NOTES OF
ASTROPHYSICAL PROCESSES

version 1.0

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Knowledge without understanding
Is but a sword stuck in its sheat.

Arthur Leywin

PREFACE

Not long ago, it occurred to me how cool it is when someone unexpectedly releases a very detailed and all-comprehensive version of their notes, especially when dealing with a course that has a handful of really different topics often interacting together in unpredictable, yet fascinating, ways—-as it's the case for the Astrophysical Processes class.

This notes will be mainly based on *my* own notes of the lectures by Professor Walter del Pozzo and Professor Marco Crisostomi during the academic year 2025-2026. However, since I take little to no pride in my messy notes, I'll be using more often than not some of the many references you can find on the course catalogue page or in the bibliography of this humble collection.

You can report errors (whatever their nature might be) and suggestions for additions at g.pannocchia3@studenti.unipi.it or through whatever convoluted way (conventional or not) you prefer[1].

Without further ado, we'd better not lose much more time on a preface and get started with it.

There was Eru, the One, who in Arda is called Ilúvatar; and he made first the Ainur [...] *But for a long while they sang only each alone, or but few together, while the rest hearkened; for each comprehended only that part of the mind of Ilúvatar from which he came, and in the understanding of their brethren they grew but slowly.*

Yet ever as they listened they came to deeper understanding, and increased in unison and harmony.

Ainulindalë, "The music of the Ainur",
Silmarillion, J. R. R. Tolkien

1 I'd like, however, not to see my house stormed by homing pidgeons.

CONTENTS

Part I

RADIATIVE TRANSPORT

1 INTERACTION OF RADIATION WITH MATTER

1.1 INTRODUCTION

Most of our knowledge about the Universe is based on the electromagnetic radiation that reaches us from far far away. EM radiation is obviusly not the only way we can probe the Universe we live in but, in respect to neutrinos, cosmic rays or even gravitational waves, it's not a long stretch to claim it is by far the most understood.

It is most important then that an astrophysicist worthy of his (or her) name has a good grasp of the theory of radiative transfer and of its applications.

Apart from a few more key differences, I'll follow the description of radiative transfer of [3] and [10], but I won't fail to emphasize whenever I'll be doing otherwise.

1.2 RELEVANT QUANTITIES FOR RADIATIVE TRANSFER

Although some books often start their description of radiative transfer from the definition of *monochromatic energy* and *monochromatic intensity*, I found that it is most misleading, since, in all but a few cases, what we experimentally measure are fundamentally *fluxes*.

We shall then consider the *monochromatic flux* F_ν ($\mathrm{erg\,s^{-1}\,Hz^{-1}\,cm^{-2}}$) produced by some source passing through a small area $\mathrm{d}A$ located somewhere in space.

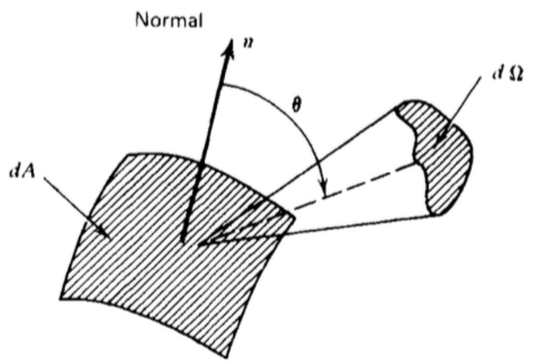

Figure 1: Schematic geometrical representation of the system.
Credits: G. Rybicki, A. Lightman [10].

If we call $\hat{k}$ the propagation direction of the flux and $\hat{n}$ the unit vector emerging from the surface $\mathrm{d}A$, it's easy to get convinced that what is actually passing through the surface is somewhat proportional to $F_\nu(\hat{k} \cdot \hat{n})$.

From the monochromatic flux we can define the *bolometric flux*, which is just the monochromatic flux integrated over all frequencies (or wavelengths)

$$F = \int_0^{+\infty} F_\nu \, \mathrm{d}\nu = \int_0^{+\infty} F_\lambda \, \mathrm{d}\lambda \tag{1}$$

This also tells us how to convert a flux per unit frequency to a flux per unit wavelength

$$F_\nu \, \mathrm{d}\nu = F_\lambda \, \mathrm{d}\lambda$$

By now it should be clear that, despite being experimentally sensible to use the flux, we're losing much information sticking with it, namely directional information.

We consider then the amount of radiation $E_\nu \, \mathrm{d}\nu$ passing through the same area in time $\mathrm{d}t$ and solid angle $\mathrm{d}\Omega$. Hence we can write

$$\mathrm{d}E_\nu \, \mathrm{d}\nu = I_\nu(\mathbf{r}, t, \hat{k})(\hat{k} \cdot \hat{n}) \, \mathrm{d}t \, \mathrm{d}\Omega \, \mathrm{d}A \, \mathrm{d}\nu \tag{2}$$

where the quantity $I_\nu(\mathbf{r}, t, \hat{k})$ is called the *specific monochromatic intensity*. If $I_\nu(\mathbf{r}, t, \hat{k})$ is specified for all directions at every point in a certain region of spacetime, then we'd have a complete prescription of the radiation field we intend on studying.

Capitalizing on the blatant similarities with distribution functions, we can evaluate the moments of the monochromatic intensity.

Definition 1.2.1. *Monochromatic mean intensity J_ν*

$$J_\nu = \frac{1}{4\pi} \int_\Omega I_\nu \, d\Omega = \frac{c}{4\pi} U_\nu$$

with U_ν the total energy density of radiation. Note that J_ν is pretty much just an average of the monochromatic intensity over all solid angles.

Definition 1.2.2. *Monochromatic flux $\vec{F}_\nu$*

$$\vec{H}_\nu = \frac{1}{4\pi} \int_\Omega I_\nu(\hat{k}) \hat{k} \, d\Omega = \frac{1}{4\pi} \vec{F}_\nu$$

I haven't explicitly proved the last equality, but it shouldn't be hard for you to convince yourself (or prove it yourself) that it is indeed true.

Definition 1.2.3. *Monochromatic radiation pressure p_ν The monochromatic pressure is defined starting from the different directions correlations of the monochromatic intensity*

$$K_\nu^{ij} = \frac{1}{4\pi} \int_\Omega I_\nu(\hat{k}) n^i n^j \, d\Omega$$

The pressure in particular is usually expressed as

$$P_\nu = \frac{1}{c} \int_\Omega I_\nu(\hat{k}) \cos^2 \theta \, d\Omega$$

where $\cos^2 \theta = (\hat{k} \cdot \hat{n})^2$.

1.3 BLACKBODY RADIATION

Even at an undergraduate level, we're all fairly familiar with *blackbody radiation*. The easiest way to deduce the expression for the energy density of photons in *thermal equilibrium* (STE) inside a cavity is by the means of statistical mechanics.

Remember the Bose-Einstein distribution

$$n = \frac{1}{\exp(h\nu/kT) - 1}$$

and the phase space density of states

$$\rho(\nu)\,\mathrm{d}\nu = \frac{4\pi g \nu^3}{c^3}\,\mathrm{d}\nu$$

from which deducing the expression from internal energy is straightforward. Remembering $g = 2$ is the quantum degeneracy of photons, a simple moltiplication of the previous expressions yields

$$U_\nu\,\mathrm{d}\nu = \frac{8\pi\nu^3}{c^3}\frac{1}{\exp(h\nu/kT) - 1}\,\mathrm{d}\nu$$

Since blackbody radiation is isotropic (it actually depends only on the absolute temperature T), the definition of mean monochromatic intensity yields

$$\boxed{B_\nu(T) = \frac{2h\nu^3}{c^2}\frac{1}{\exp(h\nu/kT) - 1}} \tag{3}$$

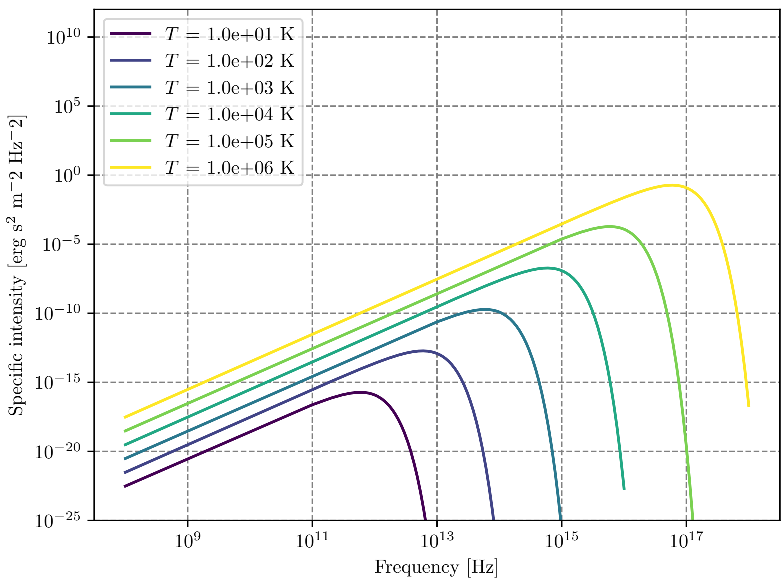

Figure 2: Blackbody frequency spectrum.

It's important to notice that, in principle, such a fundamental result holds only in *strict thermodynamic equilibrium* (STE), but we'll soon see how to generalize this formulation for less "restrictive" environments.

An incredible number of important results descends from (3), and it may be worthwhile to at least cite some of them, starting from Stefan-Boltzmann law.

We'll use the following result without proving it

$$\int_0^{+\infty} B_\nu(T)\,\mathrm{d}\nu = \frac{2h}{c^2}\frac{\pi^2}{15}\left(\frac{kT}{h}\right)^4$$

Computing the bolometric flux and the bolometric energy density by integrating over all frequencies using what we've just written down, you find the following

$$U(T) = aT^4 \quad F(T) = \sigma_{SB}T^4$$

Clearly the two constants a and σ_{SB} cannot be independent, and are actually related by the integral we've previously calculated. Using for example

$$F(T) = \pi \int_0^{+\infty} B_\nu(T)\,\mathrm{d}\nu$$

you can easily find out that the *Stefan-Boltzmann constant* is equal to

$$\sigma_{SB} = \frac{2\pi^5 k^4}{15c^2h^3}$$

and the relation with a is simply $\sigma_{SB} = ac/4$.

The equation

$$F(T) = \frac{2\pi^5 k^4}{15c^2h^3}T^4 \tag{4}$$

is what is usually known as the *Stefan-Boltzmann law*.

Let us now consider two different regimes for eq.3: $h\nu/kT \ll 1$ and $h\nu/kT \gg 1$. The first yields what is commonly known as the Rayleigh-Jeans Law which is, sadly, pretty much relevant only for radioastronomy.

Since

$$\exp\left(\frac{h\nu}{kT}\right) = 1 + \frac{h\nu}{kT} + o\left(\frac{h\nu}{kT}\right)^2$$

the blackbody radiation assumes the much simpler form of

$$B_\nu^{RJ} = \frac{2\nu^2}{c^2}kT \tag{5}$$

Another important results is achieved in the opposite regime, when the exponential term is rather larger than unity

$$B_\nu^{W} = \frac{2h\nu^3}{c^2}\exp\left(-\frac{h\nu}{kT}\right) \tag{6}$$

This expression is often known as Wien's Law.

In the presence of matter, it is not immediately obvious what changes may occur in the specific intensity as we move along a ray path. The aim of this section will be to eviscerate the matter.

Let's consider the following geometric construction

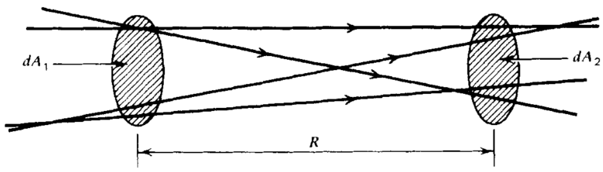

Figure 3: Geometrical construction for ray paths propagating in empty space. Credits: G. Rybicki, A. Lightman.

It won't take a lot of effort to convince yourself that in empty space the monochromatic intensity I_ν is actually conserved. Simply writing down the definitions and imposing the conservation of energy

$$I_{\nu_2}\, \mathrm{d}A_2\, \mathrm{d}t\, \mathrm{d}\Omega_2\, \mathrm{d}\nu = I_{\nu_1}\, \mathrm{d}A_1\, \mathrm{d}t\, \mathrm{d}\Omega_1\, \mathrm{d}\nu$$

the conclusion follows from observing that $\mathrm{d}A_2\, \mathrm{d}\Omega_2 = \mathrm{d}A_1\, \mathrm{d}\Omega_1$.

If we consider an affine parameter of the form $\vec{x} = \vec{x}_0 + \hat{k}s$, we may as well write the previous results in a more familiar fashion

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = 0 \implies (\hat{k}\cdot\nabla)I_\nu = 0 \tag{7}$$

What changes if matter is present along the ray path? Clearly it will no longer be true that $(\hat{k}\cdot\nabla)I_\nu = 0$, but we're not that far off. All that we need is some little work on both terms.

How the right member of the equation should change is obvious: It needs to keep track of the "creation" and "destruction" of photons in the considered volume of spacetime.

The left member requires a little more care. Consider infinitesimal time and space displacements along the ray path, respectively $\mathrm{d}t$ and $\mathrm{d}\vec{x}$

$$\Delta E_\nu\, \mathrm{d}\nu = \left(I_\nu(\vec{x}+\mathrm{d}\vec{x}, t+\mathrm{d}t, \hat{k}) - I_\nu(\vec{x}, t, \hat{k})\right)\, \mathrm{d}t\, \mathrm{d}\Omega\, \mathrm{d}A\, \mathrm{d}\nu$$

Taking a first order expansion in respect to the affine parameter s along the ray path yields

$$\left(\frac{1}{c}\partial_t I_\nu + \partial_s I_\nu\right)\, \mathrm{d}t\, \mathrm{d}s\, \mathrm{d}\Omega\, \mathrm{d}A\, \mathrm{d}\nu = \text{photon addition} - \text{photon removal}$$

This equation is clearly a generalization of eq.7 for non-stationary radiative transport and in the presence of matter. It's about time we get to know what "lives" in the right side of the equation.

1.4.1 Monochromatic emission coefficient

For the moment, we'll define the *spontaneous* monochromatic emission coefficient j_ν as

$$\mathrm{d}E_\nu \, \mathrm{d}\nu = j_\nu \, \mathrm{d}V \, \mathrm{d}t \, \mathrm{d}\Omega \, \mathrm{d}\nu \tag{8}$$

which in general has a non-zero dependence on the emission direction. Sometimes the spontaneous emission coefficient is defined by the *emissivity* ϵ_ν (**please note** that rather often the two names are used almost interchangably), which is the energy emitted spontaneously per unit frequency per unit time per unit mass

$$j_\nu = \frac{\epsilon_\nu \rho}{4\pi}$$

where ρ is the mass density of the emitting medium.

If we perform the decomposition $\mathrm{d}V = \mathrm{d}A \, \mathrm{d}s$, the contribution of spontaneous emission to the specific intensity is

$$\mathrm{d}I_\nu = j_\nu \, \mathrm{d}s$$

1.4.2 Absorption coefficient

Similarly, we can consider the energy that is absorbed from the radiation when passing through a medium. There exists similar definitions; I'll use the one we gave in class and that is incidentally the one used in [3] and [10] as well.

We define the *absorption coefficient* α_ν through the following relation

$$\mathrm{d}I_\nu = -\alpha_\nu I_\nu \, \mathrm{d}s \tag{9}$$

If we use a microscopic model, then the absorption coefficient can be understood as particles with numeric density n presenting an effective absorbing area, the *cross section*. The coefficient α_ν can thus be rewritten in terms of

$$\alpha_\nu = n\sigma_\nu = \rho\kappa_\nu$$

where κ_ν is called the mass absorption coefficient or the *mass-weighed opacity coefficient*.

I should probably point out that in eq.9, we consider "absorption" to include both "true absorption" and stimulated emission, because both are proportional to the intensity of the incoming beam. Depending on the entity of the contribution, the α_ν coefficient may be positive or even negative, giving raise to curious phenomena.

Making full use of what we've just defined, we can finally present the celebrated *equation of radiative transfer* (although in the notable absence of scattering)

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = -\alpha_\nu I_\nu + j_\nu \tag{10}$$

which is actually fairly easy to solve when one of the two coefficients vanishes.

Emission only

We set $\alpha_\nu = 0$ and the equation may be solved by direct integration

$$I_\nu(s) = I_\nu(s_0) + \int_{s_0}^{s} j_\nu(s')\,\mathrm{d}s'$$

the result is not that interesting per se.

Absorption only

This time we set $j_\nu = 0$. The equation is easily solved this time as well

$$I_\nu(s) = I_\nu(s_0)\exp\left(-\int_{s_0}^{s} \alpha_\nu(s')\,\mathrm{d}s'\right)$$

In this case, it's rather common to write down the equation in terms of a new variable, namely the *optical depth* τ_ν

$$\mathrm{d}\tau_\nu = \alpha_\nu\,\mathrm{d}s \tag{11}$$

Given this definition we'll say that if

- $\tau_\nu \gg 1$: the medium is *optically thick or opaque*
- $\tau_\nu \ll 1$: the medium is *optically thin or transparent*

this has some crucial implications we'll be going through in a moment.

