AAE 532 - Orbit Mechanics

Purdue University

PS2 Solutions

Authors:
Mitchell Dominguez
Tyler Hook
Samantha Ramsey

Email:
doming18@purdue.edu
hook9@purdue.edu
ramsey87@purdue.edu

Instructor:
Kathleen Howell

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Useful Constants

	Axial Rotaional Period (Rev/Day)	Mean Equatorial Radius (km)	Gravitational Parameter $\mu = Gm(km^3/sec^2)$	Semi-major Axis of orbit (km)	Orbital Period (sec)	Eccentricity of Orbit	Inclination of Orbit to Ecliptic (deg)
☉ Sun	0.0394011	695990	132712440017.99	-	-	-	-
☾ Moon	0.0366004	1738.2	4902.8005821478	384400 (around Earth)	2360592 27.32 Earth Days	0.0554	5.16
☾ Mercury	0.0170514	2439.7	22032.080486418	57909101	7600537 87.97 Earth Days	0.20563661	7.00497902
♀ Venus	0.0041149 (Retrograde)	6051.9	324858.59882646	108207284	19413722 224.70 Earth Days	0.00676399	3.39465605
⊕ Earth	1.0027378	6378.1363	398600.4415	149597898	31558205 365.26 Earth Days	0.01673163	0.00001531
♂ Mars	0.9747000	3397	42828.314258067	227944135	59356281 686.99 Earth Days	0.09336511	1.84969142
♃ Jupiter	2.4181573	71492	126712767.8578	778279959	374479305 11.87 Years	0.04853590	1.30439695
♄ Saturn	2.2522053	60268	37940626.061137	1427387908	930115906 29.47 Years	0.05550825	2.48599187
⛢ Uranus	1.3921114 (Retrograde)	25559	5794549.0070719	2870480873	2652503938 84.05 Years	0.04685740	0.77263783
Ψ Neptune	1.4897579	25269	6836534.0638793	4498337290	5203578080 164.89 Years	0.00895439	1.77004347
♇ Pluto	-0.1565620 (Retrograde)	1162	981.600887707	5907150229	7830528509 248.13 Years	0.24885238	17.14001206

- First three columns of the body data are consistent with GMAT 2020a default values, which are mainly from JPL's ephemerides file de405.spk

- The rest of the data are from JPL website(`https://ssd.jpl.nasa.gov/?planet_pos`, retrieved at 09/01/2020)
based on E.M. Standish's "Keplerian Elements for Approximate Positions of the Major Planets"

Problem 1

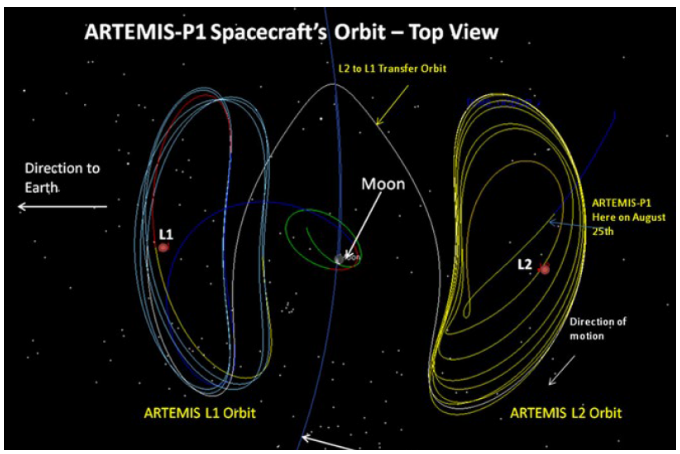
Problem Statement

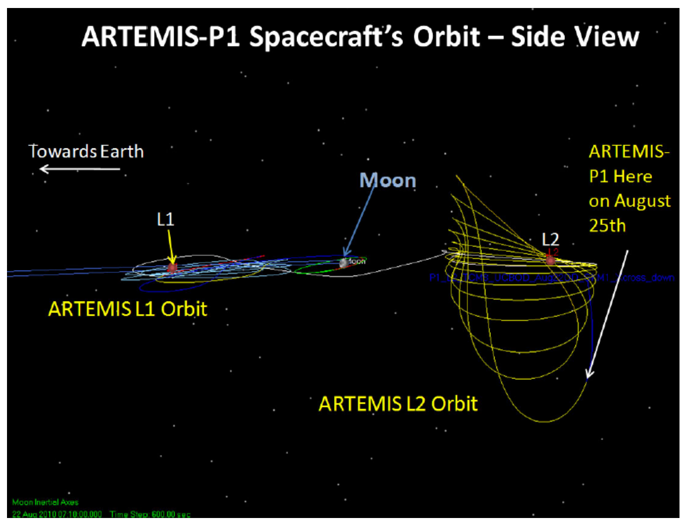
Recall that the FIRST Artemis mission was mentioned in class. After a complex route to the vicinity of the Moon, the two identical ARTEMIS spacecraft (P1 and P2) arrived in the lunar vicinity on August 23 and October 22, 2010, respectively. The spacecraft eventually inserted into lunar orbits on June 27 and July 17, 2011. The trajectories in the lunar vicinity prior to lunar orbit insertion were influenced significantly by the gravity fields of other bodies, particularly the Earth and the Sun. The ballistic P1 path from arrival to the lunar insertion point appears in the images below. Note that it is far from a classical elliptical orbit about the Moon.

