Professor S.J. Deng

ISyE/Math 6759 Stochastic Processes in Finance – I

Homework Set 4

Please write down your name in the format of 'Last name, First name'
Note: All questions are from Neftci's book unless otherwise noted.

Problem 1-2
Group I (Difference between Deterministic and Stochastic Calculus)
Chapter 3 (p75) Exercise 8, 9

Problem 3-5
Group II (Stochastic Integration)
Chapter 9 (p228) Exercise 1, 2, 3

Problem 6-8
Group III (Martingale)
Neftci Chapter 6: p154. 2 (typo in the textbook, in parts (c), (d), (e): the process shall be {X~i} instead {Vi}),
3(a), (b), (c), 4

Problem 9-11
Group IV (Ito's Lemma)
Chapter 10(p251) Exercise 1, 2, 3

Problems 12 (Brownian motion)
Suppose the standard deviation of continuously compounded annual return of stock AAA is 10%. Assume that the stock return follows a Brownian motion. What is the standard deviation of continuously compounded four-year return of stock AAA?
Hint: consider the property of independent and stationary increments.

Problem 13 (Martingale)
Let $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\geq 0}, \mathbb{P})$ be a filtered probability space and Y_n, $n \geq 0$, a sequence of absolutely integrable random variables adapted to the filtration $(\mathcal{F}_n)_{n\geq 0}$. Assume that there exist real numbers u_n, v_n, $n \geq 0$, such that

$$\mathbf{E}(Y_{n+1} \mid \mathcal{F}_n) = u_n Y_n + v_n.$$

Find two real sequences a_n and b_n, $n \geq 0$, so that the sequence of random variables $M_n := a_n Y_n + b_n$, $n > 1$, be martingale w.r.t. the same filtration.

Homework4 Part 2

Problem 1

Let M_n represents a symmetric random walk (i.e., a sum of n i.i.d. Bernoulli random variable taking value 1 or -1 with probability ½) and let τ be a bounded stopping time, there is a constant C < ∞ , such that P($\tau \leq C$) =1.

(1) What is the definition of martingale?
(2) Show that ${M_n}^2$ – n is a martingale.
(3) Explain why one shall get E(${M_\tau}^2$)=E(τ).

Problem 2

X(t) is an Ornstein-Uhlenbeck process defined by

$$dX(t)=2(4-X(t))dt+8dZ(t)$$

where Z(t) is a standard Brownian motion.

Let $Y(t) = \frac{1}{X(t)}$. You are given that $dY(t) = \alpha\big(Y(t)\big)dt + \beta\big(Y(t)\big)dZ(t)$ for some functions and $\alpha(y)$and $\beta(y)$.

Determine $\alpha(\frac{1}{2})$.

Problem 5

You are given:

(1) The true stochastic process of the short-rate is given by

$$dr(t) = \big(0.008 - 0.1r(t)\big)dt + 0.05dZ(t),$$

where Z(t) is a standard Brownian motion under the risk neutral probability measure.

(2) The risk-neutral process of the short-rate is given by

$$dr(t) = \big(0.013 - 0.1r(t)\big)dt + 0.05d\tilde{Z}(t),$$

Where $\check{Z}(t)$ is a standard Brownian motion under the risk-neutral probability measure.

(3) For t≤ T, let P(r,t,T) be the price at time t of a zero-coupon bond that pays \$1 at time T, if the short-rate at time t is r. The price of each zero-coupon bond follows an Ito process:

$$\frac{dP(r(t),t,T)}{P(r(t),t,T)} = \alpha(r(t),t,T)dt - q(r(t),t,T)dZ(t), t \leq T.$$

Calculate $\alpha(0.04,2,5)$.

Problem 7

Prove the following process is a martingale:

$$Z_t = \exp\left(\int_0^t f(s)\, dB_s - \frac{1}{2}\int_0^t f^2(s) ds\right)$$

where f is a continuous function from [0,T] to R.

Problem 8 (Prob 3.46): (Quant Job Interview Question)

If Wt is a standard Brownian motion, is $W_t{}^3$ a martingale?

Problem 9

Suppose that x is the yield to maturity with continuous compounding on a zero-coupon bond that pays off $1 at time T. Assume that x follows the process

$$dx = a(x_0 - x)\, dt + sx\, dz$$

where a, x_0, and s are positive constants and dz is a Wiener process. What is the process followed by the bond price?