In the stationary limit, the equation of radiative transport may be written as

$$(\hat{k}\cdot\nabla)I_\nu(\hat{k},\vec{x}) = j_\nu(\vec{x}) - \alpha_\nu(\vec{x})I_\nu(\hat{k},\vec{x})$$

In terms of the *source function* $S_\nu = j_\nu/\alpha_\nu$ it can now be written as

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}\tau_\nu} = -I_\nu + S_\nu \tag{12}$$

which can be integrated to yield the formal solution

$$I_\nu(\hat{k},\tau_\nu) = I_\nu(\tau_{\nu,0})\exp(-\tau_\nu) + \int_{\tau_{\nu,0}}^{\tau_\nu} d\tau'_\nu\,S_\nu\exp(-(\tau_\nu - \tau'_\nu))$$

Assume for the moment that the matter through which radiation is passing has constant properties and has no background source. Then the source function S_ν is constant and the formal equation becomes

$$I_\nu = I_\nu(\tau_{\nu,0})e^{-\tau_\nu} + S_\nu(1 - e^{-\tau_\nu})$$

If the medium and is optically thin, then the equation is reduced to

$$I_\nu = S_\nu\tau_\nu = j_\nu L \tag{13}$$

by taking the Taylor expansion of the exponential term and calling L some typical length of the medium.

If, on the other hand, the medium is optically thick, we can neglect the exponential $e^{-\tau_\nu}$ to obtain

$$I_\nu = S_\nu \tag{14}$$

1.5 KIRCHHOFF'S LAW AND LTE

The most notable implication of eq.14 is if we consider the specific intensity coming out of a small hole on a box kept in thermodynamic equilibrium. We know that what's going to come out of there is the blackbody radiation

$$I_\nu = B_\nu(T)$$

but what if we were to put an optically thick object just behind the hole?

If the object is in thermodynamic equilibrium with the surroundings (and it *will* be, given an appropriate amount of time), then the radiation coming out of the hole will still be blackbody radiation. But eq.14 tells us that the source function will tend to be equal to the specific intensity, hence

$$S_\nu = B_\nu(T) \tag{15}$$

which actually puts a constraint on the possible values of the emission coefficient in terms of the absorption coefficient. This is exactly what is expressed in Kirchhoff's law

$$j_\nu = \alpha_\nu B_\nu \tag{16}$$

Let us briefly consider what we have just derived. Often matter tends to emit and absorb at specific frequencies corresponding to what are commonly called *spectral lines*. We would expect then both j_ν and α_ν to have peaks (or depression) around these lines. But Kirchhoff's law forces their ratio to be equal to a smooth blackbody profile.

Thus we can expect to observe two very different scenarios if the medium is optically thin rather than optically thick. In the former, the radiation emerging from the medium is essentially determined by its emission coefficient; since j_ν is expected to present peaks, so will the radiation spectrum, which will be appear in spectral lines, as shown in Fig.4 and Fig.5.

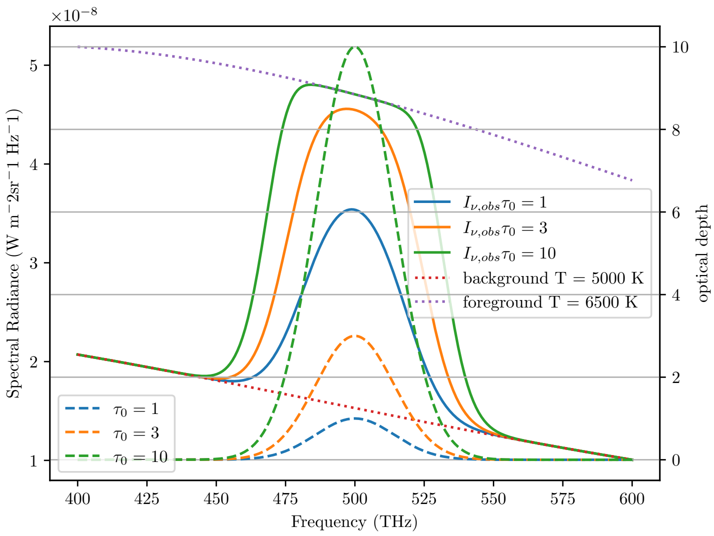

Figure 4: An example of emission features formation for different temperatures and different values of τ. Credits: Prof. Walter del Pozzo.

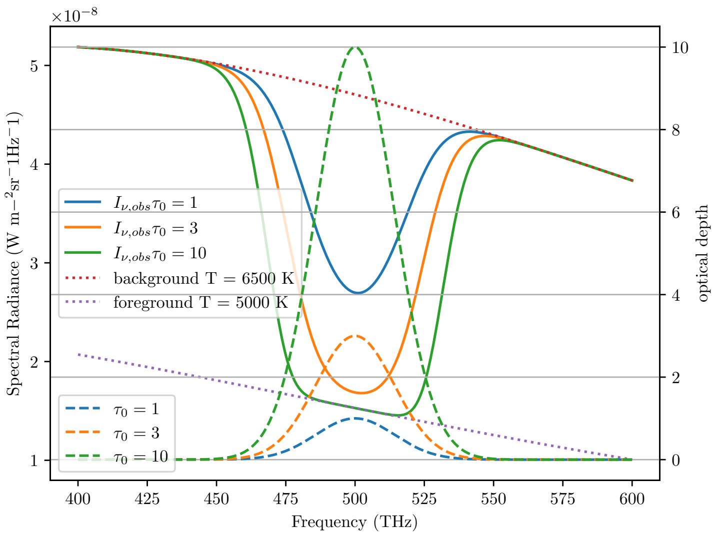

Figure 5: An example of absorption features formation for different temperatures and different values of τ. Credits: Prof. Walter del Pozzo.

On the other hand, the intensity coming out of an optically thick body is its source function, which must be equal to the blackbody function. Hence we expect the medium the emit in a continuum, pretty much like a blackbody.

All throughout this description, we've been assuming the medium to have constant properties, which has the perk of being a good approximation for many objects of interest, but still turns out to be a really poor one for many other objects. Stars, for example.

Ingenuously, we may expect stars to emit radiation like blackbodies, but they're not. Actually, stars present absorption lines–possibly many, depending on the class of star. What we cannot assume in stars is them having constant properties, starting from temperature.

In fact, we could take a guess and claim that stars are in *strict* thermodynamic equilibrium. It would be a very bad guess indeed.

1.5.1 Local Thermodynamic Equilibrium (LTE)

Let's be honest: In a realistic situation, we *rarely* have strict thermodynamic equilibrium. If a body is in thermodynamic equilibrium, we can assume a number of important physical principles to hold, like the Maxwellian distribution

$$\mathrm{d}n_v = 4\pi n \left(\frac{m}{2\pi kT}\right)^{3/2} v^2 \exp\left(-\frac{mv^2}{2kT}\right) \mathrm{d}v \tag{17}$$

where n is the total number of particles per unit volume and m is the mass of each particle.

Similarly, we can expect certain laws to hold, like Boltzmann's law for occupation numbers

$$\frac{n_E}{n_0} = \frac{g_E}{g_0} \exp\left(-\frac{E - E_0}{kT}\right) \tag{18}$$

and Saha's equation

$$\frac{N_{j+1}n_e}{N_j} = 2\frac{Z_{j+1}(T)}{Z_j(T)}\left(\frac{2\pi mkT}{h^2}\right)^{3/2}\exp\left(-\frac{\chi_{j,j+1}}{kT}\right) \tag{19}$$

where n_e is the density of electrons and $\chi_{j,j+1}$ is the ionization potential. Saha's equation in particular is expected to be crucial in interpreting the effect that ionization has on the emission/absorption spectrum.

The proverbial "one-million-dollar-question" then is: When can we expect a system to be in thermodynamic equilibrium and when can we expect the previous principles to hold?

Even if the system initially does not obey the, say, Maxwellian distribution, it will eventually relax to it after undergoing some *collisions*.

Collisions are crucial in establishing thermodynamic equilibrium.

When collisions are frequent, the mean free path of particles will be small, and particles will interact more effectively. When this happen, we can expect the principles aforementioned to hold. Since we're physicists, vague sentences like "*the mean free path of particles will be small*" are destined to elicit a deep sense of unease and distress. How small does the free path have to be? One meter? Two micrometers? Below the Planck lengthscale?

When we've defined the absorption coefficient α_ν, the sharpest among my four readers total may have noticed that α_ν has the dimension of the inverse of a length. It is safe to assume that α_ν^{-1} may define some distance over which a significant fraction of the radiation would get absorbed by matter.

Such a "mean-distance" is defined in a homogeneous medium as

$$< \tau_\nu >= \alpha_\nu l_\nu = 1$$

Thus, if l_ν is sufficiently small such that the temperatue can be taken as a constant over such distance, we can safely say that the useful relations we have defined earlier still hold, although only locally.

In such a fortunate scenario, known as *Local Thermodynamic Equilibrium* (LTE), all the important laws requiring thermodynamic equilibrium are expected to hold, provided that we use the local temperature $T(\vec{x})$.

In the interiors of stars, for example, LTE will prove to be a very good approximation, that will get progressively worse as we inch towards the "surface" of the star.

1.6 PARALLEL PLANE APPROXIMATION

One useful approximation that may be worthwhile to dedicate some of our time to is the *plane parallel atmosphere*, that will allow us to obtain notable results for describing of radiation travels through, say, the inner regions of the stellar atmosphere.

In the following, we're going to assume to be able to neglect the curvature and assume the various thermodynamic quantities to be constant over horizontal planes. Using Fig.6 as a reference, we see that

$$\mathrm{d}s = \frac{\mathrm{d}z}{\cos\theta} = \frac{\mathrm{d}z}{\mu}$$

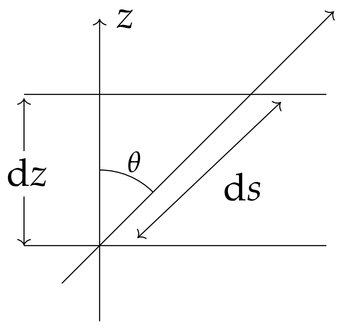

Figure 6: A ray path through a plane parallel atmosphere.

where we used the customary notation in astrophysics $\mu = \cos\theta$.

We shall consider a scattering free, stationary situation for the equation of radiative transport (10). Hence, due to planar symmetry, we expect the specific intensity to depend only on z and μ. For the sake of the current discussion, we perform a slight modification to the definition of optical depth, so that

$$d\tau_\nu = -\alpha_\nu \, dz$$

This way the equation of radiative transfer may be cast in the following form

$$\mu \partial_{\tau_\nu} I_\nu(\tau_\nu, \mu) = I_\nu - S_\nu$$

which has a formal solution easily computed

$$I_\nu \exp(-\frac{t_\nu}{\mu})|_{\tau_{\nu,0}}^{\tau_\nu} = -\int_{\tau_{\nu,0}}^{\tau_\nu} \frac{S_\nu}{\mu} e^{-\frac{t_\nu}{\mu}} \, dt_\nu \tag{20}$$

This is customarily solved considering two distinct intervals for (I) μ: $\mu \in [0,1]$ and (II) $\mu \in [-1,0]$. In case (I) we can assume the ray path to begin from a great depth inside the star, so that $\tau_{\nu,0} \to \infty$, while in case (II) we assume the ray to receive contributions beginning from with the top of the atmosphere, where $\tau_{\nu,0} \approx 0$. For case (II), we're also assuming no radiation is coming from *outside the star*[1]

Now we can assume LTE throughout the stellar atmosphere so that eq.16 is verified. The source function at some optical depth shall then be equal to $B_\nu(T(\tau_\nu))$. For the source function at a nearby optical depth we can simply compute a Taylor expansion around the optical depth τ_ν

$$S(t_\nu) = B_\nu(\tau_\nu) - (t_\nu - \tau_\nu)\frac{dB_\nu}{d\tau_\nu} + o(t_\nu^2)$$

We can use this to solve eq.(20), finding for both positive and negative values of μ a very important equation

$$I_\nu(\tau_\nu, \mu) = B_\nu(\tau_\nu) + \mu \frac{dB_\nu}{d\tau_\nu} \tag{21}$$

1 Please note that this condition may be not valid at all in close binary systems.

provided the point considered is sufficiently inside the atmosphere so that $\tau_\nu \gg 1$[2]. Using this simple result, we can compute the three momenta of the equation of transport

$$U_\nu = \frac{4\pi}{c} B_\nu(\tau_\nu) \tag{22}$$
$$F_\nu = \frac{4\pi}{3} \frac{\mathrm{d}B_\nu}{\mathrm{d}\tau_\nu} \tag{23}$$
$$P_\nu = \frac{4\pi}{3c} B_\nu(\tau_\nu) \tag{24}$$

1.6.1 The Grey Atmosphere

If we consider the absorption coefficient α_ν constant at all frequencies, then the atmosphere is called a "grey atmosphere". This implies that the value of the optical depth at some physical depth is constant for all frequencies.

Under this assumption, we could solve

$$\mu \frac{\partial I}{\partial \tau} = I - S$$

I'll skip the explicit calculation (which are actually fairly easy for once) and present just the final result. Two more assumptions are to be made, however: The first is to assume *radiative equilibrium*, which roughly translates into asking that there are not sources or sinks of energy in the atmosphere, thus $\partial_\tau F = 0$. As a further semplification, we assume the *Eddington approximation* to hold everywhere in the atmosphere, so that

$$P = \frac{1}{3} U$$

It should be evident that this last equation may be verified in presence of an isotropic source of radiation, also in its frequency-dependent form.

We then come to the following conclusion

$$I_{obs}(\tau = 0, \mu) = \frac{3F}{4\pi}\left(\mu + \frac{2}{3}\right) = \frac{S(\tau)}{(\tau + \frac{2}{3})}\left(\mu + \frac{2}{3}\right) \tag{25}$$

from which we deduce the equation for the *limb darkening*

$$\frac{I(0,\mu)}{I(0,1)} = \frac{3}{5}\left(\mu + \frac{2}{3}\right)$$

which roughly translates into saying that the radiation that we observe at the surface is the one equivalent at a source function S evaluated at $\tau = 2/3$. This is known as the *Eddington-Barbier estimation*.

A somewhat more general way to solve the problem is assuming the following functional relation for the specific intensity

$$I_\nu(\tau, \mu) = a_\nu(\tau_\nu) + b_\nu(\tau_\nu)\mu$$

2 You can see [3], §2.4.1 for more detailed calculations.

and compure the three momenta proper

$$J_\nu = \frac{1}{2}\int_{-1}^{+1} I_\nu \,\mathrm{d}\mu = a_\nu \tag{26}$$

$$H_\nu = \frac{1}{2}\int_{-1}^{+1} I_\nu \mu \,\mathrm{d}\mu = \frac{b_\nu}{3} \tag{27}$$

$$K_\nu = \frac{1}{2}\int_{-1}^{+1} I_\nu \mu^2 \,\mathrm{d}\mu = \frac{a_\nu}{3} \tag{28}$$

Now we assume a stronger version of the *Eddington approximation* $K_\nu = J_\nu/3$ so that the two following expressions can be written

$$\frac{\partial H_\nu}{\partial \tau_\nu} = J_\nu - S_\nu \tag{29}$$

$$\frac{\partial K_\nu}{\partial \tau_\nu} = H_\nu = \frac{1}{3}\frac{\partial J_\nu}{\partial \tau_\nu} \tag{30}$$

The mixing of the two gives us a second order PDE that it's still somewhat valid even in the outer regions of the stellar atmosphere.

1.7 RADIATIVE DIFFUSION APPROXIMATION

This section would probably be clearer if you take a brief detour to Chapter 3 to build some groundwork for scattering processes, but it still suits better the main subject of this chapter.

For the sake of coherence (and personal laziness) I'm going to talk about the radiative diffusion approximation right here, also because it makes use of the plane parallel approximation we just went through.

In Chapter 3 we have used random walk arguments to show that S_ν approaches B_ν at large *effective optical depths* in a homogeneous medium. Real media are seldom homogeneous, but often, as in the interiors of stars, there is a high degree of local homogeneity.

The equation of radiative transport in presence of scattering (45) may be cast in a slightly different form

$$I_\nu = S_\nu - \frac{\mu}{\alpha_\nu + \sigma_\nu}\partial_z I_\nu$$

We shall then assume that over a distance l_* (the thermalization length) I_ν is constant and at zero-th order is $I_\nu^{(0)} = S_\nu^{(0)} = B_\nu$. Plugging this in the equation of radiative transfer gives us $I_\nu^{(1)}$ by a simple iterative procedure

$$I_\nu^{(1)} = B_\nu - \frac{\mu}{\alpha_\nu + \sigma_\nu}\partial_z B_\nu \tag{31}$$

With the simple redefining $\mathrm{d}\tau_\nu = -(\alpha_\nu + \sigma_\nu)\,\mathrm{d}z$, we can put eq.(31) in a form functionally equal to (21).

Let us now compute the flux F_ν using the above form for the intensity

$$F_\nu(z) = 2\pi\int_{-1}^{+1} I_\nu^{(1)} \mu \,\mathrm{d}\mu = -\frac{4\pi}{3}\frac{\partial_z B_\nu}{\alpha_\nu + \sigma_\nu} = -\frac{4\pi}{3}\frac{\partial_T B_\nu}{\alpha_\nu + \sigma_\nu}\partial_z T$$

Recalling the result

$$\partial_T \int_0^{+\infty} B_\nu \, \mathrm{d}\nu = \frac{4\sigma_{SB} T^3}{\pi}$$

we can define a *mean absorption coefficient* using the *Rosseland approximation for radiative diffusion*

$$\frac{1}{\alpha_R} := \frac{\int_0^{+\infty} \frac{1}{\alpha_\nu + \sigma_\nu} \, \partial_T B_\nu \, \mathrm{d}\nu}{\int_0^{+\infty} \partial_T B_\nu \, \mathrm{d}\nu} \tag{32}$$

If we integrate the monochromatic flux over the frequencies and make use of the Rosseland mean, we find a useful expression often used in stellar structure models

$$F(z) = -\frac{-16\sigma_{SB} T^3}{3\alpha_R} \partial_z T \tag{33}$$

2 THE EINSTEIN COEFFICIENTS

2.1 INTRODUCTION

Kirchhoff's law (eq.16), which relates the (spontaneous) emission and the absorption coefficient, seems to imply some underlying microscopic connection between the two phenomena.

