Professor S.J. Deng

ISyE/Math 6759 Stochastic Processes in Finance – I

Homework Set 2

Please write down your name in the format of 'Last name, First name'. Turn in the hard copy.

Note: This homework is important.

Problem 1 (Probability)

You are bidding for a firm whose unknown true value is uniformly distributed between 0 and 1. Although you do not know the true value S of the firm, you do know that as soon as people learn that you have made a bid this news will cause the value to double to $2S$. Your bid, however, will be accepted only if it is at least as large as the original value of the firm. How do you bid so as to maximize your expected payoff?

Problem 2 (Probability)

You and I are to play a game. You roll a die until a number other than a one appears. When such a number appears for the first time, I pay you the same number of dollars as there are dots on the upturned face of the die, and the game ends. What is the expected payoff to this game?

Problem 3 (Probability)

You have a large jar containing 999 fair pennies and one two-headed penny. Suppose you pick one coin out of the jar and flip it 10 times and get all heads. What is your opinion on the type of penny you picked up?

Problem 4 (Probability)

Four cards are shuffled and placed face down in front of you. Their faces (hidden) display the four elements: water, earth, wind, and fire. You are to turn the cards over one card at a time until you either win or lose. You win if both the water and earth cards are turned over. You lose if the fire card is turned over. What is the probability of winning?

Problem 5 (Probability)

Assume that 10% of the population is infected by a virus. And due to some reasons, the test machine is not perfect. Errors can happen when people are tested.

Assume: A=test result is positive B= this person is infected.

P(A|B)=0.95 P(~A|B)=0.05

P(A|~B)=0.01 P(~A|~B)=0.99

Now someone did 3 times of test independently, and turns out to be positive twice and negative once, what's the probability that he is actually infected?

You may find this formula useful:

P(Bi|A)=P(Bi)*P(A|Bi)/Sum(P(Bj)*P(A|Bj)) (summation over all j)

Problem 6 (central limit theorem)

A new cinema is under construction and the total number of the seats is to be determined. Assume that in this area, 1600 people will go to cinema everyday and for the probability for each person to choose this cinema is 3/4. Following are the requirements for determining the seat quantity q:

1. q is as large as possible.
2. The probability of "there are over 200 empty seats in one day" is no more than 0.1

Determine q.

Problem 7-8 (State-Price Vector Pricing, Arbitrage condition, Future pricing)

Neftci's Book Chapter 2, P30, Exercise 2 and Exercise 3 (a), (b)

2. In an economy there are two states of the world and four assets. You are given the following prices for three of these securities in different states of the world:

	Price		Dividend	
	State 1	State 2	State 1	State 2
Security A	120	70	4	1
Security B	80	60	3	1
Security C	90	150	2	10

"current" prices for A, B, C are 100, 70, and 180, respectively.

(a) Are the "current" prices of the three securities arbitrage-free?
(b) If not, what type of arbitrage portfolio should one form?
(c) Determine a set of arbitrage-free prices for securities A, B, and C.
(d) Suppose we introduce a fourth security, which is a one-period futures contract written on B. What is its price?

(e) Suppose a put option with strike price $K = 125$ is written on C. The option expires in period 2. What is its arbitrage-free price?

3. Consider a stock S_t and a plain vanilla, at-the-money, put option written on this stock. The option expires at time $t+\Delta$, where Δ denotes a small interval. At time t, there are only two possible ways the S_t can move. It can either go *up* to $S^u_{t+\Delta}$, or go *down* to $S^d_{t+\Delta}$. Also available to traders is risk-free borrowing and lending at annual rate r.

(a) Using the arbitrage theorem, write down a three-equation system with *two* states that gives the arbitrage-free values of S_t and C_t.
(b) Now plot a two-step binomial tree for S_t. Suppose at every node of the tree the markets are arbitrage-free. How many three-equation systems similar to the preceding case could then be written for the entire tree?

Problem 9 (Binomial Tree)

Neftci's Book Chapter 2, P31, Exercise 4

Answer all questions with the initial price set to be $S_0 = 100$, strike price K = 100.

4. A four-step binomial tree for the price of a stock S_t is to be calculated using the up and down ticks given as follows:

$$u = 1.15 \qquad\qquad d = \frac{1}{u}$$

These up and down movements apply to one-month periods denoted by $\Delta = 1$. We have the following dynamics for S_t,

$$S_{t+\Delta}^{up} = uS_t \qquad S_{t+\Delta}^{down} = dS_t,$$

where *up* and *down* describe the two states of the world at each node.

Assume that time is measured in months and that $t = 4$ is the expiration date for a European call option C_t written on S_t. The stock does not pay any dividends and its price is expected (by "market participants") to grow at an annual rate of 15%. The risk-free interest rate r is known to be constant at 5%.

(a) According to the data given above, what is the (approximate) annual volatility of S_t if this process is known to have a log-normal distribution?
(b) Calculate the four-step binomial trees for the S_t and the C_t.
(c) Calculate the arbitrage-free price C_o of the option at time $t = 0$.

