NOTES OF
STELLAR PHYSICS

version 1.0

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For those who come after

PREFACE

In a sense, you might say I'm getting quite fond of writing these kind of notes.

Not long ago, it occurred to me how cool it is when someone unexpectedly releases a very detailed and all-comprehensive version of their notes, especially when dealing with a course that has a handful of topics often interacting together in unpredictable, yet fascinating, ways.

These notes will be mainly based on *my* own notes of the lectures by Professor Scilla Degl'Innocenti during the academic year 2025-2026. However, since I take little to no pride in my messy notes, I'll be often using some of the references you can find on the course catalogue page or that have been cited during class. Either way, I'll do my best keeping track of them all in the bibliography of this humble collection.

You can report errors (whatever their nature might be) and suggestions for additions at g.pannocchia3@studenti.unipi.it or through whatever convoluted way (conventional or not) you prefer[1].

You can find all the notes I've redacted here.

Without further ado, we'd better not lose much more time on a preface and get started with it.

There was Eru, the One, who in Arda is called Ilúvatar; and he made first the Ainur [...] *But for a long while they sang only each alone, or but few together, while the rest hearkened; for each comprehended only that part of the mind of Ilúvatar from which he came, and in the understanding of their brethren they grew but slowly.*
Yet ever as they listened they came to deeper understanding, and increased in unison and harmony.

Ainulindalë, "The music of the Ainur",
Silmarillion, J. R. R. Tolkien

1 I'd like, however, not to see my house stormed by homing pidgeons.

CONTENTS

1 ASTROPHYSICS' BUILDING BLOCKS

1.1 INTRODUCTION

As per the name of this chapter, in this first chapter we're going to introduce a series of important quantities and concepts that are going to be particularly useful later on, for this first introductory part I'm going to follow [1], §1.

For expressing astrophysical measurements, it should be clear that the units that we customarily use for our everyday life wouldn't be doing a really good job. For example, most of the times, it isn't particularly convient to have masses expressed in gramms or kilogramms, for the masses we'll be usually concerned with are so large that saying that a star weighs 10^{33} g isn't particularly insightful, since we don't have an everyday-life object to compare it with.

What often happens in astrophysics is that masses are usually expressed in terms of *solar masses* $M_\odot$, which has value

$$M_\odot = 1.99 \cdot 10^{33}\,\mathrm{g} \tag{1}$$

For example Vega, a bright white star in the Lyra constellation, has a mass roughly estimated to $M_{\mathrm{Vega}} = 2.15^{+0.10}_{-0.15}\,M_\odot$. The masses of most stars lie within a relatively narrow range from $0.1 - 20\,M_\odot$.

Stars are formed in clusters, which are predominantely composed of neutral Hydrogen. Thanks to a whole series of mechanisms interplaying in said clusters, it's often possible to assume that stars born within a given cluster are approximately coeval and have a similar chemical composition. What we can't assume, however, is them having the same masses.

To describe how the masses of stars are distributed within a cluster, it is customary to invoke a IMF, an *initial mass function*, which is essentially an empirical function that describes the mass distribution of stars for a given cloud.

Similarly, it isn't particularly convenient to express distances with meters, centimeters, kilometers or, God forbids, miles and feet. For relatively "close" objects, it is often used the *astronomical unit* (AU)

$$1\,\mathrm{AU} = 1.50 \cdot 10^{13}\,\mathrm{cm} \tag{2}$$

which is the average distance of the Earth from the Sun. Although this is a rather useful quantity, we're not still up there. We have to kick it up a notch once more. As the Earth moves around the Sun, the "nearby" stars seem to change their position just slightly with respect to the faraway stars, which appear to be fixed in place.

This phenomenon is called *parallax*; it is not particularly different from what you experience observing your own finger first with one eye and then with the other.

Consider the construction shown in Fig.1.

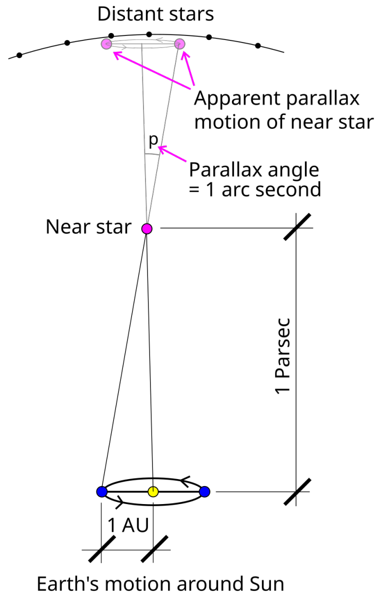

Figure 1: Defintion of parsec. Credits: Wikipedia.

Let us consider a star on the polar axis of the Earth's orbit at a distance d from the center of the orbit. The angle p in the picture is half of the angle by which this star appears to shift with the annual motion of the Earth on the distant stars' plane.

With simple geometrical considerations, if we assume Earth's orbit to be circular[1], then we can write the following relation

$$\tan(p) = \frac{1\,\mathrm{AU}}{d} \approx p$$

if we assume d to be much larger than an astronomic unit (which for all relevant scenarios is a condition that is well satisfied).

We can then define the *parsec* (pc) as the distance the star has to be so that its geometrical parallax turns out to be 1 arcsec. Hence

$$d = \frac{3.09 \cdot 10^{18}\,\mathrm{cm}}{p\,[\mathrm{arcsec}]} \implies 1\,\mathrm{pc} = 3.09 \cdot 10^{18}\,\mathrm{cm} \tag{3}$$

Obviously, for even larger distances the standard units are the *kiloparsec* (roughly the measure of galactic sizes), the *megaparsec* (the typical measure of galactic distances) and the *gigaparsec* (the typical measure of the observable Universe).

1.2 RELEVANT QUANTITIES FOR RADIATIVE TRANSFER

Although some books often start their description of radiatively relevant quantities from the definition of *monochromatic energy* and *monochromatic intensity*, I found that it is most misleading, since, in all but a few cases, what we experimentally measure are fundamentally *fluxes*.

1 Actually, it is not. Earth's orbital eccentricity is roughly $e = 0.017$, which is sufficiently small to let us approximate it as circular.

