ECE 602: LUMPED LINEAR SYSTEMS

Professor Stan Żak

Controllability Tests for Continuous-Time (CT) Linear Time-Invariant (LTI) Systems

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- **Objective**: Discuss different controllability tests for CT LTI controlled systems modeled as

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0,$$

  where $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ and $\boldsymbol{B} \in \mathbb{R}^{n \times m}$

- Recall the solution of the system

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}(0) + \int_0^t e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)d\tau$$

Reachability and Controllability Definitions

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}(0) + \int_0^t e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)d\tau$$

The system $\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t)$ is reachable if for any $\boldsymbol{x}_f$ there is $t_1 > 0$ and a control law, $\boldsymbol{u}(\cdot)$, that transfers $\boldsymbol{x}(t_0) = \boldsymbol{0}$ to $\boldsymbol{x}(t_1) = \boldsymbol{x}_f$

The system $\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t)$ is controllable if there is a control law $\boldsymbol{u}(\cdot)$ that transfers any initial state $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$ to the origin at some time $t_1 > t_0$

- For **continuous-time** LTI systems controllability and reachability are equivalent

Some Controllability Tests

The following are equivalent:

1. The system $\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t)$ is controllable
2. $\operatorname{rank}\begin{bmatrix} \boldsymbol{B} & \boldsymbol{AB} & \cdots & \boldsymbol{A}^{n-1}\boldsymbol{B} \end{bmatrix} = n$
3. The controllability Gramian

$$\boldsymbol{W}(t_0, t_1) = \int_{t_0}^{t_1} e^{-\boldsymbol{A}t}\boldsymbol{B}\boldsymbol{B}^\top e^{-\boldsymbol{A}^\top t}\, dt$$

   is non-singular for all $t_1 > t_0$
4. The Popov-Belevitch-Hautus (PBH) Test
   $\operatorname{rank}\begin{bmatrix} s\boldsymbol{I}_n - \boldsymbol{A} & \boldsymbol{B} \end{bmatrix} = n \quad$ for all $s \in \operatorname{eig}(\boldsymbol{A})$

Methods of Proof

- A sentence is a part of a language
- Sentences are used to make different statements
- Statements are either true or false
- Consider the following two statements:

  A : $x > 7$;
  B : $x^2 > 49$.

- We can combine the above two statements into one statement, called a *conditional*, that has the form: "IF A THEN B."
- A conditional is a compound statement obtained by placing the word "IF" before the first statement and inserting the word "THEN" before the second statement

Methods of Proof—Contd

- The symbol "$\Rightarrow$" is used to represent the conditional "IF A THEN B" as $A \Rightarrow B$
- The statement $A \Rightarrow B$ also reads as "A implies B," or "A only if B," or "A is sufficient for B," or "B is necessary for A"
- Statements A and B may either be true or false
- The relationship between the truth or falsity of A and B and the conditional $A \Rightarrow B$ can be illustrated by means of a diagram called a *truth table*
- In the table, T stands for "true," and F stands for "false"

A	B	$A \Rightarrow B$
F	F	T
F	T	T
T	F	F
T	T	T

More on Methods of Proof

A	B	$A \Rightarrow B$
F	F	T
F	T	T
T	F	F
T	T	T

- It is intuitively clear that if A is true, then B must also be true for the statement $A \Rightarrow B$ to be true
- If A is not true, then the sentence $A \Rightarrow B$ does not have an obvious meaning in everyday language
- Interpret $A \Rightarrow B$ to mean that we cannot have A true and B not true
- Check that the truth values of the statements $A \Rightarrow B$ and "not (A and (not B))" are the same—proof by contradiction
- Check that the truth values of $A \Rightarrow B$ and "not $B \Rightarrow$ not A" are the same—proof by contraposition

Let Us Show (2) $\Longrightarrow$ (4)

- $\operatorname{rank}\begin{bmatrix} \boldsymbol{B} & \boldsymbol{AB} & \cdots & \boldsymbol{A}^{n-1}\boldsymbol{B} \end{bmatrix} = n \Longrightarrow$
  $\operatorname{rank}\begin{bmatrix} s\boldsymbol{I}_n - \boldsymbol{A} & \boldsymbol{B} \end{bmatrix} = n$ for all $s \in \operatorname{eig}(\boldsymbol{A})$
- Proof by contraposition

$$S_1 \Longrightarrow S_2 \iff (\text{NOT } S_2) \Longrightarrow (\text{NOT } S_1)$$

- NOT S_2 means, there is a vector $\boldsymbol{q} \neq \mathbf{0}$ such that for some $s \in \operatorname{eig}(\boldsymbol{A})$,

$$\boldsymbol{q}^\top \begin{bmatrix} s\boldsymbol{I}_n - \boldsymbol{A} & \boldsymbol{B} \end{bmatrix} = \mathbf{0}^\top,$$

  that is,

$$\boldsymbol{q}^\top \boldsymbol{A} = s\boldsymbol{q}^\top, \quad \boldsymbol{q}^\top \boldsymbol{B} = \mathbf{0}^\top$$

- Then

$$\boldsymbol{q}^\top \boldsymbol{AB} = s\boldsymbol{q}^\top \boldsymbol{B} = \mathbf{0}^\top$$
$$\boldsymbol{q}^\top \boldsymbol{A}^2\boldsymbol{B} = s\boldsymbol{q}^\top \boldsymbol{AB} = \mathbf{0}^\top$$

