System analysis in process engineering

Models and predictions

Version 97

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Foreword

This is the course book of a VLAG course given by the Department of Process Engineering. This course was first given in 1991 as a special subject class in a series of 9*2 hour lectures. Since 1994, this course has been given in the form of a class block: one week with 8 hours per day of lectures and tutorials. Course participants are required to attend at least 9 of the 10 morning and afternoon sessions (an attendance list will be made up at the beginning of the course). Prerequisites for this course are: introduction to process engineering, linear differential equations, introduction to statistics, model generation and system engineering and biochemical reactor science or transfer processes. This course proceeds very rapidly, therefore it is essential to prepare in advance. This means that the course book must be read before the course begins.

> It is also possible for regular students at the end of their study to follow this course (P053-206,2 points) if they meet the above prerequisites. Examination will take place by working out a problem in a small group (2 to 3 people). The one extra study point must therefore be devoted to working out a case. The subject of the case must be chosen by your group. It may not be part of a thesis (afstudeervak), but it may be an extension thereof. The final decision about the subject is made in consultation with Marcel Zwietering or Albert van der Padt. Halfway through the solution of the case, an interim discussion will take place. The final report must be submitted no later than 1 October of the year during which the course is taken. The grade is determined after the report is discussed.

In modelling, various disciplines are used: mathematics (numerical mathematics, analytical mathematics, differential equations and optimization), statistics, regression and testing, informatics (dataprocessing, simulation and users interface), systems analysis (block diagrams).

During this course the following subjects will be treated:

1. Introduction (Introduction to modelling; Types of models)
2. Systems Analysis (Diagrams; linear differential equations; simple analytical solutions, Laplace transformations, analytical solution by means of eigenvalues, numerical solution; Control Engineering)
3. Statistics (Estimates; Linear regression; Comparison of models)
4. Epilogue
5. Philosophy
6. The theory shall be applied and explained using various cases. This will involve a discussion of scientific articles. These will include modelling of microbiological quality; modelling and optimization, extraction by using reversed micelles; modelling and optimization of hydrolysis in a membrane reactor;
7. Alternating with the lectures will be tutorials where problems will be solved using Lotus 123.

The various subjects which are to be discussed must not be seen as an exhaustive treatment of those subjects, but only as an illustration and a review, with the emphasis on the links between the various subjects. The solution methods and theories described in broad outline are therefore not the only possible ones. For example, if you would like to solve a problem numerically, you can use this course book to look into the subject and to understand more or less what it boils down to. At the same time, you will also have to use a course book or reference book about numerical mathematics in order to look for the most suitable solution methods and to understand how everything is formally structured. The same applies to

differential equations, control engineering and statistics. The methods treated here are certainly not complete and are sometimes not even correctly described in a formal sense. The first objective is to get a brief glimpse of the subjects and to see what they amount to in broad terms, so you can understand complex books and articles and are informed about the existence of various methods. The second goal is to be able to combine these methods in order to solve problems. These methods will be used in practice during the tutorials, when a number of problems must be solved using Lotus 123.

This course explains the use of mathematical tools which can be used for system modelling. The use of statistical tools to compare the values of various models is also explained. The 'abstract' theory is covered rapidly at the beginning of the course, with special emphasis on the meaning of certain terms (e.g. eigenvalues, variance) and what can be inferred from them. The cases (in Excel) are used to explain and gain insight into the theory (through self-study). During the cases involving hydrolysis, microbial quality and micelles, an analysis of the models is carried out based on scientific publications. As a result, the theory which is learned is clarified and the added value of the critical understanding of models will become apparent.

Tentative Program

Day	Subject	Instructor
Mon. A.M.	welcome introduction to modelling case 1a, b and c, *examples of modelling (computer simulation)*	Dr Pepping Dr. van der Padt Self-study/tutorial
Mon. P.M.	block diagrams, linear differential equations, non-homogenous differential equations, multiple dimensions Laplace-transformation eigenvalues, complex numbers, stability, numerical solutions case 2, *numerical solution of a differential equation*	Prof. Boom Dr. van der Padt Self-study/tutorial
Tue. A.M.	case micelles partial differential equations, von Neuman stability, controls on/off PID, optimal control, Equivalent discrete systems	Dr. van der Padt Prof. Boom
Tue. P.M.	example O_2 transfer	Prof. Boom
Wed. A.M.	case 3, *numerical solution of a set of differential equations* statistics and s^2, linear regression case 4, 1,000 normal errors case 5, *estimation of constant value*	Self-study/tutorial Dr. van der Padt Self-study/tutorial
Wed. P.M.	$y = a\,x + b$, multiple dimensions solve 3.2 and 3.3 case 6, *line through origin* (explanation own case MSc students)	Dr. van der Padt Self-study/tutorial Dr. van der Padt
Thu. A.M.	case hydrolysis comparisons of models case 7, *comparisons of models*	Dr. van der Padt Self-study/tutorial
Thu. P.M.	effect of transformations non-linear regression case 8, *transformations*	Dr. van der Padt Self-study/tutorial
Fri. A.M.	case quality 11 hrs philosophy	Dr. van der Padt Dr. H.G.J. Gremmen
Fri. P.M.	case 9, *non-linear regression* finish cases Diploma presentation	Self-study/tutorial Prof. Boom

1. Introduction

1.1 Modelling

In the chemical industry, the description and optimization of processes began around the year 1900. Biochemical and biological processes also began to be described and optimized around 1960. Although there were many doubts in the beginning about whether or not it was indeed possible to describe biological systems using models and mathematical equations, modelling has now become a common tool in biotechnology. The development of biological and biochemical models is currently beginning in the food industry as well.

Some people believe that a model is a computer program, others believe that a model is a mathematical equation, still others believe that it is a set of assumptions, while some believe that it is a simplified representation of reality. This last definition comprises most of the instances where the word 'model' is used (model set-up, model substrate, kinetic model, scale model, heat transmission model), and we shall also use this definition: a model is the simplified representation of reality. One can also assume that the complex reality in its entirety cannot be described, or can be described only with great difficulty. By representing a portion of this reality in a simplified fashion, one can attempt to obtain a simpler description. Models exist which are aimed at producing experimental simplifications, such as a model set-up (pilot plant or a beaker with enzymes as a model for a pig stomach) or a model chemical system (water with ammonia as a model for waste water, or broth as a microbiological model medium). There are also models from which mathematical equations can be derived. We will examine this type of model more extensively. In many cases a model results in a computer program, for instance in order to solve equations or to make predictions. But a model does not by definition have to be implemented in a computer.

Models can be used to make predictions, to understand the mechanism behind a process, to control a process, to optimize a process or to design a process. Depending on the purpose of a model, different approaches to modelling may be required.

1.2 Approach

Table 1: Modelling approach with example

	step	example 1.1
A	Definition of the system	Growth of bacteria in batch
B	Assumptions	Growth is proportional to number
C	Formulate equations	$dc_x/dt=\mu c_x$
D	Solve	$c_x=c_{xo}\exp(\mu t)$
E	Test	Experiment
F	Use	Predict, optimize

To begin with, the system we are going to model must be defined (A, Table 1). Assumptions for the system to be modelled must then be made (B). After this, the entire system must be described with mathematical equations (C). These equations can then be solved (D) after which the model can be tested (E) by comparing it with reality (validation). At this point it is a good idea to reconsider the description of the system and the assumptions. If the model turns out to satisfactorily describe reality (within possible limits), the model can be used further (F). However, one must always remain critical about the results. The approach

described above appears simple, but it is certainly true that during the course of modelling, steps must often be taken over again. For example, during step C it may not be possible to formulate equations. One must return then to step B. Or during the testing stage (E) it may turn out that the model is wrong, and one must sometimes start all over again from the beginning.

Example 1.1
A) We want to describe the growth of bacteria in batch culture.
B) We assume that there is no time lag, no limitation of substrate, no local gradients (ideal mixing). We also assume that the increase in the number of organisms is proportional to the number of organisms (if I have two organisms which divide, then I have two more; if I have 100 organisms which divide, then I have another 100).
C) The increase in the number of organisms is dc_x/dt, and this proportional to the number. This results in the following equation:

$$\frac{dc_x}{dt} = \mu c_x \tag{1.1}$$

where μ is the proportional constant, the specific growth rate. Notice that this equation is a direct translation of the statement 'the increase in the number of organisms is proportional to the number of organisms' into mathematical language.
D) This differential equation can be solved analytically: Integration results in:

$$c_x = c_{xo} \exp(\mu t) \tag{1.2}$$

This mathematical equation can be again translated back into English: "The number of bacteria begins at c_{xo} and increases exponentially in time." These are simplified examples and the translations are very simple to make, but it is also frequently useful in more complex cases to state the problem in English and then translate it into mathematics and the reverse.
E) This model has been tested many times.
F) This model is frequently used for predictions and optimizations.

Advantages of models:
1) One can interpolate results and (within strict limitations) also extrapolate them.
2) One obtains insight into the process.
3) One can quickly calculate the results of different conditions.
4) One can optimize processes
5) One can combine various research results.
6) It can help with the structured collection of data.

Explanation using example 1.1:
1) One can interpolate within the measured intervals. While the measurements take place only at discrete times, one can also predict the development between the points. Therefore, the number of organisms can be determined at every point in time. Extrapolation can take place if we remain within the boundary conditions; there must be an surplus of substrate available. If this is no longer the case, then μ is no longer constant. Without measurement points, we do not know if we remain within this limitation.
2) We obtain insight into the process; we learn that bacteria grow exponentially. This is therefore an autocatalytic process, and the kinetics are the same as phenomena such as first order chemical reaction, the dying off of micro-organisms and radioactive decay.
3) Laboratory experiments and pilot plant experiments are very expensive. By approaching problems systematically and by modelling the problem, the amount of experimentation can be reduced. For example, if three factors are important to a specific process, an order of

magnitude estimate might show that one factor has 1000 times as much influence on the process than the other two. Therefore, only this single factor must be explored further.
4) One can use this model, for instance, to optimize the ratio of fermentation costs/inoculation level. With a low level of inoculation, the process takes longer and we therefore incur more costs to achieve the same level of production. However, using more inoculation material also increases the costs (Figure 1.1).

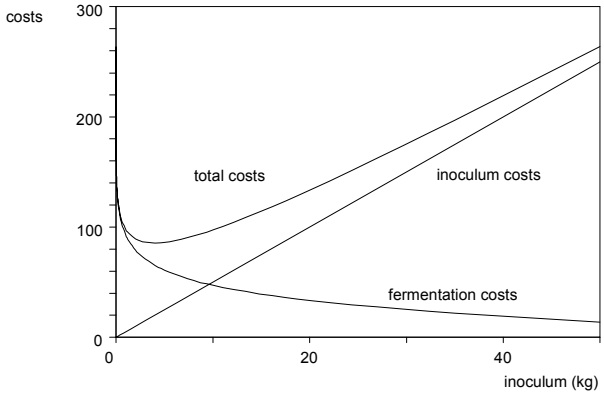

Figure 1.1: Optimization of the fermentation costs for various levels of inoculant.

5) This model can be linked to other models, such as models for lag time and models for substrate consumption.
6) A (large) quantity of data can be reduced to a single parameter (μ).

Possible problems during modelling:
1) Model errors
a) Real errors (try to avoid!)
b) Inaccuracies resulting from assumptions.
c) Mistakes resulting from arriving at an interval which lies outside the model (extrapolation).

2) Inaccuracies while calculating the solution

3) Models with too many unknown parameters can almost always be fitted so that the model appears to be good. Formally, this is not a problem with modelling, but a mistake made during the establishment of the model or during the statistical evaluation. This still makes it wrong, even though it fits.

Explanation using example 1.1:
1a) If the lag time is important, the results of this model are not valid.
1b) The model can give good predictions with very short lag times, but they will be slightly inaccurate.
1c) If substrate limitation is important, the results of the model are no longer valid as time proceeds; this model is only valid as long as the substrate is available in sufficient quantity. Extrapolation can therefore lead to errors.
2) Not applicable (exact solution: analytical)
3) Not applicable (only a single parameter)

Example 1.2
A) An object which is released in the air falls downward. We want to determine the position of the object in time.
B) The object will accelerate under the influence of gravity. If air resistance is negligible, the acceleration of the object will be constant if the gravitational acceleration is constant. We assume that the mass of the entire object is concentrated in one point (the centre of gravity).

C) $\frac{dv}{dt} = a$ (1.3)

This equations means: acceleration is the change in velocity over time.

D) Integration results in: $v = \frac{dS}{dt} = a \cdot t$ (1.4)

In words: velocity is the change of position with time.

Integrating again results in: $S = 0.5 \cdot a \cdot t^2$ (1.5)

The position changes in a quadratic (more than linear) fashion with time.

In the Netherlands, the parameter a has been determined to be 9.81 m/s^2 and is called g.
E) This model has naturally been tested many times.

Errors
1a) The model is not valid for a falling leaf, because the air resistance is not negligible.
1b) The rate of gravitational acceleration is not exactly 9.81 m/s^2 at every location in the Netherlands.
1c) If the object falls from a tall building, the velocity will become so high that air resistance is no longer negligible (extrapolation).
2) Inaccuracies during the solution do not play a role in this case (analytical).
3) This model also does not have too many parameters (only a single parameter).

Advantages
1) The position of the falling object can be calculated at every point in time (interpolating and under strict limitations also extrapolating).
2) We obtain more confidence in the fact that gravitational acceleration is constant and more confidence in the other assumptions (insight into the process).
3) We can quickly calculate the effect of various conditions.
4) We can use this model to optimize processes (e.g. falling droplets which are evaporating in a drying tower).
5) We can link various research results to each other (e.g. drying kinetics and falling droplets).
6) It can help with the structured collection of data (all falling experiments can be reduced to a single parameter g).

Concludingly, we see in examples 1.1 and 1.2 that assumptions lead to mathematical equations which we can solve. We also see that there are parameters in the model, such as μ and g, which we can determine with experiments or can find in the literature. With these models we can therefore do all kinds of useful things, but we also see that these models must be used in a critical fashion.

3. Types of models

Models can have various characteristics. Tables 2 and 3 show various examples of these characteristics. Models based on certainties (e.g. the law of conservation of energy, always in

effect except during nuclear reactions) are certainly good, if we at least don't overlook any flows (e.g. evaporation). The previously described model for batch growth was based on a good assumption (logical and validated in practice). If we want to calculate the quantity of fat oxidation in a package of butter in the distribution chain, we can begin with an order of magnitude estimate and assume that the temperature in the distribution chain is constant and is equal to 7° C. We can then calculate how much fat oxidizes in time at that temperature (using other models) and then calculate if fat oxidation is important (order of magnitude). There are also empiric models. These models (such as the Monod equation) use equations which, in practice, appear to provide good descriptions of data. With empirical models, extrapolation is especially dangerous. The examples given in Table 2 must be really considered as such. Someone could have used the Monod model just as easily as an example under 'good assumptions'.

Table 2: Models can be based on:

certainties	e.g. conservation of mass and energy
good assumptions	e.g. growth proportional with number
assumptions of order of magnitude	e.g. $T = 7°C$ in distribution chain
empirical	e.g. Monod equation

Table 3: Various characteristics of models:

black box model	grey box model
descriptive model	predictive model
unstructured model	structured model
distributed model	segregated model
verbal model	non-verbal model
mathematical model	non-mathematical model
deterministic model	stochastic model
continuous model	discrete model
non-gradient model	gradient model
stationary model	non-stationary model
linear model	non-linear model
.....	.....

BLACK BOX MODEL (EMPIRICAL, DESCRIPTIVE): A black box model is a model of a specific system in which only the input and the output are related to each other. The process/system itself is considered to be a black box within which something happens, but what happens is unknown. Example: In example 1.1 assume that measurements were made of the number of bacteria in a batch, and that we fitted a polynomial to our results without thinking further about our system. Or, we looked at the measurement points graphically and came to the conclusion that the data points rose approximately in an exponential fashion. Another check ($\ln(c_x)$ against time) showed that they were indeed increasing exponentially. Therefore, an exponential equation was chosen as a model (empirical, mathematical). Notice that the result is the same as in example 1.1! Black box models should not actually be used outside the interval from which they are derived, because one has no idea at all about what is taking place in the system, and therefore can in no way predict what the system will do outside the interval within which it is tested (validated). In practice, however, this is frequently done.

GREY BOX MODEL (EXPLANATORY, PREDICTIVE, FUNDAMENTAL, STRUCTURED): If one looks into the box and attempts to describe what is taking place there, a grey box model is established. The box will never be completely 'white' because it is impossible (?) to know everything about a system. An example of a grey box system is once

again example 1.1, but with the emphasis on the way the model was derived and not simply on fitting the results. With grey box models it is generally less dangerous to extrapolate, but one must still be careful; it is also true for these models that factors outside of the measured interval -which are not part of the model- can become important.

DISTRIBUTED MODEL: This model assumes that units are continuously distributed throughout the system (continuum). Example: a jar of mayonnaise modelled as a quantity of water with a quantity of oil.

SEGREGATED MODEL: This model assumes that there are distinct units within the system (corpuscular). Example: a jar of mayonnaise as a water phase (continuum) with a large quantity of distinct oil droplets.

VERBAL MODEL: Qualitative description of a system. Example: part B of example 1.1 or example 1.2.

MATHEMATICAL MODEL: Quantitative description of a system. Example: equation 1.1.

DETERMINISTIC MODEL: A model that always yields the same output with the same input. The input-output relationship is unambiguous. Example 1.1.

STOCHASTIC MODEL: A model where chance plays a role. Example: a model which predicts the chance that one will select a red ball out of a vase containing ten red and ten white balls. Another example is the following stochastic model (compare with equation 1.2):

$$c_x = c_{xo} \exp(\mu t) + \varepsilon \qquad (1.6)$$

Where ε is a normally distributed error

CONTINUOUS MODEL: A model of a system where events take place continuously in time. Example 1.1.

DISCRETE MODEL: A model where events take place at specific points in time. Example: a model for a synchronously dividing culture ($N=N_o*2^n$)

GRADIENT MODEL: A model of systems where gradients occur. Example: a model for the change in temperature across the external wall of a house.

STATIONARY MODEL: A model where no changes take place in time. Example: a model of a continuous culture of micro-organisms in steady state.

NON-STATIONARY MODEL: Batch culture of micro-organisms (quantity is dependent on time).

LINEAR MODEL: For linear systems, the superposition principle applies: if an input signal $u_1(t)$ results in output $y_1(t)$ and $u_2(t)$ in output $y_2(t)$, then an input signal $a_1u_1(t)+a_2u_2(t)$ results in an output of $a_1y_1(t)+a_2y_2(t)$.

In this type of model it is important to make a distinction between linearity in the parameters (important for statistics) and linearity in the variable. The model $y=ax^2$ is linear in the parameter a but non-linear in the variable x. The model $y=a*\exp(b/T)$ is non-linear in the variable T and non-linear in the parameter b (but is linear in parameter a).

There are also other subdivisions of models. A model can, of course, belong to different types. The equation 1.2 is a mathematical, deterministic, continuous, non-gradient, non-stationary, non-linear model. The text of part B of example 1.2 is an explanatory, verbal, non-mathematical model.

4 Modern developments

EXPERT SYSTEMS: Models wherein the knowledge of one or more experts is modelled. These are not calculation models but models where help can be given in making choices and decisions. Example: a program which can indicate possible diseases, based on symptoms of a patient. It usually turns out, however, that such a system is best used by another expert. The system can then supplement the knowledge of the expert (e.g. concerning unusual tropical diseases). If such a system helps an expert make decisions, it is frequently called a decision support system (DSS).

FUZZY MODELS OR LOGICAL MODELS: In some cases certain factors cannot be described quantitatively, but only in terms of high, middle and low. These models are comprised of rules such as:
'During a fermentation of micro-organisms, if the oxygen concentration is low and the sugar concentration high, there may be a case of oxygen limitation. This can be solved by increasing the agitation or the aeration.'

NEURAL NETWORKS: A neural network processes data in a totally different way than a normal computer program. Unlike a program, it is not comprised of a number of commands connected to each other in series, but is made up of a large number of mutually connected units which (following the analogy with an actual neural network) are called cells. Cells can be stimulated by the external world or by other cells; they are also capable of providing information to the external world or of stimulating other cells. The degree to which the cells stimulate other cells or provide information depends on the degree to which they themselves are stimulated. One of the most important properties of neural networks is that feedback takes place between the outputs and data inputs. As a result, the structure of the network can be dynamically adapted; this allows the same data to be processed more efficiently the second time than the first time. An artificial learning effect therefore takes place. Neural networks derive their knowledge (which is contained in their structure) from experience; they are not programmed, but trained. Proponents of neural networks see them as a very promising development because their greatest power is in exactly those areas where traditional computers are weak, such as in immediate pattern recognition (by having an 'overview') and making decisions based on incomplete information or information which is difficult to compare. Opponents of neural networks see them as a euphemism for fitting with too many parameters.

CHAOS MODELS: It frequently turns out that processes are not linear, which leads to models yielding apparently chaotic results. Nonetheless, it is possible to recognize structure in these results. An example of this is the logistic model:

$$x(k+1) = a[1 - x(k)] \cdot x(k) \tag{1.7}$$

If a simulation is carried out using various values of a where $x(0)$ is equal to, for example 0.1, stationary situations may or may not occur, depending on the value of a. If a is larger than about 3, it turns out that there are two stationary solutions. If a is larger than about 3.5 there are four stationary solutions. If a is still larger, there are more and more stationary solutions; a chaotic picture forms, but one which still has a clear structure. The theory built around

these chaos models could possibly be used in process control or to describe turbulent flows, for example.

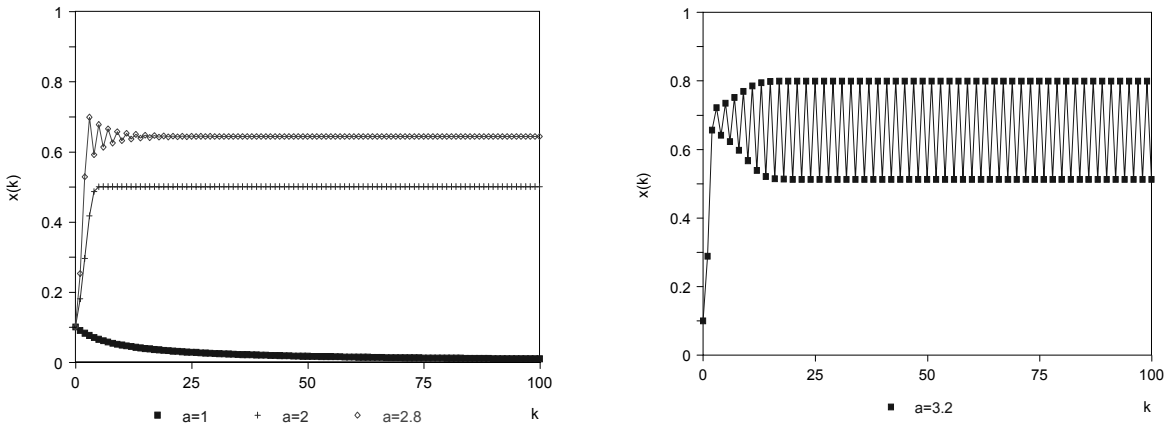

Figure 1.2: Chaos system: Changes in the simulation at various values of *a*.

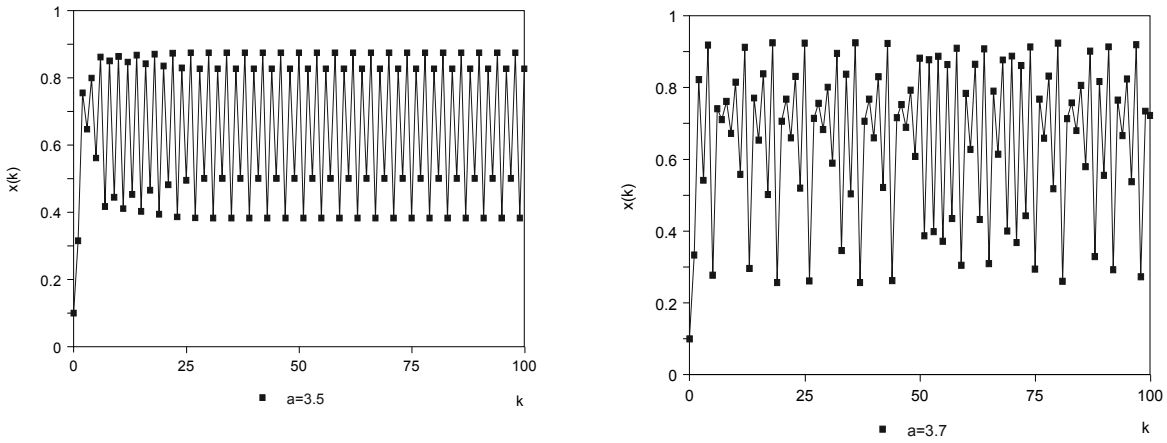

Figure 1.2: continued.

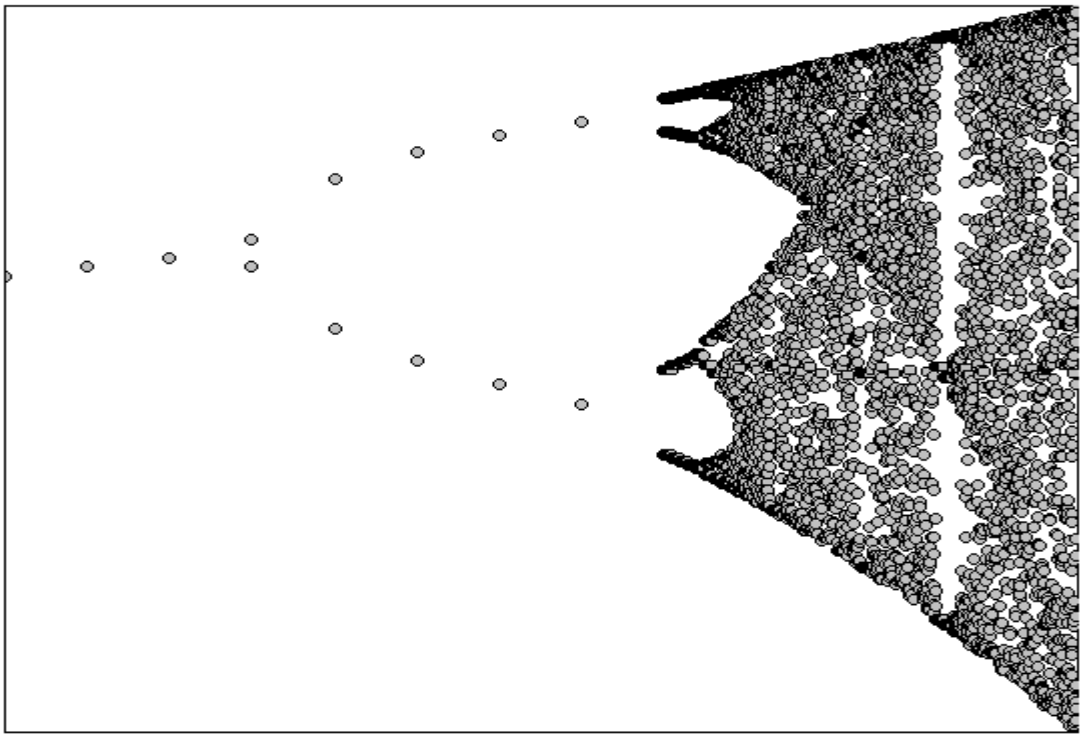

Figure 1.3: Chaos system: Stationary situations at various values of *a*.

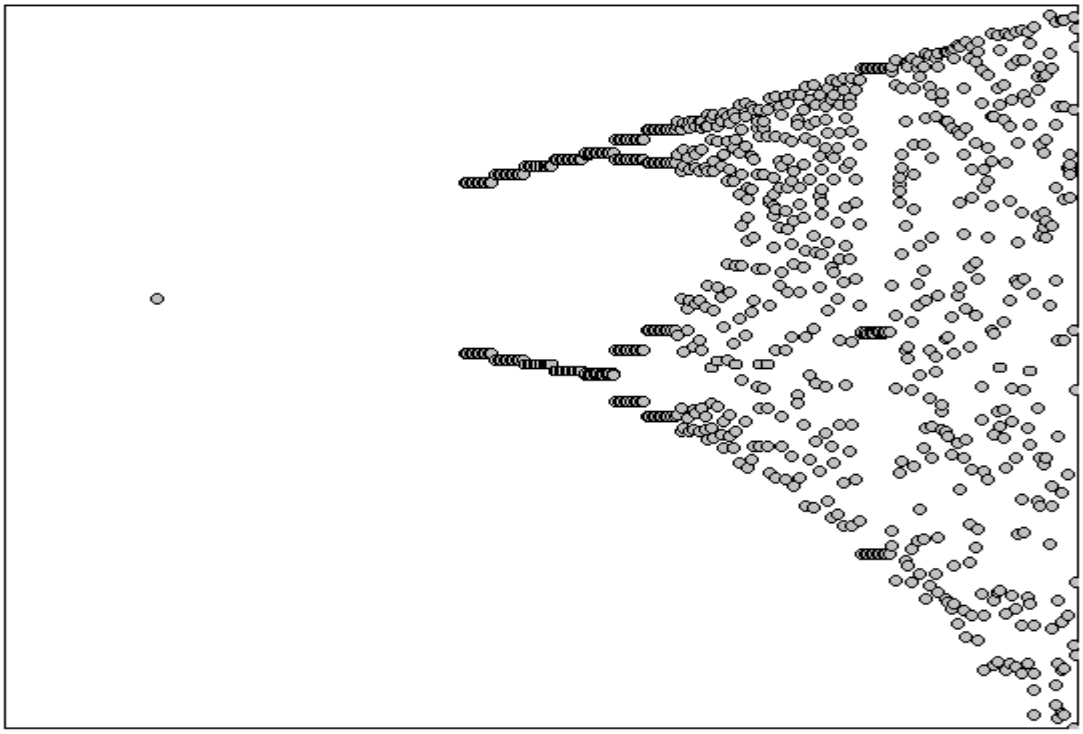

Figure 1.4: Chaos system: Enlargement of figure 1.3.

If we 'zoom in' on piece of the curve, we will again see the same pattern repeated.

5. No more experiments?

It is a common misconception that once you begin modelling, you only need a computer and that experiments are no longer necessary. The opposite is often true. Due to modelling, and the fact that you want to quantify more factors, you frequently see that you must make more measurements. You can, however, set up better experiments by using models. Modelling ultimately results in the quantification of the process to be modelled and in the quantification of confidence in the model. It is through this calculation of confidence(s) that you frequently learn that you have to make more measurements, because the models are otherwise too inaccurate. The course of events is usually: first make a model, then perform experiments, then improve the model.

6. Do models already exist or not?

If an engineer has to solve a problem and thinks that he can use models to do this, he can obtain these models in two different ways. He can search in the literature to see if his problems have already been modelled, or he can create a model himself. In any case, it is advisable to first search the literature and look for a similar problem (Figure 1.5). During this literature search, one must dare to look across the borders of scientific disciplines. For example, the model for the breakdown of Vitamin C is exactly the same as a model for the death of micro-organisms during pasteurization. Only the parameters are different. However, it is not advisable to use models from the literature indiscriminately, they must first be examined critically. If there are no similar problems described in the literature, or if the models from the literature are unsuitable, the engineer must create the models himself. In order to make a better appraisal of models from scientific literature, the following sections will examine various important aspects of modelling.

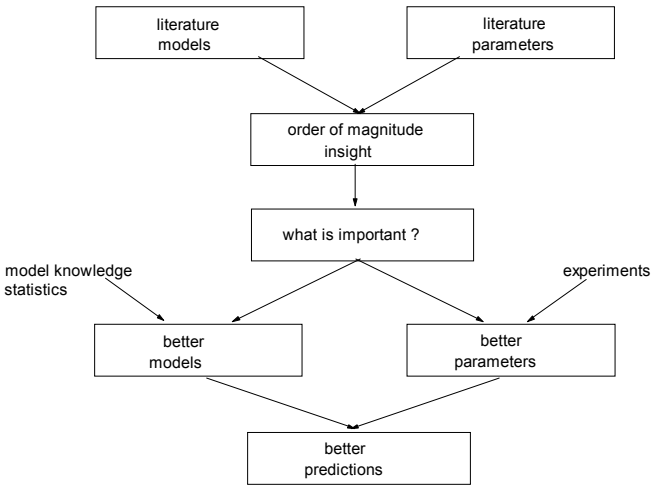

Figure 1.5: Possible procedure to set up models.

7. Help with solutions

During modelling there is usually a great deal of calculating. Computers are therefore useful for dataprocessing (making graphs, estimating parameters, performing statistical evaluation), simulation and optimization. When using computers for this, it is possible to make use of already existing programs (packages) or to write the programs oneself. Existing programs have certain advantages:

- the research and programming has already been done (time savings).
- a package will often have different solution methods, so that the best method for the existing problem can be chosen.
- the program has usually been tested extensively.
- the program is usually user-friendly.

And there are also possible problems:

- you are limited to what a program can do and must sometimes make the problem subordinate to what the program is capable of.
- you must become familiar with the program (this varies greatly, some programs are easy to learn and well organized, while others are obscure).
- you must understand the theory behind the program, otherwise it is a 'black box' and you may or may not notice when strange things happen. You can also get all kinds of output which you may think is completely undesirable (ballast) and do not pay attention to (such as statistical information), but which is actually extremely important. Because you don't understand the program, you don't look at this information (this is more a shortcoming of the user than of the program).
- programs cost money (but may still be cheaper than doing it yourself!), and it takes time to decide on the best package and to order it.
- it is not always possible to check to see if the program is doing what you think it is.
- depending on the supplier: there are often mistakes or imperfections in this type of program, which in the best case are improved in the next version, but then the new version must again be purchased.

The advantages of doing your own programming:

- because you are doing the programming, you can (within limits) calculate everything which you want to calculate (more flexible).
- because you are writing the program yourself, the value of every parameter or variable can be called up, so bugs can be easily taken out of the program and strange things that may be happening can be investigated.
- there is no purchase cost (but it takes time, and time is money!!).
- by programming yourself, you have more understanding of the program; the program is more transparent to the maker.

