Homeworks of Bayesian Inference

(B004652)

Group 1

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Week 1

Exchangeability and stochastic processes

1.1 Exercise 1

1.1.1 Data

- E_1, E_2, E_3, E_4, E_5, events of simple alternative, exchangeable
- $P(E_2) = \omega_1 = \frac{1}{2}$
- $P(E_3 \wedge E_5) = \omega_2 = \frac{1}{4}$
- $\omega_5 = \frac{\omega_3^5}{\binom{5}{3}} = \frac{\omega_1^5}{\binom{5}{1}} = \frac{1}{30}$

1.1.2 Questions

Compute:

1. $P(E_2 \wedge E_3 \wedge E_4) = \omega_3$
2. $P(E_1 \wedge E_2 \wedge E_3 \wedge E_4) = \omega_4$
3. $P(E_1 \wedge E_2 \wedge \bar{E}_3 \wedge \bar{E}_4 \wedge \bar{E}_5) = \frac{\omega_2^5}{\binom{5}{2}}$

1.1.3 Solutions

First we find ω_1^5 and ω_3^5:

$$\omega_1^5 = \frac{1}{30} \cdot \binom{5}{1} = \frac{1}{6}$$
$$\omega_3^5 = \frac{1}{30} \cdot \binom{5}{3} = \frac{1}{3}$$

Knowing that

$$\omega_h = \frac{1}{\binom{n}{h}} \sum_{r=h}^{n} \omega_r^n \binom{r}{h}$$

we can write that

$$\begin{aligned}\omega_1 &= \frac{\omega_1^5\binom{1}{1} + \omega_2^5\binom{2}{1} + \omega_3^5\binom{3}{1} + \omega_4^5\binom{4}{1} + \omega_5^5\binom{5}{1}}{\binom{5}{1}} \\ &= \frac{1}{6}\cdot\frac{1}{5} + \frac{2}{5}\omega_2^5 + \frac{1}{5}\cdot\frac{1}{3}\cdot 3 + \frac{1}{5}\cdot 4\omega_4^5 + \frac{1}{30} \\ &= \frac{8}{30} + \frac{2}{5}\omega_2^5 + \frac{4}{5}\omega_4^5\end{aligned}$$

$$\begin{aligned}\omega_2 &= \frac{\omega_2^5\binom{2}{2} + \omega_3^5\binom{3}{2} + \omega_4^5\binom{4}{2} + \omega_5^5\binom{5}{2}}{\binom{5}{2}} \\ &= \frac{1}{10}\omega_2^5 + \frac{1}{10}\cdot\frac{1}{10}\cdot 3 + \frac{1}{10}\cdot 6\omega_4^5 + \frac{1}{30} \\ &= \frac{2}{15} + \frac{1}{10}\omega_2^5 + \frac{3}{5}\omega_4^5\end{aligned}$$

Combining them:

$$\begin{aligned}&\begin{cases}\frac{2}{5}\omega_2^5 + \frac{4}{5}\omega_4^5 = \frac{1}{2} - \frac{8}{30} \\ \frac{1}{10}\omega_2^5 + \frac{3}{5}\omega_4^5 = \frac{1}{4} - \frac{2}{15}\end{cases} \\ \implies &\begin{cases}\omega_2^5 = \frac{7}{24} \\ \omega_4^5 = \frac{7}{48}\end{cases}\end{aligned}$$

Now we can obtain

$$\omega_3 = \frac{\omega_3^5\binom{3}{3} + \omega_4^5\binom{4}{3} + \omega_5^5\binom{5}{3}}{\binom{5}{3}} \qquad = \frac{1}{3}\cdot\frac{1}{10} + \frac{7}{48}\cdot 4\frac{1}{10} + \frac{1}{30} = \frac{1}{8}$$

$$\omega_4 = \frac{\omega_4^5\binom{4}{4} + \omega_5^5\binom{5}{4}}{\binom{5}{4}} \qquad = \frac{7}{48}\cdot\frac{1}{5} + \frac{1}{30} = \frac{1}{16}$$

1.2 Exercise 2

1.2.1 Data

- Process of simple alternative $\{|E_n|\}$
- $P(E_1) = \omega_1 = \frac{1}{2}$
- $P(E_1 \wedge E_2) = \omega_2 = \frac{1}{4}$
- $P(E_1 \wedge E_2 \wedge E_3) = \omega_3 = \frac{1}{7}$
- $P(E_1 \wedge E_2 \wedge E_3 \wedge E_4) = \frac{3}{28}$

1.2.2 Questions

1. Could the 4 indicators $|E_1|$, $|E_2|$, $|E_3|$ and $|E_4|$ be the starting path of an exchangeable process?
2. Could it continue for at least one step?

1.2.3 Solutions

1. An exchangeable process must satisfy the condition

$$(-1)^{n-h}\Delta^{n-h}\omega_h \geq 0,\ \forall n, h \leq n$$

Thus we compute

- $(-1)^{4-1}\Delta^{4-1}\omega_1 = (-1)\cdot\Delta^3\omega_1 = \frac{1}{14} \geq 0$

- $(-1)^{4-2}\Delta^{4-2}\omega_2 = (\ \ 1)\cdot\Delta^2\omega_2 = \frac{1}{14} \geq 0$
- $(-1)^{4-3}\Delta^{4-3}\omega_3 = (-1)\cdot\Delta\ \omega_3 = \frac{1}{28} \geq 0$
- $(-1)^{4-4}\Delta^{4-4}\omega_4 = (\ \ 1)\cdot\ \ \ \omega_4 = \frac{3}{28} \geq 0$

Thus we can affirm that the process is exchangeable.

2. If we consider $n = 5$ we can rewrite

$$
\begin{aligned}
-\Delta\ \omega_4 = \omega_4 - \omega_5 \geq 0 &\implies \omega_5 \leq \omega_4 \\
\Delta^2\omega_3 = \omega_3 - 2\omega_4 + \omega_5 \geq 0 &\implies \omega_5 \geq 2\omega_4 - \omega_3 \\
-\Delta^3\omega_2 = \omega_2 - 3\omega_3 + 3\omega_4 - \omega_5 \geq 0 &\implies \omega_5 \leq 3\omega_4 - 3\omega_3 + \omega_2 \\
\Delta^4\omega_1 = \omega_1 - 4\omega_2 + 6\omega_3 - 4\omega_4 + \omega_5 \geq 0 &\implies \omega_5 \geq 4\omega_4 - 6\omega_3 + 4\omega_2 - \omega_1
\end{aligned}
$$

And substituting ω_k with their values we obtain a system:

$$
\begin{cases}
\omega_5 \leq \frac{3}{28} \\
\omega_5 \geq \frac{2}{28} \\
\omega_5 \leq \frac{3}{28} \\
\omega_5 \geq \frac{2}{28}
\end{cases}
\implies \frac{2}{28} \leq \omega_5 \leq \frac{3}{28}
$$

Thus we can affirm that the process could continue.

