ECE 602: LUMPED LINEAR SYSTEMS

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Lyapunov functions

Lyapunov Function Definition

A function that allows one to deduce stability is termed a Lyapunov function

Lyapunov Function Properties for Continuous Time Systems

- Continuous-time system

$$\dot{x}(t) = f\left(x(t)\right)$$

- Equilibrium state of interest

$$x_e$$

Three Properties of a Lyapunov Function

We seek an aggregate summarizing function V

- V is continuous
- V has a unique minimum with respect to all other points in some neighborhood of the equilibrium of interest
- Along any trajectory of the system, the value of V never increases

Lyapunov Theorem for Continuous Systems

- Continuous-time system

$$\dot{x}(t) = f(x(t))$$

- Equilibrium state of interest

$$x_e = 0$$

Lyapunov Theorem---Negative Rate of Increase of V

- If $\boldsymbol{x}(t)$ is a trajectory, then $V(\boldsymbol{x}(t))$ represents the corresponding values of V along the trajectory
- In order for $V(\boldsymbol{x}(t))$ not to increase, we require

$$\dot{V}(x(t)) \leq 0$$

The Lyapunov Derivative

- Use the chain rule to compute the derivative of $V(\boldsymbol{x}(t))$

$$\dot{V}\left(x\left(t\right)\right)=\nabla V\left(x\right)^{\mathrm{T}}\dot{x}$$

- Use the plant model to obtain

$$\dot{V}\left(x\left(t\right)\right)=\nabla V\left(x\right)^{\mathrm{T}}f\left(x\right)$$

- Recall

$$\nabla V\left(x\right)=\begin{bmatrix}\frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} & \cdots & \frac{\partial V}{\partial x_2}\end{bmatrix}^{\mathrm{T}}$$

Lyapunov Theorem for LTI Systems

The system d$\boldsymbol{x}$/dt=$\boldsymbol{Ax}$ is asymptotically stable, that is, the equilibrium state $\boldsymbol{x}_e=\mathbf{0}$ is asymptotically stable (a.s), if and only if any solution converges to $\boldsymbol{x}_e=\mathbf{0}$ as t tends to infinity for any initial $\boldsymbol{x}_0$

Lyapunov Theorem Interpretation

- View the vector $\boldsymbol{x}(t)$ as defining the coordinates of a point in an n-dimensional state space

- In an a.s. system the point $\boldsymbol{x}(t)$ converges to $\boldsymbol{x}_e=\mathbf{0}$

Lyapunov Theorem for n=2

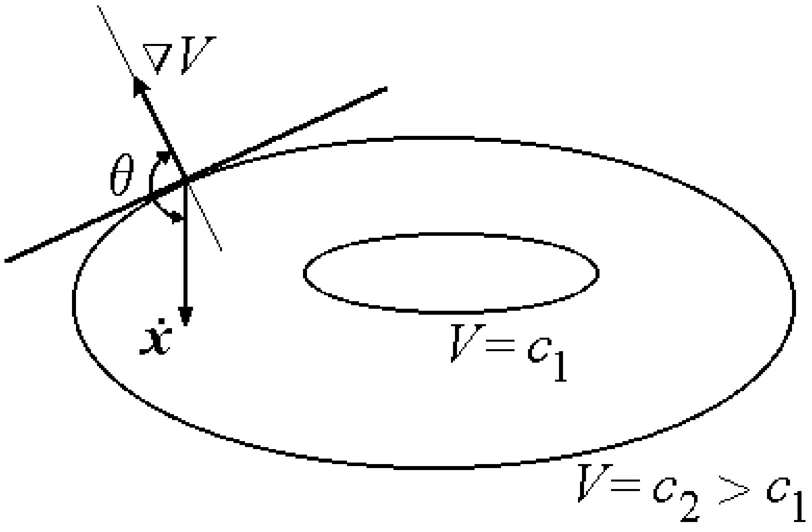
If a trajectory is converging to $\boldsymbol{x}_e = \mathbf{0}$, it should be possible to find a nested set of closed curves $V(x_1, x_2) = c$, $c \geq 0$, such that decreasing values of c yield level curves shrinking in on the equilibrium state $\boldsymbol{x}_e = \mathbf{0}$