As was first discovered by Einstein, that is exactly the case. Let's consider a two level atom interacting with radiation. As depicted in Fig.7, we'll con-

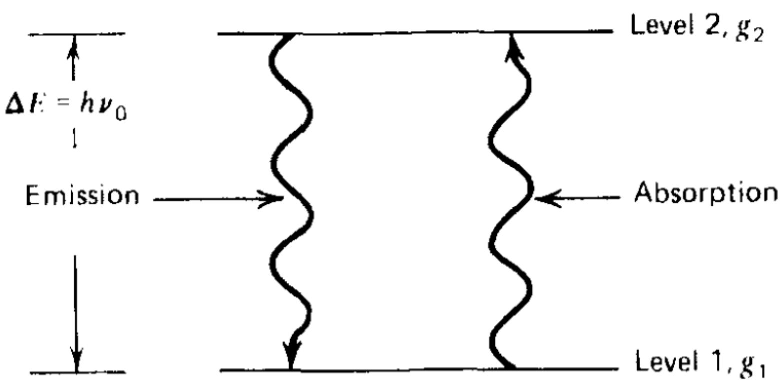

Figure 7: Photon emission and absorption in a two levels atom.
Credits: G. Rybicki, A. Lightman.

sider two discrete energy levels: the lower with energy E and degeneracy g_1, while the upper level has energy equal to $E + h\nu_0$ and degeneracy g_2. Transition between the two levels is possible only through absorption ($1 \rightarrow 2$) or emission ($2 \rightarrow 1$) of photons of energy $h\nu_0$.

Three processes can be thus indentified: *spontaneous emission*, *stimulated emission* and *absorption*.

Spontaneous Emission

The process we'll refer to as *spontaneous emission* occurs when the system transitions from the excited state 2 to the lower level 1 through the emission of a photon.

Spontaneous emission can occur even in the **absence of radiation fields** and can be assumed to be **isotropic**. The transition probability for spontaneous emission is defined through the *Einstein A-coefficient* A_{21}, which has units s^{-1}.

Absorption

In the presence of a radiation field with the right energy, the system can absorb a photon to transition from state 1 to a higher energy state. Assuming that the radiation field cannot self-interact, we expect the probability per unit time to be proportional to the density of photons (or to the mean intensity J_ν) at frequency ν_0.

However, the energy difference between the two states is not infinitely sharp, but more like a smoother curve we'll call the *line profile function* $\phi(\nu)$,

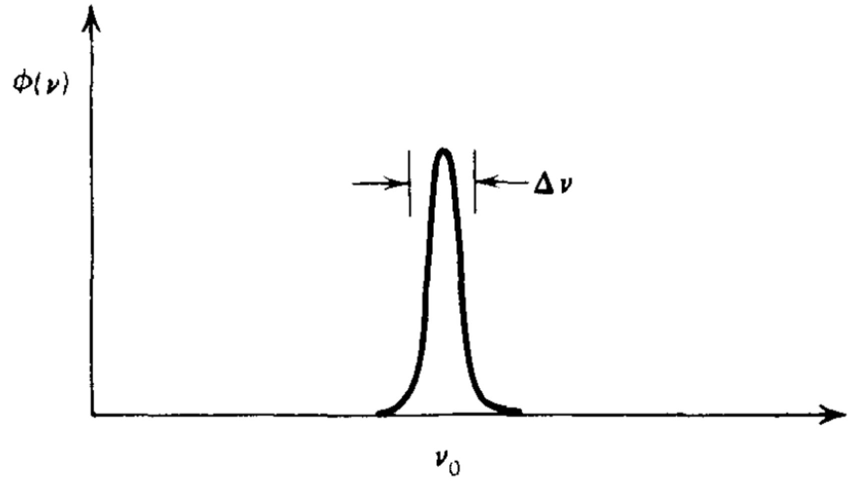

Figure 8: Line profile for a two levels atom.
Credits: G. Rybicki, A. Lightman.

which is conveniently peaked at $\nu = \nu_0$ and is correctly normalized

$$\int_0^{+\infty} \phi(\nu)\, \mathrm{d}\nu = 1$$

Not bothering for the moment with the underlying physical mechanisms that concur in determining the line profile, we'll propose the following definition for the transition probability for absorption $B_{12}\bar{J}$, where $\bar{J}$ is

$$\bar{J} = \int_0^{+\infty} J_\nu \phi(\nu)\, \mathrm{d}\nu$$

and B_{12} is the *Einstein B-coefficient*.

Stimulated Emission

As anticipated, there exists yet another way for a system to emit a photon, but this time requiring the presence of a radiation field. Taken $\bar{J}$ has the same meaning as before, we'll define the transition probability per unit time for stimulated emission as $B_{21}\bar{J}$.

If we were to assume that the mean intensity J_ν changes slowly over the width $\Delta\nu$ of the line profile, we could in principle approximate the line profile as a δ-function peaked at ν_0.

Some books use the energy density U_ν in the definitions. It's pretty much the same since they carry the same information, but be aware that the two definitions will differ of a $c/4\pi$ factor.

2.2 RELATIONS BETWEEN THE EINSTEIN COEFFICIENTS

It would be most useful if there were some kind of relations between the three different coefficients and if there were some way of relating them to what we've called α_ν and j_ν when dealing the radiative transport.

If this wasn't the case, this section would have no reason to exist, so it's safe for you to assume that such relations do in fact exist. To do so, however, we have to invoke the *quantum theorem of detailed balance* [5], which roughly says

> If states a and b of a system have the same energy, then if P_{ab} is the probability per unit time of a transition from a to b, and P_{ba} from b to a,
> $$P_{ab} = P_{ba}$$

For an atom of matter to be in equilibrium with a radiation field, then, the probability of said atom being in the ground state and absorbing a photon of a given frequency must be equal to the probability that it is in the excited state and emits the photon.

If we assume to be in a steady-state condition, the rate of transition from level 1 to level 2 has to be equal to the rate of transition from level 2 to level 1, or, more generally

$$n_j \sum_j R_{ij} - \sum_j n_j R_{ji} = 0$$

If we put in the transition rates we've written down earlier, we get

$$n_1 B_{12} \bar{J} = n_2 A_{21} + n_2 B_{21} \bar{J}$$

which means that the number of transitions per unit time per unit volume out of state 1 must be equal to the number of transitions per unit time per unit volume into state 1.

Solving for $\bar{J}$

$$\bar{J} = \frac{n_2 A_{21}}{n_1 B_{12} - n_2 B_{21}} = \frac{A_{21}/B_{21}}{(n_1/n_2)\cdot(B_{12}/B_{21}) - 1} \tag{34}$$

In thermodynamic equilibrium we can use eq.18 to relate the occupation numbers

$$\frac{n_1}{n_2} = \frac{g_1}{g_2}\frac{\exp\left(-E/kT\right)}{\exp\left[-(E+h\nu_0)/kT\right]} = \frac{g_1}{g_2}\exp\left(-h\nu_0/kT\right)$$

but in thermodynamic equilibrium we know that $J_\nu = B_\nu$, so if B_ν varies slowly on the scale of $\Delta\nu$ we can assume $\bar{J} \approx B_\nu(\nu_0)$.

That means the following relations must simultaneously hold

$$g_1 B_{12} = g_2 B_{21} \tag{35}$$

$$A_{21} = \frac{2h\nu^3}{c^2} B_{21} \tag{36}$$

which are known as *Einstein relations* and connect atomic properties A_{21}, B_{21} and B_{12} and have no reference to the temperature T (unlike Kirchhoff's law).

Although we have derived eq.36 assuming thermodynamic equilibrium, those two relations **must be always valid** whether or not the atoms are in thermodynamic equilibrium.

2.2.1 Absorption and Emission coefficients in terms of Einstein coefficients

To obtain the emission coefficient j_ν we have to make a crucial assumption about the frequency distribution of the emitted radiation: This emission is distributed with the same line profile $\phi(\nu)$ that describes absorption, which is often verified in astrophysics (good for us).

By the definition (8), we already know the amount of energy emitted in volume $\mathrm{d}V$, solid angle $\mathrm{d}\Omega$, frequency $\mathrm{d}\nu$, and time $\mathrm{d}t$. Now, since each atom contributes with $h\nu_0$ to spontaneous emission over a 4π solid angle, we can express the emission coefficient as

$$j_\nu = \frac{h\nu_0}{4\pi} n_2 A_{21} \phi(\nu) \tag{37}$$

Similarly, you can show that the energy absorbed from radiation in frequency range $\mathrm{d}\nu$, solid angle $\mathrm{d}\Omega$, time $\mathrm{d}t$ and volume $\mathrm{d}V$ is

$$\frac{h\nu_0}{4\pi} n_1 B_{12} I_\nu \phi(\nu)\, \mathrm{d}\nu\, \mathrm{d}\Omega\, \mathrm{d}t\, \mathrm{d}V$$

From here follows immediately the *true absorption coefficient*

$$\alpha_\nu = \frac{h\nu_0}{4\pi} n_1 B_{12} \phi(\nu) \tag{38}$$

Note that since stimulated emission is proportional to the specific intensity I_ν and only affects the photons along the given beam, pretty much like true absorption. We shall then define the *absorption coefficient, corrected for stimulated emission* as

$$\alpha_\nu = \frac{h\nu_0}{4\pi} \phi(\nu)(n_1 B_{12} - n_2 B_{21}) \tag{39}$$

in which we're regarding stimulated emission kinda like a *negative absorption*.

In terms of the newly defined emission and absorption coefficients, the equation of radiative transfer takes a new look

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = -\frac{h\nu_0}{4\pi}\phi(\nu)(n_1 B_{12} - n_2 B_{21}) + \frac{h\nu_0}{4\pi} n_2 A_{21} \phi(\nu) \tag{40}$$

Computing the ratio j_ν/α_ν with the new definitions holds

$$S_\nu = \frac{n_2 A_{21}}{n_1 B_{12} - n_2 B_{21}} = \frac{2h\nu^3}{c^2}\left(\frac{g_2 n_1}{g_1 n_2} - 1\right)^{-1} \tag{41}$$

where the last equality is obtained just by substitution of the Einstein relations (36). Note that eq.(41) is a generalized Kirchhoff's law. Let's consider three interesting cases sprouting from this last equality.

Thermal Emission (LTE)

If the matter is in local thermal equilibrium (LTE) with itself (but not necessarily with the radiation), eq.18 must hold locally. In this case, we correctly retrieve

$$S_\nu = B_\nu$$

Nonthermal Emission

We can't assume eq.18 to hold, and particles do not obey the Maxwellian distribution or any of the fancy properties we'd like them to.

Inverted Populations: MASERS

Let's consider the definition of the absorption coefficient corrected for stimulated emission eq.39. Whether the coefficient is positive or negative depends on the term in parentheses, which can be restated using Einstein's relations

$$\frac{n_1 g_2}{n_2 g_1} - 1 > 0$$

so that when this relation is satisfied, even out of thermal equilibrium, we say that the system has *normal populations*. However, it is possible to put enough atoms in the upper state so that we achieve what is called a *inverted population*

$$\frac{n_1}{g_1} < \frac{n_2}{g_2}$$

which is quite the odd configuration, if you think about it. We're basically saying that a higher energy level is more densely populated than a lower energy one, which, by means of eq.18, corresponds to stating that the temperature T is negative, since E is assumed to be non-negative.

And no, T is not in Celsius degrees.

In such a scenario, the absorption coefficient is negative, so, rather than being damped to extintion, the intensity of the radiation ray *increases* when passing through matter. Such a system is called MASER (Microwave Amplification by Stimulated Emission of Radiation). For visible light you'd get what we usually call "laser".

The fun part here is that we actually *observe* this phenomenon in Nature, like for example in some molecular clouds formed in the ISM. Some sources in specific molecular lines (like the OH lines) were found to be *abnormally high*. Let me quantify how much "abnormal" we're talking about.

Your typical molecular cloud has a density a bit shy of 10^9 particles per m^3 and temperature in the range of $10 - 30\,K$, so kinda chilly. Cool.

If those "abnormal" sources were assumed to be optically thick in the spectral lines and the specific intensity was equated to B_ν, you'd probably roll down you chair reading on your computer that those sources should have temperatures as high as $10^9\,K$.

Not so chilly anymore, huh?

The favored explaination for this weird phenomenon (and most likely the correct one) makes use exacly of what we've just shown: Maser action.

If you want to read something (slightly) more rigorous, you can look at [3], §6.6.3.

2.3 HYDROGENOID ATOMS

Fairly interesting objects to study are *hydrogenoid atoms*, that is atoms with Z protons but that had ended up (no matter how) with just one e^-. For this type of atoms, Quantum Mechanics predicts energy levels distributed as such

$$E_n = -\frac{Z^2\,\mathrm{Ry}}{n^2} \quad \mathrm{Ry} = \frac{m_e e^4}{2\hbar^2} \approx 13.6\,\mathrm{eV}$$

If we were to introduce relativistic corrections through perturbation theory, the expression would get a bit more complicated

$$E(n,l) = -\frac{Z^2\,\mathrm{Ry}}{n^2}\left(1 + \frac{Z^2\alpha^2}{n^2}\left[\frac{n}{j+1/2} - 3/4\right]\right)$$

which introduces a *fine splitting* of the energy levels, which now depend on the value of the angular momentum. It is probably worth pointing out that the energy levels are still degenerate in the magnetic quantum number m. Given a problem with spherical symmetry, it would be beneath us to claim that a scalar perturbation[1] could be able to completely solve the degeneracy. Degeneracy is solved completely only if we introduce in the system a preferred direction[2].

Depending on the value of the orbital angular momentum eigenvalue l we can assign a name to distinguish them:

- $l = 0 \rightarrow s$-orbitals
- $l = 1 \rightarrow p$-orbitals
- $l = 2 \rightarrow d$-orbitals
- $l = 3 \rightarrow f$-orbitals

This way, transitions may be represented in a *Grotrian diagram* (Fig.9), where the energy of a given level is plotted in function of the l eigenvalue, in which selection rules are made most evident. What happens if we are to consider atoms with more than one electron? Does our model still work? Nope, not a chance.

On top of getting stupidly difficult to even evaluate the non relativistic limit, there are *many more ways* to couple interactions, resulting in more and more sources of opacity. It is actually elements with $Z > 1$ and $\#e^- > 1$ that make the greatest contributions to opacity, especially *metals* ($Z \geqslant 3$).

Sometimes it is useful to express the electron configuration by means of the *Russel-Saunders notation*. Said S, L, J respectively the spin, the orbital angular momentum and the total angular momentum eigenvalues, we can express a given electron configuration in such a fashion

$$^{2S+1}L_J$$

where L is usually written in spectroscopic notation, that is the symbol associated to the orbital with that L value.

1 The system is still invariant under rotations.
2 cfr. *Stark-Lo Surdo Effect* and the *Zeeman effect*.

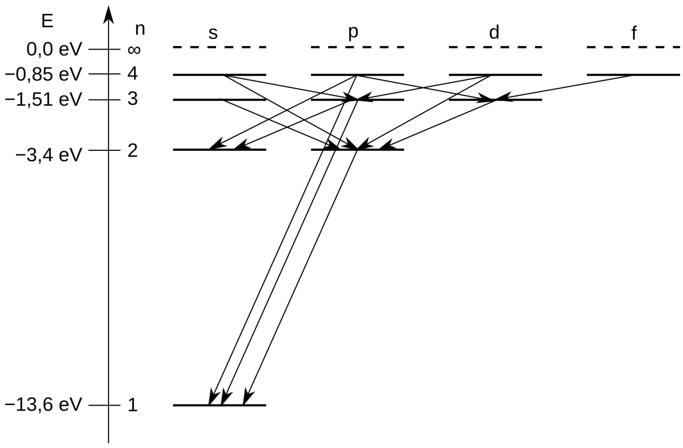

Figure 9: Example of Grotrian Diagram for Hydrogen. Credits: Wikipedia.

2.3.1 Hyperfine Transition

What we'll be most interested in considering, however, is the hyperfine splitting originating if we consider the coupling between the spin of the nucleus and the electron.

If we were to consider such interaction, we'd find a hyperfine splitting of the ground state. We'll be interested in the energy of the radiation coming from an electron spin flip: The system will transition from a state with total spin $F = 1$ to a one with $F = 0$, resulting in the emission of radiation. Ignoring higher order corrections to the Hamiltonian, the radiation resulting

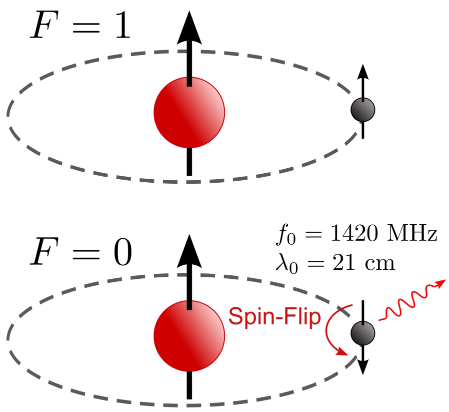

Figure 10: Schematic representation of the spin flip and the resulting emission. Credits: Wikipedia.

from the spin flip will have a wavelength

$$\lambda_{hf} = \frac{3\pi\hbar^5 m_p c^3}{g_e g_p m_e^2 e^8} \approx 21.106114054160(30)\,\text{cm} \tag{42}$$

where the expression is in CGS units and g_e, g_p are respectively the electron and the proton spin g-factors (≈ 2 for the electron, ≈ 5.59 for the proton).

