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Géométrie asymptotique sous-linéaire : hyperbolicité, autosimilarité, invariants

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ERRATUM FOR THE PHD THESIS "SUBLINEAR ASYMPTOTIC GEOMETRY: HYPERBOLICITY, SELF-SIMILARITY, INVARIANTS"

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page 16: The assertion "On appelle α la dérivation structurelle, elle n'est bien définie qu'à un multiple strictement positif près" is incorrect as soon as N is nonabelian. The correct replacement is given by Sequeira as [3, Proposition 5.2.2]. Though α is undefined, only properties depending on its Jordan form are used when we refer to it, so this does not create a problem in the thesis.

page 18: Just before Eq. (7), replace "Groupe semi-simple G" with "Groupe semi-simple de centre trivial, sans facteur compact G".

page 24: The Hölder dimension of the tree on Figure 5 is actually infinite. To build a pair of examples that are truly distinguished by Holdim one should instead start from the hyperbolic plane $\mathbb{H}^2$, fix a base-point o and remove $3 \cdot 2^n$ half spaces at a distance roughly $n \log n$. This will have Hölder dimension 1 and be distinguished from the metric tree of constant edge length.

page 28: Footnote 35 contains a misleading statement as it puts on the same level results with different strength. The conclusion in [1] is strictly weaker than that in the other cited works of Xie.

page 28-29: One should not have used the (wrong) "parabolique minimaux" and (ambiguous) "**R**-sous-groupe de Borel" terminology. Actually in both cases we mean an AN subgroup whenever KAN is an Iwasawa decomposition of the group G.

page 35: In Théorème 51, the assumption should be: "Soit X un espace de Banach et $e \in [0, 1)$".

page 114: $\mathscr{W}^{p;k}_{\ell,\mathrm{loc.}}$ may not be a Fréchet algebra. Hence one should replace "Fréchet algebra" by "normed algebra" in the conclusion of Lemma II.32.

page 116-117: Proposition II.35 and II.36 have an incomplete proof, because of the failure of Lemma II.32. Nevertheless, this does not affect the proof of Theorem II.1, which relies on Lemma II.56 rather than[1] Proposition II.36. (See [2, Section 3] for a proof of Theorem II.1 with only the necessary steps.)

page 122: Definition II.41 contains a mistake. The correct definition reads as

$$\mathrm{Cdim}^{\Gamma}_{O(u)}(\beta) = \sup \left\{ p \in \mathbf{R}_{>0} : \forall k \in \mathcal{O}^+(u),\, \exists \ell \in \mathcal{O}^+(u), \exists m \in \mathcal{O}^+(u), \mathrm{mod}^{\ell,m}_{p;k}(\Gamma, \beta) = +\infty \right\}.$$

The author thanks Samuel Colvin, Yves Cornulier, Emiliano Sequeira, and especially Tullia Dymarz, Peter Haïssinsky, Pierre Pansu and the anonymous referees of the journal articles for helping him to locate and correct numerous errors on several versions of the manuscript.

References

[1] Enrico Le Donne and Xiangdong Xie. "Rigidity of fiber-preserving quasisymmetric maps". In: *Rev. Mat. Iberoam.* 32.4 (2016), pp. 1407–1422. ISSN: 0213-2230. DOI: 10.4171/RMI/923. URL: https://doi-org.revues.math.u-psud.fr/10.4171/RMI/923.

[2] Gabriel Pallier. "Sublinear quasiconformality and the large-scale geometry of Heintze groups". In: *Conform. Geom. Dyn.* 24 (2020), pp. 131–163. DOI: 10.1090/ecgd/352. URL: https://doi-org.revues.math.u-psud.fr/10.1090/ecgd/352.

[3] Emiliano Sequera Manzino. *Relative L^p and Orlicz cohomology and Applications to Heintze groups.* Ph.D.Thesis – Universidad de la República and Université de Lille, available at http://www.theses.fr/2020LIL1I053. 2020.

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[1] The reason for Proposition II.36 to appear in the thesis is that it was required in an earlier attempt to prove Theorem II.1 and seemed of independent interest.

Géométrie asymptotique sous-linéaire :
hyperbolicité, autosimilarité, invariants

Résumé

Les équivalences sous-linéairement bilipschitziennes ont été introduites par Yves Cornulier afin de décrire les cônes asymptotiques des groupes de Lie. Elles généralisent les quasiisométries. Cette thèse construit des invariants pour l'équivalence sous-linéairement bilipschitzienne entre groupes et espaces hyperboliques au sens de Gromov, en utilisant l'analyse au bord de Gromov. Une classe d'applications généralisant les homéomorphismes quasisymétriques, et une dimension conforme associée, sont introduites. Les espaces symétriques riemanniens de type non-compact et de rang un, ainsi que certains espaces homogènes de courbure strictement négative, sont classifiés à équivalence sous-linéairement bilipschitzienne près.

Sublinear asymptotic geometry:
hyperbolicity, self-similarity, invariants

Abstract

Sublinearly biLipschitz equivalences have been introduced by Yves Cornulier as means of describing the asymptotic cones of Lie groups; they include and generalize quasiisometries. This thesis provides invariants for sublinearly biLipschitz equivalence between Gromov-hyperbolic groups and spaces using analysis on the Gromov boundary. A class of mappings generalizing quasisymmetric mappings, and a corresponding conformal dimension, are introduced as tools. The rank one Riemannian symmetric spaces of noncompact type as well as a subclass of homogeneous negatively curved Riemannian manifolds are classified up to sublinearly biLipschitz equivalence.

Se vider de sa fausse divinité, se nier soi-même, renoncer à être en imagination le centre du monde, discerner tous les points du monde comme étant des centres au même titre et le véritable centre comme étant hors du monde, c'est consentir au règne de la nécessité mécanique dans la matière et du libre choix au centre de chaque âme. Ce consentement est amour. La face de cet amour tournée vers les personnes pensantes est charité du prochain ; la face tournée vers la matière est amour de l'ordre du monde, ou, ce qui est la même chose, amour de la beauté du monde.

Simone Weil

Remerciements — Acknowledgements

Pierre Pansu encadre mon travail depuis quarante-quatre mois, sa bienveillance et son détachement ont été inestimables pour la réalisation de cette thèse et au-delà. Mystérieusement, Pierre m'écoute encore après avoir entendu bien des fantaisies et des doutes pas plus raisonnables, alors même que je n'ai pas saisi toute la finesse de ses idées. Il n'est pas facile de le suivre en tout, j'aimerais retenir ne serait-ce qu'une partie de son ouverture sur les maths.

Autre mystère que les années n'éclaircissent pas, ces théorèmes pour tous les groupes de Lie. Yves de Cornulier ne s'arrête pas là. Ses questions en petite dimension m'ont coûté un peu de sommeil mais je ne rêvais pas mieux. Romain Tessera a manifesté de l'intérêt pour mon travail et m'a débloqué.

I had the opportunity to discuss on my thesis and related subjects with Enrico le Donne, Cornelia Druţu, David Hume, John Mackay, Fréderic Paulin and I am very grateful to all of them. Tullia Dymarz et Peter Haïssinsky ont bien voulu rapporter ce travail, et m'ont permis de l'améliorer ; Yves Benoist, également, me fait l'honneur de participer au jury. Je les remercie vivement.

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Ne comptant plus les rentrées scolaires, j'ai donné du travail à beaucoup de profs ; a posteriori certaines notions valaient la peine qu'on y revienne. Serge Dupont et Bernard Randé m'ont fait comprendre, ou au moins croire,

que le birapport (que les homographies préservent) et les espaces métriques (même s'ils se plongent isométriquement dans les e.v.n.) en font partie. J'ai eu plaisir à retrouver Esther Cohen-Bacri et Charlotte Chalifour pendant cette thèse.

Charles Favre et Andrei Moroianu notamment m'ont permis de rester au contact des maths ; Harold Rosenberg a accompagné mes débuts dans la recherche, Rémi Leclercq un passage compliqué de mon orientation.

Ce manuscrit contient mon travail en tant que doctorant. Il y avait heureusement d'autres[1] thèses en cours et autant de doctorant·e·s à Orsay et plus loin. Le sourire de Jeanne a été bien plus qu'une politesse au quotidien ; Guillaume m'a semblé parfois nous amener avec lui au-delà de la justesse, ce qui aide aussi. Merci à Antoine et ses colocataires d'Ivry pour me ramener sur terre de temps en temps, Camille et Louise chez qui la pause thé ne compte pas vacances, a Irving para simpatizar con el caos, Pierre et Luc pour la confection d'énigmes et autres séminaires, Mélanie à Lyon et ses curieuses oscillations, Hugo, Lucile, Armand R., Pierre, Hugo, Thomas, Hédi, Joseph, Noémie[2], Thomas, Claire, Anthony, Martin, Sasha, François, Mor, Thibault, Linxiao, Guillaume, Gabriele, Maxime et bien d'autres.

Au labo j'ai été aidé voire secouru administrativement par Céline Farcy et Florence Rey notamment ; je tiens aussi à remercier Stéphane Nonnenmacher sans qui cette thèse ne serait peut-être pas imprimée, ainsi que Clarisse qui nous aide à lutter contre le désordre et s'occupe de nos plantes l'été. Au département MP de l'IUT d'Orsay, merci aux collègues, en particulier Maximilien Lagron et Sabine Marduel, pour leur soutien, pardon aux étudiant·e·s ayant subi mon enthousiasme trop matinal pour l'algèbre linéaire[3]. Merci à Éric, Sophie et Maxime et les élèves en particulier Ewan, Quentin, Medina, Sheryl et Tiago avec qui on a fait des ateliers et congrès sympathiques, et aux collègues du Math Park Xavier, Roger, Nina et Igor ; bon courage à Agentan.

Avec Clément j'ai commencé à réfléchir sérieusement aux maths il y a plus de dix ans, Margot et Gedeon savent ce que cela veut dire ; Anne-Cécile et Ambroise, Astrid, Jade, Romain et Mélanie avec Pablo m'ont accueilli à divers stades, je n'oublie pas Rogeni et la joyeuse bande de La Guerche en particulier Eram, Cyril et Max prêt à me laisser écrire là-bas. Je salue

[1] Qu'on ne s'y méprenne pas, j'ai (presque) tout écrit.

[2] Je crois savoir dans quel sens son oiseau vole maintenant.

[3] Certainement oui, c'est utile ; accessoirement, ça me rend utile.

Antoine et Michael avec la rigueur toute militaire qui nous caractérise, ainsi qu'Anouk, Ariles et Élodie-Jane que j'espère revoir très bientôt.

Dans une autre vie, peut-être, Xiaoqi a écrit cette thèse ; dans celle-ci, même depuis l'autre bout du monde, elle m'a encouragé.

Grâce à Geneviève, Xavier, Monique, Anne-Claude, Claudia, Samba, Joel et Nicolas notamment, je n'ai pas complètement perdu le Nord.

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Finalement, merci à mes parents pour leur soutien moral, à ma petite sœur Zoe pour les décisions petites et grandes, ou simplement pour exister, et à mon grand frère Jules qui sait reconnaître l'importance de tout ça, sans le prendre trop au sérieux non plus.

Table des matières — Contents

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Notation

- $\mathbf{Z}$ is the infinite cyclic group, $\mathbf{R}$ is the ordered field of real numbers, $\mathbf{N}$ is the ordered monoid of natural integers.

- If r, s are members of $\mathbf{R}$, $r \vee s$, resp. $r \wedge s$ means $\sup\{r, s\}$, resp. $\inf\{r, s\}$, $|r|$ denotes $r \vee -r$. The same notation applies to ($\mathbf{R}$-valued) functions.

- For $\mathbf{R}$-valued functions f, g defined on $\mathbf{N}$ or on the nonnegative reals $\mathbf{R}_{\geqslant 0}$, $f \ll g$ (sometimes written $f(r) \ll g(r)$ or $f(r) = o(g(r))$) means that for every positive real $\varepsilon > 0$ there exists R such that for every r greater than R,
$$f(r) \leqslant \varepsilon |g(r)|.$$
The notation $f \preccurlyeq g$ means that there exists a positive c such that for every x in the source of f,
$$|f(x)| \leqslant c|g(cx + c)| + c.$$
The notation $f \asymp g$ means that $f \preccurlyeq g$ and $g \preccurlyeq f$.

- To denote characteristic functions we use:
$$[\mathcal{P}] = \begin{cases} 1 & \text{if } \mathcal{P} \text{ is true} \\ 0 & \text{if } \mathcal{P} \text{ is false,} \end{cases}$$
e.g. if $(\Gamma, \mathcal{C}, d\gamma)$ is a measured family of subsets in a measurable space $(X, \mathcal{B})$ such that $\{\gamma \in \Gamma : \gamma \cap B = \emptyset\} \in \mathcal{C}$ whenever $B \in \mathcal{B}$, and $f : \Gamma \to \mathbf{R}_{\geqslant 0}$ is measurable then $\int_\Gamma [\gamma \cap B \neq \emptyset] f(\gamma) d\gamma$ denotes $\int_{\{\gamma \in \Gamma : \gamma \cap B \neq \emptyset\}} f(\gamma) d\gamma$ (See [100] for justification).

- If A is a subset of an affine space over $\mathbf{R}$, $\operatorname{Conv}(A)$ denotes its convex hull.

- $\mathbf{Q}$, $\mathbf{C}, \mathbf{H}$ are respectively the fields of rational numbers, complex numbers and Hamilton's quaternion algebra. $\Re$ and $\Im$ are the real and imaginary parts in all these algebras.

- The notation $K \Subset \Omega$ means that K is a compact subspace of the topological space Ω.

- dim is the topological dimension, and that of linear algebra. Hdim is the Hausdorff dimension for metric spaces. $\mathcal{H}^s$ is the Hausdorff measure.

- Unless stated otherwise, all Lie groups are real Lie groups, they are connected, and their Lie algebra are denoted with the same letter, gothic lowercase, e.g. $\mathfrak{g}$ denotes the Lie algebra of the Lie group G.

- For a group G resp. a Lie algebra $\mathfrak{h}$, G^{ab} resp. $\mathfrak{h}^{\mathrm{ab}}$ denotes the abelianization of G resp. $\mathfrak{h}$.

Introduction et contexte

La thèse est consacrée aux invariants asymptotiques des espaces et groupes métriques, principalement en courbure strictement négative. Leur étude, explicite depuis le recensement de Gromov en 1993 [83], est l'un des apports modernes des méthodes géométriques et analytiques à la compréhension des groupes qui s'y prêtent. Les invariants asymptotiques les plus importants pour le travail qui nous occupe sont eux-mêmes des espaces métriques : il s'agit de la sphère à l'infini des espaces de courbure strictement négative, et du cône asymptotique.

Avant d'embrasser la plus grande généralité, raison d'être de l'hyperbolicité au sens de Gromov, les exemples fondateurs gardent leur importance pour nous mener vers les propriétés essentielles. Pour nous ce sont d'une part les espaces de Heintze, riemanniens homogènes et de courbure strictement négative ; d'autre part les espaces de Carnot-Carathéodory, sous-riemanniens et s'offrant sous certains aspects à l'analyse. Les espaces symétriques de type non compact et de rang un d'une part, les groupes de Heisenberg d'autre part, se distinguent dans ces deux classes ; ce ne sont pas des objets dont l'étude est propre à la géométrie métrique.

On détaille ici le contexte et les enjeux qui ont guidé la thèse ainsi que les résultats et les perspectives auxquelles elle mène.

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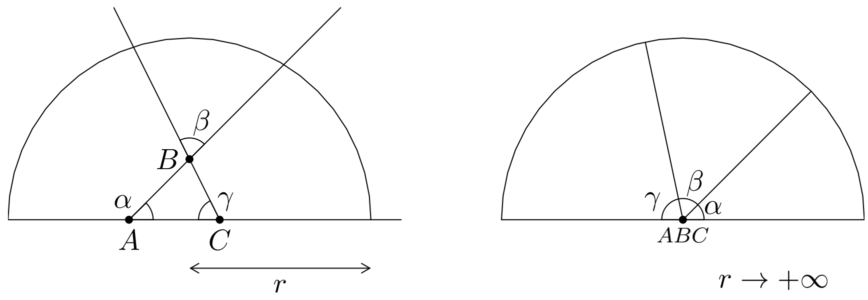


FIGURE 1 : Triangle vu depuis une grande distance.

A. Un raisonnement à grande échelle

En mai 1831, Heinrich Christian Schumacher écrit à Carl Friedrich Gauss. Les deux géodésiens ont déjà une correspondance étendue, couvrant notamment la campagne de triangulation du royaume de Hanovre et du duché de Holstein les années précédentes. Cette fois il est encore question de triangles, mais la discussion est plus théorique.

Que la somme des angles d'un triangle soit égale à deux droits est ce qu'affirme une proposition du premier livre des *Eléments*. Cependant celle-ci est reléguée avec une précaution notable parmi celles qui dépendent du cinquième postulat, légèrement au-delà de la moitié du premier livre. Le postulat et la proposition ont une certaine proximité ; à l'époque dont il est question ici, faire admettre une preuve de la proposition sans recourir à une transcription du postulat, c'est remettre en cause l'indépendance de ce dernier vis-à-vis des autres axiomes[1].

L'interrogation de Schumacher est en substance la suivante (figure 1). Considérons un triangle ABC d'un plan que nous imaginerons minimalement axiomatisé ; donnons-nous tout de même le droit d'y prolonger les rayons $[AB)$, $[CB)$ et la droite (AC) indéfiniment, et d'identifier β à $\widehat{ABC}$ qui lui est opposé par le sommet. Les angles α, β, γ se transportent comme des arcs sur un objet appelé « cercle infini ». L'erreur occasionnée par le fait que A B et C n'étaient pas tout à fait confondus, est nulle à la limite ; au besoin aussi petite que l'on voudra avec une approximation assez grande du cercle infini. La proposition est démontrée.

[1]Des réserves devraient être apportées dans une lecture moderne des axiomes, le plan non archimédien de Dehn en témoigne.

Schumacher, proche de Gauss, est conscient que celui-ci n'acceptera pas cette preuve et lui demande de localiser le passage fallacieux dans son raisonnement.

La critique de Gauss est célèbre car c'est l'un des marqueurs, très espacés dans le temps, de sa réflexion sur la géométrie non-euclidienne ; il y donne notamment la formule du périmètre d'un cercle de ce que nous appelons aujourd'hui le plan hyperbolique. On se contentera de relever deux objections.

1. Au sujet des changements d'échelle :

   „In der Euklidischen Geometrie gibt es nichts absolut grosses, wohl aber in der Nicht-Euklidischen, diess ist gerade ihr wessentlicher Charakter.“ [68, p.270]

   « Dans la géométrie euclidienne, rien n'est grand d'une manière absolue, mais il n'en est pas de même dans la géométrie non euclidienne, c'est un caractère essentiel [de la seconde]. »

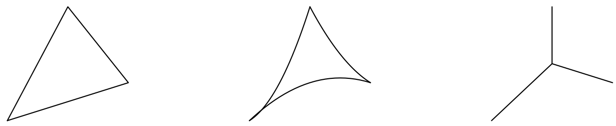
2. Concernant le cercle à l'infini :

   „In der Bildersprache des Unendlichen würde man also sagen müssen, dass die Peripherien zweier unendlichen Kreise, deren Halbmeßen um einde endliche Größe, verschieden sind, selbst um eine Größe verschieden sind, die zu ihnen ein endliches Varhältnis hat. “ [68, p.271]

   « Dans le langage figuré des infinis, il faudrait dire que les arcs de circonférence de deux cercles infinis [du plan non-euclidien] dont les rayons diffèrent d'une grandeur finie, diffèrent entre eux d'une grandeur qui partage avec chacun un rapport fini. »

Il y a là deux observations distinctes.

1. Les homothéties du plan euclidien, si l'on tient à le munir d'une distance, ramènent toutes ses figures (par exemple ses triangles) à une taille arbitrairement petite. Pour la géométrie métrique moderne, cette propriété est l'autosimilarité ; elle est caractéristique des espaces vectoriels normés, mais aussi des « fractals », du groupe de Heisenberg sous-riemannien qu'on définira plus loin.



$R \ll \delta$ $\qquad$ $R \sim \delta$ $\qquad$ $R \gg \delta$

FIGURE 2 : Allure du triangle de diamètre R dans le plan de courbure $-1/\delta^2$.

   Le plan non euclidien que conçoit Gauss est complètement dépourvu d'autosimilarité. Il possède, comme la sphère métrique, une échelle significative, dont le carré est inversement proportionnel à la courbure[2], au-delà de laquelle les triangles n'ont plus l'allure des triangles euclidiens. Dans le langage contemporain, cette échelle est la constante d'hyperbolicité, traditionnellement notée δ suivant Gromov [82, 1.1.C].

   Le raisonnement s'applique en revanche aux triangles infiniment petits ; pour eux il n'y a pas besoin de cercle infini.

2. Gauss suggère que le « cercle infini » du plan non-euclidien est d'une nature très différente de l'espace des directions du plan euclidien muni de distance angulaire.

   Il n'a pas plus de centre privilégié, mais les changements dans le décret du centre occasionnent des erreurs multiplicatives lors des mesures d'arcs infinis. En termes assez vagues, ceci est lié au fait (considérablement élaboré depuis, et qu'on explicitera) que les arcs de grands cercles sont distordus dans le sens suivant : le périmètre d'un cercle de rayon r est $2\pi \sinh r \sim \pi e^r$; la bonne normalisation est par un facteur exponentiel en fonction du rayon.

Pour une contextualisation minimale de cet échange, nous devons signaler qu'indépendamment de sa valeur exprimée par rapport aux axiomes[3], au moins la première des observations de Gauss n'est pas du tout un fait isolé. Nous renvoyons au commentaire [107, p.277] sur la même lettre, à l'introduction de [138] pour une preuve de Legendre et à [11] pour un panorama.

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[2]La courbure de Gauss ; cependant, les recherches de Gauss en géométrie non euclidienne et en géométrie différentielle (qui furent, elles, consignées) n'ont pas de lien établi.

[3]Que ce soit pour les besoin de la démonstration directe ou par l'absurde, ou bien en remplacement plus ou moins assumé de la formulation d'Euclide.

B. Vocabulaire et position du problème

B.1. Quasiisométries et équivalences sous-linéaires

Un espace métrique (Y, d) est la donnée d'un ensemble Y et d'une fonction $d : Y \times Y \to \mathbf{R}_{\geqslant 0}$ qui est symétrique, nulle sur la diagonale de $Y \times Y$ et seulement là, et vérifie l'*inégalité du triangle* :

$$\forall y_0, y_1, y_2 \in Y,\ d(y_0, y_2) \leqslant d(y_0, y_1) + d(y_1, y_2). \tag{$\triangle$}$$

Un espace métrique Y est géodésique si pour toute paire de points y_0, y_2 il existe $\gamma : I \to Y$ où I est un segment de $\mathbf{R}$, $\gamma(\inf I) = y_0$, $\gamma(\sup I) = y_2$ et pour tous $s, t \in I$, $d(\gamma(s), \gamma(t)) = |s - t|$. C'est le cas par exemple des variétés riemanniennes complètes d'après le théorème de Hopf-Rinow.

Définition 1. Soient Y et Y' deux espaces métriques. Soient λ et c des réels positifs, avec $\lambda \geqslant 1$. On dit que $f : Y \to Y'$ est une (λ, c)-quasiisométrie si les deux conditions suivantes sont réunies.

$$\forall y_1, y_2 \in Y,\ \frac{1}{\lambda} d(y_1, y_2) - c \leqslant d(f(y_1), f(y_2)) \leqslant \lambda d(y_1, y_2) + c. \tag{1}$$

$$\forall y' \in Y',\ \exists y \in Y,\ d(y', f(y)) \leqslant c. \tag{2}$$

λ est appelée *constante de Lipschitz à grande échelle*. En présence de (1) seulement, on dit que f est un plongement quasiisométrique. On note $\mathrm{QIsom}(Y)$ le groupe des quasiisométries d'un espace Y prises modulo la relation d'être à distance bornée[4]. Deux espaces métriques reliés par une quasiisométrie sont dits quasiisométriques.

Remarque 2. L'intérêt de la notion en théorie géométrique des groupes est exprimé par le lemme de Milnor et Švarc : deux espaces métriques géodésiques localement compacts munis d'actions propres, continues et cocompactes d'un même groupe topologique séparé par isométries, sont quasiisométriques, et ce groupe est localement compact et engendré par un voisinage compact du neutre. En pratique cette observation est utile car tout tel groupe admet effectivement de telles actions sur un espace Y qui peut être supposé un recollement localement fini de variétés riemanniennes complètes à bords géodésiques [28, section 2]. Le choix d'un tel Y privilégié est plus ou moins

[4] f et g de même source et image sont dites à distance bornée si $\sup_x d(f(x), g(x)) < +\infty$. C'est un exercice de vérifier que (1) et (2) garantissent que les plongements quasiisométriques se composent, et l'inversibilité des quasiisométries, modulo cette relation.

canonique selon la nature du groupe ; autour de cette thèse ce sont les espaces symétriques, homogènes de courbure négative (voire millefeuille) et les immeubles fuchsiens ; dans l'esprit général de la théorie géométrique des groupes ce peut être un graphe de Cayley ou un complexe de Rips.

Soulignons deux aspects, pas complètement indépendants, découlant de la définition 1 d'une quasiisométrie : le comportement bilipschitzien à grande échelle, et le caractère uniforme. Ce dernier peut s'exprimer sous la forme suivante : il existe Φ^-, Φ^+ des fonctions croissantes de limites infinies telles que

$$\forall y_1, y_2 \in Y, \Phi^-(d(y_1, y_2)) \leqslant d(f(y_1), f(y_2)) \leqslant \Phi^+(d(y_1, y_2)). \tag{3}$$

Les équivalences uniformes entre espaces géodésiques sont des quasiisométries[5] ; en revanche pour les plongements les deux notions diffèrent fortement : c'est le phénomène de distorsion, l'objet du § 3 du texte de Gromov [83]. Pour nous le prototype du plongement uniforme non quasiisométrique est celui d'un horocycle (ou horocercle) à l'intérieur du plan hyperbolique $\mathbb{H}^2_{\mathbf{R}}$. Il correspond à la distorsion du groupe des translations dans le groupe affine positif de $\mathbf{R}$; elle est exponentielle.

Nous étudions dans cette thèse une généralisation de la notion de quasiisométrie dans laquelle le caractère bilipschitzien à grande échelle est préservé, mais l'uniformité au sens de (3) ne l'est pas.

Définition 3 (Cornulier, [39, Section 2])**.** Soient (Y, o) et (Y', o') deux espaces métriques pointés ; on abrège $d(o, y)$, resp. $d(o', y')$ en $|y|$ resp. $|y'|$. Soit $u : \mathbf{R}_{\geqslant 0} \to \mathbf{R}_{\geqslant 1}$ vérifiant

$$\forall s, r \in \mathbf{R}_{\geqslant 0}, \quad r \leqslant s \implies u(r) \leqslant u(s) \tag{a$_1$}$$

$$u(r) \ll r \tag{a$_2$}$$

$$\limsup u(2r)/u(r) < +\infty. \tag{a$_3$}$$

Soient λ et c des réels positifs, avec $\lambda \geqslant 1$. On dit que $f : Y \to Y'$ est (ou plus exactement représente) une $O(u)$-équivalence bilispchitzienne si

$$\forall y_1, y_2 \in Y, \frac{1}{\lambda} d(y_1, y_2) - cu(|y_1| \vee |y_2|) \leqslant d(f(y_1), f(y_2)) \leqslant \lambda d(y_1, y_2) + cu(|y_1| \vee |y_2|). \tag{1'}$$

[5] Ce fait est connu sous une certaine forme en géométrie des espaces de Banach, voir par exemple Corson et Klee [44, 4.3]. Une source qui nous concerne plus directement est Gromov [83, 0.2.D] (sans preuve) ; pour une version différemment généralisée, voir Cornulier et de la Harpe [41, 3.B.9].

$$\forall y' \in Y',\, \exists y \in Y,\, d(y', f(y)) \leqslant cu(|y'|). \tag{2'}$$

λ est toujours appelée constante de Lipschitz à grande échelle. Sous la condition (1') seule on dit que f est un plongement bilipschitzien sous-linéaire. Si $\lambda = 1$ on dit que c'est une $O(u)$-isométrie, un plongement $O(u)$-isométrique s'il n'y a pas (2').

Des exemples importants pour la suite de fonctions u vérifiant (a_1), (a_2), (a_3) sont $u(r) = 1$ (où l'on retrouve les quasiisométries), $u(r) = 1 \vee \log(1+r)$, $u(r) = 1 \vee r^e$ avec $e \in (0,1)$. Deux plongements $O(u)$-bilipschitziens f et g de même source Y et de même image sont dits $O(u)$-proches s'il existe $c \in \mathbf{R}_{>0}$ tel que $d(f(y), g(y)) \leqslant cu(|y|)$ [39, p.2], ce qui remplace la relation d'être à distance bornée ; notons qu'il est équivalent de requérir l'existence de $c' \in \mathbf{R}_{>0}$ tel que $d(f(y), g(y)) \leqslant c'u(|f(y)|)$ dans la relation de $O(u)$-proximité tout comme on peut demander $d(y', f(y)) \leqslant c''u(|y|)$ pour un $c'' \in \mathbf{R}_{>0}$ dans la condition (2').

Remarque 4. Cornulier introduit aussi une catégorie d'équivalences $o(u)$-bilipschitziennes, pour laquelle la condition (a_2) est relâchée de sorte que $u(r) = r$ est admise, et (1') et (2') sont aménagées pour valoir pour tout c arbitrairement petit à condition que les y_i soient assez loin. Les résultats de cette thèse ne s'appliquent pas aux équivalences $o(r)$-bilipschitziennes ; nous les incluons parfois dans cette introduction.

Les quasiisométries et les équivalences bilipschitziennes sous-linéaires sont fonctorielles par rapport à la construction d'objets appelés cônes asymptotiques, ou au moins préservent ces derniers dans un sens que nous allons préciser ; commençons par rappeler leur définition.

Définition 5 ((Pré)cônes asymptotiques, cf. Druţu [49, 2A])**.** Soit $\mathscr{U}$ un ultrafiltre non principal (i.e. maximal et plus fin que le filtre des parties cofinies[6]) sur $\mathbf{N}$. Soit Y un espace métrique, $o_j \in Y^{\mathbf{N}}$, et r_j une suite croissante de réels de limite infinie. On forme l'ensemble

$$\operatorname{Precone}(Y, o_j, r_j) = \left\{ (y_j) \in Y^{\mathbf{N}} : d(o_j, y_j) = O(r_j) \right\}.$$

Puis $\operatorname{Cone}_{\mathscr{U}}(Y, o_j, r_j)$ est l'espace métrique formé sur le quotient de Precone, de distance

$$d([y_j], [y'_j]) = \lim_{j \to \mathscr{U}} \frac{d(y_j, y'_j)}{r_j}$$

en identifiant les points à distance nulle (Nous reléguons les exemples à C.1).

[6] Par rapport à ZF, l'existence de tels ultrafiltres est plus faible que le lemme de Zorn mais plus forte que le théorème de Hahn-Banach.

Pour Y homogène, ou au minimum de groupe d'isométries co-borné, si l'on veut désigner un cône asymptotique à isométrie près, il n'est pas nécessaire de préciser de suite (o_j), cf. [49, Remark 2.A.2].

Remarque 6. Sous ZFC, si l'hypothèse du continu ($2^{\aleph_0}$ est le plus petit cardinal plus grand que $\aleph_0$) est vraie alors pour tout groupe de Lie connexe G, $\mathrm{Cone}_{\mathscr{U}}(G, r_j)$, ne dépend pas de $\mathscr{U}$ à homéomorphisme bilipschitzien près (Cornulier [36, Corollaire 1.9]).

Proposition 7. *Soient Y et Y' deux espaces métriques, u vérifiant les conditions* (a_1)*,* (a_2)*,* (a_3) *de la Définition 3. Supposons que $f : Y \to Y'$ représente une $O(u)$-équivalence bilipschitzienne. Alors pour tout $(o, o') \in Y \times Y'$,*

i. f envoie $\mathrm{Precone}(Y, o, r_j)$ *dans* $\mathrm{Precone}(Y', o', r_j)$*.*

ii. Pour tout $\mathscr{U}$ ultrafiltre non principal sur $\mathbf{N}$*, f induit un homéomorphisme bilipschitzien* $\mathrm{Cone}_{\mathscr{U}}(Y, o, r_j) \to \mathrm{Cone}_{\mathscr{U}}(Y', o', r_j)$*.*

La proposition est impliquée par le sens direct de [34, Proposition 2.9] (Cornulier), voir aussi [103, Lemma 1.16].

En particulier, si de tels Y et Y' sont reliés par une application sous-linéairement bilipschizienne, alors leurs cônes asymptotiques à points base constants et ultrafiltre et facteurs de normalisation donné sont bilipschitziennement homéomorphes.

Nous devons mentionner que les équivalences bilipschitziennes sous-linéaires préservent encore certaines structures uniformes (ou grossières) à grande échelle, même si cet aspect a été peu travaillé dans cette thèse. Rappelons qu'une structure uniforme à grande échelle sur un ensemble Y est la donnée d'une famille $\mathcal{E}$ de sous-ensembles du produit cartésien $Y \times Y$ appelés entourages, stable par composition, passage à l'inverse, union finie et passage au sous-ensemble, contenant la diagonale. A partir d'une distance d sur X on forme une structure uniforme dans laquelle $E \subset Y \times Y$ est un entourage si $\sup \{d(y_0, y_1) : (y_0, y_1) \in E\} < +\infty$; si (Y, d) est géodésique l'archimédianité de $\mathbf{R}$ dit que quel que soit $c \in \mathbf{R}_{>0}$ c'est la plus petite structure uniforme contenant l'ensemble des paires de point à distance $\leqslant c$.

Définition 8 (Structure uniforme sous-linéaire)**.** Soit Y un espace métrique, $o \in Y$ un point-base. On définit

1. La structure uniforme sous-linéaire $\mathcal{E}_{o(r)}(Y)$ dont les entourages sont $E_v = \{(y_1, y_2) \in Y \times Y : d(x, y) \leqslant v(|x| \vee |y|)\}$ pour tout $v(r) \ll r$.

2. Etant donnée $u : \mathbf{R}_{\geqslant 0} \to \mathbf{R}_{\geqslant 1}$ vérifiant (a_1), (a_2) et (a_3), la structure uniforme $O(u)$-sous-linéaire $\mathcal{E}_{O(u)}(Y)$ dont les entourages sont $E_v = \{(y_1, y_2) \in Y \times Y : d(x,y) \leqslant v(|x| \vee |y|)\}$ pour tout $v(r) = O(u)$.

A notre connaissance, la structure $\mathcal{E}_{o(r)}$ a été explicitement introduite par Dranishnikov et Smith [47, 2] et la structure $\mathcal{E}_{O(u)}$ a été implicitement introduite par Cornulier [39]. Bien que ces définitions semblent privilégier le point o, pour toute paire de points $o_1, o_2 \in X$ l'identité $(X, o_1) \to (X, o_2)$ est un isomorphisme pour ces structures[7] donc cette dépendance est fictive. Les conditions techniques (a_1) et (a_3) permettent de s'assurer que si Y est géodésique alors $\mathcal{E}_{O(u)}$ est engendré par le seul entourage E_u ; elle est monogène au sens donné par Roe [138, p.34]. Les équivalences bilipschitziennes sous-linéaires sont des isomorphismes pour les structures sous-linéaires (Cornulier le montre implicitement [39]). Pour $\mathcal{E}_{o(r)}$ ce ne sont pas tous les isomorphismes :

Exemple 9. Soit $n \geqslant 2$ un entier naturel. Pour $Y = \mathbf{R}^n$ la structure uniforme sous-linéaire $\mathcal{E}_{o(r)}$ coïncide avec la structure uniforme topologique associée à la compactification par S^{n-1} définie dans [138, Theorem 2.27]. En particulier, $f : \mathbf{R}^n \to \mathbf{R}^n$ induit une équivalence si et seulement si elle se prolonge à S^{n-1}, après conjuguaison par $\varphi : x \mapsto \tanh \|x\| \frac{x}{\|x\|}$; c'est le cas de beaucoup d'applications qui ne sont pas bilipchitziennes à grande échelle, par exemple $f(x) = x\|x\|$.

Remarque 10. Dans sa thèse en 2002, N. Wright a étudié les structures uniformes dites C_0 [156], pour lesquelles (avec les notations de la Définition 8) les entourages sont les E_v avec $v = o(1)$. Ces structures sont plus fines que les structures usuelles, à l'opposé des structures sous-linéaire.

B.2. Position du problème

B.2.1. Invariants pour l'équivalence bilipschitzienne sous-linéaire

L'objectif général, qui sera progressivement spécifié dans cette introduction, a été de décrire les équivalences sous-linéaires bilipschitziennes entre certaines paires d'espaces métriques géodésiques, principalement en courbure strictement négative. Cela demande notamment de produire des invariants

[7] Voir Roe [138, Chapter 2] pour la définition d'un isomorphisme (coarse equivalence) entre structures uniformes à grande échelle (coarse structures).

en vue d'une classification (l'un d'entre eux, le cône asymptotique, est privilégié dès le départ par rapport aux autres). Nous sommes en cela guidés par plusieurs travaux [8].

- Les recherches analogues sur les quasiisométries autour du problème de classification (des espaces riemanniens homogènes) explicitement annoncé par Gromov en 1984 [78] et de la rigidité quasiisométrique, aussi proposé par Gromov [79] et explorée à partir de la fin des années 1980 (on en donnera quelques aperçus en C.4).

- Les recherches d'Yves Cornulier sur les cônes asymptotiques des groupes de Lie qu'on exposera en C.1 donnant des conditions suffisantes pour l'équivalence sous-linéaire bilipschitzienne entre groupes de Lie. Ils indiquent notamment que la classification devrait différer significativement de la classification à quasiisométrie près même aux endroits où cette dernière reste conjecturale.

B.2.2. Pourquoi la courbure strictement négative ? Les espaces de courbure strictement négative ne sont pas distingués par leurs cônes asymptotiques. De même que le triangle « absolument grand » de la figure 2 est un tripode (son troisième côté est contenu dans l'union des deux autres), les cônes asymptotiques des espaces de courbure strictement négative sont des arbres réels, c'est-à-dire des espaces géodésiques dans lesquels les triangles sont des tripodes[9]. Sous une hypothèse d'homogénéité supplémentaire, leurs cônes asymptotiques ont un cardinal de branchement $2^{\aleph_0}$ en tout point, ils sont donc isométriques, ceci indépendamment de l'hypothèse du continu [114, Theorem 3.5].

B.3. Espaces

On présente ici brièvement les espaces en jeu dans cette thèse (autosimilaires, puis hyperboliques) sous leurs aspects métriques.

B.3.1. Groupes nilpotents et métriques de Carnot-Carathéodory

Une algèbre de Lie $\mathfrak{g}$ de dimension finie est dite Carnot-graduable si elle admet une graduation par les entiers naturels non nuls, notée $(\mathfrak{g}_i)_{i \in \mathbf{N}}$, et

[8] Le point de départ précis de cette thèse est dans l'article [39], on l'explicitera en C.3.

[9] De manière équivalente toute paire de point y est relié par un arc dont l'image est unique et isométrique à un segment réel.

qu'elle est engendrée par $\mathfrak{g}_1$. Une algèbre Carnot-graduée est nilpotente[10]. Si $\mathfrak{g}$ est une algèbre nilpotente, on lui associe l'algèbre Carnot-graduée (en général non isomorphe à $\mathfrak{g}$)[11]

$$\mathfrak{g}_\infty = \bigoplus_{i \in \mathbf{Z}_{>0}} \mathfrak{g}^{(i)}/\mathfrak{g}^{(i+1)}, \tag{4}$$

avec les crochets induits par ceux de $\mathfrak{g}$ (noter que $\mathfrak{g}^{\mathrm{ab}}$ et $\mathfrak{g}^{\mathrm{ab}}_\infty$ sont naturellement isomorphes). La donnée de représentants vectoriels $\mathcal{V} = (V_i)$ de $\mathfrak{g}^{(i)}/\mathfrak{g}^{(i+1)}$ dans $\mathfrak{g}^{(i)}$ pour tout i détermine un morphisme linéaire bijectif, non canonique $L_\mathcal{V} : \mathfrak{g} \to \mathfrak{g}_\infty$. Etant donnée une algèbre Carnot-graduée réelle $\mathfrak{g}$, la graduation détermine une action $\mathbf{R}^\times_{>0} \curvearrowright \mathfrak{g}$ par $e^\lambda.v = e^i v$ en restriction à $\mathfrak{g}_i$. Le générateur infinitésimal est une dérivation δ de $\mathfrak{g}$ tel que $\mathfrak{g}_1 = \ker(\delta - 1)$. Le sous-espace $\mathfrak{g}_1 \subseteq \mathfrak{g}$ définit un champ de plans τ invariant à gauche (non intégrable dès que $\mathfrak{g}$ n'est pas abélienne) dans G, appelé *distribution horizontale*. Une fois munie d'une norme $\|\cdot\|$ sur $\mathfrak{g}_1$ on utilise cette distribution pour définir une distance dite de Carnot-Carathéodory d_{CC}, informellement la plus petite distance qui est plus grande que $\|\cdot\|$ pour des paires de points infiniment voisins dans la direction de $\mathfrak{g}_1$. On montre que cette distance est autosimilaire : pour tout $t \in \mathbf{R}$, $d_{\mathrm{CC}}(e^{t\delta}x, e^{t\delta}y) = e^t d(x,y)$, que $d_{\mathrm{CC}} < +\infty$ [80, 0.4], que d_{CC} induit la topologie de G mais qu'il lui attribue une dimension de Hausdorff en général différente, à savoir

$$\mathrm{Hdim}\, d_{\mathrm{CC}} = \sum_i i \dim \mathfrak{g}_i = \sum_i \dim \mathfrak{g}^{(i)}, \tag{5}$$

ceci en confrontant l'autosimilarité et le fait que $\mathrm{Jac}(e^{t\delta}) = \exp(t\,\mathrm{tr}(\delta)) = \exp\left(t \sum_i i \dim \mathfrak{g}_i\right)$ pour tout t. On l'appelle aussi dimension homogène.

Remarque 11. Le Donne a montré qu'un espace géodésique autosimilaire, localement compact et dont le groupe d'isométrie est transitif est une distance de Carnot-Carathéodory sur un groupe de Carnot gradué [104, Theorem 1.1].

Une algèbre de Lie nilpotente de degré 2 est toujours Carnot-graduable, et tout supplémentaire du centre définit avec celui-ci une graduation ; une algèbre Carnot-graduée de degré 2 est la donnée d'un triple (V_1, V_2, ω) avec $\omega \in \Lambda^2 V_1 \otimes V_2$ surjective, les conditions imposées par l'identité de Jacobi étant vides. En particulier, après s'être donné $\mathbf{K} \in \{\mathbf{C}, \mathbf{H}\}$ et n un entier naturel non-nul, $V_1 = \mathbf{K}^{n-1}$, $V_2 = \Im \mathbf{K}$, et $\omega(u,v) = \Im \sum_i \overline{u_i} v_i$, on forme l'algèbre de Lie d'un groupe dit d'Iwasawa. On considère aussi $\mathbf{R}^{n-1}$ comme un groupe d'Iwasawa (avec $V_2 = \{0\} = \Im \mathbf{R}$).

[10] Si $(\mathfrak{g}^{(i)})$ désigne sa suite centrale descendante, alors par récurrence sur i, $\mathfrak{g}^{(i)} \subseteq \oplus_{j \geqslant i} \mathfrak{g}_j$

[11] Une autre notation standard pour $\mathfrak{g}_\infty$ est $\mathsf{gr}(\mathfrak{g})$ [74, 1.3].

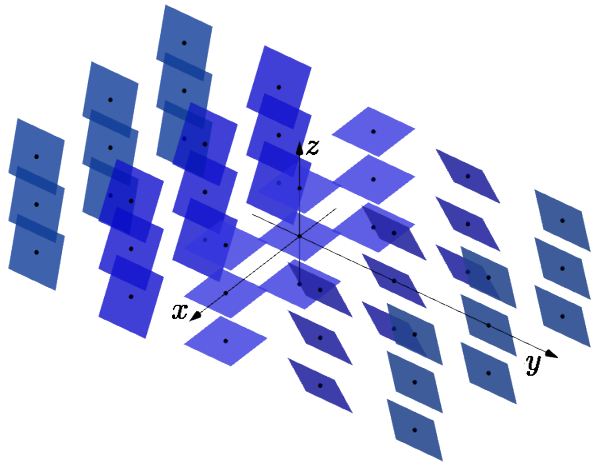


FIGURE 3 : Distribution horizontale sur le groupe de Heisenberg (image C. Falcon) $\tau = \ker(dz - ydx)$.

Exemple 12 (Groupe de Heisenberg)**.** On prend $n = 2$, $\mathbf{K} = \mathbf{C}$. La loi est (en coordonnées exponentielles, avec $p, q, p', q' \in \mathbf{R}$)

$$(p + iq, w) \bullet (p' + iq', w') = \left(p + p' + i(q + q'), w + w' + \frac{1}{2}(pq' - p'q)\right).$$

En posant $x = p, y = q, z = w + \frac{1}{2}pq$ on obtient la représentation unipotente fidèle de dimension 3,

$$\text{Heis} = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbf{R} \right\}.$$

Ce sont les coordonnées qui ont été utilisées pour dessiner la distribution horizontale sur la Figure 3.

B.3.2. Espaces symétriques de rang un et de type non compact

[12] Soit N un groupe d'Iwasawa et δ une dérivation de Carnot. On fait agir le groupe $A = \mathbf{R}$ sur N par $t.x = e^{t\delta}(x)$. Parmi les métriques riemanniennes invariantes à gauche sur le produit semidirect AN, l'une et une seule à isométrie près est symétrique de courbure sectionnelle maximale -1. On la note $\mathbb{H}^n_{\mathbf{K}}$ et on l'appelle espace hyperbolique réel (complexe, quaternionien) de dimension n. Les stabilisateurs des points sont isomorphes à $SO(n)$, $SU(n) \times S^1$

[12]Le point de vue donné ici est partial. Voir Bridson-Haefliger [23, 10.1-10.12] pour la définition à l'intéreur de $\mathbf{K}\mathbb{P}^n$ (sans le cas des octaves). Il existe un ouvrage consacré à la géométrie de $\mathbb{H}^n_{\mathbf{C}}$ [71].

ou $Sp(n) \times S^3$ selon que $\mathbf{K}$ est égal à $\mathbf{R}$, $\mathbf{C}$ ou $\mathbf{H}$ (rappelons que $Sp(n)$ est le groupe qui préserve la forme sesquilinéaire hermitienne $\sum_i \overline{u}_i v_i$ sur $\mathbf{H}^n$). Le groupe d'isométries est le groupe de Lie simple de rang réel un $SO(n,1)$, $SU(n,1)$, $Sp(n,1)$. Finalement il existe encore un plan hyperbolique, mais pas plus, sur les octaves de Cayley. Son groupe d'isométries est la forme réelle $F_4^{(-20)}$, on peut montrer que ses compacts maximaux sont isomorphes au groupe Spin(9) vu comme groupe d'automorphisme de la fibration $S^{15} \to \mathbf{O}\mathbb{P}^1$. Voir Parker [128, 9.3] pour une construction du dernier.

B.3.3. Groupes de Heintze La définition précédente de $\mathbb{H}^n_{\mathbf{K}}$ fait apparaître une décomposition du sous-groupe de Borel AN sous la forme d'un produit semi-direct où le sous-groupe $\mathbf{R} = A$ opère sur le groupe d'Iwasawa N par automorphismes engendrés par la dérivation graduante (les dilatations). Plus généralement étant donné un groupe de Lie nilpotent quelconque N, et une dérivation α dont les valeurs propres sont toutes de partie réelle strictement positive (seules les algèbres de Lie nilpotentes ont de telles dérivations, mais sur une algèbre nilpotente donnée elles n'existent pas toujours [45]), on forme une extension $S = \mathbf{R} \ltimes N$ où $\mathbf{R}$ agit par $t.n = e^{t\alpha}(n)$.

Théorème 13 (Heintze 1974, [88, Theorem 2 and 1])**.** *Soit S un groupe résoluble de la forme ci-dessus. S admet une métrique riemannienne invariante de courbure strictement négative. De plus, tout espace riemannien homogène connexe*[13] *de courbure strictement négative est une métrique invariante sur un tel groupe.*

On appelle un groupe sous la forme ci-dessus groupe de Heintze. Un groupe de Heintze est *purement réel* si le spectre de α est réel ; on dira qu'il est *de type diagonalisable* si α est diagonalisable ; il est dit *de type Carnot* si α est une dérivation graduante de N Carnot-graduable. Puisque la dérivation n'a pas de valeur propre nulle, un groupe de Heintze est métabélien[14] si et seulement si N est abélien.

[13] A posteriori ces espaces sont simplement connexes, car les groupes de Heintze de dimension > 1 sont de centre trivial. Ceci était connu avant le théorème de Heintze, d'après Kobayashi [101].

[14] Un groupe résoluble est dit métabélien si son sous-groupe dérivé est abélien.

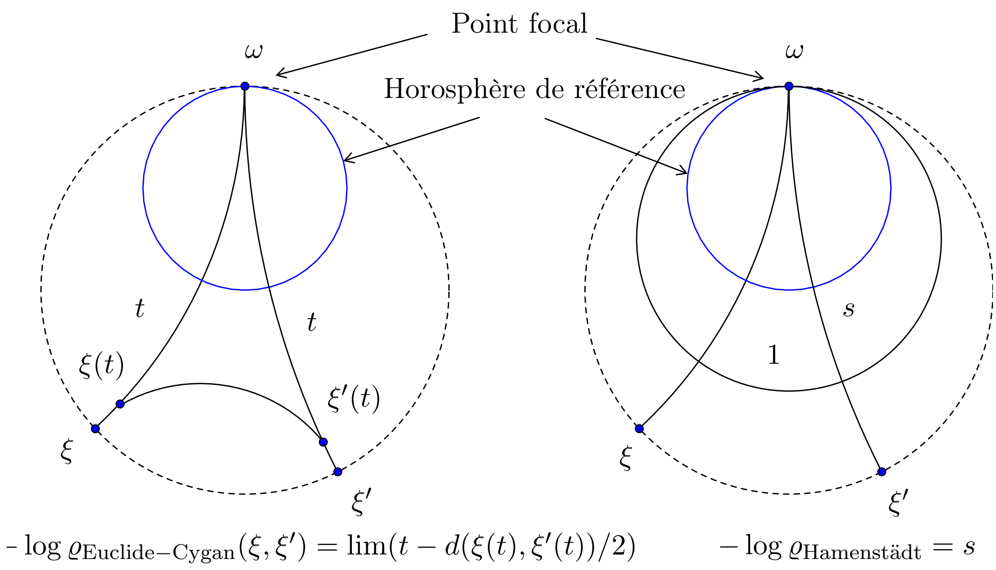


FIGURE 4 : Les noyaux d'Euclide-Cygan et d'Hamenstädt sur la sphère à l'infini d'un groupe de Heintze diminuée de son point focal sont tels que $\rho(e^{\alpha t}\xi, e^{\alpha t}\xi') = e^t \rho(\xi,\xi')$.

On peut garder à l'esprit l'échelle de généralité suivante :

$$\begin{aligned}\mathrm{Aff}^+(\mathbf{R}) &\in \{\text{unipotent maximal d'un groupe simple de rang réel un}\}\\ &\subset \{\text{groupe de Heintze de type Carnot}\}\\ &\subset \{\text{groupe de Heintze de type diagonalisable}\}\\ &\subset \{\text{groupe de Heintze purement réel}\}\,.\end{aligned}$$

Attention, deux groupes de Heintze munis de métriques riemanniennes invariantes peuvent être isométriques sans être isomorphes. Par exemple on peut tirer partie des isométries de $\mathbb{H}^3_{\mathbf{R}}$ qui fixent une géodésique pour donner une nouvelle structure de groupe opérant simplement transitivement sur le plan hyperbolique réel, $G = \mathbf{R}^2 \rtimes \mathbf{R}$ avec

$$t.\begin{pmatrix} x \\ y \end{pmatrix} = e^t \begin{pmatrix} \cos(\beta t) & \sin(\beta t) \\ -\sin(\beta t) & \cos(\beta t) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \text{ i.e.} \alpha = \begin{pmatrix} 1+i\beta & 0 \\ 0 & 1-i\beta \end{pmatrix}$$

où β est un réel non nul. Cependant, deux groupes de Heintze purement réels, s'ils ont des métriques isométriques, sont isomorphes [2], [75, Corollary 5.3]. Pour la géométrie à grande échelle on ne perd rien à se restreindre aux groupes purement réels : sur tout groupe de Heintze il existe une métrique riemannienne invariante qui est aussi portée par un groupe purement réel [2].

Le flot de $(e^{\alpha t})$ a un unique point fixe attractif au bord, on le note ω et on l'appelle point focal. La sphère à l'infini privée du point focal porte des quasidistances autosimilaires, voir la figure 4. Les groupes de Heintze sont hyperboliques au sens de Gromov. Ils sont aussi moyennables (en particulier il y a une mesure invariante sur la sphère à l'infini, elle charge le point focal). Ce comportement les différencie nettement des groupes hyperboliques discrets. Nous préciserons ceci en C.4.

B.3.4. Espaces en présence et géométrie de comparaison Dans un groupe de Heintze on appelle α la dérivation structurelle, elle n'est bien définie qu'à un multiple strictement positif près. Une convention est de normaliser α pour que la plus petite valeur propre soit 1, et la métrique pour que $\|\partial_t\| = 1$. Les paraboliques minimaux des groupes simples de rang un sont CAT(-1) de manière compatible avec cette convention, mais ce n'est pas le cas en général pour les groupes de Heintze. Un exemple important pour la suite est $G_1' = \mathbf{R}^2 \rtimes \mathbf{R}$ avec

$$t. \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e^t & te^t \\ 0 & e^t \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \text{i.e. } \alpha = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Le Donne et Nicolussi Golo montrent qu'aucune distance d sur $\mathbf{R}^2$ (géodésique ou non) ne vérifie $d(e^{t\alpha}\xi, e^{t\alpha}\xi') = e^t d(\xi, \xi')$ [46, 5.4], ce qui témoigne du fait que G_1' n'est pas CAT(-1) pour un choix de métrique normalisé.

Quant aux groupes nilpotents non abéliens, leur géométrie, qu'elle soit riemannienne ou à grande échelle, n'est pas caractérisable par une condition de courbure. D'après Wolf ils admettent des plans de courbure sectionnelle > 0 et d'autres de courbure < 0 (voir Milnor [116, Theorem 2.4]) et Pauls a montré [132, Theorem A] qu'ils ne se plongent pas même quasiisométriquement dans un espace CAT(0).

C. Invariants et rigidités

C.1. Cônes asymptotiques des groupes de Lie

On survole ici des résultats sur les cônes asymptotiques des groupes de Lie obtenus depuis les années 1970, précédant la synthèse de Cornulier commencée en 2008 (le vocabulaire ayant été en partie introduit a posteriori, il ne correspond pas nécessairement exactement aux énoncés originaux).

C.1.1. Groupes nilpotents, et un peu au-delà Soit G un groupe nilpotent de type fini sans torsion[15], resp. nilpotent de Lie simplement connexe, avec une distance propre invariante à gauche. Par la correspondance de Lie-Mal'cev [110] est associée à G une algèbre de Lie rationnelle $\mathfrak{g}$, resp. correspond à G une algèbre de Lie réelle $\mathfrak{g}$, qui est elle-même une déformation de l'algèbre Carnot-graduable définie par (4). Pour tout r_j de limite infinie et $\mathscr{U}$ non-principal, $\mathrm{Cone}_{\mathscr{U}}(G, r_j)$ est isométrique au groupe de Lie réel simplement connexe G_∞ tel que $\mathrm{Lie}(G_\infty) = \mathfrak{g}_\infty \otimes \mathbf{R}$, avec une métrique géodésique invariante par l'automorphisme gradué qui fut décrite (explicitement pour des choix de distances explicites) dans la thèse de Pansu en 1982 [124]. En particulier, ses dimensions topologique et de Hausdorff sont

$$\dim \mathrm{Cone}_{\mathscr{U}}(G, r_j) = \sum_i \dim G^{(i)}/G^{(i+1)} \otimes_{\mathbf{Z}} \mathbf{R} \text{ et} \tag{6}$$

$$\mathrm{Hdim}\, \mathrm{Cone}_{\mathscr{U}}(G, r_j) = \sum_i i \dim G^{(i)}/G^{(i+1)} \otimes_{\mathbf{Z}} \mathbf{R} \text{ d'après (5).} \tag{6'}$$

D'après une réinterprétation par Cornulier du théorème de Pansu (voir [39, Theorem 6.16] et C.2.3), G et G_∞ sont sous-linéairement bilipschitziennement équivalents, de sorte que via le théorème de Pansu-Rademacher [126, Théorème 2], on obtient l'énoncé suivant.

Théorème 14. *Deux groupes de Lie nilpotents simplement connexes G et G' sont sous-linéairement bilipschitziennement équivalents si et seulement si G_∞ et G'_∞ sont isomorphes.*

Remarque 15. L'énoncé d'origine du théorème de Pansu ne fait pas intervenir de cône asymptotique au sens de la Définition 5 ; pour ces groupes un critère de compacité pour la suite des boules renormalisées assure une convergence dite de Gromov-Hausdorff pointée. Le lien avec le cône asymptotique au sens de la Définition II.4 peut être fait par Kapovich-Leeb [96, Proposition 3.2]. Les cônes asymptotiques sont tous isométriques, qu'on suppose l'hypothèse du continu ou sa négation vraie, cf. la remarque 6 et Cornulier [36, p.7].

[15] Cette hypothèse simplificatrice n'est pas nécessaire : tout groupe nilpotent de type fini est polycyclique et, de même qu'on prouve qu'un groupe polycyclique admet un sous-groupe d'indice fini fortement polycyclique [135, chapter 4], un groupe nilpotent de type fini admet un sous-groupe d'indice fini sans torsion.

Le type (R) Rappelons le théorème de croissance polynomiale de Gromov : un groupe de type fini est à croissance polynomiale si et seulement s'il est virtuellement nilpotent ; le théorème précédent de Pansu précise alors le cône asymptotique. Parmi les groupes de Lie la caractérisation de la croissance polynomiale diffère et précède : Guivarc'h et Jenkins ont indépendamment montré en 1973 qu'il s'agit des groupes dits de type (R) [84] [94, Theorem 1.4]. Un groupe G est de type (R) si toutes les valeurs propres de ad_X pour tout $X \in \mathfrak{g}$ sont purement imaginaires. Par une construction générale dûe à Auslander et Green [4], à un groupe de type (R) on peut associer une *ombre nilpotente* en modifiant les crochets (nous renvoyons à [51, III.2] pour la construction). Breuillard observe que puisqu'il existe une métrique riemannienne simultanément invariante sur les deux groupes de Lie associés[16] via des coordonnées exponentielles de seconde espèce[17] [22, Lemma 3.11], le cône asymptotique d'un groupe résoluble de type (R) est celui de son ombre nilpotente, et relève alors de la thèse de Pansu.

