ECE 602: LUMPED LINEAR SYSTEMS

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Converting State-Space Models to Transfer Function Models

Transfer Function Models of CT LTI Systems

A continuous-time LTI system with zero initial state

- impulse response: $h(t)$, $t \geq 0$
- Output under any input $u(t)$, $t \geq 0$, is

$$y(t) = \int_0^t h(t-\tau)u(\tau)\, d\tau = h(t) \star u(t), \quad t \geq 0$$

- Taking Laplace transform, we obtain the transfer function model:

$$Y(s) = H(s)U(s) \quad \text{where } H(s) = \mathcal{L}\left[h(t)\right]$$

Example: $\dddot{y}(t) + 2\ddot{y}(t) + 3\dot{y}(t) + y(t) = \ddot{u}(t) - 2\dot{u}(t) + u(t)$

Transfer Function Models of DT LTI Systems

For a discrete-time LTI system with impulse response $h(k)$, $k = 0, 1, \ldots$, its transfer function model is

$$Y(z) = H(z)U(z) \quad \text{where } H(z) = \mathcal{Z}\left[h(k)\right]$$

Example: $y[k] - 0.5y[k-1] + y[k-2] = u[k] - 0.7u[k-1], \quad k = 0, 1, \ldots$

Input/Output Models vs. Internal Models

- Transfer function models are Input/Output (I/O) models
  - Describe how the input affects the output
  - System viewed as a black box
  - Valid only for linear, time-invariant systems
- State-space models are internal models
  - Describe how the input affects not only the output, but also all the internal state variables
  - More complete models suitable for complicated systems
  - Valid for nonlinear, time-varying systems.

Transfer Functions of State-Space Models

A continuous-time LTI system with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}, \quad t \geq 0$$

has transfer function/matrix (assuming zero initial condition $x(0) = 0$):

$$H(s) = C(sI - A)^{-1}B + D$$

where $I \in \mathbb{R}^{n \times n}$ is the identity matrix.

Transfer Functions of State-Space Models

A discrete-time LTI system with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$

$$\begin{cases} x[k+1] = Ax[k] + Bu[k] \\ y[k] = Cx[k] + Du[k] \end{cases}, \quad k = 0, 1, \ldots$$

has transfer function/matrix (assuming zero initial condition $x[0] = 0$):

$$H(z) = C(zI - A)^{-1}B + D$$

where $I \in \mathbb{R}^{n \times n}$ is the identity matrix.

Examples

Example 1:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-1 & 0\\ 0 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}1\\ 1\end{bmatrix}u$$

$$y = \begin{bmatrix}-1 & 1\end{bmatrix}x$$

Example 2:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-3 & -2\\ 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}1\\ 0\end{bmatrix}u$$

$$y = \begin{bmatrix}0 & -1\end{bmatrix}x$$