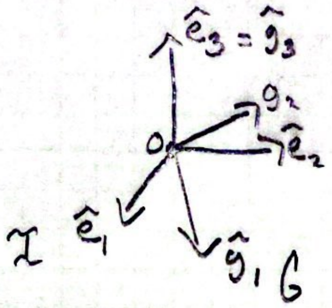
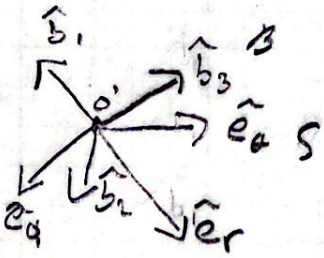
Gabriel Colangelo | HW6 | MAE 562

10.107

Reference Frames:





$\mathcal{I} = (O, \hat{e}_1, \hat{e}_2, \hat{e}_3)$

$\mathcal{G} = (O, \hat{g}_1, \hat{g}_2, \hat{g}_3)$

$\mathcal{B} = (O', \hat{b}_1, \hat{b}_2, \hat{b}_3)$

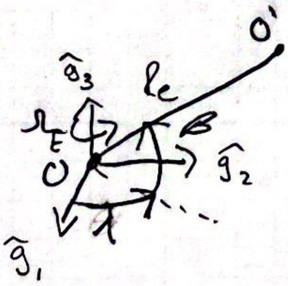
$\mathcal{S} = (O', \hat{e}_\phi, \hat{e}_\theta, \hat{e}_r)$

$^{\mathcal{I}}\vec{\omega}^{\mathcal{G}} = \Omega_E \hat{e}_3 = \Omega_E \hat{g}_3$

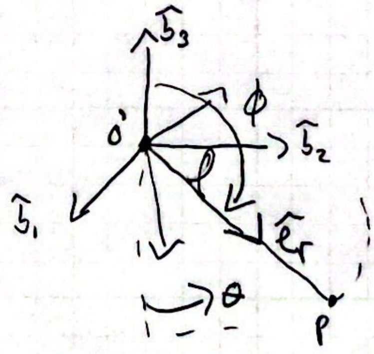
$^{\mathcal{B}}\vec{\omega}^{\mathcal{S}} = \dot{\phi}\, \hat{e}_\theta + \dot{\theta}\, \hat{b}_3$

$^{\mathcal{I}}\vec{\omega}^{\mathcal{B}} = {}^{\mathcal{I}}\vec{\omega}^{\mathcal{G}}$

Coordinates:



$(\ell_E, \lambda, \beta)_{\mathcal{G}}$



$(l, \theta, \phi)_{\mathcal{B}}$

Kinematics:

$$\vec{r}_{P/O'} = l\, \hat{e}_r$$

$$^{\mathcal{B}}\vec{v}_{P/O'} = {}^{\mathcal{S}}\frac{d}{dt}\left(\cancelto{0}{\vec{r}_{P/O'}}\right) + {}^{\mathcal{B}}\vec{\omega}^{\mathcal{S}} \times \vec{r}_{P/O'}$$

$$^{\mathcal{B}}\vec{v}_{P/O'} = l\dot{\phi}\, \hat{e}_\phi + l\dot{\theta} \sin\phi\, \hat{e}_\theta$$

$${}^B\vec{a}_{P/O'} = {}^S\frac{d}{dt}\left({}^B\vec{v}_{P/O'}\right) + {}^B\vec{\omega}^S \times {}^B\vec{v}_{P/O'}$$

$${}^B\vec{a}_{P/O'} = (l\ddot{\phi} - l\dot{\theta}^2\cos\phi\sin\phi)\hat{e}_\phi + (2l\dot{\theta}\dot{\phi}\cos\phi + l\ddot{\theta}\sin\phi)\hat{e}_\theta - (l\dot{\phi}^2 + l\dot{\theta}^2\sin^2\phi)\hat{e}_r$$