Remarque 16. Les théorèmes de Breuillard sont plus généraux : ils prennent en compte tous les groupes compactement engendrés à croissance polynomiale et décrivent leurs cônes asymptotiques sous la forme précédente. Cette classe est close par équivalence bilipschitzienne sous-linéaire, parce que la croissance polynomiale d'un espace homogène équivaut à la propreté de tous ses cônes asymptotiques (Cornulier [39, Corollary 3.5]).

C.1.2. Groupes semi-simples sans facteur compact (présentation axiomatique) Commençons par rappeler que le cône asymptotique des groupes de Lie simples de rang réel un est un $\mathbf{R}$-arbre homogène. Les cônes asymptotiques d'un groupe semi-simple G (ou de son espace symétrique riemannien associé Y) ont été décrits par Kleiner et Leeb [99]. Leur dimension topologique est toujours finie égale à

$$\dim \mathrm{Cone}_{\mathscr{U}}(G, r_j) = \mathrm{rk}_{\mathbf{R}}\, G. \tag{7}$$

Il s'agit d'immeubles euclidiens, objets construits à partir de réalisations géométriques associés aux complexes de Coxeter de type affine, non discrets, appelés appartements. Il se présentent de plusieurs manières, axiomatiquement ou plus explicitement. Kleiner et Leeb donnent des axiomes qui se

[16] A rapprocher de l'opération décrite au paragraphe précédent pour les groupes de Heintze où la partie imaginaire des valeurs propres de la dérivation peut être supprimée.

[17] De même que ces coordonnées, l'identification de l'algèbre de Lie avec celle de son ombre nilpotente résulte de choix qui ne sont pas canoniques.

trouvent équivalents à ceux de Tits pour les immeubles de type affine [148, p.162], si on y ajoute la maximalité de l'atlas des appartements, d'après Parreau [129, Proposition 2.21]. L'avantage des axiomes de Kleiner et Leeb pour ce problème est qu'ils sont plus proches de ceux vérifiés par l'espace symétrique riemannien de façon à pouvoir s'en déduire par préservation après ultraproduit [99, 4] et suffisent pour les applications qu'on va décrire. D'un autre côté, la théorie développée par Tits menait, du moins dans le cas localement fini, à une classification [148, Corollaire p.175], en se ramenant à une construction antérieure de Bruhat-Tits (en réponse à une axiomatique différente) où les données sont un groupe algébrique semi-simple et un corps valué.

La motivation de Kleiner et Leeb était l'énoncé de rigidité des quasiisométries suivant (voir C.4 pour les conséquences).

Théorème 17 ([99, Th 1.1.3])**.** *Soit Y un espace symétrique irréductible de rang supérieur, $G = \mathrm{Isom}(Y)$. Toute quasiisométrie de Y est à distance bornée d'un élément de G.*

Il y a un homomorphisme $G \to \mathrm{QIsom}(Y)$, qui se trouve être injectif [81, p.39] ; le théorème dit que c'est un isomorphisme. C'est par cette reconstruction de G dans le cas irréductible (et la préservation de la structure produit [99, 6.4.3], illustrée en [99, 9]), que Kleiner et Leeb déduisent la classification à quasiisométrie près [99, Corollary 1.1.4] : si deux espaces riemanniens symétriques de type non compact sans facteur de rang un sont quasiisométriques, alors après normalisation des distances dans chacun des facteurs de leurs décompositions de de Rham, ils peuvent être rendus isométriques. L'hypothèse d'absence de facteurs de rang un (ou même plats) peut être enlevée d'après la classification des espaces symétriques de rang un à quasiisométrie près par Mostow [119]. La source du théorème 17 est la propriété de rigidité topologique suivante des cônes asymptotiques.

Théorème 18 (Kleiner et Leeb [99, Th 6.4.4])**.** *Un homéomorphisme entre immeubles euclidiens irréductibles de rang au moins* 2 *dont la partie de translation du groupe de Weyl est d'orbites dense sur les appartements*[18] *est une homothétie.*

Depuis les théorèmes de Kleiner et Leeb, des descriptions non axiomatiques des cônes asymptotiques de Y espace symétrique de type non compact

[18] On peut remplacer l'hypothèse par « cône asymptotique d'espace riemannien symétrique irréductible de rang réel au moins 2 ».

ont été données par Thornton[19] [146] et Parreau[20] [130], la dernière faisant intervenir les données de Bruhat-Tits quand $Y = \mathfrak{P}_n$ et $\mathrm{G} = \mathrm{SL}_n$ est tel que $\mathrm{G}(\mathbf{R}) = \mathrm{Isom}(Y)$.

C.2. Travaux de Cornulier

Y. Cornulier a entrepris d'étudier les cônes asymptotiques pour tous les groupes de Lie connexes (ou bien de $\mathbf{Q}_p$-points des groupes linéaires algébriques connexes définis sur $\mathbf{Q}_p$) dans [34] ; voir aussi [36, chapitre 1]. L'objectif initial était notamment de calculer la dimension topologique des cônes, interpolant (6) et (7) ; il est atteint par [34, Corollary 1.6]. Cependant l'étude va plus loin et donne aussi dans certains cas une description des cônes [36, Proposition 1.11]. Pour les groupes de Lie ce travail a amené la Définition 3. Il est donc d'un intérêt particulier pour nous.

Remarque 19 (Sur conedim et AN asdim). La dimension topologique des cônes (qu'on notera dorénavant conedim pour les groupes de Lie suivant Cornulier) minore en général un autre invariant asymptotique, la dimension d'Assouad-Nagata asymptotique[21] (Dydak-Higes, 2008 [52, Corollary 4.3]) qu'on notera AN asdim. Pour un groupe de Lie G, Higes et Peng [91, th 6.10] ont montré[22] que la seconde égale $\dim G - \dim K$ où K est un compact maximal (qui existe et dont le choix n'importe pas). En particulier, pour G semi-simple, $\mathrm{AN\,asdim}(G)$ est la dimension topologique de l'espace symétrique et si KAN est une décomposition d'Iwasawa, l'inégalité de Dydak et Higes prend la forme

$$\operatorname{conedim} G \overset{(7)}{=} \dim A \leqslant \dim AN = \mathrm{AN\,asdim}\, G. \tag{8}$$

Pour les groupes de Lie, l'invariance de AN asdim par $o(r)$-équivalence bilipschitzienne découle d'un résultat de Dranishnikov et Smith [47, Corollary

[19] La description de Thornton est celle d'un espace homogène $\mathrm{G}({}^{\rho}\mathbf{R})/\,\mathrm{G}(\mathcal{O})$ où ${}^{\rho}\mathbf{R}$ est un corps valué de Robinson et $\mathcal{O}$ son anneau de valuation. C'est là que l'unicité (avec celle du corps de Robinson) dépend de si l'on suppose l'hypothèse du continu ou sa négation.

[20] Parreau le démontre pour $\mathrm{G} = \mathrm{SL}_n$, voir [130, Théorème 3.21, Section 3.2] et les références à Bruhat-Tits page 66. C'est aussi annoncé par Kramer et Tent sans restriction sur G [102].

[21] voir Buyalo-Schroeder pour la définition [27].

[22] Roe démontrait en 1993 l'invariance par quasiisométrie de $\dim G - \dim K$ pour les groupes semi-simples sans facteurs compacts [137, (3.41), (3.42)] : la cohomologie grossière d'un espace globalement symétrique de type non compact (en particulier, contractile) détecte sa dimension topologique via l'isomorphisme [137, (3.33)] avec sa cohomologie à support compact (laquelle est concentrée en degré maximal).

3.11], qui donne aussi un plongement des cônes épointés dans la couronne de Higson-Roe[23] sous-linéaire suggérant l'inégalité de Dydak et Higes.

C.2.1. Radical exponentiel Etudié par Guivarc'h sous le nom de « sous-groupe instable » [85], il fut réintroduit et nommé par Osin [120]. Dans un groupe de Lie connexe résoluble simplement connexe, il s'agit de l'ensemble des éléments exponentiellement distordus (ainsi que le neutre) pour une métrique invariante propre dont le choix n'importe pas. L'exemple qu'on peut garder à l'esprit est celui du groupe $\mathbf{R}$ des translations dans le groupe $\mathrm{Aff}^+(\mathbf{R}) = \mathbf{R} \rtimes \mathbf{R}$. Voici une définition (dont l'équivalence avec la caractérisation géométrique précédente est un théorème).

Définition 20 ([34, Definition 6.2])**.** Soit G un groupe de Lie résoluble[24] simplement connexe. $\mathrm{R}_{\exp} G$ est le plus petit noyau connexe d'un morphisme de G vers un groupe de type (R).

Un groupe de Lie connexe est dit triangulable, ou de classe $(\mathcal{C}_0)$, s'il est isomorphe à un sous-groupe connexe fermé de matrices triangulaires supérieures. Pour les groupes triangulables, la série centrale descendante se stabilise au radical exponentiel [43, p.85]. En particulier le quotient par celui-ci n'est pas seulement de type (R), il est nilpotent.

Remarque 21. Pour G groupe de Lie résoluble simplement connexe, en associant les formules de Higes et Peng [91, Theorem 5.5] et de Cornulier [36, Theorem 1.1] on peut exprimer la dimension du radical exponentiel comme la différence dans l'inégalité de Dydak et Higes :

$$\dim \mathrm{R}_{\exp} G = \mathrm{AN}\,\mathrm{asdim}(G) - \mathrm{conedim}\, G. \tag{8^+}$$

En particulier si G est un groupe de Heintze il s'agit de $\dim \partial_\infty G$, on donnera en C.3 une interprétation dûe à Cornulier de son invariance par $o(r)$-équivalence bilipschitzienne dans un contexte plus général.

C.2.2. Réductions Un groupe de Lie connexe est de classe $(\mathcal{C}_1)$, resp. $(\mathcal{C}_\infty)$ si c'est une extension scindée $G = R \rtimes H$, où H est un groupe nilpotent,

[23] Voir Roe [138, 2.3] pour la définition. Dranishnikov et Smith n'utilisent pas la structure uniforme usuelle mais la structure $\mathcal{E}_{o(r)}$ définie en B.1.

[24] Cette définition ne vaut que pour les groupes résolubles. Pour les groupes moyennables la croissance polynomiale équivaut à la propriété dite RD [95, 3.1.8] [31, 4.1 et 7] et Cornulier observe que pour écrire la formule de dimension des cônes il faut indirectement remplacer le type (R) par la propriété (RD) en général.

resp. un groupe nilpotent Carnot-graduable, et l'action de H sur R est $\mathbf{R}$-diagonalisable[25].

Théorème 22 (Cornulier [34], [35, Theorem 1.2 and Corollary 1.3])**.** *Tout groupe de Lie réel connexe G est*

- $O(1)$*-bilipzschitziennement équivalent à un groupe G_0 de classe* $(\mathcal{C}_0)$*,*
- $O(\log)$*-bilipschitziennement équivalent à un groupe G_1 de classe* $(\mathcal{C}_1)$ *et*
- $O(r^e)$*-bilipschitziennement équivalent à un groupe G_∞ de classe* $(\mathcal{C}_\infty)$*, avec e explicite.*

Cette réduction se fait en trois étapes :

0. Soit G un groupe de Lie connexe. Il existe un groupe de Lie triangulable G_0 tel que G et G_0 sont liés par une suite finie de morphismes d'image co-compacte ou de noyaux compacts [36, Lemme 1.3].

1. Soit $R = \mathrm{R}_{\exp} G_0$, il existe une extension scindée
$$1 \to R \to G_1 \to G_1/R \to 1 \tag{9}$$
où G_1 est de la classe $(\mathcal{C}_1)$. On a dit que $H = G/R$ est nilpotent, et que son action sur R est $\mathbf{R}$-diagonalisable.

∞. L'action de H étant $\mathbf{R}$-diagonalisable, se factorise par H^{ab}, donc définit encore une action de H_∞ à travers H_∞^{ab} ; soit $G_\infty = R \rtimes H_\infty$ le produit semi-direct correspondant. Le fait que G_∞ soit $O(r^e)$-bilipschitz équivalent à G_1 découle des estimées de distance données dans [34, Lemma 4.1] et de l'interprétation par Cornulier de la thèse de Pansu.

Exemple 23 (Groupes semi-simples sans facteur compact)**.** Soit G un groupe semi-simple sans facteur compact, écrivons une décomposition d'Iwasawa sous la forme $G = KAN$. Alors $G_0 = G_1 = G_\infty = AN$, $H = A$ est abélien[26], le radical exponentiel de G_0 est N.

[25] Explicitons. Si $\mathfrak{h}$ est nilpotente, les $\mathfrak{h}$-modules se décomposent en espaces primaires qui sont des espaces caractéristiques communs à tous les éléments qui agissent (Bourbaki Lie [14, VII § 1]). Ici dire que l'action est $\mathbf{R}$-diagonalisable, c'est dire que les espaces primaires sont des espaces propres et que les poids associés sur $\mathfrak{h}$ sont à valeurs réelles.

[26] Un tel $S = AN$ se plonge bilipschitziennement dans l'espace symétrique $\mathfrak{P}_n = \mathrm{SL}(n\mathbf{R})/\,\mathrm{SO}(n)$ d'après Mostow ; Cornulier observe plus généralement que quand H est abélien dans (9), $G_1 = G_\infty$ se plonge quasiisométriquement dans un espace CAT(0) [34, Th 1.10], [36, Th 1.6], et que c'est une condition nécessaire d'après Pauls déjà cité.

Exemple 24 (Groupes nilpotents)**.** Soit G un groupe nilpotent. Alors $G = G_0 = G_1$, le radical exponentiel est trivial, et G_∞ est le groupe Carnot-graduable associé à G.

Exemple 25 (Groupes de type (R))**.** Soit G un groupe de type (R). G_0 et G_1 sont l'ombre nilpotente, G_∞ est le gradué de l'ombre nilpotente.

Exemple 26 (Groupes de Heintze)**.** Soit G un groupe de Heintze. G_0 est purement réel, $G_1 = G_\infty$ est purement réel de type diagonalisable.

C.2.3. Sur l'exposant e Une fois le problème ramené au cas nilpotent, une première borne sur l'exposant optimal e du théorème 22 peut être donnée en combinant les travaux précurseurs de Guivarc'h [84] et Goodman [73] dans les années 1970 : $e \leqslant 1 - 1/s$, où s est le degré de nilpotence de H, i.e. le temps que la série centrale de G_0 met à se stabiliser. Cette borne fut retrouvée puis améliorée par Cornulier [39, Theorem 6.15] ; signalons seulement ici que e peut être confiné arbitrairement proche de 0, pour des groupes de grand degré "presque" Carnot graduables [39, Proposition 6.13].

C.3. Equivalences sous-linéaires en courbure strictement négative : contexte récent

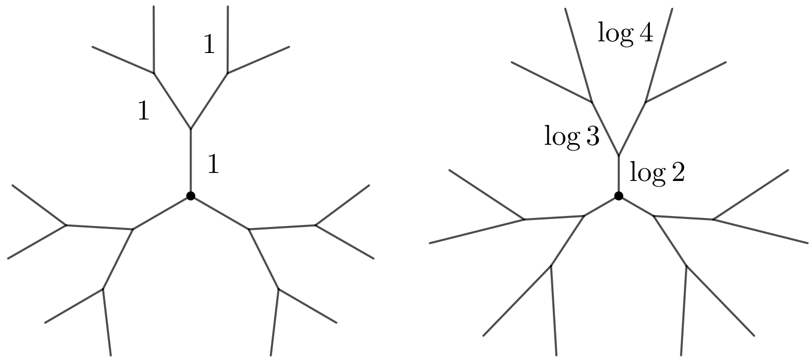
Outre la borne sur l'exposant e, l'une des idées de [39] était de formuler le problème de l'équivalence bilipschitzienne sous-linéaire hors des groupes de Lie, en particulier pour ceux qui sont dits hyperboliques au sens de Gromov[27]. Au départ de cette thèse, se trouve la constatation qu'il était envisageable de prolonger les équivalences sous-linéaires à la sphère à l'infini des variétés de courbure négative (ou au bord à l'infini des groupes hyperboliques au sens de Gromov). Celle-ci nous est venue du théorème de Cornulier [39] utilisant seulement sa structure topologique.

Théorème 27 (Cornulier 2017)**.** *Un groupe de surface et un groupe libre ne sont pas sous-linéairement bilipschitz-équivalents.*

Le prolongement en homéomorphismes au bord amène aussi, une fois associé à la robustesse de l'hyperbolicité au sens de Gromov par équivalence sous-linéaire et via les résultats classiques de Stallings, Dunwoody[28] (voir

[27] L'intersection de cette classe avec celle des groupes de Lie métriques est assez petite : elle ne va pas vraiment au-delà des groupes de Heintze, d'après Cornulier et Tessera [42, Corollary 3].

[28] Là c'est plutôt la topologie de l'espace des bouts qui intervient.



$\dim \partial_\infty T = 0$ $\dim \partial_\infty T' = 0$

Höldim $\partial_\infty T = 0$ Höldim $\partial_\infty T' = 1$

FIGURE 5 : La dimension Hölder distingue un arbre homogène d'un autre dont la reproduction ralentit à vitesse logarithmique.

Ghys et de la Harpe [70, chapitre 7] pour la preuve), resp. Tukia, Casson-Jungreis et Gabai (voir Mackay et Tyson [108, 3.4.7 et 3.6] pour les références) caractérisant les groupes hyperpoliques ayant pour bord un Cantor, resp. un cercle :

Théorème 28 (Cornulier 2018, [39, 1.10])**.** *Un groupe compactement engendré*[29] *est sous-linéairement équivalent à un groupe libre (resp. à un groupe de surface) si et seulement s'il admet une action géométrique sur un arbre homogène, resp. sur le plan hyperbolique réel.*

En fait, Cornulier a montré mieux : les équivalences sous-linéaires sont bihölderiennes au bord de Gromov, ce qui suggérait que la structure préservée au bord est plus fine. Dans cette direction, mentionnons que Colvin a récemment introduit l'invariant dimensionnel suivant.

Définition 29 (Colvin 2019, [33, Definition 1.6])**.** Soit Z un espace métrique. La dimension Hölder de Z est

$$\text{Höldim}(X) := \inf \left\{ \text{Hdim}(Z') : Z' \text{ Hölder-equivalent à Z} \right\}.$$

Pour les espaces auto-similaires, la dimension Hölder coïncide avec la dimension topologique [33, Th 1.7]. Colvin observe néanmoins qu'elle est

[29] Dans le cas groupe de surface les travaux des auteurs précédents ne permettent que d'atteindre l'hypothèse « de type fini » ; pour le cas général les détails de la preuve sont donnés par Cornulier [40, Theorem 19.109] ; il y a aussi besoin d'un théorème d'Hinkkanen.

strictement plus grande que la dimension topologique pour les Cantor dont les trous diminuent assez vite [33, Section 8]. Sur la figure 5 nous avons dessiné des arbres, l'un ayant à l'infini un Cantor autosimilaire, l'autre un Cantor de Colvin. La dimension Hölder distingue l'arbre de droite de celui de gauche (ou de tout arbre homogène) à équivalence bilipschitzienne sous-linéaire près.

C.4. Rigidités et classifications quasiisométriques (fragments)

C.4.1. Rigidité Au-delà du problème de la classification qui est central à cette thèse, pourquoi décrire les quasiisométries d'un groupe ou d'un espace donné ? Il s'avère que ce problème est ancien, lié (au moins au départ) au théorème de rigidité de Mostow tel que revu par Margulis [111], et c'est lui qui a véritablement motivé une partie de la recherche sur les invariants asymptotiques de sorte que nous devons l'évoquer. Illustrons-le sur le théorème de Kleiner et Leeb déjà cité.

Théorème 30 (Rigidité quasiisométrique des espaces symétriques de rang supérieur, Kleiner-Leeb)**.** *Soit Γ un groupe de type fini, quasiisométrique à un espace symétrique irréductible Y de rang supérieur. Alors Γ se surjecte avec noyau fini sur un réseau uniforme de $G = \operatorname{Isom}(Y)$.*

Remarque 31 (Existence)*.* Tout groupe de Lie semi-simple a effectivement un réseau uniforme d'après A. Borel [12].

Le théorème 30 se déduit du théorème 17 de la manière suivante. Soit $\Phi : \Gamma \to Y$ une quasiisométrie de quasiinverse $\widehat{\Phi}$. Alors il y a un morphisme $\mathscr{Q} : \Gamma \to \operatorname{QIsom}(Y) = G$ défini par $\mathscr{Q}(\gamma) : y \mapsto \Phi(\gamma\widehat{\Phi}(y))$; l'image opère cocompactement par quasisurjectivité de Φ, et proprement par propreté de Φ. En fait la conclusion tient si $\mathscr{Q}(\Gamma)$ est seulement conjugué à un sous-groupe de $\operatorname{Isom}(Y)$ dans $\operatorname{QIsom}(Y)$.

En rang un, la rigidité des quasiisométries n'a lieu que pour les espaces hyperboliques quaternioniens et le plan hyperbolique des octaves de Cayley où elle a été mise en évidence par Pansu [126] mais la rigidité quasiisométrique est encore là ; via le principe précédent et à travers le dictionnaire de la sphère à l'infini, elle fut obtenue implicitement par Tukia[30] [149, Corollary

[30] Précisément, Tukia a montré que les groupes d'homéomorphismes du bord de $\mathbb{H}^n_{\mathbf{R}}$, $n \geqslant 3$ qui sont dits quasiconformes et s'étendent en groupes co-compacts sur $\mathbb{H}^n$, sont conjugués à des sous-groupes du groupe des transformations Möbius par un homéomorphisme quasiconforme. Si Γ est quasiisométrique à $\mathbb{H}^n_{\mathbf{R}}$ alors $\mathscr{Q}(\Gamma)$ prolongé au bord est un

G] pour les espaces hyperboliques réels de dimension au moins 3. Chow l'a démontrée pour les espaces hyperboliques complexes [32, Theorem 3].

Pour les réseaux non uniformes dans les groupes de Lie semisimples (différents[31] de $\mathrm{SL}(2,\mathbf{R})$), la rigidité quasiisométrique tient sous une forme plus individuelle, obtenue par Schwartz en rang un, suivi de Eskin, Farb, et Druţu (voir Farb [62] pour l'énoncé général). Elle se déduit cette-fois ci de la surjectivité de l'inclusion $\mathrm{Comm}(\Lambda) \to \mathrm{QIsom}(\Lambda)$ où $\mathrm{Comm}(\Lambda)$ est le commensurateur du réseau non uniforme Λ dans G ; c'est plus délicat que dans l'argument précédent, voir Schwartz [140, 10.4] en rang un et Druţu [48, 5.4] en rang supérieur. De nos jours, la rigidité des plongements quasiisométriques est étudiée par Fisher et Whyte (entre espaces symétriques) [65] et par Fisher et Nguyen (entre les réseaux non uniformes) [64].

Le programme de classification et rigidité quasiisométrique, essentiellement achevé pour les groupes de Lie semi-simples et leurs réseaux à la fin des années 1990 par les auteurs précédents, a été poursuivi depuis dans différentes directions, notamment :

- les groupes modulaires de surfaces de complexité suffisante, rappelant les réseaux non uniformes[32] (Mosher-Whyte [118, Theorem 1], Hamenstädt, Behrstock-Kleiner-Minsky-Mosher [7, Theorem 1.2]),
- Les immeubles fuchsiens et leurs réseaux co-compacts, rappelant les réseaux uniformes en rang un (Bourdon, Xie). En particulier, Xie a montré la rigidité des quasiisométries pour les immeubles fuchsiens dont il sera question au chapitre II [157, Th 1.2].

C.4.2. Groupes de Lie résolubles et leurs réseaux En ce qui concerne spécifiquement les groupes de Lie et leurs réseaux, des progrès récents ont eu lieu pour les groupes résolubles à croissance exponentielle via la méthode dite de différentiation grossière. Rappelons que là les réseaux

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tel groupe, car les constantes de Lipschitz à grande échelle des éléments de $\mathscr{Q}(\Gamma)$ peuvent être uniformément bornées [62, Proposition 3.1] (garantissant que $\mathscr{Q}(\Gamma)$ agissant sur la sphère à l'infini est un groupe quasiconforme) et l'action de $\mathscr{Q}(\Gamma)$ est co-compacte sur $\mathbb{H}^n_{\mathbf{R}}$. Pour $n=3$ ce théorème était démontré par Sullivan dès 1981 [144].

[31] Les réseaux non uniformes de $\mathrm{SL}(2,\mathbf{R})$ sont tous quasiisométriques (à un arbre homogène) et on a vu en C.3 qu'un groupe quasiisométrique à un arbre homogène est virtuellement libre, mais cela ne se traduit pas par une relation de commensurabilité à l'intérieur de $\mathrm{SL}(2,\mathbf{R})$ (pas de rigidité de Mostow), c'est pourquoi on ne parle pas de rigidité quasiisométrique à cet endroit.

[32] Elle est maintenant conjecturée pour les groupes d'automorphismes extérieurs des groupes libres de rang assez grand.

sont uniformes[33] [135] ; en revanche il n'y en a pas toujours[34] . Rassemblons quelques conséquences en dimension trois, qui ont été dépassées par Peng et Dymarz ([133], [134], [54], [55]).

Théorème 32 (Eskin, Fisher, Whyte, [60, Theorem I.3, Theorem I.2],[61, Theorem I.2])**.** *Pour tout* $\mu \in (-\infty, -1]$ *formons* $G_\mu = \mathbf{R}^2 \rtimes \mathbf{R}$ *où* $t.(x, y) = (e^t x, e^{\mu t})$*. Alors*

(i) G_μ *et* $G_{\mu'}$ *sont quasiisométriques si et seulement si* $\mu = \mu'$*.*

Si, de plus, Γ *est un groupe de type fini, quasiisométrique à* G_μ*, alors*

(ii) $\mu = -1$*, i.e.* G_μ *est le groupe de Lie* SOL*.*

(iii) Γ *se surjecte avec noyau fini sur un réseau du groupe de Lie* SOL*.*

Le point (i) est l'énoncé de classification ; (ii) et (iii) sont ceux de rigidité. Pour un survol décrivant un état avancé du travail sur la différentiation grossière, voir Eskin et Fisher [59].

C.4.3. Retour à la courbure strictement négative Nous évoquons ici la classification des groupes de Heintze (définis en B.3.3) et la description de leurs quasiisométries. Pour situer ce problème et ses conséquences pour les groupes hyperboliques moyennables, consulter Cornulier [40] (il s'agit du cas *focal de type connexe* dans sa terminologie).

Le premier résultat, de classification, est celui de Mostow en 1970 déjà cité : les quasiisométries distinguent les espaces symétriques de type non compact et de rang un [119]. Ceci fut renforcé par Pansu :

Théorème 33 (Pansu [126, Corollaire 12.4])**.** *Deux groupes de Heintze de type Carnot quasiisométriques sont isomorphes.*

L'isomorphisme entre les algèbres graduées correspondantes est donné par une différentielle de Pansu. La rigidité des quasiisométries a lieu, notamment, quand ces algèbres ont peu d'automorphismes gradués. Elle n'est pas

[33] Ce n'est plus vrai pour les groupes résolubles localement compacts, bien que cela tienne pour les groupes moyennables linéaires [5]. Guivarc'h avait aussi noté que la moyennabilité était une condition suffisante parmi les groupes de Lie [84, Lemme 1.10].

[34] Le groupe unimodulaire Heis $\rtimes_\rho \mathbf{R}$ avec ρ engendré par la dérivation $\mathrm{diag}(1, 1, -2)$ n'a pas de réseau [134, 5.3].

réservée aux paraboliques minimaux de $\mathrm{Sp}(n,1)$ et $F_4^{(-20)}$; c'est un phénomène fréquent, à tel point que Xie conjecture désormais indices à l'appui[35] que les espaces hyperboliques réels et complexes sont les seules exceptions parmi le type Carnot.

Conjecture 34 (de rigidité des quasiisométries pour les groupes de Heintze de type Carnot [163, p.132])**.** *Toute quasiisométrie d'un groupe de Heintze de type Carnot qui n'est un $\mathbf{R}$-sous-groupe de Borel de $SO(n,1)$ ou $SU(n,1)$ pour aucun $n \geqslant 1$ est à distance bornée d'une isométrie.*

Cette conjecture entraînerait une forme inexistentielle de rigidité quasiisométrique qu'on peut aussi interpréter comme un renforcement de celle des espaces symétriques de rang un, à rapprocher de (ii) dans le théorème 32.

Conjecture 35. *Soit H un groupe de Heintze de type Carnot différent pour tout $n \geqslant 1$ d'un $\mathbf{R}$-sous-groupe de Borel de $SO(n,1)$, $SU(n,1)$, $Sp(n,1)$ ou de $F_4^{(-20)}$. Aucun groupe de type fini n'est quasiisométrique à H.*

En effet sous l'hypothèse de rigidité des quasiisométries pour H, une quasiisométrie entre H et un groupe de type fini Γ ferait agir ce dernier de façon *focale* (voir [28, Section 3] pour la définition), ce qui est exclu pour les groupes hyperboliques *discrets* non virtuellement cycliques ; cet argument est dû à Kleiner [141, p.819]. Cornulier émet une conjecture de conclusion plus faible que la rigidité des quasiisométries, qui implique aussi la conjecture 35 (cette-fois ci au-delà du type Carnot) par le même principe [40, Conjecture 19.104].

Conjecture 36 (de la sphère pointée)**.** *Soit G un groupe de Heintze de point focal ω. Alors $\mathrm{QIsom}(G)$ fixe ω si et seulement si G n'est pas un sous-groupe de Borel de $SO(n,1)$, $SU(n,1)$, $Sp(n,1)$ ou $F_4^{(-20)}$ pour $n \geqslant 1$.*

(Le point focal est fixé par les isométries, même celles qui ne sont pas des translations à gauche). La meilleure avancée vers la conjecture de la sphère pointée a été faite par Carrasco Piaggio, qui l'a établie en 2014 pour tous

[35] Pour ne citer que les meilleurs résultats actuels : Pansu [126, 13, 14], Xie pour les groupes de Heintze métabéliens différents d'un sous-groupe de Borel [160, Corollary 1.4], Xie [159] quand le nilradical est Carnot-réductible sans facteurs isomorphes, Xie [161] pour le type diagonalisable avec nilradical Heisenberg hors espaces hyperboliques complexes, Xie [162] quand le nilradical est le groupe de Heisenberg de dimension 3 hors type diagonalisable, Le Donne-Xie [105] pour le type Carnot avec action du groupe des automorphismes gradués sur l'abélianisé non irréductible.

les groupes de Heintze hors le type Carnot [29]. On peut ôter l'hypothèse de type Carnot dans la conjecture 35 si l'on utilise cela.

On termine en mentionnant quelques-uns des derniers progrès vers la classification. Outre le théorème de Pansu, un résultat important dans une direction transversale a été obtenu par Xie par la technique de Q-variation puis retrouvé par Carrasco Piaggio par le calcul de la cohomologie ℓ^ϕ en degré 1 [29, Corollaire 1.10].

Théorème 37 (Xie 2010 [160, Theorem 1.1])**.** *Deux groupes de Heintze purement réels métabéliens quasiisométriques sont isomorphes.*

Les théorèmes 33 et 37 soutiennent la conjecture de classification attribuée à Hamenstädt [40, Conjecture 19.88] :

Conjecture 38. *Deux groupes de Heintze purement réels quasiisométriques sont isomorphes.*

Dans toute sa généralité la conjecture semble lointaine. Des progrès ont été obtenus par Carrasco Piaggio et Sequeira en 2016 [30] identifiant des invariants de quasiisométrie dans la dérivation structurelle α.

D. Résultats, perspectives et questions

D.1. Résultats

On fixe ici une fonction u vérifiant (a_1), (a_2), (a_3).

Théorème I (Chapitre I, [122])**.** *Soient Y et Y' deux espaces riemanniens symétriques de type non compact et de rang un. Si Y et Y' sont $O(u)$-équivalents, alors ils sont homothétiques.*

Notre second résultat concerne les groupes de Heintze métabéliens.

Théorème II (Chapitre II, [123])**.** *Soient H et H' deux groupes de Heintze métabéliens de type diagonalisable. Si H et H' sont $O(u)$-équivalents, alors ils sont isomorphes.*

La combinaison avec le théorème 22 de Cornulier donne un résultat de classification des groupes de Heintze métabéliens à équivalence sous-linéaire près : en utilisant les notations du théorème 22,

Théorème II'. *Supposons $u(r) \geqslant \log r$. Deux groupes de Heintze métabéliens H et H' sont $O(u)$-équivalents si et seulement si H_∞ et H'_∞ sont isomorphes.*

D.2. Perspectives et questions

D.2.1. Groupes de Heintze Supposons toujours u telle que dans (a_1), (a_2), (a_3). Les théorèmes I et II seraient impliqués par une réponse positive à la question suivante, formulée par analogie avec la conjecture 38.

Question 39 (Voir chapitre II)**.** *Deux groupes de Heintze de type diagonalisable, $O(u)$-équivalents, sont-ils isomorphes ?*

Question 40 (Affaiblissement de la précédente)**.** *Soient deux groupes de Heintze purement réels, $O(u)$-équivalents, de dérivations α, α' normalisées de sorte que les plus petites valeurs propres soient* 1*. Les polynômes caractéristiques χ_α et $\chi_{\alpha'}$ sont-ils égaux ?*

Une réponse positive, ou au moins des progrès en direction d'une réponse positive à la question 40 nous paraissent atteignables en reprenant les idées de cette thèse et en les associant à celles développées par Carrasco Piaggio et Sequeira [30] (le théorème II donne une réponse positive dans le cas métabélien, tandis que [30] donne une réponse positive dans le cas $u = 1$). Une étape importante pour cela est d'étendre le théorème principal de [105] dont Medwid et Xie ont récemment produit une reformulation locale [115, Theorem 1.1], aux homéomorphismes sous-linéairement quasisymétriques introduits au chapitre I.

Question 41. *Supposons $u(r) \ll \log r$. Existe-t-il deux groupes de Heintze, $O(u)$-équivalents mais non isomorphes ?*

Nous nous attendons plutôt à une réponse négative pour la question 41, mais elle requiert des idées nouvelles. En effet K. Fässler et T. Orponen ont produit des courbes de $\mathcal{H}^1$-mesure finie non dirigées par un vecteur propre, dans le bord à l'infini du groupe de Heintze de dimension trois dont la dérivation n'est pas diagonalisable (le groupe G'_1 de B.3.4), ce qui empêche de raisonner comme au lemme I.58 où l'on contrôle la dimension de Hausdorff de l'image de presque toute telle courbe.

D.2.2. Groupes nilpotents Nous décrivons ici des parties d'un travail en cours avec C. Llosa Isenrich et R. Tessera suivant une question de Cornulier [39, 6.20].

Définition 42. Soit $\mathcal{P} = \langle S \mid \mathcal{R} \rangle$ une présentation finie symétrique d'un groupe G et $\pi : L_S \to G$ la surjection associée. On dit que $f : \mathbf{N} \to \mathbf{N}$ est

Nom	Structure[a]	[b]	[c]	Observations[d]	$\delta(n)$ [e]
$\mathfrak{h}_3$	$12=3$	2	3	Heisenberg	n^3
$\mathfrak{l}_{4,2}$	$12=3$	2	5	$\mathfrak{h}_3 \oplus \mathbf{R}$	n^3
$\mathfrak{l}_{4,3}$	$12=3,\ 13=4$	3	7	fil., gr.	n^4
$\mathfrak{l}_{5,2}$	$12=3$	2	6	$\mathfrak{h}_3 \oplus \mathbf{R}^2$	n^3
$\mathfrak{l}_{5,4}$	$12=34=5$	2	6	Heis., $\mathfrak{h}_3 \oplus_{\mathfrak{z}} \mathfrak{h}_3$	n^2 [3]
$\mathfrak{l}_{5,8}$	$12=3,\ 14=5$	2	7		n^3
$\mathfrak{l}_{5,3}$	$12=3,\ 13=4$	3	8	gr., $\mathfrak{l}_{4,3} \oplus \mathbf{R}$	n^4
$\mathfrak{l}_{5,5}$	$12=3,\ 13=25=4$	3	8	$e_{\mathfrak{g}} = 2/3$ [39]	n^4
$\mathfrak{l}_{5,9}$	$12=3,\ 13=4,\ 23=5$	3	10	gr.	n^4
$\mathfrak{l}_{5,7}$	$12=3,\ 13=4,\ 14=5$	4	11	fil., met., gr.	n^5
$\mathfrak{l}_{5,6}$	$12=3,\ 13=4,\ 14=23=5$	4	11	fil., met., $e_{\mathfrak{g}} = 3/4$ [39]	n^5

TABLE 1 : Les algèbres de Lie nilpotentes non abéliennes (réelles) de dimension $\leqslant 5$ et quelques-unes de leurs propriétés géométriques à grande échelle. Voir Cornulier ([40, 6], [38, 1.2.2]) pour d'autres et plus d'informations.

[a] Ces algèbres admettent des présentations avec constantes de structures dans $\{0,1\}$ et d'uniques formes rationnelles, on abrège $[e_i, e_j] = e_k$ en $ij = k$.

[b] Degré, [c] Dimension homogène (voir B.3.1).

[d] gr. : Carnot-graduable, fil. : filiforme, met. : métabélienne, $\oplus_{\mathfrak{z}}$: produit central (les centres, de même dimension, sont identifiés). $e_{\mathfrak{g}}$ est défini dans [39, Section 6] à partir de la structure de l'algèbre de Lie, il majore e dans le théorème 22.

[e] Type de croissance de la fonction de Dehn (Wenger a montré qu'elle n'est pas toujours polynomiale [155, Theorem 1.2]).

une fonction isopérimétrique pour G relativement à $\mathcal{P}$ si tout mot $w \in \ker \pi$ de longueur $\leqslant n$ dans L_S s'écrit comme un produit d'au plus n conjugués de relateurs dans $\mathcal{R}$.

La définition s'étend aux groupes compactement présentés (voir [145, 2.6] ou [41, chapter 8] pour le formalisme pour les groupes compactement présentés) et l'existence d'une fonction isopérimétrique de type de croissance donnée (à $\asymp$ près) est un invariant de quasiisométrie : au parcours du produit de conjugués de relateurs on associe une homotopie dans le complexe de Rips associé à $\mathcal{P}$, chaque nouveau terme correspondant au balayage d'une cellule. On peut compliquer l'invariant en contraignant les homotopies à avoir lieu dans une boule de rayon $g(n)$ donné (par exemple, de rayon $\preccurlyeq n^2$ si n est la longueur du mot) ; on montre que ce couple de types de croissance $(f(n), g(n))$ se transporte mieux par équivalence sous-linéaire bilipschitzienne, bien qu'il ne soit plus exactement un invariant. Disons que c'est un couple de remplissage ; la minimisation de f gardant g fixée est proche de la fonction de Dehn.

Rappelons qu'une algèbre de Lie est filiforme si elle est de degré maximal parmi les algèbres de sa dimension. Il existe en toute dimension $d \geqslant 3$, une algèbre filiforme $\mathfrak{l}_d$ définie comme $\mathbf{R}^{d-1} \rtimes_j \mathbf{R}$, où j est un seul bloc de Jordan de taille $d-1$ (ce sont $\mathfrak{h}_3$, $\mathfrak{l}_{4,3}$, $\mathfrak{l}_{5,7}$ en dimension 3, 4, 5, voir la table 1).

Question 43. *Soit $G = L_p \times_Z L_q$ avec $3 \leqslant p < q$ le groupe de Lie associé à $\mathfrak{l}_p \oplus_{\mathfrak{z}} \mathfrak{l}_q$. A-t-on que (n^{q-1}, n) est un couple de remplissage pour G ?*

Une réponse positive à la question permettrait de minorer l'exposant infimal e du théorème 22 sur ces exemple, par $1/(p+q)$.

D.2.3. Rang supérieur et rigidité Etant donné le théorème 17 de Kleiner et Leeb, nous posons la question suivante.

Question 44. *Une $o(r)$-équivalence bilipschitzienne (resp. une $O(u)$)-équivalence bilipschitzienne avec u admissible) d'un espace symétrique irréductible de rang supérieur est-elle sous-linéairement proche (resp. $O(u)$-proche) d'une isométrie ?*

Une réponse positive à la question permettrait la classification par le même principe que via le théorème 17, sans recourir à une description explicite des cônes asymptotiques. A l'issue de cette thèse il nous paraît probable que ces descriptions (voir C.1.2) permettent de distinguer tous les espaces symétriques sans facteur de rang un à $o(r)$-équivalence bilipschitzienne près, mais nous n'en avons pas la complète certitude[36]. En rang un, même pour les espaces hyperboliques quaternioniens et le plan de Cayley, il est moins plausible que la réponse à la question 44 soit positive ; dans sa formulation quantitative faisant intervenir la classe $O(u)$ elle est liée à la suivante (voir I.40 pour la définition d'homéomorphisme $O(u)$-quasiMöbius).

[36] Une autre voie, intermédiaire entre la présentation axiomatique et la description de toute la structure des immeubles en question, aurait été d'utiliser que l'immeuble de Tits à l'infini du cône asymptotique a les mêmes appartements que le bord de Tits de l'espace symétrique (cela découle de Kleiner-Leeb [99, Theorem 5.2.1] en prenant les bords de Tits), conjointement avec l'invariance de la dimension d'Assouad-Nagata asymptotique (remarque 19). Ce n'est pas suffisant, car pour tout (p,q) avec $2 \leqslant p < q$ les espaces symétriques de type non compact irréductibles de type III $\mathrm{SU}(p,2q)/\,\mathrm{S}(U_p \times U_{2q})$ et $Sp(p,q)/Sp(p) \times Sp(q)$ ont même système de racines restreint BC_p et même dimension d'Assouad-Nagata asymptotique $4pq$ (voir Helgason [89, Table V p.518]). Nous remercions Guy Rousseau de nous avoir signalé ces paires.

Question 45. *Tout homéomorphisme $O(u)$-quasiMöbius de la sphère à l'infini d'un espace symétrique de rang un (au bord de Gromov d'un groupe hyperbolique de type connexe) est-il le prolongement au bord d'une $O(u)$-équivalence bilipschitzienne ?*

En effet, une très grande souplesse est autorisée a priori dans la construction des homéomorphismes $O(u)$-quasiMöbius (voir II.1.2) de sorte qu'on n'espère pas que le groupe des équivalences bilipschitziennes sous-linéaires de Y puisse être de dimension finie dans ce cas.

On poursuit vers les questions de rigidité. Voici une fomulation très générale.

Question 46. *Pour quels espaces métriques Y a-t-on que pour tout G de type fini, si G est $o(r)$-bilipschitziennement équivalent à Y alors G opère géométriquement sur Y ?*

La réponse à la question 46 est

- Positive pour $Y = \mathbf{R}^n$ d'après le théorème de croissance polynomiale, (6) et (6'), et la remarque 16 (ou Cornulier [39, 3.5]).
- Positive pour $Y = \mathbb{H}^2_{\mathbf{R}}$ d'après Cornulier, voir C.3.
- Négative en général pour les groupes nilpotents, résolubles... d'après le théorème 22.

Pour $Y = \mathbb{H}^3_{\mathbf{R}}$ Cornulier note que, via son théorème de prolongement en homéomorphismes au bord, une réponse positive serait impliquée par la conjecture de Cannon, et nos travaux ne permettent pas d'envisager une autre voie. A ce stade il n'est pas exclu a priori qu'un groupe de type fini sous-linéairement bilipschitziennement équivalent au revêtement universel d'une variété compacte indécomposable de dimension 3 soit commensurable au groupe fondamental d'une telle variété (pas forcément la même).

Nous espérons que l'énoncé analogue à la rigidité quasiisométrique pourrait tenir pour les équivalences sous-linéaires, et les réseaux non-uniformes dans les groupes semi-simples. Il faut pour cela commencer à investiguer les plongements sous-linéairement bilipschitziens entre les réseaux non uniformes, dans la généralité qui est celle de Fisher et Nguyen [64].

D.2.4. Groupes de type fini (recherche d'exemples minimaux)

Voici deux paires de groupes de type fini polycycliques qui sont sous-linéairement bilipschitziennement équivalents (car réseaux uniformes dans des groupes de Lie résolubles, et via le théorème 22) mais ne sont pas quasiisométriques.

Exemple 47 (cf. Cornulier [40])**.** Soient Λ et Λ' des réseaux dans les groupes de Lie $L_{5,7}$ et $L_{5,6}$ de la table 1 ($\mathfrak{l}_{5,7}$ est $\mathfrak{g}_{5,3}$ dans la nomenclature de Magnin [109, 3.2.3]). Les algèbres de Lie-Malcev associées sont telles que $\mathfrak{l}_{5,7|\mathbf{Q}} = \mathrm{gr}(\mathfrak{l}_{5,6|\mathbf{Q}}) = \mathfrak{l}_{4,3|\mathbf{Q}} \oplus \mathbf{Q}$ mais les anneaux de cohomologie ne sont pas isomorphes bien que les nombres de Betti soient les mêmes. Donc Λ et Λ' ne sont pas quasiisométriques d'après le théorème de Sauer [139, Theorem 1.5].

Exemple 48 (abélien-par-cyclique)**.** Soit $H \in \mathrm{SL}(2, \mathbf{Z})$ une matrice hyperbolique, puis $G_2 = \mathbf{Z}^4 \rtimes_A \mathbf{Z}$ et $G_2' = \mathbf{Z}^4 \rtimes_{A'} \mathbf{Z}$ où

$$A = \begin{pmatrix} H & 0 \\ 0 & H \end{pmatrix}, \quad A' = \begin{pmatrix} H & I \\ 0 & H \end{pmatrix}.$$

Le fait que G_2 et G_2' ne soient pas quasiisométriques découle de résultats de Peng [134, Corollary 5.3.8].

Deux éléments de $\mathrm{SL}(3, \mathbf{Z})$ sans valeur propre égale à 1 et même polynôme caractéristique étant conjuguées pour des raisons arithmétiques[37], il n'y a pas d'exemple de dimension cohomologique virtuelle plus petite que 4 du type abélien-par-cyclique. Du côté nilpotent il n'y en a pas car les groupes non Carnot-graduables n'apparaissent qu'en dimension 5.

Question 49. *Existe-t-il une paire $\{G, G'\}$ de groupes polycycliques non quasiisométriques mais SBE, de dimension cohomologique virtuelle $\leqslant 4$?*

(Même pour les réseaux, en dehors du cas nilpotent où elle coïncide avec la dimension topologique des cônes, la dimension cohomologique virtuelle n'a pas de raison a priori d'être un invariant de SBE). Du côté des groupes hyperboliques discrets, on ne connaît aucune telle paire [39, Question 1.15].

Indépendamment de ce problème, pour distinguer les groupes de type fini il serait souhaitable d'utiliser d'autres invariants asymptotiques, par exemple l'espace des bouts, pas seulement sa structure topologique, mais aussi sa structure Hölder [39, Corollary 1.4].

[37] Soient $A, A' \in \mathrm{SL}(3, \mathbf{Z})$ tels que $\chi_A = \chi_{A'} = P$, non conjuguées. Nécessairement P se scinde non simplement sous la forme $(T-\xi)^2(T-\xi^{-2})$ avec $\xi \in \overline{\mathbf{Z}}$ a priori, mais en fait $\xi \in \mathbf{Q}$ car $P \in \mathbf{Z}[X]$ situe ξ dans deux corps de nombres distincts de degré 3, et 3 est un nombre premier. Finalement $\nu_2(\xi) = \nu_2(\xi^{-1}) = 0$, donc $\xi = \pm 1$.

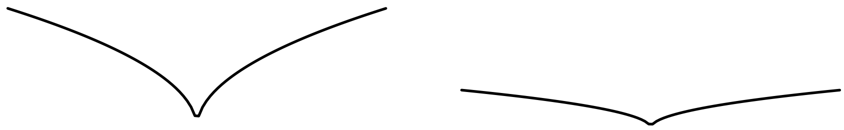


FIGURE 6 : Tout plongement $O(1)$-isométrique $\mathbf{R} \to \mathbf{R}^2$ est $O(\sqrt{r})$-proche du paramétrage d'une droite affine par abscisse curviligne. Ici le plongement $x \mapsto (x, \sqrt{|x|})$ figuré à deux échelles.

D.2.5. Stabilité des applications sous-linéairement isométriques entre espaces auto-similaires Les plongements quasiisométriques, et mêmes les quasiisométries[38], entre espaces euclidiens peuvent être assez arbitraires [99, p.115]. Les $O(u)$-isométries (dont la constante de Lipschitz est égale à 1) sont-elles proches d'isométries ? On reformule ci-dessous des résultats de la littérature.

Théorème 50 (Hyers-Ulam [93] pour les espaces de Hilbert, Gevirtz [69] pour les espaces de Banach)**.** *Soit X un espace de Banach, soit $f : X \to X$ une $O(1)$-isométrie. Alors f est à distance bornée d'une isométrie.*

Puisque les isométries sont elles-même affines d'après le théorème de Mazur et Ulam, on peut remplacer « isométrie » par « isomorphisme linéaire » dans la conclusion.

Théorème 51 (Rassias, Xiang [136])**.** *Soit X un espace de Banach $e \in [0,1)$. Soit Y un espace $L^q(\mu)$ avec[39] $q \leqslant 2$. Soit $f : X \to Y$ un plongement $O(r^e)$-isometrique. f est $O(r^{e'})$-proche d'un plongement isométrique avec $e' = (1+e)/2$.*

Si l'on compare les deux énoncés, on constate que la sous-linéarité permet de s'affranchir de l'hypothèse de quasisurjectivité, bien qu'elle amène une perte, le passage de e à e'. Voici quelques questions :

Question 52. *Existe-t-il e tel que si $f : \mathbf{R} \to N$ est un plongement $O(1)$-isométrique de $\mathbf{R}$ dans un groupe de Carnot sous-riemannien, alors f est $O(r^e)$-proche d'un plongement isométrique ?*

[38] Par une application ingénieuse du théorème de Borsuk-Ulam, les plongements quasiisométriques de $\mathbf{R}^n$ dans lui-même se trouvent être quasisurjectifs [143, Lemma 2.3] sans utiliser la théorie des cônes asymptotiques.

[39] q n'est là que pour quantifier l'uniforme convexité.

Question 53. *Soit $f : N \to N'$ un plongement $O(1)$-isometrique entre groupes de Carnot sous-riemanniens. f est-il $O(r^e)$-proche d'un plongement isométrique ? d'un plongement isométrique affine ?*

Dans ce dernier cas les deux conclusions diffèrent, voir Kivioja et Le Donne [98].

D.2.6. Hyperbolicité sous-linéaire Sans hypothèse d'homogénéité, l'hyperbolicité au sens de Gromov peut être perdue par équivalence bilipschitzienne sous-linéaire. Nous n'avons pas le sentiment d'avoir dégagé une notion d'hyperbolicité sous-linéaire convenable avec laquelle travailler. Nous rassemblons néanmoins dans la Section II.4 quelques exemples riemanniens pour lesquels la courbure riemannienne décroît (en valeur absolue) suffisamment lentement à l'infini pour que la constante d'hyperbolicité dans les grandes boules reste négligeable face à leur rayon. On espère retrouver une partie de la structure quasiconforme au bord géodésique. Nous tentons finalement une comparaison avec certaines manifestations d'hyperbolicité affaiblie observées en géométrie hyperbolique aléatoire.

E. Contenu

Cette thèse rassemble deux articles pré-publiés. Le chapitre I reprend le contenu de [122] (à l'exception d'une mise à jour des références), à paraître sous le titre *Large-scale sublinearly Lipschitz geometry of hyperbolic spaces*. Son objet est la preuve du théorème I. Le chapitre II reprend le contenu de la prépublication [123] intitulée *Sublinear quasiconformality and the large-scale geometry of Heintze groups*, augmenté de la section II.4. Son objet est la preuve du théorème II, ainsi que l'étude des propriétés des homéomorphismes sous-linéairement quasisymétriques introduits au chapitre I.

I

SBE between hyperbolic metric spaces

Sublinearly Lipschitz maps between metric spaces have been gradually made into an object of study by Y. Cornulier in a series of papers starting in 2008 [34, 35, 39]. Here is a short definition of a sublinearly biLipschitz equivalence (compare to Definition I.10):

Definition I.1. Let X and Y be pointed metric spaces. In X and Y, denote the distances by $|\cdot - \cdot|$ and distances to the base-point by $|\cdot|$. A map $f : X \to Y$ is called a sublinearly biLipschitz equivalence (SBE) if there exists a nondecreasing, doubling function $u : \mathbf{R}_{\geqslant 0} \to \mathbf{R}_{\geqslant 1}$ with $u(r) \ll r$ as $r \to +\infty$, and $(\underline{\lambda}, \overline{\lambda}) \in \mathbf{R}^2_{>0}$ such that for any x, x' in X and y in Y,

$$\underline{\lambda}|x - x'| - u(|x| \vee |x'|) \leqslant |f(x) - f(x')| \leqslant \overline{\lambda}|x - x'| + u(|x| \vee |x'|), \tag{I.1}$$

$$\inf\left\{|y - y'| : y' \in f(X)\right\} \leqslant u(|y|), \tag{I.2}$$

where $|x| \vee |x'|$ denotes $\sup\{|x|, |x'|\}$.

Note that while the function u in the definition may depend up to an additive (or multiplicative, as u takes values higher than 1) error on base-points, the large-scale Lipschitz and reverse Lipschitz data $(\underline{\lambda}, \overline{\lambda})$ do not. The technical conditions on u are required so that there is a well-behaved notion of $(\lambda, O(u))$-sublinearly biLipschitz equivalence (resp. $(\lambda, o(u))$-sublinearly biLipschitz equivalence) between nonpointed metric spaces; it is useful to retain only the class $O(u)$ or $o(u)$ for composition purposes, see Cornulier [39, Proposition 2.2] and section I.1 below. When $u = 1$, $O(u)$-sublinearly biLipschitz equivalences are the more traditional quasiisometric maps.

Sublinearly Lipschitz maps were devised in the first place so that for any nonprincipal ultrafilter ω over $\mathbf{Z}_{\geqslant 0}$ or $\mathbf{R}_{\geqslant 0}$ and scaling sequence (λ_j), $\mathrm{Con}_\omega(\cdot, \lambda_j)$ (with fixed basepoint) defines a functor from the large-scale sublinearly Lipschitz category to the Lipschitz category [35, Proposition 2.9].

The asymptotic cone characterization of hyperbolicity (Gromov [83, 2.A], Druţu [49, 3.A.1.(iii)]) ensures that within the class of quasihomogeneous, geodesic metric spaces (such as finitely generated groups), hyperbolicity is preserved by sublinearly biLipschitz equivalences (see Cornulier [39, Proposition 4.2]). However, while asymptotic cones up to biLipschitz homeomorphisms are fine SBE invariants in order to distinguish, e.g., nilpotent groups, this is not the case in the hyperbolic setting, since all complete nonpositively curved Riemannian manifolds and nonelementary Gromov-hyperbolic groups share the same asymptotic cones, namely the universal $2^{\aleph_0}$-branched $\mathbf{R}$-tree, even defined up to isometry (see for instance Erschler and Polterovich [56, Theorem 1.1.3]). This suggests to study the effects of SBEs on other asymptotic invariants instead. In this direction, Cornulier proved that sublinearly biLipschitz equivalences induce biHölder homeomorphisms between geodesic boundaries of proper geodesic hyperbolic metric spaces equipped with visual distances [39, Theorem 1.7 and Theorem 4.3]. Restated within the spaces, this says that for pairs of triples of far apart points sent to each other by a sublinearly biLipschitz equivalence, Gromov products in the source and target are within linear control of each other, a feature which may be derived from the large scale biLipschitz behavior. Similarly to Gromov products, cross-differences, or positive logarithms of cross-ratios, have an incarnation as large distances within the space, so that one can hope that the same control remains between them, with a sublinear error term. This is our main result.

Theorem I.2 (Restatement of Theorem I.41)**.** *Let $f : X \to Y$ be a $(\underline{\lambda}, \overline{\lambda}, O(u))$-sublinearly biLipschitz equivalence between hyperbolic proper geodesic spaces. Then f induces a map φ between the geodesic boundaries with the property that for all distinct $(\xi_1, \ldots \xi_4)$ on the geodesic boundary of X, all of them close enough,*

$$\underline{\lambda} \log^+[\xi_i] - v\left(\overline{\boxtimes}\{\xi_i\}\right) \leqslant \log^+[\varphi(\xi_i)] \leqslant \overline{\lambda} \log^+[\xi_i] + v\left(\overline{\boxtimes}\{\xi_i\}\right), \tag{I.3}$$

where $v = O(u)$ is a sublinear function, $\log^+(s) = \max(0, \log s)$ for all $s \in \mathbf{R}_{>0}$, $\overline{\boxtimes}\{\xi_i\}$ denotes the supremum of all Gromov products over pairs in the four ξ_i's, and the brackets $[\xi_i]$ denote the cross-ratios $[\xi_1, \ldots \xi_4]$ (see I.1.3 for definitions).

The homeomorphisms as in (I.3) are given the name of sublinearly quasiMöbius (Definition I.40). A distinctive feature of sublinearly quasiMöbius homeomorphisms is that their distortion of the moduli of small annuli (or "eccentricity" of small ellipsoids) is bounded at small, non-infinitesimal scale:

Definition I.3. Let Ξ be a metric space. An annulus A of Ξ is a difference of concentric balls $B(\xi, s) \setminus B(\xi, r)$ for some $\xi \in \Xi$ and $r, s \in \mathbf{R}_{>0}$. The real number $\mathfrak{M} = \log(s/r)$ is called a modulus[1] for A.

