ECE 68000: MODERN AUTOMATIC CONTROL

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Mathematical Modeling

Mathematical modeling—Outline

- Mathematical model
- Simulation and design models
- Review of work and energy
- The work-energy theorem for a particle
- Newton's second law
- The Lagrangian
- The Lagrange equations of motion

Mathematical model—Section 1.4

- A mathematical model is a mathematical representation of the significant, relevant, aspects of a given physical system
- Significance and relevance being in relation to an application where the model is to be used
- A physical system and its model are not the same things
- For brevity, refer to the physical system's model as the system
- A mathematical model of a system used to design a controller is one of the four essential elements of the control problem

Two models

- Simulation model also called truth model
- Design model

Simulation model

- The truth model is the simulation model that should include all the relevant characteristics of the physical system to be controlled
- The control designs are being simulated using the truth model
- The simulation model usually to complex for the controller design purpose

Design model

- A design model is developed by eliminating all unpleasant nonlinear effects while capturing all the essential features of the process
- A design model is much more amenable to use for the design of a control system than a simulation model

Review of work and energy—Newton's second law

- Suppose we are given a particle of a constant mass m subjected to a force $\boldsymbol{F}$
- By Newton's second law, we have

$$\boldsymbol{F} = m\frac{d\boldsymbol{v}}{dt} = m\boldsymbol{a},$$

where the force $\boldsymbol{F}$ and the velocity $\boldsymbol{v}$ are vectors

Newton's second law in 3D

$$\boldsymbol{F} = \begin{bmatrix} F_{x_1} \\ F_{x_2} \\ F_{x_3} \end{bmatrix} = m \begin{bmatrix} \frac{d^2 x_1}{dt^2} \\ \frac{d^2 x_2}{dt^2} \\ \frac{d^2 x_3}{dt^2} \end{bmatrix}$$

$$= m \begin{bmatrix} \frac{dv_{x_1}}{dt} \\ \frac{dv_{x_2}}{dt} \\ \frac{dv_{x_3}}{dt} \end{bmatrix} = m \begin{bmatrix} a_{x_1} \\ a_{x_2} \\ a_{x_3} \end{bmatrix}$$

Suppose that a force $\boldsymbol{F}$ is acting on a particle located at a point A and the particle moves to a point B

Work along infinitesimally small distance

- The work δW done by $\boldsymbol{F}$ along infinitesimally small distance $\delta \boldsymbol{s} = \begin{bmatrix} \delta x_1 & \delta x_2 & \delta x_3 \end{bmatrix}^\top$ is
-
$$\begin{aligned} \delta W &= \boldsymbol{F}^\top \delta \boldsymbol{s} \\ &= \begin{bmatrix} F_{x_1} & F_{x_2} & F_{x_3} \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \\ \delta x_3 \end{bmatrix} \\ &= F_{x_1} \delta x_1 + F_{x_2} \delta x_2 + F_{x_3} \delta x_3 \end{aligned}$$
- The work W_{AB} done on the path from A to B is obtained by integrating the above equation

Work W_{AB} done on the path from A to B

$$W_{AB} = \int_A^B (F_{x_1} dx_1 + F_{x_2} dx_2 + F_{x_3} dx_3)$$

- We should like to establish a relation between work and kinetic energy

Work W_{AB} in a different format

- Observe that

$$\ddot{x}\,dx = \frac{d\dot{x}}{dt}dx = \dot{x}d\dot{x}$$

- Express W_{AB} as

$$\begin{aligned} W_{AB} &= \int_A^B (m\ddot{x}_1 dx_1 + m\ddot{x}_2 dx_2 + m\ddot{x}_3 dx_3) \\ &= \int_A^B m\,(\dot{x}_1 d\dot{x}_1 + \dot{x}_2 d\dot{x}_2 + \dot{x}_3 d\dot{x}_3) \\ &= m\left(\frac{\dot{x}_1^2}{2} + \frac{\dot{x}_2^2}{2} + \frac{\dot{x}_3^2}{2}\right)\Bigg|_A^B \end{aligned}$$

Work W_{AB}—contd.

