#1) $\frac{d^4 q}{dt^4} - \sin(q) = 0$

$x_1 = q \quad x_2 = \dot{q} \quad x_3 = \ddot{q} \quad x_4 = \dddot{q}$

$\dot{x}_1 = x_2 \quad \dot{x}_2 = x_3 \quad \dot{x}_3 = 4 \quad \dot{x}_4 = \sin(x_1)$

$$A = \left.\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} & \frac{\partial f_1}{\partial x_4} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} & \frac{\partial f_2}{\partial x_4} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} & \frac{\partial f_3}{\partial x_4} \\ \frac{\partial f_4}{\partial x_1} & \frac{\partial f_4}{\partial x_2} & \frac{\partial f_4}{\partial x_3} & \frac{\partial f_4}{\partial x_4} \end{pmatrix}\right|_{x_e = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}} = \left.\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \cos(x_1) & 0 & 0 & 0 \end{pmatrix}\right|_{x_e = \vec{0}}$$

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad |\lambda I - A| = \begin{vmatrix} \lambda & -1 & 0 & 0 \\ 0 & \lambda & -1 & 0 \\ 0 & 0 & \lambda & -1 \\ -1 & 0 & 0 & \lambda \end{vmatrix} = \lambda^4 - 1$$

$$\lambda^4 - 1 = (\lambda^2+1)(\lambda^2-1) \Rightarrow \lambda_{1,2} = \pm i \quad \lambda_{3,4} = \pm 1$$

$\lambda_4 = 1 > 0 \therefore$ the non-linear system is unstable about $x_e = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

#2) $\ddot{q} + \dot{q} - q^3 = 0$

$x_1 = q \qquad x_2 = \dot{q}$

$\dot{x}_1 = x_2 \qquad \dot{x}_2 = -x_2 + x_1^3$

$$A = \left.\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}\right|_{x_e = \binom{0}{0}} = \left.\begin{pmatrix} 0 & 1 \\ 3x_1^2 & -1 \end{pmatrix}\right|_{x_e = \binom{0}{0}} = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$$

$$|\lambda I - A| = \lambda^2 + \lambda = 0 = \lambda(\lambda + 1) = 0$$

$$\lambda_1 = 0 \qquad \lambda_2 = -1$$

λ_1 has zero-real part $\therefore$ the stability properties of the non-linear system about $x_e = \vec{0}$ cannot be determined from linearization.

because $\lambda_2 < 0$

#3)

i) $\dot{x}_1 = (1+x_1^2)x_2$

$\dot{x}_2 = -x_1^3$

$$A = \left.\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}\right|_{x_e=\vec{0}} = \left.\begin{pmatrix} 2x_1x_2 & 1+x_1^2 \\ -3x_1^2 & 0 \end{pmatrix}\right|_{x_e=\vec{0}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$|\lambda I - A| = \lambda^2 = 0 \qquad \lambda_1 = \lambda_2 = 0$

$\boxed{\text{Both eigenvalues have zero-real part, } \therefore \text{ the stability of the non-linear system about } x_e \text{ can't be determined from linearization.}}$

ii) $\dot{x}_1 = \sin(x_2)$

$\dot{x}_2 = (\cos x_1)x_3$

$\dot{x}_3 = e^{x_1}x_2$

$$A = \left.\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \end{pmatrix}\right|_{x_e=\begin{pmatrix}0\\0\\0\end{pmatrix}} = \left.\begin{pmatrix} 0 & \cos(x_2) & 0 \\ -\sin(x_1)x_3 & 0 & \cos(x_1) \\ e^{x_1}x_2 & e^{x_1} & 0 \end{pmatrix}\right|_{x_e=\begin{pmatrix}0\\0\\0\end{pmatrix}}$$

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \Rightarrow |\lambda I - A| = \lambda(\lambda^2 - 1)$$

$$\therefore \lambda_1 = 0 \qquad \lambda_{2,3} = \pm 1$$

$\boxed{\lambda_3 = 1 > 0 \therefore \text{ linearized } A \text{ is unstable \& the non-linear system is unstable about } x_e = \begin{pmatrix}0\\0\\0\end{pmatrix}.}$

#4) $x_1[k+1] = x_1[k]^2 + \sin(x_2[k])$

$x_2[k+1] = 0.4\cos(x_2[k])\,x_1[k]$

$$A = \left.\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}\right|_{x_e = \vec{0}} = \left.\begin{pmatrix} 2x_1 & \cos(x_2) \\ 0.4\cos(x_2) & -0.4x_1\cos(x_2) \end{pmatrix}\right|_{x_e = \binom{0}{0}}$$

$$A = \begin{pmatrix} 0 & 1 \\ 0.4 & 0 \end{pmatrix} \qquad |\lambda I - A| = \lambda^2 - 0.4 = 0$$

$$\lambda_{1,2} = \pm\sqrt{0.4} < 1$$

Both λ_1 & λ_2 have magnitude less than 1, $\therefore$ it can be determined that the system is exponentially stable about the equilibrium state. $x_e = \binom{0}{0}$

#5) $x_1[k+1] = (1 + x_1[k]^3)\, x_2[k]$

$x_2[k+1] = x_1[k]^3 + x_2[k]^5$

$$A = \left.\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_1} \end{pmatrix}\right|_{x_e=\vec{0}} = \left.\begin{pmatrix} 3x_1[k]^2 x_2[k] & 1 + x_1^3[k] \\ 3x_1[k]^2 & 5x_2[k]^4 \end{pmatrix}\right|_{x_e=\vec{0}}$$