Week 2

Conjugate priors and posterior distributions

2.1 Exercise 2.3

2.1.1 Data

- $p(x, y, z) \propto f(x, z)\ g(y, z)\ h(z)$

2.1.2 Questions

Prove that:

1. $p(x|y, z) \propto f(x, z)$
2. $p(y|x, z) \propto g(y, z)$
3. X and Y conditionally independent, given Z.

2.1.3 Solutions

We know by definition that

$$p(x|y, z) = \frac{p(x, y, z)}{p(y, z)}$$

and also that

$$p(y, z) = \int_{S_X} p(x, y, z)\partial x \propto \int_{S_X} f(x, z)g(y, z)h(z)\partial x = g(y, z)h(z) \int_{S_X} f(x, z)\partial x$$

Where S_X is the support of the r.v. X. Then we can write

$$\begin{aligned} p(x|y, z) &= \frac{f(x, z)g(y, z)h(z)}{g(y, z)h(z) \int_{S_X} f(x, z)\partial x} \\ &= \frac{f(x, z)}{\int_{S_X} f(x, z)\partial x} \end{aligned}$$

But $\int_{S_X} f(x, z)\partial x$ is constant given z, so we can say

$$p(x|y, z) \propto f(x, z)$$

as we wanted to show.
Similarly, we can write

$$\begin{aligned} p(y|x,z) &= \frac{p(x,y,z)}{p(x,z)} \\ &= \frac{f(x,z)g(y,z)h(z)}{f(x,z)h(z)\int_{S_Y} g(y,z)\partial y} \\ &= \frac{g(y,z)}{\int_{S_Y} g(y,z)\partial y} \\ &\propto g(y,z) \end{aligned}$$

To show that $X \perp Y$ given Z we have to prove that $p(y|z,x) = p(y|z)$, so:

$$\begin{aligned} p(y|z) &= \frac{p(y,z)}{p(z)} \\ &= \frac{\int_{S_X} f(x,z)g(y,z)h(z)\partial x}{\int_{S_X}\int_{S_Y} f(x,z)g(y,z)h(z)\partial y\partial x} \\ &= \frac{g(y,z)h(z)\int_{S_X} f(x,z)\partial x}{h(z)\int_{S_X} f(x,z)\partial x \int_{S_Y} g(y,z)\partial y} \\ &= \frac{g(y,z)}{\int_{S_Y} g(y,z)\partial y} \\ &= p(y|x,z) \end{aligned}$$

2.2 Exercise 3.5

2.2.1 Data

- $p(y|\phi) = c(\phi)h(y)\exp(\phi t(y))$
- $p_1(\theta)\dots p_k(\theta)$ conjugate priors
- $\tilde{p}(\theta) = \sum_{k=1}^{K} \omega_k p_k(\theta)$ where $\omega_k > 0$ and $\sum_k \omega_k = 1$

2.2.2 Questions

1. $p(\theta|y)$ as a function of $p(y|\theta)$ and $\tilde{p}$
2. Previous question but in the case that $\theta \sim$ Pois and $p_1 \dots p_k \sim \Gamma$

2.2.3 Solution

For the Bayes rule:

$$\begin{aligned} p(\theta|y) &= \frac{p(y|\theta)\cdot p(\theta)}{p(y)} \\ &= \frac{p(y|\theta)\cdot \tilde{p}(\theta)}{p(y)} \\ &= \frac{\prod_i p(y_i|\theta)\tilde{p}(\theta)}{p(y)} \\ &= \frac{\prod_i c(\theta)h(y_i)\exp(\theta t(y_i))\cdot \tilde{p}(\theta)}{p(y)} \\ &= \frac{\prod_i h(y_i)c(\phi)^n \exp(\phi\sum_i t(y_i))\cdot \sum_k w_k p_k(\theta)}{\int_{S_\theta}\prod_i h(y_i)c(\phi)^n \exp(\phi\sum_i t(y_i))\sum_k w_k p_k(\theta)\partial\theta} \\ &= \frac{c(\phi)^n \exp(\phi\sum_i t(y_i))\cdot \sum_k w_k p_k(\theta)}{\int_{S_\theta} c(\phi)^n \exp(\phi\sum_i t(y_i))\sum_k w_k p_k(\theta)\partial\theta} \end{aligned}$$

In the particular case that $p(y|\theta)$ is a Poisson distribution and p_k are Gamma distribution, we have that

- $t(y) = y$
- $\phi = \log(\theta)$
- $c(\phi) = \exp(e^{-\phi}) = \exp(\theta^{-1})$
- $p_k(\theta) = \frac{\beta_k^{\alpha_k}}{\Gamma(\alpha_k)} \theta^{\alpha_k - 1} \exp(-\beta_k \theta) = c_k \theta^{\alpha_k - 1} \exp(-\beta_k \theta)$

So we can rewrite the posterior of the first part as

$$\begin{aligned} p(\theta|\mathbf{y}) &\propto \exp(\theta^{-1})\theta \exp\left(\sum_i y_i\right) \sum_k w_k c_k \theta^{\alpha_k - 1} \exp(-\beta_k \theta) \\ &= \sum_k w_k c_k \theta^{\alpha_k} \exp\left(-\beta_k \theta + \theta^{-1} + \sum_i y_i\right) \end{aligned}$$

Week 3

Non informative priors

3.1 Exercise 3.10

3.1.1 Data

- $\psi = g(\theta)$ where g is a monotone function
- $h(\cdot) = g^{-1}(\cdot)$, that is $\theta = h(\psi)$
- $p_\theta(\theta) = \text{PDF of } \theta \implies p_\psi(\psi) = p_\theta(h(\psi)) \cdot \left|\frac{dh}{d\psi}\right|$

3.1.2 Questions

1. Let $\theta \sim Beta(a, b)$ and $\psi = \text{logit}(\theta)$. Obtain p_ψ and plot it for the case $a = b = 1$.
2. Let $\theta \sim Gamma(a, b)$ and $\psi = \log(\theta)$. Obtain p_ψ and plot it for the case $a = b = 1$.