- Let $\boldsymbol{v}_A = \begin{bmatrix} \dot{x}_{1A} & \dot{x}_{2A} & \dot{x}_{3A} \end{bmatrix}^\top$ and $\boldsymbol{v}_B = \begin{bmatrix} \dot{x}_{1B} & \dot{x}_{2B} & \dot{x}_{3B} \end{bmatrix}^\top$
- Recall that $\|\boldsymbol{v}\|^2 = \boldsymbol{v}^\top \boldsymbol{v} = \dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2$
- Then,

$$\begin{aligned} W_{AB} &= m\left(\frac{\dot{x}_1^2}{2} + \frac{\dot{x}_2^2}{2} + \frac{\dot{x}_3^2}{2}\right)\Bigg|_A^B \\ &= m\left(\frac{\|\boldsymbol{v}_B\|^2}{2} - \frac{\|\boldsymbol{v}_A\|^2}{2}\right) \\ &= \frac{m\|\boldsymbol{v}_B\|^2}{2} - \frac{m\|\boldsymbol{v}_A\|^2}{2} \end{aligned}$$

W_{AB} and kinetic energy

- $\frac{m\|\boldsymbol{v}_B\|^2}{2}$ is the kinetic energy of the particle at the point B and $\frac{m\|\boldsymbol{v}_A\|^2}{2}$ is its kinetic energy at the point A
- $$W_{AB} = \frac{m\|\boldsymbol{v}_B\|^2}{2} - \frac{m\|\boldsymbol{v}_A\|^2}{2} = K_B - K_A$$
  is the work required to change particle's velocity from some value $\boldsymbol{v}_A$ to a final value $\boldsymbol{v}_B$
- $W_{AB} = K_B - K_A = \Delta K$ is the work-energy theorem for a particle

The work-energy theorem for a particle

- The work is equal to the change in the kinetic energy,

$$W_{AB} = K_B - K_A = \Delta K$$

- Now, if the kinetic energy K changes by ΔK, the potential energy U must change by an equal but opposite amount,

$$\Delta K + \Delta U = 0.$$

- The work done by a conservative force depends only on the starting and the end points of motion and not on the path followed between them

The work-energy theorem for a particle—contd.

- Therefore, for motion in one dimension,

$$\begin{aligned}\Delta U &= U(x) - U(x_0) \\ &= -W \\ &= -\int_{x_0}^{x} F(s)\,ds\end{aligned}$$

- Differentiate the above with respect to x noting that the derivative of the constant reference $U(x_0)$ is zero,

$$F(x) = -\frac{dU(x)}{dx}$$

Generalization to 3D

- We have

$$\Delta U = -\int_{x_{10}}^{x_1} F_{x_1}\, ds - \int_{x_{20}}^{x_2} F_{x_2}\, ds - \int_{x_{30}}^{x_3} F_{x_3}\, ds,$$

- Observe that

$$\begin{aligned} \nabla(\Delta U) &= \nabla(U(\boldsymbol{x}) - U(\boldsymbol{x}_0)) \\ &= \nabla U(\boldsymbol{x}) \end{aligned}$$

because the gradient of the constant reference $U(\boldsymbol{x}_0)$ is zero

Generalization to 3D—contd.

- Hence

$$
\begin{aligned}
\boldsymbol{F}(\boldsymbol{x}) &= -\nabla(\Delta U) \\
&= -\begin{bmatrix} \frac{\partial U}{\partial x_1} \\ \frac{\partial U}{\partial x_2} \\ \frac{\partial U}{\partial x_3} \end{bmatrix} \\
&= -\nabla U(\boldsymbol{x}),
\end{aligned}
$$

where $\boldsymbol{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^\top$.

Newton's equation in an equivalent format

- Note that

$$K = \frac{m \left\| \dot{\boldsymbol{x}} \right\|^2}{2} = \frac{m \dot{\boldsymbol{x}}^\top \dot{\boldsymbol{x}}}{2}$$

- Hence,

$$\frac{\partial K}{\partial \dot{x}_i} = m \dot{x}_i, \quad i = 1, 2, 3$$

- Thus,

$$\frac{d}{dt} \left(\frac{\partial K}{\partial \dot{x}_i} \right) + (\nabla U)_i = 0, \quad i = 1, 2, 3$$

Newton's equation in an equivalent format—contd.

