NOTES OF
ASTROPHYSICAL PROCESSES

version 1.0

Giacomo Pannocchia

a.y. 2025-2026

Knowledge without understanding
Is but a sword stuck in its sheat.

Arthur Leywin

PREFACE

Not long ago, it occurred to me how cool it is when someone unexpectedly releases a very detailed and all-comprehensive version of their notes, especially when dealing with a course that has a handful of really different topics often interacting together in unpredictable, yet fascinating, ways—-as it's the case for the Astrophysical Processes class.

This notes will be mainly based on *my* own notes of the lectures by Professor Walter del Pozzo and Professor Marco Crisostomi during the academic year 2025-2026. However, since I take little to no pride in my messy notes, I'll be using more often than not some of the many references you can find on the course catalogue page or in the bibliography of this humble collection.

You can report errors (whatever their nature might be) and suggestions for additions at g.pannocchia3@studenti.unipi.it or through whatever convoluted way (conventional or not) you prefer[1].

Without further ado, we'd better not lose much more time on a preface and get started with it.

There was Eru, the One, who in Arda is called Ilúvatar; and he made first the Ainur [...] But for a long while they sang only each alone, or but few together, while the rest hearkened; for each comprehended only that part of the mind of Ilúvatar from which he came, and in the understanding of their brethren they grew but slowly.

Yet ever as they listened they came to deeper understanding, and increased in unison and harmony.

Ainulindalë, "The music of the Ainur",
Silmarillion, J. R. R. Tolkien

---

1 I'd like, however, not to see my house stormed by homing pidgeons.

CONTENTS

Part I

RADIATIVE TRANSPORT

1

INTERACTION OF RADIATION WITH MATTER

1.1 INTRODUCTION

Most of our knowledge about the Universe is based on the electromagnetic radiation that reaches us from far far away. EM radiation is obviusly not the only way we can probe the Universe we live in but, in respect to neutrinos, cosmic rays or even gravitational waves, it's not a long stretch to claim it is by far the most understood.

It is most important then that an astrophysicist worthy of his (or her) name has a good grasp of the theory of radiative transfer and of its applications.

Apart from a few more key differences, I'll follow the description of radiative transfer of [1], but I won't fail to emphasize whenever I'll be doing otherwise.

1.2 RELEVANT QUANTITIES FOR RADIATIVE TRANSFER

Although some books often start their description of radiative transfer from the definition of *monochromatic energy* and *monochromatic intensity*, I found that it is most misleading, since, in all but a few cases, what we experimentally measure are fundamentally *fluxes*.

We shall then consider the *monochromatic flux* F_ν ($\mathrm{erg\,s^{-1}\,Hz^{-1}\,cm^{-2}}$) produced by some source passing through a small area $\mathrm{d}A$ located somewhere in space.
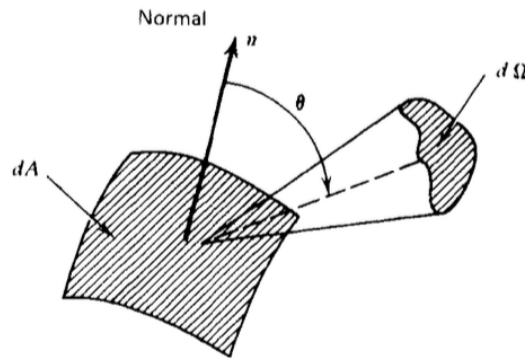

Figure 1: Schematic geometrical representation of the system.
Credits: G. Rybicki, A. Lightman [3].

If we call $\hat{k}$ the propagation direction of the flux and $\hat{n}$ the unit vector emerging from the surface $\mathrm{d}A$, it's easy to get convinced that what is actually passing through the surface is somewhat proportional to $F_\nu(\hat{k} \cdot \hat{n})$.

From the monochromatic flux we can define the *bolometric flux*, which is just the monochromatic flux integrated over all frequencies (or wavelengths)

$$F = \int_0^{+\infty} F_\nu \, d\nu = \int_0^{+\infty} F_\lambda \, d\lambda \tag{1}$$

This also tells us how to convert a flux per unit frequency to a flux per unit wavelength

$$F_\nu \, d\nu = F_\lambda \, d\lambda$$

By now it should be clear that, despite being experimentally sensible to use the flux, we're losing much information sticking with it, namely directional information.

We consider then the amount of radiation $E_\nu \, d\nu$ passing through the same area in time dt and solid angle $d\Omega$. Hence we can write

$$dE_\nu \, d\nu = I_\nu(\mathbf{r}, t, \hat{k})(\hat{k} \cdot \hat{n}) \, dt \, d\Omega \, dA \, d\nu \tag{2}$$

where the quantity $I_\nu(\mathbf{r}, t, \hat{k})$ is called the *specific monochromatic intensity*. If $I_\nu(\mathbf{r}, t, \hat{k})$ is specified for all directions at every point in a certain region of spacetime, then we'd have a complete prescription of the radiation field we intend on studying.

Capitalizing on the blatant similarities with distribution functions, we can evaluate the moments of the monochromatic intensity.

Definition 1.2.1. *Monochromatic mean intensity J_ν*

$$J_\nu = \frac{1}{4\pi} \int_\Omega I_\nu \, d\Omega = \frac{c}{4\pi} U_\nu$$

with U_ν the total energy density of radiation. Note that J_ν is pretty much just an average of the monochromatic intensity over all solid angles.

Definition 1.2.2. *Monochromatic flux $\vec{F}_\nu$*

$$\vec{H}_\nu = \frac{1}{4\pi} \int_\Omega I_\nu(\hat{k}) \hat{k} \, d\Omega = \frac{1}{4\pi} \vec{F}_\nu$$

I haven't explicitly proved the last equality, but it shouldn't be hard for you to convince yourself (or prove it yourself) that it is indeed true.

Definition 1.2.3. *Monochromatic radiation pressure p_ν The monochromatic pressure is defined starting from the different directions correlations of the monochromatic intensity*

$$K_\nu^{ij} = \frac{1}{4\pi} \int_\Omega I_\nu(\hat{k}) n^i n^j \, d\Omega$$

The pressure in particular is usually expressed as

$$P_\nu = \frac{1}{c} \int_\Omega I_\nu(\hat{k}) \cos^2 \theta \, d\Omega$$

where $\cos^2 \theta = (\hat{k} \cdot \hat{n})^2$.

1.3 BLACKBODY RADIATION

Even at an undergraduate level, we're all fairly familiar with *blackbody ra-diation*. The easiest way to deduce the expression for the energy density of photons in *thermal equilibrium* (STE) inside a cavity is by the means of statistical mechanics.

Remember the Bose-Einstein distribution

$$n = \frac{1}{\exp(h\nu/kT) - 1}$$

and the phase space density of states

$$\rho(\nu)\,d\nu = \frac{4\pi g\nu^3}{c^3}\,d\nu$$

from which deducing the expression from internal energy is straightforward. Remembering $g = 2$ is the quantum degeneracy of photons, a simple molti-plication of the previous expressions yields

$$U_\nu\,d\nu = \frac{8\pi\nu^3}{c^3}\frac{1}{\exp(h\nu/kT) - 1}\,d\nu$$

Since blackbody radiation is isotropic (it actually depends only on the absolute temperature T), the definition of mean monochromatic intensity yields

$$B_\nu(T) = \frac{2h\nu^3}{c^2}\frac{1}{\exp(h\nu/kT) - 1} \tag{3}$$

Figure 2: Blackbody frequency spectrum.

It's important to notice that, in principle, such a fundamental result holds only in *strict thermodynamic equilibrium* (STE), but we'll soon see how to gen-eralize this formulation for less "restrictive" environments.

An incredible number of important results descends from (3), and it may be worthwhile to at least cite some of them, starting from Stefan-Boltzmann law.

We'll use the following result without proving it

$$\int_0^{+\infty} B_\nu(T)\,\mathrm{d}\nu = \frac{2h}{c^2}\frac{\pi^2}{15}\left(\frac{kT}{h}\right)^4$$

Computing the bolometric flux and the bolometric energy density by integrating over all frequencies using what we've just written down, you find the following

$$U(T) = aT^4 \quad F(T) = \sigma_{SB}T^4$$

Clearly the two constants a and σ_{SB} cannot be independent, and are actually related by the integral we've previously calculated. Using for example

$$F(T) = \pi \int_0^{+\infty} B_\nu(T)\,\mathrm{d}\nu$$

you can easily find out that the *Stefan-Boltzmann constant* is equal to

$$\sigma_{SB} = \frac{2\pi^5 k^4}{15c^2 h^3}$$

and the relation with a is simply $\sigma_{SB} = ac/4$.

