# Optimal Estimation Methods

# (Lecture 14 – Kalman Filtering: Part III)

Dr. John L. Crassidis

University at Buffalo – State University of New York
Department of Mechanical & Aerospace Engineering
Amherst, NY 14260-4400
johnc@buffalo.edu
http://www.buffalo.edu/~johnc

University at Buffalo The State University of New York

# Continuous Kalman Filter (i)

- State and Measurement Model

$$\dot{\mathbf{x}}(t) = F(t)\,\mathbf{x}(t) + B(t)\,\mathbf{u}(t) + G(t)\,\mathbf{w}(t)$$
$$\tilde{\mathbf{y}}(t) = H(t)\,\mathbf{x}(t) + \mathbf{v}(t)$$

where $\mathbf{w}(t)$ and $\mathbf{v}(t)$ are zero-mean Gaussian noise processes with covariances given by (Actually spectral densities)

$$E\left\{\mathbf{w}(t)\,\mathbf{w}^T(\tau)\right\} = Q(t)\,\delta(t-\tau)$$
$$E\left\{\mathbf{v}(t)\,\mathbf{v}^T(\tau)\right\} = R(t)\,\delta(t-\tau)$$
$$E\left\{\mathbf{v}(t)\,\mathbf{w}^T(\tau)\right\} = 0$$

(Infinite noise)

where the Dirac delta function is defined by

$$\delta(t-\tau) = \begin{cases} 0 & t \neq \tau \\ \infty & t = \tau \end{cases}$$

Physically doesn't make sense because we have an infinite covariance. Discuss why this function is used later.

- Kalman filter structure given by

$$\dot{\hat{\mathbf{x}}}(t) = F(t)\,\hat{\mathbf{x}}(t) + B(t)\,\mathbf{u}(t) + K(t)[\tilde{\mathbf{y}}(t) - H(t)\,\hat{\mathbf{x}}(t)]$$
$$\hat{\mathbf{y}}(t) = H(t)\,\hat{\mathbf{x}}(t)$$

# Continuous Kalman Filter (ii)

- Define the following state error $\tilde{\mathbf{x}}(t) = \hat{\mathbf{x}}(t) - \mathbf{x}(t)$
- Take derivative and substitute true and estimate equations

$$\dot{\tilde{\mathbf{x}}}(t) = E(t)\,\tilde{\mathbf{x}}(t) + \mathbf{z}(t)$$

where

$$E(t) = F(t) - K(t)\,H(t)$$

← Dynamics of error & estimate

$$\mathbf{z}(t) = -G(t)\,\mathbf{w}(t) + K(t)\,\mathbf{v}(t)$$

- Note that $\mathbf{u}(t)$ cancels in the error state
- Since $\mathbf{w}(t)$ and $\mathbf{v}(t)$ are uncorrelated, we have

$$E\left\{\mathbf{z}(t)\,\mathbf{z}^T(\tau)\right\} = \left[G(t)\,Q(t)\,G^T(t) + K(t)\,R(t)\,K^T(t)\right]\delta(t-\tau)$$

- Using the matrix exponential solution for the error state

$$\tilde{\mathbf{x}}(t) = \Phi(t,t_0)\,\tilde{\mathbf{x}}(t_0) + \int_{t_0}^{t} \Phi(t,\tau)\,\mathbf{z}(\tau)\;d\tau$$

  - Note that $\Phi(t,t_0)$ is associated with $E(t)$ not $F(t)$

# Continuous Kalman Filter (iii)

- The state error covariance is defined by $P(t) \equiv E\left\{\tilde{\mathbf{x}}(t)\,\tilde{\mathbf{x}}^T(t)\right\}$
- Assuming that the initial state error is uncorrelated with $\mathbf{z}(t)$ gives

$$P(t) = \Phi(t,t_0)\,P(t_0)\,\Phi^T(t,t_0) + E\left\{\left[\int_{t_0}^{t}\Phi(t,\tau)\,\mathbf{z}(\tau)\;d\tau\right]\left[\int_{t_0}^{t}\Phi(t,\tau)\,\mathbf{z}(\tau)\;d\tau\right]^T\right\}$$

$$= \Phi(t,t_0)\,P(t_0)\,\Phi^T(t,t_0) + \int_{t_0}^{t}\int_{t_0}^{t}\Phi(t,\tau)\,E\left\{\mathbf{z}(\tau)\,\mathbf{z}^T(\zeta)\right\}\Phi^T(t,\zeta)\;d\tau\;d\zeta$$

- Substituting

$$E\left\{\mathbf{z}(\tau)\,\mathbf{z}^T(\zeta)\right\} = \left[G(\tau)\,Q(\tau)\,G^T(\tau) + K(\tau)\,R(\tau)\,K^T(\tau)\right]\delta(\tau-\zeta)$$

gives

$$P(t) = \Phi(t,t_0)\,P(t_0)\,\Phi^T(t,t_0)$$
$$+ \int_{t_0}^{t}\int_{t_0}^{t}\Phi(t,\tau)\left[G(\tau)\,Q(\tau)\,G^T(\tau) + K(\tau)\,R(\tau)\,K^T(\tau)\right]\Phi^T(t,\zeta)\delta(\tau-\zeta)\;d\tau\;d\zeta$$

# Delta Function (i)

- Now use the following property of the Dirac delta function

$$\int_a^b f(\tau)\,\delta(\tau-\zeta)\;d\tau = f(\zeta) \qquad (1)$$

- Discussion on why this is true; Dirac delta function is defined by

$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases}$$

with

$$\int_{t_1}^{t_2} \delta(t)\;dt = 1$$

if $0 \in [t_1, t_2]$ and zero otherwise

- It is “infinitely peaked” at $t = 0$ with the total area of unity
- Can view this function as a limit of Gaussian (remember that the total integral of the pdf is one!)

$$\delta(t) = \lim_{\sigma\to 0} \frac{1}{\sqrt{2\pi}\sigma} e^{-t^2/2\sigma^2}$$

# Delta Function (ii)

- Important property is given by

$$\int f(t)\,\delta(t)\ dt = f(0)$$

- This is easy to see
- First of all, $\delta(t)$ vanishes everywhere except $t = 0$
  - Therefore, it does not matter what values the function $f(t)$ takes except at $t = 0$
  - We can then say $f(t)\,\delta(t) = f(0)\delta(t)$
  - Then $f(0)$ can be pulled outside the integral; does not depend on $t$
  - Can easily be generalized to obtain Eq. (1)
  - The Dirac delta function is not a function, because it is too singular
    - Instead, it is said to be a "distribution"
    - It is a generalized idea of functions, but can be used only inside integrals
    - In fact, $\int \delta(t)\ dt$ can be regarded as an "operator" which pulls the value of a function at zero

# Continuous Kalman Filter (i)

- Therefore, $P(t)$ is given by

$$
\begin{aligned}
P(t) &= \Phi(t,t_0)\,P(t_0)\,\Phi^T(t,t_0) \\
&\quad + \int_{t_0}^{t} \Phi(t,\zeta)\left[G(\zeta)\,Q(\zeta)\,G^T(\zeta) + K(\zeta)\,R(\zeta)\,K^T(\zeta)\right]\Phi^T(t,\zeta)\;d\zeta
\end{aligned}
$$

