite clear: simple factors of G and G' must pair to one another, whilst preserving the $\mathbf{R}$-rank one vs. higher $\mathbf{R}$-rank distinction; the strong pointed sphere property further allows to ensure that the remaining factors, namely the Heintze groups of diagonal type with this property, are not paired with the $\mathbf{R}$-rank one factors which do not have it.

Proof of Theorem 3.16. Let n, m, n', m', S, S', P, P', G and G' be as in the assumptions of Theorem 3.16. Decomposing G and G' into products of simple factors, we find that there exists p, p', q and q' so that

$$G = \prod_{i=1}^{p+q} G_i \quad \text{and} \quad G' = \prod_{j=1}^{p'+q'} G_j,$$

where G_i has $\mathbf{R}$-rank one for $1 \leqslant i \leqslant p$ and $\mathbf{R}$-rank at least two for $p+1 \leqslant i \leqslant p+q$, similarly for G'_j. Let $\ell = p+m$ and $\ell' = p'+m'$. For $i \in \{m+1, \ell\}$ define $H_i := A_{i-m}N_{i-m}$, where $G_k = K_k A_k N_k$ for $1 \leqslant k \leqslant p$; similarly, define H'_j for j in $\{m'+1, \ell'\}$. In this way, $\{H_i\}_{i=1}^{m}$, resp. $\{H_j\}_{j=1}^{m'}$ are the Heintze groups that were originally present in the statement, and $\{H_i\}_{i=m+1}^{m+p}$, resp. $\{H_j\}_{j=m'+1}^{m'+p'}$ are the Iwasawa subgroups of the rank 1 factors of G, resp. G' (which are also diagonal type Heintze groups). Especially, the groups H_i for $1 \leqslant i \leqslant m$ and H'_j for $1 \leqslant j \leqslant m'$ have coarse type I.

It follows from Theorem 3.7(1) that the groups S and P on the one hand, S' and P' on the other hand, are $O(\log)$-bilipschitz equivalent. By assumption, S and S' are quasiisometric, especially they are $O(\log)$-bilipschitz equivalent. So P and P' are $O(\log)$-bilipschitz equivalent. Note that the coarse type I factors of P, resp. P' are the groups H_i, resp. the groups H'_j, while the coarse type II factors are of P, resp. P' are the groups $A_k N_k$ for $k \in \{p+1, \ldots, q\}$, resp. $A'_s N'_s$ for $s \in \{p'+1, \ldots, p'+q'\}$, which are respectively quasiisometric to the simple Lie groups G_k and G'_s of higher $\mathbf{R}$-rank.

Applying Theorem A to P and P' we find that $n = n'$, $q = q'$, $\ell = \ell'$, and there are two bijections σ_{I} of $\{1, \ldots, \ell\}$ and σ_{II} of $\{p+1, \ldots p+q\}$ so that

(I) H_i is $O(\log)$-bilipschitz equivalent to $H_{\sigma_{\mathrm{I}}(i)}$ for all $1 \leqslant i \leqslant \ell$;
(II) G_i is $O(\log)$-bilipschitz equivalent to $G_{\sigma_{\mathrm{II}}(i)}$ for all $p < i \leqslant p+q$;

From (II) and Lemma 3.14 we deduce that the groups

$$G_{\mathrm{II}} = \prod_{i=p+1}^{p+q} G_i \quad \text{and} \quad G'_{\mathrm{II}} = \prod_{j=p'+1}^{p'+q'} G'_j,$$

are isomorphic. We now claim that $\sigma_{\mathrm{I}}(\{1, \ldots, p\}) = \{1, \ldots, p'\}$, especially $p = p'$. If it was not the case, then some H_i, say H_{i_0}, with $i_0 > p$, would be $O(u)$-equivalent to a rank one symmetric space. But then the group $\mathrm{SBE}(H_{i_0})^{O(\log)}$ would be transitive on its Gromov boundary. This cannot be, as H_{i_0} has the strong pointed sphere property. It now follows from (I) and [Pal20a] that the factors of

$$G_{\mathrm{I}} = \prod_{i=1}^{p} G_i \quad \text{and} \quad G'_{\mathrm{I}} = \prod_{j=1}^{p} G'_j,$$

are pairwise isomorphic, so that G_{I} and G'_{I} are isomorphic. Finally, by the claim above $\sigma_{\mathrm{I}}(\{p+1, \ldots, \ell\}) = \{p+1, \ldots, \ell\}$ so that H_i is $O(\log)$-bilipschitz equivalent to $H'_{\sigma(i)}$ for all $i \in \{p+1, \ldots, \ell\}$. □