Proposition I.4 (Restatement of Proposition I.48)**.** *Let Ξ and Ψ be compact, uniformly perfect metric spaces and $\varphi : \Xi \to \Psi$ a $(\lambda^{-1}, \lambda, O(u))$-sublinearly quasiMöbius homeomorphism. Let A be an annulus of inner radius r, outer radius R and modulus $\mathfrak{M}$. There exists $w = O(u)$ such that if R is sufficiently small, $\varphi(A)$ is contained in an annulus of modulus*

$$\mathfrak{M}' = 2\lambda\mathfrak{M} + w(-\log r).$$

When $u = 1$ this is a characterization of power-quasisymmetric mappings, compare Mackay and Tyson, [108, Lemma 1.2.18]. With their scale-sensitive moduli distortion, sublinearly quasiMöbius homeomorphisms may lack the analytic properties of quasisymmetric mappings, even between Euclidean spaces. Nevertheless we prove that they preserve the Hausdorff dimension of visual metrics in a favorable setting:

Proposition I.5 (Consequence of Proposition I.60)**.** *Let Ξ^* and Ψ^* be punctured*[2] *boundaries of purely real, normalized Heintze groups of Carnot type with homogeneous dimensions p and p' (see I.5.2 for definitions). Assume there exists a homeomorphism $\varphi : \Xi^* \to \Psi^*$ which is sublinearly quasiMöbius over any compact subset (with respect to the visual metrics). Then $p = p'$.*

The Heintze groups of Carnot type form an intermediate class between hyperbolic symmetric spaces and simply connected negatively curved homogeneous spaces. The invariance of the topological dimension of the geodesic boundary is more generally granted by Cornulier's theorem on biHölder continuity. Once combined, those two asymptotic invariants allow to distinguish all hyperbolic symmetric spaces, answering a question of Druţu [39, Question 1.16 (2)]:

[1] This would be an ill-defined function if applied to the set A since ξ, r, s may vary, nevertheless we write that A is an annulus of modulus $\mathfrak{M}$. It mostly matters to bound moduli from above.

[2] The puncture is made at a distinguished point so that the remaining part of the boundary is transitively acted upon by the group; see I.5.2.

Theorem I.6. *Let X and Y be rank one Riemannian symmetric spaces of noncompact type. If there exists a sublinearly biLipschitz equivalence between X and Y, then X and Y are homothetic.*

Contents of Chapter I

I.1. Background

I.1.1. Large-scale Sublinearly Lipschitz maps

Here is a summary of Cornulier's definitions included for the reader's convenience. Call admissible any function $u : \mathbf{R}_{\geqslant 0} \to \mathbf{R}_{\geqslant 1}$ with the following properties:

1. u is nondecreasing
2. u is doubling: $\limsup_{r \to +\infty} u(2r)/u(r) < +\infty$
3. u is strictly sublinear: $\lim_{r \to +\infty} u(r)/r = 0$.

It is not really restrictive, and in fact useful in statements, to allow such a function to be only eventually defined and conditions (1), (2) to hold only on a neighborhood of $+\infty$ in $\mathbf{R}_{\geqslant 0}$. However we will frequently work with a precise admissible function u while keeping track on explicit bounds, and where they become valid. To facilitate this we introduce the following notation:

- For all $\varepsilon > 0$, $r_\varepsilon(u)$ is $\sup\{r \in \mathbf{R}_{\geqslant 0} : u(r) > \varepsilon r\}$. This is finite by (3).
- Properties (1), (2) and the fact that $\inf_r u(r) > 0$ ensure that for any $\tau > 1$, $\sup_r u(\tau r)/u(r)$ is finite. We shall denote this number $u \uparrow \tau$.

The following lemma is for our use only; it describes the way in which the constants $r_\varepsilon(u)$ and $u \uparrow \tau$ evolve when shifting function u.

Lemma I.7. *Let u be an admissible function. For any $p \in \mathbf{R}_{>0}$, define $u_p : \mathbf{R}_{\geqslant 0} \to \mathbf{R}_{\geqslant 1}$ as $u_p(t) = u(p+t)$. Then*

1. *for all $\tau \in \mathbf{R}_{>1}$, $u_p \uparrow \tau \leqslant u \uparrow \tau$.*
2. *For all $\varepsilon \in \mathbf{R}_{>0}$, if $p \geqslant r_{\varepsilon/2}(u)$ then*

$$r_\varepsilon(u_p) \leqslant \frac{u \uparrow 2}{\varepsilon} u(p).$$

Proof. Start with (1). By definition, u is nondecreasing, hence

$$u_p \uparrow \tau = \sup_r \frac{u(\tau r + p)}{u(r+p)} \leqslant \sup_r \frac{u(\tau r + \tau p)}{u(r+p)} = u \uparrow \tau.$$

As for (2), the hypothesis made on p means that for all p' greater than p, $u_p(p') \leqslant \frac{\varepsilon}{2}(p+p') \leqslant \varepsilon p'$, so $r_\varepsilon(u_p) \leqslant p$, and then $p + r_\varepsilon(u_p) \leqslant 2p$. Also note that since u_p is nondecreasing, $\varepsilon r_\varepsilon(u_p)$ is equal to $u(p + r_\varepsilon(u_p))$, so that

$$\varepsilon r_\varepsilon(u_p) = u(p + r_\varepsilon(u_p)) \leqslant u(2p) \leqslant (u \uparrow 2) u(p).$$

Finally $r_\varepsilon(u_p) \leqslant \varepsilon^{-1}(u \uparrow 2)u(p)$. □

In the following, let u be an admissible function, and let X and Y be two pointed metric spaces. Recall that whenever r and s are real numbers, $r \vee s$ denotes $\sup\{r, s\}$ and $r \wedge s$ denotes $\inf\{r, s\}$.

Definition I.8. A map $f : X \to Y$ is called $(\overline{\lambda}, O(u))$-Lipschitz if there exists $\overline{\lambda} \in \mathbf{R}_{>0}$ (called a large-scale Lipschitz constant) and a nondecreasing function $v = O(u)$ such that for all $(x_1, x_2) \in X^2$,

$$|f(x_1) - f(x_2)| \leqslant \overline{\lambda}|x_1 - x_2| + v(|x_1| \vee |x_2|).$$

We may write that f is $(\overline{\lambda}, v)$-Lipschitz to put emphasis on v, or on the contrary a $O(u)$-Lipschitz map if the actual Lipschitz constant and function v are not relevant.

Definition I.9. $f, g : X \to Y$ are $O(u)$-close if $|f(x) - g(x)| = O\left(u(|x|)\right)$.

One checks that $O(u)$-Lipschitz maps can be composed (with a multiplicative effect on large-scale Lipschitz constants), in a way compatible with $O(u)$-closeness [39, Proposition 2.2], hence there is a well-defined category $\mathcal{L}_{O(u)}$ with metric spaces as objects[3] and large-scale $O(u)$-Lipschitz maps modulo $O(u)$-closeness as morphisms.

Definition I.10 (compare Definition I.1)**.** $f : X \to Y$ is a $O(u)$-Sublinearly Bilipschitz Equivalence (SBE) if the $O(u)$-closeness class of f is an isomorphism in $\mathcal{L}_{O(u)}$. This can be metric-geometrically rephrased as follows [39, Proposition 2.4]:

1. f is $O(u)$-Lipschitz;
2. f is $O(u)$-expansive : there exists a nondecreasing $v = O(u)$ and $\underline{\lambda} \in \mathbf{R}_{>0}$ such that

$$\forall (x_1, x_2) \in X^2,\ |f(x_1) - f(x_2)| \geqslant \underline{\lambda}|x_1 - x_2| - v(|x| \vee |x'|);$$

[3] More precisely, at first, objects are pointed metric spaces. Nevertheless the notion does not really depend on a given base-point.

3. f is $O(u)$-surjective : for $y \in Y$,

$$d(y, f(X)) = O(u(|y|)).$$

Conditions (1) and (2) alone define the notion of a $O(u)$-Lipschitz embedding. Precisely a $(\underline{\lambda}, \overline{\lambda}, v)$-embedding is a map such that

$$\forall (x_1, x_2) \in X^2,\ \underline{\lambda}|x_1 - x_2| - v(|x| \vee |x'|) \leqslant |f(x_1) - f(x_2)| \leqslant \overline{\lambda}|x_1 - x_2| + v(|x_1| \vee |x_2|).$$

We will give an equivalent definition in subsection I.3.1. If there exists an admissible u such that f is a $O(u)$-sublinearly biLipschitz equivalence (resp. embedding), then f is called a sublinearly biLipschitz equivalence (resp. embedding). In some occasion, we will abbreviate $(\underline{\lambda}, \overline{\lambda})$ into a single biLipschitz constant $\lambda = \sup\{\overline{\lambda}, 1/\underline{\lambda}\}$ and call f a $(\lambda, O(u))$-sublinearly biLipschitz equivalence.

Two metric spaces X and Y such that there exists a sublinearly biLipschitz equivalence $f : X \to Y$ are called asymptotically biLipschitz in Druţu and Kapovich's book [50, 10.8].

I.1.2. Gromov products and Cornulier's estimates

Let X be a metric space. Recall that for $x_0, x_1, x_2 \in X$, the Gromov product of x_2 and x_3 seen from x_0 is by definition $(x_2 \mid x_3)_{x_0} := \frac{1}{2}(|x_1 - x_0| + |x_2 - x_0| - |x_1 - x_2|)$, and that X is δ-hyperbolic (as defined by M.Gromov [82, 1.1.C]) if there exists $\delta \in \mathbf{R}_{\geqslant 0}$ such that

$$\forall (x_0, x_1, x_2, x_3) \in X^4,\ (x_1 \mid x_3)_{x_0} \geqslant \inf\{(x_1 \mid x_2)_{x_0}, (x_2 \mid x_3)_{x_0}\} - \delta. \quad \text{(I.4)}$$

If X is δ-hyperbolic and geodesic, then in addition, the Rips inequality is available: triangles in X are 4δ-slim, [70, 2.21]. A Cauchy-Gromov sequence in X is a sequence $(x_n)_{n \in \mathbf{Z}_{\geqslant 0}}$ such that $(x_n \mid x_m) \to +\infty$ as $n, m \to +\infty$. Two Cauchy-Gromov sequences $\{x_n\}$, $\{y_n\}$ are equivalent, denoted $(x_n) \sim (y_n)$, if $(x_n \mid y_n) \to +\infty$ as $n \to +\infty$. This is an equivalence relation if X is hyperbolic thanks to (I.4), and the Gromov boundary of X is $\partial_{\mathrm{G}} X = \{\text{Cauchy-Gromov sequences}\} / \sim$. If X is in addition proper and geodesic, this is also the visual boundary, or geodesic boundary that we will denote $\partial_\infty X$. Though not stated by Cornulier in this form, the following is given by the proof of his theorem [39, 4.3].

Proposition I.11. *Let u be an admissible function. Assume X and Y are hyperbolic, that X is geodesic, and let $f : X \to Y$ be a $O(u)$-Lipschitz* *embedding. Then f induces a (set-theoretic) boundary map $\partial_G f : \partial_G X \to \partial_G Y$. If g is $O(u)$-close to f, then $\partial_G f = \partial_G g$. If f is $O(u)$-surjective, then $\partial_G f$ is a bijection.*

This can be expressed quantitatively; we restate below certain estimates from Cornulier's proof, at the stage when the points that intervene still lie within the space. Whenever δ is a hyperbolicity constant, set a parameter

$$\mu = \begin{cases} 2^{1/\delta} & \delta > 0 \\ e & \delta = 0, \end{cases} \tag{I.5}$$

fix a base-point $o \in X$ and define a kernel $\rho_\mu : X \times X \to \mathbf{R}_{\geqslant 0}$, $\rho_\mu(x, y) := \mu^{-(x|y)_o}$. The δ-hyperbolicity inequality (I.4) translates into a quasi-ultrametric inequality for ρ_μ : $\rho_\mu(x_0, x_2) \leqslant \mu^\delta \rho(x_0, x_1) \vee \rho_\mu(x_1, x_2)$ for all $(x_0, x_1, x_2) \in X^3$. This ρ_μ can be made subadditive by the chain construction:

$$\check{\rho}_\mu(x, x') := \inf \left\{ \sum_{i=1}^{n} \mu^{-(x_{i-1}|x_i)_o} : n \in \mathbf{Z}_{\geqslant 1}, x = x_0, \dots x_n = x' \right\}. \tag{I.6}$$

Lemma I.12 (Frink 1937, [66, Lemma 2][4])**.** *Let $\mathcal{X}$ be a set and $\varrho : \mathcal{X} \times \mathcal{X} \to \mathbf{R}_{\geqslant 0}$ be a $\mathbf{R}$-valued kernel on $\mathcal{X}$. Assume there is $K \in \mathbf{R}_{\geqslant 1}$ such that for all $(x_0, x_1, x_2) \in \mathcal{X}^3$, $\rho(x_0, x_2) \leqslant K\rho(x_0, x_1) \vee \rho(x_1, x_2)$. Let $\check{\varrho}$ be associated to ϱ by the chain construction (I.6). If $K \leqslant 2$, then $\check{\varrho} \leqslant \varrho \leqslant 4\check{\varrho}$.*

This allows the construction of a true distance $d_\mu = \check{\rho}_\mu$ from ρ_μ on the visual boundary, called the visual distance. Subadditivity of the resulting kernel for points within the space plays a key role in the following result.

Theorem I.13 (Cornulier)**.** *Let v be an admissible function. Let $(\underline{\lambda}, \overline{\lambda})$ be large-scale expansion and Lispschitz constants. Let $f : (X, o) \to (Y, o)$ be a large-scale $(\underline{\lambda}, \overline{\lambda}, v)$-sublinearly biLipschitz embedding. Assume there exists $\delta \in \mathbf{R}_{\geqslant 0}$ such that X and Y are δ-hyperbolic and that X is geodesic. For all $\alpha \in (0, \underline{\lambda})$ there exists a constant $M = M(\alpha, \delta) \in \mathbf{R}_{>0}$ and $R = R(\alpha, \lambda, v, \delta, |f(o)|) \in \mathbf{R}_{>0}$ such that for all $x, x' \in X$,*

$$(x \mid x')_o \geqslant R \implies (f(x) \mid f(x'))_o \geqslant \alpha(x \mid x')_o - M(\alpha, \delta). \tag{I.7}$$

Especially, if X and Y are proper geodesic, then $\partial_G f = \partial_\infty f$ is α-Hölder continuous for the metrics d_μ on the boundaries, where μ is set as in (I.5).

[4] see also Bourbaki [13, IX.6, Proposition 2].

Remark I.14. There is a dependence on μ in Cornulier's version which disappears in (I.7) because μ depends on δ according to convention (I.5).

A particular instance of theorem I.13 occurs when the source space is $\mathbf{R}_{\geqslant 0}$ or $\mathbf{Z}_{\geqslant 0}$. For the latter, constants R and M can be explicitly extracted from the beginning of Cornulier's proof:

$$\forall s, t \in \mathbf{Z}_{\geqslant t_\alpha}, \left(\widetilde{\gamma}(s) \mid \widetilde{\gamma}(t)\right)_o \geqslant \alpha \inf\{s, t\} - \log_\mu \left(\frac{2}{1 - \mu^{-(\alpha + \underline{\lambda})/2}} \right), \tag{I.8}$$

where $\widetilde{\gamma}$ replaces f of Lemma I.13, and

$$t_\alpha = \sup \left\{ s \in \mathbf{Z}_{\geqslant 0} : |\widetilde{\gamma}(0)| + v(s) \geqslant 4(\underline{\lambda} - \alpha)s \right\} \tag{I.9}$$

replaces R of I.13. This form will be of special interest in subsection I.3.2, especially the dependence of t_α on $|\widetilde{\gamma}(0)|$ is important for us.

I.1.3. Metric invariants of 4 points at infinity

Let (Y, o) be a pointed, proper geodesic hyperbolic space, and let $\partial^4_\infty Y$ denote the space of distinct 4-tuples on $\partial_\infty Y$. For $(\eta_1, \eta_2, \eta_3, \eta_4) \in \partial^4_\infty Y$, define

$$\begin{aligned} &\overline{\boxtimes} \left\{\eta_1, \eta_2, \eta_3, \eta_4\right\} := \sup \left\{ (\eta_i \mid \eta_j)_o : i \neq j \right\}, \text{ and} \\ &\underline{\boxtimes} \left\{\eta_1, \eta_2, \eta_3, \eta_4\right\} := \inf \left\{ (\eta_i \mid \eta_j)_o : i \neq j \right\}. \end{aligned}$$

More generally, let (Ξ, ϱ) be a metric space (to be thought of as a geodesic boundary with a visual distance) and let $(\xi_1, \dots \xi_4)$ be distinct points in Ξ. Define their metric cross-ratio as

$$[\xi_1, \xi_2, \xi_3, \xi_4]^\varrho = [\xi_i]^\varrho := \frac{\varrho(\xi_1, \xi_3)\varrho(\xi_2, \xi_4)}{\varrho(\xi_1, \xi_4)\varrho(\xi_2, \xi_3)}.$$

The superscript ϱ might be omitted if sufficiently clear. Observe that if ϱ has been obtained by the chain construction (I.6) from a quasi-distance $\widehat{\varrho}$ on Ξ such that

$$\exists K \in [1, 2), \forall (\xi_1, \xi_2, \xi_3) \in \Xi^3, \varrho(\xi_1, \xi_3) \leqslant K \sup \left\{ \varrho(\xi_1, \xi_2), \varrho(\xi_2, \xi_3) \right\},$$

then by Frink's theorem $\varrho \leqslant \widehat{\varrho} \leqslant 4\varrho$, and

$$\forall \nu \in \mathbf{R}_{>1}, \left| \log_\nu [\xi_i] - \log_\nu \frac{\varrho(\xi_1, \xi_3)\varrho(\xi_2, \xi_4)}{\varrho(\xi_1, \xi_4)\varrho(\xi_2, \xi_3)} \right| \leqslant \log_\nu 16. \tag{I.10}$$

Especially, if $(\Xi, \varrho) = (\partial_\infty X, \check{\rho}_\nu)$ for a δ-hyperbolic, proper geodesic, pointed space (X, o) and a parameter $\nu \in (1, \mu(\delta)]$, then by (I.10), $\log_\nu [\xi_i]^{d_\nu}$ depends on ν only up to an additive error: precisely for all $\nu, \nu' \in (1, \mu]$,

$$\begin{aligned}&\left|\log_\nu [\xi_i]^{d_\nu} - \log_{\nu'} [\xi_i]^{d_{\nu'}}\right| \\ &\leqslant \left|\log_\nu [\xi_i]^{d_\nu} - (\xi_1, \xi_4)_o - (\xi_2, \xi_3)_o + (\xi_1, \xi_3)_o + (\xi_2, \xi_4)_o\right| \\ &+ \left|\log_{\nu'} [\xi_i]^{d_{\nu'}} - (\xi_1, \xi_4)_o - (\xi_2, \xi_3)_o + (\xi_1, \xi_3)_o + (\xi_2, \xi_4)_o\right| \\ &\leqslant \log_\nu 16 + \log_{\nu'} 16.\end{aligned}$$

In the sequel we refer to $\log_\mu [\xi_i]^{d_\mu}$ as $\log[\xi_i]$, where μ follows convention (I.5). When nonnegative, this logarithm has a geometric interpretation:

Proposition I.15. *Let (X, o) be a proper geodesic, δ-hyperbolic space. There exists a constant $C = C(\delta)$ in $\mathbf{R}_{\geqslant 0}$ such that for all $(\xi_1, \dots \xi_4) \in \partial^4 X$,*

$$d_X(\chi_{14}, \chi_{23}) - C \leqslant \log^+[\xi_i] \leqslant d_X(\chi_{14}, \chi_{23}) + C.$$

where χ_{ij} are geodesic lines between ξ_i and ξ_j (whose existence is provided by the visibility property of X, see Ghys and de la Harpe [70, 7.6]).

Proposition I.15 seems well-known, yet we could not locate a proof in the literature, so we include one in subsection I.2.4. It is better understood as a statement about cross-differences, see Buyalo and Schroeder [27, 4.1].

I.2. Preliminaries from hyperbolic metric geometry

I.2.1. A lemma on right-angled quadrilaterals

Let $\delta \in \mathbf{R}_{\geqslant 0}$ be a constant, and let X be a geodesic δ-hyperbolic metric space. We shall work under the following convention. In the course of proofs or statements about X, one often needs to construct objects (e.g. a geodesic segment between two points). The rules of δ-hyperbolic geometry only allow to locate such objects in X up to a few multiples of δ. For us, as soon as an object in X has been constructed, it remains fixed until the end of the statement or proof so that forthcoming objects can be attached to it. This means that, for instance, if a geodesic segment between two points has been previously defined, then the midpoint of these points will be understood as the midpoint of this geodesic segment. Especially, if[5] $\gamma \subset X$ is a geodesically convex subspace and $b \in X$ is a point, $p_\gamma(b)$ is an orthogonal projection (closest point) of b on γ. This is well defined up to 16δ, and p_γ has a contracting behavior on distances expressed by the following lemma.

[5] We will abusively write γ when referring to $\mathrm{im}(\gamma)$ when γ is a (quasi)geodesic.

Lemma I.16 (See Shchur, [142, Lemma 1][6]). *Let γ be a geodesic, b a point in X. Then for all $c \in \gamma$, $|c-p_\gamma(b)| \leqslant |b-c|-|b-p_\gamma(b)|+16\delta$. In particular, for all $b, b' \in X$,*

$$|p_\gamma(b) - p_\gamma(b')| \leqslant |b - b'| + 16\delta. \tag{I.11}$$

Definition I.17. Let $\alpha \in \mathbf{R}_{\geqslant 0}$. Say that a metric space P is α-connected if for any $\alpha' \in \mathbf{R}_{>\alpha}$, the equivalence relation generated by $[d(x,y) \leqslant \alpha']$ over $x, y \in P$ has a unique class.

Lemma I.18. *Let $\alpha \in \mathbf{R}_{>0}$ and let $S \subset X$ be a α-connected subspace (for instance a quasigeodesic). Let γ be a geodesic of X. Then any $p_\gamma(S)$ is $(\alpha + 16\delta)$-connected. In particular if S is a geodesic then $p_\gamma(S)$ is 16δ-connected.*

Proof. Let $S' = p_\gamma(S)$ and let $\alpha' \in \mathbf{R}_{>0}$ be such that $\alpha' > \alpha + 16\delta$. If there is $s'_1 = p_\gamma(s_1)$ such that $d(s'_1, S' \setminus \{s'_1\}) \geqslant \alpha'$, then for all $s'_2 = p_\gamma(s_2)$, $|s'_1 - s'_2| > \alpha'$ implies with (I.11), that $s_1 - s_2 > \alpha'$. Thus S is not α-connected. □

Definition I.19. Let $\eta \in \mathbf{R}_{\geqslant 0}$ be a constant and let X be a geodesic space. Say that an ordered list $x_1, \dots x_r$ of points in X with $r \geqslant 3$ is η-almost lined up if there exists a geodesic segment σ such that for all i, x_i lies in the η-neighborhood $\mathcal{N}_\eta(\sigma)$ of $\operatorname{im}(\sigma)$ and the $p_\sigma(x_i)$ are lined up in this order on σ.

Lemma I.20 (Gromov product of almost lined up points). *Let $\eta \in \mathbf{R}_{\geqslant 0}$ and let x_1, x_2, x_3 be three points in a geodesic metric space X which are η-almost lined up. Then*

$$|(x_2 \mid x_3)_{x_1} - |x_1 - x_2|| \leqslant 5\eta. \tag{I.12}$$

Proof. Let σ be a geodesic segment achieving the almost-lined upness assumption. For $i \in \{1,2,3\}$, let $y_i = p_\sigma(x_i)$. By hypothesis $|x_i - y_i| \leqslant \eta$, so by the triangle inequality $||y_i - y_j| - |x_i - x_j|| \leqslant 2\eta$; then by definition of the Gromov product $\left|(x_2 \mid x_3)_{x_1} - (y_2 \mid y_3)_{y_1}\right| \leqslant 3\eta$. Finally, y_1, y_2 and y_3 are lined up, hence $(y_2 \mid y_3)_{y_1} = |y_1 - y_2|$. The conclusion follows from the triangle inequality in $\mathbf{R}$. □

Lemma I.21 (Right-angled triangles degenerate). *Let σ be a geodesic of a geodesic hyperbolic space X, $b \in X$ and $a = p_\sigma(b)$ on σ. Let c be a point of σ. Then there exists $t \in [bc]$ such that*

[6] There is a 4δ additive error term instead of our 16δ in Shchur's version, because Shchur defines a δ-hyperbolic space via Rips inequality there.

1. *$|a-t| \leqslant 28\delta$*

2. *$d(t,\sigma) \leqslant 4\delta$ and $d(t,[ba]) \leqslant 4\delta$.*

3. *for any u in the subsegment $[tc]$ of $[bc]$, $d(u,\sigma) \leqslant 4\delta$.*

In particular, if $|b-a|$, $|c-a|$ are large enough, then b,a,c are 28δ-almost lined up in this order.

Proof. Let $\triangle$ be the geodesic triangle abc with sides $[ba]$, $[bc]$ and the subsegment $[ac]$ of σ. Set $\ell = |b-c|$ and assume $\alpha : [0,\ell] \to X$ parametrizes $[bc]$ so that $\alpha(0) = c$, $\alpha(\ell) = b$. If $\sup\{d(\alpha(s),\sigma) : s \in [0,\ell]\} \leqslant 4\delta$, set $t=b$; then (3) and (2) are automatically true, while $|a-t| = d(t,\sigma) \leqslant 4\delta \leqslant 28\delta$ so that also (1) is true. Otherwise, define

$$t = \alpha(s),\ s = \inf\left\{u \in [0,\ell], d(\alpha(u),\sigma) > 4\delta\right\}.$$

As $\triangle$ is 4δ-slim, $d(t,[ba]) \leqslant 4\delta$ while $d(t,\sigma) \leqslant 4\delta$ also. Let t_b, resp. t_c be an orthogonal projection of t on σ, resp. on $[ba]$. By the triangle inequality, $|t_c - t_b| \leqslant 4\delta + 4\delta = 8\delta$. Then $|t_b - b| \leqslant |t_c - b| + 8\delta \leqslant |b-a| + 8\delta$. By the contraction Lemma I.16, $|t_b - a| \leqslant 8\delta + 16\delta = 24\delta$. By the triangle inequality, $|t-a| \leqslant 24\delta + 4\delta = 28\delta$. □

Lemma I.22 (Quadrilaterals with two consecutive right-angles degenerate)**.** *Let a_0, a_1, b_0, b_1 be four points in X. For $i \in \{0,1\}$, let γ_i be a geodesic segment between a_i and b_i. Assume that $138\delta \leqslant |a_0 - a_1|$, and that one of the following holds:*

1. *Either, $a_i = p_\sigma(b_i)$ for all $i \in \{0,1\}$, or*

2. *$a_i = p_{\gamma_i} a_{1-i}$ for all $i \in \{0,1\}$.*

Then for all $i \in \{0,1\}$, $d(a_i, [b_0 b_1]) \leqslant 56\delta$.

Proof. Let σ be a geodesic segment between a_0 and a_1, and let m be the midpoint of σ. By Lemma I.21, there exists t_0 and t_1 on $[b_0 m]$ and $[b_1 m]$ respectively such that

$$\forall i \in \{0,1\},\ |a_i - t_i| \leqslant 28\delta \tag{I.13}$$

Moreover, by (2) of Lemma I.21 and the triangle inequality,

$$|p_\sigma(t_i) - a_i| \leqslant |p_\sigma(t_i) - t_i| + |t_i - a_i| \leqslant 4\delta + 28\delta = 32\delta. \tag{I.14}$$

Thus a_i, $p_\sigma(t_i)$, m and a_{1-i} are lined up on σ as below:

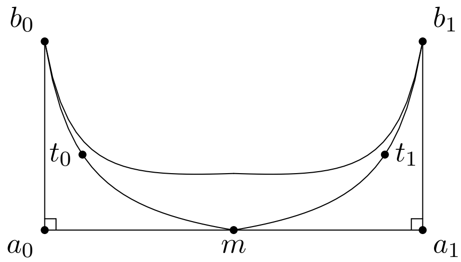
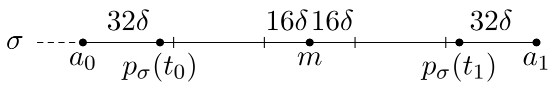


Figure 7: Main points occurring in the proof of Lemma I.22.



Next, we proceed to prove that t_i is far from $[mb_{1-i}]$. Note that since the triangles ma_ib_i are slim, one need only show that t_i is far from $[a_{1-i}b_{1-i}]$ and $[ma_{1-i}]$.

- In case (1), for all $a_i' \in \gamma_i$, since $p_\sigma(a_i') = a_i$ and by (I.14) and Lemma I.16,

$$|t_i - a_{1-i}'| \geqslant |p_\sigma(t_i) - a_{1-i}| - 16\delta \geqslant |a_i - a_{1-i}| - |p_\sigma(t_i) - a_i| - 16\delta$$
$$\geqslant 138\delta - 48\delta = 90\delta,$$

  hence $d(t_i, \gamma_{1-i}) \geqslant 90\delta$.

- In case (2), as $a_{1-i} = p_{\gamma_{1-i}}a_i$, $d(t_i, \gamma_{1-i}) \geqslant d(a_i, \gamma_{1-i}) - 28\delta \geqslant 110\delta$.

- In both cases, $d(t_i, [ma_{1-i}]) \geqslant 69\delta - 32\delta = 35\delta$.

Using the previous inequality together with the fact that the triangle $a_{1-i}mb_{1-i}$ is 4δ-slim,

$$d(t_i, [mb_{1-i}]) \geqslant 35\delta - 4\delta = 31\delta > 4\delta.$$

Finally, b_1mb_2 is 4δ-slim, hence $d(t_i, [b_1b_2]) \leqslant 4\delta$, and by the triangle inequality,

$$d(a_i, [b_0b_1]) \leqslant |a_i - t_i| + d(t_i, [b_0b_1]) \leqslant 28\delta + 4\delta \leqslant 56\delta. \qquad \square$$

I.2.2. An estimate on geodesic projections

Let X be as before a geodesic δ-hyperbolic metric space, and fix a base-point $o \in X$.

Lemma I.23. *Let $\gamma, \gamma' : \mathbf{R} \to X$ be two geodesics; define $\xi_- = [\gamma]_{-\infty}$, $\xi_+ = [\gamma]_{+\infty}$, $\xi'_- = [\gamma']_{-\infty}$ and $\xi'_+ = [\gamma']_{+\infty}$ on the boundary at infinity $\partial_\infty X$ of X. Assume that the $\xi_\pm, \xi'_\pm$ are all distinct. Then*

$$\sup\left\{|p_\gamma(b)| : b \in \gamma'\right\} \leqslant \overline{\boxtimes}\{\xi_-, \xi_+, \xi'_-, \xi'_+\} + 284\delta, \tag{I.15}$$

where we recall that $\overline{\boxtimes}\{\xi_-, \xi_+, \xi'_-, \xi'_+\}$ is an abbreviation for $\sup\,(\xi_1 \mid \xi_2)_o$ over distinct pairs $\{\xi_1, \xi_2\}$ in $\{\xi_\pm, \xi'_\pm\}$.

Proof. Change if necessary the parametrizations of γ and γ' in such a way that $\gamma(0) = p_\gamma(o)$, $\gamma'(0) = p_{\gamma'}(o)$. Let $b \in \gamma'$.

- Either $|p_\gamma(b) - \gamma(0)| < 138\delta$; then by the triangle inequality, $|p_\gamma(b)| < |\gamma(0)| + 138\delta$. Let $s \in \mathbf{R}$. Since X is δ-hyperbolic,

  $$(\gamma(s) \mid \gamma(-s))_o \geqslant \min\left\{(\gamma(-s) \mid \gamma(0))_o, (\gamma(0) \mid \gamma(s))_o\right\} - \delta. \tag{I.16}$$

  By Lemma I.21, when s is large enough o, $\gamma(0)$ and $\gamma(s)$ (resp. o, $\gamma(0)$ and $\gamma(s)$) are 28δ-almost lined up in this order, so by Lemma I.20, (I.16) becomes

  $$(\gamma(s) \mid \gamma(-s))_o \geqslant |\gamma(0)| - 5 \cdot 28\delta - \delta = |\gamma(0)| - 141\delta.$$

  Finally, $|p_\gamma(b)| < |\gamma(0)| + 138\delta \leqslant (\gamma(s) \mid \gamma(-s))_o + 138\delta + 141\delta$. Letting $s \to +\infty$,

  $$\begin{aligned} |p_\gamma(b)| \leqslant \liminf_{s \to +\infty} (\gamma(s) \mid \gamma(-s))_o &\leqslant (\xi_- \mid \xi_+)_o + 279\delta \\ &\leqslant (\xi_- \mid \xi_+)_o + 284\delta. \end{aligned}$$

- Or $|p_\gamma(b) - \gamma(0)| \geqslant 138\delta$ in which case Lemma I.22 applies so that o, $\gamma(0)$ and $p_\gamma(b), b$ are 56δ-almost lined up in this order. Let $s, s' \in \mathbf{R}$ be such that $\inf\{|s|, |s'|\} \geqslant \sup\{|p_\gamma(b) - \gamma(0)|, |b - \gamma'(0)|\}$. Then

  $$\begin{aligned} (\gamma(s) \mid \gamma'(s'))_o \geqslant \min\{&(\gamma(s) \mid \gamma(0))_o, (\gamma(0), p_\gamma(b))_o, \\ &(p_\gamma(b) \mid b)_o, (b \mid \gamma'(0))_o, (\gamma'(0) \mid \gamma'(s'))_o\} - 4\delta. \end{aligned}$$

  Applying repeatedly Lemma I.20,

  $$\begin{aligned} (\gamma(s) \mid \gamma'(s'))_o \geqslant \min\{&|\gamma(0)| - 140\delta, |\gamma(0)| - 140\delta, \\ &|p_\gamma(b)| - 5 \cdot 56\delta, |\gamma'(0)| - 140\delta, |\gamma'(0)| - 140\delta\} - 4\delta. \end{aligned}$$

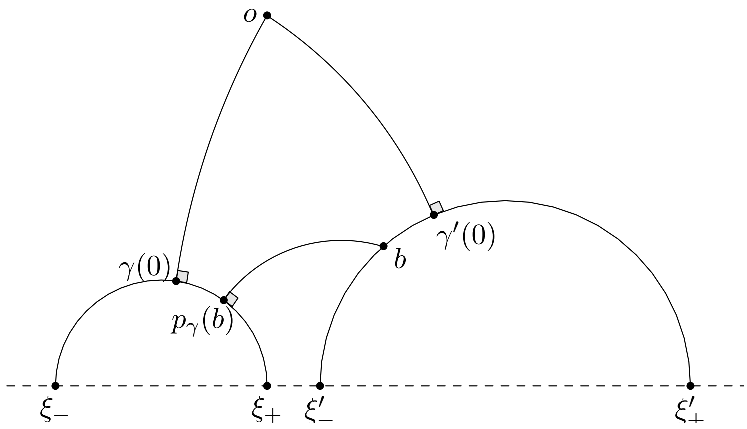


Figure 8: Configuration of Lemma I.23 in the half-plane model of $\mathbb{H}^2$.

Now letting $s, s' \to \pm\infty$,

$$\begin{aligned}|p_\gamma(b)| &\leqslant \overline{\boxtimes}\{\xi_-, \xi'_-, \xi_+, \xi'_+\} + 5 \cdot 56\delta + 4\delta \\ &= \overline{\boxtimes}\{\xi_-, \xi'_-, \xi_+, \xi'_+\} + 284\delta. \end{aligned}$$ □

I.2.3. Quantitative Morse stability

We prove here a version of the Morse lemma with an emphasis on the linear dependence of the tracking distance on the quasiisometry additive error term.

Lemma I.24 (Morse stability for quasigeodesics)**.** *Let $c, \delta \in \mathbf{R}_{\geqslant 0}$, $(\underline{\lambda}, \overline{\lambda}) \in \mathbf{R}^2_{>0}$ be constants. Let X be a geodesic, δ-hyperbolic metric space. Let $J = [a, b]$ be a closed bounded interval of $\mathbf{R}$ and let $\widetilde{\gamma} : J \to X$ be $(\underline{\lambda}, \overline{\lambda}, c)$ quasigeodesic, i.e.*

$$\forall (s,t) \in J^2, \underline{\lambda}|s-t| - c \leqslant |\widetilde{\gamma}(s) - \widetilde{\gamma}(t)| \leqslant \overline{\lambda}|s-t| + c.$$

Recall that $\lambda = \sup\{\overline{\lambda}, 1/\underline{\lambda})$, and assume that[7] *$c \geqslant 6\lambda^2\delta$. There exist functions $h, \widetilde{h} : \mathbf{R}_{>0} \to \mathbf{R}_{>0}$ such that if $\gamma : [0, |\widetilde{\gamma}(b) - \widetilde{\gamma}(a)|] \to X$ is any geodesic segment with same endpoints as $\widetilde{\gamma}$, then*

$$\forall t \in J, d(\widetilde{\gamma}(t), \mathrm{im}(\gamma)) \leqslant h(\lambda)(\delta + c) \tag{I.17}$$

$$\forall s \in [0, |\widetilde{\gamma}(b) - \widetilde{\gamma}(a)|], d(\gamma(s), \mathrm{im}(\widetilde{\gamma})) \leqslant \widetilde{h}(\lambda)(\delta + c). \tag{I.18}$$

Precisely, h and $\widetilde{h}$ can be taken as $h(\lambda) = 12(1+8\lambda^2)$ and $\widetilde{h}(\lambda) = 16(5+6\lambda^2)$.

[7] This assumption could be dropped; we make it in order to simplify h and $\widetilde{h}$, and because c is to be replaced by an unbounded function v in the next section.

Remark I.25. Our expression for $\widetilde{h}(\lambda)$ is certainly not optimal: Shchur [142, Theorem 2] claims that $\widetilde{h}(\lambda) = O(\log \lambda)$. For us in the following, only the linear dependence over the sum of additive errors $\delta + c$ in (I.17) and (I.18) matters.

Proof. A sketch of proof for the part of lemma expressed by (I.17) can be found in Thurston's exposition of the Mostow rigidity theorem, [147, 5.9.2] with non-explicit right-hand side bound; see also an early (and more explicit) proof by Efremovich and Tihomirova [58, p. 1142–1143], also taking place in $\mathbb{H}^n_{\mathbf{R}}$. When projecting onto a geodesic line in hyperbolic space, the lengths of curves situated at a distance η are contracted with a factor depending exponentially[8] on η, so that the length of portions of quasigeodesics leaving a tube of thickness η around a geodesic can be bounded. This can be carried into a general argument in δ-hyperbolic space, replacing length by a rough analogue; for this we build on Shchur's work [142]. For $\alpha \in \mathbf{R}_{>0}$, $I \subset \mathbf{R}$ a bounded interval and $\sigma : I \to X$ a curve such that $\sigma(I)$ is $\alpha/2$-connected, define the length of σ at scale α as

$$\ell_\alpha(\sigma) = \sup_{(t_i) \in T_\alpha(\sigma)} \sum_i |\sigma(t_i+1) - \sigma(t_i)|,$$

where $(t_i) \in T_\alpha(\sigma)$ if there is $r \in \mathbf{Z}_{\geqslant 0}$ such that $\inf I = t_0 < \cdots < \sup I = t_r$ and if $\{\sigma(t_i)\}$ is a α-separated net in $\mathrm{im}(\sigma)$. If σ is a $(\underline{\lambda}, \overline{\lambda}, c)$-quasi-geodesic segment (e.g. a portion of $\widetilde{\gamma}$) and α is such that $\alpha \geqslant 2c$, then

$$\ell_\alpha(\sigma) \leqslant 2\overline{\lambda}|I|, \tag{I.19}$$

see Shchur [142, Lemma 7]. Now let η be a positive real number (to be fixed later). Define $\mathcal{N}_\eta\gamma$ as the η-neighborhood of $\mathrm{im}(\gamma)$ in X, and

$$U_\eta = \{t \in J, \widetilde{\gamma}(t) \notin \mathcal{N}_\eta\gamma\}.$$

Let $I \in \pi_0(U_\eta)$, $t = \inf I$ and $t' = \sup I$. t and t' are both finite, since J is bounded and $\widetilde{\gamma}$ and γ have the same endpoints. Then $\widetilde{\gamma}_{|[t,t']}$ is outside $\mathcal{N}_\eta(\gamma)$; by Shchur's exponential contraction estimate[9] [142, Lemma 10], there

[8] It is useful to write the hyperbolic metric in cylindrical coordinates around γ to appreciate that the contraction factor is a hyperbolic cosine of η.

[9] Shchur's lemma is actually stated in a slightly different form, namely our $|p_\gamma\widetilde{\gamma}(t) - p_\gamma\widetilde{\gamma}(t')|$ is replaced by $\operatorname{diam} p_\gamma\widetilde{\gamma}(I)$, and follows a different convention on the δ hyperbolicity constant.

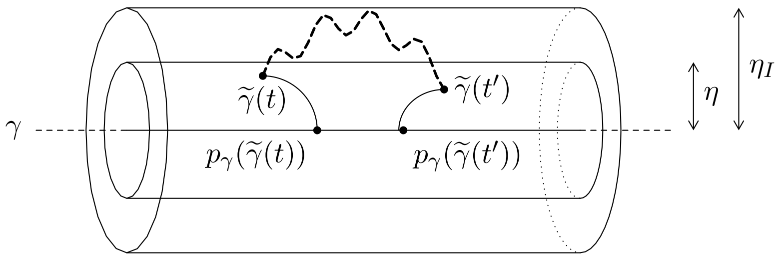


Figure 9: Proof of the Morse stability lemma I.24.

exists a constant $S \in \mathbf{R}_{>0}$ such that, as soon as $\eta \geqslant 2c + 12\delta$,

$$\begin{aligned}\left|p_\gamma\widetilde{\gamma}(t) - p_\gamma\widetilde{\gamma}(t')\right| &\leqslant \sup\left\{\frac{6\delta}{c}e^{-S\eta}\ell_{2c}\widetilde{\gamma}_{|[t,t']}, 24\delta\right\}\\ &\leqslant 24\delta + \frac{6\delta}{c}e^{-S\eta}\ell_{2c}\widetilde{\gamma}_{|[t,t']}.\end{aligned} \tag{I.20}$$

On the other hand,

$$\begin{aligned}\ell_{2c}\widetilde{\gamma}_{|[t,t']} \underset{(\mathrm{I.19})}{\leqslant} 2\overline{\lambda}|t'-t| &\leqslant 2(\overline{\lambda}/\underline{\lambda})\left[|\widetilde{\gamma}(t') - \widetilde{\gamma}(t)| + c\right]\\ &\leqslant 2\lambda^2\left[2\eta + \left|p_\gamma\widetilde{\gamma}(t) - p_\gamma\widetilde{\gamma}(t')\right| + c\right],\end{aligned} \tag{I.21}$$

where we used the triangle inequality together with the fact that $\widetilde{\gamma}(t), \widetilde{\gamma}(t') \in \partial\mathcal{N}_\eta\gamma$ for the last inequality. Combining (I.20) and (I.21),

$$\begin{aligned}\frac{1}{2\lambda^2}\ell_{2c}\widetilde{\gamma}_{|[t,t']} &\leqslant 2\eta + \left|p_\gamma\widetilde{\gamma}(t) - p_\gamma\widetilde{\gamma}(t')\right| + c\\ &\leqslant 2\eta + 24\delta + \frac{6\delta}{c}e^{-S\eta}\ell_{2c}\widetilde{\gamma}_{|[t,t']} + c,\end{aligned}$$

hence

$$\left(\frac{1}{2\lambda^2} - \frac{6\delta}{c}e^{-S\eta}\right)\ell_{2c}\widetilde{\gamma}_{|[t,t']} \leqslant 2\eta + 24\delta + c. \tag{I.22}$$

Define $\eta_I = \sup_{u\in I} d(\widetilde{\gamma}(u), \gamma)$. Then, as $c > 3\delta\lambda^2$ by hypothesis,

$$\eta_I \leqslant \eta + \frac{1}{2}\ell_{2c}\widetilde{\gamma}_{|[t,t']} \vee 2c \underset{(\mathrm{I.22})}{\leqslant} \eta + 2c + \frac{2\eta + 24\delta + c}{1/\lambda^2 - (3\delta/c)\cdot e^{-S\eta}}. \tag{I.23}$$

It remains to set η in order to explicit the bound on η_I given by the last inequality. Actually, as $c \geqslant 6\delta\lambda^2$, if $\eta = 2c + 12\delta$ (remember that $\widetilde{\gamma}_{|I}$ must be at least this far for the exponential contraction to operate),

$$\begin{aligned}\eta_I &\leqslant \eta + 2c + \frac{2\eta + 24\delta + c}{1/(2\lambda^2)}\\ &\leqslant 12\delta + 4c + \lambda^2\left(4\eta + 48\delta + 2c\right)\\ &= 12\delta + 4c + \lambda^2(96\delta + 10c) \leqslant 12(1 + 8\lambda^2)(\delta + c).\end{aligned}$$

Finally,

$$\sup\{d(\widetilde{\gamma}(t),\gamma) : t \in J\} = \eta \vee \sup_{I\in\pi_0 U_\eta} \eta_I \leqslant 12(\delta+c) \vee 12(1+8\lambda^2)(\delta+c)$$
$$= 12(1+8\lambda^2)(\delta+c).$$

This is (I.17). Now, let $s \in [0, |\widetilde{\gamma}(b)-\widetilde{\gamma}(a)|]$. Because $\widetilde{\gamma}$ is c-connected, by Lemma I.18 $p_\gamma\widetilde{\gamma}$ is $c+16\delta$-connected, so there is $s' \in [0, |\widetilde{\gamma}(b)-\widetilde{\gamma}(a)|]$ such that $|s'-s| \leqslant c+16\delta$ and $s' = p_\gamma(\widetilde{\gamma}(t))$ for a $\hat{t} \in J$. The triangle inequality in X yields

$$\begin{aligned} d(\gamma(s), \mathrm{im}(\widetilde{\gamma})) \leqslant |\gamma(s)-\widetilde{\gamma}(\hat{t})| &\leqslant |s-s'| + |\gamma(s')-\widetilde{\gamma}(\hat{t})| \\ &\underset{(\mathrm{I.17})}{\leqslant} 12(1+8\lambda^2)(\delta+c)+16\delta+c \\ &\leqslant 16(5+\lambda^2)(\delta+c). \end{aligned}$$

This is (I.18). □

Remark I.26. V. Shchur [142, Theorem 1] claims a stronger result. However the proof in [142] has a gap, noticed by S. Gouëzel and recently fixed by Gouëzel and Shchur, see [76].

I.2.4. Proof for Proposition I.15

Let $\xi_1, \dots \xi_4$ be as in the statement of Proposition I.15 and assume that the geodesic lines χ_{14} and χ_{23} are parametrized in such a way that a common perpendicular geodesic segment σ falls on $\chi_{14}(0)$ and $\chi_{23}(0)$, accordingly to Figure 10. Let H be the metric subspace of X defined as $\chi_{14} \cup \chi_{23} \cup \sigma$ and denote by $|\cdot|_{\mathsf{H}}$ the path distance in H. By Lemma I.22 (1), if $d(\chi_{14},\chi_{23}) \geqslant 138\delta$ then for all $t \in \mathbf{R}$, whenever $(\chi,\chi') \in \{\chi_{14},\chi_{23}\}^2$ and $\epsilon \in \{\pm 1\}$,

$$\big||\chi(t)-\chi'(\epsilon t)| - |\chi(t)-\chi'(\epsilon t)|_{\mathsf{H}}\big| \leqslant 4\cdot 56\delta = 212\delta. \tag{I.24}$$

For all $t \in \mathbf{R}$ (compare Buyalo and Schroeder [27, p. 37]),

$$2\left\{\begin{array}{c} (\chi_{14}(-t) \mid \chi_{14}(t))_o \\ +\,(\chi_{23}(t) \mid \chi_{23}(-t))_o \\ -\,(\chi_{14}(-t) \mid \chi_{23}(t))_o \\ -\,(\chi_{14}(t) \mid \chi_{23}(-t))_o \end{array}\right\} = \left\{\begin{array}{c} |\chi_{14}(-t)| + |\chi_{14}(t)| - 2t \\ +|\chi_{23}(-t)| + |\chi_{23}(t)| - 2t \\ -|\chi_{14}(-t)| - |\chi_{23}(t)| + |\chi_{14}(-t)-\chi_{23}(t)| \\ -|\chi_{14}(t)| - |\chi_{23}(-t)| + |\chi_{14}(t)-\chi_{23}(-t)| \end{array}\right\}$$
$$= -4t + |\chi_{14}(-t)-\chi_{23}(t)| + |\chi_{14}(t)-\chi_{23}(-t)|. \tag{I.25}$$

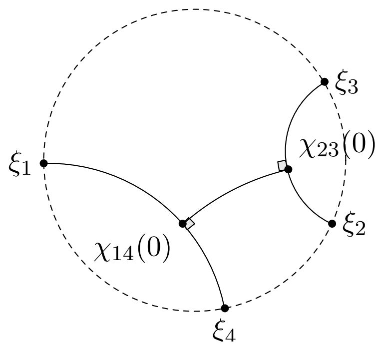


Figure 10: Geometric interpretation of the nonnegative part of the logarithm of cross-ratio: up to an additive error, this is the distance $|\chi_{14}(0) - \chi_{23}(0)|$.

By (I.24), there is Δ with $|\Delta| \leqslant 2 \cdot 212\delta = 424\delta$ such that

$$\begin{aligned} &- 4t + |\chi_{14}(-t) - \chi_{23}(t)| + |\chi_{14}(t) - \chi_{23}(-t)| \\ &= -4t + |\chi_{14}(-t) - \chi_{23}(t)|_{\mathsf{H}} + |\chi_{14}(t) - \chi_{23}(-t)|_{\mathsf{H}} + \Delta \\ &= 2d(\chi_{14}, \chi_{23}) + \Delta. \end{aligned} \tag{I.26}$$

On the other hand, by (I.10),

$$\left| \log_\mu [\xi_i] - \lim_{t \to +\infty} \left\{ \begin{array}{c} (\chi_{14}(-t) \mid \chi_{14}(t))_o + (\chi_{23}(t) \mid \chi_{23}(-t))_o \\ - (\chi_{14}(-t) \mid \chi_{23}(t))_o - (\chi_{14}(t) \mid \chi_{23}(-t))_o \end{array} \right\} \right| \leqslant 8\delta + \log_\mu 16.$$

If $d(\chi_{14}, \chi_{23})$ is large enough, letting $t \to +\infty$ in (I.25) combined with the estimate (I.26), we reach the desired inequality of Proposition I.15. This is valid for small values as well since $\log^+$ then takes small values.

Remark I.27. The right-hand side inequality of Proposition I.15 can be deduced from the elementary case of a metric tree via tree approximation [70, Theorem 2.12]. See Bourdon's remark [19, 2.3].

I.3. Sublinear tracking

Sublinearly biLipschitz embeddings of the real half-line, resp. of the real line admit trackings by geodesic rays, resp. lines; we prove this in I.3.2, resp. I.3.3. In the spirit of (I.17) and (I.18), the bound on the tracking distance can be expressed as a constant (denoted $H, \widetilde{H}$...) times the additive error function v, however at the cost of being valid only farther than a given tracking radius. The tracking constants and the tracking radii depend on v, more precisely through its large-scale features $v \uparrow \tau$, $r_\varepsilon(v)$ and $\sup\{r : v(r) \leqslant \mathrm{cst}(\lambda, \delta, \dots)\}$ described in I.1.1. While the use of tracking radii allow tracking estimates to take a particularly simple form when applied in I.3.4, their dependence upon v must not be kept entirely implicit, especially it must be taken into account for later use in section I.4 when v becomes a parameter, a task undertaken in I.3.5.

I.3.1. Preliminaries

Unless otherwise stated, geodesic rays into a pointed metric space are assumed to have their origin at the base-point. This convention will not apply to the rougher $O(v)$-rays that we define hereafter.

Definition I.28. Let u be an admissible function and X a metric space. A $O(u)$-geodesic, resp. a $O(u)$-ray in X is a $O(u)$-sublinearly biLipschitz embedding $\mathbf{R} \to X$, resp. $\mathbf{R}_{\geqslant 0} \to X$.

When $u = 1$, this is the classical notion of a quasigeodesic, resp. of a quasigeodesic ray. By definition, $O(u)$-geodesics, resp. $O(u)$-rays, are sent to $O(u)$-geodesics resp. $O(u)$-rays when one applies a $O(u)$-sublinearly biLipschitz embedding to the space. $O(u)$-geodesics behave like quasi-geodesics inside every ball, with an additive error parameter controlled by the radius; however the containing ball sits in the target space, so that the dependence of the additive error on radius only becomes apparent on the large scale. We turn this observation into a lemma, which may be considered as an alternative definition for large-scale Lipschitz embeddings, easier to handle through certain technical steps.

Lemma I.29. *Let u be an admissible function. Let $(\underline{\lambda}, \overline{\lambda})$ be Lipschitz constants, let $v = O(u)$ be nondecreasing, and let $f : (X, o) \to (Y, o)$ be a large-scale $(\underline{\lambda}, \overline{\lambda}, v)$-biLipschitz embedding. Then there exist $\widehat{v} = O(u)$, $t_{\bigcirc} \in \mathbf{R}_{\geqslant 0}$ and $R_{\bigcirc} \in \mathbf{R}_{\geqslant 0}$ (depending on f and v) such that for all $x, x_1, x_2 \in X$*

I. If $x \notin B(o, t_{\bigcirc})$ or $f(x) \notin B(o, R_{\bigcirc})$ then

$$\frac{1}{3\lambda}|x| \leqslant |x| \wedge |f(x)| \leqslant 3\lambda|x|.$$

II. If $x_1, x_2 \in X \setminus B(o, t_{\bigcirc})$ or $f(x_1), f(x_2) \in Y \setminus B(o, R_{\bigcirc})$, then

$$\begin{cases} |f(x_1) - f(x_2)| \leqslant \lambda|x_1 - x_2| + \widehat{v}\left((|x_1| \vee |x_2|) \wedge (|f(x_1)| \vee |f(x_2)|)\right), \\ |f(x_1) - f(x_2)| \geqslant \frac{1}{\lambda}|x_1 - x_2| - \widehat{v}\left((|x_1| \vee |x_2|) \wedge (|f(x_1)| \vee |f(x_2)|)\right). \end{cases}$$

Moreover $t_\bigcirc$, $R_\bigcirc$ and $\widehat{v}$ may be taken as:

$$t_\bigcirc(|f(o)|, v) = \sup\left\{r : v(r) \geqslant \frac{r}{3\lambda}\right\} \vee 3\lambda|f(o)| = r_{1/(3\lambda)}(v) \vee 3\lambda|f(o)|, \tag{I.27}$$

$$R_\bigcirc(|f(o)|, v) = 4|f(o)| \vee 2(2\lambda+1)t_\bigcirc(|f(o)|, v), \text{ and} \tag{I.28}$$

$$\widehat{v} = (v \uparrow 3\lambda)v. \tag{I.29}$$

Proof. By definition of $t_\bigcirc(|f(o)|, v)$, for all $x \in X \setminus B(o, t_\bigcirc)$, $|f(o)| \leqslant 1/(3\lambda)|x|$ and $v(|x|) \leqslant \frac{1}{3\lambda}|x|$, so $\frac{1}{3\lambda}|x| \leqslant |f(x)| \leqslant \left(\lambda + \frac{2}{3\lambda}\right)|x| \leqslant 3\lambda|x|$; this is the first case in (I). Now assume that $R_\bigcirc$ is defined as in (I.28). Note that $R_\bigcirc \geqslant 2r_{1/(3\lambda)}(v) \geqslant r_{1/2}(v)$ so that if $f(x) \in Y \setminus B(o, R_\bigcirc(|f(o)|, v))$, then

$$\begin{aligned} |x| &\geqslant \overline{\lambda}^{-1}\left(|f(x)| - |f(o)| - v(|x|)\right) \\ &\geqslant \begin{cases} \lambda^{-1}(|f(x)| - |f(o)| - v(|f(x)|) & \text{if } |x| \leqslant |f(x)|, \text{ or} \\ \lambda^{-1}(|f(x)| - |f(o)| - |x|/2) & \text{if } |x| \geqslant |f(x)|. \end{cases} \end{aligned}$$

In both cases,

$$|x| \geqslant \frac{1}{\lambda + 1/2}\left(\frac{1}{2}|f(x)| - |f(o)|\right),$$

and then $|x| \geqslant t_\bigcirc(|f(o)|, v)$ since by definition $R_\bigcirc \geqslant 2|f(o)| + (2\lambda+1)t_\bigcirc$. Hence the hypotheses in (I) actually reduce to the single first one. (II) follows from (I), the fact that f is a (λ, v)-embedding, that $\widehat{v}$ is nondecreasing, and the left distributivity of $\leqslant$ over $\wedge$. $\square$

I.3.2. Rays

Let Y be a proper geodesic hyperbolic space, and $\widetilde{\gamma} : \mathbf{R} \to Y$ a $O(u)$-geodesic ray. Inequality (I.8) says in particular that $\{\widetilde{\gamma}(t)\}_{t \in \mathbf{Z}_{\geqslant 0}}$ is a Cauchy-Gromov sequence. Since Y is proper and geodesic, its Gromov boundary is equal to $\partial_\infty Y$ and there exists a geodesic ray $\gamma : (\mathbf{R}_{\geqslant 0}, 0) \to (Y, o)$ such that $\eta := [\gamma] = \partial_\infty \widetilde{\gamma}(+\infty)$. We will prove that γ actually tracks $\widetilde{\gamma}$, in the sense that the growth of distance between them is in the $O(u)$-class. We need a preliminary lemma.

Lemma I.30. *Let $\delta \in \mathbf{R}_{\geqslant 0}$, and let (Y, o) be a proper geodesic δ-hyperbolic space. Let $\gamma : \mathbf{R} \to Y$ be a geodesic ray into Y, and let γ' be a non-pointed geodesic ray asymptotic to γ, i.e. $[\gamma] = [\gamma'] \in \partial_\infty Y$. Then for all $s \in \mathbf{R}_{\geqslant 0}$ such that $s \geqslant |\gamma'(0)| + 16\delta$,*

$$d(\gamma'(s), \operatorname{im}(\gamma)) \leqslant 8\delta.$$

Proof. This is a classical result in hyperbolic metric geometry, use for instance the proof of $(ii) \implies (iii)$ in [70, Proposition 7.1] with appropriate changes of notation, and replace Ghys and de la Harpe's D with $\sup\{|\gamma'(0)|, 16\delta\}$. $\square$

Lemma I.31 (Sublinear tracking for rays)**.** *Let v be an unbounded admissible function. Let (Y, o) be a proper, geodesic, pointed metric space. Assume there exists $\delta \in \mathbf{R}_{\geqslant 0}$ such that Y is δ-hyperbolic. Let $(\underline{\lambda}, \overline{\lambda}) \in \mathbf{R}^2_{>0}$ be Lipschitz data, and let $\widetilde{\gamma} : \mathbf{R}_{\geqslant 0} \to Y$ be a $(\underline{\lambda}, \overline{\lambda}, v)$-ray. Let $\eta \in \partial_\infty Y$ be the endpoint of $\widetilde{\gamma}$, and let γ be any geodesic ray such that $[\gamma] = \eta$. Then there exist constants $H, \widetilde{H} \in \mathbf{R}_{>0}$, $t_{\simeq}, R_{\simeq}, \in \mathbf{R}_{\geqslant 0}$ such that for all positive real t and s,*

$$t \geqslant t_{\simeq} \implies d(\widetilde{\gamma}(t), \gamma) \leqslant Hv(t) \tag{I.30}$$

$$s \geqslant R_{\simeq} \implies d(\gamma(s), \operatorname{im}(\widetilde{\gamma})) \leqslant \widetilde{H}v(s), \tag{I.31}$$

where H and $\widetilde{H}$ depend on λ and v only, while $t_{\simeq}$ and $R_{\simeq}$ can be decomposed into

$$t_{\simeq} = t^0_{\simeq}(\lambda, v, \delta) + 2\lambda|\widetilde{\gamma}(0)| \tag{I.32}$$

$$R_{\simeq} = R^0_{\simeq}(\lambda, v, \delta) + |\widetilde{\gamma}(0)|. \tag{I.33}$$

Remark I.32. In view of Lemma I.24, it does matter for us that v be unbounded. If v is bounded, though, $\widetilde{\gamma}$ is a quasi-geodesic ray and the same result classically holds, see for instance Ghys and de la Harpe [70, 5.25], with extra additive terms in the estimates (I.30) and (I.31).

