ISyE/Math 6759 Stochastic Processes in Finance – I
Homework Set 3

Problem 1-2 (Simple Asset Pricing/ Replicating Portfolio)
Neftci (3rd Ed.) Chapter 4 (p.63), Exercises 1, 2.

1. Suppose you can bet on an American presidential election in which one of the candidates is an incumbent. The market offers you the following payoffs R:

$$R = \begin{cases} \$1000 \text{ If incumbent wins} \\ -\$1500 \text{ If incumbent loses} \end{cases}$$

You can take either side of the bet. Let the true probability of the incumbent winning be denoted by $p, 0 < p < 1$.

(a) What is the expected gain if $p = .6$?
(b) Is the value of p important for you to make a decision on this bet?
(c) Would two people taking this bet agree on their assessment of p? Which one would be correct? Can you tell?
(d) Would statistical or econometric theory help in determining the p?
(e) What weight would you put on the word of a statistician in making your decision about this bet?
(f) How much would you pay for this bet?

2. Now place yourself exactly in the same setting as before, where the market quotes the above R. It just happens that you have a close friend who offers you the following separate bet, R^*:

$$R = \begin{cases} \$1500 \text{ If incumbent wins} \\ -\$1000 \text{ If incumbent loses} \end{cases} \quad (4.25)$$

Note that the random event behind this bet is the same as in R. Now consider the following:
(a) Using the R and the R^*, construct a portfolio of bets such that you get a guaranteed risk-free return (assuming that your friend or the market does not default).
(b) Is the value of the probability p important in selecting this portfolio? Do you care what the p is? Suppose you are given the R, but the payoff of R^* when the incumbent wins is an unknown to be determined. Can the above portfolio help you determine this unknown value?
(c) What role would a statistician or econometrician play in making all these decisions? Why?

Problem 3:
Assume the dynamic behavior of stock price S_t in year t satisfies the function: $S_t = S_{t-1} + B_t$. The current stock price S_0 is \$100. The continuously compounded risk free interest rate is 5%.

(1) If $B_t = \begin{cases} 10, \; with\ probability\ of\ \frac{1}{3} \\ -10, with\ probability\ of\ \frac{2}{3} \end{cases}$, what is the expected value of stock price after 10 years (S_{10}). What is the probability of $S_{10} \geq 100$?

(2) $B_t = \begin{cases} 0.5 \times S_{t-1}, & with\ probability\ of\ p \\ -0.5 \times S_{t-1}, & with\ probability\ of\ 1-p \end{cases}$.

Is this economy arbitrage free? Why? A European call option is written on the stock price with strike price of 100 and expiration time of three years later. What is the arbitrage free price of the option?

(3) If the price of above call option is \$40, what dynamic arbitrage portfolio you will construct?

(4) If $B_1 = \begin{cases} 0.5 \times S_{t-1}, & with\ probability\ of\ p_1 \\ 0, & with\ probability\ of\ p_2 \\ -0.5 \times S_{t-1}, & with\ probability\ of\ p_3 \end{cases}$, namely there are 3 possible states of the world in year 1, and B_t for t >= 2 are defined the same way as those in (2). Is the market

complete now? There is a financial product worth \$2 will produce payoff of \$0 in state 1 with probability p1, \$2 in state 2 with probability p2, and \$4 in state 3 with probability p3, after one year from now. What is the arbitrage free price of a European call option with strike price of \$100 and expiration time of one year from now?

Problem 4:

A stock has volatility $\sigma = 0.3$ and a current value of \$36. A European-style put option on this stock has a strike price of \$40 and expiration is in 5 months. The interest rate is 2% per year.

1) Find the value of this put using a binomial lattice with Δt = 1-month and u = exp(σ *sqrt(Δt)).
2) Find the value of this put using a binomial lattice with Δt = half-month and u = exp(σ *sqrt(Δt)).
3) Are the prices obtained in 1) and 2) the same? Which one is the correct price?

Problem 5:

Consider a family of European-style call options written on a non-dividend-paying stock, each option being identical except for its strike price (i.e., they all have the same expiration time). The value of the call with strike price K is denoted by C(K).

Prove the following three general relations using arbitrage arguments, assuming risk-free rate is non-negative:

1) K2 > K1 implies C(K1) > C(K2).
2) K2 > K1 implies K2 – K1 > C(K1) – C(K2).
3) K3 > K2 > K1 implies C(K2) <= (K3 - K2)/(K3 – K1) * C(K1) + (K2 – K1)/(K3 – K1) * C(K3)

Problem 6: (State-Price Vector Pricing, Arbitrage condition, Future pricing)

Neftci Chapter 2 (p30) Exercise 3 (a) (b)

3. Consider a stock S_t and a plain vanilla, at-the-money, put option written on this stock. The option expires at time $t+\Delta$, where Δ denotes a small interval. At time t, there are only two possible ways the S_t can move. It can either go *up* to $S^u_{t+\Delta}$, or go *down* to $S^d_{t+\Delta}$. Also available to traders is risk-free borrowing and lending at annual rate r.

   (a) Using the arbitrage theorem, write down a three-equation system with *two* states that gives the arbitrage-free values of S_t and C_t.

   (b) Now plot a two-step binomial tree for S_t. Suppose at every node of the tree the markets are arbitrage-free. How many three-equation systems similar to the preceding case could then be written for the entire tree?