Lyapunov Theorem and Level Curves

- The limiting level curve $V(x_1,x_2)=V(\mathbf{0})=0$ is 0 at the equilibrium state $\boldsymbol{x}_e=\mathbf{0}$
- The trajectory moves through the level curves by cutting them in the inward direction ultimately ending at $\boldsymbol{x}_e=\mathbf{0}$

The trajectory is moving in the direction of decreasing V

Note that $$\dot{V} = \|\nabla V\| \|\dot{\boldsymbol{x}}\| \cos\theta < 0$$



Level Sets

- The level curves can be thought of as contours of a cup-shaped surface
- For an a.s. system, that is, for an a.s. equilibrium state $\boldsymbol{x}_e=\mathbf{0}$, each trajectory falls to the bottom of the cup

Positive Definite Function---General Definition

The function V is positive definite in S, with respect to $\boldsymbol{x}_e$, if V has continuous partials, $V(\boldsymbol{x}_e)=0$, and $V(\boldsymbol{x})>0$ for all $\boldsymbol{x}$ in S, where $\boldsymbol{x}\neq\boldsymbol{x}_e$

Positive Definite Function With Respect to the Origin

Assume, for simplicity, $\boldsymbol{x}_e = \mathbf{0}$, then the function V is positive definite in S if V has continuous partials, $V(\mathbf{0}) = 0$, and $V(\boldsymbol{x}) > 0$ for all $\boldsymbol{x}$ in S, where $\boldsymbol{x} \neq \mathbf{0}$

Example: Positive Definite Function

Positive definite function of two variables

$$
\begin{aligned}
V(x_1, x_2) &= 2x_1^2 + 3x_2^2 \\
&= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
&= \boldsymbol{x}^T \boldsymbol{P} \boldsymbol{x} \\
&> 0 \quad \text{for all} \quad \boldsymbol{x} \neq \boldsymbol{0}
\end{aligned}
$$

Positive Semi-Definite Function---General Definition

The function V is positive semi-definite in S, with respect to $\boldsymbol{x}_e$, if V has continuous partials, $V(\boldsymbol{x}_e)=0$, and $V(\boldsymbol{x})\geq 0$ for all $\boldsymbol{x}$ in S

Positive Semi-Definite Function With Respect to the Origin

Assume, for simplicity, $\boldsymbol{x}_e = \mathbf{0}$, then the function V is positive semi-definite in S if V has continuous partials, $V(\mathbf{0})=0$, and $V(\boldsymbol{x}) \geq 0$ for all $\boldsymbol{x}$ in S

Example: Positive Semi-Definite Function

An example of positive semi-definite function of two variables

$$\begin{aligned} V(x_1, x_2) &= x_1^2 \\ &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \boldsymbol{x}^T \boldsymbol{P} \boldsymbol{x} \\ &\geq 0 \quad \text{in} \quad R^2 \end{aligned}$$

Quadratic Forms

- $V = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}$, where $\boldsymbol{P} = \boldsymbol{P}^{\mathrm{T}}$
- If $\boldsymbol{P}$ not symmetric, need to symmetrize it
- First observe that because the transposition of a scalar equals itself, we have

$$\left(x^T P x\right)^T = x^T P^T x = x^T P x$$

Symmetrizing Quadratic Form

- Perform manipulations

$$\begin{aligned} \boldsymbol{x}^T \boldsymbol{P} \boldsymbol{x} &= \frac{1}{2} \boldsymbol{x}^T \boldsymbol{P} \boldsymbol{x} + \frac{1}{2} \boldsymbol{x}^T \boldsymbol{P} \boldsymbol{x} \\ &= \frac{1}{2} \boldsymbol{x}^T \boldsymbol{P} \boldsymbol{x} + \frac{1}{2} \boldsymbol{x}^T \boldsymbol{P}^T \boldsymbol{x} \\ &= \boldsymbol{x}^T \left(\frac{\boldsymbol{P} + \boldsymbol{P}^T}{2} \right) \boldsymbol{x} \end{aligned}$$

