A REMARK ON CONFORMAL DIMENSION AND PINCHING

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Abstract. We explain how to derive from Pansu's conformal dimension a lower bound on the pinching of negatively curved left-invariant Riemannian metrics on a class of solvable Lie groups.

A Heintze group is a semi-direct product $N \rtimes \mathbf{R}$, where N is simply connected nilpotent Lie group, and $\mathbf{R}$ acts on N through the one-parameter subgroup generated by a derivation α of $\operatorname{Lie}(N)$ whose all eigenvalues have positive real part. Every Heintze group has some of its left-invariant metrics negatively curved, and conversely every connected negatively curved isometrically homogeneous space is isometric to a Heintze group with a left-invariant metric [Hei74]. Though not stated there in this form, the following is a direct consequence of [Pan89, Section 5].

Theorem (after Pansu)**.** *Let $S = N \rtimes \alpha$ as above, and set $\sigma_1 = \inf \{\Re\lambda : \lambda \in \operatorname{sp}(\alpha)\}$. Let g be a left-invariant negatively curved Riemannian metric g on S, normalized so that its sectional curvature has $-b^2 \leqslant K_g \leqslant -1$, where $b \geqslant 1$. Then*

$$b \geqslant \frac{\operatorname{tr}(\alpha)}{\sigma_1(\dim N)}.$$

Proof. Set $n = \dim S = \dim N + 1$. Let $x \in S$. By assumption $\operatorname{Ric}_g \geqslant -(n-1)b^2 g$ and then

$$\operatorname{vol}(B(x,r)) \leqslant \text{cst.} \int_0^r \sinh^{n-1}(bt)dt,$$

so that the volume-theoretic entropy $h = \limsup_{r\to+\infty} r^{-1} \log \operatorname{vol}(B(x,r))$ is not greater than $(n-1)b$. Pansu attaches to S a quasiisometry invariant, the conformal dimension $\operatorname{Cdim} \partial_\infty S$ of its boundary[1] and establishes the following facts:

(1) $\operatorname{Cdim} \partial_\infty S \leqslant h$ [Pan89, Lemme 5.2]
(2) $\operatorname{Cdim} \partial_\infty S = \operatorname{tr}(\alpha)/\sigma_1$ [Pan89, Théorème 5.5].

Combining these facts with the inequality $h \leqslant (n-1)b$, one obtains $(n-1)b \geqslant h \geqslant \operatorname{tr}(\alpha)/\sigma_1$. □

With the notation as above, a Heintze group S is said of Carnot type if $\ker(\alpha - 1)$ generates the Lie algebra of N, and then α is called the Carnot derivation (we will give an example further). For Carnot type Heintze groups S where N has nilpotency class s, one has always $\operatorname{tr}(\alpha) < s \dim N$, hence the lower bound on pinching given by the conformal dimension can never reach s^2. Using some detailed curvature computations on Carnot type Heintze groups, B. Healy, in a recent preprint [Hel20], proved that a pinching of s^2 can be attained, and observed that in combination

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[1] Beware that we use the modern notation, see [MT10] for a survey on conformal dimension; in [Pan89] this was denoted $\mathbf{q}(\partial M)$.

with a result of Belegradek and Kapovitch [BK05, Theorem 1] this pinching can be proven to be optimal when N has a lattice.

Example. Let $s \geqslant 2$. Let N be the model filiform nilpotent group of class s and dimension $n = s+1$, namely $N = \mathbf{R} \ltimes \mathbf{R}^s$ where $\mathbf{R}$ acts by a single nilpotent Jordan block. Let α be the Carnot derivation of N, namely $\operatorname{diag}(1,1,2,\dots s)$ in the decomposition above. Then, any negatively curved Riemannian metric on $S = \mathbf{N} \rtimes \mathbf{R}$ is at least b^2-pinched, where

$$b = \frac{1+s(s+1)/2}{(s+1)} = s/2 + \frac{1}{s+1}. \tag{$\star$}$$

Note that in the example, since S is Carnot type and N has lattices, the bound ($\star$) is non-optimal in view of the previous discussion.

Corollary. *Let S be a Heintze group, $N = [S,S]$. Assume that S has Riemannian metrics with pinching arbitrarily close to 1. Then α has all its eigenvalues with the same real part, and N is abelian.*

Proof. Let $\mathfrak{n} = \operatorname{Lie}(N)$. The theorem forces the equality in

$$\sigma_1 \dim \mathfrak{n} \leqslant \sum_{\lambda \in \operatorname{sp}(\alpha)} \Re(\lambda) = \operatorname{tr}(\alpha)$$

which can only occur if all the terms are equal. Now, denote by V^λ the generalized eigenspace of α with eigenvalue λ, observe that $[V^\lambda, V^\mu] \subseteq V^{\lambda+\mu}$ for any λ, μ. Consequently, if $\oplus_{\tau \in \mathbf{R}} V^{\sigma + i\tau} = \mathfrak{n}$ for a given positive σ then $[\mathfrak{n},\mathfrak{n}] \subseteq \oplus_{\tau\in\mathbf{R}} V^{2\sigma+i\tau} = 0$, and N is abelian. □

The conclusion that N is abelian remains if a single left-invariant metric on S is assumed to be strictly more than quarter-pinched, a theorem by Eberlein and Heber, who also characterized the Heintze groups with a quarter-pinched Riemannian metric [EH96].

References

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