We shall then consider the *monochromatic flux* F_ν ($\mathrm{erg\,s^{-1}\,Hz^{-1}\,cm^{-2}}$) produced by some source passing through a small area $\mathrm{d}A$ located somewhere in space.

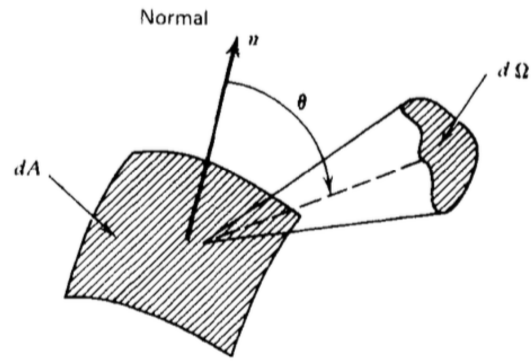

Figure 2: Schematic geometrical representation of the system.
Credits: G. Rybicki, A. Lightman [2].

If we call $\hat{k}$ the propagation direction of the flux and $\hat{n}$ the unit vector emerging from the surface $\mathrm{d}A$, it's easy to get convinced that what is actually passing through the surface is something proportional to $F_\nu(\hat{k}\cdot\hat{n})$.

From the monochromatic flux we can define the *bolometric flux*, which is just the monochromatic flux integrated over all frequencies (or wavelengths)

$$F = \int_0^{+\infty} F_\nu \,\mathrm{d}\nu = \int_0^{+\infty} F_\lambda \,\mathrm{d}\lambda \tag{4}$$

This also tells us how to convert a flux per unit frequency to a flux per unit wavelength

$$F_\nu \,\mathrm{d}\nu = F_\lambda \,\mathrm{d}\lambda$$

It should be clear that, despite being experimentally sensible for us to use the flux, we're losing much information sticking with it, namely directional information.

We consider then the amount of radiation $E_\nu \,\mathrm{d}\nu$ passing through the same area in time $\mathrm{d}t$ and solid angle $\mathrm{d}\Omega$. Hence we can write

$$\mathrm{d}E_\nu \,\mathrm{d}\nu = I_\nu(\mathbf{r}, t, \hat{k})(\hat{k}\cdot\hat{n})\,\mathrm{d}t\,\mathrm{d}\Omega\,\mathrm{d}A\,\mathrm{d}\nu \tag{5}$$

where the quantity $I_\nu(\mathbf{r}, t, \hat{k})$ is called the *specific monochromatic intensity*. If $I_\nu(\mathbf{r}, t, \hat{k})$ is specified for all directions at every point in a certain region of spacetime, then we'd have a complete prescription of the radiation field we intend on studying.

Capitalizing on the blatant similarities with distribution functions, we can evaluate the moments of the monochromatic intensity.

Definition 1.2.1. *Monochromatic mean intensity J_ν*

$$J_\nu = \frac{1}{4\pi}\int_\Omega I_\nu \,d\Omega = \frac{c}{4\pi} U_\nu$$

with U_ν the total energy density of radiation. Note that J_ν is essentially just an average of the monochromatic intensity over all solid angles.

Definition 1.2.2. *Monochromatic flux $\vec{F}_\nu$*

$$\vec{H}_\nu = \frac{1}{4\pi}\int_\Omega I_\nu(\hat{k})\hat{k}\,d\Omega \implies \frac{1}{4\pi}F_\nu = \vec{H}_\nu \cdot \hat{n}$$

I haven't explicitly proved the last equality, but it shouldn't be hard for you to convince yourself (or prove it yourself) that it is indeed true.

Definition 1.2.3. *Monochromatic radiation pressure p_ν The monochromatic pressure is defined starting from the different directions correlations of the monochromatic intensity*

$$K_\nu^{ij} = \frac{1}{4\pi}\int_\Omega I_\nu(\hat{k})k^i k^j\,d\Omega$$

The pressure in particular is usually expressed as

$$P_\nu = \frac{1}{c}\int_\Omega I_\nu(\hat{k})\cos^2\theta\,d\Omega$$

where $\cos^2\theta = (\hat{k}\cdot\hat{n})^2$.

1.2.1 Blackbody radiation

Even at an undergraduate level, we're all fairly familiar with *blackbody radiation*. The easiest way to deduce the expression for the energy density of photons in *thermal equilibrium* (STE) inside a cavity is by the means of statistical mechanics.

Remember the Bose-Einstein distribution ($\mu = 0$)

$$n = \frac{1}{\exp(h\nu/kT) - 1}$$

and the phase space density of states (per unit volume)

$$\rho(\nu)\,\mathrm{d}\nu = \frac{4\pi h g \nu^3}{c^3}\,\mathrm{d}\nu$$

from which deducing the expression from internal energy is straightforward. Remembering $g = 2$ is the quantum degeneracy of photons, a simple moltiplication of the previous expressions yields

$$U_\nu\,\mathrm{d}\nu = \frac{8\pi h}{c^3}\frac{\nu^3}{\exp(h\nu/kT) - 1}\,\mathrm{d}\nu$$

Since blackbody radiation is isotropic (it depends only on the absolute temperature T), the definition of mean monochromatic intensity yields

$$\boxed{B_\nu(T) = \frac{2h\nu^3}{c^2}\frac{1}{\exp(h\nu/kT) - 1}} \tag{6}$$

It's important to notice that, in principle, such a fundamental result holds only in *strict thermodynamic equilibrium* (STE), but we'll soon see how to generalize this formulation for less "restrictive" environments.