Showing (2) $\Longrightarrow$ (4)

- We have

$$\boldsymbol{q}^\top \boldsymbol{A}\boldsymbol{B} = s\boldsymbol{q}^\top \boldsymbol{B} = \mathbf{0}^\top$$
$$\boldsymbol{q}^\top \boldsymbol{A}^2\boldsymbol{B} = s\boldsymbol{q}^\top \boldsymbol{A}\boldsymbol{B} = \mathbf{0}^\top$$

- Continue to obtain

$$\boldsymbol{q}^\top \boldsymbol{A}^{n-1}\boldsymbol{B} = s\boldsymbol{q}^\top \boldsymbol{A}^{n-2}\boldsymbol{B} = \mathbf{0}^\top.$$

- Write the above as

$$\boldsymbol{q}^\top \begin{bmatrix} \boldsymbol{B} & \boldsymbol{A}\boldsymbol{B} & \cdots & \boldsymbol{A}^{n-1}\boldsymbol{B} \end{bmatrix} = \mathbf{0}^\top,$$

- This means that the rank of the controllability matrix of the pair $(\boldsymbol{A}, \boldsymbol{B})$ is less than n, which is

$$\text{NOT } S_1$$

QED

Showing (1) $\implies$ (2)

- We are to show $S_1 =$ The system $\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t)$ is controllable $\implies S_2 = \operatorname{rank}\begin{bmatrix} \boldsymbol{B} & \boldsymbol{AB} & \cdots & \boldsymbol{A}^{n-1}\boldsymbol{B} \end{bmatrix} = n$
- Proof by contradiction

$$S_1 \implies S_2 \iff \text{NOT } (S_1 \text{ AND } (\text{ NOT } S_2))$$

- NOT S_2 means

$$\operatorname{rank}\begin{bmatrix} \boldsymbol{B} & \boldsymbol{AB} & \cdots & \boldsymbol{A}^{n-1}\boldsymbol{B} \end{bmatrix} < n,$$

- Then, there is a constant n-vector $\boldsymbol{q} \neq \mathbf{0}$ such that

$$\boldsymbol{q}^\top \boldsymbol{B} = \mathbf{0}^\top, \quad \boldsymbol{q}^\top \boldsymbol{AB} = \mathbf{0}^\top, \ldots, \boldsymbol{q}^\top \boldsymbol{A}^{n-1}\boldsymbol{B} = \mathbf{0}^\top$$

- By the Cayley-Hamilton theorem the matrix $\boldsymbol{A}$ satisfies its own characteristic equation
- Hence

$$\boldsymbol{A}^n = -a_{n-1}\boldsymbol{A}^{n-1} - \cdots - a_1\boldsymbol{A} - a_0\boldsymbol{I}_n$$

$(1) \implies (2)$

- We have

$$\boldsymbol{q}^\top \boldsymbol{A}^n \boldsymbol{B} = \boldsymbol{q}^\top \left(-a_{n-1}\boldsymbol{A}^{n-1}\boldsymbol{B} - \cdots - a_1 \boldsymbol{A}\boldsymbol{B} - a_0 \boldsymbol{B}\right) = \boldsymbol{0}^\top$$

- By induction

$$\boldsymbol{q}^\top \boldsymbol{A}^i \boldsymbol{B} = \boldsymbol{0}^\top \quad \text{for} \quad i = n+1, n+2, \ldots$$

- Let now $\boldsymbol{x}(0) = \boldsymbol{q}$ and $\boldsymbol{x}(t_1) = \boldsymbol{0}$
- We will show that there is no control law that can transfer the system from $\boldsymbol{x}(0) = \boldsymbol{q}$ to $\boldsymbol{x}(t_1) = \boldsymbol{0}$
- From the solution formula for the controlled system, we obtain

$$-\boldsymbol{q} = \int_0^{t_1} e^{-\boldsymbol{A}t} \boldsymbol{B}\boldsymbol{u}(t)dt$$

- Premultiply by $\boldsymbol{q}^\top$

$$0 \neq -\|\boldsymbol{q}\|^2 = \int_0^{t_1} \boldsymbol{q}^\top e^{-\boldsymbol{A}t} \boldsymbol{B}\boldsymbol{u}(t)dt = 0$$

(1) $\Longrightarrow$ (2)—Contd

- We have

$$0 \neq \|\boldsymbol{q}\|^2 = \int_0^{t_1} \boldsymbol{q}^\top e^{-\boldsymbol{A}t} \boldsymbol{B}\boldsymbol{u}(t)dt = 0$$

because

$$\begin{aligned} \boldsymbol{q}^\top e^{\boldsymbol{A}(t_1-t)}\boldsymbol{B} &= \boldsymbol{q}^\top \left(\boldsymbol{B} + (t_1 - t)\boldsymbol{A}\boldsymbol{B} + \frac{(t_1-t)^2}{2!}\boldsymbol{A}^2\boldsymbol{B} + \cdots \right) \\ &= \boldsymbol{0}^\top \end{aligned}$$

- A CONTRADICTION $\Longrightarrow\Longleftarrow$
- We showed

$$(1) \Longrightarrow (2) \iff \text{NOT } ((1) \text{ AND } (\text{ NOT } (2)))$$

QED