__FBD:__

$$-mg\hat{b}_3 = -mg(-\sin\phi\,\hat{e}_\phi + \cos\phi\,\hat{e}_r)$$

	$\hat{e}_\phi$	$\hat{e}_\theta$	$\hat{e}_r$
$\hat{b}_1$	$c\theta c\phi$	$-s\theta$	$c\theta s\phi$
$\hat{b}_2$	$s\theta c\phi$	$c\theta$	$s\theta s\phi$
$\hat{b}_3$	$-s\phi$	0	$c\phi$

$$= {}^B C^S$$

$$\vec{r}_{O'/O} = R_E\cos(\lambda)\cos(\beta)\,\hat{g}_1 + R_E\sin(\lambda)\cos(\beta)\,\hat{g}_2 + R_E\sin(\beta)\,\hat{g}_3$$

$${}^I\vec{v}_{O'/O} = \underbrace{{}^G\frac{d}{dt}\left(\vec{r}_{O'/O}\right)}_{0} + {}^I\vec{\omega}^B \times \vec{r}_{O'/O}$$

$${}^I\vec{v}_{O'/O} = \Omega_E R_E\cos\lambda\cos\beta\,\hat{g}_2 - \Omega_E R_E\sin(\lambda)\cos(\beta)\,\hat{g}_1$$

$${}^I\vec{a}_{O'/O} = \underbrace{{}^G\frac{d}{dt}\left({}^I\vec{v}_{O'/O}\right)}_{0} + {}^I\vec{\omega}^G \times {}^I\vec{v}_{O'/O}$$

$${}^I\vec{a}_{O'/O} = -\Omega_E^2 R_E(\cos\lambda\cos\beta\,\hat{g}_1 + \sin\lambda\sin\beta\,\hat{g}_2)$$

$${}^I\vec{a}_{P/O} = {}^I\vec{a}_{O'/O} + {}^B\vec{a}_{P/O'} + 2\,{}^I\vec{\omega}^B \times {}^B v_{P/O'} + {}^I\vec{\omega}^B \times ({}^I\vec{\omega}^B \times r_{P/O'})$$

	$\hat{b}_1$	$\hat{b}_2$	$\hat{b}_3$
$\hat{g}_1$	$c\lambda s\beta$	$s\lambda s\beta$	$-c\beta$
$\hat{g}_2$	$-s\lambda$	$c\lambda$	0
$\hat{g}_3$	$c\lambda c\beta$	$s\lambda c\beta$	$s\beta$

$${}^I\vec{a}_{O'/O} = -\Omega_E^2 R_E\left[\cos\lambda\cos\beta(\cos\lambda\sin\beta\,\hat{b}_1 + \sin\lambda\sin\beta\,\hat{b}_2 - \cos\beta\,\hat{b}_3) + \sin\lambda\sin\beta(-\sin\lambda\,\hat{b}_1 + \cos\lambda\,\hat{b}_2)\right]$$

$${}^{I}\vec{a}_{o'/o} = -\Omega_E^2 R_E [(\cos^2\lambda \cos\beta \sin\beta - \sin\beta \sin^2\lambda)\hat{b}_1 + (\cos\lambda \cos\beta \sin\lambda \sin\beta + \sin\lambda \sin\beta \cos\lambda)\hat{b}_2 - (\cos\lambda \cos^2\beta)\hat{b}_3 \quad \underline{(1)}$$

$${}^{I}\vec{\omega}^{B} = \Omega_E (\cos\lambda \cos\beta \hat{b}_1 + \sin\lambda \cos\beta \hat{b}_2 + \sin\beta \hat{b}_3)$$

$${}^{B}\vec{v}_{P/o'} = l\dot{\phi}(\cos\theta \cos\phi \hat{b}_1 + \sin\theta \cos\phi \hat{b}_2 - \sin\phi \hat{b}_3) + l\dot{\theta}\sin\phi(-\sin\theta \hat{b}_1 + \cos\theta \hat{b}_2)$$

$$\vec{r}_{P/o'} = l(\cos\theta \sin\phi \hat{b}_1 + \sin\theta \sin\phi \hat{b}_2 + \cos\phi \hat{b}_3)$$

$${}^{B}\vec{a}_{P/o'} = (l\ddot{\phi} - l\dot{\theta}^2 \cos\phi \sin\phi)(\cos\theta\cos\phi \hat{b}_1 + \sin\theta \cos\phi \hat{b}_2 - \sin\phi \hat{b}_3) + (2l\dot{\theta}\dot{\phi}\cos\phi + l\ddot{\theta}\sin\phi)(-\sin\theta \hat{b}_1 + \cos\theta \hat{b}_2) + (l\dot{\phi}^2 + l\dot{\theta}^2 \sin^2\phi)(\cos\theta \sin\phi \hat{b}_1 + \sin\theta \sin\phi \hat{b}_2 + \cos\phi \hat{b}_3) \quad \underline{(2)}$$

