ECE 68000: MODERN AUTOMATIC CONTROL

Professor Stan Żak

Solving the ARE: The Eigenvector Method

The Hamiltonian matrix

- A major difficulty when solving the ARE is that it is a nonlinear equation
- We present a method for solving the ARE referred to as the MacFarlane and Potter method, or the eigenvector method
- Begin by representing the ARE in the form

$$\begin{aligned} & \boldsymbol{A}^\top \boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} + \boldsymbol{Q} - \boldsymbol{P}\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^\top \boldsymbol{P} \\ &= \begin{bmatrix} \boldsymbol{P} & -\boldsymbol{I}_n \end{bmatrix} \begin{bmatrix} \boldsymbol{A} & -\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^\top \\ -\boldsymbol{Q} & -\boldsymbol{A}^\top \end{bmatrix} \begin{bmatrix} \boldsymbol{I}_n \\ \boldsymbol{P} \end{bmatrix} = \boldsymbol{O} \end{aligned}$$

- The $2n \times 2n$ matrix in the middle is the *Hamiltonian matrix*
- Use the symbol $\boldsymbol{H}$ to denote the Hamiltonian matrix, that is,

$$\boldsymbol{H} = \begin{bmatrix} \boldsymbol{A} & -\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^\top \\ -\boldsymbol{Q} & -\boldsymbol{A}^\top \end{bmatrix}$$

Representing ARE in matrix product format

- The ARE can be represented as

$$\begin{bmatrix} \boldsymbol{P} & -\boldsymbol{I}_n \end{bmatrix} \boldsymbol{H} \begin{bmatrix} \boldsymbol{I}_n \\ \boldsymbol{P} \end{bmatrix} = \boldsymbol{O}$$

- Premultiply the above equation by $\boldsymbol{X}^{-1}$, and then postmultiply it by $\boldsymbol{X}$, where $\boldsymbol{X}$ is a nonsingular $n \times n$ matrix,

$$\begin{bmatrix} \boldsymbol{X}^{-1}\boldsymbol{P} & -\boldsymbol{X}^{-1} \end{bmatrix} \boldsymbol{H} \begin{bmatrix} \boldsymbol{X} \\ \boldsymbol{P}\boldsymbol{X} \end{bmatrix} = \boldsymbol{O}$$

- Suppose can find matrices $\boldsymbol{X}$ and $\boldsymbol{P}\boldsymbol{X}$ such that

$$\boldsymbol{H} \begin{bmatrix} \boldsymbol{X} \\ \boldsymbol{P}\boldsymbol{X} \end{bmatrix} = \begin{bmatrix} \boldsymbol{X} \\ \boldsymbol{P}\boldsymbol{X} \end{bmatrix} \Lambda,$$

  where Λ is an $n \times n$ matrix

Some manipulations

- We have
$$\begin{bmatrix} \boldsymbol{X}^{-1}\boldsymbol{P} & -\boldsymbol{X}^{-1} \end{bmatrix} \begin{bmatrix} \boldsymbol{X} \\ \boldsymbol{P}\boldsymbol{X} \end{bmatrix} \boldsymbol{\Lambda} = \boldsymbol{O}$$
- Reduced the problem of solving the ARE to that of constructing appropriate matrices $\boldsymbol{X}$ and $\boldsymbol{P}\boldsymbol{X}$
- To proceed further, let $\boldsymbol{v}_i$ be an eigenvector of $\boldsymbol{H}$ and s_i the corresponding eigenvalue, then
$$\boldsymbol{H}\boldsymbol{v}_i = s_i \boldsymbol{v}_i.$$
- Assume that $\boldsymbol{H}$ has at least n distinct real eigenvalues among its $2n$ eigenvalues
- The results obtained can be generalized for the case when the eigenvalues of $\boldsymbol{H}$ are complex or non-distinct

Eigenvalues and eigenvectors

- Write

$$\boldsymbol{H} \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n \end{bmatrix} = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n \end{bmatrix} \begin{bmatrix} s_1 & 0 & \cdots & 0 \\ 0 & s_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & s_n \end{bmatrix}$$

- Let

$$\begin{bmatrix} \boldsymbol{X} \\ \boldsymbol{PX} \end{bmatrix} = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n \end{bmatrix}$$

and

$$\Lambda = \begin{bmatrix} s_1 & 0 & \cdots & 0 \\ 0 & s_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & s_n \end{bmatrix}$$

Solution candidates

- The above choice of $\boldsymbol{X}$ and $\boldsymbol{PX}$ constitutes a possible solution to the equation

$$\begin{bmatrix} \boldsymbol{X}^{-1}\boldsymbol{P} & -\boldsymbol{X}^{-1} \end{bmatrix} \begin{bmatrix} \boldsymbol{X} \\ \boldsymbol{PX} \end{bmatrix} \Lambda = \boldsymbol{O}$$

