Chapter 3: Basic of complex analysis for functions in one variable

Refresher Courses in Analysis

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Complex differentiable and holomorphic functions

Definition 1

Let $U \subset \mathbb{C}$ be open and $z_0 \in U$.

(i) $f : U \to \mathbb{C}$ is called **complex differentiable** at z_0 if

(ii) f is called **holomorphic** on U if it is complex differentiable at all $z_0 \in U$.

(iii) $\mathrm{Hol}(U) := \{f : U \to \mathbb{C} \mid f \text{ holomorphic on } U\}$ denotes the space of holomorphic functions on U.

Note: that the existence of the above limit is a much stronger cond. that in the real case. In other words:

Examples

1) (polynomials)

   Fact (Lionville) Every bounded entire function (i.e. $f \in \mathrm{Hol}(\mathbb{C})$) is constant.

2) (power series)

   $\Rightarrow$ Power series (and polynomials) are implicitly often complex differentiable.
   $\Rightarrow$ Every hol. fct is locally given by a power series, i.e. it is complex analytic.

3)

Examples

4) Let $f, g \in \mathrm{Hol}(U)$, $U \subset \mathbb{C}$ open.

5) Let $U, V \subset \mathbb{C}$ open, $f \in \mathrm{Hol}(V)$ and $g : U \to V \subset \mathbb{C}$.

Cauchy-Riemann DE

Let $U \subset \mathbb{C}$ open and $f : U \to \mathbb{C}$ be a function.

Aim: Compare the notion of being hol. with differentiability of f as of 2-real variables.

Proposition 1

The following conds. are equivalent:

(i) f is hol. on U.

(ii) f is "real differentiable" on U and $\forall z \in U$, the real linear map $df(z) : \mathbb{C} \to \mathbb{C}$ is given by multipilcation with a complex nb $a_z \in \mathbb{C}$, i.e. $df(z)$ is complex linear.

(iii) f is real diff. and $\forall z \in U$

Remarks

(i) The C.-R. eq. are often often decomposed into real and imaginary part:

By writing $df(z)$ as a 2×2 matrix, we have

$(**)$ corresponds to the fact that a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is complex linear $\iff$ $a = d$ and $b = -c$.

Remarks

(ii) We can introduce a new complex basis of partial derivatives

Then $(*)$ is equivalent to

Moreover, if f is hol., then

So, the notation is compatible with the notation

Remarks

(iii) A $\mathcal{C}^2$-fct on $U \subset \mathbb{R}^2 \cong \mathbb{C}$ is called **harmonic** if

Assume that f is hol., then
But

So hol. fcts are harmonic!

(Locally) biholomorphic functions

$\Rightarrow$ Recall the Inverse function theorem (Thm. 3) in Chap. 1.

Holomorphic version of diffeomorphism

Definition 2

- Let $U, V \subset \mathbb{C}$ be open. $f : U \to V$ is called **bihol.** if it is
  (i) holomorphic,
  (ii) bijective, and
  (iii) $f^{-1} : U \to V$ is hol.
- $f : U \to \mathbb{C}$ is called **locally bihol.** at $z \in U$ if
  (i) $\exists$ an open neigh. U_z of $z \in U$,
  (ii) $f(U_z) := U_z$ is open, and
  (iii) $f|_{U_z} : U_2 \to V_z$ is bihol.

Proposition 2

(a) f is locally bihol. at $z \in U \iff f'(z) \neq 0$ for $f \in \mathrm{Hol}(U), z \in U$.

(b) $f \in \mathrm{Hol}(U)$ injective $\iff f'(z) \neq 0 \forall z \in U$ and $f : U \to V := f(U)$ is bihol.

Domain and maximum principle

Definition 3 (domain)

A subset $U \subset \mathbb{C}$ is called a **domain** if it is open and connected.