- Replace the dummy variable $\zeta$ with $\tau$

$$
\begin{aligned}
P(t) &= \Phi(t,t_0)\,P(t_0)\,\Phi^T(t,t_0) \\
&\quad + \int_{t_0}^{t} \Phi(t,\tau)\left[G(\tau)\,Q(\tau)\,G^T(\tau) + K(\tau)\,R(\tau)\,K^T(\tau)\right]\Phi^T(t,\tau)\;d\tau
\end{aligned}
$$

- This is just done to keep notation consistent

# Continuous Kalman Filter (ii)

- Taking the time derivative of $P(t)$ gives

$$\begin{aligned}
\dot{P}(t) &= \frac{\partial \Phi(t,t_0)}{\partial t} P(t_0)\Phi^T(t,t_0) + \Phi(t,t_0)P(t_0)\frac{\partial \Phi^T(t,t_0)}{\partial t} \\
&\quad + \int_{t_0}^{t} \frac{\partial \Phi(t,\tau)}{\partial t} \left[G(\tau)\,Q(\tau)\,G^T(\tau) + K(\tau)\,R(\tau)\,K^T(\tau)\right] \Phi^T(t,\tau)\,d\tau \\
&\quad + \int_{t_0}^{t} \Phi(t,\tau) \left[G(\tau)\,Q(\tau)\,G^T(\tau) + K(\tau)\,R(\tau)\,K^T(\tau)\right] \frac{\partial \Phi^T(t,\tau)}{\partial t}\,d\tau \\
&\quad + \Phi(t,t) \left[G(t)\,Q(t)\,G^T(t) + K(t)\,R(t)\,K^T(t)\right] \Phi^T(t,t)
\end{aligned}$$

- Using the properties of the matrix exponential leads to

$$\begin{aligned}
\dot{P}(t) &= E(t)\,\Phi(t,t_0)\,P(t_0)\Phi^T(t,t_0) + \Phi(t,t_0)P(t_0)\,\Phi^T(t,t_0)\,E^T(t) \\
&\quad + E(t)\int_{t_0}^{t} \Phi(t,\tau) \left[G(\tau)\,Q(\tau)\,G^T(\tau) + K(\tau)\,R(\tau)\,K^T(\tau)\right] \Phi^T(t,\tau)\,d\tau \\
&\quad + \int_{t_0}^{t} \Phi(t,\tau) \left[G(\tau)\,Q(\tau)\,G^T(\tau) + K(\tau)\,R(\tau)\,K^T(\tau)\right] \Phi^T(t,\tau)\,d\tau\,E^T(t) \\
&\quad + G(t)\,Q(t)\,G^T(t) + K(t)\,R(t)\,K^T(t)
\end{aligned}$$

# Continuous Kalman Filter (iii)

- Using the definition of $P(t)$ and $E(t)$ gives

$$\begin{aligned}\dot{P}(t) = {} & [F(t) - K(t)\,H(t)]\,P(t) + P(t)\,[F(t) - K(t)\,H(t)]^T \\ & + G(t)\,Q(t)\,G^T(t) + K(t)\,R(t)\,K^T(t)\end{aligned} \qquad (1)$$

  - Valid for any gain $K(t)$
- Choose to minimize the trace of the derivative of $P(t)$

$$\text{minimize} \quad J[K(t)] = \text{Tr}[\dot{P}(t)]$$

- Why the derivative?
  - We wish to minimize the rate of increase of $P(t)$
  - Note that we cannot determine the definiteness of the derivative of $P(t)$ for general matrices of $F(t)$, $H(t)$ and $G(t)$, even though we assume that $R(t)$ is positive definite and that $Q(t)$ is at least positive semi-definite
  - Therefore, the trace of the derivative of $P(t)$ may be positive or negative at any given time
  - Still fine because we have *chosen* (i.e., we can choose anything) to minimize the rate of increase of $P(t)$

# Continuous Kalman Filter (iv)

- The necessary conditions lead to

$$\frac{\partial J}{\partial K(t)} = 0 = 2K(t)\,R(t) - 2P(t)\,H^T(t)$$

  - Note that the second derivative is $R(t)$, which is a positive definite matrix, leading to a minimization
- Solving for the gain gives

$$K(t) = P(t)\,H^T(t)\,R^{-1}(t)$$

  - Note the resemblance to $K_k = P_k^+ H_k^T R_k^{-1}$
- Substituting the gain into Eq. (1) gives

Valid for this gain only

$$\begin{aligned}\dot{P}(t) = {} & F(t)\,P(t) + P(t)\,F^T(t) \\ & - P(t)\,H^T(t)\,R^{-1}(t) H(t)\,P(t) + G(t)\,Q(t)\,G^T(t)\end{aligned}$$

  - This is known as the <u>*continuous-time Riccati equation*</u>

used for LQR control

# Continuous Kalman Filter (v)

Not used in practice

| | |
|---|---|
| **Model** | $\dot{\mathbf{x}}(t) = F(t)\,\mathbf{x}(t) + B(t)\,\mathbf{u}(t) + G(t)\,\mathbf{w}(t),\ \ \mathbf{w}(t) \sim N(\mathbf{0}, Q(t))$<br>$\tilde{\mathbf{y}}(t) = H(t)\,\mathbf{x}(t) + \mathbf{v}(t),\ \ \mathbf{v}(t) \sim N(\mathbf{0}, R(t))$ |
| **Initialize** | $\hat{\mathbf{x}}(t_0) = \hat{\mathbf{x}}_0$<br>$P_0 = E\left\{\tilde{\mathbf{x}}(t_0)\,\tilde{\mathbf{x}}^T(t_0)\right\}$ |
| **Gain** | $K(t) = P(t)\,H^T(t)\,R^{-1}(t)$ |
| **Covariance** | $\dot{P}(t) = F(t)\,P(t) + P(t)\,F^T(t)$<br>$-P(t)\,H^T(t)\,R^{-1}(t)H(t)\,P(t) + G(t)\,Q(t)\,G^T(t)$ |
| **Estimate** | $\dot{\hat{\mathbf{x}}}(t) = F(t)\,\hat{\mathbf{x}}(t) + B(t)\,\mathbf{u}(t)$<br>$+K(t)[\tilde{\mathbf{y}}(t) - H(t)\,\hat{\mathbf{x}}(t)]$ |

