Homeworks of Bayesian Inference

(B004652)

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Last revision October 2, 2018

Week 1

Exchangeability and stochastic processes

1.1 Exercise 1

1.1.1 Data

- E_1, E_2, E_3, E_4, E_5, events of simple alternative, exchangeable
- $P(E_2) = \omega_1 = \frac{1}{2}$
- $P(E_3 \wedge E_5) = \omega_2 = \frac{1}{4}$
- $\omega_5 = \frac{\omega_3^5}{\binom{5}{3}} = \frac{\omega_1^5}{\binom{5}{1}} = \frac{1}{30}$

1.1.2 Questions

Compute:

1. $P(E_2 \wedge E_3 \wedge E_4) = \omega_3$
2. $P(E_1 \wedge E_2 \wedge E_3 \wedge E_4) = \omega_4$
3. $P(E_1 \wedge E_2 \wedge \bar{E}_3 \wedge \bar{E}_4 \wedge \bar{E}_5) = \frac{\omega_2^5}{\binom{5}{2}}$

1.1.3 Solutions

First we find ω_1^5 and ω_3^5:

$$\omega_1^5 = \frac{1}{30} \cdot \binom{5}{1} = \frac{1}{6}$$
$$\omega_3^5 = \frac{1}{30} \cdot \binom{5}{3} = \frac{1}{3}$$

Knowing that

$$\omega_h = \frac{1}{\binom{n}{h}} \sum_{r=h}^{n} \omega_r^n \binom{r}{h}$$

we can write that

$$\begin{aligned}\omega_1 &= \frac{\omega_1^5\binom{1}{1} + \omega_2^5\binom{2}{1} + \omega_3^5\binom{3}{1} + \omega_4^5\binom{4}{1} + \omega_5^5\binom{5}{1}}{\binom{5}{1}} \\ &= \frac{1}{6}\cdot\frac{1}{5} + \frac{2}{5}\omega_2^5 + \frac{1}{5}\cdot\frac{1}{3}\cdot 3 + \frac{1}{5}\cdot 4\omega_4^5 + \frac{1}{30} \\ &= \frac{8}{30} + \frac{2}{5}\omega_2^5 + \frac{4}{5}\omega_4^5\end{aligned}$$

$$\begin{aligned}\omega_2 &= \frac{\omega_2^5\binom{2}{2} + \omega_3^5\binom{3}{2} + \omega_4^5\binom{4}{2} + \omega_5^5\binom{5}{2}}{\binom{5}{2}} \\ &= \frac{1}{10}\omega_2^5 + \frac{1}{10}\cdot\frac{1}{10}\cdot 3 + \frac{1}{10}\cdot 6\omega_4^5 + \frac{1}{30} \\ &= \frac{2}{15} + \frac{1}{10}\omega_2^5 + \frac{3}{5}\omega_4^5\end{aligned}$$

Combining them:

$$\begin{aligned}&\begin{cases}\frac{2}{5}\omega_2^5 + \frac{4}{5}\omega_4^5 = \frac{1}{2} - \frac{8}{30} \\ \frac{1}{10}\omega_2^5 + \frac{3}{5}\omega_4^5 = \frac{1}{4} - \frac{2}{15}\end{cases} \\ \implies &\begin{cases}\omega_2^5 = \frac{7}{24} \\ \omega_4^5 = \frac{7}{48}\end{cases}\end{aligned}$$

Now we can obtain

$$\begin{aligned}\omega_3 &= \frac{\omega_3^5\binom{3}{3} + \omega_4^5\binom{4}{3} + \omega_5^5\binom{5}{3}}{\binom{5}{3}} &&= \frac{1}{3}\cdot\frac{1}{10} + \frac{7}{48}\cdot 4\frac{1}{10} + \frac{1}{30} = \frac{1}{8} \\ \omega_4 &= \frac{\omega_4^5\binom{4}{4} + \omega_5^5\binom{5}{4}}{\binom{5}{4}} &&= \frac{7}{48}\cdot\frac{1}{5} + \frac{1}{30} = \frac{1}{16}\end{aligned}$$

1.2 Exercise 2

1.2.1 Data

- Process of simple alternative $\{|E_n|\}$
- $P(E_1) = \omega_1 = \frac{1}{2}$
- $P(E_1 \wedge E_2) = \omega_2 = \frac{1}{4}$
- $P(E_1 \wedge E_2 \wedge E_3) = \omega_3 = \frac{1}{7}$
- $P(E_1 \wedge E_2 \wedge E_3 \wedge E_4) = \frac{3}{28}$

1.2.2 Questions

1. Could the 4 indicators $|E_1|$, $|E_2|$, $|E_3|$ and $|E_4|$ be the starting path of an exchangeable process?
2. Could it continue for at least one step?

