Notes of Astrophysical Processes

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1 Fundaments of statistical mechanics

1.1 Distribution function and collision-less Boltzmann equation

Let's briefly recall some fundamentals of statistical mechanics.
An ensemble of particles can be treated as a continuum. In this case some useful quantities are:

- average distance between particles

$$d = \left(\frac{4\pi N}{3}\right)^{-\frac{1}{3}} \tag{1.1}$$

  where N is the total number of particles

- assuming LTE we can derive the De Broglie length

$$\lambda \approx \frac{h}{\sqrt{3mK_bT}} \tag{1.2}$$

  where h and K_b are the constants of Planck and Boltzmann

- Bohr radius

$$a_0 = \frac{\hbar}{m_e c\alpha} \tag{1.3}$$

  where α is fine-structure constant

Considering distances large enough, interactions between particles, e.g. collisions, are always binary. We usually assume that we can neglect the details of these interactions.
Exceptions are: very dense environments (e.g. neutron stars), long-range forces like Coulomb interactions.
Under these conditions, a gas is described by a distribution function $f(\vec{x}, \vec{u}, t)$ so that $f(\vec{x}, \vec{u}, t)d^3\vec{x}d^3\vec{u}dt$ is the number of particles in a volume $d^3\vec{x}$ with velocity $d^3\vec{u}$ at time dt. So:

- the total number of particles is

$$N = \int d^3\vec{x}d^3\vec{u}dtf(\vec{x}, \vec{u}, t) \tag{1.4}$$

- the particle density is

$$n(\vec{x}, t) = \int d^3\vec{u}f(\vec{x}, \vec{u}, t) \tag{1.5}$$

- the mean density is

$$\rho = Am_H n(\vec{x}, t) \tag{1.6}$$

  where A is the atomic weight

- the average velocity is

$$\vec{v} = \int d^3\vec{x}f(\vec{x}, \vec{u}, t)\vec{u} \tag{1.7}$$

To understand how the distribution function evolves in the time assume that there is some force $\vec{F}$ so that each particle of mass m has an acceleration $\vec{a} = \frac{\vec{F}}{m}$.
Let's consider a set of particles in $(\vec{x_0}, \vec{u_0})$ at time t_0, they will evolve to

$$\begin{aligned} \vec{x} &= \vec{x_0} + \vec{u_0}dt \\ \vec{u} &= \vec{u_0} + \vec{a}dt \end{aligned} \tag{1.8}$$

So we want to understand how $f(\vec{x_0}, \vec{u_0}, t_0)$ evolves into $f(\vec{x}, \vec{u}, t)$. The volume elements are related by

$$d^3\vec{x}d^3\vec{u} = |J|d^3\vec{x_0}d^3\vec{u_0} \tag{1.9}$$

where J is the Jacobian of the transformation.
If the number of particles in the volumes is equal, e.g. $dN = dN_0$, and if we consider the first order, e.g. $|J| = 1 + O(dt^2) \Rightarrow d^3\vec{x}d^3\vec{u} = d^3\vec{x_0}d^3\vec{u_0}$, then

$$f(\vec{x_0} + \vec{u_0}dt, \vec{u_0} + \vec{a}dt, t_0 + dt) = f(\vec{x_0}, \vec{u_0}, t_0) \tag{1.10}$$

This implies that

$$\partial_t f + u^i\partial_i f + a^i\partial_{u_i} f = 0 \tag{1.11}$$

which is called *collisionless Boltzmann equation* or *Vlasov's equation*.
One example of a non collisional gas are galaxies. In this case, if ϕ is the gravitational potential, we have that $\vec{a} = -\frac{\vec{\nabla}\phi}{m}$ and the boundary condition is given by $\nabla^2\phi = 4\pi G\rho$, $\rho = \int d^3\vec{u}f(\vec{x}, \vec{u}, t)M$

1.2 Collisional rate balance and equilibrium distribution

We can use Vlasov's equation also for the collisional case as well by writing:

$$\partial_t f + u^i \partial_i f + a^i \partial_{u_i} f = \left(\frac{Df}{Dt}\right)_{coll}. \tag{1.12}$$

The collision integral can be put in a simple form depending on the distribution functions of in-going and out-going particles

$$\left(\frac{Df}{Dt}\right)_{coll} = \frac{Rin - Rout}{d^3\vec{x}d^3\vec{u}} \tag{1.13}$$

at equilibrium they will balance statistically, so $\left(\frac{Df}{Dt}\right)_{coll} = 0$. The collisional balance condition implies statistical equilibrium, hence thermodynamical equilibrium and one derive the Boltzmann distribution:

$$f(\vec{U}) = N\left(\frac{m}{2\pi K_b T}\right)^{\frac{3}{2}} e^{\frac{mU^2}{2K_b T}} \tag{1.14}$$

where $\vec{U}$ is the mean particle velocity.
Boltzmann H-theorem shows the Maxwellian to also be the unique form of the equilibrium distribution.
We deal with moving particles that go from regions to regions, which are subject to ionization, etc. In most cases in astrophysics how quickly we can reach the equilibrium will be set from collisions between electrons and heavier nuclei or protons. So we have to consider the Coulomb Force $F = \frac{Z_1 Z_2 e^2}{r^2} \sim \frac{1}{r^2}$ and N particles in $(r, r+dr) \sim \frac{1}{r^2}$ hence the binary collision assumption breaks down.[1]
In a very simple way we can consider some volume into this cloud of charges, homogeneously distributed. The more I look further the more each charge is shielded. The potential from shielded ion is

$$\phi(r) = \frac{Z_i e}{r} e^{-\frac{r}{D}} \tag{1.15}$$

where [2] $D = \left[\frac{K_b T}{4\pi e^2 (n_e + \sum Z^2 n_i)}\right]^{\frac{1}{2}}$, which is called *Debye lenght*, sets a cutoff in the collision integral, so

$$R_{in} = \int_{b_{min}}^{D} dIP(I) \tag{1.16}$$

If we indicate with b the impact parameter, the integral is then defined between:

- the minimum impact parameter $b_{min} = \frac{Z_i e^2}{m_e v^2}$
- the maximum impact parameter $b_{max} = D$

It is convenient to define the quantity $\Lambda = \frac{b_{max}}{b_{min}}$, which is the ratio of maximum to minimum impact parameter to be considered in the collision integral. Remember that for hydrogen plasma $\Lambda = \frac{3}{e^3}\left(\frac{K_b^3 T^3}{8\pi n_e}\right)^{\frac{1}{2}}$ and typical values are $ln\Lambda \simeq 10$.
Now we can define the relaxation time that is the average time between two collisions, e.g. the time taken by the ionized gas to return to thermodynamic equilibrium:

$$t_{relaxation} \approx \frac{m_e^2 v^3}{8\pi e^4 n_p log\Lambda} \tag{1.17}$$

and the $Spitzer\ self - relaxation\ time$, e.g. the time for trajectories to isotropise:

$$t_D = \frac{m^{\frac{1}{2}}(3K_b T)^{\frac{3}{2}}}{[8\pi e^4 Z^4 n 0.074 log\Lambda]} = \frac{11.4(AT^3)^{\frac{1}{2}}}{nZ^4 log\Lambda} \tag{1.18}$$

From the comparison of these two quantities we can define if the system we are considering can be considered at equilibrium. For example, in stellar atmospheres we find out that $t_{relaxation} \simeq 10t_D$, so the gas can be considered at all time in equilibrium. Now, if radiation processes are important, e.g. if electron densities are large enough, we can consider the velocity Maxwellian almost always.

[1] Details of calculations in Mihalas & Mihalas chapter 1

[2] numerically $D = 4.8\left(\frac{T}{n_e}\right)^{\frac{1}{2}}$ for H, e.g. fot $T = 10^4 K$, $n_e = 10^{14} cm^{-3}$ $D \sim 5 \cdot 10^{-5} cm$

1.3 Level populations and maximum entropy

In order to obtain the Boltzmann distribution:

$$\frac{n_i}{n_j} = \frac{g_i}{g_j} e^{-\frac{\epsilon_i - \epsilon_j}{K_b T}} \tag{1.19}$$

where

- n_i, n_j are the particles in levels i and j
- g_i, g_j are degeneration factors
- ϵ_i, ϵ_j are the energies of the levels i and j

let's start by defining the *Entropy*

$$S = KlogW \tag{1.20}$$

W is the probability distribution describing the state of the system. To determine it we enumerate all possibilities assuming particles are indistinguishable.

$$\begin{aligned} W(\{n_i\}) &= \frac{\prod_i g_i}{\prod_i n_i!} \\ W &= \sum_i W(\{n_i\}) \end{aligned} \tag{1.21}$$

with constraints $\sum_i n_i = N$ (that fix the normalization) and $\sum_i n_i \epsilon_i = E$, with E mean energy. In order to find the maximum value for the entropy, we have to maximize W. To do that we need to resolve $\frac{\delta S}{\delta W} = \frac{\delta KlogW}{\delta W} = 0$, and using Lagrange multipliers we obtain[3]:

$$\begin{aligned} \frac{\delta}{\delta n_i}\Big(KlogW - \alpha(\sum_i n_i - N) - \beta(\sum_i n_i \epsilon_i - E)\Big) = 0 \\ \Rightarrow n_i = \alpha g_i e^{-\beta \epsilon_i} \ , \quad \beta = \frac{1}{K_b T} \end{aligned} \tag{1.22}$$

result which leads to the Boltzmann distribution we wanted.
From the partition function

$$Z = \sum_i g_i e^{-\beta \epsilon_i} \tag{1.23}$$

instead, we get all the thermodynamic proprieties of the gas. In fact:

- the energy is $E = \frac{\partial logZ}{\partial \beta}$
- the probability that we have a state s is $P_s = \frac{1}{Z} e^{\frac{E_s}{K_b T}}$

[3] α and β are Lagrange multipliers

2 Fluid

2.1 Forces on a fluid, stress tensor and pressure

Argument is again based on the typical distance between particles and the possibility of grouping together several *cells* in the phase space. Essentially the idea is that if we are able to formulate a unique probability distribution over the phase space (hence a Maxwellian), then we can define the macroscopic quantities such as temperature, pressure and density in a well defined manner. We can also well define averages on the distribution function and treat those as if the material was a continuum.
So we are going to assume that we can model the gas as a continuum, defined starting from averages over the distribution function. For instance

$$\begin{aligned} n(\vec{x},t) &= \int f(\vec{x},\vec{v},t)d^3\vec{v} \\ \vec{v}(\vec{x},t) &= \frac{1}{n}\int \vec{u}f(\vec{x},\vec{u},t)d^3\vec{u} \end{aligned} \tag{2.1}$$

in which we suppose that the velocity $\vec{u}$ is composed by a mean velocity $\vec{U}$ and a random one $\vec{v}$ with zero mean: $\vec{u} = \vec{v} + \vec{U}$.
Before deriving the basic equations describing the dynamics of fluids, let's start understanding how to describe forces acting on a fluid:

- contact forces
- body or volume forces

Volume forces are long range forces, such as gravity, and we will always assume that the volume element is small enough that they do not appreciably vary. For instance, if we consider some volume δV the force can be written as $\vec{F} = \vec{F}(\vec{x},t)\rho\delta V$, where $\vec{F}(\vec{x},t)$ is the force for unit mass and in the case of gravity is equal to $\vec{g}$.
Contact forces are the short range forces so they act at the interface between a volume δV and the surrondings in a thin layer whose width is negligible, but still large enough so that the averaging over distribution function is still valid. In this case we can write $\vec{F} = \sum(\vec{n},\vec{x},t)\delta A$, where $\sum(\vec{n},\vec{x},t)$ is the local stress that the fluid exerts on the environment.
To visualize the surface forces we refer to the following figure

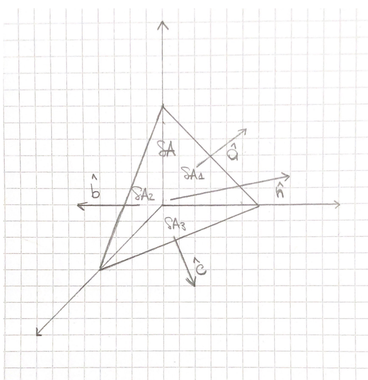

Figura 2.1: Infinitesimal tetrahedron

So the total surface force is:

$$\sum(\hat{n})\delta A + \sum(-\hat{a})\delta A_1 + \sum(-\hat{b})\delta A_2 + \sum(-\hat{c})\delta A_3 \tag{2.2}$$

and, because of the orthogonality, e.g. $\delta A_1 = \hat{a}\cdot\hat{n}\delta A$, the i-th component of the force is

$$F_i = \sum{}_i(\hat{n})\delta A - (a_j \sum{}_i(-\hat{a}) + b_j \sum{}_i(-\hat{b}) + c_j \sum{}_i(-\hat{c}))\delta A \tag{2.3}$$

Now if the size of the tetrahedron goes to zero, since $\delta V \to 0$ then $\vec{F}_{volume}$, $\vec{a}_{\delta_V} \to 0$, so from the equation $m\vec{a} = Volume\ forces + Surface\ forces$, we understand that the surface forces tend to zero as well. This condition is satisfied by

$$\sum{}_i(\hat{n}) = (a_j \sum{}_i(-\hat{a}) + b_j \sum{}_i(-\hat{b}) + c_j \sum{}_i(-\hat{c}))n_j \tag{2.4}$$

So the force on one side depends on the forces on the other sides, hence

$$\sum{}_i(\hat{n}) = \sigma_{ij} n_j \tag{2.5}$$

σ_{ij} is called *Stress Tensor* and it is a symmetric second order tensor, e.g. $\sigma_{ij} = \sigma_{ji}$. [4]
In matrix form

$$\sigma_{ij} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}$$

We indicate σ_{ii} as the *Normal stresses* and σ_{ij} as the *Shearing stresses*. Being symmetric σ_{ij} can be diagonalised, eigen vectors are the principal axes of the stress tensor. In a fluid at rest it is easy to pick the principal axes and write

$$\sigma_{ij} = \begin{pmatrix} \frac{1}{3}\sigma_{11} & & \\ & \ddots & \\ & & \ddots \end{pmatrix} + \begin{pmatrix} \sigma'_{11} - \frac{1}{3}\sigma_{11} & & \\ & \ddots & \\ & & \ddots \end{pmatrix}$$

If the volume has to remain unchanged, the stress tensor must be isotropic everywhere, hence $\sigma'_{ii} - \frac{1}{3}\sigma_{ii} = 0$. Then we can introduce the *Pressure* as $\sigma_{ij} = -p\delta_{ij} \Rightarrow p = -\frac{1}{3}\sigma_{ii}$

[4]just to easily visualize it

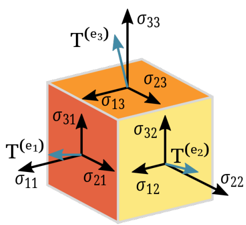

Figura 2.2: Stress Tensor

3 Hydrostatic equilibrium

For equilibrium we need to reach the case $\sum \vec{F} = 0$, where $\vec{F}$ is the total force.
In our case (indicating with $\sum$ the forces per unit area) the previous condition is explicitly written as

$$\int dV \rho \vec{F} + \int d\vec{A} \sum = 0$$

and using the *Divergence theorem*, we obtain

$$\int dV(\rho\vec{F} - \vec{\nabla}P) = 0 \quad \Rightarrow \quad \rho\vec{F} = \vec{\nabla}P \tag{3.1}$$

thus volume forces must balance pressure forces.
If $\vec{F}$ is conservative, then $\vec{F} = -\vec{\nabla}\phi \Rightarrow -\rho\vec{\nabla}\phi = \vec{\nabla}P$. If we take the curl

$$\vec{\nabla} \times (-\rho\vec{\nabla}\phi) = \vec{\nabla} \times \vec{\nabla}P \quad \Rightarrow \quad \vec{\nabla}\rho \times \vec{\nabla}\phi = 0 \tag{3.2}$$

So isopotential curves are also isodensity and isobars curves.
We still get the equation of *Hydrostatic Equilibrium*:

$$\frac{dP}{d\phi} = -\rho(\phi) \tag{3.3}$$

Let's consider again the relation $-\rho\vec{\nabla}\phi = \vec{\nabla}P$ and let's take the case of gravity, e.g. $\vec{\nabla}^2\phi = 4\pi G\rho$.
Thus:

$$\vec{\nabla}\Big(-\frac{\vec{\nabla}P}{\rho}\Big) = 4\pi G\rho \tag{3.4}$$

Notice that in order to resolve this equation, we need to specify a constitutive relation to link P and ρ, like an equation of state.

3.0.1 Plane parallel atmosphere

Consider a slab identifying the $\hat{z}$ direction as in figure

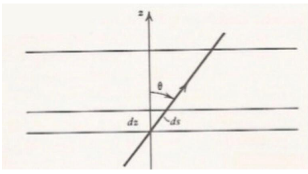

Figura 3.1: Plane parallel atmosphere approximation

Call this a local plane parallel approximation.
Now consider $\vec{g} = constant$, direct as $-\hat{z}$, so

$$\frac{dP}{dz} = -g\rho \tag{3.5}$$

We said that we need a relation between P and ρ, then we simply choose $\rho = \rho_0 = constant$. In this way we obtain the *Stelvino law*:

$$P = P_0 - \rho gz \tag{3.6}$$

Instead, if we choose an isothermal equation of state $P \sim \rho$, then

$$P = P_0 e^{-\frac{z}{H}} \tag{3.7}$$

whit H indicating the pressure scale height. Locally this quantity sets the scale over which we can consider the density constant.