The equation

$$F(T) = \frac{2\pi^5 k^4}{15c^2 h^3}T^4 \tag{4}$$

is what is usually known as the *Stefan-Boltzmann law*.

Let us now consider two different regimes for eq.3: $h\nu/kT \ll 1$ and $h\nu/kT \gg 1$. The first yields what is commonly known as the Rayleigh-Jeans Law which is, sadly, pretty much relevant only for radioastronomy.

Since

$$\exp\left(\frac{h\nu}{kT}\right) = 1 + \frac{h\nu}{kT} + o\left(\frac{h\nu}{kT}\right)^2$$

the blackbody radiation assumes the much simpler form of

$$B_\nu^{RJ} = \frac{2\nu^2}{c^2}kT \tag{5}$$

Another important results is achieved in the opposite regime, when the exponential term is rather larger than unity

$$B_\nu^{W} = \frac{2h\nu^3}{c^2}\exp\left(-\frac{h\nu}{kT}\right) \tag{6}$$

This expression is often known as Wien's Law.

1.4 RADIATIVE TRANSFER EQUATION

In the presence of matter, it is not immediately obvious what changes may occur in the specific intensity as we move along a ray path. The aim of this section will be eviscerate the matter.

Let's consider the following geometric construction

Figure 3: Geometrical construction for ray paths propagating in empty space. Credits: G. Rybicki, A. Lightman.

It won't take a lot of effort to convince yourself that in empty space the monochromatic intensity I_ν is actually conserved. Simply writing down the definitions and imposing the conservation of energy

$$I_{\nu_2} \, \mathrm{d}A_2 \, \mathrm{d}t \, \mathrm{d}\Omega_2 \, \mathrm{d}\nu = I_{\nu_1} \, \mathrm{d}A_1 \, \mathrm{d}t \, \mathrm{d}\Omega_1 \, \mathrm{d}\nu$$

the conclusion follows from observing that $\mathrm{d}A_2 \, \mathrm{d}\Omega_2 = \mathrm{d}A_1 \, \mathrm{d}\Omega_1$.

If we consider an affine parameter of the form $\vec{x} = \vec{x}_0 + \hat{k}s$, we may as well write the previous results in a more familiar fashion

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = 0 \implies (\hat{k} \cdot \nabla)I_\nu = 0 \tag{7}$$

What changes if matter is present along the ray path? Clearly it will no longer be true that $(\hat{k} \cdot \nabla)I_\nu = 0$, but we're not that far off. All that we need is some little work on both terms.

How the right member of the equation should change is obvious: It needs to keep track of the "creation" and "destruction" of photons in the considered volume of spacetime.

The left member requires a little more care. Consider infinitesimal time and space displacements along the ray path, respectively $\mathrm{d}t$ and $\mathrm{d}\vec{x}$

$$\Delta E_\nu \, \mathrm{d}\nu = \left(I_\nu(\vec{x} + \mathrm{d}\vec{x}, t + \mathrm{d}t, \hat{k}) - I_\nu(\vec{x}, t, \hat{k}) \right) \mathrm{d}t \, \mathrm{d}\Omega \, \mathrm{d}A \, \mathrm{d}\nu$$

Taking a first order expansion in respect to the affine parameter s along the ray path yields

$$\left(\frac{1}{c}\partial_t I_\nu + \partial_s I_\nu \right) \mathrm{d}t \, \mathrm{d}s \, \mathrm{d}\Omega \, \mathrm{d}A \, \mathrm{d}\nu = \text{photon addition} - \text{photon removal}$$

This equation is clearly a generalization of eq.7 for non-stationary radiative transport and in the presence of matter. It's about time we get to know what "lives" in the right side of the equation.

1.4.1 Monochromatic emission coefficient

For the moment, we'll define the *spontaneous* monochromatic emission coefficient j_ν as

$$\mathrm{d}E_\nu \, \mathrm{d}\nu = j_\nu \, \mathrm{d}V \, \mathrm{d}t \, \mathrm{d}\Omega \, \mathrm{d}\nu \tag{8}$$

which in general has a non-zero dependence on the emission direction. Sometimes the spontaneous emission coefficient is defined by the *emissivity* ϵ_ν (**please note** that rather often the two names are used almost interchangably), which is the energy emitted spontaneously per unit frequency per unit time per unit mass

$$j_\nu = \frac{\epsilon_\nu \rho}{4\pi}$$

where ρ is the mass density of the emitting medium.

If we perform the decomposition $\mathrm{d}V = \mathrm{d}A \, \mathrm{d}s$, the contribution of spontaneous emission to the specific intensity is

$$\mathrm{d}I_\nu = j_\nu \, \mathrm{d}s$$

1.4.2 Absorption coefficient

Similarly, we can consider the energy that is absorbed from the radiation when passing through a medium. There exists similar definition; I'll use the one we gave in class and that is incidentally the one used in [1] and [3] as well.

We define the *absorption coefficient* α_ν through the following relation

$$\mathrm{d}I_\nu = -\alpha_\nu I_\nu \, \mathrm{d}s \tag{9}$$

If we use a microscopic model, then the absorption coefficient can be understood as particles with numeric density n presenting an effective absorbing area, the *cross section*. The coefficient α_ν can thus be rewritten in terms of

$$\alpha_\nu = n\sigma_\nu = \rho\kappa_\nu$$

where κ_ν is called the mass absorption coefficient or the *mass-weighed opacity coefficient*.

I should probably point out that in eq.9, we consider "absorption" to include both "true absorption" and stimulated emission, because both are proportional to the intensity of the incoming beam. Depending on the entity of the contribution, the α_ν coefficient may be positive or even negative, giving raise to curious phenomena.

Making full use of what we've just defined, we can finally present the celebrated *equation of radiative transfer* (although in the notable absence of scattering)

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = -\alpha_\nu I_\nu + j_\nu \tag{10}$$

which is actually fairly easy to solve when one of the two coefficients vanishes.

Emission only

We set $\alpha_\nu = 0$ and the equation may be solved by direct integration

$$I_\nu(s) = I_\nu(s_0) + \int_{s_0}^s j_\nu(s') \, ds'$$

the result is not that interesting per se.

Absorption only

This time we set $j_\nu = 0$. The equation is easily solved this time as well

$$I_\nu(s) = I_\nu(s_0) \exp\left(-\int_{s_0}^s \alpha_\nu(s') \, ds'\right)$$

In this case, it's rather common to write down the equation in terms of a new variable, namely the *optical depth* τ_ν

$$d\tau_\nu = \alpha_\nu \, ds \tag{11}$$

Given this definition we'll say that if

- $\tau_\nu \gg 1$: the medium is *optically thick or opaque*

- $\tau_\nu \ll 1$: the medium is *optically thin or transparent*

this has some crucial implications we'll be going through in a moment.

In the stationary limit, the equation of radiative transport may be written as

$$(\hat{k} \cdot \nabla) I_\nu(\hat{k}, \vec{x}) = j_\nu(\vec{x}) - \alpha_\nu(\vec{x}) I_\nu(\hat{k}, \vec{x})$$

In terms of the *source function* $S_\nu = j_\nu/\alpha_\nu$ it can now be written as

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu \tag{12}$$

which can be integrated to yield the formal solution

$$I_\nu(\hat{k}, \tau_\nu) = I_\nu(\tau_{\nu,0}) \exp(-\tau_\nu) + \int_{\tau_{\nu,0}}^{\tau_\nu} d\tau_\nu' \, S_\nu \exp(-(\tau_\nu - \tau_\nu'))$$

Assume for the moment that the matter through which radiation is passing has constant properties and has no background source. Then the source function S_ν is constant and the formal equation becomes

$$I_\nu = I_\nu(\tau_{\nu,0}) e^{-\tau_\nu} + S_\nu(1 - e^{-\tau_\nu})$$

If the medium and is optically thin, then the equation is reduced to

$$I_\nu = S_\nu \tau_\nu = j_\nu L \tag{13}$$

by taking the Taylor expansion of the exponential term and calling L some typical length of the medium.

