Homeworks of Bayesian Inference

(B004652)

Group 1

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Last revision October 23, 2018

Week 1

Exchangeability and stochastic processes

1.1 Exercise 1

1.1.1 Data

- E_1, E_2, E_3, E_4, E_5, events of simple alternative, exchangeable
- $P(E_2) = \omega_1 = \frac{1}{2}$
- $P(E_3 \wedge E_5) = \omega_2 = \frac{1}{4}$
- $\omega_5 = \frac{\omega_3^5}{\binom{5}{3}} = \frac{\omega_1^5}{\binom{5}{1}} = \frac{1}{30}$

1.1.2 Questions

Compute:

1. $P(E_2 \wedge E_3 \wedge E_4) = \omega_3$
2. $P(E_1 \wedge E_2 \wedge E_3 \wedge E_4) = \omega_4$
3. $P(E_1 \wedge E_2 \wedge \bar{E}_3 \wedge \bar{E}_4 \wedge \bar{E}_5) = \frac{\omega_2^5}{\binom{5}{2}}$

1.1.3 Solutions

First we find ω_1^5 and ω_3^5:

$$\omega_1^5 = \frac{1}{30} \cdot \binom{5}{1} = \frac{1}{6}$$
$$\omega_3^5 = \frac{1}{30} \cdot \binom{5}{3} = \frac{1}{3}$$

Knowing that

$$\omega_h = \frac{1}{\binom{n}{h}} \sum_{r=h}^{n} \omega_r^n \binom{r}{h}$$

we can write that

$$\begin{aligned}
\omega_1 &= \frac{\omega_1^5\binom{1}{1} + \omega_2^5\binom{2}{1} + \omega_3^5\binom{3}{1} + \omega_4^5\binom{4}{1} + \omega_5^5\binom{5}{1}}{\binom{5}{1}} \\
&= \frac{1}{6}\cdot\frac{1}{5} + \frac{2}{5}\omega_2^5 + \frac{1}{5}\cdot\frac{1}{3}\cdot 3 + \frac{1}{5}\cdot 4\omega_4^5 + \frac{1}{30} \\
&= \frac{8}{30} + \frac{2}{5}\omega_2^5 + \frac{4}{5}\omega_4^5
\end{aligned}$$

$$\begin{aligned}
\omega_2 &= \frac{\omega_2^5\binom{2}{2} + \omega_3^5\binom{3}{2} + \omega_4^5\binom{4}{2} + \omega_5^5\binom{5}{2}}{\binom{5}{2}} \\
&= \frac{1}{10}\omega_2^5 + \frac{1}{10}\cdot\frac{1}{10}\cdot 3 + \frac{1}{10}\cdot 6\omega_4^5 + \frac{1}{30} \\
&= \frac{2}{15} + \frac{1}{10}\omega_2^5 + \frac{3}{5}\omega_4^5
\end{aligned}$$

Combining them:

$$\begin{aligned}
&\begin{cases} \frac{2}{5}\omega_2^5 + \frac{4}{5}\omega_4^5 = \frac{1}{2} - \frac{8}{30} \\ \frac{1}{10}\omega_2^5 + \frac{3}{5}\omega_4^5 = \frac{1}{4} - \frac{2}{15} \end{cases} \\
\implies &\begin{cases} \omega_2^5 = \frac{7}{24} \\ \omega_4^5 = \frac{7}{48} \end{cases}
\end{aligned}$$

Now we can obtain

$$\begin{aligned}
\omega_3 &= \frac{\omega_3^5\binom{3}{3} + \omega_4^5\binom{4}{3} + \omega_5^5\binom{5}{3}}{\binom{5}{3}} &&= \frac{1}{3}\cdot\frac{1}{10} + \frac{7}{48}\cdot 4\frac{1}{10} + \frac{1}{30} = \frac{1}{8} \\
\omega_4 &= \frac{\omega_4^5\binom{4}{4} + \omega_5^5\binom{5}{4}}{\binom{5}{4}} &&= \frac{7}{48}\cdot\frac{1}{5} + \frac{1}{30} = \frac{1}{16}
\end{aligned}$$