Proposition 3

(a) $U \subset \mathbb{C}$ be a domain and $f \in \mathrm{Hol}(U)$ non-constant $\Rightarrow f(U) \subset \mathbb{C}$ is also a domain.

(b) ("Maximum principle") $U \subset \mathbb{C}$ be a domain and $f \in \mathrm{Hol}(U)$ non-constant $\Rightarrow$ the fct

has no (global) maximum on U.

Remarks:

- By making U smaller it is also clear that $|f|$ has no local max. on U.
- (b) is often applied in the following way:

Accumulation point

Definition 4 (accumulation point)

Let (X, d) be a metric space.

(a) A point $x_0 \in X$ is called **isolated** in X if $\exists r > 0$ s.t.

$$B(x_0, r) = \{x_0\}.$$

In other words, $\{x_0\} \subset X$ is open.

(b) Let $A \subset X$ be a subset. Then $x_0 \in X$ is called an **accumulation point** of A in X if there is a sequence $(a_n)_{n\in\mathbb{N}}$ in $A \backslash \{x_0\}$ s.t.

$$a_n \to x_0.$$

Remark: $x_0 \in X$ is an accumulation point of A if $x_0 \in \overline{A}$ and x_0 is **not** an isolated point of $\overline{A}$. The set of accumulation points of A is closed.

Zeros of hol. fcts

Proposition 4 (zeros of hol. fcts)

Let $U \subset \mathbb{C}$ be a domain, $f \in \text{Hol}(U) \setminus \{0\}$ and $Z(f) := \{z_0 \in U | f(z_0) = 0\}$.
Then, $Z(f)$ has no accumulation point in U (equiv.: every $z_0 \in Z(f)$ is isolated in $Z(f)$) and $Z(f)$ is at most countable.

For every $z_0 \in Z(f)$ $\exists m \in \mathbb{N}$ and $g \in \text{Hol}(U)$ with $g(z_0) \neq 0$ s.t.

Remark/Example: $Z(f)$ may have accumulation point outside U in $\mathbb{C}$.
For instance: $f \in \mathbb{C} \setminus \{0\} \to \mathbb{C}$ by $f(z) = \sin(\frac{\pi}{z})$, we have

Thus, 0 is an accumulation point but $0 \in \mathbb{C} \setminus \{0\}$.

Identity theorem

Corollary 1 ("Identity theorem")

Let $U \subset \mathbb{C}$ be adomain and $f, g \in \mathrm{Hol}(\mathbb{C})$. Define

If Z has an accumulation point in U, then $f = g$.

Remarks: Identity Thm. says in particular that:

- $f \in \mathrm{Hol}(U), U$ domain, is uniquely determined by

$$f|_{B(z_0,\epsilon)}, z_0 \in U, \epsilon > 0 \text{ arbitrary.}$$

- $f \in \mathrm{Hol}(U), U$ domain, $U \cap \mathbb{R} \neq \emptyset$.
  Then $U \cap \mathbb{R} \subset \mathbb{R}$ is open, non-empty, and it has an accumulation point.
  Hence f is uniquely determined by $f|_{U \cap \mathbb{R}}$.

Isolated singularities

Motivation: $U \subset \mathbb{C}$ open, $f \in \mathrm{Hol}(U)$.
If $z_0 \neq \mathbb{C}$, then $f(z_0)$ is not defined!
$\Rightarrow$ We may think of z_0 as a "singularity" of U.

Definition 5 (isolated singularity)

Let $U \subset \mathbb{C}$ be open and $f \in \mathrm{Hol}(U)$.
A point $z_0 \in \mathbb{C} \backslash U$ is called an **isolated singularity** of f, if z_0 is an isolated point of $\mathbb{C} \backslash U$, i.e.

Example:

Types of isolated singularities

There are 3 types of isolated singularities:

Definition 6

Let $U \subset \mathbb{C}$ be open, $f \in \mathrm{Hol}(U)$ and $z_0 \in U$ isolated sing. of f.