Disadvantages of doing your own programming:

- it takes a lot of time (and therefore money: fl 100 per hour!) to figure out how to calculate what you want to calculate.
- it takes time to program everything (especially input/output such as tables and graphs).
- a self-written program will frequently have only a single solution method; the program has to be rewritten for new problems even if they are similar.
- the program can calculate things incorrectly without being noticed (the program is tested by fewer people).
- more time is required to maintain the program.
- you must, in the first place, be able to write programs.
- self-written programs are frequently difficult for others to use because you don't spend as much time on making a good user's interface as a software manufacturer does.

Whether you want to write a program for a particular problem yourself or buy a package therefore seems to be a difficult decision

I would personally recommend the following: extensive problems for which good packages are available ---> package. Examples: wordprocessing (e.g. WORD, WP), spreadsheet (e.g. LOTUS 123), flowsheet (e.g. ASPEN), finite element methods (FLUENT). 'Simple' problems can be written up yourself (e.g. in LOTUS, PASCAL, SAS, DELPHI, MATLAB or Mathcad). In a business environment it is not appreciated if someone (with the exception of a programmer) 'wastes' a few weeks of his/her time by doing programming. If it is a problem for which you can write a program in a single day, then you could, of course, do it yourself.

8. Pitfalls

One must always consider the results of models critically and not assume that the answer given by a computer will be correct. One must check to see if the order of magnitude of the predictions is correct and once again check to see if all the assumptions are valid and make sure that nothing important has been overlooked. When checking the order of magnitude, the use of dimensionless, or characteristic numbers (Fo,Bi, etc.) can be useful. With these, the results of complex calculations can be checked with simple calculations. Finally, one must of course check to see if the model provides a good description of reality.

A great deal is possible by using models, such as setting up experiments, processing data, optimization and process control. Models must, however, be used critically. They must be seen as a useful tool which should be used carefully.

2. System analysis

2.1 Diagrams: e.g. flow charts

A diagram is a **surveyable** representation of the elements of a system and the relationships between them. Schematic representations of a system can be extremely important for presenting a system. The objective of a diagram is not to impress. You must therefore not attempt to express the complexity of what you are working with; instead you should try to make the situation clear to your reader and show the most important relationships. An example of a diagram is Figure 1.5. Example 1.2 from this course book can be schematically represented as shown in Figure 2.1.

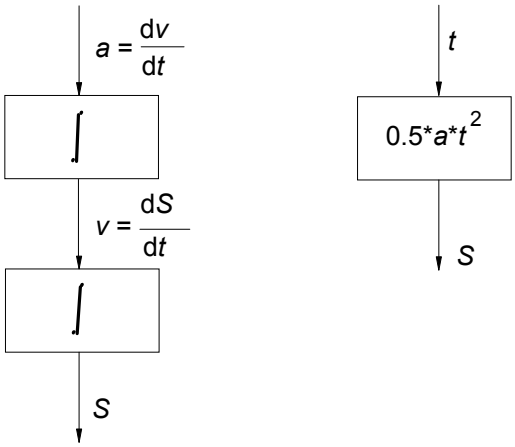

Figure 2.1: flow chart of falling cannonball.

Example 2.1:
We can again look at the growth of bacteria in a batch culture. If we assume that:
1) if there is sufficient substrate the bacteria grow exponentially (hence increase proportional to number);
2) the equations of Pirt apply to substrate use;
3) the growth rate is dependent on the substrate concentration according to Monod kinetics;
we can then set up the following equations:

$$\frac{\mathrm{d}c_x}{\mathrm{d}t} = \mu(c_s) \cdot c_x \tag{2.1}$$

$$\frac{\mathrm{d}c_s}{\mathrm{d}t} = -\frac{1}{Y_{xs}} \cdot \mu(c_s) \cdot c_x - m_s \cdot c_x \tag{2.2}$$

$$\mu = \mu_{\max} \cdot \frac{c_s}{c_s + K_s} \tag{2.3}$$

We can draw various diagrams to represent these equations (see figure 2.2). In this example the diagram on the left is quite complex. The middle diagram is somewhat clearer, but a better simplification is to leave out all the equations from the boxes (these can be found in the text) and only show the relationship between r_x, r_s, c_x, and c_s as shown in the diagram on the right.

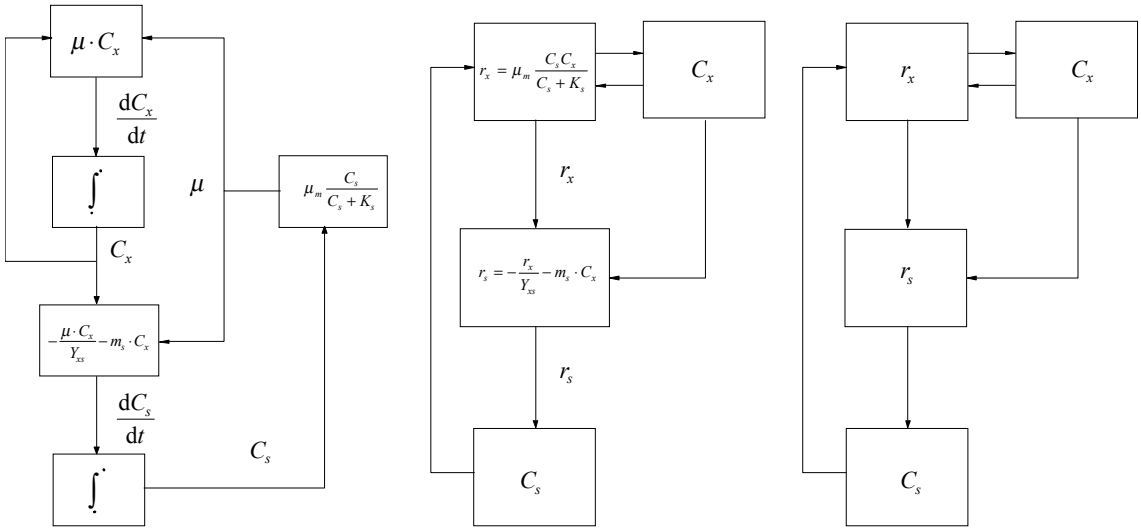

Figure 2.2: Flow charts for the growth of biomass.

2.2 Linear differential equations

2.2.1 one dimension

With many processes, the description (just as in examples 1.1, 1.2 and 2.1) will result in differential equations. In many cases the change of something in time (d*x*/d*t*) is a specific, known function (e.g. see equation 2.1). The simplest form is a linear, one-dimensional equation.

$$x' = \frac{dx}{dt} = ax + b \tag{2.4}$$

In this way, equation 2.1 can be written as:

$$c_x' = \frac{dc_x}{dt} = \mu \cdot c_x \tag{2.5}$$

In this equation, μ is dependent on the substrate concentration and is therefore also dependent on time. The equation is therefore not very easy to integrate. In a situation where the substrate concentration is so high that the growth rate is constant in time, it is possible to integrate the equation:

$$\int \frac{dc_x}{c_x} = \int \mu dt = \mu \int dt \tag{2.6}$$

$$\ln\left(\frac{c_x}{c_x(0)}\right) = \mu t \qquad c_x = c_x(0) \cdot e^{\mu t} \tag{2.7}$$

We separate the variables (c_x and t), so that on the left there is a term which is only dependent on c_x and on the right a term which is only dependent on t. Both terms can then be integrated. The same procedure is followed in examples 1.1 and 1.2, where the differential equations can also be solved simply. It becomes somewhat more difficult when the equation is non-homogeneous (then we cannot separate the variables):

$$x' = ax + be^t \tag{2.8}$$

First we solve the homogeneous part of the equation: $x=p\cdot\exp(at)$ (see 2.7) and then we propose a clever trial solution.

$$x = pe^{at} + qe^t \tag{2.9}$$

We then calculate the derivative:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = a \cdot pe^{at} + qe^t \tag{2.10}$$

and what this has to be in order to satisfy our differential equation:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = a\left(pe^{at} + qe^t\right) + b \cdot e^t \tag{2.11}$$

if these equations are made equal to each other we see that we have chosen the correct trial solution and now we can calculate q:

$$q = aq + b \quad \text{so} \quad q = \frac{b}{1-a} \tag{2.12}$$

The solution is:

$$x = pe^{at} + \frac{b}{1-a}e^t \tag{2.13}$$

$$x(0) = p + \frac{b}{1-a} \tag{2.14}$$

$$p = x(0) - \frac{b}{1-a} \tag{2.15}$$

$$x = \left(x(0) - \frac{b}{1-a}\right)e^{at} + \frac{b}{1-a}e^t \tag{2.16}$$

Always verify the solution by entering this in the differential equation and checking the initial condition. It is easy to make mistakes in these calculations but it is simple to find them.

A more general solution method is the **variation of constants**.
Take the same differential equation:

$$x'(t) = ax(t) + be^t \tag{2.17}$$

the homogenous part is again solved first:

$$x'(t) = ax(t) \quad \Rightarrow \tag{2.18}$$

$$x(t) = c \cdot e^{at} \tag{2.19}$$

assume that c is dependent on time (this is therefore the variation of the constant) and that

$$x(t) = c(t) \cdot e^{at} \tag{2.20}$$

is the solution. If the derivative can be determined from this solution, it must be equal to (2.17)

$$x'(t) = c'(t)e^{at} + a \cdot c(t)e^{at} = a \cdot c(t)e^{at} + be^{t} \quad \Rightarrow \tag{2.21}$$

$$c'(t)e^{at} = be^{t} \tag{2.22}$$

this equation can now be integrated:

$$c'(t) = be^{(1-a)t} \tag{2.23}$$

$$c(t) = \frac{b}{1-a} \cdot e^{(1-a)t} + K \tag{2.24}$$

where K is the integration constant. c(t) can now be filled in (in 2.20):

$$x(t) = \left(\frac{b}{1-a} \cdot e^{(1-a)t} + K \right) e^{at} = \frac{b}{1-a} \cdot e^{t} + Ke^{at} \tag{2.25}$$

with the boundary condition $x(0)$ at time 0, the following solution results:

$$K = x(0) - \frac{b}{1-a} \tag{2.26}$$

$$x(t) = \frac{b}{1-a} e^{t} + \left(x(0) - \frac{b}{1-a} \right) e^{at} \tag{2.27}$$

which, of course, is the same solution as provided by (2.16)

2.2.2 Multiple dimensions

In addition to single equations, one frequently encounters a system of equations. Matrix vector notation is then useful for reaching a solution. Again take equation 2.1 and also 2.2. If we again assume that μ is constant, this system of differential equations becomes:

$$\frac{\mathrm{d}c_x}{\mathrm{d}t} = \mu c_x \tag{2.28}$$

$$\frac{\mathrm{d}c_s}{\mathrm{d}t} = -\frac{1}{Y_{xs}} \cdot \mu \cdot c_x - m_s \cdot c_x \tag{2.29}$$

If we now write this as a matrix vector we obtain:

$$\frac{\mathrm{d}\begin{pmatrix} c_x \\ c_s \end{pmatrix}}{\mathrm{d}t} = \begin{pmatrix} \mu & 0 \\ -\frac{1}{Y_{xs}} \cdot \mu - m_s & 0 \end{pmatrix} \cdot \begin{pmatrix} c_x \\ c_s \end{pmatrix} \tag{2.30}$$

There are now various ways we can solve this, which will be handled in the following paragraphs.

2.2.2.1 Specific solution for this case

In this specific instance, $\mathrm{d}c_x/\mathrm{d}t$ is not dependent on c_s and we can simply solve this part. This results in equation 2.7, so for equation 2.29 we then have:

$$\frac{\mathrm{d}c_s}{\mathrm{d}t} = \left(-\frac{1}{Y_{xs}} \cdot \mu - m_s \right) \cdot c_x(0) \cdot e^{\mu t} \tag{2.31}$$

All terms except the *e* power are independent of time, therefore this equation is also very simple to integrate:

$$c_s - c_s(0) = \left(-\frac{1}{Y_{xs}} - \frac{m_s}{\mu} \right) \cdot c_x(0) \left[e^{\mu t} \right]_0^t \tag{2.32}$$

$$c_s - c_s(0) = \left(-\frac{1}{Y_{xs}} - \frac{m_s}{\mu} \right) \cdot c_x(0) \cdot \left(e^{\mu t} - 1 \right) \tag{2.33}$$

In most cases, this will not be so simple and we must use other methods.

2.2.2.2 Laplace transformation

By means of the Laplace transformation (see textbook on differential equations) it is possible to convert a system of differential equations into a system of algebraic equations. The Laplace transformation of function *y(t)*is defined as:

$$L\left[y(t) \right] = Y(s) = \int_0^\infty y(t) e^{-st} \mathrm{d}t \tag{2.34}$$

For example, for *y(t)*=1 we have:

$$L\left[y(t) \right] = Y(s) = \int_0^\infty e^{-st} \mathrm{d}t = \left[-\frac{1}{s} e^{-st} \right]_0^\infty = \frac{1}{s} \tag{2.35}$$

and for *y(t)*=exp(*at*) we have:

$$L\left[e^{at} \right] = Y(s) = \int_0^\infty e^{(a-s)t} \mathrm{d}t = \left[\frac{1}{a-s} e^{(a-s)t} \right]_0^\infty = \frac{1}{s-a} \tag{2.36}$$

Moreover, $L(\mathrm{d}x/\mathrm{d}t)=s.X-x(0)$. In mathematics handbooks there are extensive tables with Laplace transformations, but they can also be found, for example, in Perry's chemical engineering handbook, which contains a table with the most common Laplace transformations.

After first converting a system of differential equations into a system of algebraic equations, these can be solved and then transformed back into differential equations. The solution for the differential equation system can then be found. This will be worked out for the previous example, where we will call the Laplace variable s. We will use small letters for normal variables or vectors and capital letters for Laplace transformations. System 2.30 then becomes:

$$\begin{pmatrix} s\cdot C_x - c_x(0) \\ s\cdot C_s - c_s(0) \end{pmatrix} = \begin{pmatrix} \mu & 0 \\ -\mu / Y_{xs} - m_s & 0 \end{pmatrix} \begin{pmatrix} C_x \\ C_s \end{pmatrix} \tag{2.37}$$

We now have an algebraic system of equations (two equations with two unknowns) which we can solve.

$$(s-\mu)C_x = c_x(0) \tag{2.38}$$

$$sC_s = -(\mu / Y_{xs} + m_s)C_x + c_s(0) \tag{2.39}$$

$$C_x = \frac{c_x(0)}{s-\mu} \tag{2.40}$$

$$C_s = -(\mu / Y_{xs} + m_s)\frac{c_x(0)}{(s-\mu)s} + \frac{c_s(0)}{s} \tag{2.41}$$

The first fraction in equation 2.41 must be separated:

$$\frac{1}{(s-\mu)s} = \frac{A}{s-\mu} + \frac{B}{s} = \frac{As + Bs - \mu B}{(s-\mu)s} \tag{2.42}$$

$$A + B = 0 \quad \text{and} \quad -\mu \cdot B = 1 \quad \text{so} \quad B = -\frac{1}{\mu} \quad A = \frac{1}{\mu} \tag{2.43}$$

$$\text{so} \quad \frac{1}{(\mathrm{s}-\mu)\cdot s} = \frac{1/\mu}{s-\mu} - \frac{1/\mu}{s} \left(= \frac{s/\mu - s/\mu + 1}{s(s-\mu)} \right) \tag{2.44}$$

Always check the results! Equation 2.41 now becomes:

$$C_s = -\left(1/Y_{xs} + \frac{m_s}{\mu}\right)c_x(0)\left(\frac{1}{s-\mu} - \frac{1}{s}\right) + \frac{c_s(0)}{s} \tag{2.45}$$

If we now reverse transform by using $L[\exp(at)]=1/(s-a)$ and $L(1)=1/s$ this becomes:

$$c_x = c_x(0)e^{\mu t} \tag{2.46}$$

$$c_s = -\left(1/Y_{xs} + \frac{m_s}{\mu}\right)c_x(0)\left(e^{\mu t} - 1\right) + c_s(0) \tag{2.47}$$

This of course provides the same result as 2.7 and 2.33.

2.2.2.3 Solution using eigenvalues and eigenvectors

We can generally write equation 2.30 as:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = Ay \tag{2.48}$$

The general solution for this system is:

$$y = \sum \upsilon_i e^{\lambda_i t} \tag{2.49}$$

where v_i are the eigenvectors and λ_i the eigenvalues. We now determine the eigenvalues of the matrix A by solving det $(A-\lambda I)=0$:

$$\begin{vmatrix} \mu - \lambda & 0 \\ -\frac{\mu}{Y_{xs}} - m_s & -\lambda \end{vmatrix} = 0 \tag{2.50}$$

It follows from this that $-\lambda(\mu - \lambda) = 0$, therefore $\lambda_1 = 0$ and $\lambda_2 = \mu$ are eigenvalues. For an eigenvector v, $Av = \lambda v$. For the eigenvector which goes with the eigenvalue 0:

$$\begin{pmatrix} \mu & 0 \\ -\frac{1}{Y_{xs}}\cdot\mu - m_s & 0 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} \mu \cdot b_1 \\ \left(-\frac{1}{Y_{xs}}\cdot\mu - m_s\right)\cdot b_1 \end{pmatrix} \overset{def}{=} 0\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.51}$$

From this it follows that b_1 must be equal to 0 and b_2 can be anything, for example q. The eigenvector is now $v_1 = (0,q)^t$ The general solution which accompanies this eigenvalue is therefore:

$$\upsilon_1 \cdot e^{\lambda_1 \cdot t} = \begin{pmatrix} 0 \\ q \end{pmatrix} e^{0t} = \begin{pmatrix} 0 \\ q \end{pmatrix} \tag{2.52}$$

Regarding the second eigenvector:

$$\begin{pmatrix} \mu & 0 \\ -\frac{1}{Y_{xs}}\cdot\mu - m_s & 0 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} \mu \cdot b_1 \\ \left(\frac{-\mu}{Y_{xs}} - m_s\right)\cdot b_1 \end{pmatrix} \overset{def}{=} \mu\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \tag{2.53}$$

From this it follows that b_1 can be anything, for example p. For b_2:

$$\left(-\frac{1}{Y_{xs}}\cdot\mu - m_s\right)b_1 = \mu b_2 \tag{2.54}$$

$$b_2 = \left(-\frac{1}{Y_{xs}} - \frac{m_s}{\mu} \right) p \tag{2.55}$$

The eigenvector which accompanies the second eigenvalue $\lambda_2 = \mu$ is $\upsilon_2^t = \left(p, \left(-\frac{1}{Y_{xs}} - \frac{m_s}{\mu} \right) \cdot p \right)$ This part of the solution then becomes:

$$\begin{pmatrix} p \\ p\left(-\frac{1}{Y_{xs}} - \frac{m_s}{\mu} \right) \end{pmatrix} e^{\mu t} \tag{2.56}$$

The total solution becomes:

$$\begin{pmatrix} c_x \\ c_s \end{pmatrix} = \begin{pmatrix} 0 \\ q \end{pmatrix} + \begin{pmatrix} p \\ p\left(-\frac{1}{Y_{xs}} - \frac{m_s}{\mu} \right) \end{pmatrix} e^{\mu t} \tag{2.57}$$

By computing the initial conditions, we naturally arrive again at 2.7 and 2.33 (2.46 and 2.47)

$$\begin{pmatrix} c_x(0) \\ c_s(0) \end{pmatrix} = \begin{pmatrix} 0 \\ q \end{pmatrix} + \begin{pmatrix} p \\ p\left(-\frac{1}{Y_{xs}} - \frac{m_s}{\mu} \right) \end{pmatrix} \tag{2.58}$$

$$c_x(0) = p \quad \text{en} \quad q = c_s(0) - p\left(-\frac{1}{Y_{xs}} - \frac{m_s}{\mu} \right) \tag{2.59}$$

When we fill in these initial conditions in 2.57:

$$\begin{pmatrix} c_x \\ c_s \end{pmatrix} = \begin{pmatrix} 0 \\ c_s(0) - c_x(0)\left(-\frac{1}{Y_{xs}} - \frac{m_s}{\mu} \right) \end{pmatrix} + \begin{pmatrix} c_x(0) \\ c_x(0)\left(-\frac{1}{Y_{xs}} - \frac{m_s}{\mu} \right) \end{pmatrix} e^{\mu t} \tag{2.60}$$

$$c_x = c_x(0) e^{\mu t} \quad c_s = c_s(0) + c_x(0)\left(-\frac{1}{Y_{xs}} - \frac{m_s}{\mu} \right)\left(e^{\mu t} - 1 \right) \tag{2.61}$$

When we determine the eigenvalues of a two dimensional system, we generally obtain a quadratic equation.

$$\lambda^2 + b\lambda + c \tag{2.62}$$

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4c}}{2} \tag{2.63}$$

If $b^2 > 4c$ we obtain real eigenvalues, if $b^2 = 4c$ we obtain two equivalent eigenvalues and if $b^2 < 4c$ we obtain imaginary (complex) eigenvalues (the solutions then show oscillations). With equivalent eigenvalues the general form of the solution is:

$$\bar{x} = \bar{x}(0)e^{\lambda t} + \bar{v}te^{\lambda t} \tag{2.64}$$

where $\bar{v}$ is the eigenvector, $x(0)$ the initial values and λ the eigenvalue.

Exercise 2.1: Solve the following system by using all the above methods:

$$\frac{\mathrm{d}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}{\mathrm{d}t} = \begin{pmatrix} 5 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

When solving the quadratic equation for the eigenvalues, it can happen that the term $b^2 - 4ac$ is negative. It is then 'impossible' to take the root of that number. At this point we have to deal with complex numbers. When you learned arithmetic in grammar school, you first worked with natural numbers (five apples - three apples = two apples). The number 0 (zero apples) was one of the first abstract concepts. Three apples minus five apples was impossible. When you began to understand the principles of overdrawing your account at the bank, you also learned about the existence of negative numbers and therefore the set of whole numbers. But five apples divided by three was impossible. When you had learned more social skills (and especially because you had your own pocket knife) you learned that you could also divide apples and that five divided by three was equal to one plus two-thirds. This resulted in the set of rational numbers. You could already take the square root of nine but the square root of three still caused problems. At this point you were ready to learn about real numbers. After this, calculating the square root of -4 again caused problems. Meanwhile you had already arrived at university. There you learned that the root of -1 was equal to i, and with this you were prepared for many more problems. You still have to wait until you can divide by 0 .....

$\mathbb{N}^+$	natural numbers	1,2,3,4...
$\mathbb{N}_0$	natural numbers including 0	0,1,2,3,4....
Z	whole numbers	...-3,-2,-1,0,1,2,3...
Q	rational numbers	-2,-3,-1/2,-1/3,0,1/3,1/2...
$\mathbb{R}$	real numbers	$-\pi,-\sqrt{3},-\sqrt{2},-1,0,1,\sqrt{2},\sqrt{3},2,\pi$
$\mathbb{C}$	complex numbers	$a + bi$

A complex number is comprised of a real part and an imaginary part (the part with the i). All complex calculations can be carried out by using only two definitions:

$$i^2 = -1 \quad \text{which is} \quad i = \sqrt{-1} \tag{2.65}$$

$$e^{ai} = \cos(a) + i\sin(a) \tag{2.66}$$

and therefore:

$$e^{(p+qi)t} = e^{pt} \cdot e^{qit} = e^{pt} \cdot \left[\cos(qt) + i\sin(qt)\right] \tag{2.67}$$

We can, for example, calculate the quadratic of 5+3*i*:
$(5+3i)^2=25+30i+9i^2=25+30i-9=16+30i$
We can also calculate the zero points of $y=5x^2+5x+5$.

$$y_{12} = \frac{-5 \pm \sqrt{25-100}}{10} = -\frac{1}{2} \pm \frac{1}{10}\sqrt{-75} = -\frac{1}{2} \pm \frac{1}{10}\sqrt{25}\sqrt{3}\sqrt{-1} = -\frac{1}{2} \pm \frac{\sqrt{3}i}{2} \tag{2.68}$$

Since mistakes can also be made with this sort of calculation it is useful to check your results:

$$5\left(-\frac{1}{2}+\frac{\sqrt{3}i}{2}\right)^2 + 5\left(-\frac{1}{2}+\frac{\sqrt{3}i}{2}\right) + 5 = 0 \tag{2.69}$$

$$5\left(\frac{1}{4}-\frac{\sqrt{3}i}{2}-\frac{3}{4}-\frac{1}{2}+\frac{\sqrt{3}i}{2}+1\right) = 0 \tag{2.70}$$

and this is correct. Also check:

$$5\left(-\frac{1}{2}-\frac{\sqrt{3}i}{2}\right)^2 + 5\left(-\frac{1}{2}-\frac{\sqrt{3}i}{2}\right) + 5 = 0 \tag{2.71}$$

Exercise 2.2: Find the solution of the system using eigenvalues and eigenvectors:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Interim solution: eigenvalues are 1±*i*. eigenvectors are:

$$\upsilon_1 = \begin{pmatrix} p(1-i) \\ p \end{pmatrix} \quad \upsilon_2 = \begin{pmatrix} q(1+i) \\ q \end{pmatrix}$$

The total solution is:

$$\begin{pmatrix} p(1-i) \\ p \end{pmatrix} e^t\left[\cos(t) + i\sin(t)\right] + \begin{pmatrix} q(1+i) \\ q \end{pmatrix} e^t\left[\cos(t) - i\sin(t)\right]$$

After implementing the initial conditions $x_1(0)$ and $x_2(0)$, it follows that:

$$x_1 = e^t\left\{\left[x_1(0)\right]\cos(t) + \left[2x_2(0) - x_1(0)\right]\sin(t)\right\}$$
$$x_2 = e^t\left\{\left[x_2(0)\right]\cos(t) + \left[x_2(0) - x_1(0)\right]\sin(t)\right\}$$

After making such a calculation it is essential to fill in t=0 and to check the initial conditions. This is very simple and is a good first check. It is also essential to calculate $\mathrm{d}x_1/\mathrm{d}t$ and $\mathrm{d}x_2/\mathrm{d}t$ and

to check if dx_1/dt is equal to $2x_2$ and if dx_2/dt is equal to $-x_1+2x_2$ (according to the given differential equation). The next step is to look at what happens at an infinite time. In this case the solutions also go to infinity. This is an unstable, increasingly oscillating system. This happens because the real part of the eigenvalues is larger than zero (we will return to this later). Since few things in nature go to infinity, you are already certain that this solution -at least for large values of time- is not realistic.

2.2.2.4 Stability of systems

The term 'stable system' means: after a disturbance from outside, a system in a stationary situation returns to that same stationary situation at time ∞. Following the disturbance there can be no other influences upon the system from outside. In other words, a system is stable if it returns to the same state of equilibrium following a single external disturbance.

An example of a stable system the sudden heating of a block in a space with a constant temperature. Assume that the block and the space around it initially have the same temperature. At a certain point in time, the temperature of the block is raised instantaneously (pulse). After a certain length of time, the block will again be the same temperature as its surroundings (equilibrium) and nothing more will change. The system is stable in temperature. The change of the temperature of the block in time is the pulse response. The specific phenomenon being observed is important in determining whether a system is stable or not. If the observer is only interested in the position of the block, then nothing changes in the system. This is a completely different system, this system does not have to be stable in the horizontal plane. If the block is given a push (pulse) it can slide and come to rest in another state of equilibrium. The choice of the initial variable of the system is therefore important.

Most biological transformations and mass transfer processes result in stable systems. Chemical transformations and electrical apparatus can lead to unstable systems. We can make qualitative arguments about whether various systems are stable or not. As the systems become more elaborate, this form of reasoning becomes more and more difficult. A mathematical examination of the model can then provide an answer.

The analytical solution to a system was shown earlier in this chapter. This analytical solution provides insight into the stability of the system. The system of equations does not even have to be solved entirely. With linear first order systems, the eigenvalues λ of matrix A are equal to the negative of the reciprocal of the time constants of the system. In this example the eigenvalues are real numbers, but the eigenvalues can also be complex. For a system to be stable, the real part of every eigenvalue must be negative, the eigenvalue itself must lie in the left half of the complex plane.

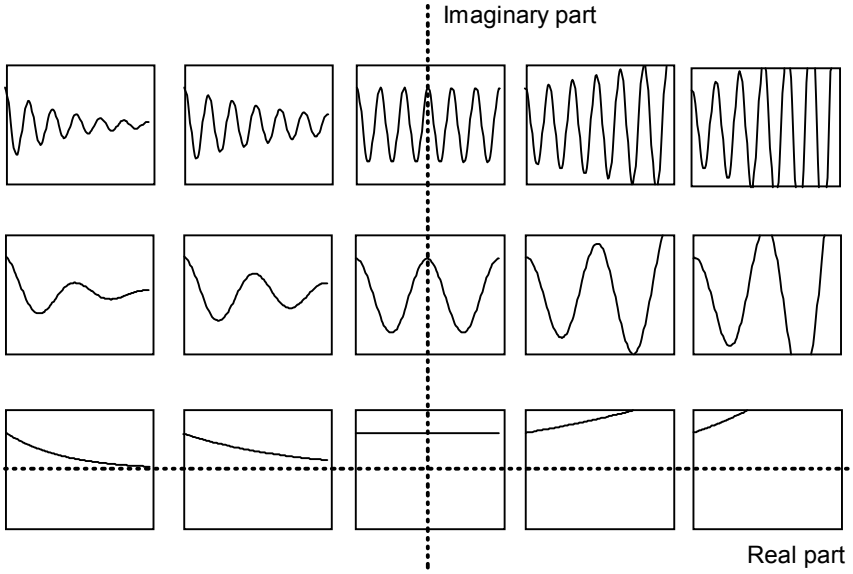

Figure 2.3: Pulse response at various eigenvalues.

Figure 2.3 shows the pulse response at various eigenvalues. If the imaginary part is zero and the real part negative, we obtain an exponentially falling curve, while if the real part is larger than zero we obtain an exponentially rising curve. It is striking that the response begins to oscillate as soon as the complex term in the eigenvalue is no longer equal to 0. If the real part is 0, a response is obtained which oscillates around the equilibrium value with a constant amplitude. If the real part is larger than 0 the oscillation increases, and with a real part smaller than 0 the oscillation decreases. As the imaginary part becomes larger, the frequency of the oscillation increases. If $\lambda = 0 + 0i$, then the system is indifferent (it remains constant, stationary, even following a disturbance there is no change over time). This is the case with the above example of the sliding block. In example 2.1 the eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = \mu$. The latter is larger than 0, so we have an unstable system. This is also apparent in the solution to 2.7 and 2.30, for $t \rightarrow \infty$ a biomass concentration of ∞ results and for the substrate, a concentration of $-\infty$. The cause of this instability is the hypothesis that the growth rate μ remains constant regardless of the substrate concentration. This is only true if the substrate concentration is so high that the maximum growth rate can be maintained. In the modelled system the substrate concentration decreases, and at a certain point in time the growth rate will also decrease. One way to maintain this substrate concentration at a high level is to continually add substrate. If the volume of the reactor is held constant by using an overflow, model 2.30 changes to:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} c_x \\ c_s - c_s(0) \end{pmatrix} = \begin{pmatrix} \mu - \frac{\phi}{V} & 0 \\ -\frac{1}{Y_{xs}}\mu - m_s & -\frac{\phi}{V} \end{pmatrix} \cdot \begin{pmatrix} c_x \\ c_s - c_s(0) \end{pmatrix} \tag{2.72}$$

Where ϕ is the flux (supply and removal flow rate) and V is the volume. The eigenvalues are now:

$$\lambda_1 = -\frac{\phi}{V} \tag{2.73}$$

$$\lambda_2 = \mu - \frac{\phi}{V} \tag{2.74}$$

λ_1 is always smaller than 0 because both V and ϕ are greater than 0. λ_2 is only smaller than 0 if $\mu < \phi / V$. If this condition is satisfied then we have a stable system.

If a system is stable that does not guarantee a physically operating system. In the example, a stable system is obtained if the flux is so high that all micro-organisms are washed out; $c_x = 0$ and $c_s = c_s(0)$. This system is formally stable, but there is no longer any production taking place. The only productive stable system is when $\mu = \phi / V$. In this case λ_2 is equal to 0.

As soon as we determine the eigenvalues, we can therefore draw conclusions about a system. You must, of course, consider these conclusions (such as stability, instability, oscillations, etc.) in the light of your knowledge about biology and physics. Mathematics is an abstraction of reality, but it must agree with reality. Interim evaluation is therefore both useful and necessary.