3.1.3 Solutions

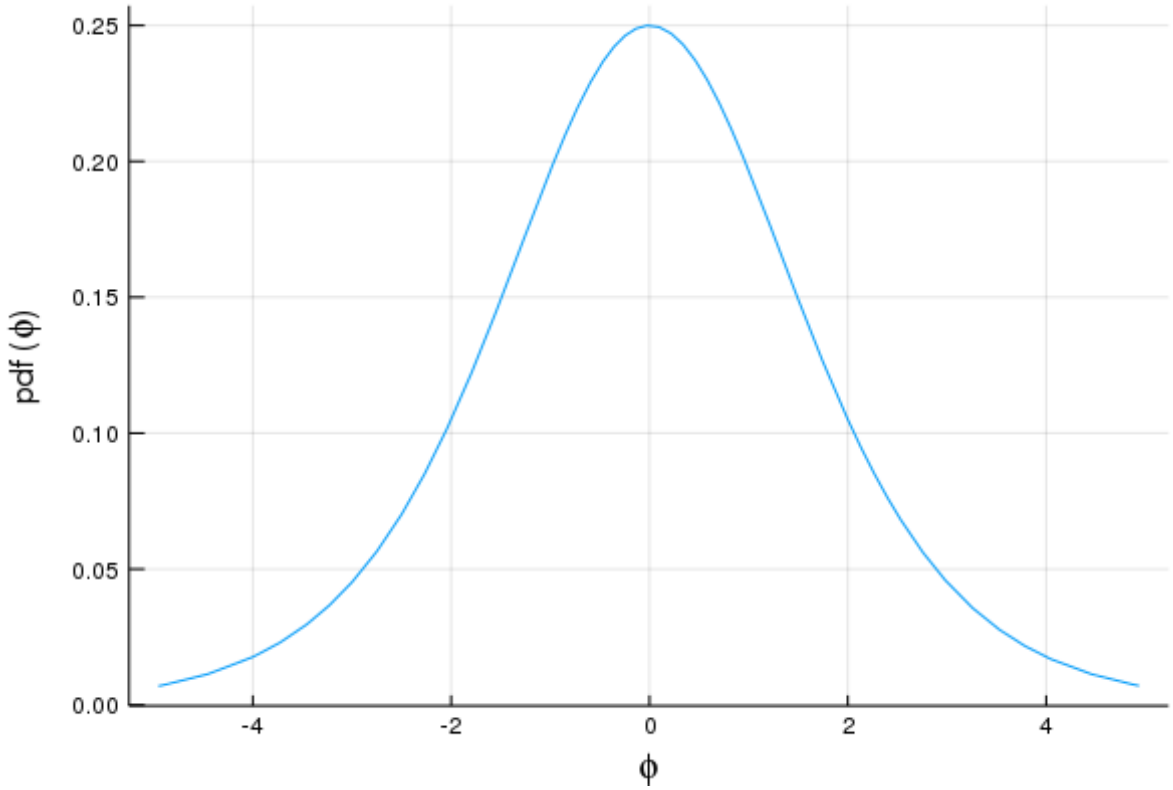
1. The inverse function of $\text{logit}(\cdot)$ is known to be $h(\psi) = \frac{\exp(\psi)}{1+\exp(x\psi)}$, and the derivative w.r.t. ψ of $h(\psi)$ is

$$\begin{aligned}\frac{\partial h(\psi)}{\psi} &= \frac{\exp(\psi)(1+\exp(\psi) - \exp(2\psi)}{(1+\exp(\psi))^2} \\ &= \frac{\exp(\psi)}{(1+\exp(\psi))^2}\end{aligned}$$

So we can write the PDF of ψ as

$$\begin{aligned}p_\psi(\psi) &= \frac{1}{B(a,b)}\left(\frac{\exp(\psi)}{1+\exp(\psi)}\right)^{a-1}\left(1 - \frac{\exp(\psi)}{1+\exp(\psi)}\right)^{b-1}\frac{\exp(\psi)}{(1+\exp(\psi))^2} \\ &= \frac{1}{B(a,b)}\frac{\exp(\psi)^{a-1}}{(1-\exp(\psi))^{a-1}}\frac{1}{(1+\exp(\psi))^{b-1}}\frac{\exp(\psi)}{(1+\exp(\psi))^2} \\ &= \frac{1}{B(a,b)}\frac{\exp(\psi)^a}{(1+\exp(\psi))^{a+b}}\end{aligned}$$

And in the case that $a = b = 1$ the plot is:



