(1)

$X_t = e^{Y_t}$

$Y_t \sim N(\mu t, \sigma^2 t)$

$E(X_t | I_s) = X_s e^{(\mu + \frac{1}{2}\sigma^2)(t-s)}$

When would $e^{-rt} X_t$

$e^{-rt} X_t$ martingale

$\Leftrightarrow E(e^{-rt} X_t | I_s) = e^{-rs} X_s$

$\Leftrightarrow e^{-rt} X_s e^{(\mu + \frac{1}{2}\sigma^2)(t-s)} = e^{-rs} X_s$

$\Leftrightarrow \mu = r - \frac{1}{2}\sigma^2$

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$Z_t = e^{-rt} X_t$

$X_t = e^{W_t}$

$Z_t = e^{-rt + W_t}$

(a) $E(dZ)$

$F = e^x$ $\quad\quad$ $dS = -r\,dt + dW_t$

$F_x = e^x$ $\quad\quad$ $a = -r$

$\quad\quad$ $b = 1$

By Ito's lemma,

$dZ = \left(Z(-r) + \frac{1}{2} Z\right) dt + Z\,dW_t$

$= \left(\frac{1}{2} - r\right) Z\,dt + Z\,dW_t$

$\therefore E(dZ) = \left(\frac{1}{2} - r\right) Z\,dt$

(b) Z martingale $\Leftrightarrow$ $r = \frac{1}{2}$

2

(c) $E(Z)$

$= E(e^{-rt+W_t})$

$= e^{-rt} E(e^{W_t})$

$e^{W_t} \sim \text{Lognormal}(0, t)$

$\Rightarrow E(e^{W_t}) = e^{\frac{t}{2}}$

$\therefore E(Z) = e^{-rt+\frac{t}{2}}$

If $Z' = e^{-rt+\sqrt{2r}W_t}$,

then $dZ' = Z\,dW_t$ by Itô's lemma

(d) $E(e^{-rt+\sqrt{2r}W_t})$

$= e^{-rt} E(e^{\sqrt{2r}W_t})$

$e^{\sqrt{2r}W_t} \sim \text{lognormal}(0, 2rt)$

$\Rightarrow E(e^{\sqrt{2r}W_t}) = \frac{2rt}{2}$

$\therefore E(Z') = e^{-rt+rt} = 1$

13

a) $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$

$\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} \xi(x)$

$= \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x-r)^2}$

$\Rightarrow \xi(x) = e^{-\frac{1}{2\sigma^2}((x-r)^2 - (x-\mu)^2)}$

(a) $r = 0.05 \Rightarrow \xi(x) = e^{-\frac{1}{2\sigma^2}((x-0.05)^2 - (x-\mu)^2)}$

(b) Take $r = 0$

$\Rightarrow \xi(x) = e^{-\frac{1}{2\sigma^2}(x^2 - (x-\mu)^2)}$

(c) When the distribution has zero mean

(d) No

(4)

(a) $C - P = e^{-r(T-t)} E\left((S_T - K)^+ - (K - S_T)^+ \mid \mathcal{F}_t \right)$

$= e^{-r(T-t)} E(S_T - K \mid \mathcal{F}_t)$

$= S_t - K e^{-r(T-t)}$

(b) Obvious if we use (a)

(c) $H(S_t, t)$

$= \max\left[C(S_t, t),\ C(S_t, t) - S_t + K e^{-r(T-t)} \right]$

$H(S_0, 0)$

$= e^{-rt} E\left[C(S_t, t) + \left(K e^{-r(T-t)} - S_t \right)^+ \right]$

$= C(S_0, 0) + e^{-rT} E\left[\left(K - e^{r(T-t)} S_t \right)^+ \right]$

(Since $S_t = S_0 e^{(r - \frac{1}{2}\sigma^2)t + \sigma W_t}$),

$= C(S_0, 0) + e^{-rT} E\left[\left(K - S_0 e^{rT} e^{\sigma W_t - \frac{1}{2}\sigma^2 t} \right)^+ \right]$