2.2.2.5 Numerical Solutions

In certain cases it is very difficult or impossible to solve a system of differential equations analytically. In that case, the equations can be solved numerically. This means that the differentials must be seen as very small changes $\partial x = \Delta x$. According to the definition, they are very small changes, in fact they are infinitely small steps. If we try to solve the problem numerically with the computer, it is impossible to calculate with infinitely small steps, because that would take an infinitely long time. Moreover, the computer has a finite resolution and infinitely small steps result in infinitely small changes which fall outside of this resolution. However, it is possible to calculate with small but finite steps, while keeping the error within limits. For this purpose we shall convert the differential equation to a difference equation. There are various methods for converting differentials into differences (see numerical mathematics). Various examples will be explained briefly in this section. For these conversions we can make good use of the Taylor series.

$$f(x+\mathrm{d}x) = f(x) + \frac{\partial f}{\partial x}\cdot \mathrm{d}x + \frac{1}{2!}\frac{\partial^2 f}{\partial x^2}\cdot \mathrm{d}x^2 + \frac{1}{3!}\frac{\partial^3 f}{\partial x^3}\cdot \mathrm{d}x^3 + \ldots\ldots \tag{2.75}$$

If we take only the first two terms, using k for the step (dx), we obtain:

$$\frac{f(x+k)-f(x)}{k} \approx \frac{\partial f}{\partial x} \tag{2.76}$$

This is the first possibility for rewriting the term $\partial f / \partial x$ (Euler forwards). We can also calculate $f(x\text{-}\mathrm{d}x)$:

$$f(x-\mathrm{d}x) \approx f(x) - \frac{\partial f}{\partial x}\cdot \mathrm{d}x \tag{2.77}$$

$$\frac{f(x)-f(x-k)}{k} \approx \frac{\partial f}{\partial x} \tag{2.78}$$

This is Euler backwards. We can also calculate $f(x+\mathrm{d}x)\text{-}f(x\text{-}\mathrm{d}x)$, disregarding all terms above the third:

$$\begin{aligned} f(x+\mathrm{d}x) &\approx f(x)+\frac{\partial f}{\partial x}\cdot \mathrm{d}x+\frac{1}{2!}\frac{\partial^2 f}{\partial x^2}\cdot \mathrm{d}x^2 \\ f(x-\mathrm{d}x) &\approx f(x)-\frac{\partial f}{\partial x}\cdot \mathrm{d}x+\frac{1}{2!}\frac{\partial^2 f}{\partial x^2}\cdot \mathrm{d}x^2 \end{aligned} \qquad (2.79)$$

$$\frac{f(x+k)-f(x-k)}{2k} \approx \frac{\partial f}{\partial x} \qquad (2.80)$$

This is a central difference. We can use the same procedure if we want to calculate a second derivative. We again take three terms and add the two equations together:

$$\begin{aligned} f(x+\mathrm{d}x) &\approx f(x)+\frac{\partial f}{\partial x}\cdot \mathrm{d}x+\frac{1}{2!}\frac{\partial^2 f}{\partial x^2}\cdot \mathrm{d}x^2 \\ f(x-\mathrm{d}x) &\approx f(x)-\frac{\partial f}{\partial x}\cdot \mathrm{d}x+\frac{1}{2!}\frac{\partial^2 f}{\partial x^2}\cdot \mathrm{d}x^2 \end{aligned} \qquad (2.81)$$

$$\frac{f(x+k)-2f(x)+f(x-k)}{k^2} \approx \frac{\partial^2 f}{\partial x^2}$$

Exercise 2.3: Write the system 2.28 and 2.29 so it can be solved numerically.

We will now compare the numerical and analytic solutions of the following equation:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -y \qquad (2.82)$$

The analytical solution is:

$$y = y_0 e^{-t} \qquad (2.83)$$

The numerical solution with Euler forwards is:

$$y(t+k) = y(t) - k\cdot y(t) = (1-k)\cdot y(t) \qquad (2.84)$$

(see also case 2.wk1 and the files growth.wk1 and xlnx.wk1 for numerical solutions)

time	analytical	k = 0.01	k = 0.1	k = 0.5	k = 1.5	k = 2.1
0	100	100	100	100	100	100
0.01	99.00498	99				
0.02	98.01986	98.01				
0.03	97.04455	97.0299				
0.04	96.07894	96.05960				
0.05	95.12294	95.09900				
0.1	90.48374	90.43820	90			
0.2	81.87307	81.79069	81			
0.3	74.08182	73.97003	72.9			
0.4	67.03200	66.89717	65.61			
0.5	60.65306	60.50060	59.049	50		
1	36.78794	36.60323	34.86784	25		
1.5	22.31301	22.14517	20.58911	12.5	-50	
2	13.53352	13.39796	12.15766	6.25		
2.1	12.24564	12.11688	10.94189			-110
2.5	8.208499	8.105851	7.178979	3.125		
3	4.978706	4.904089	4.239115	1.5625	25	
3.5	3.019738	2.967003	2.503155	0.78125		
4	1.831563	1.795055	1.478088	0.390625		
4.2	1.499557	1.468188	1.197251			121
4.5	1.110899	1.086019	0.872796	0.195312	-12.5	
5	0.673794	0.657048	0.515377	0.097656		
5.5	0.408677	0.397518	0.304325	0.048828		
6	0.237875	0.240500	0.179701	0.024414	6.25	
6.3	0.183630	0.177898	0.131002			-133.1
6.5	0.150343	0.145504	0.106111	0.012207		
7	0.091188	0.088031	0.062657	0.006103		
7.5	0.055308	0.053259	0.036998	0.003051	-3.125	
8	0.033546	0.032222	0.021847	0.001525		
8.4	0.022486	0.021555	0.014334			146.41
8.5	0.020346	0.019494	0.012900	0.000762		
9	0.012340	0.011794	0.007617	0.000381	1.5625	
9.5	0.007485	0.007135	0.004498	0.000190		
10	0.004539	0.004317	0.002656	0.000095		
10.5	0.002753	0.002611	0.001568	0.000047	-0.78125	-161.051
11	0.001670	0.001580	0.000926	0.000023		
11.5	0.001013	0.000956	0.000546	0.000011		
12	0.000614	0.000578	0.000322	0.000005	0.390625	

We see that k=0.01 provides a good solution and k=0.1 a solution which is somewhat further away. k=0.5 results in a very large error, k=1.5 results in an oscillating but stable solution, k=2.1 results in a non-stable, oscillating solution and will ultimately lead to overflow errors

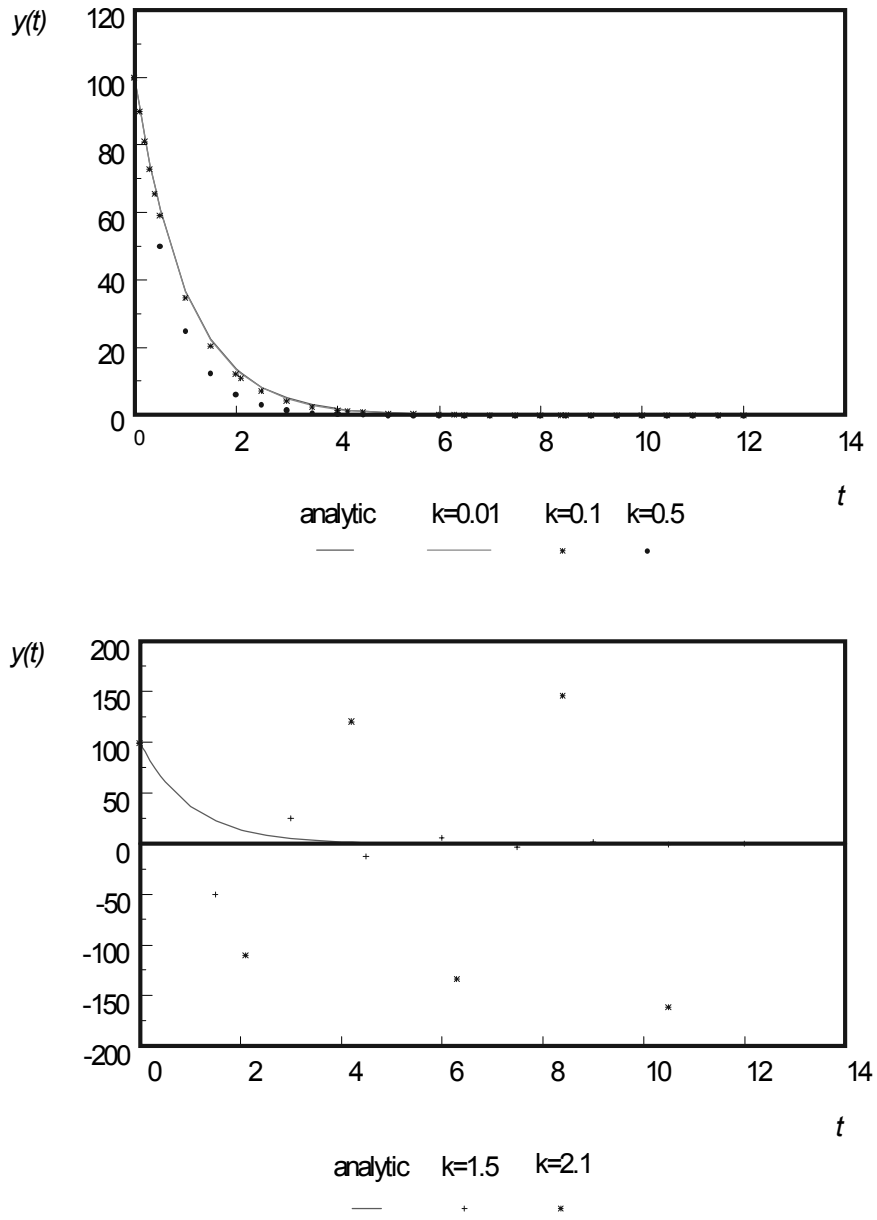

Figure 2.4: Result of the numerical solution dy/dt=-y

Exercise 2.4: Calculate the first five time steps of the solution where k=0.1 for both centered in time and backwards in time. For centered in time, the solution at t=-0.1 is also required, for this use 110.517. With backwards in time, an equation occurs where $y(t)$ is a function $y(t-k)$. Write this so that $y(t^*+k)$ is a function of $y(t^*)$ by assuming that $t=t^*+k$.

We see here that a small value of k results in the smallest deviation from the analytical solution. However, choosing very small values for k has two disadvantages. Firstly, the calculation time increases greatly and secondly, rounding off errors begin to cause problems (Figure 2.5). This graph is not drawn to scale. In general, the steps must be very small in order to cause problems with rounding off.

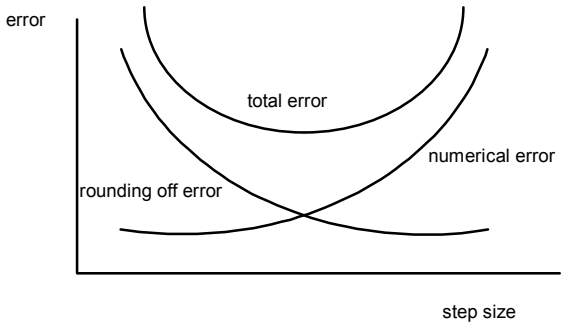

Figure 2.5: The effect of step size on numerical deviation (not drawn to scale).

In addition to the Euler method there are also other methods available for the numerical solution of a differential equation, e.g. the Runge-Kutta method (see numerical mathematics).

2.2.2.6 Partial Differential Equations

There are many means available to us for solving partial differential equations. The Laplace transformation can also be useful here, and the Fourier transformation can sometimes be used as well (see textbook on partial differential equations):

$$F\left[f(t)\right] = g(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-ist} \mathrm{d}t \tag{2.85}$$

This Fourier transformation is somewhat similar to the Laplace transformation, and can also be found in books of mathematical tables and tables in other handbooks. For general cases (e.g. the second law of Fick for non-stationary mass transfer without flows, equation 2.86) graphs are often available (see physical transport phenomena of mass transfer processes).

$$\frac{\partial c}{\partial t} = D \frac{\partial^2 c}{\partial x^2} \tag{2.86}$$

By using dimensionless indicators (Fo, Bi), these graphs can be used for finding a graphic solution for a wide range of problems. For the solution of partial differential equations we can also use numerical solutions (see exercise 2.5). Bessel functions are sometimes also necessary. These are series from zero to infinity. Previously, these were used a great deal because these functions (as with logarithms, for example) could be found in tables. After the computer -and the use of numerical solutions- became popular, Bessel functions were not used much any longer. However, since many calculation packages (such as Mathcad) have these Bessel functions available as a standard feature, the use of Bessel functions has again become simple and in some cases can be recommended. If the partial differential equation is solved numerically, the numerical stability must be examined for every distinct differential equation. However, if the equation is solved in a another way (resulting in Bessel functions, for example) one only has to pay attention to the stability of the determination of the Bessel functions, and the manufacturer of the computer package has usually taken care of this. With numerical solution methods, the chance is much larger (also with commercial packages) that something strange will happen (see the files masstran.wk1 and mass2.wk1 for the numerical solution of the second law of Fick).

Exercise 2.5: Write the second law of Fick (equation 2.86) by using forward in time, centered in time, and backward in time, and in all cases, centered in space.

2.2.2.7 Stability of numerical solutions

During the numerical solution of a linear differential equation, a time step must be chosen. The previous example showed that the step size cannot be chosen freely. An excessively large time step results in a system that oscillates uncontrollably. If you are using a simulation package, you must make sure that a step size is chosen which is small enough to obtain stability. A strange result of a simulation can occur when the wrong time step is chosen. A rule of thumb is that if a time step larger than $|0.2 / \lambda|$ is chosen an inaccurate, as well as numerically unstable, system is obtained (with λ the largest eigenvalue). For complex eigenvalues the time step must be quite a bit smaller than the period of oscillation, for example thirty steps per oscillation. The frequency of the oscillation is $\mathrm{Im}(\lambda)/2\pi$ and the period $1/f$, therefore a time step should be chosen which is smaller than $2/30*\pi / \mathrm{Im}(\lambda) \approx 0.2 /\mathrm{Im}(\lambda)$. With partial differential equations the step size cannot be determined in a simple fashion. One method is to study the error propagation behaviour by using the so-called Von Neumann stability. This method examines which solutions of the form:

$$u_{n,k} = \alpha^n e^{i\beta x_k} \tag{2.87}$$

are stable. If $|\alpha| \leq 1$ for all α, the solution is Von Neumann stable. If we apply this to, for example, the equation:

$$\frac{\partial u}{\partial t} + b\frac{\partial u}{\partial x} = 0 \tag{2.88}$$

where $b>0$, and solve this equation numerically with:

$$\frac{u_{n+1,k} - u_{n,k}}{h_t} = -b\frac{u_{n,k+1} - u_{n,k-1}}{2h_x} \tag{2.89}$$

we obtain:

$$\frac{\alpha^{n+1} e^{i\beta x_k} - \alpha^n e^{i\beta x_k}}{h_t} = -b\frac{\alpha^n e^{i\beta x_{k+1}} - \alpha^n e^{i\beta x_{k-1}}}{2h_x} \tag{2.90}$$

$$\frac{\alpha - 1}{h_t} = -b\frac{e^{i\beta h_x} - e^{-i\beta h_x}}{2h_x} \tag{2.91}$$

if we use:

$$e^{ip} = \cos p + i \sin p \tag{2.92}$$

we obtain:

$$e^{ip} - e^{-ip} = \cos(p) + i\sin(p) - \cos(-p) - i\sin(-p) = 2i\sin(p) \tag{2.93}$$

If we use this, we obtain:

$$\alpha = 1 - \frac{bh_t i \sin(\beta h_x)}{h_x} \tag{2.94}$$

The modulus can be larger than 1, therefore this does not provide a stable solution. With this numerical solution the simulation will not go towards the real solution; the system is numerically unstable for every step size. To solve this model numerically, it would better to choose another conversion to differences. Once again, you must check that this conversion leads to a numerically stable system.

Exercise 2.6: Compare the Von Neumann stability of the equation: $\frac{\partial u}{\partial t} = b^2 \frac{\partial^2 u}{\partial x^2}$

with Euler forwards and Euler backwards of the time derivative.
Take note: $1 - \cos(2\phi) = 2\sin^2 \phi$

2.2.2.8 Mathematical notation for differential equations

This section explains various forms of mathematical notation:
There are various kinds of derivatives:

partial derivative:

$$\frac{\partial f(x,t)}{\partial t} = \lim_{\Delta t \to 0} \frac{f(x,t+\Delta t) - f(x,t)}{\Delta t} \tag{2.95}$$

In words: the partial derivative is the change of *f* with an infinitely small change of *t*, while all other variables (such as *x*, *y* and *z*) remain constant. Therefore, everything remains constant except *t* !

Take note: the symbol for partial derivative is a 'bent d' (∂) and not a delta (δ)!!! A delta is the mathematical symbol for a (Dirac) pulse!!!

the normal derivative:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \lim_{\Delta t \to 0} \frac{f(x,t+\Delta t) - f(t)}{\Delta t} \tag{2.96}$$

In words: the derivative is the change of *f* with an infinitely small change in *t* (where all other possible variables can also change).

Take note: the d for derivative is a mathematical operator (like sin, ln, exp) and is given in roman case; variables and functions (*x,t,f*) are given in italics.

It is clear from the above that these two derivatives are very different things, although they can be equal to each other. They can also be written together:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} \cdot \frac{\mathrm{d}x}{\mathrm{d}t} \quad \left(+ \frac{\partial f}{\partial y} \cdot \frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial f}{\partial z} \cdot \frac{\mathrm{d}z}{\mathrm{d}t} \right) \tag{2.97}$$

Until now we have only looked at the variables *t*, *x*, *y* and *z* but there can, of course, be all kinds of other variables (e.g. temperature and pressure).

Let us explain this with an example. I am standing against a wall in a room, and someone who is standing next to the opposite wall opens a jar that smells like caramel. If I remain at initial location, ($\partial c / \partial t = \mathrm{d}c / \mathrm{d}t$ because *x* does not change), it will gradually begin to smell more and more like caramel. However, if I walk slowly towards the other wall, my observation will be much stronger than if I remain still, because I am walking into the increasing concentration gradient. If written as a formula:

$$\frac{\mathrm{d}c}{\mathrm{d}t} = \frac{\partial c}{\partial t} + \frac{\partial c}{\partial x} \cdot \frac{\mathrm{d}x}{\mathrm{d}t} \tag{2.98}$$

In words: the observation of caramel fragrance is that which I would have if I remained still, plus how quickly this would change if I moved * how quickly I moved. If I remained still (dx/dt=0), I observe the concentration changing in time (partial derivative = normal derivative). Be aware that normal d's are used for dx/dt in this formula. The position is dependent only on time, and is not dependent on c,y, or z (therefore $\partial x / \partial t$ = dx/dt).

In many cases, the derivative according to time is also indicated by placing a dot above the variable. For example:

$$\dot{y} = y' = \frac{\mathrm{d}y}{\mathrm{d}t} \tag{2.99}$$

Partial derivatives are sometimes indicated with a subscript:

$$u_{xx} = \frac{\partial^2 u}{\partial x^2} \tag{2.100}$$

The derivative of something in the x, y and z direction is frequently indicated with the symbol 'nabla', an upside-down capital letter Delta:

$$\nabla f = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} \tag{2.100a}$$

equation 2.97 can then be rewritten as:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} \cdot \frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y} \cdot \frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial f}{\partial z} \cdot \frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \nabla f \begin{pmatrix} \frac{\mathrm{d}x}{\mathrm{d}t} \\ \frac{\mathrm{d}y}{\mathrm{d}t} \\ \frac{\mathrm{d}z}{\mathrm{d}t} \end{pmatrix} = \frac{\partial f}{\partial t} + \nabla f \cdot \vec{v} \tag{2.97a}$$

In equations with this nabla, one is therefore looking in all three directions. It can best be understood if you first read it as simply the 'derivative in the x direction'.

2.3 Control systems

2.3.1 Off/on and PID controllers

Many systems can be controlled, i.e. we can carry out an action from outside the system where we try to bring the system into the state we want to be in or try to keep it there. For example, we want to maintain a constant temperature inside a large vessel. In order to do this we must first measure the temperature in the vessel and have a way of changing this from outside, such as placing a heat exchanger in the vessel. We can then regulate the temperature in the vessel by means of the flow rate and the temperature of the water in the heat exchanger. In order to do this, we must have a control which does not over-react or under-react. In the first case that can lead to an undesired temperature and in the second case it can cause too much wear on the pump or the burners or the bimetal. In order to avoid this, one must know approximately what the

effect of a control action on the system is. There are many control strategies. Only a few different techniques will be briefly explained here.

On/off controller

One of the simplest controllers is the on/off controller (e.g. the simple thermostat in a home). The disadvantage of this controller is that it does not usually control very precisely. For example, the temperature in a home can vary between 17°-19°C (the thermostat goes on at 17° and off at 19°). If it is set more precisely it becomes excessively sensitive. This is usually not very beneficial (e.g. higher energy costs, more wear and tear).

PID controller

One of the most widely used controllers in practice is the PID controller, or a special form of this (e.g. the PI controller). PID stands for Proportional Integrating Differentiating Controller. This controller requires a regulation parameter for every action (for action P a parameter K_p). The controller sends out a control signal (u) based on the measured error (ε). This measured error is multiplied by the proportional factor (e.g. if the temperature is 10°C too low, I must add approximately 2 kJ energy if it is 2°C too low I must add 0.4 kJ). The error signal is also integrated in time and this is multiplied by the integrating parameter (K_i). This integrating action reduces small deviations in the long term. In addition, the derivative of the error signal can be multiplied by the differentiating parameter (K_d). This action 'looks' to see how strongly you are going in the right direction or the wrong direction. If the derivative of the error is positive, the error is becoming larger and larger, and you should control more strongly. If the derivative of the error is negative, the error is becoming smaller, so things are going in the right direction and the controller can begin to react somewhat less.

$$u(t) = K_p \cdot \varepsilon + K_i \int_0^t \varepsilon \mathrm{d}t + K_d \frac{\mathrm{d}\varepsilon}{\mathrm{d}t} \tag{2.101}$$

Frequently, measurements take place only at discrete time intervals. We can then use, for instance, the following control law:

$$u(k) = K_p \cdot \varepsilon(k) + K_i \tau \sum_{i=0}^{k} \varepsilon(i) + K_d \frac{\varepsilon(k) - \varepsilon(k-1)}{\tau} \tag{2.102}$$

where τ is the control time step.

Now take two simple systems which can be described with
system 1:

$$x(k+1) = x(k) - 0.1 \cdot u(k) \tag{2.103}$$

system 2:

$$x(k+1) = x(k) - 0.1 \cdot u(k) + 0.1 \tag{2.104}$$

Use the following values: $K_p = 5$, $K_i = 0.5$, $K_d = 0.5$, $\tau = 1$. If we then look at the reaction of the two systems with a P controller, a PI controller and a PID controller, we obtain Figures 2.6 and 2.7 at an initial disturbance of 2.

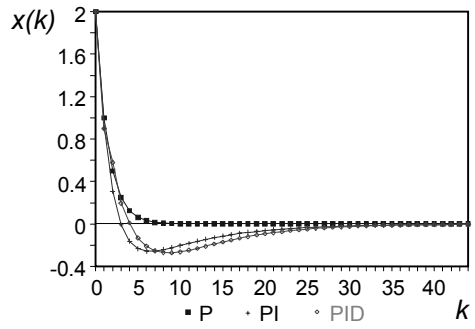

Figure 2.6: System 1

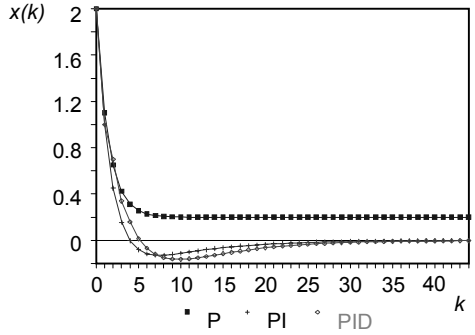

Figure 2.7: System 2

We see in these graphs that the P controller works well in compensating and reducing a disturbance in system 1, but that if there is a constant disturbance (0.1) in the system there will be a constant deviation. By adding an I action, this deviation can be controlled and removed. The D action can be useful, but frequently results in controls which react nervously (especially if there are sudden, large disturbances, this results in a strong control action, such as with strong noise).

Take note: with controls, you should realize that you don't always have to eliminate the disturbance instantly; at the next time interval the value will be measured again and another control action will be carried out. Models which predict the effect of a control action therefore do not have to be exact, but they must provide a good representation of the dynamic behaviour of the system.

2.3.2 Equivalent discrete systems

Most actual systems will be continuous, they can therefore be better described with a continuous (mathematical) system. Control takes place with the help of computers, which must carry out measurements at fixed points in time and carry out a control action. The control algorithm is therefore discrete. We can convert a continuous system of differential equations into a system of discrete difference equations.

Assume the following general system:

$$\underline{\dot{x}}(t) = A\,\underline{x}(t) + B\,\underline{u}(t) \tag{2.105}$$

$$\underline{y}(t) = C\,\underline{x}(t) + D\,\underline{u}(t) \tag{2.106}$$

$$\underline{u}(t) = \underline{u}(t_i) \tag{2.107}$$

In these equations, x is the state variable. The first equation is the description of this system (differential equation). In addition, y is the measured output, which is a function of x and u (the control signal). We can solve the differential equation between two samples (between nT and $(n+1)T$, with T being the sample time) because the control signal between these two times is constant.

$$\underline{x}(i+1) = \Phi\,\underline{x}(i) + \Gamma\,\underline{u}(i) \tag{2.108}$$

$$\underline{y}(i) = C\,\underline{x}(i) + D\,\underline{u}(i) \tag{2.109}$$

$$\Phi = \exp(AT) \tag{2.110}$$

$$\Gamma = \int_0^T \exp(A\lambda) B \mathrm{d}\lambda \tag{2.111}$$

Take note:

$$\exp(At) = I + At + A\frac{t^2}{2!} + A^3\frac{t^3}{3!} + \ldots \tag{2.112}$$

This is the definition, not the way to solve it. For a simple example of an EDS, see example 2.2.

2.3.3 Optimal Control

If we want to control a system optimally, we must first decide what 'optimal' is. For this purpose we can define a cost function which we will minimize. This cost function is a trade-off between how much a deviation from the state variable costs with respect to the control. For example, when controlling the oxygen concentration in a reactor, the deviation from the oxygen concentration results in a loss of production, but a desired change in the oxygen concentration also results in costs (for compression, physical limitations, energy). A widely used cost function for such process management is the quadratic cost function.

For a continuous system:

$$J = \int_0^\infty \left[\underline{x}^t(t) Q \underline{x}(t) + \underline{u}^t(t) R \underline{u}(t) \right] \mathrm{d}t \tag{2.113}$$

For a discrete system:

$$J = \sum_{i=0}^{\infty} \left[\underline{x}^t(i+1) Q \underline{x}(i+1) + \underline{u}^t(i) R \underline{u}(i) \right] \tag{2.114}$$

In this derivation $\underline{x}$ =0 is assumed to be optimal. As the deviation of the control signal becomes larger, the costs increase in a quadratic fashion. This quadratic increase has the additional benefit that negative as well as positive deviations have the same consequences. (Take note: this does not always have to be beneficial.) The matrices Q and R are the weighing matrices and their relationship indicates the trade-off between the costs of control and the loss of production caused by deviations from the state variable. We can choose a linear control law, i.e. the control action (u) is linearly proportional to the condition (x) of the system. The cost function is then minimal ($\mathrm{d}J/\mathrm{d}F$=0) if the following equations apply:

continuous:

$$\underline{u}(t) = -F \underline{x}(t) \tag{2.115}$$

$$F = R^{-1} B^t P \tag{2.116}$$

$$Q - PBR^{-1}B^t P + PA + A^t P = 0 \tag{2.117}$$

discrete:

$$\underline{u}(i) = -L \underline{x}(i) \tag{2.118}$$

$$L = \left(\Gamma^t S \Gamma + R \right)^{-1} \Gamma^t S \Phi \tag{2.119}$$

$$S = \Phi^t S \Phi - \Phi^t S \Gamma \left(\Gamma^t S \Gamma + R \right)^{-1} \Gamma^t S \Phi + Q \quad \wedge \quad S > 0 \tag{2.120}$$

This is the algebraic matrix Riccati equation. These equations are difficult to solve, computers can be of help. However, in order to obtain some idea of what this involves, we shall optimize a simple system and prove that it is optimal.

Example 2.2: continuous optimal control.

Assume the following system description:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 0 + u(t) \tag{2.121}$$

(the change in x is equal to the control signal). Use these values for this system: A=0 and B=1 For the weighted matrices use Q=R=1. We then obtain the following control law:

$$\underline{u}(t) = -F\,\underline{x}(t) \qquad F = R^{-1}B^t P = P \tag{2.122}$$

$$0 = Q - PBR^{-1}B^t P + PA + A^t P = 1 - P^2 \quad \Rightarrow \quad P = \pm 1 \tag{2.123}$$

Because the system must be physically stable, the only satisfactory value is F=P=1. Therefore the optimal linear control law is:

$$u(t) = -x(t) \quad \rightarrow \quad \frac{\mathrm{d}x}{\mathrm{d}t} = -x(t) \tag{2.124}$$

We see that the eigenvalue of the controlled system has become equal to -1. The eigenvalue of the uncontrolled system was 0 (see equation 2.121). We can now prove that this is optimal (This is only used as an illustration. With most systems this is too complicated and also not necessary). The formulas provide the optimal solution:

$$u(t) = -Fx(t) \tag{2.125}$$

The system description is:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u(t) = -Fx(t) \tag{2.126}$$

The solution to this equation is:

$$x = x(0)\exp(-Ft) \tag{2.127}$$

The control action is therefore:

$$u(t) = -Fx(t) = -Fx(0)\exp(-Ft) \tag{2.128}$$

For the cost function, we can fill in:

$$J = \int_0^\infty x^2(0)\exp(-2Ft) + F^2 x^2(0)\exp(-2Ft)\mathrm{d}t \tag{2.129}$$

$$J = x^2(0)\frac{(1+F^2)}{-2F}\left[\exp(-2Ft)\right]_0^\infty = x^2(0)\frac{(1+F^2)}{2F} \tag{2.130}$$

(Only if F>0!) This is optimal if dJ/dF=0:

$$\frac{\mathrm{d}J}{\mathrm{d}F} = x^2(0)\frac{2F\cdot 2F - \left(2+2F^2\right)}{4F^2} = 0 \tag{2.131}$$

$$F^2 = 1 \quad \rightarrow \quad F = \pm 1 \tag{2.132}$$

F = -1 is unsatisfactory, when the upper limit following integration is filled in, the e power is not 0.

We can also establish an equivalent discrete system (A=0;B=1) for this system (with sample time T):

$$\Phi = \exp(AT) = \exp(0) = 1 \tag{2.133}$$

$$\Gamma = \int_0^T \exp(A\lambda)B\mathrm{d}\lambda = \int_0^T \mathrm{d}\lambda = T \tag{2.134}$$

The system description then becomes (think about how this is logical):

$$x(k+1) = x(k) + u(k)T \tag{2.135}$$

The change in x is equal to the control signal (equation 2.121). Therefore, if we want to know the new x, that is equal to the old x plus the change, multiplied by the time span.

If we again want to find an optimal linear control law, this becomes:

$$u(k) = -Lx(k) \qquad L = \frac{TS}{T^2S + R} \tag{2.136}$$

$$S = S - \frac{T^2S^2}{T^2S + R} + Q \tag{2.137}$$

Assume that the time interval T = 1s and $R = Q = 1$. We now solve the second equation first:

$$S = S - \frac{S^2}{S+1} + 1 \qquad S^2 - S - 1 = 0 \qquad S = \frac{1 \pm \sqrt{1+4}}{2} \tag{2.138}$$

The positive root is equal to:

$$S = \frac{1+\sqrt{1+4}}{2} = 1.618 \qquad L = \frac{S}{S+1} = 0.618 \tag{2.139}$$

The optimal digital control law is:

$$u(k) = -0.618 \cdot x(k) \tag{2.140}$$

We can now calculate the expected results if x becomes 1 due to a disturbance:
$x(0)$=1.000: $u(0)$=-0.618; $x(1)$=0.382: $u(1)$=-0.236; $x(2)$=0.146: $u(2)$=-0.090; $x(3)$=0.056: etcetera

We see that we can describe $x(k+1)$ as:

$$x(k+1) = x(k) - Lx(k) = (1-L)\cdot x(k) = 0.382 \cdot x(k) \tag{2.141}$$

The result after a disturbance is:

$$x(n) = (0.382)^n \cdot x(0) \tag{2.142}$$

Exercise 2.7: Calculate an expression for the cost function of this system where you do not yet fill in the optimal value of L.

Exercise 2.8: Use the series:

$$a + ar + ar^2 + ar^3 + ar^4 + \ldots = \sum_{i=0}^{\infty} ar^i = \frac{a}{1-r}$$

$$|r| < 1$$

to write the cost function in a simpler fashion. Now calculate the optimal value of L by setting dJ/dL equal to zero.

Exercise 2.9: Calculate the optimal control law for example 2.2 where Q=4 and R=1. The error in this condition is then weighted more heavily. Do this for both the continuous and the discrete case. Calculate the optimal control law by using the formulas and by optimizing the cost function.