If, on the other hand, the medium is optically thick, we can neglect the exponential $e^{-\tau_\nu}$ to obtain

$$I_\nu = S_\nu \tag{14}$$

The most notable implication of eq.14 is if we consider the specific intensity coming out of a small hole on a box kept in thermodynamic equilibrium. We know that what's going to come out of there is the blackbody radiation

$$I_\nu = B_\nu(T)$$

but what if we were to put an optically thick object just behind the hole?

If the object is in thermodynamic equilibrium with the surroundings (and it *will* be, given an appropriate amount of time), then the radiation coming out of the hole will still be blackbody radiation. But eq.14 tells us that the source function will tend to be equal to the specific intensity, hence

$$S_\nu = B_\nu(T) \tag{15}$$

which actually puts a constraint on the possible values of the emission coefficient in terms of the absorption coefficient. This is exactly what is expressed in Kirchhoff's law

$$j_\nu = \alpha_\nu B_\nu \tag{16}$$

Let us briefly consider what we have just derived. Often matter tends to emit and absorb at specific frequencies corresponding to what are commonly called *spectral lines*. We would expect then both j_ν and α_ν to have peaks (or depression) around these lines. But Kirchhoff's law forces their ratio to be equal to a smooth blackbody profile.

Thus we can expect to observe two very different scenarios if the medium is optically thin rather than optically thick. In the former, the radiation emerging from the medium is essentially determined by its emission coefficient; since j_ν is expected to present peaks, so will the radiation spectrum, which will be appear in spectral lines, as shown in Fig.4 and Fig.5.
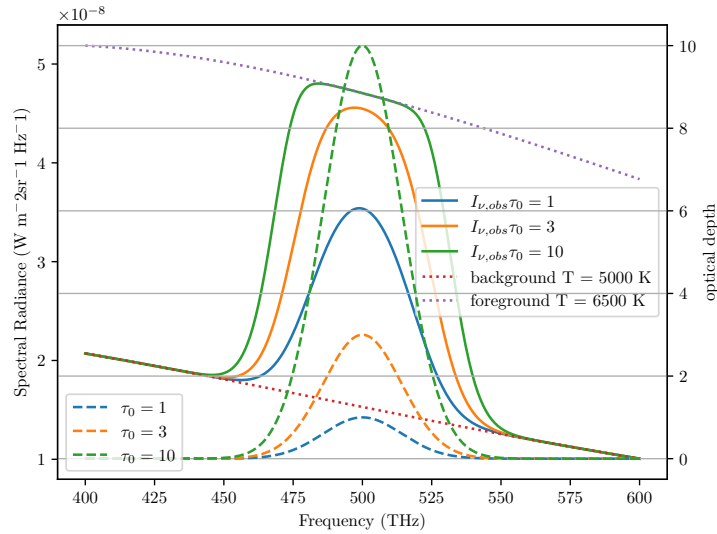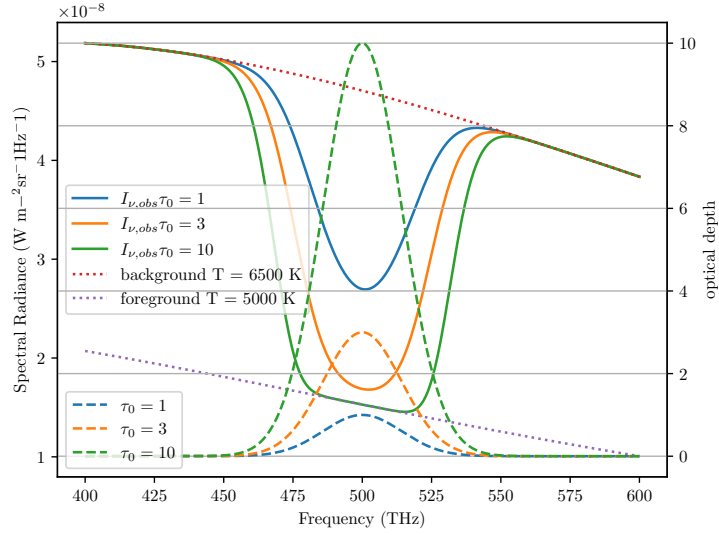

Figure 4: An example of emission features formation for different temperatures and different values of τ. Credits: Prof. Walter del Pozzo.

Figure 5: An example of absorption features formation for different temperatures and different values of τ. Credits: Prof. Walter del Pozzo.

On the other hand, the intensity coming out of an optically thick body is its source function, which must be equal to the blackbody function. Hence we expect the medium the emit in a continuum, pretty much like a blackbody.

All throughout this description, we've been assuming the medium to have constant properties, which has the perk of being a good approximation for many objects of interest, but still turns out to be a really poor one for many other objects. Stars, for example.

Ingenuously, we may expect stars to emit radiation like blackbodies, but they're not. Actually, stars present absorption lines–possibly many, depending on the class of star. What we cannot assume in stars is them having constant properties, starting from temperature.

In fact, we could take a guess and claim that stars are in *strict* thermodynamic equilibrium. It would be a very bad guess indeed.

1.5.1 Local Thermodynamic Equilibrium (LTE)

Let's be honest: In a realistic situation, we *rarely* have strict thermodynamic equilibrium. If a body is in thermodynamic equilibrium, we can assume a number of important physical principles to hold, like the Maxwellian distribution

$$\mathrm{d}n_v = 4\pi n \left(\frac{m}{2\pi kT}\right)^{3/2} v^2 \exp\left(-\frac{mv^2}{2kT}\right) \mathrm{d}v \tag{17}$$

where n is the total number of particles per unit volume and m is the mass of each particle.

Similarly, we can expect certain laws to hold, like Boltzmann's law for occupation numbers

$$\frac{n_E}{n_0} = \frac{g_E}{g_0} \exp\left(-\frac{E - E_0}{kT}\right) \tag{18}$$

and Saha's equation

$$\frac{N_{j+1}n_e}{N_j} = 2\frac{Z_{j+1}(T)}{Z_j(T)}\left(\frac{2\pi mkT}{h^2}\right)^{3/2}\exp\left(-\frac{\chi_{j,j+1}}{kT}\right) \tag{19}$$

where n_e is the density of electrons and $\chi_{j,j+1}$ is the ionization potential. Saha's equation in particular is expected to be crucial in interpreting the effect that ionization has on the emission/absorption spectrum.

The proverbial "one-million-dollar-question" then is: When can we expect a system to be in thermodynamic equilibrium and when can we expect the previous principles to hold?

Even if the system initially does not obey the, say, Maxwellian distribution, it will eventually relax to it after undergoing some *collisions*.

Collisions are crucial in establishing thermodynamic equilibrium.

When collisions are frequent, the mean free path of particles will be small, and particles will interact more effectively. When this happen, we can expect the principles aforementioned to hold. Since we're physicists, vague sentences like "*the mean free path of particles will be small*" are destined to elicit a deep sense of unease and distress. How small does the free path have to be? One meter? Two micrometers? Below the Planck lengthscale?

When we've defined the absorption coefficient α_ν, the sharpest among my four readers total may have noticed that α_ν has the dimension of the inverse of a length. It is safe to assume that α_ν^{-1} may define some distance over which a significant fraction of the radiation would get absorbed by matter.

Such a "mean-distance" is defined in a homogeneous medium as

$$< \tau_\nu >= \alpha_\nu l_\nu = 1$$

Thus, if l_ν is sufficiently small such that the temperatue can be taken as a constant over such distance, we can safely say that the useful relations we have defined earlier still hold, although only locally.

In such a fortunate scenario, known as *Local Thermodynamic Equilibrium* (LTE), all the important laws requiring thermodynamic equilibrium are expected to hold, provided that we use the local temperature $T(\vec{x})$.

In the interiors of stars, for example, LTE will prove to be a very good approximation, that will get progressively worse as we inch towards the "surface" of the star.

1.6 PARALLEL PLANE APPROXIMATION

One useful approximation that may be worthwhile to dedicate some of our time to is the *plane parallel atmosphere*, that will allow us to obtain notable results for describing of radiation travels through, say, the inner regions of the stellar atmosphere.