- Note that

$$\left(\frac{\boldsymbol{P} + \boldsymbol{P}^T}{2} \right)^T = \frac{\boldsymbol{P} + \boldsymbol{P}^T}{2}$$

Tests for Positive and Positive Semi-Definiteness of Quadratic Form

- $V = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}$, where $\boldsymbol{P} = \boldsymbol{P}^{\mathrm{T}}$, is positive definite if and only if all eigenvalues of $\boldsymbol{P}$ are positive
- $V = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}$, where $\boldsymbol{P} = \boldsymbol{P}^{\mathrm{T}}$, is positive semi-definite if and only if all eigenvalues of $\boldsymbol{P}$ are non-negative

Comments on the Eigenvalue Tests

- These tests are only good for the case when $\boldsymbol{P}=\boldsymbol{P}^{\mathrm{T}}$. You must symmetrize $\boldsymbol{P}$ before applying the above tests
- Other tests, the Sylvester's criteria, involve checking the signs of principal minors of $\boldsymbol{P}$

Negative Definite Quadratic Form

$V=\boldsymbol{x}^{\top}\boldsymbol{P}\boldsymbol{x}$ is negative definite if and only if

$$-\boldsymbol{x}^{\top}\boldsymbol{P}\boldsymbol{x}=\boldsymbol{x}^{\top}(-\boldsymbol{P})\boldsymbol{x}$$

is positive definite

Negative Semi-Definite Quadratic Form

$V = \boldsymbol{x}^{\top}\boldsymbol{P}\boldsymbol{x}$ is negative semi-definite if and only if

$$-\boldsymbol{x}^{\top}\boldsymbol{P}\boldsymbol{x} = \boldsymbol{x}^{\top}(-\boldsymbol{P})\boldsymbol{x}$$

is positive semi-definite

Example: Checking the Sign Definiteness of a Quadratic Form

- Is ***P***, equivalently, is the associated quadratic form, $V = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}$, pd, psd, nd, nsd, or neither?

$$\boldsymbol{P} = \begin{bmatrix} 2 & -6 \\ 0 & 2 \end{bmatrix}$$

- The associated quadratic form

$$V = \boldsymbol{x}^T \boldsymbol{P}\boldsymbol{x} = 2x_1^2 - 6x_1x_2 + 2x^2$$

Example: Symmetrizing the Underlying Matrix of the Quadratic Form

- Applying the eigenvalue test to the given quadratic form would seem to indicate that the quadratic form is pd, which turns out to be false
- Need to symmetrize the underlying matrix first and then can apply the eigenvalue test

Example: Symetrized Matrix

- Symmetrizing manipulations

$$
\begin{aligned}
\frac{1}{2}\left(\boldsymbol{P}+\boldsymbol{P}^T\right) &= \frac{1}{2}\left(\begin{bmatrix} 2 & -6 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ -6 & 2 \end{bmatrix}\right) \\
&= \begin{bmatrix} 2 & -3 \\ -3 & 2 \end{bmatrix}
\end{aligned}
$$

- The eigenvalues of the symmetrized matrix are: 5 and -1
- The quadratic form is indefinite!

Example: Further Analysis

- Direct check that the quadratic form is indefinite
- Take $\boldsymbol{x}=[1 \;\; 0]^T$. Then

$$x^T P x = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -6 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2 > 0$$

- Take $\boldsymbol{x}=[1 \;\; 1]^T$. Then

$$x^T P x = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -6 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = -2 < 0$$

Stability Test for $\boldsymbol{x}_e=\mathbf{0}$ of d$\boldsymbol{x}$/dt=$\boldsymbol{Ax}$

- Let $V=\boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}$ where $\boldsymbol{P}=\boldsymbol{P}^{\mathrm{T}}>0$
- For V to be a Lyapunov function, that is, for $\boldsymbol{x}_e=\mathbf{0}$ to be a.s.,

$$\dot{V}(x(t)) < 0$$

- Evaluate the time derivative of V on the solution of the system d$\boldsymbol{x}$/dt=$\boldsymbol{Ax}$---Lyapunov derivative