Remark I.33. It is important to make the dependence of the tracking radius $R^0_{\simeq}$ upon the function v explicit, at least to some extent. However, in order not to overload the current proof, we reconstruct it separately (but along with other tracking radii) in subsection I.3.5, and only keep record of the steps needed for its definition here, with enough details to ensure that it only depends on λ, v and δ.

Sketch of proof for Lemma I.31. For every $t \in \mathbf{R}_{\geqslant 0}$, set a real positive T large enough according to t so that (I.8) ensures the Gromov product $(\widetilde{\gamma}(T), \eta)_o$ is significantly greater than $|\widetilde{\gamma}(t)|$, and use the stability lemma I.24 to prove that $\widetilde{\gamma}(t)$ is not far from the geodesic segment γ_T between o and $\widetilde{\gamma}(T)$. Here keeping an efficient inequality requires that T stay within linear control of t, which can be done consistently with the antagonist constraint of (I.8). Further, show that the projection of $\widetilde{\gamma}(t)$ on γ_T is close to γ, using the slim triangle $o\widetilde{\gamma}(T)\eta$, see figure 11. Finally, (I.31) is deduced from

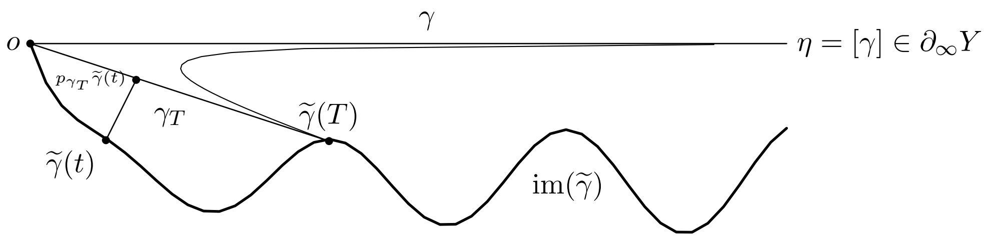


Figure 11: Sublinear tracking for $O(u)$-rays (Step 1: $o = \widetilde{\gamma}(0)$).

(I.30) with a metric connectedness argument in the same way that (I.18) was deduced from (I.17) in the proof of Lemma I.24.

Proof of lemma I.31. Setting $\alpha = \underline{\lambda}/2$ in the Gromov product estimate (I.8) and letting $s \to +\infty$,

$$\forall T \in \mathbf{Z}_{\geqslant t_\alpha},\ ([\gamma] \mid \widetilde{\gamma}(T))_o \geqslant \underline{\lambda} T/2 - M'(\underline{\lambda}, \delta), \tag{I.34}$$

where $M'(\underline{\lambda}, \delta) = \log_\mu \left(\frac{2}{1-\mu^{-(3\underline{\lambda})/4}} \right)$. We will first prove the lemma in the case $|\widetilde{\gamma}(0)| = 0$, i.e. $\widetilde{\gamma}(0) = o$ (this is pictured on Figure 11), and then use I.30 to extend the result to the general case.

Step 1: $\widetilde{\gamma}(0) = o$. — Let $(t, T) \in \mathbf{R}^2_{\geqslant 0}$ be such that $t \leqslant T$. Since v is nondecreasing and unbounded, there is $T_2 \in \mathbf{R}_{>0}$ such if $T \geqslant T_2$, then $v(T) \geqslant 6\lambda^2\delta$. This is the condition required to apply Lemma I.24. By inequality (I.17) of this lemma applied to $\gamma_T = [o\widetilde{\gamma}(T)]$ and $\widetilde{\gamma}_{|[0,T]}$,

$$\text{if } T \geqslant T_2,\ d(\widetilde{\gamma}(t), \mathrm{im}(\gamma_T)) \leqslant h(\lambda)\,(\delta + v(T))\,. \tag{I.35}$$

Similarly, by (I.18), if $T \geqslant T_2$ then

$$\forall S \in [0, |\widetilde{\gamma}(T)|],\ d(\gamma_T(S), \mathrm{im}(\widetilde{\gamma})) \leqslant \widetilde{h}(\lambda)\,(\delta + v(T))\,. \tag{I.36}$$

By (I.9) and our definition of α, $t_\alpha = r_{2\underline{\lambda}}(v)$; start assuming that $t \geqslant t_\alpha \vee T_2$. We look for T greater than t (hence, greater than t_α and T_2) such that $(\widetilde{\gamma}(T) \mid \eta)_o \geqslant 2|\widetilde{\gamma}(t)|$. Thanks to (I.34) this holds when $T = \lceil t \rceil \vee \lceil (4/\underline{\lambda})\,(|\widetilde{\gamma}(t)| + M(\underline{\lambda}, \delta)) \rceil$; we keep this dependence of T with respect to t from now on. Let $\triangle_T$ be a (geodesic, semi-ideal) triangle with vertices o, $\widetilde{\gamma}(T)$ and η (Recall that by convention, γ_T is the side of $\triangle_T$ between o and $\widetilde{\gamma}(T)$). By (I.35) and the triangle inequality,

$$|p_{\gamma_T}(\widetilde{\gamma}(t))| \leqslant h(\lambda)\,(\delta + v(T)) + |\widetilde{\gamma}(t)|.$$

Again by the triangle inequality,

$$\begin{aligned} d\left(p_{\gamma_T}\widetilde{\gamma}(t), [\widetilde{\gamma}(T)\eta)\right) &\geqslant \left|p_{[\widetilde{\gamma}(T)\eta)}o\right| - |p_{\gamma_T}(\widetilde{\gamma}(t))| \\ &\geqslant \left|p_{[\widetilde{\gamma}(T)\eta)}o\right| - h(\lambda)\left(\delta + v(T)\right) - |\widetilde{\gamma}(t)|. \end{aligned}$$

By the triangle inequality $\left|p_{[\widetilde{\gamma}(T)\eta)}o\right| \geqslant (\widetilde{\gamma}(T) \mid \eta)_o$, so that the previous inequality becomes

$$\begin{aligned} d\left(p_{\gamma_T}\widetilde{\gamma}(t), [\widetilde{\gamma}(T)\eta)\right) &\geqslant (\widetilde{\gamma}(T) \mid \eta)_o - h(\lambda)\left(\delta + v(T)\right) - |\widetilde{\gamma}(t)| \\ &\geqslant |\widetilde{\gamma}(t)| - h(\lambda)\left(\delta + v(T)\right), \end{aligned} \tag{I.37}$$

where we have replaced the Gromov product according to the definition of T. Let us now bound $v(T)$. By definition,

$$\begin{aligned} T &\leqslant 4\lambda|\widetilde{\gamma}(t)| + 4\lambda M'(\underline{\lambda}, \delta) + 1 \\ &\leqslant 4\lambda\left(\lambda t + v(t)\right) + 4\lambda M'(\underline{\lambda}, \delta) + 1, \end{aligned}$$

hence for $t \geqslant t_0 = \sup\{t_\alpha, T_2, 4\lambda M'(\underline{\lambda}, \delta) + 1\}$, $T \leqslant (1 + 8\lambda^2)t$, and

$$v(T) \leqslant (v \uparrow 1 + 8\lambda^2)v(t). \tag{I.38}$$

Substituting this in inequality (I.37), for all t such that $t \geqslant t_0$,

$$d\left(p_{\gamma_T}\widetilde{\gamma}(t), [\widetilde{\gamma}(T)\eta)\right) \geqslant |\widetilde{\gamma}(t)| - h(\lambda)(\delta + (v \uparrow 1 + 8\lambda^2)v(t)) \tag{I.39}$$

Define

$$t_1 := t_\bigcirc \vee \sup\{s : v(s) \leqslant \delta\} \vee 3\lambda r_{1/(12\lambda h(\lambda) v\uparrow 1+8\lambda^2)}(v) \vee 24\lambda\delta \vee t_0$$

where we used a notation introduced in I.1.1 in the last term involved in the definition of t_1; we recall that it ensures that for $t \geqslant t_1$, $(24\lambda h(\lambda) v \uparrow 1 + 8\lambda^2)v(t) \leqslant t$, so that by (I.39)

$$\begin{aligned} d\left(p_{\gamma_T}\widetilde{\gamma}(t), [\widetilde{\gamma}(T)\eta)\right) &\geqslant t/(3\lambda) - h(\lambda)(\delta + (v \uparrow 1 + 8\lambda^2)v(t)) \\ &\geqslant t/(3\lambda) - h(\lambda)(v(t) + (v \uparrow 1 + 8\lambda^2)v(t)) \\ &= t/(3\lambda) - h(\lambda)v(t) - h(\lambda)(v \uparrow 1 + 8\lambda^2)v(t) \\ &\geqslant t/(3\lambda) - 2h(\lambda)(v \uparrow 1 + 8\lambda^2)v(t) \\ &\geqslant t/6\lambda, \end{aligned}$$

where in the first line we used that $t_1 \geqslant t_\bigcirc$ and Lemma I.29, in the second line we used $t_1 \geqslant \sup\{s : v(s) \leqslant \delta\}$, in the fourth line we used $v \uparrow 1+8\lambda^2 \geqslant 1$, and in the last line we used $2h(\lambda)v \uparrow 1 + 8\lambda^2)v(t) \leqslant t/(6\lambda)$. Finally, since $t_1 \geqslant 24\delta$,

$$\forall t \in \mathbf{R}_{>t_1},\ d\left(p_{\gamma_T}\widetilde{\gamma}(t), [\widetilde{\gamma}(T)\eta)\right) > 4\delta.$$

But $\triangle_T$ is 4δ-slim, so

$$\forall t \in \mathbf{R}_{>t_1},\ d\left(p_{\gamma_T}\widetilde{\gamma}(t), \gamma\right) \leqslant 4\delta. \tag{I.40}$$

By the triangle inequality,

$$\begin{aligned}\forall t \in \mathbf{R}_{>t_1},\ d\left(\widetilde{\gamma}(t), \mathrm{im}(\gamma)\right) &\leqslant |\widetilde{\gamma}(t) - p_{\gamma_T}\widetilde{\gamma}(t)| + d\left(p_{\gamma_T}\widetilde{\gamma}(t), \mathrm{im}(\gamma)\right)\\ &\underset{(\mathrm{I.35}),(\mathrm{I.40})}{\leqslant} h(\lambda)\left(\delta + v(T)\right) + 4\delta\\ &\underset{(\mathrm{I.38})}{\leqslant} h(\lambda)\left(\delta + (v \uparrow 1 + 8\lambda^2)v(t)\right).\end{aligned}$$

Define $t_3 = \sup\{s : v(s) \leqslant h(\lambda)\delta\} \vee t_1$. The last inequality implies

$$\forall t \in \mathbf{R}_{\geqslant t_3},\ d(\widetilde{\gamma}(t), \mathrm{im}(\gamma)) \leqslant \left(2h(\lambda)(v \uparrow 1 + 8\lambda^2) + 1\right) v(t). \tag{I.41}$$

We have proved (I.30) in the special case $|\widetilde{\gamma}(0)| = 0$; set $H_0(\lambda, v) = 2h(\lambda)(v \uparrow 1 + 8\lambda^2) + 1$.

Step 2: $\widetilde{\gamma}(0)$ arbitrary. — Let γ' be a non-pointed geodesic ray $[\widetilde{\gamma}(0)\eta)$. Apply (I.41) to $\widetilde{\gamma}$ and γ'. This gives the existence, for all $t \in \mathbf{R}_{\geqslant t_3}$, of $s' \in \mathbf{R}_{\geqslant 0}$ such that $d(\widetilde{\gamma}(t), \gamma'(s')) \leq H_0(\lambda, v)v(t)$. Moreover $s' = d(\widetilde{\gamma}(0), \gamma'(s')) \geqslant d(\widetilde{\gamma}(0), \widetilde{\gamma}(t)) - H_0(\lambda, v)v(t) \geqslant \underline{\lambda}t - (1 + H_0(\lambda, v))\, v(t)$. Hence for all $t \in \mathbf{R}$ such that $t \geqslant t_4 := \sup\left\{t_3, r_{\underline{\lambda}/(2+2H_0(\lambda,v))}(v)\right\}$,

$$s' \geqslant t/(2\lambda). \tag{I.42}$$

Set $t_5 := t_4 \vee \sup\{r : v(r) \leqslant 8\delta\}$, $t^0_{\backsimeq} := t_5 \vee 16\delta$ and then $t_{\backsimeq} = t^0_{\backsimeq} + 2\lambda|\widetilde{\gamma}(0)|$. By (I.42), $s' \geqslant |\widetilde{\gamma}(0)| + 16\delta$. Moreover γ' and γ are asymptotic, so that by Lemma I.30 on asymptotic geodesic rays, $d(\gamma'(s'), \mathrm{im}(\gamma)) \leqslant 8\delta$. By the triangle inequality and the definition of $t_{\backsimeq}$ we conclude that

$$d(\widetilde{\gamma}(t), \gamma) \leqslant |\widetilde{\gamma}(t) - \gamma'(s')| + d(\gamma'(s'), \mathrm{im}(\gamma)) \leqslant (1 + H_0(\lambda, v))\, v(t).$$

By construction, $t^0_{\backsimeq}$ only depends on λ, v, δ, so (I.30) is reached in the general case.

From now on we proceed to attain (I.31). As before start by assuming $|\widetilde{\gamma}(0)| = 0$. For all $t \in \mathbf{R}_{\geqslant 0}$, since $\widetilde{\gamma}_{|[0,t]}$ is $v(t)$-connected, $p_\gamma(\widetilde{\gamma}_{|[0,t]})$ is $v(t) + 16\delta$-connected by Lemma I.18, in particular it is $2v(t)$-connected as soon as $t \geqslant t_6 := \sup\{r : v(r) \leqslant 16\delta\}$. On the other hand, by Lemma I.29, if $t \geqslant t_{\bigcirc}$

then $|\widetilde{\gamma}(t)| \geqslant (\underline{\lambda}/3)t$. Hence, if $t \geqslant t_7 := \sup\{t_6, t_\bigcirc\}$, the convex hull of $p_\gamma(\widetilde{\gamma}_{|[0,t]})$ contains $\gamma([0, (\underline{\lambda}/3)t - Hv(t)])$ where H is the constant from (I.30) (note that t_7 only depends on v, λ, δ since we are assuming $\widetilde{\gamma}(0) = o$).

Hence for all $t \geqslant t_8 = \sup\{t_7, r_{\underline{\lambda}/(6H)}(v)\}$, every $s \in [0, (\underline{\lambda}/6)t]$ lies between two orthogonal projections of points of $\widetilde{\gamma}_{|[0,t]}$ on γ. Define $R_8 := t_8/(6\lambda)$. For all $s \in \mathbf{R}$ such that $s \geqslant R_8$, there is $t_s \in [0, 6\lambda s]$ such that

$$|\gamma(s) - p_\gamma(\widetilde{\gamma}(t_s))| \leqslant 2v(6\lambda s) \leqslant 2(v \uparrow 6\lambda)v(s). \tag{I.43}$$

By the triangle inequality,

$$\begin{aligned}|\gamma(s) - \widetilde{\gamma}(t_s)| &\leqslant |\gamma(s) - p_\gamma(\widetilde{\gamma}(t_s))| + |p_\gamma(\widetilde{\gamma}(t_s)) - \widetilde{\gamma}(t_s)| \\ &\leqslant H_0 v(t_s) + 2(v \uparrow 6\lambda)v(s) \\ &\leqslant 2(v \uparrow 6\lambda)(H_0 + 1)v(s) \text{ for } s \geqslant R_8,\end{aligned} \tag{I.44}$$

where we used that $v(t_s) \leqslant (v \uparrow 6\lambda)v(s)$ for the last inequality. Set $\widetilde{H}_0(\lambda, v) := 2(v \uparrow 6\lambda)(H_0 + 1)$, and assume from now that $\widetilde{\gamma}(0)$ is arbitrary. Define $R^0_{\simeq} = R_8 \vee \sup\{r : \widetilde{H}_0 v(r) \leqslant 8\lambda\} \vee 16\delta$ and $\widetilde{H} = 2\widetilde{H}_0$. Then by Lemma I.30 applied to $\gamma = o\eta$ and $\gamma' = \widetilde{\gamma}(0)\eta$, (I.44) and the triangle inequality, for all $s \geqslant R^0_{\simeq} + |\widetilde{\gamma}(0)|$, $d(\gamma(s), \mathrm{im}(\widetilde{\gamma})) \leqslant \widetilde{H} v(s)$. □

I.3.3. Geodesics

Our next aim consists in tracking $O(u)$-geodesics $\widetilde{\gamma}$. For this we need two steps:

1. Control the Gromov product of ends $\partial_\infty\widetilde{\gamma}(-\infty)$ and $\partial_\infty\widetilde{\gamma}(+\infty)$ with respect to $|\widetilde{\gamma}(0)|$. This is achieved by Lemma I.34.
2. Track $\widetilde{\gamma}$ near both ends, starting at a distance linearly controlled by their Gromov product, and interpolate in between using the classical version of the stability lemma. This strategy is set up in Lemma I.35.

Beware that, in contrast to the situation with (quasi)geodesics, one cannot re-parametrize a (λ, v)-geodesic (e.g. to assume that $\widetilde{\gamma}(0)$ is the closest[10] point $\widetilde{b}$ to o in $\mathrm{im}(\widetilde{\gamma})$) without changing the function v. For this reason, and in order to simplify bounds on the tracking distance in step (2), we introduce an additional constant L and, from Lemma I.35 on, make the assumption that $|\widetilde{\gamma}(0)| \leqslant L\widetilde{b}$.

[10]Such a point $\widetilde{b}$ exists since $\widetilde{\gamma}$ is proper.

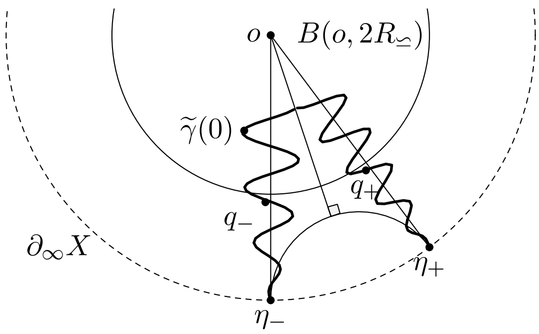


Figure 12: Main objects occurring in the proof of Lemma I.34. The geodesic ray $\gamma_\pm$ from o to $\eta_\pm$ intersects the sphere $\partial B(o, 2R_{\backsimeq})$ at point $p_\pm$ (not depicted). Beware that the reasoning is by contradiction: this picture is not realistic.

Lemma I.34. *Let $\delta \in \mathbf{R}_{\geqslant 0}$, $\lambda \in \mathbf{R}_{\geqslant 1}$ be constants, and let (Y, o) be a pointed proper geodesic δ-hyperbolic space. Let v be an admissible function. Let $\widetilde{\gamma}$ be a (λ, v)-geodesic into Y. Denote $\eta_\pm$ in $\partial_\infty X$ its endpoints, precisely $\eta_\pm = \partial_\infty \widetilde{\gamma}(\pm\infty)$. Then there exist $K = K(\lambda, v, \delta)$ and $R_\sqcap = R_\sqcap(\lambda, v, \delta)$, both in $\mathbf{R}_{>0}$ such that if $|\widetilde{\gamma}(0)| \geqslant R_\sqcap$,*

$$(\eta_- \mid \eta_+)_o \leqslant K|\widetilde{\gamma}(0)|. \tag{I.45}$$

Proof. The proof uses that $O(u)$-geodesics cannot make large round trips; see figure 12. Assume by contradiction that $(\eta_- \mid \eta_+)_o \geqslant 3(R^0_{\backsimeq} + |\widetilde{\gamma}(0)|) + 4\delta = 3R_{\backsimeq} + 4\delta$ for $\widetilde{\gamma}(0)$ arbitrarily far. Track the rays $\widetilde{\gamma}_- : t \mapsto \widetilde{\gamma}(-t)$ and $\widetilde{\gamma}_+ : t \mapsto \widetilde{\gamma}(t)$ with geodesic rays γ_- and γ_+. Let $\gamma = (\eta_- \eta_+)$ be a geodesic line. Define $p_\pm$ as the intersection point of $\gamma_\pm$ and $\partial B(o, 2R_{\backsimeq})$, i.e. $p_\pm = \gamma_\pm(2R_{\backsimeq})$. The twice-ideal triangle $o\eta_-\eta_+$ is 4δ-thin, and by the triangle inequality

$$\begin{aligned} d(p_\pm, \gamma) &\geqslant d(o, p_\gamma(o)) - d(o, p_\pm) \\ &\geqslant (\eta_- \mid \eta_+)_o - 2R_{\backsimeq} \geqslant R_{\backsimeq} + 4\delta > 4\delta, \end{aligned}$$

so $d(p_\pm, \gamma_\mp) < 4\delta$ and $|p_- - p_+| \leqslant 8\delta$ (where we used that both points p_+ and p_- lie on the same sphere centered at o). By sublinear tracking lemma I.31, there is $q_\pm$ on $\mathrm{im}(\widetilde{\gamma}_\pm)$ such that $|p_\pm - q_\pm| \leqslant \widetilde{H} v(2R_{\backsimeq})$, and thanks to the triangle inequality,

$$|q_+ - q_-| \leqslant |p_+ - p_-| + 2\widetilde{H} v(2R_{\backsimeq}) \leqslant 8\delta + 2\widetilde{H} v(2R_{\backsimeq}). \tag{I.46}$$

Let t_+, t_- in $\mathbf{R}$ be such that $q_\pm = \widetilde{\gamma}(t_\pm)$, and write $T = \sup\{|t_+|, |t_-|\}$. The portion of $\widetilde{\gamma}$ between t_- and t_+ is a $(\lambda, v(T))$ quasi-geodesic segment. By a length-distance estimate for quasi-geodesics, for α large enough,

$$\begin{aligned}\ell_\alpha\left(\widetilde{\gamma}_{|[t_-,t_+]}\right) &\leqslant 2\lambda(t_+ - t_-) \leqslant 2\lambda\left(\lambda|q_- - q_+| + v(T)\right)\\ &\underset{(\text{I.46})}{\leqslant} 4\lambda^2 Hv(2R_\simeq) + 2\lambda v(T) + 16\lambda^2\delta. \qquad (\text{I.47})\end{aligned}$$

T can be bounded from above for $|\widetilde{\gamma}(0)|$ large enough:

$$\begin{aligned}\underline{\lambda}T - v(T) &\leqslant \sup\left\{|\widetilde{\gamma}(0) - q_-|, |\widetilde{\gamma}(0) - q_+|\right\}\\ &\leqslant 2R^0_\simeq + |\widetilde{\gamma}(0)| + 2\widetilde{H}v(T),\end{aligned}$$

so that since $v(T) \ll T$, there is a constant T_0 depending on v, λ (explicitly $T_0 = r_{1/(8\lambda\widetilde{H})}(v)$) such that $T \leqslant \inf\{T_0, \lambda(2R^0_\simeq + |\widetilde{\gamma}(0)|)\}$. On the other hand, $\ell_\alpha\left(\widetilde{\gamma}_{|[t_-,t_+]}\right)$ is greater than $|q_+ - \widetilde{\gamma}(0)| + |q_- - \widetilde{\gamma}(0)|$, and

$$\begin{aligned}|q_+ - \widetilde{\gamma}(0)| + |q_- - \widetilde{\gamma}(0)| &\geqslant |p_+ - \widetilde{\gamma}(0)| + |p_- - \widetilde{\gamma}(0)| - 2Hv(2R_\simeq)\\ &\geqslant 2R^0_\simeq + |\widetilde{\gamma}(0)| - 2Hv(2R_\simeq).\end{aligned}$$

Substitute this in (I.47) and make all dependences over $|\widetilde{\gamma}(0)|$ explicit:

$$\begin{aligned}2R^0_\simeq + |\widetilde{\gamma}(0)| - 2Hv(2R_\simeq) \leqslant & 4\lambda^2 Hv(2R_\simeq) + 2\lambda v(T) + 16\lambda^2\delta.\\ \leqslant & 4\lambda^2 Hv(2R_\simeq) + 2\lambda v(T_0)\\ & + 2\lambda v((\lambda/2)(2R_\simeq)) + 16\lambda^2\delta.\end{aligned}$$

The last inequality rewrites under the form

$$\begin{aligned}|\widetilde{\gamma}(0)| &\leqslant \left[4\lambda^2 H + 2\lambda\left(v \uparrow \lambda\right)\right]\left[v \uparrow 2\right] v(R_\simeq) + 2\lambda v(T_0) + 16\lambda^2\delta + 2R^0_\simeq\\ &\leqslant H_3 v\left(R^0_\simeq + |\widetilde{\gamma}(0)|\right) + \frac{\lambda}{4\widetilde{H}} r_{1/(8\lambda\widetilde{H})}(v) + 16\lambda^2\delta + 2R^0_\simeq, \qquad (\text{I.48})\end{aligned}$$

where $H_3 = \left[4\lambda^2 H + 2\lambda\left(v \uparrow \lambda\right)\right]\left[v \uparrow 2\right]$. If $|\widetilde{\gamma}(0)| \geqslant 3R^0_\simeq$ then (I.48) yields

$$|\widetilde{\gamma}(0)| \leqslant 3(v \uparrow 2)H_3 v(|\widetilde{\gamma}(0)|) + \frac{3\lambda}{4\widetilde{H}} r_{1/(8\lambda\widetilde{H})}(v) + 48\lambda^2\delta.$$

This inequality would lead to a contradiction for $|\widetilde{\gamma}(0)|$ larger than

$$R_\sqcap := 3R^0_\simeq \vee \left(\frac{3\lambda}{2\widetilde{H}} r_{1/(8\lambda\widetilde{H})}(v) + 96\lambda^2\delta\right) \vee r_{1/(6(v\uparrow 2)H_3(\lambda,\delta,v))}(v), \qquad (\text{I.49})$$

precisely, if $|\widetilde{\gamma}(0)| \geqslant R_\sqcap$, then $(\eta_- \mid \eta_+)_o \leqslant 3(R^0_\simeq + |\widetilde{\gamma}(0)|) + 4\delta \leqslant 5|\widetilde{\gamma}(0)|$ as $R_\sqcap \geqslant R^0_\simeq \vee 4\delta$. One may take $K = 5$. $\square$

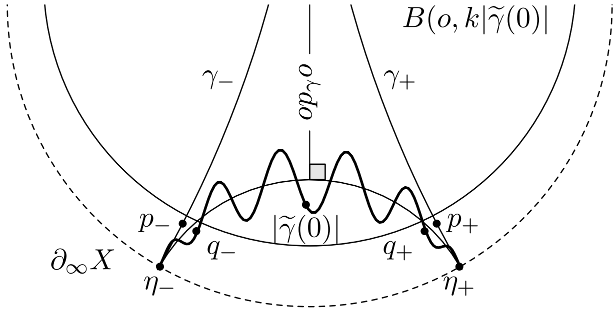


Figure 13: Main objects of proof of Lemma I.35. Tracking is achieved by the classical Morse Lemma I.24 between q_- and q_+ and by the ray tracking Lemma I.31 beyond those points.

Lemma I.35 (Tracking for $O(u)$-geodesics)**.** *Let $\delta \in \mathbf{R}_{\geqslant 0}$, $\lambda \in \mathbf{R}_{\geqslant 1}$, let u be an admissible function and let $v = O(u)$ be nondecreasing. Let (Y, o) be a proper geodesic pointed δ-hyperbolic space, and let $\widetilde{\gamma} : \mathbf{R} \to Y$ be a (λ, v)-geodesic. Define $\widetilde{b}$ as a closest point to o in $\operatorname{im}(\widetilde{\gamma})$. Let $L \in \mathbf{R}_{\geqslant 1}$ be a real constant and assume that the Gromov product $(\partial_\infty\widetilde{\gamma}(+\infty) \mid \partial_\infty\widetilde{\gamma}(-\infty))_o$ is larger than 60δ. There exist constants $\widetilde{R} = \widetilde{R}(\lambda, \delta, v, L)$, H_2 and $\widetilde{H}_2$ in $\mathbf{R}_{>0}$ (depending on λ, v and L) such that if*

$$\widetilde{R} \leqslant |\widetilde{\gamma}(0)| \leqslant L|\widetilde{b}|, \tag{I.50}$$

then for any geodesic $\gamma : \mathbf{R} \to Y$ with $[\gamma]_{\pm\infty} = \partial_\infty\widetilde{\gamma}(\pm\infty)$ and $\gamma(0) = p_\gamma o$,

$$\forall t \in \mathbf{R}, d(\widetilde{\gamma}(t), \gamma) \leqslant H_2 v(|\widetilde{\gamma}(t)|), \text{ and} \tag{I.51}$$

$$\forall s \in \mathbf{R}, d(\gamma(s), \operatorname{im}(\widetilde{\gamma})) \leqslant \widetilde{H}_2 v(|\gamma(s)|). \tag{I.52}$$

Proof. We divide the proof into 4 steps.

Step 1: Setting. — As before, write $\eta_\pm = \partial_\infty\widetilde{\gamma}(\pm\infty)$, cut $\widetilde{\gamma}$ in two (λ, v)-geodesic rays $\widetilde{\gamma}_\pm$ starting at $\widetilde{\gamma}(0)$, and track $\widetilde{\gamma}_\pm$ with geodesic rays $\gamma_\pm$. Let $\gamma = (\eta_-\eta_+)$, parametrized in such a way that $p_\gamma o = \gamma(0)$. Define $\widetilde{R}_0 = R_\sqcap$ and start assuming $|\widetilde{\gamma}(0)| \geqslant \widetilde{R}_0$. Let k be a real parameter whose value should be fixed later; only assume for now that $k \geqslant 2K + 1$, where K is the constant from Lemma I.34. Define

$$p_\pm := \gamma_\pm\left(k\left[|\widetilde{\gamma}(0)| + R^0_\backsimeq\right] \vee 2(2\lambda+1)r_{1/3\lambda}(v) \vee r_{1/(2\widetilde{H})}(v)\right), \tag{I.53}$$

where $R^0_\backsimeq$ is the constant from Lemma I.31 applied to $\widetilde{\gamma}_+$ or $\widetilde{\gamma}_-$ and let $q_\pm$ be a closest point to $p_\pm$ on $\operatorname{im}(\widetilde{\gamma}_\pm)$. Since $k \geqslant 2K$ and $|\widetilde{\gamma}(0)| \geqslant R_\sqcap$, by Lemma I.34, $k|\widetilde{\gamma}(0)| \geqslant 2(\eta_- \mid \eta_+)_o$, and $|p_+| = |p_-| \geqslant 2(\eta_- \mid \eta_+)_o$ as well. Let σ be a geodesic segment from o to $p_\gamma o$. By the triangle inequality, $d(p_\pm, \sigma) \geqslant 2(\eta_- \mid \eta_+)_o - \sup_{c \in \sigma} |c| \geqslant (\eta_- \mid \eta_+)_o - 56\delta$. As $(\eta_- \mid \eta_+)_o > 60\delta$ by hypothesis,

$$d(p_\pm, \sigma) > 60\delta - 56\delta = 4\delta, \text{ hence } d(p_\pm, \gamma) \leqslant 4\delta, \tag{I.54}$$

where we used that the once-ideal right-angled triangles $o\eta_\pm(p_\gamma o)$ are 4δ-slim (recall that $p_\pm$ lies on the side $\gamma_\pm$ by definition). Further, because $k \geqslant 1$, inequality (I.31) of the tracking lemma I.31 allows to bound $|q_- - p_-|$ and $|q_+ - p_+|$:

$$|p_\pm - q_\pm| \leqslant \widetilde{H} v(|p_\pm|), \tag{I.55}$$

so that by the triangle inequality and the definition (I.53) of $p_\pm$,

$$|q_\pm| \geqslant |p_\pm| - \widetilde{H} v(|p_\pm|) \underset{(I.53)}{\geqslant} \frac{1}{2}|p_\pm|. \tag{I.56}$$

Step 2: Selection of k. — At this point, in order to control the quasi-geodesic additive error term of $\widetilde{\gamma}$ between q_- and q_+ we need to select k large enough so that $|p_\pm| \geqslant R_\bigcirc$, where $R_\bigcirc$ is associated to $\widetilde{\gamma}$ as in Lemma I.29. Recall from the expression (I.28) of $R_\bigcirc$ that $R_\bigcirc = 4|\widetilde{\gamma}(0)| \vee 2(2\lambda + 1)(3\lambda|\widetilde{\gamma}(0)| \vee r_{1/(3\lambda)} v)$. Thus from now on we fix $k = (2K+1) \vee 8 \vee 12\lambda(2\lambda+1)$. By inequality (I.55), this is sufficient to ensure $|q_\pm| \geqslant R_\bigcirc$, and then using the estimates and notation of Lemma I.29, the portion of $\widetilde{\gamma}$ situated between q_+ and q_- is a (λ, c)-quasigeodesic segment, with $c = \widehat{v}(|q_+| \vee |q_-|)$.

Step 3: Tracking between q_- and q_+, and estimation of H_2. — Let $\overline{\gamma}$ be a geodesic segment between q_+ and q_-. Let $t_\pm \in \mathbf{R}$ be such that $\widetilde{\gamma}(t_\pm) = \widetilde{\gamma}_\pm(\pm t_\pm) = q_\pm$. By Lemma I.24 $\operatorname{dist}_{\mathrm{H}}(\overline{\gamma}, \widetilde{\gamma}_{|[t_-,t_+]}) \leqslant (h(\lambda) \vee \widetilde{h}(\lambda))(\delta + c)$, and by hyperbolic geometry, letting $s_\pm$ be such that $\gamma(s_\pm) = p_\gamma(q_\pm)$, $\operatorname{dist}_{\mathrm{H}}(\overline{\gamma}, \gamma_{|[s_-,s_+]})$ cannot be much greater than the pairwise distance between the endpoints of these geodesic segments:

$$\begin{aligned}\operatorname{dist}_{\mathrm{H}}(\overline{\gamma}, \gamma_{|[s_-,s_+]}) &\leqslant |q_\pm - \gamma(s_\pm)| + 8\delta \\ &\leqslant 4\delta + \widetilde{H} v(|p_\pm|) + 8\delta, \end{aligned} \tag{I.57}$$

where we combined (I.54) and (I.55) by means of the triangle inequality. Hence

$$\begin{aligned}\forall t \in [t_-, t_+],\ d(\widetilde{\gamma}(t), \gamma) &\leqslant (h(\lambda) \vee \widetilde{h}(\lambda))(\delta + (v \uparrow 3\lambda) v(|q_+| \vee |q_-|)) \\ &\quad + 8\delta + 4\delta + \widetilde{H} v\,(|q_+| \vee |q_-|) \\ &\leqslant \left(12 + \widetilde{h}(\lambda)\right)\delta + (\widetilde{h}(\lambda)(v \uparrow 3\lambda) + \widetilde{H}) v \left(|p_\pm| + \widetilde{H} v(|p_\pm|\right). \end{aligned} \tag{I.58}$$

(here $\widetilde{h}(\lambda)$ is used alone as it is equal to $\widetilde{h}(\lambda) \vee h(\lambda)$). If $|\widetilde{\gamma}(0)| \geqslant R_{\sqcap}$, then in view of the definition of $p_{\pm}$ (I.53),

$$\begin{aligned} |p_{\pm}| &= k\left[|\widetilde{\gamma}(0)| + R^0_{\simeq}\right] \vee 2(2\lambda+1)r_{1/3\lambda}(v) \vee r_{1/(2\widetilde{H})}(v) \\ &\leqslant \left(k + \frac{R^0_{\simeq}}{R_{\sqcap}}\right)|\widetilde{\gamma}(0)| \vee 2(2\lambda+1)r_{1/3\lambda}(v) \vee r_{1/(2\widetilde{H})}(v) \\ &\underset{(\text{I.49})}{\leqslant} (2k|\widetilde{\gamma}(0)|) \vee 2(2\lambda+1)r_{1/3\lambda}(v) \vee r_{1/(2\widetilde{H})}(v) \end{aligned} \tag{I.59}$$

where we used $k \geqslant 1$ so that $k + 1/3 \leqslant 2k$ in the last inequality. Define $\widetilde{R}_1 = \widetilde{R}_0 \vee 2(2\lambda+1)r_{1/3\lambda}(v) \vee r_{1/(2\widetilde{H})}(v)$. By (I.59), if $|\widetilde{\gamma}(0)| \geqslant \widetilde{R}_1$,

$$v\left(|p_{\pm}| + \widetilde{H}v(|p_{\pm}|)\right) \leqslant v\left(\frac{3}{2}|p_{\pm}|\right) \leqslant v(3k|\widetilde{\gamma}(0)|).$$

Recall that by the right-hand side of assumption (I.50), $|\widetilde{\gamma}(0)| \leqslant L|\widetilde{b}| = L\inf\{|\widetilde{\gamma}(t)| : t \in \mathbf{R}\}$. Plugging (I.59) in (I.58), one obtains that for all t in $[t_-, t_+]$,

$$\begin{aligned} d(\widetilde{\gamma}(t), \gamma) &\leqslant (12 + \widetilde{h}(\lambda))\delta + \left(\widetilde{h}(\lambda)(v \uparrow 3\lambda) + \widetilde{H}\right) v(3k|\widetilde{\gamma}(0)|) \\ &\leqslant (12 + \widetilde{h}(\lambda))\delta + 2\widetilde{H}(v \uparrow 3\lambda)v(3Lk|\widetilde{\gamma}(t)|), \end{aligned}$$

where we used that $\widetilde{H} \geqslant \widetilde{h}(\lambda)$ on the second line; this is because by definition, $\widetilde{H} = 4(v \uparrow 6\lambda)(2h(\lambda)(v \uparrow 1 + 8\lambda^2) + 1) \geqslant 8h(\lambda) \geqslant \widetilde{h}(\lambda)$. Define $\widetilde{R}_3 = \widetilde{R}_2 \vee L^{-1}\sup\{r : v(r) \leqslant (12 + \widetilde{h}(\lambda))\delta\}$. If $|\widetilde{\gamma}(0)| \geqslant \widetilde{R}_3$, the right-hand side of assumption (I.50) ensures that $v(|\widetilde{\gamma}(t)|) \geqslant (12 + \widetilde{h}(\lambda))\delta$ for all t, so we have proved

$$\forall t \in [t_-, t_+],\ d(\widetilde{\gamma}(t), \gamma) \leqslant (1 + 2(v \uparrow 3\lambda)\widetilde{H}(v \uparrow 3Lk))v(|\widetilde{\gamma}(t)|). \tag{I.60}$$

On the other hand, in view of the tracking lemma I.31, for all $t \in (-\infty, t_-)$, $d(\widetilde{\gamma}(t), \gamma_-) \leqslant (v \uparrow 3\lambda)Hv(|\widetilde{\gamma}(t)|)$ and similarly for all $t \in (t_+, +\infty), d(\widetilde{\gamma}(t), \gamma_+) \leqslant (v \uparrow 3\lambda)v(|\widetilde{\gamma}(t)|)$. Since the twice-ideal triangle $o\eta_-\eta_+$ is 4δ-slim, using the triangle inequality and the fact that $v(|\widetilde{\gamma}(t)|) \geqslant 12\delta$ for all t provided $|\widetilde{\gamma}(0)| \geqslant \widetilde{R}_3$ by definition of $\widetilde{R}_3$,

$$\forall t \in \mathbf{R} \setminus [t_-, t_+],\ d(\widetilde{\gamma}(t), \gamma) \leqslant ((v \uparrow 3\lambda)H + 4\delta/(12\delta))v(|\widetilde{\gamma}(t)|). \tag{I.61}$$

Putting (I.60) and (I.61) together yields the expected tracking inequality (I.51) for the provisional $\widetilde{R}_3$. Precisely H_2 may be taken as

$$H_2 = 4(v \uparrow 3Lk)(v \uparrow 3\lambda)\widetilde{H} \vee 2(v \uparrow 3\lambda)H. \tag{I.62}$$

Step 4: Estimation of the tracking constant $\widetilde{H}_2$. — From here, one could deduce (I.52) using (I.51) for $\widetilde{R}$ large enough by a metric connectedness argument as in Lemma I.24 or Lemma I.31, but let us rather use the former estimates from the current proof. Define $\widetilde{R}_4 = \widetilde{R}_3 \vee r_{1/(2LH_2)}(v)$; then if $|\widetilde{\gamma}(0)| \geqslant \widetilde{R}_4$, it follows from the tracking inequality just obtained for $\widetilde{\gamma}$ that $|p_\gamma o| = |\gamma(0)| \geqslant (1/2)|\widetilde{\gamma}(0)|$. Then for all $s \in [s_-, s_+]$, by (I.57),

$$d(\gamma(s), \overline{\gamma}) \leqslant \widetilde{H} v(|p_\pm|) + 12\delta \leqslant 2\widetilde{H} v(2k|\gamma(s)|) \vee 24\delta. \tag{I.63}$$

On the other hand, recall that by Lemma I.24, for all $c \in \overline{\gamma}$, $d(c, \widetilde{\gamma}) \leqslant \widetilde{h}(\lambda)(\delta + (v \uparrow 3\lambda) v(|q_+| \vee |q_-|))$. Combining this with (I.63) by means of the triangle inequality while remembering the bound on $|q_\pm|$ implied by (I.55), one obtains

$$\forall s \in [s_-, s_+],\ d(\gamma(s), \widetilde{\gamma}) \leqslant (2\widetilde{H} + \widetilde{h}(\lambda)(\delta + (v \uparrow 3\lambda)) v(3k|\gamma(s)|). \tag{I.64}$$

Finally, if $s \in \mathbf{R}$ is such that $s \leqslant s_-$ or $s \geqslant s_+$, since o, $p_\gamma o$ and $\gamma(s)$ are 28δ-almost lined up, $|\gamma(s)| \geqslant |s| - |p_\gamma o| \geqslant |s|/2$. $\gamma(s)$ is at most 4δ away from its orthogonal projection on $\gamma_{\epsilon(s)}$, where $\epsilon(s)$ is the sign of s. Given the definition of $p_\pm$, $p_{\gamma_{\epsilon(s)}} \gamma(s)$ is at a distance at least $R_{\backsimeq}$ from the origin, and inequality (I.31) from Lemma I.31 bounds its distance to $\widetilde{\gamma}$ so that

$$\begin{aligned} d(\gamma(s), \widetilde{\gamma}) &\leqslant |\gamma(s) - p_{\gamma_{\epsilon(s)}} \gamma(s)| + d(p_{\gamma_{\epsilon(s)}} \gamma(s), \widetilde{\gamma}) \\ &\leqslant 4\delta + \widetilde{H} v(|p_{\gamma_{\epsilon(s)}} \gamma(s)|) \leqslant 2\widetilde{H} v(|\gamma(s)|). \end{aligned}$$

Together with (I.64), this proves (I.52) with $\widetilde{R} = \widetilde{R}_4$ and

$$\widetilde{H}_2 = \left(2\widetilde{H} + \widetilde{h}(\lambda)\right)(\delta + (v \uparrow 3\lambda))(v \uparrow 3k). \tag{I.65}$$

□

I.3.4. Distance between $O(u)$-geodesics

Lemma I.36. *Let $\delta \in \mathbf{R}_{\geqslant 0}$ be a constant. Let γ_1 and γ_2 be geodesic lines into a δ-hyperbolic space, with four pairwise distinct endpoints. Define $\Delta = d(\mathrm{im}(\gamma_1), \mathrm{im}(\gamma_2))$. Then for all $s_1, s_2 \in \mathbf{R}$,*

$$|\gamma_1(s_1) - \gamma_2(s_2)| \geqslant \Delta + d(\gamma_1(s_1), p_{\gamma_1}\mathrm{im}(\gamma_2)) \vee d(\gamma_2(s_2), p_{\gamma_2}\mathrm{im}(\gamma_1)) - 56\delta. \tag{I.66}$$

Proof. The distance on the left is symmetric relatively to $\gamma_i(s_i)$, so it suffices to prove $|\gamma_1(s_1) - \gamma_2(s_2)| \geqslant \Delta + d(\gamma_1(s_1), p_{\gamma_1}\mathrm{im}(\gamma_2)) - 56\delta$. The points $\gamma_1(s_1), p_{\gamma_2}(\gamma_1(s_1))$ and $\gamma_2(s_2)$ are the vertices of a right-angled hyperbolic

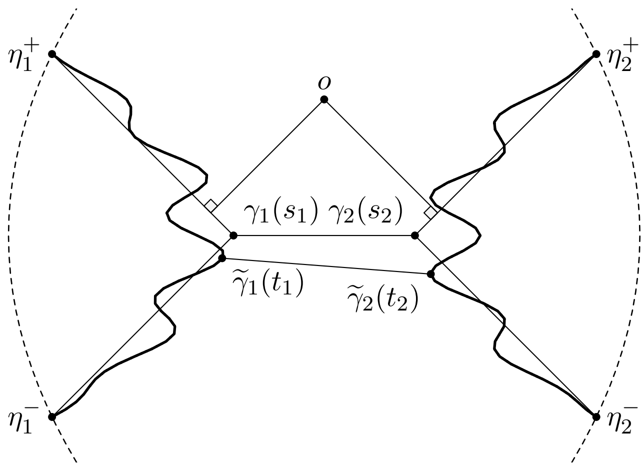


Figure 14: Main points occuring in the proof of Lemma I.37. Straight, resp. wavy lines depict geodesic, resp. $O(u)$-geodesic lines; the boundary is dashed.

triangle so that by Lemma I.21, they are 28δ-almost lined up. By the triangle inequality,

$$\begin{aligned} d(\gamma_1(s_1), \gamma_2(s_2)) + 2 \cdot 28\delta &\geqslant d(\gamma_1(s_1), p_{\gamma_2}(\gamma_1(s_1))) + d(p_{\gamma_2}(\gamma_1(s_1)), \gamma_2(s_2)) \\ &\geqslant \Delta + d(\gamma_1(s_1), p_{\gamma_1}\mathrm{im}(\gamma_2)). \end{aligned}$$ □

Lemma I.37. *Let v^1 and v^2 be admissible functions, and define $v = v^1 \vee v^2$. Let $L \in \mathbf{R}_{>1}$ be a constant. Let δ be a hyperbolicity constant and let $\lambda = (\underline{\lambda}, \overline{\lambda}) \in \mathbf{R}^2_{>0}$ be expansion and Lipschitz constants. There exist $J = J(\lambda, v, L)$, $R = R(\delta, \lambda, v, L)$ and, for $i \in \{1,2\}$, $\widetilde{R}^i = \widetilde{R}^i(\delta, \lambda, v^i, L)$ in $\mathbf{R}_{>0}$ such that for any proper geodesic, pointed δ-hyperbolic space (Y, o), if $(\gamma_1, \widetilde{\gamma}_1)$ and $(\gamma_2, \widetilde{\gamma}_2)$ are such that*

i. *γ_1, γ_2 are geodesics $\mathbf{R} \to Y$ with four distinct endpoints $\eta_i^\pm = \gamma_i(\pm\infty)$,*

ii. *for $i \in \{1,2\}$, $\widetilde{\gamma}_i$, is a (λ, v^i)-geodesics $\mathbf{R} \to Y$,*

iii. *for $i \in \{1,2\}$, $\partial_\infty \widetilde{\gamma}_i(\pm\infty) = [\gamma_i]_\pm$,*

iv. *$\underline{\boxtimes}\left\{\eta_1^\pm, \eta_2^\pm\right\} \geqslant 60\delta$, and $\overline{\boxtimes}\left\{\eta_1^\pm, \eta_2^\pm\right\} \geqslant R$,*

v. *for all $i \in \{1,2\}$, $\widetilde{R}^i \leqslant |\widetilde{\gamma}_i(0)| \leqslant L \inf_{t \in \mathbf{R}} |\widetilde{\gamma}_i(t)|$,*

then

$$|d(\gamma_1, \gamma_2) - d(\widetilde{\gamma}_1, \widetilde{\gamma}_2)| \leqslant Jv\left(\boxtimes\left\{\eta_1^\pm, \eta_2^\pm\right\}\right). \tag{I.67}$$

Sketch of proof for Lemma I.37. See figure 14. The main tool is the geodesic tracking lemma I.35; however the tracking between $\widetilde{\gamma}_i$ and γ_i becomes inefficient far from the origin. Thus we need to prove that shortest geodesic segments between γ_1 and γ_2 on the one hand, and between $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ on the other hand, are close to the origin (at most not significantly farther than the largest Gromov product). The part concerning γ_1 and γ_2 was already expressed by Lemma I.23; as for the other part we show (inequality (I.71)) that letting t_1, t_2 be such that $d(\widetilde{\gamma}_1, \widetilde{\gamma}_2) = |\widetilde{\gamma}_1(t_1) - \widetilde{\gamma}_2(t_2)|$, $|\widetilde{\gamma}_1(t_1)| \vee |\widetilde{\gamma}_2(t_2)|$ is linearly controlled by $\boxtimes\{\eta_1^\pm, \eta_2^\pm\}$ on the large-scale. This uses the well-known behavior described by Lemma I.36: geodesic rays spread apart linearly from each other after the Gromov products are reached; since they track $O(u)$-geodesics at a distance growing sublinearly, $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ also spread away from each other, which prevents $\widetilde{\gamma}_i(t_i)$ from being much farther than all the Gromov products.

Proof. For $i \in \{1, 2\}$, let $s_i \in \mathbf{R}$ be such that $|\gamma_1(s_1) - \gamma_2(s_2)| = d(\gamma_1, \gamma_2)$. As $\gamma_1(s_1) \in p_{\gamma_1}(\gamma_2)$, and similarly $\gamma_2(s_2) \in p_{\gamma_2}(\gamma_1)$, by the projection lemma I.23, $\sup_i |\gamma_i(s_i)| \leqslant \boxtimes\left\{\eta_1^\pm, \eta_2^\pm\right\} + 284\delta$. Further, set $\widetilde{R}^i = \widetilde{R}^i(\lambda, \delta, v^i, L)$ according to the tracking lemma I.35. Note that by the assumptions (i) to (iii), the first inequality in assumption (iv) and the right-hand side inequality in assumption (v), applied to the pairs $(\gamma_i, \widetilde{\gamma}_i)$, by Lemma I.35,

$$\begin{aligned} \forall i \in \{1,2\},\ d(\gamma_i(s_i), \widetilde{\gamma}_i) &\leqslant \widetilde{H}_2 v(|\gamma_i(s_i)|) \\ &\leqslant \widetilde{H}_2 v(\boxtimes\left\{\eta_1^\pm, \eta_2^\pm\right\} + 284\delta). \end{aligned}$$

By the triangle inequality, setting $J^+ = 2\widetilde{H}_2(v \uparrow 2)$ and $R_0 = \sup\{r : v(r) \leqslant 284\delta\}$, as soon as $\boxtimes\left\{\eta_1^\pm, \eta_2^\pm\right\} \geqslant R_0$,

$$d(\widetilde{\gamma}_1, \widetilde{\gamma}_2) - d(\gamma_1, \gamma_2) \leqslant d(\gamma_1(s_1), \widetilde{\gamma}_1) + d(\gamma_2(s_2), \widetilde{\gamma}_2) \leqslant J^+ v\left(\boxtimes\left\{\eta_1^\pm, \eta_2^\pm\right\}\right). \tag{I.68}$$

This is one half of inequality (I.67).

For $i \in \{1, 2\}$ let $t_i \in \mathbf{R}$ be such that $d(\widetilde{\gamma}_1, \widetilde{\gamma}_2) = |\widetilde{\gamma}_1(t_1) - \widetilde{\gamma}_2(t_2)|$. Let $\widetilde{s}_i$ be such that $\gamma_i(\widetilde{s}_i) = p_{\gamma_i}\widetilde{\gamma}_i(t_i)$. By the triangle inequality and the tracking lemma I.35,

$$|\gamma_1(\widetilde{s}_1) - \gamma_2(\widetilde{s}_2)| \leqslant d(\widetilde{\gamma}_1, \widetilde{\gamma}_2) + 2H_2 v(|\widetilde{\gamma}_1(t_1)| \vee |\widetilde{\gamma}_2(t_2)|). \tag{I.69}$$

Inequality (I.66) of Lemma I.36 gives a lower bound on $|\gamma_1(\widetilde{s}_1) - \gamma_2(\widetilde{s}_2)|$, which can be plugged into (I.69) yielding

$$\begin{aligned} d(\gamma_1, \gamma_2) + &d\left(\gamma_1(\widetilde{s}_1), p_{\gamma_1}\mathrm{im}(\gamma_2)\right) \vee d\left(\gamma_2(\widetilde{s}_2), p_{\gamma_2}\mathrm{im}(\gamma_1)\right) \\ &\leqslant d(\widetilde{\gamma}_1, \widetilde{\gamma}_2) + 2H_2 v(|\widetilde{\gamma}_1(t_1)| \vee |\widetilde{\gamma}_2(t_2)|) + 56\delta. \end{aligned} \tag{I.70}$$

On the other hand, using twice the triangle inequality and Lemma I.23,

$$\begin{aligned}d\left(\gamma_1(\tilde{s}_1), p_{\gamma_1}\mathrm{im}(\gamma_2)\right) \vee d\left(\gamma_2(\tilde{s}_2), p_{\gamma_2}\mathrm{im}(\gamma_1)\right) &\geqslant |\gamma_1(\tilde{s}_1)| \vee |\gamma_2(\tilde{s}_2)| \\ &\quad - \boxtimes\left\{\eta_1^{\pm}, \eta_2^{\pm}\right\} - 284\delta \\ &\geqslant |\tilde{\gamma}_1(t_1)| \vee |\tilde{\gamma}_2(t_2)| - \boxtimes\left\{\eta_1^{\pm}, \eta_2^{\pm}\right\} \\ &\quad - H_2 v(|\tilde{\gamma}_1(t_1)| \vee |\tilde{\gamma}_2(t_2)|) - 284\delta,\end{aligned}$$

where we have used the tracking inequality (I.51) from Lemma I.35 for the last line. Reorganizing (I.70),

$$\begin{aligned}|\tilde{\gamma}_1(t_1)| \vee |\tilde{\gamma}_2(t_2)| &\leqslant \boxtimes\{\eta_1^{\pm}, \eta_2^{\pm}\} + 340\delta + 3H_2 v(|\tilde{\gamma}_1(t_1)| \vee |\tilde{\gamma}_2(t_2)|) \\ &\quad + d(\tilde{\gamma}_1, \tilde{\gamma}_2) - d(\gamma_1, \gamma_2) \\ &\underset{(\mathrm{I}.68)}{\leqslant} \boxtimes\{\eta_1^{\pm}, \eta_2^{\pm}\} + 340\delta + 3H_2 v(|\tilde{\gamma}_1(t_1)| \vee |\tilde{\gamma}_2(t_2)|) \\ &\quad + J^+ v\left(\boxtimes\{\eta_1^{\pm}, \eta_2^{\pm}\}\right)\end{aligned}$$

when $\boxtimes\left\{\eta_1^{\pm}, \eta_2^{\pm}\right\} \geqslant R_0$. Hence,

$$\begin{aligned}|\tilde{\gamma}_1(t_1)| \vee |\tilde{\gamma}_2(t_2)| \leqslant \inf \big\{ & r_{1/(6H_2)}(v), \\ & 2\left[\boxtimes\{\eta_1^{\pm}, \eta_2^{\pm}\} + 340\delta + J^+ v\left(\boxtimes\{\eta_1^{\pm}, \eta_2^{\pm}\}\right)\right]\big\}.\end{aligned}$$

Set $R_1 = \sup\{r : v(r) \geqslant 584\delta/J^+\}$ and $R_2 = \sup\{R_0, R_1, r_{1/(2J^+)}(v)\}$. Then if $\boxtimes\left\{\eta_1^{\pm}, \eta_2^{\pm}\right\} \geqslant R_2$,

$$|\tilde{\gamma}_1(t_1)| \vee |\tilde{\gamma}_2(t_2)| \leqslant \inf\left\{r_{1/(6H_2)}(v), 4\boxtimes\{\eta_1^{\pm}, \eta_2^{\pm}\}\right\}. \tag{I.71}$$

Thus if $R_3 = r_{1/(4H_2)}(v)$, and if $t_1, t_2 \in \mathbf{R}$ are such that $|\tilde{\gamma}_1(t_1)| \vee |\tilde{\gamma}_2(t_2)| \geqslant R_3$, then

$$H_2 v(\tilde{\gamma}_1(t_1)| \vee |\tilde{\gamma}_2(t_2)|) \leqslant H_2(v \uparrow 4) v\left(\boxtimes\{\eta_1^{\pm}, \eta_2^{\pm}\}\right).$$

Finally by the triangle inequality, writing $J^- = 2H_2(v \uparrow 4)$,

$$\begin{aligned}d(\gamma_1, \gamma_2) - d(\tilde{\gamma}_1, \tilde{\gamma}_2) &\leqslant d(\gamma_1(\tilde{s}_1), \gamma_2(\tilde{s}_2)) - d(\tilde{\gamma}_1(t_1), \tilde{\gamma}_2(t_2)) \\ &\leqslant J^- v\left(\boxtimes\{\eta_1^{\pm}, \eta_2^{\pm}\}\right).\end{aligned} \tag{I.72}$$

To reach the conclusion of Lemma I.37, define $J = J^- \vee J^+$ and then combine (I.68) with (I.72). □

I.3.5. Tracking radii

While there are four relevant parameters (λ, v, δ, L) to express $R^0_{\simeq}$, $R_{\sqcap}$, $\widetilde{R}$ and R, only the dependence on v is of interest for what follows. Consequently, a constant depending on the remaining parameters λ, δ, L can be written as, e.g., $C(\lambda,\delta)$ or $C(\lambda,\delta,L)$.

