Chapter 0: Some brief recalls (“Sandbox”)

Refresher Courses in Analysis

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Definitions and properties of metric spaces

Definition 1

- (distance) Let X be a non-empty space. A **distance** $d : X \times X \to [0, \infty)$ verifies the 3 properties: $\forall x, y, z \in X$:
  (i) symmetry: $d(x, y) = d(x, y)$.
  (ii) pos. definiteness: $d(x, y) = 0 \iff x = y$.
  (iii) Δ-inequality: $d(x, y) \leq d(x, z) + d(z, y)$.
- (metric space) If d is a distance on X, then (X, d) is a **metric space**.
- (Cauchy seq.) $(x_n)_{n \in \mathbb{N}}$ is a **Cauchy seq.** : $\forall \epsilon > 0\ \exists N \in \mathbb{N}$ s.t. $\forall n, m \in \mathbb{N}$
$$d(x_n, x_m) < \epsilon.$$
- (completeness) A metric space (X, d) is **complete** if every Cauchy seq. in X converges in X.

Proposition 1

Let (X, d) be a complete metric space.
If Y is a closed subset of X then (Y, d) is a complete metric space.

Continuity and Lipschitz continuity

Definition 2

Let (X, d_x) and (Y, d_y) be 2 metric spaces.

- (continuity) A function $f : X \to Y$ is **continuous** in $x_0 \in X$ if $\forall \epsilon > 0 \; \exists \delta > 0$ s.t. $d_x(x_0, x) < \delta$ then
$$d_y(f(y_0), f(x)) < \epsilon.$$
If f is continuous in x_0, $\forall x_0 \in X$, we say that f is continuous on X.
- (Lipschitz cont.) A function $f : X \to Y$ is **Lipschitz continuous** $\iff \exists$ a constant $L \in [0, \infty)$ s.t. $\forall x_1, x_2 \in X$, we have
$$d_y(f(x_1), f(x_2)) \leq L d_x(x_1, x_2).$$
L is called the **Lipschitz constant**.

Definitions and properties of normed vector spaces

Definition 3

Let V be a non-empty vector space (v.s.).

- (norm) A **norm** on V is a function $||\cdot|| : V \to [0, \infty)$ satisfying the 3 axioms: $\forall x, y \in V$ and $\lambda \in \mathbb{R}$
  (i) pos. definiteness: $||x|| = 0 \Rightarrow x = 0$.
  (ii) absolute homogeneity: $||\lambda x|| = |\lambda| ||x||$.
  (iii) Δ-ineq. : $||x + y|| \leq ||x|| + ||y||$.
- (normed v.s.) If $||\cdot||$ is a norm on V, then $(V, ||\cdot||)$ is a **normed v.s.**.
- (Banach space) A **Banach space** is complete normed v.s.
- (linear maps and operators) Let $(V, ||\cdot||_V), (W, ||\cdot||_W)$ be 2 finite-dimensional normed v.s. over $\mathbb{R}$ or $\mathbb{C}$. Denote by $\mathcal{L}(V, W)$ be the **v.s. of linear map** $A : V \to W$. We define the **operator** of A by

$$||A||_{\text{op}} = ||A|| := \sup\{||Av||_W | v \in V, ||v|| = 1\}.$$

Examples

(1) Typical distances:

$$d_2 : (x,y) \mapsto \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2}, \; n \in \mathbb{N} \text{ (Euclidean distance on } \mathbb{R}^n)$$

$$d_p : (x,y) \mapsto \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{1/p}, \; p \in [1,\infty) \text{ (generalization)}$$

$$d_\infty : (x,y) \mapsto \max_{i \in \{1,\dots,n\}} |x_i - y_i|, \; n \in \mathbb{N}.$$

(2) The corresponding normed distances:

$$||\cdot||_2 = d_2 : x \mapsto \sqrt{\sum_{i=1}^{n} |x_i|^2}, \; n \in \mathbb{N} \text{ (Euclidean norm on } \mathbb{R}^n)$$

$$||\cdot||_p = d_p : x \mapsto \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}, \; p \in [1,\infty)$$

$$||\cdot||_\infty = d_\infty : x \mapsto \max_{i \in \{1,\dots,n\}} |x_i|, \; n \in \mathbb{N}.$$

Some important properties

- Property of normed operators

Proposition 2

Let $(V, ||\cdot||_V), (W, ||\cdot||_W)$ 2 f.-d. normed v.s. over $\mathbb{R}$ or $\mathbb{C}$ and $||A||_{\mathcal{L}(V,W)} < \infty \ \forall A \in \mathcal{L}(V, W)$. **Then**, for every $v \in V$, we have

$$||Av||_W \leq ||A||_{\mathcal{L}(V,W)} \, ||v||_W.$$

- Connection between diff. and Lipschitz continuity

Lemma 1

Let $U \subset \mathbb{R}^n$ be open, $f : U \to \mathbb{R}^k$ diff. fct. and $A \subset U$ convex s.t. $df|_A : A \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^k)$ is bounded, i.e. $\exists C \ ||df(x)||_{\mathcal{L}(\mathbb{R}^n,\mathbb{R}^k)} \leq C \ \forall x \in A$. **Then**, f is Lipschitz cont. on A with Lipschitz constant

$$L = \sup_{x \in A} ||df(x)||_{\mathcal{L}(\mathbb{R}^n,\mathbb{R}^k)}.$$

In particular, if f is cont. diff., then f is Lipschitz cont on each **compact** subset $A \subset U$, e.g. on closed balls.