1)

a) $P(X=0)=0.4$, $P(X=1)=0.6$

$P(Y=0)=0.65$, $P(Y=1)=0.35$

b) No

c) $E(X)=0.6$, $E(Y)=0.35$

d) $P(X=0 \mid Y=1)=\frac{3}{7}$, $P(X=1 \mid Y=1)=\frac{4}{7}$

e) $E(X \mid Y=1)=\frac{4}{7}$.

$E(X^2 \mid Y=1)=\frac{4}{7}$

$Var(X \mid Y=1)=\frac{4}{7}-\left(\frac{4}{7}\right)^2=\frac{12}{49}$

(2)

$P(X_4 = 0) = (1-p)^4$

a) $P(X_4 = 1) = 4p(1-p)^3$

$P(X_4 = 2) = 6p^2(1-p)^2$

$P(X_4 = 3) = 4p^3(1-p)$

$P(X_4 = 4) = p^4$

$\therefore P(X_4 > 0) = 1 - (1-p)^4$

$P(X_4 > 1) = 6p^2(1-p)^2 + 4p^3(1-p) + p^4$

$P(X_4 > 2) = 4p^3(1-p) + p^4$

$P(X_4 > 4) = 0.$

b) $E(B_i) = p, \quad Var(B_i) = p - p^2$

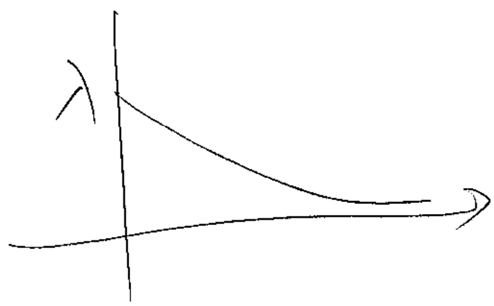
$\therefore E(X_3) = 3p$

$Var(X_3) = 3p(1-p)$

③ − 1

$F(z) = 1 - e^{-\lambda z}, \quad z > 0$

(a) $f(z) = \lambda e^{-\lambda z}, \quad z > 0$



(b) $E(Z) = \int_0^\infty z \cdot \lambda e^{-\lambda z} dz = \frac{1}{\lambda}$

(c) $E(Z^2) = \int_0^\infty z^2 \lambda e^{-\lambda z} dz = \frac{2}{\lambda^2}$

$\therefore Var(Z) = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$

(d) $Z_1, Z_2 \sim exp(\lambda)$, indep

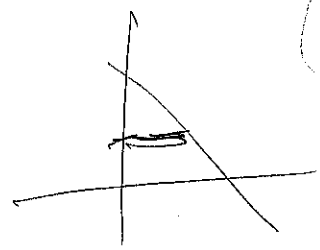
$S = Z_1 + Z_2$

$P(S \leq s) = P(Z_1 + Z_2 \leq s)$

③ — 2

$$\int_0^s \int_0^{s-y} \lambda e^{-\lambda x} \cdot \lambda e^{-\lambda y} \, dx\,dy$$

~~Y+Y≤s~~ ~~X+Y≤s~~



$(X+Y \le s)$

$-\lambda s + \lambda y$

$$= \int_0^s -\lambda e^{-\lambda y} \, e^{-\lambda x}\Big|_0^{s-y} \, dy$$

$$= \int_0^s -\lambda e^{-\lambda y} \left(e^{-\lambda(s-y)} - 1 \right) dy$$

$$= -\lambda \int_0^s e^{-\lambda s} - e^{-\lambda y} \, dy$$

$$= -\lambda \left(s e^{-\lambda s} + \frac{1}{\lambda} e^{-\lambda y}\Big|_0^s \right)$$

$$= -\lambda \left(s e^{-\lambda s} + \frac{1}{\lambda} e^{-\lambda s} - \frac{1}{\lambda} \right)$$