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(d)

$H(S_0, 0)$

$= C(S_0, 0) + e^{-rT} E[(K - e^{r(T-t)} S_t)^+]$

$= C(S_0, 0) + e^{-rt} E[(Ke^{-r(T-t)} - S_t)^+]$

$= C(S_0, 0, K, T) + P(S_0, 0, Ke^{-r(T-t)}, t)$

(e) By Black-Scholes formula,

$C(S_0, 0, K, T) = S_0 N(d_1^1) - Ke^{-rT} N(d_2^1)$,

$P(S_0, 0, Ke^{-r(T-t)}, t) = N(-d_2^2) Ke^{-rT} - N(-d_1^2) S_0$,

where $d_1^1 = \frac{1}{\sigma\sqrt{T}} \left(\ln \frac{S_0}{K} + \left(r + \frac{\sigma^2}{2}\right) T \right)$,

$d_2^1 = d_1^1 - \sigma\sqrt{T}$,

$d_1^2 = \frac{1}{\sigma\sqrt{t}} \left(\ln \left(\frac{S_0}{Ke^{-r(T-t)}} \right) + \left(r + \frac{\sigma^2}{2}\right) t \right)$,

$d_2^2 = d_1^2 - \sigma\sqrt{t}$

$\therefore H(S_0, 0) = S_0 (N(d_1^1) - N(-d_1^2)) + Ke^{-rT} (N(-d_2^2) - N(d_2^1))$

5

(a) $S_t = S_0 e^{(r-f-\frac{1}{2}\sigma^2)t + \sigma W_t}$

$f = S_0 e^x$, $a = r - f - \frac{1}{2}\sigma^2$, $b = \sigma$

By Ito's lemma,

$$dS_t = (S_t(r-f-\tfrac{1}{2}\sigma^2) + \tfrac{1}{2} S_t \sigma^2)\,dt + S_t \sigma\, dW_t$$

$$= (r-f) S_t\, dt + \sigma S_t\, dW_t$$

(b) $X = e^{\sigma W_t - \frac{1}{2}\sigma^2 t}$

$f = e^x$, $a = -\frac{1}{2}\sigma^2$, $b = \sigma$

By Ito's lemma,

$$dX = (X(-\tfrac{1}{2}\sigma^2) + \tfrac{1}{2} X \sigma^2)\,dt + X\sigma\, dW_t$$

$$= \sigma X\, dW_t$$

$\therefore$ X is a martingale

5

(d) $Z = \frac{1}{S_0} e^{-(r-f-\frac{1}{2}\sigma^2)t - \sigma W_t}$

$f = e^x$, $a = -(r-f-\frac{1}{2}\sigma^2)$, $b = -\sigma$

By Itô's lemma,

$dZ = \left(Z(-(r-f-\frac{1}{2}\sigma^2) + \frac{1}{2} Z\sigma^2 \right) dt + Z(-\sigma) dW_t$

$= (f - r + \sigma^2) Z\, dt - \sigma Z\, dW_t$

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$a+b=1000$

$0.2a-0.25b=0$

$\Rightarrow a=556,\ b=444$

$\therefore$ Allocate 556 S_1.

17)

$$dS_1 = \alpha_1 S_1 dt + \sigma_1 S_1 dW_t$$

$$dS_2 = \alpha_2 S_2 dt + \sigma_2 S_2 dW_t$$

$$S_1 = S_1(0) e^{(\alpha_1 - \frac{1}{2}\sigma_1^2)t + \sigma_1 W_t}$$

$$S_2 = S_2(0) e^{(\alpha_2 - \frac{1}{2}\sigma_2^2)t + \sigma_2 W_t}$$

$\alpha_1 - \frac{1}{2}\sigma_1^2 = 0.1, \quad \sigma_1 = 0.2,$

$\alpha_2 - \frac{1}{2}\sigma_2^2 = 0.125, \quad \sigma_2 = 0.3$

$\alpha_1 = 0.12, \quad \alpha_2 = 0.17$

They should have the same Sharpe ratio.