1.2 Exercise 2

1.2.1 Data

- Process of simple alternative $\{|E_n|\}$
- $P(E_1) = \omega_1 = \frac{1}{2}$
- $P(E_1 \wedge E_2) = \omega_2 = \frac{1}{4}$
- $P(E_1 \wedge E_2 \wedge E_3) = \omega_3 = \frac{1}{7}$
- $P(E_1 \wedge E_2 \wedge E_3 \wedge E_4) = \frac{3}{28}$

1.2.2 Questions

1. Could the 4 indicators $|E_1|$, $|E_2|$, $|E_3|$ and $|E_4|$ be the starting path of an exchangeable process?
2. Could it continue for at least one step?

1.2.3 Solutions

1. An exchangeable process must satisfy the condition

$$(-1)^{n-h}\Delta^{n-h}\omega_h \geq 0,\ \forall n, h \leq n$$

Thus we compute

- $(-1)^{4-1}\Delta^{4-1}\omega_1 = (-1)\cdot\Delta^3\omega_1 = \frac{1}{14} \geq 0$

- $(-1)^{4-2}\Delta^{4-2}\omega_2 = (\ \ \ 1) \cdot \Delta^2\omega_2 = \frac{1}{14} \geq 0$
- $(-1)^{4-3}\Delta^{4-3}\omega_3 = (-1) \cdot \Delta\ \omega_3 = \frac{1}{28} \geq 0$
- $(-1)^{4-4}\Delta^{4-4}\omega_4 = (\ \ \ 1) \cdot \quad \omega_4 = \frac{3}{28} \geq 0$

Thus we can affirm that the process is exchangeable.

2. If we consider $n = 5$ we can rewrite

$$\begin{aligned}
-\Delta\ \ \omega_4 &= \omega_4 - \omega_5 \geq 0 \implies \omega_5 \leq \omega_4 \\
\Delta^2\omega_3 &= \omega_3 - 2\omega_4 + \omega_5 \geq 0 \implies \omega_5 \geq 2\omega_4 - \omega_3 \\
-\Delta^3\omega_2 &= \omega_2 - 3\omega_3 + 3\omega_4 - \omega_5 \geq 0 \implies \omega_5 \leq 3\omega_4 - 3\omega_3 + \omega_2 \\
\Delta^4\omega_1 &= \omega_1 - 4\omega_2 + 6\omega_3 - 4\omega_4 + \omega_5 \geq 0 \implies \omega_5 \geq 4\omega_4 - 6\omega_3 + 4\omega_2 - \omega_1
\end{aligned}$$

And substituting ω_k with their values we obtain a system:

$$\begin{cases} \omega_5 \leq \frac{3}{28} \\ \omega_5 \geq \frac{2}{28} \\ \omega_5 \leq \frac{3}{28} \\ \omega_5 \geq \frac{2}{28} \end{cases} \implies \frac{2}{28} \leq \omega_5 \leq \frac{3}{28}$$

Thus we can affirm that the process could continue.

Week 2

Conjugate priors and posterior distributions

2.1 Exercise 2.3

2.1.1 Data

- $p(x, y, z) \propto f(x, z)\ g(y, z)\ h(z)$

2.1.2 Questions

Prove that:

1. $p(x|y, z) \propto f(x, z)$
2. $p(y|x, z) \propto g(y, z)$
3. X and Y conditionally independent, given Z.

