ECE 68000: MODERN AUTOMATIC CONTROL

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Necessary condition for a given functional to achieve a maximum or minimum

Maximizing a functional

Definition

A functional v is maximal on some function x_0 if the value of the functional on any function x close to x_0 is less than or equal to that on x_0, that is,

$$\Delta v = v(x) - v(x_0) \leq 0$$

First-order necessary condition for a given functional to achieve a maximum or minimum

Theorem

If for a functional v there exists a variation and the functional reaches a maximum or a minimum on a curve x_0, then

$$\delta v\left(x_0\right) = 0$$

Proof of the first order necessary condition for a given functional to achieve a maximum or minimum

- Suppose x_0 and δx are fixed
- Then
$$v\left(x_0 + \alpha\delta x\right) = \Phi(\alpha)$$
- That is,
$$v\left(x_0 + \alpha\delta x\right)$$
is a function of α that for $\alpha = 0$ achieves its maximum or minimum

Proof—contd.

- We have
$$v\left(x_0 + \alpha\delta x\right)$$
is a function of α that for $\alpha = 0$ achieves its maximum or minimum
- This implies that $\frac{d}{d\alpha}\Phi = 0$, or equivalently,
$$\frac{d}{d\alpha} v\left(x_0 + \alpha\delta x\right)\bigg|_{\alpha=0} = 0$$
- Hence, by the Lemma, $\delta v = 0$
- Therefore, a functional that achieves its maximum or a minimum on a function x_0, has its variation evaluated at x_0 equal to zero. $\square$

Fundamental lemma of calculus of variations

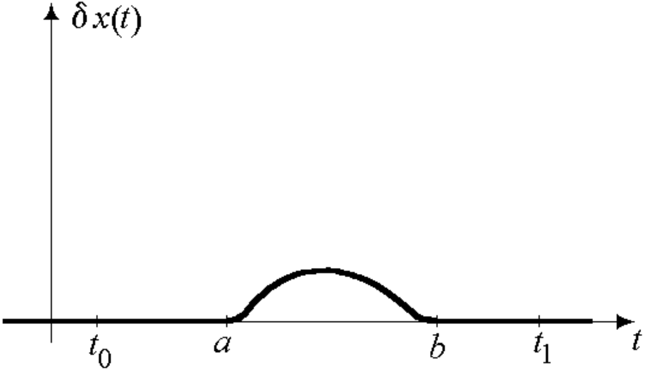
Lemma

Suppose that a function $\phi : \mathbb{R} \to \mathbb{R}$ *is continuous on the interval* $[t_0,\ t_1]$. *Then, for all* δx,

$$\int_{t_0}^{t_1} \phi(t)\delta x(t)dt = 0$$

if and only if $\phi(t) = 0$ *at every point of the interval* $[t_0,\ t_1]$.

Illustration of the Lemma



Proof of the lemma

- $(\Longleftarrow)$ Sufficiency is obvious because we can satisfy $\int_{t_0}^{t_1} \phi(t)\delta x(t)dt = 0$ by choosing ϕ that vanishes at every point of the interval $[t_0,\ t_1]$
- $(\Longrightarrow)$ Necessity by contraposition
- We thus prove that if ϕ is nonzero on some subinterval of $[t_0,\ t_1]$, then $\int_{t_0}^{t_1} \phi(t)\delta x(t)dt \neq 0$
- Specifically, assume, without loss of generality, that ϕ is positive on the interval $[a,\ b]$, where $t_0 \leq a \leq b \leq t_1$
- Note that if ϕ were nonzero at only one point, then because it is continuous, it would have to be nonzero in some neighborhood of that point. Hence, we can assume that $a < b$

Proof of the lemma—contd.

- Consider now a variation given by

$$\delta x = \delta x(t) = \begin{cases} k^2(t-a)^2(t-b)^2 & \text{for} \quad a \le t \le b \\ 0 & \text{elsewhere.} \end{cases}$$

- If we assumed ϕ to be negative on the interval $[a,\ b]$, then we would use δx to be negative
- Because δx is positive in $[a,\ b]$ and zero elsewhere in the interval $[t_0,\ t_1]$, the integrand of $\int_{t_0}^{t_1} \phi(t)\delta x(t)dt$ will have the same sign as ϕ in $[a,\ b]$ and will be zero outside this subinterval
- Thus

$$\int_{t_0}^{t_1} \phi(t)\delta x(t)dt = \int_a^b \phi(t)\delta x(t)dt \neq 0,$$

and the proof is complete. □