ECE 602: LUMPED LINEAR SYSTEMS

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Reachability and Controllability of Continuous-Time (CT) Linear Time-Invariant (LTI) Systems

Reachability and Controllability of Continuous-Time (CT) Linear Time-Invariant (LTI) Systems

- **Objective**: Introduce notions of reachability and controllability for CT LTI controlled systems modeled as

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0,$$

  where $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ and $\boldsymbol{B} \in \mathbb{R}^{n \times m}$

- Recall the solution of the system

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}(0) + \int_0^t e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)d\tau$$

Solving CT LTI system modeling equation—General case

Important Solution Formula

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}(t-t_0)}\boldsymbol{x}(t_0) + \int_{t_0}^{t} e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)d\tau$$

Special case, $t_0 = 0$

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}(0) + \int_{0}^{t} e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)d\tau$$

Reachability Definition

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}(0) + \int_0^t e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)d\tau$$

The system $\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t)$ is reachable if for any $\boldsymbol{x}_f$ there is $t_1 > 0$ and a control law, $\boldsymbol{u}(\cdot)$, that transfers $\boldsymbol{x}(t_0) = \mathbf{0}$ to $\boldsymbol{x}(t_1) = \boldsymbol{x}_f$

Controllability Definition

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}(0) + \int_0^t e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)d\tau$$

The system $\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t)$ is controllable if there is a control law $\boldsymbol{u}(\cdot)$ that transfers any initial state $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$ to the origin at some time $t_1 > t_0$

- For **continuous-time** LTI systems controllability and reachability are equivalent

For CT LTI systems controllobaility $\iff$ reachability

- Let, for simplicity, $t_0 = 0$, then

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}(0) + \int_0^t e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)d\tau$$

- For reachability, $\boldsymbol{x}_0 = \boldsymbol{0}$ and $\boldsymbol{x}_f = \boldsymbol{x}(t_1)$ is arbitrary
- Hence

$$\boldsymbol{x}_f = \int_0^{t_1} e^{\boldsymbol{A}(t_1-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)d\tau = e^{\boldsymbol{A}t_1}\int_0^{t_1} e^{-\boldsymbol{A}\tau}\boldsymbol{B}\boldsymbol{u}(\tau)d\tau$$

- Premultiply both sides by $e^{-\boldsymbol{A}t_1}$ to obtain

$$\boldsymbol{v} = e^{-\boldsymbol{A}t_1}\boldsymbol{x}_f = \int_0^{t_1} e^{-\boldsymbol{A}\tau}\boldsymbol{B}\boldsymbol{u}(\tau)d\tau$$

Controllability $\iff$ reachability

- We have
$$\boldsymbol{v} = \int_0^{t_1} e^{-\boldsymbol{A}\tau} \boldsymbol{B}\boldsymbol{u}(\tau) d\tau,$$
where $\boldsymbol{v} \in \mathbb{R}^n$ is arbitrary
- That is, the system is reachable if we can construct a control law $\boldsymbol{u}(\cdot)$ for which the above holds for any $\boldsymbol{v} \in \mathbb{R}^n$
- The system is controllable if we can construct a control law that transfers the system from arbitrary initial state $\boldsymbol{x}_0$ to the origin

$$\begin{aligned} \boldsymbol{0} = \boldsymbol{x}(t_1) &= e^{\boldsymbol{A}t_1}\boldsymbol{x}_0 + \int_0^{t_1} e^{\boldsymbol{A}(t_1-\tau)} \boldsymbol{B}\boldsymbol{u}(\tau) d\tau \\ &= e^{\boldsymbol{A}t_1} \int_0^{t_1} e^{-\boldsymbol{A}\tau} \boldsymbol{B}\boldsymbol{u}(\tau) d\tau \end{aligned}$$