# Derivation from Discrete (i)

- Can also be derived from the discrete-time Kalman filter
  - Need to convert the measurement and process noise covariances
  - Work on process noise covariance first

$$\Upsilon_k E\left\{\mathbf{w}_k \mathbf{w}_k^T\right\} \Upsilon_k^T = \Upsilon_k Q_k \Upsilon_k^T$$

$$= E\left\{\left[\int_{t_k}^{t_{k+1}} \Phi(t_{k+1},\tau)\, G(\tau)\, \mathbf{w}(\tau)\ d\tau\right]\left[\int_{t_k}^{t_{k+1}} \Phi(t_{k+1},\varsigma)\, G(\varsigma)\, \mathbf{w}(\varsigma)\ d\varsigma\right]^T\right\}$$

$$= \int_{t_k}^{t_{k+1}} \int_{t_k}^{t_{k+1}} \Phi(t_{k+1},\tau)\, G(\tau)\, E\left\{\mathbf{w}(\tau)\, \mathbf{w}^T(\varsigma)\right\} G^T(\varsigma)\Phi^T(t_{k+1},\varsigma)\ d\tau\ d\varsigma$$

  - Substituting $E\{\mathbf{w}(\tau)\,\mathbf{w}^T(\zeta)\} = Q(\tau)\,\delta(\tau-\zeta)$ and using the property of the Dirac delta function leads to

$$\Upsilon_k Q_k \Upsilon_k^T = \int_{t_k}^{t_{k+1}} \Phi(t_{k+1},\tau)\, G(\tau)\, Q(\tau) G^T(\tau)\Phi^T(t_{k+1},\tau)\ d\tau$$

# Derivation from Discrete (ii)

- The integral is difficult to evaluate even for simple systems
- Look at a first order approximation with $\Phi \approx (I + \tau F)$
- Substitute this into the integral and define $\Delta t \equiv t_{k+1} - t_k$, and retain only first-order terms in $\Delta t$

$$\Upsilon_k Q_k \Upsilon_k^T = \Delta t \, G(t) \, Q(t) \, G^T(t)$$

- We should note here that the matrix $Q_k$ is a covariance matrix; however, the matrix $Q(t)$ is a *spectral density matrix*
  - Multiplying $Q(t)$ by the delta function converts it into a covariance matrix
- For time-invariant systems an exact solution is possible
- Form the following augmented matrix

$$\mathcal{A} = \begin{bmatrix} -F & G\,Q\,G^T \\ 0 & F^T \end{bmatrix} \Delta t$$

# Derivation from Discrete (iii)

- Then compute the matrix exponential of it

$$\mathcal{B} = e^{\mathcal{A}} \equiv \begin{bmatrix} \mathcal{B}_{11} & \mathcal{B}_{12} \\ 0 & \mathcal{B}_{22} \end{bmatrix} = \begin{bmatrix} \mathcal{B}_{11} & \Phi^{-1}\mathcal{Q} \\ 0 & \Phi^T \end{bmatrix}$$

where $\mathcal{Q} \equiv \Upsilon\, Q \Upsilon^T$ (where this $Q$ is the discrete-time covariance)

- The state transition matrix is then given by

$$\boxed{\Phi = \mathcal{B}_{22}^T}$$

- The discrete-time process noise covariance is given by

$$\boxed{\mathcal{Q} = \Phi\, \mathcal{B}_{12}}$$

- Use the discrete-time covariance propagation now

$$P_{k+1}^{-} = \Phi_k P_k^{+} \Phi_k^T + \Upsilon_k Q_k \Upsilon_k^T$$

- Note that this approach can adequately work for time-varying systems when the sampling interval is "small" enough

# Derivation from Discrete (iv)

- The relationship between the discrete measurement covariance and continuous measurement covariance is not as obvious as the process noise covariance case
- Consider the following linear model

$$\tilde{y}_k = x + v_k$$

- Suppose that the time interval $\Delta t$ is broken into equal samples, denoted by $\delta$
- The least-squares estimate for $x$ with $m$ observations is given by

$$\hat{x} = \frac{1}{m} \sum_{j=1}^{m} \tilde{y}_j$$

- The relationship between the discrete-time process $v_k$ and the continuous-time process must surely involve the sampling interval

# Derivation from Discrete (v)

- Consider the following relationship

$$E\left\{v_k v_j^T\right\} = \begin{cases} 0 & k \neq j \\ \delta^d R & k = j \end{cases}$$

  for some value of $d$

- Then, the estimate error variance is given by

$$E\left\{(x-\hat{x})^2\right\} = \frac{\delta^d R}{m}$$

- The limit $m \to \infty$, $\delta \to 0$, and $m\delta \to \Delta t$ gives

$$E\left\{(x-\hat{x})^2\right\} = \begin{cases} \infty & d < -1 \\ 0 & d > -1 \\ \dfrac{R}{\Delta t} & d = -1 \end{cases}$$

# Derivation from Discrete (vi)

- If the continuous model is to be meaningful in the sense that the error variance is nonzero but finite, we must choose $d = -1$
- Toward this end in the sampling process, the continuous-time measurement process must be averaged over the sampling interval $\Delta t$ in order to determine the equivalent discrete sample
- Then, we have

$$\tilde{\mathbf{y}}_k = \frac{1}{\Delta t}\int_{t_k}^{t_{k+1}} \tilde{\mathbf{y}}(t)\ dt = \frac{1}{\Delta t}\int_{t_k}^{t_{k+1}} \left[H(t)\,\mathbf{x}(t) + \mathbf{v}(t)\right]\ dt$$

$$\approx H_k \mathbf{x}_k + \frac{1}{\Delta t}\int_{t_k}^{t_{k+1}} \mathbf{v}(t)\ dt$$

- The discrete-time measurement covariance is determined using

$$E\left\{\mathbf{v}_k \mathbf{v}_k^T\right\} \equiv R_k = \frac{1}{\Delta t^2}\int_{t_k}^{t_{k+1}}\int_{t_k}^{t_{k+1}} E\left\{\mathbf{v}(\tau)\,\mathbf{v}^T(\varsigma)\right\}\ d\tau\ d\varsigma$$

# Derivation from Discrete (vii)

- Substituting $E\{\mathbf{v}(\tau)\,\mathbf{v}^T(\zeta)\} = R(\tau)\,\delta(\tau-\zeta)$ and using the property of the Dirac delta function leads to

$$\boxed{R_k = \frac{R(t)}{\Delta t}}$$

- The implication of this relationship is that the discrete-time covariance approaches infinity in the continuous representation
  - This may be counterintuitive at first, but as shown before the inverse time dependence of the discrete-time covariance and the continuous-time equivalent is the *only* relationship that yields a well-behaved process
  - Also, it seems to make sense that as the sampling interval decreases the noise, and thus its covariance increases
  - The limit when $\Delta t \to 0$ shows that the covariance approaches the continuous-time covariance, which is infinite. This is exactly why the covariance in the continuous-time has a delta function in it

# Derivation from Discrete (viii)

- Go back to discrete-time Kalman filter equations

$$\hat{\mathbf{x}}_{k+1} = \Phi_k \hat{\mathbf{x}}_k + \Gamma_k \mathbf{u}_k + \Phi K_k [\tilde{\mathbf{y}}_k - H_k \hat{\mathbf{x}}_k]$$

$$K_k = P_k H_k^T [H_k P_k H_k^T + R_k]^{-1}$$

$$P_{k+1} = \Phi_k P_k \Phi_k^T - \Phi_k K_k H_k P_k \Phi_k^T + \Upsilon_k Q_k \Upsilon_k^T$$

