ISyE/Math 6759 Stochastic Processes in Finance – I
Homework Set 5

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Please remember to write down your name and GTID in the submitted homework.

Problem 1 Neftci's Book Chapter 13, P228, Exercise 2

Suppose X_t is a geometric Brownian motion process with drift μ and diffusion parameter σ. When would the $e^{-rt}X_t$ be a martingale? That is, when would the following equality hold:

$$\mathbb{E}\left[X_t \mid X_s, s < t\right] = e^{r(t-s)}X_s$$

Problem 2 Neftci's Book Chapter 13, P228, Exercise 3

Consider

$$Z_t = e^{-rt}X_t$$

where X_t is an exponential Wiener process:

$$X_t = e^{W_t}$$

(a) Calculate the expected value of the increment $dZ(t)$.

(b) Is Z_t a martingale?

(c) Calculate $\mathbb{E}\left[Z_t\right]$. How would you change the definition of X_t to make Z_t a martingale?

(d) How would $\mathbb{E}\left[Z_t\right]$ then change?

Problem 3 Neftci's Book Chapter 14, P252, Exercise 2

Assume that the return R_t of a stock has the following log-normal distribution for fixed t:

$$\log\left(R_t\right) \sim \mathcal{N}\left(\mu, \sigma^2\right)$$

Suppose we let the density of $\log\left(R_t\right)$ be denoted by $f\left(R_t\right)$ and hypothesize that $\mu = 0.17$. We further estimate the variance as $\sigma^2 = 0.09$.

(a) Find a function $\xi\left(R_t\right)$ such that under the density, $\xi\left(R_t\right) f\left(R_t\right)$, $\log\left(R_t\right)$ has a mean equal to the risk-free rate $r = 0.05$.

(b) Find a $\xi(R_t)$ such that $\log(R_t)$ has mean zero.

(c) Under which probability is it "easier" to calculate

$$\mathbb{E}\left[\log(R_t)^2\right]$$

(d) Is the variance different under these probabilities?

Problem 4 Neftci's Book Chapter 15, P267, Exercise 1

In this exercise we use the Girsanov theorem to price the chooser option. The chooser option is an exotic option that gives the holder the right to choose, at some future date, between a call and a put written on the same underlying asset. Let the T be the expiration date, S_t be the stock price, K the strike price. If we buy the chooser option at time t, we can choose between call or put with strike K, written on S_t. At time t the value of the call is

$$C(S_t, t) = e^{-r(T-t)}\mathbb{E}\left[\max(S_T - K, 0) \mid I_t\right]$$

whereas the value of the put is:

$$P(S_t, t) = e^{-r(T-t)}\mathbb{E}\left[\max(K - S_T, 0) \mid I_t\right]$$

and thus, at time t, the chooser option is worth:

$$H(S_t, t) = \max\left[C(S_t, t), P(S_t, t)\right]$$

(a) Using these, show that:

$$C(S_t, t) - P(S_t, t) = S_t - Ke^{-r(T-t)}$$

Does this remind you of a well-known parity condition?

(b) Next, show that the value of the chooser option at time t is given by

$$H(S_t, t) = \max\left[C(S_t, t), C(S_t, t) - S_t \right.$$
$$\left. + Ke^{-r(T-t)}\right]$$

(c) Consequently, show that the option price at time zero will be given by

$$H(S_0, 0) = C(S_0, 0) + e^{-rT}\mathbb{E}$$
$$\left[\max\left[K - S_0 e^{rT} e^{\sigma W_t - \frac{1}{2}\sigma^2 t}, 0\right]\right]$$

where S_0 is the underlying price observed at time zero.

(d) Now comes the point where you use the Girsanov theorem. How can you exploit the Girsanov theorem and evaluate the expectation in the above formula easily?

(e) Write the final formula for the chooser option.

Problem 5 Neftci's Book Chapter 15, P268, Exercise 2

In this exercise we work with the BlackScholes setting applied to foreign currency denominated assets. We will see a different use of Girsanov theorem. [For more details see Musiela and Rutkowski (1997).] Let r, f denote the domestic and the foreign risk-free rates. Let S_t be the exchange rate, that is, the price of one unit of foreign currency in terms of domestic currency. Assume a geometric process for the dynamics of S_t :

$$dS_t = (r - f)S_t dt + \sigma S_t dW_t$$

(a) Show that

$$S_t = S_0 e^{\left(r-f-\frac{1}{2}\sigma^2\right)t+\sigma W_t}$$

where W_t is a Wiener process under probability $\mathbb{P}$.