Controllability $\iff$ reachability—Contd

- The system is reachable if we can construct a control law $\boldsymbol{u}(\cdot)$ such that for any $\boldsymbol{v} \in \mathbb{R}^n$

$$\boldsymbol{v} = \int_0^{t_1} e^{-\boldsymbol{A}\tau} \boldsymbol{B}\boldsymbol{u}(\tau) d\tau$$

- The system is controllable if

$$-e^{-\boldsymbol{A}t_1}\boldsymbol{x}_0 = e^{\boldsymbol{A}t_1} \int_0^{t_1} e^{-\boldsymbol{A}\tau} \boldsymbol{B}\boldsymbol{u}(\tau) d\tau$$

- That is, the system is controllable if for any $\boldsymbol{x}_0 \in \mathbb{R}^n$

$$-\boldsymbol{x}_0 = \int_0^{t_1} e^{-\boldsymbol{A}\tau} \boldsymbol{B}\boldsymbol{u}(\tau) d\tau$$

- Comparing conditions for reachability and controllability, we conclude that they are equivalent

A Test for Controllability

- Recall that the system is reachable if we can construct a control law $\boldsymbol{u}(\cdot)$ such that for any $\boldsymbol{v} \in \mathbb{R}^n$

$$\boldsymbol{v} = \int_0^{t_1} e^{-\boldsymbol{A}\tau} \boldsymbol{B} \boldsymbol{u}(\tau) d\tau = \int_0^{t_1} \boldsymbol{M}(\tau) \boldsymbol{u}(\tau) d\tau$$

- Let's construct $\boldsymbol{u}$ such that the above holds
- First, note that $\boldsymbol{M}(t) = e^{-\boldsymbol{A}t}\boldsymbol{B}$ is an n-by-m matrix whose elements are functions of time
- Let

$$\boldsymbol{u}(t) = \boldsymbol{M}(t)^\top \left(\int_0^{t_1} \boldsymbol{M}(\tau) \boldsymbol{M}(\tau)^\top d\tau \right)^{-1} \boldsymbol{v}$$

The Controllability Gramian

- Substitute our u into $\boldsymbol{v} = \int_0^{t_1} \boldsymbol{M}(\tau)\boldsymbol{u}(\tau)d\tau$

$$\boldsymbol{v} = \int_0^{t_1} \boldsymbol{M}(\tau)\boldsymbol{M}(\tau)^\top d\tau \left(\int_0^{t_1} \boldsymbol{M}(\tau)\boldsymbol{M}(\tau)^\top d\tau\right)^{-1} \boldsymbol{v}$$

- The invertibility of

$$\boldsymbol{W}(0, t_1) = \int_0^{t_1} e^{-\boldsymbol{A}\tau}\boldsymbol{B}\boldsymbol{B}^\top e^{-\boldsymbol{A}^\top\tau} d\tau \in \mathbb{R}^{n\times n}$$

  for all $t_1 > 0$ is necessary and sufficient for the system to be controllable

- We call $\boldsymbol{W}(0, t_1)$ the controllability Gramian

Some Controllability Tests

The following are equivalent:

1. The system $\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t)$ is controllable
2. $\operatorname{rank} \begin{bmatrix} \boldsymbol{B} & \boldsymbol{AB} & \cdots & \boldsymbol{A}^{n-1}\boldsymbol{B} \end{bmatrix} = n$
3. The controllability Gramian

$$\boldsymbol{W}(t_0, t_1) = \int_{t_0}^{t_1} e^{-\boldsymbol{A}t}\boldsymbol{B}\boldsymbol{B}^\top e^{-\boldsymbol{A}^\top t} dt$$

   is nonsingular for all $t_1 > t_0$

4. The Popov-Belevitch-Hautus (PBH) Test
   $\operatorname{rank} \begin{bmatrix} s\boldsymbol{I}_n - \boldsymbol{A} & \boldsymbol{B} \end{bmatrix} = n \quad \text{for all } s \in \operatorname{eig}(\boldsymbol{A})$