Lyapunov Derivative for d$\boldsymbol{x}$/dt=$\boldsymbol{Ax}$

- Note that $V(\boldsymbol{x}(t))=\boldsymbol{x}(t)^{\mathsf{T}}\boldsymbol{Px}(t)$
- Use the chain rule

$$\begin{aligned}\dot{V}(x(t)) &= \dot{x}^T P x + x^T P \dot{x} \\ &= x^T A^T P x + x^T P A x \\ &= x^T \left(A^T P + P A \right) x\end{aligned}$$

- We used

$$\dot{x}^T = x^T A^T$$

Lyapunov Matrix Equation

- Denote

$$A^T P + P A = -Q$$

- Then the Lyapunov derivative can be represented as

$$\dot{V} = \frac{d}{dt} V = -x^T Q x$$

where

$$Q = Q^T > 0$$

Terms to Our Vocabulary

- Theorem---a major result of independent interest
- Lemma---an auxiliary result that is used as a stepping stone toward a theorem
- Corollary---a direct consequence of a theorem, or even a lemma

Lyapunov Theorem

The real matrix ***A*** is a.s., that is, all eigenvalues of ***A*** have negative real parts if and only if for any $Q = Q^T > 0$ the solution P of the continuous matrix Lyapunov equation

$$A^T P + PA = -Q$$

is (symmetric) positive definite

How Do We Use the Lyapunov Theorem?

- Select an arbitrary symmetric positive definite $\boldsymbol{Q}$, for example, an identity matrix, $\boldsymbol{I}_n$
- Solve the Lyapunov equation for $\boldsymbol{P}=\boldsymbol{P}^T$
- If $\boldsymbol{P}$ is positive definite, the matrix $\boldsymbol{A}$ is a.s. If $\boldsymbol{P}$ is not p.d. then $\boldsymbol{A}$ is not a.s.

How NOT to Use the Lyapunov Theorem

- It would be no use choosing $\boldsymbol{P}$ to be positive definite and then calculating $\boldsymbol{Q}$
- For unless $\boldsymbol{Q}$ turns out to be positive definite, nothing can be said about a.s. of $\boldsymbol{A}$ from the Lyapunov equation

Example: How NOT to Use the Lyapunov Theorem

- Consider an a.s. matrix

$$A = \begin{bmatrix} -1 & 3 \\ 0 & -1 \end{bmatrix}$$

- Try

$$\boldsymbol{P} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- Compute $\boldsymbol{Q} = -\left(\boldsymbol{A}^T\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}\right)$

Example: Computing Q

$$\begin{aligned}
\boldsymbol{Q} &= -\left(\boldsymbol{A}^T\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}\right) \\
&= -\left(\begin{bmatrix} -1 & 0 \\ 3 & -1 \end{bmatrix} + \begin{bmatrix} -1 & 3 \\ 0 & -1 \end{bmatrix}\right) \\
&= \begin{bmatrix} 2 & -3 \\ -3 & 2 \end{bmatrix}
\end{aligned}$$

The matrix $\boldsymbol{Q}$ is indefinite!---recall the previous example

Solving the Continuous Matrix Lyapunov Equation Using MATLAB

- Use the MATLAB's command lyap
- Example:

$$A = \begin{bmatrix} -1 & 3 \\ 0 & -1 \end{bmatrix}$$

- $\boldsymbol{Q}=\boldsymbol{I}_2$
- P=lyap(A',Q)

$$\boldsymbol{P}= \begin{bmatrix} 0.50 & 0.75 \\ 0.75 & 2.75 \end{bmatrix}$$

- Eigenvalues of $\boldsymbol{P}$ are positive: 0.2729 and 2.9771; $\boldsymbol{P}$ is positive definite

Benefits of the Lyapunov Theory

- Solution to differential equation are not needed to infer about stability properties of equilibrium state of interest

☞ **Lyapunov functions** are useful in designing robust and adaptive controllers