Case Study

For the plant,

$$\dot{x} = Ax + bu = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u,$$
$$y = cx + du = \begin{bmatrix} 0 & 1 \end{bmatrix} x + 3u,$$

design a combined observer-controller compensator such the control law, $u = -kx + r$, yields the controller's poles located at $\{-1, -1\}$, and the observer poles are to be located at -3 and -4.

Please take a moment to work through this problem on your own. When you are ready, scroll down the page to view the solution.

Explanation: We can use Ackermann's formula applied to the pair $\left(A^\top, c^\top\right)$ to obtain the estimator gain vector l. We form the controllability matrix of the dual pair $A^\top, c^\top)$, then find the last row of its inverse and call it q_1. We have

$$\begin{bmatrix} c^\top & A^\top c^\top \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}.$$

Hence, $q_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}$. The desired characteristic polynomial of $A - lc$ is

$$\det(sI_2 - A + lc) = (s+3)(s+4) = s^2 + 7s + 12.$$

Therefore, the estimator gain l is

$$\begin{aligned} l^\top &= q_1 \left(\left(A^\top\right)^2 + 7A^\top + 12I_2\right) \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 20 & 10 \\ 0 & 30 \end{bmatrix} \\ &= \begin{bmatrix} 20 & 10 \end{bmatrix}. \end{aligned}$$

Hence,

$$l = \begin{bmatrix} 20 \\ 10 \end{bmatrix}.$$

The observer dynamics can be represented in different forms as follows\

$$\begin{aligned} \dot{\tilde{x}} &= A\tilde{x} + bu + l\left(y - \tilde{y}\right) \\ &= \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \tilde{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 20 \\ 10 \end{bmatrix} \left(y - \tilde{y}\right), \end{aligned}$$

where

$$\tilde{y} = c\tilde{x} + du.$$

Equivalently,

$$\begin{aligned}\dot{\tilde{x}} &= (A - lc)\tilde{x} + bu + ly - ldu \\ &= (A - lc)\tilde{x} + (b - ld)u + ly \\ &= \begin{bmatrix} 1 & -20 \\ 1 & -8 \end{bmatrix}\tilde{x} + \begin{bmatrix} -59 \\ -30 \end{bmatrix}u + \begin{bmatrix} 20 \\ 10 \end{bmatrix}y.\end{aligned}$$

Again, we can use Ackermann's formula applied this time to the pair (A, b) to design a state-feedback control law, $u = -kx + r$, such that the closed-loop system driven by the state-feedback controller has a double pole at -1. We obtain

$$u = -\begin{bmatrix} 5 & 9 \end{bmatrix}x + r.$$

Then, the combined observer-controller compensator has the form

$$\begin{aligned}\dot{\tilde{x}} &= (A - lc)\tilde{x} + (b - ld)u + ly \\ &= \begin{bmatrix} 1 & -20 \\ 1 & -8 \end{bmatrix}\tilde{x} + \begin{bmatrix} -59 \\ -30 \end{bmatrix}u + \begin{bmatrix} 20 \\ 10 \end{bmatrix}y \\ u &= -\begin{bmatrix} 5 & 9 \end{bmatrix}\tilde{x} + r.\end{aligned}$$