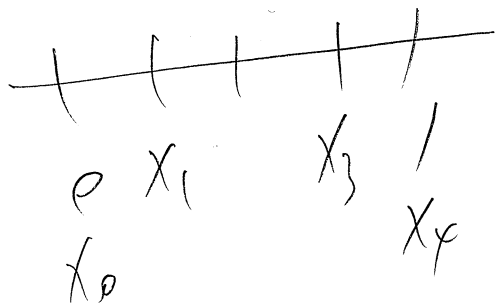
Part 1-1 (Ch 3-8)

(a) $\int_0^1 x^3 dx = \frac{1}{4} x^4 \Big|_0^1 = \frac{1}{4}$

(b)



$X_i = \frac{i}{4}$

$$\sum f(X_i)(X_i - X_{i-1}) = \frac{1}{4}\left[f(\tfrac{1}{4}) + \cdots + f(1)\right]$$
$$= \frac{1}{4}\left[(\tfrac{1}{4})^3 + (\tfrac{2}{4})^3 + (\tfrac{3}{4})^3 + 1^3\right]$$
$$= 0.39$$

$$\sum f(X_{i-1})(X_i - X_{i-1}) = \frac{1}{4}\left[0^3 + (\tfrac{1}{4})^3 + (\tfrac{2}{4})^3 + (\tfrac{3}{4})^3\right]$$
$$= 0.14$$

(c) Both of them approches to the integral as $n \to \infty$.

Part 1-2 (Ch 3-9)

(a) $\int_0^1 x \sin\frac{\pi}{x}\, dx = -0.183$

(b) $\sum_{i=1}^{4} f(x_i)(x_i - x_{i-1})$

$= \frac{1}{4}\left[\frac{1}{4}\sin(4\pi) + \frac{1}{3}\sin(3\pi) + \frac{1}{2}\sin(2\pi) + \sin\pi\right]$

$= 0$

$\sum_{i=1}^{4} f(x_{i-1})(x_i - x_{i-1})$

$= \frac{1}{4}\left[0 + \frac{1}{4}\sin 4\pi + \frac{1}{3}\sin 3\pi + \frac{1}{2}\sin 2\pi\right]$

$= 0$

(c) Both of them approach to the integral

as $n \to \infty$.

Part 1-3 (Ch 9-1)

(a) 1) $\sum_{i=1}^{n} W(t_{i-1})^2 (W(t_i) - W(t_{i-1}))$

2) $\sum_{i=1}^{n} W(t_i)^2 \; (W(t_i) - W(t_{i-1}))$

3) $\sum_{i=1}^{n} \left(\frac{W(t_{i-1}) + W(t_i)}{2}\right)^2 (W(t_i) - W(t_{i-1}))$

(b) $\sum_{i=1}^{n} W(t_{i-1})^2 \; (W(t_i) - W(t_{i-1}))$

(d) $E\left[\sum_{i=1}^{n} W(t_{i-1})^2 \; (W(t_i) - W(t_{i-1}))\right]$

$= \sum_{i=1}^{n} E(W(t_{i-1})^2) \, E(W(t_i) - W(t_{i-1}))$

$= 0.$

Part 1 - 4 (Ch 9 - 2)

$$t_j W(t_j) - t_{j-1} W(t_{j-1})$$

$$= \left(t_j W(t_j) - t_j W(t_{j-1})\right) + \left(t_j W(t_{j-1}) - t_{j-1} W(t_{j-1})\right)$$

$$\therefore \Sigma\, t_j W(t_j) - t_{j-1} W(t_{j-1})$$

$$= \Sigma\, t_j (W(t_j) - W(t_{j-1})) + \Sigma\, (t_j - t_{j-1}) W(t_{j-1})$$

Part 1-5 (Ch 9-3)

$$\int_0^T t\,dW(t)$$

$$\sum_{i=0}^{n-1} t_i(W(t_{i+1})-W(t_i))$$

$$= \sum_{i=0}^{n-1}\left(t_{i+1}W(t_{i+1})-t_iW(t_i)\right) - \sum_{i=0}^{n-1} W(t_{i+1})(t_{i+1}-t_i)$$

$$= TW(T) - \sum_{i=0}^{n-1} W(t_{i+1})(t_{i+1}-t_i)$$

$$\therefore \int_0^T t\,dW(t) = TW(T) - \int_0^T W(t)\,dt.$$

Part 1 - 6 (Ch 6. -2)