$$\frac{\alpha_1 - r}{\sigma_1} = \frac{\alpha_2 - r}{\sigma_2}$$

$$\frac{0.12 - r}{0.2} = \frac{0.17 - r}{0.3}$$

$\therefore r = 0.02$

8

By problem 17, when $\frac{dS}{S} = \mu dt + \sigma dW_t$,

$$S(t)^n = S(0)^n e^{(n\mu + \frac{1}{2}n(n-1)\sigma^2 - \frac{1}{2}n^2\sigma^2)t + n\sigma W_t} \quad \cdots \circledast$$

Using $\circledast$ with $n = a$, $\mu = 0.05$, $\sigma = 0.2$, $t = 1$

$$\Rightarrow S(1)^a = 0.5^a e^{0.03a + 0.2aW_1}$$

$$1.4 = E((S(1))^a)$$

$$= 0.5^a e^{0.03a} E(e^{0.2aW_1})$$

$$= 0.5^a e^{0.03a + 0.02a^2}$$

$\Rightarrow a = -0.5$

(8) Now, under Q-measure,

$$\frac{dS}{S} = 0.03\, dt + 0.2\, dW_t$$

$$F = E^Q\left(e^{-0.03} S(1)^a\right)$$

Using (*) with $S(0) = 0.5$, $n = a$, $\mu = 0.03$,

$\sigma = 0.2$, $t = 1$

$$S(1)^a = 0.5^a\, e^{a(0.03) - \frac{1}{2} a (0.2)^2 + a(0.2) W_1}$$

$$= 0.5^a\, e^{0.01a + 0.2 a W_1}$$

$$F = e^{-0.03}\, 0.5^a\, e^{0.01a}\, E^Q\left(e^{0.2 a W_1}\right)$$

$$= e^{-0.03}\, 0.5^a\, e^{0.01a + 0.02a^2}$$

$$= e^{-0.03}\, 0.5^{-0.5}\, e^{0.01(-0.5) + 0.02(-0.5)^2}$$

$$= 1.37$$

10

$$F_{GL}(t) = E^{Q}\left(e^{-r(T-t)} \ln S_T\right)$$

$$\ln \frac{S_T}{S_t} \sim N\left(\left(r - \frac{1}{2}\sigma^2\right)(T-t),\ \sigma^2 (T-t)\right)$$

$$\therefore F_{GL}(t) = e^{-r(T-t)}\left[\ln S_t + \left(r - \frac{1}{2}\sigma^2\right)(T-t)\right]$$

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(12)

$$F_t = e^{-r(T-t)} E^{Q}(S_T^{\beta})$$

Using the result of problem 17,

$$S_T^{\beta} = S_t^{\beta} e^{(\beta r - \frac{1}{2}\beta\sigma^2)(T-t) + \beta\sigma(W_T - W_t)}$$

$$\therefore F_t = e^{-r(T-t)} S_t^{\beta} e^{(\beta r - \frac{1}{2}\beta\sigma^2)(T-t)} E(e^{\beta\sigma(W_T - W_t)})$$

$$= e^{-r(T-t)} S_t^{\beta} e^{(\beta r - \frac{1}{2}\beta\sigma^2)(T-t)} e^{\frac{1}{2}\beta^2\sigma^2(T-t)}$$

13

$$F_t = e^{-r(T-t)} E^Q\left(K I(\alpha < S_T < \beta)\right)$$

$$= e^{-r(T-t)} K \quad P^Q(\ln\alpha < \ln S_T < \ln\beta)$$

$$\ln S_T \sim N\left(\ln S_t + (r - \tfrac{1}{2}\sigma^2)(T-t),\ \sigma^2(T-t)\right)$$

$$d_2 = \frac{\ln\alpha - (\ln S_t + (r - \frac{1}{2}\sigma^2)(T-t))}{\sigma\sqrt{T-t}}$$

$$d_1 = \frac{\ln\beta - (\ln S_t + (r - \frac{1}{2}\sigma^2)(T-t))}{\sigma\sqrt{T-t}}$$