In the following, we're going to assume to be able to neglect the curvature and assume the various thermodynamic quantities to be constant over horizontal planes. Using Fig.6 as a reference, we see that
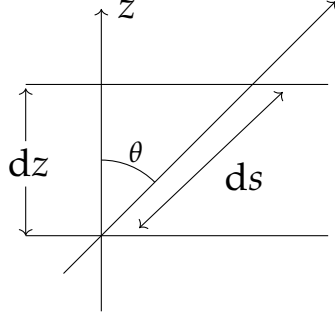
$$ds = \frac{dz}{\cos\theta} = \frac{dz}{\mu}$$

Figure 6: A ray path through a plane parallel atmosphere.

where we used the customary notation in astrophysics $\mu = \cos\theta$.

We shall consider a scattering free, stationary situation for the equation of radiative transport (10). Hence, due to planar symmetry, we expect the specific intensity to depend only on z and μ. For the sake of the current discussion, we perform a slight modification to the definition of optical depth, so that

$$\mathrm{d}\tau_\nu = -\alpha_\nu \,\mathrm{d}z$$

This way the equation of radiative transfer may be cast in the following form

$$\mu\partial_{\tau_\nu} I_\nu(\tau_\nu,\mu) = I_\nu - S_\nu$$

which has a formal solution easily computed

$$I_\nu \exp\left(-\frac{t_\nu}{\mu}\right)\Big|_{\tau_{\nu,0}}^{\tau_\nu} = -\int_{\tau_{\nu,0}}^{\tau_\nu} \frac{S_\nu}{\mu} e^{-\frac{t_\nu}{\mu}} \,\mathrm{d}t_\nu \tag{20}$$

This is customarily solved considering two distinct intervals for (I) μ: $\mu \in [0,1]$ and (II) $\mu \in [-1,0]$. In case (I) we can assume the ray path to begin from a great depth inside the star, so that $\tau_{\nu,0} \to \infty$, while in case (II) we assume the ray to receive contributions beginning from with the top of the atmosphere, where $\tau_{\nu,0} \approx 0$. For case (II), we're also assuming no radiation is coming from *outside the star*[1]

Now we can assume LTE throughout the stellar atmosphere so that eq.16 is verified. The source function at some optical depth shall then be equal to $B_\nu(T(\tau_\nu))$. For the source function at a nearby optical depth we can simply compute a Taylor expansion around the optical depth τ_ν

$$S(t_\nu) = B_\nu(\tau_\nu) - (t_\nu - \tau_\nu)\frac{\mathrm{d}B_\nu}{\mathrm{d}\tau_\nu} + \mathrm{o}(t_\nu^2)$$

We can use this to solve eq.(20), finding for both positive and negative values of μ a very important equation

$$I_\nu(\tau_\nu,\mu) = B_\nu(\tau_\nu) + \mu\frac{\mathrm{d}B_\nu}{\mathrm{d}\tau_\nu} \tag{21}$$

---

[1] Please note that this condition may be not valid at all in close binary systems.

provided the point considered is sufficiently inside the atmosphere so that $\tau_\nu \gg 1^2$. Using this simple result, we can compute the three momenta of the equation of transport

$$U_\nu = \frac{4\pi}{c} B_\nu(\tau_\nu) \tag{22}$$

$$F_\nu = \frac{4\pi}{3} \frac{\mathrm{d}B_\nu}{\mathrm{d}\tau_\nu} \tag{23}$$

$$P_\nu = \frac{4\pi}{3c} B_\nu(\tau_\nu) \tag{24}$$

1.6.1 The Grey Atmosphere

If we consider the absorption coefficient α_ν constant at all frequencies, then the atmosphere is called a "grey atmosphere". This implies that the value of the optical depth at some physical depth is constant for all frequencies.

Under this assumption, we could solve

$$\mu \frac{\partial I}{\partial \tau} = I - S$$

I'll skip the explicit calculation (which are actually fairly easy for once) and present just the final result. Two more assumptions are to be made, however: The first is to assume *radiative equilibrium*, which roughly translates into asking that there are not sources or sinks of energy in the atmosphere, thus $\partial_\tau F = 0$. As a further semplification, we assume the *Eddington approximation* to hold everywhere in the atmosphere, so that

$$P = \frac{1}{3}U$$

It should be evident that this last equation may be verified in presence of an isotropic source of radiation, also in its frequency-dependent form.

We then come to the following conclusion

$$I_{obs}(\tau = 0, \mu) = \frac{3F}{4\pi} \left(\mu + \frac{2}{3} \right) = \frac{S(\tau)}{\left(\tau + \frac{2}{3} \right)} \left(\mu + \frac{2}{3} \right) \tag{25}$$

from which we deduce the equation for the *limb darkening*

$$\frac{I(0, \mu)}{I(0, 1)} = \frac{3}{5} \left(\mu + \frac{2}{3} \right)$$

which roughly translates into saying that the radiation that we observe at the surface is the one equivalent at a source function S evaluated at $\tau = 2/3$. This is known as the *Eddington-Barbier estimation*.

A somewhat more general way to solve the problem is assuming the following functional relation for the specific intensity

$$I_\nu(\tau, \mu) = a_\nu(\tau_\nu) + b_\nu(\tau_\nu)\mu$$

---

2 You can see [1], §2.4.1 for more detailed calculations.

and compure the three momenta proper

$$J_\nu = \frac{1}{2} \int_{-1}^{+1} I_\nu \, \mathrm{d}\mu = a_\nu \tag{26}$$

$$H_\nu = \frac{1}{2} \int_{-1}^{+1} I_\nu \mu \, \mathrm{d}\mu = \frac{b_\nu}{3} \tag{27}$$

$$K_\nu = \frac{1}{2} \int_{-1}^{+1} I_\nu \mu^2 \, \mathrm{d}\mu = \frac{a_\nu}{3} \tag{28}$$

Now we assume a stronger version of the *Eddington approximation* $K_\nu = J_\nu/3$ so that the two following expressions can be written

$$\frac{\partial H_\nu}{\partial \tau_\nu} = J_\nu - S_\nu \tag{29}$$

$$\frac{\partial K_\nu}{\partial \tau_\nu} = H_\nu = \frac{1}{3}\frac{\partial J_\nu}{\partial \tau_\nu} \tag{30}$$

The mixing of the two gives us a second order PDE that it's still somewhat valid even in the outer regions of the stellar atmosphere.

1.7 RADIATIVE DIFFUSION APPROXIMATION

This section would probably be clearer if you take a brief detour to Chapter 3 to build some groundwork for scattering processes, but it still suits better the main subject of this chapter.

For the sake of coherence (and personal laziness) I'm going to talk about the radiative diffusion approximation right here, also because it makes use of the plane parallel approximation we just went through.

In Chapter 3 we have used random walk arguments to show that S_ν approaches B_ν at large *effective optical depths* in a homogeneous medium. Real media are seldom homogeneous, but often, as in the interiors of stars, there is a high degree of local homogeneity.

The equation of radiative transport in presence of scattering (45) may be cast in a slightly different form

$$I_\nu = S_\nu - \frac{\mu}{\alpha_\nu + \sigma_\nu}\partial_z I_\nu$$

We shall then assume that over a distance l_* (the thermalization length) I_ν is constant and at zero-th order is $I_\nu^{(0)} = S_\nu^{(0)} = B_\nu$. Plugging this in the equation of radiative transfer gives us $I_\nu^{(1)}$ by a simple iterative procedure

$$I_\nu^{(1)} = B_\nu - \frac{\mu}{\alpha_\nu + \sigma_\nu}\partial_z B_\nu \tag{31}$$

With the simple redefining $\mathrm{d}\tau_\nu = -(\alpha_\nu + \sigma_\nu)\,\mathrm{d}z$, we can put eq.(31) in a form functionally equal to (21).

