Chapter 2: Ordinary Differential Equations

Refresher Courses in Analysis

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Explicit and implicit form

Definition 1 (Ordinary differential equation)

Let $\Omega \subset \mathbb{R} \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n$ be open and $F : \Omega \to \mathbb{R}^n$ be a continuous function. An **ordinary differential equation** (ODE) of order $k \in \mathbb{N}$ is an equation of the following kind:

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Definition 2

Let $I \subset \mathbb{R}$ be an open interval and Ω, F as in Def. 1.
A **solution** of (1) is a function $\varphi : I \to \mathbb{R}^n$ s.t. $\forall x \in I$:

(i)

(ii)

(iii)

Examples

Remarks:

- Often, we take $n = 1$. If $n \geq 2$, one calls (1) also a system of ODEs.
- The Implicit function theorem tells us that under certain conditions (2) is equivalent to

  at least locally!

Examples:

- Explicit form:
- Implicit form:

Cauchy problem

Aim: We are interested in **finding solutions** of

satisfying initial conditions.

Defintion 3 (Initial value problem or Cauchy problem (CP))

Given $x_0 \in \mathbb{R}$ and $a_0, a_1, \ldots, a_{k-1} \in \mathbb{R}^n$. We have to find an open interval I_0 containing $x_0 \in \mathbb{R}$ and a function $\varphi \in \mathcal{C}^k(I_0, \mathbb{R}^n)$ s.t.

(i) φ is a solution of (1).

(ii) $\varphi^{(l)}(x_0) = a_l, \; l = 0, \ldots, k-1.$

We say that $(x_0, a_0, a_1, \ldots, a_{k-1})$ is the **initial condition** of the Cauchy problem or initial value problem.

Some examples

1.

Some examples

2. (Newton's law in classical mechanics, $k = 2, n = 3$)

Problem

Framework: $U \subset \mathbb{R}^n$ open subset, $I \subset \mathbb{R}$ interval s.t. $I \times U \subset \Omega$ and

We define a **new** function

Then Y satisfies the following system of diff. eqs.:

Fact: If y solves the CP associate with (4) with $y^{(l)}(x_0) = a_l, l = 0, \cdots, n-1$, then Y solves the CP with $Y(x_0) = (a_0, a_1, \cdots, a_{k-1})$.
Vice versa: If Y solves (5) then $y := Y_0$ solves (4).

Reduction of order

Proposition 1

Let $k \in \mathbb{N}$ and $\Omega \subset \mathbb{R} \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n$ be an open interval. Consider a function $F : \Omega \to \mathbb{R}^n$.
There is a **one-to-one correspondence** between the solutions (1) and solutions of the equation

Moral: For the understanding of differential equations (DEs) of arbitrary order it is (in principle) sufficient to understand (system of) DEs of first order. One has to pay the price that one has to introduce more variables!

Reduction of order

<u>Remarks</u>:

- Most of the diff. eqs. are not explicitly solvable. Nevertheless, we want to understand
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    -
    -
- Prop. 1 tells us that it is sufficient to understand 1. order eqs.
- In Example 2., by applying the reduction of order to (3) (which is of order 2 on $\mathbb{R}^3$), we obtain an eq. of 1. order in

Elementary solutions of the Cauchy problem

Aim: We are interested in **finding elementary solutions** of differential equations of first order

satisfying initial conditions.

We will focus on the following:

- Separation of variables
- Linear equations and variation of constant
- "Homogeneous" differential equation

Separation of variables

Consider a diff. eq.

a) If $g(y_0) \neq 0$, then

b) If $g(y_0) = 0$, then

Example

Consider a simple non-linear eq. of first order ($k = 1$) with $n = 1$:

Such an eq. in one variable can be solved by the method of separation of variables.

- **Case** $a = 0$:
- **Case** $a \neq 0$:

For non-linear eqs. solutions need not to be defined everywhere. The (maximal) domain of definition may depend on the initial condition.

Linear equations and variation of constants

(i) **Homogeneous eq.**:

(ii) **Inhomogeneous eq.**:

"Homogeneous" differential equation

Diff. eq. of the form

Problem

Consider our general Cauchy problem:

$\Omega \subset \mathbb{R} \times \mathbb{R}^n$ open, $(x_0, y_0) \in \Omega$ initial value and $F : \Omega \to \mathbb{R}^n$ cont. fct.

Aim: We are interested in finding a solution of the Cauchy problem

Questions:

(a) Has (CP) always a solution (at least in a small neighb. of x_0 in $\mathbb{R}$)?

(b) Is this solution unique?

Remark: By Prop. 1 the answer to (a) and (b) also gives corresp. answers for DEs of higher order.