$$\therefore F_t = e^{-r(T-t)} K\left(\Phi(d_1) - \Phi(d_2)\right)$$

(14)

$$F_t = e^{-r(T_1 - t)} E^{Q}\left(\frac{S_{T_1}}{S_{T_0}}\right)$$

$$\ln \frac{S_{T_1}}{S_{T_0}} \sim N\left((r - \tfrac{1}{2}\sigma^2)(T_1 - T_0),\ \sigma^2(T_1 - T_0)\right)$$

$$\therefore F_t = e^{-r(T_1 - t)} \cdot e^{(r - \frac{1}{2}\sigma^2)(T_1 - T_0) + \frac{1}{2}\sigma^2(T_1 - T_0)}$$

$$= e^{-r(T_0 - t)}$$

[1(b)]

$$F = S^{-\frac{2r}{\sigma^2}}$$

$$rSF_S + \frac{1}{2}S^2\sigma^2 F_{SS} - rF = 0 \ ?$$

$$F_S = -\frac{2r}{\sigma^2} S^{-\frac{2r}{\sigma^2}-1}$$

$$F_{SS} = -\frac{2r}{\sigma^2}\left(-\frac{2r}{\sigma^2}-1\right) S^{-\frac{2r}{\sigma^2}-2}$$

$$(\text{LHS}) = rS\left(-\frac{2r}{\sigma^2}\right) S^{-\frac{2r}{\sigma^2}-1}$$

$$+\frac{1}{2}S^2\sigma^2 \frac{2r}{\sigma^2}\left(\frac{2r}{\sigma^2}+1\right) S^{-\frac{2r}{\sigma^2}-2}$$

$$-rS^{-\frac{2r}{\sigma^2}}$$

$$= S^{-\frac{2r}{\sigma^2}}\left[-\frac{2r^2}{\sigma^2} + \frac{1}{2}\sigma^2\frac{2r}{\sigma^2}\left(\frac{2r}{\sigma^2}+1\right) - r\right]$$

$$= 0$$

$\therefore$ It satisfies the Black-Scholes Equation.

(1)

$$dS = \mu S\,dt + \sigma S\,dW_t$$

$f = x^n$, $a = \mu S$, $b = \sigma S$

$f_x = nx^{n-1}$, $f_{xx} = n(n-1)x^{n-2}$, $f_t = 0$

$$d(S^n) = \left(nS^{n-1}\mu S + \frac{1}{2}n(n-1)S^{n-2}\sigma^2 S^2\right)dt$$
$$+ nS^{n-1}\sigma S\,dW_t$$

$$\therefore \frac{d(S^n)}{S^n} = \left(n\mu + \frac{1}{2}n(n-1)\sigma^2\right)dt + n\sigma\,dW_t$$

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$$\Delta S_A = \mu_A S_A \Delta t + \sigma_A S_A \varepsilon_A \sqrt{\Delta t}$$

$$\Delta S_B = \mu_B S_B \Delta t + \sigma_B S_B \varepsilon_B \sqrt{\Delta t}$$

$$\Delta S_A + \Delta S_B = (\mu_A S_A + \mu_B S_B)\Delta t + (\sigma_A S_A \varepsilon_A + \sigma_B S_B \varepsilon_B)\sqrt{\Delta t}$$

This cannot be written as

$$\Delta S_A + \Delta S_B = \mu (S_A + S_B)\Delta t + \sigma (S_A + S_B)\varepsilon \sqrt{\Delta t}$$

$\therefore$ The value of the portfolio does not follow geometric brownian motion.

12-0

$B = e^{-x(T-t)}$

$f(x) = e^{-(T-t)x}$

$f_t = xf, \quad f_x = -(T-t)f, \quad f_{xx} = (T-t)^2 f$

By Itô's lemma,

$$dB = \left(x - (T-t)a(x_0 - x) + \frac{1}{2}(T-t)^2 s^2 x^2\right) B\, dt - (T-t) s x B\, dz.$$