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad |\lambda I - A| = \lambda^2 = 0$$

$|\lambda_1| = |\lambda_2| = 0 < 1$ $\therefore$ the non-linear system is exponentially stable about $x_e = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

#6) $x_1[k+1] = x_2[k]$

$x_2[k+1] = \sin(x_1[k]) + x_2[k]^5$

$$A = \left.\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}\right|_{x_e=\vec{0}} = \left.\begin{pmatrix} 0 & 1 \\ \cos(x_1[k]) & 5x_2[k]^4 \end{pmatrix}\right|_{x_e=\vec{0}}$$

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$|\lambda I - A| = \lambda^2 - 1 = 0 \therefore \lambda_{1,2} = \pm 1$

$\boxed{|\lambda_1| = |\lambda_2| = 1 \therefore \text{the stability of the non-linear system cannot be determined.}}$

#7) $\dot{x}_1 = \sigma(x_2 - x_1)$

$\dot{x}_2 = rx_1 - x_2 - x_1 x_3$

$\dot{x}_3 = -bx_3 + x_1 x_2$

$\sigma, r, b > 0$

$V(x) = rx_1^2 + \sigma x_2^2 + \sigma(x_3 - 2r)^2$

For bounded solutions: $V(x)$ is radially unbounded & $\dot{V} \leq 0$ for $\|x\| \geq R$

For radially unbounded: $\lim_{\|x\| \to \infty} V(x) = \infty$

$\|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$

$\|x\| = \infty$ for $x = \begin{pmatrix} \infty \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \infty \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \infty \end{pmatrix}, \begin{pmatrix} \infty \\ \infty \\ 0 \end{pmatrix}, \begin{pmatrix} \infty \\ 0 \\ \infty \end{pmatrix}, \begin{pmatrix} 0 \\ \infty \\ \infty \end{pmatrix}, \& \begin{pmatrix} \infty \\ \infty \\ \infty \end{pmatrix}$

Check limits:

$\lim_{x \to \begin{pmatrix} \infty \\ 0 \\ 0 \end{pmatrix}} V(x) = r\infty^2 = \infty$ ✓ $\quad r > 0$

$\lim_{x \to \begin{pmatrix} 0 \\ \infty \\ 0 \end{pmatrix}} V(x) = \sigma\infty^2 = \infty$ ✓ $\quad \sigma > 0$

$\lim_{x \to \begin{pmatrix} 0 \\ 0 \\ \infty \end{pmatrix}} V(x) = \sigma(\infty)^2 = \infty$ ✓ $\quad \sigma > 0$

$\therefore \lim_{\|x\| \to \infty} V(x) = \infty$ for all combinations of $\|x\| = \infty$ $\therefore$

$V(x) = rx_1^2 + \sigma x_2^2 + \sigma(x_3 - 2r)^2$ is radially unbounded as

$\lim_{\|x\| \to \infty} V(x) = \infty.$

$$\frac{\partial V}{\partial x} = \begin{bmatrix} 2rx_1 & 2\sigma x_2 & 2\sigma(x_3 - 2r) \end{bmatrix}$$

$$\dot{V} = \frac{\partial V}{\partial x}\dot{x} = \begin{bmatrix} 2rx_1 & 2\sigma x_2 & 2\sigma(x_3 - 2r) \end{bmatrix} \begin{bmatrix} \sigma(x_2 - x_1) \\ rx_1 - x_2 - x_1 x_3 \\ -bx_3 + x_1 x_2 \end{bmatrix}$$

$$\dot{V} = 2rx_1\sigma(x_2 - x_1) + \cancel{2\sigma x_2 r x_1} - 2\sigma x_2^2 - \cancel{2\sigma x_1 x_2 x_3}$$
$$+ \cancel{2\sigma x_3 x_1 x_2} - 2\sigma x_3^2 b + 4\sigma r b x_3 - \cancel{4\sigma r x_1 x_2}$$

$$\dot{V} = -2rx_1^2\sigma - 2\sigma x_2^2 - 2\sigma x_3^2 b + 4\sigma r b x_3$$
$$\dot{V} = -2rx_1^2\sigma - 2\sigma x_2^2 - 2\sigma x_3 b(x_3 - 2r)$$

$-2rx_1^2\sigma \leq 0$ for all x

$-2\sigma x_2^2 \leq 0$ for all x

$-2\sigma x_3 b(x_3 - 2r) \leq 0$ for $|x_3| \geq 2r$

If $x_3 < -2r$: $x_3 < 0$ & $-2\sigma x_3 b > 0$ & $(x_3 - 2r) < 0$

$\therefore -2\sigma x_3 b(x_3 - 2r) < 0$

If $x_3 > 2r$: $x_3 > 0$, & $-2\sigma x_3 b < 0$ & $(x_3 - 2r) > 0$

$\therefore -2\sigma x_3 b(x_3 - 2r) < 0$

$\therefore \dot{V} \leq 0$ for $\|x_3\| \geq 2r$

$V(x) = rx_1^2 + \sigma x_2^2 + \sigma(x_3 - 2r)^2$ is radially unbounded & $\dot{V} \leq 0$ for $\|x\| \geq 2r$ $\therefore$ all solutions to the system are bounded.