3.0.2 Spherical rotating object

First of all we rewrite the *Equation* (3.4) in spherical coordinates

$$\frac{1}{r^2}\frac{d}{dr}\Big(\frac{r^2}{\rho}\frac{dP}{dr}\Big) = -4\pi G\rho \tag{3.8}$$

The easiest way to solve this equation is to take a polytropic equation of state, e.g $P = C\rho^{1+\frac{1}{n}}$. We know how to calculate analytically the solution for $n = 0$ and $n = 5$. In particular

$$\begin{aligned} n = 0 \quad &\Rightarrow \quad P = \frac{2\pi}{3}G\rho_0^2(R^2 - r^2) \\ n = 5 \quad &\Rightarrow \quad P = \frac{27R^3C^{\frac{5}{2}}}{(2\pi G)^{\frac{3}{2}}(R^2 + r^2)^3} \end{aligned} \tag{3.9}$$

Consider now a fluid rotating rigidly around an axis, call it z.

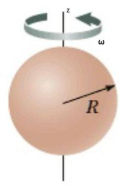

Figura 3.2: Spherical rotating object

In the corotating frame, with ω constant, the centrifugal potential is $\phi_c = -\frac{1}{2}\omega^2 r^2$, so

$$-\rho\vec{\nabla}(\phi + \phi_c) = \vec{\nabla}P \quad \Rightarrow \quad -\rho\vec{\nabla}(\phi - \frac{1}{2}\omega^2 r^2) = \vec{\nabla}P \tag{3.10}$$

Now take the two components ($\vec{\nabla} = \Big(\frac{\partial}{\partial r}, \frac{\partial}{\partial z}\Big)$)

$$\begin{aligned} -\rho\Big[\frac{\partial}{\partial r}\phi - \omega^2 r\Big] = \frac{\partial P}{\partial r} \quad &\Rightarrow \quad \frac{\partial}{\partial r}\phi = -\frac{1}{\rho}\frac{\partial P}{\partial r} + \omega^2 r \\ -\rho\Big[\frac{\partial}{\partial z}\phi\Big] = \frac{\partial P}{\partial z} \quad &\Rightarrow \quad \frac{\partial}{\partial z}\phi = -\frac{1}{\rho}\frac{\partial P}{\partial z} \end{aligned} \tag{3.11}$$

and substituting in the *Poisson equation* we obtain:

$$\frac{1}{r}\frac{\partial}{\partial r}\Big[r\Big(-\frac{1}{\rho}\frac{\partial P}{\partial r} + \omega^2 r\Big)\Big] + \frac{\partial}{\partial z}\Big[-\frac{1}{\rho}\frac{\partial P}{\partial z}\Big] = 4\pi G\rho \tag{3.12}$$

If we specify an equation of state to relate P and ρ we can arrive to a single PDE for ρ that we can solve numerically. In particular:

- if $\rho = \rho_0$ we get as solutions the *Maclaurin spheroid equations*
- else the *Equation* (3.12) is not analytically solvable, but they are numerically solvable

4 Moments of the Vlasov equation

Let us go back to the distribution function $f(\vec{x},\vec{v},t)$ and the Boltzmann equation.
At local statistical equilibrium the rate collisions balance, e.g. $\left(\frac{df}{dt}\right)_{coll} = 0$, and if we take the general moment of a function $g(\vec{x},\vec{v},t)$ with respect to $f(\vec{x},\vec{v},t)$ we can easily rewrite and solve each piece of the $Vlasov's\ equation$.

$$\begin{aligned} &\frac{\partial f}{\partial t} + \vec{v}\ \vec{\nabla} f + \dot{\vec{v}}\vec{\nabla}_{\vec{v}} f = 0 \\ &\Downarrow \\ 1^{st}\ piece: &\int d^3\vec{v}\frac{\partial f}{\partial t} g = \int d^3\vec{v}\Big(\frac{\partial (fg)}{\partial t} - f\frac{\partial g}{\partial t}\Big) \\ 2^{nd}\ piece: &\int d^3\vec{v} g\vec{v}\cdot\vec{\nabla} f = \int d^3\vec{v}\Big(\vec{\nabla}\cdot(g\vec{v}f) - gf\vec{\nabla}\cdot\vec{v} - f\vec{v}\vec{\nabla} g\Big) \\ 3^{rd}\ piece: &\int d^3\vec{v} g\dot{\vec{v}}\vec{\nabla}_{\vec{v}} f = \int d^3\vec{v}\Big(\vec{\nabla}_{\vec{v}}(g\dot{\vec{v}}f) - gf\vec{\nabla}_{\vec{v}}\cdot\dot{\vec{v}} - f\dot{\vec{v}}\vec{\nabla}_{\vec{v}} g\Big) \end{aligned} \tag{4.1}$$

Let's simplify a few pieces:

- because of canonical variables $\int d^3\vec{v} gf\vec{\nabla}\vec{v} = 0$
- if we transform via divergence theorem the integral $\int d^3\vec{v}\vec{\nabla}_{\vec{v}}(g\dot{\vec{v}}f)$ to an integral at infinity, where there is no particles, it goes to zero
- since $\dot{\vec{v}} = \frac{\vec{F}}{m}$ and for conservative or magnetic force $\vec{\nabla}_{\vec{v}}\frac{\vec{F}}{m} = 0$, then $\int d^3\vec{v} gf\vec{\nabla}_{\vec{v}}\cdot\dot{\vec{v}} = 0$.

Now let's recall that $\int d^3\vec{v} f \equiv n$ and define $< g >= \frac{1}{n}\int d^3\vec{v} gf$. So the three terms can be rewritten as

$$\begin{aligned} 1^{st}\ piece = &\ \frac{\partial}{\partial t} n < g > -n\Big\langle\frac{\partial g}{\partial t}\Big\rangle \\ 2^{nd}\ piece = &\ \vec{\nabla}(n < g\vec{v} >) - n < \vec{v}\vec{\nabla} g > \\ 3^{rd}\ piece = &\ -n < \dot{\vec{v}}\vec{\nabla}_{\vec{v}} g > \end{aligned} \tag{4.2}$$

4.1 Different results

4.1.1 Mass conservation

Impose $g = 1$ and remember that $\rho = Am_A n(\vec{x})$. Then *continuity equation* or *mass conservation equation* is:

$$\begin{aligned} &\frac{\partial}{\partial t} n + \vec{\nabla} n < \vec{v} >= 0 \\ &\Downarrow \\ &\frac{\partial}{\partial t}\rho + \vec{\nabla}(\rho < \vec{v} >) = 0 \end{aligned} \tag{4.3}$$

Since $\vec{v} = \vec{V} + \vec{u}$ [5] and $< u >= 0 \Rightarrow \vec{\nabla}(\rho < \vec{v} >) = \vec{\nabla}(\rho\vec{V})$. Locally $\rho\vec{V}$ is the mass density flux.
Let's take the integral over a volume

$$\int d^3\vec{x}\frac{\partial\rho}{\partial t} + \int d^3\vec{x}\ \vec{\nabla}\cdot(\rho\vec{V}) = 0 \quad\Rightarrow\quad \dot{M} + \int_{\partial S}\hat{n} dS\rho\vec{V} = 0 \tag{4.4}$$

where $\dot{M}$ is the Rate of mass changing while the integral represents the mass flux through a surface ∂S.

4.1.2 Euler equation

Impose $g = m\vec{v}$, then our equation is rewritten as

$$\frac{\partial}{\partial t} n < g > -n\Big\langle\frac{\partial g}{\partial t}\Big\rangle + \vec{\nabla}(n < g\vec{v} >) - n < \vec{v}\vec{\nabla} g > -n < \dot{\vec{v}}\vec{\nabla}_{\vec{v}} g >= 0.$$

Let's analyze some pieces separately:

$$\begin{aligned} 1^{st}\ piece = &\ \frac{\partial}{\partial t} < \vec{v} > -\frac{\partial\vec{v}}{\partial t} \\ 2^{nd}\ piece = &\ < \vec{v}\otimes\vec{v} > - < \vec{v}\cdot\vec{\nabla}\vec{v} > \\ 3^{rd}\ piece = &\ \vec{a}\cdot\vec{\nabla}_{\vec{v}}\vec{v} \end{aligned} \tag{4.5}$$

[5] We define the velocity as $v_i = V_i + u_i$, *where V_i is the mean velocity and u_i is a random zero−mean variable*

The situation is simplified considering that not having direct dependence on t and $\vec{x}$ then $\frac{\partial \vec{v}}{\partial t}$ and $\vec{v} \cdot \vec{\nabla} \vec{v}$ are null. Instead, the last piece depends on the actual force under consideration.
So

$$\frac{\partial}{\partial t} < \rho \vec{v} > + \vec{\nabla} \cdot < \rho \vec{v} \otimes \vec{v} > = external\ forces$$

$$\Downarrow$$

$$In\ component\ \ \frac{\partial}{\partial t} < \rho V_i > + \partial_j < \rho v_i v_j > =$$

$$\frac{\partial}{\partial t} \rho V_i + \partial_j < \rho (u_i + V_i)(u_j + V_j) > =$$

$$\frac{\partial}{\partial t} \rho V_i + \partial_j (\rho < u_i u_j > + \rho V_i V_j) = external\ forces \tag{4.6}$$

where $< u_i u_j >$ and $V_i V_j$ indicate the *Isotropic Pressure* and the *Stress Tensor*.

Introducing the 2x2 *Reynolds stress tensor* T_{ij} and the *external force per unit mass* $\vec{f}$, we obtain the *Euler equation*

$$\frac{\partial}{\partial t} \rho \vec{V} + \vec{\nabla} \bar{T} - \rho \vec{f} = 0 \tag{4.7}$$

It is useful to distinguish two different reference frame:

- Laboratory frame, also named *Eulerian frame*, where the fluid moves
- Comoving frame, also named *Lagrangian frame*, where we sit on a fluid.

The two frames are essentially equivalent, but carry an important physical distinction in relativistic flows. Indeed, in the second case it is convenient to define the *covariant derivative* $\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla}$ to describe the dynamics of the fluid. Note that in the relativistic case this is the actual covariant derivative[6], instead in this contest this is reffered to as a material or convective derivative.
In an *incompressible* fluid we must have $\frac{D\rho}{Dt} = 0$. This statement is equivalent to setting $\vec{\nabla} \cdot \vec{v} = 0$, hence that the velocity field has no divergence.

4.1.3 Kinetic Energy

Impose [7]$g = \frac{1}{2} m v^2$ and note that $\frac{\partial g}{\partial t} = \vec{\nabla} g = 0$.
Considering the same equation

$$\frac{\partial}{\partial t} n < g > - n \left\langle \frac{\partial g}{\partial t} \right\rangle + \vec{\nabla}(n < g \vec{v} >) - n < \vec{v} \vec{\nabla} g > - n < \dot{\vec{v}} \vec{\nabla}_{\vec{v}} g > = 0$$

we may rewrite its terms as:

$$1^{st}\ term\ \ \frac{\partial}{\partial t} n < g > = \frac{\partial}{\partial t} \Big(\frac{\rho}{2} < v^2 > \Big) = \frac{\partial}{\partial t} \Big(\frac{\rho}{2} < (V+u)(V+u) > \Big) = \frac{\partial}{\partial t} \Big(\frac{\rho}{2} (V^2 + < u^2 >) \Big)$$

$$2^{nd}\ term\ \ \vec{\nabla}(n < g \vec{v} >) = \vec{\nabla} \rho < \frac{1}{2} v^2 \vec{v} > = \vec{\nabla} \frac{\rho}{2} < (u^2 + V^2 + 2 \vec{u} \cdot \vec{V})(\vec{u} + \vec{V}) > =$$

$$\vec{\nabla} \Big[\frac{\rho}{2} < u^2 \vec{u} > + \frac{\rho}{2} < u^2 > \vec{V} + \frac{\rho}{2} V^2 \vec{V} + \rho < \vec{u} \cdot (\vec{u} \cdot \vec{V}) > \Big]$$

$$3^{rd}\ term\ \ - n < \dot{\vec{v}} \cdot \vec{\nabla}_{\vec{v}} g > = - \vec{F} \cdot \vec{V} \tag{4.8}$$

Now let's define the *heat flux* as $\phi_h = \frac{\rho}{2} < u^2 \vec{u} >$, let's recognize that $\rho < \vec{u} \cdot (\vec{u} \cdot \vec{V}) > =$ $\rho < u_i u_j V_j > = \rho < u_i u_j > V_j = \rho P V_j$ [8], then consider the *internal energy* $\epsilon \equiv \frac{\rho < u^2 >}{2}$ [9], so we get

$$\frac{\partial}{\partial t} \Big[\frac{\rho}{2} V^2 + \epsilon \Big] + \vec{\nabla} \Big[\rho \vec{V} \Big(\frac{V^2}{2} + \epsilon \Big) \Big] = \vec{F} \cdot \vec{V} - \vec{\nabla} \cdot [\vec{\phi_h} + P \vec{V}] \tag{4.9}$$

[6] In covariant form the *continuity equation* is written as $\frac{D\rho}{Dt} - \rho \vec{\nabla} \cdot \vec{v} = 0$
[7] recall that v is the *microscopic velocity*
[8] recall that $< u_i u_j >$ is the *isotropic pressure* P
[9] e.g. $\frac{3}{2} n K_b T$

4.2 Particle paths, streamlines, streaklines

Let's introduce some quantities:

- **Particle path**: it is the trajectory followed by a volume element (particle) in time. Quantitatively $\vec{x}(t) = \vec{x}_0 + \int_0^t \vec{v}(\vec{x}, t^{'})dt^{'}$
- **Streamlines**: lines tangent to the velocity field at a fixed instant of time. If we introduce an $affine\ parameter\ s$, the relation $\frac{d\vec{x}}{ds} = \vec{v}(\vec{x}, t)$ defines the streamline. [10]
- **Streakline**: curve traced by all fluid elements passing through a given fixed point.

In general the three curves are distinct and coincide only in steady flows, for which $\vec{v}(\vec{x})$ does not depend on time.

[10] Note the similarity with the definition of *geodetic*

5 Virial theorem

The standard statement of the *Virial Theorem* states that any system (described by a Vlasov like equation) must obey to

$$\frac{1}{2}\frac{d^2I}{dt^2} = 2T + W \tag{5.1}$$

with $T = kinetic\ energy = \int \frac{1}{2}\dot{x}_i\dot{x}_i dm$ and $W = potential\ energy = \int x_i G_i dm$[11].
Hence the virial theorem connects bulk changes in the matter distribution with variations in kinetic and potential energies.
For a steady state system one can derive useful constraints that apply globally to the system, e.g. $2T = -W$.
Note that T and W are defined by taking averages over the distribution function, hence we must be in the fluid condition.
Let's take *Euler equation* and let's write it in the form we will use teh most:

$$\frac{\partial v_i}{\partial t} + v_j\frac{\partial v_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} - \frac{\partial \phi}{\partial x_i} \tag{5.2}$$

Assuming that $f_i = -\partial_i\phi$. Take the first moment with respect to position to obtain a version of the virial theorem accounting for the pressure as

$$2T + \Omega - 3\int PdV = 0 \tag{5.3}$$

where the second term represents the *gravitational contribute*, while the third one is the *isotropic thermodynamic work*.

5.1 Energy conservation

Let's go back to $g = \frac{1}{2}mv^2$. We get

$$\frac{\partial}{\partial t}\Big(\frac{\rho}{2}v^2\Big) + \vec{\nabla}\Big(\rho\vec{v}\Big[\frac{1}{2}v^2 + \frac{P}{\rho}\Big]\Big) + \vec{v}\cdot\vec{F} = 0 \tag{5.4}$$

In this contest $\frac{P}{\rho} \equiv \epsilon$ is the *internal thermal energy*, so that $\rho\vec{v}\Big[\frac{1}{2}v^2 + \epsilon\Big]$ is the *heat flux* $\vec{F}_h$.
Note that for stationary flows and potential forces we can integrate over the volume and recover *Bernoulli's theorem*

$$\frac{1}{2}\rho v^2 + P + \rho\phi = \ constant \tag{5.5}$$

Now, if we realise that the heat flux must be balance by the rate of change of internal energy, we can write a continuity equation as

$$\tfrac{\partial\rho\epsilon}{\partial t} + \vec{\nabla}\cdot\vec{F}_h = 0$$

with ϵ used as the *internal energy density*.
Called E the *total energy*, it is useful recall the relation

$$\frac{dE}{dt} = T\frac{dS}{dt} + \frac{P}{\rho^2}\frac{d\rho}{dt} \tag{5.6}$$

because now we have

$$\frac{\partial}{\partial t}\rho\Big(\frac{\rho}{2}v^2 + \epsilon\Big) + \vec{\nabla}\Big[\rho\vec{v}\Big(\frac{1}{2}v^2 + \epsilon + \frac{P}{\rho}\Big)\Big] = \frac{dE}{dt} - \rho T\frac{dS}{dt} + \Lambda \tag{5.7}$$

with $\Lambda(T,\rho)$ includes all energy losses, also the ones not included in Vlasov's equation.

5.1.1 Some useful variables

Since we spoke about energy, it is worth defining some variables of interest

- **Polytropic equation of state**: $P = k\rho^\Gamma$
  where k is the *entropy constant* and Γ is the *characteristic exponent of polytropic*[12].
  Anyway ,in general equations like $P \simeq \rho^\alpha$ are called *Barotropic*, we will see that these are important for vorticity.