$$2\,{}^{I}\vec{\omega}^{B} \times {}^{B}\vec{v}_{P/o'} = 2\Omega_E \begin{pmatrix} c\lambda c\beta \\ s\lambda c\beta \\ s\beta \end{pmatrix}_B \times \begin{pmatrix} l\dot{\phi} c\theta c\phi - l\dot{\theta} s\phi s\theta \\ l\dot{\phi} s\theta c\phi + l\dot{\theta} s\phi c\theta \\ -l\dot{\phi} s\phi \end{pmatrix}_B$$

$$= 2\Omega_E [c\lambda c\beta(l\dot{\phi} s\theta c\phi + l\dot{\theta} s\phi c\theta)\hat{b}_3 + c\lambda c\beta l\dot{\phi} s\phi \hat{b}_2 - s\lambda c\beta(l\dot{\phi} c\theta c\phi - l\dot{\theta} s\phi s\theta)\hat{b}_3 - s\lambda c\beta l\dot{\phi} s\phi \hat{b}_1 + (s\beta l\dot{\phi} c\theta c\phi - l\dot{\theta} s\phi s\theta s\beta)\hat{b}_2 - (s\beta l\dot{\phi} s\theta c\phi + l\dot{\theta} s\phi c\theta s\beta)\hat{b}_1] \quad (\underline{3})$$

$${}^{I}\vec{\omega}^{B} \times \vec{r}_{P/o'} = \Omega_E \begin{pmatrix} c\lambda c\beta \\ s\lambda c\beta \\ s\beta \end{pmatrix}_B \times l \begin{pmatrix} c\theta s\phi \\ s\theta s\phi \\ c\phi \end{pmatrix}_B$$

$$= \Omega_E l [(c\lambda c\beta s\theta s\phi)\hat{b}_3 - c\lambda c\beta c\phi \hat{b}_2 - s\lambda c\beta c\theta s\phi \hat{b}_3 + s\lambda c\beta c\phi \hat{b}_1 + s\beta c\theta s\phi \hat{b}_2 - s\beta s\theta s\phi \hat{b}_1]$$

$${}^{I}\vec{\omega}^{0} \times ({}^{I}\vec{\omega}^{0} \times \vec{r}_{P/0'}) = \Omega_E \begin{pmatrix} c\lambda c\beta \\ s\lambda c\beta \\ s\beta \end{pmatrix} \times \Omega_E l \begin{pmatrix} s\lambda c\beta c\phi - s\beta s\theta s\phi \\ s\beta c\theta s\phi - c\lambda c\beta c\phi \\ c\lambda c\beta s\theta s\phi - s\lambda c\beta c\theta s\phi \end{pmatrix}$$

$$= \Omega_E^2 \lambda [c\lambda c\beta(s\beta c\theta s\phi - c\lambda c\beta c\phi)\hat{b}_3 - c\lambda c\beta(c\lambda c\beta s\theta s\phi - s\lambda c\beta c\theta s\phi)\hat{b}_2$$
$$- s\lambda c\beta(s\lambda c\beta c\phi - s\beta s\theta s\phi)\hat{b}_3 + s\lambda c\beta(c\lambda c\beta s\theta s\phi - s\lambda c\beta c\theta s\phi)\hat{b}_1$$
$$+ (s\beta s\lambda c\beta c\phi - s\beta^2 s\theta s\phi)\hat{b}_2 - (s\beta^2 c\theta s\phi - c\lambda c\beta c\phi s\beta)\hat{b}_1] \quad (\underline{4})$$

Combine (1), (2), (3), (4)

$${}^{I}\vec{a}_{P/0} = [-\Omega_E^2 R_E(c^2\lambda c\beta s\beta - s\beta s^2\lambda) + (c\theta c\phi)(\lambda\ddot{\phi} - l\dot{\theta}^2 c\phi s\phi)$$
$$- s\theta(2\lambda\dot{\theta}\dot{\phi}c\phi + l\ddot{\theta}s\phi) - (c\theta s\phi)(\lambda\dot{\phi}^2 + \lambda\dot{\theta}^2 s^2\phi)$$
$$+ 2\Omega_E(-s\lambda c\beta\lambda\dot{\phi}s\phi - s\beta\lambda\dot{\phi}s\theta c\phi - l\dot{\theta}s\phi c\theta s\beta)$$
$$+ \Omega_E\lambda(s\lambda c\beta c\phi - s\beta s\theta s\phi)]\hat{b}_1$$