$$= 1 - e^{-\lambda s} - \lambda s e^{-\lambda s}$$

pdf: $f(s) = \lambda e^{-\lambda s} - \lambda \left(e^{-\lambda s} - \lambda s e^{-\lambda s} \right)$

$$= \lambda^2 s e^{-\lambda s} \quad (s > 0)$$

(e) $E(S) = \frac{2}{\lambda}$, $\quad Var(S) = \frac{2}{\lambda^2}$

9

$$p(k) = \frac{\lambda^k e^{-\lambda}}{k!}, \quad k \geq 0 \; (\lambda > 0)$$

(a) $$\sum_{k=0}^{\infty} p(k) = \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = 1.$$

(b) $$E(Z) = \sum_{k=0}^{\infty} k\, p(k)$$
$$= \sum_{k=1}^{\infty} \frac{\lambda^k e^{-\lambda}}{(k-1)!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^{k+1}}{k!}$$
$$= \lambda e^{-\lambda} e^{\lambda} = \lambda$$

$$E(Z^2) = \sum_{k=0}^{\infty} k^2 p(k)$$
$$= \sum_{k=1}^{\infty} (k(k-1)+k)\, p(k)$$
$$= \sum_{k=2}^{\infty} k(k-1)p(k) + \underbrace{\sum_{k=1}^{\infty} k\,p(k)}_{\lambda}$$
$$= \sum_{k=0}^{\infty} \frac{\lambda^{k+2} e^{-\lambda}}{k!} + \lambda$$
$$= \lambda^2 + \lambda \qquad \therefore Var(Z) = \lambda$$

15)

a) $\frac{6}{10}$

b) $\dfrac{\frac{2}{5}}{\frac{6}{10}} = \frac{2}{3}$

c) $\frac{2}{3} \times 1 + \frac{1}{3} \times \frac{1}{2} = \frac{5}{6}$

d) $\dfrac{\frac{2}{5} \times 1}{\frac{2}{5} \times 1 + \frac{2}{5} \times \frac{1}{4}} = \frac{4}{5}$

6-1 At first roll, $\geq 5 \to$ stop, $< 5 \to$ keep going

a) At second roll, $\geq 4 \to$ stop, $< 4 \to$ keep going

Below is reasoning.

① ② ③

$1 \sim 6$

$\frac{21}{6} = \frac{7}{2}$

4↑ stop

3, 2, 1 $\to \frac{7}{2}$

4↓ 8↑

$\frac{1}{6}(4+5+6) + \frac{7}{2}\cdot\frac{1}{2}$

$\frac{15}{6} + \frac{7}{4} = \frac{60+42}{24} = \frac{102}{24} = \frac{51}{12} = \frac{17}{4}$

---

b) Expectation of payoff is $\frac{6}{6} + \frac{5}{6} + \frac{2}{3} \times \frac{17}{4} = \underline{\underline{\frac{14}{3}}}$.

16 - 2

c) $\max(X, Y, Z) = 1 : \frac{1}{6^3}$

$2 : \frac{1}{3^3} - \frac{1}{6^3}$

$3 : \frac{1}{2^3} - \frac{1}{3^3}$

$4 : \left(\frac{2}{3}\right)^3 - \frac{1}{2^3}$

$5 : \left(\frac{5}{6}\right)^3 - \left(\frac{2}{3}\right)^3$

$6 : 1 - \left(\frac{5}{6}\right)^3$

$\therefore E(\max(X, Y, Z)) = \frac{119}{24} > \frac{14}{3}$

$\therefore$ Amended game has the higher expected payoff.

⑦

prize position X

choice Y

empty door Z

$$Pr(X=2 \mid Y=1, Z=3)$$

$$= \frac{P(Z=3 \mid X=2, Y=1)\, P(X=2 \mid Y=1)}{P(Z=3 \mid Y=1)}$$

$P(Z=3 \mid X=2, Y=1) = \frac{1}{8}$

$P(Y=2 \mid Y=1) = \frac{1}{10}$

$P(Z=3 \mid Y=1) = \frac{1}{10}\left(\frac{1}{9} + 0 + \frac{1}{8} \times 8\right) = \frac{1}{9}$

$\therefore \dfrac{\frac{1}{8} \times \frac{1}{10}}{\frac{1}{9}} = \frac{9}{80} > \frac{1}{10}$

↳ If you don't change

$\therefore$ You should choose another.