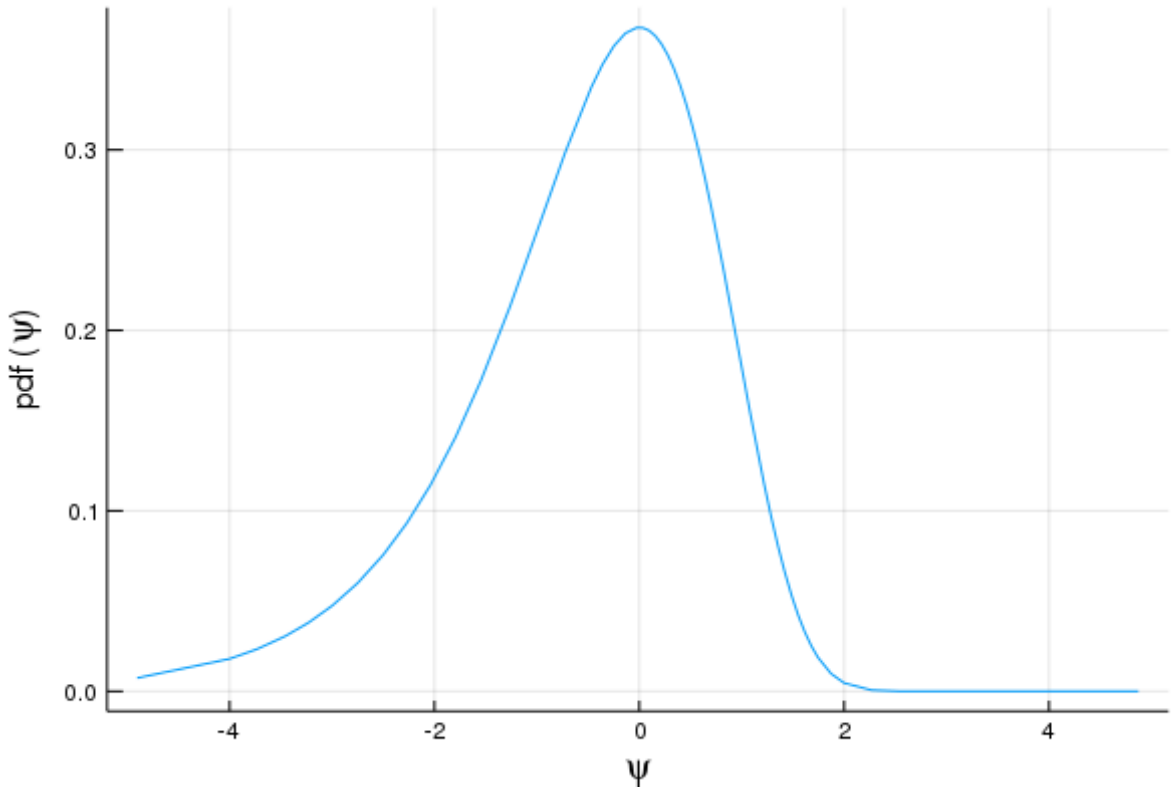
2. The inverse function of $\log(\theta)$ is known to be $h(\psi) = \exp(\psi)$, and the derivative w.r.t. ψ of $h(\psi)$ is

$$\frac{\partial h(\psi)}{\partial \psi} = \exp(\psi)$$

So we can write the PDF of ψ as

$$\begin{aligned} p_\psi(\psi) &= \frac{b^a}{\Gamma(a)} \exp(\psi)^{a-1} \exp(-b \exp(\psi)) \exp(\psi) \\ &= \frac{b^a}{\Gamma(a)} \exp(a\psi - b \exp(\psi)) \end{aligned}$$

And in the case that $a = b = 1$ the plot is:



3.2 Exercise 3.14

3.2.1 Data

3.2.2 Questions

1. Given $Y_1 \dots Y_n \sim \text{Bernoulli}(\theta)$ find the MLE of θ and $\frac{J(\theta)}{n}$.
2. Find a PDF $p_U(\theta)$ such that $\log p_U(\theta) = l(\theta|\mathbf{y})/n + c$ where c is a constant that does not depend on θ. Compute $-\partial^2 \log p_U(\theta)/\partial^2\theta$.
3. Obtain a PDF for θ that is proportional to $p_U(\theta)p(\mathbf{y}|\theta)$. Can this be considered a posterior for θ?
4. Repeat previous points with $Y_1 \dots Y_n \sim \text{Poisson}(\theta)$.

3.2.3 Solutions

1.

$$
\begin{aligned}
\hat{\theta}_{MLE} &= \arg\min_{\theta} \; l(\theta|\mathbf{y}) \\
&= \arg\min_{\theta} \; \log\left(\prod_i \theta^{y_i}(1-\theta)^{1-y_i}\right) \\
&= \arg\min_{\theta} \; \sum_i \log\left(\theta^{y_i}(1-\theta)^{1-y_i}\right) \\
&= \arg\min_{\theta} \; \log(\theta)\sum_i y_i + \log(1-\theta)\sum_i (1-y_i) \\
&= \arg\min_{\theta} \; \log(\theta)\sum_i y_i + n\log(1-\theta) - \log(1-\theta)\sum_i (y_i)
\end{aligned}
$$

To find the minimum we compute the zeros of $\partial l(\theta|\mathbf{y})/\partial\theta$

$$\begin{aligned}
0 &= \frac{\partial l(\theta|\mathbf{y})}{\partial\theta} \\
&= \frac{\sum_i y_i}{\theta} - \frac{n}{1-\theta} + \frac{\sum_i y_i}{1-\theta} \\
&= \frac{\sum_i y_i - \theta\sum_i y_i - \theta n + \theta\sum_i y_i}{\theta(1-\theta)} \\
&= \frac{\sum_i y_i - \theta n}{\theta(1-\theta)}
\end{aligned}$$

Thus, if $\theta \notin \{0,1\}$, $\hat{\theta}_{MLE} = \frac{\sum_i y_i}{n}$.

$$\begin{aligned}
-\frac{\partial^2 l(\theta|\mathbf{y})}{n\partial^2\theta} &= -\frac{-n\theta(1-\theta) - (\sum_i y_i - n\theta)(1-2\theta)}{n\theta^2(1-\theta)^2} \\
&= -\frac{n\theta^2 - n\theta - \sum_i y_i + 2\theta\sum_i y_i + n\theta - 2n\theta^2}{n\theta^2(1-\theta^2)} \\
&= \frac{n\theta^2 - 2\theta\sum_i y_i + \sum_i y_i}{n\theta^2(1-\theta)^2} \\
&= \frac{\theta^2 - 2\theta\hat{\theta}_{MLE} + \hat{\theta}_{MLE}}{\theta^2(1-\theta)^2}
\end{aligned}$$

2. The constrains on $p_U(\theta)$ imply that

$$\begin{aligned}
p_U(\theta) &= c\sqrt[n]{\prod_i \theta^{y_i}(1-\theta)^{1-y_i}} \\
&= c\prod_i \theta^{y_i/n}(1-\theta)^{(1-y_i)/n} \\
&= c\,\theta^{\sum_i y_i/n}(1-\theta)^{\sum_i(1-y_i)/n} \\
&= c\,\theta^{\sum_i y_i/n}(1-\theta)^{1-\sum_i y_i/n}
\end{aligned}$$

where c is the normalization constant.

$$\begin{aligned}
-\frac{\partial^2 \log p_U(\theta)}{\partial^2\theta} &= -\frac{\partial^2 l(\theta|\mathbf{y})/n + c}{\partial^2\theta} \\
&= -\frac{\partial^2 l(\theta|\mathbf{y})}{\partial^2\theta}/n \\
&= \frac{\theta^2 - 2\theta\hat{\theta}_{MLE} + \hat{\theta}_{MLE}}{\theta^2(1-\theta)^2}
\end{aligned}$$

