Lie groups with a small space of metric structures

joint with E. Le Donne and X. Xie

Q: Let G be a group. d_1, d_2 are left-invariant distances on G. How do they compare?

Examples: . G finitely generated, d_1 and d_2 word metrics.

. G connected Lie group, d_1 and d_2 Riemannian.

Facts: 1) If G is locally compact and d_1 and d_2 are proper, then $(G, d_1) \xrightarrow{id} (G, d_2)$ is a coarse equivalence: $\rho_-(d_1) \le d_2 \le \rho_+(d_1)$ $\rho_-, \rho_+ : [0,\infty)$

2) If G is locally compact and compactly generated and if d_1 and d_2 are proper and geodesic, then $(G, d_1) \xrightarrow{id} (G, d_2)$ are quasiisometric:

$\rho_-(s) = \lambda_- s - c$
$\rho_+(s) = \lambda_+ s + c$ for some λ_-, λ_+, $c \geq 0$.

Define $d(d_1, d_2) = \inf \left\{ \log \frac{\lambda_+}{\lambda_-} : \lambda_-, \lambda_+ \text{ as above} \right\}$

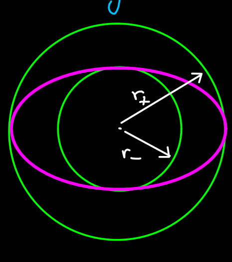
and let $\mathcal{D}(G) = \{ \text{Proper geodesic left-invariant distances} \} / \sim$

equipped with d. $d_1 \sim d_2$ if $\lambda_- = \lambda_+$ (say that they are roughly similar)

Examples: 1) $G = \mathbb{R}^d$, $d \geq 2$.

$\mathcal{D}_{\text{Riemannian}}(G) \simeq \{ \text{symmetric pos. def matrices} \}$
$\simeq SL(n, \mathbb{R}) / SO(n)$

(symmetric space)



$d(d_1, d_2) = \log(r_+/r_-)$

2) $G = \pi_1 \Sigma_g$, $g \geq 2$ (Σ_g closed hyperbolic surface)

Then $\mathcal{D}(G) \supsetneq$ Teichmüller space T_g with symmetrized Thurston metric

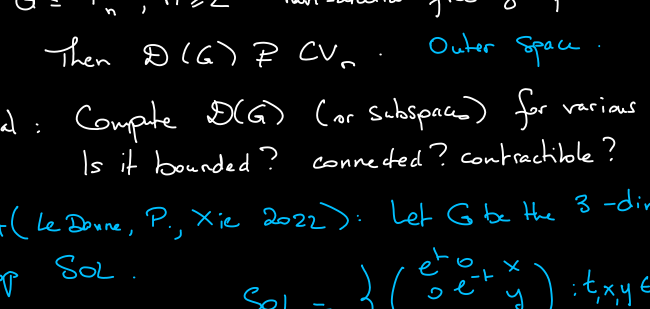
Let g be a hyperbolic metric

Pick $x_0 \in \Sigma_g$; $G.x_0 \subset \widetilde{\Sigma}_g$; restrict $\tilde{g}$ to $G.x_0$ and get a left-invariant Geometric metric on G.

$G \curvearrowright \widetilde{\Sigma}_g$
$G.x_0 \subset \widetilde{\Sigma}_g$

NB: $\tilde{g}_1$ and $\tilde{g}_2$ roughly similar $\Rightarrow$ $\tilde{g}_1$ and $\tilde{g}_2$ roughly isometric
$\Rightarrow g_1$ and g_2 isometric

(marked length spectrum rigidity)



(I owe this picture to E. Oregon-Reyes)

3) $G = F_n$, $n \geq 2$ non-abelian free group

Then $\mathcal{D}(G) \supsetneq CV_n$. Outer space.

Goal: Compute $\mathcal{D}(G)$ (or subspaces) for various G. Is it bounded? connected? contractible?

Thm A (Le Donne, P., Xie 2022): Let G be the 3-dim group SOL.

$$\text{SOL} = \left\{ \begin{pmatrix} e^t & 0 & x \\ 0 & e^{-t} & y \\ 0 & 0 & 1 \end{pmatrix} : t, x, y \in \mathbb{R} \right\}$$

Then $\mathcal{D}_{\text{Riemannian}}(G) = \{*\}$.
That is, all the left-invariant Riemannian metrics are roughly similar.

Application: Let $g_0 = dt^2 + e^{-2t} dx^2 + e^{2t} dy^2$

Eskin-Fisher-Whyte 2013 (reformulated):
any self-quasiisometry of (SOL, g_0) is a rough isometry.

Thm B: Equip SOL with any left-invariant Riemannian metric g. Then any self-qi of (SOL, g) is a rough isometry.