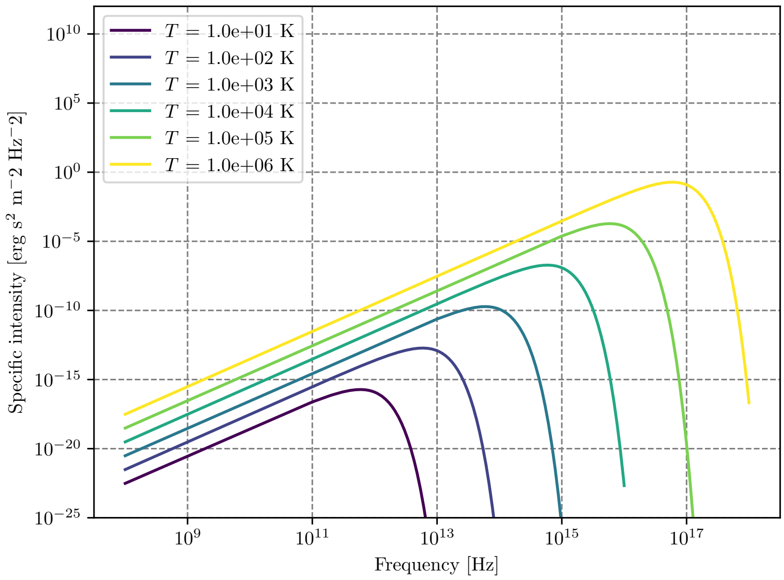

Figure 3: Blackbody frequency spectrum.

An incredible number of important results descends from (6), and it may be worthwhile to cite at least some of them, starting from Stefan-Boltzmann law. We'll use the following result without proving it

$$\int_0^{+\infty} B_\nu(T)\,\mathrm{d}\nu = \frac{2h}{c^2}\frac{\pi^4}{15}\left(\frac{kT}{h}\right)^4$$

Computing the bolometric flux and the bolometric energy density by integrating over all frequencies using what we've just written down, you find the following

$$U(T) = aT^4 \quad F(T) = \sigma_{SB}T^4$$

Clearly the two constants a and σ_{SB} cannot be independent, and are actually related by the integral we've previously calculated. Using for example[2]

$$F(T) = \pi \int_0^{+\infty} B_\nu(T)\,\mathrm{d}\nu$$

you can easily find out that the *Stefan-Boltzmann constant* is equal to

$$\sigma_{SB} = \frac{2\pi^5 k^4}{15c^2h^3}$$

and the relation with a is simply $\sigma_{SB} = ac/4$.

The equation

$$F(T) = \frac{2\pi^5 k^4}{15c^2h^3}T^4 \tag{7}$$

is what is usually known as the *Stefan-Boltzmann law*.

2 The emergent flux from an isotropically emitting surface (such as a blackbody) is $\pi \cdot$ brightness which is none other than the specific intensity.

Let us now consider two different regimes for eq.6: $h\nu/kT \ll 1$ and $h\nu/kT \gg 1$. The first yields what is commonly known as the Rayleigh-Jeans Law which is, sadly, pretty much relevant only for radioastronomy.

Since

$$\exp\left(\frac{h\nu}{kT}\right) = 1 + \frac{h\nu}{kT} + o\left(\frac{h\nu}{kT}\right)^2$$

the blackbody radiation assumes the much simpler form of

$$B_\nu^{RJ} = \frac{2\nu^2}{c^2}kT$$

Another important results is achieved in the opposite regime, when the exponential term is rather larger than unity

$$B_\nu^{W} = \frac{2h\nu^3}{c^2}\exp\left(-\frac{h\nu}{kT}\right)$$

This expression is known as Wien's Law.

1.2.2 Characteristic Temperatures

Starting from the blackbody spectrum we can give various definitions of temperature, that are going to be more and less useful when dealing with stars.

Brightness Temperature

One way of characterizing brightness (specific intensity) at a certain frequency is to give the temperature of the blackbody having the same brightness at that frequency. That is, for any value I_ν we define $T_b(\nu)$ by the relation

$$I_\nu = B_\nu(T_b)$$

this way of specifying brightness has the advantage of being closely connected with the physical properties of the emitter, and has the simple unit (K).

This procedure is used especially in radio astronomy, where the Rayleigh-Jeans law is usually applicable.

$$T_b = \frac{c^2}{2\nu k}I_\nu \quad h\nu/kT \ll 1 \tag{8}$$

Note that the uniqueness of the definition of brightness temperature relies on the monotonicity property of Planck's law. Also note that, in general, the brightness temperature is a function of ν.

Color Temperature

Often a spectrum is measured to have a shape more or less of blackbody form, but not necessarily of the proper absolute value. For example, by measuring F_ν, from an unresolved source we cannot find I_ν, unless we know the distance to the source and its physical size.

By fitting the data to a blackbody curve without regard to vertical scale, a color temperature T_c, is obtained. Often the "fitting" procedure is nothing more than estimating the peak of the spectrum and applying Wien's displacement law to find a temperature.

Effective Temperature

The effective temperature of a source T_{eff} is derived from the total amount of flux, integrated over all frequencies, radiated at the source. We obtain T_{eff} by equating the actual flux F to the flux of a blackbody at temperature T_{eff}

$$F = \int \cos\theta \, I_\nu \, d\nu \, d\Omega := \sigma T_{eff}^4 \tag{9}$$

1.3 MAGNITUDE SCALES

When coming down to determining the properties of a star, there are two quantities that shine a little brighter than the others: The luminosity and the flux.

The luminosity of a star is generally defined as the power emitted by a given star over time (potentially over a given wavelength) and its expression is given by the relation

$$L = -\frac{dE}{dt} \quad \text{erg} \cdot \text{s}^{-1} \tag{10}$$

As we'll see, stars often emit isotropically in space. It goes without saying, however, that an observer at distance d from that star won't be able to collect all the light that is emitted. What we can measure is a flux, the fraction of energy that impinges on our detector. The flux is related to the luminosity through the following relation

$$\phi = \frac{L}{4\pi r^2} \quad \text{erg} \cdot \text{s}^{-1} \cdot \text{cm}^{-2} \tag{11}$$

The flux, however, doesn't tel us much about the characteristics of the star, differently from the luminosity. But since what we can measure are fluxes, it is then crucial that we find a way to accurately measure distances so to infer the luminosity of an object.

Since ancient times, mankind has always tried to measure the brightness of stars. The efforts of those that came before us are nowadays crystallized in a quantities that is still in use: The *magnitude*.

The concept of magnitude is strictly related to the way the human eye was believed to perceive the differences in intensities.

