then $F(x_0) = \frac{1}{2}(L+R)$.

Remark $F(x)$ converges to $f(x)$ (?)

in the sense of $\|f-F\|_{L^2} = \left(\int_{-\pi}^{\pi} (f(x)-F(x))^2 dx\right)^{\frac{1}{2}}$

: L^2- norm (MA 544)

11.2 Arbitrary period.

Q $p \neq 2\pi$? $(p>0)$. $F(x) = ?$

Let $L = \frac{p}{2}$: half period. ($p = 2L$)

(idea) Use $\cos(nMx)$ & $\sin(nMx)$

$M = ?$

$\cos(nMx)$ have period $p = 2L$:

$n = 1, 2, \cdots$

$$\cos(nM(x+2L)) = \cos(nMx + nM2L) = \cos(nMx).$$

$n=1$: $\cos(Mx + \underbrace{M2L}_{=2\pi}) = \cos(Mx)$

$M = \frac{2\pi}{2L} = \frac{\pi}{L}$: $\cos\left(\frac{n\pi x}{L}\right), \sin\left(\frac{n\pi x}{L}\right)$

$$F(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

$a_n, b_n = ?$

$\left\{1, \cos\left(\frac{n\pi x}{L}\right), \sin\left(\frac{n\pi x}{L}\right)\right\}$: orthogonal on $(-L, L)$

$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$, $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$

$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$.

(Ex) $f(x) = \begin{cases} 0, & -\frac{1}{2} \le x < 0 \\ x, & 0 \le x < \frac{1}{2} \end{cases}$: periodic in $\mathbb{R}$ with $p=1$

$L = \frac{1}{2}$