  - Use the first-order approximation $\Phi \approx (I + \Delta t\, F)$ and $\Upsilon_k Q_k \Upsilon_k^T = \Delta t\, G(t)\, Q(t)\, G^T(t)$ to give

$$\begin{aligned} P_{k+1} = {} & [I + \Delta t\, F(t)] P_k [I + \Delta t\, F(t)]^T + \Delta t\, G(t)\, Q(t)\, G^T(t) \\ & - [I + \Delta t\, F(t)] K_k H_k P_k [I + \Delta t\, F(t)]^T \end{aligned}$$

  - Dividing by $\Delta t$ and collecting terms gives

$$\begin{aligned} \frac{P_{k+1} - P_k}{\Delta t} = {} & F(t)\, P_k + P_k F^T(t) + \Delta t\, F(t)\, P_k F^T(t) \\ & - F(t)\, K_k H_k P_k - K_k H_k P_k F^T(t) - \frac{1}{\Delta t} K_k H_k P_k \\ & - \Delta t\, F(t)\, K_k H_k P_k F^T(t) + G(t)\, Q(t)\, G^T(t) \end{aligned} \qquad (1)$$

# Derivation from Discrete (ix)

- From the definition of the gain $K_k$ and using the previously derived relationship for $R_k$ we have

$$K_k = P_k H_k^T \left[ H_k P_k H_k^T + \frac{R(t)}{\Delta t} \right]^{-1}$$
$$= \Delta t \, P_k H_k^T [\Delta t \, H_k P_k H_k^T + R(t)]^{-1}$$

- Therefore, the limiting condition on $K_k$ gives

$$\lim_{\Delta t \to 0} K_k = 0$$

- However, when $K_k$ is divided by $\Delta t$ we have

$$\lim_{\Delta t \to 0} \frac{K_k}{\Delta t} = P(t) \, H^T(t) \, R^{-1}(t)$$

- Hence, in the limit as $\Delta t \to 0$ Eq. (1) reduces exactly to the continuous-time covariance propagation

$$\dot{P}(t) = F(t) \, P(t) + P(t) \, F^T(t) - P(t) \, H^T(t) \, R^{-1}(t) H(t) \, P(t) + G(t) \, Q(t) \, G^T(t)$$

# Derivation from Discrete (x)

- Using the first-order approximations of $\Gamma = \Delta t\, B$ and $\Phi = (I + \Delta t\, F)$, the discrete-time state estimate becomes

$$\hat{\mathbf{x}}_{k+1} = [I + \Delta t\, F(t)]\hat{\mathbf{x}}_k + \Delta t\, B(t)\,\mathbf{u}_k + [I + \Delta t\, F(t)]K_k[\tilde{\mathbf{y}}_k - H_k\hat{\mathbf{x}}_k]$$

- Dividing by $\Delta t$ and collecting terms gives

$$\frac{\hat{\mathbf{x}}_{k+1} - \hat{\mathbf{x}}_k}{\Delta t} = F(t)\,\hat{\mathbf{x}}_k + B(t)\,\mathbf{u}_k + \left[\frac{K_k}{\Delta t} + F(t)\,K_k\right][\tilde{\mathbf{y}}_k - H_k\hat{\mathbf{x}}_k]$$

- Hence, in the limit as $\Delta t \to 0$ this reduces exactly to the continuous-time state estimate equation

$$\dot{\hat{\mathbf{x}}}(t) = F(t)\,\hat{\mathbf{x}}(t) + B(t)\,\mathbf{u}(t) + K(t)[\tilde{\mathbf{y}}(t) - H(t)\,\hat{\mathbf{x}}(t)]$$

$$K(t) = P(t)\,H^T(t)\,R^{-1}(t)$$

- Continuous-time equations are “simpler” for analysis purposes than the discrete-time equations
  - For example, substitute $R(t) = \Delta t\, R_k$ in continuous-time version to determine the effects of sampling in the discrete-time covariance

- Stability proven through a Lyapunov function

$$V[\tilde{\mathbf{x}}(t)] = \tilde{\mathbf{x}}^T(t)\, P^{-1}(t)\, \tilde{\mathbf{x}}(t), \qquad \tilde{\mathbf{x}}(t) \equiv \hat{\mathbf{x}}(t) - \mathbf{x}(t) \qquad (1)$$

- Substitute the following into $\dot{\tilde{\mathbf{x}}}(t) = \dot{\hat{\mathbf{x}}}(t) - \dot{\mathbf{x}}(t)$

$$\dot{\hat{\mathbf{x}}}(t) = F(t)\, \hat{\mathbf{x}}(t) + B(t)\, \mathbf{u}(t) + K(t)[\tilde{\mathbf{y}}(t) - H(t)\, \hat{\mathbf{x}}(t)]$$
$$\dot{\mathbf{x}}(t) = F(t)\, \mathbf{x}(t) + B(t)\, \mathbf{u}(t) + G(t)\, \mathbf{w}(t)$$
$$\tilde{\mathbf{y}}(t) = H(t)\, \mathbf{x}(t) + \mathbf{v}(t)$$

to give

$$\dot{\tilde{\mathbf{x}}}(t) = \overbrace{[F(t) - K(t)\, H(t)]}^{\text{Error and estimate dynamics}}\tilde{\mathbf{x}}(t) + K(t)\, \mathbf{v}(t) - G(t)\, \mathbf{w}(t)$$

- Ignore the inputs since the matrix $F(t) - K(t)H(t)$ defines the stability of the filter, so use

$$\dot{\tilde{\mathbf{x}}}(t) = [F(t) - K(t)\, H(t)]\tilde{\mathbf{x}}(t) \qquad (2)$$

  - Note, just as the discrete-time case, in reality a stochastic stability analysis should be done since random inputs exist, but we'll ignore this here (does not change final result)

# Stability (ii)

- Need the time derivative of $P(t)$
  - Take the derivative of $P(t)P^{-1}(t) = I$

$$\frac{d}{dt}\left[P(t)\,P^{-1}(t)\right] = \dot{P}(t)\,P^{-1}(t) + P(t)\,\dot{P}^{-1}(t) = 0$$

  - Solving for the inverse time derivative gives

$$\dot{P}^{-1}(t) = -P^{-1}(t)\,\dot{P}(t)\,P^{-1}(t)$$

- Substituting the following

$$\dot{P}(t) = F(t)\,P(t) + P(t)\,F^T(t) - P(t)\,H^T(t)\,R^{-1}(t)H(t)\,P(t) + G(t)\,Q(t)\,G^T(t)$$

gives

$$\begin{aligned}\dot{P}^{-1}(t) = -P^{-1}(t)\,F(t) - F^T(t)\,P^{-1}(t) + H^T(t)\,R^{-1}(t)\,H(t)\\ - P^{-1}(t)\,G(t)\,Q(t)\,G^T(t)\,P^{-1}(t)\end{aligned} \quad (3)$$