2.3.4 Solutions

Solution 2.1

$$x_1(t) = \left(x_1(0) + \frac{1}{4}x_2(0)\right)\exp(5t) - \frac{1}{4}x_2(0)\exp(t) \quad x_2(t) = x_2(0)\exp(t)$$

Solution 2.3

$$c_x(t+k) = c_x(t) + k\mu c_x(t) \quad c_s(t+k) = c_s(t) + k\left(-\frac{1}{Y_{xs}}\cdot\mu\cdot c_x(t) - m_s c_x(t)\right)$$

Solution 2.4 (see also case2.wk1)

time	analytical	FT	CT	BT
-0.1	110.517092			
0	100	100	100	100
0.1	90.48374	90	90.517092	90.909
0.2	81.87307	81	81.89658	82.6446
0.	74.08182	72.9	74.13778	75.13148
0.4	67.03200	65.61	67.06903	68.30135
0.5	60.65306	59.049	60.72397	62.09213

Solution 2.5

$$c(x,t+k) = c(x,t) + \frac{kD}{h^2}\left[c(x+h,t) - 2c(x,t) + c(x-h,t)\right]$$

$$-\frac{kD}{h^2}c(x+h,t) + c(x,t)\left(1 + \frac{2kD}{h^2}\right) - \frac{kD}{h^2}c(x-h,t) = c(x,t-k)$$

Realize that this is difficult to solve. It is possible with a matrix vector system.

$$c(x,t+k) = c(x,t-k) + \frac{2kD}{h^2}\left[c(x+h,t) - 2c(x,t) + c(x-h,t)\right]$$

Solution 2.6

First Euler forwards:

$$u_t = b^2 u_{xx} \quad \rightarrow \quad \frac{u_{n+1,k} - u_{n,k}}{h_t} = b^2 \frac{u_{n,k+1} - 2u_{n,k} + u_{n,k-1}}{h_x^2}$$

$$\frac{\alpha^{n+1} e^{i\beta x_n} - \alpha^n e^{i\beta x_k}}{h_t} = \frac{b^2 \alpha^n}{h_x^2}\left(e^{i\beta x_{k+1}} - 2e^{i\beta x_k} + e^{i\beta x_{k-1}}\right) \quad \rightarrow \quad \frac{\alpha - 1}{h_t} = \frac{b^2}{h_x^2}\left(e^{i\beta h_x} - 2 + e^{-i\beta h_x}\right)$$

Now:

$$e^{ip} + e^{-ip} = 2\cos p \quad \rightarrow \quad \frac{\alpha - 1}{h_t} = 2\frac{b^2}{h_x^2} \cdot \left[\cos(\beta h_x) - 1\right] \quad \rightarrow \quad \alpha = 1 - 4h_t \frac{b^2}{h_x^2} \cdot \sin^2\left(1/2\beta h_x\right)$$

This term is always smaller than 1 and must still be larger that -1. Therefore, regarding stability:

$$h_t \frac{b^2}{h_x^2} \leq 1/2$$

After reverse difference:

$$u_t = b^2 u_{xx} \quad \rightarrow \quad \frac{u_{n,k} - u_{n-1,k}}{h_t} = b^2 \frac{u_{n,k+1} - 2u_{n,k} + u_{n,k-1}}{h_x^2}$$

$$\frac{\alpha^n e^{i\beta x_k} - \alpha^{n-1} e^{i\beta x_{kn}}}{h_t} = \frac{b^2 \alpha^n}{h_x^2}\left(e^{i\beta x_{k+1}} - 2e^{i\beta x_k} + e^{i\beta x_{k-1}}\right) \quad \rightarrow \quad 1 - \frac{1}{\alpha} = -4\frac{h_t b^2}{h_x^2} \cdot \sin^2\left(0.5\beta h_x\right)$$

$$\frac{1}{\alpha} = 1 + 4\frac{h_t b^2}{h_x^2} \cdot \sin^2\left(0.5\beta h_x\right)$$

This term is always larger than or equal to 1, therefore α always lies between 0 and 1 and is always stable. (Note that this is the inverse of α !) A very large time interval can therefore be chosen. Note that this solution is numerically stable but can be very imprecise!

Solution 2.7

$$J = \left(1 - 2L + 2L^2\right)\left[x(0)\right]^2 \sum_{i=0}^{\infty}\left[(1-L)^{2i}\right]$$

Solution 2.8

$$J = \left(1 - 2L + 2L^2\right)\left[x(0)\right]^2 \frac{1}{1-(1-L)^2} \quad 0 = 2L^2 + 2L - 2$$

The positive root of this equation is: $L = \frac{-1+\sqrt{5}}{2} = 0.618$

Solution 2.9

Continuous: $F = P = 2$. The negative solution is not satisfactory. Note that the control becomes stronger (this is logical) if the error has been weighted more heavily with respect to the control action.

Discrete system: $L = -2 + 2\sqrt{2} = 0.828$. Note that the control also becomes stronger here!

3. Statistics

After we have established a model, we want to be able to predict how a system will behave. To this end it is essential to estimate, to 'fit', the parameters which are used in the model. In statistics, methods have been developed to estimate parameters and to determine the confidence interval of the estimation. The confidence interval of the prediction can also be calculated. Finally, this chapter explains how you can compare models with each other.

We shall consider a model to be a mathematical relationship which relates one or more independent variables x to a single dependent variable y. For example:

$$y = b_0 + b_1 x_1 + b_2 x_2$$

where x_1 and x_2 are the explanatory variables and b_1 and b_2 are the parameters.

3.1. Estimations

We are frequently interested in the mean of a series of measurements. If all values y_i of the population are known (measured), we can **calculate** the mean and the variance (mean squared error) of this population. The mean is calculated as follows:

$$\bar{y} = \frac{\sum_{i=1}^{n} (y_i)}{n} \tag{3.1}$$

The variance is calculated as follows:

$$\sigma^2 = \frac{RSS}{n} = \frac{\sum_{i=1}^{n} (y_i - \bar{y})^2}{n} \tag{3.2}$$

In this formula *RSS* = *Residual Sum of Squares*, the sum of the squares of the differences between the predicted average and the measured values. This is the variance for this particular series of measurements. Until now we have not had to estimate anything, we divide by n. We have calculated the average of the series of measurements $\bar{y}$ with the accompanying standard deviation σ. You can use this, for example, to describe the weight of 1000 samples in the refrigerator, if you have actually determined the weight of all 1000.

Frequently, however, (random) samples are taken and we approximate the average value. Sometimes this technique is chosen because the set is too large (weighing 1000 samples is a big job) and sometimes because no discrete samples can be taken. An example of this would be the determination of the pH, temperature, concentration etc. of a solution. You can measure these values as often as you like. A widely used approximation for the average of the population from which a series of measurements has been selected is (if the measurement error is normally distributed and there is the same standard deviation for all measurements):

$$\hat{y} = \frac{\sum_{i=1}^{n} (y_i)}{n} \tag{3.3}$$

In this formula $\hat{y}$ is the estimated mean (note that in this case, this is equal to $\bar{y}$), n the number of measurement points and y_i the measured values.

The variance of the population from which this measurement series originates can be **estimated** with:

$$s^2 = \frac{RSS}{n-1} = \frac{\sum_{i=1}^{n}(y_i - \hat{y})^2}{n-1} \tag{3.4}$$

In this formula *RSS* = *Residual Sum of Squares*, the sum of the squares of the differences between the model approximation and the measured values. The confidence interval of $\hat{y}$ can be calculated with the *t* test. In this test the pure standard deviation of the estimated mean is:

$$s_{\hat{y}} = \frac{s}{\sqrt{n}} = \sqrt{\frac{s^2}{n}} = \sqrt{\frac{RSS}{n(n-1)}} \tag{3.5}$$

The two-sided confidence interval then becomes:

$$y = \hat{y} \pm t_{DF,1-1/2\alpha} s_{\hat{y}} = \hat{y} \pm t_{DF,1-1/2\alpha} \cdot \sqrt{\frac{RSS}{n(n-1)}} \tag{3.6}$$

with t_{DF} the (Student's) *t* value with *DF* degrees of freedom with confidence level α. Degrees of freedom are actually the number of measurement points which are in excess of the minimum required to approximate the necessary parameters. Here we want to estimate the mean value of *y*, just one parameter. If we can perform one measurement of *y*, we can already estimate $\hat{y}$. However, we would then know nothing about the confidence. If we can perform five measurements of *y*, we can make a better estimate of $\hat{y}$ and we can also make a judgement about the confidence. We then have 4 degrees of freedom (5 measurement points - 1 parameter).

Example 3.1
Suppose we measure the pH of a solution, and do this 5 times: 5.24, 5.30, 5.28, 5.28, 5.30
The best estimate of the pH is then:

$$\hat{y} = \frac{\sum_{i=1}^{n}(x_i)}{n} = 26.40 / 5 = 5.28$$

The calculated variance of this series of measurements is:
$\sigma^2 = (0.04 \cdot 0.04 + 0.02 \cdot 0.02 + 0.02 \cdot 0.02)/5 = 0.0024/5 = 0.00048$.
The estimated variance of the population is:
$s^2 = (0.04 \cdot 0.04 + 0.02 \cdot 0.02 + 0.02 \cdot 0.02)/4 = 0.0024/4 = 0.0006$
and the standard deviation of the population is:
$s = 0.0245$

We have 4 degrees of freedom, therefore the Student's *t* value with 95% confidence and 4 degrees of freedom is 2.78 (two-sided test). The standard deviation of the mean is $s_{\hat{y}} = 0.0245 / \sqrt{5} = 0.01096$, therefore the confidence interval of the estimate of the pH is:

$$pH_{\min} = 5.28 - 2.78 \cdot 0.010956 = 5.250; \qquad pH_{\max} = 5.310$$

we can therefor say with 95% confidence that the actual pH lies between 5.25 and 5.31. If we perform a new measurement, we do not necessarily have to measure the actual pH, because the new

measurement once again has measurement error. The variance of the new measurement is the sum of the error in $\hat{y}$ and the error made during this new measurement:

$$\text{var}(y_{new}) = \text{var}(\hat{y}) + s^2 = \frac{1}{n} \cdot \frac{RSS}{n-1} + s^2 = \left(\frac{1}{n} + 1\right) \cdot s^2 \tag{3.7}$$

Take note: if n is large, 1 shall be >> $1/n$ and $var(y_{new})$ will approach s^2.

Analogously to (3.6), the confidence interval for the new measured value is:

$$y_{new} = \hat{y} + t_{df,1-1/2\alpha} \cdot \sqrt{\text{var}(y_{new})} = \hat{y} + t_{df,1-1/2\alpha} \cdot \sqrt{\left(\frac{1}{n} + 1\right) s^2} \tag{3.8}$$

If we measure the pH yet again, the value will then lie between:

$$pH_{\text{new,min}} = 5.28 - 2.78 \cdot \sqrt{(1/5+1) \cdot 0.0006} = 5.205; \quad pH_{\text{new,max}} = 5.355$$

3.2. Linear regression

Linear regression is a technique to find the best estimate of the parameters of a linear equation (model). In this section the solution method and the statistical calculations will be explained. For this purpose we shall use the dependent variable y, which is a function of x. The measurement points of y shall be called y_i and the model prediction $y_{s,i}$, sometimes also written as $\hat{y}$. In linear regression, one begins with the following assumptions:

1. a single y goes with every x, the variance σ^2 is the same for every y independent of the x value.
2. every y is independent of every other y.
3. every measurement x_i is accompanied by a potential population of y_i values which are normally distributed.

Notice that nothing is said about x. This is because it is assumed that x is true, that it has no error.

These conditions must be satisfied to make linear regression possible. We will first show linear regression using the formula $y = bx$ and later by using an arbitrary linear equation. This can also be found in books on statistics. In this context linear refers to linearity in the parameters: $y = b_0 + b_1x^2 + b_2x^3$ is linear in b_0, b_1 and b_2 (not in x). We will discuss how you can decide if you have satisfied the conditions of linear regression. If you have not satisfied these conditions, you can sometimes use transformations to satisfy the assumptions for linear regression. Finally we will discuss one method of non-linear regression.

3.2.1. Straight line through the origin (e.g. calibration line)

Linear regression with: $y = bx$.

where $\vec{x} \neq 0$. We can see this as: $\vec{y}_s = b \cdot \vec{x}$ (3.9)

$$eg \quad \begin{pmatrix} 0 \\ 2 \\ 4 \\ 6 \end{pmatrix} = 2 \cdot \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}$$

Where x is a vector filled with the points which have been measured (the x values).

We now have a vector x and y (data points on a line) and we would like to know what b is. To find this we choose a minimization criterium, in this case a minimal sum of squares of the deviations (residual sum of squares = *RSS*), where the deviation (dy) is the difference between the measured y value (y) and the predicted y value (y_s):

$$\mathrm{d}\,\vec{y} = \vec{y} - \vec{y}_s \tag{3.10}$$

Therefore we want to minimize $RSS = \sum_i \left(\mathrm{d}\, y_i^2\right)$ to b.

We can write this as:

$$RSS = \sum_i \left[\left(y_i - \hat{y}_i\right)^2\right] = \sum_i \left[\left(y_i - b \cdot x_i\right)^2\right] \tag{3.11}$$

We can minimize this *RSS* by determining the derivative to b and setting this to 0.

$$\frac{\mathrm{d}RSS}{\mathrm{d}b} = \frac{\mathrm{d}}{\mathrm{d}b} \sum_i \left(y_i^2 - 2b \cdot y_i \cdot x_i + b^2 \cdot x_i^2\right) = \sum_i \left(-2y_i \cdot x_i + 2 \cdot b \cdot x_i^2\right) \tag{3.12}$$

The second derivative is $\sum_i \left(2x_i^2\right)$ and is therefore positive, so we have a minimum. For this minimum the following applies:

$$0 = \frac{\mathrm{d}RSS}{\mathrm{d}b} = \sum_i \left(-2y_i \cdot x_i + 2b \cdot x^2\right) \Rightarrow \sum_i \left(2y_i \cdot x_i\right) = \sum_i \left(2b \cdot x_i^2\right) \quad \Rightarrow \quad b = \frac{\sum y_i \cdot x_i}{\sum x_i^2} \tag{3.13}$$

We can also derive this in another way. In matrix vector calculation this amounts to the perpendicular projection of y-y_s on the vector x. This is because y_s lies on this vector (because it is a linear combination of the vector) and we thus minimize the distance between y and y_s. In other words, we have a vector with data points in space. We also have a model, namely that the reality lies on (another) vector. We know the direction (x) of this vector but not its length (b). With a little spatial insight one can imagine that the best value for b is that where the perpendicular of y intersects x (Figure 3.1).

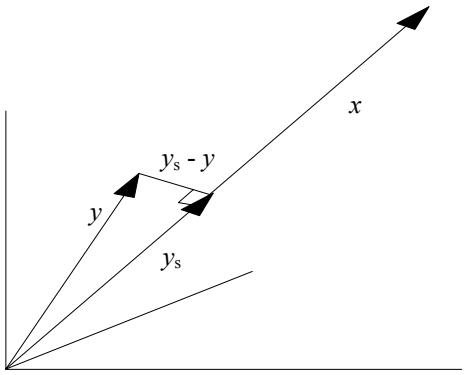

Figure 3.1: spatial representation of linear regression

If y-y_s is perpendicular to the vector (x), the inproduct of y-y_s with x is zero.

Written in formulas:

$(y\text{-}bx,x) = 0 \quad \Rightarrow \quad (y,x) = (bx,x) \quad \Rightarrow \quad y^t x = bx^t x$

$$b = \left(x^t x\right)^{-1} \cdot y^t x = \frac{\sum y_i \cdot x_i}{\sum x_i^2} \tag{3.14}$$

This last equation is the same as equation 3.13. We are also interested in the confidence level of the estimated parameter and the confidence interval of the model predictions.

Definitions (see books on statistics)

$$\underline{s}^2 = \frac{(\mathrm{d}y, \mathrm{d}y)}{DF} = \frac{\mathrm{d}y^2}{DF} = \frac{RSS}{DF} \tag{3.15}$$

with DF = dim R = n points - n param

Assume $\upsilon = \left(x^t \cdot x\right)^{-1}$

$$\mathrm{var}(b) = \upsilon \cdot s^2 = \upsilon \cdot \frac{RSS}{DF} \tag{3.16}$$

In the case $y_s = bx$, x is a vector, $(x^t.x)$ is in this case equal to the inproduct of the vector x with itself. This is equivalent to $\sum x^2$. The variance of b then becomes:

$$\mathrm{var}(b) = \upsilon \frac{RSS}{DF} = \frac{1}{\sum x^2} \cdot \frac{RSS}{DF} \tag{3.17}$$

1) Confidence interval of the parameter *b*:

$$b = \hat{b} \pm t_r \cdot \sqrt{\mathrm{var}(b)} = \hat{b} \pm t_r \cdot \sqrt{\upsilon \cdot \frac{RSS}{n \text{ points} - n \text{ param}}} \tag{3.18}$$

where t is the Student's t value. This comes down to:

$$b = \hat{b} \pm t_r \cdot \sqrt{\frac{1}{\sum x^2} \cdot \frac{RSS}{n\,\text{points} - n\,\text{param}}} \tag{3.19}$$

2) Confidence interval of the estimate (y_p) at point $x = p$:

$$y = p \cdot b \tag{3.20}$$

$$\text{var}\left(y_p\right) = p^2 \cdot \upsilon \cdot s^2 \tag{3.21}$$

$$y_p = \hat{y}_p \pm t_r \sqrt{\text{var}\left(y_p\right)} = \hat{y}_p \pm t_r \sqrt{p^2 \cdot \upsilon \cdot \frac{RSS}{n\,\text{points} - n\,\text{param}}} \tag{3.22}$$

Here again, $(x^t \cdot x)$ is equal to $\sum x^2$, so that:

$$y_p = \hat{y}_p \pm t_r \sqrt{\frac{p^2}{\sum x^2} \cdot \frac{RSS}{n\,\text{points} - n\,\text{param}}} \tag{3.23}$$

3) Confidence interval of the new measurement (y_q) at point $x=q$

The variance for the estimate (y_q) at a new measurement point $x=q$ is made up of the pure standard deviation of the estimate plus the variance in the new measurement:

$$\text{var}\left(y_q\right) = \text{var}\left(\hat{y}_q\right) + s^2 \tag{3.24}$$

With $s^2 = RRS/DF$

$$\text{var}\left(y_q\right) = q^2 \cdot \upsilon \cdot s^2 + s^2 = \left(q^2 \cdot \upsilon + 1\right)s^2 \tag{3.25}$$

the confidence interval then becomes:

$$y_q = \hat{y}_q \pm t_r \sqrt{\text{var}\left(y_q\right)} = \hat{y}_q \pm t_r \sqrt{\left(q^2 \cdot \upsilon + 1\right) \cdot \frac{RSS}{n\ \text{points} - n\,\text{param}}} \tag{3.26}$$

With $(x^t \cdot x)$, $\sum x^2$ is:

$$y_q = \hat{y}_q \pm t_r \sqrt{\left(\frac{q^2}{\sum x^2} + 1\right) \cdot \frac{RSS}{n\,\text{points} - n\,\text{param}}} \tag{3.27}$$

Take note: if q lies in the measurement interval then in many cases $1 >> q^2/\sum x^2$ and $var(y_q)$ will approach s^2 .

Exercise 3.1: We will now take a line of n's for x and the values from Example 3.1 for y. With the formulas given here, calculate the best estimate with confidence interval and compare this with the results found in Example 3.1.

Example 3.2:
Suppose we have collected the following measurement points for a falling cannonball:

time (s)	measure 1 (m)	measure 2 (m)
0.0	0.000	0.000
0.1	0.049	0.048
0.2	0.195	0.196
0.3	0.441	0.442
0.4	0.785	0.784
0.5	1.225	1.226
0.6	1.765	1.766
0.7	2.404	2.405
0.8	3.140	3.139
0.9	3.972	3.970
1.0	4.000	4.000

The model is $y = 0.5 \cdot g \cdot t^2$. The cannonball falls from a 4 meter height to the ground (see also example 1.1).
We now assume that $x = t^2$ and $y =$ is the position of the cannonball. We then have an equation $y = b \cdot x$, where $b = 0.5 \cdot g$. If we examine these data points further, it turns out that the data point after 0.9 seconds lies outside of the system limits. We have assumed that the cannonball is falling, at time 1s the cannon ball is on the ground and the position of the cannonball can no longer be described with the present model. If we place all the points on a graph, we see immediately that the last data point indeed deviates (Figure 3.2).
We can therefore disregard this last data point (with a clear motivation!).

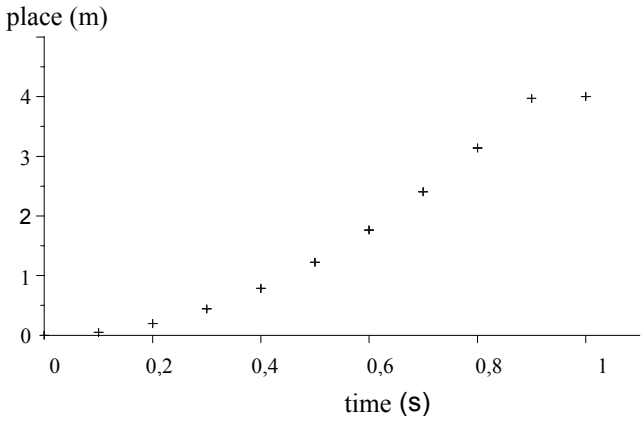

Figure 3.2: data points for a falling cannonball.

We will now calculate:

$$\left.\begin{array}{l} \sum x^2 = 3.0666 \\ \\ \sum y_i \cdot x_i = 15.03902 \end{array}\right\} \Rightarrow \quad b = \frac{\sum y_i \cdot x_i}{\sum x^2} = 4.904134$$

Hence we approximate the rate of acceleration during the fall as: 9.808268. We now have 20 data points and have estimated 1 parameter, in other words we have 19 degrees of freedom. If we had only 1 data point, we could have also estimated b with 0 degrees of freedom. In this case we would have had absolutely no certainty that our model was correct. With 20 data points we have now estimated 1 parameter, and with the help of this 1 parameter our prediction fits very well with our 20 data points. We require only a single data point to estimate b, but the other 19 give us more certainty about the correctness of the model and give us more certainty that we have accurately estimated b. We can now

compare our model predictions with our measurement data.It is frequently useful to do this on a graph.

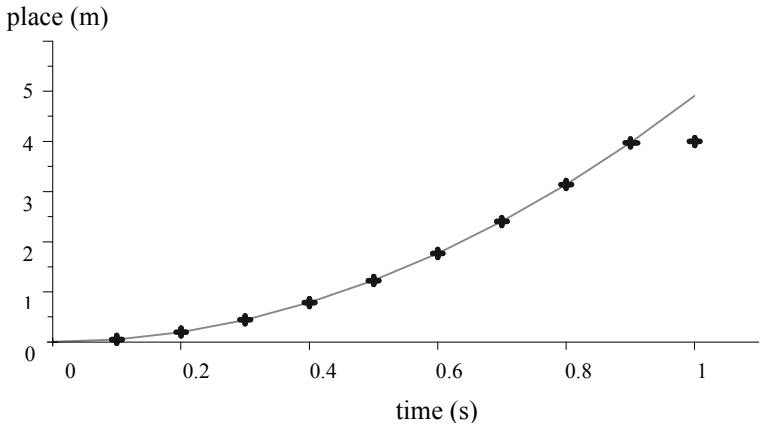

Figure 3.3: Model of falling cannonball. Note that the prediction for $x > 0.9$ is wrong,this is an incorrect extrapolation.

We can also make a table with x and measured y values, the model prediction, the deviation, the squares of the deviation and the sum of these squares (*RSS*).

t^2	y_i	y_s	dy	dy^2
0.00	0.000	0.000	0.000	0.00
0.01	0.049	0.049041	-0.00004	1.71E-09
0.04	0.195	0.196165	-0.00116	1.36E-06
0.09	0.441	0.441372	-0.00037	1.38E-07
0.16	0.785	0.784661	0.000338	1.15E-07
0.25	1.225	1.226033	-0.00103	1.07E-06
0.36	1.765	1.765488	0.00048	2.39E-07
0.49	2.404	2.403026	0.000973	9.49E-07
0.64	3.140	3.138646	0.001353	1.83E-06
0.81	3.972	3.972349	-0.00034	1.22E-07
0.00	0.000	0.000	0.000	0.00
0.01	0.048	0.049041	-0.00104	1.08E-06
0.04	0.196	0.196165	-0.00016	2.74E-08
0.09	0.442	0.441372	0.000627	3.94E-07
0.16	0.784	0.784661	-0.00066	4.38E-07
0.25	1.226	1.226033	-0.00003	1.14E-09
0.36	1.766	1.765488	0.000511	2.62E-07
0.49	2.405	2.403026	0.001973	3.90E-06
0.64	3.139	3.138646	0.000353	1.25E-07
0.81	3.970	3.972349	-0.00234	5.52E-06
			RSS:	1.76E-05

We also estimate the variance $s^2 = RSS/(n \text{ points} - n \text{ param})$=1.76E-5/19=9.25E-07
We can also calculate the confidence interval of the estimated b by using:

$$\upsilon = \left(x^t x\right)^{-1} = \left(\sum x^2\right)^{-1} = \frac{1}{\sum x^2} \quad \text{and} \quad \operatorname{var} b = \upsilon \cdot s^2$$

$$b = \hat{b} \pm t_{19}\sqrt{\operatorname{var}(b)} = b \pm 2.09\sqrt{\frac{s^2}{\sum\left(x^2\right)}}$$

this is estimated as:
4.904134 + 2.09·0.000549 = 4.905282 $\Rightarrow$ g_{max} = 9.810565
4.904134 - 2.09·0.000549 = 4.902987 $\Rightarrow$ g_{min} = 9.805974
(In the Netherlands g is normally 9.80665)

We can now predict the position of the object at 0.25 s, namely
$y = 4.904134 \cdot (0.25)^2 = 0.306508$

The confidence interval of this estimate can now be calculated:
var($y(0.25)$) = $p \cdot v \cdot p \cdot RSS/(n$ points - n param) = $0.25^4/3.0666 \cdot 1.76\text{E-}5 / 19 = 1.178\text{E-}9$

Therefore $y_{min} = 0.306508 - 2.09 \cdot \text{sqrt}(1.178\text{E-}8) = 0.30644$
and $y_{max} = 0.306508 + 2.09 \cdot \text{sqrt}(1.178\text{E-}8) = 0.30658$

By using a model, we therefore see that we can, after performing measurements, estimate a parameter (with confidence interval). Moreover, we can also use this model to make predictions (with confidence interval).

3.2.2. Straight line with intercept

Linear regression with: $y = b_o + b_1 \cdot x$

We can see this as: $$\vec{y}_s = b_0 \cdot \vec{x}_0 + b_1 \cdot \vec{x}_1 \tag{3.28}$$

with $\vec{y}_s =$ estimation for $\vec{y}$

$$eg \begin{pmatrix} 1 \\ 3 \\ 5 \\ 7 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + 2 \cdot \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}$$

with x_0 being a vector filled with ones and x_1 a vector filled with data (the x values).

We now have a matrix x and a vector y (datum points on a line) and we would like to know b_0 and b_1. For this we can choose the same minimization criterium:
Minimizing $\sum_i \left(dy^2\right)$ to b_0 and b_1 .
In matrix vector calculation this amounts to the perpendicular projection of y-y_s on the surface formed by the vectors x_o and x_1. y_s lies namely on this surface (because it is a linear combination of the two vectors which extend across space) and we minimize in this way the distance between y and y_s. This distance is minimal if y-y_s is perpendicular to the surface (<x_o>,<x_1>).

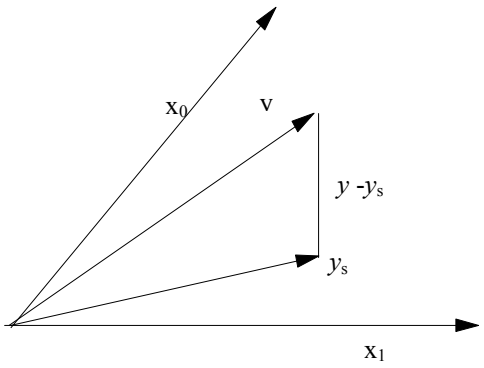

Figure 3.4: 'Spatial' representation of linear regression.

If y-y_s is perpendicular to the surface (<x_0>,<x_1>), the inproduct of y-y_s with x_1 and the inproduct of y-y_s with x_0 are both zero. In formulas:

$$\begin{aligned}\left(y - b_0 x_0 - b_1 x_1, x_0\right) &= 0\\ \left(y - b_0 x_0 - b_1 x_1, x_1\right) &= 0\end{aligned} \tag{3.29}$$

$$\begin{aligned}\left(y, x_0\right) &= \left(b_0 x_0, x_0\right) + \left(b_1 x_1, x_0\right)\\ \left(y, x_1\right) &= \left(b_0 x_0, x_1\right) + \left(b_1 x_1, x_1\right)\end{aligned} \tag{3.30}$$

$$\begin{aligned}y^t x_0 &= b_0 x_0^t x_0 + b_1 x_0^t x_1\\ y^t x_1 &= b_0 x_0^t x_1 + b_1 x_1^t x_1\end{aligned} \tag{3.31}$$

$$\begin{pmatrix} y^t \cdot x_0 \\ y^t \cdot x_1 \end{pmatrix} = \begin{pmatrix} x_0^t \cdot x_0 & x_0^t \cdot x_1 \\ x_0^t \cdot x_1 & x_1^t \cdot x_1 \end{pmatrix} \cdot \begin{pmatrix} b_0 \\ b_1 \end{pmatrix} \tag{3.32}$$

$$\begin{pmatrix} b_0 \\ b_1 \end{pmatrix} = \begin{pmatrix} x_0^t x_0 & x_0^t x_1 \\ x_0^t x_1 & x_1^t x_1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} y^t x_0 \\ y^t x_1 \end{pmatrix} = \frac{\begin{pmatrix} x_1^t x_1 & -x_0^t x_1 \\ -x_0^t x_1 & x_0^t x_0 \end{pmatrix}}{x_0^t x_0 \cdot x_1^t x_1 - \left(x_0^t \cdot x_1\right)^2} \cdot \begin{pmatrix} y^t x_0 \\ y^t x_1 \end{pmatrix} \Rightarrow$$

$$\begin{pmatrix} b_0 \\ b_1 \end{pmatrix} = \frac{\begin{pmatrix} \sum x^2 & -\sum x \\ -\sum x & n \end{pmatrix}}{n\sum x^2 - \left(\sum x\right)^2} \cdot \begin{pmatrix} \sum y \\ \sum xy \end{pmatrix} \tag{3.33}$$

N.B.: $\begin{pmatrix} b_0 \\ b_1 \end{pmatrix} = \begin{pmatrix} x_0^t x_0 & x_0^t x_1 \\ x_0^t x_1 & x_1^t x_1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} y^t x_0 \\ y^t x_1 \end{pmatrix} = V \cdot \begin{pmatrix} y^t x_0 \\ y^t x_1 \end{pmatrix}$

when written out: intercept: $b_0 = \dfrac{\sum x^2 \cdot \sum y - \sum x \cdot \sum (x \cdot y)}{n \cdot \sum x^2 - \left(\sum x\right)^2}$ (3.34)

slope: $b_1 = \dfrac{-\sum x \cdot \sum y + n\sum (x \cdot y)}{n \cdot \sum x^2 - \left(\sum x\right)^2}$ (3.35)

The confidence interval can also be calculated in this case for b_o, b_1, y_p or y_q where point p is a data point from the data series (measurement series) and q is a new measurement (DF is now n - 2).

In the case of the model: $y = b_o + b_1 \cdot x$ the variance of the parameters is described by:

$$\operatorname{var}(b_0) = \upsilon_{11} \cdot s^2 = \frac{\sum x^2}{n\sum x^2 - \left(\sum x\right)^2} \cdot \frac{RSS}{n-2} \tag{3.36}$$

$$\mathrm{var}(b_1) = \upsilon_{22} \cdot s^2 = \frac{n}{n\sum x^2 - \left(\sum x\right)^2} \cdot \frac{RSS}{n-2} \tag{3.37}$$

The confidence interval for y_p and y_q can also be calculated by using summations because:

$$(1,p)(V)\begin{pmatrix}1\\p\end{pmatrix} = \begin{pmatrix}\upsilon_{11}+p\upsilon_{12} & \upsilon_{12}+p\upsilon_{22}\end{pmatrix}\begin{pmatrix}1\\p\end{pmatrix} = \upsilon_{11}+2p\upsilon_{12}+p^2\upsilon_{22}$$

From this it follows that:

$$\mathrm{var}(y_p) = (1,p)\cdot V \cdot \begin{pmatrix}1\\p\end{pmatrix} \cdot s^2 = \frac{\sum x^2 - 2p\cdot\sum x + n\cdot p^2}{n\sum x^2 - \left(\sum x\right)^2} \cdot \frac{RSS}{n-2} \tag{3.38}$$

$$\mathrm{var}(y_q) = \mathrm{var}(\hat{y}_q) + s^2 = (1,q)\cdot V \cdot \begin{pmatrix}1\\q\end{pmatrix} \cdot s^2 + s^2 \Rightarrow$$

$$\mathrm{var}(y_q) = \left\{\frac{\sum x^2 - 2q\cdot\sum x + n\cdot q^2}{n\sum x^2 - \left(\sum x\right)^2} + 1\right\} \cdot \frac{RSS}{n-2} \tag{3.39}$$

Example 3.3:

$x_0^t = (1,1,1,1)$ $\quad x_0^t x_0 = 4$ $\quad (=n)$

$x_1^t = (0,1,2,3)$ $\quad x_1^t x_1 = 14$ $\quad \left(=\sum x^2\right)$ $\quad x_0^t x_1 = 6$ $\quad \left(=\sum x\right)$

$y^t = (1,3,5,7)$ $\quad y^t x_0 = 16$ $\quad \left(=\sum y\right)$ $\quad y^t x_1 = 34$ $\quad \left(=\sum x\cdot y\right)$

Therefore:

$$\begin{pmatrix}b_0\\b_1\end{pmatrix} = \begin{pmatrix}4 & 6\\6 & 14\end{pmatrix}^{-1} \cdot \begin{pmatrix}16\\34\end{pmatrix} = \frac{\begin{pmatrix}14 & -6\\-6 & 4\end{pmatrix}}{4\cdot 14 - 36} \cdot \begin{pmatrix}16\\34\end{pmatrix} = \frac{\begin{pmatrix}7 & -3\\-3 & 2\end{pmatrix}}{14-9} \cdot \begin{pmatrix}8\\17\end{pmatrix} = \frac{\begin{pmatrix}56-51\\-24+34\end{pmatrix}}{5} = \begin{pmatrix}1\\2\end{pmatrix}$$

$$y = 1\cdot x_0 + 2\cdot x_1 \rightarrow \begin{matrix}1 = 1+2\cdot 0\\3 = 1+2\cdot 1\\5 = 1+2\cdot 2\\7 = 1+2\cdot 3\end{matrix} \quad \text{or} \quad \begin{pmatrix}1\\3\\5\\7\end{pmatrix} = \begin{pmatrix}1\\1\\1\\1\end{pmatrix} + 2\cdot\begin{pmatrix}0\\1\\2\\3\end{pmatrix}$$

Remark: ($\sum x^2 \cdot \sum y$) and ($\sum x \cdot \sum xy$) and ($n\sum xy$ and $\sum xy$) are frequently of the same order of magnitude. Since they are subtracted from each other, it is important that you use many digits after the decimal point to achieve sufficient accuracy.

Exercise 3.2: Also calculate an estimate for the parameters a and b for the models $y = a + b*t^2$ and $y = b \cdot t^2$ with the following data set:

time	y
0.0	0.000
0.3	0.441
0.6	1.765
0.9	3.972

This could be useful if we did not know the exact initial value of the position and wanted to estimate this.