3. Such a PDF would have the form

$$\begin{aligned}
p(\theta|\mathbf{y}) &= c\cdot p_U(\theta)p(\mathbf{y}|\theta) \\
&= c\,\theta^{\sum_i y_i/n}(1-\theta)^{1-\sum_i y_i/n}\cdot\theta^{\sum_i y_i}(1-\theta)^{\sum_i(1-y_i)} \\
&= c\,\theta^{\sum_i y_i(1+\frac{1}{n})}(1-\theta)^{\sum_i(1-y_i)(1+\frac{1}{n})} \\
&= c\,\theta^{\sum_i y_i(1+\frac{1}{n})}(1-\theta)^{(n-\sum_i y_i)(1+\frac{1}{n})}
\end{aligned}$$

Where $c = \int_{S_\theta} p(\theta,\mathbf{y})\partial\theta$ to guarantee that the PDF is proper. We can observe that the obtained PDF is a Beta distribution with parameters $\sum_i y_i(1+\frac{1}{n})+1$ and $(n-\sum_i y_i)(1+\frac{1}{n})+1$. It is a posterior because it is the product of a prior and a conditioned probability. Moreover, it is a case of conjugate prior.

4. Same steps: simplify the log-likelihood, find the first derivative and constrain to zero.

$$\begin{aligned}\hat{\theta}_{MLE} &= \arg\min_{\theta}\ l_{\text{Poisson}}(\theta|\mathbf{y}) \\ &= \arg\min_{\theta}\ \log\left(\prod_i \frac{\exp(-\theta)\theta^{y_i}}{y_i!}\right) \\ &= \arg\min_{\theta}\ \sum_i \log\frac{\exp(-\theta)\theta^{y_i}}{y_i!} \\ &= \arg\min_{\theta}\ -n\theta + \log\theta\sum y_i - \sum \log y_i!\end{aligned}$$

$$\begin{aligned}0 &= \frac{\partial l(\theta|\mathbf{y})}{\partial\theta} \\ &= -n + \frac{\sum_i y_i}{\theta}\end{aligned}$$

Thus $\hat{\theta}_{MLE} = \sum_i y_i/n$.

$$\begin{aligned}-\frac{\partial^2 l(\theta|\mathbf{y})}{n\partial^2\theta} &= \frac{\sum_i y_i}{n\theta^2} \\ &= \frac{\hat{\theta}_{MLE}}{\theta^2}\end{aligned}$$

In this case the PDF p_U would be

$$\begin{aligned}p_U(\theta) &\propto \sqrt[n]{\prod_i \exp(-\theta)\theta^{y_i}} \\ &= \exp(-\theta)\theta^{\sum_i y_i/n}\end{aligned}$$

Thus

$$-\frac{\partial^2 \log p_U(\theta)}{\partial^2\theta} = \frac{\sum_i y_i}{n\theta^2}$$

While the posterior is

$$\begin{aligned}p(\theta|\mathbf{y}) &\propto \exp(-\theta)\theta^{\sum_i y_i/n}\prod_i \exp(-\theta)\theta^{y_i} \\ &= \exp(-(n+1)\theta)\theta^{\sum_i y_i(1+\frac{1}{n})}\end{aligned}$$

This time the posterior is a Gamma distribution with parameters $\sum_i y_i(1+\frac{1}{n})$ and $n+1$.

Week 4

Monte Carlo simulations

4.1 Exercise 1

4.1.1 Data

- Data sequence:
  `c(1, 1, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0)`

4.1.2 Questions

Check that the sequence comes from an exchangeable process using the Bayesian p-value and the number of switch as statistic test.

4.1.3 Solutions

Solution: 0.016
Code:

```
set.seed(10)

count_switch <- function(x) x %>% diff() %>% abs() %>% sum()

y <- c(1, 1, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0)
t_obs <- sum(abs(diff(y)))

# posterior hyperparameters
alpha <- 0.5 + sum(y)
beta  <- 0.5 + length(y) - sum(y)

niter <- 10000
ssize <- 20

theta <- rbeta(niter, alpha, beta)
t_rep <- map_dbl(theta, ~ rbinom(ssize, 1, .) %>% count_switch())

# Result
pval <- mean(t_rep <= t_obs)
cat("Bayesian p-val:", pval)
```

4.2 Exercise 2

4.2.1 Questions

We want a sample from a distribution $f(x)$ using another distribution $g(x)$ through an AR algorithm. Being assumed the value of M, proof that this algorithm is equivalent to a *standard* AR.

a) Generate $x \sim$ G

b) Generate $u \sim \mathrm{U}(0, Mg(x))$

c) Accept $y \stackrel{def}{=} x$ if $u < f(x)$

d) Otherwise go back to a).

4.2.2 Solutions

We can see that the two algorithms differ only for steps b) and c), where the difference is the Uniform distribution from which we sample u. To verify that they are equivalent we have to see if:

1. The proportion of values accepted near a generic x is the same, that is equal to $\frac{f(x)}{Mg(x)}$,
2. The efficiency is the same: the mean number of replications before rejecting a point is the same and it's equal to M.

In a generic u in the support of $\mathrm{U}(0, Mg(x))$ the proportion of values below u is

$$\mathbb{P}(U < u) = \frac{u}{Mg(x)}$$

therefor if $u \stackrel{def}{=} f(x)$ the proportion of accepted values given x is $\frac{f(x)}{Mg(x)}$, so 1 is verified.

Let K be the number of replications before accepting a value. So

$$K \sim \mathrm{Geom}(p) \text{ with } p \text{ probability of accepting at each replication}$$

We can compute the expected value of acceptance probability:

$$p = \mathbb{P}(U \leq f(x)) = \int_{S_X} g(x) \int_0^{f(x)} \frac{1}{Mg(x)} \partial u \partial x = \int_{S_X} g(x) \frac{f(x)}{Mg(x)} \partial x = \frac{1}{M} \int_{S_X} f(x) \partial x = \frac{1}{M}$$

So the expected value $\mathrm{E}[K] = \frac{1}{p} = M$ and even 2 is verified, and we can affirm that the two procedures are equivalent.