- **Sound speed**[13]: $c_S = \Big(\frac{\partial P}{\partial\rho}\Big)^{\frac{1}{2}} = \Big(\frac{\Gamma P}{\rho}\Big)^{\frac{1}{2}}$
  the last step is true if we consider a polytropic

[11] G_i is accounting for contributions from internal and external forces
[12] $\Gamma = 1$ represents *isothermal processes*, $\Gamma = \frac{c_p}{c_v} = \frac{5}{3}$ is instead the expression for an ideal and adiabatic gas
[13] we will properly derive it when we will study perturbations

5.2 Bernoulli flow: de Laval nozzle.

Take the $Euler's\ equation$ subject to a conservative force

$$\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} - \frac{\partial \phi}{\partial x_i} \tag{5.8}$$

Let's consider the steady state, e.g. $\frac{\partial}{\partial t} = 0$, one dimensional and isolated case ($\phi = 0$), then we have

$$\rho v \frac{\partial v}{\partial x} = -\frac{\partial P}{\partial x} \tag{5.9}$$

with the *continuity equation* becoming $\frac{d\rho v}{dx} = 0$, that by the divergence theorem implies that $\rho\vec{v}\cdot\vec{A} =\ constant$, where A represents any surface.
Assume an isothermal EOS $P = c_S^2\rho$. The $Euler's\ equation$ and the *continuity equation* can be rewritten as

$$\begin{gathered} v\frac{dv}{dx} = -\frac{c_S^2}{\rho}\frac{d\rho}{dx} \\ \frac{d\rho}{\rho dx} + \frac{dv}{vdx} + \frac{dA}{Adx} = 0 \end{gathered} \tag{5.10}$$

by substituting the second relation in the first one

$$v\frac{dv}{dx} = c_S^2\Big(\frac{1}{v}\frac{dv}{dx} + \frac{1}{A}\frac{dA}{dx}\Big) \tag{5.11}$$

A possible solution is

$$\tfrac{v^2}{2} - c_S^2\Big(logv + logA\Big) = constant$$

that tell us that as the area increases, the velocity must decrease and viceversa. Actually this equation is a rewrite of Bernoulli's theorem which links the velocity of the fluid with the area it flows through.
An alternative way to write the equation is

$$v\frac{dv}{dx} - \frac{c_S^2}{v}\frac{dv}{dx} = \frac{1}{A}\frac{dA}{dx} \quad \Rightarrow \quad \frac{dv}{dx}\Big(1 - \frac{c_S^2}{v^2}\Big) = \frac{1}{vA}\frac{dA}{dx} \tag{5.12}$$

It is important to define the $Mach\ number$ $M \equiv \frac{c_S}{v^2}$. We see that, as the flow becomes transonic, the behaviour changes

- if $M < 1,\ dA > 0 \ \Rightarrow \ dv < 0$
- if $M > 1,\ dA > 0 \ \Rightarrow \ dv > 0$

This happens because we have to keep $\rho v A = \dot{m}$ constant in a compressible medium. However, if the density changes, the speed of sound cannot be really considered constant, hence we should write

$$v\frac{dv}{dx} = -c_S^2\frac{d\rho}{\rho dx} - \frac{dc_S^2}{dx} \tag{5.13}$$

As before, we can use the *continuity equation* to get

$$\begin{gathered} v\frac{dv}{dx} = c_S^2\Big(\frac{1}{v}\frac{dv}{dx} + \frac{1}{A}\frac{dA}{dx}\Big) - \frac{dc_S^2}{dx} \\ v\frac{dv}{dx} - \frac{c_S^2}{v}\frac{dv}{dx} = \frac{c_S^2}{A}\frac{dA}{dx} - \frac{dc_S^2}{dx} \\ (1 - M^2)\frac{dv}{dx} = \frac{c_S^2}{vA}\frac{1}{v}\frac{dc_S^2}{dx} \end{gathered} \tag{5.14}$$

that relates the cross-section gradient to the sound speed gradient and to the velocity gradient.

5.3 Viscosity and diffusion

In our derivation of the equations of motion we neglected the collision integral by saying that at equilibrium it is zero. Fluids, however, display friction, hence some molecular-microscopic effect must come into play. The fluid friction is called *Viscosity*.

We want to understand what happens when a fluid moves. We have to consider:

- **Dilatation or Compression**: are changes in the volume of the fluid element that can be thought as "*isotropic*" *stresses*. Note that in the incompressible fluids they would be absent.

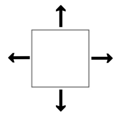

Figura 5.1: Sketch of dilation or compression

- **Rotation**: we consider a solid body rotation, hence it will cause no net force on the fluid element

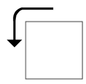

Figura 5.2: Sketch of rotation

- **Shear**: is the most interacting case, in fact this involves relative motion of different sides of the fluid.

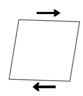

Figura 5.3: Sketch of shear

Let's take a fluid parcel at position $\vec{x}$ and velocity $\vec{v}$ and a neighboring one at $\vec{x}+\Delta\vec{x}$ and velocity $\vec{v}+\delta\vec{v}$.
We can expand the change in velocity which at first order and in component is

$$\delta v_i = \Delta x_j \frac{\partial v_i}{\partial x_j}$$

This variation in velocity can be simply decompose as $\delta v_i = \delta v_i^s + \delta v_i^a$, where $\delta v_i^s = \Delta x_j e_{ij}$ represents the *symmetrical part*, while $\delta v_i^a = \Delta x_j \epsilon_{ij}$ represents the $anti-symmetrical$ one.
Let's take a closer look at the terms of these latter relations.

- ϵ_{ij} is the curl of the velocity field $\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right)$. Since $\vec{\nabla}\times\vec{v} \equiv \vec{w} \Rightarrow \epsilon_{ij}$ is related to the vorticity of the fluid

- e_{ij} is the shear and it is where friction effects are going to manifest, explicitly $e_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$.
  So e_{ij} is symmetric and depends linearly on the gradients of $\vec{v}$, hence it can depend only on symmetric part of $\vec{\nabla}\vec{v}$.
  Recall that the stress tensor can be decomposed as $\sigma_{ij} = p\delta_{ij} + \sigma_{ij}^{'}$ and that, at rest, the only acting part is the isotropic part, that we called the pressure.
  For what we have said so far, we can write $\sigma_{ij}^{'}$, e.g. the stress tensor after we removed its isotropic part, as

$$\sigma_{ij}^{'} = C_0\delta_{ij}\frac{\partial v_k}{\partial x_k} + C_1\frac{\partial v_i}{\partial x_j} + C_2\frac{\partial v_j}{\partial x_i}$$

  We know that the off-diagonal parts should be responsible for the *friction* and that $Tr[\sigma_{ij}^{'}] = 0$, hence $3C_0 + C_1 + C_2 = 0$. Since the symmetry $C_1 = C_2$, so $3C_0 + 2C_1 = 0$. Thus

$$\sigma'_{ij} = C_0\Big[\delta_{ij}\tfrac{\partial v_k}{\partial x_k} - \tfrac{2}{3}\Big(\tfrac{\partial v_i}{\partial x_j} + \tfrac{\partial v_j}{\partial x_i}\Big)\Big]$$

Now identifying $2C_0$ with the *viscosity* η and defining the *Kinematic viscosity* $\nu = \frac{\eta}{\rho}$, after substituting in the momentum equation, we arrive at the **Navier-Stokes equation**

$$\begin{gathered}\rho\Big(\frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\vec{\nabla}\vec{v}\Big) = -\vec{\nabla}P + \vec{f} + \rho\nu\Big(\nabla^2\vec{v} - \frac{2}{3}\vec{\nabla}(\vec{\nabla}\cdot\vec{v})\Big)\\ In\ component: \quad \rho\Big(\frac{\partial}{\partial t} + v_j\frac{\partial}{\partial x_j}\Big)v_i = -\frac{\partial P}{\partial x_i} + \eta\frac{\partial^2}{\partial x_i \partial x_j}v_i\end{gathered} \tag{5.15}$$

$Navier-Stokes\ equation$ is a force balance equation, hence with dimensions $MT^{-2}L$. [14] This implies that the dimensions of the viscosity is ML^2T^{-1}.
We can rewrite the $Navier-Stokes\ equation$ to make it dimensionless simply by dividing by[15] U^2L^{-1}

$$\Big(\frac{\partial}{\partial t'} + \vec{v}' \cdot \vec{\nabla}'\Big)\vec{v}' = \vec{f}' + \frac{\nu}{UL}(\vec{\nabla}')^2\vec{v}' \tag{5.16}$$

We can define the *Reynolds number* $Re = \frac{UL}{\nu}$ where L is the any typical length scale of the problem, U is the typical velocity and ν is the kinematic viscosity.
It is important to notice two things: the first one is that given U there is always a scale over which the viscosity dominates, these are the regimes of *small Re* and the flow is *laminar*; the second one is that at large scale the viscosity is not important, but it was observed (by Reynolds) that when $Re >> 1$ the fluid goes through a *phase transition* and becomes *turbolent*.

5.4 Energy dissipation

So shear and strain (hence viscosity) act as friction, thus we do expect that they will dissipate energy. In order to understand how this happen, it is convenient to approach the problem in the following way: let's take a volume of fluid V and let's consider the forces that act on it. The rate at which work is done on this volume will be:

$$\int v_i F_i \rho dV + \int v_i \sigma_{ij} n_j dS \tag{5.17}$$

where the first term is due to body forces, the second one to surface forces. In particular, the last term can be rewritten, using the divergence theorem, as $\int \frac{\partial}{\partial x_j}(v_i\sigma_{ij})dV$.
Thus the total rate of work per unit volume will be given by:

$$v_iF_i + \frac{v_i}{\rho}\frac{\partial \sigma_{ij}}{\partial x_j} + \frac{\sigma_{ij}}{\rho}\frac{\partial v_i}{\partial x_j} \equiv \frac{\frac{work}{time}}{volume} \tag{5.18}$$

In this last equation the second term is the contribution arising from small differences in stress in opposite directions, the third one represent the contributions arising from differences in velocity in opposite directions. So this third term is responsible for the $deformation$. We will assume that the deformation dissipation will wholly go into internal energy. Assuming that the heat transfer in the fluid happens due to molecular conduction, we have that the rate of heat gain per unit volume is given by

$$\frac{1}{\rho}\frac{\partial}{\partial x_i}\Big(K\frac{\partial T}{\partial x_i}\Big) \tag{5.19}$$

with K being the *thermal conductivity*.
From the first principle, written per unit mass

$$\begin{gathered}\frac{DE}{Dt} = \frac{DW}{Dt} + \frac{DQ}{Dt}\\ W = \frac{\sigma_{ij}}{\rho}\frac{\partial v_i}{\partial x_j}, \quad Q = \frac{1}{\rho}\frac{1}{\partial x_i}\Big(K\frac{\partial T}{\partial x_i}\Big)\end{gathered} \tag{5.20}$$

we obtain

$$\frac{DE}{Dt} = \frac{1}{\rho}\Big[\sigma_{ij}\frac{\partial v_i}{\partial x_j} + \frac{\partial}{\partial x_i}\Big(K\frac{\partial T}{\partial x_i}\Big)\Big] \tag{5.21}$$

which can be further explicitated in terms of the quantities defined before.
Here we just note that $\frac{DE}{Dt}$ is non negative, thus showing that in response to shear and heat transfer the internal energy of the fluid element can only increase.

[14] M is for mass, L for length and T for time
[15] U is for velocity

5.5 Vorticity and rotation

When deriving the expression of the stress tensor, we noted that the antisymmetric part of the tensor itself can be express as the components of a (pseudo)vector :

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right)$$

hence we introduced the vorticity

$$\vec{\omega} = \vec{\nabla} \times \vec{v} \quad \Rightarrow \quad \epsilon_{ijk}\partial_j v_k = \omega_i$$

The vorticity represents the *rigid body* rotation of a parcel of fluid.
To understand its dynamics, a couple of vector identities are useful:

$$\text{I)}\ (\vec{v} \cdot \vec{\nabla})\vec{v} = \vec{\nabla}\left(\frac{1}{2}\vec{u} \cdot \vec{u}\right) - \vec{v} \times \vec{\omega}$$

$$\text{II)}\ \vec{\nabla} \times (\vec{v} \times \omega) = -\vec{\omega}(\vec{\nabla} \cdot \vec{v}) + (\vec{\omega} \cdot \vec{\nabla})\vec{v} - (\vec{v} \cdot \vec{\nabla})\vec{\omega}(+\vec{v}(\vec{\nabla} \cdot \vec{\omega})\ \textit{identically}\ 0)$$

Let's begin from $Navier-Stokes\ equation$ written in vector form, for conservative forces, ignoring the Bulk viscosity and assuming a constant shear viscosity

$$\begin{gathered}\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} = -\frac{\vec{\nabla}P}{\rho} - \vec{\nabla}\phi + \nu\nabla^2\vec{v} \\ Take\ the\ curl \\ \frac{\partial \vec{\omega}}{\partial t} + \vec{\nabla} \times (\vec{v} \times \vec{\omega}) = -\frac{\vec{\nabla}P \times \vec{\nabla}\rho}{\rho^2} - \nu\nabla^2\vec{\omega}\end{gathered} \tag{5.22}$$

Using the second of the previous identities, we arrive at

$$\frac{\partial \vec{\omega}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{\omega} = \vec{\omega}(\vec{\nabla} \cdot \vec{v}) - \frac{\vec{\nabla}P \times \vec{\nabla}\rho}{\rho^2} - \nu\nabla^2\vec{\omega} \tag{5.23}$$

Let's assume, for the moment, that we have an incompressible fluid, e.g. $\vec{\nabla} \cdot \vec{v} = 0$, $\nu = 0$. So

$$\frac{D\vec{\omega}}{Dt} = -\frac{\vec{\nabla}P \times \vec{\nabla}\rho}{\rho^2} \tag{5.24}$$

If we assume, moreover, an equation of state $P = P(\rho)$, we obtain the barotropic condition $\vec{\nabla}P \times \vec{\nabla}\rho = 0$. This implies

$$\frac{\partial \vec{\omega}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{\omega} = 0 \quad \text{or equivalently} \quad \frac{D\vec{\omega}}{Dt} = 0$$

This means that in ideal, barotropic fluids vorticity is convected conservatively.
Defining the *circulation*, e.g. the circuitation of the velocity

$$\Gamma = \oint \vec{v} \cdot d\vec{x} = \int \vec{\omega} \cdot d\vec{A} \quad [16] \tag{5.25}$$

we arrive at $Kelvin's\ theorem$ for the circulation in an ideal, barotropic fluid

$$\frac{D\Gamma}{Dt} = 0 \tag{5.26}$$

Then in an ideal, barotropic fluid the circulation is conserved, hence if at the beginning is zero it must remain zero.
Instead, if the fluid is not barotropic and the viscosity $\nu \neq 0$, we have

$$\frac{D\Gamma}{Dt} = -\oint_\gamma \frac{dP}{\rho} + \nu \oint_\gamma \nabla^2\vec{v} \cdot d\vec{x} \tag{5.27}$$

5.6 Vortex stretching and Tilting

Let's write the vorticity equation in the material derivative:

$$\frac{D\vec{\omega}}{Dt} = (\vec{\omega} \cdot \vec{\nabla})\vec{v} \tag{5.28}$$

[16] in the last step we used the *Stokes theorem*

where $(\vec{\omega} \cdot \vec{\nabla})\vec{v}$ is the *stretching of the vorticity.*
As the velocity increases in the direction of $\vec{\omega}$, the vorticity increases. Just look at the water flowing through a sink in the bathtub.
For the same reason vorteces can tilt. For instance, in *cartesian components*, assuming no z direction vorticity and some x direction instead, we have

$$\begin{aligned} \frac{D\omega_x}{Dt} &= \omega_x \frac{\partial v_x}{\partial x} \\ \frac{D\omega_z}{Dt} &= \omega_x \frac{\partial v_z}{\partial x} \end{aligned} \tag{5.29}$$

So the x *component* of the vorticity will derive the z *component*, together with the z *component* of the velocity.
At a certain point, we will reach a condition where

$$\begin{aligned} \frac{D\omega_x}{Dt} &= \omega_x \frac{\partial v_x}{\partial x} + \omega_z \frac{\partial v_x}{\partial z} \\ \frac{D\omega_z}{Dt} &= \omega_x \frac{\partial v_z}{\partial x} + \omega_z \frac{\partial v_z}{\partial z} \end{aligned} \tag{5.30}$$

and the vortex will stretch or expand.
In absence of viscosity this is nothing more than conservation of angular momentum and its redistribution along different components.

5.7 Ensotrophy

Take the vorticity equation evolution for a barotropic and incompressible non viscous fluid

$$\frac{D\vec{\omega}}{Dt} = \vec{\omega} \cdot \vec{\nabla}\vec{v} \tag{5.31}$$

Multiply by $\vec{\omega}$ and integrate over the volume:

$$\begin{aligned} \vec{\omega} \cdot \frac{D\vec{\omega}}{Dt} &= \vec{\omega} \cdot (\vec{\omega} \cdot \vec{\nabla}\vec{v}) \\ \int \frac{1}{2} \frac{D\omega^2}{Dt} dV &= \int \vec{\omega} \cdot (\vec{\omega} \cdot \vec{\nabla}\vec{v}) dV \end{aligned} \tag{5.32}$$

Now we can define the *Ensotrophy* as

$$Ensotrophy \equiv \int \frac{1}{2}\omega^2 dV \tag{5.33}$$

which measures the energy associated with the rotations in the fluid.