Let us now compute the flux F_ν using the above form for the intensity

$$F_\nu(z) = 2\pi \int_{-1}^{+1} I_\nu^{(1)} \mu \, \mathrm{d}\mu = -\frac{4\pi}{3}\frac{\partial_z B_\nu}{\alpha_\nu + \sigma_\nu} = -\frac{4\pi}{3}\frac{\partial_T B_\nu}{\alpha_\nu + \sigma_\nu}\partial_z T$$

Recalling the result

$$\partial_T \int_0^{+\infty} B_\nu \, \mathrm{d}\nu = \frac{4\sigma_{SB}T^3}{\pi}$$

we can define a *mean absorption coefficient* using the *Rosseland approximation for radiative diffusion*

$$\frac{1}{\alpha_R} := \frac{\int_0^{+\infty} \frac{1}{\alpha_\nu + \sigma_\nu} \partial_T B_\nu \, \mathrm{d}\nu}{\int_0^{+\infty} \partial_T B_\nu \, \mathrm{d}\nu} \tag{32}$$

If we integrate the monochromatic flux over the frequencies and make use of the Rosseland mean, we find a useful expression often used in stellar structure models

$$F(z) = -\frac{-16\sigma_{SB}T^3}{3\alpha_R} \partial_z T \tag{33}$$

2 | THE EINSTEIN COEFFICIENTS

2.1 INTRODUCTION

Kirchhoff's law (eq.16), which relates the (spontaneous) emission and the absorption coefficient, seems to imply some underlying microscopic connection between the two phenomena.

As was first discovered by Einstein, that is exactly the case. Let's consider a two level atom interacting with radiation. As depicted in Fig.7, we'll con-
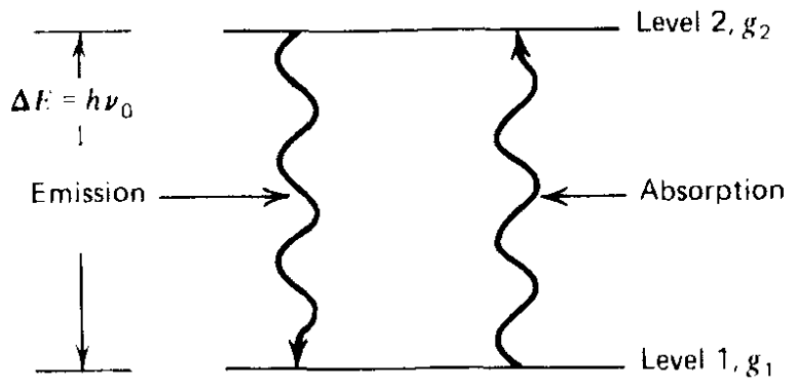

Figure 7: Photon emission and absorption in a two levels atom.
Credits: G. Rybicki, A. Lightman.

sider two discrete energy levels: the lower with energy E and degeneracy g_1, while the upper level has energy equal to $E + h\nu_0$ and degeneracy g_2. Transition between the two levels is possible only through absorption $(1 \rightarrow 2)$ or emission $(2 \rightarrow 1)$ of photons of energy $h\nu_0$.

Three processes can be thus indentified: *spontaneous emission*, *stimulated emission* and *absorption*.

Spontaneous Emission

The process we'll refer to as *spontaneous emission* occurs when the system transitions from the excited state 2 to the lower level 1 through the emission of a photon.

Spontaneous emission can occur even in the **absence of radiation fields** and can be assumed to be **isotropic**. The transition probability for spontaneous emission is defined through the *Einstein A-coefficient* A_{21}, which has units s^{-1}.

Absorption

In the presence of a radiation field with the right energy, the system can absorb a photon to transition from state 1 to a higher energy state. Assuming that the radiation field cannot self-interact, we expect the probability per unit time to be proportional to the density of photons (or to the mean intensity J_v) at frequency v_0.

However, the energy difference between the two states is not infinitely sharp, but more like a smoother curve we'll call the *line profile function $\phi(v)$*,
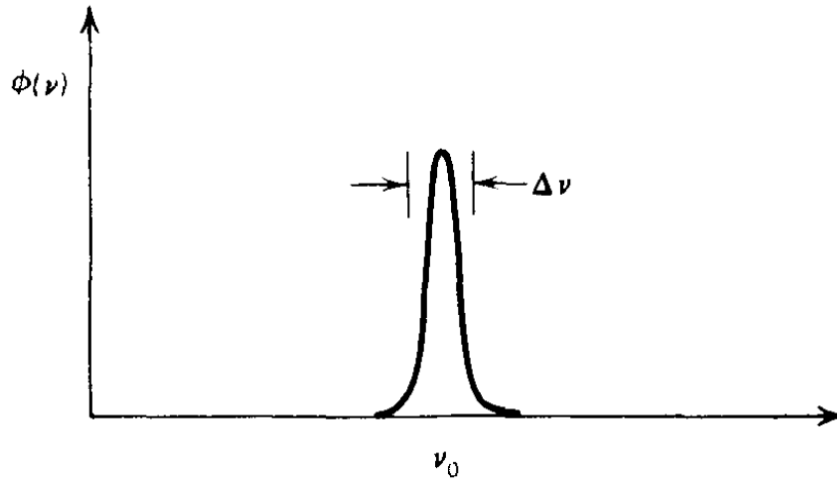

Figure 8: Line profile for a two levels atom.
Credits: G. Rybicki, A. Lightman.

which is conveniently peaked at $v = v_0$ and is correctly normalized

$$\int_0^{+\infty} \phi(v)\, dv = 1$$

Not bothering for the moment with the underlying physical mechanisms that concur in determining the line profile, we'll propose the following definition for the transition probability for absorption $B_{12}\bar{J}$, where $\bar{J}$ is

$$\bar{J} = \int_0^{+\infty} J_v \phi(v)\, dv$$

and B_{12} is the *Einstein B-coefficient*.

Stimulated Emission

As anticipated, there exists yet another way for a system to emit a photon, but this time requiring the presence of a radiation field. Taken $\bar{J}$ has the same meaning as before, we'll define the transition probability per unit time for stimulated emission as $B_{21}\bar{J}$.

If we were to assume that the mean intensity J_v changes slowly over the width Δv of the line profile, we could in principle approximate the line profile as a δ-function peaked at v_0.

Some books use the energy density U_ν in the definitions. It's pretty much the same since they carry the same information, but be aware that the two definitions will differ of a $c/4\pi$ factor.

2.2 RELATIONS BETWEEN THE EINSTEIN COEFFICIENTS

It would be most useful if there were some kind of relations between the three different coefficients and if there were some way of relating them to what we've called α_ν and j_ν when dealing the radiative transport.

If this wasn't the case, this section would have no reason to exist, so it's safe for you to assume that such relations do in fact exist. To do so, however, we have to invoke the *quantum theorem of detailed balance* [2], which roughly says

> If states a and b of a system have the same energy, then if P_{ab} is the probability per unit time of a transition from a to b, and P_{ba} from b to a,
> $$P_{ab} = P_{ba}$$

For an atom of matter to be in equilibrium with a radiation field, then, the probability of said atom being in the ground state and absorbing a photon of a given frequency must be equal to the probability that it is in the excited state and emits the photon.

If we assume to be in a steady-state condition, the rate of transition from level 1 to level 2 has to be equal to the rate of transition from level 2 to level 1, or, more generally

$$n_j \sum_j R_{ij} - \sum_j n_j R_{ji} = 0$$

If we put in the transition rates we've written down earlier, we get

$$n_1 B_{12} \bar{J} = n_2 A_{21} + n_2 B_{21} \bar{J}$$

which means that the number of transitions per unit time per unit volume out of state 1 must be equal to the number of transitions per unit time per unit volume into state 1.

Solving for $\bar{J}$

$$\bar{J} = \frac{n_2 A_{21}}{n_1 B_{12} - n_2 B_{21}} = \frac{A_{21}/B_{21}}{(n_1/n_2) \cdot (B_{12}/B_{21}) - 1} \tag{34}$$

In thermodynamic equilibrium we can use eq.18 to relate the occupation numbers

$$\frac{n_1}{n_2} = \frac{g_1}{g_2} \frac{\exp\left(-E/kT\right)}{\exp\left[-(E + h\nu_0)/kT\right]} = \frac{g_1}{g_2} \exp\left(-h\nu_0/kT\right)$$

but in thermodynamic equilibrium we know that $J_\nu = B_\nu$, so if B_ν varies slowly on the scale of $\Delta\nu$ we can assume $\bar{J} \approx B_\nu(\nu_0)$.

That means the following relations must simultaneously hold

$$g_1 B_{12} = g_2 B_{21} \tag{35}$$

$$A_{21} = \frac{2h\nu^3}{c^2} B_{21} \tag{36}$$

which are known as *Einstein relations* and connect atomic properties A_{21}, B_{21} and B_{12} and have no reference to the temperature T (unlike Kirchhoff's law).

Although we have derived eq.36 assuming thermodynamic equilibrium, those two relations **must be always valid** whether or not the atoms are in thermodynamic equilibrium.