(b) Is the process

$$\frac{S_t e^{ft}}{S_0 e^{rt}} = e^{\sigma W_t - \frac{1}{2}\sigma^2 t}$$

a martingale under measure $\mathbb{P}$?

(c) Let $\mathbb{Q}$ be the probability

$$\mathbb{Q}(A) = \int_A e^{\sigma W_T - \frac{1}{2}\sigma^2 T} dP$$

What does Girsanov theorem imply about the process, $W_t - \sigma t$, under $\mathbb{Q}$?

(d) Show, using Ito formula, that

$$dZ_t = Z_t \left[\left(f - r + \sigma^2\right) dt - \sigma dW_t\right]$$

where $Z_t = 1/S_t$.

(e) Under which probability is the process $\mathrm{Z}_t e^{rt}/e^{ft}$ a martingale?

(f) Can we say that $\mathbb{Q}$ is the arbitrage-free measure of the foreign economy?

Problem 6

Two non-dividend-paying assets have the following price processes:

$$\frac{dS_1(t)}{S_1(t)} = 0.08dt + 0.2dZ(t)$$
$$\frac{dS_2(t)}{S_2(t)} = 0.0925dt - 0.25dZ(t)$$

Where $Z(t)$ is a standard Brownian motion. An investor is to synthesize the risk-free asset by allocating 1000 between the two assets.

Determine the amount to be invested in the first asset, S_1.

Problem 7

Consider an arbitrage-free securities market model, in which the risk-free interest rate is constant. There are two non-dividend-paying stocks whose price processes are

$$S_1(t) = S_1(0)e^{0.1t+0.2Z(t)}$$
$$S_2(t) = S_2(0)e^{0.125t+0.3Z(t)}$$

Where $Z(t)$ is a standard Brownian motion and $t \geq 0$. Determine the continuously compounded risk-free interest rate.

Problem 8

Assume the Black-Scholes framework. For $t \geq 0$, let $S(t)$ be the time-t price of a non-dividend-paying stock. You are given:

1. $S(0) = 0.5$
2. The stock price process is $\frac{dS(t)}{S(t)} = 0.05dt + 0.2dZ(t)$, where $Z(t)$ is a standard Brownian motion.
3. $\mathbb{E}\left[S(1)^a\right] = 1.4$, where a is a negative constant.
4. The continuously compounded risk-free interest rate is 3%. Consider a contingent claim that pays $S(1)^a$ at time 1.

Calculate the time-0 price of the contingent claim.

Problem 9 Exercise 6.1

Consider the standard Black-Scholes model and a T-claim $\mathcal{X}$ of the form $\mathcal{X} = \Phi(S(T))$. Denote the corresponding arbitrage free price process by $\Pi(t)$.

(a) Show that, under the martingale measure $\mathbb{Q}$, $\Pi(t)$ has a local rate of return equal to the short rate of interest r. In other words show that $\Pi(t)$ has a differential of the form

$$d\Pi(t) = r \cdot \Pi(t)dt + g(t)dW(t)$$

Hint: Use the $\mathbb{Q}$-dynamics of S together with the fact that F satisfies the pricing PDE.

- (b) Show that, under the martingale measure $\mathbb{Q}$, the process $Z(t) = \frac{\Pi(t)}{B(t)}$ is a martingale. More precisely, show that the stochastic differential for Z has zero drift term, i.e. it is of the form

$$dZ(t) = Z(t)\sigma_Z(t)dW(t)$$

Determine also the diffusion process $\sigma_Z(t)$ (in terms of the pricing function F and its derivatives).

Problem 10 Exercise 6.2

Consider the standard Black-Scholes model. An innovative company, F&H INC, has produced the derivative "the Golden Logarithm", henceforth abbreviated as the GL. The holder of a GL with maturity time T, denoted as $GL(T)$, will, at time T, obtain the sum $\ln S(T)$. Note that if $S(T) < 1$ this means that the holder has to pay a positive amount to F&H INC. Determine the arbitrage free price process for the $GL(T)$.