$$+[-\Omega_E^2 R_E(c\lambda c\beta s\lambda s\beta + s\lambda s\beta c\lambda) + (s\theta c\phi)(\lambda\ddot{\phi} - \lambda\dot{\theta}^2 c\phi s\phi)$$
$$+ c\theta(2\lambda\dot{\theta}\dot{\phi}c\phi + \lambda\ddot{\theta}s\phi) - (s\theta s\phi)(\lambda\dot{\phi}^2 + \lambda\dot{\theta}^2 s^2\phi)$$
$$+ 2\Omega_E(c\lambda c\beta\lambda\dot{\phi}s\phi + s\beta\lambda\dot{\phi}c\theta c\phi - l\dot{\theta}s\phi s\theta s\beta)$$
$$+ \Omega_E\lambda(c\lambda c\beta s\theta s\phi - s\lambda c\beta c\theta s\phi)]\hat{b}_2$$

$$+[-\Omega_E^2 R_E c\lambda c^2\beta - s\phi(\lambda\ddot{\phi} - l\dot{\theta}^2 c\phi s\phi) - c\phi(l\dot{\phi}^2 + l\dot{\theta}^2 s^2\phi)$$
$$+ 2\Omega_E(c\lambda c\beta(\lambda\dot{\phi}s\theta c\phi + l\dot{\theta}s\phi c\theta) - s\lambda c\beta(\lambda\dot{\phi}c\theta c\phi - l\dot{\theta}s\phi s\theta))$$
$$+ c\lambda c\beta s\theta s\phi - s\lambda c\beta c\theta s\phi]\hat{b}_3$$

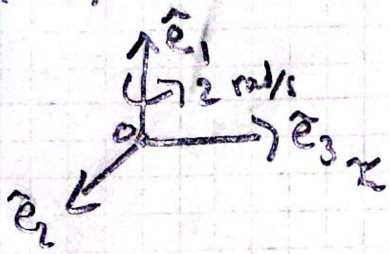
$$\vec{F}_P = m\,{}^{I}\vec{a}_{P/0} = -mg\hat{b}_3 - T(c\theta s\phi\hat{b}_1 + s\theta s\phi\hat{b}_2 + c\phi\hat{b}_3)$$

b) Rate of Rotation of plane $= \Omega_E \sin\beta = \left(\frac{360^\circ}{24hr}\right)\left(\sin(48)\right)$

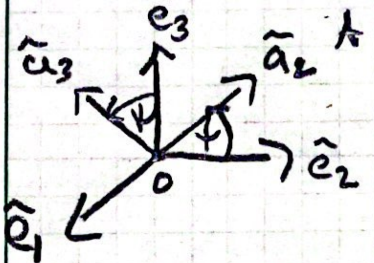
$$= \boxed{11.15\ ^\circ/_{hr}}$$

c) If constrcuted at North or South pole, ($\beta = \pm 90^\circ$) Pendulum rotates as same rate as Earth. If constructed at equator ($\beta = 0^\circ$), no motion occurs.

10.13)

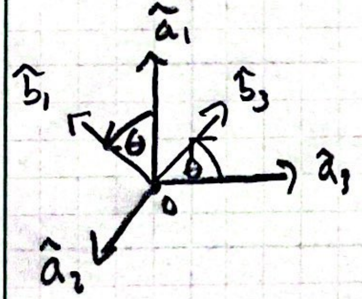


$$\mathcal{I} = (0, \hat{e}_1, \hat{e}_2, \hat{e}_3)$$



$$\begin{array}{c|ccc} & \hat{a}_1 & \hat{a}_2 & \hat{a}_3 \\ \hline \hat{e}_1 & 1 & 0 & 0 \\ \hat{e}_2 & 0 & C\psi & -S\psi \\ \hat{e}_3 & 0 & S\psi & C\psi \end{array}$$

$${}^{I}\vec{\omega}^{A} = \dot{\psi}\,\hat{e}_1 = 2 \text{ rad/s}$$



$$\begin{array}{c|ccc} & \hat{b}_1 & \hat{b}_2 & \hat{b}_3 \\ \hline \hat{a}_1 & C\theta & 0 & S\theta \\ \hat{a}_2 & 0 & 1 & 0 \\ \hat{a}_3 & -S\theta & 0 & C\theta \end{array}$$