(i) z_0 is called **removable** if $\tilde{f} \in \mathrm{Hol}(U \cup \{z_0\})$ s.t. $\tilde{f}|_U = f$.

(ii) We say that f has a **pole** at z_0, if z_0 is not removable but

has a removable singularity at z_0 for some $m > 0$. The smallest m with this property is called the **order** of the pole of f at z_0.

(iii) z_0 is called **essential** singularity of f if it is neither removable nor a pole.

Types of isolated singularities

Examples:

(1)

(2)

Definition 7 (meromorphic function)

Let $U \subset \mathbb{C}$ be open.
A **meromorphic function** on U is a holomorphic fct $f : U_0 \to \mathbb{C}$ on an open subset $U_0 \subset U$ s.t.

Let $M(U)$ be the set of meromorphic fcts in U.

Remarks:

- S consists of isolated points, therefore it is at most countable.
- Strictly speaking a meromorphic fct f on U is **not** a fct on U, only a fct on $U_0 = U \backslash S$. Moreover, one can think of f as a fct

  by setting $f(z) = \infty, z \in S$.
- If $U \subset \mathbb{C}$ is a domain, then $(M(U), +, \cdot)$ is a field with multiplicative unit given by the constant fct 1. In particular, for $f, g \in M(U), g \neq 0$, we have

Residue

Recall: For $f \in \mathrm{Hol}(U)$, Laurent series with center $z_0 \in U \backslash \mathbb{C}$ is a series of the form $f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$, where

$$a_n = \frac{1}{2\pi i} \int_{|z-z_0|=\epsilon} \frac{f(z)}{(z - z_0)^{n+1}} \, \mathrm{d}z.$$

Focus: on the coefficient a_{-1}.

Definition 8 (residue)

Let $f \in \mathrm{Hol}(U), z_0 \in U \backslash \mathbb{C}$ an isolated sing. Choose $\epsilon > 0$ sufficiently small s.t. $\overline{B}(z_0, \epsilon) \backslash \{z_0\} \subset U$.
Then, the **residue** $Res_{z_0}(f) \in \mathbb{C}$ of f at z_0 is defined as

Example: $Res_{z_0}(e^{1/z}) = 1$.

Ways to compute $Res_{z_0}(f)$

(i) z_0 removable sing.: $Res_{z_0}(f) = 0$.

(ii) z_0 a pole of order 1 ("principale pole") $\Rightarrow Res_{z_0}(f) = \lim_{z\to z_0}(z-z_0)f(z)$.

(iii) z_0 a pole of order at most m:
$Res_{z_0}(f) = \frac{1}{(m-1)!}\lim_{z\to z_0}\left(\frac{d}{dz}\right)^{m-1}(z-z_0)^m f(z)$.

(iv) f has a primitive $F \in \mathrm{Hol}(B(z_0,\mathbb{C})\setminus\{z_0\}) \iff Res_{z_0}(f) = 0$.

(v) f, g hol. around z_0 and g has a zero of order 1 at $z_0 \Rightarrow Res_{z_0}(f/g) = \frac{f(z_0)}{g(z_0)}$.

Definition 9 (path and cycle)

Let $U \subset \mathbb{C}$ be open.

(a) A **curve** in U is a cont. map $\gamma : [a, b] \to U$, $[a, b] \subseteq \mathbb{R}$ closed bounded intervall.

(b) A **closed curve** is a curve with $\gamma(a) = \gamma(b)$.

(c) A **path** is a piecewise cont. diff. curve γ, i.e.

(d) A **closed path** is a path which is a closed curve.

(e) A **cycle** γ in U is a formal finite linear combination of closed paths in U with integer coeff:

where $\gamma_i : [a_i, b_i] \to \mathbb{C}$ closed path, $\lambda_i \in \mathbb{Z}$.

Notation: $\gamma^* := \{\gamma(t) | t \in [a, b]\} \subset U$ denotes the image of the curve γ.