4. Applications to 5-dimensional solvable Lie groups

We recall our motivation for the product theorem and the strategy of our work. Let G and H be two completely solvable Lie groups which we want to determine whether they are quasiisometric or not. If they were quasiisometric, we would have obtained an $O(\log)$-bilipschitz equivalence between $\rho_1(G)$ and $\rho_1(H)$. If it so happens that $\rho_1(G)$ or $\rho_1(H)$ decompose as products, and if the factors of these products admit the conditions of Theorem A, we obtain an $O(\log)$-bilipschitz equivalence between each of the respective factors. In favorable situations, we are able to rule out these factor maps by various reasons which were not applicable to the product groups. This rules out the existence of the original quasiisometry between G and H.

To conclude, our strategy is beneficial whenever:

(1) $\rho_1(G)$ or $\rho_1(H)$ decompose as direct products, whereas G and H did not.
(2) There is some obstruction for $O(\log)$-bilipschitz equivalence between the factors of $\rho_1(G)$ and $\rho_1(H)$.

An important quasiisometry invariant is the Dehn function. In a subsequent paper [GP][2], using the work of Cornulier and Tessera [CT17], we compute the Dehn functions of all completely solvable groups of exponential growth up to dimension 5.

Based on these computations, we can summarize the contribution of our strategy to these groups.

First, we extract the simply connected solvable Lie groups G such that $\rho_1(G)$ decomposes as a product, and then divide them according to their Dehn function. The result is Table 1. The names of the groups are from the classification found in [Mub63] (see also [PSWZ76]); the Lie group are named $G_{d,i}^{\alpha_1,\dots,\alpha_r}$ where d is the dimension, i is a positive integer, and $\alpha_1,\dots,\alpha_r$ are parameters (we keep the parameters in the same order as in [Mub63, PSWZ76], but we sometimes use different letters, in accordance with [GP])).

Image by ρ_1	**Dehn function**			**conedim**
	exponential	quadratic	cubic	
$\mathbf{R}^2 \times G_{3,3}$		$G_{5,16}^{0,\tau}$, $G_{5,17}^{\tau,0,1}$		3
$\mathbf{R}^2 \times G_{3,5}^{1/\alpha}$	$G_{5,13}^{\alpha<1,0,1}$	$G_{5,13}^{\alpha>1,0,1}$		3
$\mathbf{R} \times G_{4,5}^{\gamma,1}$ (†)	$G_{5,19}^{1,\beta<0}$, $G_{5,35}^{0,\beta<0}$	$G_{5,19}^{1,\beta>0}$, $G_{5,35}^{0,\beta>0}$ $G_{5,27}$, $G_{5,28}^{1}$, $G_{5,32}^{\alpha}$		2
$\mathbf{R} \times G_{4,8}$	$G_{5,20}^{0}$			2
$\mathbf{Heis} \times A_2$			$G_{5,25}^{1,0}$	4
$\mathbf{R} \times G_{4,9}^{1}$		$G_{5,30}^{1}$, $G_{5,37}$		2

(†) γ is either equal to β, or to 1 if there is no parameter called β.

TABLE 1. Indecomposable, completely solvable groups G such that $\rho_1(G)$ is decomposable, their Dehn functions, and the dimensions of their asymptotic cones.

The next step is to determine which of the factors that appear in the decompositions in the ρ_1 column admit the assumptions of Theorem C or at the very least Theorem 3.16. We do this in the following Lemma.

Lemma 4.1. *Let $\alpha \in (0,1), \beta \in (0,1)$. The following hold true:*

(1) The groups $G_{3,5}^{\alpha}, G_{4,5}^{\beta,1}$ are diagonal type Heintze groups with abelian derived subgroups.
(2) $P_1 = G_{3,3}$, $P_2 = G_{4,5}^{1,1}$ and $P_3 = G_{4,9}^{1}$ can be written as $P_i = A_iN_i$, $1 \leqslant i \leqslant 3$, where $G_i = K_iA_iN_i$ is among the rank one simple Lie groups $G_1 = \mathrm{SO}(3,1)$, $G_2 = \mathrm{SO}(4,1)$ and $G_3 = \mathrm{SU}(2,1)$.
(3) $G_{4,9}^{\beta}$ is a diagonal type Heintze group with the strong pointed sphere property.

[2]The full list of Lie groups and computations can currently be found in Section 5 of the arXiv preprint arXiv:2410.05042.