$${}^{A}\vec{\omega}^{B} = \dot{\theta}\,\hat{a}_2 = 10 \text{ rad/s}$$

$${}^{I}\vec{\omega}^{B} = {}^{I}\vec{\omega}^{A} + {}^{A}\vec{\omega}^{B} = \dot{\psi}\,\hat{a}_1 + \dot{\theta}\,\hat{a}_2$$

$$\vec{r}_{B/0} = 25\text{mm}\,\hat{b}_3$$

$$\vec{r}_{A/0} = 50\text{mm}\,\hat{a}_2 = .05 \text{ m}$$

$${}^{I}\vec{v}_{A/0} = {}^{B}\frac{d}{dt}\left(\cancel{\vec{r}_{A/0}}^{\,0}\right) + {}^{I}\vec{\omega}^{A} \times \vec{r}_{A/0}$$

$${}^{I}\vec{v}_{A/0} = \dot{\psi}\,\hat{a}_1 \times \vec{r}_{A/0} = 2\hat{a}_1 \times .05\,\hat{a}_2$$

$$\boxed{{}^{I}\vec{v}_{A/0} = 0.10 \text{ m/s}\ \hat{a}_3}$$

$$^{I}\vec{v}_{B/O} = {}^{B}\frac{d}{dt}\left(\cancel{\vec{r}_{B/O}}\right)^{0} + {}^{I}\vec{\omega}^{B} \times \vec{r}_{B/O}$$

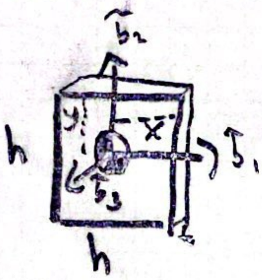
$$^{I}\vec{v}_{B/O} = (2\hat{a}_1 + 10\hat{a}_2) \times (.025\hat{b}_3)$$

$$\hat{b}_3 = \sin\theta\,\hat{a}_1 + \cos\theta\,\hat{a}_3$$

$$^{I}\vec{v}_{B/O} = (2\hat{a}_1 + 10\hat{a}_2) \times (.025\sin\theta\,\hat{a}_1 + .025\cos\theta\,\hat{a}_3)$$

$$\boxed{^{I}\vec{v}_{B/O} = -0.05\cos\theta\,\hat{a}_2 - 0.25\sin\theta\,\hat{a}_3 + 0.25\cos\theta\,\hat{a}_1}$$

11.7)



$h \gg t$ $\qquad \rho = \frac{m}{V} = \frac{m_P}{h^2 t}$

$(x, y, z)_B$

a) $[I_G]_B = \rho \int_B \begin{pmatrix} y^2+z^2 & -xy & -xz \\ -yx & x^2+z^2 & -yz \\ -zx & -zy & x^2+y^2 \end{pmatrix}_B dx\, dy\, dz$

$$\int_{-t/2}^{t/2}\int_{-h/2}^{h/2}\int_{-h/2}^{h/2} y^2+z^2 \, dx\, dy\, dz = \int_{-t/2}^{t/2}\int_{-h/2}^{h/2} \left[y^2 x + z^2 x \Big|_{-h/2}^{h/2} \right] dy\, dz$$

$$\int_{-t/2}^{t/2}\int_{-h/2}^{h/2} y^2 h + z^2 h \, dy\, dz = \int_{-t/2}^{t/2} \left[\frac{y^3}{3} h + z^2 h y \Big|_{-h/2}^{h/2} \right] dz$$

$$\frac{\left(\frac{h}{2}\right)^3}{3} h + z^2 h \frac{h}{2} - \frac{\left(\frac{-h}{2}\right)^3}{2} h - z^2 h \left(\frac{-h}{2}\right)$$

$$= \frac{h^4}{24} + z^2 \frac{h^2}{2} + \frac{h^4}{24} + z^2 \frac{h^2}{2} = \frac{h^4}{12} + z^2 h^2$$

$$\int_{-t/2}^{t/2} \frac{h^4}{12} + z^2 h^2 \, dz = \frac{h^4}{12} \frac{t}{2} + \frac{\left(\frac{t}{2}\right)^3}{3} h^2 - \frac{h^4}{12} \frac{t}{2} - \frac{\left(\frac{t}{2}\right)^3 h^2}{3}$$