Week 5

Unit information prior (again)

5.1 Exercise 5.5, Hoff

5.1.1 Questions

1. Reparametrize the normal model as $p(y|\theta, \psi)$ where $\psi = 1/\sigma^2$. Write the log-likelihood in terms of θ and ψ.

2. Find a probability density $p_U(\theta, \psi)$ so that $\log(\theta, \psi) = l(\theta, \psi|\mathbf{y})/n + c$ where c is a constant and does not depend on θ and ψ.

5.1.2 Solutions

1.

$$\begin{aligned} l(\theta, \psi|\mathbf{y}) &= \sum_{i=1}^{n} \log \left(\frac{\psi}{2\pi}\right)^{\frac{1}{2}} \exp\left(\frac{\psi(y_i - \theta)^2}{2}\right) \\ &= \frac{n}{2} \log\left(\frac{\psi}{2\pi}\right) + \frac{\psi}{2} \sum_{i=1}^{n} (y_i - \theta)^2 \\ &= \frac{n}{2} \log\left(\frac{\psi}{2\pi}\right) + \frac{\psi}{2} \sum_{i=1}^{n} (y_i - \bar{y})^2 + \frac{\psi}{2} \sum_{i=1}^{n} (\bar{y} - \theta)^2 \end{aligned}$$

2.

$$l_U(\theta, \psi|\mathbf{y}) \propto \frac{1}{2} \log\left(\frac{\psi}{2\pi}\right) +$$

Week 6

Gibbs sampler

6.1 Exercise 6.1, Hoff

6.1.1 Data

- Let's reconsider the number of children data in Exercise 4.8.
- Assume Poisson sampling models for the two groups as before, but parameterize θ_A and θ_B as $\theta_A = \theta$ and $\theta_B = \theta_\gamma$, where γ is the relative rate θ_B/θ_A. Let $\theta \sim \text{Gamma}(a_\theta, b_\theta)$ and $\gamma \sim \text{Gamma}(a_\gamma, b_\gamma)$.

6.1.2 Questions

1. Are θ_A and θ_B independent or dependent under this prior distribution? In which situations is such a joint prior distribution justified?
2. Obtain the form of the full conditional distribution of θ given $\mathbf{y}_A$, $\mathbf{y}_B$ and γ.
3. Obtain the form of the full conditional distribution of γ given $\mathbf{y}_A$, $\mathbf{y}_B$ and θ.
4. Set $a_\theta = 2$ and $b_\theta = 1$. Let $a_\gamma = b_\gamma \in \{8, 16, 32, 64, 128\}$. For each of these five values, run a Gibbs sampler of at least 5,000 iterations and obtain $\text{E}[\theta_B - \theta_A | \mathbf{y}_A, \mathbf{y}_B]$. Describe the effects of the prior distribution for γ on the results.

6.1.3 Solutions

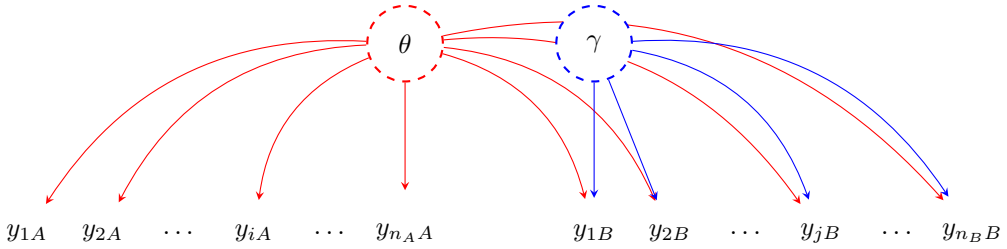
1. We evaluate the covariance expression of θ_A and θ_B:

$$\begin{aligned}
\text{Cov}[\theta_A, \theta_B] &= \text{E}[\theta_A \theta_B] - \text{E}[\theta_A]\,\text{E}[\theta_B] \\
&= \text{E}[\theta^2 \gamma] - \text{E}[\theta]\,\text{E}[\theta\gamma] \\
&= \text{E}[\theta^2]\,\text{E}[\gamma] - \text{E}[\theta]^2\,\text{E}[\gamma] && \text{(for the independence of } \theta \text{ and } \gamma\text{)} \\
&= \text{E}[\gamma](\text{E}[\theta^2] - \text{E}[\theta]^2) \\
&= \text{E}[\gamma]\,\text{Var}[\theta] = \frac{a_\gamma}{b_\gamma}\frac{a_\theta}{b_\theta^2} > 0
\end{aligned}$$

   Thus the two variables are not independent.

   This kind of prior joint distribution could be useful in situations where we are modeling the number of events in two different contests A and B in a certain time interval and the mean number of events is respectively θ_A and θ_B. The parameter γ can be interpreted as an acceleration factor caused by a phenomenon that change in the contest B, but that remains constant in the "rest of the world" A. In this way of thinking θ represents the *standard* and γ represents the element that, ceteris paribus, deviates from it. So, this parameterization of the prior joint is justified when we want to measure the intensity of the change caused to a starting population A. As a consequence it lends itself well in experimental rather than observational scopes.

2. First of all we observe the DAG that represents the relational model of variables.



Full conditional of θ The *Markov blanket* of θ is

$$\text{bl}(\theta) = \{Y_{1A}, Y_{2A}, \dots, Y_{n_A A}, Y_{1B}, \dots, Y_{n_B B}, \gamma\}$$

in fact the θ node does not have parents and its children are all the observations. Moreover, for the B sample, γ is another parent.

$$\begin{aligned}
p(\theta \mid \text{bl}(\theta)) &\propto p(\theta|a_\theta, b_\theta) \prod_{i=1}^{n_A} p(Y_{iA}|\theta) \prod_{j=1}^{n_B} p(Y_{jB}|\theta, \gamma) \\
&= \frac{b_\theta^{a_\theta}}{\Gamma(a_\theta)} \theta^{a_\theta - 1} \exp(-b_\theta \theta) \prod_{i=1}^{n_A} \theta^{y_{iA}} \frac{\exp(-\theta)}{y_{iA}!} \prod_{j=1}^{n_B} (\theta \cdot \gamma)^{y_{jB}} \frac{\exp(-\theta\gamma)}{y_{jB}!} \\
&\propto \theta^{a_\theta - 1} \exp(-b_\theta \theta) \theta^{\sum_i y_{iA}} \exp(-n_A \theta) (\theta \cdot \gamma)^{\sum_j y_{jB}} \exp(-n_B \theta) \\
&= \theta^{a_\theta + \sum_i y_{iA} + \sum_j y_{jB} - 1} \exp(-(b_\theta + n_A + n_B)\theta)
\end{aligned}$$