6 Rotating frames

Although non-inertial, it is often convinient to work in co-rotating frames to understand the dynamics of bodies.
When dealing with a rotating star or a binary system in which mass transfer is happening, in a co-rotating frame we will need to include the effect of non-inertial forces like the *centrifugal* and *Coriolis* force. These will have a serious impact on the fluid, especially the second one which will induce necessarily circulation in the body.
The *Coriolis acceleration* is $\vec{a}_c = 2\vec{\Omega} \times \vec{v}$, with $\vec{\Omega}$ the *constant* rotational velocity.
In general the non-inertial body force, per unit mass, will be given by

$$-\Big[2\vec{\Omega} \times \vec{v} + \frac{d\vec{\Omega}}{dt} \times \vec{r} + \vec{\Omega} \times (\vec{\Omega} \times \vec{r})\Big] = \vec{f}_{ni} \tag{6.1}$$

$Euler's\ equation$ in a $co-rotating\ frame$ is

$$\frac{D\vec{v}}{Dt} + 2\vec{\Omega} \times \vec{v} = -\frac{\vec{\nabla}P}{\rho} + \frac{\vec{f}}{\rho} \tag{6.2}$$

and going to the *vorticity equation*, without assuming incompressibility:

$$\frac{D\vec{\omega}}{Dt} + \vec{\nabla} \times (2\vec{\Omega} \times \vec{v}) = \frac{1}{\rho^2}\vec{\nabla}P \times \vec{\nabla}\rho + \vec{\nabla} \times \frac{\vec{f}}{\rho} \tag{6.3}$$

Assuming $\vec{\Omega}$ is constant in time, but it can vary in space, we can define an *absolute vorticity*

$$\tilde{\omega} = 2\vec{\Omega} + \vec{\omega} \tag{6.4}$$

and arrive at

$$\frac{D}{Dt}\frac{\tilde{\omega}}{\rho} = \frac{\tilde{\omega}}{\rho}\vec{\nabla}\vec{v} + \frac{1}{\rho^3}\vec{\nabla}\rho \times \vec{\nabla}P + \frac{1}{\rho}\vec{\nabla} \times \frac{\vec{f}}{\rho} \quad {}^{17} \tag{6.6}$$

The equation above is useful since it allows to study the redistribution of any scalar quantity due to the rotation compared to the non-rotating case. Alternatively, the same equation can tell us how the gradient of a quantity is going to drive the flow. To see that, expand

$$\frac{\tilde{\omega}}{\rho} \cdot \frac{D}{Dt}\vec{\nabla}Q = \frac{D}{Dt}\Big(\frac{\tilde{\omega}}{\rho} \cdot \vec{\nabla}Q\Big) - \vec{\nabla}Q \cdot \frac{D}{Dt}\frac{\tilde{\omega}}{\rho} \tag{6.7}$$

Substitute teh equation of motion and rearrange the terms. In particular we will get a term like

$$\tfrac{\tilde{\omega}}{\rho} \cdot \vec{\nabla}(\vec{v} \cdot \vec{\nabla}Q) \quad \rightarrow \quad \tfrac{\tilde{\omega}}{\rho}\vec{\nabla}\dot{Q}$$

So that we have

$$\frac{D}{Dt}\Big(\frac{\tilde{\omega}}{\rho} \cdot \vec{\nabla}Q\Big) = \frac{\tilde{\omega}}{\rho}\vec{\nabla}\dot{Q} + \frac{1}{\rho^3}\vec{\nabla}Q \cdot (\vec{\nabla}\rho \times \vec{\nabla}P) + \frac{1}{\rho}\vec{\nabla}Q \times \Big(\vec{\nabla} \times \frac{1}{\rho}\vec{f}\Big) \tag{6.8}$$

where $\frac{\tilde{\omega}}{\rho} \cdot \vec{\nabla}Q$ is the *potential vorticity* and Q is any scalar variable.

6.1 Taylor-Proudman theorem

The $Taylor-Proudman\ theorem$ essentially states that in a steady rotating flow the motion is two dimensional and orthogonal to the axis of rotation.
In order to see this, take the equation of motion

$$\frac{D\vec{v}}{Dt} + 2\vec{\Omega} \times \vec{v} = \frac{\vec{\nabla}P}{\rho} \tag{6.9}$$

consider a steady flow and take the curl:

$$\vec{\nabla} \times (2\vec{\Omega} \times \vec{v}) = -\frac{1}{\rho^2}(\vec{\nabla}\rho \times \vec{\nabla}P) \tag{6.10}$$

Now, if

[17] to derive it, start from the Euler's equation written as

$$\frac{\partial\vec{v}}{\partial t} + \vec{v} \times \vec{\omega} = -\frac{\vec{\nabla}P}{\rho} + \frac{\vec{f}}{\rho} \tag{6.6}$$

substitute $\vec{f} = \vec{f}_i + \vec{f}_{ni}$, collect and take the curl and finally use the continuity equation

- the fluid is barotropic, then

$$\begin{aligned} \vec{\nabla}P \times \vec{\nabla}\rho = 0 \quad \Rightarrow \quad \vec{\nabla} \times (\vec{\Omega} \times \vec{v}) = 0 \\ Equivalently \;\; \vec{\Omega}\vec{\nabla} \cdot \vec{v} - \vec{v}\, \vec{\nabla} \cdot \vec{\Omega} + \vec{v} \cdot \vec{\nabla}\vec{\Omega} - \vec{\Omega} \cdot \vec{\nabla}\vec{v} = 0 \end{aligned} \tag{6.11}$$

- the fluid is incompressible and Ω is *approximately* constant, e.g. $\vec{\Omega} \cdot \vec{\nabla}\vec{v} = 0$, the velocity does not change in the direction of $\vec{\Omega}$. This means that the flow is essentially two dimensional, or better said $\vec{v}$ cannot depend on z if $\vec{\Omega} = \Omega\hat{z}$

Anyway the rotational velocity defines a $Taylor - Proudman\ column$.
For system in radial symmetry (for instance a star), this implies that large vortical motion, like *convection*, establishes meridional motions.

6.2 Geostrophic approximation and coordinate system

It is convenient when studying the atmospheres the equivalent of a shallow water approximation. This assumes slow flows, essentially two-dimensional, in the geostrophic system.
For illustrative purposes we can consider the following figure

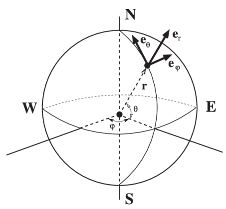

Figura 6.1: schematic figure for a geostrophic system

Actually we want some approximations to apply, for instance if we can locally approximate the sphere as a plane tangent to it, then we can consider the atmosphere as a plane, thin layer where $\rho =\ constant$.
In spherical symmetry

$$\begin{aligned} 2\rho\Omega v_\phi sin\theta = \frac{\partial P}{\partial r} \\ \frac{Dv_\theta}{Dt} - 2\Omega v_\phi cos\theta = -\frac{1}{\rho r}\frac{\partial P}{\partial \theta} - \frac{1}{r}\frac{\partial \Phi}{\partial r} \\ \frac{Dv_\phi}{Dt} - 2\Omega v_\theta cos\theta = -\frac{1}{\rho r sin\theta}\frac{\partial P}{\partial \phi} - \frac{1}{r sin\theta}\frac{\partial \Phi}{\partial \phi} \end{aligned} \tag{6.12}$$

where Φ is the *Gravitational potential*.
These equations are complicated, but give some insights on the motion: a parcel of fluid in radial motion will change its angular momentum, hence we will have vorticity and meridional motion as we saw before.
In the *geostrophic approximation* the above equations simplify dramatically.
Let's define a *geostrophic system* properly

z: UP
x: AZIMUTHAL
y: MERIDIONAL

The position of the plane on the sphere is accounted for by its inclination λ so that is $z = rsin\lambda$. The *vertically velocity*, if we assume the pressure to be constant over some typical scale H and we restrict to that scale, is simply $vtan\lambda$.
In this coordinate system, if we assume incompressibility, e.g. $\vec{\nabla} \cdot \vec{v} = 0$, we can write

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{vtan\lambda}{H} = 0 \tag{6.13}$$

Assuming barotropicity and that only the vorticity is responsable for the motion along $\hat{z}$, we arrive at

$$\begin{aligned} \frac{\partial v_x}{\partial t} + 2\Omega v_y = -\frac{1}{\rho}\frac{\partial P}{\partial x} \\ \frac{\partial v_y}{\partial t} - 2\Omega v_x = -\frac{1}{\rho}\frac{\partial P}{\partial y} \end{aligned} \tag{6.14}$$

from which we can derive the $Rossby - waves$.

6.3 Rossby number and Rossby waves

In a co-rotating frame we have the competition of two forces:

- the inertial one, varying like $\frac{V^2}{L}$
- Coriolis one, varying like ΩV

Define the dimensionless parameter, called *Rossby number*

$$R_0 = \frac{V}{\Omega L} \tag{6.15}$$

that quantifies the relative importance of rotation and inertia.
Small R_0 flows are called *geostrophic.*
R_0 provides a convenient way to assess how important rotation is and finds applications in the context of the study if stellar structure and magnetic field generation by dynamos where several processes, like rotation and convection, come into play.

6.4 Rayleigh stability criterion for rotating fluid

We want to understand when, in a differentially rotating medium, the fluid is stable. In other words: what are the conditions for which when a fluid element is displaced, we find that it will go back where it was?
Let's take the $Kepler's\ law$:

$$\Omega^2 r^3 = \ constant \tag{6.16}$$

The specific angular momentum is

$$j = r^2\Omega \quad \Rightarrow \quad j \sim r^{\frac{1}{2}} \tag{6.17}$$

If the fluid moves in an effective potential $\phi = \frac{1}{2}r^2\Omega^2 = \frac{1}{2}\frac{j^2}{r^2}$, we can ask ourselves whether angular momentum alone can restore a displaced parcel of fluid.
From $\vec{f} = -\vec{\nabla}\phi = \vec{a}$, we get for the displacement δr, at first order

$$\delta\ddot{r} = -\frac{1}{r^3}\frac{\partial j^2}{\partial r}\bigg|_{r=0}\delta r \tag{6.18}$$

so we get an equation recalling an harmonic oscillator

$$\delta\ddot{r} + \omega^2\delta r = 0$$

that is stable when $\omega^2 > 0$, so when the angular momentum increases outwards.
In general the *Rayleigh stability criterion* states that if the $Rayleigh\ frequency$ $f_r = \frac{1}{r^3}\frac{\partial(r^2\Omega)^2}{\partial r} < 0$, then the fluid is unstable to redistribution of angular momentum. Moreover, the angular momentum must not depend on displacement along the rotation axis, e.g. $\frac{\partial}{\partial t}r^2\Omega = 0$, hence the stability imposes that rotation is stable on cylinders.
Let's take a planar motion and consider the vorticity equation

$$\frac{D\vec{\omega}}{Dt} = \vec{\omega}\cdot\vec{\nabla}\vec{v} + \nu\nabla^2\vec{\omega} \tag{6.19}$$

Because we are on a plane $\vec{\omega} = \omega\hat{z}$, hence, in cylindrical coordinates we get

$$\frac{\partial\omega}{\partial t} = \nu\frac{1}{r}\frac{\partial^2\omega}{\partial t^2} \tag{6.20}$$

which is a $diffusion\ equation$, having solution

$$\omega = \frac{C}{(\pi\nu t)^{\frac{1}{2}}}e^{-\frac{r}{4\nu t}} \tag{6.21}$$

So the initial vorticity decays as $t^{-\frac{1}{2}}$ and it is transferred outwards. Since vorticity is nothing else than angular momentum, angular momentum is transferred outwards and the fluid migrates inward.
Viscosity is then the primary mechanism allowing for acceleration.
Note also that ν will dissipate energy, in internal energy, as

$$\frac{DE}{Dt} = \frac{1}{\rho}\Big[\sigma_{ij}\frac{\partial v_i}{\partial x_j} + \frac{\partial}{\partial x_i}K\frac{\partial T}{\partial x_i}\Big] \tag{6.22}$$

Hence the fluid will also get hotter as it loses angular momentum.

7 Fluid perturbations: sound waves

Let's talk about the effects of perturbation on a fluid and let's try to establish the criteria for fluid stability.
Let's start with the simplest one: the propagation of *pressure waves*, or *sound waves*, in a fluid. Obviously, the fluid needs to be compressible.
It is easy to start by writing the *continuity* and the *Euler's equation* in the absence of extrernal forces

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \vec{\nabla}\rho\vec{v} = 0 \\ \frac{\partial \rho\vec{v}}{\partial t} + \vec{v}\cdot\vec{\nabla}\rho\vec{v} = -\vec{\nabla}P \end{aligned} \tag{7.1}$$

Now we assume *barotropicity*, that $\rho = \rho_0 + \delta\rho$, $\vec{v} = \vec{v}_0 + \delta\vec{v} = \delta\vec{v}$ and restrict to the one dimensional case

$$\begin{aligned} \frac{\partial \delta\rho}{\partial t} + \rho_0 \frac{\partial}{\partial x}\delta v = 0 \\ \rho_0 \frac{\partial \delta v}{\partial t} + \left(\frac{\partial P}{\partial \rho}\right)\frac{\partial \delta\rho}{\partial x} = 0 \end{aligned} \tag{7.2}$$

Now, let's take the derivative $\frac{\partial}{\partial t}$ of the first and $\frac{\partial}{\partial x}$ of the second equation

$$\begin{aligned} \frac{\partial^2 \delta\rho}{\partial t^2} + \rho_0 \frac{\partial}{\partial t}\frac{\partial}{\partial x}\delta v = 0 \\ \rho_0 \frac{\partial^2 \delta v}{\partial t \partial x} + \left(\frac{\partial P}{\partial \rho}\right)\frac{\partial^2 \delta\rho}{\partial x^2} = 0 \end{aligned} \tag{7.3}$$

where we use the *Schwarz theorem* to exchange the order of differentiation.
Putting them together, we get

$$\frac{\partial^2 \delta\rho}{\partial t^2} + \left(\frac{\partial P}{\partial \rho}\right)\frac{\partial^2 \delta\rho}{\partial x^2} = 0 \tag{7.4}$$

which is a wave equation for $\delta\rho$, the density perturbation, propagating with a velocity $c_S = \left(\frac{\partial P}{\partial \rho}\right)^{\frac{1}{2}}$, that we already define as the sound speed.
Since this is a wave equation, we can solve it assuming the solution is a plane wave

$$\delta\rho \sim e^{i(kx - wt)} \tag{7.5}$$

by susbstitution in the equation, we obtain the *dispersion relation* $c_S^2 k^2 = \omega^2$ So at this order sound waves are not dispersive and propagate at the speed of sound c_S.
Note that, at the first order, sound waves are not dissipative since the viscosity contribution is second order.
Thermal conductivity will however enter and damp waves.
We can understand at least qualitatively that due to viscosity sound waves will be damped and their energy will eventually be converted into internal energy of the gas.[18]
Let us now relax the assumption that the perturbation is small. In this case, we cannot neglect higher orders and have to work with the full equations. So the equations turn non-linear and what one observes is that the front steepens. This can be understood as the piling up of material at the wavefront: as the density increases, so does the velocity due to the fact that $dv \sim c_S \frac{d\rho}{\rho}$. Hence the lower density material stays behind and high density material piles up. In terms of wavefront this implies that the front must steepen until eventually the solution becomes multivalued and stops making sense: we form a *shock*.
Effectively we have formed a discontinuity in the density, hence in pressure and velocity, between the $pre-shock$ and $post-shock$ material.
At this point our differential equations stop making sense, their integrals, however, do. In particular the fluxes across the shock will have to be conserved

$$\begin{aligned} mass\ flux: \quad \rho v\Big|_{\Sigma} = 0 \\ momentum\ flux: \quad \rho v^2 + P\Big|_{\Sigma} = 0 \\ energy\ flux: \quad \frac{v^2}{2} + \epsilon\Big|_{\Sigma} = 0 \end{aligned} \tag{7.6}$$

[18] Look at 51 in Mihalas Mihalas

These are called the *Rankine − Hugoniot conditions*.
Let's look at these in a bit more details. First let's go back to the linear equation $\frac{\partial^2}{\partial t^2}\delta\rho + c_S^2\frac{\partial^2\delta\rho}{\partial x^2} = 0$
Any solution could be written as $f(x + c_S t) + f(x - c_S t)$, hence with single argument $f(x \pm c_S t)$. Reimann found that also the non-linear equation admits solutions depending on a single argument $x \pm vt$, but v now is not the sound speed, but a function of the fluid velocity in that space-time point. This implies that any function can be uniquely identifies as a function of any other, hence of the fluid velocity itself. In particular, we can assign $\rho = \rho(u)$, where u is the fluid velocity.
The continuity and the momentum equation thus become:

$$\begin{gathered}\frac{\partial\rho}{\partial u}\frac{\partial u}{\partial t} + \Big(u\frac{D\rho}{Du} + \rho\Big)\frac{\partial u}{\partial x} = 0 \quad or \quad \frac{\partial u}{\partial t} + \Big[u + \rho\frac{Du}{D\rho}\Big]\frac{\partial u}{\partial x} = 0 \\ \frac{\partial u}{\partial t} + \Big[u + \frac{1}{\rho}\Big(\frac{\partial P}{\partial\rho}\Big)\frac{D\rho}{Du}\Big]\frac{\partial u}{\partial x} = 0\end{gathered} \tag{7.7}$$

Compering the two we find that:

$$\frac{Du}{D\rho} = \pm\Big(\frac{\partial P}{\partial\rho}\Big)\frac{1}{\rho} = \pm\frac{c_S}{\rho} \tag{7.8}$$

Assuming $P = P(\rho)$, therefore, fluid velocity and density are related as

$$u = \pm\int_{\rho_0}^{\rho}\frac{c_S}{\rho}d\rho = \pm\int_{P_0}^{P}\frac{dP}{\rho c_S} \tag{7.9}$$

Going back to the original equations, they now become

$$\begin{aligned}\frac{\partial u}{\partial t} + (u \pm c_S)\frac{\partial u}{\partial x} = 0 \\ \frac{\partial\rho}{\partial t} + (u \pm c_S)\frac{\partial\rho}{\partial x} = 0\end{aligned} \tag{7.10}$$

that admit general solutions of the kind

$$\begin{aligned}u = f(x - (u \pm c_S)t) \\ \rho = g(x - (u \pm c_S)t)\end{aligned} \tag{7.11}$$

with *phase speed* $v_P = u \pm c_S(u)$.

