Symplectic approach to canonical transformations

Derived assuming a time-variant Hamiltonian, but it applies in the time varying case as well.

$$K(Q,P,t) = H(q,p,t) + \cancel{\frac{\partial H}{\partial t}}$$

0

Let $\eta = [q_1, \ldots, q_m, p_1, \ldots, p_m]^T \in \mathbb{R}^{2m}$

Recall,

$$\dot{\eta} = J \frac{\partial H}{\partial \eta} \qquad J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \in \mathbb{R}^{2m \times 2m}$$

$\nabla_\eta H$

$$\frac{\partial H}{\partial \eta} = \left[\frac{\partial H}{\partial q_1}, \frac{\partial H}{\partial q_2} \cdots \frac{\partial H}{\partial q_m}, \frac{\partial H}{\partial p_1} \cdots \frac{\partial H}{\partial p_m}\right]^T$$

Look at the transformed variables:

Let $\zeta = [Q_1, \ldots, Q_m, P_1, \ldots P_m]^T \in \mathbb{R}^{2m}$

where $Q_i = Q_i(q,p)$

$P_i = P_i(q,p)$

note: $\zeta = \zeta(\eta)$

Take the time derivative of ζ

$$\dot{\zeta}_i = \sum_{j=1}^{2m} \frac{\partial \zeta_i}{\partial \eta_j} \dot{\eta}_j$$

(chain rule) for $i = 1, \ldots, 2m$

Which can be written as

$$\dot{\xi} = A\dot{\eta}$$

where

$$A = \begin{bmatrix} \frac{\partial \xi_1}{\partial \eta_1} & \cdots & \frac{\partial \xi_1}{\partial \eta_{2m}} \\ \vdots & \ddots & \vdots \\ \frac{\partial \xi_{2m}}{\partial \eta_1} & \cdots & \frac{\partial \xi_{2m}}{\partial \eta_{2m}} \end{bmatrix}$$

Jacobian of the Transformation

As a result,

$$\dot{\xi} = A\dot{\eta} = AJ\frac{\partial H}{\partial \eta}$$

We can also look at $H = H(\xi) = H(\xi(\eta))$,

$$\frac{\partial H}{\partial \eta_i} = \sum_{j=1}^{2m} \frac{\partial H}{\partial \xi_j}\frac{\partial \xi_j}{\partial \eta_i}$$ (chain rule) for $i = 1, \ldots, 2m$

$$\Rightarrow \frac{\partial H}{\partial \eta} = A^T \frac{\partial H}{\partial \xi}$$

Combining these expressions,

$$\dot{\xi} = AJA^T\frac{\partial H}{\partial \xi}$$

However for a canonical transformation, we want

$$\dot{\xi} = J\frac{\partial H}{\partial \xi}$$

We therefore require that $AJA^T = J$

"symplectic condition"

Symplectic Matrices

If A satisfies the symplectic condition, then $A \in Sp(2m)$

$Sp(n) \triangleq \{ R \in \mathbb{R}^{n \times n} \mid RJR^T = J \}$ where $J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$

is a set of all symplectic matrices.

$Sp(n)$ is a Lie Group under matrix multiplication, the Symplectic Group.

How do we implement the symplectic approach to canonical transformations?

1) Pick a transformation $\xi = \xi(\eta)$
2) Calculate $A = \frac{\partial \xi}{\partial \eta}$ Jacobian
3) Check $AJA^T \stackrel{?}{=} J$

↳ If so, canonical transformation

Note: A generating function was not needed.