Lemma I.38. *Let v be an admissible function. Let $\lambda \in \mathbf{R}_{\geqslant 1}$ be a biLipschitz constant. Let δ be a hyperbolicity constant. There exist a positive integer n and constants $C(\lambda)$, $C(\lambda,\delta)$, $C(\lambda,\delta,L)$ such that in Lemma I.31, Lemma I.34, Lemma I.35 and Lemma I.37, the tracking radii may be taken as*

$$R^0_{\simeq} = r_{C(\lambda)(v\uparrow 1+\lambda)^{-n}}(v) \vee C(\lambda,\delta)\left(1+\sup\left\{r : v(r) \leqslant C(\lambda,\delta)\right\}\right) \tag{I.73}$$

$$\widetilde{R} = C(\lambda,\delta) r_{C(\lambda)(v\uparrow L)^{-1}(v\uparrow 1+\lambda)^{-n}}(v) \vee \left(1+\sup\left\{r : v(r) \leqslant C(\lambda,\delta,L)\right\}\right) \tag{I.74}$$

$$R = r_{C(\lambda,\delta)(v\uparrow 1+\lambda)^{-n}}(v) \vee C(\lambda,\delta)\left(1+\sup\left\{r : v(r) \leqslant C(\lambda,\delta,L)\right\}\right) \tag{I.75}$$

Proof. It will be used without further notice that $r_\alpha(v) \vee r_\beta(v) = r_{\alpha\wedge\beta}(v)$, for all $\alpha, \beta \in \mathbf{R}_{>0}$, and that $\lambda \geqslant 1$, especially $1/\lambda \leqslant \lambda \leqslant \lambda^2$. The bounds we obtain need not be excessively precise, and we allow losing multiplicative factors frequently. Start with (I.73), and notation as in the proof of Lemma I.31. Recall from the definition of t_1 in this proof that it was defined as

$$\begin{aligned} t_1 &= r_{2\underline{\lambda}}(v) \vee r_{1/(3\lambda)}(v) \vee r_{1/(12\lambda h(\lambda) v\uparrow 1+8\lambda^2)}(v) \vee \sup\{s : v(s) \leqslant \delta\} \\ &\quad \vee 24\lambda\delta \vee (4\lambda M'(\underline{\lambda},\delta)+1) \vee \sup\{r : v(r) \leqslant 6\lambda^2\delta\} \\ &\leqslant r_{1/(12\lambda h(\lambda) v\uparrow 1+8\lambda^2)}(v) \vee C(\lambda,\delta). \end{aligned} \tag{I.76}$$

Next, $t_3 = t_1 \vee \sup\{r : v(r) \leqslant h(\lambda)\delta\}$ since $h(\lambda)\delta \geqslant 6\lambda^2\delta$. After that, $t \geqslant t_4 := \sup\left\{t_3, r_{\underline{\lambda}/(2+2H_0(\lambda,v))}(v)\right\}$, where $H_0(\lambda,v) = 2h(\lambda)(v\uparrow 1+8\lambda^2)+1$. From this and (I.76) we deduce

$$t_4 \leqslant r_{1/(12\lambda H_0)}(v) \vee \sup\{r : v(r) \leqslant h(\lambda)\delta\} \vee C(\lambda,\delta) \tag{I.77}$$

and then

$$\begin{aligned} t_5 &\leqslant t_4 \vee \sup\{r : v(r) \leqslant 8\delta\} \\ &\leqslant r_{1/(12\lambda H_0)}(v) \vee \sup\{r : v(r) \leqslant 8h(\lambda)\delta\} \vee C(\lambda,\delta); \end{aligned}$$

$$\begin{aligned} t_{\simeq 0} &\leqslant t_5 \vee 16\delta = t_5 \vee C(\lambda,\delta); \\ t_6 &= t_{\simeq 0} \vee \sup\{r : v(r) \leqslant 8\delta\} \\ &\leqslant r_{1/(12\lambda H_0)}(v) \vee \sup\{r : v(r) \leqslant 16h(\lambda)\delta\} \vee C(\lambda,\delta). \end{aligned}$$

As $t_7 = t_6 \vee t_{\bigcirc}$, $t_{\bigcirc} = r_{1/(3\lambda)}(v)$ and $H_0 \geqslant 4$ (since $h(\lambda) \geqslant 12$ by Lemma I.24), the same bound applies to t_7. Next,

$$t_8 = t_7 \vee r_{\underline{\lambda}/(6H)}(v) \leqslant r_{\underline{\lambda}/(6H)}(v) \vee \sup\{r : v(r) \leqslant 16h(\lambda)\delta\} \vee C(\lambda, \delta) \quad \text{(I.78)}$$

(remember that $H = 1 + H_0$ by definition). Thus

$$\begin{aligned} R^0_{\backsimeq} &= t_8/(6\lambda) \vee \sup\{r : 2(v \uparrow 6\lambda)(H_0 + 1)v(r) \leqslant 8\lambda\} \\ &\leqslant t_8 \vee \sup\{r : 2(H_0 + 1)v(6\lambda r) \leqslant 8\lambda\} \\ &\leqslant t_8 \vee \sup\left\{r : 2\left(2h(\lambda)(v \uparrow 1 + 8\lambda^2) + 1\right) v(6\lambda r) \leqslant 8\lambda\right\} \\ &\leqslant t_8 \vee \sup\left\{r : 6h(\lambda)v(6\lambda(1 + 8\lambda^2)r) \leqslant 8\lambda\right\} \\ &\underset{\text{(I.78)}}{\leqslant} r_{1/(6\lambda H)}(v) \vee C(\lambda, \delta)\,(1 + \sup\{r : v(r) \leqslant C(\lambda, \delta)\}) . \qquad \text{(I.79)} \end{aligned}$$

This inequality implies (I.73) (one may take $n = 4$ there), since $H = 2h(\lambda)(v \uparrow 1 + 8\lambda^2) + 2 \leqslant C(\lambda)(v \uparrow 1 + \lambda)^4$. Let us turn to (I.74). Start establishing a similar bound for $R_{\sqcap}$, with notation as in the proof of Lemma I.34. By (I.49),

$$R_{\sqcap} = 3R^0_{\backsimeq} \vee \frac{3\lambda}{2\widetilde{H}} r_{1/(8\lambda\widetilde{H})}(v) \vee 192\lambda^2\delta \vee r_{1/(6(v\uparrow 2)H_3(\lambda,v,\delta))}(v),$$

where $H_3 = \left(4\lambda^2 H + 2\lambda(v \uparrow \lambda)\right)(v \uparrow 2) \leqslant 6\lambda^2 H(v \uparrow 2) \leqslant 6\lambda^2\widetilde{H}$, hence by (I.79),

$$R_{\sqcap} \leqslant 3\lambda r_{1/(8\lambda^2\widetilde{H}(v\uparrow 2))}(v) \vee C(\lambda, \delta)\,(1 + \sup\{r : v(r) \leqslant C(\lambda, \delta)\}) . \qquad \text{(I.80)}$$

In the proof of Lemma I.35, $\widetilde{R}$ was defined as a supremum of four terms:

$$\begin{aligned} \widetilde{R} &= R_{\sqcap} \vee 2(2\lambda + 1)r_{1/(3\lambda)}(v) \vee r_{1/(2\widetilde{H})}(v) \vee r_{1/(2LH_2)}(v) \\ &\quad \vee L^{-1}\sup\{r : v(r) \leqslant (12 + \widetilde{h}(\lambda))\delta\} \\ &\leqslant R_{\sqcap} \vee 5\lambda^2 r_{1/(3\lambda)\wedge 1/(2\widetilde{H})\wedge 1/(2LH_2)}(v) \vee C(\lambda, \delta). \end{aligned}$$

We need to bound the tracking constants $\widetilde{H}$ and H_2. By definition of $\widetilde{H}$ in the proof of Lemma I.31, $\widetilde{H} = 4(v \uparrow 6\lambda)\left(2h(\lambda)(v \uparrow 1 + 8\lambda^2) + 1\right) \leqslant C(\lambda)\,(v \uparrow 1 + \lambda)^{n_0}$, where n_0 is large enough, and by (I.62) with $k = 11 \vee 12\lambda(2\lambda + 1) \leqslant 36\lambda^2$,

$$H_2 \leqslant 4\left[(v \uparrow 3Lk)(v \uparrow 3\lambda)\widetilde{H}\right] \vee 2(v \uparrow 3\lambda)H \leqslant C(\lambda)(v \uparrow L)(v \uparrow 1 + \lambda)^{n_1},$$

where n_1 is large enough. By (I.80) and the previous bounds, $\widetilde{R}$ may be taken as

$$\widetilde{R} = 5\lambda^2 r_{1/(C(\lambda)(v\uparrow L)(v\uparrow 1+\lambda)^{n_0})} \vee C(\lambda, \delta)(1 + \sup\{r : v(r) \leqslant C(\lambda, \delta)\}).$$

This is a precise form of (I.74). Finally, we must prove (I.75). With notation as in the proof of Lemma I.37, $R = r_{1/(2J^+)}(v) \vee \sup\{r : v(r) \leqslant 284\delta \vee 584\delta/J^+\}$, where $J^+ = 2\widetilde{H}_2(v \uparrow 2)$ so that

$$R \leqslant r_{1/(4\widetilde{H}_2(v\uparrow 2))}(v) \vee \sup\{r : v(2r) \leqslant 584\delta/(2\widetilde{H}_2)\}, \tag{I.81}$$

and we need to bound $\widetilde{H}_2$. With notation as in the proof of Lemma I.35, recall from (I.62) that $\widetilde{H}_2$ can be bounded by

$$\begin{aligned}\widetilde{H}_2 = (2\widetilde{H} + \widetilde{h}(\lambda))(\delta + (v \uparrow 3\lambda)) &\leqslant C(\lambda)\widetilde{H}C(\lambda,\delta)(v \uparrow 3\lambda)(v \uparrow 3k)\\ &\leqslant C(\lambda,\delta)(v \uparrow 1+\lambda)^7(v \uparrow 3\lambda)(v \uparrow 3\cdot 36\lambda^2)(v \uparrow 3\lambda).\end{aligned}$$

Plugging this inequality in (I.81) yields the expected (I.75). □

Lemma I.39 (Sublinear growth of tracking radii)**.** *Let w be an admissible function. For all $p \in \mathbf{R}_{\geqslant 0}$, define $w_p(r) = w(p+r)$, and then denote by R_p, resp. $\widetilde{R}_p$ the constants $R(\lambda,\delta,w_p)$ and $\widetilde{R}(\lambda,\delta,w_p)$ of Lemma I.37. There exist $\widetilde{K} = \widetilde{K}(\lambda,\delta,w,L)$ and $K = K(\lambda,\delta,w,L)$ in $\mathbf{R}_{>0}$ such that*

$$\widetilde{R}_p \leqslant \widetilde{K}w(p), \text{ and} \tag{I.82}$$

$$R_p \leqslant Kw(p). \tag{I.83}$$

Proof. By Lemma I.38, there exists a positive integer n such that $\widetilde{R}_p$ and R_p may be taken as

$$\begin{aligned}\widetilde{R}_p &= C(\lambda,\delta)r_{C(\lambda)(w_p\uparrow L)^{-1}(w_p\uparrow 1+\lambda)^{-n}}(w_p)\\ &\qquad \vee C(\lambda,\delta)\left(1+\sup\left\{r : w_p(r) \leqslant C(\lambda,\delta,L)\right\}\right)\end{aligned} \tag{I.84}$$

$$R_p = r_{C(\lambda)(w_p\uparrow 1+\lambda)^{-n}}(w_p) \vee C(\lambda,\delta)\left(1+\sup\left\{r : w_p(r) \leqslant C(\lambda,\delta,L)\right\}\right). \tag{I.85}$$

The rightmost terms $C(\lambda,\delta)\left(1+\sup\left\{r : w_p(r) \leqslant C(\lambda,\delta,L)\right\}\right)$ are nonincreasing functions of p, since $\{w_p\}$ is a nondecreasing sequence of functions, so that their dependence over p can be removed. Further, $w_p \uparrow 1+\lambda$ is a nonincreasing function of p by Lemma I.7 (1), hence $(w_p \uparrow 1+\lambda)^{-n}$ is a nondecreasing function of p. Thus (I.84) and (I.85) may be simplified as

$$\begin{aligned}\widetilde{R}_p &= C(\lambda,\delta)r_{C(\lambda)(w_p\uparrow L)^{-1}(w\uparrow 1+\lambda)^{-n}}(w_p) \vee C(\lambda,\delta,L,w)\\ R_p &= r_{C(\lambda)(w\uparrow 1+\lambda)^{-n}}(w_p) \vee C(\lambda,\delta,L,w).\end{aligned}$$

Then by Lemma I.7 (2), $\widetilde{R}_p \leqslant C(\lambda,\delta,L,w) \vee C(\lambda)(w\uparrow 2)(w \uparrow 1+\lambda)^n w(p)$. This proves (I.82) for a constant $\widetilde{K} = \widetilde{K}(\lambda,\delta,L,w)$, and similarly there exists $K = K(\lambda,\delta,L,w)$ such that $R_p \leqslant Kw(p)$, which is (I.83). □

I.4. On the sphere at infinity

I.4.1. Sublinearly quasiMöbius homeomorphisms

With geodesic boundaries of hyperbolic spaces in mind, we abstractly define sublinearly quasiMöbius homeomorphisms between compact metric spaces:

Definition I.40. Let u be an admissible function. Let $(\underline{\alpha}, \overline{\alpha}) \in \mathbf{R}^2_{>0}$ be a couple of constants. Let (Ξ, ϱ) and (Ψ, ϑ) be metric spaces and let $\varphi : \Xi \to \Psi$ be a homeomorphism. φ is a $(\underline{\alpha}, \overline{\alpha}, O(u))$-sublinearly quasiMöbius homeomorphism if there exist $v = O(u)$, $\nu \in \mathbf{R}_{>1}$ and $\mathcal{E} \in \mathbf{R}_{>0}$ such that for all $(\xi_1, \dots \xi_4) \in \Xi^4$ with $0 < \inf_{i \neq j} \varrho(\xi_i, \xi_j) \leqslant \sup_{i \neq j} \varrho(\xi_i, \xi_j) < \mathcal{E}$,

$$\underline{\alpha} \log_\nu^+ [\xi_i] - v \left(\sup_{i \neq j} [-\log_\nu \varrho(\xi_i, \xi_j)] \right) \leqslant \log_\nu^+ [\varphi(\xi_i)]$$
$$\overline{\alpha} \log_\nu^+ [\xi_i] + v \left(\sup_{i \neq j} [-\log_\nu \varrho(\xi_i, \xi_j)] \right) \geqslant \log_\nu^+ [\varphi(\xi_i)] .$$

Note that one would only need a change of function v within the $O(u)$-class to compensate a different choice of ν. We call $\underline{\alpha}$, $\overline{\alpha}$ and $\alpha = \sup\{\overline{\alpha}, 1/\underline{\alpha}\}$ the Lipschitz-Möbius constants of φ.

Although this is not a direct consequence of Definition I.40, sublinearly quasi-Möbius homeomorphisms between uniformly perfect spaces are stable under composition; we postpone the proof to subsection I.4.2. Also note that in the definition one could replace the source and target distances with any equivalent real-valued kernels $\widehat{\varrho}$ and $\widehat{\vartheta}$, or even, if no special attention is required on precise Lipschitz-Möbius constants, with kernels such that $\widehat{\varrho}^{\gamma_1}$ and $\widehat{\vartheta}^{\gamma_2}$ are equivalent to ϱ and ϑ for a pair of exponents $\gamma_1, \gamma_2 \in \mathbf{R}_{>0}$. This occurs on geodesic boundaries when $\widehat{\varrho}$ and $\widehat{\vartheta}$ are visual quasimetrics while ϱ and ϑ are visual distances.

Recall that, by Proposition I.11, any large-scale sublinearly Lipschitz embedding f between proper geodesic Gromov-hyperbolic spaces induces a boundary map, which only depends on the $O(u)$-closeness class of f so that it can be denoted $\partial_\infty [f]_{O(u)}$.

Theorem I.41. *Let u be an admissible function. Let $(\underline{\lambda}, \overline{\lambda}) \in \mathbf{R}^2_{>0}$ be expansion and Lipschitz constants. Let $f : X \to Y$ be a $(\underline{\lambda}, \overline{\lambda}, O(u))$-sublinearly biLipschitz equivalence betwen proper, geodesic hyperbolic spaces. Then $\partial_\infty [f]_{O(u)}$ is a $(\underline{\lambda}, \overline{\lambda}, O(u))$-sublinearly quasiMöbius homeomorphism.*

Sketch of proof for Theorem I.41. Our argument is inspired from the lecture notes by Bourdon [18, Theorem 2.2] on Mostow rigidity and Tukia's theorem; the main ingredient is Lemma I.37, which ensures that the geometric interpretation of the cross-difference (see Proposition I.15 and Figure 10) subsists with a sublinear error when applying a sublinearly biLipschitz equivalence and measuring distances between $O(u)$-geodesics in the target space. Lemma I.37 must be applied with care, though, since the control functions and tracking radii deteriorate as the Gromov products of endpoints grow. This is where Lemma I.39 intervenes and certifies that the growth of tracking radii is sublinear with respect to Gromov products, so that the tracking estimates and their consequences are ultimately valid.

Proof. Fix basepoints o in X and Y, and let $w = O(u)$ be an admissible function such that f is a $(\underline{\lambda}, \overline{\lambda}, w)$-sublinearly biLipschitz equivalence from (X, o) to (Y, o). For any quadruple $(\xi_1, \ldots \xi_4) \in \partial^4_\infty X$, write for short $\eta_i = \partial f(\xi_i)$ for all i in $\{1, \ldots 4\}$, and for all $\varepsilon, \mathcal{E} \in \mathbf{R}_{>0}$ such that $\varepsilon < \mathcal{E}$, let $F(\varepsilon, \mathcal{E})$ be the subspace of $\partial^4 X$ defined by

$$\begin{cases} \overline{\boxtimes}\{\xi_i\} > -\log_\mu \varepsilon, \\ \underline{\boxtimes}\{\xi_i\} > -\log_\mu \mathcal{E}. \end{cases}$$

Note that, since $\partial_\infty X$ is compact the space defined by the first inequality is a neighborhood of the ends in $\partial^4_\infty X$, hence it suffices to prove the inequality

$$\underline{\lambda}[\xi_i] - v\left(\boxtimes\{\xi_i\}\right) \leqslant [\eta_i] \leqslant \overline{\lambda}[\xi_i] + v\left(\boxtimes\{\xi_i\}\right)$$

for all $(\xi_i) \in F(\varepsilon, \mathcal{E})$, for some small ε and $\mathcal{E}$ and $v = O(u)$. For any pair $\{i,j\} \in \{\{1,4\},\{2,3\}\}$ let χ_{ij} be a geodesic in X with endpoints ξ_i and ξ_j, resp. γ_{ij} a geodesic in Y with endpoints η_i and η_j such that $\chi_{ij}(0) = p_{\chi_{ij}}(o)$ and $\gamma_{ij}(0) = p_{\gamma_{ij}}(o)$. Finally, write $\widetilde{\gamma}_{ij}(t) = f \circ \chi_{ij}(t)$, and observe that $\widetilde{\gamma}_{ij}$ is a (λ, w')-geodesic, where $w'(r) := w(|\chi_{14}(r)| \vee |\chi_{23}(r)|) \leqslant w\left((\xi_1 \mid \xi_4)_o \vee (\xi_2 \mid \xi_3)_o + r\right)$. Especially, $\widetilde{\gamma}_{ij}$ is a (λ, w_p) geodesic, where

$$w_p(r) := w(p + r),$$

and $p = (\xi_i \mid \xi_j)_o$. We shall apply Lemma I.37 with $v^1 = w_{(\xi_1|\xi_4)}$, $v^2 = w_{(\xi_2|\xi_3)}$ and $v = w_{\boxtimes\{\xi_i\}}$. Assumptions (i), (ii) and (iii) follow from the definitions of γ_{ij} and $\widetilde{\gamma}_{ij}$. Then, recall from inequality (I.27) in Lemma I.29 that if $|\chi_{ij}(0)| \geqslant t_{\bigcirc}(|f(o)|, w)$, then for all $t \in \mathbf{R}$, $\frac{1}{3\lambda}|\chi_{ij}(t)| \leqslant |\widetilde{\gamma}_{ij}(t)| \leqslant 3\lambda|\chi_{ij}(t)|$, and then

$$\forall t \in \mathbf{R},\ |\widetilde{\gamma_{ij}}(0)| \leqslant 3\lambda|\chi_{ij}(0)| \leqslant 3\lambda|\chi_{ij}(t)| \leqslant 9\lambda^2|\widetilde{\gamma_{ij}}(t)|.$$

This is the right-hand side inequality of (v) with $L = 9\lambda^2$, that we fix for the rest of the proof. Observe that the lower bound needed on the radii $|\chi_{ij}(0)|$ is guaranteed as soon as $\boxtimes\{\xi_i\} \geqslant t_\bigcirc(|f(o)|, w) = r_{1/(3\lambda)}(w) \vee 3\lambda|f(o)|$. On the other hand, by Cornulier's theorem I.13 $\partial_\infty[f]$ is uniformly continuous on $\partial_\infty X$, so there exists $R_\square \in \mathbf{R}_{\geqslant 0}$ such that $\boxtimes\{\xi_i\} \geqslant R_\square \implies \boxtimes\{\eta_i\} \geqslant 60\delta$. Let $\widetilde{K} = \widetilde{K}(\lambda, w, \delta, L)$ be the constant from Lemma I.39, and define

$$\mathcal{E} = \mu^{-(R_\square \vee t_\bigcirc(|f(o)|,w) \vee r_{1/(3\lambda\widetilde{K})}(w))}.$$

Then as soon as $\boxtimes\{\xi_i\} > -\log_\mu \mathcal{E}$,

$$\begin{cases} \boxtimes\{\eta_i\} \geqslant 60\delta. & \text{as } \boxtimes\{\xi_i\} \geqslant R_\square \\ |\widetilde{\gamma}_{ij}(0)| \leqslant L \inf_{t\in\mathbf{R}} |\widetilde{\gamma_{ij}}(t)| & \text{as } \boxtimes\{\xi_i\} \geqslant t_\bigcirc(|f(o)|, w) \\ \widetilde{R}_{(\xi_i|\xi_j)_o} \leqslant \frac{(\xi_i|\xi_j)_o}{3\lambda} \leqslant |\widetilde{\gamma}_i(0)| & \text{as } (\xi_i \mid \xi_j)_o \geqslant \boxtimes\{\xi_i\} \geqslant t_\bigcirc(|f(o)|, w) \vee r_{1/(3\lambda\widetilde{K})}(w). \end{cases}$$

The first line is the first condition in (iv), the second and third one are the assumption (v); we used (I.82) from Lemma I.39 in the third line. By the conclusion of Cornulier's theorem I.13 applied to both $\partial_\infty[f]$ and to $\partial_\infty[f]^{-1}$, there exists $\varepsilon_0 \in \mathbf{R}_{>0}$ such that

$$\overline{\boxtimes}\{\xi_i\} > -\log_\mu \varepsilon_0 \implies 2\lambda\overline{\boxtimes}\{\xi_i\} \geqslant \overline{\boxtimes}\{\eta_i\} \geqslant \frac{1}{2\lambda}\overline{\boxtimes}\{\xi_i\}.$$

Let K be the constant from Lemma I.39. Define $\varepsilon = \varepsilon_0 \wedge \mathcal{E} \wedge \mu^{-2\lambda r_{(1/3\lambda K)}(w)}$. Then by (I.83) of Lemma I.39, $\overline{\boxtimes}\{\xi_i\} > -\log_\mu \varepsilon \implies \overline{\boxtimes}\{\eta_i\} \geqslant R_{\overline{\boxtimes}\{\xi_i\}}$. Thus if $(\xi_i) \in F(\varepsilon, \mathcal{E})$ then Lemma I.37 applies to $(\gamma_{ij}, \widetilde{\gamma}_{ij})$, and

$$\begin{aligned} |d_Y(\gamma_{23}, \gamma_{14}) - d_Y(\widetilde{\gamma}_{23}, \widetilde{\gamma}_{14})| &\leqslant J w_{\overline{\boxtimes}\{\xi_i\}}(\overline{\boxtimes}\{\eta_i\}) \\ &\leqslant J(w \uparrow 2\lambda) w(\overline{\boxtimes}\{\eta_i\}). \end{aligned} \tag{I.86}$$

Thanks to Proposition I.15, there exists $C = C(\delta)$ in $\mathbf{R}_{\geqslant 0}$ such that

$$\begin{cases} d_X(\chi_{14}, \chi_{23}) - C(\delta) & \leqslant \log^+[\xi_i] \leqslant d_X(\chi_{14}, \chi_{23}) + C(\delta). \\ d_Y(\gamma_{14}, \gamma_{23}) - C(\delta) & \leqslant \log^+[\eta_i] \leqslant d_Y(\gamma_{14}, \gamma_{23}) + C(\delta). \end{cases}$$

In view of (I.86) and the previous set of inequalities, it suffices to prove

$$\underline{\lambda} d_X(\chi_{14}, \chi_{23}) - v\left(\overline{\boxtimes}\{\xi_i\}\right) \leqslant d_Y(\gamma_{14}, \gamma_{23}) \leqslant \overline{\lambda} d_X(\chi_{14}, \chi_{23}) + v\left(\overline{\boxtimes}\{\xi_i\}\right) \tag{I.87}$$

for some function $v = O(u)$. Start with the left-hand side inequality. Letting $\widetilde{s}_1, \widetilde{s}_2 \in \mathbf{R}$ be such that $|f \circ \chi_{14}(\widetilde{s}_1) - f \circ \chi_{23}(\widetilde{s}_2)| = d(\widetilde{\gamma}_{14}, \widetilde{\gamma}_{23})$,

$$\begin{aligned}\underline{\lambda} d(\chi_{14}, \chi_{23}) - w\left(\boxtimes\{\xi_i\}\right) &\leqslant \underline{\lambda}|\chi_{14}(\widetilde{s}_1) - \chi_{23}(\widetilde{s}_2)| - w\left(|\chi_{14}(\widetilde{s}_1)| \vee |\chi_{23}(\widetilde{s}_2)|\right) \\ &\leqslant |f \circ \chi_{14}(\widetilde{s}_1) - f \circ \chi_{23}(\widetilde{s}_2)| \\ &= d(\widetilde{\gamma}_{14}, \widetilde{\gamma}_{23}) \\ &\underset{(\text{I.86})}{\leqslant} d(\gamma_{14}, \gamma_{23}) + J(w \uparrow 2\lambda)\left(\boxtimes\{\eta_i\}\right),\end{aligned}$$

hence

$$\underline{\lambda} d(\chi_{14}, \chi_{23}) \leqslant d(\gamma_{14}, \gamma_{23}) + (1 + J(w \uparrow 2\lambda))\, w\left(\boxtimes\{\eta_i\} + \boxtimes\{\xi_i\}\right). \tag{I.88}$$

Let us proceed in the same way for the right-hand side of (I.87). By Lemma I.37, letting $s_1, s_2 \in \mathbf{R}$ be such that $|\chi_{14}(s_1) - \chi_{23}(s_2)| = d(\chi_{14}, \chi_{23})$,

$$\begin{aligned}d(\gamma_{14}, \gamma_{23}) &\leqslant d(\widetilde{\gamma}_{14}, \widetilde{\gamma}_{23}) + J(w \uparrow 2\lambda)\left(\boxtimes\{\eta_i\}\right) \\ &\leqslant |\widetilde{\gamma}_{14}(s_1) - \widetilde{\gamma}_{23}(s_2)| + J(w \uparrow 2\lambda)\left(\boxtimes\{\eta_i\}\right) \\ &\leqslant \overline{\lambda} d(\chi_{14}, \chi_{23}) + \left(1 + (w \uparrow 2\lambda)^2\right) w\left(\boxtimes\{\xi_i\}\right).\end{aligned} \tag{I.89}$$

Setting $v = \left(1 + (w \uparrow 2\lambda)^2\right) w$ this proves (I.87) and the theorem. □

I.4.2. Properties of sublinearly quasiMöbius homeomorphisms

After simplifying the cross-ratio estimates when two, resp. one points are far away, one obtains that sublinearly quasiMöbius homeomorphisms between appropriate spaces are Hölder, resp. almost quasisymmetric, see figure 15. Precisely we work under the following assumption (Buyalo and Schroeder [27, 7.2] or Mackay and Tyson [108, 1.3.2]); see however Remark I.47.

Definition I.42. Let Ξ be a metric space. Then X is uniformly perfect if there exists $\tau \in (0,1)$ such that for every ball $B \subsetneq \Xi$, the annulus $B \setminus \tau B$ is non-empty.

Note that in the definition, for any positive integer k, up to replacing τ with τ^k one can assume for free that $B \setminus \tau B$ has k points. Uniform perfectness is granted for boundaries of non-elementary hyperbolic groups, or for connected spaces.

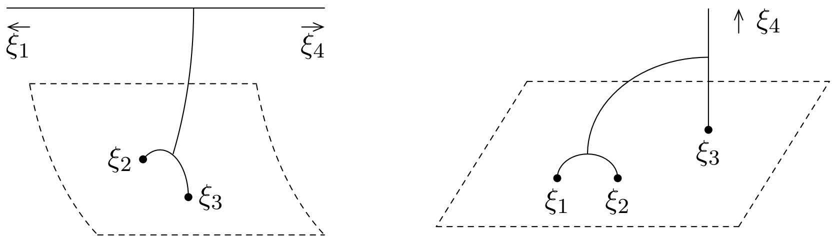


Figure 15: Hyperbolic ideal tetrahedra. On the left two points are far away from the remaining pair; on the right one point is far from the remaining triple.

Proposition I.43 ("almost" Hölder continuity)**.** *Let (Ξ, ϱ) and (Ψ, ϑ) be compact uniformly perfect metric spaces and let $\varphi : \Xi \to \Psi$ be a $(\underline{\lambda}, \overline{\lambda}, O(u))$-sublinearly quasiMöbius homeomorphism. Then φ admits a modulus of continuity*

$$\omega(t) = \exp\left(\underline{\lambda} \log t + v(-\log t)\right), \tag{I.90}$$

with $v = O(u)$.

Remark I.44. As a consequence, under the same assumptions and for all $\alpha \in (0, \underline{\lambda})$, φ is α-Hölder continuous. Thus Theorem I.41 may be seen as a strengthening of Cornulier's theorem (with the restriction made on spaces).

Proof. Let $\mathcal{E}$ be the constant from Definition I.40 associated to φ, and let τ be such that Ξ is τ-uniformly perfect. Define

$$\mathcal{D}_1 := \frac{\tau^4}{4}\left(\mathcal{E} \wedge \frac{\operatorname{diam} \Xi}{3}\right) \text{ and}$$
$$\mathcal{D}_1' = \inf\left\{\vartheta(\varphi(\xi_1), \varphi(\xi_2)) : \xi_1, \xi_2 \in \Xi, \varrho(\varphi(\xi_1), \varphi(\xi_2)) \geqslant \left(\tau^{-1} - 1\right)\mathcal{D}_1\right\}.$$

Let ξ_1 and ξ_2 in Ξ be such that $\varrho(\xi_1, \xi_2) < \mathcal{D}_1$. The ball $B := \tau^{-4} B(\xi_1, \mathcal{D}_1)$ is not equal to Ξ (this would indeed contradict the definition of $\mathcal{D}_1$), so there exists $\alpha \in B \setminus \tau B$ and $\beta \in \tau^2 B \setminus \tau^3 B$. By the triangle inequality

$$\varrho(\alpha, \beta) \geqslant \left(\tau^{-3} - \tau^{-2}\right)\mathcal{D}_1 \text{ and } \varrho(\beta, \xi_2) \geqslant \left(\tau^{-1} - 1\right)\mathcal{D}_1,$$

for short

$$\inf_i \varrho(\alpha, \xi_i) \wedge \inf_i \varrho(\beta, \xi_i) \geqslant \left(\tau^{-1} - 1\right)\mathcal{D}_1. \tag{I.91}$$

Further, by definition of $\mathcal{D}_1'$, a similar inequality holds in the target space:

$$\inf_i \vartheta(\varphi(\alpha), \varphi(\xi_i)) \wedge \inf_i \vartheta(\varphi(\beta), \varphi(\xi_i)) \geqslant \mathcal{D}_1'. \tag{I.92}$$

By definition of the metric cross ratios,

$$\frac{\left(\tau^{-1}-1\right)^2\mathcal{D}_1^2}{\operatorname{diam}(\Xi)}\frac{1}{\varrho(\xi_1,\xi_2)} \leqslant [\alpha,\xi_1,\xi_2,\beta] \leqslant \frac{\operatorname{diam}(\Xi)^2}{\mathcal{D}_1}\frac{1}{\varrho(\xi_1,\xi_2)}.$$

$$\frac{{\mathcal{D}_1'}^2}{\operatorname{diam}(\Psi)}\frac{1}{\vartheta(\varphi(\xi_1),\varphi(\xi_2))} \leqslant [\alpha',\varphi(\xi_1),\varphi(\xi_2);\beta'] \leqslant \frac{\operatorname{diam}(\Psi)^2}{\mathcal{D}_1'}\frac{1}{\vartheta(\varphi(\xi_1),\varphi(\xi_2))}.$$

thus $\log^+[\alpha,\xi_1,\xi_2,\beta] - \log^+\frac{1}{\varrho(\xi_1,\xi_2)}$ and $\log^+[\alpha',\varphi(\xi_1),\varphi(\xi_2);\beta'] - \log^+\frac{1}{\vartheta(\varphi(\xi_1),\varphi(\xi_2))}$ are bounded by

$$\mathcal{L} = 2\left(\left|\log\frac{1-\tau}{\tau}\right| + |\log\mathcal{D}_1| + |\log\operatorname{diam}(\Xi)|\right)$$

and

$$\mathcal{L}' = \left(|\log\mathcal{D}'| + |\log\operatorname{diam}(\Psi)|\right)$$

respectively. Now by hypothesis φ is $(\underline{\lambda},\overline{\lambda},v_0)$-sublinearly quasiMöbius for some $v_0 = O(u)$. By definition, setting $v = v_0 + \mathcal{L}$, for all ξ_1,ξ_2 such that $\varrho(\xi_1,\xi_2) < \mathcal{D}_1 \wedge 1$ (note that $\{\xi_1,\xi_2\}$ is the closest pair among ξ_1,ξ_2,α,β),

$$-\log\vartheta(\varphi(\xi_1),\varphi(\xi_2)) \leqslant \overline{\lambda}(-\log\varrho(\xi_1,\xi_2)) + v(-\log\varrho(\xi_1,\xi_2)), \tag{I.93}$$

$$-\log\vartheta(\varphi(\xi_1),\varphi(\xi_2)) \geqslant \underline{\lambda}(-\log\varrho(\xi_1,\xi_2)) - v(-\log\varrho(\xi_1,\xi_2)). \tag{I.94}$$

In particular the conclusion (I.90) is equivalent to the second inequality. □

The Hölder continuity (I.94) intervenes in the following analog of Lemma I.29, a technical refinement of definition I.40.

Lemma I.45. *Let u be an admissible function. Let $(\underline{\alpha},\overline{\alpha})$ be Lipschitz-Möbius data. Let φ be a $(\underline{\alpha},\overline{\alpha},v)$ sublinearly quasiMöbius homeomorphism between compact uniformly perfect spaces $(\Xi,\widehat{\varrho})$ and $(\Psi,\widehat{\vartheta})$ with $v = 0(u)$. There exist $\widehat{v} = O(u)$ (depending on $(\underline{\alpha},\overline{\alpha},v)$ and $\mathcal{E}_2 \in \mathbf{R}_{>0}$ such that for all $(\xi_1,\dots\xi_4) \in \Xi^4$ with $0 < \inf_{i\neq j}\varrho(\xi_i,\xi_j) \leqslant \sup_{i\neq j}\varrho(\xi_i,\xi_j) < \mathcal{E}_2$,*

$$\underline{\alpha}\log_\nu^+[\xi_i] - \widehat{v}\left(\sup_{i\neq j}\left[-\log_\nu\widehat{\varrho}(\xi_i,\xi_j)\right] \wedge \sup_{i\neq j}\left[-\log_\nu\widehat{\vartheta}(\varphi(\xi_i),\varphi(\xi_j))\right]\right) \leqslant \log_\nu^+\left[\varphi(\xi_i)\right]$$

$$\overline{\alpha}\log_\nu^+[\xi_i] + \widehat{v}\left(\sup_{i\neq j}\left[-\log_\nu\widehat{\varrho}(\xi_i,\xi_j)\right] \wedge \sup_{i\neq j}\left[-\log_\nu\widehat{\vartheta}(\varphi(\xi_i),\varphi(\xi_j))\right]\right) \geqslant \log_\nu^+\left[\varphi(\xi_i)\right].$$

Proof. By Proposition I.43 and the fact that v is sublinear, there is $\mathcal{E}_{\text{Hölder}} \in \mathbf{R}_{>0}$ such that for all $(\xi_1,\dots\xi_4) \in \Xi^4$ distinct and such that $\sup\widehat{\varrho}(\xi_i,\xi_j) \leqslant \mathcal{E}_{\text{Hölder}} \wedge e^{-(\log\nu)r_{\underline{\alpha}/2}(v)}$, the following holds:

$$\sup_{i\neq j}\left[-\log_\nu\widehat{\vartheta}(\varphi(\xi_i),\varphi(\xi_j))\right] \geqslant (\underline{\alpha}/2)\sup_{i\neq j}\left[-\log_\nu\widehat{\varrho}(\xi_i,\xi_j)\right]. \tag{I.95}$$

The conclusion follows. □

Proposition I.46. *Let u be an admissible function. The collection of $O(u)$-sublinearly quasiMöbius homeomorphisms form a groupoid $\mathcal{M}_{O(u)}$ with uniformly perfect compact metric spaces as objects. Composition in $\mathcal{M}_{O(u)}$ has a multiplicative effect on Lipschitz-Möbius and reverse Lipschitz-Möbius constants.*

Proof. Let $(\Omega, \widehat{\varpi}), (\Xi, \widehat{\varrho})$ and $(\Psi, \widehat{\vartheta})$ be compact metric spaces and let $\varphi : \Xi \to \Psi$ and $\psi : \Omega \to \Xi$ be $O(u)$-quasiMöbius homeomorphisms, with respective parameters $(\underline{\alpha}_\varphi, \overline{\alpha}_\varphi, v_\varphi)$ and $(\underline{\alpha}_\psi, \overline{\alpha}_\psi, v_\psi)$ where we assume that v_φ and v_ψ are nondecreasing; let $\nu > 1$ be such as in Definition I.40 for φ and ψ. Let us prove that $\varphi \circ \psi$ is a $(\underline{\alpha}_\varphi \underline{\alpha}_\psi, \overline{\alpha}_\varphi \overline{\alpha}_\psi, w)$-sublinearly quasiMöbius homeomorphism for some $w = O(u)$. Let $\widehat{v_\varphi}$ and $\widehat{v}_\psi$ be associated to v_φ and v_ψ by Lemma I.45. Let $(\omega_1, \ldots, \omega_4)$ be a 4-tuple of distinct points in Ω ; for $i \in \{1, \ldots, 4\}$ set $\xi_i = \psi(\omega_i)$ and $\eta_i = \varphi(\xi_i)$. If all the ω_i are close enough then by Proposition I.43 the ξ_i will be close enough so that applying Lemma I.45 to φ and ψ and combining the resulting estimates,

$$\begin{aligned} &\underline{\alpha}_\psi \left(\underline{\alpha}_\varphi \log_\nu^+ [\omega_i] - \widehat{v_\varphi} \left(\sup_{i \neq j} [-\log_\nu \widehat{\varpi}(\omega_i, \omega_j)] \wedge \sup_{i \neq j} \left[-\log_\nu \widehat{\vartheta}(\xi_i, \xi_j) \right] \right) \right) \\ &- \widehat{v_\psi} \left(\sup_{i \neq j} [-\log_\nu \widehat{\varrho}(\xi_i, \xi_j)] \wedge \sup_{i \neq j} \left[-\log_\nu \widehat{\vartheta}(\eta_i, \eta_j) \right] \right) \leqslant \log_\nu^+ [\eta_i] . \end{aligned}$$

Now since v_φ and v_ψ are nondecreasing,

$$\begin{aligned} \underline{\alpha}_\psi \underline{\alpha}_\psi \log_\nu^+ [\omega_i] - \underline{\alpha}_\psi \widehat{v_\varphi} \left(\sup_{i \neq j} - \log_\nu \widehat{\varpi}(\omega_i, \omega_j) \right) \\ - \widehat{v_\psi} \left(\sup_{i \neq j} - \log_\nu \widehat{\varrho}(\xi_i, \xi_j) \right) \leqslant \log_\nu^+ [\eta_i] . \end{aligned} \tag{I.96}$$

Applying Proposition I.43 to φ (precisely, the estimate (I.94)), one obtains that for all distinct i, j,

$$\begin{aligned} -\log \widehat{\vartheta}(\xi_i, \xi_j) &\geqslant \underline{\alpha}_\varphi (-\log \widehat{\varpi}(\omega_i, \omega_j)) - v_\varphi (-\log \widehat{\varpi}(\omega_i, \omega_j)) \\ &\geqslant (\underline{\alpha}_\varphi / 2)(-\log \widehat{\varpi}(\omega_i, \omega_j)) \end{aligned}$$

when all the ω_i are all close enough. Taking the supremum over pairs $\{i, j\}$, applying $\widehat{v}_\psi$ and inserting this in (I.96)

$$\begin{aligned}\log_\nu^+[\eta_i] &\geqslant \underline{\alpha}_\psi\underline{\alpha}_\psi \log_\nu^+[\omega_i] - \underline{\alpha}_\psi\widehat{v_\varphi}\left(\sup_{i\neq j} - \log_\nu \widehat{\varpi}(\omega_i,\omega_j)\right) \\ &\quad - \widehat{v_\psi}\left(\frac{\underline{\alpha}_\varphi}{2}\sup_{i\neq j} - \log_\nu \widehat{\varpi}(\xi_i,\xi_j)\right) \\ &\geqslant \underline{\alpha}_\psi\underline{\alpha}_\psi \log_\nu^+[\omega_i] - w_1\left(\sup_{i\neq j} - \log_\nu \widehat{\varpi}(\omega_i,\omega_j)\right) \qquad \text{(I.97)}\end{aligned}$$

for an appropriate $w_1 = O(u)$. Similarly

$$\log_\nu^+[\eta_i] \leqslant \underline{\alpha}_\psi\underline{\alpha}_\psi \log_\nu^+[\omega_i] + w_2\left(\sup_{i\neq j} - \log_\nu \widehat{\varpi}(\omega_i,\omega_j)\right) \qquad \text{(I.98)}$$

for an appropriate $w_2 = O(u)$. Setting $w = w_1 \vee w_2$ and bringing (I.97) and (I.98) together proves the claim. □

Remark I.47. The assumption of uniform perfectness (Definition I.42) could be dropped in Proposition I.46 if one adopts the heavier form of Definition I.40 given by the inequalities of Lemma I.45. It follows from the proof of Theorem I.41 that this more restrictive definition is still valid for boundary maps of sublinearly biLipschitz equivalences.

We now turn to the scale-sensitive moduli distortion property of sublinearly quasiMöbius homeomorphisms. Recall that for any ξ in a metric space Ξ, the annulus $A = B(\xi,s)\setminus B(\xi,r)$ is said to have a modulus $\mathfrak{M} = \log(s/r)$.

Proposition I.48. *Let φ be a $(\underline{\lambda},\overline{\lambda},O(u))$-sublinearly quasiMöbius homeomorphism between spaces (Ξ,ϱ) and (Ψ,ϑ). Assume that Ξ is uniformly perfect. There exist $\mathcal{D}_1 \in \mathbf{R}_{>0}$ and $w = O(u)$ such that the following holds: let $A \subset \Xi$ be an annulus of inner radius $r \in \mathbf{R}_{>0}$ and outer radius $R \in (r,\mathcal{D}_1]$. Then $\varphi(A)$ is contained in an annulus of modulus $2\lambda\mathfrak{M} + w(-\log r))$.*

Proof. Define $\mathcal{D}_1$ and $\mathcal{D}_1'$ as in the proof of Proposition I.43. For any triple $(\xi_1,\xi_2,\xi_3) \in \Xi^3$ such that $\{\xi_1,\xi_2,\xi_3\}$ has diameter less than $\mathcal{D}_1$ one can find $\omega \in \tau^{-4}B(\xi_1,\mathcal{D}_1)\setminus\tau^{-3}B(\xi_1,\mathcal{D}_1)$. Define $\omega' = \varphi(\omega)$. By the triangle inequality and the definition of $\mathcal{D}_1'$

$$\begin{cases}\inf_i \varrho(\omega,\xi_i) & \geqslant (\tau^{-3}-1)\mathcal{D}_1 \geqslant \frac{1-\tau}{\tau}\mathcal{D}_1, \text{ and} \\ \inf_i \vartheta(\omega',\eta_i) & \geqslant \mathcal{D}_1',\end{cases}$$

where $\eta_i = \varphi(\xi_i)$ for $i \in \{1,2,3\}$. Define $\mathcal{D}_2 = \frac{1-\tau}{\tau}\mathcal{D}_1$. Applying the definition of the metric cross-ratio we deduce from the previous inequalities

$$\left|\log[\omega,\xi_1,\xi_2,\xi_3] - \log\frac{\varrho(\xi_1,\xi_3)}{\varrho(\xi_1,\xi_2)}\right| \leqslant 2|\log\operatorname{diam}(\Xi)| \vee |\log\mathcal{D}_2| \tag{I.99}$$

$$\left|\log[\omega',\eta_1,\eta_2,\eta_3] - \log\frac{\vartheta(\eta_1,\eta_3)}{\vartheta(\eta_1,\eta_2)}\right| \leqslant 2|\log\operatorname{diam}(\Xi)| \vee |\log\mathcal{D}_1'|. \tag{I.100}$$

Denote by $\mathcal{L}$, resp. $\mathcal{L}'$ the right-hand side bounds of (I.99), resp. (I.100). Let $r \in \mathbf{R}_{>0}$ and $\mathfrak{M} \in \mathbf{R}_{\geqslant 0}$ be such that $R = r\exp(\mathfrak{M}) \leqslant \mathcal{D}_1$. Fix ξ_1 and write $B = B(\xi_1, r)$. Fix ξ_2 in $\tau B \setminus \tau^2 B$. For any $\xi_3 \in A = B(R)\setminus B(r)$ the triangle inequality gives

$$\varrho(\xi_1,\xi_2) \wedge \varrho(\xi_1,\xi_3) \wedge \varrho(\xi_2,\xi_3) \geqslant \left((1-\tau)\wedge\tau^2\right) r.$$

Let v_0 be such that φ is $(\underline{\lambda}, \overline{\lambda}, v_0)$-quasiMöbius. Define $v_1 = v_0 + \mathcal{L}\vee\mathcal{L}'$ and then $v_2 = (v_1 \uparrow (1-\tau)\wedge\tau^2)v_1$. Applying Definition I.40 to φ for $(\omega,\xi_1,\xi_2,\xi_3)$ together with (I.99) and (I.100), one obtains the set of inequalities

$$\begin{aligned}
\log\frac{\vartheta(\eta_1,\eta_3)}{\vartheta(\eta_1,\eta_2)} \leqslant \log^+\frac{\vartheta(\eta_1,\eta_3)}{\vartheta(\eta_1,\eta_2)} &\leqslant \overline{\lambda}\log^+\frac{\varrho(\xi_1,\xi_3)}{\varrho(\xi_1,\xi_2)} + v_1\left(-\log\left((1-\tau)\wedge\tau^2\right)r\right)\\
&\leqslant \overline{\lambda}\mathfrak{M} + v_2(-\log r) - 2\overline{\lambda}\log\tau,\\
-\log\frac{\vartheta(\eta_1,\eta_3)}{\vartheta(\eta_1,\eta_2)} \leqslant \log^+\frac{\vartheta(\eta_1,\eta_2)}{\vartheta(\eta_1,\eta_3)} &\leqslant \overline{\lambda}\log\frac{\varrho(\xi_1,\xi_2)}{\varrho(\xi_1,\xi_3)} + v_2(-\log r).
\end{aligned}$$

Hence for any $\xi_3, \xi_3' \in A$, by the triangle inequality in $\mathbf{R}$, using $\vartheta(\eta_1,\eta_2)$ as an intermediate point,

$$\left|\log\frac{\vartheta(\eta_1,\eta_3)}{\vartheta(\eta_1,\eta_3')}\right| \leqslant 2\overline{\lambda}\mathfrak{M} + 2v_2(-\log r) - 4\overline{\lambda}\log\tau \leqslant 2\lambda\mathfrak{M} + w(-\log r), \tag{I.101}$$

where $w = O(u)$. The proposition follows from the last statement. The expansion constant $\underline{\lambda}$ would intervene in lower bounds on $\inf\frac{\vartheta(\varphi(\xi_1),\varphi(\xi_3))}{\vartheta(\varphi(\xi_1),\varphi(\xi_2))}$ for ξ_2 in the internal ball, and ξ_3 outside the external ball, centered at ξ_1. □

This last property of sublinearly Möbius maps will be of use in section I.5 where we implement some measure theory on the boundary. There is still a need to reformulate it slightly, however, since we will be then working with balls rather than annuli, and quasimetrics rather than true distances. In that purpose, we introduce the following terminology: for any $s \in \mathbf{R}_{>0}$, if B is a quasiball $B = B^{\widehat{\varrho}}(\Xi, r)$ where $\widehat{\varrho}$ is a kernel equivalent to the distance in X, then sB is $B^{\widehat{\varrho}}(\Xi, sr)$.

Proposition I.49. *Assume that* (Ξ, ϱ) *and* (Ψ, ϑ) *are metrized compact connected topological manifolds such that any small enough metric sphere is an embedded topological sphere of maximal dimension, and that* $\varphi : \Xi \to \Psi$ *is a* $(\underline{\lambda}, \overline{\lambda}, O(u))$*-sublinearly quasiMöbius homeomorphism. Let* $Q \in \mathbf{R}_{\geqslant 1}$ *be a constant. Let* $\widehat{\varrho}$*, resp.* $\widehat{\vartheta}$ *be an equivalent kernel on* Ξ*, resp. on* Ψ*. Then for any* $\alpha \in (0, \underline{\lambda})$ *and* $\beta \in (\overline{\lambda}, +\infty)$ *there exists* $w = O(u)$ *(depending on* Q*) such that for any* $\widehat{\varrho}$*-quasiball* $B \subset \Xi$ *with center* ξ *and small enough radius* r *there exists a* $\widehat{\vartheta}$*-quasiball* B' *in* Ψ*, and*

$$\begin{cases} r^\beta \leqslant \mathrm{radius}(B') \leqslant r^\alpha \\ B' \subseteq \varphi\left(Q^{-1}B\right) \subset \varphi(B) \subseteq Q^\lambda e^{w(-\log r)} B'. \end{cases} \tag{I.102}$$

Remark I.50. Though this would be valid, we do not include in the conclusion that B have center $\varphi(\xi)$, since it will not be required in section I.5.

Proof. The statement for any equivalent kernel follows from the particular case when $\widehat{\varrho} = \varrho$ and $\widehat{\vartheta} = \vartheta$. Let $B'' \setminus B'$ be an annulus containing $\varphi(B \setminus Q^{-1}B)$. Since φ is a homeomorphism, images of spheres by φ are topological spheres. By the Jordan-Brouwer separation theorem, $\varphi(Q^{-1}B)$ is one of the two connected components of $\Psi \setminus \varphi(\partial(Q^{-1}B))$, and by Proposition I.43, if r is small enough its diameter is bounded by r^α. Since $\varphi(\partial(Q^{-1}B)) \subset B'' \setminus B'$, $B' \subseteq \varphi(Q^{-1}B)$ and $\mathrm{radius}(B') \leqslant r^\alpha$. By Proposition I.48 B'' can be written $Q^\lambda e^w(-\log r)B'$. Finally, by Proposition I.43, for all $\beta' \in (\overline{\lambda}, \beta)$, $\mathrm{diam}(B'') \geqslant \mathrm{diam}\, \varphi(B) \geqslant r^{\beta'}$ if r is small enough. This implies the lower bound on $\mathrm{radius}(B')$ for r small enough. □

I.5. Riemannian negatively curved homogeneous spaces

I.5.1. Setting

Simple Lie groups of real rank one with left invariant metrics are mentioned early in Gromov's essay as important examples of δ-hyperbolic spaces [82, 1.5(2)] and it is natural to ask to which extent they – or their quasiisometrically related symmetric spaces of noncompact type – differ on the large scale. Beyond these examples, it was proved in 1974 by E. Heintze [88, § 2] that any connected homogeneous negatively curved Riemannian manifold is the principal space of a solvable Lie group $S = N \rtimes_\alpha \mathbf{R}$, where N is nilpotent with Lie algebra $\mathfrak{n}$ and $\alpha \in \mathrm{Der}(\mathfrak{n})$ is such that for any compact neighborhood K of 1 in N, $\cup_{t \geqslant 0} \exp(t\alpha)K = N$. Such an S is called a Heintze group.

For a principal space X of the Heintze group S, denote by ω the endpoint on $\partial_\infty X$ (in positive time) of the orbits of the $\mathbf{R}$ factor, and by $\partial_\infty^* X$ the punctured boundary $\partial_\infty X \setminus \{\omega\}$. Any choice of a basepoint $o \in X$ will determine a chart $\Phi : \partial_\infty^* X \to N$ by letting $(\omega\xi)$ be the $\Phi(\xi)$-left translate of the $\mathbf{R}$ factor in $(X, o) \simeq (S, 1)$, and a horofunction $-t : X \to \mathbf{R}$ from ω and such that $t(o) = 0$.

I.5.2. Quasimetrics and measures on the punctured boundary

From now on, we make an assumption that S is purely real, i.e. α has only positive real eigenvalues. This is not restrictive as far as large-scale properties are concerned, due to the following fact:

Proposition I.51. *Any Heintze group is quasiisometric to a purely real Heintze group.*

Proposition I.51 follows from a special case of a result by D.Alekseevskiĭ [2, Theorem 3.3]. See also Cornulier, [37, Corollary 5.16] for a generalized form.

For any $s \in \mathbf{R}_{>0}$ there is a homomorphism $N \rtimes_{s\alpha} \mathbf{R} \to N \rtimes_\alpha \mathbf{R}$, $(n, t) \mapsto (n, ts)$. Up to rescaling the operation of $\mathbf{R}$, we will work under a normalization assumption:

Definition I.52. A purely real Heintze group $N \rtimes_\alpha \mathbf{R}$ is normalized if the smallest eigenvalue of α is equal to 1. In this case, the eigenvalues are ordered in increasing order, $1 = \lambda_1, \ldots \lambda_r$ and one defines $p = \operatorname{tr} \alpha$.

Lemma I.53. *Choose a horofunction β from ω in X, and let $\widehat{\varrho}$ be the visual quasimetric on $\partial_\infty^* X$ with parameter e with respect to β. $\widehat{\varrho}$ is a N-invariant, S-equivariant adapted kernel on $\partial_\infty^* X$; precisely*

$$\forall \xi_1, \xi_2 \in \partial_\infty^* X,\ \widehat{\varrho}(s\xi_1, s\xi_2) = e^t \widehat{\varrho}(\xi_1, \xi_2), \tag{I.103}$$

if $s = (n, t)$ in the semidirect product decomposition $N \rtimes \mathbf{R}$.

Proof. Applying s is equivalent to removing t to the horofunction β. □

We refer to $\widehat{\varrho}$ as the homogeneous quasimetric on the punctured boundary; it is indeed a quasimetric (see e.g. Buyalo and Schroeder [27, 3.3]). Different, equally natural choices for $\widehat{\varrho}$ are possible; under the constraint of satisfying (I.103) and a quasiultrametric inequality they would lead to equivalent kernels.

We shall give later (Lemma I.55) a sufficient condition for $\widehat{\varrho}$ to be equivalent to a true distance. For the moment however we only draw measure-theoretic conclusions.

By definition, N operates on $\partial_\infty^* X$, and then on the space of measures on $\partial_\infty^* X$; the subspace of invariant measures is an affine line $\mathcal{L}$, by uniqueness of the Haar measure of N up to scaling. This operation extends to $S \curvearrowright \mathcal{L}$ via its modular function: for any $\mu \in \mathcal{L}$, for any $\widehat{\varrho}$-quasiball B,

$$\forall s \in S,\ \mu(sB) = \Delta(s)\mu(B), \tag{I.104}$$

where $\Delta(s) = \exp(t \cdot \operatorname{tr} \alpha) = e^{pt}$ if $s = (n,t)$, and we recall that p is the trace of α.

I.5.3. Horizontal lines and horizontal curves

Let $\mathfrak{n}_1(\alpha) = \ker(\alpha - 1)$. In the tangent space of $\partial_\infty^* X$, the distribution $\Phi^* \mathfrak{n}_1(\alpha)$ does not depend on the chart $\Phi : \partial_\infty^* X \to N$. We refer to it as the horizontal distribution, and denote it by τ (not forgetting the left action of N). For any N-invariant line L in τ, denote by Γ_L the family of horizontal L-lines in $\partial_\infty^* X$, that is, smooth horizontal curves γ tangent to L. The space Γ_L can be identified with the homogeneous space N/R, where R is a one-parameter subgroup of N whose infinitesimal generator represents $\Phi_* L$. Since N is a nilpotent Lie group it is unimodular, especially Δ_N is constant along R so that Γ_L possesses a Haar measure ρ, following A.Weil [154, § 9].

Lemma I.54. *Let L be as above, and let μ be a N-invariant measure on $\partial_\infty^* X$. Then for any $Q \in \mathbf{R}_{>0}$ there exists $c \in \mathbf{R}_{>0}$ (depending on μ, Q and L) such that for any $\widehat{\varrho}$-quasiball B,*

$$\rho\left\{\gamma \in \Gamma_L : \gamma \cap B \neq \emptyset\right\} = c\mu\left(Q^{-1}B\right)^{(p-1)/p}. \tag{I.105}$$

Proof. S operates simply transitively on the space of $\widehat{\varrho}$-quasiballs, while N operates simply transitively on their centers preserving radii, and $\theta : B \mapsto \{\gamma : \gamma \cap B \neq \emptyset\}$ defines a S-equivariant map. Hence it suffices to show (I.105) for a one-parameter family of balls $\{e^t B\}_{t \in \mathbf{R}}$. Let $v \in \mathfrak{n}_1(\alpha)$ be a nonzero vector such that $[\Phi^* v] = L$. Since $v \in \mathfrak{n}_1(\alpha)$, the linear map α operates on $\mathfrak{n}/(\mathbf{R}v)$ with trace $p-1$, and $\rho\left\{\gamma \in \Gamma_L : \gamma \cap e^t B \neq \emptyset\right\}$ is proportional to $e^{t(p-1)}$. On the other hand, by Lemma I.53 and (I.104), $\mu(Q^{-1}tB)$ is proportional to e^{tp}, hence $\mu(Q^{-1}tB)^{(p-1)/p}$ is proportional to $e^{t(p-1)}$ as well. $\square$

Lemma I.55. *Assume that S is normalized (Definition I.52) and that the operation of the derivation α on $\mathfrak{n}^{\mathrm{ab}}$ is scalar, hence the identity. Then*

1. *(N, α) is a Carnot graded group, i.e.*

   (a) $\mathfrak{n}$ admits a grading $(\mathfrak{n}_i)$ by $\mathbf{Z}_{>0}$ such that $\mathfrak{n}_i = \ker(\alpha - i)$

   (b) $\mathfrak{n}$ is generated by $\mathfrak{n}_1$.