Proof. Let us check (1) with the help of the structure constants given in Table 2 and using Heintze's characterisation [Hei74]. For $G = G^{\alpha}_{3,5}$ we may write that the Lie algebra is $\mathfrak{g}^{\alpha}_{5,3} = \mathbf{R}^2 \rtimes_\delta \mathbf{R}$ where $\delta = \mathrm{ad}_{e_3}$ has strictly positive spectrum $\{1, \alpha\}$. Similarly $\mathfrak{g}^{\beta,1}_{4,5} = \mathbf{R}^2 \rtimes_\delta \mathbf{R}$ where $\delta = \mathrm{ad}_{e_4}$ has strictly positive spectrum $\{1, \beta\}$ (with the eigenvalue 1 having multiplicity 2). As for (2), we see again from the structure constants that $\mathfrak{g}_{3,3} = \mathbf{R}^2 \rtimes_\delta \mathbf{R}$ where $\delta = \mathrm{ad}_{e_3} = 1$, $\mathfrak{g}^{1,1}_{4,5} = \mathbf{R}^3 \rtimes_\delta \mathbf{R}$ where $\delta = 1$, and $\mathfrak{g}^1_{4,9} = \mathfrak{heis} \rtimes_\delta \mathbf{R}$ where $\mathfrak{heis}$ is the 3-dimensional Heisenberg Lie algebra and δ is its Carnot derivation (see e.g. [Pan89, §9.3]). As for (3), note that $\beta > 0$ and $\beta \neq 1$ so that it is a diagonal Heintze group with derived subgroup isomorphic to **Heis**, the foliation in its Gromov boundary along left cosets of $\langle e_2 \rangle$ are preserved by the self-sublinear bilipschitz equivalences [Pal20b, Lemma 3.9], and so it has the strong pointed sphere property by using the exact same mechanism as in the proof of [Pal22, Lemma 4.1]. □

A_2	$[e_2, e_1] = e_1$.
Heis	$[e_1, e_2] = e_3$.
$G_{3,3}$	$[e_3, e_1] = e_1$, $[e_3, e_2] = e_2$.
$G^{\alpha}_{3,5}$	$[e_3, e_1] = e_1$, $[e_3, e_2] = \alpha e_2$, $-1 \leqslant \alpha < 1$, $\alpha \neq 0$
$G^{\alpha,\beta}_{4,5}$	$[e_4, e_1] = e_1$, $[e_4, e_2] = \alpha e_2$, $[e_4, e_3] = \beta e_3$, $-1 \leqslant \alpha \leqslant \beta \leqslant 1$, $\alpha\beta \neq 0$.
$G_{4,8}$	$[e_1, e_2] = e_3$, $[e_4, e_1] = e_1$, $[e_4, e_2] = -e_2$
$G^{\beta}_{4,9}$	$[e_1, e_2] = e_3$, $[e_4, e_1] = e_1$, $[e_4, e_2] = \beta e_2$, $[e_4, e_3] = (1+\beta) e_3$, $-1 < \beta \leqslant 1$.

TABLE 2. Structure constants for the factor groups appearing in $\rho_1(G)$, G from Table 1 (whenever we do not write a bracket, it means it is 0).

We conclude that if a group G appearing in Table 1 has quadratic Dehn function then $\rho_1(G)$ admits the hypothesis of Theorem C.

Corollary 4.2 (of Theorem C)**.** *Denote:*

- $\mathcal{G}^3_{3,3} := \{G^{0,\tau}_{5,16}, G^{\tau,0,1}_{5,17}\}$
- $\mathcal{G}^3_{3,5} := \{G^{\alpha>1,0,1}_{5,13}\}$
- $\mathcal{G}^2_{4,5} := \{G^{1,\beta>0}_{5,19}, G^{0,\beta>0}_{5,35}, G_{5,27}, G^1_{5,28}, G^{\alpha}_{5,32}\}$
- $\mathcal{G}^2_{4,9} = \{G^1_{5,30}, G_{5,37}\}$

Let $c \in \{2, 3\}$. If $G \in \mathcal{G}^c_{i,j}, H \in \mathcal{G}^c_{k,l}$ are quasi-isometric, then $(i,j) = (k,l)$. Further, $\rho_1(G)$ and $\rho_1(H)$ are isomorphic.

Remark 4.3. The index c in the statement serves to record the dimension of the asymptotic cone, so that if $G \in \mathcal{G}^c_{i,j}, H \in \mathcal{G}^{c'}_{k,l}$ are quasiisometric, then $c = c'$; we mention this in order to emphasize in which respect our corollary brings new information. If $(i,j) = (3,5)$, then one can further deduce that G and H are isomorphic, but at this stage, considering ρ_0 was actually sufficient to reach such a conclusion. All the groups in $\mathcal{G}^3_{3,3}$ are quasiisometric to each other (since they have the same image under ρ_0) so one should not hope to have the conclusion that G and H are isomorphic when $(i,j) = (3,5)$. On the other hand, it would be highly desirable to know if the conclusion of the corollary can be improved to a group isomorphism between G and H when $(i,j) \in \{(4,5), (4,9)\}$; in the latter case when $(i,j) = (4,9)$, this is related to Question A.2 in our Appendix, since $\mathbf{R} \times G^1_{4,9}$ has a Riemannian symmetric metric with a non-trivial Euclidean factor.