Unfortunately, such a transition cannot be observed in lab-experiments. The extimated half-life time of the transition from the Einstein A coefficient[3] is of the order of a few Myrs. Which means that unless you have *a lot* of time to spare to fight against death, you'll probably never experience even a single one of these transitions.

This, however, has a strikingly beautiful consequence: Although "rare", we should be able to detect the 21-cm emission line if we were to look somewhere where the transition may have happened millions of years ago, namely somewhere in the sky.

That is exactly the case. If you were to steal a radiotelescope[4] and point it towards the Galactic plane, you'll see a really sharp line in proximity to the 21-cm line ($\sim 1420\,\mathrm{MHz}$).

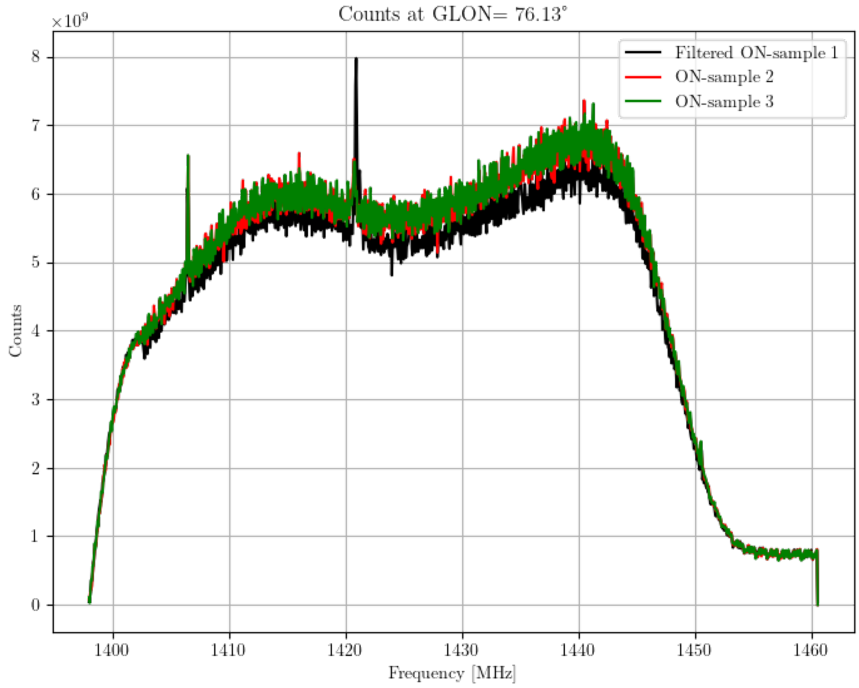

Figure 11: Sampling of the 21-cm emission line with the new Spider500 Radio telescope of the Physics Department of the University of Pisa.
Credits: AMLab, Group 8, a.y. 2025/2026.

2.4 LINE BROADENING

It would be most dumb for us to assume that energy levels, or the lines connecting them, are infinitely sharp. Even a quick look at Heisenberg's uncertainty principle should convince you otherwise.

When we defined the Einstein's coefficients, we had to introduce the line profile $\phi(\nu)$ to account for the non-zero width of the line, and it's about time we consider some of the phenomena that concur in determining the line's actual shape.

3 Remember that A has the dimensions of the inverse of a time, so $\tau_{HL} \approx A^{-1}$.
4 Do not recommend. They get awfully big if you want to see something cool.

2.4.1 Natural Broadening

For this one, you'll have to recall some of the basics results of Quantum Mechanics. The calculation is not particularly difficult, but it's surely long, so I'm going to omit that and take a simpler route.

Note that the spontaneous decay of an atomic state n proceeds at a rate

$$\Gamma = \sum_m A_{nm}$$

where the sum is over all states m of lower energy (which may be a lot).If radiation is present, we should add the induced rates to this. The coefficient of the wave function of state n, therefore, is of the form $e^{-\Gamma t/2}$ and leads to a decay of the electric field by the same factor. We have an emitted spectrum determined by the decaying sinusoid type of electric field, so the line profile must be a *Lorentz or Natural profile*

$$\phi(\nu) = \frac{\Gamma}{4\pi^2} \frac{1}{(\nu - \nu_0)^2 + \left(\frac{\Gamma}{4\pi}\right)^2}$$

More often than not, however, we're going to assume that the effect of natural broadening is rather peaked around ν_0, similar to a δ-function, since, at least as far as astrophysics is concerned, there are more relevant sources of broadening.

2.4.2 Doppler Broadening

An atom is in thermal motion, so that the frequency of emission or absorption in its own frame corresponds to a different frequency for an observer. Each atom has its own Doppler shift, so that the net effect is to spread the line out, but not to change its total strength.

Recall the classic Doppler effect

$$\nu - \nu_0 = \frac{\nu_0 v_z}{c} \tag{43}$$

Here ν_0 is the rest frequency. Note that this can be extended to *bulk motion* as well. We can decompose the velocity as

$$\vec{v} = \langle \vec{v} \rangle + \delta\vec{v}$$

which is often called *Reynolds' decomposition* in Fluid-dynamics books when dealing with turbulence. Generally, an ensamble of atoms will have different $\delta\vec{v}$, so each atom will absorb photons at different frequencies.

What we have to do is then convolute the Doppler effect over some velocity distribution. Assuming LTE to hold, such distribution will be the Maxwellian distribution. Switching into a reference frame where $\langle \vec{v} \rangle = 0$ (which is always possible), then the small fluctuations $\delta\vec{v}$ shall follow the Maxwellian distribution.

Please note that the one dimensional version of the Maxwellian distribution must be used, which is a Gaussian distribution with $\mu = 0$ and a given variance.

The convoluted line profile will then be

$$\phi(\nu) = \int \delta(\nu - \nu_0)\nu_0 \left(1 + \frac{v}{c}\right) \exp\left(-\frac{m_a v^2}{2kT}\right) \mathrm{d}v$$

where, as anticipated, we identified the natural broadening as a δ-function. This is easily computed

$$\phi(\nu) = \frac{1}{\Delta\nu_D\sqrt{\pi}} e^{-(\nu-\nu_0)^2/(\Delta\nu_D)^2} \qquad \Delta\nu_D = \frac{\nu_0}{c}\sqrt{\frac{2kT}{m_a}} \tag{44}$$

Here $\Delta\nu_D$ represents the *Doppler width*.

Generally speaking, the minimum broadening we can expect is given by a convolution of Doppler Broadening and Natural Broadening in its proper Lorentzian form. This hasn't a fancy analytical solution, but it's rather expressed from what is called a *Voigt profile*.

Roughly speaking, the Voigt profile is something that looks like a Gaussian at the center of the line[5] (where Doppler effect dominates) but has the *wings* of the Lorentzian distribution.

$$H(a, u) = \frac{a}{\pi} \int_{-\infty}^{+\infty} \frac{e^{-y^2}}{a^2 + (u - y)^2} \mathrm{d}y \quad a = \frac{\Gamma}{4\pi\Delta\nu_D} \quad u = \frac{\nu - \nu_0}{\Delta\nu_D}$$

In terms of this compact definition of the Voigt profile, the overall line profile will be

$$\phi(\nu) = \frac{1}{\Delta\nu_D\sqrt{\pi}} H(a, u)$$

5 Sometimes it's called *kernel* of the distibution.

3 SCATTERING PROCESSES

3.1 TRANSPORT THROUGH SCATTERING

When we've first written down the equation for radiative transport (eq.10) we've neglected the effects of a possibly much relevant source of photons, which is *scattering*, another fairly common emission process. Scattering depends completely on the amount of radiation falling on the spacetime element we're considering.

How do we include it into the picture of radiative transport?

For the present discussion we assume *isotropic* scattering, so that the scattered radiation is emitted equally into equal solid angles. We also assume that the total amount of radiation emitted per unit frequency range is just equal to the total amount absorbed in that same frequency range (*coherent scattering*).

This are some fair requirements if you think about it. Isotropy is most likely assured if the main type of scattering is the (non-relativistic) Thomson scattering on electrons, while coherence of the scattering is just requiring the scattering to be elastic.

Such conditions are not always met, but for the moment we'll turn our heads away from the obviou problem and see where this brings us.

The emission coefficient for coherent, isotropic scattering can be found simply by equating the power absorbed per unit volume and frequency ranges to the corresponding power emitted. Here we define the *scattering coefficient* σ_ν so that

$$j_\nu = \sigma_\nu J_\nu$$

Dividing by the scattering coefficient, we find that the source function for pure scattering is simply equal to the mean intensity within the emitting material

$$S_\nu = J_\nu = \frac{1}{4\pi} \int_\Omega I_\nu \, \mathrm{d}\Omega$$

The equation of radiative transport is thus modified[1]

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = -\sigma_\nu (I_\nu - J_\nu) = -\sigma_\nu (I_\nu - \frac{1}{4\pi} \int_\Omega I_\nu \, \mathrm{d}\Omega) \tag{45}$$

Solving this equation is a bloodbath. The source function is not known a priori and depends on the solution I_ν, at all directions through a given point, making this a *integro-differential equation*, also known as "bloody mess".

1 Note that we're assuming for the scattering version of RT a similar form of the absorption version (10).

3.2 RANDOM WALKS

A particularly useful way of looking at scattering, which leads to important order-of-magnitude estimates, is by means of random walks. We shall now develope a formalism that interprets the processes of absorption, emission, and propagation in probabilistic terms for a single photon rather than the average behavior of large numbers of photons.

Consider a photon emitted in an infinite, homogeneous scattering region. It travels a displacement $\vec{r}_1$ before being scattered, then travels in a new direction over a displacement $\vec{r}_2$ before being scattered, and so on. After N free paths is, the total displacement will look like

$$\vec{R} = \sum_{i=1}^{N} \vec{r}_i$$

Since this is a vector, the average total displacement will be identically null. Therefore, we must evaluate the mean square of the total displacement which, in the assumption of independent and isotropic scattering, must be expressed as

$$l_*^2 = \langle \vec{R}^2 \rangle = \sum_{i=1}^{N} \langle \vec{r}_i^2 \rangle$$

The quantity l_*^2 is the root mean square net displacement of the photon, while l^2 denotes the mean square of the free path of a photon, which within a factor of order unity, it's simply the mean free path of a photon.

Since the mean square displacements of each scattering iteration have no reason to be different, we'll just write

$$l_* = \sqrt{N} l$$

Said L the linear dimension of the object photons are trying to "escape" from, for regions of large optical depth the number of scatterings required to actually escape is roughly determined by setting $l_* \approx L$. Then $N \sim L^2/l^2$. But since the optical thickness of the medium τ is of order L/l, our previous results yield

$$\tau^2 \approx N \quad \tau \gg 1$$

For regions of small optical thickness, the mean number of scatterings is small, so that we can approximate $N \approx \tau$. For most order-of-magnitude estimates we could then use

$$N \approx \tau^2 + \tau$$

which has the evident perk of well-behaving in the two limits we've discussed.

3.2.1 Combined scattering and absorption

What happens if we had both scattering and absorption? We'd have two terms on the right hand side of the tranfer equation

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = -\alpha_\nu (I_\nu - B_\nu) - \sigma_\nu (I_\nu - \frac{1}{4\pi} \int_\Omega I_\nu \, \mathrm{d}\Omega) \tag{46}$$

Please note that we're assuming Kirchhoff's law to hold, and the medium to be optically thick, so that $j_\nu = \alpha_\nu B_\nu$. In terms of a suitably defined source function S_ν

$$S_\nu = \frac{\alpha_\nu B_\nu + \sigma_\nu J_\nu}{\alpha_\nu + \sigma_\nu}$$

we can rewrite eq.46 as

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = -(\alpha_\nu + \sigma_\nu)(I_\nu - S_\nu)$$

We may then define an *effective optical depth*[2] τ_* so that

$$\mathrm{d}\tau_* = (\alpha_\nu + \sigma_\nu)\,\mathrm{d}s$$

so that the mean free path will be just $l_\nu = (\alpha_\nu + \sigma_\nu)^{-1}$.

In our random walk approximation, the probability of a scattering process to end in a true absorption will be

$$\epsilon_\nu = \frac{\alpha_\nu}{\alpha_\nu + \sigma_\nu}$$

The probability of for scattering will then be $1 - \epsilon_\nu$, which is known as *single-scattering albedo*. In terms of ϵ_ν, the source function becomes

$$S_\nu = (1 - \epsilon_\nu) J_\nu + \epsilon_\nu B_\nu$$

Let us consider first an infinite homogeneous medium. A random walk starts with the thermal emission of a photon (creation) and ends, possibly after a number of scatterings, with a true absorption (destruction). Since the walk can be terminated with probability ϵ at the end of each free path, the mean number of free paths is $N = \epsilon^{-1}$. We then have $l_* = l\epsilon^{-1/2}$.

So if we were to substitute in the definition of l_ν, we'd get

$$l_* \approx [\alpha_\nu(\alpha_\nu + \sigma_\nu)]^{-1/2}$$

This length represents a measure of the net displacement between the points of creation and destruction of a typical photon; it often goes by the name of *diffusion length*, *thermalization length*, or *effective mean path*. Usually it is also frequency dependent.

Essentially, it is the average length over which emitting and absorbing elements are radiatevely coupled.

The behavior of a finite medium can be explained in terms of what we've been creating so far. Its properties will depend (strongly) on whether its linear extension L is larger or smaller than the effective free path.

In terms of the effective optical depth $\tau_* = L/l_*$, we can restate its definition as follows

$$\tau_* \approx [\tau_a(\tau_a + \tau_s)]^{1/2} \quad \tau_a = \alpha_\nu L \quad \tau_s = \sigma_\nu L$$

When the effective free path is large compared to L, we have $\tau_* \ll 1$ and the medium is *effectively thin*. Most photons will then escape by random walking their way out of the medium before being destroyed by a true absorption.

2 Also called *extinction coefficient*.

Conversely, a medium for which $\tau_* \gg 1$ is *effectively thick*. Most photons thermally emitted at depths larger than the effective path length will be destroyed by absorption before they get out.

This new definition allows us to reformulate when for a medium will be possible to be in LTE: Over a distance of order l_*, photons will do many scattering events, unable to leave the medium if $\tau_* \gg 1$, assuring that LTE is established *locally*.

Were this to happen, we could safely claim then that

$$I_\nu \to B_\nu \quad S_\nu \to B_\nu$$

This way is perhaps more clear why l_* is called thermalization length.

Part II

FLUID DYNAMICS

4 FUNDAMENTALS OF FLUID DYNAMICS

4.1 PHYSICAL PROPERTIES OF FLUIDS

A "simple" fluid might be defined as a material such that the relative positions of the elements that make it change by an amount which is not small when suitably chosen forces are applied.

Given this definition, we could be tempted to say there's not much difference between gases and liquids, but that wouldn't be a clever guess. The main difference between gases and liquids lays not in density, but in *compressibility*.

To build a coherent and "nice" description of a fluid, we have to be able to attach a definite meaning to the notion of value "at a point" for the various fluid properties. What we'd like is, essentially, to be able to treat a fluid like a continuum.

We know that in the real world an abstract concept like a continuum cannot possibly exists: Matter is discrete and discrete alone. However, it may have occurred to you, that normally we don't exactly see matter broken down to its fundamental components. It looks rather smooth and continuous to me, not at all discrete.

We are able to regard a fluid as a continuum when the measured fluid property is constant for sensitive volumes small on the macroscopic scale but large in respect to the microscopic ones.

When looking at a fluid, we can fairly distinguish two kinds of forces that act on matter in bulk

- long-range forces or *body forces* (like gravity): $\delta\vec{F}_v = \vec{F}(\vec{x},t)\rho\delta V$
- short-range forces that arise from reactions with matter

The latter can be expressed as: $\delta\vec{F}_s = \vec{\sigma}(\hat{n},\vec{x},t)\delta A$, where $\vec{\sigma}$ is the stress exerted by the fluid on the surface element to which $\hat{n}$ points.

The force exerted across the surface element on the fluid on the side to which $\hat{n}$ points is such that

$$-\vec{\sigma}(\hat{n},\vec{x},t)\delta A = \vec{\sigma}(-\hat{n},\vec{x},t)\delta A$$

The classical procedure to obtain a functional expression for $\vec{\sigma}$ comes from considering all the forces acting instantaneously on the fluid within a δV volume in the shape of a tetrahedron (Fig.12). The calculation is rather tedious but at least straightforward. At the end of it, you'll find out that the i-th component of the stress vector can be written as

$$\sigma_i = \sigma^{ij} n_j$$

where the index placement is quite the overkill, but we'll be clearer in a moment. Here and in the following, we'll be often adopting Einstein convention for summation over repeated indices.

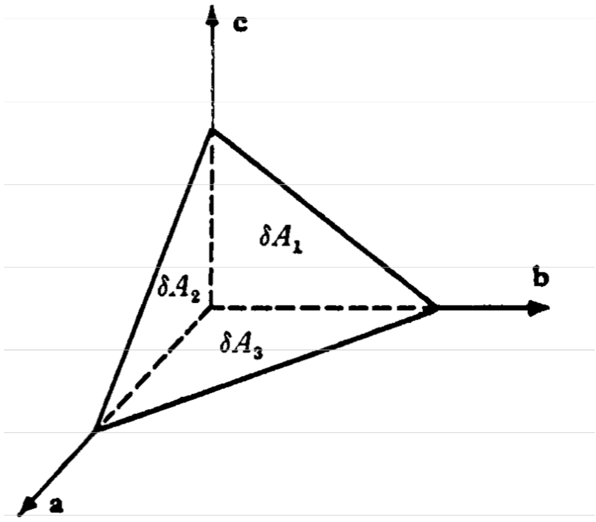

Figure 12: Tetrahedron construction for the stress tensor. Credits: Batchelor [1]

It is customary to bestow a name on σ^{ij}: the *stress tensor*. Curiously enough, the conservation of total angular momentum implies that the the stress tensor must be *symmetric*. This rather nice properties should possibly be ringing a bell.