2.1.3 Solutions

We know by definition that

$$p(x|y, z) = \frac{p(x, y, z)}{p(y, z)}$$

and also that

$$p(y, z) = \int\limits_{S_X} p(x, y, z)\partial x \propto \int\limits_{S_X} f(x, z)g(y, z)h(z)\partial x = g(y, z)h(z) \int\limits_{S_X} f(x, z)\partial x$$

Where S_X is the support of the r.v. X. Then we can write

$$\begin{aligned} p(x|y, z) &= \frac{f(x, z)g(y, z)h(z)}{g(y, z)h(z) \int_{S_X} f(x, z)\partial x} \\ &= \frac{f(x, z)}{\int_{S_X} f(x, z)\partial x} \end{aligned}$$

But $\int_{S_X} f(x, z)\partial x$ is constant given z, so we can say

$$p(x|y, z) \propto f(x, z)$$

as we wanted to show.
Similarly, we can write

$$\begin{aligned}
p(y|x,z) &= \frac{p(x,y,z)}{p(x,z)} \\
&= \frac{f(x,z)g(y,z)h(z)}{f(x,z)h(z)\int_{S_Y} g(y,z)\partial y} \\
&= \frac{g(y,z)}{\int_{S_Y} g(y,z)\partial y} \\
&\propto g(y,z)
\end{aligned}$$

To show that $X \perp Y$ given Z we have to prove that $p(y|z,x) = p(y|z)$, so:

$$\begin{aligned}
p(y|z) &= \frac{p(y,z)}{p(z)} \\
&= \frac{\int_{S_X} f(x,z)g(y,z)h(z)\partial x}{\int_{S_X}\int_{S_Y} f(x,z)g(y,z)h(z)\partial y\partial x} \\
&= \frac{g(y,z)h(z)\int_{S_X} f(x,z)\partial x}{h(z)\int_{S_X} f(x,z)\partial x \int_{S_Y} g(y,z)\partial y} \\
&= \frac{g(y,z)}{\int_{S_Y} g(y,z)\partial y} \\
&= p(y|x,z)
\end{aligned}$$

2.2 Exercise 3.5

2.2.1 Data

- $p(y|\phi) = c(\phi)h(y)\exp(\phi t(y))$
- $p_1(\theta)\dots p_k(\theta)$ conjugate priors
- $\tilde{p}(\theta) = \sum_{k=1}^{K} \omega_k p_k(\theta)$ where $\omega_k > 0$ and $\sum_k \omega_k = 1$

2.2.2 Questions

1. $p(\theta|y)$ as a function of $p(y|\theta)$ and $\tilde{p}$
2. Previous question but in the case that $\theta \sim \text{Pois}$ and $p_1 \dots p_k \sim \Gamma$

2.2.3 Solution

For the Bayes rule:

$$\begin{aligned}
p(\theta|y) &= \frac{p(y|\theta)\cdot p(\theta)}{p(y)} \\
&= \frac{p(y|\theta)\cdot \tilde{p}(\theta)}{p(y)} \\
&= \frac{\prod_i p(y_i|\theta)\tilde{p}(\theta)}{p(y)} \\
&= \frac{\prod_i c(\theta)h(y_i)\exp(\theta t(y_i))\cdot \tilde{p}(\theta)}{p(y)} \\
&= \frac{\prod_i h(y_i)c(\phi)^n \exp(\phi\sum_i t(y_i))\cdot \sum_k w_k p_k(\theta)}{\int_{S_\theta}\prod_i h(y_i)c(\phi)^n \exp(\phi\sum_i t(y_i))\sum_k w_k p_k(\theta)\partial\theta} \\
&= \frac{c(\phi)^n \exp(\phi\sum_i t(y_i))\cdot \sum_k w_k p_k(\theta)}{\int_{S_\theta} c(\phi)^n \exp(\phi\sum_i t(y_i))\sum_k w_k p_k(\theta)\partial\theta}
\end{aligned}$$

In the particular case that $p(y|\theta)$ is a Poisson distribution and p_k are Gamma distribution, we have that

- $t(y) = y$
- $\phi = \log(\theta)$
- $c(\phi) = \exp(e^{-\phi}) = \exp(\theta^{-1})$
- $p_k(\theta) = \frac{\beta_k^{\alpha_k}}{\Gamma(\alpha_k)}\theta^{\alpha_k - 1}\exp(-\beta_k\theta) = c_k\theta^{\alpha_k-1}\exp(-\beta_k\theta)$