2.2.1 Absorption and Emission coefficients in terms of Einstein coefficients

To obtain the emission coefficient j_ν we have to make a crucial assumption about the frequency distribution of the emitted radiation: This emission is distributed with the same line profile $\phi(\nu)$ that describes absorption, which is often verified in astrophysics (good for us).

By the definition (8), we already know the amount of energy emitted in volume dV, solid angle $d\Omega$, frequency $d\nu$, and time dt. Now, since each atom contributes with $h\nu_0$ to spontaneous emission over a 4π solid angle, we can express the emission coefficient as

$$j_\nu = \frac{h\nu_0}{4\pi}n_2 A_{21}\phi(\nu) \tag{37}$$

Similarly, you can show that the energy absorbed from radiation in frequency range $d\nu$, solid angle $d\Omega$, time dt and volume dV is

$$\frac{h\nu_0}{4\pi}n_1 B_{12} I_\nu \phi(\nu)\, d\nu\, d\Omega\, dt\, dV$$

From here follows immediately the *true absorption coefficient*

$$\alpha_\nu = \frac{h\nu_0}{4\pi}n_1 B_{12}\phi(\nu) \tag{38}$$

Note that since stimulated emission is proportional to the specific intensity I_ν and only affects the photons along the given beam, pretty much like true absorption. We shall then define the *absorption coefficient, corrected for stimulated emission* as

$$\alpha_\nu = \frac{h\nu_0}{4\pi}\phi(\nu)(n_1 B_{12} - n_2 B_{21}) \tag{39}$$

in which we're regarding stimulated emission kinda like a *negative absorption*.

In terms of the newly defined emission and absorption coefficients, the equation of radiative transfer takes a new look

$$\frac{dI_\nu}{ds} = -\frac{h\nu_0}{4\pi}\phi(\nu)(n_1 B_{12} - n_2 B_{21}) + \frac{h\nu_0}{4\pi}n_2 A_{21}\phi(\nu) \tag{40}$$

Computing the ratio j_ν/α_ν with the new definitions holds

$$S_\nu = \frac{n_2 A_{21}}{n_1 B_{12} - n_2 B_{21}} = \frac{2h\nu^3}{c^2}\left(\frac{g_2 n_1}{g_1 n_2} - 1\right)^{-1} \tag{41}$$

where the last equality is obtained just by substitution of the Einstein relations (36). Note that eq.(41) is a generalized Kirchhoff's law. Let's consider three interesting cases sprouting from this last equality.

Thermal Emission (LTE)

If the matter is in local thermal equilibrium (LTE) with itself (but not necessarily with the radiation), eq.18 must hold locally. In this case, we correctly retrieve

$$S_\nu = B_\nu$$

Nonthermal Emission

We can't assume eq.18 to hold, and particles do not obey the Maxwellian distribution or any of the fancy properties we'd like them to.

Inverted Populations: MASERS

Let's consider the definition of the absorption coefficient corrected for stimulated emission eq.39. Whether the coefficient is positive or negative depends on the term in parentheses, which can be restated using Einstein's relations

$$\frac{n_1 g_2}{n_2 g_1} - 1 > 0$$

so that when this relation is satisfied, even out of thermal equilibrium, we say that the system has *normal populations*. However, it is possible to put enough atoms in the upper state so that we achieve what is called a *inverted population*

$$\frac{n_1}{g_1} < \frac{n_2}{g_2}$$

which is quite the odd configuration, if you think about it. We're basically saying that a higher energy level is more densely populated than a lower energy one, which, by means of eq.18, corresponds to stating that the temperature T is negative, since E is assumed to be non-negative.

And no, T is not in Celsius degrees.

In such a scenario, the absorption coefficient is negative, so, rather than being damped to extinction, the intensity of the radiation ray *increases* when passing through matter. Such a system is called MASER (Microwave Amplification by Stimulated Emission of Radiation). For visible light you'd get what we usually call "laser".

The fun part here is that we actually *observe* this phenomenon in Nature, like for example in some molecular clouds formed in the ISM. Some sources in specific molecular lines (like the OH lines) were found to be *abnormally high*. Let me quantify how much "abnormal" we're talking about.

Your typical molecular cloud has a density a bit shy of 10^9 particles per m^3 and temperature in the range of $10 - 30\,\mathrm{K}$, so kinda chilly. Cool.

If those "abnormal" sources were assumed to be optically thick in the spectral lines and the specific intensity was equated to B_ν, you'd probably roll down you chair reading on your computer that those sources should have temperatures as high as $10^9\,\mathrm{K}$.

Not so chilly anymore, huh?

The favored explaination for this weird phenomenon (and most likely the correct one) makes use exacly of what we've just shown: Maser action.

If you want to read something (slightly) more rigorous, you can look at [1], §6.6.3.

Fairly interesting objects to study are *hydrogenoid atoms*, that is atoms with Z protons but that had ended up (no matter how) with just one e^-. For this type of atoms, Quantum Mechanics predicts energy levels distributed as such

$$E_n = -\frac{Z^2 \, \text{Ry}}{n^2} \quad \text{Ry} = \frac{m_e e^4}{2\hbar^2} \approx 13.6 \, \text{eV}$$

If we were to introduce relativistic corrections through perturbation theory, the expression would get a bit more complicated

$$E(n, l) = -\frac{Z^2 \, \text{Ry}}{n^2} \left(1 + \frac{Z^2 \alpha^2}{n^2} \left[\frac{n}{j + 1/2} - 3/4 \right] \right)$$

which introduces a *fine splitting* of the energy levels, which now depend on the value of the angular momentum. It is probably worth pointing out that the energy levels are still degenerate in the magnetic quantum number m. Given a problem with spherical symmetry, it would be beneath us to claim that a scalar perturbation[1] could be able to completely solve the degeneracy. Degeneracy is solved completely only if we introduce in the system a preferred direction[2].

Depending on the value of the orbital angular momentum eigenvalue l we can assign a name to distinguish them:

- $l = 0 \to$ *s*-orbitals

- $l = 1 \to$ *p*-orbitals

- $l = 2 \to$ *d*-orbitals

- $l = 3 \to$ *f*-orbitals

This way, transitions may be represented in a *Grotrian diagram* (Fig.9), where the energy of a given level is plotted in function of the l eigenvalue, in which selection rules are made most evident. What happens if we are to consider atoms with more than one electron? Does our model still work? Nope, not a chance.

On top of getting stupidly difficult to even evaluate the non relativistic limit, there are *many more ways* to couple interactions, resulting in more and more sources of opacity. It is actually elements with $Z > 1$ and $\#e^- > 1$ that make the greatest contributions to opacity, especially *metals* ($Z \geqslant 3$).

Sometimes it is useful to express the electron configuration by means of the *Russel-Saunders notation*. Said S, L, J respectively the spin, the orbital angular momentum and the total angular momentum eigenvalues, we can express a given electron configuration in such a fashion

$$^{2S+1}L_J$$

where L is usually written in spectroscopic notation, that is the symbol associated to the orbital with that L value.

---

1 The system is still invariant under rotations.
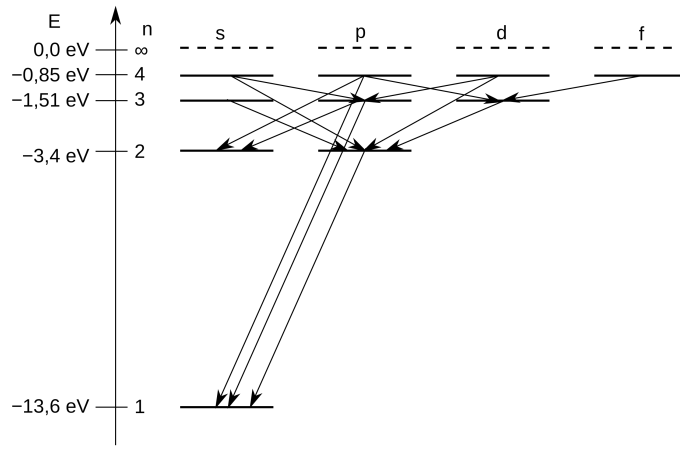2 cfr. *Stark-Lo Surdo Effect* and the *Zeeman effect*.

Figure 9: Example of Grotrian Diagram for Hydrogen. Credits: Wikipedia.