7.1 Development of shocks

Consider for clarity a concrete case for an ideal gas.
In this case

$$c_S^2 \sim \frac{P}{\rho} \sim \rho^{\gamma-1} \quad \Rightarrow \quad (\gamma-1)\frac{d\rho}{\rho} = 2\frac{dc_S}{c_S}, \quad dc_S = \frac{(8\gamma-1)}{2}\frac{c_S}{\rho}d\rho \tag{7.12}$$

integrating for u

$$\begin{gathered}u = \pm\int_{\rho_0}^{\rho}\frac{c_S}{\rho}d\rho = \pm2\frac{(c_S - c_{S_0})}{\gamma-1} \\ \Downarrow \\ c_S = c_{S_0} \pm \frac{1}{2}(\gamma-1)u\end{gathered} \tag{7.13}$$

$$\textit{that implies a phase velocity} \;\; v_P(u) = \frac{1}{2}(\gamma+1) \pm c_{S_0}$$

Note that the *phase velocity* is not constant which leads to the *front distortion* (as we mentioned the front steepens).
Now consider a *polytropic gas*

$$\begin{aligned}\rho = \rho_0\Big[1 \pm \frac{1}{2}(\gamma-1)\Big(\frac{u}{c_{S_0}}\Big)\Big]^{\frac{2}{\gamma-1}} \\ P = P_0\Big[1 \pm \frac{1}{2}(\gamma-1)\Big(\frac{u}{c_{S_0}}\Big)\Big]^{\frac{2}{\gamma-1}} \\ T = T_0\Big[1 \pm \frac{1}{2}(\gamma-1)\Big(\frac{u}{c_{S_0}}\Big)\Big]^{2}\end{aligned} \tag{7.14}$$

So the most compressed regions are faster and hotter. This leads to a shock development of the type

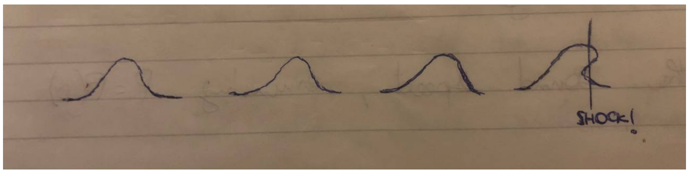

Figura 7.1: outline of a shock development

7.2 Planar shocks

Let's go back to the treatment of shocks and let's begin with the simplest case, e.g. *planar case*. This is better dealt with in the frame of the shock, where material flows with velocity $\vec{v_1}$ towards the shock and leaves the post-shock with velocity $\vec{v_2}$.
Conservation of mass imposes

$$v_2 = \frac{\rho_1}{\rho_2} v_1 \tag{7.15}$$

We can go to the momentum flux and write

$$\rho_1 v_1^2 + P_1 = \rho_2 v_2^2 + P_2 \tag{7.16}$$

where we can eliminate a variable using the continuity equation and using the energy conservation $\left.\frac{v^2}{2} + \frac{\gamma}{\gamma-1}\frac{P}{\rho}\right]_{\Sigma} = 0$, we obtain

$$\frac{\rho_1}{\rho_2} = \frac{v_1}{v_2} = \frac{(\gamma+1)P_2 + (\gamma-1)P1}{(\gamma-1)P_2 + (\gamma+1)P1} \tag{7.17}$$

This last equation relates jumps in the thermodynamic variables given the *shock speed*, or the *shock compression*.
In particular, for very strong compression $P_1 >> P_2$ we have $\frac{\rho_2}{\rho_1} \to \frac{\gamma+1}{\gamma-1}$. This for an *ideal monocromatic gas* leads to $\frac{\rho_2}{\rho_1} = 4 \; for \; \gamma = \frac{5}{3}$.
In general, however, we have no idea of the compression in astrophysical environments, but we have access to velocities via spectroscopy. This is useful also to assess the effect of a shock traveling at a certain velocity.
Notice that all we did assumes that γ is the same before and after the shock. If the temperature jump is big enough we might get ionization as well, so different considerations must be made.
Nevertheless, define $M_1 = \frac{v_1}{c_{S1}}$ and $M_2 = \frac{v_2}{c_{S2}}$ and leaving out some algebraic passages, we obtain:

$$\begin{aligned}
\frac{\rho_1}{\rho_2} &= \frac{\left(\frac{1}{2}\gamma+1\right)M_1^2}{\frac{1}{2}(\gamma-1)M_1^2+1} = \frac{v_1}{v_2} \\
\frac{P_1}{P_2} &= \frac{2\gamma M_1^2 - (\gamma-1)}{\gamma+1} \\
\frac{T_1}{T_2} &= \frac{\left[2\gamma M_1^2 - (\gamma-1)\right]\left[(\gamma-1)M_1^2+2\right]}{\left[(\gamma+1)M_1\right]^2} \\
M_2^2 &= \frac{(\gamma-1)M_1^2+2}{2\gamma M_1^2-(\gamma-1)}
\end{aligned} \tag{7.18}$$

7.3 Brunt-Väisälä frequency and Schwarzschild criterion

Take a parcel of fluid in a stratified medium. Displace it and assume the whole process adiabatic.

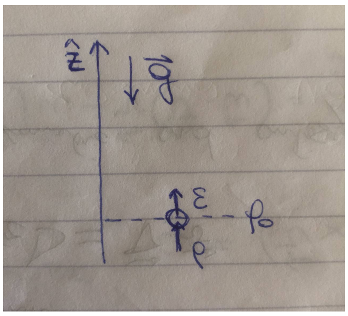

Figura 7.2

Referring to the sketch above, we indicate with ϵ the displacement, with ρ the density of parcel in the displaced position and with ρ_0 the density of ambient materia. Now notice that $\vec{\nabla}\rho = \frac{\partial \rho}{\partial z} \neq 0$. Obviously, when we displace the parcel, it will experience a force ($Archimedes'\ force$) which we can write $-g(\rho - \rho_0)$. So the equation of the motion for the displacement is

$$\rho \frac{\partial^2 \epsilon}{\partial t^2} = -g(\rho - \rho_0) \tag{7.19}$$

Assume the displacement to be small [19] 559560 matricola

$$\begin{aligned} \rho &= \rho(0) + \vec{\nabla}_{ad}\rho\epsilon \\ \rho_0 &= \rho(0) + \vec{\nabla}_{at}\rho\epsilon \end{aligned} \tag{7.20}$$

Substituting

$$\frac{\partial^2 \epsilon}{\partial t^2} = -\frac{g}{\rho}(\vec{\nabla}_{ad}\rho - \vec{\nabla}_{at}\rho)\epsilon \tag{7.21}$$

Let's define the *Brunt-Väisälä frequency*

$$\omega_{BV}^2 \equiv \frac{g}{\rho}(\vec{\nabla}_{ad}\rho - \vec{\nabla}_{at}\rho) \quad \Rightarrow \quad \frac{\partial^2 \epsilon}{\partial t^2} + \omega_{BV}^2 \epsilon = 0 \tag{7.22}$$

that has solution

$$\epsilon(t) \sim e^{i\omega_{BV} t} \tag{7.23}$$

We have the first case of different possible outcomes to a perturbation.
Begin with the trivial $\omega_{BV} = 0$ solution. The atmosphere in this case is *neutral* and no motion is possible.
If $\omega_{BV} \in R$, the resulting solution has an oscillatory character: there is a global circulation and the period for such motion is $\frac{2\pi}{\omega_{BV}}$.
The most interesting case is $\omega_{BV} \in I$: the perturbation grows in size exponentially with time. The atmosphere is *convectively unstable*.
From the definition of ω_{BV}, we find the *Schwarzschild stability criterion*

$$\vec{\nabla}_{ad}\rho < \vec{\nabla}_{at}\rho \tag{7.24}$$

This pose the basis of the convective mixing lenght theory, which we will see when speaking about turbolence.

7.4 Sound waves in startified atmospheres

Let's consider again a stratified atmosphere

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \vec{\nabla}\rho\vec{v} &= 0 \\ \frac{\partial \rho\vec{v}}{\partial t} + \vec{v}\cdot\vec{\nabla}\rho\vec{v} &= -\vec{\nabla}P + \vec{g} \end{aligned} \tag{7.25}$$

Assume we have a constant $\vec{g} = -g\hat{z}$ along the $z - direction$ and let's restrict to the $z - direction\ propagation$

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial z}\rho v &= 0 \\ \frac{\partial \rho v}{\partial t} + v\frac{\partial}{\partial z}\rho v &= -\frac{\partial}{\partial z}P - \vec{g} \end{aligned} \tag{7.26}$$

Expand the variables to first order

$$\begin{aligned} \rho &= \rho_0 + \delta\rho \\ v &= v_0 + \delta v \end{aligned} \tag{7.27}$$

and assume that the unperturbed medium is static. Also, we already knew that $\rho_0 = \tilde{\rho}e^{-\frac{z}{H}}$, with H being the scale height.
The problem is easier in the $Lagragian\ frame$. Call the Lagragian perturbations

$$\begin{aligned} \Delta\rho &= \delta\rho + \delta v\frac{\partial \rho_0}{\partial z} \\ \Delta P &= \delta P + \delta v\frac{\partial P_0}{\partial z} \\ \Delta v &= \delta v \end{aligned} \tag{7.28}$$

[19] Remember that we assumed no energy exchange with the surrounding

Substitute in the continuity and momentum equations to get

$$
\begin{aligned}
\frac{\partial \Delta\rho}{\partial t} + \rho_0 \frac{\partial \Delta v}{\partial z} = 0 \\
\frac{\partial \Delta v}{\partial t} = -\frac{c_S^2}{\rho_0}\frac{\partial \Delta\rho}{\partial z}
\end{aligned}
\tag{7.29}
$$

Putting them together, we arrive at

$$
\frac{\partial^2 \Delta\rho}{\partial t^2} - \rho_0 \frac{\partial}{\partial z}\Big(\frac{c_S^2}{\rho_0}\frac{\partial \Delta\rho}{\partial z}\Big) = 0\ , \quad c_S^2 = \Big(\frac{\partial P}{\partial \rho}\Big)\ \textit{in general depends on } z \tag{7.30}
$$

For semplicity, assume that the medium is *isothermal* then c_S is fixed.
Differentiating with respect to z, we get

$$
\begin{aligned}
\frac{\partial^2 \Delta\rho}{\partial t^2} + \frac{\rho_0 c_S^2}{\rho_0^2}\frac{\partial \rho_0}{\partial z}\frac{\partial \Delta\rho}{\partial z} - c_S^2 \frac{\partial^2 \Delta\rho}{\partial z^2} = 0 \\
\frac{\partial \rho_0}{\partial z} = -\frac{\rho_0}{H} \\
\Downarrow \\
\frac{\partial^2 \Delta\rho}{\partial t^2} - c_S^2 \frac{\partial^2 \Delta\rho}{\partial z^2} - \frac{c_S^2}{H}\frac{\partial \Delta\rho}{\partial z} = 0
\end{aligned}
\tag{7.31}
$$

Looking at the last written equation, we notice that the first two terms represent the *standard wave*, while the last is due to *stratification*.
Looking for plane waves solutions $\Delta\rho \sim e^{i(kz-\omega t)}$, we find

$$
\omega^2 = c_S^2\Big(K^2 - \frac{iK}{H}\Big) \tag{7.32}
$$

So sound waves are *dispersive*, in this case.
Solving for $K(\omega)$

$$
K = \frac{i}{2H} \pm \Big(\frac{\omega^2}{c_S^2} - \frac{1}{4H^2}\Big)^{\frac{1}{2}} \tag{7.33}
$$

Let's take $\omega \in R$, so we have two distinct cases:

- $\omega > \frac{c_S}{2H}$ $\rightarrow$ in this case we have $ReK = \frac{1}{2H}$, $ImK = \pm\Big(\frac{\omega^2}{c_S^2} - \frac{1}{4H^2}\Big)^{\frac{1}{2}}$. The perturbation is of the form

  $$
  \Delta\rho \sim e^{-\frac{z}{2H}} e^{i\Big(\pm\Big(\frac{\omega^2}{c_S^2} - \frac{1}{4H^2}\Big)^{\frac{1}{2}} - \omega t\Big)}
  $$

  the phase velocity is

  $$
  v_{ph} = \frac{\omega}{\pm\Big(\frac{\omega^2}{c_S^2} - \frac{1}{4H^2}\Big)^{\frac{1}{2}}} \quad \textit{so wave shape gets distorted}
  $$

  Looking at the velocity, we find that

  $$
  \Delta v = \frac{\Delta\rho}{\rho_0}\frac{\omega}{K} \quad \Rightarrow \quad \Delta v \sim e^{\frac{z}{2H}}\,, \quad \frac{\Delta\rho}{\rho_0} \sim e^{\frac{z}{2H}}
  $$

  so both the velocity and the compression increase with z, hence in the absence of dissipation a sound wave eventually will develop a shock

- $\omega < \frac{c_S}{2H}$ $\rightarrow$ in this case K is imaginary, hence things do not propagate.

8 Instabilities

8.1 Jeans instability

We want to explore what happens to a pressure wave in a self-gravitating medium.
Let's take the linearised continuity and momentum equation and complete with Poisson's equation.
So we consider

$$\begin{aligned} \rho &= \rho_0 + \rho_1 \\ v &= v_0 + v_1 \\ \phi &= \phi_0 + \phi_1 \end{aligned} \tag{8.1}$$

with the 0 *quantities* referring to the imperturbed ones and the 1 *quantities* to the first order perturbations.
The equations become:

$$\begin{aligned} \frac{\partial \rho_1}{\partial t} + \rho_0 \frac{\partial v_{1i}}{\partial x_i} &= 0 \\ \rho_0 \frac{\partial v_{1i}}{\partial t} &= -\frac{\partial P_1}{\partial x_i} - \rho_0 \frac{\partial \phi_1}{\partial x_i} \\ \nabla^2 \phi_1 &= 4\pi \rho_1 G \end{aligned} \tag{8.2}$$

Let's rewrite these relations in a more convenient form.
First of all, let's define $c_S = \left(\frac{\partial P}{\partial \rho}\right)^{\frac{1}{2}}$, so we arrive at

$$\rho_0 \frac{\partial v_i}{\partial t} = -c_S^2 \frac{\partial \rho_1}{\partial x_i} - \rho_0 \frac{\partial \phi_1}{\partial x_i}. \tag{8.3}$$

Now from the continuity equation, we get

$$\frac{\partial v_{1i}}{\partial x_i} = -\frac{1}{\rho_0}\frac{\partial \rho_1}{\partial t} \quad {}^{20} \longrightarrow \frac{\partial^2 v_{1i}}{\partial x_i \partial t} = -\frac{1}{\rho_0}\frac{\partial^2 \rho_1}{\partial t^2} \tag{8.4}$$

and from the momentum equation

$$\rho_0 \frac{\partial^2 v_{1i}}{\partial x_i \partial t} = -c_S^2 \frac{\partial^2 \rho_1}{\partial x_i^2} - \rho_0 \frac{\partial^2 \phi_1}{\partial x_i^2} \tag{8.5}$$

By substituting and considering the Poisson's equation, we arrive at

$$\begin{gathered} \frac{\partial^2 \rho_1}{\partial t^2} = -c_S^2 \frac{\partial^2 \rho_1}{\partial x_i^2} - \rho_0 4\pi G \rho_1 \\ \textit{vector} \Downarrow \textit{notation} \\ \left(\frac{\partial^2}{\partial t^2} - c_S^2 \nabla^2 - 4\pi G \rho_0\right)\rho_1 = 0 \end{gathered} \tag{8.6}$$

This last relation admits solutions of the kind

$$\rho_1 \sim e^{i(\vec{k}\cdot\vec{x} - \omega t)}, \quad \textit{with } \omega^2 = c_S^2 (K^2 - K_J^2), \quad K_J \equiv \textit{Jeans wavenumber} = \left(\frac{4\pi G \rho_0}{c_S^2}\right) \tag{8.7}$$

Similarly to the case of the Brunt-Väisälä case, we have to distinguish two cases:

- $K > K_J$: we have simple sound waves, which are dispersed due to the effect of self-gravity
- $K < K_J$: the frequency becomes imaginary, hence we develop on instability: the compression is such that self-gravity is strong enough that the the perturbation collapses.

The transition between stable and unstable is sets by a characteristic wavelength and the associated mass

$$\begin{aligned} \lambda_J \equiv \textit{Jeans length} &= \frac{c_S^2}{2G\rho_0} \\ M_J \equiv \textit{Jeans mass} &= \frac{4}{3}\pi \rho_0 \lambda_J^3 \end{aligned} \tag{8.8}$$

The *Jeans mass* gives the mass scale, for a fixed perturbation wavelength, that will undergo gravitational collapse under its own self-gravity.