1.2.3 Solutions

1. An exchangeable process must satisfy the condition

$$(-1)^{n-h}\Delta^{n-h}\omega_h \geq 0,\ \forall n, h \leq n$$

   Thus we compute

   - $(-1)^{4-1}\Delta^{4-1}\omega_1 = (-1) \cdot \Delta^3\omega_1 = \frac{1}{14} \geq 0$
   - $(-1)^{4-2}\Delta^{4-2}\omega_2 = (\quad 1) \cdot \Delta^2\omega_2 = \frac{1}{14} \geq 0$
   - $(-1)^{4-3}\Delta^{4-3}\omega_3 = (-1) \cdot \Delta\ \omega_3 = \frac{1}{28} \geq 0$
   - $(-1)^{4-4}\Delta^{4-4}\omega_4 = (\quad 1) \cdot \quad \omega_4 = \frac{3}{28} \geq 0$

   Thus we can affirm that the process is exchangeable.

2.

Week 2

Conjugate priors and posterior distributions

2.1 Exercise 2.3

2.1.1 Data

- $p(x, y, z) \propto f(x, z)\ g(y, z)\ h(z)$

2.1.2 Questions

Prove that:

1. $p(x|y, z) \propto f(x, z)$
2. $p(y|x, z) \propto g(y, z)$
3. X and Y conditionally independent, given Z.

2.1.3 Solutions

We know by definition that

$$p(x|y, z) = \frac{p(x, y, z)}{p(y, z)}$$

and also that

$$p(y, z) = \int_{S_X} p(x, y, z)\partial x \propto \int_{S_X} f(x, z)g(y, z)h(z)\partial x = g(y, z)h(z) \int_{S_X} f(x, z)\partial x$$

Where S_X is the support of the r.v. X. Then we can write

$$\begin{aligned} p(x|y, z) &= \frac{f(x, z)g(y, z)h(z)}{g(y, z)h(z) \int_{S_X} f(x, z)\partial x} \\ &= \frac{f(x, z)}{\int_{S_X} f(x, z)\partial x} \end{aligned}$$

But $\int_{S_X} f(x,z)\partial x$ is constant given z, so we can say

$$p(x|y,z) \propto f(x,z)$$

as we wanted to show.
Similarly, we can write

$$\begin{aligned} p(y|x,z) &= \frac{p(x,y,z)}{p(x,z)} \\ &= \frac{f(x,z)g(y,z)h(z)}{f(x,z)h(z)\int_{S_Y} g(y,z)\partial y} \\ &= \frac{g(y,z)}{\int_{S_Y} g(y,z)\partial y} \\ &\propto g(y,z) \end{aligned}$$

To show that $X \perp Y$ given Z we have to prove that $p(y|z,x) = p(y|z)$, so:

$$\begin{aligned} p(y|z) &= \frac{p(y,z)}{p(z)} \\ &= \frac{\int_{S_X} f(x,z)g(y,z)h(z)\partial x}{\int_{S_X}\int_{S_Y} f(x,z)g(y,z)h(z)\partial y\partial x} \\ &= \frac{g(y,z)h(z)\int_{S_X} f(x,z)\partial x}{h(z)\int_{S_X} f(x,z)\partial x\int_{S_Y} g(y,z)\partial y} \\ &= \frac{g(y,z)}{\int_{S_Y} g(y,z)\partial y} \\ &= p(y|x,z) \end{aligned}$$

2.2 Exercise 3.5

2.2.1 Data

- $p(y|\phi) = c(\phi)h(y)\exp(\phi t(y))$
- $p_1(\theta)\dots p_k(\theta)$ conjugate priors
- $\tilde{p}(\theta) = \sum_{k=1}^{K} \omega_k p_k(\theta)$ where $\omega_k > 0$ and $\sum_k \omega_k = 1$

2.2.2 Questions

1. $p(\theta|y)$ as a function of $p(y|\theta)$ and $\tilde{p}$
2. Previous question but in the case that $\theta \sim$ Pois and $p_1 \dots p_k \sim \Gamma$

2.2.3 Solution

For the Bayes rule:

$$p(\theta|y) \propto p(y|\theta) \cdot p(\theta) = p(y|\theta) \cdot \tilde{p}(\theta)$$

In the particular case that it's the sample comes from a Poisson distribution and the prior is a mixture of Gamma distributions:

$$\begin{aligned} p(\theta|y) &\propto p(y|\theta) \cdot \tilde{p}(\theta) \\ &\propto \theta^k \exp(\theta) \sum_k w_k x^{\alpha_k - 1} \exp(-\frac{x}{\beta_k}) \end{aligned}$$