Hypparchus, back in ancient Greece, classified all stars into six classes according to their apparent brightness. This classes went from class I, the brightest stars, to class VI, the less bright. According to Hypparchus, the brightest and faintest stars had an apparent brightness that is in the ratio of 100.

Let us consider a star with flux F_* and some reference flux F_0. We define the *apparent magnitude* as

$$m_* = x \log_{10}\left(\frac{F_*}{F_0}\right)$$

Calculating the difference in apparent magnitude between the brigthest and faintest stars and making use of the aforementioned characteristics, we can find the correct values for the constant x and define the apparent magnitude as

$$m_* = -2.5 \log_{10} \left(\frac{F_*}{F_0} \right) \tag{12}$$

Obviously, unless a bolometer is used, you can't perform a measurement covering all wavelengths. What you're going to measure is actually a convolution of the flux emitted by the star (as a function of the wavelength) and the transmission curve of the detector you're using

$$S_{det} = \int_0^\infty \mathrm{d}\lambda \, F(\lambda) T(\lambda)$$

More often than not, it is customary to measure magnitudes in given wavelengths' ranges, from which a "bolometric" magnitude can be inferred by adding a *bolometric correction* that is given from stellar atmosphere models.

Similarly, we can define the *absolute magnitude* as the magnitude a star would have if it was located 10 pc away.

Using the fact that $F \propto r^{-2}$, we find a relation for the absolute magnitude M

$$m - M = 5 \log_{10} \left(\frac{d}{10\,[\mathrm{pc}]} \right) \tag{13}$$

The difference $m - M$ is often called *distance module* (DM) and is sometimes used as an indirect way to express a distance.

For classifying stars according to their color it's often used the so-called UBV photometric system (from *Ultraviolet*, *Blue*, *Visual*), also called the Johnson system (or Johnson-Morgan system), which makes use of filters so that the mean wavelengths of response functions (at which magnitudes are measured to mean precision) are 364 nm for U, 442 nm for B, 540 nm for V.

Given this setup, we can easily calculate the magnitudes, say, in the Blue and Visible light-bands, so that the relative magnitudes are

$$m_B - m_V = -2.5 \log_{10} \left(\frac{F_B}{F_V} \right) := B - V$$

where $B - V$ is said *color index*.

When passing through interstellar dust[3] the intensity of radiation gets modified. The dust that makes the ISM absorbs light in the V-UV bands to later re-emit it as InfraRed radiation or scatters the "short" wavelength, producing a net-effect that essentially "increases" the wavelength of the incoming radiation and decreases its intensity.

These two phenomena are known as *reddening* (do not confuse it with *redshift*) and *extinction*.

The absolute magnitude in the, say, blue and visible bands, will thus be modified

$$M_V = M_V^0 + A_V$$
$$M_B = M_B^0 + A_B$$

3 The interstellar dust, which is usually referred to as ISM, consists in small particles of silicates, graphite, amorphous carbon, ices and organical mixtures of size $10^{-2} - 1\,\mu$m

where the quantities with the "o" superscript are the un-reddened magnitudes. We can define the new color index

$$B - V = (B - V)^0 + E[B - V]$$

where $E[B - V] = A_B - A_V$ is the extinction. Within our galaxy, the ratio of reddening to the extinction is a well-known quantity

$$R = \frac{A}{E[B - V]} \approx 3.0$$

2 STELLAR STRUCTURES

2.1 RADIATIVE TRANSFER

Most of our knowledge about the Universe is based on the electromagnetic radiation that reaches us from far far away. EM radiation is obviusly not the only way we can probe the Universe we live in but, in respect to neutrinos, cosmic rays or even gravitational waves, it's not a long stretch to claim it is by far the most understood.

It is most important then that an astrophysicist worthy of his (or her) name has a good grasp of the theory of radiative transfer and of its applications.

Apart from a few more key differences, I'll follow the description of radiative transfer of [1] and [2], but I won't fail to emphasize whenever I'll be doing otherwise.

2.1.1 Radiative transfer equation

In the presence of matter, it is not immediately obvious what changes may occur in the specific intensity as we move along a ray path. The aim of this section will be to eviscerate the matter.

Let's consider the following geometric construction

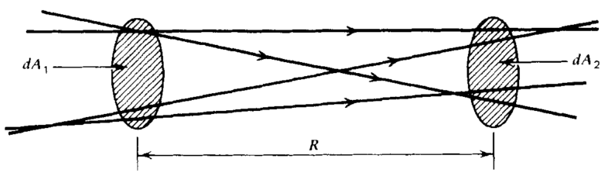

Figure 4: Geometrical construction for ray paths propagating in empty space. Credits: G. Rybicki, A. Lightman.

It won't take a lot of effort to convince yourself that in empty space the monochromatic intensity I_ν is actually conserved. In fact, from simply writing down the definitions and imposing the conservation of energy

$$I_{\nu_2}\, \mathrm{d}A_2\, \mathrm{d}t\, \mathrm{d}\Omega_2\, \mathrm{d}\nu = I_{\nu_1}\, \mathrm{d}A_1\, \mathrm{d}t\, \mathrm{d}\Omega_1\, \mathrm{d}\nu$$

the conclusion follows observing that $\mathrm{d}A_2\, \mathrm{d}\Omega_2 = \mathrm{d}A_1\, \mathrm{d}\Omega_1$.

If we consider an affine parameter of the form $\vec{x} = \vec{x}_0 + \hat{k}s$, we may as well write the previous results in a more familiar fashion

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = 0 \implies (\hat{k}\cdot\nabla)I_\nu = 0 \tag{14}$$

What changes if matter is present along the ray path? Clearly it will no longer be true that $(\hat{k}\cdot\nabla)I_\nu = 0$, but we're not that far off. All that we need is some little work on both terms.

How the right hand side of the equation should change is obvious: It needs to keep track of the "creation" and "destruction" of photons in the considered volume of spacetime.