Corollary 4.4. *The groups $G_{5,19}^{1,\beta}$, $\beta \in (0,+\infty)$, whose Lie algebra have structure constants*

$$[e_1, e_2] = e_3,\ [e_5, e_1] = e_1,\ [e_5, e_3] = e_3,\ [e_5, e_4] = \beta e_4,$$

are pairwise non-quasiisometric.

Proof. We computed that $\rho_1(G_{5,19}^{1,\beta})$ is isomorphic to $\mathbf{R} \times G_{4,5}^{1,\beta}$ (see Table 1) so that $\rho_1(G)$ records the parameter β for $G \in \mathcal{G}_{4,5}^2$, hence by Corollary 4.2, β is a quasiisometry invariant in this family. □

Remark 4.5. Using Theorem 3.16 and part (3) in Lemma 4.1, we can prove that $\mathbf{R}\times G_{4,9}^{\beta}$ is never quasiisometric to $G_{5,30}^{1}$ or $G_{5,37}$ unless $\beta = 1$. This was not possible with Theorem C only.

Appendix A. Completely solvable groups quasiisometric to symmetric spaces

The theorem below follows from the combined works of many authors on the quasiisometric rigidity of symmetric spaces in the 1990s, complemented by an improvement of Kleiner-Leeb [KL09] and synthetized in [Cor18, Theorem 19.25].

Theorem A.1. *Let G and H be two completely solvable groups. Assume that*

(1) G and H are quasiisometric, and
(2) G or H admits a symmetric left-invariant Riemannian metric with no Euclidean factor.

Then G and H are isomorphic.

Note that Theorem A.1 without assumption (2) would be [Cor18, Conjecture 19.113].

Proof. Without loss of generality, we can assume that H admits a left-invariant symmetric metric g, so that (H, g) is isometric to the symmetric space X. Since H is completely solvable, X must be of non-compact type. H acts simply transitively by isometries on X, so any larger connected Lie group H' of isometries of X will contain non-trivial point stabilizers; the latter are compact, and a completely solvable group does not have non-trivial compact subgroups. So H is maximal among the completely solvable groups of isometries of X. Now, G is quasiisometric to X, therefore by [Cor18, Theorem 19.25] it has a continuous, proper, cocompact action by isometries on X. The kernel of this action is a compact, therefore trivial, subgroup of G so that we may consider G as a subgroup of $\operatorname{Isom}(X)$; it is a closed subgroup by properness of the action. Let $\widehat{G}$ be a maximal completely solvable subgroup of $\operatorname{Isom}(X)$ containing G. By the combination of [GW88, Theorem 1.11] and [GW88, Theorem 4.3], H and $\widehat{G}$ are isomorphic. It remains to show that $G = \widehat{G}$. By [HP13] we know that

$$\dim G = \operatorname{asdim}_{\mathrm{AN}} G = \operatorname{asdim}_{\mathrm{AN}} H = \dim H = \dim \widehat{G},$$

so G and $\widehat{G}$ have the same dimension. Since $G \subseteq \widehat{G}$ and G and $\widehat{G}$ are both completely solvable, $G = \widehat{G}$. □

Question A.2. In the theorem above, can one allow Euclidean factors in assumption (2)?

The answer to this question is not obviously yes since in [Cor18, Lemma 19.29], the assumption that there is no Euclidean factors is necessary.

References

[AW76] Robert Azencott and Edward N Wilson. Homogeneous manifolds with negative curvature. i. *Transactions of the American Mathematical Society*, 215:323–362, 1976.

[Bou75] N. Bourbaki. *Éléments de mathématique. Fasc. XXXVIII: Groupes et algèbres de Lie. Chapitre VII: Sous-algèbres de Cartan, éléments réguliers. Chapitre VIII: Algèbres de Lie semi-simples déployées.* Actualités Scientifiques et Industrielles, No. 1364. Hermann, Paris, 1975.

[BR23] Marc Bourdon and Bertrand Rémy. de Rham l^p-cohomology for higher rank spaces and groups: Critical exponents and rigidity, 2023.

