Relative orientation & Euler Angles

→ Euler angles are one method for describing the orientation or attitude of a rigid body

→ Elementary rotation matrices can be multiplied to give more complex, 3D rotations. The angles that are used in this construction are Euler Angles

→ There are many conventions for Euler angles.

→ Two major classifications:

- Intrinsic Method: Each rotation in a series of rotations occurs about the current axes (intermediate axes)

- Extrinsic Method: The rotations occur about fixed or initial axes.

→ We select a convention with a set of 3 numbers, which are the axes that are rotate about when we do the elementary rotations.

Ex. 3 - 2 - 1

↑
Number of the axis you rotate around

Question: How many types of Euler angle conventions are there?

1-2-1	3-1-2
1-2-3	3-1-3
1-3-1	3-2-1
1-3-2	3-2-3
2-1-2	
2-1-3	
2-3-1	
2-3-2	

$\hat{e}_1$

12 possible for intrinsic

\+

12 possible for extrinsic

24 possible conventions

Notes on Rotation Matrices

1) $({}^{B}C^{F})^{T} = ({}^{B}C^{F})^{-1} = {}^{F}C^{B}$ (orthogonality property)
2) Rotation matrices preserve length of a vector during rotation
3) Rotation matrices can be used to transform components (coordinates) of a vector between reference frames, or $[\vec{r}_{P/O}]_{I} = {}^{I}C^{B}\,[\vec{r}_{P/O}]_{B}$
4) Rotation matrices can also be used to specify attitude (orientation).
5) Determinant is always +1 ("special")

Properties 1) and 5) lead us to define a set of all "special" orthogonal 3x3 matrices:

$$\underbrace{SO(3)}_{\text{Lie Group}} \overset{\text{Def'n}}{\triangleq} \underset{\text{set}}{\{} R \underset{\text{"member of"}}{\in} \mathbb{R}^{3\times3} \overset{\text{"such that"}}{|} \underbrace{RR^T = I}_{\substack{\text{orthogonality} \\ R^T = R^{-1}}}, \underbrace{\det(R) = +1}_{\text{"special" property}} \}$$

Key idea: Every orientation of a rigid body which is free to rotate relative to a fixed/inertial frame can be identified with an element of $SO(3)$.

Q: How many degrees of freedom are in $R \in SO(3)$?

9 Entries in 3x3 matrix $\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$

$$\#DOF = 9 - \#\text{constraints}$$
$$= 9 - 6 = \boxed{3}$$

Constraints:

- Any column dotted with another is zero (3 equations)
- Any column dotted with itself is one (3 equations)

(Similarly, we could have done this for rows)

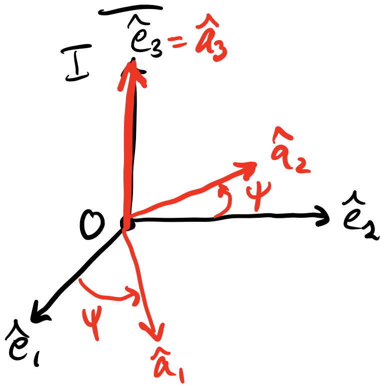
There are a number of ways to "parameterize" $SO(3)$

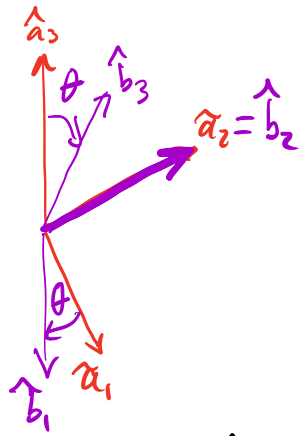
(a) Full $SO(3)$ matrix $\longrightarrow$ 9 numbers
(b) Euler angles $\longrightarrow$ 3 numbers (incomplete)
(c) Quaternions $\longrightarrow$ 4 numbers
↳ Generalization of complex numbers

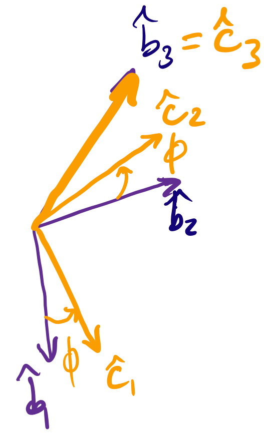
Intrinsic method of Euler angle construction

Confusing pictures: "fixed-camera perspective"

Ex. 3-2-3







	$\hat{a}_1$	$\hat{a}_2$	$\hat{a}_3$
$\hat{e}_1$	$c\psi$	$-s\psi$	0
$\hat{e}_2$	$s\psi$	$c\psi$	0
$\hat{e}_3$	0	0	1

$\underbrace{}_{{}^{I}C^{A}}$

	$\hat{b}_1$	$\hat{b}_2$	$\hat{b}_3$
$\hat{a}_1$	$c\theta$	0	$s\theta$
$\hat{a}_2$	0	1	0
$\hat{a}_3$	$-s\theta$	0	$c\theta$