The left hand side of the equation requires a little more care. Consider infinitesimal time and space displacements along the ray path, respectively $\mathrm{d}t$ and $\mathrm{d}\vec{x}$

$$\Delta E_\nu \,\mathrm{d}\nu = \left(I_\nu(\vec{x} + \mathrm{d}\vec{x}, t + \mathrm{d}t, \hat{k}) - I_\nu(\vec{x}, t, \hat{k}) \right) \mathrm{d}t \,\mathrm{d}\Omega \,\mathrm{d}A \,\mathrm{d}\nu$$

Taking a first order expansion in respect to the affine parameter s along the ray path yields

$$\left(\frac{1}{c}\partial_t I_\nu + \partial_s I_\nu \right) \mathrm{d}t \,\mathrm{d}s \,\mathrm{d}\Omega \,\mathrm{d}A \,\mathrm{d}\nu = \text{photon addition} - \text{photon removal}$$

This equation is a generalization of eq.14 for non-stationary radiative transport and in the presence of matter. It's about time we get to know what "lives" in the right hand side of the equation.

2.1.2 Monochromatic emission coefficient

For the moment, we'll define the *spontaneous* monochromatic emission coefficient j_ν as

$$\mathrm{d}E_\nu \,\mathrm{d}\nu = j_\nu \,\mathrm{d}V \,\mathrm{d}t \,\mathrm{d}\Omega \,\mathrm{d}\nu \tag{15}$$

which in general has a non-zero dependence on the emission direction. Sometimes the spontaneous emission coefficient is defined by the *emissivity* ϵ_ν (**please note** that often the two names are used almost interchangeably), which is the energy emitted spontaneously per unit frequency per unit time per unit mass

$$j_\nu = \frac{\epsilon_\nu \rho}{4\pi}$$

where ρ is the mass density of the emitting medium.

If we perform the decomposition $\mathrm{d}V = \mathrm{d}A \,\mathrm{d}s$, the contribution of spontaneous emission to the specific intensity is

$$\mathrm{d}I_\nu = j_\nu \,\mathrm{d}s$$

2.1.3 Absorption coefficient

Similarly, we can consider the energy that is absorbed from the radiation when passing through a medium. There exists various definitions; I'll use the one we gave in class and that is incidentally the one used in [1] and [2] as well.

We define the *absorption coefficient* α_ν through the following expression

$$\mathrm{d}I_\nu = -\alpha_\nu I_\nu \,\mathrm{d}s \tag{16}$$

If we use a microscopic model, then the absorption coefficient can be understood as particles with numeric density n presenting an effective absorbing area, the *cross section* σ_ν. The coefficient α_ν can thus be rewritten in terms of

$$\alpha_\nu = n\sigma_\nu = \rho\kappa_\nu$$

where κ_ν is called the mass absorption coefficient or the *mass-weighed opacity coefficient*.

Note that in eq.16, we consider "absorption" to include both "true absorption" and stimulated emission, because both are proportional to the intensity of the incoming beam. Depending on the entity of the contribution, the α_ν coefficient may be positive or even negative, giving raise to curious phenomena.

Making full use of what we've just defined, we can finally present the celebrated *equation of radiative transfer* (although in the notable absence of scattering)

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = -\alpha_\nu I_\nu + j_\nu \tag{17}$$

which is actually fairly easy to solve when one of the two coefficients vanishes.

Emission only

We set $\alpha_\nu = 0$ and the equation may be solved by direct integration

$$I_\nu(s) = I_\nu(s_0) + \int_{s_0}^{s} j_\nu(s')\,\mathrm{d}s'$$

the result is not that interesting per se.

Absorption only

This time we set $j_\nu = 0$. The equation is easily solved this time as well

$$I_\nu(s) = I_\nu(s_0)\exp\left(-\int_{s_0}^{s} \alpha_\nu(s')\,\mathrm{d}s'\right)$$

In this case, it's rather common to write down the equation in terms of a new variable, the *optical depth* τ_ν

$$\mathrm{d}\tau_\nu = \alpha_\nu\,\mathrm{d}s \tag{18}$$

Given this definition we'll say that if

- $\tau_\nu \gg 1$: the medium is *optically thick or opaque*
- $\tau_\nu \ll 1$: the medium is *optically thin or transparent*

This has some crucial implications we'll be going through in a moment.

In the stationary limit, the equation of radiative transport may be written as

$$(\hat{k}\cdot\nabla)I_\nu(\hat{k},\vec{x}\,) = j_\nu(\vec{x}\,) - \alpha_\nu(\vec{x}\,)I_\nu(\hat{k},\vec{x}\,)$$

In terms of the *source function* $S_\nu = j_\nu/\alpha_\nu$ it becomes

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}\tau_\nu} = -I_\nu + S_\nu \tag{19}$$

which can be integrated to yield the formal solution

$$I_\nu(\hat{k},\tau_\nu) = I_\nu(\tau_{\nu,0})\exp(-\tau_\nu) + \int_{\tau_{\nu,0}}^{\tau_\nu} d\tau'_\nu\, S_\nu \exp(-(\tau_\nu - \tau'_\nu))$$

Assume for the moment that the matter through which radiation is passing has constant properties and has no background source. Then the source function S_ν is constant and the formal equation becomes

$$I_\nu = S_\nu(1 - e^{-\tau_\nu})$$

If the medium and is optically thin, then the equation is reduced to

$$I_\nu = S_\nu \tau_\nu = j_\nu L \tag{20}$$

by taking the Taylor expansion of the exponential term and calling L some typical length of the medium.

If, on the other hand, the medium is optically thick, we can neglect the exponential $e^{-\tau_\nu}$ to obtain

$$I_\nu = S_\nu \tag{21}$$

2.2 KIRCHHOFF'S LAW AND LTE

The most notable implication of eq.21 is if we consider the specific intensity coming out of a small hole on a box kept in thermodynamic equilibrium. We know that what's going to come out of there is the blackbody radiation

$$I_\nu = B_\nu(T)$$

but what if we were to put an optically thick object just behind the hole?

If the object is in thermodynamic equilibrium with the surroundings (and it *will* be, given an appropriate amount of time), then the radiation coming out of the hole will still be blackbody radiation. But eq.21 tells us that the source function will tend to be equal to the specific intensity, hence

$$S_\nu = B_\nu(T) \tag{22}$$

which actually puts a constraint on the possible values of the emission coefficient in terms of the absorption coefficient. This is exactly what is expressed in Kirchhoff's law

$$j_\nu = \alpha_\nu B_\nu \tag{23}$$

Let us briefly consider what we have just derived. Matter often tends to emit and absorb at specific frequencies corresponding to what are commonly called *spectral lines*. We would expect then both j_ν and α_ν to have peaks (or depression) around these lines. But Kirchhoff's law forces their ratio to be equal to a smooth blackbody profile.