3.2.3. Multi-dimensional systems

Linear regression with: $y = Xb$

Instead of using two different vectors x, we can also see this as a matrix X with the columns holding the various x vectors. Therefore in the previous example the matrix X would have been:

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ . & . \\ . & . \\ . & . \end{pmatrix}$$

If we want to minimize the *RSS*, y-Xb must be perpendicular to the space spanned by X.

y-Xb $\perp$ D (<x_1>,<x_2>,<x_3>.....)

$$\forall\ i\ \ \left(y - Xb, x_i\right) = 0$$

$$\left(y - Xb\right)^t \cdot X = 0$$

$$y^t \cdot X = \left(Xb\right)^t \cdot X \qquad \left(y^t \cdot X\right)^t = \left(\left(Xb\right)^t \cdot X\right)^t$$

$$X^t \cdot y = X^t \cdot Xb$$

$$b = \left(X^t \cdot X\right)^{-1} X^t \cdot y \tag{3.40}$$

The inverse of matrix of $X^t.X$ must exist. If we have a singular matrix (not invertible, therefore dependent rows or columns and determinant=0) then the regression parameters are dependent (e.g. $y = ax + bx + c$) and another set of equations must be chosen. If too many parameters are chosen, the matrix $X^t.X$ will be difficult to invert (e.g. $y = ax + bx^{1.1} + c$).

If the matrix is invertable, we can now minimize the minimization criterium for every linear combination of x vectors and thus determine the best set of parameters b. By using data points we can therefore determine the best set of parameters in every linear model. We are also interested in the confidence interval of the approximated parameters and in the confidence interval of the model predictions.

Definitions (see statistics books)

$$\underline{s}^2 = \frac{(\mathrm{d}y, \mathrm{d}y)}{DF} = \frac{\mathrm{d}y^2}{DF} = \frac{RSS}{DF} \text{ with } DF\text{r} = \dim R = n \text{ poimts - } n \text{ param} \tag{3.41}$$

Suppose $V = \left(X^t \cdot X\right)^{-1}$

$$\mathrm{cov}\big((p,b),(q,b)\big) = \upsilon_{pq} s^2, \text{ with } \upsilon_{pq} = p^t V q \tag{3.42}$$

e.g.: $p^t = (1,0,0) \qquad q^t = (1,0,0)$

$$\mathrm{var}(b_1) = (1,0,0)\, V \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \sigma^2 = \upsilon_{11} \cdot \sigma^2 = \upsilon_{11} \cdot \frac{RSS}{r}$$

e.g.: $p^t = (1,0,0) \qquad q^t = (0,1,0)$

$$\mathrm{cov}(b_1, b_2) = (1,0,0)\, V \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \sigma^2 = \upsilon_{12} \cdot \sigma^2$$

e.g.: $p^t = (1,-1,0) \qquad q^t = (1,-1,0)$

$$\mathrm{var}(b_1 - b_2) = (1,-1,0)\, V \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \sigma^2 = (1,-1,0) \cdot \begin{pmatrix} \upsilon_{11} - \upsilon_{12} \\ \upsilon_{21} - \upsilon_{22} \\ \upsilon_{31} - \upsilon_{32} \end{pmatrix} \cdot \sigma^2 = (\upsilon_{11} - 2\upsilon_{12} + \upsilon_{22})\sigma^2$$

Take note: $v_{12} = v_{21}$

1) Confidence interval of the parameters:

e.g. b_1

$$b_1 = \hat{b}_1 \pm t_r \cdot \sqrt{\mathrm{var}(b_1)} = \hat{b}_1 \pm t_r \cdot \sqrt{\upsilon_{11} \cdot \frac{RSS}{n \text{ points} - n \text{ param}}} \tag{3.43}$$

because $var(b_1) = v_{11} \cdot RSS/DF$

where t is the Student's t value.

2) Confidence interval of the estimate (y_p) op punt x_i:

$y = Xb \;\rightarrow\; y_i = p^t b$ with p^t being the i^e row of X

$$\mathrm{var}(y_p) = p^t \cdot V \cdot p \cdot s^2 = \sum_l \sum_k X_{il} \cdot X_{ik} \upsilon_{kl} \cdot s^2$$

$$y_i = \hat{y}_p \pm t_r \sqrt{\mathrm{var}(y_p)} = \hat{y}_p \pm t_r \sqrt{\frac{\sum_l \sum_k (X_{il} \cdot X_{ik} \cdot \upsilon_{kl}) \cdot RSS}{n \text{ points} - n \text{ param}}} \tag{3.44}$$

3) Confidence interval of a new measurement (y_q) at $x = q$:

$$\mathrm{var}\left(y_q\right) = \mathrm{var}\left(\hat{y}_q\right) + s^2 = \left\{q^t \cdot V \cdot q + 1\right\} \cdot s^2 \tag{3.45}$$

$$y_q = \hat{y}_q \pm t_r \sqrt{\mathrm{var}\left(\hat{y}_q\right)} = \hat{y}_q \pm t_r \sqrt{\frac{\left\{q^t \cdot V \cdot q + 1\right\} \cdot RSS}{n\,\mathrm{points} - n\,\mathrm{param}}} \tag{3.46}$$

Exercise 3.3: Do exercise 3.2 again using these formulas and calculate the confidence interval of the parameters and the position estimate at 0.25 s.

3.2.4. Transformations

Linear regression may only be used if the conditions (see 3.2) are satisfied. You must make sure that the variance of y is constant, that y is independent of every other y and that the error has a standard normal distribution. If a single one of these conditions is not satisfied, you may not use linear regression. You must choose of another regression technique, a weighted linear regression or a transformation. In this section we shall consider the use of transformations. Transformations must be chosen in such a way that the variance of the transformed y is constant, the transformed y is independent from every other transformed y and the error in the transformed y is normally distributed, so that all conditions for linear regression are satisfied. For instance, if there is a situation with a relative error, y must be transformed. However, a transformation is frequently chosen because the model is not linear. In Arrhenius kinetics, for example, linear transformation occurs by taking the logarithm left and right (this is worked out in an example). As a result, the error is also transformed, and we must check to see if we still have a normally distributed error with a constant variance after the transformation takes place. If not, then we may not use linear regression. Instead we could use, for instance, non-linear regression (see 3.3). In this section we shall discuss methods which allow you to see if a transformation is necessary/allowable, and how you must interpret the estimated values.

3.2.4.1. Verification of the initial assumptions

Constant variance of the y values can be checked easily by graphing the variance against x. This should result in a horizontal straight line. After all, σ^2 is constant. Frequently, no information will be available about the variance of the y value. The variance can be estimated by measuring several values of y_i for every x_i (preferably more than 5 replicas). The variance of y_i is independent of the model which is going to be used for fitting. If no replicas have been measured, one can still make a conclusion about the variance by looking at the residuals; this is a point approximation for the error in the y value, at least when the model is correct! If the y values have a constant variance, the residuals will lie around zero and form a horizontal band as shown in Figure 3.5.A. This must also be true if you graph the residuals in another fashion, for instance as a function of the estimated $y_{i,s}$, or in the sequence in which y is measured.

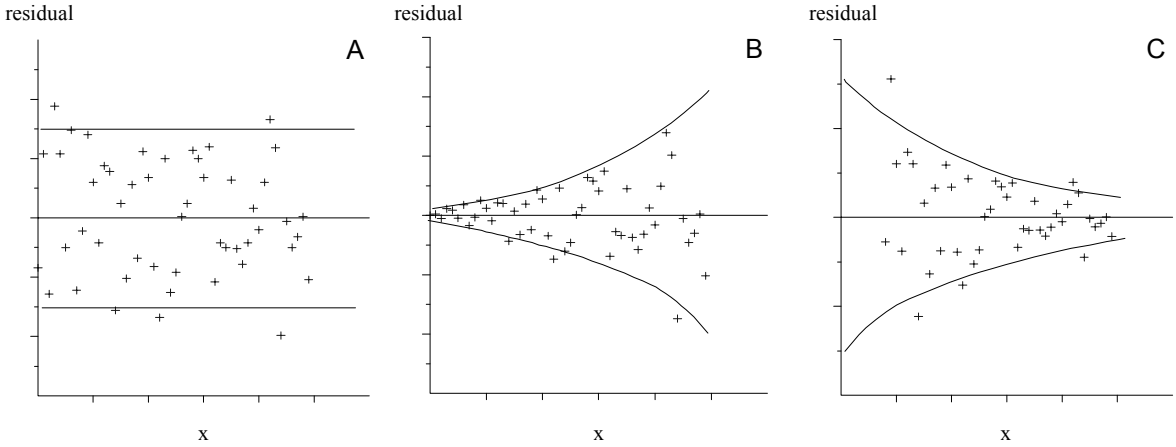

Figure 3.5: Graphs of the residuals of three different fits. A: normally distributed error; B: error increases as *x* increases; C: error decreases as *x* increases.

If the residuals do not form a band around 0, we can interpret the residual graph. If the residuals fan out such as in Figure 3.5.B, we often have a situation with a relative error which increases with an increasing *x* value. Of course, the error can also decrease as *x* increases as shown in Figure 3.5.C. In these cases normal linear regression may not be used.

The second condition is that the *y* values must be independent of every other *y*. This sounds very logical, but with measurement series it may happen that the sample is dependent on the previous history; this is known as auto-correlation. An example of this is when a measurement is carried out automatically which appears to be dependent on light. The measurements carried out during the day are then structurally different in value from those taken at night. This is also shown in the residual graph. The residuals lie in groups, first below, and then above zero. (Figures 3.6.B. and 3.6.C.). This is positive auto-correlation, where a positive deviation is followed by a positive deviation, and the reverse. In the case of our example, this error can be eliminated by choosing another measurement method or including the influence of light in the model. Sometimes a positive error even results in a negative error. This is known as negative auto-correlation (take the example of glassware in a cabinet; if there is too little glassware in a cabinet, frequently 'too much' is often purchased; if there is a lot of glassware in inventory, then 'too little' is often purchased). Auto-correlation can be tested, for example, with the Durbin-Watson test. If there is a situation with auto-correlation, one must frequently choose another model or expand the model. In any case, transformations do not provide a solution.

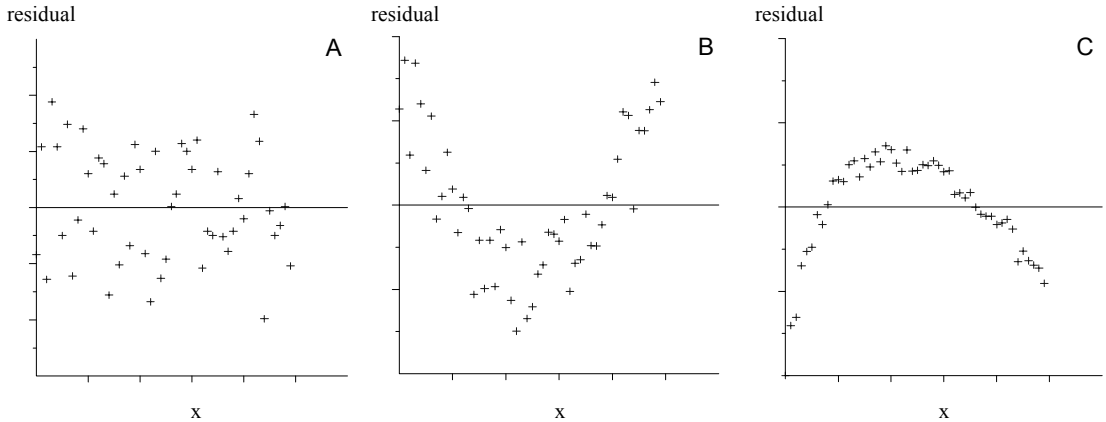

Figure 3.6: Residuals. A: normally distributed error (Figure 3.5.A); B and C: positive auto-correlation.

The problem of looking at residuals is that they are caused by measurement error and also by error in the model (lack of fit). Trends in the residuals can therefore also occur due to structural errors in the model.

The third condition is that the error in *y* must be normally distributed. The simplest way to look at this is to make a histogram of the residuals. If the residuals are normally distributed, the histogram has a bell shape and is symmetrical around zero. One problem is that we frequently do not have enough data to make a histogram. This is shown in Figure 3.7. In Figure 3.7.A., a histogram is made from the graph of the residuals as shown in Figure 3.5.A. This figure is drawn from 50 normally distributed errors. By definition we therefore have a normally distributed error, but this is not so obvious from the histogram. Histogram 3.7.B is drawn from 50 relative errors; we cannot determine this from the histogram, however. The histogram appears to be virtually symmetrical. With small numbers of data points it remains difficult to decide if we have a normal distribution. However, it is sometimes very clear that the distribution is **not** symmetrical. The histogram of Figure 3.7.C is clearly non-symmetrical. Other tests are described in the literature to test for normally distributed error, for instance by testing the standardized residuals (residual/*s*) for normal distribution, or by making a so-called normal plot. If the error is not normally distributed, transformations can provide an answer.

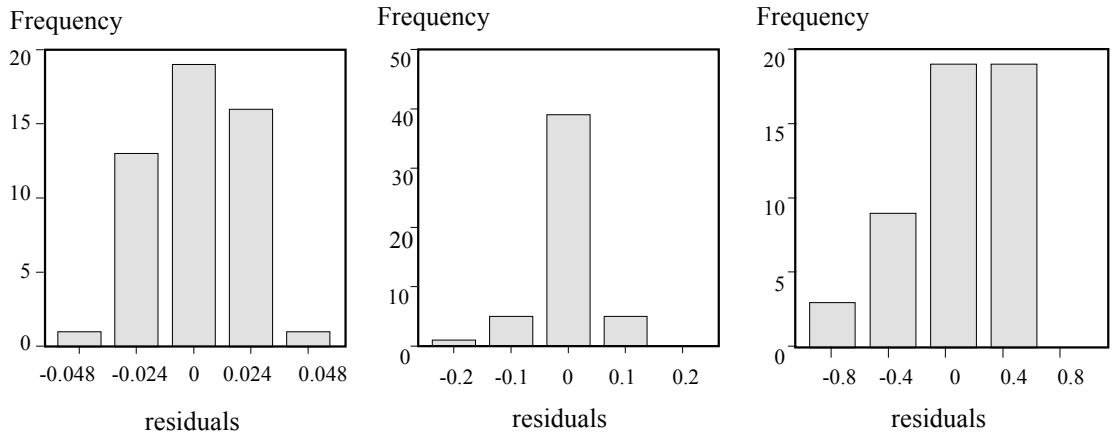

Figure 3.7. Histogram of the residuals from figures 3.5 and 3.6.
A: figure 3.5.A: B:figure 3.5.B: C:figure 3.5.C

3.2.4.2. Transformation

If you have chosen to apply a transformation: after transformation has taken place and the transformed data have been fitted, you must make sure that the requirements for linear regression have been satisfied as described in the previous section (or better still, the requirements for transformation with the help of measured replicas).

If the variance of the *y* values is known, we can use this to derive the transformation for these *y* values:

1) If the variance σ_{yi}^2 is a linear function of $\bar{y}_i$, then $\sqrt{y_i}$ is frequently a good transformation: $\sqrt{y_i} = g(x)$. To put it briefly, $\mathrm{var}(y_i) / \bar{y}_i$ must then be constant.

2) If the standard deviation σ_{yi} increases linearly, with increasing $\bar{y}_i$ (relative error), then $\ln(y_i)$ is frequently a good transformation: $\ln(y_i) = g(x)$. In this case, $\mathrm{var}(y_i) / \bar{y}_i^2$ is constant.

If we have a model which relates y to x: $y = f(x)$, when we transform y we must also transform the model.

$$y = f(x) \quad \Rightarrow \quad \sqrt{y} = \sqrt{f(x)}$$

and

$$y = f(x) \quad \Rightarrow \quad \ln(y) = \ln[f(x)]$$

Several transformations are frequently possible. For example, if there is a relative error, the logarithm can often be taken left and right:

$y = b_1 x_1 \varepsilon \Rightarrow \ln(y) = \ln(b_1) + \ln(x_1) + \ln(\varepsilon)$

where ε is the error. In this case, $1n(\varepsilon)$ must be normally distributed. Another option is to divide left and right by x:

$y = b_1 x_1 + \varepsilon x_1 \Rightarrow y/x_1 = b_1 + \varepsilon$

here ε must be normally distributed (see example 3.4).

Sometimes a transformation is chosen simply because the function is not linear, for example, with the first order kinetics or Arrhenius kinetics (see example 3.5). This is only allowed if, following transformation, the conditions for linear regression are satisfied: 1) a constant variance, 2) independent y values and 3) a normally distributed error (check!).

Finally, it is also possible to use a weighted least-squares method. Another possibility is to fit with another error assumption, for example, a Poisson or Gamma distribution. See statistic books for these methods.

3.2.4.3. Reverse transformation

If we have used a transformed function for fitting, we usually want to know the confidence intervals of the non-transformed values. This amounts to the reverse transformation of the estimated value and the confidence interval. However, the approximated value can no longer lie in the middle of the confidence interval; both the estimated value and the interval must therefore always be given. There is a single exception, however, i.e. when the standard deviation is a function of x: $\sigma_{yi} = x^c \sigma$, where c is a proportional constant. This occurs when there is a relative error, among other occasions. In this case, the variance of the estimate of a new measurement (y_q) on point $x = q$ is again equal to the pure, standard deviation of the estimate $\mathrm{var}(\hat{y}_q)$ plus the variance in the new measurement $x^{2c}s^2$ (see also equation 3.24):

$$\mathrm{var}(y_q) = \mathrm{var}(\hat{y}_q) + x_q^{2c} s^2 = \left\{ q^t \cdot V \cdot q + x_q^{2c} \right\} \cdot s^2 \tag{3.47}$$

So that the confidence interval of the approximation of a new measurement (y_q) on point $x=q$ becomes equal to:

$$y_q = \hat{y}_q \pm t_r \sqrt{\left(\operatorname{var} \hat{y}_q\right)} = \hat{y}_q \pm t_r \sqrt{\frac{\left\{q^t \cdot V \cdot q + x_q^{2c}\right\} \cdot RSS}{n\,\text{points} - n\,\text{param}}} \tag{3.48}$$

Example 3.4: relative error:
Suppose we have measured a calibration line. We assume an absolute error and fit the calibration line with $y = b_0 + b_1 x_1$. We also calculate the confidence interval for a new measurement:

$$y_q = \hat{y}_q \pm t_r \sqrt{\operatorname{var}\left(\hat{y}_q\right)} = \hat{y}_q \pm t_r \sqrt{\frac{\left\{q^t \cdot V \cdot q + 1\right\} \cdot RSS}{n\,\text{points} - n\,\text{param}}}$$

(Figure 3.8). From the calculations it follows that 0 lies before the intercept in the confidence interval of the estimate: $-0.11 < b_0 < 0.16$, from which it can be concluded that this parameter is not truly significant and can perhaps be left out (see section 3.5). If we look at the graph of the residuals (Figure 3.8.B) we see that the variance increases as x increases. We must therefore find a transformation to make the variance of y constant.

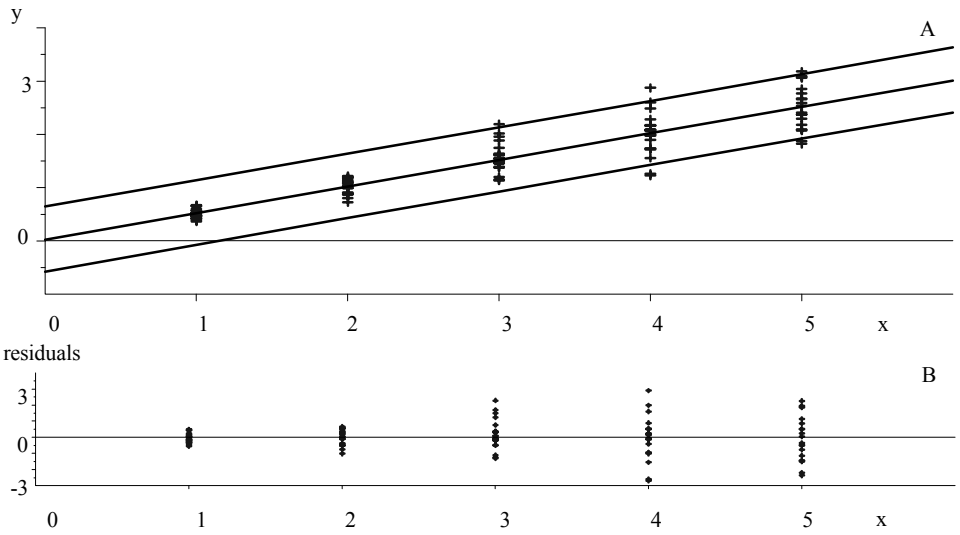

Figure 3.8.A: A calibration line with relative error fitted with $y = b_0 + b_1 x_1$ with confidence interval for the new measurement; B: the residuals.

Because every value is measured 10 times, we can calculate the variance for every population of y_i which is measured. Figure 3.9.A shows that the variance increases as x increases. If $\left(\operatorname{var} y_i\right) / \hat{y}_i$ is set against x, no constant value is obtained (Figure 3.9.B). $\sqrt{y_i}$ is therefore not a good transformation. Because the variance increases as x increases, the following transformation is chosen:

$y = b_0 + b_1 + \varepsilon x_1 \Rightarrow \quad y/x_1 = b_0/x_1 + b_1 + \varepsilon.$

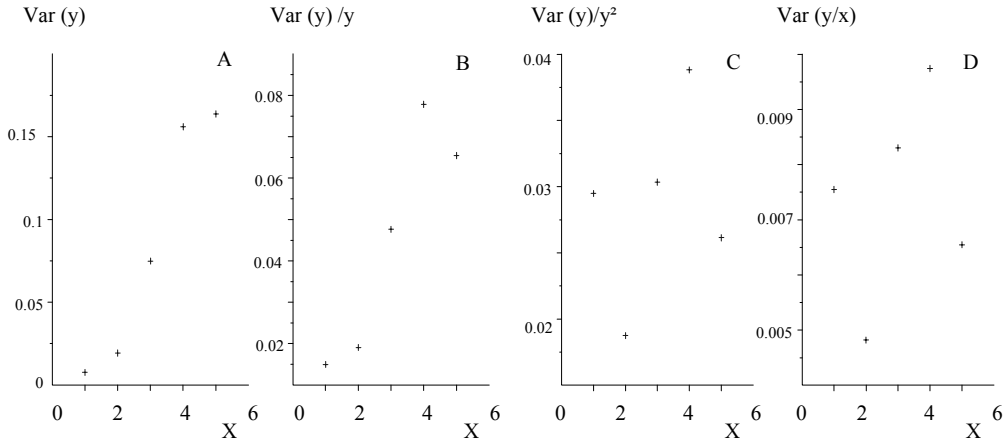

Figure 3.9.A: The variance of y; B: var(y)/y; C: var(y)/y^2; D: var(y/x)

Figure 3.10 shows the results for the transformed data. All calculations are performed with $y' = y/x_1$ and $x' = 1 / x$. The residuals stand out in Figure 3.10.B, this time no trend can be discovered and we assume that following this transformation the variance of y' is constant and the error is normally distributed. From the fit it follows that the intercept of the transformed function b_1 lies between 0.47 and 0.53, for the slope of the transformed function: $-0.05 < b_0 < 0.06$, therefore this is probably not significant.

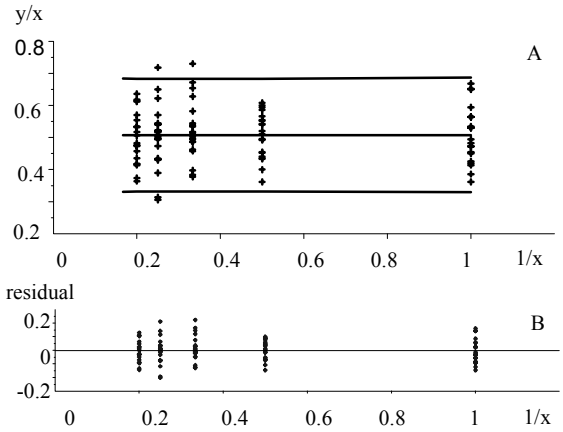

Fig 3.10 A: The transformed calibration line fitted with $y/x_1 = b_0/x_1 + b_1$ with the confidence interval for the new measurement; B: the residuals.

Finally, the values are reversed transformed. Notice that b_0 and b_1 do not have to be transformed and that $\hat{y}$ can therefore be directly calculated with $\hat{y} = b_0 + b_1 x_1$. For the calculation of the confidence interval for a new measurement one must, however, take into account the fact that the variance is a function of x, so that:

$$y_q = \hat{y}_q \pm t_r \sqrt{\frac{\left\{q^t \cdot V \cdot q + 1\right\} \cdot RSS}{n\,\text{points} - n\,\text{param}}}$$

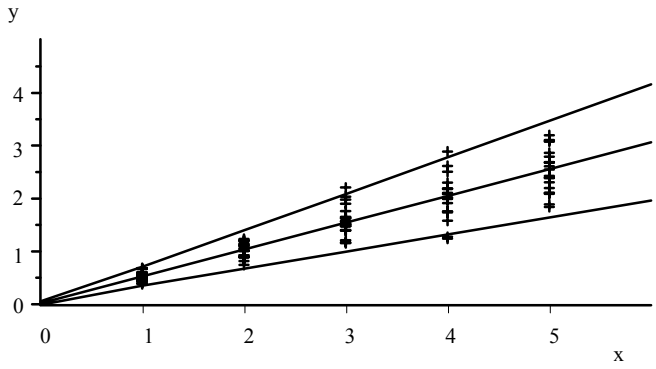

Figure 3.11: The reverse transformed calibration line with confidence interval for the new measurement from figure 3.10 .

Example 3.5: Arrhenius kinetics:

It is frequently assumed that the reaction constant k is dependent on the temperature according to an Arrhenius relationship:

$$k = k_{\infty} e^{-\Delta E/RT} \tag{3.47}$$

where k is the reaction rate constant (s^{-1}), k_{∞} the frequency factor (s^{-1}), ΔE the activation energy ($kJ.mol^{-1}$), R the gas constant 8.314 $kJ.mol^{-1}.K^{-1}$ and T the absolute temperature (K). To determine the frequency factor and the activation energy, k values are measured at various temperatures. To fit the Arrhenius kinetics equation with linear regression, the equation is transformed by taking the logarithm left and right of the = sign:

$$\ln(k) = \ln(k_{\infty}) - \frac{\Delta E}{R} \cdot \frac{1}{T}$$

By setting $\ln(k)$ against $1/T$, a straight line is obtained with intercept $\ln(k_{\infty})$ and slope $-\Delta E/R$. Following transformation, the error must, of course, be normally distributed with a constant variance. Figure 3.12 shows two fits with residuals, the left with normally distributed error in 1n(k) and the right with normally distributed error in k.

Figure 3.12 shows that if the error of k is normally distributed, the fit with small values of $1/T$ is underestimated and with large values of $1/T$ fails completely. In this case another method should be chosen, such as non-linear regression (see section 3.3), weighting the y value or another error assumption. With normally distributed error in 1n(k), the confidence interval for 1n(k), the intercept, and the slope can be calculated.

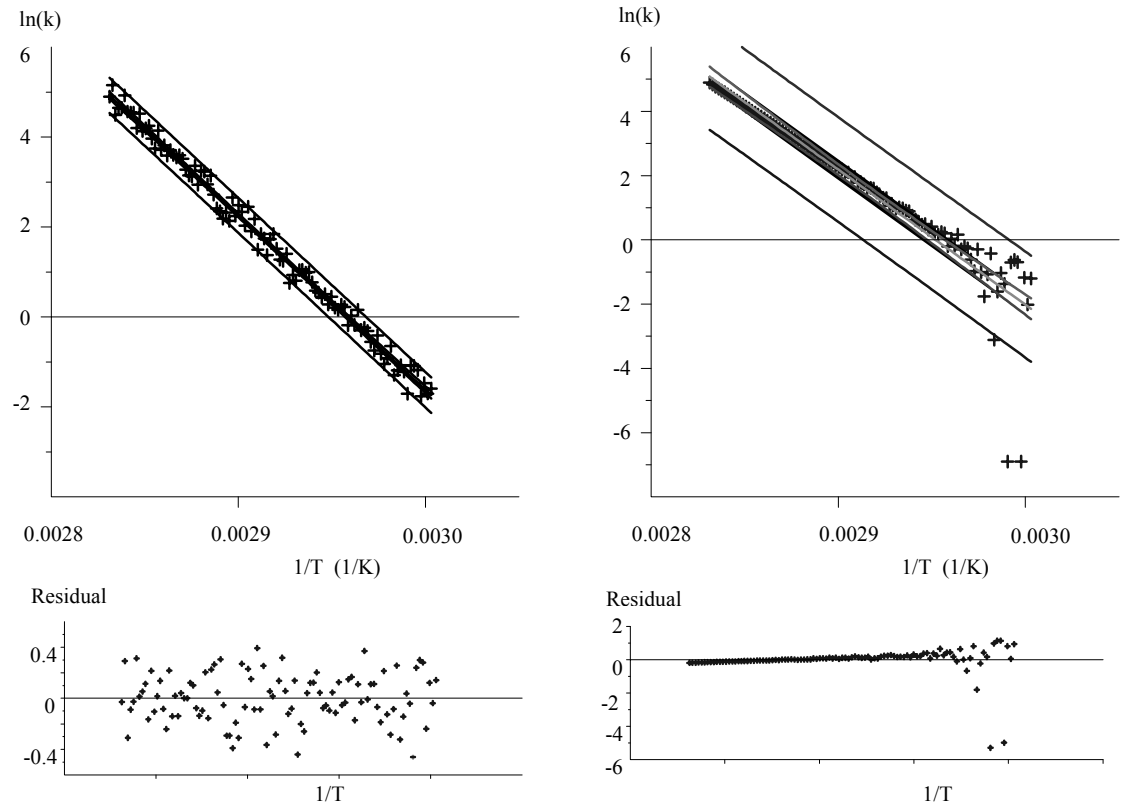

Fig. 3.12: An Arrheniusplot with the residuals;
left with normally distributed error in 1n(k) and right in k.

Figure 3.13 is obtained by reverse transformation, the confidence interval also remains within limits when the temperature is around 80°C. Regarding the intercept of the transformed function: $112 < \ln(k) < 117$ so that for k: 7.5E48 < $\hat{k}$ = 6.9E49 < 6.4E50 s^{-1}. Note that $\hat{k}$ does not lie in the middle of the confidence interval. For the slope: $-39558 < -\Delta\hat{E}/R < -38030$ K so that $38.030 \cdot 8.314 = 316 < \Delta\hat{E} = 323 < 329$ kJ.mol^{-1}.

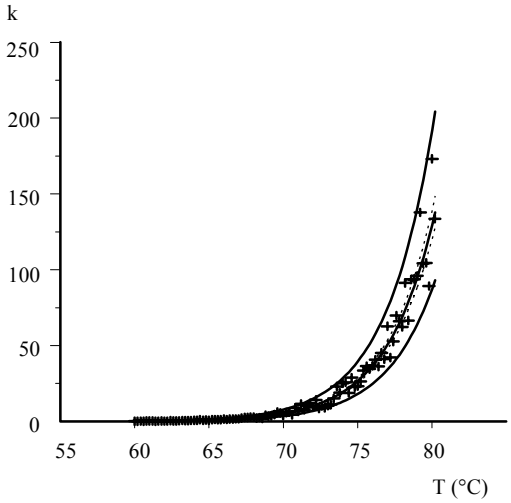

Figure 3.13: k values as a function of temperature

3.2.5. Solutions

Solution 3.1: This gives the same results as example 3.1.

Solution 3.2: Worked out for the model $y = a + b \cdot t^2$.