Example

Consider the following Cauchy problem:

- First: $y(x) = 0 \; \forall x \in \mathbb{R}$ is a solution of $(*)$.
- Ignore for a moment the initial conditions and look for general solutions with $y(x) \neq 0$ for x in a certain time interval.

Peano theorem

<u>Moral</u>: For general F, solutions of (CP) may not be unique. However, concerning (a) there is the following general theorem:

Theorem 1 (Peano)

Let $F : I \times U \to \mathbb{R}^n$ be a continuous fct. and $(x_0, y_0) \in I \times U = \Omega$ be initial cond.
Then, there exists $\epsilon > 0$ and a solution

of the Cauchy problem (CP).

<u>Remarks:</u>

- There is no uniqueness assertation.
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Preparation for the Picard-Lindelhöf theorem

Now consider $F : I \times U \to \mathbb{R}^n$ cont. fct. and $(x_0, y_0) \in I \times U$.

Aim: Establish Picard-Lindelhöf theorem which guarantees the existence and uniqueness of the solutions of the Cauchy problem above.
We need the following lemma:

Lemma 1

Let $F : \Omega \to \mathbb{R}^n$ be a continuous fct. and $(x_0, y_0) \in \Omega$ be initial cond.
A continuous fct $y : I \to U \subset \mathbb{R}^n$ is a solution of the Cauchy problem (CP) $\iff$ $\forall x \in I$

Preparation for the Picard-Lindelhöf theorem

Hence: we need to **find** a complete metric space of fcts s.t. the map T

is a contracting map from this space to itself. This means that we can apply Banach's fixed point theorem.

We need a Lipschitz character for the function F:

Corollary 1

Let $F : I \times U \to \mathbb{R}^n$ cont. diff. fct w.r.t. y-variable (i.e. all F_x are diff.). Assume further that U is convex and that $d_2 F : I \times U \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ is bounded.
Then, F satisfies $(L_{I,U})$.

Picard-Lindelhöf theorem

Theorem 2 (Picard-Lindelhöf, Version A)

Let $F : \Omega \to \mathbb{R}^n$ be a cont. fct satisfying $(L_{I,U})$ (w.r.t. second variable) and let $(x_0, y_0) \in \Omega = I \times U$.
Then, there exist $\epsilon > 0$ s.t. $I_\epsilon := (x_0 - \epsilon, x_0 + \epsilon) \subseteq I$ and a unique solution

Moreover, for every $\epsilon' \leq \epsilon$ ($\epsilon' > 0$), the restriction

is the unique solution of (CP) defined on I_ϵ.

<u>Note</u>: Since the proof is based on Banach's fixed point theorem, we can extract an iterative method to construct the unique solution of (CP).

Improved error estimates for the Picard-Lindelhöf iteration

Define inductively

Then $y(x) := \lim_{k\to\infty} y_k(x)$ exists for $x \in I_\epsilon$ for $\epsilon > 0$ and is a solution of (CP). This procedure is called the P.-L. iteration method of successive approx.:

Propostion 2 (Improved error estimates for the P.-L. iteration)

Consider $F : I \times U \to \mathbb{R}^n$ a cont. fct satisfying $(L_{I,U})$. Let $A \subset U$ be closed and $(x_0, y_0) \in I \times A$.
We set $y_0(x) := y_0$ and inductively $\forall x \in I$ satisfying the following cond.:

Let $I_0 \subset I$ be open s.t. $x_0 \in I$ and $(A_{k,x})$ holds $\forall x \in I_0, k \in \mathbb{N}$. **Then**, we have $\forall x \in I_0$:

(a)

(b)

Moreover: $y_0 : I_0 \to A \subset \mathbb{R}^n$ is cont. diff. and solves (CP).

Some facts about linear ODEs

Next we review some facts about:

- Local and global solutions
- Dependence of solutions on initial values and parameters
- Autonomous DE and vector fields

Local solutions

Maximal solutions

Dependence on a parameter

First-order DE depending on a parameter is a DE of the form

where $F : \Omega \times M \to \mathbb{R}^n$ cont., $\Omega \subset \mathbb{R} \times \mathbb{R}^n$ open and (M, d) metric spcase.

Solution: It is a fct $y : I \times M \to \mathbb{R}^n$ s.t. for every $\eta_0 \in M$:

Example: $n = 1, \Omega = \mathbb{R} \times (0, \infty), M = (0, \infty)$.