- Taking the time derivative of Eq. (1) gives

$$\dot{V}[\tilde{\mathbf{x}}(t)] = \dot{\tilde{\mathbf{x}}}^T(t)\,P^{-1}(t)\,\tilde{\mathbf{x}}(t) + \tilde{\mathbf{x}}^T(t)\,P^{-1}(t)\,\dot{\tilde{\mathbf{x}}}(t) + \tilde{\mathbf{x}}^T(t)\,\dot{P}^{-1}(t)\,\tilde{\mathbf{x}}(t)$$

# Stability (iii)

- Substituting Eqs. (2) and (3), and simplifying yields

$$\dot{V}[\tilde{\mathbf{x}}(t)] = -\tilde{\mathbf{x}}^T(t)\left[H^T(t)\,R^{-1}(t)\,H(t) + P^{-1}(t)\,G(t)\,Q(t)\,G^T(t)\,P^{-1}(t)\right]\tilde{\mathbf{x}}(t)$$

- Clearly, if $R(t)$ is positive definite and $Q(t)$ is at least positive semi-definite, then the Lyapunov condition is satisfied and the continuous-time Kalman filter is stable
- Same analogies as the discrete-time Kalman filter
  - This means that the filter will "track" the measurements even if the measurements are unbounded!
  - These conditions are usually always met
  - Sometimes $Q(t)$ is zero, for example when estimating for a constant parameter

# Steady-State Kalman Filter

- Say we have an autonomous system
  - Filter state and output matrices as well as measurement and process noise covariance matrices are all time-invariant
  - Then the error-covariance reaches steady-state quickly
  - Covariance found by solving the *algebraic Riccati equation* (ARE)

| | |
|---|---|
| **Model** | $\dot{\mathbf{x}}(t) = F\,\mathbf{x}(t) + B\,\mathbf{u}(t) + G\,\mathbf{w}(t),\ \ \mathbf{w}(t) \sim N(\mathbf{0}, Q)$ <br> $\tilde{\mathbf{y}}(t) = H\,\mathbf{x}(t) + \mathbf{v}(t),\ \ \mathbf{v}(t) \sim N(\mathbf{0}, R)$ |
| **Initialize** | $\hat{\mathbf{x}}(t_0) = \hat{\mathbf{x}}_0$ |
| **Gain** | $K = P\,H^T R^{-1}$ |
| **Covariance** | $F\,P + P\,F^T - P\,H^T R^{-1} H\,P + G\,Q\,G^T = 0$ |
| **Estimate** | $\dot{\hat{\mathbf{x}}}(t) = F\,\hat{\mathbf{x}}(t) + B\,\mathbf{u}(t) + K[\tilde{\mathbf{y}}(t) - H\,\hat{\mathbf{x}}(t)]$ |

ARE

- Goal is to solve for the algebraic Riccati equation

$$F\,P + P\,F^T - P\,H^T\,R^{-1}H\,P + G\,Q\,G^T = 0$$

  - First need to show that the propagation can be factored into

$$P(t) = S(t)\,Z^{-1}(t), \quad \text{or} \quad P(t)\,Z(t) = S(t) \qquad (1)$$

    for some $n \times n$ matrices $S(t)$ and $Z(t)$
  - Taking the time derivative gives

$$\dot{P}(t)\,Z(t) + P(t)\dot{Z}(t) = \dot{S}(t) \qquad (2)$$

  - Substituting the following

$$\dot{P}(t) = F(t)\,P(t) + P(t)\,F^T(t) - P(t)\,H^T(t)\,R^{-1}(t)H(t)\,P(t) + G(t)\,Q(t)\,G^T(t)$$

    gives

$$\begin{aligned} &P(t)[F^T Z(t) - H^T R^{-1} H\,S(t) + \dot{Z}(t)] \\ &+ [G\,Q\,G^T Z(t) + F\,S(t) - \dot{S}(t)] = 0 \end{aligned}$$

# Riccati Equation (ii)

- Therefore, the following two differential equations must be true

$$\dot{Z}(t) = -F^T Z(t) + H^T R^{-1} H\, S(t)$$
$$\dot{S}(t) = G\, Q\, G^T Z(t) + F\, S(t)$$

- In order to satisfy Eqs. (1) and (2), initial conditions of $Z(t_0) = I$ and $S(t_0) = P(t_0)$ can be used
- Separating the columns of the $Z(t)$ and $S(t)$ matrices gives

$$\begin{bmatrix} \dot{\mathbf{z}}_i(t) \\ \dot{\mathbf{s}}_i(t) \end{bmatrix} = \mathcal{H} \begin{bmatrix} \mathbf{z}_i(t) \\ \mathbf{s}_i(t) \end{bmatrix}$$

where $\mathbf{z}_i(t)$ and $\mathbf{s}_i(t)$ are the $i^{\text{th}}$ columns of $Z(t)$ and $S(t)$, respectively, and $\mathcal{H}$ is the *Hamiltonian matrix*, defined by

$$\mathcal{H} \equiv \begin{bmatrix} -F^T & H^T R^{-1} H \\ G\, Q G^T & F \end{bmatrix}$$

- It can be shown that if $\lambda$ is an eigenvalue of $\mathcal{H}$, then $-\lambda$ is also an eigenvalue of $\mathcal{H}$

# Riccati Equation (iii)

- Thus, the eigenvalues can be arranged in a diagonal matrix with

$$\mathcal{H}_\Lambda = \begin{bmatrix} \Lambda & 0 \\ 0 & -\Lambda \end{bmatrix}$$

  where $\Lambda$ is a diagonal matrix of the $n$ unstable eigenvalues
- An eigenvalue/eigenvector decomposition of $\mathcal{H}$ gives

$$\mathcal{H}_\Lambda = W^{-1}\mathcal{H}\,W$$

  where $W$ is the matrix of eigenvectors, which can be represented in block form as

$$W = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}$$

- The solutions for $\mathbf{z}_i(t)$ and $\mathrm{s}_i(t)$ can be found in terms of their eigensystems

$$\mathbf{z}_i(t) = \mathbf{w}_1 e^{\lambda t}$$

$$\mathbf{s}_i(t) = \mathbf{w}_2 e^{\lambda t}$$

# Riccati Equation (iv)

where $\mathbf{w}_1$ and $\mathbf{w}_2$ are eigenvectors that satisfy

$$(\lambda I - \mathcal{H}) \begin{bmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \end{bmatrix} = \mathbf{0}$$

- Going forward in time the unstable eigenvalues dominate, so that

$$\mathbf{z}_i(t) \to W_{11}\, e^{\Lambda t}\, \mathbf{c}_i$$

$$\mathbf{s}_i(t) \to W_{21}\, e^{\Lambda t}\, \mathbf{c}_i$$

  where $\mathbf{c}_i$ is an arbitrary constant, and $W_{11}$ and $W_{21}$ are the eigenvectors associated with the unstable eigenvalues