Integration along paths and cycle

Definition 10 (integration along paths and cycle)

(i) Let $\gamma : [a, b] \to U \subset U$ be a path and $f : \gamma^* \to \mathbb{C}$ cont.
We define

(ii) Let γ be a cycle. For any cont. $f : U \to \mathbb{C}$ we define

Remarks:

- Integral can be taken as Riemann or Lebesgue, more in the sense of an oriented integral.
- The length of γ is defined as $l(\gamma) = \int_a^b |\gamma'(t)| \, \mathrm{d}t \in [0, \infty)$.

Examples

Here $z_0 \in \mathbb{C}$ is fixed.

(1) **Constant path**: $\gamma : [a, b] \to \mathbb{C}, \gamma(t) = z_0$.
We have that $\gamma'(t) = 0$ and thus

(2) **Circles** (positively oriented): Given $r > 0$, define
$\gamma : [0, 2\pi] \to \mathbb{C}, \gamma(t) = z_0 + re^{it}$ closed path. We have that $\gamma'(t) = ire^{it}$ and thus

Winding number

Definition 11 (winding number)

Let $\gamma : [a, b] \to \mathbb{C}$ be a closed curve and $z \in \mathbb{C} \backslash \gamma^*$.
We define the **winding number** of γ around z by

$$n_\gamma(z) := \frac{1}{2\pi i} \int_\gamma \frac{1}{\xi - z} \, \mathrm{d}\xi \in \mathbb{Z}.$$

In particular, for $z \in \mathbb{C} \backslash \gamma^*$ with γ a cycle in U, the winding number is

$$n_\gamma(z) := \sum_{i=1}^{r} \lambda_i n_{\gamma_i}(z) \in \mathbb{Z}$$

is well-defined.

Examples:

Residue theorem

Theorem 1 (residue theorem)

Let $U \subset \mathbb{C}$ be open, f hol. on U up to isolated sing., i.e. $\exists U_0 \subset U$ open, $f \in \mathrm{Hol}(U_0)$ and every $a \in S := U \backslash U_0$ is an isolated sing. of f.
Let γ be a cycle in $U_0 = U \backslash S$ s.t. $n_\gamma(z) = 0, \forall z \in \mathbb{C} \backslash U$, i.e. γ only around isolated sings. of f.
Then, the set $S_1 = \{a \in S | n_\gamma(a) \neq 0\}$ is finite and we have the residue formula:

Example Compute $\int_0^\infty \frac{1}{1+x^n}\, \mathrm{d}x$.
Let $f(z) = \frac{1}{1+z^n}, n \geq 2$. Observe that $1 + z^n = 0 \iff z = \theta^l, l$ odd with $\theta = e^{\pi i/n}$. Then $S_1 = \{0\}$ and $n_\gamma(\theta) = 1$ indep. of $r > 1$.
Hence $\int_{\gamma_r} \frac{1}{1+z^n}\, \mathrm{d}z = 2\pi i \; Res_\theta(\frac{1}{1+z^n})$ with $Res_\theta(\frac{1}{1+z^n}) = \frac{1}{(1+z^n)'}\Big|_\theta = [\dots] = \frac{-e^{\pi i/n}}{n}$.

$$\Rightarrow \int_0^\infty \frac{1}{1+x^n} = \frac{2\pi i}{n} \cdot \frac{e^{-\pi i/n}}{1 - e^{2\pi i/n}} = \frac{\pi}{n} \cdot \frac{1}{\sin(\pi/n)}.$$

Application1

Framework: Let $P, Q \in \mathbb{C}[x]$ polynomials s.t. $Q(x) \neq 0 \; \forall x \in \mathbb{R}$ and define $R(x) := \frac{P(x)}{Q(x)}$ rational and meromorphic function on $\mathbb{C}$.