[20] we derive both members by $\frac{\partial}{\partial t}$

8.2 Surface instability

Let's review some of the most important and relevant instabilities onsetting at the interface between fluids.
We begin by setting up the problem in an *ideal case*.
Let's start with the most idealised case of two incompressible fluids in a gravitational potential, moving at two different velocities

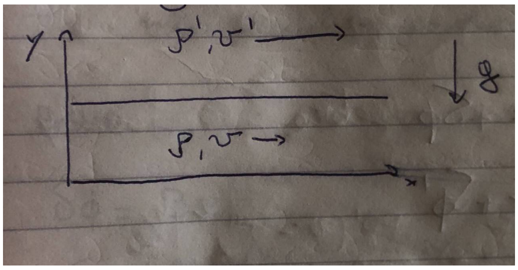

Figura 8.1

Note that in each fluid initially we have $\vec{\nabla} \times \vec{v} = 0$, $\vec{\nabla} \times \vec{v}^{'} = 0$, but the vorticity at the interface will not be $\vec{\omega} = 0$.
Since we assume the fluids to be incompressible and irrotational, we can define the velocities using only potentials, so that $\vec{v} = -\vec{\nabla}\phi$, $\vec{v}^{'} = -\vec{\nabla}\phi^{'}$.
If we call Ψ the *gravitational potential*, the momentum equation becomes

$$-\vec{\nabla}\frac{\partial \phi}{\partial t} + \vec{\nabla}\Big(\frac{v^2}{2}\Big) = -\frac{\vec{\nabla}P}{\rho} - \vec{\nabla}\Psi \tag{8.9}$$

This has a $Bernoulli - like\ integral\ form$

$$-\frac{\partial \phi}{\partial t} + \frac{v^2}{2} + \frac{P}{\rho} + \Psi = F(t) \tag{8.10}$$

Take a small perturbation ϵ at the interface

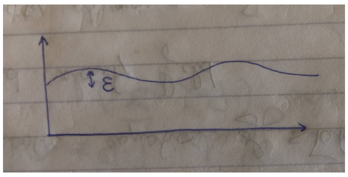

Figura 8.2

so that the interface has an equation $y = \epsilon(x, t)$.
The two potentials are:

$$\begin{aligned} \phi &= -vx + \delta\phi \\ \phi^{'} &= -v^{'}x + \delta\phi^{'} \end{aligned} \tag{8.11}$$

and from incompressibility $\nabla^2\delta\phi = \nabla^2\delta\phi^{'} = 0$.
In order to follow the evolution of $\epsilon(x, t)$, let's sit on a $Lagragian\ frame$:

$$\begin{aligned} \frac{D\epsilon}{Dt} &= \frac{\partial \epsilon}{\partial t} + v\frac{\partial \epsilon}{\partial x} = -\frac{\partial \delta\phi}{\partial y} \\ \frac{D\epsilon}{Dt} &= \frac{\partial \epsilon}{\partial t} + v^{'}\frac{\partial \epsilon}{\partial x} = -\frac{\partial \delta\phi^{'}}{\partial y} \end{aligned} \tag{8.12}$$

Let's assume that we can treat ϵ as a *plane wave*

$$\epsilon(x, t) = Ae^{i(kx-\omega t)} \tag{8.13}$$

The *incompressibility condition* $\nabla^2\delta\phi = \nabla^2\delta\phi^{'} = 0$ gives for the velocity potentials

$$\begin{aligned} \delta\phi &= Be^{i(kx-\omega t)+ky} \\ \delta\phi^{'} &= B^{'}e^{i(kx-\omega t)-ky} \end{aligned} \tag{8.14}$$

where the $\pm ky$ parts have been added to ensure the perturbations tend to zero at the boundaries at infinity.
Now let us focus on gravity, let's assume we can write

$$\Psi = g\epsilon \;\; with \;\; \frac{\partial \Psi}{\partial y}\Big|_{y=0} \equiv g \tag{8.15}$$

Since $\epsilon = \epsilon(x,t)$, then the force $\vec{F} = -\vec{\nabla}\Psi$ can be written as $\vec{F} \equiv \vec{F}(t) + g\epsilon$.
Going back to the original integral form equation and solving for the *pressure*, requiring pressure equilibrium at the interface:

$$\begin{aligned} \rho\Big(-\frac{\partial \delta\phi}{\partial t} + \frac{v^2}{2} + g\epsilon\Big) &= \rho^{'}\Big(-\frac{\partial \delta\phi^{'}}{\partial t} + \frac{(v^{'})^2}{2} + g\epsilon\Big) + K(t) \\ K(t) = \rho F(t) - \rho^{'} F^{'}(t) &= \frac{1}{2}\rho v^2 - \frac{1}{2}\rho^{'}(v^{'})^2 \quad {}^{21} \end{aligned} \tag{8.16}$$

Let's look at

$$\begin{aligned} v^2 = (u_1 + \vec{\nabla}\delta\phi)^2 \quad &\Rightarrow \quad v^2 - 2v\frac{\partial \delta\phi}{\partial x} \\ (v^{'})^2 = (u_1^{'} + \vec{\nabla}\delta\phi^{'})^2 \quad &\Rightarrow \quad (v^{'})^2 - 2v^{'}\frac{\partial \delta\phi^{'}}{\partial x} \end{aligned} \tag{8.17}$$

Substitute and make some algebra, then we finally arrive to

$$\frac{\omega}{k} = \frac{\rho v + \rho^{'} v^{'}}{\rho^{'} + \rho} \pm \Big[\frac{g}{k}\frac{\rho - \rho^{'}}{\rho + \rho^{'}} - \frac{\rho\rho^{'}(v - v^{'})^2}{(\rho + \rho^{'})^2}\Big]^{\frac{1}{2}} \tag{8.18}$$

that defines the generic dispersion relation for these kind of perturbations.
Begin with the case

$$\begin{gathered} v = v^{'} = 0, \quad \rho > \rho^{'} \\ \Downarrow \\ \frac{\omega}{k} = \pm\sqrt{\frac{g}{k}\frac{\rho - \rho^{'}}{\rho + \rho^{'}}} \end{gathered} \tag{8.19}$$

In the limit $\rho >> \rho^{'}$ we get the *deep water approximation*

$$\omega = \pm\sqrt{gk} \tag{8.20}$$

8.3 Rayleigh-Taylor instability

If instead we consider

$$v = v^{'} = 0, \quad \rho < \rho^{'} \tag{8.21}$$

ω becomes complex, hence this condition is *unstable*.
Note that instead of g we could have used any acceleration $\vec{a}$ and we still would have got the same result.

8.4 Kelvin-Helmoltz instability

Take a *RT stable fluid*, but with $v \neq v^{'} \neq 0$. So the fluid is unstable if

$$\frac{g}{k}\frac{\rho - \rho^{'}}{\rho + \rho^{'}} - \frac{\rho\rho^{'}(v - v^{'})^2}{(\rho + \rho^{'})^2} < \; 0 \tag{8.22}$$

Then gravity helps stabilising the fluid, since if $g = 0$ this is always unstable. This instability is called the *Kelvin − Helmotz instability*.

8.5 Thermal instability

Thermal instabilities may occur whenever a fluid is pushed out of equilibrium locally.
Imagine a uniform medium subject to cooling, for instance via radiation.
In general the cooling is encoded in the *cooling function* $\Lambda(\rho, t)$ and the heating due to external energy injection or conduction. Imagine that a medium is characterised by a broken cooling law (like law density plasmas), if a temperature or density fluctuation moves the gas to a new cooling regime, the system might become unstable. The instability is not hydrodynamical per se, but it can drive a flow: a fluctuation lead to excess cooling, hence the temperature drops, the density

increases, further increasing the cooling and dropping the pressure, so the pressure inbalance between perturbated region and unperturbate environment drives the condensation.
Thermal instabilities shape the ISM and maybe responsible for the formation of clouds, without invoking self-gravity. They may further push towards a self-gravitating regime and seed Jeans instability.
To derive it the basic starting points are the equations:

$$\begin{gathered}
\frac{D\rho}{Dt} + \rho\vec{\nabla}\cdot\vec{v} = 0 \\
\rho\frac{D\vec{v}}{Dt} + \vec{\nabla}P = 0 \\
\frac{1}{\gamma-1}\frac{DP}{Dt} - \frac{\gamma}{\gamma-1}\frac{P}{\rho}\frac{D\rho}{Dt} + \rho L - \vec{\nabla}\cdot(K\vec{\nabla}T) = 0 \\
with\ equation\ of\ state\ \ P = \rho K_b T
\end{gathered} \tag{8.23}$$

Expanding at first order in the perturbations written as plane waves, the equations become

$$\begin{gathered}
\omega\rho_1 + \rho_0 i\vec{K}\cdot\vec{v}_1 = 0 \\
\omega\rho_0\vec{v}_1 + i\vec{K}P_1 = 0 \\
\frac{\omega}{\gamma-1}P_1 - \frac{\omega\gamma P_0}{(\gamma-1)\rho_0}\rho_1 + \left[\rho_0\frac{\partial L}{\partial\rho}\rho_1 + \rho_0\frac{\partial L}{\partial T}T_1 + K_0K^2T_1\right]_{evalued\ in\ the\ unperturbate\ fluid} = 0 \\
with\ the\ condition\ \ \frac{P_0}{P_1} - \frac{\rho_1}{\rho_0} - \frac{T_1}{T_0} = 0
\end{gathered} \tag{8.24}$$

We can then verify that the dispersion relation is cubic, hence ω has at least one real root, that if it is positive indicates the growth of the mode.

9 Turbulence

We have seen that flows where the *Reynolds number* $Re = \frac{UL}{\nu}$ is large, transition from a state in which viscosity maintains the orderly laminar flow into turbulent motion.
Note, again, that turbulence is chaotic, but not in the Hamiltonian sense.
Turbulence is *always dissipative*. Hence, sustained turbulence require a constant input, a source, that will ultimately set the largest scale of the turbulence.
Finally note that for a fixed viscosity every flow is turbulent at same scale. Hence turbulence shapes every scale in the Universe.
Due to its unpredictable nature, turbulence is studied stochastically and statistically.
We begin by the incompressible case, e.g. $\vec{\nabla} \cdot \vec{v} = 0$. In a steady state $\frac{\partial}{\partial t} = 0$ so

$$v_j \frac{\partial v_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \nu \nabla^2 v_i \tag{9.1}$$

Taking the divergence and using the continuity equation, we get

$$\nabla^2 P = -\frac{\partial^2 \tau_{ij}}{\partial x_i \partial x_j}, \quad with\ \tau_{ij} = \rho v_i v_j \ \ the\ Reynolds\ stress \tag{9.2}$$

So in incompressible turbulent flows, changes in pressure only alter the velocity field, pushing around fluid parcels.
Before we go on, we need to review a few contcepts.
As we did for the *Vlasov equation*, suppose that we can break the fluid velocity $\vec{v} = \vec{u} + \vec{U}$, with $<\vec{v}>= \vec{U}$, $<\vec{u}>= 0$, but $<\vec{u}^2> \neq 0$.
Being expectation values, they need to be computed within some give hypoteses, that is spatio-temporal intervals, etc.
In the fully developed (stationary) turbulence, the *Taylor hypothesis* states that time and space averages are the same, hence implying that the correlation structure of turbulence is the same.
Take two points x and $x' = x + r$, then the expectation value

$$< v_i(x) v_j(x+r) >= U(x)U(x+r) + R_{ij}(r) \tag{9.3}$$

defines the *correlation tensor* R_{ij}.
Note that we assumed stationarity explicitly by looking in displacement independent of x

$$R_{ij}(r) =< u_i(x) u_j(x+r) > \tag{9.4}$$

We demand that $R_{ij} \to 0$ as $r \to \infty$, hence its *Fourier trasform* is defined.
The *power spectrum* of the process is

$$\phi_{ij}(K) = \frac{1}{2\pi^3} \int d^3r R_{ij}(\vec{r}) e^{-i\vec{K}\cdot\vec{r}} \tag{9.5}$$

We just found *wiener − khintchine theorem*.
It is also worth noting that

$$\begin{gathered} R_{ij}(0)\ is\ an\ energy \\ R_{ij}(0) = \int \phi_{ij}(\vec{K}) d^3K =< u_i u_j > \end{gathered} \tag{9.6}$$

If we require *isotropy*, so that $< u_i u_j >= 3 < u^2 >$ and $d^3K = K^2 dK$, then we can define the energy spectrum via

$$\int E(K) K^2 dK = \int \phi_{ii}(\vec{K}) d\vec{K} \tag{9.7}$$

Within these assumptions we cannot really move forward without making assumptions over the velocity fluctuation distribution. Whatever it is, we know it is going to go to zero at infinity, hence it is integrable. Additionally, it should have a maximum at zero and, locally, decay

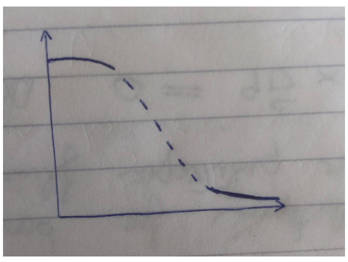

Figura 9.1: Very simplified trend of velocity fluctuation distribution

Near the maximum, hence small displacements, we can take a quadratic approximation. Because of symmetry and isotropy $R_{ij}(\vec{r}) = R_{ij}(-\vec{r})$, hence it must be quadratic and it can be represented as a longitudinal and transverse sum

$$R_{ij}(r) = F(r)r_i r_j + G(r)\delta_{ij} \tag{9.8}$$

Near zero $\frac{\partial R_{ij}}{\partial r_i} = \frac{\partial R_{ij}}{\partial r_j} = 0$, $\frac{\partial^2 R_{ij}}{\partial r_i \partial r_j} < 0$ because *zero* is a maximum.
This defines a scale length

$$\frac{\partial^2 R}{\partial r^2} = -\frac{1}{\lambda^2} \tag{9.9}$$

called the *Taylor scale*, representing the curvature near the zero of the correlation function, defining where most of the energy is located. λ is not fixed by any property of the fluid, hence is not really useful to understand turbulence, but it is useful to understand that it can be described in terms of scales.
The realisation of Kolmogorov and following breakthrough is that any scale is important and that each scale has the same statistical proprieties (scale invariant) all the way down to the molecular viscosity level. At each scale, turbulence itself is the source of dissipation to the lower scales. Ultimately, thus turbulence should be governed by the rate at which energy is dissipated (hence introduced in the system) and last in microscopic viscosity.
Let's call ϵ the energy dissipation rate and ν the viscosity. Each scale L will have a different *Reynolds number* $Re = \frac{UL}{\nu}$, but if we require the process to be stationary then it must be excatly equal to what is necessary for ϵ to remain unchanged from a scale to another. This implies that the spectrum of the fluctuations must be *universal* and *scale free*. This is the *inertial* part of the spectrum.
Now if $[\epsilon] = L^2T^{-3} = U^3L^{-1}$ is constant for any scale l, then $u_l = \epsilon^{\frac{1}{3}} l^{\frac{1}{3}}$.
The effective viscosity is $u_l l = \epsilon^{\frac{1}{3}} l^{\frac{4}{3}}$.
The *kinetic viscosity* introduces a scale at which the viscous time is equal to the fluid element dynamical time

$$\begin{aligned} l_K &\equiv Kolmogorov\ length = \left(\frac{\nu^3}{\epsilon}\right)^{\frac{1}{4}} \\ v_K &= \left(\nu\epsilon\right)^{\frac{1}{4}} \quad typical\ viscosity \\ t_K &= \left(\frac{\nu}{\epsilon}\right)^{\frac{1}{2}} \quad timescale \end{aligned} \tag{9.10}$$

If the cascade is universal, it should not depend on ν, characteristic of the specific fluid, hence ν should be *hidden*.
Note that we can indeed write

$$\nu = l^{\frac{4}{3}} \epsilon^{\frac{1}{3}} \tag{9.11}$$

Assume we can write a dimensionless universal spectrum

$$E_*(l_K, K) = \frac{E(K,t)}{v_K^2} \tag{9.12}$$

Now, the energy dissipation rate can be put as

$$\epsilon = -\frac{3}{2}\frac{d^2u}{dt^2} = 2\nu \int_0^\infty E(K,t)K^2 dK \tag{9.13}$$

leading to

$$E(K,t) \sim \epsilon^{\frac{3}{2}} K^{-\frac{5}{3}} \quad Kolmogorov\ spectrum \tag{9.14}$$

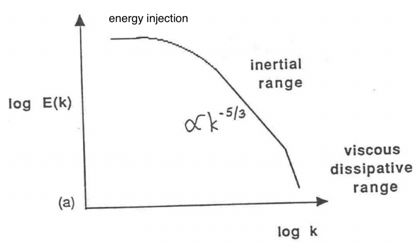

Figura 9.2: Kolmogorov spectrum

9.1 Compressible turbulence

The main feature in compressible media is the *compressibility*, hence density and pressure perturbation propagate as sound waves.
So in the compressible case vorticity and shear will feedback in the density, then via the equation of state, in the pressure that drives the emission of acoustic waves.
Let's consider the continuity equation and the momentum equation

$$\begin{aligned} \frac{\partial \rho}{\partial t} &= -\rho_0 \frac{\partial v_i}{\partial x_i} \\ \frac{\partial \rho v_i}{\partial t} + \frac{\partial \tau_{ij}}{\partial x_j} &= -\frac{\partial P}{\partial x_i} = -c_S^2 \vec{\nabla} \rho \end{aligned} \tag{9.15}$$

Differentiating and combining

$$\Big(\frac{\partial^2}{\partial t^2} - c_S^2 \nabla^2\Big)\rho = -\frac{\partial^2 \tau_{ij}}{\partial x_i \partial x_j} \tag{9.16}$$

This is a wave equation with a source, then

$$\delta\rho \sim -\frac{1}{c_S^2} \int \frac{dr}{r} \frac{\partial^2}{\partial x_i \partial x_j} \tau_{ij}\Big(t - \frac{r}{c_S}, r\Big) \tag{9.17}$$

Assuming that time derivatives are the same as space derivatives, so that $\frac{\partial}{\partial t} \to c_S \frac{\partial}{\partial x}$, then we get

$$\delta\rho \sim -\frac{1}{c_S^4} \int \frac{(r_i - r_i^{'})(r_j - r_j^{'})}{|\vec{r} - \vec{r}^{'}|^3} \frac{\partial^2 \tau_{ij}}{\partial t^2} d\vec{r}^{'} \tag{9.18}$$

So the radiation pattern is determined by $\ddot{\tau}_{ij} n_i n_j$. This implies that the

$$< \delta\rho^2 > \sim \frac{< \ddot{\tau}_{ij} \ddot{\tau}_{ij} >}{c_S^8} \tag{9.19}$$

Now, each $\ddot{\tau}_{ij}$ carries a $\frac{v^4}{L^2}$ and the volume integral for the average brings a L^3, hence

$$\frac{< \ddot{\tau}_{ij} \ddot{\tau}_{ij} >}{c_S^8} \sim \rho \frac{v^8}{c_S^5 L} \quad \Rightarrow \quad Rate\ of\ energy\ dissipation\ \ \epsilon \sim \rho v^3 M^5 \tag{9.20}$$

with M *Mach number*.
In the isotropic case

$$E = 40 \frac{\rho v^8}{c_S^5 L} \tag{9.21}$$

Notice that this is the just the energy emitted in sound waves and says nothing about the way in which the energy is dissipated in the medium.
An interesting application could be in the role turbulence may play in heating the chromosphere: the Sun photosphere is convective and turbulent, the turbulence in a compressive medium generates sound waves that then propagate in a thin medium with $\frac{\partial \rho}{\partial z} < 0$. Albeit not hydrostatic, the waves might steepen in shocks and contribute to the raise in T from 6000 K to 20000 K seen in the chromosphere and also play a part in the launching of the solar wind.
Turbulence is also observed in the *interstellar medium* from $sub-pc$ to Kpc scales. The driving terms seems to be associated with external sources rather than the local star formation activity.
In particular we can talk about *supernovae*. Assuming $E_{SN} = 10^{51}\ erg$ and a frequency[22] of $20\ SN\ per\ 10^6\ yr\ per\ Kpc^2$, we get an energy injection of

$$\int \epsilon_{SN} dz \simeq \frac{20 \cdot 10^{51}\ erg}{3 \cdot 10^{13}\ s \times 9 \cdot 10^{43}\ cm^2} \simeq 7 \cdot 10^{-5}\ erg\ cm^{-2}\ s^{-1} \tag{9.22}$$

Energy dissipation in turbulence can be estimated as

$$\int \epsilon dz \simeq 0.5 \rho v^3 \simeq 10^{-24}\ g\ cm^{-3} (10^6\ cm\ s^{-1})^3 \simeq 10^{-6}\ erg\ cm^{-2}\ s^{-1} \tag{9.23}$$

The energy in supernovae is enough to sustain turbulence in the disk, assuming a typical vertical scale height $H \sim 70\ pc$ and turbulent velocities $\sim 15\ Km\ s^{-1}$.
Note that the typical turn-around time is $O(5\ Myr)$.
We will see that turbulence is expected to play a key role in accretion disks as well.