So we can rewrite the posterior of the first part as

$$\begin{aligned} p(\theta|\mathbf{y}) &\propto \exp(\theta^{-1})\theta\exp\left(\sum_i y_i\right)\sum_k w_k c_k \theta^{\alpha_k-1}\exp(-\beta_k\theta) \\ &= \sum_k w_k c_k \theta^{\alpha_k}\exp\left(-\beta_k\theta + \theta^{-1} + \sum_i y_i\right) \end{aligned}$$

Week 3

Non informative priors

3.1 Exercise 3.10

3.1.1 Data

- $\psi = g(\theta)$ where g is a monotone function
- $h(\cdot) = g^{-1}(\cdot)$, that is $\theta = h(\psi)$
- $p_\theta(\theta) = \text{PDF of } \theta \implies p_\psi(\psi) = p_\theta(h(\psi)) \cdot \left|\frac{dh}{d\psi}\right|$

3.1.2 Questions

1. Let $\theta \sim Beta(a, b)$ and $\psi = \text{logit}(\theta)$. Obtain p_ψ and plot it for the case $a = b = 1$.
2. Let $\theta \sim Gamma(a, b)$ and $\psi = \log(\theta)$. Obtain p_ψ and plot it for the case $a = b = 1$.

3.1.3 Solutions

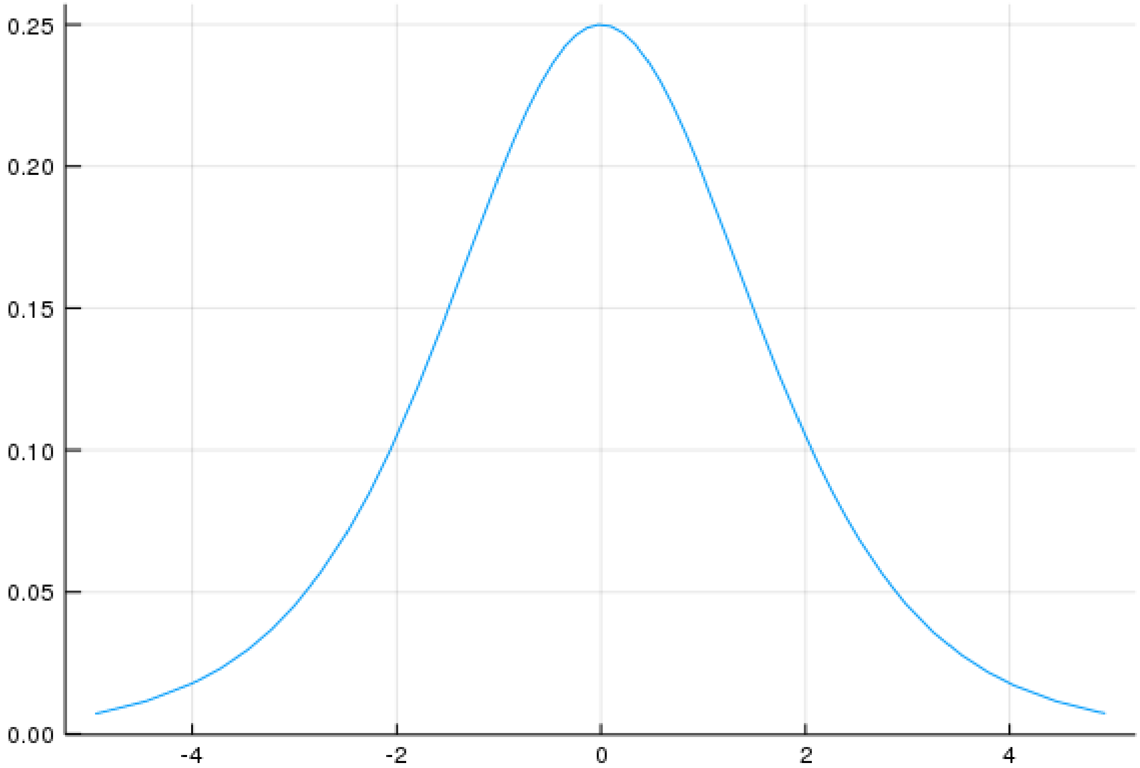
1. The inverse function of $\text{logit}(\cdot)$ is known to be $h(\psi) = \frac{\exp(\psi)}{1+\exp(x\psi)}$, and the derivative w.r.t. ψ of $h(\psi)$ is