We can note the kernel of a Gamma distribution, so

$$\theta \mid \text{bl}(\theta) \sim \text{Gamma}(a_\theta + \textstyle\sum_i y_{iA} + \sum_j y_{jB},\ b_\theta + n_A + n_B)$$

3. **Full conditional of γ** The *Markov blanket* of this random variable is

$$\text{bl}(\gamma) = \{Y_{1B}, Y_{2B}, \dots, Y_{n_B B}, \theta\}$$

$$\begin{aligned}
p(\gamma \mid \text{bl}(\gamma)) &\propto p(\gamma|a_\gamma, b_\gamma) \prod_{j=1}^{n_B} p(y_{jB}|\theta, \gamma) \\
&= \frac{b_\gamma^{a_\gamma}}{\Gamma(a_\gamma)} \gamma^{a_\gamma - 1} \exp(-b_\gamma \gamma) \prod_{j=1}^{n_B} (\theta \cdot \gamma)^{y_{iB}} \frac{\exp(-\theta\gamma)}{y_{jB}!} \\
&\propto \gamma^{a_\gamma - 1} \exp(-b_\gamma \gamma) (\theta \cdot \gamma)^{\sum_j y_{jB}} \exp(-n_B(\theta \cdot \gamma)) \\
&\propto \gamma^{a_\gamma + \sum_j y_{jB} - 1} \exp(-(b_\gamma + n_B \theta)\gamma)
\end{aligned}$$

We can note the kernel of a Gamma distribution, so

$$\gamma \mid \text{bl}(\gamma) \sim \text{Gamma}(a_\gamma + \textstyle\sum_j y_{jB},\ b_\gamma + n_B \theta)$$

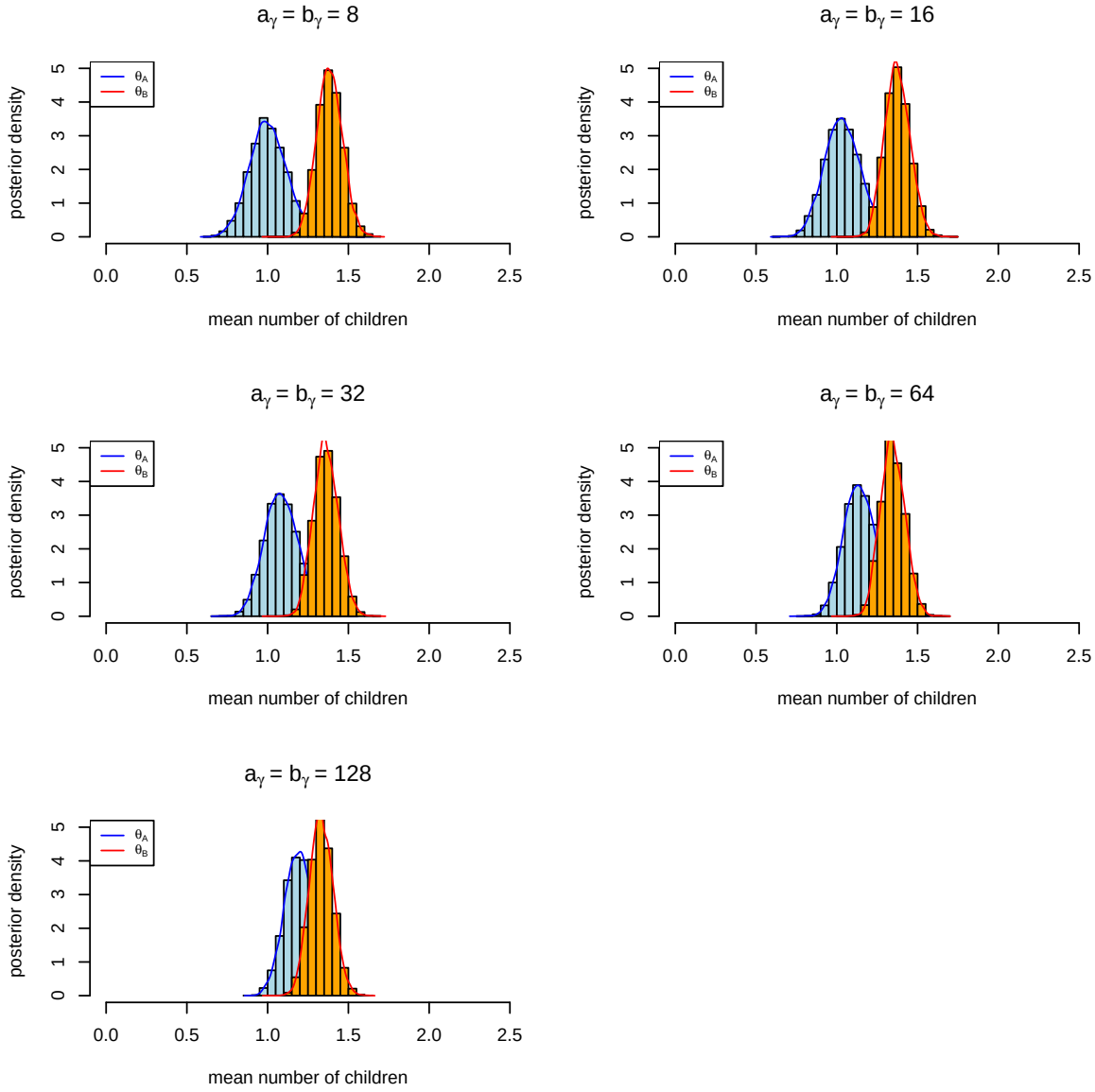
4. At this point we can proceed with a simulation through the Gibbs algorithm following the MCMC approach.

```
library(tidyverse)

# Load data
yobs <- list(
    A = scan("data/menchild30bach.dat"),
    B = scan("data/menchild30nobach.dat")
)
```

```r
# Hyperparameters
a_theta <- 2
b_theta <- 1
gammapriors <- 2 ^ (3:7)

# Sample statistics
ytot <- map(yobs, sum)
n <- map(yobs, length)

# Simulation main function
library(compiler) # to accelerate
gibbs_sim <- function(a_gamma, b_gamma = a_gamma,
                      theta0 = 1, gamma0 = 1, nsim = 5000, seed = 10) {
    set.seed(seed)
    # Initialize
    result <- matrix(NA, nsim, 2, dimnames = list(NULL, c("theta", "gamma")))
    result[1, ] <- c(theta0, gamma0)
    # Main loop
    for (r in 2:nsim) {
        result[r, "theta"] <- rgamma(1, a_theta + ytot$A + ytot$B,
                                     b_theta + n$A + n$B * result[r - 1, "gamma"])
        result[r, "gamma"] <- rgamma(1, a_gamma + ytot$B, b_gamma + n$B * result[r, "theta"])
    }
    # Return
    return(as_tibble(result))
}

# Simulation
simulations <- map(gammapriors, gibbs_sim, nsim = 1E4)
statistics <- map_dbl(simulations, ~ with(., mean(theta * gamma - theta)))
# [1] 0.3720311 0.3344181 0.2713526 0.2000735 0.1327382
```