(d) Using the above setting, work out all hedging portfolios at each node of the first three periods, specifically, period t=0 => t=1, period t=1 => t=2 and period t=2 => t=3.

Problem 10

Neftci's Book Chapter 2, P32, Exercise 5

Change r=5% to r=0.4%

5. You are given the following information concerning a stock denoted by S_t.

- Current value = 102.
- Annual volatility = 30%.
- You are also given the spot rate $r = 5\%$, which is known to be constant during the next 3 months.

It is hoped that the dynamic behavior of S_t can be approximated reasonably well by a binomial process if one assumes observation intervals of length 1 month.

(a) Consider a European call option written on S_t. The call has a strike price $K = 120$ and an expiration of 3 months. Using the S_t and the risk-free borrowing and lending, B_t, construct a portfolio that replicates the option.
(b) Using the replicating portfolio price this call.
(c) Suppose you sell, over-the-counter, 100 such calls to your customers. How would you hedge this position? Be precise.
(d) Suppose the market price of this call is 5. How would you form an arbitrage portfolio?

Problem 11

Neftci's Book Chapter 2, P32, Exercise 6

1) $S_{t+1} - S_t = \mu S_t + \sigma S_t \varepsilon_t$; $\Delta t = 1$ **year.**

6. Suppose you are given the following data:

- Risk-free yearly interest rate is $r = 6\%$.
- The stock price follows:

$$S_t - S_{t-1} = \mu S_t + \sigma S_t \epsilon_t,$$

where the ϵ is a serially uncorrelated binomial process assuming the following values:

$$\epsilon = \begin{cases} +1 & \text{with probability } p \\ -1 & \text{with probability } 1 - p. \end{cases}$$

The $0 < p < 1$ is a parameter.

- Volatility is 12% a year.
- The stock pays no dividends and the current stock price is 100.

Now consider the following questions.

(a) Suppose μ is equal to the risk-free interest rate:

$$\mu = r$$

and that the S_t is arbitrage-free. What is the value of p?

(b) Would a $p = 1/3$ be consistent with arbitrage-free S_t?

(c) Now suppose μ is given by:

$$\mu = r + \text{risk premium}$$

What do the p and ϵ represent under these conditions?

(d) Is it possible to determine the value of p?

2) using the value of r and σ in part 1)

Neftci's Book Chapter 2, P32, Exercise 7

7. Using the data in the previous question, you are now asked to approximate the current value of a European call option on the stock S_t. The option has a strike price of 100, and a maturity of 200 days.

(a) Determine an appropriate time interval Δ, such that the binomial tree has 5 steps.
(b) What would be the implied u and d?
(c) What is the implied "up" probability?
(d) Determine the tree for the stock price S_t.
(e) Determine the tree for the call premium C_t.

Problem 12 (State-Price Vector)

Now we consider an economy of three states and four assets with prices in different states as follows:

	Price		
	State 1	**State 2**	**State 3**
Security A	120	70	80
Security B	80	60	50
Security C	90	150	190
Security D	30	20	30

"Current" price for A, B, C are 100, 70 and 180, respectively

Question: Calculate the no arbitrage price of D and give the replicating portfolio

Remark: As you have noticed, this problem is similar to Problem 7. I put this problem here to help you become very comfortable with portfolio replication and matrix calculation**.**

Problem 13

Let H (Heads) and T (Tails) denote the two outcomes of a random experiment of tossing a fair coin. Suppose I toss the coin infinite many times and divide the outcomes (which are infinite sequences of Heads and Tails) into two types of events: (a) the portion of H or T of is exactly one half (e.g. HTHTHTHT… or HHTTHHTT...) (b) the portion of H or T is not one half (i.e. the complement of event (a). e.g. HTTHTTHTT…). What are the probabilities for events (a) and (b), respectively?

Problem 14

Suppose there are 100 strings, each of the string, of course, has 2 ends. Then you randomly choose 2 of the ends and tie them together. Each end will be tied only once and the process repeats until there are no free ends left. (i.e. It will lead to 100 randomly chosen pairs of tied ends.) Let L be the number of resulting loops. What is E(L)?

Problem 15 Russian Roulettes

a. Suppose two players play a traditional Russian Roulettes game. One bullet is put into a 6-revolver and the barrel is randomly spun so that there is equal chance for each chamber to be under the hammer. Two players take turns to pull the trigger against themselves until one kills him or herself. Under such rules, would you rather go first or second? What's the probability of survival to go first?
b. Now the rule is changed, the barrel gets spun after each shot. Now would you rather go first or second? What's the probability of survival for each choice?
c. Suppose two bullets instead of one are randomly put into the chamber. Your opponent went first and survived the shot. Now you are given the chance to spin the barrel.

Should you do it or not?

d. What if the two bullets are consecutively put into the chamber and the barrel is spun (i.e. two bullets are next to each other), should you spin again after your opponent survived the first round?