$$= \frac{h^4}{12} t - \frac{t^3}{12} h^2$$

$h >> t \therefore$ Neglect h.O.T of t

$$\therefore \int_B y^2+z^2 \, dx\,dy\,dz \approx \frac{h^4 t}{12}$$

$$I_{11} = \rho \int_B y^2+z^2 \, dx\,dy\,dz = \frac{m\rho}{h^2 t} \cdot \frac{h^4 t}{12}$$

$$I_{11} = \frac{m\rho h^2}{12}$$

$$I_{22} = \rho \int_B x^2+z^2 \, dx\,dy\,dz$$

$$\int_{-t/2}^{t/2}\int_{-h/2}^{h/2}\int_{-h/2}^{h/2} x^2+z^2 \, dx\,dy\,dz$$

Because x & y have same bounds of integration it can be shown that $\int_B x^2+z^2 \, dV = \int_B y^2+z^2 \, dV \therefore$

$\underline{I_{11} = I_{22}}$

$$I_{33} = \rho \int_B x^2+y^2 \, dV$$

$$\int_{-t/2}^{t/2}\int_{-h/2}^{h/2}\int_{-h/2}^{h/2} x^2+y^2 \, dx\,dy\,dz = \int_{-t/2}^{t/2}\int_{-h/2}^{h/2} \left[\frac{x^3}{3} + y^2 x \Big|_{-h/2}^{h/2} \right] dy\,dz$$

$$= \int\int_{-h/2}^{h/2} \frac{h^3}{12} + y^2 h \, dy\,dz = \int_{-t/2}^{t/2} \frac{h^4}{12} + \frac{h^4}{12} dz = \int_{-t/2}^{t/2} \frac{h^4}{6} \, dz$$

$$I_{33} = \rho \frac{h^4 t}{6} = \frac{m_p}{h^2 t} \frac{h^4 t}{6}$$

$$I_{33} = \frac{m_p h^2}{6}$$

$$I_{12} = \rho \int_B -xy \, dV = I_{21}$$

$$\int_{-t/2}^{t/2} \int_{-h/2}^{h/2} \int_{-h/2}^{h/2} -xy \, dx\, dy\, dz = \iint \left[-\frac{x^2}{2} y \Big|_{-h/2}^{h/2} \right] dy\, dz$$

$$-\frac{\left(\frac{h}{2}\right)^2}{2} y + \frac{\left(\frac{h}{2}\right)^2}{2} y = 0$$

$$\int_{-t/2}^{t/2} \int_{-h/2}^{h/2} 0 \, dy\, dz = 0 = I_{12} = I_{21}$$

$$I_{31} = I_{13} = \rho \int_B -xz \, dx\, dy\, dz$$

$$= \iint \int_{-h/2}^{h/2} -xz \, dx\, dy\, dz = \int_{-t/2}^{t/2} \int_{-h/2}^{h/2} \left[-\frac{x^2}{2} z \Big|_{-h/2}^{h/2} \right] dy\, dz$$

$$= \iint 0 \, dy\, dz = 0 = I_{31} = I_{13}$$

$\therefore$ It can be shown all products of inertia are zero.

$$\therefore \boxed{[\mathbb{I}_G]_B = \begin{pmatrix} \frac{m_p h^2}{12} & 0 & 0 \\ 0 & \frac{m_p h^2}{12} & 0 \\ 0 & 0 & \frac{m_p h^2}{6} \end{pmatrix}}$$

5)

$$\vec{r}_{O/G} = \frac{h}{2}\hat{b}_3$$

$$\|\vec{r}_{O/G}\| I - [\vec{r}_{O/G}]_B [\vec{r}_{O/G}]_B^T$$

$$\begin{pmatrix} \frac{h^2}{4} & 0 & 0 \\ 0 & \frac{h^2}{4} & 0 \\ 0 & 0 & \frac{h^2}{4} \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ h/2 \end{pmatrix} \begin{pmatrix} 0 & 0 & h/2 \end{pmatrix}$$

$$\begin{pmatrix} \frac{h^2}{4} & 0 & 0 \\ 0 & \frac{h^2}{4} & 0 \\ 0 & 0 & \frac{h^2}{4} \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{h^2}{4} \end{pmatrix} = \begin{pmatrix} \frac{h^2}{4} & 0 & 0 \\ 0 & \frac{h^2}{4} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$[I_O]_B = [I_G]_B + m_p \begin{pmatrix} \frac{h^2}{4} & 0 & 0 \\ 0 & \frac{h^2}{4} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$[I_O]_B = \begin{pmatrix} \frac{m_p h^2}{12} + \frac{m_p h^2}{4} & 0 & 0 \\ 0 & \frac{m_p h^2}{12} + \frac{m_p h^2}{4} & 0 \\ 0 & 0 & \frac{m_p h^2}{6} \end{pmatrix}$$

$$\boxed{[I_O]_B = \begin{pmatrix} \frac{m_p h^2}{3} & 0 & 0 \\ 0 & \frac{m_p h^2}{3} & 0 \\ 0 & 0 & \frac{m_p h^2}{6} \end{pmatrix}}$$

c)