$$\begin{aligned}\frac{\partial h(\psi)}{\psi} &= \frac{\exp(\psi)(1 + \exp(\psi) - \exp(2\psi)}{(1 + \exp(\psi))^2} \\ &= \frac{\exp(\psi)}{(1 + \exp(\psi))^2}\end{aligned}$$

So we can write the PDF of ψ as

$$\begin{aligned}p_\psi(\psi) &= \frac{1}{B(a,b)}\left(\frac{\exp(\psi)}{1+\exp(\psi)}\right)^{a-1}\left(1 - \frac{\exp(\psi)}{1+\exp(\psi)}\right)^{b-1}\frac{\exp(\psi)}{(1+\exp(\psi))^2} \\ &= \frac{1}{B(a,b)}\frac{\exp(\psi)^{a-1}}{(1-\exp(\psi))^{a-1}}\frac{1}{(1+\exp(\psi))^{b-1}}\frac{\exp(\psi)}{(1+\exp(\psi))^2} \\ &= \frac{1}{B(a,b)}\frac{\exp(\psi)^a}{(1+\exp(\psi))^{a+b}}\end{aligned}$$

And in the case that $a = b = 1$ the plot is:



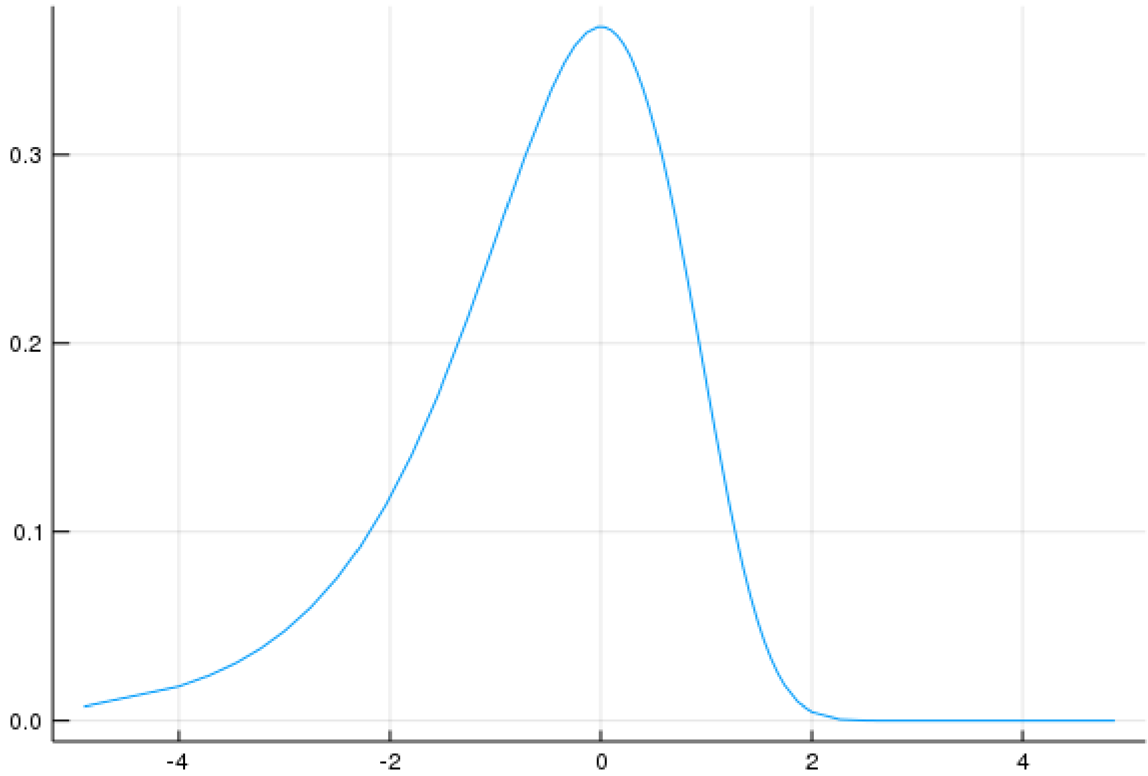
2. The inverse function of $\log(\theta)$ is known to be $h(\psi) = \exp(\psi)$, and the derivative w.r.t. ψ of $h(\psi)$ is