Dependence on a parameter

Theorem 3 (P.-L. Version B)

Consider $I \subset \mathbb{R}$ open interval, $U \subset \mathbb{R}^n$ open, (M, d)-metric space.
Let $F : I \times U \times M \to \mathbb{R}^n$ cont. fct s.t. $F(\cdot, \cdot, \eta) : I \times U \to \mathbb{R}^n$ satisfies $(L_{I,U})$ $\forall \eta \in M$ with Lipschitz const. indep. of η, and $(x_0, y_0, \eta_0) \in I \times U \times M$ the initial condition.
Then, $\exists \epsilon > 0$ s.t. $\forall z \in U$ with $||z - y_0|| < \epsilon$ and $\eta \in M$ with $d(\eta, \eta_0) < \epsilon$ there exists a unique solution $y_{z,\eta} : I_\epsilon \to U$ of

Moreover, the fct $y : I_\epsilon \times B(y_0, \epsilon) \times B_d(\eta_0, \epsilon) \to U$ is given by

Dependence of solutions on initial values

Aim: Under certain assumps., we want to establish diff. dependence on initial values.

Consider

1. $F : I \times U \to \mathbb{R}^n$ cont. and cont. diff. w.r.t. y-variable, i.e.
   Under this cond., F satisfies local Lipschitz cond. on $I \times U$.
   Fix $x_0 \in I, U_0 \subset U$ open and consider the family of initial value problem:

2. Assume that we have a family of solutions:

Objective: Any family of solutions 1:

Known: It is cont. diff. w.r.t. x (by def. of a solution).

Problem: Existence and continuity of $d_2 y$.

Dependence of solutions on initial values

Proposition 3 (dep. of initial value)

Let $I \subset \mathbb{R}$ be an open interval s.t. $x_0 \in I$ and $U_0 \subset U \subset \mathbb{R}^n$ be open subsets. Assume that $F : I \times U \to \mathbb{R}^n$ satisfies ❶ and let $y : I \times U_0 \to \mathbb{R}^n$ be a cont. family of solutions of ❷.
Then, $d_2 y : I \times U_0 \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ exists and

In particular, $d_2 y$ is continuous.

Theorem 4 (generalization of the Prop. 3)

Let $I \subset \mathbb{R}$ be an open interval s.t. $x_0 \in I$ and $U_0 \subset U \subset \mathbb{R}^n$ be open subsets. Let $F : I \times U \to \mathbb{R}^n$ be of class $\mathcal{C}^k, k \in N \cup \{\infty\}$ and $y : I \times U_0 \to \mathbb{R}^n$ be a family of solutions of

Then y is of class $\mathcal{C}^k$.

Autonomous DE and vector fields

- **Autonomous DE**:
  e.g.

- **Vector fields**: A (cont., diff., $\mathcal{C}^k$-) vector field on U is a (cont., diff., $\mathcal{C}^k$-) function $F : U \to \mathbb{R}^n$.
  (a) $F(x) = v \in \mathbb{R}^n, \forall x \in U$ is a constant vector field.
  (b) $\text{grad}(f) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \cdots, \frac{\partial f}{\partial x_n}\right)$, where $f : U \to \mathbb{R}$ is cont. diff.
  (c) $y' = F(y)$, where $F : U \to \mathbb{R}^n$ is a cont. vector field on U.
  $\Rightarrow$ find the so-called **integral curves** of the v.f.
  $\Rightarrow$ find a family of curves (parametrized by i.v.) s.t. each vector $F(z)$ of the v.f. is tangent at z to the curve of the family passing through z
  $\Rightarrow$ $y'(x)$ is the tangent vector to the curve $x \mapsto y(x)$ at $z = y(x)$ and has to be equal to $F(z) = F(y(x))$
  (d) <u>Derivative</u>: $X : U \to \mathbb{R}^n$ cont. v.f. ($U \subset \mathbb{R}^n$ open)

Conserved quantity for an autonomous DE

Consider $U \subset \mathbb{R}^n$ open and $X : U \to \mathbb{R}^n$ a cont. vector field.
Let

and $g : U \to \mathbb{R}$ a fct of class $\mathcal{C}^1$.
$\Rightarrow$ g is called **conserved quantity** of $(*)$ (or of X) if for all solutions $y : I \to U$ of $(*)$ the function

Properties

(a) If $Xg(y) = 0 \; \forall y \in U \Rightarrow g$ is a conserved quantity of X.

(b) Assume that $\forall y_0 \in U \; \exists$ a solution $y : I \to U$ of $(*)$ and $x_0 \in I$ s.t. $y(x_0) = y_0$ (this is always the case due Peano and P.-L. theorems). **Then**, every conserved quantity of X satisfies $Xg(y) = 0 \; \forall y \in U$.

Examples

(i) Special case of Newton's law:

(ii)

Autonomous DE and vector fields

Uniqueness and existence: Let $U \subset \mathbb{R}^n$ be open and

$\phi : \mathbb{R} \times U \to U$ be a **one parameter transformation group**, i.e.

Note that by using Thm. 4:

- ϕ is of class $\mathcal{C}^k$
- ϕ_t is $\mathcal{C}^k$-differentiable.

Overview of this chapter