- Then, from Eq. (1) it follows that at steady-state, we have

$$\boxed{P = W_{21} W_{11}^{-1}}$$

- Therefore, the gain $K$ can be computed off-line and remains constant
  - This can significantly reduce the on-board computational load

# Example (i)

- First-Order System

$$\dot{x}(t) = f\,x(t) + w(t)$$
$$\tilde{y}(t) = x(t) + v(t)$$

where $f$ is a constant, and the spectral densities of $w(t)$ and $v(t)$ are given by $q$ and $r$, respectively

- Scalar version of the Riccati equation is given by

$$\dot{p}(t) = 2\,f\,p(t) - r^{-1}p^2(t) + q, \quad p(t_0) = p_0$$

- The Hamiltonian system is given by

$$\begin{bmatrix} \dot{z}(t) \\ \dot{s}(t) \end{bmatrix} = \begin{bmatrix} -f & r^{-1} \\ q & f \end{bmatrix} \begin{bmatrix} z(t) \\ s(t) \end{bmatrix}, \quad \begin{bmatrix} z(t_0) \\ s(t_0) \end{bmatrix} = \begin{bmatrix} 1 \\ p_0 \end{bmatrix}$$

- The characteristic equation of this system is given by $s^2 - (f^2 + r^{-1}q) = 0$, which means the solutions for $z(t)$ and $s(t)$ involve hyperbolic functions

# Example (ii)

- Assume that the solutions are given by

$$z(t) = \cosh(a\,t) + c_1 \sinh(a\,t)$$
$$s(t) = p_0 \cosh(a\,t) + c_2 \sinh(a\,t)$$

where $a = (f^2 + r^{-1} q)^{1/2}$, and $c_1$ and $c_2$ are constants

- To determine the other constants, take time derivatives of $z(t)$ and $s(t)$ and compare them to the Hamiltonian system, giving

$$c_1 = \frac{p_0 r^{-1} - f}{a}, \quad c_2 = \frac{p_0 f + q}{a}$$

- Hence, using Eq. (1) the solution for $p(t)$ is given by

$$p(t) = \frac{p_0 a + (p_0 f + q)\tanh(a\,t)}{a + (p_0 r^{-1} - f)\tanh(a\,t)}$$

- Clearly, even for this simple first-order system the solution to the Riccati equation involves complicated functions
  - Analytical solutions are extremely difficult (if not impossible!) to determine for higher-order systems, so numerical procedures are typically required to integrate the Riccati differential equation

# Example (iii)

- The steady-state value for $p(t)$ is given by

$$\lim_{t\to\infty} p(t) \equiv p = \frac{(a+f)p_0 + q}{r^{-1}p_0 + a - f} = r\,(a+f)$$

- This result is verified by solving the algebraic Riccati equation
- Hence, the continuous-time Kalman filter equations are given by

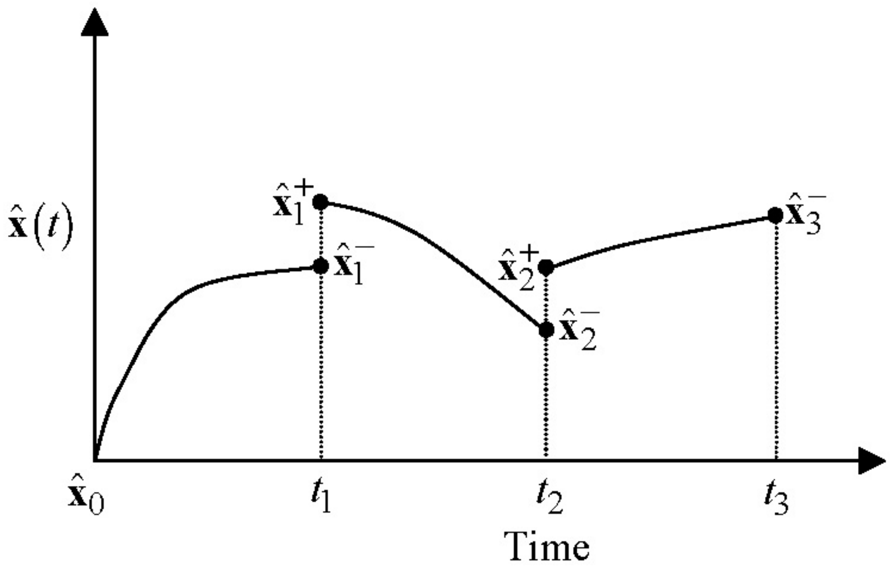
$$\dot{\hat{x}}(t) = -a\,\hat{x}(t) + (a+f)\tilde{y}(t)$$
$$\hat{y}(t) = \hat{x}(t)$$

- Note that the filter dynamics are always stable
- Also, when $q = 0$ the solution for the steady-state gain is given by zero (note, ARE solution does not exist in this case), and the measurements are completely ignored in the state estimate
  - This is only true at steady-state, not when using the full equations
- Furthermore, the individual values for $r$ and $q$ are irrelevant; only their ratio is important in the filter design

# Continuous-Discrete KF (i)

- Most physical dynamical systems involve continuous-time models and discrete-time measurements taken from a digital signal processor
  - The system model and measurement model are given by

$$\dot{\mathbf{x}}(t) = F(t)\,\mathbf{x}(t) + B(t)\,\mathbf{u}(t) + G(t)\,\mathbf{w}(t)$$
$$\tilde{\mathbf{y}}_k = H_k \mathbf{x}_k + \mathbf{v}_k$$




The state estimate model is propagated forward in time until a measurement occurs

Then a discrete-time state update occurs, which updates the propagated state

# Continuous-Discrete KF (ii)

| | |
|---|---|
| **Model** | $\dot{\mathbf{x}}(t) = F(t)\mathbf{x}(t) + B(t)\mathbf{u}(t) + G(t)\mathbf{w}(t),\ \mathbf{w}(t) \sim N(\mathbf{0}, Q(t))$ <br> $\tilde{\mathbf{y}}_k = H_k\mathbf{x}_k + \mathbf{v}_k, \quad \mathbf{v}_k \sim N(\mathbf{0}, R_k)$ |
| **Initialize** | $\hat{\mathbf{x}}(t_0) = \hat{\mathbf{x}}_0$ <br> $P_0 = E\left\{\tilde{\mathbf{x}}(t_0)\,\tilde{\mathbf{x}}^T(t_0)\right\}$ |
| **Gain** | $K_k = P_k^- H_k^T[H_k P_k^- H_k^T + R_k]^{-1}$ |
| **Update** | $\hat{\mathbf{x}}_k^+ = \hat{\mathbf{x}}_k^- + K_k[\tilde{\mathbf{y}}_k - H_k\hat{\mathbf{x}}_k^-]$ <br> $P_k^+ = [I - K_k H_k]P_k^-$ |
| **Propagation** | $\dot{\hat{\mathbf{x}}}(t) = F(t)\,\hat{\mathbf{x}}(t) + B(t)\,\mathbf{u}(t)$ <br> $\dot{P}(t) = F(t)\,P(t) + P(t)\,F^T(t) + G(t)\,Q(t)\,G^T(t)$ |