We have evidence of the fact that the two means converge when the parameters of the prior distribution get higher. This imply that the difference gets lower and that the mean number of children in the two groups tends to be the same when we take higher hyper-parameters.

We can see it even graphically comparing the posterior distributions of θ_A and θ_B in the 5 configurations:



Code:

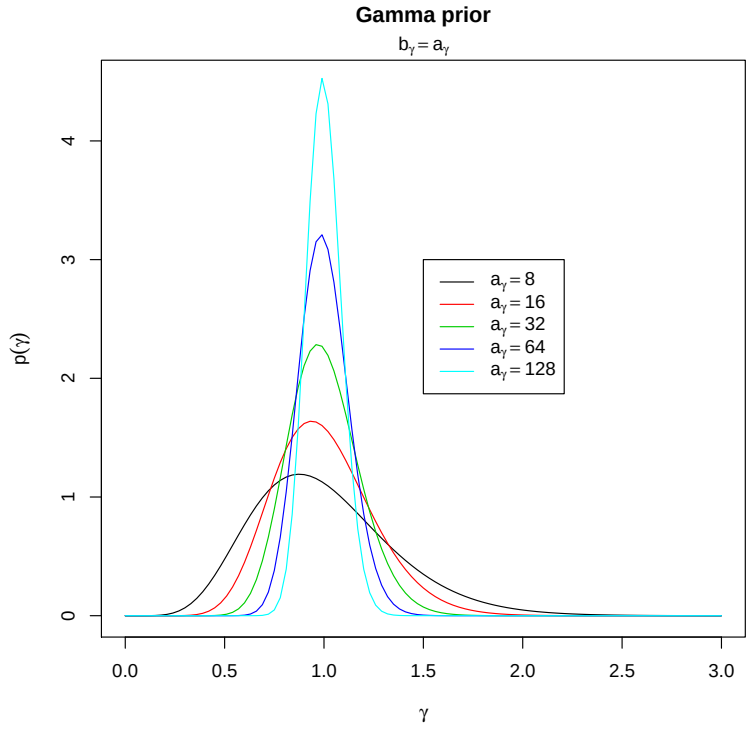
```
opar <- par(mfrow = n2mfrow(6))

for (j in seq_along(gammapriors)) {
    theta <- pluck(simulations, j, "theta")
    gamma <- pluck(simulations, j, "gamma")
    hist(theta, prob = TRUE, col = "lightblue", ylim = c(0, 5), xlim = c(0, 2.5),
         ylab = "posterior density", xlab = "mean number of children",
         main = substitute(paste(a[gamma], " = ", b[gamma], " = " , prior),
    list(prior = gammapriors[j])))
    lines(density(theta), col = "blue")
    hist(gamma * theta, prob = TRUE, col = "orange", add = TRUE)
    lines(density(gamma * theta), col = "red")
    legend("topleft", c(expression(theta[A]), expression(theta[B])),
           col = c("blue", "red"), lty = 1, cex = 0.7)
}

par(opar)
```

It's apparent even graphically that when the hyper-parameters of γ increase, the posterior distri-

butions get nearer, going even to overlap.

This is due to the fact that the higher we sat the priors of γ, the more we weight the information that we already have about the variable, and consequently the observed information gain less importance.

It's interesting even to observe how the Gamma distribution change when we set the different a_γ and b_γ.



Code:

```
plot(NULL, xlim = c(0, 3), ylim = c(0, 4.5),
     xlab = expression(gamma), ylab = expression(p(gamma)),
     main = "Gamma prior")
for (i in seq_along(gammapriors)) {
    curve(dgamma(x, gammapriors[i], gammapriors[i]), add = TRUE, col = i)
}
legend_txt <- parse(text = paste("a[gamma] ==", gammapriors))
legend(x = 1.5, y = 3, legend_txt, col = seq_along(gammapriors), lty = 1)
mtext(expression(b[gamma] == a[gamma]))
```

As we can see the in the plot, the higher the hyper parameters we set, the more the distribution shrinks to 1, that is also the expected value of the Gamma. Vice versa, the variance decreases proportionally.

Going toward the limit case in which a_γ and b_γ go to infinity we are implicitly affirming that we do not want to *learn* anything from the new observations, and we already have a strong opinion about the Gamma that we don't want to change. However, if are talking from a Bayesian perspective, this situation is not sane neither useful; it's just a way to see how much the inference can change on the base of the choice of a priori distribution of parameters.

Observation It would be appropriate to adequately assess the convergence of the Gibbs algorithm, but in this case it was not done because it was not explicitly requested by the exercise. We just calculate the effective sample size and make sure it is high enough to make sure we have reached the equilibrium distribution and have it very close to it.

```r
library(coda)
map(simulations, effectiveSize)

# [[1]]
# theta    gamma
# 1175.254 1198.183
#
# [[2]]
# theta    gamma
# 1471.870 1353.061
#
# [[3]]
# theta    gamma
# 1840.738 1695.119
#
# [[4]]
# theta    gamma
# 2424.931 2238.538
#
# [[5]]
# theta    gamma
# 3259.088 2942.661
```