$X_n = \sum B_i$

(a) $E(X_4 | \mathcal{I}_1) = X_1$

$E(X_4 | \mathcal{I}_2) = X_2$

$E(X_4 | \mathcal{I}_4) = X_4$

(b) Yes, by (a)

(c) $\tilde{X}_i$ is not a martingale

Since $\begin{pmatrix} E(\tilde{X}_2 | \mathcal{I}_1) = B_1 + 1 + \sqrt{2} \\ \tilde{X}_1 = B_1 + 1 \end{pmatrix} \Rightarrow E(\tilde{X}_2 | \mathcal{I}_1) \neq \tilde{X}_1$.

Part 1-7 (Ch 6-3)

(a) $X_t = 2W_t + t$

$s<t$, $E(X_t | \mathcal{F}_s) = E(2W_t + t | \mathcal{F}_s)$

$= 2W_s + t$

$X_s = 2W_s + s$

$\therefore$ Not a martingale

(b) We can use the fact that $W_t^2 - t$ is a martingale.

$s<t$, $E(X_t | \mathcal{F}_s) = E(W_t^2 - t + t | \mathcal{F}_s)$

$= W_s^2 - s + t$

$X_s = W_s^2$

$\therefore$ Not a martingale

(c) $dX_t = (2tW_t dt + t^2 dW_t) - 2tW_t dt$ (Ito's lemma)

$= t^2 dW_t$

$\therefore$ Martingale

Part 1 - 8 (Ch 6-4)

(1) $M_T = W_T$

$W_T = \int_0^T dW_t$

$\therefore g = 1$

(2) $M_T = W_T^2 - T$

$d(W_t^2 - t) = 2W_t dW_t$

$\Rightarrow W_T^2 - T = \int_0^T 2W_t dW_t$

$\therefore g = 2W_t$

(3) $M_T = e^{W_T - \frac{1}{2}T}$

$d(e^{W_t - \frac{1}{2}t}) = e^{W_t - \frac{1}{2}t} dW_t$

$\Rightarrow e^{W_T - \frac{1}{2}T} = 1 + \int_0^T e^{W_t - \frac{1}{2}t} dW_t$

$\therefore g = e^{W_t - \frac{1}{2}t}$

Pare 1-9 (Ch10-1)

(a) 1) $f = W_t^2$

$f = x^2$, $f_t = 0$, $f_x = 2x$, $f_{xx} = 2$

$a = 0$, $b = 1$

Ito → $df = \frac{1}{2} \cdot 2\, dt + 2W_t\, dW_t$

2) $f = \sqrt{W_t}$

$f = x^{\frac{1}{2}}$, $f_t = 0$, $f_x = \frac{1}{2}x^{-\frac{1}{2}}$, $f_{xx} = -\frac{1}{4}x^{-\frac{3}{2}}$.

$a = 0$. $b = 1$

Ito → $df = \frac{1}{2}\left(-\frac{1}{4}\right) W_t^{-\frac{3}{2}} dt + \frac{1}{2} W_t^{-\frac{1}{2}} dW_t$