For vertical plate: $[\mathbb{I}_O]_{\mathcal{C}} = \begin{pmatrix} \frac{M_p h^2}{3} & 0 & 0 \\ 0 & \frac{M_p h^2}{3} & 0 \\ 0 & 0 & \frac{M_p h^2}{6} \end{pmatrix}$

Cube

For Top/Bottom plate: $[\mathbb{I}_O]_{\mathcal{C}} = \begin{pmatrix} \frac{M_p h^2}{3} & 0 & 0 \\ 0 & \frac{M_p h^2}{6} & 0 \\ 0 & 0 & \frac{M_p h^2}{3} \end{pmatrix}$

cube

For side plate: $[\mathbb{I}_O]_{\mathcal{C}} = \begin{pmatrix} \frac{M_p h^2}{6} & 0 & 0 \\ 0 & \frac{M_p h^2}{3} & 0 \\ 0 & 0 & \frac{M_p h^2}{3} \end{pmatrix}$

Cube

Cube is 2x Vertical, 2x side, 1 Top, 1 bottom:

Therefore $[\mathbb{I}_G]_{\mathcal{C}} = 2\begin{pmatrix} \frac{M_p h^2}{3} & 0 & 0 \\ 0 & \frac{M_p h^2}{3} & 0 \\ 0 & 0 & \frac{M_p h^2}{6} \end{pmatrix} + 2\begin{pmatrix} \frac{M_p h^2}{3} & 0 & 0 \\ 0 & \frac{M_p h^2}{6} & 0 \\ 0 & 0 & \frac{M_p h^2}{3} \end{pmatrix}$

cube

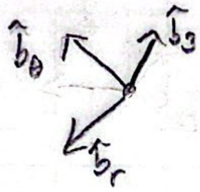
$$+ 2\begin{pmatrix} \frac{M_p h^2}{6} & 0 & 0 \\ 0 & \frac{M_p h^2}{3} & 0 \\ 0 & 0 & \frac{M_p h^2}{3} \end{pmatrix}$$

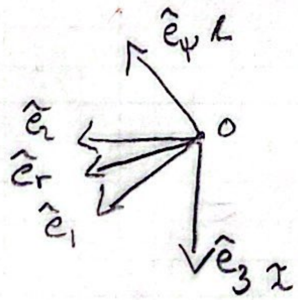
$\frac{2}{3} + \frac{2}{3} + \frac{1}{3} = \frac{5}{3}$

$$\boxed{[\mathbb{I}_G] = \begin{pmatrix} \frac{5}{3} h^2 M_p & 0 & 0 \\ 0 & \frac{5}{3} M_p h^2 & 0 \\ 0 & 0 & \frac{5}{3} M_p h^2 \end{pmatrix}}$$

11.11)

Reference Frames:





$\mathcal{I} = (O, \hat{e}_1, \hat{e}_2, \hat{e}_3)$

$\mathcal{R} = (O, \hat{e}_r, \hat{e}_\psi, \hat{e}_3)$

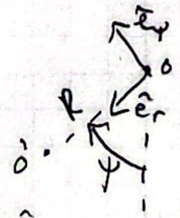
$\mathcal{B} = (O', \hat{b}_r, \hat{b}_\theta, \hat{b}_3)$ $\quad\quad \hat{b}_3 = -\hat{e}_r$

${}^{\mathcal{I}}\vec{\omega}^{\mathcal{R}} = \dot{\psi}\,\hat{e}_3$

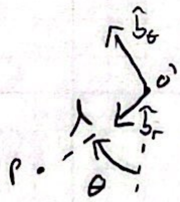
${}^{\mathcal{R}}\vec{\omega}^{\mathcal{B}} = \dot{\theta}\,\hat{b}_3$

Coordinates:

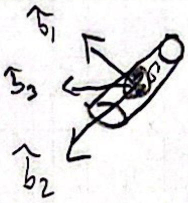
Top:



