Professor S.J. Deng

ISyE/Math 6759 Stochastic Processes in Finance – I

Homework Set 1

Please remember to write down your name and GTID in the submitted homework.

Problems 1-4 Neftci's book (Second Edition) Chapter 5, exercise 1 - 4 (p117-p118)
Clarifications and Hints:

1) Problem 1(e): We can calculate conditional variance in similar way as we do for unconditional variance. In this question, $Var[X \mid Y=1] = E[X - E(X) \mid Y=1]^2$
2) For Problem 2(a), let p=1/2 when you plot the distribution function. The matlab function for binomial distribution are binopdf( ) and binocdf( ).
3) In Problem 3, the cdf of an exponential distribution should be

   $$P(Z \le z) = \begin{cases} 1 - e^{-\lambda z} ....z \ge 0 \\ 0.......................z < 0 \end{cases}$$

   In Problem 3(a), interpret 'density function' as 'probability density function'
   In 3(d), Assume the parameter for both Z1 and Z2 is λ
4) As you may have noticed, there is an obvious typo (actually two typos) about Poisson distribution, correct it yourself.

Problem 5. Let us now try something more interesting.
A person has 5 coins in his pocket. Two have both sides being heads, one has both sides being tails, and two are normal. The coins cannot be distinguished unless one looks at them.
a) The person closes his eyes, picks a coin from pocket at random, and tosses it. What is the probability that the down-side of the coin is heads?
b) He opens his eyes and sees that the up-side of the coin is heads. What is the probability that the downside is also heads (namely, this is a two-heads coin).
c) Without looking at the other side of the coin, he tosses it again. What is the probability that the downside is heads?
d) Now he looks at the upside of the coin and it is heads. What is the probability that the downside of the coin is heads?

Hints: 1) To get yourself started, let *D* denote the event that a two-heads coin is picked, *N* denote the event that a normal coin is picked, and *Z* be the event that a two-tails coin is picked. Let H_{L_i} (and H_{U_i}) denote the event that the down-side (and the up-side) of the coin on the *i*th toss is heads.

2) You may find $\mathrm{P}[\mathrm{B}_1 \mid \mathrm{A}] = \dfrac{P(B_1 \cap A)}{P(A)} = \dfrac{P[A \mid B_1]P(B_1)}{P[A \mid B_1]P(B_1) + P[A \mid B_2]P(B_2)}$

useful

Problem 6. Below is a classical interview question from investment banks. Only a few people have got it completely correct. Now that you have learnt enough to work it out, think it through and… enjoy!

I will roll a single die no more than three times. You can stop me immediately after the first roll, or immediately after the second, or you can wait for the third. I will pay you the same amount of dollars as the number turned up on my last roll (roll number three unless you stop me sooner).
a) What is your playing strategy?
b) If you were running this game, how much would you charge players for repeated plays of the game?
(Hint: take a note at how 'the law of large numbers' come into play here)
c) Suppose instead an amended game is played: I roll a single die three times without pause, and the payoff to player is the maximum of the three rolls. What is the expected payoff to the player? Can you tell up front whether the original or amended game has the higher expected payoff?

Problem 7 Suppose you are in a game show and there are 10 doors in front of you. You know that there is a prize behind one of them, and nothing behind the other 9. You have to choose a door containing the prize in order to win the prize. However, before you choose, the game show host promises that rather than immediately opening the door of your choice to reveal its contents, he will open one of the other 9 doors to reveal that it is an empty door. He will then give you the option to change your choice. You may assume that the host is completely impartial – not malicious in any way. For instance, if you choose door 3, he will open one door, say door 5, to reveal that it is empty. Should you continue with door 3 or choose another door? Please compute the probability of finding the prize behind your chosen door before the game show host reveals that one door is empty, and the probability of you finding the prize by changing to a different door after seeing the revealed empty door.

Problem 8 Suppose that Mr. Warren Buffet and Mr. Zhao Danyang agree to meet at a specified place between 12 pm and 1 pm. Suppose each person arrives between 12 pm and 1 pm at random with uniform probability. What is the distribution function for the length of the time that the first to arrive has to wait for the other?

Problem 9 Take a stick of unit length and break it into three pieces, choosing the break point at random. (The break points are assumed to be chosen simultaneously). What is the probability that the three pieces can be used to form a triangle?

Problem 10 Let X, Y be continuous random variables with following joint probability density function:

$$f(x,y) = Cxy^2 \qquad \forall 0 \le x \le y \le 1$$

1. What is the acceptable value of C?
2. Are X and Y independent?
3. Compute the following:
   a) f(x) and E(X)
   b) f(y) and E(Y)
   c) E(X | Y = y)
   d) E(Y | X = x)

Problem 11 Given a continuous random variable X, find the value of c for which $E[(X – c)^2]$ is minimized.

Problem 12 A random variable X has probability density function:

$$f(x) = \begin{cases} cx\,e^{-k^2x^2}, x \ge 0 \\ 0, x < 0 \end{cases}$$

(1) Find the value of c which makes f(x) a valid probability density function;
(2) Calculate the E(x) and Var(X).

Hint: The standard normal distribution has probability density function:

$$f(u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2}$$

Problem 13 Two players A and B play a marble game. Each player has both a red and blue marble. They present one marble to each other. If both present red, A wins \$3. If both present blue, A wins \$1. If the colors of the two marbles do not match, B wins \$2. Is it better to be A, or B, or does it matter?

Problem 14 There is a bag of 100 coins, 99 of them are fair coins and the rest one is a coin with both sides being heads. You randomly drew a coin from the bag and tossed it 10 times. The result was 10 heads. What’s the probability that you drew the coin with both sides to be heads?

Problem 15 There is a deck of 50 cards. Each card has a number from 1 to 50 and nothing else. The dealer will shuffle the deck and reveal the top 4 cards one by one (donate the numbers on the top four cards to be A, B, C, D, A being the first card and D being the forth). If the numbers on the 4 cards revealed are in ascending order (not necessarily consecutive, as long as A<B<C<D) you

will win $10, otherwise you lose $1. Would you play? How about a deck of 100 cards?

Problem 16 Three people, A, B and C are in a three-man duel. A is a bad shooter that shoots with 1/3 accuracy. B is moderate that shoots with 2/3 accuracy. C is a perfect shooter that shoots on target all the time. In this duel A will shoot first, then B and then C. The cycle goes on until only one person is left. All of the three people are perfectly smart and rational. They'll do their best to optimize their chance of survival. What is A's best course of action? And what's the chance of survival?

(Hint: A can choose to shoot at either a person or deliberately miss the target.)

Problem 17 Suppose in a society where there are equal numbers of men and women. There is a 50% chance for each child that a couple gives birth to is a girl and the genders of their children are mutually independent. Suppose in this strange and primitive society every couple prefers a girl and they will continue to have more children until they get a girl and once they have a girl they will stop having more children, what will eventually happen to the gender ratio of population in this society?

Problem 18 A person shoots basketball 100 times and scores 1 point if he makes one shot, 0 point otherwise. He has already made the first shot and missed the second one. For the following shots, the probability of making each shot is his score before the shot divided by total shots before the shot, i.e. if he has scored 13 points out of first 20 shots, then his next shot has a probability of 13/20 to score. What's the probability of scoring exactly 66 points after making 100 shots (including the first two)?