$$\frac{\partial h(\psi)}{\partial \psi} = \exp(\psi)$$

So we can write the PDF of ψ as

$$\begin{aligned} p_\psi(\psi) &= \frac{b^a}{\Gamma(a)} \exp(\psi)^{a-1} \exp(-b \exp(\psi)) \exp(\psi) \\ &= \frac{b^a}{\Gamma(a)} \exp(a\psi - b \exp(\psi)) \end{aligned}$$

And in the case that $a = b = 1$ the plot is:



3.2 Exercise 3.14

3.2.1 Data

3.2.2 Questions

1. Given $Y_1 \dots Y_n \sim \text{Bernoulli}(\theta)$ find the MLE of θ and $\frac{J(\theta)}{n}$.
2. Find a PDF $p_U(\theta)$ such that $\log p_U(\theta) = l(\theta|\mathbf{y})/n + c$ where c is a constant that does not depend on θ. Compute $-\partial^2 \log p_U(\theta)/\partial^2\theta$.
3. Obtain a PDF for θ that is proportional to $p_U(\theta)p(\mathbf{y}|\theta)$. Can this be considered a posterior for θ?
4. Repeat previous points with $Y_1 \dots Y_n \sim \text{Poisson}(\theta)$.

3.2.3 Solutions

1.

$$
\begin{aligned}
\hat{\theta}_{MLE} &= \arg\min_\theta \; l(\theta|\mathbf{y}) \\
&= \arg\min_\theta \; \log\left(\prod_i \theta^{y_i}(1-\theta)^{1-y_i}\right) \\
&= \arg\min_\theta \; \sum_i \log\left(\theta^{y_i}(1-\theta)^{1-y_i}\right) \\
&= \arg\min_\theta \; \log(\theta)\sum_i y_i + \log(1-\theta)\sum_i(1-y_i) \\
&= \arg\min_\theta \; \log(\theta)\sum_i y_i + n\log(1-\theta) - \log(1-\theta)\sum_i(y_i)
\end{aligned}
$$

To find the minimum we compute the zeros of $\partial l(\theta|\mathbf{y})/\partial\theta$

$$
\begin{aligned}
0 &= \frac{\partial l(\theta|\mathbf{y})}{\partial\theta} \\
&= \frac{\sum_i y_i}{\theta} - \frac{n}{1-\theta} + \frac{\sum_i y_i}{1-\theta} \\
&= \frac{\sum_i y_i - \theta\sum_i y_i - \theta n + \theta\sum_i y_i}{\theta(1-\theta)} \\
&= \frac{\sum_i y_i - \theta n}{\theta(1-\theta)}
\end{aligned}
$$

Thus, if $\theta \notin \{0,1\}$, $\hat{\theta}_{MLE} = \frac{\sum_i y_i}{n}$.