$\underbrace{}_{{}^{A}C^{B}}$

	$\hat{c}_1$	$\hat{c}_2$	$\hat{c}_3$
$\hat{b}_1$	$c\phi$	$-s\phi$	0
$\hat{b}_2$	$s\phi$	$c\phi$	0
$\hat{b}_3$	0	0	1

$\underbrace{}_{{}^{B}C^{C}}$

$${}^{I\to A}\vec{\omega} = \dot{\psi}\,\hat{a}_3$$

$${}^{A\to B}\vec{\omega} = \dot{\theta}\,\hat{b}_2$$

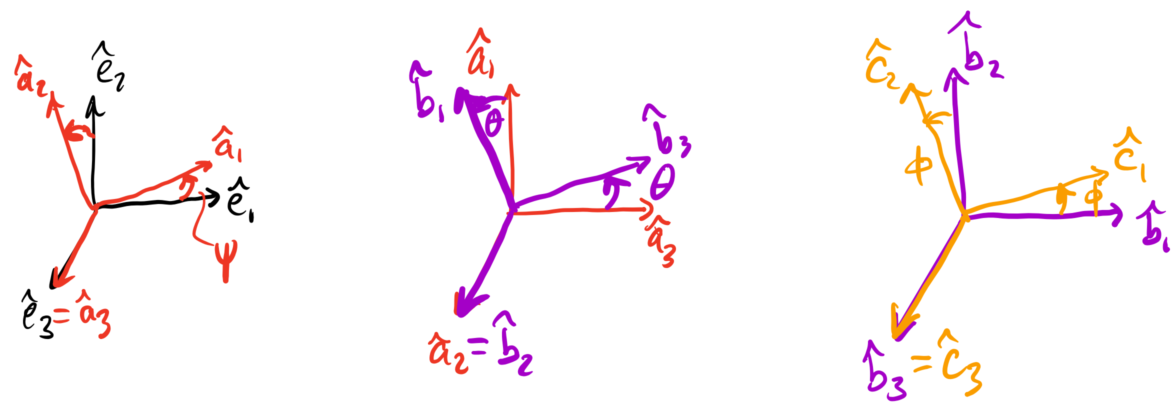
$${}^{B\to C}\vec{\omega} = \dot{\phi}\,\hat{c}_3$$

Compose the overall transformation by multiplication:

$$ {}^{I}C^{c} = ({}^{I}C^{A})({}^{A}C^{B})({}^{B}C^{C}) $$

Common point of confusion: If you want ${}^{c}C^{I}$, do not alter the pictures (i.e. the definition of your angles) simply re-do your transformation tables interchanging the axes titles, OR simply take a transpose of ${}^{I}C^{c}$.

3-2-3 "Elementary Rotation Perspective"



- The Elementary Rotation Perspective requires the use of the right-hand rule at every step in the making of the pictures, but it makes the transformation tables much easier to form.

Euler rate kinematics (For 3-2-3 Euler angles)

Important link between what you measure in the body frame (i.e. the angular rates) and the rate of change of orientation (i.e. Euler angles).

$$
\begin{aligned}
{}^{I}\vec{\omega}^{C} &= {}^{I}\vec{\omega}^{A} + {}^{A}\vec{\omega}^{B} + {}^{B}\vec{\omega}^{C} \quad \text{(Addition of angular velocities)} \\
&= \dot{\psi}\hat{a}_3 + \dot{\theta}\hat{b}_2 + \dot{\phi}\hat{c}_3 \\
&= \dot{\psi}(-s\theta\hat{b}_1 + c\theta\hat{b}_3) + \dot{\theta}(s\phi\hat{c}_1 + c\phi\hat{c}_2) + \dot{\phi}\hat{c}_3 \\
&= -\dot{\psi}s\theta(c\phi\hat{c}_1 - s\phi\hat{c}_2) + \dot{\psi}c\theta\hat{c}_3 + \dot{\theta}s\phi\hat{c}_1 + \dot{\theta}c\phi\hat{c}_2 + \dot{\phi}\hat{c}_3
\end{aligned}
$$

Set this equal to

$$
{}^{I}\vec{\omega}^{C} = \omega_1\hat{c}_1 + \omega_2\hat{c}_2 + \omega_3\hat{c}_3
$$

$$
\boxed{
\begin{aligned}
\omega_1 &= \dot{\theta}s\phi - \dot{\psi}s\theta c\phi \\
\omega_2 &= \dot{\theta}c\phi + \dot{\psi}s\theta s\phi \\
\omega_3 &= \dot{\phi} + \dot{\psi}c\theta
\end{aligned}
}
$$

Euler rate Kinematics (or Rotational Kinematics)

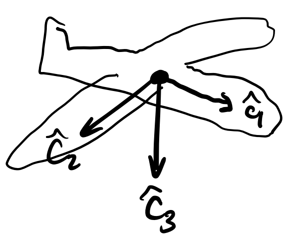
Rewrite to solve for the Euler Angular rates:

$$\dot{\psi} = (-\omega_1 c\phi + \omega_2 s\phi)\csc\theta$$
$$\dot{\theta} = \omega_1 s\phi + \omega_2 c\phi$$
$$\dot{\phi} = (\omega_1 c\phi - \omega_2 s\phi)\cot\theta + \omega_3$$