2. *Let $\|\cdot\|$ be a norm on $\mathfrak{n}_1$, and let $\Phi : \partial_\infty^* X \to N$ be a chart. Then $\widehat{\varrho}$ is equivalent to a subRiemannian Carnot-Carathéodory metric*

$$d_{\mathrm{CC}}(\xi_1, \xi_2) = \inf \{\ell(\gamma) : \gamma \in \Gamma(\xi_1, \xi_2)\},$$

   where $\Gamma(\xi_1, \xi_2)$ denotes the space of absolutely continuous curves $[0,1] \to \partial_\infty^* X$ between ξ_1 and ξ_2 with derivative almost everywhere in the horizontal distribution τ, and $\ell(\gamma) = \int_{[0,1]} \|\Phi_* \gamma'\|$ is the length of γ.

In this case, X is said to be of Carnot type (following Cornulier's terminology).

If X is of Carnot type, condition (1b) ensures that $\Gamma(\xi_1, \xi_2)$ is never empty and d_{CC} takes finite values.

Proof. See the survey of Cornulier [40, 2.G.1 and 2.G.2] for (1). Further, $s = (n, t) \in S$ acts on the space of horizontal curves sending $\Gamma(\xi_1, \xi_2)$ on $\Gamma(s\xi_1, s\xi_2)$ and multiplying lengths by e^t, hence

$$d_{\mathrm{CC}}(s\xi_1, s\xi_2) = e^t d_{\mathrm{CC}}(\xi_1, \xi_2) \tag{I.106}$$

for all $\xi_1, \xi_2 \in \partial_\infty^* X$. Select $\xi \in \partial_\infty^* X$. Since d_{CC} and $\widehat{\varrho}$ are both quasimetrics, they are continuous, hence bounded, over unit quasiballs of each other centered at ξ. Finally, S operates transitively on the spaces of quasiballs of $\widehat{\varrho}$ and d_{CC}. Hence $\widehat{\varrho}$ and d_{CC} are equivalent (the control constants depend on $\|\cdot\|$). □

I.5.4. Volumes of quasiballs and intersecting horizontal lines

Lemma I.56. *Let $q \in \mathbf{R}_{\geqslant 1}$ be a constant and let $\mathcal{X}$ be a proper metric space. Let $\widehat{\varrho}$ be an equivalent kernel on $\mathcal{X}$ with quasi-ultrametric constant q. There exists a constant Q depending on q, such that for any countable covering $\mathcal{B}$ of $\mathcal{X}$ by $\widehat{\varrho}$-quasi-balls, there exists an extraction $\mathcal{B}'$ of $\mathcal{B}$ whose elements are disjoint and such that $\{QB\}_{B \in \mathcal{B}'}$ is a covering of X.*

Proof. See A.P. Morse, [117, Theorem 3.4], or Federer [63, 2.8.4-2.8.6]. □

In the following, whenever $q \in \mathbf{R}_{\geqslant 1}$ is a constant, Q is another constant depending on q defined by the previous lemma.

Lemma I.57 (adapted from P.Pansu, [126, Lemme 6.3])**.** *Let $\mathcal{X}$ be a proper metric space, and let Γ a measured space of curves on $\mathcal{X}$ (denote its measure by ρ). Let $p \in \mathbf{R}_{>1}$ and $q \in \mathbf{R}_{\geqslant 1}$ be constants. Let $\widehat{\varrho}$ be a kernel on $\mathcal{X}$, equivalent to the original distance and with a q-quasiultrametric inequality. Let U be an open, bounded subset of $\mathcal{X}$, endowed with Borel measures μ and ν, such that for any $\widehat{\varrho}$-quasiball B contained in U,*

$$\rho \left\{\gamma \in \Gamma : \gamma \cap B \neq \emptyset\right\} \leqslant \mu \left(Q^{-1} B\right)^{(p-1)/p}. \tag{H}$$

For all $\gamma \in \Gamma$ and for all $r > 0$, set

$$\Phi_r^1(\gamma) = \inf_{\mathcal{F}} \sum_{B \in \mathcal{F}} \phi(B),$$

where $\phi(B) := \nu\left(Q^{-1}B\right)^{1/p}$, the infimum taken over countable coverings $\mathcal{F}$ of $\gamma \cap U$ with balls of radius r exactly, contained in U. Then

$$\int_\Gamma \Phi_r^1(\gamma) d\rho \leqslant \nu(U)^{1/p} \mu(U)^{(p-1)/p}. \tag{I.107}$$

For Lemma I.57, Pansu's proof can be reproduced almost verbatim [126, Lemme 6.3], with the only differences of using Lemma I.56 instead of the covering lemma used by Pansu, having r fixed and not going to the limit at the end. The argument is based on the Hölder inequality; in a more general setting it is aimed at bounding a discretized version of the conformal modulus, and then to obtain lower bounds for the conformal dimension, [125, § 2 and 3].

Lemma I.58 (compare [126, Proposition 6.5])**.** *Let (N, α) and (N', α') be Carnot groups with grading derivations α, α', normalized, with positive eigenvalues, of traces p and p'. Let X and X' be principal spaces of $N \rtimes_\alpha \mathbf{R}$ and $N' \rtimes_{\alpha'} \mathbf{R}$ respectively, and assume there exists a homeomorphism $\varphi : \partial_\infty^* X \to \partial_\infty^* X'$ which is sublinearly quasiMöbius over every compact subset. Then $p \leqslant p'$.*

Sketch of the proof of the Lemma I.58. Define τ as p'/p and let Γ_L be a family of horizontal lines in the boundary of X. We follow the lines of Pansu [126, Proposition 6.5], despite loosing strength in the conclusion. Precisely this amounts to comparing two facts:

1. Without any assumption on N and α, for any $\sigma \in (\tau, +\infty)$, the image of almost every horizontal curve $\gamma \in \Gamma_L$ has locally finite σ-dimensional $\widehat{\varrho}$-Hausdorff measure. Hence almost every curve has $\widehat{\varrho}$-Hausdorff dimension less than τ.

2. Since X is of Carnot type, $\widehat{\varrho}$ is equivalent to the subRiemannian distance d_{CC} by Lemma I.55, hence any nonconstant curve should have $\widehat{\varrho}$-Hausdorff-dimension greater than 1.

This proves that $\tau \geqslant 1$, i.e. $p \leqslant p'$.

Proof. Let U be a open, relatively compact subset of $\partial_\infty^* X$. Define $U' = \varphi(U)$. Let Γ_L^U be the (non-empty) set $\{\gamma \cap U : \gamma \in \Gamma_L\}$ measured with

$$(\cap_U)_\star \left(\rho\lfloor\{\gamma \in \Gamma_L : \gamma \cap U \neq \emptyset\}\right),$$

where ρ has been defined in I.5.2, and $\cap_U(\gamma) = U \cap \gamma$. We still denote this measure ρ. Let μ, resp. μ' be a N-invariant measure on $\partial_\infty^* X$, resp. on $\partial_\infty^* X'$, restricted to U, resp. to U'. Define a measure ν on U as

$$\nu(B) = \mu'(\varphi(B))$$

for any Borel subset $B \subset U$. Let $\widehat{\varrho}$ be the homogeneous quasimetric on $\partial_\infty^* X$, let q be its ultrametric constant and define Q accordingly (see Lemma I.56). Let $r \in \mathbf{R}_{>0}$ be a radius that will be repeatedly assumed as small as needed. Choose $\gamma \in \Gamma_L^U$, and let $\mathcal{F}$ be any covering of γ with quasiballs of the same $\widehat{\varrho}$-radius r (we emphasize that all quasiballs must have radius r). By assumption, the quasiballs $\{\varphi(B), B \in \mathcal{F}\}$ cover $\varphi(\gamma)$. By Theorem I.41 and Proposition I.49, there exists $v = O(u)$, and if r is small enough, a collection $\mathcal{F}'$ of quasiballs and $\mathcal{F} \to \mathcal{F}'$, $B \mapsto B'$ such that

$$\forall B \in \mathcal{F},\, B' \subset \varphi\left(Q^{-1} B\right) \subset \varphi(B) \subset Q^{2\lambda} e^{v(-\log r)} B' =: B''. \tag{I.108}$$

Define $\mathcal{F}'' = \{B''\}$ together with a map $\mathcal{F} \to \mathcal{F}'', B \mapsto B''$. This is a quasiball covering of $\varphi(\gamma)$.

Next, define a gauge function $\phi(B) := \nu\left(Q^{-1}B\right)^{1/p} = \mu'\left(\varphi\left(Q^{-1}B\right)\right)^{1/p}$. There exists a constant $c_0 \in \mathbf{R}_{>0}$, not depending on r and such that

$$\begin{aligned} \phi(B) &\overset{(\mathrm{I}.108)}{\geqslant} \mu'(B')^{1/p} = c_0^\tau \mathrm{diam}(B')^\tau \\ &\overset{(\mathrm{I}.108)}{\geqslant} \left(\frac{c_0}{Q^{2\lambda} e^{v(-\log r)}}\right)^\tau \mathrm{diam}(B'')^\tau. \end{aligned} \tag{I.109}$$

Define $r'' = r^{1/(2\lambda)}$. Using Cornulier's theorem I.13, if r is small enough, then

$$\begin{aligned} \forall B \in \mathcal{F},\, \operatorname{diam} B'' \leqslant e^{v(-\log r)} Q^{2\lambda} \operatorname{diam} B' &\leqslant e^{v(-\log r)} Q^{2\lambda} \operatorname{diam} \varphi(B) \\ &\leqslant e^{v(-\log r)} Q^{2\lambda} r^{2/(3\lambda)} \\ &\leqslant r'', \end{aligned}$$

where we used $v(s) \ll s$ and took r small enough in the last line. On the other hand, using (I.93) from the proof of Proposition I.43, one obtains a reverse inequality:

$$\forall B'' \in \mathcal{F},\, \log \operatorname{diam} B'' \geqslant 2\lambda \log r = 4\lambda^2 \log r''. \tag{I.110}$$

One can rewrite $Q^{2\lambda} e^{v(-\log r)}$ as $e^{w(-\log r'')}$ with $w = O(u)$. Taking logarithms in (I.109),

$$\begin{aligned} \log \phi(B) &\geqslant \tau \log c_0 - w(-\log r'') + \tau \log \operatorname{diam} B'' \\ &\underset{(\mathrm{I.110})}{\geqslant} \tau \log c_0 - w\left(-\frac{1}{4\lambda^2} \log \operatorname{diam} B''\right) + \tau \log \operatorname{diam} B''. \end{aligned}$$

The function w is strictly sublinear, so for any $\sigma \in (\tau, +\infty)$, there is $r_\sigma \in \mathbf{R}_{>0}$ such that

$$\forall r \in (0, r_\sigma), \forall B \in \mathcal{F},\, \phi(B) \geqslant (r'')^\sigma \geqslant \left(\operatorname{diam} B''\right)^\sigma. \tag{I.111}$$

Recall that for all $\mathcal{F}$ the quasiballs $B'' \in \mathcal{F}''$ cover $\varphi(\gamma)$. By definition of the $\widehat{\varrho}$-Hausdorff premeasure at scale r'',

$$\Phi^1_r(\gamma) = \inf_{\mathcal{F}} \sum_{B \in \mathcal{F}} \phi(B) \geqslant \sum_{B''} \operatorname{diam}(B'')^\gamma \geqslant \mathscr{H}^\sigma_{r''} \varphi(\gamma). \tag{I.112}$$

By Lemma I.54, the hypothesis (H) of Lemma I.57 is fullfilled. Hence, for all $r \in (0, r_\sigma)$,

$$\int_{\Gamma^U_L} \Phi^1_r(\gamma) d\rho \leqslant \nu(U)^{1/p} \mu(U)^{(p-1)/p}.$$

By monotone convergence, for ρ-almost every γ, $\sup_r \Phi^1_r(\gamma)$ is finite, and then by (I.112), $\mathscr{H}^\sigma \varphi(\gamma)$ is finite. Considering this fact for all terms of a decreasing sequence $\{\sigma_j\}$ converging to τ, one deduces that, still for ρ-almost every γ,

$$\operatorname{Hdim} \varphi(\gamma) \leqslant \inf_j \sigma_j = \tau. \tag{I.113}$$

Finally, X has been assumed of Carnot type, hence $\widehat{\varrho}$ is equivalent to the Carnot-Carathéodory metric d_{CC} by Lemma I.55. By the triangle inequality, the 1-dimensional d_{CC}-Hausdorff measure of any nonconstant curve is nonzero, in particular its d_{CC}-Hausdorff dimension must be greater than 1. This dimension does not change when replacing d_{CC} with the equivalent quasimetric $\widehat{\varrho}$. By (I.113) there exists $\gamma \in \Gamma_L^U$ such that $1 \leqslant \operatorname{Hdim} \varphi(\gamma) \leqslant \tau$. Hence $1 \leqslant \tau$. $\square$

Remark I.59. In the statement of Lemma I.58, the assumption that $\varphi : \partial_\infty^* X \to \partial_\infty^* X'$ be sublinearly quasiMöbius over every compact subset is due do the fact that theorem I.41 is stated for visual quasimetric based at points of X only (and not for horofunctions). Over any compact subset of $\partial_\infty X \setminus \{\omega\}$, the homogeneous quasimetric is within bounded multiplicative error to a fixed visual quasimetric, compare e.g. [29, p.461].

Lemma I.58 is applied to show that p is a SBE invariant between spaces of Carnot type. In fact this can be made slightly more general:

Proposition I.60. *Let X_1 and X_2 be principal spaces of purely real, normalized Heintze groups $N_1 \rtimes_{\alpha_1} \mathbf{R}$ and $N_2 \rtimes_{\alpha_2} \mathbf{R}$. Assume that for all $i \in \{1, 2\}$ the operation defined by α_i on $\mathfrak{n}_i^{\mathrm{ab}}$ is unipotent*[11]*. If there exists a sublinearly biLipschitz equivalence between X_1 and X_2, then $\operatorname{tr}(\alpha_1) = \operatorname{tr}(\alpha_2)$.*

Proof. For every $i \in \{1, 2\}$, decompose α_i into $\alpha_i^\sigma + \alpha_i^\nu$, where α_i^σ is semisimple and α_i^ν is nilpotent. By hypothesis, $\mathfrak{n}_i^\sigma$ operates as the identity on $\mathfrak{n}_i^{\mathrm{ab}}$, hence N_i are Carnot gradable groups, and α_i are grading derivations of their Lie algebra. A particular instance of a theorem by Cornulier implies that there exists $O(\log)$-sublinearly biLipschitz equivalences $\psi_i : N \rtimes_{\alpha_i^\sigma} \mathbf{R} \to N \rtimes_{\alpha_i} \mathbf{R}$ (see [35, Theorem 4.4]: in our very special case the exponential radical is N, and the Cartan subgroup is $\mathbf{R}$). The groups $N \rtimes_{\alpha_i^\sigma} \mathbf{R}$ are of Carnot type, so by Theorem I.41 and Lemma I.58, $\operatorname{tr}(\alpha_1^\sigma) = \operatorname{tr}(\alpha_2^\sigma)$. Finally, $\operatorname{tr}(\alpha_1) = \operatorname{tr}(\alpha_1^\sigma) = \operatorname{tr}(\alpha_2^\sigma) = \operatorname{tr}(\alpha_2)$. $\square$

Note that if sublinearly biLipschitz equivalences are replaced by quasi-isometries in the last statement, known invariants are much finer than the trace. In this direction, M. Carrasco Piaggio and E. Sequeira obtained that for normalized purely real Heintze groups, resp. for normalized purely real Heintze groups with a fixed Heisenberg group as exponential radical N, the

[11] Recall that $\mathfrak{g}^{\mathrm{ab}}$ denotes the abelianization of the Lie algebra $\mathfrak{g}$; to any derivation α of $\mathfrak{g}$ one can associate an endomorphism α^{ab} of $\mathfrak{g}^{\mathrm{ab}}$.

characteristic polynomial, resp. the full Jordan form of α, are quasiisometric invariants [30, Theorem 1.1, resp. Theorem 1.3]. By contrast, the Jordan form of the normalized derivation is not a SBE invariant, precisely it is not a $O(\log)$-SBE invariant by Cornulier's theorem [35, Theorem 4.4].

I.5.5. Proof of Theorem I.6

Notation is as before. When X is a rank one symmetric space of noncompact type, several restrictions appear (see Heintze, [88, Proposition 4 and Corollary]):

1. X is of Carnot type.
2. The Lie algebra $\mathfrak{n}$ is two-step, $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$ where $\mathfrak{n}_2$ is possibly zero.
3. Save for one case, namely the Cayley hyperbolic plane, there exists a division algebra structure on $\mathbf{R} \oplus \mathfrak{n}_2$, and $\mathfrak{n}_1$ is a module over this division algebra. The structure of $\mathfrak{n}$ is completely determined by these data.

The Frobenius classification of division algebras over $\mathbf{R}$ reduces considerably the list of candidates thanks to (3): the two relevant parameters are the division algebra $\mathbf{K} \in \{\mathbf{R}, \mathbf{C}, \mathbf{H}\}$ and a positive integer, the rank of $\mathfrak{n}_1$ over $\mathbf{K}$. The Cayley hyperbolic plane fits in this list, setting $\mathfrak{n}_1 = \mathbf{O}$. The homogeneous dimension is computed as

$$\begin{aligned} \operatorname{tr}(\alpha) = \dim \mathfrak{n}_1 + 2 \dim \mathfrak{n}_2 &= \dim \mathfrak{n} + \dim \mathfrak{n}_2 \\ &= \dim X - 1 + \dim \Im(\mathbf{K}), \end{aligned}$$

and $\mathbf{K}$ is completely determined by $\dim \Im(\mathbf{K}) \in \{0, 1, 3, 7\}$. By Theorem I.13 and Proposition I.60, $\mathbf{K}$ is a SBE invariant, as

$$\dim \Im(\mathbf{K}) = \operatorname{Hdim}(\partial_\infty^* X, \widehat{\varrho}) - \dim \partial_\infty X.$$

The rank n of $\mathfrak{n}_1$ over $\mathbf{K}$ is a SBE invariant as well, since it can be computed by the formula

$$(1+n) \dim_{\mathbf{R}} \mathbf{K} = 1 + \dim \partial_\infty X.$$

II

On sublinear quasiconformality

An embedding f between metric spaces is quasisymmetric if there is an increasing homeomorphism $\eta : [0, +\infty) \to [0, +\infty)$ such that for any x, y, z in the source space and positive real t,

$$d(x, y) \leqslant t d(x, z) \implies d(f(x), f(y)) \leqslant \eta(t) d(f(x), f(z)). \qquad \text{(II.1)}$$

The properties of sufficiently well-behaved compact metric spaces that are invariant under quasisymmetric homeomorphisms are known to be counterparts of the coarse (or quasiisometrically invariant) properties of proper geodesic Gromov-hyperbolic spaces, the two categories being related by the Gromov boundary and hyperbolic cone functors ([10], [138, 2.5]). Instances are the conformal dimension [125] and the ℓ_p or L_p cohomology [21].

This chapter is part of our aim to transpose this equivalence by replacing quasiisometries with sublinearly biLipschitz equivalences, which originated from the work of Cornulier on the asymptotic cones of connected Lie groups[1] [34]. Here the sublinear feature is described by an asymptotic class $O(u)$, where u is a strictly sublinear nondecreasing positive function on the half line such that $\limsup_r u(2r)/u(r) < +\infty$, e.g. $u(r) = \log r$ (we call such a function admissible).

In Chapter I the Gromov-boundary behavior of sublinearly biLipschitz equivalences between Gromov-hyperbolic spaces was characterized (Theorem II.15). It differs from that of quasisymmetric homeomorphisms sublinearly in a certain sense; we shall indicate how in II.1.2. The purpose of the present chapter is to push further the analysis of those boundary mappings and identify the structure preserved on the boundary. A numerical invariant is derived. It is denoted by $\mathrm{Cdim}_{O(u)}$; Pansu's conformal dimension introduced in [125, 3] and usually denoted Cdim corresponds to $\mathrm{Cdim}_{O(1)}$. We compute

[1] Beware that we use the terminology of [39].

this invariant and prove that it equals Cdim on the examples originally studied by Pansu and Bourdon (that we recall below). Function spaces of locally bounded energy, that are carried by sublinearly quasisymmetric mappings up to shifts in parameters, are also constructed. These form algebras reminiscent of the Sobolev spaces $W^{1,p}_{\mathrm{loc.}}$. One can sometimes have access to the topological dimension of the spectrum of these algebras; its dependence over p measures the degree of energy needed to break invariance along certain foliations on the boundary, and provides further invariants. This latter approach is inspired from Bourdon [17, p.248], Bourdon-Kleiner [20, Section 10] and Carrasco Piaggio [29, p.465].

A purely real Heintze group is a simply connected solvable group which splits as an extension of $\mathbf{R}$ by its nilradical N, associated to $\rho : \mathbf{R} \to \mathrm{Aut}(\mathrm{Lie}(N))$ with positive real roots. From such a group H one can make another one H_Σ by forgetting the unipotent part of ρ. Since the nilradical of H is uniformly exponentially distorted, following Cornulier one can prove that this does not alter the logarithmic sublinear large-scale structure (see [35, Th 1.2] recalled here in II.3.1.1). We prove a partial converse.

Theorem II.1. *Let H and H' be purely real Heintze groups with abelian nilradicals. Let u be any sublinear, admissible function. If H and H' are $O(u)$-sublinearly biLipschitz equivalent then H_Σ and H'_Σ are isomorphic.*

This answers positively to Cornulier [39, 1.16(1)] who raised the question for $\dim H = 3$. For comparison, it is known that two purely real Heintze groups with abelian nilradicals are quasiisometric if and only if they are isomorphic by the work of Xie [160] (also obtained by Carrasco Piaggio [29, 1.10]). In the vein of conjecture [40, 6C2], we ask:

Question II.2. *Let H and H' be purely real Heintze group. Assume that H and H' are sublinearly biLipschitz equivalent. Are H_Σ and H'_Σ isomorphic?*

A positive answer would imply both Theorem II.1 and [122, Theorem 2]. The classification problem can be motivated beyond Lie groups by the fact that the purely real Heintze groups are known to parametrize other objects:

- The commability[2] classes of compactly generated locally compact groups that are hyperbolic with a topological sphere at infinity [37, 5.16].

[2]Namely, to such a group G one can associate the purely real core of the unique focal-universal group commable to G. Commability is a variant of weak commensurability adapted to the locally compact setting, see [37]. For the definition of a hyperbolic locally compact group we refer the reader to [28].

- Together with orbits of scalar products, the connected Riemannian negatively curved homogeneous spaces [88] [75, Corollary 5.3].

Unlike Heintze groups, hyperbolic buildings become rare in large dimension [67]. The two-dimensional case displays a vast subfamily with local finiteness properties, that of Fuchsian buildings, for which the dimension at infinity $\operatorname{Cdim} \partial_\infty$ is known: it was computed by Bourdon in 1997 [15] for some of them and 2000 in full generality [16]. We check that $\operatorname{Cdim}_{O(u)} \partial_\infty$ equals the former in this case, distinguishing pairs of Fuchsian buildings up to sublinear biLipschitz equivalence. Here is the statement for the Bourdon buildings.

Proposition II.3 (Strengthening of [15, Théorème 1.1])**.** *Let $p, q \in \mathbf{Z}$ with $p \geqslant 5$ and $q \geqslant 2$. Let I_{pq} be a Bourdon building (right-angled Fuchsian, with constant thickness q). For all strictly sublinear admissible u,*

$$\operatorname{Cdim}_{O(u)} \partial_\infty I_{pq} = \operatorname{Cdim}_{O(1)} \partial_\infty I_{pq} = 1 + \frac{\log(q-1)}{\operatorname{argch}((p-2)/2)}. \tag{II.2}$$

Conventions, notation Through all the Chapter, $u : \mathbf{R}_{\geqslant 0} \to \mathbf{R}_{\geqslant 1}$ is a nondecreasing, strictly sublinear, doubling function, i.e. $u(r) \ll r$ as $r \to +\infty$ and $\sup_r u(2r)/u(r) < +\infty$. Examples are: $u(r) = \sup(1, r^\gamma)$ with $0 \leqslant \gamma < 1$ and $u(r) = \sup(1, \log(r))$. The combinatorial moduli and certain associated measures are multiply parametrized; we stick to Pansu's and Tyson's notation [151] [125], but in order to emphasize certain monotonicities with respect to the parameters the following convention will be applied: when m in a poset M is parametrized over p in a poset P, we write $(m_p)_{p \in P}$ if $p \leqslant p' \implies m_{p'} \leqslant m_p$, and $(m^p)_{p \in P}$ if $p \leqslant p' \implies m^p \leqslant m^{p'}$.

Contents of Chapter II

II.1. Sublinear quasiconformality

II.1.1. $O(u)$-quasisymmetric structures

The notion of a quasisymmetric structure is a reformulation of that of a space with a quasidistance, where the emphasis is made on balls, their inclusion relations and relative sizes, rather than on a given quasidistance function. Related notions are: b-metric topological spaces [112, IV.1], Margulis structures [81, p.62].

II.1.1.1. Definition

Definition II.4 (Compare[3] [125, 1.1 and 2.7] for $u = 1$)**.** Let Z be a set. A $O(u)$-quasisymmetric structure on Z is a set β of abstract balls[4] together with a realization map $\beta \to \mathcal{P}(Z) \setminus \{\emptyset\}$, $b \mapsto \widehat{b}$, a map $\delta : \beta \to \mathbf{Z}$ and a shift map $\mathbf{Z}_{\geqslant 0} \times \beta \to \beta$, $(k, b) \mapsto k.b$ such that

(SC0) The shift is an action and δ is equivariant with respect to the shift: $\forall k, k' \in \mathbf{Z}_{\geqslant 0}$, $k'.k.b = (k'+k).b$ and $\forall k \in \mathbf{Z}, \forall b \in \beta$, $\delta(k.b) = \delta(b) - k$.

(SC1) $\forall k \in \mathbf{Z}_{\geqslant 0}$, $\forall b, b' \in \beta$,

- (i) $\widehat{k.b} \supseteq \widehat{b}$
- (ii) if $\widehat{b} \subseteq \widehat{b'}$ then $\widehat{k.b} \subseteq \widehat{k.b'}$
- (iii) if $\delta(b) < \delta(b')$ then $\widehat{b} \not\subseteq \widehat{b'}$.

(SC2) There exists $n_0 \in \mathbf{Z}_{\geqslant 0}$ and a function $q : \mathbf{Z}_{\geqslant n_0} \to \mathbf{Z}_{\geqslant 0}$, $q = O(u)$ and such that

$$\forall b, b' \in \beta, \left(n_0 \leqslant \delta(b) \leqslant \delta(b'), \widehat{b} \cap \widehat{b'} \neq \emptyset \right) \implies \widehat{q(\delta(b)).b} \supset \widehat{b'}.$$

(SC3) $\forall x \in Z, \forall y \in Z \setminus \{x\}, \forall n \in \mathbf{Z}, \exists b \in \beta : \delta(b) \geqslant n, x \in \widehat{b}, y \notin \widehat{b}$.

Example II.5 (Space with a quasidistance)**.** Recall that a quasidistance on a set $\mathcal{Z}$ is a kernel $\varrho : \mathcal{Z} \times \mathcal{Z} \to \mathbf{R}$ with the axioms of a distance, the triangle inequality being replaced by

$$\forall (x, y, z) \in \mathcal{Z}^3,\ \varrho(x, z) \leqslant K \left(\varrho(x, y) \vee \varrho(y, z) \right) \qquad (\triangle_K)$$

[3] In [125, 1.1 and 2.7] they are called “bonnes structures quasiconformes”. The “bonne” axiom is a disguised form of the quasi-triangle inequality, here (SC2).

[4] This formalism is here to avoid referring directly to centers and radii, which are preferable to diameters, but may not be uniquely defined. The notion of a constituent (see [57, Definition 2]) circumvents the problem of radii, but it still makes use of centers.

where $K \in \mathbf{R}_{\geqslant 1}$ is a constant. Given a dense[5] subspace (to be thought of as a set of centers) $X \subseteq \mathcal{Z}$, a quasidistance gives to $\mathcal{Z}$ a $O(1)$-quasisymmetric structure in which $\beta = X \times \mathbf{Z}$ and for $b = (x, n)$ in β and $k \in \mathbf{Z}$, $\delta(b) = n$, $k.b = (x, n-k)$ and $\widehat{b} = \{z \in Z : \varrho(x, z) \leqslant e^{-n}\}$. ($\triangle_K$) is responsible for (SC2) with $q = K^2$, the separation axiom for (SC3).

Sub-example II.5.1. $Z = \mathbf{R}$ and β is $\mathbf{R} \times \mathbf{Z}$; for $b = (x, n)$, $\delta(b) = -n$. For all $b = (s, n)$ in β, $\widehat{b} = s + e^{-n}[0, 1]$ (One can replace $[0, 1]$ by any bounded closed interval). One can take $q = 3/2$ in (SC2). The shift is such that $\widehat{k.b} = e^k \widehat{b}$.

Proposition II.6. *Let (Z, β, q, δ) be a $O(u)$-quasisymmetric structure. For all $n \in \mathbf{Z}$, define*

$$E_n = \bigcup_{b \in \beta : \delta(b) \geqslant n} \widehat{b} \times \widehat{b}.$$

Then E_n forms a fundamental system of entourages, endowing Z with a uniform structure.

Proof. Let $\mathcal{E} \subset \mathcal{P}(Z \times Z)$ be the set of subsets containing one of the E_n. It follows from the definition that $\mathcal{E}$ is a filter. Let $E \in \mathcal{E}$ and let $n \in \mathbf{Z}$ be such that $E \supset E_n$. To check Weil's axiom (U'$_{\mathrm{III}}$) [153, p.8] one needs find $m \in \mathbf{Z}$ such that

$$\forall x, y, z \in E,\ \{(x, y)\} \cup \{(y, z)\} \subset E_m \implies (x, z) \in E_n. \tag{II.3}$$

This can be rephrased as follows: for any pair of distinct $x, y \in Z$, set

$$\varrho(x, y) = \exp\left(-\inf\left\{\delta(b) : b \in \beta, \{x\} \cup \{y\} \subset b\right\}\right)$$

and $\varrho(x, x) = 0$. Especially $(x, y) \in E_n \iff -\log \varrho(x, y) \geqslant n$. Then for all $x, y, z \in Z^3$

$$\varrho(x, z) \leqslant e^{v(\inf\{-\log \varrho(x,y), -\log \varrho(y,z)\})} \left[\varrho(x, y) \vee \varrho(y, z)\right] \tag{$\triangle_{O(u)}$}$$

where $v = O(u)$ (one may take $v(n) = q(n)$ at least for $n \geqslant n_0$). Set $m_0 = 2n \vee 2 \sup \left\{m' \in \mathbf{Z}_{\geqslant 0} : v(m') \geqslant \frac{m'}{2}\right\}$. Then for every $m \geqslant m_0$, $m - v(m) \geqslant m - m/2 = m/2 \geqslant n$. (II.3) is achieved. □

Remark II.7. An open subspace Ω of a $O(u)$-quasisymmetric structure (Z, β) inherits a $O(u)$-quasisymmetric structure $(\Omega, \beta_{|\Omega})$ where

$$\beta_{|\Omega} = \left\{b \in \beta : \forall k \in \mathbf{Z}_{\geqslant 0}, \widehat{k.b} \cap \Omega \neq \emptyset\right\},$$

the shift is restricted to $\beta_{|\Omega}$, and the realization is $\widehat{b}_{|\Omega} = \widehat{b} \cap \Omega$.

[5]A quasidistance induces a topology, see [121, 1.99].

II.1.1.2. Hyperbolic cones and sublinear large-scale geometry The boundary of a Gromov-hyperbolic space has a Margulis structure, see e.g. [81]; further, the boundary construction can be reversed as suggested by M.Gromov [82, 1.8.A(b)] and elaborated by M. Bonk and O. Schramm ([10, § 7], see also [131]), so that in the current formalism any $O(1)$-quasisymmetric structure occurs at the boundary of a Gromov-hyperbolic space[6]. It is a classical fact that quasiisometries between Gromov hyperbolic groups extend to biHölder, quasisymmetric homeomorphism between their boundaries, i.e. they do so in a way that preserves the features of the $O(1)$-quasisymmetric structure. This paper is rather concerned with sublinearly biLipschitz maps, for which we recall the definition:

Definition II.8 (Cornulier, [39])**.** Let (Y, o) and (Y', o') be metric spaces. A $O(u)$-sublinearly biLipschitz equivalence (SBE) is a map $f : Y \to Y'$ for which there exists $\lambda \in \mathbf{R}_{\geqslant 1}$ and $v = O(u)$ such that

1. $\forall y_1, y_2 \in Y, \frac{1}{\lambda} d(y_1, y_2) - v(\sup\{d(o, y_1), d(o, y_2)\}) \leqslant d(f(y_1), f(y_2))$
2. $\forall y_1, y_2 \in Y, \lambda d(y_1, y_2) + v(\sup\{d(o, y_1), d(o, y_2)\}) \geqslant d(f(y_1), f(y_2))$
3. $\forall y' \in Y', \exists y \in Y, d(y', f(y)) \leqslant v(d(y, o))$.

Unlike quasiisometries (which are the $O(u)$-SBE with $u = 1$), SBEs are not coarse equivalences in general. However they do preserve certain coarse sublinear structures in the sense of Dranishnikov and Smith [47, 2], or large-scale sublinear structures in the sense of Dydak and Hoffland [53, p.1014]. $O(u)$-quasisymmetric structures are boundary analogs of the former, in a more specific way where u is explicit. In all our applications Y and Y' will be Gromov-hyperbolic, proper geodesic metric spaces, and when the boundaries come under consideration the function $d(o, \cdot)$ could be replaced by the positive part of a fixed Busemann function h on Y (the resulting requirements are weaker and fulfilled by SBE maps, even if function v may be changed). Boundary maps of sublinearly biLipschitz equivalences are still homeomorphisms, however a notion more general than quasiconformality needs to be defined.

[6]Namely a certain quotient space of β, two abstract balls being close if close for δ and if their realizations intersect, compare e.g. [138, chapter 2]. Abstract, resp. concrete balls are turned into geodesic segments, resp. their endpoints. The metric hyperbolicity is implied by (SC1) and (SC2).

II.1.2. $O(u)$-quasisymmetric homeomorphisms

II.1.2.1. Definition and comparison with quasisymmetric mappings Denote by $\mathcal{O}^+(u)$ the semigroup of germs of functions v valued in $\mathbf{Z}_{\geqslant 0}$, defined on large enough integers, such that $v = O(u)$, with the composition law $\dotplus$ defined as

$$(v_1 \dotplus v_2)(n) = v_2(n) + v_1(n - v_2(n)) \tag{II.4}$$

for $n \in \mathbf{Z}$ large enough. The reason for this composition law is the requirement that $(\mathrm{Id} - v_1) \circ (\mathrm{Id} - v_2) = \mathrm{Id} - (v_1 \dotplus v_2)$. $\mathbf{Z}_{\geqslant 0}$ embeds in $\mathcal{O}^+(u)$ as the commutative subsemigroup[7] of constant functions. $\mathcal{O}^+(u)$ acts on small enough abstract balls: for every v in $\mathcal{O}^+(u)$ there exists $n_0 \in Z$ such that $\mathbf{Z}_{\geqslant n_0}$ lies in the domain of v and if $\delta(b) \geqslant n_0$ then $v.b$ is defined as $v.b = v(\delta(b)).b$.

Definition II.9 (round sets and rings, compare [151, 3.4])**.** Let $\beta \to \mathcal{P}(Z)$ be a $O(u)$-quasisymmetric structure. Given $k \in O(u)_{\geqslant 0}$ and $n \in \mathbf{Z}$, a subset $a \in \mathcal{P}(X)$ is a (k,n)-round set (or simply a k-round set) if there exists $b \in \beta$ such that $\delta(b) = n$ and $\widehat{b} \subseteq a \subseteq \widehat{k.b}$. A couple of subsets $(a^-, a^+) \in \mathcal{P}(X)^2$ is a (k,n)-ring if there exists $b \in \beta$ such that $\delta(b) \geqslant n$ and $\widehat{b} \subseteq a^- \subseteq a^+ \subseteq \widehat{k.b}$. Denote by $\mathscr{B}_n^k(\beta)$ resp. $\mathscr{R}_n^k(\beta)$ the collection of (k,n)-round sets, resp. of (k,n)-rings, and $\mathscr{B}^k(\beta)$ resp. $\mathscr{R}^k(\beta)$ their union over $n \in \mathrm{domain}(k)$.

Definition II.10 (outer rings)**.** Let $\beta \to \mathcal{P}(Z)$ be a $O(u)$-quasisymmetric structure. Given $j \in \mathcal{O}^+(u)$, a pair of subsets $(a^-, a^+) \in \mathcal{P}(X)^2$ is a (j,n)-outer ring if there exists $n \in \mathbf{Z}$ and $b \in \beta$ such that $a^- \subseteq \widehat{b} \subseteq \widehat{j.b} \subseteq a^+$ and $\delta(b) \geqslant n$. Denote by $\mathscr{O}_{j;n}(\beta)$ the collection of (j,n) outer rings.

The reader may think of k as a parameter of asphericity[8] (akin to $\log t$ in (II.1)) that depends on the scale. Whereas quasisymmetric mappings preserve bounded asphericities, $O(u)$-quasisymmetric homeomorphisms will be asked to preserve asphericities within the $O(u)$ class. We define them in two steps.

[7]Noncommutativity of $\dotplus$ should not be a concern; one can check that for some $C \geqslant 1$, $C^{-1}(v_1 + v_2)(n) \leqslant (v_1 \dotplus v_2)(n) \leqslant C(v_1 + v_2)(n)$ for n large enough.

[8]We borrow the term "asphericity" from the survey [81, p.88]. Another choice is "modulus" adopted in [122] but it would be misleading here since for our purposes in section II.2, moduli are global rather than infinitesimal conformal invariants. Still another term found in the modern litterature is "eccentricity". We prefer not to define asphericity for subsets since we would face the same issues as with radii and centers.

Definition II.11 (Equivalent $O(u)$-quasisymmetric structures)**.** Let β and β' be two $O(u)$-quasisymmetric structures on a set Z. β' is finer than β if there exists $\lambda \in \mathbf{R}_{>0}$ and $n_0 \in \mathbf{Z}$ such that

$$\forall k \in \mathcal{O}^+(u),\, \exists k' \in \mathcal{O}^+(u) : \forall n \in \mathbf{Z}_{\geqslant n_0},\, \mathscr{R}_n^k(\beta) \subseteq \mathscr{R}_{\lfloor \lambda n \rfloor}^{k'}(\beta') \tag{II.5}$$

$$\forall j' \in \mathcal{O}^+(u),\, \exists j \in \mathcal{O}^+(u) : \forall n \in \mathbf{Z}_{\geqslant n_0},\, \mathscr{O}_{j;n}(\beta) \subseteq \mathscr{O}_{j';\lfloor \lambda n \rfloor}(\beta') \tag{II.6}$$

β and β' are said equivalent if both finer than each other. Up to taking logarithms k' plays with respect to k in (II.5) the rôle of $\eta(t)$ with respect to t in (II.1), so that we will still denote $\eta : \mathcal{O}^+(u) \to \mathcal{O}^+(u)$ a map such that one may take $k' = \eta(k)$ in (II.5). Similarly, denote $\overline{\eta} : \mathcal{O}^+(u) \to \mathcal{O}^+(u)$ a map such that one may take $j = \overline{\eta}(j')$ in (II.6). λ is analogous to a Hölder exponent comparing snowflake-equivalent metrics.

Definition II.12 ($O(u)$-quasisymmetric homeomorphism)**.** Let $\varphi : Z \to Z'$ be a bijection between two sets endowed with $O(u)$-quasisymmetric structures β and β'. One can pull-back β' to Z by means of φ. The map φ is a $O(u)$-quasisymmetric homeomorphism if β and $\varphi^*\beta'$ are $O(u)$-equivalent.

Two $O(u)$-equivalent structures on Z define the same uniform structure on Z so that $O(u)$-conformal homeomorphisms are uniform homeomorphisms. This can be made more quantitative: they are biHölder continuous when this makes sense [122, 4.4]. Not every quasi-symmetric homeomorphism is $O(1)$-quasisymmetric, but every power-quasisymmetric homeomorphisms[9] is. Note that a consequence of Definition II.12 is that

$$\forall k \in \mathcal{O}^+(u),\, \exists k' \in \mathcal{O}^+(u), \mathscr{B}^k(\beta) \subseteq \mathscr{B}^{k'}(\varphi^*\beta')$$

since k-balls may be identified with the k-annuli (a^-, a^+) for which there is equality $a^- = a^+$. This does not suffice for all our needs, nevertheless it is simpler and we shall use it when possible.

Remark II.13. A reformulation of (II.5) and (II.6) is

$$\forall K \in [1, +\infty),\, \exists K' \in [1, +\infty),\, \mathscr{R}_n^{\lceil Ku \rceil}(\beta) \subseteq \mathscr{R}_{\lfloor \lambda n \rfloor}^{\lfloor K'u \rfloor}(\beta').$$

$$\forall J' \in [1, +\infty),\, \exists J \in [1, +\infty),\, \mathscr{O}_n^{\lfloor Ju \rfloor}(\beta) \subseteq \mathscr{O}_{\lfloor \lambda n \rfloor}^{\lceil J'u \rceil}(\beta').$$

Remark II.14. The requirement (II.6) will be needed only when we deal with packings.

[9] A power quasisymmetric embedding is an embedding for which one can take $\eta(t) = \sup\{t^\alpha, t^{1/\alpha}\}$ for some $\alpha \in (0, +\infty)$ in (II.1); this is not restrictive between uniformly perfect metric spaces (called "homogeneously dense" by Tukia and Väisälä) [150, 3.12] [87, 11.3].

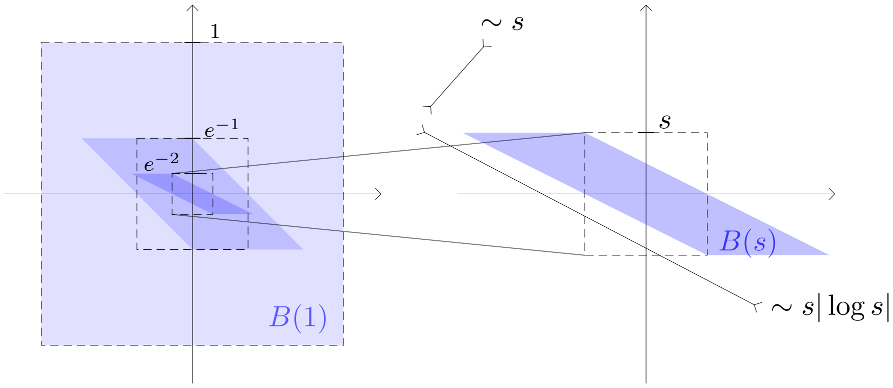


Figure 16: Concentric balls of a quasidistance on $\mathbf{R}^2$ that is invariant under translation and dilation by $\exp(t\alpha')$ with unipotent, non identity α', and coincides with the ℓ^∞ distance for pairs of points at distance 1. For comparison, dashed ℓ^∞ spheres of equal radii. Compare Figure 18.

II.1.2.2. $O(u)$-quasisymmetric homeomorphisms as boundary mappings If Y is a proper geodesic Gromov-hyperbolic space, we call visual kernel on $Z = \partial_\infty Y$ a function $\rho : Z \times Z \to \mathbf{R}_{\geqslant 0}$ such that $\rho(\xi, \eta) = \exp -(\xi, \eta)_o$ for some $x, x' \in Z$, where $(\xi, \eta)_o$ denotes the Gromov product of ξ and η seen from $o \in Y$ (this is $\sup \liminf_{i,j} (\xi_i, \eta_j)_o$ for over all sequences $\xi_i \to \xi$, $\eta_j \to \eta$).

Theorem II.15. *Let Y and Y' be Gromov-hyperbolic, geodesic, proper metric spaces with uniformly perfect Gromov boundaries. Let $f : Y \to Y'$ be a $O(u)$-sublinearly biLipschitz equivalence (Definition II.8). Let β and β' be the $O(u)$-quasisymmetric structures on the Gromov boundaries of Z and Z' associated to visual kernels. Then $\partial_\infty f : Z \to Z'$ is a $O(u)$-quasisymmetric homeomorphism.*

Since the original statement is not this one, we give details on how to deduce it from [122].

How to deduce Theorem II.15 from Chapter I. Fix visual kernels d on Z and Z', start assuming for simplicity that every metric sphere of positive radius in Z and Z' has at least one point, denote $\varphi = \partial_\infty f$; note that φ and φ^{-1} are both sublinearly quasiMöbius (Definition I.40 and Theorem I.41), especially they are biHölder; up to snowflaking Z or Z' let $\gamma \in (0,1)$ be a Hölder exponent for both. By Proposition[10] I.48 sufficiently small rings of inner

[10] Beware that one must translate “annulus” into “ring” and “modulus” into “asphericity”

radius r and asphericity $\log(R/r)$ are sent by φ to rings with asphericity $\log R/r + O(u(-\log r))$ and inner radii greater than $r^{1/\gamma}$; this implies (II.5) translating $-\log r$ into n, $\log R/r$ into k and noting that $u(\gamma n) = O(u(n))$ since u is doubling. Let us prove (II.6). Fix $\ell' \in O(u)$ a positive function. We need ℓ such that if A contains a ℓ'-outer ring then $f(A)$ will contain a ℓ-outer ring. Fix $\zeta \in Z$ and $r > 0$. Let $r' = \sup \{d(\varphi(\zeta), \varphi(\xi)) : d(\zeta, \xi) \leqslant r\}$. Let $\xi_0 \in Z$ be such that $d(\varphi(\xi_0), \varphi(\zeta)) = r'$. Let $\xi_1 \in Z$ be such that $\varphi(\xi_1) \in B(\varphi(\zeta), r' \exp(\ell'(-\log r'))$. By quasiMöbiusness of φ^{-1}, there exists $\lambda \in \mathbf{R}_{>0}$ and $v \in O(u)$ a positive function such that

$$\begin{aligned}\log^+ \frac{d(\zeta, \xi_1)}{d(\zeta, \xi_0)} &\geqslant \lambda \log^+ \frac{d(\varphi(\zeta), \varphi(\xi_1))}{d(\varphi(\zeta), \varphi(\xi_0))} \\ &\quad - v(-\log \inf \{d(|\varphi(\zeta), \varphi(\xi_0)), d(|\varphi(\zeta), \varphi(\xi_1))\}) \\ &\geqslant \ell'(\lfloor -\log r' \rfloor) - v(-\log r').\end{aligned}$$

Setting $\ell(n) = \ell'(n/\gamma) + v(n/\gamma)$ this proves (II.6) for the quasisymmetric structure β and the pullback $\varphi^* \beta'$ on Z. Finally, uniform perfectness of Z and Z' allows to carry the proof up to bounded approximations should certain points not exist. □

Example II.16 (The plane and the twisted plane)**.** Let $Y = \mathbf{R}^2 \rtimes_\alpha \mathbf{R}$ and $Y' = \mathbf{R}^2 \rtimes_{\alpha'} \mathbf{R}$ where

$$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \alpha' = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and the semi-direct products are formed with $t \in \mathbf{R}$ acting on $\mathbf{R}^2$ as $e^{t\alpha}$ and $e^{t\alpha'}$ respectively. Equip Y and Y' with left invariant metrics; they are Gromov-hyperbolic and $-t$ is a Busemann function. Identify both Gromov boundaries $\partial_\infty^* Y = \partial_\infty Y \setminus [-t]$ and $\partial_\infty^* Y' = \partial_\infty Y' \setminus [-t]$ to $\mathbf{R}^2$, and equip them with the quasisymmetric structures associated to quasidistances ρ and ρ' such that $\rho(e^{t\alpha}\xi_1, e^{t\alpha}\xi_2) = e^t \rho(\xi_1, \xi_2)$ for all $\xi_1, \xi_2 \in \partial_\infty Y$. The map $\iota : Y \to Y'$ which is the identity in coordinates is a $O(\log)$-sublinearly biLipschitz equivalence [35]. On $\partial_\infty Y$ and $\partial_\infty Y$ The identity map $\partial_\infty^* \iota$ of $\mathbf{R}^2$ is a $O(\log)$-quasisymmetric homeomorphism, as Figure 16.

II.1.2.3. Examples and non-properties of $O(u)$-quasisymmetric homeomorphisms The following indicates a way to produce $O(u)$-quasisymmetric homeomorphisms of the Euclidean plane starting from the

to conform to our current terminology.

observation that products of biLipschitz homeomorphisms are quasisymmetric homeomorphisms.

The first step of the construction is to build a homeomorphism of the circle with controlled (almost Lipschitz in a precise sense) modulus of continuity. Let T be a rooted infinite binary tree, whose set of vertices V is identified with the set of finite words over the alphabet $\{0,1\}$. Let $(\epsilon_j) \in (0,1/2)^{\mathbf{N}}$ be a decreasing sequence with limit 0. To every $\eta \in \{-1,1\}^V$ we associate a homeomorphism Φ_η of the circle as follows:

1. for each v of length $|v|$ one associates a real number τ^v with the binary expansion $v : \tau^v = \sum_{i=1}^{|v|} v_i 2^{-i}$.

2. Let $M_\eta(v)$ be the uniform measure on $[0, 2^{-|v|}]$ with total mass
$$\|M_\eta(v)\| = \prod_{w \in \mathrm{Pref}(v)\setminus\{v\}} \left(\frac{1}{2} + \eta(w)\epsilon_{|w|}\right),$$
where $\mathrm{Pref}(v)$ denotes the set of prefixes of v (including the empty one).

3. For any nonnegative integer ℓ, $M_\eta^t := \sum_{v \in V : |v| = t} \tau_*^v M_\eta(v)$, where τ_*^v is the pushforward by the translation $x \mapsto x + \tau^v$.

4. Let Φ_η^t be the repartition function of M_η^t; then $\Phi_\eta^t(\tau_v)$ is constant for $t \geqslant |v|$, so $\|\Phi_\eta^t - \Phi_\eta^{t+1}\|_\infty \leqslant \sup_{v:|v|=t} \|M_\eta(v)\| \leqslant (2/3)^t$ for t large enough. By normal convergence, there exists a uniform limit $\Phi_\eta \in \mathrm{Homeo}^+([0,1])$ of the Φ_η^t as $t \to +\infty$. Realizing S^1 as $[0,1]/\sim$ where $0 \sim 1$ and considering η a random variable one may view Φ_η as a random homeomorphism of the circle.

Proposition II.17. *If $\epsilon_j \notin \ell^1(\mathbf{N})$ then Φ_η is not absolutely continuous.*

Proof. Let λ be the Haar measure on S^1, and for $t \in \mathbf{N}_{\geqslant 1}$, let Φ_η^t be the approximation of Φ_η at time t given by $(\Phi^t)' = M^t$. Note that whenever k is an integer with $0 \leqslant k \leqslant 2^t$, one has $\Phi(2^{-t}k) = \Phi^t(2^{-t}k)$. To every $x \in S^1$ one can associate a geodesic $\gamma_x \subset T$ representing its base 2 expansion (the finite one for dyadic x). Fix $\rho \in (0,1)$. Define $A_\eta(\rho) = \left\{x \in [0,1] : \forall t \in \mathbf{N}, 2^t \|M_\eta(\gamma_x(t))\| \geqslant \rho\right\}$. This is the complementary set in $[0,1]$ of

$$\begin{aligned} B_\eta(\rho) &= \left\{x \in [0,1] : \exists t \in \mathbf{N}, 2^t \|M_\eta(\gamma_x(t))\| \leqslant \rho\right\} \\ &= \bigcup_{v \in V : \|M_\eta(v)\| \leqslant 2^{-|v|}\rho} \left[\tau^v, \tau^v + 2^{-|v|}\right] \\ &= \bigsqcup_{v \in V : \forall w \in \mathrm{Pref}(v) \left(\|M_\eta(w)\| \leqslant 2^{-|w|}\rho \implies w = v\right)} \left[\tau^v, \tau^v + 2^{-|v|}\right] \end{aligned}$$

where we used that $\left[\tau^w, \tau^w + 2^{-|w|}\right] \supseteq \left[\tau^v, \tau^v + 2^{-|v|}\right]$ if and only if $w \in \mathrm{Pref}(v)$, with equality if and only if $v = w$. Note that the λ-measure of $\Phi_\eta(B(\rho))$ is smaller than ρ for all ρ, since

$$\begin{aligned} \lambda(\Phi(B_\eta(\rho))) &= \sum_{v \in V : \forall w \in \mathrm{Pref}(v) \left(\|M_\eta(w)\| \leqslant 2^{-|w|}\rho \implies w = v\right)} \lambda\left(\Phi^{|v|}\left[\tau^v, \tau^v + 2^{-|v|}\right]\right) \\ &\leqslant \sum_{v \in V : \forall w \in \mathrm{Pref}(v) \left(\|M_\eta(w)\| \leqslant 2^{-|w|}\rho \implies w = v\right)} 2^{-|v|}\rho \\ &\leqslant \rho, \end{aligned}$$

where we used[11] that the intervals $\left[\tau^v, \tau^v + 2^{-|v|}\right]$ under consideration are disjoint so that the sum of their measures is $\leqslant 1$. On the other hand, if $\epsilon_j \notin \ell^1(\mathbf{N})$ then

$$\lambda(B_\eta(\rho)) = 1 - \lambda\left(A_\eta(\rho)\right) = 1 - 0 = 1,$$

since for almost every x, the sequence $(2^t \|M_\eta(\gamma_x(t))\|)$ is not bounded away from $0:$ up to a null set (the dyadics) one may identify $([0,1], \lambda)$ with the shift space of geodesics rays in T and consider $A_\eta(\rho)$ as an event of probability zero. Especially $\lambda\left(\bigcap_{\rho \downarrow 0} B_\omega(\rho)\right) = 1$, whereas the image of this set by Φ has λ-measure 0. $\square$

From now on assume that $\epsilon_j \notin \ell^1(\mathbf{N})$ but decays sufficiently fast so that the partial sums remain controlled by $u:$

$$\sum_{j \leqslant t} \epsilon_j = O(u(t)), \tag{II.7}$$

[11] Let us provide additional intermediate steps below. For any $v \in V$ in the set indexing the sums, one has, especially, that $2^{|v|}\|M_\eta(v)\| \leqslant \rho$. Now by definition

$$2^{|v|}\|M_\eta(v)\| = \frac{\lambda\left(\Phi^{|v|}[\tau^v, \tau^v + 2^{-|v|}]\right)}{\lambda\left([\tau^v, \tau^v + 2^{-|v|}]\right)}$$

so that (omitting the indexation) $\sum_v \lambda\left(\Phi^{|v|}[\tau^v, \tau^v + 2^{-|v|}]\right) \leqslant \rho \sum_v \lambda\left([\tau^v, \tau^v + 2^{-|v|}]\right)$.

where we recall that u is strictly sublinear. For instance if $\epsilon_j = (3+j)^{-\alpha}$ with $\alpha \in (0,1)$ one may take $u(t) = t^{1-\alpha}$.

Proposition II.18. *Assume that ϵ_j decays sufficiently fast so that* (II.7) *holds. Then there exists $v \in O(u)$ such that for all $\eta \in \{0,1\}^V$*

$$\log l(\Phi_\eta, s) \leqslant \log s + v(-\log s) \tag{II.8}$$

and

$$\log s - v(-\log s) \leqslant \log L(\Phi_\eta, s) \tag{II.9}$$

where $l(\Phi_\eta, s) = \sup\{|\Phi_\eta(x) - \Phi_\eta(y)| : |x-y| \leqslant s\}$ and
$L(\Phi_\eta, s) = \inf\{|\Phi_\eta(x) - \Phi_\eta(y)| : |x-y| \geqslant s\}$.

Proof. Define $t = -\lceil \log_2 s \rceil$. If $|x-y| \leqslant s$, then $[x,y]$ is contained in the union of two adjacent dyadic intervals of length 2^{-t}. Let γ and γ' be the corresponding geodesic segments in T. Then

$$|\Phi_\eta(x) - \Phi_\eta(y)| \leqslant \|M_\eta(\gamma(t))\| + \|M_\eta(\gamma'(t))\| \leqslant 2\prod_{j=0}^{t-1}\left(\frac{1}{2} + \epsilon_j\right),$$

Hence $\log|\Phi_\eta(x) - \Phi_\eta(y)| \leqslant (1-t)\log 2 + \sum_{j=0}^{t-1}\log(1+2\epsilon_j) \leqslant \log s + v(-\log s)$ where $v = O(u)$. Similarly, if $|x-y| \geqslant s$ then $[x,y]$ contains a dyadic interval of length 2^{-1-t} with associated geodesic segment γ so that

$$|\Phi_\eta(x) - \Phi_\eta(y)| \geqslant \|M_\eta(\gamma)\| \geqslant \prod_{j=0}^{t-1}\left(\frac{1}{2} - \epsilon_j\right),$$

providing (II.8). □

Remark II.19. The aim of Proposition II.18 is only to give a modulus of continuity (and a reverse modulus of continuity) for Φ_η. However we expect the deviation of $\log|\Phi_\eta(x) - \Phi_\eta(y)|$ from $\log|x-y|$ to be typically much lower because of Lindeberg's version of the central limit theorem [106, Satz II].

Remark II.20. M_η is homogeneously multifractal in the sense of Buczolich and Seuret [24], and its multifractal spectrum is concentrated at $\{1\}$. Especially Proposition II.18 provides examples for [24, Proposition 9].

We can now produce homeomorphisms of $\mathbf{R}$ in the following way: for every $k \in \mathbf{Z}$, choose $\eta_k \in \{-1,1\}^V$, produce a measure M_{η_k} on $[0,1]$, and then set $\mu = \sum_{k \in \mathbf{Z}} k_* \mu_{\eta_k}$. Finally $\psi : \mathbf{R}^2 \to \mathbf{R}^2$ is such that $\psi(s) = \int_0^s d\mu$. This may be considered a random process if η_k are considered random variables.