Suppose $x = t^2 \Rightarrow y = a + bx$

{PRIVATE }t	x	y
0	0	0
0.3	0.09	0.441
0.6	0.36	1.765
0.9	0.81	3.972

model:

$$y = a + b \cdot x = \begin{pmatrix} 0 \\ 0.441 \\ 1.765 \\ 3.972 \end{pmatrix} = a \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + b \begin{pmatrix} 0 \\ 0.09 \\ 0.36 \\ 0.81 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0.09 \\ 1 & 0.36 \\ 1 & 0.81 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = X \begin{pmatrix} a \\ b \end{pmatrix}$$

$$V = \left(X^t \cdot X \right)^{-1} = \left(\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0.09 & 0.36 & 0.81 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 1 & 0.09 \\ 1 & 0.36 \\ 1 & 0.81 \end{pmatrix} \right)^{-1} \Rightarrow$$

$$V = \begin{pmatrix} 4 & 1.26 \\ 1.26 & 0.7938 \end{pmatrix}^{-1} = \begin{pmatrix} 0.500 & -0.7937 \\ -0.7937 & 2.5196 \end{pmatrix}$$

determinant = 1.588

$$\begin{pmatrix} a \\ b \end{pmatrix} = V \cdot X^t \cdot y = \begin{pmatrix} 0.5 & -0.7937 \\ -0.7937 & 2.5196 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0.09 & 0.36 & 0.81 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0.441 \\ 1.765 \\ 3.972 \end{pmatrix} \Rightarrow$$

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0.5 & -0.7937 \\ -0.7937 & 2.5196 \end{pmatrix} \cdot \begin{pmatrix} 6.178 \\ 3.892 \end{pmatrix} = \begin{pmatrix} -0.00021 \\ 4.904 \end{pmatrix}$$

Solution 3.3: confidence intervals of the parameters

$a = \hat{a} \pm t_r \sqrt{\mathrm{var}\, a}$, $DF = 4 - 2 = 2$, $t_{DF,0.975} = 4.3$

variance of the measurement series (data series):

{PRIVATE }t	x	y	$\hat{y}$	$y - \hat{y}$
0	0	0	-0.00021	45.9E-9
0.3	0.09	0.441	0.4411	17.6E-9
0.6	0.36	1.765	1.7652	30.1E-9
0.9	0.81	3.972	3.9719	8.4E-9

$RSS =$ 102.0E-9

$s^2 = RSS / DF$ = 102E-9 / 2 = 51.0E-9
var$a = v_{11} \cdot s^2 = 0.5 \cdot 51.0\text{E-}9 = 25.5\text{E-}9$

$$a = \hat{a} \pm t_r \sqrt{\text{var}\, a} = -0.00021 \pm 4.3 \cdot \sqrt{25.5\text{E}-9} = -0.00021 \pm 0.00069 \quad \Rightarrow$$

-0.00090 < a < 0.00047

Take note: zero is in the confidence interval, a is probably not necessary for the model (section 3.5)

analogously for b:

$$\text{var}\, b = \upsilon_{22} \cdot s^2 = 2.5196 \cdot 51.0\text{E}-9 = 129\text{E}-9$$

$$b = \hat{b} \pm t_r \sqrt{\text{var}\, a} = 4.904 \pm 4.3 \cdot \sqrt{129\text{E}-9} = 4.904 \pm 0.00154 \Rightarrow$$

4.902 < b < 4.905

confidence interval of the estimate $\hat{y}$ for a new measurement at t = 0.25
x_i = 0.0625

$$\hat{y} = p^t \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 & 0.0625 \end{pmatrix} \cdot \begin{pmatrix} -0.00021 \\ 4.904 \end{pmatrix} = 0.3063$$

$$\text{var}\, y = \left\{ p^t \cdot V \cdot p + 1 \right\} \cdot s^2 = \left\{ \begin{pmatrix} 1 & 0.0625 \end{pmatrix} \cdot \begin{pmatrix} 0.5 & -0.7937 \\ -0.7937 & 2.5196 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0.0625 \end{pmatrix} + 1 \right\} \cdot 51.0\text{E}-9 = 72E-9$$

$$y = \hat{y} \pm t_r \sqrt{\text{var}\, y} = 0.3063 \pm 4.3 \cdot \sqrt{72\text{E}-9} = 0.3063 \pm 0.00097 \quad \Rightarrow$$

0.3053 < y < 0.3073

3.3. Non-linear regression

This section will use an example to explain the difference between linear regression and **non**-linear regression. Only one method will be explained, which is certainly not the only one.
If a model is **non**-linear in the parameters, one must switch to **non**-linear regression if the error in y has a standard normal distribution (if this is not the case, a transformation must be found so that the error in transformed y: y' is normally distributed). **Non**-linear regression proceeds analogously to the procedure which applies to a linear model: minimizing the sum of the squares of the deviation (*RSS*) to the parameters d*RSS*/d*parameter* = 0. With a linear model, however, after differentiation the parameter vector can be separated from the matrix with the independent variables X and the measurement result $\overline{y}_i$. As a result, d*RSS*/d*parameter* can be solved analytically in the linear model. This will be demonstrated with an example:

Assume the linear model:

$$\hat{y} = b_0 + b_1 \cdot x \tag{3.49}$$

We want to minimize the $RSS = \sum_{i=1}^{n} \left(y_i - \hat{y}_s \right)^2$ to b_0 en b_1. For the sake of simplicity, we shall leave out the boundaries above and below to the summation notation.

$$\begin{aligned} RSS &= \sum \left(\hat{y}_i - y_i \right)^2 = \sum \left(\hat{y}_i^2 - 2\hat{y}_i y_i + y_i^2 \right)^2 \\ RSS &= \sum \left\{ \left(b_0 + b_1 x_i \right)^2 - 2\left(b_0 + b_1 x_i \right) y_i + y_i^2 \right\} \\ RSS &= \sum \left\{ b_0^2 + 2b_0 b_1 x_i + b_1^2 x_i^2 - 2b_0 y_i - 2b_1 x_i y_i + y_i^2 \right\} \end{aligned} \tag{3.50}$$

Minimize to b_0 and b_1:

$$\frac{\mathrm{d}RSS}{\mathrm{d}b_0} = \sum \left(2b_0 + 2b_1 x_i - 2y_i \right) = 0 \quad \Rightarrow \quad \sum \left(b_0 + b_1 x_i \right) = \sum \left(y_i \right) \tag{3.51}$$

$$\frac{\mathrm{d}RSS}{\mathrm{d}b_1} = \sum \left(2b_0 x_i + 2b_1 x_i^2 - 2x_i y_i \right) = 0 \Rightarrow \sum \left(b_0 x_i + b_1 x_i^2 \right) = \sum \left(x_i y_i \right) \tag{3.52}$$

In matrix vector notation this becomes:

$$\begin{pmatrix} \sum 1 & \sum x_i \\ \sum x_i & \sum x_i^2 \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \end{pmatrix} = \begin{pmatrix} \sum y_i \\ \sum x_i y_i \end{pmatrix} \tag{3.53}$$

From this it follows that the parameters can be calculated:

$$\begin{pmatrix} b_0 \\ b_1 \end{pmatrix} = \begin{pmatrix} \sum 1 & \sum x_i \\ \sum x_i & \sum x_i^2 \end{pmatrix}^{-1} \begin{pmatrix} \sum y_i \\ \sum x_i y_i \end{pmatrix} \tag{3.54}$$

Note that this equation is the same as equation 3.33. With a **non**-linear model, this goes wrong. The parameter vector $\vec{b}$ cannot be separated from the x_i and y_i. Another strategy must be followed. This can best be illustrated by an example:

Assume the **non**-linear model:

$\hat{y} = 3b_0 + \exp\left(b_1 x \right)$

We want to minimize the $RSS = \sum \left(\hat{y}_i - y_i \right)^2$ to b_0 en b_1.

$RSS = \sum \left(\hat{y}_i - y_i \right)^2 = \sum \left(\hat{y}_i^2 - 2\hat{y}_i y_i + y_i^2 \right) \quad \Rightarrow$

$RSS = \sum \left(\left[3b_0 + \exp\left(b_1 x_i \right) \right]^2 - 2\left[3b_0 + \exp\left(b_1 x_i \right) \right] y_i + y_i^2 \right) =$

$\sum \left\{ 9b_0^2 + 6b_0 \exp\left(b_1 x_i \right) + \left[\exp\left(b_1 x_i \right) \right]^2 - 6b_0 y_i - 2\exp\left(b_1 x_i \right) y_i + y_i^2 \right\}$

Minimizing *RSS* to the parameters is the same as calculating:

$\mathrm{d}RSS / \mathrm{d}b_0 = 0 \quad \wedge \quad \mathrm{d}^2 RSS / \mathrm{d}b_0^2 > 0$ and $\mathrm{d}RSS / \mathrm{d}b_1 = 0 \quad \wedge \quad \mathrm{d}^2 RSS / \mathrm{d}b_1^2 > 0$:

$$\frac{\mathrm{d}RSS}{\mathrm{d}b_0} = \sum\left\{18b_0 + 6\exp(b_1x_i) - 6y_i\right\} = 0 \;\Rightarrow\; \sum\left\{3b_0 + \exp(b_1x_i)\right\} = \sum y_i$$

$$\frac{\mathrm{d}RSS}{\mathrm{d}b_1} = \sum\left\{6b_0x_i\exp(b_1x_i) + 2x_i\left[\exp(b_1x_i)\right]^2 - 2x_i\exp(b_1x_i)y_i\right\} = 0 \;\Rightarrow$$

$$\sum\left\{3b_0x_i\exp(b_1x_i) + x_i\left[\exp(b_1x_i)\right]^2\right\} = \sum x_i\exp(b_1x_i)y_i$$

in matrix vector notation:

$$\begin{pmatrix} \sum 3 & \sum\frac{1}{b_1}\exp(b_1x_i) \\ \sum 3x_i\exp(b_1x_i) & \sum\frac{x_i}{b_1}\left[\exp(b_1x_i)\right]^2 \end{pmatrix}\begin{pmatrix} b_0 \\ b_1 \end{pmatrix} = \begin{pmatrix} \sum y_i \\ \sum x_i\exp(b_1x_i)y_i \end{pmatrix}$$

This equation cannot be solved analytically because both the matrix and the term on the right are dependent on the parameters b_0 and b_1.

Nevertheless, we have a solvable system since there are two unknowns, b_0 and b_1 and two equations. We could use *TRIAL AND ERROR*, but because we must look for both b_0 and b_1, this is nearly impossible. Therefore, several optimization methods have been developed, including the Gauss-Newton method, the gradient method, or a combination of both such as the method of Marquardt.

As an example, the Gauss-Newton method will be used. With this method, one looks at the improvement of the estimate of $\hat{y}_i = f(x_i, b_0, b_1)$ when the parameter values b_0 and b_1 are changed by a step, Δb_0 and Δb_1 respectively. This improvement is optimized to the step size Δb_0 en Δb_1.

The approximation for the new $\hat{y}_i = f(x_i, b_0 + \Delta b_0, b_1 + \Delta b_1)$ can be calculated by using a Taylor series (expansion) where the higher terms are disregarded:

$$f(x_i, b_0 + \Delta b_0, b_1 + \Delta b_1) = f(x_i, b_0, b_1) + \frac{\partial f(x_i, b_0, b_1)}{\partial b_0}\Delta b_0 + \frac{\partial f(x_i, b_0, b_1)}{\partial b_1}\Delta b_1 \;\Rightarrow$$

$$f(x_i, b_0 + \Delta b_0, b_1 + \Delta b_1) = f(x_i, b_0, b_1) + \begin{pmatrix} \frac{\partial f(x_i, b_0, b_1)}{\partial b_0} & \frac{\partial f(x_i, b_0, b_1)}{\partial b_1} \end{pmatrix}\begin{pmatrix} \Delta b_0 \\ \Delta b_1 \end{pmatrix}$$

for the sake of simplicity we will replace $f(x_i, b_0, b_1)$ by f so that:

$$\hat{y}_i(x_i, b_0 + \Delta b_0, b_1 + \Delta b_1) = f + \begin{pmatrix} \frac{\partial f}{\partial b_0} & \frac{\partial f}{\partial b_1} \end{pmatrix}\begin{pmatrix} \Delta b_0 \\ \Delta b_1 \end{pmatrix} = f + \frac{\partial f}{\partial b_0}\Delta b_0 + \frac{\partial f}{\partial b_1}\Delta b_1 \qquad (3.55)$$

Note that with the given b_0 and b_1, $\hat{y}_i$ is linear in Δb_0 and Δb_1.
Minimize of d*RSS*/d*parameters*.

First minimize to b_0:

$$\frac{\mathrm{d}RSS}{\mathrm{d}\Delta b_0} = \frac{\mathrm{d}\sum\left(\hat{y}_i^2 - 2\hat{y}_iy_i + y_i^2\right)}{\mathrm{d}\Delta b_0} = \sum\left\{2\hat{y}_i\frac{\mathrm{d}\hat{y}_i}{\mathrm{d}\Delta b_0} - 2y_i\frac{\mathrm{d}\hat{y}_i}{\mathrm{d}\Delta b_0}\right\} = 0$$

equation 3.55 is filled in for $\hat{y}$:

$$\sum\left\{2\left(f+\frac{\partial f}{\partial b_0}\Delta b_0+\frac{\partial f}{\partial b_1}\Delta b_1\right)\frac{\mathrm{d}\left(f+\frac{\partial f}{\partial b_0}\Delta b_0+\frac{\partial f}{\partial b_1}\Delta b_1\right)}{\mathrm{d}\Delta b_0}-2y_i\frac{\mathrm{d}\left(f+\frac{\partial f}{\partial b_0}\Delta b_0+\frac{\partial f}{\partial b_1}\Delta b_1\right)}{\mathrm{d}\Delta b_0}\right\}=0 \quad \Rightarrow$$

$$\sum\left\{2\left(f+\frac{\partial f}{\partial b_0}\Delta b_0+\frac{\partial f}{\partial b_1}\Delta b_1\right)\frac{\partial f}{\partial b_0}-2y_i\frac{\partial f}{\partial b_0}\right\}=0 \tag{3.56}$$

separate variables and divide by 2:

$$\sum\left\{\left(\frac{\partial f}{\partial b_0}\right)^2\Delta b_0+\frac{\partial f}{\partial b_0}\cdot\frac{\partial f}{\partial b_1}\Delta b_1\right\}=\sum\left\{\left(y_i-f\right)\frac{\partial f}{\partial b_0}\right\} \tag{3.57}$$

analogously, for b_1:

$$\sum\left\{\frac{\partial f}{\partial b_0}\cdot\frac{\partial f}{\partial b_1}\Delta b_0+\left(\frac{\partial f}{\partial b_1}\right)^2\Delta b_1\right\}=\sum\left\{\left(y_i-f\right)\frac{\partial f}{\partial b_1}\right\} \tag{3.58}$$

in matrix vector notation this becomes:

$$\begin{pmatrix}\sum\left(\frac{\partial f}{\partial b_0}\right)^2 & \sum\left(\frac{\partial f}{\partial b_0}\cdot\frac{\partial f}{\partial b_1}\right)\\ \sum\left(\frac{\partial f}{\partial b_0}\cdot\frac{\partial f}{\partial b_1}\right) & \sum\left(\left(\frac{\partial f}{\partial b_1}\right)^2\right)\end{pmatrix}\begin{pmatrix}\Delta b_0\\ \Delta b_1\end{pmatrix}=\begin{pmatrix}\sum\frac{\partial f}{\partial b_0}\cdot\left(y_i-f\right)\\ \sum\frac{\partial f}{\partial b_1}\cdot\left(y_i-f\right)\end{pmatrix} \tag{3.59}$$

this can be solved analogously to 3.33:

$$\begin{pmatrix}\Delta b_0\\ \Delta b_1\end{pmatrix}=\begin{pmatrix}\sum\left(\frac{\partial f}{\partial b_0}\right)^2 & \sum\left(\frac{\partial f}{\partial b_0}\cdot\frac{\partial f}{\partial b_1}\right)\\ \sum\left(\frac{\partial f}{\partial b_0}\cdot\frac{\partial f}{\partial b_1}\right) & \sum\left(\left(\frac{\partial f}{\partial b_1}\right)^2\right)\end{pmatrix}^{-1}\begin{pmatrix}\sum\frac{\partial f}{\partial b_0} & \sum\frac{\partial f}{\partial b_1}\end{pmatrix}\left(y_i-f\right) \tag{3.60}$$

Note: if we fill in this matrix for equation 3.49, 3.53 also follows from this!

If Δb_0 and Δb_1 are known, the new b_0 en b_1 can be calculated. Using these new values for b_0 and b_1, the procedure is repeated until Δb_0 and Δb_1 approach 0. Sometimes Δb_0 and Δb_1 will be begin to fluctuate. In this case the steps must be somewhat smaller.

This non-linear regression will be illustrated based on the example $\hat{y} = 3b_0 + \exp(b_1 x)$ with the measured values:

y(0) = 0; *y(0.5)* = 0.35 and *y(1)* = 1⇒

$$\frac{\partial f}{\partial b_0} = 3 \quad \text{and} \quad \frac{\partial f}{\partial b_1} = x_i \exp(b_1 x_i)$$

so that:

$$\begin{pmatrix} 9 & \sum 3x_i \exp(b_1 x_i) \\ \sum 3x_i \exp(b_1 x_i) & \sum x_i^2 \exp^2(b_1 x_i) \end{pmatrix} \begin{pmatrix} \Delta b_0 \\ \Delta b_1 \end{pmatrix} = \begin{pmatrix} 3 & x_i \exp[b_1 x_i] \end{pmatrix} (y_i - f_{xi})$$

for the initial values, choose for example $b_0 = 10$ en $b_1 = 10$, now we can calculate Δb_0 en Δb_1. (Take note: the values given here have been rounded off, for the calculations more decimals will be used):

x_i	y_i	$\hat{y}_i$	Δy $y_i - \hat{y}_i$	Δy^2	$\frac{\partial f}{\partial b_0}$	$\frac{\partial f}{\partial b_1}$	$\left(\frac{\partial f}{\partial b_0}\right)^2$	$\frac{\partial f}{\partial b_1} \cdot \frac{\partial f}{\partial b_1}$	$\left(\frac{\partial f}{\partial b_1}\right)^2$	$\frac{\partial f}{\partial b_0}\Delta y$	$\frac{\partial f}{\partial b_1}\Delta y$
0	0	31	-31	961	3	0	9	0	0	-93	0
0.5	0.35	178	-178	31706	3	74	9	222	5506	-534	-1321
1	1	22056	-22055	4.9E8	3	22026	9	66079	4.9E8	-66166	-4.9E8
			RSS =	4.9E8		sum	27	66302	4.9E8	-66793	-4.9E8

$$\left(X^t X\right)^{-1} = \begin{pmatrix} 27 & 66302 \\ 66302 & 4.9\text{E}+8 \end{pmatrix}^{-1} = \begin{pmatrix} 0.056 & -7.6\text{E-}6 \\ -7.6\text{E-}6 & 3.1\text{E-}9 \end{pmatrix}$$

and $X^t y = \begin{pmatrix} -66794 \\ -4.9\text{E}8 \end{pmatrix}$ so that: $\begin{pmatrix} \Delta b_0 \\ \Delta b_1 \end{pmatrix} = \left(X^t X\right)^{-1} X^t y = \begin{pmatrix} -22.477 \\ -0.998 \end{pmatrix}$

therefore, the new values become $b_0 = -12.47$ en $b_1 = 9.00$. The procedure must now be repeated, which results in the following series of b_0 en b_1:

Try	1	2	3	4	5	6	7	.....	13
b_0	10	-12.5	-7.8	-4.6	-2.8	-1.8	-1.2		-0.34231
b_1	10	9.0	8.0	7.0	6.0	5.0	4.0		0.69992
RSS	4.9E8	6.5E7	8.8E6	1.2E6	1.6E5	2.1E4	2896		0.002671

Check: $f(0) = 3b_0 + \exp(b_1 x) = -0.027 \approx 0$
$f(0.5) = 0.392 \approx 0.35$
$f(1) = 0.986 \approx 1$

3.4. Comparisons of estimates

If we perform two series of measurements under different conditions, we often want to determine if there is a significant difference between the two series. This can be done by comparing the confidence intervals of the two measurement series. If these two do not overlap each other, the values are in any case significantly different. If one of the two averages lies in the confidence interval of the other, there is usually no significant difference. If neither is the case (the confidece intervals overlap but not in such a way that one of the averages lies in the area of the other), it becomes somewhat more difficult. We must then see if the 'surface of the overlapping area' is larger than 5%. We can do this in two ways (for this we assume that the variance of the two populations is the same):

$$t = \frac{\bar{y}_1 - \bar{y}_2}{s_p \sqrt{1/n_1 + 1/n_2}} \tag{3.61}$$

where n is the number of measured points and s_p the pooled variance:

$$s_p^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 - 1 + n_2 - 1} \tag{3.62}$$

We must compare this t with the t value at n_1+n_2-2 degrees of freedom. We measure, for example, (3, 2, 1) and (4, 6, 8). Then $\bar{y}_1 = 2$ and $\bar{y}_2 = 6$.

$$RSS_1 = 2 \;\rightarrow\; s_1^2 = 2/2 = 1 \quad \wedge \quad RSS_2 = 8 \;\rightarrow\; s_2^2 = 8/2 = 4$$

$$s_p^2 = \frac{2 \cdot 1 + 2 \cdot 4}{4} = \frac{10}{4} = \frac{5}{2}$$

$$t = \frac{2 - 6}{\sqrt{\frac{5}{2} \cdot (1/3 + 1/3)}} = -\frac{4}{\sqrt{5/3}} = -3.1 < -2.78$$

(with 4 degrees of freedom)

Therefore the averages are different. We can derive this by making use of linear regression with, for example, three points at every treatment:

$$y = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} b_1 + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} b_2 \qquad \text{Therefore the X matrix is: } X = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}$$

$$X^t X = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \sum x_1 & 0 \\ 0 & \sum x_2 \end{pmatrix} = \begin{pmatrix} n_1 & 0 \\ 0 & n_2 \end{pmatrix}$$

This generally results in:

$$X^t X^{-1} = V = \begin{pmatrix} 1/n_1 & 0 \\ 0 & 1/n_2 \end{pmatrix}$$

$$\mathrm{var}(b_1) = \frac{1}{n_1} \frac{RSS}{DF} \quad \wedge \quad \mathrm{var}(b_2) = \frac{1}{n_2} \frac{RSS}{DF}$$

$RSS = RSS_1 + RSS_2$

Therefore the variance of the difference between the means is:

$$\mathrm{var}(b_1 - b_2) = \mathrm{var}(b_1) + \mathrm{var}(b_2) - \mathrm{cov}(b_1, b_2)$$

covar(b_0 , b_1) = 0 because the non-diagonal elements of the V matrix are zero (see also 3.2.3 under equation 3.42).

$$\mathrm{var}(b_1 - b_2) = \mathrm{var}(b_1) + \mathrm{var}(b_2)$$

$$t = \frac{\bar{b}_1 - \bar{b}_2}{\sqrt{\mathrm{var}(b_1 - b_2)}} = \frac{\bar{b}_1 - \bar{b}_2}{\sqrt{\mathrm{var}(b_1) + \mathrm{var}(b_2)}}$$

$$t = \frac{\bar{b}_1 - \bar{b}_2}{\sqrt{\frac{RSS}{DF}\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = \frac{\bar{b}_1 - \bar{b}_2}{\sqrt{\frac{RSS_1 + RSS_2}{n_1 - 1 + n_2 - 1}\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = \frac{\bar{b}_1 - \bar{b}_2}{\sqrt{s_p^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

Therefore, using the above example this becomes:

$$X^t y = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 1 \\ 4 \\ 6 \\ 8 \end{pmatrix} = \begin{pmatrix} 6 \\ 18 \end{pmatrix} = \begin{pmatrix} \sum y_1 \\ \sum y_2 \end{pmatrix}$$

$$b = \left(X^t X\right)^{-1} X^t y = \begin{pmatrix} 1/3 & 0 \\ 0 & 1/3 \end{pmatrix} \begin{pmatrix} 6 \\ 18 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \end{pmatrix} = \begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \end{pmatrix}$$

$RSS = RSS_1 + RSS_2 = 1+0+1+4+0+4=10$, DF=6-2=4, $RSS/DF=5/2$
$\mathrm{var}(b_1)=5/6=\mathrm{var}(b_2)$, $\mathrm{var}(b_1)+\mathrm{var}(b_2)=10/6=5/3$

$$t = -\frac{4}{\sqrt{5/3}} = -3.1$$

If we carry out a two-sided test on this, the t value is 2.78. Therefore the hypothesis $y_1=y_2$ is rejected.

3.5. Comparisons of models/accuracy of the fit

3.5.1. Graphic representation

An important tool for comparing models is graph paper and the human eye in combination with the brain (*basic eyeball technology*). When considering and comparing models on a graph, you often see very quickly what the shortcomings of a model are, in which areas a model is weak, etc. Therefore, always graph the data points with the model prediction. Also make a graph of the residuals (the deviations of the data points and the model). The residuals must be neatly distributed. If the residuals deviate strongly during a particular interval, something strange is going on. Generally speaking, the best method is to take a good look at the graph. Since the eye is sometimes prejudiced, statistics are available to act as a referee, especially in cases of doubt.

3.5.2. t-test

The above mentioned t test provides confidence intervals for parameters. These confidence intervals can be used for comparing models. For example, if the value 0 for a particular parameter lies in the

confidence interval, this parameter is not really significant and can perhaps be left out. The word 'perhaps' was used in this last sentence because statistical grounds do not have to be the only reasons for retaining or rejecting a parameter.

Example 3.6:
In Chapter 3, example 3.2, we described the data of the falling canonball with two models.

model 1: $y = ax + b$
model 2: $y = ax$

If we look at the confidence interval of parameter b in model 1, we see that 0 does/does not lie in the confidence interval. Which model can therefore be best used for describing the data?

3.5.3. F- test

Models can also be compared with each other by using the F test. In an F test, variances are compared with each other. Frequently, the variance of the model var_1 is compared with the model $y = \bar{y}$, var_2.

$$f_{df_2}^{df_1} = \frac{\text{var}_1}{\text{var}_2} \tag{3.63}$$

If the f value is smaller than the tabulated F value, one assumes that there is no difference between the two models and the model $y = \bar{y}$, is accepted.

To make an orderly comparison, one makes use of an ANOVA table which shows the *RSS*, the number of degrees of freedom and the variance in an orderly fashion for the various models. Finally the f values are calculated and compared with the F values.

	Model	*RSS*	*DF*	*var*	*f*	*F*
1	$y = \bar{y}$	$RSS_1 = \sum (y_i - \bar{y})^2$	n - 1	$\frac{RSS_1}{DF_1}$		
2	$y = f(x)$	$RSS_2 = \sum (y_i - f(x_i))^2$	$n - p_2$	$\frac{RSS_2}{DF_2}$	$f_{DF_2}^{DF_1} = \frac{\text{var}_1}{\text{var}_2}$	F_{DF2}^{DF1}

You should be aware that some packages test with respect to the model $y = 0$. It is therefore always important to check which test is being used. It may be clear that an improvement is quickly obtained with regard to the model $y = \bar{y}$, (and a great improvement with respect to the model $y = 0$). A more sensitive procedure is to compare the *lack of fit* of models. In this process, a 'lower' model is compared with a 'coordinated' model.

The added value of the extra parameters, *the lack of fit*, is tested by comparing the variance of the difference between the model and the general model:

(RSS_{model}-$RSS_{\text{general model}}$)/($DF_{\text{model}}$-$DF_{\text{general model}}$) (variance of the *lack of fit*) with the variance of the general model itself, $RSS_{\text{general model}}/DF_{\text{general model}}$ (estimated measurement error). These two variances can be computed using the f-statistic as follows:

$$f_{DF_{\text{general model}}}^{DF_{\text{model}}-DF_{\text{general model}}} = \frac{\left(\frac{RSS_{\text{model}} - RSS_{\text{general model}}}{DF_{\text{model}} - DF_{\text{general model}}}\right)}{\left(\frac{RSS_{\text{general model}}}{DF_{\text{general model}}}\right)}$$

(where DF= degrees of freedom).

If the added value of the extra parameters is much smaller than the measurement error, the extra parameters are not significant. One then chooses the model with the smallest number of parameters (Ockham's razor).

Examples of general models are models 1 and 2 from example 3.6. For instance, if we have two models and want to compare these:

$y = ax + b$ (model 1) en $y = a\ln x + b$ (model 2)

as general model we can then use:

$y = a(x^c - 1)/c + b$ (model 3)

Exercise 3.4: work this out for yourself.

The procedure is as follows:
The data points (assume, for example, 10 data points) are fitted with models 1, 2 and 3. For each model the residual sum of squares (*RSS*) is calculated.

Model 3 (general) is considered to be correct and the deviations which are left over are assumed to be measurement errors. The variance is RSS_3, divided by the number of degrees of freedom (number of measurement points - number of parameters, see also equation 3.4). Therefore, in this case, $RSS_3/(10\text{-}3)$.

The added value of the general model (through the addition of the parameter c) is expressed in a lower *RSS* ($RSS_3<RSS_1$ and $RSS_3<RSS_2$).The variance of this difference is equal to (RSS_1-RSS_3)/(DF_1-DF_3) and (RSS_2-RSS_3)/(DF_2-DF_3). DF_1 and DF_2 are equal to 10-2=8, and DF_3 was equal to 7, therefore DFx-DF_3=1. Therefore the test variable becomes:

$$f_7^1 = \frac{\frac{RSS_1 - RSS_3}{DF_1 - DF_3}}{\frac{RSS_3}{DF_3}} = \frac{\frac{RSS_1 - RSS_3}{1}}{\frac{RSS_3}{7}} \quad \text{and} \quad \frac{\frac{RSS_2 - RSS_3}{1}}{\frac{RSS_3}{7}} \tag{3.64}$$

where $\frac{RSS_1 - RSS_3}{1}$ and $\frac{RSS_2 - RSS_3}{1}$ is the 'profit parameter' and $\frac{RSS_3}{7}$ represents the measurement error.

The limits are tabulated in the same way as the *t* test variable. An example is worked out in the following section.

It is also advisable to make a table of the data analogous to the ANOVA table:

	Model	*RSS*	*DF*	*var*	*f*	*F*
1:	$y = 0$	$RSS_1 = \sum (y_i - 0)^2$	n - 1	$\frac{RSS_1 - RSS_{gm}}{DF_1 - DF_{gm}}$	$f_{DF_{om}}^{DF_1 - DF_{om}} = \frac{\text{var}_1}{\text{var}_{gm}}$	$F_{DF_{gm}}^{DF_1 - DF_{gm}}$
2:	$y = \bar{y}$	$RSS_2 = \sum (y_i - \bar{y})^2$	n - 1	$\frac{RSS_2 - RSS_{gm}}{DF_2 - DF_{gm}}$	$f_{DF_{gm}}^{DF_2 - DF_{om}} = \frac{\text{var}_2}{\text{var}_{gm}}$	$F_{DF_{gm}}^{DF_2 - DF_{gm}}$
3:	$y = f(x)$	$RSS_3 = \sum (y_i - f(x_i))^2$	n - p_3	$\frac{RSS_3 - RSS_{gm}}{DF_3 - DF_{gm}}$	$f_{DF_{gom}}^{DF_3 - DF_{gm}} = \frac{\text{var}_3}{\text{var}_{gm}}$	$F_{DF_{gm}}^{DF_3 - DF_{gm}}$
gm:	$y = g(x)$	$RSS_{gm} = \sum (y_i - g(x_i))^2$	n - p_4	$\frac{RSS_{gm}}{DF_{gm}}$		

The 'best' general model is based on the use of replicas: for every setting *x*, the mean of the measured *y* value is used as a model prediction. However, this model has no predictive value for *x* values other than the measured values, but it is the best way to determine the measurement error.

Exercise 3.5: Use this method to compare the two models from exercise 3.2 with each other. Change all values of *y* by adding 0.000902, calculate the confidence interval of *a* and then determine the *f*-statistic there.

Solutions

3.4:

$$\lim_{c \downarrow 0}\left[a\frac{\left(x^c - 1\right)}{c} + b \right] = a \lim_{c \downarrow 0}\left[\frac{\left(e^{c\ln(x)} - 1\right)}{c} \right] + b$$

$$\overset{l'Hopital}{=} a \lim_{c \downarrow 0}\left[\frac{\left(e^{c\ln(x)} \cdot \ln(x)\right)}{1} \right] + b = a \ln x + b$$

so for $c = 0$ we have model 2, and if $c = 1$ we have model 1.

3.5:
t tests: $b = 0$ lies in the confidence interval and is therefore not significant. It is best to use the model $y = ax$.
F tests: var_2=5.102E-8; var_1=9.176E-8 $\Rightarrow f(1,2)=1.8 << 18.52$

After addition: $0 < b < 0.001372$
f=18.49

Therefore, the proposition that *a* is not significant lies just on the border between acceptance and rejection in both tests.

3.5.4.

3.5.4 r^2 coefficient of determination and correlation coefficient

After carrying out a regression, statistical (computer) programs frequently produce the term r^2 (or r). This is a possible standard for the correctness of the fit. This r^2 can be calculated in various ways.

$$r^2 = \frac{\text{regression } SS}{\text{total } SS} \tag{3.65}$$

In this formula, regression *SS* is equal to total *SS* minus the residual sum of squares (*RSS*), and the total *SS* (*TSS*) is the sum of the squares of the deviation from the mean (therefore *RSS* for the model y = constant). Regression *SS* is therefore the 'profit' in *SS* which we have achieved with our regression function (with respect to the mean). r^2 is sometimes also called the percentage of explained variance.

$$r^2 = \frac{TSS - RSS}{TSS} \tag{3.66}$$

$$r^2 = \frac{\sum\left[(y-\bar{y})^2 - (y-\hat{y})^2\right]}{\sum(y-\bar{y})^2} \tag{3.67}$$

According to the 'partition theorem' (see chapter 7.2 appendix 2):

$$\sum(y-\bar{y})^2 \quad = \quad \sum(y-\hat{y})^2 \quad + \quad \sum(\hat{y}-\bar{y})^2 \tag{3.68}$$

total variation = unexplained variation + explained variation

TSS = *RSS* + unexplained variation

it follows that:

$$r^2 = \frac{\sum(\hat{y}-\bar{y})^2}{\sum(y-\bar{y})^2} = \frac{\text{explained variation}}{\text{total variation}} \tag{3.69}$$

$$r^2 = \frac{\sum\hat{y}^2 - 2\bar{y}\sum\hat{y} + n\bar{y}^2}{\sum y^2 - 2\bar{y}\sum y + n\bar{y}^2} = \frac{\sum\hat{y}^2 - n\bar{y}^2}{\sum y^2 - n\bar{y}^2} \tag{3.70}$$

This assumes that (see chapter 7.2 appendix 2):

$$\sum\hat{y} = \sum y = n\bar{y} \tag{3.71}$$

For linear regression with $y = ax+b$, the regression coefficients are:

$$a = \frac{n\sum xy - \sum x\sum y}{n\sum x^2 - \left(\sum x\right)^2} = \frac{SXY}{SSX} = \frac{\sum xy - \frac{1}{n}\sum x\sum y}{\sum x^2 - \frac{1}{n}\left(\sum x\right)^2} \tag{3.72}$$

$$b = \frac{\sum y\sum x^2 - \sum x\sum xy}{n\sum x^2 - \left(\sum x\right)^2} = \bar{y} - a\bar{x} = \frac{\bar{y}SSX - SXY\bar{x}}{SSX} \tag{3.73}$$

because:

$$\bar{y}SSX - SXY\bar{x} = \bar{y}\left(n\sum x^2 - \left(\sum x\right)^2\right) - \left(n\sum xy - \sum x\sum y\right)\bar{x} = \sum y\sum x^2 - \bar{y}\left(\sum x\right)^2 - \sum x\sum xy + \bar{x}\sum x\sum y = \sum y\sum x^2 - \sum x\sum xy \tag{3.74}$$

In that case:

$$r^2 = \frac{\sum \hat{y}^2 - n\bar{y}}{\sum y^2 - n\bar{y}^2} = \frac{\sum (ax+b)^2 - n\bar{y}^2}{\sum y^2 - n\bar{y}^2} = \frac{\sum\left[\frac{SXY}{SSX}(x-\bar{x})+\bar{y}\right]^2 - n\bar{y}^2}{\sum y^2 - n\bar{y}^2} \tag{3.75}$$

$$r^2 = \frac{\frac{SXY^2}{SSX^2}\sum\left(x^2 - 2x\bar{x} + \bar{x}^2\right) + \frac{SXY}{SSX}\bar{y}\sum(x-\bar{x})}{\sum y^2 - n\bar{y}^2} = \frac{\frac{SXY^2}{SSX^2}\sum\left(x^2 - n\bar{x}^2\right)}{SSY} = \frac{SXY^2}{SSX\cdot SSY} \tag{3.76}$$

$$r = \frac{SXY}{\sqrt{SSX\cdot SSY}} = \frac{\sum xy - \frac{1}{n}\sum x\sum y}{\sqrt{\left(x^2 - \frac{1}{n}\left(\sum x\right)^2\right)\left(y^2 - \frac{1}{n}\left(\sum y\right)^2\right)}} \tag{3.77}$$

Correlation coefficient

If both x and y are stochastic functions, the correlation coefficient can be calculated:

$$r = \frac{\text{covariance}(x,y)}{\sqrt{\text{variance}\,x\cdot\text{variance}\,y}} \tag{3.78}$$

$$r = \frac{\sum \mathrm{d}x\mathrm{d}y}{\sqrt{\sum \mathrm{d}x^2\mathrm{d}y^2}} \tag{3.79}$$

$$\mathrm{d}x = x - \bar{x} \quad \mathrm{d}y = y - \bar{y} \tag{3.80}$$

$$r = \frac{\sum_{i=1}^{n}\left(x_i - \bar{x}\right)\left(y_i - \bar{y}\right)}{\sqrt{\Sigma\left(x_i - \bar{x}\right)^2\,\Sigma\left(y_i - \bar{y}\right)^2}} = \frac{\sum_{i=1}^{n} x_i y_i - n\cdot\bar{x}\cdot\bar{y}}{\sqrt{\left(\Sigma\, x_i^2 - n\bar{x}^2\right)\left(\Sigma\, y_i^2 - n\bar{y}^2\right)}} = \frac{\frac{1}{n}\sum_{i=1}^{n} x_i y_i - \bar{x}\cdot\bar{y}}{\sqrt{\left(\Sigma\,\frac{x_i^2}{n} - \bar{x}^2\right)\left(\Sigma\,\frac{y_i^2}{n} - \bar{y}^2\right)}} \tag{3.81}$$

This equation is exactly the same as for the above r, but means something different! Correlation can only exist between two stochastic variables. With linear regression (such as being discussed here) x is exact and y is a stochastic function. The r in linear regression could be seen as the correlation between the measured (y) values and the predicted values. These are both stochastic functions (measured versus predicted).