Problem 7: (Binomial Tree)

(a)-(c): Neftci Chapter 2 (p31) Exercise 4 (a)-(c). Answer all questions with the initial price set to
be S_0 = 100, strike price K = 100.

4. A four-step binomial tree for the price of a stock S_t is to be calculated using the up and down ticks given as follows:

$$u = 1.15 \qquad\qquad d = \frac{1}{u}$$

These up and down movements apply to one-month periods denoted by $\Delta = 1$. We have the following dynamics for S_t,

$$S_{t+\Delta}^{up} = uS_t \qquad S_{t+\Delta}^{down} = dS_t,$$

where *up* and *down* describe the two states of the world at each node.

Assume that time is measured in months and that $t = 4$ is the expiration date for a European call option C_t written on S_t. The stock does not pay any dividends and its price is expected (by "market participants") to grow at an annual rate of 15%. The risk-free interest rate r is known to be constant at 5%.

(a) According to the data given above, what is the (approximate) annual volatility of S_t if this process is known to have a log-normal distribution?
(b) Calculate the four-step binomial trees for the S_t and the C_t.
(c) Calculate the arbitrage-free price C_o of the option at time $t = 0$.

(d) Using the above setting, work out all hedging portfolios at each node of the first three periods, specifically, period t=0 => t=1, period t=1 => t=2 and period t=2 => t=3.

Problem 8:

Neftci Chapter 2 (p32) Exercise 5. Change r=5% to r=0.4%

5. You are given the following information concerning a stock denoted by S_t.

- Current value = 102.
- Annual volatility = 30%.
- You are also given the spot rate $r = 5\%$, which is known to be constant during the next 3 months.

It is hoped that the dynamic behavior of S_t can be approximated reasonably well by a binomial process if one assumes observation intervals of length 1 month.

(a) Consider a European call option written on S_t. The call has a strike price $K = 120$ and an expiration of 3 months. Using the S_t and the risk-free borrowing and lending, B_t, construct a portfolio that replicates the option.
(b) Using the replicating portfolio price this call.
(c) Suppose you sell, over-the-counter, 100 such calls to your customers. How would you hedge this position? Be precise.
(d) Suppose the market price of this call is 5. How would you form an arbitrage portfolio?

Problem 9:

1) Neftci Chapter 2 (p32) Exercise 6. $S_{t+1} - S_t = \mu S_t + \sigma S_t \varepsilon_t$; $\Delta t = 1$ year.

6. Suppose you are given the following data:

- Risk-free yearly interest rate is $r = 6\%$.
- The stock price follows:

$$S_t - S_{t-1} = \mu S_t + \sigma S_t \epsilon_t,$$

where the ϵ is a serially uncorrelated binomial process assuming the following values:

$$\epsilon = \begin{cases} +1 & \text{with probability } p \\ -1 & \text{with probability } 1 - p. \end{cases}$$

The $0 < p < 1$ is a parameter.

- Volatility is 12% a year.
- The stock pays no dividends and the current stock price is 100.

Now consider the following questions.

(a) Suppose μ is equal to the risk-free interest rate:

$$\mu = r$$

and that the S_t is arbitrage-free. What is the value of p?

(b) Would a $p = 1/3$ be consistent with arbitrage-free S_t?

(c) Now suppose μ is given by:

$$\mu = r + \text{risk premium}$$

What do the p and ϵ represent under these conditions?

(d) Is it possible to determine the value of p?

2) Neftci Chapter 2 (p32) Exercise 7, using the value of r and σ in part 1).

7. Using the data in the previous question, you are now asked to approximate the current value of a European call option on the stock S_t. The option has a strike price of 100, and a maturity of 200 days.

(a) Determine an appropriate time interval Δ, such that the binomial tree has 5 steps.

(b) What would be the implied u and d?

(c) What is the implied "up" probability?

(d) Determine the tree for the stock price S_t.

(e) Determine the tree for the call premium C_t.

Problem 10:

Use a 2-period binomial tree to price an American Option with the following parameters:
Strike Price K = 22, continuously compounding annualized risk-free rate $r_f = 5\%$.
Current price $S_0 = 20$.
Time to Expiration T = 2 years, each period of the tree represents one year.
u = 1.2840, d = 0.8607.

Problem 11:

Consider a stock which pays no dividend. The current stock price is \$62 and the annualized volatility for the stock is $\sigma = 0.20$. The annual continuously compounding risk-free rate is 2.5%. Consider a five-month option with a strike price of \$60. After 3 months, the purchaser will have the right to choose this option to be either an European call option or an European put option. Please use a 5-step (monthly) binomial lattice model to price this exotic option.

(Note: use $u = \exp\left(\sigma \cdot \sqrt{\Delta t}\right)$ and $d = \exp\left(-\sigma \cdot \sqrt{\Delta t}\right)$).

Problem 12:

A stock price is currently \$50. It is known that at the end of 2 months it will be either \$53 or \$48. The risk-free interest rate is 10% per annum with continuous compounding. What is the value of a 2-month European call option with a strike price of \$49? Use no-arbitrage arguments.

Problem 13:

A stock price is currently \$25. It is known that at the end of 2 months it will be either \$23 or \$27. The risk-free interest rate is 10% per annum with continuous compounding. Suppose S(T) is the stock price at the end of 2 months. What is the value of a derivative that pays off S(T)^ 2 at this time?