If we put ourselves in the fluid's rest frame, where there's no net flux of any of the components of momentum in any of the three orthogonal directions, it's not a far stretch to be claiming that the fluid is indeed isotropic at rest.

Isotropy implies that, at least in this reference frame, the stress tensor must be diagonal. Since the symmetry of the system is essentially spherical, all the spatial components σ^{ii} must therefore be equal. The most general form fot the 4-dimensional stress tensor is simply the energy-momentum tensor for a perfect fluid (as far as special relativity is concerned)

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + p\eta^{\mu\nu} \tag{47}$$

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

Note that we'll be using the same metric convention of [2], which is $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ and will work in units $c = 1$[1]. The 4-vector u^μ is the 4-velocity $u^\mu = \mathrm{d}x^\mu/\mathrm{d}s$. Please note that requiring the energy-momentum tensor to be locally conserved $\partial_\mu T^{\mu\nu} = 0$ would allow us to deduce the three fundamental equations of fluid dynamics: The mass conservation, the energy conservation and Euler's equation. We will, however, get to those equation taking another route, passing briefly from statistical mechanics. For those interested, you can see [2], §1.9.

1 To properly reintroduce c factors, you have to multiply ρ by c^2 and then look carefully at the definition of u^μ.

The stress tensor we've been discussing so far is the solely spatial component of $T^{\mu\nu}$, which is $\sigma^{ij} = T^{ij} = p\delta^i_j$. Since the stress tensor is isotropic, all its diagonal elements are equal and then we may just write

$$p = -\frac{1}{3}\sigma^i_i$$

as a definition of *static fluid pressure*.

4.1.1 Mechanical equilibrium

A necessary condition for equilibrium of a fluid requires that body and surface forces compensate

$$\int_V \rho\vec{F}\,\mathrm{d}V - \int_{\partial V} p\hat{n}\,\mathrm{d}A = 0$$

Applying the divergence theorem and requiring that what we find holds for all possible volumes V, last equation yields

$$\rho\vec{F} = \nabla p \tag{48}$$

If $\vec{F}$ is a conservative force $\vec{F} = -\nabla\phi$, then the condition for mechanical equilibrium may be cast in the equivalent form by taking the curl of both members

$$\nabla\rho \wedge \nabla\phi = 0 \implies \frac{\mathrm{d}p}{\mathrm{d}\phi} = -\rho(\phi)$$

So we find out that equilibrium in a fluid is possible if isochoric and equipotential surfaces are aligned.

For example, for a self-gravitating medium, the following equation holds (in special relativity at least)

$$\nabla^2\phi = 4\pi G\rho \tag{49}$$

which can be suitably rearranged recalling $\nabla\phi = -\nabla p/\rho$ into (we assume spherical symmetry)

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{\rho}\frac{\mathrm{d}p}{\mathrm{d}r}\right) = -4\pi G\rho \tag{50}$$

which is of crucial importance for stellar structure models. Note that this is a prime example of the *closure problem*: To be properly solved (either numerically or analitically) we need to specify an equation of state of some kind.

Lane-Emden equation

Assume for example a barotropic relation of the form

$$p(\rho) = \kappa\rho^{1+1/n} \tag{51}$$

$$\rho(r) = \rho_c\phi^n(r) \tag{52}$$

with ϕ some adimensional density profile with the following characteristics: $\phi(0) = 1$ and $\partial_r\phi|_{r=0} = 0$.

In terms of an adimensional radius $r = a\eta$, with a constant depending only on n

$$a = \left[\frac{K\rho_c^{1/n-1}(n+1)}{4\pi G}\right]^{1/2}$$

eq.50 takes a much simpler and elegant form

$$\frac{1}{\eta^2}\partial_\eta(\eta^2\partial_\eta\phi) = -\phi^n \tag{53}$$

which is the general *Lane-Emden equation*. Solutions for this equation are calculated numerically in Fig.13. Albeit not a topic covered by the course,

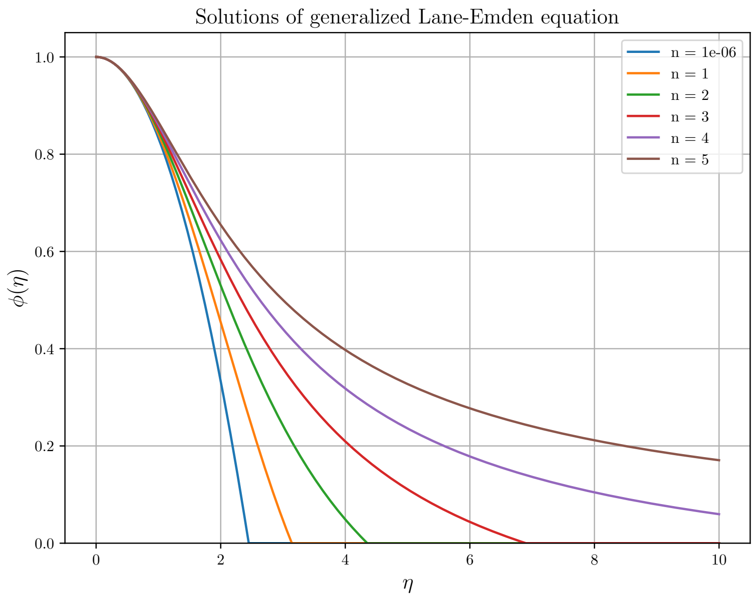

Figure 13: Numerical solutions to the Lane-Emden equation.

I'd still like to point out that from eq.53 it's possible to find the fixed value of mass able to satisfy the equation in the ultrarelativistic limit of degenerate electrons ($n = 3$). The mass is given by

$$M = \int_0^R 4\pi r^2\rho\,\mathrm{d}r = 4\pi a^2\rho_c\int_0^{\eta_1}\eta^2\phi^2\,\mathrm{d}\eta$$

Note that in the equation above, the integral will evaluate to the same result for all stars with a given n. This is the first time we encounter a *self-similar solution*, but we'll have time to get acquaitanted with those in future chapters.

The value of that integral is actually fixed by the original equation itself to $-\eta_1^2\partial_\eta\phi|_{\eta_1}$.

If we consider now a specific value of n, namely $n = 3$, we'd find out that the mass of the star does not depend in any way from the centeral density of the star. That means that in such a regime, only one fixed value of mass is allowed.

Numerically we find that for $n = 3$, $|\eta_1^2\partial_\eta\phi|(\eta_1)| \approx 2.018$, so that the limiting mass is of the order of

$$M_{Ch} \approx 1.46\left(\frac{2}{\mu_e}\right)M_\odot \tag{54}$$

This is the celebrated *Chandrasekhar mass limit* that constrains white dwarves from having larger masses.

Spherically symmetric rotating objects

Another most interesting example is what we observe if we consider a rotating spherical fluid with radius R and angular velocity $\vec{\Omega}$. In the comoving frame, there are obviously fictious forces arising to account for the rotation: *Coriolis' force* $\vec{\Omega} \wedge \vec{v}$ and *Centrifugal acceleration* $\vec{\Omega} \wedge (\vec{\Omega} \wedge \vec{r})$. For the time being, we'll neglect the former.

Centrifugal acceleration can be expressed by the gradient of a scalar potential

$$\vec{\Omega} \wedge (\vec{\Omega} \wedge \vec{r}) = -\frac{1}{2}\nabla(\vec{\Omega}^2 \vec{r}^2) = -\frac{1}{2}\nabla\phi_C \tag{55}$$

This means that the effective potential is no longer that that granted spherical symmetry (say, the usual Newtonian gravitational potential). The equation for mechanical equilibrium will have to be modified into

$$-\rho\nabla(\phi - \phi_C) = \nabla p$$

which in polar coordinates reads something like

$$-\rho\partial_r \left[\phi - \Omega^2 r^2\right] = \partial_r p \tag{56}$$

$$-\rho\partial_z\phi = \partial_z p \tag{57}$$

In conjunction with eq.49 we could, at least theoretically, study the deviation from spherical symmetry, given that we're provided an equation of state, that is. But in general it's safe to assume that rotation **affects the structure of a star**. Since the star is no longer spherically symmetric, we'll no longer

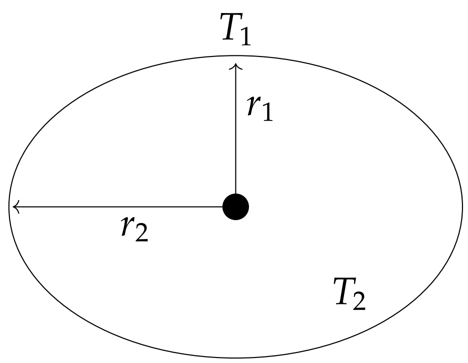

Figure 14: The effect of rotation on a spherically symmetric object. Symmetry is broken and gradients arise.

see the same surface temperature everywhere, but it will change from point to point, giving rise to temperature and pressure gradients.

Now, if we are in presence of a temperature gradient, a sensible call would be remembering the existence of *thermal conduction*, which is temperature-gradients driven. This implies the presence of blobs of hot gas (or whatever) moving towards the cooler regions (Fig.14). At this point, we'll no longer be able to neglect the Coriolis force, which will further twist the surface of the star.

It's clear that hydrostatic equilibrium is not sustainable in these conditions.

4.2 VLASOV EQUATION

What we'd like to do in the present section is trying to build a formalism that allows us to connect the microscopical interpretation of matter through statistical mechanics to the macroscopic space.

To do so, we'll define a distribution function $f = f(\vec{x}, \vec{v}, t)$ so that the number of particles in range $[\vec{x}, \vec{x} + \mathrm{d}\vec{x}]$ and $[\vec{v}, \vec{v} + \mathrm{d}\vec{v}]$ is given by

$$\mathrm{d}n(t) = f(\vec{x}, \vec{v}, t)\, d\vec{x}\, \mathrm{d}\vec{v} \tag{58}$$

To be properly defined, f must meet the usual requirements for distribution functions

$$f \geqslant 0 \qquad \int f\, \mathrm{d}\vec{x} < +\infty \qquad \int f\, \mathrm{d}\vec{v} < +\infty \tag{59}$$

Thanks to this properties we can easily define the number and mass density

$$n(\vec{x}, t) = \int f(\vec{x}, \vec{v}, t)\, \mathrm{d}\vec{v} \implies \rho(\vec{x}, t) = A m_H n(\vec{x}, t)$$

as well as the concept of *averaged quantities*, for example the average velocity

$$\langle \vec{v} \rangle = \frac{1}{N} \int \vec{u} f(\vec{x}, \vec{u}, t)\, \mathrm{d}\vec{u}$$

This expression is most useful to decompose the velocity as $\vec{v} = \langle \vec{v} \rangle + \delta\vec{v}$, which is known as Reynolds' decomposition. This decomposition implies $\langle \delta\vec{v} \rangle = 0$.

To build the equations of fluid dynamics, we need to ask ourselves what is it of the distribution function f if we let the system evolve in time?

Liouville's theorem tells us that the six dimensional volume in phase space is conserved $\mathrm{d}^3x\mathrm{d}^3v \approx \mathrm{d}^3x_0\mathrm{d}^3v_0$ up to first order in $\mathrm{d}t$[2]. This way we can write the initial infinitesimal number of particles in volume $\mathrm{d}^3x_0\mathrm{d}^3v_0$ and the number of particles in volume $\mathrm{d}^3x\mathrm{d}^3v$

$$\begin{aligned} \mathrm{d}N_0 &= f(\vec{x}_0, \vec{v}_0, t)\, \mathrm{d}^3x_0\mathrm{d}^3v_0 \\ \mathrm{d}N &= f(\vec{x}, \vec{v}, t)\, \mathrm{d}^3x\mathrm{d}^3v \end{aligned}$$

Requesting the number of particles to be conserved $\mathrm{d}N_0 = \mathrm{d}N$, we end up with

$$f(\vec{x}_0, \vec{v}_0, t) = f(\vec{x}_0 + \vec{u}_0\, \mathrm{d}t, \vec{v}_0 + \vec{a}_0\, \mathrm{d}t, t) \approx f(\vec{x}_0, \vec{v}_0, t) + \frac{\mathrm{d}f}{\mathrm{d}t}$$

which implies $\mathrm{d}f/\mathrm{d}t = 0$. Writing down what this implies component by component we get

$$\boxed{\partial_t f + v^i \partial_{x^i} f + a^i \partial_{v^i} f = 0} \tag{60}$$

This is called *Collision-less Boltzmann equation* or *Vlasov's equation*. Please, don't mind askying why of the weird indices placement. Whichever way you put the indices is (here) absolutely irrelevant.

2 Properly speaking, we'd have to consider the transformation $\vec{x} = \vec{x}_0 + \vec{u}_0\, \mathrm{d}t$, $\vec{v} = \vec{v}_0 + \vec{a}_0\, \mathrm{d}t$, which has a Jacobian $\|J\| = 1 + \mathrm{o}(\mathrm{d}t^2)$.

Collisions may be re-introduced by adding a term of the form $\partial_t f_{\text{coll}}$, but in the following we'll assume the detailed balance principle to hold, so that there are as many particles getting kicked out of the volume as those that are pulled in.

4.3 FROM BOLTZMANN TO EULER

Let us consider eq.60 along with Reynolds's decomposition for the velocity $v_i = V_i + u_i$. We have three different terms to evaluate if we integrate over velocities to calculate the momenta of a given quantity $g(\vec{x}, \vec{v}, t)$

$$(i)\ \ \partial_t f \rightarrow \int \mathrm{d}^3 v \partial_t f\, g(\vec{x}, \vec{v}, t) = \int \mathrm{d}^3 v \left(\partial_t (fg) - f \partial_t g\right)$$

$$(ii)\ \ v_i \partial_i f \rightarrow \int \mathrm{d}^3 v g \vec{v} \nabla f = \int \mathrm{d}^3 v \left(\nabla (g \vec{v} f - f \vec{v}\, \nabla g)\right)$$

$$(iii)\ \ a_i \partial_{v_i} f \rightarrow - \int \mathrm{d}^3 v f \dot{\vec{v}}\, \nabla_{\vec{v}} g$$

Recalling the definition of the average value of g, the three terms above may be cast in the following form

$$\begin{aligned} (i) \quad & \partial_t (n \langle g \rangle) - n \langle \partial_t g \rangle \\ (ii) \quad & \nabla (n \langle g \vec{v} \rangle) - n \langle \nabla g\, \vec{v} \rangle \\ (iii) \; & - n \langle \nabla_{\vec{v}} g\, \dot{\vec{v}} \rangle \end{aligned}$$

At this point, consider the following expressions for g

$$g = 1 \implies \text{Mass conservation} \tag{61}$$

$$\partial_t n + \nabla (n \vec{v}) = 0 \tag{62}$$

$$\vec{g} = m \vec{v} \implies \text{Euler's equation} \tag{63}$$

$$\partial_t (\rho \vec{v}) + \vec{v} \cdot \nabla (\rho \vec{v}) = -\nabla p - \rho \vec{F} \tag{64}$$

$$g = \frac{1}{2} m v^2 \implies \text{Energy conservation} \tag{65}$$

$$\partial_t (\frac{1}{2} \rho v^2 + \varepsilon) + \nabla (\rho \vec{v} \left[\frac{v^2}{2} + \varepsilon \right]) = \vec{F} \cdot \vec{v} - \nabla (\vec{\phi}_H + p \vec{v}) \tag{66}$$

which are all nice and dandy[3].

Sometimes it is useful to cast the equation of energy conservation in a slightly different form. Called E the total energy, we recall the first law of thermodynamics

$$\frac{\mathrm{d}E}{\mathrm{d}t} = T \frac{\mathrm{d}S}{\mathrm{d}t} + \frac{p}{\rho^2} \frac{\mathrm{d}\rho}{\mathrm{d}t} \tag{67}$$

because now we have

$$\partial_t (\frac{1}{2} \rho v^2 + \varepsilon) + \nabla (\rho \vec{v} \left[\frac{v^2}{2} + \varepsilon \right]) = \frac{\mathrm{d}E}{\mathrm{d}t} - \rho T \frac{\mathrm{d}S}{\mathrm{d}t} + \Lambda (T, \rho) \tag{68}$$

where Λ includes all energy losses, also the ones not included in Vlasov's equation.

3 In the last equation, $\vec{\phi}_H$ is the heat flux.

In the following, we'll often use something called the *Lagrangian derivative* which is just

$$\frac{\mathrm{D}}{\mathrm{D}t} = \partial_t + \vec{v} \cdot \nabla \tag{69}$$

which allows us to rewrite two equations we'll often be using in a more compact form

$$\frac{\mathrm{D}\vec{v}}{\mathrm{D}t} = -\frac{\nabla p}{\rho} \tag{70}$$

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\rho \nabla \cdot \vec{v} \tag{71}$$

From this form of the continuity equation, we can give a definition of *compressibility*: we'll say that a fluid is *incompressible* if $\nabla \cdot \vec{v} = 0$, otherwise we'll say that it's compressible.

So far we've been ignoring the effects of viscosity, which can be actually retrieved if we include collisions in the otherwise collision-less Boltzmann equation. If we were to do that, we would get the celebrated *Navier-Stokes equation*.