Adjusted r^2

Since r^2 always continues to increase with an increasing number of parameters, r^2 is not a good measure of the correctness of the fit. An n-1 degree polynomial gives an *RSS* of zero and therefore an r^2 of 1. With an increase in the number of fitting parameters, *RSS* shall always decrease, or at the very most, remain the same. Therefore, r^2 shall rise or perhaps remain the same (if the parameter does not provide any improvement at all). It can happen that a parameter does not produce any significant improvement. According to the principle of Ockham's razor (parsimony/simplicity) a large number of parameters is not desirable. Therefore, the better criterium is the 'adjusted' r^2 or corrected r^2.

$$\bar{r}^2 = \frac{\frac{TSS}{DF} - \frac{RSS}{DF}}{\frac{TSS}{DF}} = \frac{\frac{TSS}{n-1} - \frac{RSS}{n-k}}{\frac{TSS}{n-1}} = 1 - \frac{n-1}{n-k} \cdot \frac{RSS}{TSS} \tag{3.82}$$

$$\bar{r}^2 = 1 - \frac{n-1}{n-k} \cdot \left(1 - r^2\right) = 1 - \frac{n-1}{n-k} + \frac{n-1}{n-k} \cdot r^2 = \frac{n-k-n+1}{n-k} + \frac{n-1}{n-k} \cdot r^2 \tag{3.83}$$

$$\bar{r}^2 = \frac{n-1}{n-k} \cdot r^2 - \frac{k-1}{n-k} = \frac{n-1}{n-k}\left(r^2 - \frac{k-1}{n-1}\right) \tag{3.84}$$

This value therefore gives a better standard for comparing models. If the number of parameters is increased, RSS does decrease, but at the same time the number of degrees of freedom also decreases. Therefore, the increase in the number of parameters is taken into account in the ratio RSS/(n-k). There are also other methods where a 'penalty term' punishes the increase in the number of parameters, such as the Jackknife, Sshwartz, Mallows and Akaike criterium (see statistics textbooks).

Multi-collinearity

Finally, packages frequently give the coefficient of determination matrix. In this matrix (often incorrectly named the correlation matrix), the coefficients of determination are given for all pairs of independent variables x_i:

$$r_{x_n,x_m} = \frac{\sum\left\{\left(x_{n,i} - \bar{x}_n\right) \cdot \left(x_{m,i} - \bar{x}_m\right)\right\}}{\sqrt{\sum\left(x_{n,i} - \bar{x}_n\right)^2 \cdot \left(x_{m,i} - \bar{x}_m\right)^2}} \tag{3.85}$$

Note that the coefficient of determination of the pair x_i,x_i is always equal to 1. The matrix with the coefficient of determination therefore generally appears as follows:

$$r = \begin{pmatrix} 1 & r_{x_1,x_2} & r_{x_1,x_3} & . & . & r_{x_1,x_m} \\ r_{x_2,x_1} & 1 & r_{x_2,x_3} & . & . & r_{x_2,x_m} \\ r_{x_3,x_1} & r_{x_3,x_2} & 1 & . & . & r_{x_3,x_m} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ r_{x_m,x_1} & r_{x_m,x_2} & r_{x_m,x_3} & . & . & 1 \end{pmatrix} \tag{3.86}$$

If there are values of r_{x_n,x_m} approximately equal to 1, this means that the variables x_n,x_m are dependent: $x_n = a\, x_m$. This is known as multi-collinearity. In the case of multi-collinearity, the model should be rewritten without one of the variables, x_n of x_m, in other words, with one fewer parameter.

4 Epilogue

4.1 Assumptions

When Newton sat under a tree daydreaming and saw an apple fall, he could have followed two lines of thought. He could have thought:

An apple is falling. The earth is pulling on the apple because this apple has a mass. The apple is a round object and the local mass of the apple is therefore different across the diameter of the apple (more mass in the middle and increasingly less towards the edge). As the apple comes closer to the earth, the gravitational force increases, therefore the force changes with time. If the earth pulls on the apple, the sun and the moon naturally pull on the apple as well, but then from the other direction. The apple also experiences friction due to collisions with air molecules and there is wind which moves the apple sideways. Never mind, that is far too complicated. I think I'll just take a nap.

But no, Newton thought: this apple is a point mass (not true) which is attracted by the earth. There is therefore a constant force (not true) which gives the apple a constant acceleration.

This led to a wonderful model, largely as the result of its simplicity and broad applicability. With this model we can (and must) investigate the validity of all the assumptions. For example: the increase in gravitational force is negligible across short distances. The model can also be expanded with factors which are important in some cases (such as friction).

It is therefore ironic that you make assumptions (such as point mass) in modelling which you know are not true. After you have established a model, you try to 'shoot down' the assumptions and see if they are not overly exaggerated.

Frowning and saying that 'it's all much more complicated' does not solve any problems. Moreover, making lists of everything that **may** have an influence brings you no further to predicting what actually will happen with a system. It is important that you first find the most important effects and understand them, and then use the above named lists to criticize what you have developed. (Which takes more time and which is more useful: 1) stating that the growth of micro-organisms is primarily determined by the temperature, precisely quantifying the effect of the temperature and then using that model to make discerning predictions, or 2) make a list of everything except for temperature which can influence the growth and then rejecting all the predictions of model 1.)

Another idea held by many people is that their own discipline is much more complicated than other disciplines, and that those in the other disciplines can also calculate and predict things much better (because there are not so many complexities involved).

4.2 Critical consideration

According to Popper, a scientist must go to work in such a way that all statements can be refuted. Therefore, after establishing models, we clearly show all the assumptions so that others, and ourselves as well, can criticize the assumptions. We must not present the models as absolute truth, we must remain critical of them. After that, we must strive to 'bowl over' our own theory, for example by performing all kinds of experiments.

Anyway, in many cases it is not at all necessary to do a great deal of work to refute models: if the model builder performs an experiment to see if the model works, this is frequently sufficient to show all kinds of gaps in the model. Note that the objectives in these two paragraphs are different. In the first paragraph the objective was bowling over the model, and in the second the objective was to show that the model works! The latter method (showing that it works) is used secretly by scientists. They should actually use the first method, but experiments so often give unexpected results that with refutation they still appear to be successful.

4.3 Proof

If a model can make a good description of the observations, this does not necessarily mean that the model is correct. A simple example of this is the following:

Yeast divides by forming buds. If we assume that the number of buds -and therefore the number of newly forming yeasts- is equivalent to the number of yeasts, it follows that the increase in yeasts is proportional to the total number of organisms. Following experimental verification, it turns out that this model provides a good description of the observations. Later it turned out that the model also provided a good description for the increase in the number of bacteria. The conclusion is that bacteria also divide by budding, and we know this is not correct. Accordingly, various assumptions can lead to the same equation and can thus make a good description of the data. The fact that the model turns out to work well in practice does not prove the assumptions which have been made.

A model predicts something, such as a number. If the measured number is the same as predicted one, this doesn't mean that the mechanism is necessarily correct. A dog is an animal with four legs. Look, there is a pasture full of animals with four legs, therefore they are all dogs.

4.4 Examples of various models

Examples of various models which result in the same equation are:

the Langmuir isotherm (for adsorption) (with Γ the absorbed quantity, c the concentration and b an affinity coefficient

$$\Gamma = \frac{\Gamma_m bc}{bc+1} = \frac{\Gamma_m c}{c+1/b} \tag{4.1}$$

the Michaelis-Menten model (for enzyme kinetics) (where V is the reaction speed (rate), c_s the substrate concentration and K_m the Michaelis-Menten constant

$$V = \frac{V_m c_s}{c_s + K_m} \tag{4.2}$$

or the Monod model (growth kinetics) where μ is the specific growth rate and K_s is the Monod constant or substrate affinity coefficient.)

$$\mu = \frac{\mu_m c_s}{c_s + K_s} \tag{4.3}$$

These equations are exactly the same. Therefore, if we assume in enzyme kinetics that the step which limits the reaction rate is the adsorption of substrate on the enzyme, and this proceeds according to a Langmuir isotherm, then we obtain exactly the same results as those which we would get if we said that the enzyme kinetics proceed according to Michaelis-Menten. The fact that we can describe enzyme kinetics with a model which shows adsorption kinetics is no proof that the rate-limiting step is the adsorption of substrate on an enzyme.

Another example concerns two different models which result in two different equations, but can describe the same effect. In this example, we are going to describe the lag phase in bacterial growth. For example, we can assume that: 1) the cells are in such a physical state that they cannot yet divide, and that component *B* must first be made which is converted to *C* before they can divide; 2) component *A*, from which *B* is formed, is available in excess (zero-order in *A*); 3) the number of molecules *C* per cell (*p*) is constant (therefore the more *C* which is formed, the more cells (*x*) there are): $c_c=p*c_x$ or $c_x=c_c/p$; 4) if I have more cells, the reactions also proceed proportionally much more quickly (First order in number). This results in the following model:

$$A \xrightarrow{k_a} B \xrightarrow{k_b} C \tag{4.4}$$

$$\frac{\mathrm{d}c_b}{\mathrm{d}t} = (k_a - k_b c_b)\cdot c_x = k_a c_x - k_b c_b c_x \tag{4.5}$$

$$\frac{\mathrm{d}c_c}{\mathrm{d}t} = p\frac{\mathrm{d}c_x}{\mathrm{d}t} = k_b c_b c_x \tag{4.6}$$

We can solve this model, for example numerically (see LOTUSFILE growmod.wk1). We can also try to interpret this equation. We can expect that if there is too little *B* available, this must first be produced before the cell can really begin to grow; this is therefore the adaptation phase/lag phase. But after a time, a stationary situation ($\mathrm{d}c_b/\mathrm{d}t=0$) could occur for *B*. Then $k_a=k_b c_b$, and therefore:

$$p\frac{\mathrm{d}c_x}{\mathrm{d}t} = k_b c_b c_x = k_a c_x \tag{4.7}$$

If *B* consequently reaches a stationary quantity ($c_b=k_a/k_b$), the final specific growth rate becomes k_a/p. In the (numerical) solution it turns out that these statements tally. This is a good check of the solution and of our understanding of the system. This model can make a good description of the lag phase during bacterial growth.

Another approach could be that we assume that: 1) bacteria grow exponentially (following the lag phase); and that 2) every individual cell has a lag phase and that this lag phase is normally distributed in the population of cells (therefore with a mean and a standard deviation). If we calculate the change in the total population with these assumptions, we can use this to make a good description of the curvature of the change in the number of bacteria in time: 1) bacteria grow exponentially following the lag phase:

$$\ln\left(\frac{N}{N_0}\right) = \mu\cdot(t-\lambda) \quad \text{with} \quad t > \lambda \tag{4.8}$$

where *N* is the number of micro-organisms, N_o the number at $t=0$, μ the specific growth rate and λ the lag time. 2)the lag phase is normally distributed:

$$\lambda = \overline{\lambda} + \sigma u \tag{4.9}$$

with the following used for the normal distribution:

$$f(u) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) \tag{4.10}$$

$$F(x) = \int_{-\infty}^{x} f(u)\mathrm{d}u \tag{4.11}$$

Using this normally distributed time lag for a single cell results in :

$$\ln(N) = \mu\left(t - \overline{\lambda} - \sigma u\right) \text{ with } t > \overline{\lambda} + \sigma u \tag{4.12}$$

and for the entire population:

$$\ln\left(\frac{N}{N_0}\right) = \int_{-\infty}^{\infty} \mu \cdot \left(t - \overline{\lambda} - \sigma u\right) \cdot f(u)\mathrm{d}u \text{ with t} > \overline{\lambda} + \sigma \mathrm{u} \tag{4.13}$$

The time is larger than the time lag if:

$$u < \left(t - \overline{\lambda}\right) / \sigma \tag{4.14}$$

We can therefore also write the integral as:

$$\ln\left(\frac{N}{N_0}\right) = \mu \int_{-\infty}^{(t-\overline{\lambda})/\sigma} \left(t - \overline{\lambda} - \sigma u\right) \cdot f(u)\mathrm{d}u \tag{4.15}$$

$$\ln\left(\frac{N}{N_0}\right) = \mu\left(t - \overline{\lambda}\right) \int_{-\infty}^{(t-\overline{\lambda})/\sigma} f(u)du - \mu\sigma \int_{-\infty}^{(t-\overline{\lambda})/\sigma} u \cdot f(u)\mathrm{d}u \tag{4.16}$$

$$\ln\left(\frac{N}{N_0}\right) = \mu\left(t - \overline{\lambda}\right)F\left[\left(t - \overline{\lambda}\right) / \sigma\right] - \frac{\mu\sigma}{\sqrt{2\pi}} \int_{-\infty}^{(t-\overline{\lambda})/\sigma} u \cdot \exp\left(-\frac{u^2}{2}\right)\mathrm{d}u \tag{4.17}$$

$$\ln\left(\frac{N}{N_0}\right) = \mu\left(t - \overline{\lambda}\right)F\left[\left(t - \overline{\lambda}\right) / \sigma\right] - \frac{\mu\sigma}{\sqrt{2\pi}} \int_{-\infty}^{(t-\overline{\lambda})^2/2\sigma^2} \exp\left(-\frac{u^2}{2}\right)\mathrm{d}\left(\frac{u^2}{2}\right) \tag{4.18}$$

$$\ln\left(\frac{N}{N_0}\right) = \mu\left(t - \overline{\lambda}\right)F\left[\left(t - \overline{\lambda}\right) / \sigma\right] + \frac{\mu\sigma}{\sqrt{2\pi}} \exp\left[-\left(t - \overline{\lambda}\right)^2 / 2\sigma^2\right] \tag{4.19}$$

We can also attempt to interpret this equation. For larger times, the *e* power will approach zero and the value of *F* will go to 1 (the normal distribution becomes 1 with larger numbers) and

therefore the equation approaches exponential growth with mean lag time for larger times. We can also make a good description using this model of the course of the lag phase in bacterial growth. By making a suitable choice of the parameters, we can even let the two models fall almost precisely over each other (see Figure 4.1), from which we can conclude that the experimental confirmation of a model does not prove the correctness of a model, it only shows that the assumed mechanism is possible. Both models can describe the observed results, although the starting points and the ultimate equations are totally different. In order to distinguish between the two models, experiments must be done to test the assumptions, such as tracing component *B* and following this in time. Or we could place a single micro-organism in each of 1000 test tubes and see if the time-to-turbidity is normally distributed, for example.

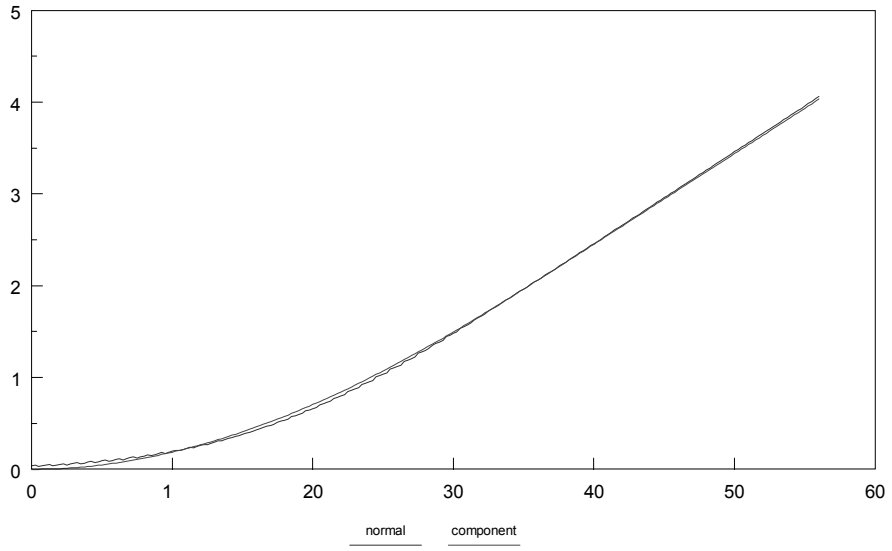

Figure 4.1: Two different models for describing the lag phase

5 Cases

5.1 Microbiological quality of foods

A) We want to describe the growth of bacteria in foods.
B) We address ourselves to spoilage-causing organisms, a defined product composition (pH, concentration of various ingredients etc. are held constant) and a certain type of packaging (gas composition, extra contamination). Under these circumstances, temperature is an important parameter for influencing quality (microbiological growth). If we know how the growth of organisms is dependent on the temperature and also know a maximum allowable level of micro-organisms for a particular product, we can calculate how certain temperature developments satisfy our quality requirement.
C) We must know how the growth of micro-organisms is dependent on the temperature. Various relationships for this are known in the literature.
D) The solution to these equations is not a problem. If we know the parameters we can calculate the growth rate of micro-organisms at every temperature.
E) When testing the model we must therefore compare various models.

We divide this problem into two parts:
-description of the growth curve (Zwietering et al. 1990)
-description of temperature dependency (Zwietering et al. 1991)

From these articles we therefore have a model for the growth of *Lactobacillus plantarum* as a function of temperature. These models can be integrated with the heat transmission model and we can solve this numerically:

$$\rho c_p \frac{\partial T}{\partial t} = \lambda \frac{\partial^2 T}{\partial x^2}$$

$$T_{x,t+k} = T_{x,t} + \frac{\lambda}{\rho c_p} \cdot \frac{T_{x+h,t} - 2T_{x,t} + T_{x-h,t}}{h^2} \cdot k$$

In this way we can calculate the heat transmission in a product and the influence of temperature on microbial growth. By using a simulation program we can then make the following graph:

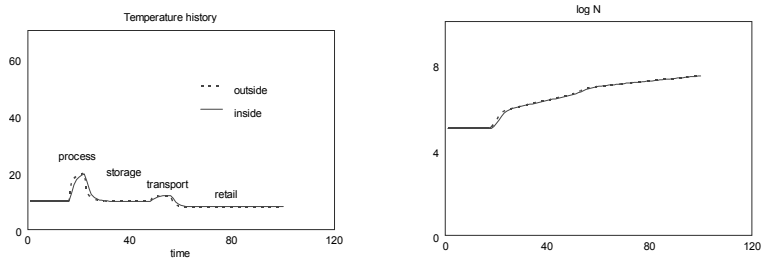

Figure 5.1: Development of temperature and number of micro-organisms

If we look at this figure we see that the number of organisms with this temperature history does not increase very strongly, but that the number at the beginning of the process is very high (high contamination level of ingredients). It could therefore be useful to subject the ingredients to a heat treatment:

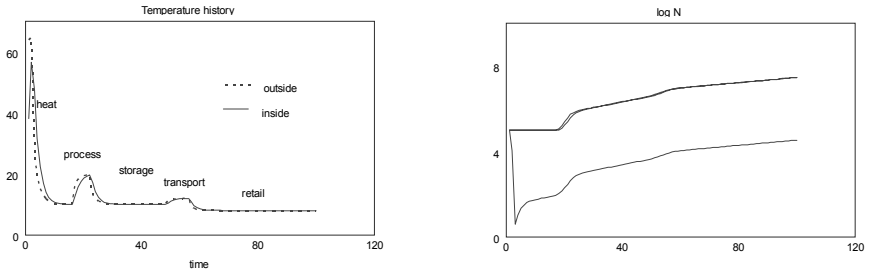

Figure 5.2: Development of temperature and number of micro-organisms with heat treatment

5.2 Modelling of extraction with the help of reversed micelles

The accompanying article describes the modelling of a reversed micelle system for the extraction of the enzyme amylase. Reversed micelles are formed in an organic phase. This is a thermodynamically stable system where a surface-active material forms a water droplet in an organic solvent (figure 1 of the article). Proteins can be taken up in a micelle. It is not yet fully understood how this occurs on a molecular level. If there is an enzyme inside the micelle, this enzyme can remain active. Another possibility is to transport the enzyme in the micelle and to extract it during a second water phase. In this way, an enzyme can be purified. A process diagram is shown in figure 2 (of the article). During the discussion of this case, the extraction system will be modelled. The system will be divided into subsystems, the equations will be established for certain assumptions. The emphasis shall be placed on the system matrix: what does it look like, what can you see from it?

5.3 The modelling of hydrolysis in a membrane reactor

This article describes the hydrolysis of fatty acids in a membrane reactor. Vegetable oils (triglycerides) can be hydrolysed to fatty acids and glycerol. Fatty acids are used for many purposes, such as the preparation of soap, paint, coatings and printing ink. The hydrolysis reaction is a collective name for the hydrolysis of triglycerides into diglycerides, diglycerides into monoglycerides and monoglycerides into glycerol. During every reaction step, water is consumed and fatty acids are produced. As with nearly all reactions, there are two substrates (water and glyceride) which react and form two products. Frequently, one of the substrates is assumed to be present in excess; this is known as a pseudo first order reaction. In this case, however, the reaction rate is determined by the presence of both substrates. This can be called a second order reaction. This system is not subject to a simple analytical solution. Therefore a physical model is established and simulated (Euler forwards). We shall look at the simulation results and measured values and determine if it possible to extrapolate the model. Another approach is to consider the process as a black box. An empirical model will be established in order to determine the fatty acid concentration. A linking of reactors will be computed.

6 Excel-Cases

In case1, several possibilities for modelling are explained using four computer programs. The other cases will be done in Excel. Cell names are referred to in the text. To prevent confusion, the function-keys are placed between braces {F1} - {F12}, and cell addresses between parentheses (C11..F12).

6.1 Case 1: Examples of modelling

I. Type **a:** followed by **enter**. Start the program by typing **case1a**. You will then start the program GROWA, a microbial growth curve fitting program. This program is used to show how models can help in data treatment. A number of growth data measurements (e.g. 5 or 20 of 252) are reduced to three biological parameters, the growth rate (*mu*), the lag time (*lambda*), and the asymptote (*A*). Answer **yes** for printing to printer (not relevant). Select **fit**, choose *g* (for Gompertz), choose 30 iterations and then look at fitting of the growth data. The next screen gives some statistical information, the screen after that gives the final parameter values with confidence bands etc. Select **no printing** and **no plotting**. Load the data file a:grow2.dat into the program (never save your data, always select **no**!), and fit it in the same manner. Load the file a:grow3.dat, fit it in the same manner. You will see that this causes problems. Computer programs do not always give the correct answers, and you must be critical of the answers given by the program. In this case it is clear that the answers are not correct. Since this is non-linear regression, the fitting can go wrong. By obtaining better initial estimates of the parameters, better fitting will be possible. If you select **begin estimation** from the menu, the fitting will go better. You can also enter some of your own data, and experiment a little. After this, **quit** the program.

This program is an example a program which performs data treatment. This type of program can provide graphical representations, statistical calculations, regression, parameter estimation, confidence, accuracy of the fit, etc.

II. The next program is a simulation program for bacterial growth in foods. Several phases in food production and distribution have to be entered and a simulation of heat transfer and bacterial growth is calculated. Type **case1b** and perform a simulation (option **5**). Do not save the results. In the top graph you see the bacterial growth and in the bottom graph the temperature. The colour green is used to represent the surface of the product, red the centre and blue that which lies in between. After the simulation, look at the chain by selecting **change data** (option **2**). At this point you can change things and look at the effect this has on the simulation. For example, try to achieve a better microbial quality. Make sure you look at the effect of selecting **4** (heat transfer). You will then have to supply the program with more information. You can get help by pressing the {F1} key. You can see how such a system can help to support decisions in the food chain; this is why these programs are often called decision support systems. You can also call up chain2.sim (option 7) and chain3.sim. After experimenting some more, quit the program.

7 Appendix

7.1 Appendix 1: Computation with matrices

Matrices offer the possibility to represent a system in a more abstract fashion. This section contains a few definitions and computations with matrices. A matrix A of mxn is:

$$A = \left(a_{ij}\right) = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ a_{21} & \dots & a_{2n} \\ . & . & . \\ . & . & . \\ a_{m1} & \dots & a_{mn} \end{pmatrix}$$

where i is the row number and j the column number.

Two matrices are equal to each other if they have the same dimensions and all elements are the same. If two matrices must be added together (or subtracted), the dimensions must also be the same. The sum of the matrices is determined by adding all corresponding elements:

$$A + B = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 1 & 3 & 4 \end{pmatrix} + \begin{pmatrix} 2 & 3 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 5 & 4 \\ 3 & 1 & 0 \\ 2 & 3 & 5 \end{pmatrix}$$

Multiplication
If matrix A must be multiplied by matrix B, the number of columns of $A(n_1)$ must be the same as the number of rows of $B(m_2)$. The product is the sum of the element-by-element products of one row of A with one column of B:

$$p_{ij} = \sum_{k=1}^{n_1} a_{ik} b_{kj}$$

$$A \cdot B = \begin{pmatrix} 3 & 2 \\ 1 & 1 \\ 5 & 4 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 5 & 6 \\ -2 & 0 & 1 & 3 \end{pmatrix} = \begin{pmatrix} -4 & 3 & 17 & 24 \\ -2 & 1 & 6 & 9 \\ -8 & 5 & 29 & 42 \end{pmatrix}$$

Note that with normal numbers (scalars) a multiplied by b is the same as b multiplied by a. This is not the case with matrices. In the above example, B cannot even be multiplied by A, but even if the number of columns of B was equal to the number of rows of A, the results of the multiplication would still be different.

$A \cdot B \neq B \cdot A$!!!!!

Transposition
The transposed matrix A^T of A is formed by exchanging all row
and column elements (mirroring):

$$a_{ij} \rightarrow a_{ji}$$

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 3 & 0 & -1 \end{pmatrix} \quad A^T = \begin{pmatrix} 1 & 3 \\ 2 & 0 \\ 4 & -1 \end{pmatrix}$$

Determinant (only for square matrices):

For a 2*2 matrix, the determinant is:

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \cdot a_{22} - a_{21} \cdot a_{12}$$

For a multidimensional matrix, the determinant is calculated by multiplying the elements along a row or column by the under-determinant. If the sum of the row and the column number of the element is even, the term is given a positive sign. With an uneven sum a negative sign is given. The under-determinants are found by drawing a horizontal and vertical line through the element which one is considering at that time, and not taking those row and column elements. This can best be explained with an example:

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Therefore for element a_{12}, there is a negative sign because the sum of the row and the column is odd. The above matrix can also be developed according to the first column instead of the first row (or for any randomly chosen row or column). This always gives the same result.

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

Inversion

An inverse of (a square matrix) A exists if there is a matrix B ($B=A^{-1}$) for which $AB=BA=I$ where I is the unit matrix. In the unit matrix, all diagonal elements are equal to 1, the other elements are zero. For example:

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The inverse of matrix *B* is again *A*. If A is square and the determinant is not zero, then *A* is non-singular. If the determinant is zero, the matrix is called singular. In this case there is no inverse matrix (there are dependent rows and columns in the matrix).

There are computational methods (very laborious for multi-dimensional matrices) to work out inverse matrices, it is also possible to use numerical methods. For a two-by-two matrix, the inverse is easy to determine by exchanging the diagonal elements, switching the signs of the non-diagonal elements and dividing by the determinant:

$$A=\begin{pmatrix}1 & 2\\ -1 & 3\end{pmatrix}\quad \det(A)=3+2=5\quad A^{-1}=\frac{\begin{pmatrix}3 & -2\\ 1 & 1\end{pmatrix}}{5}=\begin{pmatrix}3/5 & -2/5\\ 1/5 & 1/5\end{pmatrix}$$

Check this yourself, it can never do any harm (it doesn't take very much time and can bring many problems to light):

$$A\cdot A^{-1}=\begin{pmatrix}1 & 2\\ -1 & 3\end{pmatrix}\begin{pmatrix}3/5 & -2/5\\ 1/5 & 1/5\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$$A^{-1}\cdot A=\begin{pmatrix}3/5 & -2/5\\ 1/5 & 1/5\end{pmatrix}\begin{pmatrix}1 & 2\\ -1 & 3\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

Eigenvalues, Eigenvectors:

The matrix $B = \lambda I - A$ is called the eigenmatrix. The characteristic equation which occurs by setting $\det(B)=0$ is called the eigenequation. This equation is of the same order as the matrix, therefore a 2*2 matrix results in a second order (quadratic) equation, a 3*3 matrix gives a third order equation. The solutions to these equations give the eigenvalues:

$$A=\begin{pmatrix}1 & 2\\ -3 & 6\end{pmatrix}\quad B=\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}-\begin{pmatrix}1 & 2\\ -3 & 6\end{pmatrix}=\begin{pmatrix}\lambda-1 & -2\\ 3 & \lambda-6\end{pmatrix}$$

$$\begin{vmatrix}\lambda-1 & -2\\ 3 & \lambda-6\end{vmatrix}=(\lambda-1)(\lambda-6)+6=0\quad \lambda^2-7\lambda+12=0$$

$$\lambda^2-7\lambda+12=0\quad \lambda=\frac{7\pm\sqrt{49-48}}{2}\quad \lambda_1=4\quad \lambda_2=3$$

The vector X_i

$(\lambda_i I - A)X_i = 0$

is called the i^{de} eigenvector. This can also be written as:

$AX_i = \lambda_i X_i$

Therefore, for the above example:

$$\begin{pmatrix} 1 & 2 \\ -3 & 6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 4\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \begin{matrix} x_1 + 2x_2 = 4x_1 \\ -3x_1 + 6x_2 = 4x_2 \end{matrix} \quad \begin{matrix} 2x_2 = 3x_1 \\ -3x_1 = -2x_2 \end{matrix}$$

$$X_1 = \begin{pmatrix} p \\ 3/2p \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ -3 & 6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 3\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \begin{matrix} x_1 + 2x_2 = 3x_1 \\ -3x_1 + 6x_2 = 3x_2 \end{matrix} \quad \begin{matrix} 2x_2 = 2x_1 \\ -2x_1 = -2x_2 \end{matrix}$$

$$X_2 = \begin{pmatrix} q \\ q \end{pmatrix}$$

Eigenvalues and eigenvectors are useful for solving systems of equations and for examining the stability of these systems. The word eigen comes from the German word Eigen, and is a so called halftranslation.