18

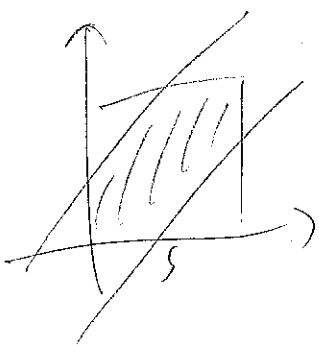
$X \sim U(0,1)$ $\quad f(x) = 1 \quad 0 < x < 1$

$Y \sim U(0,1)$ $\quad f(y) = 1 \quad 0 < y < 1$

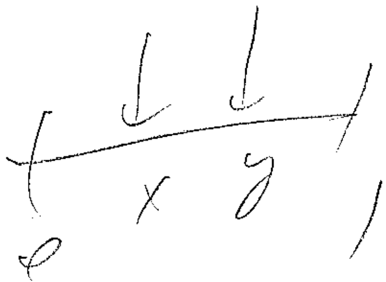
distribution of $|X-Y|$

$$\Pr(|X-Y| \le s) = \iint_{|x-y| \le s} 1 \, dx\,dy = 1-(1-s)^2$$

$$= \underline{\underline{2s - s^2}}$$

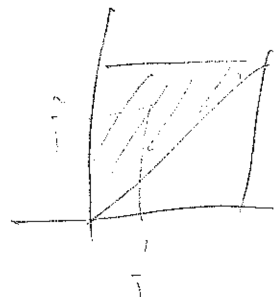


$\boxed{19}$



$0<x<y<1$

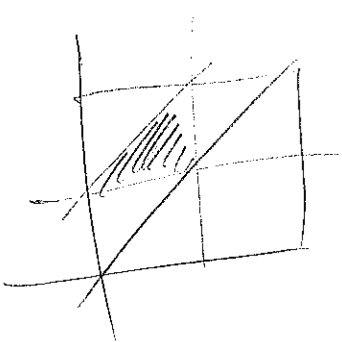
prob of forming triangle



$f(x,y) = 2$ if $0<x<y<1$

max should less than $\frac{1}{2}$.

That means, $x<\frac{1}{2}$, $y-x<\frac{1}{2}$, $y>\frac{1}{2}$



$$\therefore \frac{\frac{1}{8}}{\frac{1}{2}} = \frac{1}{4}$$

10 -1

$f(x,y) = Cxy^2, \quad 0 \le x \le y \le 1$

(1) $\int_0^1 \int_0^y Cxy^2 \, \cancel{dy\,dx}\ \cancel{dy\,dx}\ dx\,dy$

$\frac{1}{2}x^2y^2$

$= C\int_0^1 \frac{1}{2}x^2y^2 \Big|_0^y \, dy$

$= C\int_0^1 \frac{1}{2}y^4 \, dy = \frac{1}{10}C = 1 \to \boxed{C=10}$

(2) No

(3) a) $\cancel{P(X\le a) = \int_0^a \int_x^1 10xy^2 \, dy\,dx = \frac{5}{3}a^2(1-a^3)}$

