6.1 Laplace transform.

(Laplace transform) $L(f) = \int_0^\infty e^{-st} f(t)\,dt$

$L(1) = \frac{1}{s}, \quad s > 0$

$L(e^{at}) = \frac{1}{s-a}, \quad s > a$

$L(t^n) = \frac{n!}{s^{n+1}}, \quad s > 0 \quad (n: \text{an integer})$

$n = 0, 1, 2, \cdots$

$L(t^a) = \frac{\Gamma(a+1)}{s^{a+1}}, \quad s > 0 \quad (a > -1)$

a real number

$\Gamma(z)$: Gamma function.

$\Gamma(z) = \int_0^\infty e^{-u} u^{z-1}\,du$

$L(t^a) = \int_0^\infty e^{-st} t^a\,dt \quad (a > -1)$

$\left(\begin{array}{l} u = st, \\ du = s\,dt \end{array}\right. \quad t = \frac{u}{s} \quad (s > 0)$

$= \int_0^\infty e^{-u} \left(\frac{u}{s}\right)^a \cdot \frac{1}{s}\,du \quad : \quad \left(\frac{u}{s}\right)^a = \frac{u^a}{s^a}$

$= \frac{1}{s^{a+1}} \int_0^\infty e^{-u} u^a\,du \quad : \quad a = a+1-1$

$= \Gamma(a+1)$

$= \frac{\Gamma(a+1)}{s^{a+1}}, \quad s > 0$

(Properties)

1. $\Gamma(1) = 1$

2 $\Gamma(z+1) = z\Gamma(z)$ $\qquad n = n-1+1$

3 $\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = \cdots$

$= n!\,\Gamma(1) = n!$

$\underline{\Gamma(n+1) = n!}$

(Linear property of $L(f)$)

$L(af+bg) = aL(f) + bL(g).$

(pf) $L(af+bg) = \int_0^\infty e^{-st}(af(t)+bg(t))\,dt$

$= a\int_0^\infty e^{-st}f(t)\,dt + b\int_0^\infty e^{-st}g(t)\,dt$

$= aL(f) + bL(g)$

$\cosh(kt) = \dfrac{e^{kt}+e^{-kt}}{2}$: the hyperbolic cosine

$\sinh(kt) = \dfrac{e^{kt}-e^{-kt}}{2}$: " sine.

$L(\cosh(kt)) = \frac{1}{2}L(e^{kt}) + \frac{1}{2}L(e^{-kt})$

$= \frac{1}{2}\cdot\frac{1}{s-k}\ (s>k) + \frac{1}{2}\frac{1}{s-(-k)}\ (s>-k)$

$$L(\cosh(kt)) = \frac{1}{2}\left(\frac{1}{s-k} + \frac{1}{s+k}\right), \quad s > |k|$$

$$= \frac{1}{2}\left(\frac{s+\cancel{k} + s-\cancel{k}}{(s-k)(s+k)}\right) = \frac{s}{s^2-k^2}, \quad s > |k|$$

$$L(\sinh(kt)) = \frac{k}{s^2-k^2}, \quad s > |k|.$$

Q Does every function have its Laplace transform? Ⓐ No.

(Ex) (1) $L(t^{-1}) = L\left(\frac{1}{t}\right) = \infty$: not defined.

$$L\left(\frac{1}{t}\right) = \int_0^\infty \underbrace{e^{-st}}_{>0} \frac{1}{t}\,dt \geq \int_0^1 e^{-st} \frac{1}{t}\,dt$$

① $s > 0$: $e^{-st} \searrow$ $\quad \min_{0<t<1} e^{-st} = e^{-s}$

② $s < 0$: $e^{-st} \nearrow$ $\quad \min_{0<t<1} e^{-st} = 1$

$$L\left(\frac{1}{t}\right) \geq \min\{e^{-s}, 1\} \int_0^1 \frac{1}{t}\,dt = [\ln|t|]_0^1$$

$$= \ln 1 - \ln 0 = \infty$$

$\therefore L\left(\frac{1}{t}\right) = \infty$

$$L(t^{-m}) = \infty \quad (m = 1, 2, 3, \cdots).$$

(2) $f(t) = e^{t^2}$ $\quad (t^2 > t, \quad t > 1)$

$$L(e^{t^2}) = \infty : L(e^{t^m}) = \infty, \; m = 2, 3, \cdots$$

(3) $f(t) = \begin{cases} 1, & t \text{ is irrational} \\ 0, & t \text{ is rational} \end{cases}$

$L(f)$: (MA544)

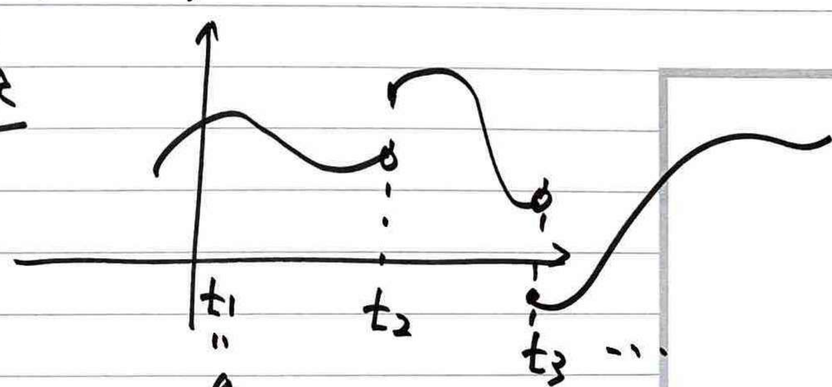
Q Which function has Laplace transform?