7.2 Appendix 2: Proofs for statistic chapter

Proof 'partition theorem': $\sum (y-\bar{y})^2 \quad = \quad \sum (y-\hat{y})^2 \quad + \quad \sum (\hat{y}-\bar{y})^2$

In linear regression the residual is perpendicular to the prediction, therefore the inproduct of *d*y and the predicted *y* is zero:

$$\sum (y-\hat{y})\hat{y} = 0 \tag{App 2.1}$$

$$\sum y\hat{y} = \sum \hat{y}^2 \tag{App 2.2}$$

$$\begin{aligned} &\sum (y-\hat{y})^2 + \sum (\hat{y}-\bar{y})^2 = \sum \left(y^2 - 2y\hat{y} + 2\hat{y}^2 - 2\hat{y}\bar{y} + \bar{y}^2\right) = \\ &\sum \left(y^2 - 2\hat{y}\bar{y} + \bar{y}^2\right) = \sum y^2 - 2n\bar{y}^2 + n\bar{y}^2 = \sum y^2 - n\bar{y}^2 \end{aligned} \tag{App 2.3}$$

$$\sum (y-\bar{y})^2 = \sum \left(y^2 - 2y\bar{y} + \bar{y}^2\right) = \sum y^2 - 2n\bar{y}^2 + n\bar{y}^2 = \sum y^2 - n\bar{y}^2 \tag{App 2.4}$$

Prove that the sum of the predictions is equal to the sum of the measured values.
For all models with a 'level' term (a constant):

$$\hat{y} = b_0 + Xb \tag{App 2.5}$$

$$\frac{\mathrm{d}\hat{y}}{\mathrm{d}b_0} = 1 \tag{App 2.6}$$

In linear regression, the derivative of the residual to all parameters is zero, therefore:

$$\frac{\mathrm{d}\sum (y_i-\hat{y})^2}{\mathrm{d}b_0} = \sum \left[\frac{\mathrm{d}y_i^2}{\mathrm{d}b_0} - 2\frac{\mathrm{d}y_i\hat{y}}{\mathrm{d}b_0} + \frac{\mathrm{d}\hat{y}^2}{\mathrm{d}b_0}\right] = \sum \left[-2y_i\frac{\mathrm{d}\hat{y}}{\mathrm{d}b_0} + 2\hat{y}\frac{\mathrm{d}\hat{y}}{\mathrm{d}b_0}\right] = 0 \tag{App 2.7}$$

$$\sum \left[-2y_i\frac{\mathrm{d}\hat{y}}{\mathrm{d}b_0} + 2\hat{y}\frac{\mathrm{d}\hat{y}}{\mathrm{d}b_0}\right] = -2\sum y_i + 2\sum \hat{y} = 0 \tag{App 2.8}$$

$$\sum y_i = \sum \hat{y} \tag{App 2.9}$$

1. Appendix 3: t and F tables

Table: *t* values (one sided, right-hand points)

DF	$t_{0.6}$	$t_{0.7}$	$t_{0.8}$	$t_{0.9}$	$t_{0.95}$	$t_{0.975}$	$t_{0.99}$	$t_{0.995}$
1	0.325	0.727	1.376	3.078	6.314	12.706	31.821	63.657
2	0.289	0.617	1.061	1.886	2.920	4.303	6.965	9.925
3	0.277	0.584	0.978	1.638	2.353	3.182	4.541	5.841
4	0.271	0.569	0.941	1.533	2.132	2.776	3.747	4.604
5	0.267	0.559	0.920	1.476	2.015	2.571	3.365	4.032
6	0.265	0.553	0.906	1.440	1.943	2.447	3.143	3.707
7	0.263	0.549	0.896	1.415	1.895	2.365	2.998	3.499
8	0.262	0.546	0.889	1.397	1.860	2.306	2.896	3.355
9	0.261	0.543	0.883	1.383	1.833	2.262	2.821	3.250
10	0.260	0.542	0.879	1.372	1.812	2.228	2.764	3.169
11	0.260	0.540	0.876	1.363	1.796	2.201	2.718	3.106
12	0.259	0.539	0.873	1.356	1.782	2.179	2.681	3.055
13	0.259	0.538	0.870	1.350	1.771	2.160	2.650	3.012
14	0.258	0.537	0.868	1.345	1.761	2.145	2.624	2.977
15	0.258	0.536	0.866	1.341	1.753	2.131	2.602	2.947
16	0.258	0.535	0.865	1.337	1.746	2.120	2.583	2.921
17	0.257	0.534	0.863	1.333	1.740	2.110	2.567	2.898
18	0.257	0.534	0.862	1.330	1.734	2.101	2.552	2.878
19	0.257	0.533	0.861	1.328	1.729	2.093	2.539	2.861
20	0.257	0.533	0.860	1.325	1.725	2.086	2.528	2.845
21	0.257	0.532	0.859	1.323	1.721	2.080	2.518	2.831
22	0.256	0.532	0.858	1.321	1.717	2.074	2.508	2.819
23	0.256	0.532	0.858	1.319	1.714	2.069	2.500	2.807
24	0.256	0.531	0.857	1.318	1.711	2.064	2.492	2.797
25	0.256	0.531	0.856	1.316	1.708	2.060	2.485	2.787
26	0.256	0.531	0.856	1.315	1.706	2.056	2.479	2.779
27	0.256	0.531	0.855	1.314	1.703	2.052	2.473	2.771
28	0.256	0.530	0.855	1.313	1.701	2.048	2.467	2.763
29	0.256	0.530	0.854	1.311	1.699	2.045	2.462	2.756
30	0.256	0.530	0.854	1.310	1.697	2.042	2.457	2.750
40	0.255	0.529	0.851	1.303	1.684	2.021	2.423	2.704
60	0.254	0.527	0.848	1.296	1.671	2.000	2.390	2.660
80	0.254	0.526	0.846	1.292	1.664	1.990	2.374	2.639
100	0.254	0.526	0.845	1.290	1.660	1.984	2.364	2.626
120	0.254	0.526	0.845	1.289	1.658	1.980	2.358	2.617
∞	0.253	0.524	0.842	1.282	1.645	1.960	2.327	2.576
%								
1-side	40	30	20	10	5	2.5	1	0.5
2-sides	80	60	40	20	10	5	2	1

Interpolation must be done with the reciprocal of the number of degrees of freedom:

$$\frac{1}{t_2} = \frac{1}{t_1} + \frac{1/t_3 - 1/t_1}{n_3 - n_2}\left(n_2 - n_1\right)$$

F table: 5% points: $F^n_m 0.95$

n / m	1	2	3	4	5	6	7	8	9	10	12	15	20	30	40	60	120	∞
1	161	200	216	225	230	234	237	239	241	242	244	246	248	250	251	252	253	254
2	18.5	19.0	19.2	19.2	19.3	19.3	19.4	19.4	19.4	19.4	19.4	19.4	19.4	19.5	19.5	19.5	19.5	19.5
3	10.1	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.74	8.70	8.66	8.62	8.59	8.57	8.55	8.53
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.91	5.86	5.80	5.75	5.72	5.69	5.66	5.63
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.68	4.62	4.56	4.50	4.46	4.43	4.40	4.37
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.00	3.94	3.87	3.81	3.77	3.74	3.70	3.67
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.57	3.51	3.44	3.38	3.34	3.30	3.27	3.23
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.28	3.22	3.15	3.08	3.04	3.01	2.97	2.93
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.07	3.01	2.94	2.86	2.83	2.79	2.75	2.71
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.91	2.85	2.77	2.70	2.66	2.62	2.58	2.54
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.69	2.62	2.54	2.47	2.43	2.38	2.34	2.30
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.48	2.40	2.33	2.25	2.20	2.16	2.11	2.07
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.28	2.20	2.12	2.04	1.99	1.95	1.90	1.84
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16	2.09	2.01	1.93	1.84	1.79	1.74	1.68	1.62
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08	2.00	1.92	1.84	1.74	1.69	1.64	1.58	1.51
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.92	1.84	1.75	1.65	1.59	1.53	1.47	1.39
120	3.92	3.07	2.68	2.45	2.29	2.18	2.09	2.02	1.96	1.91	1.83	1.75	1.66	1.55	1.50	1.43	1.35	1.26
∞	3.84	3.00	2.61	2.37	2.21	2.10	2.01	1.94	1.88	1.83	1.75	1.67	1.57	1.46	1.40	1.32	1.22	1.00

F table: 1% points: $F^n_m 0.99$

n / m	1	2	3	4	5	6	7	8	9	10	12	15	20	30	40	60	120	∞
1	4052	5000	5403	5625	5764	5859	5928	5981	6022	6056	6106	6157	6209	6261	6287	6313	6339	6366
2	98.5	99.0	99.2	99.2	99.3	99.3	99.4	99.4	99.4	99.4	99.4	99.4	99.4	99.5	99.5	99.5	99.5	99.5
3	34.1	30.8	29.5	28.7	28.2	27.9	27.7	27.5	27.3	27.2	27.1	26.9	26.7	26.5	26.4	26.3	26.2	26.1
4	21.2	18.0	16.7	16.0	15.5	15.2	15.0	14.8	14.7	14.5	14.4	14.2	14.0	13.8	13.7	13.7	13.6	13.5
5	16.3	13.3	12.1	11.4	11.0	10.7	10.5	10.3	10.2	10.1	9.89	9.72	9.55	9.38	9.29	9.20	9.11	9.02
6	13.7	10.9	9.78	9.15	8.75	8.47	8.26	8.10	7.98	7.87	7.72	7.56	7.40	7.23	7.14	7.06	6.97	6.88
7	12.2	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.72	6.62	6.47	6.31	6.16	5.99	5.91	5.82	5.74	5.65
8	11.3	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.91	5.81	5.67	5.52	5.36	5.20	5.12	5.03	4.95	4.86
9	10.6	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.35	5.26	5.11	4.96	4.81	4.65	4.57	4.48	4.40	4.31
10	10.0	7.56	6.55	5.99	5.64	5.39	5.20	5.06	4.94	4.85	4.71	4.56	4.41	4.25	4.17	4.08	4.00	3.91
12	9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.39	4.30	4.16	4.01	3.86	3.70	3.62	3.54	3.45	3.36
15	8.68	6.36	5.42	4.89	4.56	4.32	4.14	4.00	3.89	3.80	3.67	3.52	3.37	3.21	3.13	3.05	2.96	2.87
20	8.10	5.85	4.94	4.43	4.10	3.87	3.70	3.56	3.46	3.37	3.23	3.09	2.94	2.78	2.69	2.61	2.52	2.42
30	7.56	5.39	4.51	4.02	3.70	3.47	3.30	3.17	3.07	2.98	2.84	2.70	2.55	2.39	2.30	2.21	2.11	2.01
40	7.31	5.18	4.31	3.83	3.51	3.29	3.12	2.99	2.89	2.80	2.66	2.52	2.37	2.20	2.11	2.02	1.92	1.81
60	7.08	4.98	4.13	3.65	3.34	3.12	2.95	2.82	2.72	2.63	2.50	2.35	2.20	2.03	1.94	1.84	1.73	1.60
120	6.85	4.79	3.95	3.48	3.17	2.96	2.79	2.66	2.56	2.47	2.34	2.19	2.03	1.86	1.76	1.66	1.53	1.38
∞	6.64	4.61	3.78	3.32	3.02	2.80	2.64	2.51	2.41	2.32	2.19	2.04	1.88	1.70	1.59	1.48	1.33	1.00

Interpolation must be done with the reciprocal of the number of degrees of freedom.

III. Type **case1c**. You will now enter a micro-organism detection system for foods. This is an example of a program which combines databases (quantitative knowledge) with models and with decision rules (qualitative yes/no-rules=knowledge). First, choose **automatic detection**, select a food (listed in Dutch), let the computer make a selection, look at the list (**show list**) of microorganisms which can spoil the food. Then use **decision rules**, and look again at the list. You see that this combines two databases (foods and micro-organisms) with models (to calculate growth rate) and qualitative knowledge.

6.2 Case 2: Numerical solution of a differential equation

The model we are going to work with is: $\frac{dy}{dt} = -y$

a) Calculate (on paper) the analytical solution (answer: $y = y_0\, e^{-t}$)
b) Determine a formula for calculating the numerical solution with Euler forward [answer:$y(t+k) = (1-k)*y(t)$]. These equations have been implemented in Excel.
c) Start Excel and load the file CASE2 Num Sol (Enable Macros).
d) Check how everything has been implemented. If cells have been named, look at the address by pressing **edit** {F2}. The ranges have been given names in the upper left box.
e) Change the time step k and look at the effect this has in the graph.
f) Implement backward in time and centered in time numerical solutions yourself. Use the analytical solution for the necessary extra input parameter.
g) Look at differences in stability and accuracy.

6.3 Case 3: Numerical solution of a set of differential equations

We are now going to look at the following system, where $x(0)=10$ and $y(0)=5$. In this system, we are going to look at the stability.

$$\frac{\mathrm{d}\begin{pmatrix} x \\ y \end{pmatrix}}{\mathrm{d}t} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

a) First calculate formulas for the eigenvalues of the matrix, and check these calculations with the equations in Excel in the file CASE3 Num Sol of a set (enable macros).
b) Look at the method of solution (numerically). Which method is chosen?
c) Check what happens if the time step (k) is near the smallest time constant (slightly larger or smaller). (The analytical solution implemented is only correct for $q = r = 0$)
d) Change the parameters p, q, r, and s in the matrix, and look at the eigenvalues and the solutions. Find a set with eigenvalues (look at the graphs and experiment with the k-value):

d1) both negative: one large, one small
d2) both negative; both of same magnitude
d3) one is positive
d4) the other is positive
d5) both are positive.
d6) both are a complex number

e) Calculate (on paper) the analytical solution of the following system and implement the analytical solution.

$$\frac{\mathrm{d}\begin{pmatrix} x \\ y \end{pmatrix}}{\mathrm{d}t} = \begin{pmatrix} -3 & 2 \\ 3 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Solution:

$$x = \left(\frac{2}{5}y_0 + \frac{3}{5}x_0\right)e^{-t} + \left(-\frac{2}{5}y_0 + \frac{2}{5}x_0\right)e^{-6t} \qquad y = \left(\frac{2}{5}y_0 + \frac{3}{5}x_0\right)e^{-t} + \left(\frac{3}{5}y_0 - \frac{3}{5}x_0\right)e^{-6t}$$

f) Compare the numerical solution in LOTUS with the analytical solution during the time span 0-3 (with various *k*-values).

g) If you still have time, calculate (on paper) the analytical solution of the following system and implement it.

$$\frac{\mathrm{d}\begin{pmatrix} x \\ y \end{pmatrix}}{\mathrm{d}t} = \begin{pmatrix} -0.1 & -1 \\ 1 & -0.1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

(Solution: $x = \left[x_0 \cos(t) - y_0 \sin(t)\right]e^{-0.1t} \qquad y = \left[y_0 \cos(t) + x_0 \sin(t)\right]e^{-0.1t}$ **)**

6.4 Case 4: Frequency of 1000 errors

Load the file case4.wk1. In this file 1000 normally distributed errors are given.

a) Make a frequency distribution (with **/data distribution**, return, return)

b) Look at the graph {F10}. The step size is 0.3. You can change this (e.g. to 0.15 or 1).

c) Implement the equation (in E8..E44) for the normal distribution:

$$\frac{1}{\sqrt{2 \cdot \pi}} \cdot \exp\left(-\frac{x^2}{2}\right)$$

This equation has to be multiplied by 1000*$step to get a prediction of the number of occasions the errors are within a certain interval. Again compare the data and the model at several steps.

d) Find out what the error is and when it happens.

6.5 Case 5: Estimation of a constant value

Load the file CASE5 Estimation of a constant value. We now add these errors to a fixed value. This is not what we would normally do in science, *i.e.* that we already know the answer and add errors; normally we only get erroneous data and want to estimate reality. For understanding statistics, however, this is a very useful procedure. We therefore have the model: $\tilde{x} = \mu + \varepsilon$. In the example μ = 5. We have 1000 ε‘s, and with that we can calculate 1000 simulated measurements (C15..C1014). We can estimate the μ-value by taking the mean of the 'measurements' (*x*-mean in F3). This mean is an estimator for μ (which we know in this case, but which we normally do not know). You can see that the mean is estimated properly. Now we can calculate the residuals (E15..E1014), the residuals squared (F15..F1014) and the residual sum of squares (RRS in F12). With this RSS we can calculate, for instance, the population standard

error: σ(popu) = √[RSS/n]. (F4 and calculated by Excel in H4). This is the error **within** the population. This population **has** a certain mean, which we can calculate (*x*-mean) and we do not have to estimate, so the DF=*n*. If we assume these 1000 measurements to be a sample from reality, we make an **estimate** of the real value of μ (the same *x*-mean!), therefore DF=*n*-1. The standard deviation of the sample is then s(sample) = √[RSS/(n-1)] (see F5). The standard deviation of the average is equal to s(sample)/√[n] (see F6 and G6). With this we can calculate the confidence interval of the average with the *t*-testing value (see F9 and F10). We can also calculate the confidence limits for a new measurement. This is built up from the uncertainty of the mean plus the error we will get for the new measurement.

$$\mathrm{var}(x_{new}) = \mathrm{var}(\bar{x}) + s^2 = \frac{1}{n}s^2 + s^2$$

$$s(x_{new}) = \sqrt{s(\bar{x})^2 + s^2} = \sqrt{\frac{1}{n}s^2 + s^2} = \sqrt{\left(\frac{1}{n}+1\right)s^2} = \sqrt{\left(\frac{1}{n}+1\right)\frac{RSS}{n-1}}$$

This value is given in cells F7, G7, and H7. With this value the confidence limits for a new measurement can be calculated (G9 and G10).

a) Look at the two different confidence bands {Tab Data}. The smaller one is the confidence of the mean. Since we have collected 1000 data points, we can estimate the mean very accurately. If we now perform a new experiment, it will often not lie in this band, since there is a new measurement error for every new point. This band is much larger. The sample average will tend to cluster about μ more closely than individual observations.
b) Look at the number of not accepted measurements (out of the new measurement band) in the graph and in I15..I1014 (sum in I13). This will be about 5% of the 1000 cases.
c) By changing the *t*-value (in I9), look at the number of accepted measurements and the width of the confidence bands (other *t*-values are given in K2..N6)
d) Look at the effect of changing sigma (from 0.1 to 2). *s*(sample) (F5) is en estimator for sigma!
e) Look at the residuals (**/graph name use residue**). These residuals should be randomly distributed.
f) Make a frequency distribution of the residuals (E15..E1014) from -4 till 4 in (M16..N56). The ranges have already been selected. Look at the graph named FREQ.

6.6 Case 6: Line through origin

Load the file CASE6 Line through origin. The first 100 epsilons of the former example are used. Now fill in the file yourself for the following model:
y=*ax*, with *a* equal to 5.

a) Make a list of *y* values (data in C21..C120): calculate reality (*y* = 5*x*) and add the error:
b) $y = a \cdot x + e \cdot \sigma$. This is the list of measured points. Look at the graph.
c) Calculate x^2, $x\,y$, and y^2 (in D21..F120).
d) Calculate the estimator for *a* (in F3). Check that this is a good estimator. Calculate the estimated line (G21..G120). Look at the graph!
e) Calculate the residuals (H21..H120) and the square of the residuals (I21..I120) and the RSS in I18.

f) Calculate *s*(model) (F4). This is the estimator for sigma! Calculate *s*(*a*) (F5), the confidence of *a* (F7..F8) *s*(*y*) (J21..J120) and the confidence of the estimated *y*'s (K21..L120). Look at the graph. Compare the results with the regression of Excel (I6..L9). Find out what all the output means.
g) Calculate the confidence of a new *y*-value (M21..N120) and look at the graph.
h) Calculate the number of cases where a measurement is not within this prediction interval.
i) Fill in the ANOVA-table (N8..S13). The ANOVA-table is a standard form for statistical results. It shows the RSS, DF and mean square for various models, e.g. *y*=0 (uncorrected total, zero parameters), *y*=*c* (corrected total, one parameter), and all other models used (residuals). In this way it can be seen how well this model behaves in comparison with the other models: there is nothing (*y*=0) and there is something but it does not change (*y*=*c*).

$$\sum (y_i - \bar{y})^2 = \sum y_i^2 - 2\sum y_i \bar{y} + \sum \bar{y}^2 = \sum y_i^2 - 2\bar{y}\sum y_i + n \cdot \bar{y}^2$$

$$= \sum y_i^2 - 2\frac{\left(\sum y_i\right)^2}{n} + n \cdot \left(\frac{\sum y_i}{n}\right)^2 = \sum y_i^2 - \frac{\left(\sum y_i\right)^2}{n}$$

6.7 Case 7: Model comparison

Load the file CASE7. In this file the following models will be compared for a data set: $y = ax + b$ **and** $y = c$. The parameters *a* and *b* are given (A3 and B3), the model equation plus the error is calculated (C40..C94).

a) Perform regression (**Tools Data analysis regression** or press the button) and calculate the sum of the data (D36) and the average (D36). Calculate the model predictions (D40..E94). Look at the graphs.
b) Calculate the square of the residuals (G40..H94) and the RSS (G35..H35).
c) Calculate the confidence interval for the slope. For these cases *t* is equal to 2.
d) Perform an *F*-test to see if *y*=*c* is significantly worse than *y*=*ax*+*b*.
e) Change the sigma. Look at the result of the *t* and *F*-tests. Try to find an *a*-value at which the *t* and *F* values are again at the limit.
f) Go to the right in the worksheet (K40..K94) where the variance of the measuring error is calculated with the RSS of averages between replicas (general model). Compare this variance with the one found by the model *y*=*ax*+*b*.
g) Perform an *F*-test with *y*=*ax*+*b* and *y*=*a* in comparison with the general model.

6.8 Case 8: Transformations

We again use the 100 normally distributed errors. We now have a case where the errors are not normally distributed on *y* but on the root of *y* (column B):

The *y* is calculated for different *x* values $\sqrt{y} = 5\sqrt{x} + 2\varepsilon$, in this case we have 5 *y* values for each *x* value.

a) Look at the graph of the data (F10).
b) Look at the calculated variances (G4..H103), calculated for the replicas at every constant *x*-value. c) Look at the plot of the variance against *x* (/**graph name use** 'VAR(Y)'). There is clearly a trend, so the variance is not constant, hence no normal linear regression is allowed.

c) Calculate the variance divided by the mean against the mean and the variance divided by the square of the mean (I4..J103) and look at the plots against x (**/graph name use** 'VAR(Y)/Y', 'VAR(Y)/Y^2', respectively). These plots should guide your decision about which transformation should be applied, which is, of course, √y, one of the transformations to reduce the variance at larger values of y.

d) To see the difference between the various transformations, perform:

e)
- a regression $y = ax$ and calculate the residuals (N4..Q103). Remember $a = \sum\left(y_i \cdot x_i\right) / \sum\left(x_i^2\right)$. Look at the plot of y and $\hat{y}$ versus x (/graph name use 'F(Y)=AX') and the residual plot (Tab 'RESY=AX').
- a regression √y = ax and calculate the residuals (S4..V103). Look at the plot of √y and $\sqrt{\hat{y}}$ versus x (Tab 'F(ROOT(Y))=AX') and the residual plot (Tab 'RESROOT(Y)=AX').
- a regression ln(y) = ax and calculate the residuals (X4..AA103). Look at the plot of ln(y) and ln($\hat{y}$) against x (Tab 'F(LN(Y))=AX') and the residual plot (Tab 'RESLN(Y)=AX').

f) Although the plots of var(y), var(y)/y and var(y)/y^2 showed that a √y transformation should be applied, the transformation didn't work out. Why?

g) Let us try a different set of transformations, now we will transform both the left and right sides of equation $y = ax$:
- perform a regression $\sqrt{y} = a\sqrt{x}$ and calculate the residuals (AL4..AH103). Look at the plot of $\sqrt{y}$ and $\sqrt{\hat{y}}$ against $\sqrt{x}$ (Tab 'F(RT(Y))=ART(X)') and the residual plot (Tab 'RESRT(Y)=ART(X)').
- perform a regression ln(y) = a + ln(x) and calculate the residuals (AK4..AO103). Note that $a = \sum\left\{\ln\left(y_i\right) - \ln\left(x_i\right)\right\} / n$. Look at the plot of ln(y) and ln($\hat{y}$) against ln(x) (Tab 'F(LN(Y))=ALN(X)') and the residual plot (Tab 'RESLN(Y)=ALN(X)').

  Indeed, the $\sqrt{y} = a\sqrt{x}$ transformation fits best.

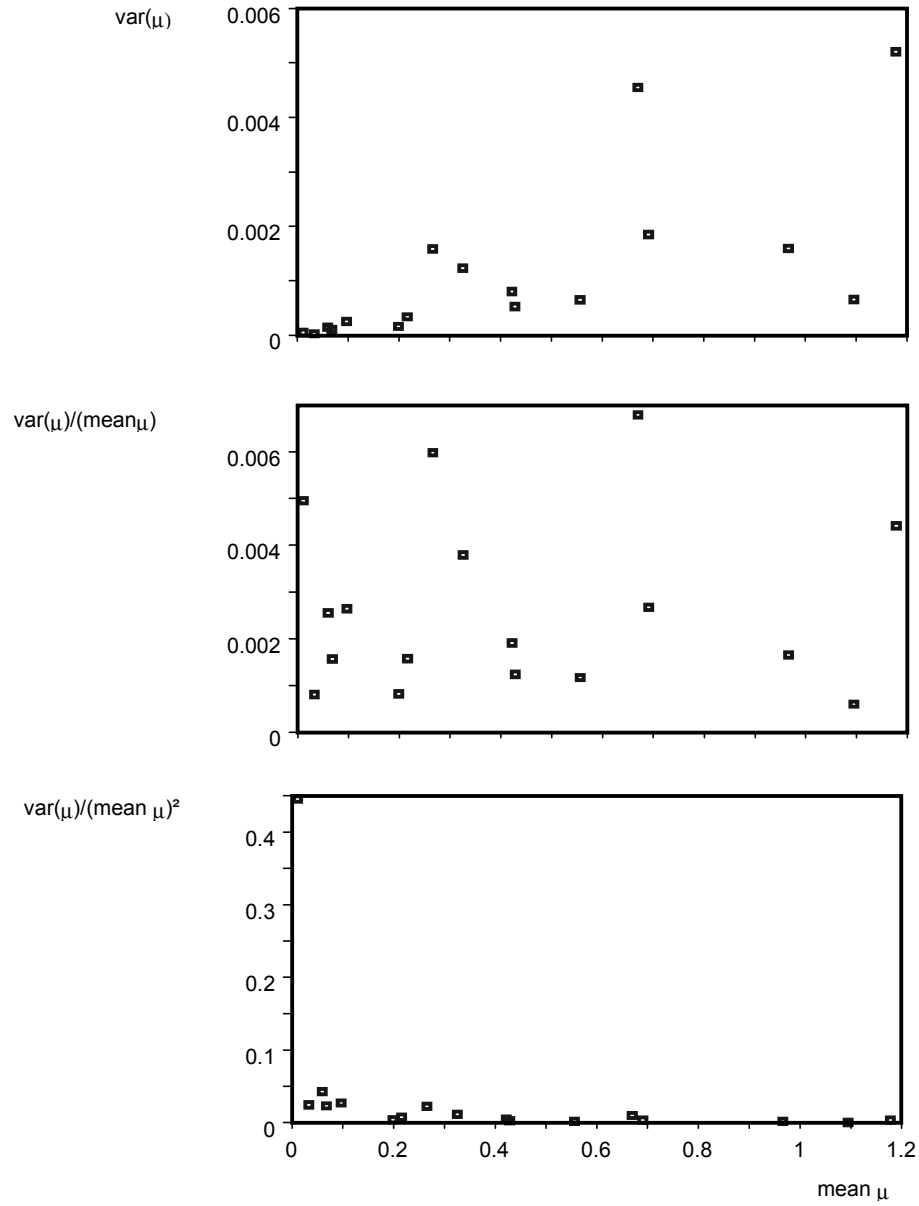

Fig. 1. Variance of μ, variance divided by the mean and divided by the square of the mean, plotted against the mean of μ.

h) Another way to chose a transformation to equalize the variance is to calculate the transformed y values and calculate the variance thereof. The variance of the transformed y values should be constant. Calculate the variances of $\sqrt{y}$ (AU4..AW103) and ln(y) (AY4..BA104) and plot them against y.

Figure 1 shows the variance of the measured growth rate μ at 17 different temperatures as a function of the mean growth rate measured at each temperature. In total, 95 growth rate measurement have been performed. The variance increases with increasing μ. The variance over the mean growth rate is more or less constant (shows no trend), so a root transformation should be applied (These are actual experimental results, to show that the things learned in this course really happen!).

6.9 Case 9: Non-linear regression

We are going to work with the equation $y = a \cdot \exp(bx)$.

a) Calculate on paper the derivative of the equation with respect to *a* and to *b*.

b) Load the file case9.wk1. The *x*- and *y*-data are in (A13..B15). In B19 and B20 are the initial estimators for *a* and *b*. In C13..C15 the equation is calculated with the estimators for *a* and *b*. In D13..E15 the residuals and the squared residuals are calculated, resulting in the RSS (E17). In A6..G8 the necessary calculations are made for the (X^tX)-matrix and X^tY-vector, using the *x*, *a, b* and residual data. In C10..G10 these data are summed and put into the X^tX-matrix (E20..F21) and X^tY vector (H23..H24). $(X^tX)^{-1}$ is calculated (E23..F24) and the appropriate changes in *a* and *b* are calculated (E26..E27) with $(X^tX)^{-1}X^tY$ resulting in new estimates for *a* and *b* (B25..B27). With these new parameters, new y_s-data and RSS are calculated in G13..H17. The adjustment of the parameters is thus advantageous. Look at the graph {F10} and the first and second fit-line.

c) After pressing {alt-a}, the new estimates of the parameters *a* and *b* are copied to the initial estimates, and everything is updated. Look at the reduction of the RSS and the graphical results of fitting. Continue fitting.

d) Go down the worksheet until A31 is located at the upper left. The same is done here for the linear transformation of the model: $\ln(y)=\ln(a)+bx=c+bx$. Calculate the derivative with respect to the parameters. Note that the derivatives are no longer a function of the parameters! Everything is implemented in the same way. Check this. We can now obtain the solution in one "iteration" step. Look at the graph (/GNU) BOTH. Notice that at high *y*-values the linear model performs badly. Now use the graph LN(Y). Here we see that the nonlinear model performs badly at small *y*-values.

e) Look at the results summary in A63..G76.

f) Put *a* and *b* in A19 and A20 on 10 (?) and look at the fit in the logarithmic LN(Y) graph.

g) Change *y*(0.5) to 9, perform the fitting and look at the graphs

6.10 Other problems

On your floppy disk there are also some other files with problems, which were mentioned during the course:

GROWTH:	numerical solution of $dN/dt = \mu N$
XLNX:	numerical solution of $x + \ln(x)$
PID:	example PID controller
MASSTRAN:	numerical solution of Fick's law (Euler forward)
MASS2:	numerical solution of Fick's law (Euler backwards)
INOCULUM:	production with different amounts of batches/day
NLR:	non-linear regression of the equation $y = 3a + \exp(bx)$ as given in the course
GROWTHMO:	two growth models from chapter 4
PARABOL:	regression with a parabola

6.11 Oxygen transfer cases

oxy1

In this file, oxygen transfer in an ideally mixed fermentor is calculated (with no depletion of oxygen in the gas phase).

$$\frac{\mathrm{d}c}{\mathrm{d}t} = -k_l a c + k_l a \frac{Pn}{H}$$

In B12..B16 the parameters are given. The eigenvalue is equal to $-k_l a$ (C6), the time constant is $1/k_l a$ (E6). This equation can be solved numerically:

$$c(t+k) = c(t) + \Delta t\left(-k_l a c(t) + k_l a \frac{Pn}{H}\right) = c(t)(1 - k_l a \Delta t) + k_l a \Delta t \frac{Pn}{H}$$

In B22..D82 this solution is given. The equation can also be solved analytically (E22..E82):

$$c(t) = \frac{Pn}{H} + \left(c(0) - \frac{Pn}{H}\right)\exp(-k_l a t)$$

An EDS can also be built:

$$\Phi = \exp(AT) \quad \Gamma = \int_0^T \exp(A\lambda) B d\lambda = \left[\frac{B}{A}\exp(A\lambda)\right]_0^T = \frac{B}{A}\left[\exp(A\lambda) - 1\right]$$

$$= -\frac{P}{H}\left[\exp(-k_l a T) - 1\right]$$

a) Look at the graph {F10}. In B21 the time step is given; change it to 5,16,17,20, and 50. The EDS remains exact, whatever time step is used. Use the graph (/GNU), scaling for the larger time steps.

b) Change the time step to 10 and look the column G22..G82. The relative part that remains to be done (in 50 s) is always equal to 1/e.

oxy2.wk1

For the system

$$\frac{\mathrm{d}c}{\mathrm{d}t} = -k_l a c + k_l a \frac{Pn}{H} = Ac + Bn$$

the P is calculated (B13) and the F is calculated (B18). The controlled system then becomes:

$$\frac{\mathrm{d}c}{\mathrm{d}t} = Ac - BFc = (A - BF)c$$

A-BF is given in A23 and the time constant is in B23 (notice that it was 50 in the uncontrolled system). The analytical solution of the controlled system:

$$c(t) = c(0)\exp\left[(A - BF)t\right]$$

is calculated in B26..B68. Beginning at time 10, a disturbance is introduced.

a) Look at the graph. For other *Q*-valued the solutions are also given (check these by changing *Q* to 0 and 5). Look also at the time constant. Also change *Q* to 10.

b) There is something very strange about this system. Try to find the error!

oxy3

We now have a controller with a setpoint. *P* and *F* are calculated (B11 and B16) and the time is again constant (E16). The setpoint is given in E20 and the disturbance is given in E21. At time 10, a disturbance is again introduced. The *c*, *x* and *n* are given in B24..D66.

a) Look at the graph. See what happens if you change *Q* to 5, 10 and 50.

b) Make *Q*=1. Why is the time constant not equal to 25.72 as it was in the previous case? Also compare the time constants with the other *Q*-values. In the previous case the time constants were 50 (*Q*=0), 25.72 (*Q*=1) 12.96 (*Q*=5) and 9.32 (*Q*=10).

oxy4

Copy A25..C25 until 2024 by pressing /C {right} {right} {enter} . {end} {down} {enter}. In this file the depletion of the gas phase is also taken into account {PgUp}.

$$\frac{\mathrm{d}\begin{pmatrix} c \\ n \end{pmatrix}}{\mathrm{d}t} = A\begin{pmatrix} c \\ n \end{pmatrix} + Bu$$

The matrix *A* is given in A11..A12. The eigenvalues are in B18..B19 and the time constants in E18..E19. The time step is in B21. The numerical solution is given in A24..C2024.

a) Look at the graph. Experiment with the time step (0.05-0.198, 0.199,.0.200).

b) Reset the time step to 0.15. In the graph CN the mole fraction in the gas phase is also given. Change the *n*(0) value to 0, and increase the time step.

c) Reset time step to 0.15 and *n*(0) to 0.2. In the graph CNC the concentration as calculated without depletion is also given (D2025..D2055). Try to find situations where there is a lot of difference between cases with and without depletion, and try to explain this.