Thus we can expect to observe two very different scenarios if the medium is optically thin rather than optically thick. In the former, the radiation emerging from the medium is essentially determined by its emission coefficient; since j_ν is expected to present peaks, so will the radiation spectrum, which will be appear in spectral lines, as shown in Fig.5 and Fig.6.

On the other hand, the intensity coming out of an optically thick body is its source function, which must be equal to the blackbody function. Hence we expect the medium to emit in a continuum, like a blackbody.

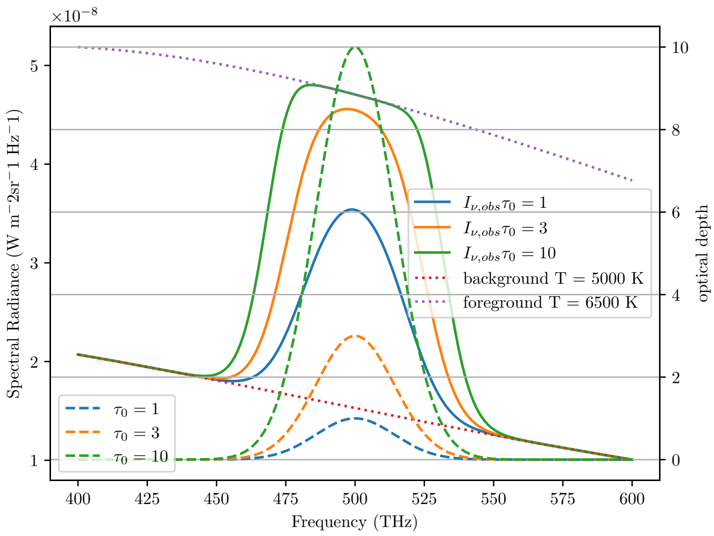

Figure 5: An example of emission features formation for different temperatures and different values of τ. Credits: Prof. Walter del Pozzo.

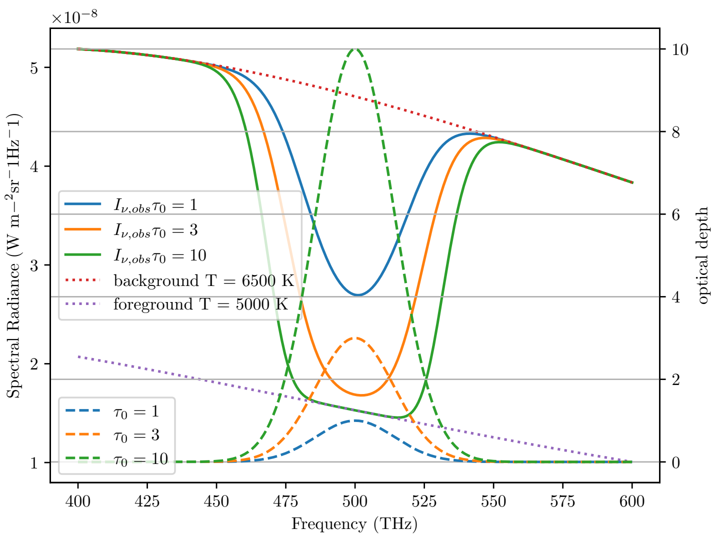

Figure 6: An example of absorption features formation for different temperatures and different values of τ. Credits: Prof. Walter del Pozzo.

All throughout this description, we've been assuming the medium to have constant properties, which has the perk of being a good approximation for many objects of interest, but still turns out to be a really poor one for many other objects. Stars, for example.

Ingenuously, we may expect stars to emit radiation like blackbodies, but they're not. Actually, stars present absorption lines, many, even, depending on the class of star. What we cannot assume in stars is them having constant properties, starting from temperature.

In fact, we could take a guess and claim that stars are in *strict* thermodynamic equilibrium. It would be a very bad guess indeed.

2.2.1 Local Thermodynamic Equilibrium (LTE)

Let's be honest: In a realistic situation, we *rarely* have strict thermodynamic equilibrium. If a body is in thermodynamic equilibrium, we can assume a number of important physical principles to hold, like the Maxwellian distribution

$$dn_v = 4\pi n \left(\frac{m}{2\pi kT}\right)^{3/2} v^2 \exp\left(-\frac{mv^2}{2kT}\right) dv \tag{24}$$

where n is the total number of particles per unit volume and m is the mass of each particle. Similarly, we can expect certain laws to hold, like Boltzmann's law for occupation numbers

$$\frac{n_E}{n_0} = \frac{g_E}{g_0} \exp\left(-\frac{E - E_0}{kT}\right) \tag{25}$$

and Saha's equation

$$\frac{N_{j+1} n_e}{N_j} = 2\frac{Z_{j+1}(T)}{Z_j(T)} \left(\frac{2\pi mkT}{h^2}\right)^{3/2} \exp\left(-\frac{\chi_{j,j+1}}{kT}\right) \tag{26}$$

where n_e is the density of electrons and $\chi_{j,j+1}$ is the ionization potential. Saha's equation in particular is expected to be crucial in interpreting the effect that ionization has on the emission/absorption spectrum.

The proverbial "one-million-dollar-question" then is: When can we expect a system to be in thermodynamic equilibrium and when can we expect the previous principles to hold?

Even if the system initially does not obey the, say, Maxwellian distribution, it will eventually relax to it after undergoing some *collisions*.

Collisions are crucial in establishing thermodynamic equilibrium.

When collisions are frequent, the mean free path of particles will be small, and particles will interact more effectively. When this happens, we can expect the principles aforementioned to hold. Since we're physicists, vague sentences like "*the mean free path of particles will be small*" are destined to elicit a deep sense of unease and distress. How small does the free path have to be? One meter? Two micrometers? Below the Planck lengthscale?