$$
\begin{aligned}
-\frac{\partial^2 l(\theta|\mathbf{y})}{n\partial^2\theta} &= -\frac{-n\theta(1-\theta) - (\sum_i y_i - n\theta)(1-2\theta)}{n\theta^2(1-\theta)^2} \\
&= -\frac{n\theta^2 - n\theta - \sum_i y_i + 2\theta\sum_i y_i + n\theta - 2n\theta^2}{n\theta^2(1-\theta^2)} \\
&= \frac{n\theta^2 - 2\theta\sum_i y_i + \sum_i y_i}{n\theta^2(1-\theta)^2} \\
&= \frac{\theta^2 - 2\theta\hat{\theta}_{MLE} + \hat{\theta}_{MLE}}{\theta^2(1-\theta)^2}
\end{aligned}
$$

2. The constrains on $p_U(\theta)$ imply that

$$
\begin{aligned}
p_U(\theta) &= c\sqrt[n]{\prod_i \theta^{y_i}(1-\theta)^{1-y_i}} \\
&= c\prod_i \theta^{y_i/n}(1-\theta)^{(1-y_i)/n} \\
&= c\,\theta^{\sum_i y_i/n}(1-\theta)^{\sum_i(1-y_i)/n} \\
&= c\,\theta^{\sum_i y_i/n}(1-\theta)^{1-\sum_i y_i/n}
\end{aligned}
$$

where c is the normalization constant.

$$
\begin{aligned}
-\frac{\partial^2 \log p_U(\theta)}{\partial^2\theta} &= -\frac{\partial^2 l(\theta|\mathbf{y})/n + c}{\partial^2\theta} \\
&= -\frac{\partial^2 l(\theta|\mathbf{y})}{\partial^2\theta}/n \\
&= \frac{\theta^2 - 2\theta\hat{\theta}_{MLE} + \hat{\theta}_{MLE}}{\theta^2(1-\theta)^2}
\end{aligned}
$$

3. Such a PDF would have the form

$$
\begin{aligned}
p(\theta|\mathbf{y}) &= c \cdot p_U(\theta)p(\mathbf{y}|\theta) \\
&= c\,\theta^{\sum_i y_i/n}(1-\theta)^{1-\sum_i y_i/n} \cdot \theta^{\sum_i y_i}(1-\theta)^{\sum_i(1-y_i)} \\
&= c\,\theta^{\sum_i y_i(1+\frac{1}{n})}(1-\theta)^{\sum_i(1-y_i)(1+\frac{1}{n})} \\
&= c\,\theta^{\sum_i y_i(1+\frac{1}{n})}(1-\theta)^{(n-\sum_i y_i)(1+\frac{1}{n})}
\end{aligned}
$$

Where $c = \int_{S_\theta} p(\theta,\mathbf{y})\partial\theta$ to guarantee that the PDF is proper. We can observe that the obtained PDF is a Beta distribution with parameters $\sum_i y_i(1+\frac{1}{n})+1$ and $(n-\sum_i y_i)(1+\frac{1}{n})+1$. It is a posterior because it is the product of a prior and a conditioned probability. Moreover, it is a case of conjugate prior.

4. Same steps: simplify the log-likelihood, find the first derivative and constrain to zero.

$$
\begin{aligned}
\hat{\theta}_{MLE} &= \arg\min_{\theta}\ l_{\text{Poisson}}(\theta|\mathbf{y}) \\
&= \arg\min_{\theta}\ \log\left(\prod_i \frac{\exp(-\theta)\theta^{y_i}}{y_i!}\right) \\
&= \arg\min_{\theta}\ \sum_i \log\frac{\exp(-\theta)\theta^{y_i}}{y_i!} \\
&= \arg\min_{\theta}\ -n\theta + \log\theta\sum y_i - \sum\log y_i!
\end{aligned}
$$

$$
\begin{aligned}
0 &= \frac{\partial l(\theta|\mathbf{y})}{\partial\theta} \\
&= -n + \frac{\sum_i y_i}{\theta}
\end{aligned}
$$

Thus $\hat{\theta}_{MLE} = \sum_i y_i/n$.