(1) If $\deg(Q) \geq \deg(P) + 2$, then

$$\int_{-\infty}^{\infty} R(x)\,\mathrm{d}x = 2\pi i \sum_{Im(a)>0, Q(a)=0} Res_a(R(z)) = -2\pi i_{Im(a)<0, Q(a)=0} Res_a(R).$$

In particular, if R is even $(R(x) = R(-x))$ then

$$\int_{0}^{\infty} R(x)\,\mathrm{d}x = 2\pi i \sum_{Im(a)>0} Res_a(R(z)).$$

Application 2

(2) If $\deg(Q) \geq \deg(P) + 1$, then

$$\lim_{r\to\infty} \int_{-r}^{r} R(x)e^{ix}\, dx = 2\pi i \sum_{Im(a)>0} Res_a(R(z)e^{iz}).$$

In particular, if $P, Q \in \mathbb{R}[x]$ the

$$\lim_{r\to\infty} \int_{0}^{r} R(x)\cos(x)\, dx = \pi i \sum_{Im(a)>0} Res_a(R(z)e^{iz}) \quad \text{if } R \text{ even}$$

$$\lim_{r\to\infty} \int_{0}^{r} R(x)\sin(x)\, dx = \pi \sum_{Im(a)>0} Res_a(R(z)e^{iz}) \quad \text{if } R \text{ odd}$$

Example: $\lim_{r\to\infty} \int_0^r \frac{\sin(x)}{x}\, dx = \frac{\pi}{2} Res_0(\frac{e^{iz}}{z}) = \frac{\pi}{2}$.

Application 3 and 4

(3) If $Q(x) \neq 0 \; \forall x \in [0, \infty)$ and $\deg(Q) \geq \deg(P) + 2$, then

$$\int_0^\infty R(x) \, \mathrm{d}x = -\sum_a Res_a(R(z)\overline{\log}(z)),$$

with $\overline{\log} : \mathbb{C} \backslash [0, \infty) \to \mathbb{C}, \overline{\log}(re^{i\varphi}) = \log(r) + i\varphi$, for $\varphi \in (0, 2\pi)$.

(4) If $Q(x) \neq 0 \; \forall x \in [0, \infty)$ s.t. $x = 0$ is a zero of most first order and $\deg(Q) \geq \deg(P) + 2$, then

$$\int_0^\infty R(x) x^\lambda \, \mathrm{d}x = \frac{2\pi i}{1 - e^{2\pi i \lambda}} \sum_{a \neq [0, \infty)} Res_a(R(z) z^\lambda),$$

where $\lambda \in (0, 1)$.

Application 5

Framework: Let $f : \mathbb{R} \to \mathbb{C}$ be a 2π- periodic

(5) Assume that f is given by the restriction of a hol. fct to the unit circle $S^1 = \{z \in \mathbb{C} \| z| = 1\}$.
Then $\exists F : B(0, R) \to \mathbb{C}$ hol. up to isolated sings. in the disc of $B(0, 1)$ with $f(x) = F(e^{ix})$ and

$$\int_0^{2\pi} f(x)\,\mathrm{d}x = 2\pi \sum_{|a|<1} Res_a\left(\frac{F(z)}{z}\right).$$

(5*) (special case) If f is rational fct in $\cos(x), \sin(x)$:

$$f(x) = R(\cos(x), \sin(x))$$

with $R(y_1, y_2) = \frac{P(y_1,y_2)}{Q(y_1,y_2)} \neq \infty$ if $y_1, y_2 \in \mathbb{R}$ and $y_1^2 + y_2^2 = 1$, then

$$\int_0^{2\pi} R(\cos(x), \sin(x))\,\mathrm{d}x = 2\pi \sum_{|a|<1} Res_a\left\{\frac{1}{z}\left[\frac{1}{2}\left(z + \frac{1}{z}\right), \frac{1}{2i}\left(z - \frac{1}{z}\right)\right]\right\}.$$

Overview of this chapter