- To construct $\boldsymbol{P}$, partition the $2n \times n$ eigenvector matrix $\begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n \end{bmatrix}$ into two $n \times n$ submatrices as follows

$$\begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n \end{bmatrix} = \begin{bmatrix} \boldsymbol{W} \\ \boldsymbol{Z} \end{bmatrix}$$

- Thus

$$\begin{bmatrix} \boldsymbol{X} \\ \boldsymbol{PX} \end{bmatrix} = \begin{bmatrix} \boldsymbol{W} \\ \boldsymbol{Z} \end{bmatrix}$$

- Take $\boldsymbol{X} = \boldsymbol{W}$, $\boldsymbol{PX} = \boldsymbol{Z}$, and assuming that $\boldsymbol{W}$ is invertible we obtain

$$\boxed{\boldsymbol{P} = \boldsymbol{Z}\boldsymbol{W}^{-1}}$$

The solution

- Need to decide what set of n eigenvalues should be chosen from those of $\boldsymbol{H}$ to construct the particular $\boldsymbol{P}$
- In the case when all $2n$ eigenvalues of $\boldsymbol{H}$ are distinct, the number of different matrices $\boldsymbol{P}$ generated by the above described method is
$$\frac{(2n)!}{(n!)^2}$$
- Let $\boldsymbol{Q} = \boldsymbol{C}^\top \boldsymbol{C}$ be a full rank factorization of $\boldsymbol{Q}$
- The Hamiltonian matrix $\boldsymbol{H}$ has n eigenvalues in the open left-half complex plane and n in the open right-half plane if and only if the pair $(\boldsymbol{A}, \boldsymbol{B})$ is stabilizable and the pair $(\boldsymbol{A}, \boldsymbol{C})$ is detectable
- The matrix $\boldsymbol{P}$ that we seek corresponds to the asymptotically stable eigenvalues of $\boldsymbol{H}$

The closed-loop system

- The eigenvalues of the Hamiltonian matrix come in pairs $\pm s_i$
- The characteristic polynomial of $\boldsymbol{H}$ contains only even powers of s

Theorem

The poles of the closed-loop system

$$\dot{\boldsymbol{x}}(t) = \left(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^\top \boldsymbol{P}\right) \boldsymbol{x}(t)$$

are those eigenvalues of $\mathbf{H}$ *having negative real parts*

Proof of the theorem

- Because $\begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n \end{bmatrix} = \begin{bmatrix} \boldsymbol{W} \\ \boldsymbol{Z} \end{bmatrix}$, we can write

$$\begin{bmatrix} \boldsymbol{A} & -\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^\top \\ -\boldsymbol{Q} & -\boldsymbol{A}^\top \end{bmatrix} \begin{bmatrix} \boldsymbol{W} \\ \boldsymbol{Z} \end{bmatrix} = \begin{bmatrix} \boldsymbol{W} \\ \boldsymbol{Z} \end{bmatrix} \Lambda$$

- Multiplying appropriate of $n \times n$ block matrices yields

$$\boldsymbol{A}\boldsymbol{W} - \boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^\top\boldsymbol{Z} = \boldsymbol{W}\Lambda,$$

  or

$$\boldsymbol{A} - \boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^\top\boldsymbol{Z}\boldsymbol{W}^{-1} = \boldsymbol{A} - \boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^\top\boldsymbol{P} = \boldsymbol{W}\Lambda\boldsymbol{W}^{-1},$$

  since $\boldsymbol{P} = \boldsymbol{Z}\boldsymbol{W}^{-1}$

- $\boldsymbol{A} - \boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^\top\boldsymbol{P}$ and Λ are similar matrices, so they have the same eigenvalues
- The eigenvalues of Λ are the asymptotically stable eigenvalues of $\boldsymbol{H}$. $\square$

Example

- The system
$$\dot{x} = 2x + u$$
and the associated performance index
$$J = \int_0^\infty (x^2 + ru^2)\, dt.$$
- Determine r such that the optimal closed-loop system has its pole at -3
- The associated Hamiltonian matrix
$$\boldsymbol{H} = \begin{bmatrix} \boldsymbol{A} & -\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^\top \\ -\boldsymbol{Q} & -\boldsymbol{A}^\top \end{bmatrix} = \begin{bmatrix} 2 & -\frac{1}{r} \\ -1 & -2 \end{bmatrix}.$$
- The characteristic equation of $\boldsymbol{H}$
$$\det(s\boldsymbol{I}_2 - \boldsymbol{H}) = s^2 - 4 - \frac{1}{r} = 0$$
- Hence, $r = \frac{1}{5}$ and $\det(s\boldsymbol{I}_2 - \boldsymbol{H}) = (s-3)(s+3)$