$\cancel{f(x) = \frac{5}{3}(2x-5x^4)}$

$\cancel{E(X) =}$

$P(X\le a) = \int_0^a \int_x^1 10xy^2 \, dy\,dx = \frac{1}{3}(5a^2-2a^5)$

$f(x) = \frac{10}{3}(x-x^4)$

$E(X) = \frac{5}{9}$

10 - 2

(3) b) $P(Y \le a) = \int_0^a \int_0^y 10xy^2 \, dx\, dy = a^5$

$f(y) = 5y^4$

$E(Y) = \frac{5}{6}$

c) $E(X \mid Y = y)$

$= \int_0^y x \frac{f(x,y)}{f(y)} dx = \frac{2}{3} y$

d) $E(Y \mid X = x)$

$= \int_x^1 y \frac{f(x,y)}{f(x)} dy$

$= \frac{3}{4} \frac{1-x^4}{1-x^3}$

(1)

$$\underset{c}{\arg\min}\ E\left[(X-c)^2\right]$$

$$E\left[(X-c)^2\right] = \int (x-c)^2 f(x)\, dx$$

$$= \int (c^2 - 2xc + x^2) f(x)\, dx$$

$$= c^2 - 2c\,E(X) + E(X^2)$$

$$\therefore \underline{c = E(X)}$$

12

$$f = c x e^{-k^2 x^2}, \quad x > 0$$

$$\int_0^\infty f(x)\,dx = c\int_0^\infty x e^{-k^2x^2}\,dx = 1$$

Standard: $\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$ $N(0,1)$

$$E(X^2) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x^2 e^{-\frac{1}{2}x^2}\,dx = 1$$

$$\cancel{\int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\,dx = \frac{1}{2}}$$

(1) $\int_0^\infty x e^{-k^2x^2}\,dx = -\frac{1}{2k^2} e^{-k^2x^2}\Big|_0^\infty$

$$= -\frac{1}{2k^2}(0-1) = \frac{1}{2k^2}$$

$\frac{c}{2k^2} = 1 \quad \therefore \underline{\underline{c = 2k^2}}$

(2) $E(X) = \int_0^\infty c x^2 e^{-k^2x^2}\,dx = \frac{\sqrt{\pi}}{2|k|}$

$E(X^2) = \int_0^\infty c x^3 e^{-k^2x^2}\,dx = \frac{1}{k^2}$ (tricky)

$\therefore Var(X) = \left(1 - \frac{\pi}{4}\right)\frac{1}{k^2}$

(13)

A expectation: $3 \times \frac{1}{4} + 1 \times \frac{1}{4} - 2 \times \frac{1}{2}$

$= 1 - 1 = 0$

$\therefore$ It's a fair game.

(14)

100 coins

99 | 1
 | H.H

draw, 10 times toss

Given 10 heads → unfair coin

$$\frac{\frac{1}{100} \times 1}{\frac{99}{100}\left(\frac{1}{2}\right)^{10} + \frac{1}{100} \times 1} = 91\%$$

(15)

50 cards

$$pr(A<B<C<D) = ?$$

$$= \frac{{}_{50}C_4}{{}_{50}P_4} = \frac{\frac{50 \cdots}{4!}}{50 \cdot 49 \cdot 48 \cdot 47} = \frac{1}{24}$$

expectation: $\frac{10}{24} - \frac{23}{24} < 0$ **No**

If 100 cards: **Same**

16

A ~~Pr~~ deliberately miss the target

B $\begin{cases} P \to 0 \\ X \to \frac{25}{63} \end{cases}$

C $\begin{cases} P \to \frac{1}{7} \\ X \to \frac{25}{63} \end{cases}$

X $\to$ $\frac{25}{63}$ $\leftarrow$ best!

17

girl 1 $\frac{1}{2}$

girl 1 boy 1 $\frac{1}{4}$

girl 1 boy 2 $\frac{1}{8}$

⋮

$E(girl) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1$

$E(boy) = \frac{1}{4} + \frac{2}{8} + \frac{3}{16} + \cdots = 1$

$\therefore$ 50 : 50

18

$1/1 \to 66/34$

$$p = {}_{98}C_{33} \times \frac{65!\,33!}{99!}$$

$$= \frac{98!}{33!\,65!}\,\frac{65!\,33!}{99!} = \underline{\underline{\frac{1}{99}}}$$