Proposition II.21. *Let $\Psi : \mathbf{R}^2 \to \mathbf{R}^2$ be defined by $\Psi(x_1, x_2) = (\psi_1(x_1), \psi_2(x_2))$ where ψ_1 and ψ_2 are as above. Then Ψ is a $O(u)$-quasisymmetric*[12] *homeomorphism.*

Proof. Equip $\mathbf{R}^2$ with the sup norm. Rephrasing Definitions II.11 and II.12 we need to prove that for every $K \in \mathbf{R}_{\geqslant 1}$ and $k \in O(u)$ there exists $L \in \mathbf{R}_{\geqslant 1}$ and $\ell \in O(u)$ such that for any sequence (x^n, y^n, z^n) of points in $\mathbf{R}^2$,

$$\begin{cases} K^{-1} n \leqslant -\log \|y^n - x^n\| \\ -\log \|y^n - x^n\| \leqslant Kn \\ \left| \log \frac{\|y^n - x^n\|}{\|z^n - x^n\|} \right| \leqslant k(n), \end{cases} \implies \begin{cases} L^{-1} n \leqslant -\log \left(\|\Psi(y^n) - \Psi(x^n)\| \right) \\ -\log \left(\|\Psi(y^n) - \Psi(x^n)\| \right) \leqslant Ln \\ \left| \log \frac{\|\Psi(y^n) - \Psi(x^n)\|}{\|\Psi(z^n) - \Psi(x^n)\|} \right| \leqslant \ell(i). \end{cases}$$

Write $x^n = (x_1^n, x_2^n)$, similarly for y^n and z^n. Let $v \in O(u)$ be such that (II.9) holds for every ψ_α, i.e.

$$\forall \alpha \in \{1, 2\}, \ |\log |\psi_\alpha(y) - \psi_\alpha(x)| - \log |y - x|| \leqslant v(-\log |y - x|). \quad \text{(II.10)}$$

Split $\mathbf{N}$ into three index subsets:

$$I_1^y = \{ n \in \mathbf{N} : -\log |y_2^n - x_2^n| > -\log |y_1^n - x_1^n| + 2v(n) \}$$

$$I_2^y = \{ n \in \mathbf{N} : -\log |y_2^n - x_2^n| < -\log |y_1^n - x_1^n| - 2v(n) \}$$

$$I_0^y = \{ n \in \mathbf{N} : |\log |y_2^n - x_2^n| - \log |y_1^n - x_1^n|| \leqslant 2v(n) \}.$$

Also, define I_α^z and J_α^z in the same way for $\alpha \in \{0, 1, 2\}$. Note that since u is non-negative, if $\alpha \neq 0$

$$\forall n \in I_\alpha^y, \begin{cases} \|y^n - x^n\| = |y_\alpha^n - x_\alpha^n| \\ \|\Psi(y^n) - \Psi(x^n)\| = |\psi_\alpha(y_\alpha^n) - \psi_\alpha(x_\alpha^n)| \end{cases} \quad \text{(II.11)}$$

and similar equalities hold for n in I_α^z, whereas if $n \in I_0^y$, resp. $n \in I_0^z$ then $\log \|y^n - x^n\| - \log |y_\alpha^n - x_\alpha^n| \leqslant 2v(Kn + 2v(n))$, resp. $\log \|z^n - x^n\| - \log |z_\alpha^n - x_\alpha^n| \leqslant 2v(Kn + 2v(n))$ for any $\alpha \in \{1, 2\}$. By (II.11), if $\alpha, \beta \in \{1, 2\}$ then for $n \in I_\alpha^y \cap I_\beta^z$

$$\frac{\|y^n - x^n\|}{\|z^n - x^n\|} = \frac{|y_\alpha^n - x_\alpha^n|}{|z_\beta^n - x_\beta^n|} \text{ and } \frac{\|\Psi(y^n) - \Psi(x^n)\|}{\|\Psi(z^n) - \Psi(x^n)\|} = \frac{|\psi_\alpha(y_\alpha^n) - \psi_\alpha(x_\alpha^n)|}{|\psi_\beta(z_\beta^n) - \psi_\beta(x_\beta^n)|}$$

[12]The $O(1)$-quasisymmetric structure, and then the $O(u)$-quasisymmetric structure on $\mathbf{R}^2$, will not depend on the norm, compare [87, p.78].

so that, taking logarithms and by (II.11) and (II.10) and (II.11) again

$$\begin{aligned}\left|\log \frac{\|\Psi(y^n)-\Psi(x^n)\|}{\|\Psi(z^n)-\Psi(x^n)\|} : \frac{\|y^n-x^n\|}{\|z^n-x^n\|}\right| &\leqslant \left|\log \frac{\|\Psi(y^n)-\Psi(x^n)\|}{\|\Psi(z^n)-\Psi(x^n)\|} - \log \frac{|y^n_\alpha - x^n_\alpha|}{|z^n_\beta - x^n_\beta|}\right| \\ &\leqslant 2v\left(-\inf\{\log|y^n_\alpha - x^n_\alpha|, \log|z^n_\beta - x^n_\beta|\}\right) \\ &\leqslant 2v\left(-\inf\{\log\|y^n-x^n\|, \log\|z^n-x^n\|\}\right) \\ &\leqslant 2v(Kn+k(n)).\end{aligned}$$

It remains to treat the case $n \in I^y_\alpha \cap I^z_\beta$ with $\inf\{\alpha,\beta\} = 0$; in this event define $\gamma = \sup\{1,\alpha,\beta\}$. Then

$$\begin{aligned}\left|\log \frac{\|\Psi(y^n)-\Psi(x^n)\|}{\|\Psi(z^n)-\Psi(x^n)\|} : \frac{\|y^n-x^n\|}{\|z^n-x^n\|}\right| &\leqslant \left|\log \frac{\|\Psi(y^n)-\Psi(x^n)\|}{\|\Psi(z^n)-\Psi(x^n)\|} - \log \frac{|y^n_\gamma - x^n_\gamma|}{|z^n_\gamma - x^n_\gamma|}\right| \\ &\quad + 4v(Kn+2v(n)) \\ &\leqslant 2v(Kn+k(n)) + 2v(Kn+2v(n)).\end{aligned}$$

Setting $L = K$ and $\ell(n) = k(n)+v(Kn+k(n))+4v(Kn+2v(n))$ this finishes the proof. □

Whereas quasiconformal mappings between open domains of[13] $\mathbf{R}^2$ have the ACL property (see Väisälä [152, 32.4]; this is instrumental for Mostow rigidity in rank one [119, § 21]), Propositions II.17 and II.21 imply that it fails for general $O(u)$-quasisymmetric homeomorphisms. This is why our main efforts in this paper are rather directed to global invariants.

II.1.3. Covering and measures

II.1.3.1. Covering lemma: extracting disjoint balls Let (Z,β) be a $O(u)$-quasisymmetric structure (Definition II.4) and let $A \in \mathcal{P}(Z)$ be a subset. Say that a countable collection of abstract balls $\mathcal{B}$ is a covering of A if the realizations of the members of $\mathcal{B}$ cover A. We adapt a classical covering lemma for metric spaces [63, 2.8.4 – 2.8.8], [113, p.24][14] to $O(u)$-quasisymmetric structures; (SC2) may be considered the case with 2 balls. The lemma says that out of any covering $\mathcal{B}$ one can extract a disjoint sub-covering $\mathcal{C}$ such that $q.\mathcal{C} = \{q.b : b \in \mathcal{B}\}$ is still a covering, where q is a positive function in the $O(u)$-class; for metric spaces it is known as the “ $5r$ covering lemma ” since one can take 5 as an exponential analog of q.

[13] Quasisymmetric homeomorphisms of the circle that are not absolutely continuous do exist [1, IV.B, Remark 2].

[14] We cite both since Federer's statement is more general, but the filtration of balls according to the logarithms of their radii is noticeable in Mattila's proof.

Lemma II.22. *Let (Z,β) be a $O(u)$-quasisymmetric structure. Let $A \in \mathcal{P}(Z)$ be a subset and let $\mathcal{B} \subseteq \beta$ be a countable covering of A; assume that* $\inf_{\mathcal{B}} \delta > -\infty$. *There exists $\mathcal{C} \subset \mathcal{B}$ such that $q.\mathcal{C}$ covers A and for every $b, b' \in \mathcal{C}$, $\widehat{b} \cap \widehat{b'} = \emptyset$ unless $b = b'$.*

Proof. Set $n_0 = \inf_{b\in\mathcal{B}} \delta(b)$. For every $n \in \mathbf{Z}$, let $\mathcal{B}_n = \{b \in \mathcal{B} : \delta(b) = n\}$. By induction on $n \in \mathbf{Z}_{\geqslant n_0}$, choose for each n (by Zorn's lemma or Hausdorff's maximality principle, see [97, 0.24]) a maximal subfamily $\mathcal{C}_n \subset \mathcal{F}_n$ whose realizations are pairwise disjoint and do not intersect the previously chosen balls, that is:

- $\forall (b,b') \in \mathcal{C}_n \times \mathcal{C}_n$, $\widehat{b} \cap \widehat{b'} \neq \emptyset \implies b = b'$.
- $\forall b \in \mathcal{C}_n$, $\forall m \in \{n_0, \dots n-1\}$, $\forall b' \in \mathcal{C}_m$, $\widehat{b} \cap \widehat{b'} = \emptyset$.
- $\forall b \in \mathcal{B}_n \setminus \mathcal{C}_n, \exists b' \in \mathcal{C}_n : \widehat{b} \cap \widehat{b'} \neq \emptyset$.

By construction, the realizations of members of $\mathcal{C} = \cup_n \mathcal{C}_n$ are disjoint. Let $x \in A$; since $\mathcal{B}$ covers A there is $b' \in \mathcal{B}$ such that $\widehat{b'} \ni x$. Either $b' \in \mathcal{C}$ or, setting $n = \delta(b)$, $b' \in \mathcal{B}_n$ and there is $b \in \mathcal{C}_m$ such that $\widehat{b} \cap \widehat{b'} \neq \emptyset$ with $m \leqslant n$. By (SC2), $\widehat{q.b} \supseteq \widehat{b'}$ so that $\widehat{q.b} \ni x$. $\square$

It follows from the lemma that as soon as a $O(u)$-quasisymmetric structure has a countable covering, then it also has a countable packing $\mathcal{C} \subset \beta$ such that $q.\mathcal{C}$ covers. This holds for instance, if the quasisymmetric structure comes from a separable metric space.

II.1.3.2. Gauges Let (Z,β) be a $O(u)$-quasisymmetric structure. We call any function $\phi : \mathcal{P}(Z) \to [0,+\infty)$ a gauge on (Z,β), and we denote by $\mathcal{G}(Z)$ the set of gauges. For every $\ell \in \mathcal{O}^+(u)$, define a shifted gauge $\widetilde{\phi}^\ell : \mathcal{P}(Z) \to [0,\infty)$ by

$$\widetilde{\phi}^\ell(a) = \sup\{\phi(\widetilde{a}) : (a,\widetilde{a}) \in \mathscr{R}^\ell(\beta)\}.$$

It is important that no restriction is made on ϕ. We define the gauge on $\mathcal{P}(Z)$ rather than β in order to ease the comparisons when changing structure.

II.1.3.3. Carathéodory measures Let (Z,β) be a $O(u)$-quasisymmetric structure. For all $k, \ell \in \mathcal{O}^+(u)$, for all $A \in \mathcal{P}(Z)$, define

$$\Phi^n_{p,k}(A) = \inf\left\{\sum_{b\in\mathscr{F}} \phi(b)^p : \mathscr{F} \subset \mathscr{B}^k_n(\beta), |\mathscr{F}| \leqslant \aleph_0,\, \mathscr{F} \text{ covers } A\right\}$$

$$\widetilde{\Phi}^{\ell;n}_{p,k}(A) = \inf\left\{\sum_{b\in\mathscr{F}} \widetilde{\phi}^\ell(b)^p : \mathscr{F} \subset \mathscr{B}^k_n(\beta), |\mathscr{F}| \leqslant \aleph_0,\, \mathscr{F} \text{ covers } A\right\}$$

and $\Phi_{p,k}(A) = \lim_{n\to+\infty} \Phi^n_{p,k}(A)$, $\widetilde{\Phi}^\ell_{p,k}(A) = \lim_{n\to+\infty} \widetilde{\Phi}^{\ell;n}_{p,k}(A)$. The $O(u)$-quasisymmetric structure β is not specified, however if β and β' are two equivalent $O(u)$-quasisymmetric structures on Z and if λ, η, η' are such that any (ℓ, n)-ring for β (resp. for β') is a $(\eta(\ell), \lfloor \lambda n \rfloor)$-ring for β' (resp. for β), then denoting Φ and Φ' the measures that correspond to ϕ for β and β' then

$$\left(\Phi'\right)_{p,\eta(k)} \leqslant \Phi_{p,k} \text{ and } \left(\widetilde{\Phi}'\right)^\ell_{p,\eta(k)} \leqslant \widetilde{\Phi}^{\eta'(\ell)}_{p,k} \tag{II.12}$$

since any covering by (k, n) round sets with respect to β is a covering by $(\eta(k), \lfloor \lambda n \rfloor)$ round sets with respect to β', and any ℓ-ring with respect to β' is a $\eta'(\ell)$-ring with respect to β (note that η or η' appears on superscript when on the right of $\leqslant$ and on subscript when on the left).

Remark II.23 (Comparisons with Hausdorff measures). When the quasisymmetric structure is that of a metric space, $s \in \mathbf{R}_{>0}$ and $\phi(\widehat{b}) = e^{-s\delta(b)}$, the Carathéodory measures Φ and $\widetilde{\Phi}$ can be compared to Hausdorff measures; namely since (k, n) round sets contain balls of radii e^{-n} and have diameter bounded by $2e^{-n+k(n)}$, one has for every $p \in \mathbf{R}_{>0}$, for every $\varepsilon \in (0, p)$

$$\mathcal{H}^{sp+\varepsilon} \ll \Phi_{p;k} \leqslant \widetilde{\Phi}^\ell_{p;k} \ll \mathcal{H}^{sp-\varepsilon} \tag{II.13}$$

for every $k, \ell \in \mathcal{O}^+(u)$.

II.1.3.4. Packing Pre-measure Let (Z, β) be a quasisymmetric structure and let $A \in \mathcal{P}(Z)$ be a subset. Let $\mathscr{P}$ be a collection of (k, n)-outer rings; say that $\mathscr{P}$ is a (k, n)-packing centered on A, denoted $\mathscr{P} \in \mathrm{Packings}_{k,n}(A)$ if inner sets meet A and outer sets are disjoint; formally

- For every $\mathbf{a} = (a^-, a^+)$ in $\mathscr{P}$, $a^- \cap A \neq \emptyset$.
- For every $\mathbf{a}_0$, $\mathbf{a}_1$ in $\mathscr{P}$, $a_0^+ \cap a_1^+ \neq \emptyset \implies \mathbf{a}_0 = \mathbf{a}_1$.

Similarly to the shifted packing measure $\widetilde{\Phi}$, define a shifted packing premeasure

$$\mathrm{P}\widetilde{\Phi}^\ell_{p;k}(A) = \lim_{n\to+\infty} \sup\left\{\sum_{\mathbf{a}\in\mathscr{P}} \widetilde{\phi}^\ell(a^-)^p : \mathscr{P} \in \mathrm{Packings}_{k,n}(A)\right\} \tag{II.14}$$

or 0 if there exists no packing indexing the sums.

Remark II.24. Let $\phi = \lambda \cdot {}^0\phi + {}^1\phi$ with $\lambda \in \mathbf{R}_{\geqslant 0}$ and ${}^i\phi \in \mathcal{G}(Z)$ for $i \in \{0,1\}$. Associate ${}^i\,\mathrm{P}\,\widetilde{\Phi}^{\ell}_{p;k}$ to ${}^i\phi$ by (II.14). Then by the Minkowski inequality

$$\left(\mathrm{P}\,\widetilde{\Phi}^{\ell}_{p;k}\right)^{1/p} \leqslant \lambda \cdot \left({}^0\,\mathrm{P}\,\widetilde{\Phi}^{\ell}_{p;k}\right)^{1/p} + \left({}^1\,\mathrm{P}\,\widetilde{\Phi}^{\ell}_{p;k}\right)^{1/p}. \tag{II.15}$$

Remark II.25. When changing $O(u)$-quasisymmetric structure from β to β', the analogs of the comparisons (II.12) are

$$(\mathrm{P}\widetilde{\Phi}')^{\eta(\ell)}_{p;k} \geqslant \mathrm{P}\widetilde{\Phi}^{\ell}_{p;\overline{\eta}(k)}. \tag{II.16}$$

Indeed (II.6) implies that $\mathrm{Packings}_{\overline{\eta}(k'),n}(\beta) \subset \mathrm{Packings}_{k',\lfloor \lambda n \rfloor}(\beta')$ whereas, every ℓ-ring for β being a $\eta(\ell)$-rings with respect to β', the supremum in (II.14) is taken over larger sums.

Remark II.26. Pansu uses a notion of packing with bounded multiplicity [127]. However it is not convenient here because even on doubling spaces, if $b \in \beta$ is such that $\delta(b) = n$ then $\widehat{\ell.b}$ cannot be covered by a uniformly bounded number of concrete balls $\widehat{b'}$ with $\delta(b') = n$.

II.2. Conformal invariants

By conformal invariants we mean real numbers attached to $O(u)$-quasisymmetric structures, possibly parametrized (for instance by asphericities) and respecting invariance under conformal equivalence. This invariance should not be understood too strictly: the vanishing, or infinitude, for some choice of parameters is considered an invariant, though those parameters may vary.

II.2.1. Combinatorial moduli and functions of bounded energy

II.2.1.1. Carathéodory and packing combinatorial moduli The modulus is obtained minimizing $\widetilde{\Phi}$ under a normalization constraint on the gauge functions, compare Pansu [125, 2.4] and Tyson [151, 3.23]: all members of Γ should have measure (to be thought of as a length[15]) greater than 1.

[15] This is similar in spirit to requiring a Riemannian metric in a given conformal class to confer sufficient length to any curve in a family as in the definition of the classical moduli.

Definition II.27. Let Γ be a family of subsets in a conformal structure (Z,β), $p \in (0,+\infty)$, k, ℓ and m in $\mathcal{O}^+(u)$. Define

$$\mathrm{mod}_{p,k}^{\ell;m}(\Gamma,\beta) = \inf\left\{\widetilde{\Phi}_{p,k}^{\ell}(Z) : \phi \in \mathcal{G}_m(\Gamma,\beta)\right\},$$

where $\mathcal{G}_m(\Gamma,\beta) = \{\phi \in \mathcal{G}(\beta) : \forall\gamma \in \Gamma, \Phi_{1,m}(\gamma) \geqslant 1\}$ is called a set of admissible gauges for Γ.

Definition II.28 (Packing variant)**.** Let Γ be a family of subsets in a conformal structure (Z,β), $p \in (0,+\infty)$, ℓ and m in $\mathcal{O}^+(u)$. Define

$$\mathrm{pmod}_{p;k}^{\ell,m}(\Gamma,\beta) = \inf\left\{\mathrm{P}\widetilde{\Phi}_{p;k}^{\ell}(Z) : \phi \in \mathcal{G}_m(\Gamma,\beta)\right\}.$$

When changing conformal structure the moduli change accordingly:

Lemma II.29. *Let β and β' be two $O(u)$-equivalent $O(u)$-quasisymmetric structures on Z. Let $\Gamma \subset \mathcal{P}(Z)$. Set η, η' and $\overline{\eta}$ so that $\mathscr{R}^k(\beta) \subset \mathscr{R}^{\eta(k)}(\beta')$, $\mathscr{R}^k(\beta') \subset \mathscr{R}^{\eta'(k)}(\beta)$ and $\mathscr{O}_{\overline{\eta}(j')}(\beta) \subset \mathscr{O}_{j'}(\beta')$ for every $k, j' \in \mathcal{O}^+(u)$. Then*

$$\mathrm{mod}_{p,\eta'(k)}^{\ell;\eta'(m)}(\Gamma,\beta) \leqslant \mathrm{mod}_{p,k}^{\eta(\ell);m}(\Gamma,\beta') \text{ and} \tag{II.17}$$

$$\mathrm{pmod}_{p,\overline{\eta}(k)}^{\ell;\eta'(m)}(\Gamma,\beta) \leqslant \mathrm{pmod}_{p,k}^{\eta(\ell);m}(\Gamma,\beta'). \tag{II.18}$$

Proof. By (II.12), for all $\phi \in \mathcal{G}(Z)$, $\left(\widetilde{\Phi}'\right)_{p,k}^{\eta(\ell)} \geqslant \widetilde{\Phi}_{p,\eta'(k)}^{\ell}$, and $\mathcal{G}_m(\beta') \subseteq \mathcal{G}_{\eta'(m)}(\beta)$. Hence, for the modulus computed with respect to β on (II.17), the infimum in Definition II.27 is taken over more gauges, while common admissible gauges contribute to lower values. This proves (II.17). Concerning the packing moduli, using (II.16) and the same observation:

$$\begin{aligned}
\mathrm{pmod}_{p;\overline{\eta}(k)}^{\ell,\eta'(m)}(\Gamma) &= \inf\left\{\mathrm{P}\widetilde{\Phi}_{p;\overline{\eta}(k)}^{\ell}(Z) : \phi \in \mathcal{G}_{\eta'(m)}(\beta)\right\} \\
&\leqslant \inf\left\{\left(\mathrm{P}\widetilde{\Phi}'\right)_{p;k}^{\eta(\ell)}(Z) : \phi \in \mathcal{G}_{\eta'(m)}(\beta)\right\} \\
&\leqslant \inf\left\{\left(\mathrm{P}\widetilde{\Phi}'\right)_{p;k}^{\eta(\ell)}(Z) : \phi \in \mathcal{G}_m(\beta')\right\} = \mathrm{pmod}_{p,k}^{\eta(\ell);m}(\Gamma,\beta'). \square
\end{aligned}$$

II.2.1.2. Functions of locally bounded energy The Dirichlet energies of functions defined on domains of the plane are preserved by conformal mappings; in fact one can prove that they are preserved with multiplicative errors by quasiconformal mappings. We investigate here analogs of the local $W^{1,p}$-spaces (without an a priori differentiable structure on domains) that are carried by $O(u)$-quasisymmetric homeomorphisms with a shift in an asphericity parameter. The notion of energy we use here is inspired by Pansu's [125, 6.1] but it is actually more closely related to Kleiner and Xie's Q-variation ([158, Definition 3.2], [160, 4]). For quasimetric spaces they are the same, the reader familiar with Q-variation may translate $E^{\ell}_{p,k}(f)(-)$ into $V_{Q,K}(f_{|-})$ with $\ell = 0$, $p = Q$ and $k = \log K$.

Let β be a $O(u)$-quasisymmetric structure on a set Z, and let $f : Z \to \mathbf{C}$ be a continuous function. Given $p \in [1, +\infty)$ and $k, \ell \in \mathcal{O}^+(u)$ one can associate to f a pre-measure on Z by $\mathbf{E}^{\ell}_{p;k}(f) = \mathrm{P}\widetilde{\Phi}^{\ell}_{p;k}$ using the gauge

$$\phi(a) = \operatorname{diam} f(a) =: \operatorname{osc}(f, a).$$

Fix $k, \ell \in \mathcal{O}^+(u)$. Say that a continuous function f has bounded $(p; k, \ell)$-energy if $\mathbf{E}^{\ell}_{p;k}(f)$ is locally finite. If $\Omega \subset Z$ is an open subset, denote the space of functions of bounded energy by

$$\mathscr{W}^{p,k}_{\ell;\mathrm{loc.}}(\Omega) = \Big\{ f \in \mathscr{C}^0(\Omega) : \forall K \in \mathcal{P}(\Omega), K \Subset \Omega \implies \mathbf{E}^{\ell}_{p;k}(f)(K) < +\infty \Big\}.$$

For all K compact in Ω, $k \in \mathcal{O}^+(u)$, $\ell \in \mathcal{O}^+(u)$ and $p \in [1, +\infty)$ define

$$\|f\|^{K;\ell}_{p;k} := \|f\|_{C^0(K)} + \mathbf{E}^{\ell}_{p;k}(f)(K)^{1/p}. \tag{II.19}$$

Lemma II.30. *Let Ω be an open subset of a $O(u)$-quasisymmetric structure β. For every $p \in [1, +\infty)$ and $\ell, k \in \mathcal{O}^+(u)$, $\mathscr{W}^{p,k}_{\ell;\mathrm{loc.}}(\Omega)$ is an algebra for pointwise multiplication and for every $K \Subset \Omega$, $f \mapsto \|f\|^{K;\ell}_{p;k}$ defines a multiplicative seminorm on $\mathscr{W}^{p,k}_{\ell;\mathrm{loc.}}(\Omega)$.*

Proof. By (II.15) and the triangle inequality in $\mathbf{C}$, for any $f, g \in \mathscr{C}(\Omega)$ and $\lambda \in \mathbf{C}$ one has $\mathbf{E}^{\ell}_{p;k}(\lambda f + g) \leqslant \mathbf{E}^{\ell}_{p;k}(f) + |\lambda| \mathbf{E}^{\ell}_{p;k}(g))$, so that $\mathscr{W}^{p,k}_{\ell;\mathrm{loc.}}(\Omega)$ is a vector space. Further, for every $A \subseteq \Omega$,

$$\operatorname{osc}(fg, a) \leqslant \sup_A |f| \operatorname{osc}(g, a) + \sup_A |g| \operatorname{osc}(f, a), \tag{II.20}$$

while, by definition

$$\mathbf{E}^{\ell}_{p;k}(fg)(K) = \lim_{n \to +\infty} \sup_{\mathscr{P} \in \mathrm{Packings}_{k,n}(K)} \sum_{\mathbf{a} \in \mathscr{P}} \sup_{(a^-, A) \in \mathscr{R}^{\ell}(\beta)} \operatorname{osc}(fg, A)^p. \tag{II.21}$$

At this point, note that since K has been assumed compact, since the topology associated to β is uniform, since f is continuous and since

$\limsup_n \cup_{\mathbf{a}\in\mathscr{P}_n} \sup_{(a^-,A)\in\mathscr{R}^\ell(\beta)} A \subseteq K$ for every sequence of $(k;n)$ packings $\mathscr{P}_n$,

$$\lim_{n\to+\infty} \sup_{\mathscr{P}\in\mathrm{Packings}_{k,n}(K)} \sup_{\mathbf{a}\in\mathscr{P}} \sup_{(a^-,A)\in\mathscr{R}^\ell(\beta)} \sup_A |f| \leqslant \|f\|_{C^0(K)}$$

and the same inequality holds for g so that inserting (II.20) in (II.21) and letting $n\to+\infty$ using this estimate and the Minkowski inequality yields

$$\mathbf{E}^\ell_{p;k}(fg)(K)^{1/p} \leqslant \|f\|_{C^0(K)}\mathbf{E}^\ell_{p;k}(g)^{1/p} + \|g\|_{C^0(K)}\mathbf{E}^\ell_{p;k}(f)^{1/p}. \tag{II.22}$$

From there (recall that $\|-\|^{K;\ell}_{p;k}$ was defined in (II.19)),

$$\begin{aligned}\|fg\|^{K;\ell}_{p;k} &= \|fg\|_{C^0(K)} + \mathbf{E}^\ell_{p;k}(fg)(K)^{1/p}\\ &\overset{(\mathrm{II.22})}{\leqslant} \|f\|_{C^0(K)}\|g\|_{C^0(K)} + \|f\|_{C^0(K)}\mathbf{E}^\ell_{p;k}(g)^{1/p} + \|g\|_{C^0(K)}\mathbf{E}^\ell_{p;k}(f)^{1/p}\\ &\leqslant \left(\|f\|_{C^0(K)} + \mathbf{E}^\ell_{p;k}(f)(K)^{1/p}\right)\left(\|g\|_{C^0(K)} + \mathbf{E}^\ell_{p;k}(g)(K)^{1/p}\right)\\ &= \|f\|^{K;\ell}_{p;k}\|g\|^{K;\ell}_{p;k}.\end{aligned}$$

□

From now on, in order to define a Fréchet algebra structure on $\mathscr{W}^{p,k}_{\ell;\mathrm{loc.}}(\Omega)$, we will need to assume more on the topology associated with β.

Definition II.31 (hemicompactness)**.** Let X be a Hausdorff topological space. An admissible exhaustion of X is an increasing sequence of compact subspaces $(K_n)_{n\geqslant 0}$ of X such that for every compact K of X there exists n such that $K\subset K_n$. A space is hemicompact if it has an admissible exhaustion.

If Z is a locally compact, second countable topological space, then any open subset of Z is hemicompact. Indeed by Lindelöf's lemma in a second countable space, every open subset is a Lindelöf space (meaning that any open cover of it has a countable subcover) [97, Chapter 1, Theorem 15], and a locally compact Lindelöf space is hemicompact.

Lemma II.32. *Let (Z,β) be a $O(u)$-quasisymmetric structure with locally compact, secound countable topology. For all non-empty open $\Omega\subset Z$, $\mathscr{W}^{p,k}_{\ell;\mathrm{loc.}}(\Omega)$ defines a unital commutative Fréchet algebra. Further, if $\varphi:(Z',\beta')\to(Z,\beta)$ is a $O(u)$-quasisymmetric homeomorphism then for every open $\Omega'\subset Z'$, letting $\Omega=\varphi(\Omega')$ the identity map defines linear continuous algebra homomorphisms*

$$\mathscr{W}^{p;k}_{\eta\circ\eta'(\ell)}(\Omega') \hookrightarrow \varphi^*\mathscr{W}^{p;\overline{\eta}(k)}_{\eta'(\ell)}(\Omega) \hookrightarrow \mathscr{W}^{p;\overline{\eta}'\circ\overline{\eta}(k)}_{\ell}(\Omega'). \tag{II.23}$$

Proof. By the observation above each open subset Ω being hemicompact, has an admissible exhaustion (K_n). The countable family of seminorms $\left(\| - \|_{\ell,\mathrm{loc.}}^{K_n,k}\right)$ defines the Fréchet algebra structure on $\mathscr{W}^{p,k}_{\mathrm{loc.}}$; the hemicompactness ensures that it does not depend on the choice of the sequence (K_n). To prove the part about $O(u)$-quasisymmetric homeomorphisms we can assume that β' is an $O(u)$-equivalent structure on the same set Z. Denote $\mathbf{E}$, resp. $\mathbf{E}'$ the energies computed with respect to β, resp. β'. By (II.16),

$$\forall k, \ell \in \mathcal{O}^+(u), \mathbf{E}^{\eta'(\ell)}_{p;k}(f) \geqslant (\mathbf{E}')^{\ell}_{p;\overline{\eta}'(k)}(f)$$

(this may be compared to Xie [158, Lemma 3.1]) so that $\mathscr{W}^{p,k}_{\mathrm{loc.}}(\Omega, \beta_{|\Omega})$ continuously embeds in $\mathscr{W}^{p,\overline{\eta}'(k)}_{\mathrm{loc.}}(\Omega \mid \beta'_{|\Omega})$. (II.23) is obtained by applying this twice and reversing the rôles of β and β'. □

Lemma II.33. *Let Ω be an open set in a $O(u)$-quasisymmetric structure (Z, β). For any compact set $K \Subset \Omega$, any $\ell \geq 1$ and any $f \in \mathscr{W}^{p,k}_{\ell;\mathrm{loc.}}(\Omega)$,*

$$\lim_{j \to +\infty} \left(\|f^j\|^{K,\ell}_k \right)^{1/j} = \|f\|_{C^0(K)}.$$

Proof. In view of (II.19), for every $f \in \mathscr{W}^{p,k}_{\ell;\mathrm{loc.}}(\Omega)$,

$$\|f^j\|_{C^0(K)} \leqslant \|f^j\|^{K,\ell} \leqslant 2 \sup \left\{ \|f^j\|_{C^0(K)}, \mathbf{E}^{\ell}_{p;k}(f^j)(K)^{1/p} \right\},$$

hence it suffices to show that $\mathbf{E}^{\ell}_{p;k}(f^j)(K)^{1/p} = O\left(\|f\|^j_{C^0(K)}\right)$ as $j \to +\infty$. Precisely this will be implied by the inequality

$$\mathbf{E}^{\ell}_{p;k}(f^j)^{1/p} \leqslant j \|f\|^{j-1}_{C^0(K)} \mathbf{E}^{\ell}_{p;k}(f)^{1/p}. \tag{II.24}$$

Let us prove (II.24). Let $a \subset \Omega$ be a round set intersecting K.

$$\operatorname{osc}(f^j, a) \leqslant j (\sup_a |f|)^{j-1} \operatorname{osc}(f, a) \leqslant j \|f\|^{j-1}_{C^0(K)} \operatorname{osc}(f, a), \tag{II.25}$$

where we have used the inequality $|x^j - y^j| \leqslant j \sup\{x, y\}^{j-1} |x - y|$ for any positive real numbers x and y. Let n be a large integer. For any $\mathscr{P} \in \mathrm{Packing}_{k;n}(K)$,

$$\sum_{a \in \mathscr{P}} \operatorname{osc}(f^j, a)^p \overset{(\mathrm{II}.25)}{\leqslant} \sum_{a \in \mathscr{P}} j^p \|f\|^{(j-1)p}_{C^0(K)} \operatorname{osc}(f, a)^p.$$

This implies (II.24) by letting $n \to +\infty$, taking supremum and applying the definition of the energies. □

Let F denote a Fréchet **C**-algebra. A character of F is a continuous nonconstant homomorphism to **C**. The space of characters on F equipped with the weak star topogy is denoted by $\mathbf{M}(\mathrm{F})$; **M** stands for "maximal closed ideals", the equivalence with characters being provided by the Gelfand-Mazur theorem for Fréchet algebras [72, 3.2.11].

Lemma II.34. *Let Ω be an open subset of a $O(u)$-quasisymmetric structure (Z,β). A character on $\mathscr{W}^{p,k}_{\mathrm{loc.}}(\Omega)$ is continuous with respect to the topology induced by $\mathscr{C}^0(\Omega)$.*

Proof. Let $\varphi \in \mathbf{M}(\mathscr{W}^{p,k}_{\mathrm{loc.}}(\Omega))$. For every compact $K \Subset \Omega$ and $\ell \geq 1$, there exists $C(K,\ell)$ such that for every $f \in \mathscr{W}^{p,k}_{\mathrm{loc.}}(\Omega)$, $|\varphi(f)| \leqslant C(K,\ell)\|f\|_k^{K,\ell}$. Notice that for every integer $j \geqslant 0$, for every $f \in \mathscr{W}^{p,k}_{\mathrm{loc.}}(\Omega)$, $|\varphi(f)| = |\varphi(f^j)|^{1/j}$ so that applying Lemma II.33,

$$|\varphi(f)| \leqslant \lim_{j\to+\infty} \left(C(K,\ell)\|f^j\|_k^{K,\ell}\right)^{1/j} = \|f\|_{C^0(K)}. \qquad \square$$

The lemma ensures that the continuous map $\mathbf{M}(\mathscr{C}^0(\Omega)) \to \mathbf{M}(\mathscr{W}^{p,k}_{\mathrm{loc.}}(\Omega))$ obtained by restricting characters is actually surjective. The next proposition uses this to describe the latter spectrum:

Proposition II.35. *Let (Z,β) be a locally compact, second countable $O(u)$-quasisymmetric structure. Let Ω be an open subspace of Z. Let $\mathcal{R}$ be a closed equivalence relation on Ω. Denote by Λ the quotient space, and $\pi : \Omega \to \Lambda$ the surjection. Assume that $\mathscr{W}^{p,k}_{\ell;\mathrm{loc.}}(\Omega)$ factors through $\mathcal{R}$, and that $\mathcal{R}$ is maximal for this property so that there is a well-defined continuous embedding of $\mathscr{W}^{p,k}_{\ell;\mathrm{loc.}}(\Omega)$ as a separating subalgebra of $C(\Lambda)$. Then*

$$\begin{aligned} \vartheta : \Lambda &\to \mathbf{M}(\mathscr{W}^{p,k}_{\ell;\mathrm{loc.}}(\Omega)) \\ L &\mapsto \{f \mapsto f(L)\} \end{aligned}$$

is a homeomorphism.

Proof. We will use the representation of the Fréchet algebras $\mathscr{W}^{p,k}_{\mathrm{loc.}}(\Omega)$ and $C(\Lambda)$ as projective limits of Banach algebras and its consequences on the associated spectra, compare the textbook by Goldmann [72, 3.2]. Let (K_n) be an admissible exhaustion of Ω. Introduce a sequence of closed ideals

$$\mathcal{I}_n = \left\{ f \in \mathscr{W}^{p,k}_{\mathrm{loc.}}(\Omega) : f_{|K_n} \equiv 0 \right\}.$$

The quotient $\mathrm{A}^n := \mathscr{W}^{p,k}_{\ell;\mathrm{loc.}}(\Omega)/\mathcal{I}_n$ becomes a Banach algebra when endowed with the norm $\|f + \mathcal{I}_n\| = \|f\|_k^{K_n,\ell}$, and embeds in $\mathrm{B}^n := C(\pi(K_n))$, by mapping the class $[f] \in \mathrm{A}^n$ to the image of $f_{|K_n}$. Further A^n is a $*$-invariant algebra in B^n, separating points by assumption. By the Stone-Weierstrass theorem, $\iota_n : \mathrm{A}^n \to \mathrm{B}^n$ has a dense range so that there is a continuous injective $\mathbf{M}(\iota_n) : \mathbf{M}(\mathrm{B}^n) \to \mathbf{M}(\mathrm{A}^n)$, which is surjective by Lemma II.34. $\mathbf{M}(\mathrm{B}^n)$ is compact (it is actually homeomorphic to $\pi(K_n)$) and $\mathbf{M}(\mathrm{A}^n)$ is Hausdorff by Hausdorffness of the weak star topology, so that $\mathbf{M}(\iota_n)$ is a homeomorphism. Now, under natural identifications

$$\mathbf{M}\left(\mathscr{W}^{p,k}_{\ell;\mathrm{loc.}}(\Omega)\right) = \varinjlim \mathbf{M}(\mathrm{A}^n)$$
$$\mathbf{M}\left(C(\Lambda)\right) = \varinjlim \mathbf{M}(\mathrm{B}^n).$$

The maps $\mathbf{M}(\iota_n)$ are compatible with the inductive limits. Denote by ι their glueing. Since $C(\Lambda)$ is a uniform Fréchet algebra, $\mathbf{M}(C(\Lambda))$ is homeomorphic to Λ through the Gelfand map $\vartheta \circ \mathbf{M}(\iota)$ [72, 4.1.7]. □

The shifts in parameters k, ℓ defining the algebra when changing $O(u)$-quasisymmetric structure (by (II.23)) are troublesome and one would prefer to define a single algebra and the topological dimension of its spectrum as an invariant. The dependence with respect to ℓ can be removed within the category of Fréchet algebras by taking an additional projective limit (one may restrict to countably many ℓ in $\mathcal{O}^+(u)$ for the seminorms). This is not the case with the parameter k since the seminorms $\| - \|_k^{K,\ell}$ decrease with respect to k. Nevertheless, under an additional assumption that the space of leaves separated by the algebras $\mathscr{W}^{p;k}_{\ell,\mathrm{loc.}}$ stabilizes when k is fixed and ℓ increases, a homeomorphism can be recovered using Proposition II.35:

Proposition II.36. *Let $\varphi : (Z', \beta') \to (Z, \beta)$ be a $O(u)$-quasisymmetric homeomorphism and let Ω, Ω' be open subsets of Z and Z'. Assume that there exists closed relations $\mathcal{R}, \mathcal{R}'$ on Ω, Ω' with quotient spaces Λ, Λ' such that the following holds: there exists $\lambda : \mathcal{O}^+(u) \to \mathcal{O}^+(u)$ such that for every $k \in \mathcal{O}^+(u)$, for all $\ell \in \mathcal{O}^+(u)$, if $\ell \geqslant \lambda(k)$ then $\mathscr{W}^{p,k}_{\ell;\mathrm{loc.}}(\Omega)$ resp. $\mathscr{W}^{p,k}_{\ell;\mathrm{loc.}}(\Omega')$ factors through $\mathcal{R}$ resp. $\mathcal{R}'$, and $\mathcal{R}$ and $\mathcal{R}'$ are maximal for this property. Then, there exists a homeomorphism between the spaces of leaves Λ and Λ'.*

Proof. Let $k \in \mathcal{O}^+(u)$. Recall that by (II.23), there is a linear continuous embedding $\varphi^* \mathscr{W}^{p;\overline{\eta}(k)}_{\eta'(\ell)}(\Omega) \hookrightarrow \mathscr{W}^{p;\overline{\eta}' \circ \overline{\eta}(k)}_{\ell}(\Omega')$ to which one associates a linear continuous map between spectra $\mathbf{M}\mathscr{W}^{p;\overline{\eta}' \circ \overline{\eta}(k)}_{\ell}(\Omega') \to \mathbf{M}\mathscr{W}^{p;\overline{\eta}(k)}_{\eta'(\ell)}(\Omega)$, that we call $\mathbf{M}\varphi_*$. In the same way, there is $\mathbf{M}\mathscr{W}^{p;\overline{\eta} \circ \overline{\eta}'(k)}_{\ell}(\Omega) \to \mathbf{M}\mathscr{W}^{p;\overline{\eta}'(k)}_{\eta(\ell)}(\Omega')$ that we call $\mathbf{M}\varphi^*$. Now set $\ell \in \mathcal{O}^+(u)$ large enough so that

$$\inf\{\ell, \eta'(\ell), \eta(\ell)\} \geqslant \sup\{\lambda \circ \overline{\eta}(k), \lambda \circ \overline{\eta} \circ \overline{\eta}'(k), \lambda \circ \overline{\eta}'(k), \lambda \circ \overline{\eta}' \circ \overline{\eta}(k)\}.$$

By Proposition II.35, $\mathbf{M}\varphi_*$ and $\mathbf{M}\varphi^*$ can be completed by maps that we call ϑ_1, ϑ_2, ϑ'_1, ϑ'_2 into the following cycle:

$$\begin{tikzcd}
 & \mathbf{M}\mathscr{W}_{\ell}^{p;\overline{\eta}'\circ\overline{\eta}(k)}(\Omega') \arrow[r, "\mathbf{M}\varphi_*"'] & \mathbf{M}\mathscr{W}_{\eta'(\ell)}^{p;\overline{\eta}(k)}(\Omega) \arrow[dr, "\vartheta_2^{-1}", "\sim"'] & \\
\Lambda' \arrow[ur, "\vartheta'_1", "\sim"'] & & & \Lambda \arrow[dl, "\vartheta_1", "\sim"'] \\
 & \mathbf{M}\mathscr{W}_{\eta(\ell)}^{p;\overline{\eta}'(k)}(\Omega') \arrow[ul, "{\vartheta'}_2^{-1}", "\sim"'] & \mathbf{M}\mathscr{W}_{\ell}^{p;\overline{\eta}\circ\overline{\eta}'(k)}(\Omega). \arrow[l, "\mathbf{M}\varphi^*"'] &
\end{tikzcd}$$

It follows from their definitions that the maps composed along the cycle give the identity on any spectra, and then on the leaf spaces. Then $\vartheta_2^{-1}\circ\mathbf{M}\varphi_*\circ\vartheta'_1$ is a homeomorphism between Λ' and Λ. $\square$

We do not use Proposition II.36 and favor a more direct approach instead in II.3.1.4, but we believe that it may be of independent interest.

II.2.1.3. Condensers and capacities For $p \in [1, +\infty)$, $k, \ell \in \mathcal{O}^+(u)$ and Ω an open subset in a $O(u)$-quasisymmetric structure, denote by $\mathscr{W}^{p,k}_{\ell;\mathrm{loc.}}(\Omega, \mathbf{R})$ the $\mathbf{R}$-subspace of $\mathscr{W}^{p,k}_{\ell;\mathrm{loc.}}(\Omega)$ of $\mathbf{R}$-valued functions.

Definition II.37 (Condenser, capacity)**.** Let Z be a $O(u)$-quasisymmetric structure and let Ω be an open subspace. A condenser in Ω is a triple of subspaces $(C, \partial_0 C, \partial_1 C)$ such that C is relatively compact, $\partial_0 C$ and $\partial_1 C$ are closed disjoint, and contained in $\overline{C} \setminus C$. Its capacity is

$$\mathrm{cap}^{\ell}_{p;k}(C) = \inf\left\{\mathbf{E}^{\ell}_{p;k}(f)(C) : f \in \mathscr{W}^{p,k}_{\ell;\mathrm{loc.}}(\Omega, \mathbf{R}),\ f_{|\partial_0 C} \leqslant 0,\ f_{|\partial_1 C} \geqslant 1\right\}.$$

Lemma II.38. *Let $(C, \partial_0 C, \partial_1 C)$ be a condenser in Ω, open subset of a $O(u)$-quasisymmetric structure β. For all $k, \ell, m \in \mathcal{O}^+(u)$, if Γ is any family of curves joining $\partial_0 C$ and $\partial_1 C$ in C then*

$$\mathrm{pmod}^{\ell,m}_{p;k}(\Gamma) \leqslant \mathrm{cap}^{\ell}_{p;k}(C). \tag{II.26}$$

Proof. Let $\varepsilon > 0$. Let $f \in \mathscr{W}^{p,k}_{\ell;\mathrm{loc.}}(\Omega, \mathbf{R})$ be such that $f_{|\partial_0 C} \leqslant 0, f_{|\partial_1 C} \geqslant 1$. Let us prove that the gauge $\phi : a \mapsto \mathrm{diam} f(a)$ is in $\mathcal{G}_m(\Gamma, \beta)$; the conclusion will follow by applying the definition of capacites and energies. By the intermediate value theorem, for every γ in Γ, $f(\gamma)$ contains $[0, 1]$. Consequently, whenever $\mathscr{F}$ is a covering of γ by m-round sets, by countable subadditivity of the outer measure $\mathcal{H}^1$ on $\mathbf{R}$

$$\sum_{a\in\mathscr{F}}\phi(a)=\sum_{a\in\mathscr{F}}\operatorname{diam} f(a)=\sum_{a\in\mathscr{F}}\mathcal{H}^1\operatorname{Conv}(f(a))\geqslant\sum_{a\in\mathscr{F}}\mathcal{H}^1 f(a)$$
$$\geqslant\mathcal{H}^1\left(\bigcup_{a\in\mathscr{F}}f(a)\right)\geqslant 1.\square$$

II.2.2. Diffusivity

The following is a central result in conformal dimension theory [108, 4.1.3]. The guiding principle is a length-volume estimate for a Riemannian parallelotope [125, 2.2]; in order to transpose this to the combinatorial moduli, one has to retain a diffusivity condition expressing that a family of curves is sufficiently spread out in the space, (D(p, r)) below. We give two variants: the first is Pansu's original; the second one is a packing variant.

II.2.2.1. Carathéodory variant

Proposition II.39. *Let (Z,β,δ,q) be a $O(u)$-quasisymmetric structure. Let Γ be a collection of subsets in Z, endowed with a positive measure $d\gamma$ such that for any $b\in\beta$, $\{\gamma\in\Gamma:\gamma\cap\widehat{b}\neq\emptyset\}$ is measurable. For each $\gamma\in\Gamma$, let m_γ be a probability Borel measure on γ. Let $p\in(1,+\infty)$. Assume that there exists a constant $\tau\in(0,+\infty)$ and $r\in\mathcal{O}^+(u)$ such that*

$$\limsup_{n\to+\infty}\sup_{\underline{b}\in\beta:\delta(\underline{b})\geqslant n}\int_\Gamma m_\gamma(\gamma\cap\widehat{r.\underline{b}})^{1-p}\left[\gamma\cap\widehat{\underline{b}}\neq\emptyset\right]d\gamma\leqslant\tau. \tag{D(p, r)}$$

Then for every $k,m\in\mathcal{O}^+(u)$,

$$\operatorname{mod}^{\ell,m}_{p;k}(\Gamma)\geqslant\frac{1}{\tau}\int_\Gamma d\gamma, \tag{II.27}$$

where[16] $\ell=q\dotplus r\dotplus k$ (We recall that the operation $\dotplus$ was defined in II.1.2.1).

Proof. Up to the formalism, the proof is due to Pansu [125, 2.9] and we do not depart from it. Inequality (II.27) will actually be obtained through a stronger one: for any 0-admissible gauge ϕ,

$$\widetilde{\Phi}^{\ell}_{p,k}(\Gamma)\geqslant\tau^{-1}\int_\Gamma\Phi_{1,0}(\gamma)^p d\gamma. \tag{II.28}$$

[16]The conclusion of the lemma (as the assumption (D(p, r)) is all the more weaker that r is large. In subsection II.3.1 we can arrange the quasisymmetric structure so that r can be assumed 1, however in subsection II.3.2 it is really necessary.

(To see why (II.28) implies (II.27) with $m = 0$ note that since $p \geqslant 1$ and ϕ is admissible the right-hand side is greater than $\int_\Gamma d\gamma$; finally $\mathrm{mod}_{p,k}^{\ell,m}$ increases with m). Set an admissible gauge ϕ. Define, for all n,

$$\tau_n := \sup_{\underline{b} \in \beta : \delta(b) \geqslant n} \int_\Gamma m_\gamma(\gamma \cap \widehat{r.\underline{b}})^{1-p} \left[\gamma \cap \widehat{\underline{b}} \neq \emptyset\right] d\gamma.$$

Fix $n \in \mathbf{Z}$. Let $k \in \mathcal{O}^+(u)$. Let $\mathscr{F}$ be a countable covering of Z by (k,n)-round sets of β; taking inner ball $\underline{b} \in \beta$ for each round set $b \in \mathscr{F}$ gives a countable $\mathcal{B} \subset \beta$ such that $k.\mathcal{B}$ covers Z. For $\gamma \in \Gamma$ define $\mathcal{B}_\gamma = \{\underline{b} \in \mathcal{B} : b \cap \gamma \neq \emptyset\}$. For every γ, $k.\mathcal{B}_\gamma$ is a covering of γ, since every $x \in \gamma$ is contained in a $b \in \mathscr{F}$ such that $\underline{b}$ has been selected in $\mathcal{B}_\gamma$. All the more, $r.k.\mathcal{B}_\gamma$ is a covering of γ and by Lemma II.22 one can extract $\mathcal{C}_\gamma$ from $\mathcal{B}_\gamma$ such that $q.r.k.\mathcal{C}_\gamma$ covers γ and have disjoint realizations. Note that $(\widehat{\underline{b}}, \widehat{q.r.k.\underline{b}}) \in \mathscr{R}^{q \dotplus r \dotplus k}(\beta)$ (as $\delta(q.r.k.b) = \delta(b) - (q \dotplus r \dotplus k)(\delta(b))$), hence

$$\phi(\widehat{q.r.k.\underline{b}}) \leqslant \sup\left\{\phi(\widetilde{b}) : (b, \widetilde{b}) \in \mathscr{R}^\ell(\beta)\right\} = \widetilde{\phi}^\ell(b).$$

Recall that $q.r.k.\mathcal{B}_\gamma$ covers γ. Thus

$$\Phi_{1,0}^{2n-\ell(n)}(\gamma) \leqslant \sum_{\underline{b} \in \mathcal{C}_\gamma} \phi\left(\widehat{q.r.k.\underline{b}}\right) \leqslant \sum_{\underline{b} \in \mathcal{C}_\gamma} \widetilde{\phi}^\ell(b). \tag{II.29}$$

Next, apply Hölder's inequality to $\alpha, \zeta : \mathcal{C}_\gamma \to \mathbf{R}$ defined by

$$\alpha(\underline{b}) = \widetilde{\phi}^\ell(\widehat{\underline{b}}) m_\gamma(\widehat{r.k.\underline{b}} \cap \gamma)^{(1-p)/p} \text{ and } \zeta(\underline{b}) = m_\gamma(\widehat{r.k.b} \cap \gamma)^{(p-1)/p}$$

so that

$$\begin{aligned}
\Phi_{1,0}^{n-\ell(n)}(\gamma)^p &\underset{\text{(II.29)}}{\leqslant} \left(\sum_{b \in \mathcal{C}_\gamma} \alpha(\underline{b})^p\right)\left(\sum_{b \in \mathcal{C}_\gamma} \zeta(\underline{b})^{p/(p-1)}\right)^{p-1} \\
&\leqslant \left(\sum_{b \in \mathcal{C}_\gamma} \widetilde{\phi}^\ell(b)^p m_\gamma(\widehat{r.k.\underline{b}} \cap \gamma)^{1-p}\right)\left(\sum_{b \in \mathcal{C}_\gamma} m_\gamma(\widehat{r.k.\underline{b}} \cap \gamma)\right)^{p-1} \\
&\leqslant \left(\sum_{b \in \mathcal{C}_\gamma} \widetilde{\phi}^\ell(b)^p m_\gamma(\widehat{r.k.\underline{b}} \cap \gamma)^{1-p}\right)(m_\gamma(\gamma))^{p-1}.
\end{aligned} \tag{II.30}$$

The last inequality comes from the fact that the $\widehat{r.k.\underline{b}}$ for $b \in \mathcal{C}_\gamma$ are disjoint by construction, hence their intersections with γ are disjoint, and m_γ is subadditive. Further, since m_γ is a probability measure, (II.30) rewrites

$$\Phi_{1,0}^{n-\ell(n)}(\gamma)^p \leqslant \sum_{b \in \mathcal{C}_\gamma} \widetilde{\phi}^\ell(b)^p m_\gamma(\widehat{r.k.\underline{b}} \cap \gamma)^{1-p}.$$

Integrating over Γ yields

$$\int_\Gamma \Phi_{1,0}^{n-\ell(n)}(\gamma)^p d\gamma \leqslant \int_\Gamma \sum_{\underline{b}\in\mathcal{C}_\gamma} \widetilde{\phi}^\ell(b)^p m_\gamma(\widehat{r.k.\underline{b}}\cap\gamma)^{1-p} d\gamma$$
$$\leqslant \sum_{\underline{b}\in\mathcal{C}} \widetilde{\phi}^\ell(b)^p \int_\Gamma [\underline{b}\in\mathcal{C}_\gamma] m_\gamma(\widehat{r.k.\underline{b}}\cap\gamma)^{1-p} d\gamma \leqslant \tau_n \sum_{b\in\mathscr{F}} \widetilde{\phi}^\ell(b)^p.$$

Infimizing over every countable $\mathscr{F}\subset\mathscr{B}_n^k(\beta)$ that covers X one obtains:

$$\widetilde{\Phi}_{p,k}^{\ell;n-\ell(n)}(X) \geqslant \tau_n^{-1}\int_\Gamma \Phi_{1,0}^{n-\ell(n)}(\gamma)^p d\gamma. \tag{II.31}$$

By monotone convergence, if $\phi\in\mathcal{G}_m(\beta)$ then

$$\lim_{n\to+\infty}\int_\Gamma \Phi_{1,0}^{n-\ell(n)}(\gamma)^p d\gamma = \int_\Gamma \Phi_{1,0}(\gamma)^p \geqslant \int_\Gamma d\gamma > 0.$$

Since ℓ is sublinear, $n-\ell(n)$ goes to $+\infty$ as $n\to+\infty$. Especially, $\widetilde{\Phi}_p^{k,\ell;n}(X)$ is bounded below by (D(p,r)). The conclusion is reached by applying the Definition II.27 of the modulus. □

II.2.2.2. Packing variant

Proposition II.40. *Same assumptions as in Proposition II.39. Assume in addition that the quasisymmetric structure is that of a separable quasimetric space. For every $k,m\in\mathcal{O}^+(u)$, setting $\ell = q\dotplus r\dotplus k$,*

$$\mathrm{pmod}_{p;k}^{\ell,m}(\Gamma) \geqslant \frac{1}{\tau}\int_\Gamma d\gamma. \tag{II.32}$$

Proof. Fix n, pick a countable $(k\dotplus r, n)$ packing $\mathscr{P}$ of Z with the following condition: for every $\mathbf{a}\in\mathscr{P}$ write $\mathbf{a}=(a^-,a^+)$, enclosing $(\widehat{\underline{b}}, \widehat{k.r.\underline{b}})$ in $\mathbf{a}$ the $q.r.k.\underline{b}$ cover. Such packings exist by II.22. This gives a countable $\mathcal{B}\subset\beta$ (the collection of $\underline{b}$) such that the realizations of $k.r.\mathcal{B}$ are disjoint. Define $\mathcal{Q}_\gamma = \left\{\underline{b}\in\mathcal{B} : \widehat{\ell.\underline{b}}\cap\gamma\neq\emptyset\right\}$. The realization of $\ell.\mathcal{Q}_\gamma$ will cover γ if $\ell\geqslant q\dotplus r\dotplus k$ and then, by definition of the Carathéodory measure, $\Phi_{1,0}^{n-\ell(n)}(\gamma)\leqslant\sum_{b\in\mathcal{Q}_\gamma}\widetilde{\phi}^\ell(b)$. This gives an inequality equivalent to (II.29) with $\mathcal{Q}_\gamma$ instead of $\mathcal{C}_\gamma$. The rest of the proof follows the same lines as for Proposition II.39 but instead of (II.31) one obtains:

$$\tau_n \mathrm{P}\widetilde{\Phi}_{p,k}^{\ell;n-\ell(n)}(X) \geqslant \tau_n \sum_{b\in\cup_\gamma\mathcal{Q}_\gamma} \widetilde{\phi}^\ell(b)^p \geqslant \int_\Gamma \Phi_{1,0}^{n-\ell(n)}(\gamma)^p d\gamma, \tag{II.33}$$

before infimizing over every admissible gauge, which gives a lower bound on $\mathrm{pmod}_{p;k}^{\ell;0}$ and then on $\mathrm{pmod}_{p;k}^{\ell;m}$ for every m. □

II.2.3. Conformal dimensions

Definition II.41. Let (Z,β) be a $O(u)$-quasisymmetric structure, and let Γ be a family of subsets in Z. The $O(u)$-conformal dimension of β with respect to Γ is

$$\operatorname{Cdim}^{\Gamma}_{O(u)}(\beta) = \sup \Big\{ p \in \mathbf{R}_{>0} : \forall k \in \mathcal{O}^+(u),\ \exists \ell \in \mathcal{O}^+(u) \\ \forall m \in \mathcal{O}^+(u),\ \operatorname{mod}^{\ell,m}_{p;k}(\Gamma,\beta) = +\infty \Big\}$$

or 0 if this set is empty. Similarly, define

$$\operatorname{PCdim}^{\Gamma}_{O(u)}(\beta) = \sup \Big\{ p \in \mathbf{R}_{>0} : \forall k \in \mathcal{O}^+(u),\ \exists \ell \in \mathcal{O}^+(u) \\ \forall m \in \mathcal{O}^+(u),\ \operatorname{pmod}^{\ell,m}_{p;k}(\Gamma,\beta) = +\infty \Big\}$$

or 0 if this set is empty.