$$
\begin{aligned}
-\frac{\partial^2 l(\theta|\mathbf{y})}{n\partial^2\theta} &= \frac{\sum_i y_i}{n\theta^2} \\
&= \frac{\hat{\theta}_{MLE}}{\theta^2}
\end{aligned}
$$

In this case the PDF p_U would be

$$
\begin{aligned}
p_U(\theta) &\propto \sqrt[n]{\prod_i \exp(-\theta)\theta^{y_i}} \\
&= \exp(-\theta)\theta^{\sum_i y_i/n}
\end{aligned}
$$

Thus

$$
-\frac{\partial^2 \log p_U(\theta)}{\partial^2\theta} = \frac{\sum_i y_i}{n\theta^2}
$$

While the posterior is

$$
\begin{aligned}
p(\theta|\mathbf{y}) &\propto \exp(-\theta)\theta^{\sum_i y_i/n}\prod_i \exp(-\theta)\theta^{y_i} \\
&= \exp(-(n+1)\theta)\theta^{\sum_i y_i(1+\frac{1}{n})}
\end{aligned}
$$

This time the posterior is a Gamma distribution with parameters $\sum_i y_i(1+\frac{1}{n})$ and $n+1$.

Week 4

Monte Carlo simulations

4.1 Exercise 1

4.1.1 Data

- Data sequence:
  `c(1, 1, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0)`

4.1.2 Questions

Check that the sequence comes from an exchangeable process using the Bayesian p-value and the number of switch as statistic test.

4.1.3 Solutions

Solution: 0.016
Code:

```r
set.seed(10)

count_switch <- function(x) x %>% diff() %>% abs() %>% sum()

y <- c(1, 1, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0)
t_obs <- sum(abs(diff(y)))

# posterior hyperparameters
alpha <- 0.5 + sum(y)
beta  <- 0.5 + length(y) - sum(y)

niter <- 10000
ssize <- 20

theta <- rbeta(niter, alpha, beta)
t_rep <- map_dbl(theta, ~ rbinom(ssize, 1, .) %>% count_switch())

# Result
pval <- mean(t_rep <= t_obs)
cat("Bayesian p-val:", pval)
```

4.2 Exercise 2

4.2.1 Questions

We want a sample from a distribution $f(x)$ using another distribution $g(x)$ through an AR algorithm. Being assumed the value of M, proof that this algorithm is equivalent to a *standard* AR.

a) Generate $x \sim \mathrm{G}$

b) Generate $u \sim \mathrm{U}(0, Mg(x))$

c) Accept $y \stackrel{def}{=} x$ if $u < f(x)$

d) Otherwise go back to a).

4.2.2 Solutions

We can see that the two algorithms differ only for steps b) and c), where the difference is the Uniform distribution from which we sample u. To verify that they are equivalent we have to see if:

1. The proportion of values accepted near a generic x is the same, that is equal to $\frac{f(x)}{Mg(x)}$,
2. The efficiency is the same: the mean number of replications before rejecting a point is the same and it's equal to M.

In a generic u in the support of $\mathrm{U}(0, Mg(x))$ the proportion of values below u is

$$\mathrm{P}(U < u) = \frac{u}{Mg(x)}$$

therefor if $u \stackrel{def}{=} f(x)$ the proportion of accepted values given x is $\frac{f(x)}{Mg(x)}$, so 1 is verified.

Let K be the number of replications before accepting a value. So

$$K \sim \mathrm{Geom}(p) \text{ with } p \text{ probability of accepting at each replication}$$

We can compute the expected value of acceptance probability:

$$p = \mathrm{P}(U \leq f(x)) = \int_{S_X} g(x) \int_0^{f(x)} \frac{1}{Mg(x)} \partial u \partial x = \int_{S_X} g(x) \frac{f(x)}{Mg(x)} \partial x = \frac{1}{M} \int_{S_X} f(x) \partial x = \frac{1}{M}$$

So the expected value $\mathrm{E}[K] = \frac{1}{p} = M$ and even 2 is verified, and we can affirm that the two procedures are equivalent.