We'll scarcely make explicit use of these notions but it's worth to write those down, at least for the sake of completeness.

We define **streamline** the convolution of the tangents to the velocity field and **streakline** the line traced by all the particles of the fluid that are passing from a specific, fixed point.

Note that the two definitions overlap only if motion is static/stationary.

4.3.1 Viscosity and diffusion

So far, we've been blatantly ignoring the collision integral in Vlasov's equation saying that by the detailed equilibrium principle, at equilibrium it's contribution is approximately zero. Clearly, this can't be always the case, even more so in fluids.

Notably, fluids display some kind of friction arising from some microscopic molecular effect. We'll call this "friction" *viscosity* [4]. We want to understand what happens when a fluid moves. We have to consider:

- **Dilatation or Compression**: are changes in the volume of the fluid element that can be thought as *isotropic stresses*. Note that in the incompressible fluids they would be absent.

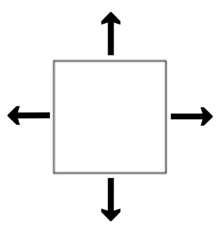

Figure 15: Sketch of dilation or compression.

4 For this section, I'm following Dott. Ricciardi's notes [9] and my own notes from the Fluid dynamic class.

- **Rotation**: we consider a solid body rotation, hence it will cause no net force on the fluid element

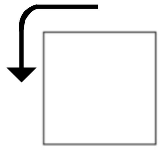

Figure 16: Sketch of rotation.

- **Shear**: is the most interacting case, in fact this involves relative motion of different sides of the fluid.

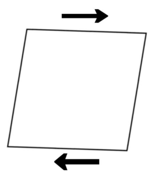

Figure 17: Sketch of shear.

In principle, we could decompose the (purely spatial) stress tensor as follows

$$\sigma_{ij} = -p\delta_{ij} + d_{ij}$$

Here d_{ij} is the *deviatoric stress tensor*, representing the deviation from isotropy. A phenomenological approach may suggest us to consider the deviatoric stress proportional to the local velocity gradient, regarded then as the principal parameter (read as: The more relevant one). To corroborate this approach, we could recall that if a fluid is stationary, velocity gradients vanish, and so would the d_{ij} tensor.

Generally, the deviatoric stress is expressed by means of a fourth rank tensor

$$d_{ij} = A_{ijkl}\partial_l u_k$$

Note that since the stress tensor is symmetric, d_{ij} must be as well. This implies that A_{ijkl} must be symmetric in i, j. After a long and tedious decomposition in symmetric and antisymmetric part, we finally get to the expression for the stress tensor that is customarily used

$$\sigma_{ij} = -p\delta_{ij} + 2\mu(\frac{1}{2}\left[\partial_j u_i + \partial_i u_j\right] - \frac{1}{3}\nabla\cdot\vec{u}\delta_{ij}) \tag{72}$$

Here μ is always positive and it's called the viscosity of the fluid. Generally, we'll be interested in the *kinematic viscosity* of the fluid $\nu = \mu/\rho$.

From eq.72 we reckon that if the fluid is uncompressible ($\nabla\cdot\vec{u} = 0$), the Navier-Stokes equation has the form

$$\rho\frac{\mathrm{D}u_i}{\mathrm{D}t} = \rho F_i - \partial_i p + \mu\partial_j^2 u_i \tag{73}$$

hence the viscosity enters the equation with the Laplacian of the velocity, which is often (but definitely not always) negligible. It is often useful to define a quantity called *Reynolds' number*

$$Re = \frac{(\vec{v} \cdot \nabla)\vec{v}}{\nu \nabla^2 \vec{v}} \approx \frac{U \cdot L}{\nu} \tag{74}$$

which quantifies the importance of viscous dissipation in respect to inertial forces. Here U, L are some characteristic scales of the system.

Bernoulli's theorem

If we neglect for the moment dissipative effects in the equation of conservation of energy, we may write

$$\partial_t(\frac{1}{2}\rho v^2 + \varepsilon) + \nabla(\rho \vec{v}\left[\frac{v^2}{2} + \varepsilon\right]) - \vec{F} \cdot \vec{v} = 0$$

where ε is some expression for the internal energy of the fluid. If we consider a stationary, incompressible velocity field and only conservative forces

$$\nabla\left(\rho\vec{v}\left[\frac{v^2}{2} + \varepsilon\right] + \rho\phi\right) = 0$$

If we take p/ρ as a measure of the internal energy of the fluid, we obtain *Bernoulli's theorem*

$$\frac{\rho\vec{v}^2}{2} + p + \rho\phi = \text{const.} \tag{75}$$

which is just a restating of conservation of energy. In the following section we'll try examining one simple application of eq.75.

4.3.2 The de Laval nozzle

Consider Bernoulli's theorem in the notable absence of an external potential ϕ, and we shall assume that $\vec{v} \cdot \vec{A}$ is constant. Here $\vec{A}$ is the outwards pointing normal to the section of a nozzle of some kind. If the fluid is

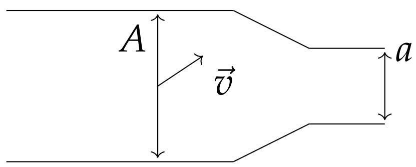

Figure 18: Schematic depiction of the De Lavalle nozzle.

incompressible, Bernoulli's theorem requires that

$$\frac{\rho v^2}{2} + p = \text{const.}$$

which has as a natural consequence that given two sections of areas A and a and velocity $\vec{v}$ at A (see Fig.18), then upon reaching the point where the section becomes a, the velocity becomes

$$\vec{V} = \frac{A}{a}\vec{v}$$

You may then be wondering what happens if we relax the condition of the fluid to be incompressible. Consider Euler's equation and the continuity equation (71) in their 1D and stationary form

$$v\partial_x v = -\frac{1}{\rho}\partial_x p \tag{76}$$

$$\partial_x(\rho v) = 0 \tag{77}$$

If we integrate the latter and make use of the divergence theorem, we find out that

$$\int \rho\vec{v}\,\mathrm{d}\vec{A} = \rho\vec{v}\cdot\vec{A} = \text{const.}$$

This can be implicitly differentiated to yield

$$\frac{\mathrm{d}\rho}{\mathrm{d}x}\frac{1}{\rho} + \frac{\mathrm{d}v}{\mathrm{d}x}\frac{1}{v} + \frac{\mathrm{d}A}{\mathrm{d}x}\frac{1}{A} = 0$$

To properly solve the equations, we need to introduce and Equation of State of some sort. Here (and rather often in the following) we'll assume a barotropic relation between pressure and density

$$p(\rho) = c_s^2\rho \tag{78}$$

where c_s is the *speed of sound* in the fluid.

Plugging in what we've just found in the 1D equation of motion

$$\begin{aligned}
\rho v\partial_x v &= -c_s^2\rho \\
v\partial_x &= c_s^2\left(\frac{\mathrm{d}v}{\mathrm{d}x}\frac{1}{v} + \frac{\mathrm{d}A}{\mathrm{d}x}\frac{1}{A}\right) \\
\partial_x v\left(v - \frac{c_s^2}{v}\right) &= \frac{c_s^2}{A}\partial_x A
\end{aligned}$$

This can be cast in a slightly more suggestive form

$$\boxed{\partial_x v = \frac{c_s^2}{Av}\frac{1}{1-\frac{c_s^2}{v^2}}\partial_x A} \tag{79}$$

Consider a nozzle with a position decreasing section ($\partial_x A < 0$). Two scenarios may arise

- (i) the fluid is *subsonic* ($v < c_s$). This implies that the fluid accelerates $\partial_x v > 0$
- (ii) the fluid is *supersonic* ($v > c_s$). This implies that the fluid decelerates $\partial_x v < 0$

If the section gradient is flipped, the implications are reversed. In principle you could then imagine a nozzle with decreasing section up to a certain x_* where the fluid becomes supersonic, and with increasing section after that point.

In such a configuration, the fluid is always accelerating.

4.4 SCHWARTZSCHILD STABILITY CONDITION

Suppose we have a stratified plane parallel atmosphere approximately in hydrostatic equilibrium. Consider now a blob of fluid which as been displaced upwards as shown in the picture below.

Initially, the blob had the same density and pressure as the fluid surrounding, respectively ρ_0, P. After the displacement, the external fluid density and pressure are ρ', P' which are, in principle, different than density and pressure inside the bubble, ρ^*, P^*.

However, we'll work under the assumption that the bubble is always in pressure equilibrium with the surrounding fluid and does not exchange energy with it[5].

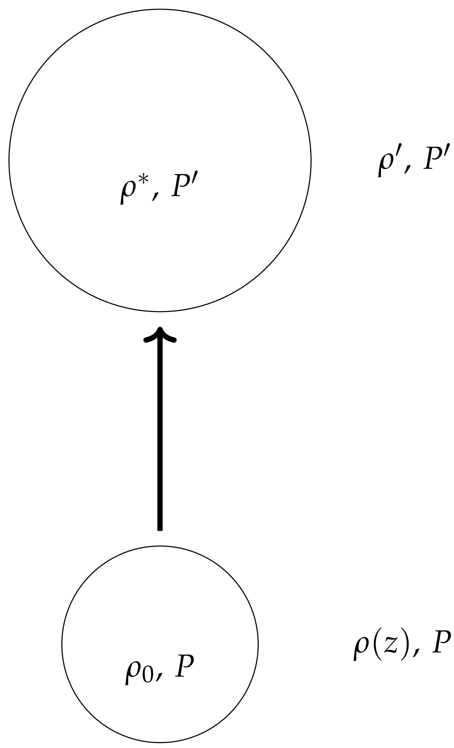

Figure 19: Vertical displacement of a blob of gas in a stratified atmosphere.

Let's consider the infinitesimal displacement ε, so that the equation of motion is simply just

$$\rho\ddot{\varepsilon} = -g(\rho' - \rho)$$

We shall now take two independent Taylor expansions: One for the density of the blob, one for the density of the surrounding fluid.

$$\rho^* = \rho_0 + (\nabla_{ad}\,\rho)\varepsilon$$
$$\rho' = \rho_0 + (\nabla_{amb}\,\rho)\varepsilon$$

Plugging this two equations into the equation of motion yields a rather familiar expression

$$\rho\ddot{\varepsilon} = -g(\nabla_{ad}\,\rho - \nabla_{amb}\,\rho)\varepsilon \tag{80}$$

5 Typically, pressure imbalances are removed rather quickly by *acoustic waves*, while heat exchange takes more time.

which is that of a harmonic oscillator. We identify the quantity

$$\omega_{BV}^2 = g(\nabla_{ad}\,\rho - \nabla_{amb}\,\rho)$$

as the *Brunt-Väisälä frequency*, which regulates the buyouancy of the blob. In particular we observe that if $\omega_{BV}^2 > 0$, the bubble is buyouantly stable. Conversely, if $\omega_{BV}^2 < 0$ the atmosphere is unstable and *convective instability* will arise.

Up to a factor of g (that would cancel out anyway), studying the sign of ω_{BV}^2 for $\rho \propto T^{-1}$ does retrieve the famous *Schwartzchild stability condition*.

How good is the "bubble picture" we've just used? Not much actually, as we'll see in the next chapter, but it stil allows to get a grasp of how convection may work.

One thing we have neglected (but is worth considering when dealing with stars) is that the bubble is likely to be made of hot plasma. That implies that the bubble is able to radiate and affect different layers of the star through means that are not only its physical displacement.

4.4.1 Convection in presence of rotation

We've seen that density gradients in a stratified atmosphere are expected to generate convection. What happens if we add rotation to the picture?

To make things a little simpler, we're going to assume that the star rotates with angular velocity $\vec{\Omega}$ **as a rigid body**. This is a rather poor assumption, that may be true only for the core region of stars; in general, *differential rotation* applies, so $\Omega = \Omega(r, t)$.

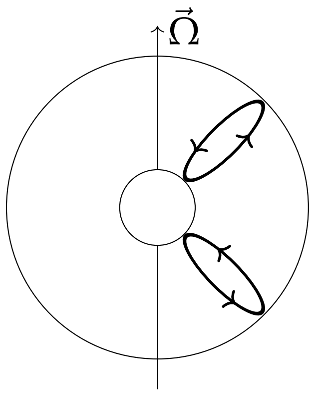

Figure 20: Convective motion in a rotating stellar atmosphere.

As we've already seen, in presence of rotation, the potential of a self-gravitating object must be modified to include the contribution of centrifugal acceleration

$$\phi \to \phi + \phi_C$$

so that when we write down Euler's equation we find

$$\frac{D\vec{v}}{Dt} + 2\Omega \wedge \vec{v} = -\frac{\nabla p}{\rho} + \vec{F} \tag{81}$$

Here $2\Omega \wedge \vec{v}$ represents Coriolis' forces [6]. At this point it's convenient to define a new quantity, the *vorticity* as $\vec{\omega} = \nabla \wedge \vec{v}$.

In a sense, vorticity represents the amount of angular momentum per unit mass transported by the fluid. If we decompose the stress tensor in a symmetric and antisymmetric part

$$\sigma_{ij} = \sigma_{ij}^S + \sigma_{ij}^A = \frac{1}{2}(\partial_j v_i + \partial_i v_j) + \frac{1}{2}(\partial_j v_i - \partial_i v_j)$$

where we recognize that the vorticity generates the antisymmetric part.

We can take the curl of Euler's equation, finding a differential equation for the transport of vorticity

$$\frac{\mathrm{D}\vec{\omega}}{\mathrm{D}t} + \nabla \wedge (2\Omega \wedge \vec{v}) = -\frac{1}{\rho^2}\nabla p \wedge \nabla \rho + \nabla \wedge \vec{F} \tag{82}$$

Hence, vorticity is always non-zero in presence of rotation. In hydrostatic equilibrium vorticity is always stable and conserved.

6 Coriolis' forces arise only in presence of latitudinal motion and imply the presence of shear.

5 TURBULENCE

"When I meet God, I am going to ask him two questions: why relativity? And why turbulence? I really believe he will have an answer for the first."

Werner Heisenberg

Turbulence is quite an interesting feature arising in fluids. Turbulent flow is fluid motion characterized by chaotic changes in pressure and flow velocity, both spatially and temporally.

Turbulence's ability to transport and mix fluid is unparalleled, comparing it to stardard laminar flows. But it comes with a price: It's bloody difficult to deal with it.

There's not a unique definition of when a fluid becomes turbulent rather than staying in a chill (and much less chaotic) laminar flow.

Usually, we say that a flow is turbulent when the Reynolds' number (74) is much larger than 1. Essentially, what happens is that viscosity is no longer able to damp and constrain the fluid motion, which ends up going nuts. It should be clear, however, that for that to happen, you need energy (ε) to be injected in the fluid

Before going through the essential features of turbulence, we shall briefly consider again our naive bubble description, only to find out that what happens is actually much more complex.

5.1 INSTABILITIES

Consider the system depicted in Fig.19. As the bubble rises, it will have a non-zero vertical velocity. Let's see what this implies.

Consider a stratified, ideal, incompressible fluids with no vorticity [1] as those shown in Fig.21. Given these assumptions, the velocity fields may be

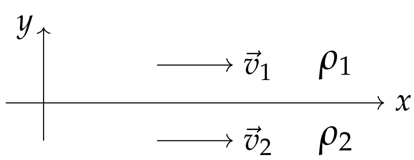

Figure 21: Stratified fluids may give raise to instabilities.

written in terms of a scalar potential

$$\vec{v}_i = -\nabla \phi_i$$

1 Actually, vorticity is divergent on the boundary.

Plugging this definition in Euler's equation yields (for example for fluid 1)

$$-\nabla(\partial_t \phi_1) + \nabla\left(\frac{v^2}{2}\right) = -\frac{\nabla p}{\rho}$$

Let's assume that the scalar potentials are composed of a constant term (temporally speaking) and a small space and time dependent fluctuation

$$\phi_i = -v_i x + \delta\phi_i$$

The condition for the fluid to be incompressible then becomes

$$\nabla^2 \delta\phi_i = 0$$

Say that the boundary between the two fluids can be parametrized by $y = \xi(x,t)$, then in the Lagrangian frame we have that

$$\frac{\mathrm{D}\xi}{\mathrm{D}t} = \partial_y \delta\phi_1 = -\partial_y \delta\phi_2$$

If we assume the pressure to be continuous at the interface and $\delta\phi_i$ to be plane waves of the form

$$\delta\phi_i = B_i \exp(i(kx - \omega t) + k_y)$$

we're able to find a dispersion relation for the system

$$\frac{\omega}{k} = \frac{\rho_1 v_1 + \rho_2 v_2}{\rho_1 + \rho_2} \pm \left[-\frac{\rho_1\rho_2(v_1 - v_2)^2}{(\rho_1 + \rho_2)}\right]^{1/2} \tag{83}$$

Unless $v_1 = v_2$, the argument of the square root is always negative, so the oscillation frequency has a purely imaginary component. When you plug that into the plane wave solution, you get an exponentially increasing amplitude for the wave, meaning that the system is unstable.

This is called *Kelvin-Helmoltz instability*. The KH instability has characteristic vortices arising, which normally are damped at some point by viscosity, stopping the endless formation of vortices.

If we introduce a gravitational field in the vertical direction, we introduce a stabilizing factor in eq.83

$$\frac{\omega}{k} = \frac{\rho_1 v_1 + \rho_2 v_2}{\rho_1 + \rho_2} \pm \left[\frac{g}{k}\frac{\rho_2 - \rho_1}{\rho_1 + \rho_2} - \frac{\rho_1\rho_2(v_1 - v_2)^2}{(\rho_1 + \rho_2)}\right]^{1/2} \tag{84}$$

Please note that even if $v_1 = v_2$, the system may still be unstable. In fact, in such a scenario, the system is stable only if $\rho_1 < \rho_2$ (referring to Fig.21), which actually makes sense: The less dense fluid must float over the denser fluid.