2.3.1 Hyperfine Transition

What we'll be most interested in considering, however, is the hyperfine splitting originating if we consider the coupling between the spin of the nucleus and the electron.

If we were to consider such interaction, we'd find a hyperfine splitting of the ground state. We'll be interested in the energy of the radiation coming from an electron spin flip: The system will transition from a state with total spin $F = 1$ to a one with $F = 0$, resulting in the emission of radiation. Ignoring higher order corrections to the Hamiltonian, the radiation resulting
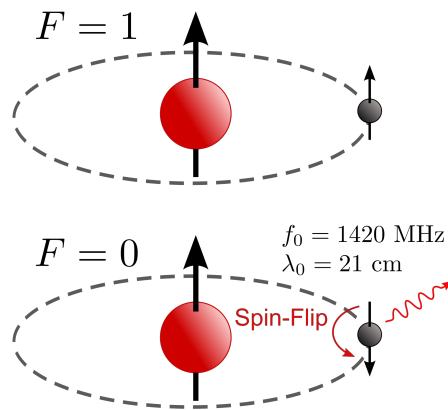

Figure 10: Schematic representation of the spin flip and the resulting emission. Credits: Wikipedia.

from the spin flip will have a wavelength

$$\lambda_{hf} = \frac{3\pi\hbar^5 m_p c^3}{g_e g_p m_e^2 e^8} \approx 21.106114054160(30)\,\text{cm} \tag{42}$$

where the expression is in CGS units and g_e, g_p are respectively the electron and the proton spin g-factors (≈ 2 for the electron, ≈ 5.59 for the proton).

Unfortunately, such a transition cannot be observed in lab-experiments. The extimated half-life time of the transition from the Einstein A coefficient[3] is of the order of a few Myrs. Which means that unless you have *a lot* of time to spare to fight against death, you'll probably never experience even a single one of these transitions.

This, however, has a strikingly beautiful consequence: Although "rare", we should be able to detect the 21-cm emission line if we were to look somewhere where the transition may have happened millions of years ago, namely somewhere in the sky.

That is exactly the case. If you were to steal a radiotelescope[4] and point it towards the Galactic plane, you'll see a really sharp line in proximity to the 21-cm line ($\sim 1420\,\text{MHz}$).
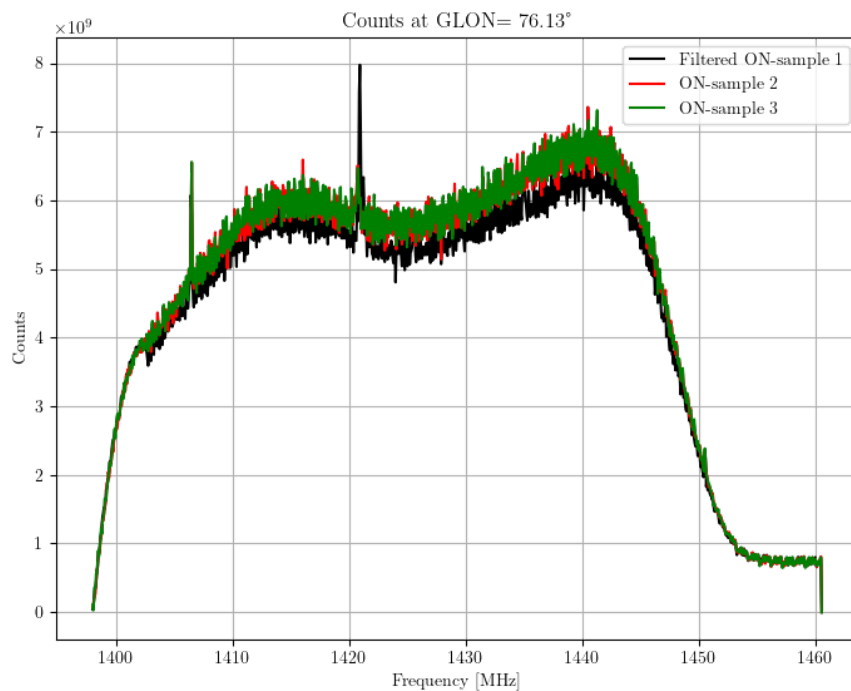
Figure 11: Sampling of the 21-cm emission line with the new Spider500 Radio telescope of the Physics Department of the University of Pisa.
Credits: AMLab, Group 8, a.y. 2025/2026.

2.4 LINE BROADENING

It would be most dumb for us to assume that energy levels, or the lines connecting them, are infinitely sharp. Even a quick look at Heisenberg's uncertainty principle should convince you otherwise.

When we defined the Einstein's coefficients, we had to introduce the line profile $\phi(\nu)$ to account for the non-zero width of the line, and it's about time we consider some of the phenomena that concur in determining the line's actual shape.

---

3 Remember that A has the dimensions of the inverse of a time, so $\tau_{HL} \approx A^{-1}$.
4 Do not recommend. They get awfully big if you want to see something cool.

2.4.1 Natural Broadening

For this one, you'll have to recall some of the basics results of Quantum Mechanics. The calculation is not particularly difficult, but it's surely long, so I'm going to omit that and take a simpler route.

Note that the spontaneous decay of an atomic state n proceeds at a rate

$$\Gamma = \sum_m A_{nm}$$

where the sum is over all states m of lower energy (which may be a lot).If radiation is present, we should add the induced rates to this. The coefficient of the wave function of state n, therefore, is of the form $e^{-\Gamma t/2}$ and leads to a decay of the electric field by the same factor. We have an emitted spectrum determined by the decaying sinusoid type of electric field, so the line profile must be a *Lorentz or Natural profile*

$$\phi(\nu) = \frac{\Gamma}{4\pi^2} \frac{1}{(\nu - \nu_0)^2 + \left(\frac{\Gamma}{4\pi}\right)^2}$$

More often than not, however, we're going to assume that the effect of natural broadening is rather peaked around ν_0, similar to a δ-function, since, at least as far as astrophysics is concerned, there are more relevant sources of broadening.

2.4.2 Doppler Broadening

An atom is in thermal motion, so that the frequency of emission or absorption in its own frame corresponds to a different frequency for an observer. Each atom has its own Doppler shift, so that the net effect is to spread the line out, but not to change its total strength.

Recall the classic Doppler effect

$$\nu - \nu_0 = \frac{\nu_0 v_z}{c} \tag{43}$$

Here ν_0 is the rest frequency. Note that this can be extended to *bulk motion* as well. We can decompose the velocity as

$$\vec{v} = \langle \vec{v} \rangle + \delta \vec{v}$$

which is often called *Reynolds' decomposition* in Fluid-dynamics books when dealing with turbulence. Generally, an ensamble of atoms will have different $\delta \vec{v}$, so each atom will absorb photons at different frequencies.

What we have to do is then convolute the Doppler effect over some velocity distribution. Assuming LTE to hold, such distribution will be the Maxwellian distribution. Switching into a reference frame where $\langle \vec{v} \rangle = 0$ (which is always possible), then the small fluctuations $\delta \vec{v}$ shall follow the Maxwellian distribution.

Please note that the one dimensional version of the Maxwellian distribution must be used, which is a Gaussian distribution with $\mu = 0$ and a given variance.

The convoluted line profile will then be

$$\phi(\nu) = \int \delta(\nu - \nu_0)\nu_0 \left(1 + \frac{v}{c}\right) \exp\left(-\frac{m_a v^2}{2kT}\right) dv$$

where, as anticipated, we identified the natural broadening as a δ-function. This is easily computed

$$\phi(\nu) = \frac{1}{\Delta\nu_D\sqrt{\pi}} e^{-(\nu-\nu_0)^2/(\Delta\nu_D)^2} \quad \Delta\nu_D = \frac{\nu_0}{c}\sqrt{\frac{2kT}{m_a}} \tag{44}$$

Here $\Delta\nu_D$ represents the *Doppler width*.

Generally speaking, the minimum broadening we can expect is given by a convolution of Doppler Broadening and Natural Broadening in its proper Lorentzian form. This hasn't a fancy analytical solution, but it's rather expressed from what is called a *Voigt profile*.