[Cor08] Yves Cornulier. Dimension of asymptotic cones of Lie groups. *J. Topol.*, 1(2):342–361, 2008.

[Cor11] Yves Cornulier. Asymptotic cones of Lie groups and cone equivalences. *Illinois J. Math.*, 55(1):237–259 (2012), 2011.

[Cor18] Yves Cornulier. On the quasi-isometric classification of locally compact groups. In *New directions in locally compact groups*, volume 447 of *London Math. Soc. Lecture Note Ser.*, pages 275–342. Cambridge Univ. Press, Cambridge, 2018.

[Cor20] Yves Cornulier. Aspects de la géométrie des groupes, 2020.

[CT17] Yves Cornulier and Romain Tessera. Geometric presentations of Lie groups and their Dehn functions. *Publ. Math. Inst. Hautes Études Sci.*, 125:79–219, 2017.

[GP] Ido Grayevsky and Gabriel Pallier. Dehn functions: computations, lower bound, and the quasiisometric rigidity of SOL_5. *In preparation.*

[Gra23] Ido Grayevsky. Sublinear rigidity of lattices in semisimple Lie groups, 2023.

[GW88] Carolyn S. Gordon and Edward N. Wilson. Isometry groups of Riemannian solvmanifolds. *Trans. Amer. Math. Soc.*, 307(1):245–269, 1988.

[Hei74] Ernst Heintze. On homogeneous manifolds of negative curvature. *Math. Ann.*, 211:23–34, 1974.

[HP13] J. Higes and I. Peng. Assouad-Nagata dimension of connected Lie groups. *Math. Z.*, 273(1-2):283–302, 2013.

[Jab19] Michael Jablonski. Maximal symmetry and unimodular solvmanifolds. *Pacific J. Math.*, 298(2):417–427, 2019.

[KKL98] Michael Kapovich, Bruce Kleiner, and Bernhard Leeb. Quasi-isometries and the de Rham decomposition. *Topology*, 37(6):1193–1211, 1998.

[KL97] Bruce Kleiner and Bernhard Leeb. Rigidity of quasi-isometries for symmetric spaces and Euclidean buildings. *Inst. Hautes Études Sci. Publ. Math.*, 86:115–197, 1997.

[KL01] Bruce Kleiner and Bernhard Leeb. Groups quasi-isometric to symmetric spaces. *Comm. Anal. Geom.*, 9(2):239–260, 2001.

[KL09] Bruce Kleiner and Bernhard Leeb. Induced quasi-actions: a remark. *Proc. Amer. Math. Soc.*, 137(5):1561–1567, 2009.

[Mub63] G. M. Mubarakzjanov. Classification of solvable Lie algebras of sixth order with a non-nilpotent basis element. *Izv. Vysš. Učebn. Zaved. Matematika*, 1963(4 (35)):104–116, 1963.

[Osi02] D. V. Osin. Exponential radicals of solvable Lie groups. *J. Algebra*, 248(2):790–805, 2002.

[Pal20a] Gabriel Pallier. Large-scale sublinearly Lipschitz geometry of hyperbolic spaces. *J. Inst. Math. Jussieu*, 19(6):1831–1876, 2020.

[Pal20b] Gabriel Pallier. Sublinear quasiconformality and the large-scale geometry of Heintze groups. *Conform. Geom. Dyn.*, 24:131–163, 2020.

[Pal22] Gabriel Pallier. On the logarithmic coarse structures of Lie groups and hyperbolic spaces. *Geom. Dedicata*, 216(5):Paper No. 56, 2022.

[Pan83] Pierre Pansu. Croissance des boules et des géodésiques fermées dans les nilvariétés. *Ergodic Theory Dynam. Systems*, 3(3):415–445, 1983.

[Pan89] Pierre Pansu. Métriques de Carnot-Carathéodory et quasiisométries des espaces symétriques de rang un. *Ann. of Math. (2)*, 129(1):1–60, 1989.

[Pen11a] Irine Peng. Coarse differentiation and quasi-isometries of a class of solvable Lie groups I. *Geom. Topol.*, 15(4):1883–1925, 2011.

[Pen11b] Irine Peng. Coarse differentiation and quasi-isometries of a class of solvable Lie groups II. *Geom. Topol.*, 15(4):1927–1981, 2011.

[PSWZ76] J. Patera, R. T. Sharp, P. Winternitz, and H. Zassenhaus. Invariants of real low dimension Lie algebras. *J. Mathematical Phys.*, 17(6):986–994, 1976.

[Wen05] S. Wenger. Isoperimetric inequalities of Euclidean type in metric spaces. *Geom. Funct. Anal.*, 15(2):534–554, 2005.

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