Conversely, the fluid is unstable if $\rho_1 > \rho_2$ (*Rayleigh-Taylor instability*). The RT instability presents characteristic "fingers" (or jets) extending from the less dense layer to the denser one.

5.2 PROPERTIES OF TURBULENCE

To try giving a description of turbulence, we'll often perform a Reynolds' decomposition

$$\vec{v} = \langle\vec{v}\rangle + \delta\vec{v}$$

This allows us to write down an "averaged" version of the Navier-Stokes equation (73), in which a new term is coming out

$$\partial_t\langle v_i\rangle + (\langle v_j\rangle \cdot \partial_j)\langle v_i\rangle = -\frac{\langle\partial_i p\rangle}{\rho} - \partial_i\phi + \nu\partial_j^2\langle v_i\rangle + \nu\langle\delta v_i\delta v_j\rangle$$

The last term $\langle\delta v_i\delta v_j\rangle$ representing correlations between velocity fluctuations is notably absent in laminar flows. It adds a third source of stress to the stress tensor. If we assume the fluid to be incompressible, then eq.72 becomes

$$\sigma_{ij} = -\langle p\rangle\delta_{ij} + \mu\left(\left[\partial_j\langle v_i\rangle + \partial_i\langle v_j\rangle\right] - \rho\langle\delta v_i\delta v_j\rangle\right)$$

The term containing the pressure is the *isotropic stress*, the term containing the gradients of v is the *viscous stress* while the third new term is the *Reynolds stress*, caused by fluctuations.

Making use of Wiener-Chinčin's theorem, we can link velocity fluctuations to fluctuations in kinetic energy. Usually the Reynolds' stress term is unpacked in a symmetric and an antisymmetric part defining the *turbulent kinetic energy tensor*

$$\kappa(\vec{x}, t) = \frac{1}{2}\langle\delta v_i\delta v_i\rangle$$

representing the mean kinetic energy (per unit mass) in the fluctuating velocity field. The *deviatoric anisotropic part* is

$$a_{ij} = \langle\delta v_i\delta v_j\rangle - \frac{2}{3}\kappa\delta_{ij}$$

which is the only component effectively transporting momentum. For a more detailed discussion about turbulence, you can check out the Fluid Dynamics course or give a look to [8].

5.2.1 Self-Similarity

Suppose to have a quantity dependent on two independent variables $Q(x, y)$. Characteristic scales $Q_0(x)$ and $\delta(x)$ are defined for Q, y respectively. Then the scaled variables are just

$$\xi = \frac{y}{\delta(x)} \qquad \tilde{Q}(\xi, x) = \frac{Q(x, \xi)}{Q_0(x)}$$

If the scaled dependent varibale is independent of x, i.e.

$$\exists\hat{Q}(\xi) : \tilde{Q}(\xi, x) = \hat{Q}(\xi)$$

then $Q(x, y)$ is self-similar.

Self-similar fluids have really interesting properties that more often than not are able to greatly simplify what we're working with. Since understanding what a self-similar solution actually is is often more complicated than using it, we'll give two examples.

The first we've already seen: It's eq.53. If you recall, we've defined *scales* for both the density and the radial distance and proceeded to tranform our messy second order ODE in a still messy but at least adimensional second order ODE.

Doing this has the perk that once you've found a solution for $\phi(\eta)$, you already have a solution for whatever density and/or radial distance scale may come to your mind.

Another example, you can find in the Schwartzschild metric for non-rotating, spherically symmetric black holes. General relativity and Birkhoff's theorem grants us that

$$\mathrm{d}s^2 = -\left(1 - \frac{R_s}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{R_s}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2_{(2)} \tag{85}$$

is the only possible spherically symmetric solution to the Einstein's equation in vacuum [2] (well, assuming there actually *is* something somewhere in space-time inducing this metric).

From (85) is perhaps even simpler understanding what a self-simlar solution is: Just plug $y = R_s/r$ and you'll find that the metric is completely specified up to a factor of R_s, which contains all the details of the source of the curvature.

5.2.2 Kolmogorov's scales

In a revolutionary article of 1941, the Russian physicist A. V. Kolmogorov developed one of the most incredible theories about the structure of turbulence in incompressible fluids [6].

we shall start assuming we're in presence of a fully turbulent flow with $Re = UL/\nu \gg 1$ and Richardson's theory about the "energy cascade" to hold: That means turbulence can be considered to be composed of eddies of different sizes which can be described through the tern of parameters l, $u(l)$, $\tau(l) = l/u(l)$.

The eddies of the largest scales are characterized by lengthscale l_0 comparable to L (the dimension of the system housing the turbulent liquid) and have characteristic velocity $u(l_0) \approx (2\kappa/3)^{1/2} \approx U$.

Larger eddies are unstable. Pretty much like boybands in the prime of their careers, they end up breaking apart, transferring energt to "smaller" eddies. This *energt cascade* continues until the Reynolds' number $Re(l)$ is sufficiently small that the eddy motion is stable.

The energy dissipation rate ε is determined by the transfer of energy from the largest eddies and is **independent of the kinematic viscosity** ν.

Here comes to play Kolmogorov's key hypothesis. Ar sufficiently high Re, the small-scale turbulent motions are claimed to be *isotropic*. Let's call l_I the lengthscale demarcation between anisotropic large and isotropic small eddies.

Isotropy in this case means that all directional information is lost, as well as all information about the geometry of larger eddies. In a sense then, the statistics of small-scale motions are in a sense *universal*.

2 Please note that we're using Carroll's [2] convention for the signature of the metric, which, incidentally, isn't the same used in the GR course.

1st hypothesis of self-similarity

In every turbulent flow with $Re \gg 1$, the statistics of the small-scale range have a universal form determined by ε, ν alone. We can define *Kolmogorov's scale* so that $Re = 1$

$$
\begin{aligned}
l_\eta &= \left(\frac{\nu^3}{\varepsilon}\right)^{1/4} \\
u_\eta &= (\varepsilon\nu)^{1/4} \\
\tau_\eta &= \left(\frac{\nu}{\varepsilon}\right)^{1/2}
\end{aligned}
$$

characterizing the smallest dissipative eddies.

The implications are astounding. Consider a point $\vec{x}_0$ at a time t_0 for a fully-developed turbulent flow. In terms of Kolmogorov's scales, we can define adimensional coordinates and velocities

$$
\begin{aligned}
\vec{y} &= \frac{\vec{x} - \vec{x}_0}{\eta} \\
\omega(\vec{y}) &= \frac{\vec{U}(\vec{x}, t_0) - \vec{U}(\vec{x}_0, t_0)}{u_\eta}
\end{aligned}
$$

Since it is not possible to form a non-dimensional quantity with ε, ν, the universal form of the statistics of the non-dimensional field $\omega(\vec{y})$ **cannot depend on** ε, ν. That means that on not-too-large scales, $\omega(\vec{y})$ is statistically isotropic and identical at all points.

That implies that all fully-developed turbulent flows have velocity fields statistically similar.

2nd hypothesis of self-similarity

In every turbulent flow with $Re \gg 1$, the statistics of motion of scale l in the range $l_0 \gg l \gg \eta$ have a universal form determined by ε alone, independent of ν which means the effect of viscosity is relevant only when we're deal-

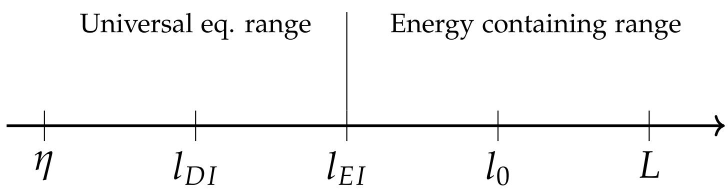

Figure 22: The different lengthscale's interplay in Kolmogorov's theory.

ing with lengthscales smaller than η. It seems like a honest assumption: a Reynolds' number smaller than 1 (74) requires viscous forces to be greater than the inertial driving. By construction, Kolmogorov's scales identify the conditions for which $Re = 1$, so all is consistent.

We're not going to show explicitly, but from the two hypothesis of self-similarity we can conclude something really important. In the equilibrium range ($l < l_{EI}$, see Fig.22) the spectrum is a universal function of ε, ν. But then, the second hypothesis of self-similarity tells us that in the inertial sub-range statistics cannot depend on ν.

If we were to calculate explicitly velocity correlations to plug them into the Wiener-Chinčin's theorem, we'd get an expression for how energy is distributed into modes κ (the wavenumber)

$$E(\kappa) = C\varepsilon^{2/3}\kappa^{-5/3} \tag{86}$$

Here C is a universal constant. What's most fascinating about (86) is that not only it has been tested to be actually true for many types of different fluids, but it is also valid if we relax the condition of incompressibility[3].

The typical velocity fluctuation in a fluid can be evaluated from the root mean square

$$\delta\bar{v} = \sqrt{\langle \delta v^2 \rangle} > 0$$

introducing a non-thermal broadening in the spectrum lines (see "Line Broadening" in Chapter 2). In principle, we have only access to velocity-fluctuations on the line of sight (since that's what we can directly measure in observations), hence the difficulty of recognizing the occuring of turbulence in most objects of astrophysical interest. And that is most frustrating since, for example, the ISM[4] has a Reynolds' number of roughly 10^8.

To actually resolve the spatial structure of turbulence we require a resolution of astrophysical proportions (pun intended), at least proportional to Re^{-1}.

As a concluding note, in the presence of magnetic fields, eq.86 is modified $E_M \propto \kappa^{-2}$. In principle, both dependencies can be present at once, allowing us to differentiate between neutral and magnetized phases.

5.3 SOUND WAVES AND SHOCKS

At this point, it should come to no surprise to you that we're coming back once more to our initial, easy rising-bubble model to further complicate it. We've already seen that as a consequence of its vertical motion, instabilities arise (84), tranforming our cool little bubble in a swirly mess.

Another fairly important consequence of the bubble's motion is the arising of pressure differences between the material itself and the wake[5] coming after the bubble.

For this section (and many more in the Gravitation Part) I'll be following [4]. Relevant paragraphs for this section are §2.4 and §3.8.

Let's write Euler's equation and the continuity equation (71) in a conservative form

$$\partial_t(\rho\vec{v}) + (\vec{v}\cdot\nabla)(\rho\vec{v}) = -\nabla p$$
$$\partial_t\rho + \nabla(\rho\vec{v}) = 0$$

and look for perturbations around the equilibrium solution.

$$\vec{v} = \vec{v}_0 + \vec{v}_1$$
$$p = p_0 + p_1$$
$$\rho = \rho_0 + \rho_1$$

3 The κ dependence is unchanged, but a dependence on the *Mach number* M is introduced.
4 i.e. *InterStellar Medium*.
5 *N.d.T.*: in Italian, it should sound something like "scia".

Since Euler's equation (and by extension, Navier-Stokes' equation) is invariant under Galileian transformations, we can set $\vec{v}_0 = 0$. We shall also assume a one dimensional motion and a barotropic equation of state, allowing us to linearize our equations.

Quantities with the "o" subscript are assumed to be uniform and stationary. Keeping only first order quantities, we find

$$\partial_t \rho_1 + \rho_0 \partial_x v_1 = 0$$
$$\rho_0 \partial_t v_1 + \left(\frac{\partial p}{\partial \rho}\right)_0 \frac{\partial \rho_1}{\partial x} = 0$$

Cross derivating the two equations and plugging in one equation into the other, we end up with

$$\partial_t^2 \rho_1 - \left(\frac{\partial p}{\partial \rho}\right)_0 \partial_x^2 \rho_1 = 0 \tag{87}$$

which is a hyperbolic equation describing propagating density waves!

We can identify

$$c_s^2 = \left(\frac{\partial p}{\partial \rho}\right)_0$$

as the *speed of sound* in the fluid. Sometimes c_s is also said to be the compressibility of the fluid.

Since c_s is the speed at which pressure (or density) perturbations travel through the gas, it limits the rapidity with which the fluid can react to such changes. For example, if the pressure in one part of a region of the fluid of characteristic size L is suddenly changed, the other parts of the region cannot respond to this change until a time of order L/c_s, the sound crossing time, has elapsed.

In a sense, c_s is also the velocity at which causality travels in the fluid. Thus, if we were to consider a *supersonic flow*, then the fluid cannot respond on the flow time $L/|\vec{v}|$, so pressure gradients have little effect on the flow. At the other extreme, for subsonic flow the fluid can adjust in less than the flow time, so to a first approximation the fluid behaves as if in hydrostatic equilibrium.

The density dependence of the sound speed implies that regions of higher than average density have higher than average sound speeds, a fact which gives rise to the possibility of *shock waves*.

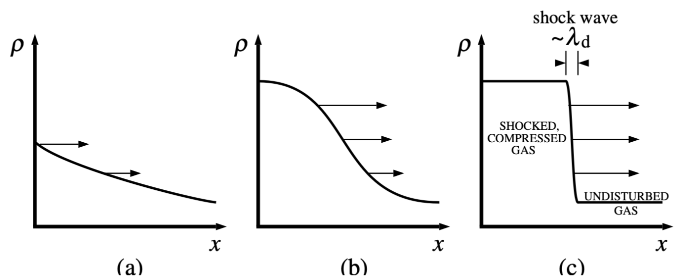

Figure 23: The formation of a shock wave. Credits: Frank, King and Raine.

In a shock relevant fluid quantities change on lengthscales of the order of the mean free path and this is represented as a discontinuity in the fluid.

5.3.1 Rankine-Hugoniot junction conditions

Since discontinuities are introduced into the picture, we can no longer use differential quantities to describe conservations. The solution to this problem is easily found looking into *integral quantities*.

To do so, it is convenient to choose a reference frame in which the shock is at rest. Since the shock thickness is small, we can regard it as locally plane; moreover, since the fluid flows through the shock so quickly, changes in the fluid conditions cannot affect the details of the transition across the shock, so we can regard the flow into and out of the shock as steady.

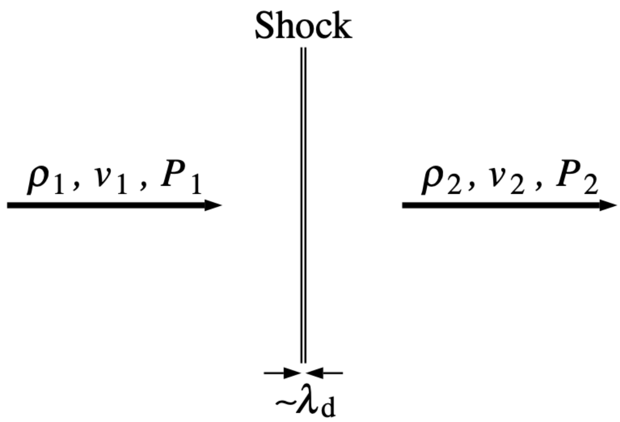

Figure 24: Calculation of the Rankine-Hugoniot junction conditions. Credits: Frank, King and Raine.

All we have to do is apply the various conservation laws in their integral form

$$\rho_1 v_1 = \rho_2 v_2 \tag{88}$$
$$p_1 + \rho_1 v_1^2 = p_2 + \rho_2 v_2^2 \tag{89}$$
$$\frac{1}{2} v_1^2 + \varepsilon_1 = \frac{1}{2} v_2^2 + \varepsilon_2 \tag{90}$$

which are respectively mass flux, momentum flux and energy flux conservation. Especially in the latter, we've assumed radiative losses and thermal conduction, etc., across the shock front to be negligible.

This is the adiabatic assumption, and the resulting junction conditions are known as the adiabatic shock conditions or the *Rankine-Hugoniot junction conditions*.

Keeping Fig.24 in mind, we can proceed to find a unique relation connecting the ratio of the downstream density (or velocity) ρ_2 and upstream density to the pressure

$$\frac{\rho_2}{\rho_1} = \frac{v_1}{v_2} = \frac{(\gamma+1)p_1 + (\gamma-1)p_2}{(\gamma-1)p_1 + (\gamma+1)p_2}$$

Note that it is not assured for the adiabatic coefficient γ to come out unscathed from the shock, even more so if it is energetic enough to ionize the fluid.

Usually, RH junction conditions are expressed in terms of the *Mach number* $\mathcal{M}_i = v_i/c_{s,i}$. For a monoatomic gas $\gamma = 5/3$, and the RH junction conditions yield

$$1 + \frac{v_2}{v_1} = \frac{5}{4}\left[\frac{3}{5\mathcal{M}_1^2 + 1}\right]$$

In the limit of a strong shock $\mathcal{M}_1 \gg 1$, we find out that

$$\frac{v_2}{v_1} = \frac{\rho_1}{\rho_2} = \frac{1}{4}$$

i.e. the velocity drops to one-quarter of its upstream value across the shock while the gas is compressed by a factor 4 in a strong shock.

For more general gases, assuming adiabatic compression, the limit compression rate is

$$\frac{v_2}{v_1} = \frac{\rho_1}{\rho_2} = \frac{\gamma - 1}{\gamma + 1}$$

If the shock is *isothermal* rather than adiabatic, there's no limit to compressibility.

Note that without proper injection of energy, shocks are eventually stalled by viscosity, then degenerating into normal soundwaves. As we have anticipated in the last section, Kolmogorov's spectrum has the same κ dependency, but a proportionality to $\mathcal{M}^8$ is introduced, meaning that shocks dissipate energy very efficiently.

Most of the times, assuming shocks to be adiabatic is a questionable take *at best* since there's a strong dependence on the cooling timescales through radiative losses.