[22] we are neglecting the disk thickness

10 Self-similar solutions

10.1 Spherical accretion and winds

We begin our review of large scale flows by studying the problems of steady accretion and outflows. The two problems are the same modulo a change in the sign of the velocities.
Consider a *spherical symmetry configuration* with the gas at rest at ∞. We model the central object as a point. We are in steady state, e.g. $\frac{\partial}{\partial t} = 0$, hence the continuity equation gives

$$\begin{aligned} \dot{M} &= \rho v A = \ constant \\ \dot{M} &= 4\pi r^2 v \rho \end{aligned} \tag{10.1}$$

The momentum equation, neglecting the gas self-gravity, is

$$v\frac{dv}{dr} = -\frac{1}{\rho}\frac{dP}{dr} - \frac{GM}{r^2} \tag{10.2}$$

Taking the usual $P = c_S^2 \rho$ assumption, the last equation becomes:

$$v^2 \frac{dlogv}{dr} = -c_S^2 \frac{dlog\rho}{dr} - \frac{GM}{r^2} \tag{10.3}$$

From the continuity equation

$$\begin{aligned} \frac{dlog\dot{M}}{dr} &= 0 \\ \frac{dlog\rho}{dr} &= -\frac{dlogv}{dr} - \frac{2}{r} \end{aligned} \tag{10.4}$$

Substituting we arrive at

$$\begin{aligned} v^2 \frac{dlogv}{dr} &= c_S^2 \Big(\frac{dlogv}{dr} + \frac{2}{r}\Big) - \frac{GM}{r^2} \\ &\Downarrow \\ (v^2 - c_S^2)\frac{dlogv}{dr} &= \frac{2c_S^2}{r}\Big(1 - \frac{GM}{2c_S^2 r}\Big) \end{aligned} \tag{10.5}$$

Note that analogy with the *de Laval Nozzle*. Just like in that case, we have a point where the flow transitions from subsonic to super-sonic

$$r_* = \frac{GM}{2c_S^2} \tag{10.6}$$

with the exception that in this flow there are no boundaries. The transition is done by the dilution. in the $v - \frac{r}{r_*}$ plane, we get

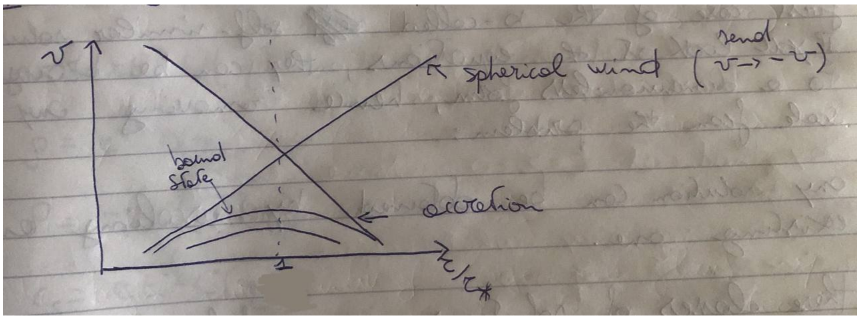

Figura 10.1: $v - \frac{r}{r_*}$ plane

In the spherical wind case we want the flow to asymptotically reach an unbound state. This implies that the terminal velocity must be the *escape velocity* $\sqrt{\frac{GM}{R}}$[23], from which we can derive terminal density, pressure, etc.
The spherical wind case is called the *Parker wind solution*. The spherical accretion case is the *Bondi solution*, generalised to the $Bondi - Hoyle - Littleton$ for non-null bulk gas velocity.
For isothermal gas in accretion

$$\begin{aligned} \dot{M} &\sim \frac{\rho G^2 M^2}{c_S^3} \\ \dot{M} &\sim \frac{(GM)^2 \rho_\infty}{(c_{S_\infty}^2 + v_\infty^2)^{\frac{3}{2}}} \end{aligned} \tag{10.7}$$

[23] R is the star radius

10.2 Self-similar flows

The spherical wind or accretion problems are a first case of the so-called $self-similar\ solutions$. If we look at the equations, they can be brought to a dimensionless form, hence removing any scale from the problem, any solution can be obtained by rescaling an existing one.
These classes of solutions are useful since they give analytical models to analyse astrophysical flows. Note that for the spherical accretion or wind case, we assumed

- Stationarity $\rightarrow$ no $\frac{\partial}{\partial t}$
- Spherical symmetry $\rightarrow$ remove all variables except r
- Constant mass flux $\rightarrow$ link $\rho,\ r,\ v$

and these conditions determined completely the solution.
In the presence of much high degree of symmetry, we are tempted to look for some combination of r and t so that everything can be expressed in terms of that variable.
Define $\eta \equiv r^{\lambda} t^{-\mu}$, λ e μ will be set by the particular constrains we have, but in general we can express

$$\begin{aligned} v &= rt^{-1}U(\eta) \\ \rho &= r^{-3}D(\eta) \\ P &= r^{-1}t^{-2}\Pi(\eta) \\ c_S &= rt^{-1}C(\eta) \\ \frac{\partial}{\partial t} &= -\mu\eta t^{-1}\frac{d}{d\eta} \\ \frac{\partial}{\partial r} &= \lambda\eta r^{-1}\frac{d}{d\eta} \end{aligned} \tag{10.8}$$

Call $n = 1, 2, 3...$ the dimensions of the problem.
Doing some algebra and assuming $P = K\rho^{\gamma}$, we can write the continuity momentum and energy equations as

$$\begin{aligned} (\lambda U - \mu)\eta\frac{d}{d\eta}D + (n-2)DU + \lambda D\eta\frac{d}{d\eta}U = 0 \\ (\lambda U - \mu)\eta\frac{d}{d\eta}D = D^{-1}\Big(\lambda\eta\frac{d}{d\eta}\Pi - \Pi\Big) - U(U-1) \\ (\lambda U - \mu)\eta\frac{d}{d\eta}(\Pi D^{-\gamma}) + ((3\gamma-1)U-1)\Pi D^{-\alpha} = 0 \end{aligned} \tag{10.9}$$

that are ordinary differential equations that can be far more easily solved.

10.3 Blast waves

Let's look at an application of similarity methods for a blast wave. This is called the $Sedov\ problem$. Say that at $t = r = 0$ some energy E is injected in the system, e.g. we have an *explosion*. This is essentially what happens in a supernovae or in a nuclear explosion.
Assume that we have spherical symmetry and the blast propagates in a uniform medium with density ρ.

$$\begin{gathered} [E] = ML^2T^{-1} \\ [\rho] = ML^{-3} \\ \Downarrow \\ \tfrac{[E]}{[\rho]} = L^5T^{-2} \quad \rightarrow \quad this\ quantity\ is\ independent\ of\ the\ mass \end{gathered}$$

Since we have no other characteristic scales, the combination $R^5t^{-2} \equiv \eta$ is our $selfsimilar\ variable$. Without going through the solution details, we can guess that, since E is fixed, the temperature will have to decrease with radius in time.
Just from our dimensional analysis, we can infer that

$$\begin{gathered} R(t) \sim \left(\tfrac{E}{\rho}\right)^{\frac{1}{5}} t^{\frac{2}{5}} \\ V(t) \sim \tfrac{2}{5}Rt^{-1} \end{gathered}$$

For typical values such as $E \sim 10^{50}\ erg$ and $\rho \sim 10^{-24}\ g\ cm^{-3}$ we find $R(t) \sim 10^{0.7}\left(\frac{E_{50}}{\rho_{24}}\right)^{\frac{1}{5}} t_{yr}^{\frac{2}{5}}\ pc$. So a SNR with $R \sim 1\ pc$ has an age of $\sim 100\ yr$ and should be expanding with $v \sim 1000\ \frac{Km}{s}$. These values are in a remarkable agreement with observations.

In the end, there are a few things that is worth noticing: we are assuming that the blast is *adiabatic* and this is certainly true when the gas is *optically thick*, but as the density decreases, so will the optical depth, hence there must exist a radius at which the blast becomes optically thin and hence not adiabatic anymore. This will be also the moment at which the solution stops being valid. Therefore, the cooling introduces a time scale in the problem.
The pressure in the wave scales as

$$P \sim \rho v^2 \sim R^{-1} t^{-2} \tag{10.10}$$

In the adiabatic case we can eliminate the time and find the pressure as a function of R only

$$\begin{gathered} t \sim \left(\frac{E}{\rho}\right)^{\frac{1}{2}} R^{\frac{5}{2}} \\ \Downarrow \\ P \sim E R^{-3} \end{gathered} \tag{10.11}$$

and, as we want, the pressure has the dimension of an energy density.
This implies that

- the radius at which any pressure is reached $R_* \sim E^{\frac{1}{3}}$
- the shock wave will eventually stall as its pressure reaches equilibrium with the surrounding medium:

$$\begin{gathered} R \sim E^{\frac{1}{5}} t^{\frac{2}{5}} \\ P_{env} \sim M R^{-1} t^{-2} \\ \Rightarrow \frac{P}{\rho} \sim R^2 t^{-2} \\ \Rightarrow R(t) \sim \left(\frac{P}{\rho}\right)^{\frac{1}{2}} t \end{gathered}$$

hence a stalled shock will expand with $V \sim constant = \alpha c_S$.

10.4 Snowplow phase

As the shock expands, it cools hence E is not constant anymore. This causes the shock to slow down and accumulate material in its courses. Eventually, the expansion will have to become subsonic. Since E is not conserved anymore, the evolution should be governed by conservation of momentum. The expansion law becomes

$$R \sim \left(\frac{MV}{\rho}\right)^{\frac{1}{4}} t^{\frac{1}{4}} \tag{10.12}$$

10.5 Ionisation fronts

We have seen that in *HII regions* the ionisation creates a pressure unbalance between the bubble and the outside. It is not a difficult exercise to derive the expansion law for the ionisation front: since the rate of expansion of the front is at least the sound speed in the neutral phase, the momentum conservation reads

$$\rho_1 \dot{R}^2 = \rho_2 c_{S_2}^2 \tag{10.13}$$

which is a way of seeing the $Rankine - Hugonoit$ *jump conditions* at work.
From the ionisation balance we find

$$\alpha \rho_2^2 R^3 = \beta \rho_1 \tag{10.14}$$

The recombinations per unit area are $\alpha(T) n_e^2 R^3$ and should balance the outside density to assure the mass conservation.
One find that

$$\rho_2 \sim \rho_1^{-\frac{1}{2}} R^{-\frac{3}{2}}$$

and from the momentum equation $R \sim t^{\frac{4}{7}}$

10.6 The Lane-Emden equation

Not strictly a self-similar solution, the $Lane - enden\ equation$ is derived along similar reasoning. Let's see how: we start from spherically symmetric hydrostatic condition

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho, \quad \frac{dP}{dr} = -\rho \frac{GM(r)}{r^2} \tag{10.15}$$

assume now that $P = K\rho^m$, this allows to combine the equations in

$$\frac{1}{r^2}\frac{d}{dr}\left[r^2 m\rho^{m-2}\frac{d\rho}{dr}\right] = -\frac{4\pi G\rho}{K} \tag{10.16}$$

If we exploit the relation $\rho^{m-2}\frac{d\rho}{dr} = \frac{1}{m-1}\frac{d\rho^{m-1}}{dr}$, we get

$$\frac{1}{r^2}\frac{d}{dr}\left[r^2 \frac{m}{m-1}\frac{d\rho^{m-1}}{dr}\right] = -\frac{4\pi G\rho}{K} \tag{10.17}$$

Let's define $\rho = \rho_0\theta^n$ and so

$$\frac{1}{r^2}\frac{d}{dr}\left[r^2 \frac{m}{m-1} n\theta^{n-1}\frac{d\theta}{dr}\right] = -\frac{4\pi G\theta^n}{K} \tag{10.18}$$

11 Accretion physics and Eddington limit

We have seen and dealt with the problem of stationary self-similar accretion. Let's go back to it and study it in a bit more detail.
Take a point mass M and relative potential $\phi = -\frac{GM}{r}$. Material falling acquires kinetic energy $\Delta\phi = \frac{GM}{r}$, assuming it starts from infinity.

- If the material ends up at rest, for instance on the surface of the star, the energy dissipated will be $e = \frac{GM}{r}$
- if, instead, the material goes in Keplerian orbit we have $e = \frac{GM}{2r}$

Anyway the dissipated energy is going in internal energy or radiated away.
Let's start with the *adiabatic accretion* case. Take an *ideal gas*, then we have

$$e = \frac{P}{(\gamma-1)\rho}$$
$$\gamma = \frac{C_p}{C_v}$$
$$P = \frac{R}{\mu}\rho T$$ [24]

We find that the gas temperature after the dissipation is

$$T = \frac{1}{2}(\gamma-1)T_{vir}, \quad where\ T_{vir} = \frac{GM\mu}{Rr}\ is\ the\ Virial\ temperature \tag{11.1}$$

and that the sound speed is $c_S = \left(\frac{\gamma RT}{\mu}\right)^{\frac{1}{2}}$.
In system having $T \sim T_{vir}$, the sound speed is comparable with the escape velocity, hence no acceleration is possible.
Adiabatic accretion takes place on the dynamical $free\ fall\ scale$

$$\tau_d = \frac{r^3}{(GM)^{\frac{1}{2}}} \tag{11.2}$$

Note that if we include radiative losses, the gas is able to remain cool and the temperature of the accreted gas is $T << T_{vir}$ and the accretion can continue. The optical depth increases with $\dot{M}$, so that at a given $\dot{M}_c$ photons are just advected with the flow. Hence, for large enough $\dot{M} > \dot{M}_c$ every accretion flow is adiabatic.

Consider a volume of gas on which a flux of photons is incident on one side. The force exerted on a unit mass is $\frac{Fk}{c}$, where F is the flux and k is the opacity. At the same time gravity is $\frac{GM}{r^2}$. The two forces balance for a flux

$$F_E = \frac{c}{k}\frac{GM}{r^2} \tag{11.3}$$

For spherically symmetric fluxes luminosity and flux are related by $L = 4\pi r^2 F$, hence we can define the *Eddington luminosity*

$$L_E = \frac{4\pi GM_c}{k} \tag{11.4}$$

that for *electron scattering* is $L_E \sim 4\cdot 10^4 \frac{M}{ML}$.
If the luminosity is due to accretion, then we derive the *Eddington rate*

$$\dot{M}_E = \frac{4\pi rc}{k} \tag{11.5}$$

Note that while L_E is an actual bound, no such bound exist for $\dot{M}_E$, meaning that $\dot{M}$ can be greater than $\dot{M}_E$.
$\dot{M}_E$ just marks the transition to advection dominated accretion flows (ADAF).

11.1 Roche lobes and mass transfer

Including angular momentum implies that accretion cannot proceed directly and we have a new timescale in the process, i.e. the timescale for outward transport of angular momentum.