# Extended Kalman Filter (i)

- A large class of estimation problems involve nonlinear models (possibly both the state and measurements)
  - For several reasons, state estimation for nonlinear systems is considerably more difficult and admits a wider variety of solutions than the linear problem
  - Several filters have been developed to handle this case
  - Common one is the Extended Kalman Filter (EKF)
- Extended Kalman Filter
  - This is a quasi-linear Kalman filter version
  - Retains same form for the covariance update and propagation as the linear Kalman filter
    - Assumes that the state errors are "small" so that a first-order Taylor series approximation is valid
    - Steady-state forms are usually not possible though
  - Model propagation and output estimates are done using the nonlinear models

# Extended Kalman Filter (ii)

- Continuous-Discrete Case

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t),\, \mathbf{u}(t),\, t) + G(t)\,\mathbf{w}(t),\ \mathbf{w}(t) \sim \mathcal{N}(\mathbf{0}, Q(t))$$
$$\tilde{\mathbf{y}}_k = \mathbf{h}(\mathbf{x}_k) + \mathbf{v}_k,\ \mathbf{v}_k \sim \mathcal{N}(\mathbf{0}, R_k)$$

- First-order Taylor series expansion about some $\bar{\mathbf{x}}(t)$

$$\mathbf{f}(\mathbf{x}(t),\, \mathbf{u}(t),\, t) \cong \mathbf{f}(\bar{\mathbf{x}}(t),\, \mathbf{u}(t),\, t) + \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\bar{\mathbf{x}}(t)} [\mathbf{x}(t) - \bar{\mathbf{x}}(t)]$$

  - Standard EKF uses current estimate, so that $\bar{\mathbf{x}}(t) = \hat{\mathbf{x}}(t)$
- Assuming an unbiased estimate, $E\left\{\hat{\mathbf{x}}(t)\right\} = \mathbf{x}(t)$, gives

$$E\left\{\mathbf{f}(\mathbf{x}(t),\, \mathbf{u}(t),\, t)\right\} = \mathbf{f}(\hat{\mathbf{x}}(t),\, \mathbf{u}(t),\, t)$$

  - Used to developed covariance propagation
- Same approach is applied to output equation
- Filter stability is not guaranteed!
  - Must be very careful or filter can diverge

# EKF Summary

| | |
|---|---|
| **Model** | $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t),\, \mathbf{u}(t),\, t) + G(t)\,\mathbf{w}(t),\ \mathbf{w}(t) \sim \mathcal{N}(\mathbf{0}, Q(t))$<br>$\tilde{\mathbf{y}}_k = \mathbf{h}(\mathbf{x}_k) + \mathbf{v}_k,\ \mathbf{v}_k \sim \mathcal{N}(\mathbf{0}, R_k)$ |
| **Initialize** | $\hat{\mathbf{x}}(t_0) = \hat{\mathbf{x}}_0$<br>$P_0 = E\left\{\tilde{\mathbf{x}}(t_0)\,\tilde{\mathbf{x}}^T(t_0)\right\}$ |
| **Gain** | $K_k = P_k^- H_k^T(\hat{\mathbf{x}}_k^-)[H_k(\hat{\mathbf{x}}_k^-)P_k^- H_k^T(\hat{\mathbf{x}}_k^-) + R_k]^{-1}$<br>$H_k(\hat{\mathbf{x}}_k^-) \equiv \left.\frac{\partial \mathbf{h}}{\partial \mathbf{x}}\right\rvert_{\hat{\mathbf{x}}_k^-}$ |
| **Update** | $\hat{\mathbf{x}}_k^+ = \hat{\mathbf{x}}_k^- + K_k[\tilde{\mathbf{y}}_k - \mathbf{h}(\hat{\mathbf{x}}_k^-)]$<br>$P_k^+ = [I - K_k H_k(\hat{\mathbf{x}}_k^-)]P_k^-$ |
| **Propagation** | $\dot{\hat{\mathbf{x}}}(t) = \mathbf{f}(\hat{\mathbf{x}}(t),\, \mathbf{u}(t),\, t)$<br>$\dot{P}(t) = F(\hat{\mathbf{x}}(t),\, t)\,P(t) + P(t)\,F^T(\hat{\mathbf{x}}(t),\, t) + G(t)\,Q(t)\,G^T(t)$<br>$F(\hat{\mathbf{x}}(t),\, t) \equiv \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right\rvert_{\hat{\mathbf{x}}(t)}$ |

# EKF Example (i)

- Consider Van der Pol's equation

$$m\,\ddot{x} + 2\,c\,(x^2 - 1)\,\dot{x} + k\,x = 0$$

  - Convert to state space using $\mathbf{x} = \begin{bmatrix} x & \dot{x} \end{bmatrix}^T$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -2\,(c/m)(x_1^2 - 1)\,x_2 - (k/m)\,x_1$$

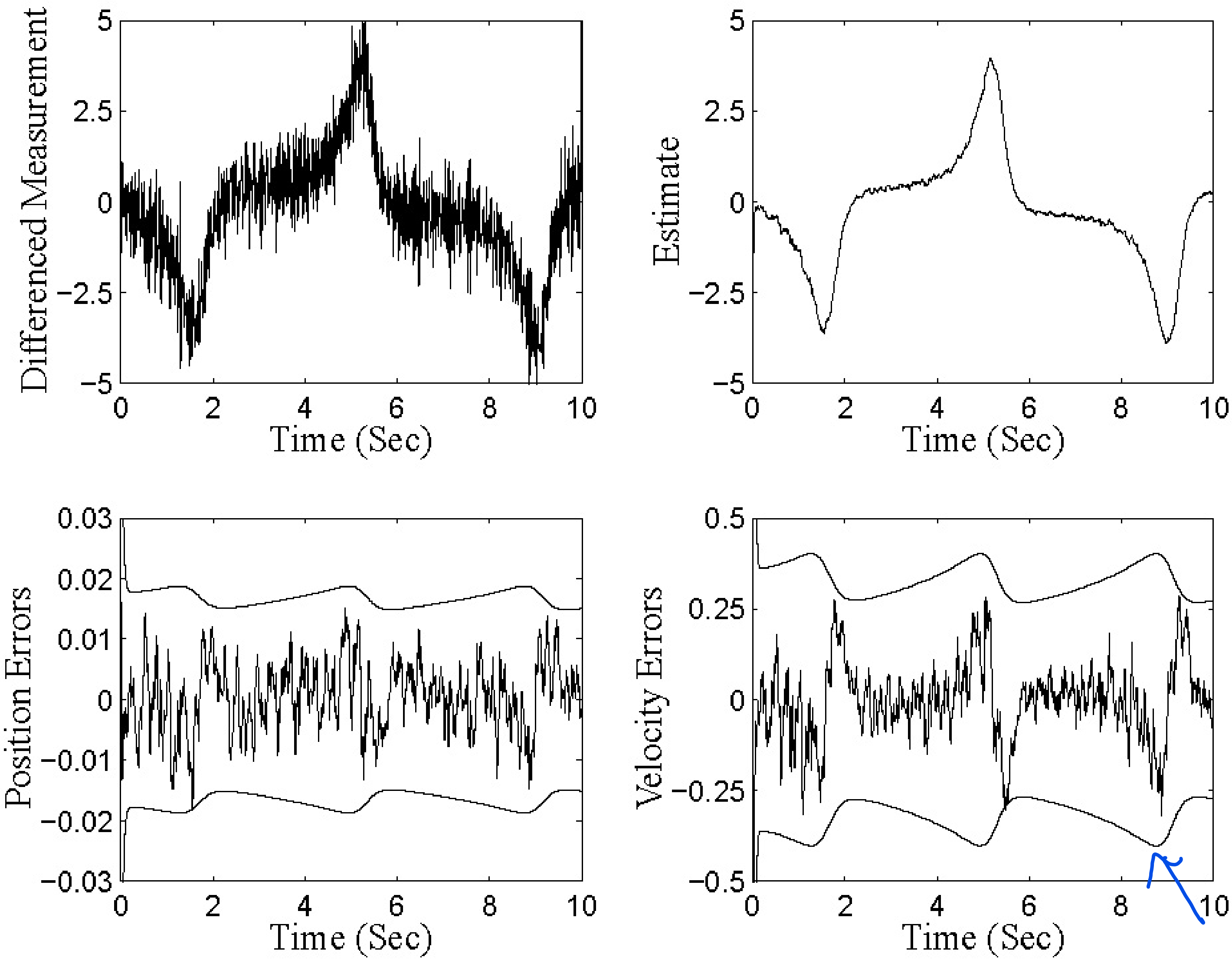
  - The measurement output is position only, so $H = [1 \;\; 0]$
  - Synthetic states are generated using $m = c = k = 1$, with an initial condition of $\mathbf{x}_0 = [1 \;\; 0]^T$
  - The sampling interval is at $0.01$ second intervals and the measurement noise standard deviation is set to $0.01$
  - The linearized model matrices are given by