(b) $f = e^{W_t^2}$

$f = e^{x^2}$, $f_t = 0$, $f_x = 2xe^{x^2}$,

$f_{xx} = 2e^{x^2} + 4x^2e^{x^2}$

$a = 0$, $b = 1$

Ito → $df = \frac{1}{2} 2 W_t e^{W_t^2} dt$

$+ \left(2e^{W_t^2} + 4W_t^2 e^{W_t^2}\right) dW_t$

Plane 2-9 (Ch 10 - 1)

c) 1) $f = e^{\sigma W_t - \frac{1}{2}\sigma^2 t}$

$f = e^x,\ f_t = 0,\ f_x = f_{xx} = e^x,$

$a = -\frac{1}{2}\sigma^2,\ b = \sigma$

$$df = \left[-\frac{1}{2}\sigma^2 e^{\sigma W_t - \frac{1}{2}\sigma^2 t} + \frac{1}{2}\sigma^2 e^{\sigma W_t - \frac{1}{2}\sigma^2 t}\right] dt$$

$$+ \sigma e^{\sigma W_t - \frac{1}{2}\sigma^2 t} dW_t$$

$$= \sigma e^{\sigma W_t - \frac{1}{2}\sigma^2 t} dW_t$$

2) $f = e^{\sigma W_t}$

$f = e^x,\ f_t = 0,\ f_x = f_{xx} = e^x$

$a = 0,\ b = \sigma$

$$df = \frac{1}{2}\sigma^2 e^{\sigma W_t} dt + \sigma e^{\sigma W_t} dW_t$$

d) $dg = W_t\, dt$

Part 1 - 10 (Ch 10-2)

(a) $a=0,\ b=1,\ f=x^4,\ f_x=4x^3,\ f_{xx}=12x^2$

$$dX_t=\left(\frac{1}{2}\cdot 12W^2\right)dt+4W^3\,dW_t$$

$$=6W_{t1}^2\,dt+4W_{t1}^3\,dW_{t1}$$

(b) $dX_t=2(W_{t1}+W_{t2})(dW_{t1}+dW_{t2})+2\,dt$

(c) $a=0,\ b=1,\ f=t^2+e^x,\ f_t=2t,\ f_x=e^x,\ f_{xx}=e^x$

$$dX_t=\left(2t+\frac{1}{2}e^{W_{t2}}\right)dt+e^{W_{t2}}dW_{t2}$$

(d) $a=0,\ b=1,\ f=e^{t^2+x},\ f_t=2tf,\ f_x=f_{xx}=f$

$$dX_t=\left(2tX_t+\frac{1}{2}X_t\right)dt+X_t\,dW_{t2}$$

Part 1-11 (Ch 10-3)

a) $S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}$

$f = S_0 e^x, \quad f_t = 0, \quad f_x = f_{xx} = S_0 e^x$

$a = \mu - \frac{1}{2}\sigma^2, \quad b = \sigma$

$$df = \left[(\mu - \tfrac{1}{2}\sigma^2) S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t} + \tfrac{1}{2}\sigma^2 S_0 e^{\boxed{\ }}\right] dt$$

$$+ \sigma S_0 e^{\boxed{\ }} dW_t$$

$$= \cancel{S_t}\, S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \cancel{\ }\sigma W_t} (\mu\, dt + \sigma\, dW_t)$$

$dS_t = \mu S_t\, dt + \sigma S_t\, dW_t$

b) μ

c) If $S_t = S_0 e^{\mu t + \sigma W_t}$

$f = S_0 e^x,$

$a = \mu, \quad b = \sigma$

Page 1-11 (Ch 10-3)

$$df = \left(\mu S_0 e^{\mu t + \sigma W_t} + \frac{1}{2}\sigma^2 S_0 e^{\square}\right) dt$$
$$+ \sigma S_0 e^{\square} dW_t$$

$$= \left(\mu + \frac{1}{2}\sigma^2\right) S_t\, dt + \sigma S_t\, dW_t$$

In this case, $\mu + \frac{1}{2}\sigma^2$.