5.3.2 Sedov-Taylor blastwave solution

There's a (arguably) cool and educational application of shockwaves, which incidentally made the fortune of a rather famous film producer.

Explosions.

An explosion is essentially a localized injection of a given quantity of energy that induces an adiabatic expansion of the gas (or the fluid). To find a proper description of explosions we're one theorem short, which we'll briefly discuss in the following paragraphs.

Buckingham-Pi theorem

Assume to have a set of n variables so that $q_1 - f(q_2, ...q_n) = 0$. That means we can define a function g that satisfies

$$g(q_1, ...q_n) = 0$$

Given a relation like the one above, the independent variables of the system can be reduced to $n - m$ with m typically of the order of the independent dimensions (called Π).

The theorem then claims that we may pass from function $g(q) = 0$ to $G(\Pi) = 0$ following this procedure

- List all parameters describing the system;

- Choose the primary dimensions (e.g. M, L, T and so on);
- Perform dimension analysis of the parameters;
- Choose parameters that contain all the primary dimensions;
- Set dimensional equations and find the explicit relations for the Π parameters.

On top of helping us create a model for explosions, this allows us to cast Euler's equation into a self-similar ODE form which can be solved much more easily (numerically, of course). Defining $\eta \equiv r^{\lambda} t^{-\mu}$, λ and μ will be set by the particular constrains we have, but in general we can express

$$\begin{aligned} v &= r t^{-1} U(\eta) \\ \rho &= r^{-3} D(\eta) \\ p &= r^{-1} t^{-2} \Pi(\eta) \\ c_S &= r t^{-1} C(\eta) \end{aligned}$$

so that relevant derivatives take the following form

$$\begin{aligned} \frac{\partial}{\partial t} &= -\mu \eta t^{-1} \frac{\mathrm{d}}{\mathrm{d}\eta} \\ \frac{\partial}{\partial r} &= \lambda \eta r^{-1} \frac{\mathrm{d}}{\mathrm{d}\eta} \end{aligned}$$

With this set of equations we can cast Euler's, continuity and energy conservation equations in a self-similar form as anticipated

$$\begin{aligned} (\lambda U - \mu)\eta \frac{\mathrm{d}D}{\mathrm{d}\eta} + (n-2)DU + \lambda D \eta \frac{\mathrm{d}U}{\mathrm{d}\eta} &= 0 \\ (\lambda U - \mu)\eta \frac{\mathrm{d}D}{\mathrm{d}\eta} = D^{-1}\left(\lambda \eta \frac{\mathrm{d}\Pi}{\mathrm{d}\eta} - \Pi\right) &- U(U-1) \\ (\lambda U - \mu)\eta \frac{\mathrm{d}(\Pi D^{-\gamma})}{\mathrm{d}\eta} + ((3\gamma - 1)U - 1)\Pi D^{-\alpha} &= 0 \end{aligned}$$

ODEs have the nice convenience of being generally easier to solve than PDEs.

As promised, here comes at long last the description of blastwaves.

The relevant parameters are just three if we assume spherical symmetry

$$[\rho] = \mathrm{M\,L^{-3}} \quad [t] = \mathrm{T} \quad [E_0] = \mathrm{M\,L^2\,T^{-2}}$$

which means our η parameter is just $\eta = r^5 t^{-2}$, since $[E_0]/[\rho] = \mathrm{L}^5\,T^{-2}$. Hence, simply inverting the expression yields

$$R \sim \left(\frac{E_0}{\rho_0}\right)^{1/5} t^{2/5} \quad v = \dot{R} \sim \frac{2}{5} R t^{-1} \tag{91}$$

incidentally, the proportionality constant is of order unity. Note that velocity is slowing down. This can be roughly explained recalling the conservation of momentum: As the wavefront expands it "gains" more mass [6] hence needing to slow down to conserve momentum.

6 The proper expression would probably be "entrains more material".

So far we've been assuming the process to be adiabatic, but is it actually a sensible guess? It certainly depends on timescales: If the cooling timescale is much larger than the dynamical timescale, than we're good.

On top of that, especially in the first instants of explosions, the medium can be considered *optically thick* thanks to the density compression induced by shockwaves. Since no radiation is coming out of there, we can assume that no energy is lost to radiation and is conserved.

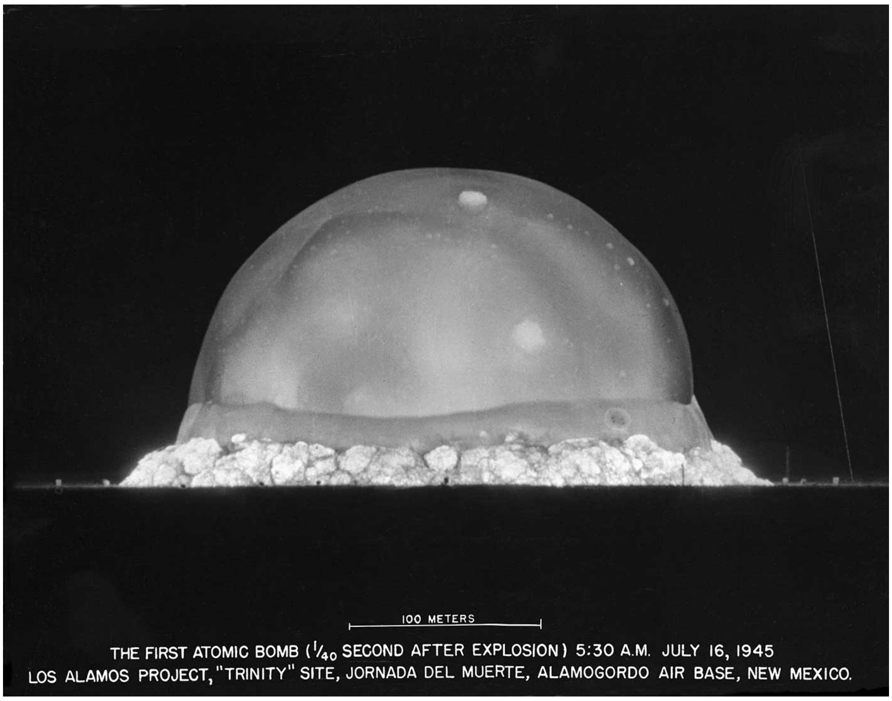

Figure 25: A real photograph portraying the first instants of the Trinity atomic test conducted in 1945 in New Mexico, Alamogordo.
From the image we see two things: After 1/40 of a second, the blastwave still retains (partial) spherical symmetry and the medium is still optically thick.

We can follow a similar line of reasoning and find out that the pressure of the blastwave scales as[7]

$$p \sim E_0 R^{-3}$$

That means that there is a radius for which each pressure is reached. That holds until the shockwave's pressure reaches ambient pressure[8] and equilibrium is established. When that happens, the shock stalls and propagates with constant velocity, usually some multiple of the sound speed.

As the shock expands, it cools down, hence E is not constant anymore. This causes the shock to slow down and *entrain* material in its course.

Eventually, the expansion will have to become subsonic due the arising of viscous forces, which we had completely ignored about in our model. But that was not without a logic. Viscous forces enter at second order in velocity, so in a first order model they can be initially neglected.

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7 It's even more simple if you recall that pressures are just energy densities. I know. It's disturbing.

8 Which is fated to happen sometime. As the blastwave expands, pressure continues to drop.

Since E is not conserved anymore, the evolution is governed by conservation of momentum, and the expansion law becomes

$$R \sim \left(\frac{MV}{\rho}\right)^{1/4} t^{1/4} \tag{92}$$

this is usually called the *Snowplow phase*.

The expression we've found to describe the Sedov-Taylor blastwave (91) also allows us to predict up to a honestly frightening degree the evolution of the expansion of Supernovae explosions. For this highly energetic phenomena, we can consider an initial injection of energy of order $E_0 \sim 10^{50}$ erg and a typical density of $\rho_0 \sim 10^{-24}\,\mathrm{g/cm^3}$. This means that the typical radius of the explosion is given by

$$R(t) \sim 10^{-0.7} \left(\frac{E_{50}}{\rho_{24}}\right)^{1/5} t_{yr}^{2/5} \;\; [\mathrm{pc}]$$

where the subscripts indicate that quantities are to be expressed in terms of the typical order of magnitude for SN events. So for example, assuming a radius of 1 parsec after a 100 years long expansion, you'd get a typical velocity for the wavefront of order $v \sim 10^3$ kms, which is a non-negligible fraction of the speed of light.

Part III

GRAVITATION

6 ACCRETION PHYSICS

Everything we've seen in the last few chapters will come into play here.

In this chapter we're considering the two main possibilities for accelerated plasma in presence of a gravitational field: It's either they receive a kick and then fall back on the surface (*accretion*) or just make it out safely from the gravitational field (*stellar winds*).

In the next sections, we'll try to give a punctual and precise description so to not underplay the important physical processes lying underneath.

6.1 STELLAR WINDS

Last century it had become clearer and clearer that an outgoing corpuscolar flow from the Sun was needed to explain certain phenomena, like cometary tail's disruptions.

Let us approximate the Sun (or a star, in general) as an autogravitating sphere of mass M with spherical symmetry, ignoring whatever drift may be caused by magnetic fields or rotation.

We're going to neglect the self-gravitating potential of the stellar corona hovering around the central star.

Following the original article by Parker [7] we may first discuss whether the stellar corona can be in hydrostatic equilibrium at all distances. The easy answer is that it (probably) cannot. If we were to do the full compu-

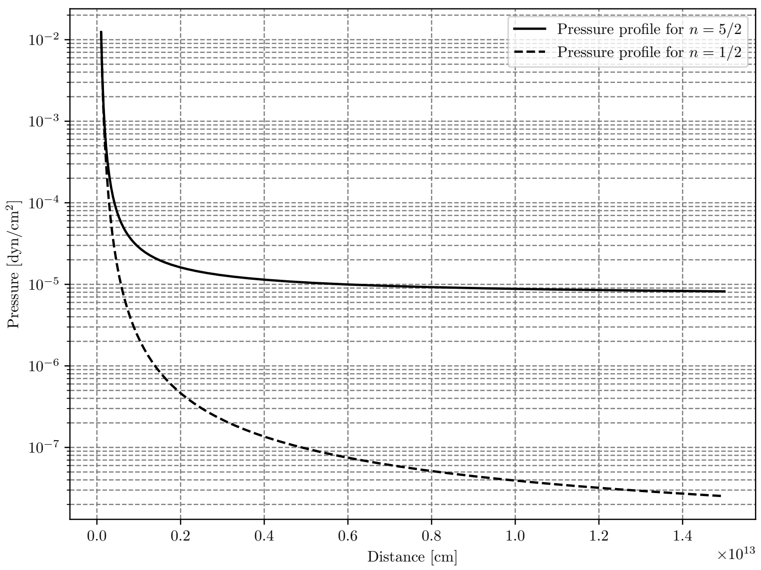

Figure 26: Pressure profile under the assumption of hydrostatic equilibrium.

tation, we'd find a pressure profile like the one in Fig.26, which presents asymptotic values for both cases considered (neutral and ionized hydrogen). Such asymptotic pressure is much larger than any other source of pressure we may expect finding out there. The pressure exerted by the ISM is about five order or magnitudes less than what is needed to mantain hydrostatic equilirbium at all distances.

Stellar coronas must then be expanding. Whether inwards or outwards that remains to be seen.

Since it is not possible to find a stable solution for the coronal gas to be in hydrostatic equilibrium at large distances from the central star, we now turn our eye towards a *dynamical model* able to describe the process.

The starting point are as always Euler's equation and the continuity equation, both expressed under the assumption of spherical symmetry and *stationarity*, so that we can put all time derivatives to zero. Eq.73 may be then written as

$$\rho v \frac{\mathrm{d}v}{\mathrm{d}r} = -\frac{\mathrm{d}p}{\mathrm{d}r} - \frac{G\rho M}{r^2} \tag{93}$$

The continuity equation in spherical symmetry yields

$$\frac{d}{dr}(r^2 \rho v) = 0 \tag{94}$$

which may be solved exactly

$$\dot{M} = 4\pi r^2 \rho v(r) = \mathrm{const.} \tag{95}$$

Assuming a barotropic equation of state (78)[1]:

$$v^2 \partial_r \log(v) = -c_s^2 \partial_r \log(\rho) - \frac{GM}{r^2}$$

which can be promptly restated as

$$(v^2 - c_s^2)\partial_r \log(v) = \frac{2c_s^2}{r}\left(1 - \frac{GM}{2c_s^2 r}\right)$$

In this form it's easy to see that if $v > c_s$ the fluid must be accelerating.

It may be worth pointing out that the latter equation may be cast in a self-similar fashion. This is easier seen if we assume the plasma to behave like a perfect gas: Under the assumption of constant temperature in the region we're observing, the equation of state is actually barotropic

$$p = nkT = \frac{kT}{\mu m_H}\rho$$

where μ is the mean molecular weight

$$\mu = \left(2X + \frac{3}{4}Y + \frac{1}{2}Z\right)^{-1}$$

1 This is not a bad assumption: The energy transported by the wind is much greater than that transported by heat conduction mechanisms [11]. This way we can assume temperature to be negligible beyond a given distance b and constant up to that point.

Expressed in this fashion, the sound speed c_s is simply equal to

$$c_s = \left(\frac{\mathrm{d}p}{\mathrm{d}\rho}\right)^{1/2} = \left(\frac{kT}{\mu m_H}\right)^{1/2} \approx \left(\frac{kT_0}{\mu m_H}\right)^{1/2}$$

at least to leading order. Plugging all this in the original expression for Euler's equation reads something like

$$\partial_\xi \psi \left(1 - \frac{\tau}{\psi}\right) = -2\xi^2 \partial_\xi \left(\frac{\tau}{\xi^2}\right) - \frac{2\lambda}{\xi^2} \tag{96}$$

This last equation was obtained setting $\xi = r/a$, $\tau = T(r)/T_0$, $\lambda = GMm_H/2kT_0 a$, $\psi = m_H v^2/2kT_0{}^2$.

To integrate (96) we shall assume that temperature is uniform and equal to T_0 in the range $r \in (a, b)$, with b a distance beyond which heating mechanism are negligible.

The interesting physics is in the region $r < b$.

Here temperature is constant, so $\tau = 1$ and the ODE is immediately solved

$$\psi - \ln(\psi) = \psi_0 - \ln(\psi_0) + 4\ln(\xi) - 2\lambda\left(1 - \frac{1}{\xi}\right) \tag{97}$$

where the integration constant has been chosen so that $\psi(\xi = 1) = \psi_0$.

Discarding all non-physical solutions (negative magnitudes and complex solutions) requires $\lambda > 2$. Once you've got rid of all the unwanted nasty solutions, the final expression is something like

$$\psi - \ln\psi = -3 - 4\ln\frac{\lambda}{2} + 4\ln\xi + \frac{2\lambda}{\xi} \tag{98}$$

which is plotted for different values of temperature T_0 in Fig.27. Closed this self-complacent excursus, let's turn back to the expression shown in class

$$(v^2 - c_s^2)\partial_r \log(v) = \frac{2c_s^2}{r}\left(1 - \frac{GM}{2c_s^2 r}\right)$$

If $v > 0$ we have at first bounded motion, pretty much like an oscillation of some sort, but then, as the fluid accelerates, it gets closer and closer to the escape velocity[3] until the fluid finally escapes.

But what if the the fluid is *ingoing* rather than outgoing? We won't have winds, proper, but rather *spherical accretion*, which is discussed in the next section.

6.2 ACCRETION

Coming soon.

2 I do admit that I've pasted the expression I'd obtained when I wrote my Batchelor thesis. I tried to adapt my previous notation to the one of this notes. Please forgive me if there's some subscript not *subscripting* properly.

3 Note that in this regard, thermal fluctuations of the velocity often give the plasma the needed bump to escape the gravitational pull.

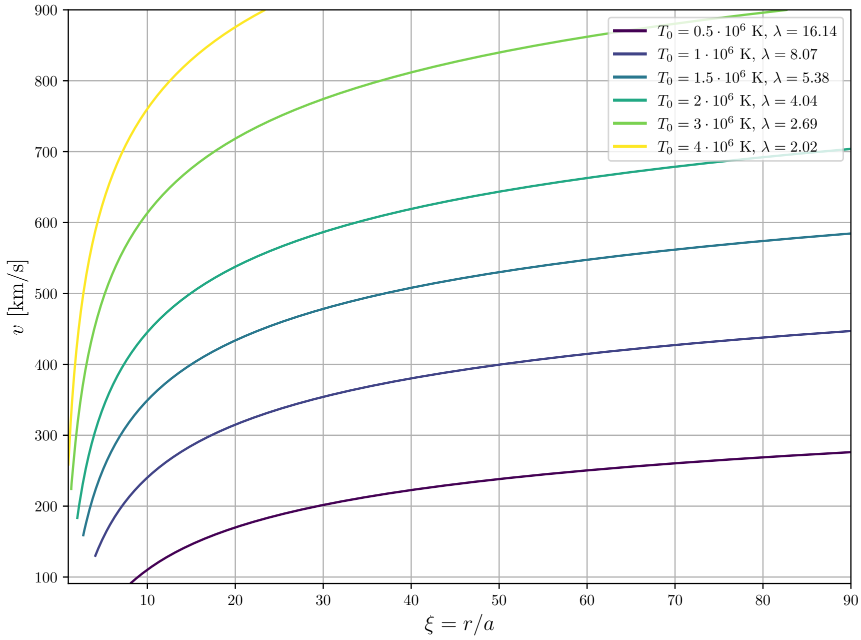

Figure 27: Velocity profile for eq.98 for different values of T_0.

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