Problem 11 Exercise 6.3

Consider the standard Black-Scholes model. Derive the Black-Scholes formula for the European call option.

Problem 12 Exercise 6.4

Consider the standard Black-Scholes model. Derive the arbitrage free price process for the T-claim $\mathcal{X}$ where $\mathcal{X}$ is given by $\mathcal{X} = \{S(T)\}^{\beta}$. Here β is a known constant.

Answers:

Problem 13 Exercise 6.5

A so called binary option is a claim which pays a certain amount if the stock price at a certain date falls within some pre-specified interval. Otherwise nothing will be paid out. Consider a binary option which pays K SEK to the holder at date T if the stock price at time T is in the interval $[\alpha, \beta]$. Determine the arbitrage free price. The pricing formula will involve the standard Gaussian cumulative distribution function N.

Problem 14 Exercise 6.6

Consider the standard Black-Scholes model. Derive the arbitrage free price process for the claim $\mathcal{X}$ where $\mathcal{X}$ is given by $\mathcal{X} = \frac{S(T_1)}{S(T_0)}$. The times T_0 and T_1 are given and the claim is paid out at time T_1.

Problem 15 Exercise 6.7

Consider the American corporation ACME INC. The price process S for ACME is of course denoted in US $ and has the P-dynamics

$$dS = \alpha S dt + \sigma S d\bar{W}_1,$$

where α and σ are known constants. The currency ratio SEK/US $ is denoted by Y and Y has the dynamics

$$dY = \beta Y dt + \delta Y d\bar{W}_2,$$

where $\bar{W}_2$ is independent of $\bar{W}_1$. The broker firm F&H has invented the derivative "Euler". The holder of a T-Euler will, at the time of maturity T, obtain the sum

$$\mathcal{X} = \ln\left[\{Z(T)\}^2\right]$$

in SEK. Here $Z(t)$ is the price at time t in SEK of the ACME stock. Compute the arbitrage free price (in SEK) at time t of a T-Euler, given that the price (in SEK) of the ACME stock is z. The Swedish short rate is denoted by r.

Problem 16

Show that S_t^{-2r/σ^2} can be the price of a traded derivative security at time t, where S_t denotes the price of the underlying asset at t.

Problem 17

Suppose that a stock price, S, follows geometric Brownian motion with expected return μ and volatility σ :

$$dS = \mu S dt + \sigma S dz$$

What is the process followed by the variable S^n? Show that S^n also follows geometric Brownian motion.

Problem 18

Stock A and stock B both follow geometric Brownian motion. Changes in any short interval of time are uncorrelated with each other. Does the value of a portfolio consisting of one of stock A and one of stock B follow geometric Brownian motion? Explain your answer.

Problem 19

You are given:

(1) The true stochastic process of the short-rate is given by

$$dr(t) = (0.008 - 0.1r(t))dt + 0.05dZ(t)$$

where $Z(t)$ is a standard Brownian motion under the risk neutral probability measure.

(2) The risk-neutral process of the short-rate is given by

$$dr(t) = (0.013 - 0.1r(t))dt + 0.05d\tilde{Z}(t),$$

Where $\tilde{Z}(t)$ is a standard Brownian motion under the risk-neutral probability measure.

(3) For $t \leq T$, let $P(r, t, T)$ be the price at time t of a zero-coupon bond that pays \$1 at time T, if the short-rate at time t is r. The price of each zero-coupon bond follows an Ito process:

$$\frac{dP(r(t), t, T)}{P(r(t), t, T)} = \alpha(r(t), t, T)dt - q(r(t), t, T)dZ(t), t \leq T.$$

Calculate $\alpha(0.04, 2, 5)$.

Problem 20

Suppose that x is the yield to maturity with continuous compounding on a zero-coupon bond that pays off \$1 at time T. Assume that x follows the process

$$dx = a\,(x_0 - x)\,dt + sxdz$$

where a, x_0, and s are positive constants and dz is a Wiener process. What is the process followed by the bond price?