Case Study

For the following dynamical system,

$$\begin{aligned}\dot{x} &= Ax + bu \\ &= \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \\ y &= cx + du \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} x + 2u,\end{aligned}$$

design an asymptotic observer with its poles located at -3 and at -4. Let $\tilde{x}$ denote the estimate of x and let $u = -\begin{bmatrix} 3 & 2 \end{bmatrix} \tilde{x} + r$. Find the transfer function of the closed-loop system, $Y(s)/R(s)$.

Please take a moment to work through this problem on your own. When you are ready, scroll down the page to view the solution.

Explanation: We first find the observer gain l such that the matrix $(A - lc)$ has its eigenvalues -3 and -4. The desired characteristic polynomial of $(A - lc)$ is

$$\det[sI - A + lc] = s^2 + 7s + 12.$$

Hence,

$$l = \begin{bmatrix} 7 \\ 14 \end{bmatrix}.$$

Let $\tilde{y} = c\tilde{x} + du$. Then the dynamics of the observer are

$$\begin{aligned}\dot{\tilde{x}} &= A\tilde{x} + bu + l(y - \tilde{y}) \\ &= A\tilde{x} + bu + ly - lc\tilde{x} - ldu \\ &= (A - lc)\tilde{x} + ly + bu - ldu \\ &= (A - lc)\tilde{x} + ly + (b - ld)u \\ &= \begin{bmatrix} -7 & 1 \\ -12 & 0 \end{bmatrix} \tilde{x} + \begin{bmatrix} 7 \\ 14 \end{bmatrix} y + \begin{bmatrix} -13 \\ -28 \end{bmatrix} u.\end{aligned}$$

The equations describing the closed loop system driven by the combined controller-observer compensator after taking into account that $y = cx + du$ and that $\tilde{y} = c\tilde{x} + du$, where $u = -k\tilde{x} + r$, are

$$\begin{aligned}\dot{x} &= Ax - bk\tilde{x} + br \\ \dot{\tilde{x}} &= (A - lc)\tilde{x} + lcx - bk\tilde{x} + br \\ y &= cx - dk\tilde{x} + dr.\end{aligned}$$

In matrix form, the above equations become

$$\begin{bmatrix} \dot{x} \\ \dot{\tilde{x}} \end{bmatrix} = \begin{bmatrix} A & -bk \\ lc & A - lc - bk \end{bmatrix} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix} + \begin{bmatrix} b \\ b \end{bmatrix} r$$

$$y = \begin{bmatrix} c & -dk \end{bmatrix} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix} + dr.$$

To compute the transfer of the closed-loop system, we use the following similarity coordinate transformation to represent the above system in the new coordinates,

$$\begin{bmatrix} x \\ x - \tilde{x} \end{bmatrix} = \begin{bmatrix} x \\ e \end{bmatrix} = \begin{bmatrix} I_n & O \\ I_n & -I_n \end{bmatrix} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix}.$$

Note that the inverse of the above transformation equals itself. In the new coordinates the closed-loop system has the form

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A - bk & bk \\ O & A - lc \end{bmatrix} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix} + \begin{bmatrix} b \\ 0 \end{bmatrix} r$$

$$y = \begin{bmatrix} c - dk & dk \end{bmatrix} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix} + dr.$$

The transfer function of the closed-loop system is\

$$\boxed{\frac{Y(s)}{R(s)} = (c - dk)(sI - A + bk)^{-1} b + d = \frac{-5s - 8}{s^2 + 3s + 2} + 2}$$