When we've defined the absorption coefficient α_ν, the sharpest among my four readers total may have noticed that α_ν has the dimension of the inverse of a length. It is safe to assume that α_ν^{-1} may define some distance over which a significant fraction of the radiation would get absorbed by matter.

Such a "mean-distance" is defined in a homogeneous medium as

$$< \tau_\nu >= \alpha_\nu l_\nu = 1$$

Thus, if l_ν is sufficiently small such that the temperature can be taken as a constant over such distance, we can safely say that the useful relations we have defined earlier still hold, although only locally.

In such a fortunate scenario, known as *Local Thermodynamic Equilibrium* (LTE), all the important laws requiring thermodynamic equilibrium are expected to hold, provided that we use the local temperature $T(\vec{x})$.

In the interiors of stars, for example, LTE will prove to be a very good approximation, that will get progressively worse as we approach the "surface" of the star.

2.3 PARALLEL PLANE APPROXIMATION

One useful approximation that may be worthwhile to dedicate some of our time to is the *plane parallel atmosphere*, that will allow us to obtain notable results for describing how radiation travels through, say, the inner regions of the stellar atmosphere.

In the following, we're going to neglect the curvature of the stellar atmosphere and assume the various thermodynamic quantities to be constant over horizontal planes.

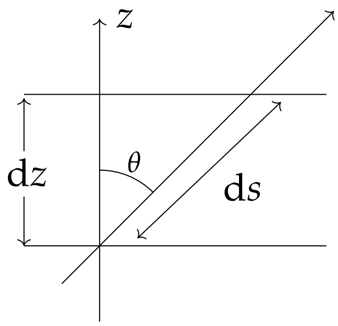

Figure 7: A ray path through a plane parallel atmosphere.

Using Fig.7 as a reference, we see that

$$\mathrm{d}s = \frac{\mathrm{d}z}{\cos\theta} = \frac{\mathrm{d}z}{\mu}$$

where we used the customary notation in astrophysics $\mu = \cos\theta$.

We shall consider a scattering free, stationary situation for the equation of radiative transport (17). Hence, due to planar symmetry, we expect the specific intensity to depend only on z and μ. For the sake of the current discussion, we perform a slight modification to the definition of optical depth, so that

$$\mathrm{d}\tau_\nu = -\alpha_\nu \, \mathrm{d}z$$

This way the equation of radiative transfer may be cast in the following form

$$\mu \partial_{\tau_\nu} I_\nu(\tau_\nu, \mu) = I_\nu - S_\nu$$

which has a formal solution easily computed

$$I_\nu \exp\left(-\frac{t_\nu}{\mu}\right) \Big|_{\tau_{\nu,0}}^{\tau_\nu} = -\int_{\tau_{\nu,0}}^{\tau_\nu} \frac{S_\nu}{\mu} \exp\left(-\frac{t_\nu}{\mu}\right) \mathrm{d}t_\nu \tag{27}$$

This is customarily solved considering two distinct intervals for μ: (I) $\mu \in [0,1]$ and (II) $\mu \in [-1,0]$. In case (I) we can assume the ray path to begin from a great depth inside the star, so that $\tau_{\nu,0} \to \infty$, while in case (II) we assume the ray to receive contributions beginning from the top of the atmosphere, where $\tau_{\nu,0} \approx 0$. For case (II), we're also assuming no radiation to be coming from *outside the star*[1].

Now we can assume LTE throughout the stellar atmosphere so that eq.23 is verified. The source function at some optical depth shall then be equal to

1 Please note that this condition may be not valid at all in close binary systems.

$B_\nu(T(\tau_\nu))$. For the source function at a nearby optical depth we can simply compute a Taylor expansion around the optical depth τ_ν

$$S(t_\nu) = B_\nu(\tau_\nu) - (t_\nu - \tau_\nu)\frac{\mathrm{d}B_\nu}{\mathrm{d}\tau_\nu} + \mathrm{o}(t_\nu^2)$$

We can use this to solve (27), finding for both positive and negative values of μ a very important equation

$$I_\nu(\tau_\nu, \mu) = B_\nu(\tau_\nu) + \mu\,\frac{\mathrm{d}B_\nu}{\mathrm{d}\tau_\nu} \tag{28}$$

provided the point considered is sufficiently inside the atmosphere so that $\tau_\nu \gg 1$[2]. Using this simple result, we can compute the three momenta of the equation of transport

$$U_\nu = \frac{4\pi}{c} B_\nu(\tau_\nu) \tag{29}$$

$$F_\nu = \frac{4\pi}{3}\frac{\mathrm{d}B_\nu}{\mathrm{d}\tau_\nu} \tag{30}$$

$$P_\nu = \frac{4\pi}{3c} B_\nu(\tau_\nu) \tag{31}$$

2.3.1 The Grey Atmosphere

If we consider the absorption coefficient α_ν constant over all frequencies, then the atmosphere is called a "grey atmosphere". This implies that the value of the optical depth at some physical depth is constant for all frequencies. Under this assumption, we could solve

$$\mu\frac{\partial I}{\partial\tau} = I - S$$

I'll skip the explicit calculation (which is actually fairly easy for once) and present just the final result. Two more assumptions are to be made, however: The first is to assume *radiative equilibrium*, which roughly translates into requiring that there are no sources nor sinks of energy in the atmosphere, thus $\partial_\tau F = 0$; as a further semplification, we assume the *Eddington approximation* to hold everywhere in the atmosphere[3], so that

$$P = \frac{1}{3}U$$

It should be evident that this last equation is verified automatically in presence of an isotropic source of radiation, also in its frequency-dependent form.

We then come to the following conclusion

$$I_{obs}(\tau = 0, \mu) = \frac{3F}{4\pi}\left(\mu + \frac{2}{3}\right) \tag{32}$$

2 You can see [1], §2.4.1 for more detailed calculations.

3 I find most intriguing that Eddington's approximation essentially assumes that $T^{\mu}{}_{\mu} = 0$, with $T^{\mu\nu}$ the electromagnetic energy-momentum tensor if we are to treat electromagnetic radiation as a perfect fluid.

from which we deduce the equation for the *limb darkening*

$$\frac{I(0,\mu)}{I(0,1)} = \frac{3}{5}\left(\mu + \frac{2}{3}\right)$$

which roughly translates into saying that the radiation that we observe at the surface is the one equivalent at a source function S evaluated at $\tau = 2/3$. This is known as the *Eddington-Barbier estimation*.