[24] $\frac{R}{\mu}$ represents in this equation the atomic weight

Consider two stars with masses M_1 and M_2 $\rightarrow$ $q = \frac{M_1}{M_2}$.
If they are in orbit around each other

$$\Omega^2 = G\frac{(M_1 + M_2)}{a^3} \tag{11.6}$$

with a their separation.
Going to a corotating frame, we can write the potential

$$\phi(r) = -\frac{GM_1}{r_1} - \frac{GM_2}{r_2} - \frac{1}{2}\Omega^2 r^2 \tag{11.7}$$

Note that we are not including non-corotating effects, such as the $Coriolis\ force$.
Equipotentials for $\phi(r)$ determine the shape of the stars

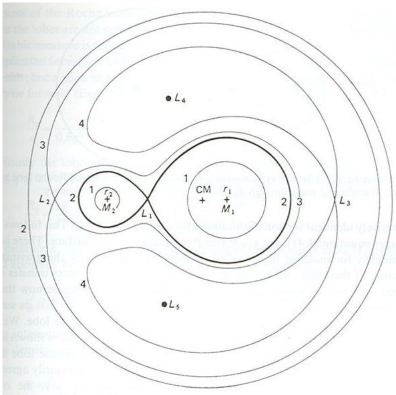

Figura 11.1: Roche lobe

Referring to the figure above, if the star size is comparable to the Roche lobe size then they are distorted. Winds drive the loss of angular momentum in a binary whose separation decreases and when one of the two stars, say for example M_2, will fill its *Roche lobe* material transfer will begin. As soon as the gas spills, it is not corotating anymore and it experiences a *Coriolis acceleration*. When passing through L_1, the gas has $T << T_{vir}$, so $c_S <<$ velocity coming from the acceleration from the gravity of the companion, hence the flow is highly supersonic. The evolution of the gas is hence subsontially ballistic. It falls towards M_1 rotates around it and finally bangs onto itself, developing a shock and settles in a ring. The size of the ring can be estimated by requiring that $M_1 \simeq conserved$ and using the *circularistation radius*, which solved returns

$$(GM_1 r_C)^{\frac{1}{2}} = j \tag{11.8}$$

with j the specific angular momentum for an orbit starting at L_1.
The ring then evolves in a disk via viscosity.

12 Accretion disks

12.1 Thin disks

Ignoring viscosity, the equation of motion in the potential of a point mass is

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \vec{\nabla}\vec{v} = -\frac{1}{\rho}\vec{\nabla}P - \frac{GM}{r^2}\hat{r} \tag{12.1}$$

and for an *ideal gas*

$$\frac{1}{\rho}\vec{\nabla}P = \frac{R}{\mu}T\vec{\nabla}logP \tag{12.2}$$

Define the following dimensionless quantities

$$\begin{aligned}
\tilde{r} &= \frac{r}{r_0} \\
\tilde{v} &= \frac{v}{\omega_0 r_0} \\
\tilde{t} &= \Omega_0 t \\
\tilde{\vec{\nabla}} &= r_0 \vec{\nabla}
\end{aligned} \tag{12.3}$$

the equation of motion is

$$\frac{\partial \tilde{v}}{\partial t} + \tilde{v} \cdot \tilde{\nabla} \tilde{v} = -\frac{T}{T_{vir}} \tilde{\nabla} log P - \frac{1}{\tilde{r}^2} \hat{r} \tag{12.4}$$

If cooling is important, then $T << T_{vir}$ hence pressure is negligible. This is equivalent to stating that the disk is *thin* or that the *gas moves on Keplerian orbits*.
The disk thickness can be derived simply looking at the balance of forces in the $z-direction$.

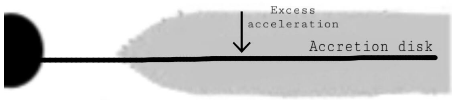

Figura 12.1: Sketch of a thin accretion disk

At first order, the excess acceleration is $g_z = -\Omega_0^2 z$.
Assuming an *isothermal gas* and *hydrostatic equilibrium*, we find out that

$$\begin{gathered} \rho = \rho_0 e^{-\frac{z^2}{2H^2}} \\ H = \frac{c_S}{\Omega_0} \quad is\ the\ disk\ scale\ height \\ c_S = \left(\frac{RT}{\mu}\right)^{\frac{1}{2}} \end{gathered} \tag{12.5}$$

In the end we can define the *aspect ratio* $\delta = \frac{H}{r}$, so we get

$$\delta = \frac{H}{r} = \frac{c_S}{\Omega r} = \frac{1}{M} = \left(\frac{T}{T_{vir}}\right)^{\frac{1}{2}} \tag{12.6}$$

In between neighboring orbits we must have shear. In the presence of viscosity we have a net torque on the disk, implying angular momentum transfer. In turn, a momentum transfer implies mass transfer.

Now, about viscosity, we can say that observations require $\nu \sim 10^{15}\ \frac{cm^2}{s}$, while $\nu_{molecular} \sim 10\ \frac{cm^2}{s}$.
Let's try to understand how to get this values. First of all we have to say that the actual process is not identified yet. So the prescription is to introduce a parameter α and assume

$$\nu = \alpha \frac{c_S^2}{\Omega} \tag{12.7}$$

The idea is the following: neighboring orbits are shearing and develop instabilities, like the $Kelvin-Helmoltz$ ones, hence a natural assumption is that hydrodynamic turbulence is responsible for the viscosity.
Take an eddy of size l developing due to shear instabilities. Its rotation rate will be given by the shear

$$\sigma = r\frac{\partial \Omega}{\partial r} \sim -\frac{3}{2}\Omega \tag{12.8}$$

The velocity for the eddy is $V \sim \sigma l$, hence the turbulent viscosity is

$$\nu_{turb} = l^2 \Omega \tag{12.9}$$

Because of compressibility, causality sets the maximum rate of rotation to c_S, hence their size must be at most $H \simeq \frac{c_S}{\sigma}$.
The largest contribution to viscosity will come from the largest eddies, hence $\nu \sim H^2\Omega\ or\ \alpha \sim 1$.

Let's describe now the disk in cylindrical coordinates (r, ϕ, z). Since the disk is thin, define the surface density as

$$\sum(r) = \int_{-\infty}^{\infty} dz \rho(z, r) \simeq 2H_0\rho_0 \tag{12.10}$$

where subscript 0 refers to the midplane.

- Continuity $\Rightarrow \frac{\partial}{\partial r} r \sum + \frac{\partial}{\partial r}(r \sum v_r) = 0$
- axysymmetry+Keplerian $\Rightarrow v_\phi^2 = \frac{GM}{r}$

and

$$\frac{\partial}{\partial t} v_\phi + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_r v_\phi}{r} = F_\phi \tag{12.11}$$

with F_ϕ viscous force.
Let's integrate the last equation over z and substitute the continuity to get the balance of the angular momentum

$$\frac{\partial}{\partial t}(r \sum \Omega r^2) + \frac{\partial}{\partial r}(r \sum v_r \Omega r^2) = \frac{\partial}{\partial r}\Big[S r^3 \frac{\partial \Omega}{\partial r}\Big] \tag{12.12}$$

and, if $\nu \neq \nu(z)$, we get

$$S = \int_{-\infty}^{\infty} \rho \nu dz \simeq \sum \nu \tag{12.13}$$

Taking an isothermal disk, α independent of z and using $\Omega \sim r^{-\frac{3}{2}}$, we arrive at the *thin disk equation*

$$r \frac{\partial \sum}{\partial t} = 3 \frac{\partial}{\partial r}\Big[r^{\frac{1}{2}} \frac{\partial}{\partial r}(\nu \sum r^{\frac{1}{2}})\Big] \tag{12.14}$$

From the azimutal equation, we get the mass transfer equation

$$\dot{M} = -2\pi r \sum v_r = 6\pi r^{\frac{1}{2}} \frac{\partial}{\partial r}(\nu \sum r^{\frac{1}{2}}) \tag{12.15}$$

which is a diffusion equation.
Note that in the thin disk model, the whole time dependence is set by ν, making this model *attractive*.

12.2 Steady state disks

In a steady state $\frac{\partial}{\partial t} = 0 \Rightarrow \dot{M} =$ *constant* and everything is determined.
For instance

$$\nu \sum = \frac{1}{3\pi} \dot{M} \Big[1 - \beta \Big(\frac{r_i}{r}\Big)^{\frac{1}{2}}\Big] \tag{12.16}$$

where β is an integration constant, related to the flux of angular momentum $F_J = -\dot{M}\beta\Omega_i r_i^2$ and r_i is the inner disk edge.
For accretion onto an object rotating with $\Omega_* < \Omega_i$, we find that $\beta = 1$, independent of Ω_*, hence F_J is inward and tends to spin up the accretion object.
For $\Omega_* \sim \Omega_i$, or for *magnetospheres*, things are differents and the object can spin down due to accretion.
Let's briefly discuss the implications of accretion onto a *solid* body. The velocity of the disk and the star will differ, hence for the gas to settle onto the star we must develop a boundary layer with a velocity gradient. This can potentially lead to instabilities that can enhance mixing in the upper layer of the star.

12.3 Disk temperature

Let's concentrate on slowly rotating stars, so that we can fix $\beta = 1$.
We are looking to estimate the disk surface temperature that, in LTE, will determine the radiation losses. The latter will be determined by the dissipation rate that, in turn, is determined by accretion.
From the first law oh thermodynamics

$$\rho T \frac{dS}{dt} = -\vec{\nabla} \cdot \vec{F} + Q_v \tag{12.17}$$

where S is the entropy, $\vec{F}$ represents the heat flux and Q_v the viscous dissipation rate.
Take almost stationary changes, e.g. changes on timescales longer than the dynamical timescale $\sim \Omega^{-1}$.

$$\begin{gathered} -\int_{-\infty}^{\infty} \vec{\nabla} \cdot \vec{F} dz + \int_{-\infty}^{\infty} Q_v dz \simeq 0 \\ surface \Downarrow integral \\ 2\sigma_r T_s^4 = \int_{-\infty}^{\infty} Q_v dz \end{gathered} \tag{12.18}$$

[25]
So, for slow changes the energy balance is local: what is dissipated locally is also radiated locally.

We need to estimate the viscous heating. Remember that this is given by $\sigma_{ij}\frac{\partial v_i}{\partial x_i}$. In our case, this is $Q_v = \frac{9}{4}\nu\rho\Omega^2$, hence we have, assuming ν is independent of z

$$\sigma_r T_s^4 = \frac{9}{8}\Omega^2\nu\sum \tag{12.19}$$

Substituting the various quantities

$$\sigma_r T_s^4 = \frac{GM}{r^3}\frac{3\dot{M}}{8\pi}\Big[1-\Big(\frac{r_i}{r}\Big)^{\frac{1}{2}}\Big] \tag{12.20}$$

Then T_s is independent of ν and in steady state depends only on $M \cdot \dot{M}$.
Far from the inner disk edge $T_s \sim r^{-\frac{3}{4}}$. This is the surface temperature gradient.
To estimate the inner temperature we have to model in details the transport mechanisms.
We make some idea by making several approximations:

- Radiative energy transport
- LTE

In addition, if we consider *plane parallel approximation*, then

$$\frac{d}{d\tau}\sigma_r T^4 = \frac{3}{4}F \tag{12.21}$$

Assuming no incident flux from outside, we can impose the approximate boundary condition

$$\sigma_r T^4\Big(\tau = \frac{2}{3}\Big) = F \tag{12.22}$$

where the *optical depth* is given by $\tau = \int_z^\infty K\rho dz$.
If most of the heat is generated near the plane of symmetry, $F \simeq$ *constant with* z, since ν is independent of z, and we get

$$F \simeq \sigma_r T_s^4 = \frac{GM}{r^3}\frac{3\dot{M}}{8\pi}\Big[1-\Big(\frac{r_i}{r}\Big)^{\frac{1}{2}}\Big] \tag{12.23}$$

integrating

$$\sigma_r T_s^4 = \frac{3}{4}\Big(\tau + \frac{2}{3}\Big)F \tag{12.24}$$

If $k =$ *constant with* z, the optical depth in the midplane is

$$\tau = k\frac{\sum}{2} \tag{12.25}$$

and for $\tau >> 1$ the temperature in the middle plane is given by

$$T^4 = \frac{27}{64}\sigma_r^{-1}\Omega^2\nu\sum{}^2 k \tag{12.26}$$

For an *ideal gas*[26] we can obtain the disk scale height

$$\begin{gathered}\frac{H}{r} = \Big(\frac{R}{\mu}\Big)^{\frac{2}{5}}\Big(\frac{3}{64\pi^2\sigma_r}\Big)^{\frac{1}{10}}\Big(\frac{k}{\alpha}\Big)^{\frac{1}{10}} GM^{-\frac{7}{20}} r^{\frac{1}{20}} (f\dot{M})^{\frac{1}{5}} \\ f = 1-\Big(\frac{r_i}{r}\Big)^{\frac{1}{2}}\end{gathered} \tag{12.27}$$

The thickness of the disk is relatively insensitive to α,k and r.
The viscous dissipation per unit area is given by

$$W_G = \frac{1}{2\pi r}\frac{GM\dot{M}}{2r^2} \tag{12.28}$$

So that

$$\frac{W_V}{W_G} = 3\Big[1-\Big(\frac{r_i}{r}\Big)^{\frac{1}{2}}\Big] \tag{12.29}$$

Then the viscous heating rate is, for $r \to \infty$ three times the gravitational heating.
Integrating over the whole disk

$$\int_{r_i}^{\infty} dr 2\pi r W_V = \frac{GM\dot{M}}{2r_i} \tag{12.30}$$

so that globally the total amount of energy released gravitationally goes to heating. From the result we found before, this does not happen locally.

[25] in the last relation, we must multiply by two the first term since the disk has two faces
[26] hence ignoring the radiation contribution to the pressure

12.4 Radiation pressure dominated disks

Especially in the inner regions, radiation pressure can become significant.
In this case the pressure becomes

$$P = P_r + P_g = \frac{1}{3}aT^4 + P_g \tag{12.31}$$

where P_r and P_g are respectively *radiation pressure* and *gravitational* one.
Define an effective sound speed $c_t^2 = \frac{P}{\rho}$, the relation $c_t = \Omega H$ keeps holding and for $P_r >> P_g$.
Now taking the expression of the temperature in the midplane (12.26) and combining it with the expression of the surface temperature and in the limit of $\tau >> 1$, we finally find

$$\begin{gathered} cH = \frac{3}{8\pi}kf\dot{M} \\ \Downarrow \\ \frac{H}{R} \simeq \frac{3}{8\pi}\frac{k}{cR}f\dot{M} = \frac{3}{2}f\frac{\dot{M}}{\dot{M}_{edd}} \end{gathered} \tag{12.32}$$

with R the stellar radius.
Near the star, the disk becomes thick if the accretion rate is close to the *Eddington rate*

$$\dot{M}_{edd} = \frac{4\pi rc}{k} \tag{12.33}$$

In this case the disk cannot be considered thin anymore and photon drag effects on the angular momentum transport must be taken into account.

12.5 Timescales

It is convenient to define timescale to order the importance of various processes the first is the dynamical timescale, set by the orbital timescale

$$t_d = \Omega^{-1} = \left(\frac{GM}{r^3}\right)^{-\frac{1}{2}} \tag{12.34}$$

The timescale for the radial drift is set by the viscous time scale

$$t_\nu = \frac{r}{|v_2|} = \frac{2}{3}\frac{f}{\alpha\Omega}\left(\frac{r}{H}\right)^2 \tag{12.35}$$

and finally we get the thermal timescale from $W_\nu = \frac{9}{4}\Omega^2\nu\sum$ and defining E_t as the *thermal energy content* $\equiv$ *enthalpy*

$$t_h = \frac{E_t}{W_\nu} \tag{12.36}$$

Similarly we can define a cooling time

$$t_c = \frac{E_t}{2\sigma_r T_s^4} \tag{12.37}$$

In a thin disk $t_c = t_h$ because the cooling rate is balanced by construction by the dissipation rate, hence with some manipulations and neglecting numerical factors

$$t_t \simeq \frac{1}{\alpha\Omega} \tag{12.38}$$

Note that in thick disk $t_c \neq t_h$, in general $t_c > t_h$ as in the case of advection dominated accretion flows (ADAF).
Also if α is not constant in the disk, t_t will restore the dependence on disk proprieties.
Since we expect $\alpha < 1$, for thin disks we get the ordering $t_\nu >> t_t > t_d$.

12.6 Irradiated disks

If the central object is very bright (as in accreting neutron stars) and the disk is concave, then the surface boundary conditions of the disk are going to be modified.
The irradiation flux, for a central point source, is given by

$$F_{irr} = \epsilon\frac{GM\dot{M}}{4\pi Rr^2} \tag{12.39}$$

with $\epsilon = \frac{dH}{dr} - \frac{H}{r}$ the angle between the disk surface and the direction from a point on the disk surface to the central source

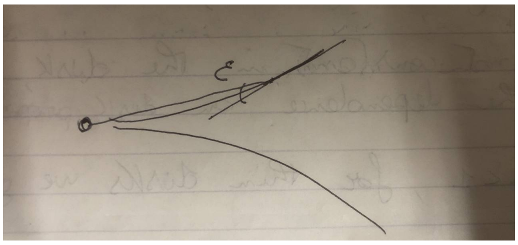

Figura 12.2: Sketch of the ϵ angle

If $\epsilon > 0$ then the disk is concave and can be illuminated by the central object.
The boundary condition for the surface temperature becomes

$$\sigma_r T^4 = F + (1-a)F_{irr} \tag{12.40}$$

where a is the $X - ray$ *albedo*.
So the surface temperature must increase to preserve the flux balance. It turns out that the dependence of the disk thickness on F_{irr} is small, as long as the disk is optically thick.
Integration of

$$\frac{d}{d\tau}\sigma_r T^4 = \frac{2}{3}F \tag{12.41}$$

with the new flux, yields

$$\sigma_r T^4 = \frac{3}{4}F\Big(\tau + \frac{2}{3}\Big) + \frac{(1-a)F_{irr}}{F} \tag{12.42}$$

which indeed implies a small additive value to $T(z)$.
In general the midplane temperature will be affected only if $\frac{F_{irr}}{F} \geq \tau$.
If we have convection in the disk the situation changes since convection couples the midplane to the external surface much more effectively.

12.7 Disk instabilities

We worked in the stationary case, but real astrophysical sources show time dependence in their luminosity. These are explained in terms of disk instabilities.
For instance in the model of $Osaki$ 74[27] the instability is driven by a temperature dependence of α.
Say $\alpha(t)$ is an increasing function of T, a perturbation in T increases α which leads to an increase in $\dot{M}$. The $\dot{M}$ burst empties the disk that cools and will have to replenish and leading to a decrease in $\dot{M}$. The cycle then repeats.

[27] See King 1995