Chapter 1: Basic of real analysis of several variables

Refresher Courses in Analysis

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Motivation example

We already know that the equation

$$x^2 = 2$$

has a unique solution $x \in (0, \infty)$, called

$$\sqrt{2} = 1.41421356237...$$

In fact, we know that

There is an alternative way to **construct** $\sqrt{2}$:

Step 1: Reformulate the eq. $x^2 = 2$ as "*fixed point problem*"

Motivation example

Step 2: Choose $x_0 \in [1, 2]$ an define recursively

Step 3: Estimate. let $x, y \in [1, 2]$, then

Contracting function

Definiton 1 (contracting function)

Let (X, d) be a metric space and let $f : X \to X$ be a map.
We say that f is **contracting** if there exists a constant $C \in [0, 1)$ such that $\forall x, y \in X$ we have

$$d(f(x), f(y)) \leq C \cdot d(x, y).$$

Remarks

- Def. 1 can be generalized as follows:

- f is contracting $\Rightarrow$ f is continuous (even Lipschitz continuous).
- f is Lipschitz continuous $\Rightarrow$ f is continuous.

Banach's fixed point theorem

Theorem 1 (Banach's fixed point theorem)

Let (X, d) be a complete metric space and let $f : X \to X$ be a contracting map.
Then f has a unique fixed point $\overline{x} \in X$.
In other words, the equation $f(x) = x$ has exactly one solution $\overline{x} \in X$.

Moreover, for any $x_0 \in X$, the sequence $(f^n(x_0))_n$, where $f^n(x) = f \circ \cdots \circ f(x)$, converge to the fixed point $\overline{x} \in X$.

Sketch of the proof

Banach's fixed point theorem

Banach's fixed point theorem

Remarks

- Thm. 1 shows that our "construction" of $\sqrt{2}$ was correct.
- Thm. 1 also provides a practical method to solve equations like $f(x) = x$ (at least approx. by computers).
- Thm. 1 does NOT work for "Lipschitz constant" $L = 1$ in general.

  Simplest example:

Preparation

Let

$U \subset \mathbb{R}^p \times \mathbb{R}^q$, $(p, q \geq 1)$, be open and

$F : U \to \mathbb{R}^q$ be a cont. diff. map.

We consider the equation

$$ \tag{1}$$

Given x, we want to find y s.t. (1) is satisfied.
For fixed x, (1) is a system of p equations in q variables.

General question:

Under which conditions can we solve (1) uniquely if possible?

<u>Find</u> a solution $y =: g(x)$ depending on x s.t. $F(x, g(x)) = 0$.
Ideally, we want that g is a "nice" function, that means cont. diff.
We may ask:

What can be said about it differential $dg(x)$?

Preparation

Recall the solution: $U \subset \mathbb{R}^p \times \mathbb{R}^q$, $F : U \to \mathbb{R}^q$ of class $\mathcal{C}^1$.
We want to **find** a solution function ("implicit function") defined on some open subset $U_1 \subset \mathbb{R}^p$ with values in another open subset $U_2 \subset \mathbb{R}^q$

A necessary condition for the existence of a $\mathcal{C}^1$-solution
Let $(x, y) \in U$ be fixed.
We consider the differential of F at (x, y)

$$dF(x, y) : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R} \text{ linear map.}$$

$dF(x, y)$ can be written as a sum of 2 maps. Let

$p_1 : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^p$ and

$p_1 : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^q$ be the projections.

A necessary condition for the existence of a $\mathcal{C}^1$-solution

Then

In matrix notation, we have

A necessary condition for the existence of a $\mathcal{C}^1$-solution

Let $U_1 \times U_2 \subset U$, and $g : U_1 \to U_2$ be differentiable.
We consider the function $f : U_1 \to \mathbb{R}^q$ given by

In other words: $\Rightarrow f$ is differentiable, and by the chain rule we obtain

If g is a solution of (1a), then $f(x) = 0 \; \forall x \in U_1$, therefore $df(x) = 0 \; \forall x \in U_2$.
$\Rightarrow dg(x)$ is a solution of

- If the linear map $d_2F(x, g(x))$ is invertible, then we get a unique solution:

- If $d_2F(x, g(x))$ is NOT invertible, then it could happen that (2a) has NO solution.

Implicit function theorem

Theorem 2 (Implicit function theorem)

Assume: $U \subset \mathbb{R}^p \times \mathbb{R}^q$ open and $F : U \to \mathbb{R}^q$ cont. diff. (i.e. $F \in \mathcal{C}^1(U, \mathbb{R}^q)$).
If $(x_0, y_0) \in U$ s.t.

(a) $F(x_0, y_0) = 0$,

(b) $d_2F(x_0, y_0)$ is invertible, i.e. $\det(d_2F(x_0, y_0)) \neq 0$,

then $\exists$ open neighb. $U_1 \subset \mathbb{R}^p$ of x_0 and $U_2 \subset \mathbb{R}^q$ of y_0 with $U_1 \times U_2 \subset U$ s.t.

(i) for each $x \in U_1$ there is a unique $y = g(x) \in U_2$ with $F(x, g(x)) = 0$,

(ii) the function $g : U_1 \to U_2$ defined by (1) is cont. diff.

Hence $d_2F(x, g(x))$ is invertible $\forall x \in U_1$ and

$$dg(x) = -[d_2F(x, g(x))]^{-1} \cdot d_1F(x, g(x)).$$

Sketch of the proof: Resets of Banach's fixed point theorem (Thm. 1). $\square$

Problem

Framework: $U \subset \mathbb{R}^q$ be open and $f : U \to \mathbb{R}^q$ be a $\mathcal{C}^1$-function.

We **look** for an inverse function

We can rewrite (i) as an "implicit function problem", namely

Inverse function theorem

Theorem 3 (Inverse function theorem)

Assume: $U \subset \mathbb{R}^n$ open, $f : U \to \mathbb{R}^n$ cont. diff., $x_0 \in U$ s.t. $df(x_0)$ is invertible.
Then: $\exists$ open neighbs. $U_0 \subset U$ of x_0 and $V = f(U) \subset \mathbb{R}^n$ of $f(x_0)$, and a cont. diff. function

$$g = (f|_{U_0})^{-1} : V \to U_0.$$

Moreover,

$$dg(f(x)) = [df(x)]^{-1} \; \forall x \in U_0.$$

Remark:

Application of Thm. 3: Diffeomorphismus

Definition 2 (diffeomorphism)

Let $U, V \subset \mathbb{R}^n$ be open and $f : U \to V$ be a map of class $\mathcal{C}^k$, $k \in \mathbb{N}^*$ (or of class $\mathcal{C}^\infty$).
f is called **diffeomorphism** of class $\mathcal{C}^k$ (or $\mathcal{C}^\infty$) $\iff$

(i) f is bijective and

(ii) $f^{-1} : V \to U$ is also of class $\mathcal{C}^k$ (or $\mathcal{C}^\infty$).

Proposition 1

Let $U, V \subset \mathbb{R}^n$ be open and $f : U \to V$ bijective of class $\mathcal{C}^k$ (or $\mathcal{C}^\infty$).
Then, the following conditions are equivalent:

(a) $df(x)$ is an invertible map $\forall x \in U$.

(b) $f^{-1} : V \to U$ is differentiable.

(c) $f : U \to V$ is a diffeomorphism of $\mathcal{C}^k$.

Examples

(1) (example of a bijective $\mathcal{C}^\infty$ map that is **not** a diffeomorphism)

(2) (polar coord. in the plane $\mathbb{R}^2$)

Overview of this chapter