Remark II.42. Given that moduli decrease with respect to p, the conformal dimension $\operatorname{Cdim}^{\Gamma}_{O(u)}(\beta)$ can be bounded above by

$$\inf \Big\{ p \in \mathbf{R}_{>0} : \exists k \in \mathcal{O}^+(u), \forall \ell, m \in \mathcal{O}^+(u),\ \operatorname{mod}^{\ell,m}_{p;k}(\Gamma,\beta) < +\infty \Big\}$$

or $+\infty$ if this set is empty, and similarly, $\operatorname{PCdim}^{\Gamma}_{O(u)}(\beta)$ by

$$\inf \Big\{ p \in \mathbf{R}_{>0} : \exists k \in \mathcal{O}^+(u), \forall \ell, m \in \mathcal{O}^+(u),\ \operatorname{pmod}^{\ell,m}_{p;k}(\Gamma,\beta) < +\infty \Big\}.$$

Proposition II.43 (Conformal invariance of the conformal dimensions)**.** *Let $\varphi : (Z,\beta) \to (Z',\beta')$ be a $O(u)$-quasisymmetric homeomorphism and let Γ, resp. Γ' be a family of subsets in Z, resp. Z', such that for all $\gamma \in \Gamma$ there exists a unique $\gamma' \in \Gamma'$ such that $\varphi(\gamma) = \gamma'$. Then*

$$\operatorname{Cdim}^{\Gamma}_{O(u)}(\beta) = \operatorname{Cdim}^{\Gamma'}_{O(u)}(\beta') \tag{II.34}$$

$$\operatorname{PCdim}^{\Gamma}_{O(u)}(\beta) = \operatorname{PCdim}^{\Gamma'}_{O(u)}(\beta'). \tag{II.35}$$

Proof. One can assume $Z = Z'$, $\Gamma = \Gamma'$ and that φ is the identity map. Let us start with (II.34). By symmetry we need only prove $\operatorname{Cdim}^{\Gamma}_{O(u)}(\beta) \leqslant \operatorname{Cdim}^{\Gamma}_{O(u)}(\beta')$ and $\operatorname{PCdim}^{\Gamma;N}_{O(u)}(\beta) \leqslant \operatorname{PCdim}^{\Gamma;N}_{O(u)}(\beta')$. The conformal dimension $\operatorname{Cdim}^{\Gamma}_{O(u)}(\beta)$ can be rewritten

$$\operatorname{Cdim}^{\Gamma}_{O(u)}(\beta) = \sup \Big\{ p \in \mathbf{R}_{>0} : \exists L : \mathcal{O}^+(u) \to \mathcal{O}^+(u) \\ \forall k \in \mathcal{O}^+(u),\ \operatorname{mod}^{L(k),0}_{p,k}(\Gamma,\beta) = +\infty \Big\},$$

and then

$$\operatorname{Cdim}^{\Gamma}_{O(u)}(\beta) = \sup \Big\{ p \in \mathbf{R}_{>0} : \exists L : \mathcal{O}^+(u) \to \mathcal{O}^+(u) \\ \forall k, m \in \mathcal{O}^+(u),\ \operatorname{mod}^{L(k),m}_{p,k}(\Gamma, \beta) = +\infty \Big\}.$$

Now assume that a real number p is in the set defined on the right and let L be the corresponding map from $\mathcal{O}^+(u)$ to itself. Define $L' = \eta \circ L \circ \eta'$. By Lemma II.29, for every k and m in $\mathcal{O}^+(u)$,

$$0 < \operatorname{mod}^{L(\eta'(k)),\eta'(m)}_{p,\eta'(k)}(\Gamma, \beta) \leqslant \operatorname{mod}^{L'(k),m}_{p,k}(\Gamma, \beta').$$

and the left-hand side is infinite, thus $\operatorname{Cdim}^{\Gamma}_{O(u)}(\beta') > p$, finishing the proof. (II.35) is obtained in the same way. □

In the following, we may omit Γ in $\operatorname{Cdim}^{\Gamma}_{O(u)}$ and write $\operatorname{Cdim}_{O(u)}$; this means that Γ must be considered the family of nonconstant curves in Z. Note that homeomorphisms preserve nonconstant curves.

II.2.4. Upper bound on $\operatorname{Cdim}_{O(u)}$

Lemma II.44 (Conformal dimension is less or equal than Hausdorff dimension)**.** *Let Z be a metric space with Hausdorff dimension q. Let Γ be the family of nonconstant curves in Z. Then $\operatorname{Cdim}^{\Gamma}_{O(u)} Z \leqslant q$.*

Proof. In view of remark II.42 this will be proved if we can show that for every $\varepsilon \in (0, q)$,

$$\exists k \in \mathcal{O}^+(u), \forall \ell, m \in \mathcal{O}^+(u), \operatorname{mod}^{\ell,m}_{q+\varepsilon;k}(\Gamma) = 0. \tag{II.36}$$

For $s \in (0,1)$ consider $\phi_s \in \mathcal{G}(\beta)$ such that $\phi(\widehat{b}) = e^{-s\delta(b)}$ on concrete balls. By comparison with the Hausdorff measures (II.13), $\Phi_{1;m} \gg \mathcal{H}^1$. The nonconstant curves have positive $\mathcal{H}^1$ measure by the triangle inequality, so $\phi_s \in \mathcal{G}_m(\Gamma)$ for all s. On the other hand, again by (II.13), $(\widetilde{\Phi}_s)^{\ell}_{q+\varepsilon;k} \ll \mathcal{H}_{qs+\varepsilon s}$ for every $\varepsilon' \in (0, qs)$. For s sufficiently close to 1, $qs + \varepsilon s > q$, so (II.36) is attained. □

II.3. Applications to large-scale geometry

Here two metric spaces Y and Y' are said sublinearly biLipschitz equivalent if there exists a sublinearly biLipschitz equivalence $f : Y \to Y'$ (Definition II.8).

II.3.1. Heintze groups

II.3.1.1. Definition

Definition II.45. A connected solvable group S is a purely real Heintze group if its Lie algebra sits in a split extension

$$1 \to \mathfrak{n} \to \mathfrak{s} \to \mathfrak{a} \to 1 \tag{II.37}$$

where $\mathfrak{n}$ is the nilradical of $\mathfrak{s}$, $\dim \mathfrak{a} = 1$ and the roots associated to $\mathfrak{a} \to \mathrm{Der}(\mathfrak{n})$ are real and positive multiples of each other. In addition, we say it is of diagonalizable type if $\mathrm{ad}_{\mathfrak{a}}$ is $\mathbf{R}$-diagonalizable.

It is convenient to encode a purely real Heintze group type as a pair (N, α) where N is a nilpotent Lie group and α is a derivation of its Lie algebra with real spectrum and lowest eigenvalue 1, realizing $\mathfrak{a} \to \mathrm{Der}(\mathfrak{n})$ once an infinitesimal generator $\partial_t \in \mathfrak{a}$ has been fixed. Such an α being nonsingular, N is the derived subgroup and (N, α) is metabelian if and only if N is abelian. Every Heintze group admits left-invariant negatively curved Riemannian metrics[17] and hence is Gromov-hyperbolic.

The nilradical of a connected solvable group contains an other characteristic subgroup $\mathrm{Exprad}(S)$, defined as the set of exponentially distorted elements (which does not depend on the choice of a left-invariant proper metric) together with 1. For purely real Heintze groups both are equal[18].

Theorem II.46 (Implied by Cornulier, [35, Th 1.2])**.** *Let H be a purely real Heintze group with data (N, α). Decompose $\alpha = \sigma + \nu$ where σ is semisimple and ν is a nilpotent derivation of $\mathfrak{n}$ such that $[\sigma, \nu] = 0$. Denote by H_Σ the purely real Heintze group of diagonalizable type with data (N, σ). Then H and H_Σ are $O(\log)$-SBE.*

II.3.1.2. Punctured boundary From now on, under the auspices of Theorem II.46 we work with a purely real Heintze group of diagonalizable type S with data (N, α), that is $S = N \rtimes \mathbf{R}$ where, denoting by t the $\mathbf{R}$ coordinate, $t.x = e^{t\alpha}(x)$ for $x \in N$ and we recall that α is diagonalisable with real positive eigenvalues. It is known that this eases the computation of conformal dimension. Because of Bourdon's reformulation of the diffusivity lemma the latter is attained, indeed by an Ahlfors regular metric (whereas

[17] Though all left-invariant metrics may not be negatively curved.

[18] One reason for this is that α is nonsingular, compare Peng [133, 2.1] keeping in mind that the Cartan subgroup has rank one here.

for the twisted plane of Example II.16 it is not [9, 6]; also, one can prove elementarily that no distance has this scaling [46, 5.4]).

The vertical geodesics with tangent vector ∂_t all end at time $+\infty$ at a distinguished point ω, and at time $-\infty$ on the punctured boundary $\partial_\infty^* S$ so that we can identify the punctured boundary with N; through this identification the one-parameter subgroup generated by α is the dilation subgroup of $\partial_\infty^* S$. Note that if ρ and ρ' are any two proper left-invariant continuous real-valued kernels on $\partial_\infty^* S$ such that $\rho(\xi, \eta) = 0 \iff \xi = \eta$ and $\rho(e^{t\alpha}\xi, e^{t\alpha}\eta) = e^t \rho(\xi, \eta)$ for all t, ξ, η and similarly for ρ', then ρ and ρ' will only differ by multiplicative constants[19]. There are several ways to construct such kernels; one is the Euclid-Cygan kernel of Paulin and Hersonsky [90, appendix] which depends[20] on a negatively curved metric on S. Another one is Hamenstädt's [86, p.456] (see Dymarz-Peng for its use on boundaries of Heintze groups [55, 2]). Given the formalism developed in II.1.1 we will rather use $O(1)$-quasisymmetric structures on the punctured boundary of the form below, which may vary according to our needs.

Definition II.47. Let B be a compact subset of N containing 1_N in its interior. We say that a $O(1)$-quasisymmetric structure β^* is generated by B if $\beta^* = N \times \mathbf{Z}$ and for all $b = (x, n) \in \beta$ in this product decomposition, $\widehat{b} = xe^{-\alpha n}(B)$ (note that $\widehat{k.b} = xe^{\alpha k}x^{-1}\widehat{b}$).

We do not fix B, nevertheless the resulting structures for B, B' are equivalent since one can find $t > 0$ such that $e^{-t\alpha}(B') \subseteq B \subseteq e^{\alpha t}(B')$. We denote by β^* such a structure on $\partial_\infty^* S$.

Lemma II.48. *Let Ω be a relatively compact subset of $\partial_\infty^* S$. Let β be the quasisymmetric structure on $\partial_\infty S$ associated with a visual kernel with base-point $o \in S$ (as in Example II.5). Then $\beta_{|\Omega}$ and $\beta^*_{|\Omega}$ are equivalent.*

Proof. See Figure 17. The Euclid-Cygan kernel of $\xi, \eta \in \Omega$ with reference horosphere $\mathcal{H}$ centered at ω is, up to a bounded additive error (only depending on the hyperbolicity constant), the distance between a geodesic segment $(\xi\eta)$ and the cloud $\top\Omega \subset \mathcal{H}$ casting its geodesic shadow from ω over Ω. Now

[19] This follows from the same compactness argument which proves that all norm topologies on a finite-dimensional vector space are uniformly equivalent.

[20] This kernel was originally made for boundaries of CAT(-1) spaces and might not always be a distance in our setting, but its quasimetric constant will be bounded by $2^{\lambda/\kappa}$ for any pair of positive numbers (λ, κ) such that sect.$(g_\lambda) \leqslant -\kappa^2$, where g_λ is the 1-parameter family of metrics described by Heintze just before stating his theorem 2 [88].

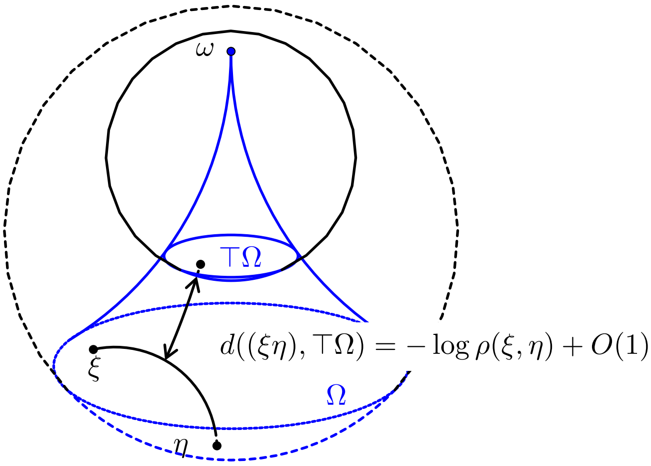


Figure 17: Quasisymmetric structures on a relatively compact open subset of the punctured boundary.

since Ω has been assumed relatively compact in $\partial_\infty^* S$, $\top\Omega$ is bounded, so that by the triangle inequality

$$(\xi,\eta)_o = d((\xi\eta), o) + O(1) = d((\xi\eta), \top\Omega) + O(1).$$

Finally, the Euclid-Cygan kernel induces the structure β^*. $\square$

Eigencurves For any nonzero eigenvector v of α, let Γ_v denote the collection of smooth curves in N everywhere tangent to the eigenspace generated by v. A curve $\gamma \in \Gamma_v$ can be parametrized by $\gamma(s) = \gamma(0)e^{sv}$, and thus Γ_v is the space of left cosets $N/\{e^{vt}\}$. The homogeneous space Γ_v has a N-invariant, α-equivariant measure ω_v [154, § 9]: for any λ and nonzero $v \in \ker(\alpha - \lambda)$, for any Borel subset A of Γ_v,

$$\omega_v(e^{\alpha t}A) = e^{\mathrm{tr}(\alpha)-\lambda}\omega_v(A). \tag{II.38}$$

II.3.1.3. Moduli of families of eigencurves and conformal dimension Let S be a purely real Heintze group of diagonalizable type with data (N,α). If Ω is an open subset of $\partial_\infty^* S$ and v is an eigenvector of α, denote by $\Gamma_v(\Omega)$ the set $\{\gamma \cap \Omega : \gamma \in \Gamma_v, \gamma \cap \Omega \neq \emptyset\}$; let us abusively denote ω_v the measure on $\Gamma_v(\Omega)$. The following Lemma corresponds to [125, 2.10 Exemple].

Lemma II.49 (Lower bound)**.** *Let $\lambda \in \mathbf{R}_{>0}$. Let $v \in \ker(\alpha - \lambda)$ be nonzero. Let W be a α-invariant subspace such that $W \oplus \mathbf{R}v = \mathfrak{n}$. Let $\beta^*_{v,W}$ be the $O(1)$-quasisymmetric structure generated by $B_0 = \{\exp P \exp sv\}_{s \in [0,1]}$, where $P \subset W$ is a compact convex subset. Let $\Omega \subset \partial^*_\infty$ be an open subset and let Ω^- be an open subset of Ω such that $\overline{\Omega^-}$ is a concrete ball of $\beta^*_{v,W}$. For every $\varepsilon > 0$, for every $k \in \mathcal{O}^+(u)$, there exists $\ell \in \mathcal{O}^+(u)$ such that for every $m \in \mathcal{O}^+(u)$,*

$$\operatorname{mod}^{\ell,m}_{(\operatorname{tr}(\alpha)/\lambda)-\varepsilon,k}(\Gamma_v(\Omega^-), \beta^*_{v,W|\Omega}) = +\infty, \tag{II.39}$$

and

$$\operatorname{pmod}^{\ell,m}_{(\operatorname{tr}(\alpha)/\lambda)-\varepsilon,k}(\Gamma_v(\Omega^-), \beta^*_{v,W|\Omega}) = +\infty. \tag{II.40}$$

Especially $\operatorname{Cdim}_{O(u)}(\beta^*_{|\Omega}) \geqslant \operatorname{tr}(\alpha)/\lambda$.

Proof. Set $p = (\operatorname{tr}(\alpha)/\lambda) - \varepsilon$. For every $\gamma \in \Gamma_v(\Omega^-)$ let m_γ be the Lebesgue measure supported on γ with total mass 1 (the existence is provided by the fact that Ω^- is relatively compact). For every $b \in \beta^*_{v,W|\Omega}$, letting $n = \delta(b)$, by (II.38), $\omega_v \left\{\gamma \in \Gamma_v(\Omega^-) : \gamma \cap \widehat{b} \neq \emptyset\right\} \leqslant \exp\{-n(\operatorname{tr}\alpha - \lambda)\}$ while for every $\gamma \in \Gamma_v(\Omega^-)$, $m_\gamma(\gamma \cap \widehat{1.b}) \geqslant \text{const.}[\gamma \cap \widehat{b} \neq \emptyset]e^{-\lambda n}$. Consequently,

$$\begin{aligned} \log \int_{\Gamma_v} m_\gamma(\gamma \cap \widehat{b})^{1-p}[\gamma \cap \widehat{1.b} \neq \emptyset] d\gamma &\leqslant -n(\operatorname{tr}\alpha - \lambda) - (1-p)\lambda n \\ &= -\varepsilon \lambda n. \end{aligned}$$

Thus (D(p,r)) is fullfilled for $r = 1$ and for every $\tau \in (0, +\infty)$; Propositions II.39 and II.40 then yield (II.39) and (II.40) respectively. The lower bound on the conformal dimension follows from the definition, viewing $\Gamma_v(\Omega^-)$ as a subcollection of the full collection of nonconstant curves in Ω. □

Proposition II.50. *Let S be a purely real Heintze group of diagonalizable type with data (N, α); assume that the lowest eigenvalue of α is 1. Let β^* denote a quasisymmetric structure as provided by Definition II.47. Let Ω be any open subset of $\partial^*_\infty S$. Then*

$$\operatorname{Cdim}_{O(u)}(\beta^*_{|\Omega}) = \operatorname{tr}(\alpha).$$

Proof. Denote by Γ the family of nonconstant curves in Ω, so that $\operatorname{Cdim}^\Gamma_{O(u)}(\beta^*_{|\Omega}) = \operatorname{Cdim}^\Gamma_{O(u)}(\beta^*_{|\Omega})$. Lemma II.49 provides one inequality, choosing v in $\ker(\alpha - 1)$. As for the reverse inequality, we need to find a gauge that confers nonzero $\Phi_{1,m}$-measure to members of Γ for all m, and then evaluate $\widetilde{\Phi}$. We will use the quasisymmetric structure generated by the exponential of a unit measure polytope in $\mathfrak{n}$ adapted to a diagonalization basis of α. Observe that for every $n \in \mathbf{Z}_{>0}$,

$$B_n \subset e^{-n}(B_0) \tag{II.41}$$

since α is diagonalizable and 1 is its lowest eigenvalue. Let d be a Riemannian left-invariant distance on N giving a diameter smaller than 1 to B_0. By (II.41) and since any left translate of $e^{-n}B_0$ has diameter $\leqslant e^{-n}$ (d being Riemannian), $\{\Phi_s\}_1^m \geqslant \mathcal{H}_d^{1/s}$. Let $q > \operatorname{tr}(\alpha)$. Then $\phi_s^{qs}(\exp(B_n)) = e^{-qn} \ll_{n \to +\infty} e^{-\operatorname{tr}(\alpha)n} = \mu(\exp(B_n))$, where μ is a Haar measure on N. Especially, μ is locally finite, so $\Phi_{qs,0} = 0$. Since $qs + s - 1 > qs$, for every $k, \ell \in \mathcal{O}^+(u)$, $\widetilde{\Phi}^{\ell}_{qs+(s-1),k} \leqslant \Phi_{qs,0}$, so that

$$\forall s > 1, \operatorname{mod}^{\ell;0}_{qs+(s-1),k} = 0 :$$

the moduli vanish in degree $> q$. Applying Definition II.11 of the conformal dimension, one obtains $\operatorname{Cdim}^{\Gamma}_{O(u)}(\beta) \leqslant qs$ for every $q > \operatorname{tr}(\alpha)$ and $s > 1$, hence $\operatorname{Cdim}^{\Gamma}_{O(u)}(\beta^*_{|\Omega}) \leqslant \operatorname{tr}(\alpha)$. □

Proposition II.51 (Generalization of I.60)**.** *Let S and S' be purely real Heintze groups, write $S = N \rtimes_\alpha \mathbf{R}$ and $S' = N \rtimes_{\alpha'} \mathbf{R}$ with normalized α and α'. If S and S' are sublinearly biLipschitz equivalent then* $\operatorname{tr}(\alpha) = \operatorname{tr}(\alpha')$.

Proof. By the previously stated theorem II.46 of Cornulier we may assume that S and S' are of diagonalizable type. If $\varphi : \partial_\infty S \to \partial_\infty S'$ is the boundary mapping of the sublinearly biLipschitz equivalence, one can also assume without loss of generality that φ preserves the focal points [40, 6D1] (this is stated for quasisymmetric mappings but the proof applies without change). Let Ω be a relatively compact subset of $\partial_\infty^* S$. Then by Lemma II.43, Theorem II.15 and Lemma II.50, letting β^* and β'^* be the quasisymmetric structures on $\partial^*_\infty S$ and $\partial_\infty S'$ respectively,

$$\begin{aligned}\operatorname{tr}(\alpha) = \operatorname{Cdim}^{\Gamma}_{O(u)}(\beta^*_{|\Omega}) &= \operatorname{Cdim}^{\Gamma'}_{O(u)}(\beta_{|\Omega}) \\ &= \operatorname{Cdim}^{\Gamma'}_{O(u)}(\beta'_{|\Omega}) = \operatorname{Cdim}^{\Gamma'}_{O(u)}(\beta'^*_{|\Omega}) = \operatorname{tr}(\alpha').\end{aligned}$$ □

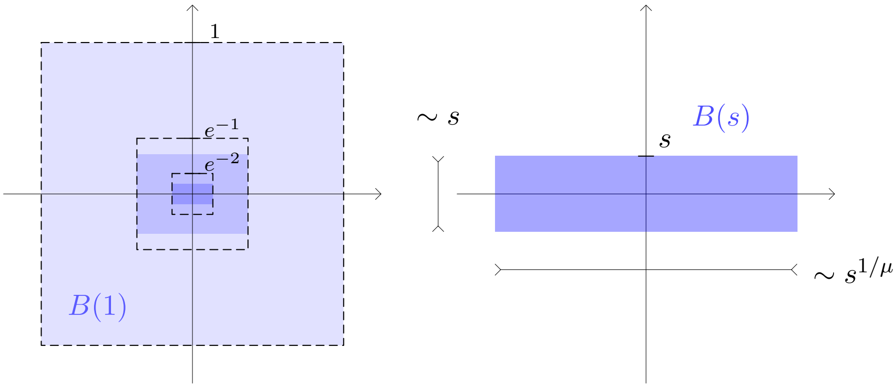


Figure 18: Concentric balls of a quasidistance on $\mathbf{R}^2$ that is invariant under translation and dilation by $\exp(t\,\mathrm{diag}(1,4/3))_{t\in\mathbf{R}}$, and coincides with the ℓ^∞ distance for pairs of points at distance 1. Compare Figure 16.

II.3.1.4. Proof of Theorem II.1

Lemma II.52 (Compare [125, 6.1] for $u=1$)**.** *Let S be a Heintze group of diagonalizable type with data (N,α). Let Ω be an open supspace of $\partial_\infty^* S$ identified with N and equipped with a quasisymmetric stucture (β,δ,q). Let $k\in\mathcal{O}^+(u)$. For every $p\in[1,+\infty)$, if $f\in\mathscr{W}^{p,k}_{\ell;\mathrm{loc.}}(\Omega)$ with $\ell\geqslant q\dotplus k$ then f is locally invariant along the left cosets of H, where*

$$\mathfrak{h}=\operatorname{Liespan}\left\{\ker(\alpha-\mu):\mu<\frac{\operatorname{tr}(\alpha)}{p}\right\}. \tag{II.42}$$

Proof. Start assuming f is in $\mathscr{W}^{p,k}_{\ell,\mathrm{loc.}}(\Omega,\mathbf{R})$ with $\ell\geqslant q\dotplus k$. Let $\mu\in(0,\operatorname{tr}(\alpha)/p)$ and let $v\in\ker(\alpha-\mu)$; up to pre-composing f with dilations and translations assume by contradiction that $f(\exp(\varepsilon v))\neq f(1)$ for arbitrarily small ε and that $1\in\Omega$. Up to post-composing f by translations and dilations of $\mathbf{R}$ one can further assume $f(1)=0$ and $f(\exp(\varepsilon v))>1$. Construct a condenser $(C,\partial_0 C,\partial_1 C)$ in Ω as follows: W is a supplementary α-invariant subspace of v in $\mathfrak{n}$, F is a Borel subset of $\exp W$, $C=\{we^{sv}:s\in(0,\varepsilon),w\in F\}$ and $\partial_i C=\{we^{i\varepsilon v}\}$. By Lemma II.38, for every $\ell\in\mathcal{O}^+(u)$, $\operatorname{pmod}^{\ell,0}_{p;k}(\Gamma)\leqslant\operatorname{cap}^{\ell}_{p;k}(C)$, where Γ is the family of curves between $\partial_0 C$ and $\partial_1 C$, which includes Γ_v. By Lemma II.49, $\operatorname{pmod}^{\ell,0}_{p;k}(\Gamma_v)=+\infty$ if $\ell\geqslant q\dotplus k$, and then $\mathbf{E}^{\ell}_{p;k}(f)(C)=+\infty$, a contradiction. So f was indeed $\langle v\rangle$-invariant, and then locally invariant on the left cosets of H. Finally, allow f to take complex values. Note that f is in $\mathscr{W}^{p,k}_{\ell;\mathrm{loc.}}(\Omega)$ if and only if $\Re f,\Im f\in\mathscr{W}^{p,k}_{\ell;\mathrm{loc.}}(\Omega,\mathbf{R})$ as $\operatorname{osc}(\Re f,a)^p\vee\operatorname{osc}(\Im f,a)^p\leqslant\operatorname{osc}(f,a)^p\leqslant 2^p\sup\{\operatorname{osc}(\Re f,a)^p,\operatorname{osc}(\Im f,a)^p\}$ for every $a\in\mathcal{P}(\Omega)$, which brings the argument back to the previous case. $\square$

We assume from now on that N is abelian, identify it (as well as $\mathfrak{n}$) with $\mathbf{R}^d$ and decompose $\mathbf{R}^d = \bigoplus_{i=1}^r \ker(\alpha - \mu_i) = \bigoplus_{i=1}^r \langle e_i^1 \ldots e_i^{d_i} \rangle$. Let $f_i^j \in (\mathbf{R}^d)^\vee$ denote the dual basis of linear forms.

Lemma II.53. *Let β be the quasisymmetric structure on $\mathbf{R}^d$ generated by $B = [-1/2, 1/2]^d$. For all $i \in \{1, \ldots r\}$, for all $j \in \{1, \ldots d_i\}$, for all $k, \ell \in \mathcal{O}^+(u)$, $f_i^j \in \mathscr{W}^{p,k}_{\ell;\mathrm{loc.}}(\beta, \mathbf{R})$ for $p > \operatorname{tr}(\alpha)/\mu_i$.*

Proof. Let ν be a Haar measure on N, normalized so that $\nu(B) = 1$. Set $p = (1+\epsilon)\operatorname{tr}(\alpha)/\mu_i$ with $\epsilon > 0$. We need prove that $\mathbf{E}^\ell_{p;k}(f_i^j)$ is locally finite for every ϵ and $\ell \in \mathcal{O}^+(u)$. We may as well prove that $\mathbf{E}^\ell_{p;k}(f_i^j)(B) < +\infty$. Let $n \in \mathbf{Z}_{\geqslant 0}$. Recall that by definition $\mathbf{E}^\ell_{p;k}(f_i^j)(B)$ is $\mathrm{P}\,\Phi^\ell_{p;k}(B)$ for $\phi(b) = \operatorname{osc}(f_{|b})$, so that $\phi(e^{-\alpha n} B)^p = (e^{-\mu_i n})^p = e^{-\operatorname{tr}(\alpha)(1+\epsilon)n}$ and ϕ increases with respect to inclusion. If $\mathscr{P} \in \mathrm{Packings}_{k,n}(B)$, enclose into each (a^-, a^+) of $\mathscr{P}$ a pair $(\widehat{\underline{b}}, \widehat{k.\underline{b}})$ and note that the $\widehat{\underline{b}}$ are disjoint; for n large enough they are also contained in $[-1,1]^d$ (since the $\widehat{\underline{b}}$ all intersect B) so

$$\sum_{\mathbf{a} \in \mathscr{P}} \nu(\widehat{\underline{b}}) = \nu\left(\bigcup_{\mathbf{a} \in \mathscr{P}} \widehat{k.\underline{b}} \right) \leqslant \nu([-1,1]^d) = 2^d.$$

From there, and using that $\nu(\widehat{b}) = e^{-\operatorname{tr}(\alpha)\delta(b)}$ for every $b \in \beta$, and that ℓ is sublinear, for n large enough

$$\sum_{\mathbf{a} \in \mathscr{P}} \widetilde{\phi}^\ell(a^-)^p \leqslant e^{p\ell(n)} \sum_{\mathbf{a} \in \mathscr{P}} \phi(\widehat{\underline{b}})^p \leqslant \sum_{\mathbf{a} \in \mathscr{P}} \nu(\widehat{\underline{b}}) \leqslant 2^d. \tag{II.43}$$

This is a uniform bound for all packings so $\mathbf{E}^\ell_{p;k}(f_i^j)(B) < +\infty$. $\square$

Remark II.54. Actually, the p-energy of coordinates (or even Lipschitz) functions in the corresponding directions is zero, as can be obtained by replacing ν with $\mathcal{H}^d$ with d slightly greater than $\operatorname{tr}(\alpha)$ in the previous proof. To get functions with nonzero yet finite energy one should form linear combinations of the examples constructed in II.1.2.3 composed with coordinates.

Remark II.55. The lower bound on energies obtained in the proof of Lemma II.52, resp. the upper bound given by Lemma II.53 can be compared to Xie's [160, Lemma 4.2] resp. [160, Lemma 4.5]. Xie's technique for the lower bound is essentially different.

Let S and S' be two purely real Heintze groups and let $\varphi : \partial_\infty^* S \to \partial_\infty^* S'$ be the extension of a sublinearly biLipschitz equivalence $f : S \to S'$ preserving the focal points; equip $\partial_\infty^* S$ with its abelian Lie group structure and split it into $E_1 = \operatorname{span}_{\mu<\operatorname{tr}(\alpha)/p}\{\ker(\alpha-\mu)\}$ and a complementary subspace E_2, and similarly decompose $\partial_\infty^* S' = E_1' \oplus E_2'$. For $z \in \partial_\infty^* S$, denote by z_1 and z_2 the projections onto E_1 and E_2. Write $\varphi(z_1,z_2) = (\varphi_1(z_1,z_2), \varphi_2(z_1,z_2))$ where $\varphi_i : E_1 \times E_2 \to E_i'$ for $i \in \{1,2\}$. For every $(z_1,z_2) \in \partial_\infty^* S$, introduce

$$\mathcal{C}(z) = \{y_1 \in E_1 : \varphi_2(y_1,z_2) = \varphi_2(z_1,z_2)\}$$

and note that $\mathcal{C}(z)$ is nonempty (as it contains $\{z_1\}$) and closed.

Lemma II.56. *For all $z \in \partial_\infty^* S$, $\mathcal{C}(z)$ (as defined above) is open in E_1.*

Proof. As $\mathcal{C}(z) = \mathcal{C}(y_1,z_2)$ for every $y_1 \in \mathcal{C}(z)$, it suffices to prove that $\mathcal{C}(z)$ is a neighborhood of z_1. Let Ω be a relatively compact open set containing z. Denote $\Omega' = \varphi(\Omega)$. Denote by β and β' respectively the quasisymmetric structures on Ω and Ω' constructed from a Gromov kernel based at $1 \in S, S'$ and denote by β^* and β'^* quasisymmetric structures on Ω and Ω' associated with Definition II.47. Since Ω and Ω' have been assumed relatively compact, β and β^* are equivalent by Lemma II.48 and there is a sequence of $O(u)$-quasisymmetric homeomorphisms

$$(\Omega,\beta^*) \xrightarrow{\mathrm{id}} (\Omega,\beta) \xrightarrow{\varphi} (\Omega',\beta') \xrightarrow{\mathrm{id}} (\Omega',\beta'^*)$$

Let $\eta, \eta', \overline{\eta}, \overline{\eta}'$ be associated to the $O(u)$-quasisymmetric homeomorphism $\varphi^{-1} : (\Omega', \beta'^*) \to (\Omega, \beta^*)$ as in II.1.2. Introduce the following sets: $F = (z+E_1)\cap\Omega$, $F' = (\varphi(z)+E_1')\cap\Omega'$, and let F_0, resp. F_0' be the connected component of F, resp. F' containing z, resp. $\varphi(z)$. F is defined inside Ω by the vanishing of coordinate functions $g_1, \ldots g_s$ with $s = \dim E_2$. Denote $g_1', \ldots g_s'$ with $g_i = \varphi_* g_i$; $\varphi(F)$ is defined in Ω' by the vanishing of $g_1', \ldots g_s'$. Let q be such that axiom (SC2) holds for β^*. Fix $k, \ell \in \mathcal{O}^+(u)$ such that $\ell \geqslant q \dotplus \overline{\eta}' \circ \eta(k)$. Using the second embedding in the sequence (II.23) applied to φ^{-1}, and the fact that $g_i \in \mathscr{W}^{p;\overline{\eta}(k)}_{\eta'(\ell)}$ for all $i \in \{1,\ldots,s\}$, one has that $g_i' \in \mathscr{W}^{p;\overline{\eta}'\circ\overline{\eta}(k)}_{\ell}(\Omega,\beta_*)$ for all $i \in \{1,\ldots,s\}$. By Lemma II.52, g_i' is locally constant on F', hence zero on its connected component containing $\varphi(z)$. This proves that $\varphi(F_0) \subseteq F_0'$ and the lemma as F_0 is open in $z+E_1$. □

By connectedness of E_1, Lemma II.56 implies that $\mathcal{C}(z) = E_1$ for all z, φ_2 only depends on the second coordinate z_2 and the foliation of $\partial_\infty^* S$ by subspaces parallel to E_1 is preserved. As φ_2 is necessarily injective, $s = \dim E_2 \leqslant \dim E_2'$. By symmetry, $\dim E_2 = \dim E_2'$. From there one deduces that

$$\forall p \in [1,+\infty), \sum_{\mu \geqslant \operatorname{tr}(\alpha)/p} \dim\ker(\alpha-\mu) = \sum_{\mu \geqslant \operatorname{tr}(\alpha')/p} \dim\ker(\alpha'-\mu) \quad \text{(II.44)}$$

which implies that α and α' have the same characteristic polynomial. Since they have been assumed diagonalizable with all eigenvalues real and greater or equal than 1, they are conjugated and the groups S, S' are isomorphic.

II.3.1.5. Comparisons There are other algebras on the boundary of hyperbolic spaces, the extensions (modulo $\mathbf{R}$) of representatives of $\ell^p H^1(X)$ to $\partial_\infty X$. Bourdon and Kleiner have studied the corresponding equivalence relations, called the ℓ^p-equivalence relations see e.g. [20, 10]. For Heintze groups of diagonalizable type, comparing our result with that provided by Carrasco Piaggio [29], the ℓ^p-equivalence relations coincides with those we obtain for $\mathscr{W}^{p,k}_{\ell;\mathrm{loc.}}$ algebras for adequate k and ℓ, except perhaps at the critical degrees.

II.3.2. Fuchsian buildings

The point here is to show that $\mathrm{Cdim}_{O(u)}$ equals Cdim in this case, following Bourdon's proof; we provide a few details of this proof.

II.3.2.1. Fuchsian buildings We recall below a definition according to Bourdon [16, 2]. Let $r \geqslant 3$ be an integer, let R be a polygon in $\mathbb{H}^2$ with r vertices labeled by $\mathbf{Z}/r\mathbf{Z}$ and angles π/m_i where $m_i \geqslant 2$ for every $i \in \mathbf{Z}/r\mathbf{Z}$. R is the fundamental domain for a cocompact Fuchsian representation of the Coxeter group

$$\mathrm{W} = \langle s_i \mid s_i^2, (s_i s_{i+1})^{m_i} \rangle,$$

where $\langle s_i \rangle$ stabilizes the edge between vertices i and $i+1$. For every $i \in \mathbf{Z}/r\mathbf{Z}$, let $q_i \geqslant 2$ be an integer. Let $\mathbf{m}, \mathbf{q} : \mathbf{Z}/r\mathbf{Z} \to \mathbf{Z}_{\geqslant 0}$ be the corresponding data. A cell 2-complex Δ is the geometric realization of a Fuchsian building (we will not distinguish between them) if

(FB1) Each 2-cell is isomorphic to the labelled R, and each 1-cell with label i lies in exactly $(1 + q_i)$ 2-cells, those are called chambers.

(FB2) Each pair of distinct 2-chambers is contained in a subcomplex isomorphic (as a labelled cell complex) to the Coxeter complex of $(\mathrm{W}, \{s_i\})$, those are called apartments.

(FB3) Given two apartments A and A' with at least one common 2-cell C, the identity map of C extends to an isomorphism of labelled complexes $A \to A'$.

The Bourdon buildings are those for which $m = 2$ (they are called right-angled) and q_i are constants. A building of such type always exists provided $p \geqslant 5$, and is uniquely defined[21]; it is usually denoted by I_{pq}, where the thickness q designates the constant[22] $q_i + 1$ and p designates r. Once the chambers are equipped with the hyperbolic metric, Fuchsian building are CAT(-1) spaces in view of the description of their links and Ballmann's criterion, we refer to [16] and reference therein for these facts as well as many examples.

Weighted combinatorial distance Starting from a Fuchsian building Δ one can associate to it a dual graph $\mathscr{G}(\Delta)$ whose vertices are the chambers of Δ, edges record adjacency, and they are assigned length $\log q$ for edges of type q. Choosing any embedding of the Cayley graph of W with respect to the $\{s_i\}$ as a subgraph of $\mathscr{G}(\Delta)$ yields a distance on W; for $w \in$ W, $|w|_{\mathbf{q}}$ denotes the length of w for this distance. The growth rate of W with respect to $\mathbf{q}$ is $\mathscr{T} := \limsup_n \frac{1}{n} \log \sharp \{w \in \mathrm{W} : |w|_{\mathbf{q}} \leqslant n\}$; this can be made more explicit [16, 3.1.1] (for the Bourdon building the growth rate with no weight is $\operatorname{argch}((p-2)/2)$) so that $\mathscr{T} = \operatorname{argch}((p-2)/2)/\log(q-1)$ for I_{pq}). The distance between two chambers d, d' in Δ is denoted by $|d - d'|_{\mathbf{q}}$, this is $|w|_{\mathbf{q}}$ for w such that $d = w.d'$ in any common apartment. The distance $|\cdot - \cdot|_{\mathbf{q}}$ on $\mathscr{G}(\Delta)$ is quasiisometric to the CAT(-1) metric on Δ, especially it is Gromov-hyperbolic.

Measure on marked apartments Given a chamber c in Δ, let $\mathcal{F}_c$ denote the space of embeddings of the Coxeter complex marked at c into Δ. There is a unique probability measure ν on $\mathcal{F}_c$ such that for any chamber d, $\nu[\pi \in \mathcal{F}_c : \pi \ni d] = e^{-|d-c|_{\mathbf{q}}}$ [16, 2.2.4].

Geodesic metric on the boundary The Gromov product on $\partial_\infty\Delta$ associated to $|\cdot|_{\mathbf{q}}$ is denoted by $(\xi, \eta) \mapsto \{\xi, \eta\}_c$. For ξ, η in $\partial_\infty\Delta$, $\varrho(\xi, \eta) = \exp(-\mathscr{T}\{\xi, \eta\}_c)$ and then $\delta(\xi, \eta) = \inf \sum \varrho(\xi_i, \xi_{i+1})$ over chains $\xi = \xi_0 \ldots \xi_s = \eta$ in $\partial_\infty\Delta$. Bourdon proves that δ and ϱ are comparable (this is the most involved part of the proof; the details for this point are given in [15, p.362]), and that $\mathrm{Hdim}(\partial_\infty\Delta)$ equals $1 + 1/\mathscr{T}$ [16, 2.2.7]. Once this is

[21] In general a building of type $(r, \mathbf{m}, \mathbf{q})$ may or may not exist, and may or may not be unique up to isomorphism of labelled complexes.

[22] The shift between q and q_i is here to conform with the building of $\mathrm{SL}(3, \mathbf{Q}_\ell)$ where links are projective planes over the residue field so that edges are incident to $1 + \ell$ cells.

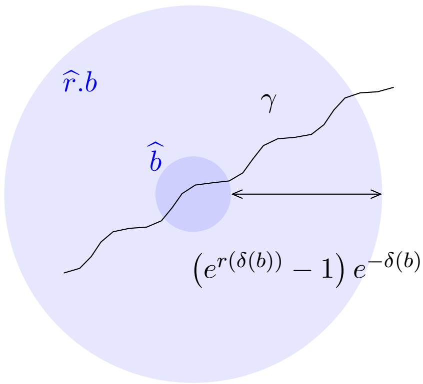


Figure 19: Inequality (II.45).

proven, δ induces the same quasisymmetric structure on the boundary, and by Lemma II.44, $\operatorname{Cdim}_{O(u)} \partial_\infty \Delta \leqslant 1 + 1/\mathscr{T}$.

II.3.2.2. Diffusivity condition and lower bound

Lemma II.57 (After Bourdon [16, 2.2.2])**.** *Let (Z, d) be an Ahlfors-regular metric space. Let β be the associated quasisymmetric structure. Let Γ be a family of rectifiable curves in Z whose lengths are nonzero and bounded above by a uniform constant. Let $d\gamma$ be a measure on Γ. Let p' be greater than 1. If there exists $\eta < +\infty$ such that*

$$\forall b \in \beta,\ \log \int_\Gamma [\gamma \cap \widehat{b} \neq \emptyset] d\gamma - (1-p')\delta(b) \leqslant \eta, \tag{D$'(p')$}$$

then $\operatorname{Cdim}^\Gamma_{O(u)}(\beta) \geqslant p'$.

Let us check that (D$'(p')$) implies (D(p,r)) provided $p > p'$ and $r \in \mathcal{O}^+(u)$ is nonzero. Since $\gamma \in \Gamma$ has been assumed rectifiable, they bear normalized arclength measures m_γ of total mass 1.

By the reverse triangle inequality, for every $\gamma \in \Gamma$ (see Figure 19),

$$m_\gamma(\gamma \cap \widehat{r.b}) \geqslant [\gamma \nsubseteq \widehat{r.b}][\gamma \cap \widehat{b} \neq \emptyset] \left(e^{r(\delta(b))} - 1\right) e^{-\delta(b)} \operatorname{length}(\gamma)^{-1}, \tag{II.45}$$

hence if $\delta(b)$ is large enough to ensure that $\gamma \nsubseteq \widehat{r.b}$, one has:

$$\begin{aligned} m_\gamma \left(\gamma \cap \widehat{r.b}\right)^{1-p} [\gamma \cap \widehat{b} \neq \emptyset] &\leqslant \left(e^{r(\delta(b))} - 1\right)^{1-p} e^{(p-1)\delta(b)} \operatorname{length}(\gamma)^{p-1} \\ &\leqslant C\,(1 - 1/e)^{1-p} \exp\left((p-1)(\delta(b) - r(\delta(b)))\right) \end{aligned}$$

where $C = \sup_{\gamma \in \Gamma} \operatorname{length}(\gamma)^{p-1}$ is finite by hypothesis. Now, using ($\mathrm{D}'(p')$) with $p' < p$,

$$\int_\Gamma m_\gamma(\gamma \cap \widehat{r.b})^{1-p}[\gamma \cap \widehat{b} \neq \emptyset] d\gamma \leqslant C\, e^\eta e^{(p'-p)\delta(b)+r(\delta(b))}.$$

The right-hand side goes to 0 because r is sublinear, so ($\mathrm{D}(p,r)$) holds for every $\tau \in \mathbf{R}_{>0}$.

Going back to Fuchsian buildings it remains to specify Γ, $d\gamma$ and p'. Following Bourdon, given a reference chamber in Δ, Γ is the collection of boundaries of apartments containing the reference chamber c :

$$\Gamma = \{\partial_\infty \mathrm{im}(\pi) : \pi \in \mathcal{F}_c\},$$

$d\gamma$ is the measure on Γ corresponding to ν on $\mathcal{F}_c$. The fact that the $\gamma \in \Gamma$ are rectifiable follows from [16, 2.2.6(ii)]. The condition ($\mathrm{D}'(p')$) for $p' = 1+1/\mathscr{T}$ is checked by Bourdon [16, 2.3.8]. By Lemma II.39, $\mathrm{Cdim}_{O(u)}(\partial_\infty \Delta) > 1 + 1/\mathscr{T} - \varepsilon$ for every positive real ε arbitrarily small. This finishes the proof that $\mathrm{Cdim}_{O(u)}\, \partial_\infty \Delta = 1 + 1/\mathscr{T}$.

II.4. Sublinear hyperbolicity?

One of the strengths of Gromov hyperbolicity is that it is a coarse notion, in the sense that if two geodesic metric spaces Y and Y' are quasiisometric, then Y is hyperbolic if and only if Y' is. This can be proved with the Morse lemma. For general (non-homogeneous) geodesic metric spaces, however, Gromov hyperbolicity is not preserved by Sublinearly biLipschitz Equivalence. Nevertheless, the Morse constant bounding the Hausdorff distance between a quasigeodesic segment and a geodesic segment with the same endpoints depends linearly on the additive error of the quasigeodesic (see Chapter I). Using this observation one can prove that if Y and Y' are geodesic metric spaces, then the following replacement of hyperbolicity is preserved by $O(u)$-Sublinearly biLipschitz Equivalence.

Definition II.58 (Tentative definition)**.** Y is $O(u)$-sublinearly hyperbolic if for every $o \in Y$ there exists $\delta = O(u)$ such that each geodesic triangle contained in $B(o,r)$ is $\delta(r)$-slim.

We have not been able to generalize Theorem I.2 to such spaces. In an attempt to identify some of their properties, we discuss examples of Hadamard manifolds satisfying the definition.

II.4.1. Decay of negative curvature and the visibility property

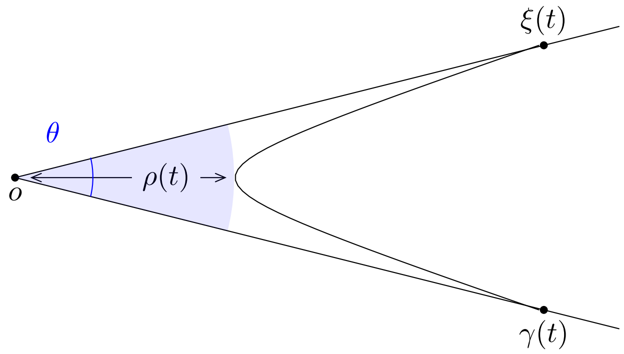


Figure 20: Two radial geodesics in the plane with metric $dr^2 + \psi(r)^2 d\theta^2$. As soon as $\psi'(t) \to +\infty$, $\sup_t \rho(t) < +\infty$ and the geodesic between $\xi(t)$ and $\gamma(t)$ comes back to a bounded neighborhood of the origin o.

Proposition II.59. *Let $\psi : \mathbf{R}_{>0} \to \mathbf{R}_{>0}$ be a C^2, convex function such that $\psi(s) \sim s$ for $s \to 0$ and consider the rotation invariant metric on the plane given in polar coordinates as*

$$ds^2 = dr^2 + \psi(r)^2 d\theta^2. \tag{II.46}$$

Then the associated metric space (X, o) is a Hadamard space; moreover

1. *Let ξ and γ be two radial geodesic rays. Set $\theta = \angle_o([\xi], [\eta])$. Then*

$$\limsup_{t \to +\infty} (\xi(t) \mid \gamma(t)) \leqslant \inf \left\{ s \in \mathbf{R}_{>0} : \psi'(s) > \frac{\pi}{\theta} - 1 \right\}. \tag{II.47}$$

2. *If ψ' is unbounded, then*

   a. *X is a visibility space.*

   b. *The orthogonal projection of $\xi(t)$ on γ stays in a bounded neighborhood of o (with explicit diameter) as $t \to +\infty$.*

Proof. The Gaussian curvature can be computed as $K(r) = -\psi''(r)/\psi(r)$. This is nonpositive since ψ has been assumed convex, especially X is a Hadamard space.

1. For any $t \in \mathbf{R}_{>0}$ define $\rho(t)$ as the distance from o to the geodesic segment $[\gamma(t)\xi(t)]$. By the Gauß-Bonnet formula applied to the Riemannian geodesic triangle $\triangle = o\gamma(t)\xi(t)$,

$$\theta\psi'(\rho(t)) = -\theta \int_0^{\rho(t)} \psi(s)K(s)ds \leqslant -\int_\triangle K d\sigma \leqslant \pi - \theta. \qquad \text{(II.48)}$$

   Inserting the orthogonal projection of o on $[\xi(t)\gamma(t)]$ in the Gromov product yields estimate (II.47).

2. a. Since ψ' is nondecreasing and unbounded by assumption, by (II.48) ρ is bounded.

   b. The metric (II.46) is invariant under reflection σ with invariant line γ. Apply the preceding discussion to ξ and $\gamma' = \sigma \circ \xi$. □

Remark II.60. An equivalent, better formulation of the condition that ψ' be unbounded is that the integral curvature of the geodesic sector spanned by ξ and γ be $-\infty$, compare Ballman, Gromov and Schroeder [6, Exercise (i) p.57]. Rotation invariance has no special importance and is here only to simplify the statement and proof.

We are especially interested in functions $\psi(r)$ such that for every $\alpha \in \mathbf{R}_{>1}$ and $\beta \in \mathbf{R}_{>0}$,

$$r^\alpha \ll \psi(r) \ll \exp(\beta r) \qquad \text{(II.49)}$$

as $r \to +\infty$ (the comparison on the left combined with convexity implies that ψ' is unbounded, the main condition for Lemma II.59 to apply). The two limiting cases in (II.49) correspond to curvature $K(r) = \alpha(1-\alpha)/r^2$ (quadratic decay) and $-\beta^2$ (constant curvature). Here are some examples; we may only define ψ on a neighborhood of $+\infty$ since we are interested in the behavior of $\sup \rho$ for $\theta \to 0$, and this goes to $+\infty$ as the inequality $\rho(t) \geqslant t - d(\gamma(t), \xi(t)) \geqslant t - \theta\psi(t)$ shows.

$\psi(r)$	$K(r)$	$\sqrt{-K(r)}$	Observations
$\sinh r$	-1	1	Hyperbolic plane
$\exp(r/\log r)$	$\sim -(\log r)^{-2}$	$\log r$	$1 \ll \delta(r) \ll r$
$\exp(r^{1-e})$, $e \in (0,1)$	$\sim -(1-e)^2 r^{-2e}$	$\asymp r^e$	$1 \ll \delta(r) \ll r$
r^α, $\alpha > 1$	$\alpha(1-\alpha)r^{-2}$	$\asymp r$	
r	0	∞	Euclidean plane

Table II.1: Flat, sublinearly hyperbolic, and hyperbolic metrics on the plane.

In both cases, the geodesically convex ball $B(o,r)$ is a CAT$(K(r))$-space, hence a $v(r)$-hyperbolic space with $v(r) = \frac{1}{\sqrt{-K(r)}}$. This is $O(\log r)$ in the first case, $O(r^e)$ in the second case.

Remark II.61. As a variant of (II.46) consider $\mathbf{R}^2$ with the Riemannian metric

$$ds^2 = dy^2 + \varphi(y)^2 dx^2,$$

where $\varphi : \mathbf{R} \to \mathbf{R}$ is a C^2 convex function such that $\varphi(y) \sim \exp y$ for $y \to -\infty$. Under condition (II.49) where $\psi(r)$ is replaced with $\varphi(y)$ as $y \to +\infty$, the analogue of Proposition II.59 holds, replacing radial geodesics with vertical ones. The difference with (II.46) is that, compared to a hyperbolic plane the curvature decay takes place out of a horodisk rather than a disk.

II.4.2. Comparison and comments

II.4.2.1. Riemannian examples As long as the curvature decay lies strictly under the threshold $K(r) \asymp -1/r^2$, it translates into a sublinear growth of the hyperbolicity constant and one might expect some of the features of the sphere at infinity of hyperbolic spaces to remain valid. In the limiting case $\psi(r) = r^\alpha$, $\alpha > 1$, the slimness of the triangle $\xi(t)\gamma(t)\zeta(t)$ occurs when o is in the interior of the convex hull of the directions of rays. However triangles lying entirely on one side of a half space through o are not slim; their δ may be comparable to their diameter. Beyond the threshold, the Riemannian examples become scarce. Especially

- By a theorem of Greene and Wu, if X has odd dimension, is nonpositively curved and has $\limsup \sup_{\pi(P) \in B(0,r)} |K(P)| = o(r^{-2})$, then X is actually flat [77, Theorem 2].

- Back to surfaces of negative curvature, by the Hong immersion theorem, lying beyond the threshold (under smoothness assumption and in a slightly strengthened way involving derivatives of the curvature) is a sufficient condition to admit a smooth immersion in Euclidean 3-space [92, Theorem A].

II.4.2.2. Random metric graphs In this speculative paragraph we collect two examples of random metric graphs exhibiting a weak hyperbolic behavior, and attempt a comparison. The first is obtained by Benjamini

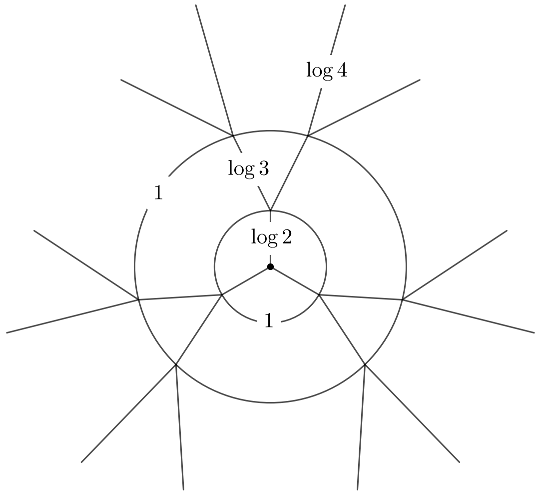


Figure 21: A metric graph (The underlying tree is that of figure 5, but we do not mean to relate them beyond this. The resulting graph is one-ended).

and Tessera after first passage percolation on a hyperbolic graph, the second one by Budzinski after turning certain Galton-Watson trees into planar maps.

Theorem II.62 (Benjamini-Tessera , [8, Corollary 1.5])**.** *Let ℓ be a positive random variable with finite first moment and $\mathbb{P}(\ell = 0) = 0$. Let X be a hyperbolic graph with bounded valency. Assume that X has a bi-infinite geodesic. Change the metric on X by setting the length of each edge as a realization of a i.i.d copy of ℓ. Then almost surely, the resulting random graph $\widetilde{X}$ has a bi-infinite geodesic. Moreover, if $q : \mathbf{Z} \to X$ is any bi-infinite quasigeodesic, then almost surely there exists a finite subset $A \subset \widetilde{X}$ such that for all $n \in \mathbf{Z}$ every quasigeodesic in $\widetilde{X}$ between $q(n)$ and $q(-n)$ goes through A.*

The process is called first-passage percolation. Benjamini and Tessera's result is actually more general as it applies with graphs with a Morse quasigeodesic (see [8]). The concatenation of the two rays ξ and η on figure 20 furnishes an example of a quasigeodesic.

Question II.63. *Are metric graphs obtained after first passage percolation on a hyperbolic graph (possibly with the restriction that certain exponential moments of ℓ be finite) almost surely sublinearly hyperbolic?*

The last result we would like to bring here comes from a sample of properties identified by Budzinski on certain random causal maps [26]. Contrarily to the previous example, the construction of these spaces does not require a pre-existing Gromov-hyperbolic space.

Theorem II.64 (Budzinski, [26, Theorem 1, 1])**.** *Let T be a rooted Galton-Watson tree conditioned to survive in a supercritical regime (where this happens with positive probability). Turn T into a planar map by embedding it into the plane and connecting metric spheres centered at the root keeping planarity. Then almost surely, there exists a constant k such that for any geodesic triangle $\triangle$ surrounding the root vertex $d(o, \triangle) \leqslant k$.*

The conclusion is checked by all the non-Euclidean planes on table II.1, including when $\psi(r) = r^{\alpha}$; Budzinski calls it weak anchored hyperbolicity ([25, Definition 0.1], [26, Definition 1]).

The distinctive phenomenon is that after two geodesic rays start spreading apart significantly, a tree will spring between them, survive and grow fast with high probability so that a geodesic segment between them can only cross it near the root (though the actual argument requires considerably more elaborate quantification). Note that there are almost surely arbitrarily large portions of the square lattice in the graph (especially it is not Gromov hyperbolic), but they are infrequent, one must go far from the origin to observe them.

It is unclear to us whether Budzinski's random graphs (or maps[23]) are almost surely sublinearly hyperbolic or almost surely not. A first attempt (suggested to us by Budzinski) in this direction would be to prove this for graphs built from deterministic trees, e.g. the one on figure 21. For the latter we expect that the hyperbolicity constant in concentric balls of radius r should be $O(\log r \cdot \log \log r)$.

[23] See [26, Theorem 1bis].

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universite PARIS-SACLAY

École doctorale de mathématiques Hadamard (EDMH)

Titre : Géométrie asymptotique sous-linéaire : hyperbolicité, autosimilarité, invariants

Mots Clefs : Géométrie à grande échelle, groupes hyperboliques, homéomorphismes quasisymétriques

Résumé : Les équivalences sous-linéairement bilipschitziennes ont été introduites par Yves Cornulier afin de décrire les cônes asymptotiques des groupes de Lie. Elles généralisent les quasiisométries. Cette thèse construit des invariants pour l'équivalence sous-linéairement bilipschitzienne entre groupes et espaces hyperboliques au sens de Gromov, en utilisant l'analyse au bord de Gromov. Une classe d'application généralisant les homéomorphismes quasisymétriques, et une dimension conforme associée, sont introduites. Les espaces symétriques riemanniens de type non-compact et de rang un, ainsi que certains espaces homogènes de courbure strictement négative, sont classifiés à équivalence sous-linéairement bilipschitzienne près.

Title : Large-scale sublinear geometry: hyperbolicity, self-similarity, invariants

Keys words : Large-scale geometry, Hyperbolic groups, Quasisymmetric mappings

Abstract : Sublinearly biLipschitz equivalences have been introduced by Yves Cornulier as a means of describing the asymptotic cones of Lie groups; they include and generalize quasiisometries. This thesis provides invariants for sublinearly biLipschitz equivalence between Gromov-hyperbolic groups and spaces using analysis on the Gromov boundary. A class of mapping generalizing quasisymmetric mappings, and a corresponding conformal dimension, are introduced as tools. The Riemannian symmetric spaces of noncompact type as well as a subclass of homogeneous negatively curved Riemannian manifolds are classified up to sublinearly biLipschitz equivalence.