Model

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Introduction

Every trading day is divided into T periods during which the trading takes place. There is a noise trader whose demand is $u_t \sim \mathcal{N}(0, \Sigma_u)$ where $t = \{0, 1, 2, ..., T\}$. There is also a "informed" trader whose terminal price distribution is $v \sim \mathcal{N}(p_0, \Sigma_0)$. The noise trader and the informed trader submit their orders x_t and u_t respectively to the market maker. The market maker observes total demand $y_t = x_t + u_t$ and sets the price at which the orders are executed.

The price impact rule for the market maker is

$$p_t = p_{t-1} + \lambda\left(\frac{y_t}{y^*}\right)^{k_y} + \phi\left(\frac{I_{t-1}}{I^*}\right)^{k_I} \tag{1}$$

Informed Trader

The informed trader assumes that the market maker's price rule is given by: $p_t = p_{t-1} + \widetilde{\lambda}^{IT}\frac{x_t+u_t}{\bar{y}} + \widetilde{\phi}^{IT}\frac{I_t}{\bar{I}}$. The trader assumes that on average the market maker has zero inventory outstanding. The expected value and variance of the market price from the point of view of the informed trader, is:

$$\begin{aligned} E_{IT}[p_t|x_t] &= p_{t-1} + \widetilde{\lambda}^{IT}\frac{x_t}{\bar{y}} \\ VAR[p_t|x_t] = \Sigma_{MM}^{IT} &= \left(\frac{\widetilde{\lambda}^{IT}}{\bar{y}}\right)^2 \widetilde{\Sigma_u}^{IT} + \left(\frac{\widetilde{\phi}^{IT}}{\bar{I}}\right)^2 \widetilde{\Sigma_I}^{IT} \end{aligned} \tag{2}$$

The informed trader's wealth at time W_t is given by $W_t = x_t(v - p_t)$. We have that the wealth is distributed with mean $\mu_{W_t} = x_t(p_0 - E_{IT}[p_t])$ and variance $\Sigma_{W_t} = x_t^2\Sigma_0 + x_t^2\Sigma_{MM}^{IT}$. I assume that the informed trader has an exponential utility given by $-e^{-\alpha W_t}$. Under normally distributed W_t the problem reduces to maximizing $\mu_{W_t} - \frac{1}{2}\alpha\Sigma_{W_t}$. If we substitute the moments of the wealth process we have that the problem if the informed trader is to maximize:

$$x_t(p_0 - E_{IT}[p_t|x_t]) - \frac{1}{2}\alpha(x_t^2\Sigma_0 + x_t^2\Sigma_{MM}^{IT}) \tag{3}$$

which achieves its maximum at:

$$x_t^* = \frac{p_0 - p_{t-1}}{\frac{2\widetilde{\lambda}^{IT}}{\bar{y}} + \alpha(\Sigma_0 + \Sigma_{MM}^{IT})} \tag{4}$$

Likelihood

What is the likelihood that we observe a price change $p_t - p_{t-1} = \delta$ The price change, conditional on the parameters, is normally distributed: $\Delta p_t \sim \mathcal{N}(\mu_{\Delta p_t}, \Sigma_{\Delta p_t})$

$$\begin{aligned} \mu_{\Delta p_t} &= \frac{\lambda}{\bar{y}}x_t^* + \phi\frac{I_{t-1}}{\bar{I}} \\ \Sigma_{\Delta p_t} &= \left(\frac{\lambda}{\bar{y}}\right)^2\Sigma_u \end{aligned} \tag{5}$$

Therefore we can write the likelihood function for a given sample of observed price changes

$$\begin{gathered} f\Big(\Delta p_t \,\Big|\, \mu_{\Delta p_t}, \Sigma_{\Delta p_t}\Big) = \\ \prod \frac{(2\pi)^{-1/2}}{\Sigma_{\Delta p_t}^{1/2}} \exp\Big(- \frac{(\Delta p_t - \mu_{\Delta p_t})^2}{2\Sigma_{\Delta p_t}}\Big) \end{gathered} \tag{6}$$

And the log-likelihood:

$$\ln f = -\frac{1}{2}\ln(2\pi) - \ln\Sigma_{\Delta p_t}^{1/2} - \frac{(\Delta p_t - \mu_{\Delta p_t})^2}{2\Sigma_{\Delta p_t}} \tag{7}$$

Likelihood for the moments

In this section we develop the model likelihood when instead of fitting the data to the actual price series we use the moments of the price series for some period. For the periods $\{1, 2, ..., T\}$ we observe a series of price changes $\{\Delta_{p_1}, \Delta_{p_2}, ..., \Delta_{p_T}\}$. The variable $\bar{\Delta} = \frac{1}{T}\sum_t^T \Delta_{p_t}$ is normally distributed (conditional on the information set at time t) and has a mean $\mu_{\bar{\Delta}} = \frac{1}{T}\sum \mu_{\Delta_{p_t}}$ and variance $\Sigma_{\bar{\Delta}} = \frac{1}{T^2}\sum \Sigma_{\Delta_{p_t}}$. Therefore we have that the mean and variance are given by:

$$
\begin{aligned}
\mu_{\bar{\Delta}} = (\frac{\lambda}{\bar{y}})\frac{p_0 - \sum p_{t-1}/T}{\frac{2\widetilde{\lambda}^{IT}}{\bar{y}} + \alpha(\Sigma_0 + \Sigma_{MM}^{IT})} + \phi\frac{\sum I_{t-1}/T}{\bar{I}} \\
\Sigma_{\bar{\Delta}} = \left(\frac{\lambda}{\bar{y}}\right)^2 \frac{\Sigma_u}{T}
\end{aligned}
\tag{8}
$$

Second moment

Let's take a look at the random variable $Z_t = \Delta_{p_t} - \mu_{\Delta_{p_t}}$. Z_t has a mean zero and variance $\Sigma_{\Delta_{p_t}} = \left(\frac{\lambda}{\bar{y}}\right)^2 \Sigma_u$ which doesn't depend on the time subscript t.

Plan for paper

- Generate data and check if traditional likelihood approach can estimate the parameters of the model (assume known inventories)
- Same as above but without assuming knowledge of inventories. Need to calculate $P(I_{t-1}|\Delta p_{t-1}, \lambda, \phi, I_0)$
- Check if GA approach returns same results as ML approach (need to modify the ML function for rounded variables maybe)
- if above true - illustrate how you can use GA and why it makes sense
- Fit with real Forex data

Implications for Trading

- If no price change
  - If there are huge volumes (total dollar value of $|buys|+|sells|$) with no price change that would suggest that trading is net zero and trades are by noise traders. If the volatility of the price is large that suggests that level of noise trading is high – mean reversion

Other Notes

Rounding

(not sure whether to round - makes calculations of variances complicated) I model a dealer market with a market maker who sets the price. Every trading day is divided into T periods during which the trading takes place. There is a noise trader whose demand is $u_t = \lfloor u'_t \rfloor$ where $\lfloor u'_t \rfloor = max\{m \in \mathbb{Z} | m \leq u'_t\}$ where $\mathbb{Z}$ is the set of integer numbers. We also have that $u'_t \sim \mathcal{N}(0, \Sigma_u)$ where $t = \{0, 1, 2, ..., T\}$

$$\ln f = -\frac{1}{2} T \ln(2\pi) - T \ln \Sigma_{\Delta p_t}^{1/2} - \frac{\sum_i^T (\Delta p_i - \mu_{\Delta p_t})^2}{2\Sigma_{\Delta p_t}} \tag{9}$$