Roughly speaking, the Voigt profile is something that looks like a Gaussian at the center of the line[5] (where Doppler effect dominates) but has the *wings* of the Lorentzian distribution.

$$H(a,u) = \frac{a}{\pi}\int_{-\infty}^{+\infty} \frac{e^{-y^2}}{a^2 + (u-y)^2}\, dy \quad a = \frac{\Gamma}{4\pi\Delta\nu_D} \quad u = \frac{\nu - \nu_0}{\Delta\nu_D}$$

In terms of this compact definition of the Voigt profile, the overall line profile will be

$$\phi(\nu) = \frac{1}{\Delta\nu_D\sqrt{\pi}} H(a,u)$$

---

5 Sometimes it's called *kernel* of the distibution.

3 | SCATTERING PROCESSES

3.1 TRANSPORT THROUGH SCATTERING

When we've first written down the equation for radiative transport (eq.10) we've neglected the effects of a possibly much relevant source of photons, which is *scattering*, another fairly common emission process. Scattering depends completely on the amount of radiation falling on the spacetime element we're considering.

How do we include it into the picture of radiative transport?

For the present discussion we assume *isotropic* scattering, so that the scattered radiation is emitted equally into equal solid angles. We also assume that the total amount of radiation emitted per unit frequency range is just equal to the total amount absorbed in that same frequency range (*coherent scattering*).

This are some fair requirements if you think about it. Isotropy is most likely assured if the main type of scattering is the (non-relativistic) Thomson scattering on electrons, while coherence of the scattering is just requiring the scattering to be elastic.

Such conditions are not always met, but for the moment we'll turn our heads away from the obviou problem and see where this brings us.

The emission coefficient for coherent, isotropic scattering can be found simply by equating the power absorbed per unit volume and frequency ranges to the corresponding power emitted. Here we define the *scattering coefficient* σ_ν so that

$$j_\nu = \sigma_\nu J_\nu$$

Dividing by the scattering coefficient, we find that the source function for pure scattering is simply equal to the mean intensity within the emitting material

$$S_\nu = J_\nu = \frac{1}{4\pi} \int_\Omega I_\nu \, \mathrm{d}\Omega$$

The equation of radiative transport is thus modified[1]

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = -\sigma_\nu(I_\nu - J_\nu) = -\sigma_\nu\left(I_\nu - \frac{1}{4\pi} \int_\Omega I_\nu \, \mathrm{d}\Omega\right) \tag{45}$$

Solving this equation is a bloodbath. The source function is not known a priori and depends on the solution I_ν, at all directions through a given point, making this a *integro-differential equation*, also known as "bloody mess".

---

[1] Note that we're assuming for the scattering version of RT a similar form of the absorption version (10).

A particularly useful way of looking at scattering, which leads to important order-of-magnitude estimates, is by means of random walks. We shall now develop a formalism that interprets the processes of absorption, emission, and propagation in probabilistic terms for a single photon rather than the average behavior of large numbers of photons.

Consider a photon emitted in an infinite, homogeneous scattering region. It travels a displacement $\vec{r}_1$ before being scattered, then travels in a new direction over a displacement $\vec{r}_2$ before being scattered, and so on. After N free paths is, the total displacement will look like

$$\vec{R} = \sum_{i=1}^{N} \vec{r}_i$$

Since this is a vector, the average total displacement will be identically null. Therefore, we must evaluate the mean square of the total displacement which, in the assumption of independent and isotropic scattering, must be expressed as

$$l_*^2 = \langle \vec{R}^2 \rangle = \sum_{i=1}^{N} \langle \vec{r}_i^2 \rangle$$

The quantity l_*^2 is the root mean square net displacement of the photon, while l^2 denotes the mean square of the free path of a photon, which within a factor of order unity, it's simply the mean free path of a photon.

Since the mean square displacements of each scattering iteration have no reason to be different, we'll just write

$$l_* = \sqrt{N} l$$

Said L the linear dimension of the object photons are trying to "escape" from, for regions of large optical depth the number of scatterings required to actually escape is roughly determined by setting $l_* \approx L$. Then $N \sim L^2/l^2$. But since the optical thickness of the medium τ is of order L/l, our previous results yield

$$\tau^2 \approx N \quad \tau \gg 1$$

For regions of small optical thickness, the mean number of scatterings is small, so that we can approximate $N \approx \tau$. For most order-of-magnitude estimates we could then use

$$N \approx \tau^2 + \tau$$

which has the evident perk of well-behaving in the two limits we've discussed.

3.2.1   Combined scattering and absorption

What happens if we had both scattering and absorption? We'd have two terms on the right hand side of the tranfer equation

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = -\alpha_\nu (I_\nu - B_\nu) - \sigma_\nu \left(I_\nu - \frac{1}{4\pi} \int_\Omega I_\nu \, \mathrm{d}\Omega \right) \tag{46}$$

Please note that we're assuming Kirchhoff's law to hold, and the medium to be optically thick, so that $j_\nu = \alpha_\nu B_\nu$. In terms of a suitably defined source function S_ν

$$S_\nu = \frac{\alpha_\nu B_\nu + \sigma_\nu J_\nu}{\alpha_\nu + \sigma_\nu}$$

we can rewrite eq.46 as

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = -(\alpha_\nu + \sigma_\nu)(I_\nu - S_\nu)$$

We may then define an *effective optical depth*[2] τ_* so that

$$\mathrm{d}\tau_* = (\alpha_\nu + \sigma_\nu)\,\mathrm{d}s$$

so that the mean free path will be just $l_\nu = (\alpha_\nu + \sigma_\nu)^{-1}$.

In our random walk approximation, the probability of a scattering process to end in a true absorption will be

$$\epsilon_\nu = \frac{\alpha_\nu}{\alpha_\nu + \sigma_\nu}$$

The probability of for scattering will then be $1 - \epsilon_\nu$, which is known as *single-scattering albedo*. In terms of ϵ_ν, the source function becomes

$$S_\nu = (1 - \epsilon_\nu)J_\nu + \epsilon_\nu B_\nu$$

Let us consider first an infinite homogeneous medium. A random walk starts with the thermal emission of a photon (creation) and ends, possibly after a number of scatterings, with a true absorption (destruction). Since the walk can be terminated with probability ϵ at the end of each free path, the mean number of free paths is $N = \epsilon^{-1}$. We then have $l_* = l\epsilon^{-1/2}$.

So if we were to substitute in the definition of l_ν, we'd get

$$l_* \approx [\alpha_\nu(\alpha_\nu + \sigma_\nu)]^{-1/2}$$

This length represents a measure of the net displacement between the points of creation and destruction of a typical photon; it often goes by the name of *diffusion length*, *thermalization length*, or *effective mean path*. Usually it is also frequency dependent.

Essentially, it is the average length over which emitting and absorbing elements are radiatevely coupled.

The behavior of a finite medium can be explained in terms of what we've been creating so far. Its properties will depend (strongly) on whether its linear extension L is larger or smaller than the effective free path.

In terms of the effective optical depth $\tau_* = L/l_*$, we can restate its definition as follows

$$\tau_* \approx [\tau_a(\tau_a + \tau_s)]^{1/2} \quad \tau_a = \alpha_\nu L \quad \tau_s = \sigma_\nu L$$

When the effective free path is large compared to L, we have $\tau_* \ll 1$ and the medium is *effectively thin*. Most photons will then escape by random walking their way out of the medium before being destroyed by a true absorption.

---

2 Also called *extinction coefficient*.

Conversely, a medium for which $\tau_* \gg 1$ is *effectively thick*. Most photons thermally emitted at depths larger than the effective path length will be destroyed by absorption before they get out.

This new definition allows us to reformulate when for a medium will be possible to be in LTE: Over a distance of order l_*, photons will do many scattering events, unable to leave the medium if $\tau_* \gg 1$, assuring that LTE is established *locally*.

Were this to happen, we could safely claim then that

$$I_\nu \to B_\nu \quad S_\nu \to B_\nu$$

This way is perhaps more clear why l_* is called thermalization length.

Part II

FLUID DYNAMICS

4 | FUNDAMENTALS OF FLUID DYNAMICS

4.1 INTRODUCTION

Coming soon.

Part III

GRAVITATION

BIBLIOGRAPHY

[1] A. R. Choudhuri. *Astrophysics for Physicists*. Astrophysics-Testbooks. Cambridge University Press, 2010.

[2] D. L. Goodstein. *States of Matter*. Dover Books on Physics. Dover Publications, 2014.

[3] George B. Rybicki and Alan P. Lightman. *Radiative Processes in Astrophysics*. 1986.