"Gimbal lock"
Note: singularity at $\theta = 0$ or π

(Varies based on choice of Euler angles)

Ex. Airplane Convention (3-2-1)



ψ yaw angle about

θ pitch angle about

ϕ roll angle about the

Translational kinematics:

$$[{}^I\vec{V}_{G/O}]_I = {}^I C^C [{}^I\vec{V}_{G/O}]_C$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}_I = {}^I C^C \begin{bmatrix} u \\ v \\ w \end{bmatrix}_C$$

Rotational kinematics:

Euler rate kinematics for 3-2-1.

$$[{}^I\vec{\omega}^C]_C = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \rightarrow \text{relate to } \dot{\psi}, \dot{\theta}, \dot{\phi}$$

These are called $\begin{bmatrix} p \\ q \\ r \end{bmatrix}$

Euler Theorem of Rotation

↳ Another way to describe orientation of a Rigid Body

The orientation of a body frame C can be obtained by a single rotation of C (initially aligned with I) about an axis that is fixed in C

How do we find the Euler axis $\hat{k}$?

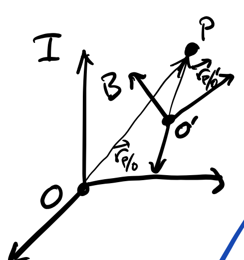
$$[\hat{k}]_I = \underbrace{{}^I C^C [\hat{k}]_C = [\hat{k}]_C}_{\substack{\text{Eigenvalue/Eigenvector equation} \\ \text{for } \lambda = 1}}$$

$[\hat{k}]_C$ is an e-vector of ${}^I C^C$, corresponding to the eigenvalue $\lambda = 1$.

Chapter 11: Multiparticle and Rigid Body Dynamics in 3D

- Since we took a cordinate free vector treatment previously, many equations are unchanged in 3D. Many important principles still hold that lead to the same derivations:
  1) Definition of center of mass
  2) Center of mass corollary
  3) Transport Eqn in 3D Kinematics
  4) Applying N2L to each particle in a multiparticle system.
  5) Internal moment assumption
  6) Separation principle for angular momentum and kinetic energy

Recall from Ch 10, Relative motion kinematics in 3D:



$$\begin{cases} \vec{r}_{P/O} = \vec{r}_{O'/O} + \vec{r}_{P/O'} \\ {}^{I}\vec{v}_{P/O} = {}^{I}\vec{v}_{O'/O} + \frac{{}^{B}d}{dt}(\vec{r}_{P/O'}) + {}^{I}\vec{\omega}^{B} \times \vec{r}_{P/O'} \\ {}^{I}\vec{a}_{P/O} = {}^{I}\vec{a}_{O'/O} + {}^{B}\vec{a}_{P/O'} + {}^{I}\vec{\alpha}^{B} \times \vec{r}_{P/O'} + 2\,{}^{I}\vec{\omega}^{B} \times {}^{B}\vec{v}_{P/O'} \\ \qquad + {}^{I}\vec{\omega}^{B} \times ({}^{I}\vec{\omega}^{B} \times \vec{r}_{P/O'}) \end{cases}$$

Note: No changes. Compare 8.21 to 10.52.

Rigid body dynamics in 3D

Euler's 1st law $\quad \vec{F}_G = m_G \, {}^{I}\vec{a}_{G/O}$

Euler's 2nd law $\quad \vec{M}_G = \frac{{}^{I}d}{dt}\left({}^{I}\vec{h}_G\right)$

Kinetic energy of a rigid body

$$T_O = T_{G/O} + T_G$$

$$\boxed{\frac{1}{2} m_G \left\| {}^{I}\vec{v}_{G/O} \right\|^2}$$

$$T_G = \frac{1}{2} \sum_{i=1}^{N} m_i \left\| {}^{I}\vec{v}_{i/G} \right\|^2$$

$\vdots$ (Manipulations using transport, scalar triple product, vector triple products, that essentially re-derive moment of inertia tensor).

$$T_G = \frac{1}{2} \, {}^{I}\vec{\omega}^{B} \cdot \mathbb{I}_G \circ {}^{I}\vec{\omega}^{B} = \frac{1}{2} \, {}^{I}\vec{h}_G \cdot {}^{I}\vec{\omega}^{B}$$

3D Rotational Equations of Motion for a Rigid Body

Alternative form of Euler's 2nd law:

$$\frac{{}^B d}{dt}\left({}^I\vec{h}_G\right) + {}^I\vec{\omega}^B \times {}^I\vec{h}_G = \vec{M}_G$$

We also saw that

$${}^I\vec{h}_G = \mathbb{I}_G \circ {}^I\vec{\omega}^B$$

Combining.

$$\mathbb{I}_G \circ \frac{{}^B d}{dt}\left({}^I\vec{\omega}^B\right) + {}^I\vec{\omega}^B \times {}^I\vec{h}_G = \vec{M}_G$$