$$F = \begin{bmatrix} 0 & 1 \\ -4\,(c/m)\,\hat{x}_1\hat{x}_2 - (k/m) & -2\,(c/m)\,(\hat{x}_1^2 - 1) \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

# EKF Example (ii)

- Note that no process noise (i.e., no error) is introduced into the first state
  - This is due to the fact that the first state is a kinematical relationship that is correct in theory and in practice (i.e., velocity is always the derivative of position)
- In the EKF the model parameters are assumed to be given by $m = 1$, $c = 1.5$, and $k = 1.2$, which introduces errors in the assumed system, compared to the true system
  - Overcome by tuning the process noise covariance matrix
- Since we know the truth we can tune $Q$ until the estimate errors are within their respective $3\sigma$ bounds
  - A value of $0.2$ is found to give good results
- Initial covariance is set to $P_0 = 1000\, I$

# EKF Example (iii)




Note that using a simple finite difference results in a very noisy "estimate" for velocity

The EKF provides a much better estimate

error bounds oscillate due to loss of observability

# EKF Example (iv)

```
% State and Initialize
dt=0.01;t=[0:dt:10]';m=length(t);
h=[1 0];r=0.01^2;
xe=zeros(m,2);x=zeros(m,2);p_cov=zeros(m,2);ym=zeros(m,1);
x0=[1;0];x(1,:)=x0';xe(1,:)=x0';
p0=1000*eye(2);p=p0;p_cov(1,:)=diag(p0)';

% True and Assumed Parameters
c=1;k=1;
cm=1.5;km=1.2;

% Process Noise (note: there is coupling but is ignored)
q=0.2*[0 0;0 1];
```

# EKF Example (v)

```
% Main Routine
for i=1:m-1;

% Truth and Measurements
f1=dt*polfun(x(i,:),c,k);f2=dt*polfun(x(i,:)+0.5*f1',c,k);
f3=dt*polfun(x(i,:)+0.5*f2',c,k);f4=dt*polfun(x(i,:)+f3',c,k);
x(i+1,:)=x(i,:)+1/6*(f1'+2*f2'+2*f3'+f4');ym(i)=x(i,1)+sqrt(r)*randn(1);

% Kalman Update
gain=p*h'*inv(h*p*h'+r);p=(eye(2)-gain*h)*p;
xe(i,:)=xe(i,:)+gain'*(ym(i)-xe(i,1));

% Propagation
f1=dt*polfun(xe(i,:),cm,km);f2=dt*polfun(xe(i,:)+0.5*f1',cm,km);
f3=dt*polfun(xe(i,:)+0.5*f2',cm,km);f4=dt*polfun(xe(i,:)+f3',cm,km);
xe(i+1,:)=xe(i,:)+1/6*(f1'+2*f2'+2*f3'+f4');
fpart=[0 1;-4*cm*xe(i,1)*xe(i,2)-km -2*cm*(xe(i,1)^2-1)];phi=c2d(fpart,[0;1],dt);
p=phi*p*phi'+q*dt;p_cov(i+1,:)=diag(p)';

end
```

# EKF Example (vi)

```
% 3-Sigma Outlier
sig3=p_cov.^(0.5)*3;

% Difference Measurement
ymd=diff(ym)/dt;ymd(m)=ym(m-1);

% Plot Results
subplot(221)
plot(t,ymd)
set(gca,'Fontsize',12);
axis([0 10 -5 5]);
set(gca,'Xtick',[0 2 4 6 8 10]);set(gca,'Ytick',[-5 -2.5 0 2.5 5]);
xlabel('Time (Sec)');ylabel('Differenced Measurement')

subplot(222)
plot(t,xe(:,2))
set(gca,'Fontsize',12);
axis([0 10 -5 5]);set(gca,'Xtick',[0 2 4 6 8 10]);
set(gca,'Ytick',[-5 -2.5 0 2.5 5]);
xlabel('Time (Sec)');ylabel('Estimate')
```

# EKF Example (vii)

```
subplot(223)
plot(t,xe(:,1)-x(:,1),t,sig3(:,1),t,-sig3(:,1))
set(gca,'Fontsize',12);
axis([0 10 -0.03 0.03]);
set(gca,'Xtick',[0 2 4 6 8 10]);set(gca,'Ytick',[-0.03 -0.02 -0.01 0 0.01 0.02 0.03]);
xlabel('Time (Sec)');ylabel('Position Errors')

subplot(224)
plot(t,xe(:,2)-x(:,2),t,sig3(:,2),t,-sig3(:,2))
set(gca,'Fontsize',12);
axis([0 10 -0.5 0.5]);
set(gca,'Xtick',[0 2 4 6 8 10]);set(gca,'Ytick',[-0.5 -0.25 0 0.25 0.5]);
xlabel('Time (Sec)');ylabel('Velocity Errors')
```

# EKF Example (viii)

```
function f=polfun(x,c,k)

% Function Routine for Van der Pol's Equation

f=[x(2);-2*c*(x(1)^2-1)*x(2)-k*x(1)];
```