Side:

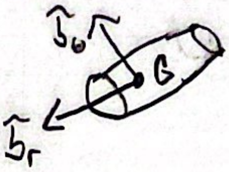


Moment of Inertia of Rod:



$$[\mathbb{I}_G] = \frac{m\ell^2}{12}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

in $\mathcal{B}$ Frame: $[\mathbb{I}_G]_{\mathcal{B}} = \dfrac{m\ell^2}{12}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$



Angular momentum of rod: ${}^{\mathcal{I}}h_G = \mathbb{I}_G \, {}^{\mathcal{I}}\vec{\omega}^B$

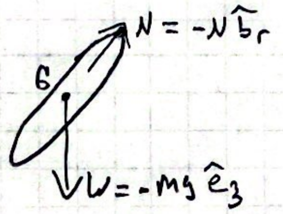
${}^{\mathcal{I}}\vec{\omega}^B = \dot{\psi}\,\hat{e}_3 + \dot{\theta}\,\hat{b}_3$

$\hat{e}_3 = \hat{b}_r \cos\theta - \hat{b}_\theta \sin\theta$

$${}^{\mathcal{I}}\vec{h}_G = \frac{ml^2}{12}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \dot{\psi}\cos\theta \\ -\dot{\psi}\sin\theta \\ \dot{\theta} \end{pmatrix}$$

$${}^{\mathcal{I}}\vec{h}_G = -\frac{ml^2}{12}\dot{\psi}\sin\theta\,\hat{b}_\theta + \frac{ml^2}{12}\dot{\theta}\,\hat{b}_3$$

FBD:



Moment about G: $\vec{M}_G = \cancelto{0}{\vec{r}_{G/G} \times \vec{W}} + \vec{r}_{O/G} \times \vec{N}$

$\frac{l}{2}\hat{b}_r \times N\hat{b}_r = 0$

$\therefore\ \vec{M}_G = 0$

Eulers Equation: ${}^B\frac{d}{dt}[{}^{\mathcal{I}}\vec{h}_G]_B + [{}^{\mathcal{I}}\vec{\omega}^B \times {}^{\mathcal{I}}\vec{h}_G]_B = \vec{M}_G$

$$\left[-\frac{ml^2}{12}\ddot{\psi}\sin\theta - \frac{ml^2}{12}\dot{\psi}\dot{\theta}\cos\theta\right]\hat{b}_\theta + \frac{ml^2\ddot{\theta}}{12}\hat{b}_3$$

$$+ (\dot{\psi}\cos\theta\,\hat{b}_r - \dot{\psi}\sin\theta\,\hat{b}_\theta + \dot{\theta}\,\hat{b}_3) \times \left(-\frac{ml^2}{12}\dot{\psi}\sin\theta\,\hat{b}_\theta + \frac{ml^2}{12}\dot{\theta}\,\hat{b}_3\right)$$

$$= \left(-\frac{ml^2}{12}\ddot{\psi}\sin\theta - \frac{ml^2}{12}\dot{\psi}\dot{\theta}\cos\theta\right)\hat{b}_\theta + \frac{ml^2}{12}\ddot{\theta}\,\hat{b}_3 + \dot{\psi}^2\cos\theta\sin\theta\,\frac{ml^2}{12}\hat{b}_3$$

$$- \dot{\psi}\cos\theta\,\frac{ml^2}{12}\dot{\theta}\,\hat{b}_\theta - \dot{\psi}\dot{\theta}\sin\theta\,\frac{ml^2}{12}\hat{b}_r + \dot{\theta}\dot{\psi}\sin\theta\,\frac{ml^2}{12}\hat{b}_r$$

$$0 = \left[-\frac{ml^2}{12} \ddot{\psi} \sin\theta - \frac{2ml^2}{12} \dot{\psi}\dot{\theta} \cos\theta \right] \hat{b}_\theta + \left[\frac{ml^2}{12} \ddot{\theta} - \dot{\psi}^2 \cos\theta \sin\theta \frac{ml^2}{12} \right] \hat{b}_3$$

$$\boxed{\begin{aligned} \hat{b}_\theta:&\quad \ddot{\psi} = \frac{-2\dot{\psi}\dot{\theta}\cos\theta}{\sin\theta} \\ \hat{b}_3:&\quad \ddot{\theta} = -\dot{\psi}^2 \cos\theta \sin\theta \end{aligned}}$$