Pf: $\exists \lambda$ such that $\lambda d_{g_0} - c \le d_g \le \lambda d_{g_0} + c$.

Let $\phi : \text{SOL} \to \text{SOL}$ be a quasiisometry

$d_{g_0}(x,y) - k \le d_{g_0}(\phi(x), \phi(y)) \le d_{g_0}(x,y) + k$
EFW 2013

$\lambda d_{g_0}(x,y) - c - k \le d_g(\phi(x), \phi(y)) \le \lambda d_{g_0}(x,y) + c + k$
$d_g(x,y) - 2c - k \le$ " $\le d_g(x,y) + 2c + k$

So ϕ is also a rough isometry of G. □

Rk: Other thms are reformulated in this way:
Carrasco Piaggio 2016. Ferragut 2022 (thesis)
↑ hyperbolic solvable groups ↑ non-unimodular solvable groups that look like SOL.

Rk: there is a "large-scale" dictionary

closed manifold	universal cover
homotopy equivalence	quasi-isometry (lift of the h.e.)
h.e. identifying the marked length spectra	rough-isometry

translation is "correct" under some assumptions (Fujiwara, Nguyen-Wang)

On the closed manifold side,
within a class of manifold (e.g. locally symmetric)

homotopy equivalence
⇑
isomorphic marked length spectra — MLS rigidity — Mostow rigidity
⇑
isometry

So, Thm B is a large-scale translation of « Mostow rigidity minus marked length spectrum rigidity. »
(provided that the translation makes sense for SOL.)

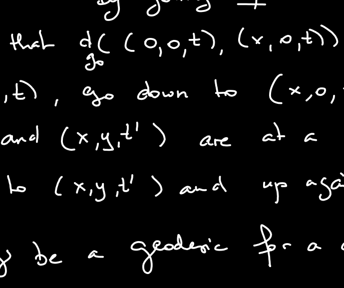
Comparison with word-hyperbolic groups
Th (Oregon-Reyes 2022): Let G be a Gromov-hyperbolic finitely generated group. Then $\mathcal{D}(G)$ is unbounded.

Idea of proof:

SOL lies in $\mathbb{H}^2 \times \mathbb{H}^2$

Pick a Busemann function b on $\mathbb{H}^2$ and define $\mathcal{S} = \{ (x,y) \in \mathbb{H}^2 \times \mathbb{H}^2 : b(x) + b(y) = 0 \}$
Equipped with the induced Riemannian metric, this is (SOL, g_0).

two vertical geodesics in the (y,t) plane are asymptotic downwards



two vertical geodesics in the (x,t) plane are asymptotic upwards

⇝ get a cartoon geodesic between $(0,0,0)$ and $(x,y,0)$ by going up to some $(0,0,t)$ such that $d_{g_0}((0,0,t),(x,0,t)) \lesssim 1$, then go to $(x,0,t)$, go down to $(x,0,t')$ such that $(x,0,t')$ and (x,y,t') are at a distance $\lesssim 1$, then go to (x,y,t') and up again to $(x,y,0)$.

Fact: let γ be a geodesic for a different metric g. Then γ stays in a tubular neighborhood fixed in advance of the cartoon geodesic.

the proof of this fact involves a key observation that the projections on the (x,t) plane and (y,t)-plane are Lipschitz.

In fact a more general version of Thm A holds in groups with this property (we call them SOL-type).

Spaces of metric structures for solvable groups: open questions.

$\mathcal{D}_{\text{Riem}}(G) = \{*\}$ — $\text{SOL}(\mathbb{R})$ — $(\mathbb{Q}_p \times \mathbb{R}) \rtimes \mathbb{Z}$ — $(\mathbb{Q}_p \times \mathbb{Q}_p) \rtimes \mathbb{Z}$

∨ like bl cocompact — ∨ as a lattice cocompact — ∨ as a lattice cocompact

SOL: $\mathbb{Z}^2 \rtimes_H \mathbb{Z}$
$\mathbb{Z}$ acts through $H \in SL(2, \mathbb{Z})$ translation length >1
e.g. $H = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$
the best current candidate to have $\mathcal{D}(G) = \{*\}$ and G finitely generated

$BS(1,p)$, $p \geq 2$
$\langle t, a \mid tat^{-1} = a^p \rangle$
???

$L_p = \mathbb{Z}/p\mathbb{Z} \wr \mathbb{Z}$
↖ diam $\mathcal{D}(G) \geq \log 2$
(belief: $\mathcal{D}(G)$ should be unbounded)

All these groups have the same asymptotic cones

Rk: if $\mathcal{D}(\Gamma) = \{*\}$ and $\Gamma < G$ as a uniform lattice then $\mathcal{D}(G) = \{*\}$ as well
So in the "diagram" above, the property (hypothetical) that $\mathcal{D}(\cdot) = \{*\}$ goes up.
But it does not go down.