- Let
$$L = K - U$$
- The function L is called the *Lagrangian function* or just the Lagrangian
- Note that
$$\frac{\partial L}{\partial \dot{x}_i} = \frac{\partial K}{\partial \dot{x}_i}$$
because $\frac{\partial U}{\partial \dot{x}_i} = 0$
- Also
$$\frac{\partial L}{\partial x_i} = -\frac{\partial U}{\partial x_i}$$
because $\frac{\partial K}{\partial x_i} = 0$

The Lagrange equations of motion in Cartesian coordinates

- Combining the above gives

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0, \quad i = 1, 2, 3$$

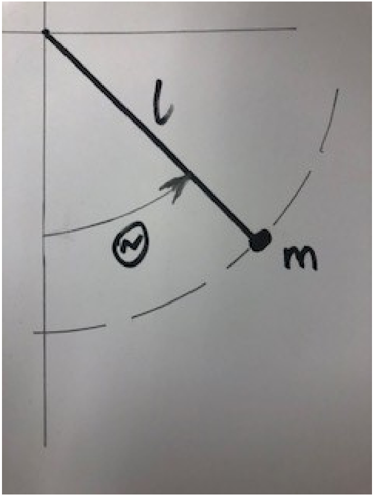
- The above is an equivalent representation of Newton's equations

$$m\ddot{x}_i - F_{x_i} = 0, \quad i = 1, 2, 3$$

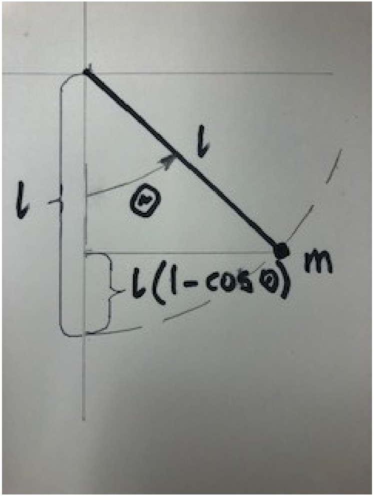
Example—the simple pendulum

- The simple pendulum is an idealized body consisting of a point mass m, suspended by a weightless inextensible cord of length l
- The simple pendulum is an example of a one degree of freedom system with a generalized coordinate being the angular displacement θ

The simple pendulum



Simple pendulum analysis



The Lagrangian for the simple pendulum

- The pendulum kinetic energy

$$K = \frac{1}{2} m l^2 \dot{\theta}^2$$

- The potential energy of the pendulum is zero when the pendulum is at rest, that is, $\theta = 0$
- Hence, its potential energy

$$U = mgl(1 - \cos\theta),$$

  where g is the acceleration due to gravity
- The Lagrangian function

$$L = K - U = \frac{1}{2} m l^2 \dot{\theta}^2 - mgl(1 - \cos\theta)$$

The simple pendulum equations of motion—manipulations

- We compute

$$\frac{\partial L}{\partial \theta} = -mgl \sin\theta$$

and

$$\frac{\partial L}{\partial \dot{\theta}} = ml^2 \dot{\theta}$$

- Hence

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = ml^2 \ddot{\theta}$$

The simple pendulum equations of motion—more manipulations

- Combining the above terms together, we obtain the Lagrange equation describing the pendulum motion,

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} &= ml^2\ddot{\theta} + mgl\sin\theta \\
&= 0
\end{aligned}
$$

- Solving for $\ddot{\theta}$, we obtain

$$
\ddot{\theta} = -\frac{g}{l}\sin\theta
$$

State-space model of the simple pendulum

- Let $x_1 = \theta$ and $x_2 = \dot{x}_1 = \dot{\theta}$
- Recall the simple pendulum equation of motion
$$\ddot{\theta} = -\frac{g}{l}\sin\theta$$
- Then, we have
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{g}{l}\sin x_1 \end{bmatrix}$$