Part 1-12

$\sigma = 0.1$

$\Rightarrow$ Answer $= (4\sigma^2)^{\frac{1}{2}} = \underline{\underline{0.2}}$

Part 1-13

$E(Y_{n+1} | \mathcal{F}_n) = U_n Y_n + V_n$

Find a_n, b_n s.t. $M_n := a_n Y_n + b_n$ is martingale

$E(M_{n+1} | \mathcal{F}_n) = M_n$

$(LHS) = E(a_{n+1} Y_{n+1} + b_{n+1} | \mathcal{F}_n)$

$= a_{n+1} (U_n Y_n + V_n) + b_{n+1}$

$(RHS) = a_n Y_n + b_n$

$(LHS) = (RHS) \iff a_{n+1} U_n = a_n \ , \ a_{n+1} V_n + b_{n+1} = b_n$

$a_{n+1} U_n = a_n \Rightarrow a_n = a_0 \prod_{k=1}^{n-1} U_k^{-1} \quad (n \geq 1)$

$a_{n+1} V_n + b_{n+1} = b_n \Rightarrow b_n = b_0 - \sum_{k=0}^{n-1} a_{k+1} V_k \quad (n \geq 1)$

Part 2-1

(1) $E(X_{n+1} | X_1, \cdots, X_n) = X_n \quad \forall n$

(2) $E(M_t^2 - t | \mathcal{I}_s) = M_s^2 - s$?

$(LHS) = E((M_t - M_s + M_s)^2 - t | \mathcal{I}_s)$

$= E[(M_t - M_s)^2 | \mathcal{I}_s] + 2E[(M_t - M_s)M_s | \mathcal{I}_s]$

$+ E(M_s^2 | \mathcal{I}_s) - t$

$= t - s + 2 \times 0 + M_s^2 - t$

$= M_s^2 - s$

$\therefore$ Martingale

(3) Since τ is almost surely bounded,

$E(M_\tau^2 - \tau) = E(M_1^2 - 1) = 1 - 1 = 0$

$\therefore E(M_\tau^2) = \tau$

Part 2-2

$dX = (8-2X)\,dt + 8\,dW_t$

$f = \frac{1}{x}$, $f_X = -\frac{1}{X^2}$, $f_{XX} = \frac{2}{X^3}$

$dY = \left(-\frac{1}{X^2}(8-2X) + \frac{1}{2}\frac{2}{X^3}8^2\right)dt - \frac{1}{X^2}8\,dW_t$

$= (-8Y^2 + 2Y + 64Y^3)\,dt - 8Y^2\,dW_t$

$\therefore \alpha(y) = -8y^2 + 2y + 64y^3$

$\alpha(\frac{1}{2}) = 7$

Part 2-5

$$\alpha(r(t), t, T) = r(t)$$

$$\therefore \alpha(0.04, 2, 5) = \underline{\underline{0.04}}$$

Part 2-7

$$Z = e^{\int_0^t f\,dW - \frac{1}{2}\int_0^t f^2\,ds}$$

$$X = \int_0^t f\,dW - \frac{1}{2}\int_0^t f^2\,ds$$

$$dX = f(t)\,dW_t - \frac{1}{2}f(t)^2\,dt$$

$$a = -\frac{1}{2}f(t)^2, \quad b = f(t)$$

$$f = e^x, \quad f' = e^x, \quad f'' = e^x$$

$$dZ = \Big(\underbrace{e^x \cdot \left(-\frac{1}{2}f(t)^2\right) + \frac{1}{2}f(t)^2 e^x}_{=0}\Big)\,dt + \square\,dW_t$$

$\therefore$ Martingale

Part 2-B

$a=0, b=1$

$f=x^3$, $f_x=3x^2$, $f_{xx}=6$

$d(W_t^3)=\underbrace{\left(\frac{1}{2}6W_t\right)}_{\neq 0}dt+3W_t^2dW_t$

$\therefore$ Not a martingale

Part 2-9

$$B = e^{-x(T-t)}$$

$$f(x) = e^{-(T-t)x}$$

$$f_t = xf, \quad f_x = -(T-t)f, \quad f_{xx} = (T-t)^2 f$$

By Ito's lemma,

$$dB = \left(x - (T-t) \cdot a(x_0 - x) + \frac{1}{2}(T-t)^2 s^2 x^2\right) B\, dt$$
$$- (T-t)\, s\, x\, B\, dz$$