(A) 1. exponential order
2. piecewise continuity.

<u>Def</u> 1. $f(t)$ is called of exponential order if $|f(t)| \le Me^{\beta t}$, $t > t_0$ for some constants M & β.

2. $f(t)$ is called piecewise continuous on $[a, b]$ if there are finitely many points $x_1, x_2, \cdots, x_n$ s.t. $a \le x_1 < x_2 < \cdots < x_n \le b$ and $f(t)$ is continuous on (x_{i-1}, x_i), $i = 2, 3, \cdots, n$

<u>Remark</u>



Thm (Existence)

Ⓗ $\left(\begin{array}{l} f(t) \text{ is of exponential order on } [0,\infty) \\ f(t) \text{ is piecewise continuous on } [0,\infty) \end{array} \right)$

Ⓒ $\underline{L(f) \text{ exists}}$ for $s > \beta$ & $\underline{s > \alpha}$ ← f.

(Inverse Laplace transform).

$$\text{we} \underset{L^{-1}}{\overset{L}{\rightleftarrows}} L(\cdot)$$

$$L^{-1}(F) = \frac{1}{2\pi i} \lim_{T\to\infty} \int_{\gamma - iT}^{\gamma + iT} e^{st} F(s)\, ds$$

contour integral

(Ex) 1. $L^{-1}\left(\frac{3!}{s^4}\right) = L^{-1}\left(\frac{3!}{s^{3+1}}\right) = t^3.$

2. $L^{-1}\left(\frac{5}{s-4}\right) = 5L^{-1}\left(\frac{1}{s-4}\right) = 5e^{4t}$

3 $L^{-1}\left(\frac{2s+5}{s^2+4}\right) = 2L^{-1}\left(\frac{s}{s^2+\underbrace{4}_{2^2}}\right) + \frac{5}{2}L^{-1}\left(\frac{1\cdot 2}{s^2+\underbrace{4}_{2^2}}\right)$

$= 2\cos(2t) + \frac{5}{2}\sin(2t).$

4. $L^{-1}\left(\frac{1}{(s-2)(s+1)}\right)$ $\quad \frac{1}{\underline{(s-2)(s+1)}} = \frac{1}{s^2-s-2}$

factoring

(Partial fraction)

$$\frac{1}{(s-2)(s+1)} \overset{\text{let.}}{=} \frac{A}{s-2} + \frac{B}{s+1}$$

$1 = A(s+1) + B(s-2)$

$= As + A + Bs - 2B$

$1 = \underbrace{(A+B)}_{=0}s + \underbrace{A-2B}_{=1}$:

$A + B = 0$

$-\underline{|A - 2B = 1}$

$3B = -1$

$\underline{B = -\frac{1}{3}, \quad A = -B = \frac{1}{3}.}$

$$\frac{1}{(s-2)(s+1)} = \frac{1/3}{s-2} - \frac{1/3}{s+1}$$

$$L^{-1}\left(\frac{1}{(s-2)(s+1)}\right) = \frac{1}{3}L^{-1}\left(\frac{1}{s-2}\right) - \frac{1}{3}L^{-1}\left(\frac{1}{s-(-1)}\right)$$

$$= \frac{1}{3}e^{2t} - \frac{1}{3}e^{-t}.$$

5. $f(t) = \begin{cases} 1, & 0 \le t < 5 \\ 0, & t \ge 5 \end{cases}$ → unit step function

$$L(f) = \int_0^\infty e^{-st} f(t)\,dt = \int_0^5 e^{-st}\,dt + \int_5^\infty 0$$

$u = -st$

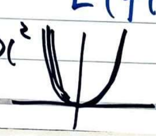
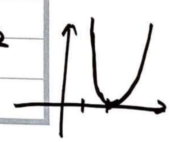
$du = -s\,dt$

$$= \int_0^{-5s} e^u \cdot \frac{1}{-s}\,du$$

$$= -\frac{1}{s}\left[e^u\right]_0^{-5s} = -\frac{1}{s}\left(e^{-5s} - 1\right)$$

(formula) $\underline{L(e^{at}f(t)) = F(s-a)}$

$L(f(t)) = F(s)$

ex) $y = x^2$  $y = (x-2)^2$ 

(pf) $L(e^{at}f(t)) = \int_0^\infty e^{-st} e^{at} f(t)\,dt$

$$= \int_0^\infty e^{-(s-a)t} f(t)\,dt = \underline{F(s-a)}.$$

(Ex) $L(e^{2t}\cos(4t)) = F(s-2)$

$$\left(F(s) = L(\cos(4t)) = \frac{s}{s^2+4^2}\right) \quad s > 0$$

$$= \frac{s-2}{(s-2)^2+16}, \quad s-2>0.$$

$(s>2)$