A somewhat more general way to solve the problem is assuming the following functional relation for the specific intensity

$$I_\nu(\tau, \mu) = a_\nu(\tau_\nu) + b_\nu(\tau_\nu)\mu$$

and compute the three momenta proper

$$J_\nu = \frac{1}{2}\int_{-1}^{+1} I_\nu \, \mathrm{d}\mu = a_\nu \tag{33}$$

$$H_\nu = \frac{1}{2}\int_{-1}^{+1} I_\nu \mu \, \mathrm{d}\mu = \frac{b_\nu}{3} \tag{34}$$

$$K_\nu = \frac{1}{2}\int_{-1}^{+1} I_\nu \mu^2 \, \mathrm{d}\mu = \frac{a_\nu}{3} \tag{35}$$

Now we assume a stronger version of the *Eddington approximation* $K_\nu = J_\nu/3$ so that the two following expressions can be written

$$\frac{\partial H_\nu}{\partial \tau_\nu} = J_\nu - S_\nu \tag{36}$$

$$\frac{\partial K_\nu}{\partial \tau_\nu} = H_\nu = \frac{1}{3}\frac{\partial J_\nu}{\partial \tau_\nu} \tag{37}$$

The mixing of the two gives us a second order PDE that it's still (approximately) valid even in the outer regions of the stellar atmosphere.

2.4 RADIATIVE DIFFUSION APPROXIMATION

This section would probably be clearer if you take a brief detour to Chapter 3 to build some groundwork for scattering processes, but it still suits better the main subject of this chapter.

For the sake of coherence (and personal laziness) I'm going to talk about the radiative diffusion approximation right here, also because it makes use of the plane parallel approximation we just went through.

In Chapter 3 we will use random walk arguments to show that S_ν approaches B_ν at large *effective optical depths* in a homogeneous medium. Real media are seldom homogeneous, but often, as in the interiors of stars, there is a high degree of local homogeneity.

The equation of radiative transport in presence of scattering (**??**) may be cast in a slightly different form

$$I_\nu = S_\nu - \frac{\mu}{\alpha_\nu + \sigma_\nu}\partial_z I_\nu$$

We shall then assume that over a distance l_* (the thermalization length) I_ν is constant and at zero-th order is $I_\nu^{(0)} = S_\nu^{(0)} = B_\nu$. Plugging this in the equation of radiative transfer gives us $I_\nu^{(1)}$ by a simple iterative procedure

$$I_\nu^{(1)} = B_\nu - \frac{\mu}{\alpha_\nu + \sigma_\nu}\partial_z B_\nu \tag{38}$$

With the simple redefining $\mathrm{d}\tau_\nu = -(\alpha_\nu + \sigma_\nu)\,\mathrm{d}z$, we can put eq.(38) in a form functionally equal to (28).

Let us now compute the flux F_ν using the above form for the intensity

$$F_\nu(z) = 2\pi \int_{-1}^{+1} I_\nu^{(1)} \mu \,\mathrm{d}\mu = -\frac{4\pi}{3}\frac{\partial_z B_\nu}{\alpha_\nu + \sigma_\nu} = -\frac{4\pi}{3}\frac{\partial_T B_\nu}{\alpha_\nu + \sigma_\nu}\partial_z T$$

Recalling the result

$$\partial_T \int_0^{+\infty} B_\nu \,\mathrm{d}\nu = \frac{4\sigma_{SB} T^3}{\pi}$$

we can define a *mean absorption coefficient* using the *Rosseland approximation for radiative diffusion*

$$\frac{1}{\alpha_R} := \frac{\int_0^{+\infty} \frac{1}{\alpha_\nu + \sigma_\nu}\,\partial_T B_\nu \,\mathrm{d}\nu}{\int_0^{+\infty} \partial_T B_\nu \,\mathrm{d}\nu} \tag{39}$$

If we integrate the monochromatic flux over the frequencies and make use of the Rosseland mean, we find a useful expression used in stellar structure models

$$F(z) = -\frac{16\sigma_{SB} T^3}{3\alpha_R}\partial_z T \tag{40}$$

3 STELLAR MODELS

3.1 INTRODUCTION

4 NUCLEAR REACTIONS

4.1 INTRODUCTION

5 STELLAR EVOLUTION

5.1 INTRODUCTION

6 | STELLAR CLUSTERS

7 STANDARD CANDLES AND VARIABILITY MECHANISMS

7.1 INTRODUCTION

ABOUT ME

Hi there, I'm Giacomo, sometimes known as Cael by a very small niche of people, the author of the notes you just finished reading.

I'm a Batchelor-graduate student at the University of Pisa now in the process of undertaking a major (Master degree) in Astronomy and Astrophysics at the same university. In my (meager) free time I'm an avid reader, gamer, anime-enthusiast as well as an independent author in the very process of publishing his first novel.

For those of you that may be interested I'll leave both the link to the novel's page as well to my instagram profile in the QRcodes right below.

Figure 8: On the left is the link to my instagram profile, on the right the one to my novel's page (Python really has a library for anything, huh?).

Come say hi if you fancy action-fantasy stories with magic, swords, some comedy to garnish and a lot of other stuff :)

At last but not the least, I want to thank all my collegues and fellow students that helped (and hopefully will help me again) during the drafting of this text, pointing out errors and occasions for further expansion.

If you'd like your names to be included and take your fair share of credits, feel free to contact me and I'll add your name right away in the few lines that are left of this page.

I hope my understading of the complex interplayings going on in our Universe has managed to facilitate your own understading.

If not, well... oopsie.

See you, space cowboys...

Giacomo

BIBLIOGRAPHY

[1] A. R. Choudhuri. *Astrophysics for Physicists*. Astrophysics-Textbooks. Cambridge University Press, 2010.

[2] George B. Rybicki and Alan P. Lightman. *Radiative Processes in Astrophysics*. 1986.