Define a system that is comprised of five particles. The law of gravity between each pair is the familiar inverse square law. Obviously, the planets are not truly aligned simultaneously, but assume that the Sun, spacecraft (s/c), and other bodies are collinear and positioned as indicated below:

$$Earth - Moon - s/c - Sun - Jupiter$$

Assume that the spacecraft is instantaneously located such that the distance between the Moon and the s/c is 77,500 km. The total mass of each ARTEMIS spacecraft is about 130 kg. The distances of the other planets from the Sun are assumed to be equal to the semi-major axis as listed in the Table of Constants under Supplementary Documents on Brightspace.





(a) Locate the center of mass of the 5- particle system. Identify it on a sketch. Add unit vectors and appropriate position vectors

(b) Write the vector differential equation for motion of the s/c with respect to the center of mass, i.e., $\bar{r}_i = \bar{r}_{s/c}$. You should obtain an expression for the accelerations on the s/c, i.e., $\bar{r}_{s/c} =$ (sum of 4 terms). Assuming the alignment above, compute the accelerations on the s/c due to each of the other bodies. Include the directions. Which body produces the largest acceleration on the s/c? smallest? What is the descending order? Net acceleration in km/s^2? [Did you use a consistent number of significant digits in your computations?]

(c) Compare the relative size of the acceleration terms (gravitational forces) and their directions on the s/c. Is the order of influence what you expected? Which gravity term dominates? Do the acceleration terms seem consistent with your expectations?

Part (a)

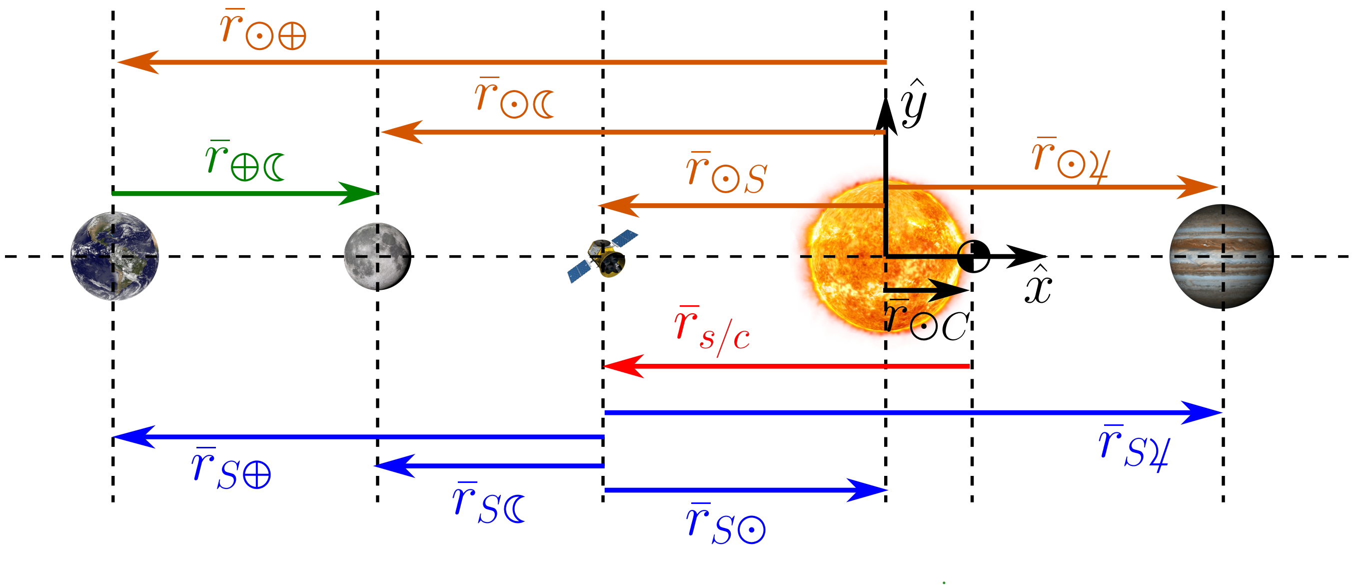


Figure 2: Vector definitions for problem 1

To compute the center of mass of the 5-particle system, position vectors are defined for each body with respect to the Sun. The coordinate system is defined such that the center of the Sun coincides with the origin. Leveraging values from the table of constants and vector addition, the position vectors are illustrated in Figure 2 and have the values

$$
\begin{aligned}
\bar{r}_{\odot\odot} &= 0\hat{x}\ km \\
\bar{r}_{\odot\jupiter} &= 778279959\hat{x}\ km \\
\bar{r}_{\odot\oplus} &= -149597898\hat{x}\ km \\
\bar{r}_{\odot\leftmoon} = \bar{r}_{\odot\oplus} + \bar{r}_{\oplus\leftmoon} &= -149213498\hat{x}\ km \\
\bar{r}_{\odot S} = \bar{r}_{\odot\leftmoon} + \bar{r}_{\leftmoon S} &= -149135998\hat{x}\ km
\end{aligned}
$$

The vector position of the center of mass relative to the center of the Sun is then computed as

$$\bar{r}_{\odot C} = \frac{\displaystyle\sum_{i=1}^{5} \mu_i \bar{r}_{\odot i}}{\displaystyle\sum_{i=1}^{5} \mu_i} = \boxed{741930\hat{x}\ km}$$

where the index i is used to index the masses and positions of each body within the summations. The location of the center of mass is also marked in Figure 2.

Part (b)

The 5-particle system experiences zero external forces, so the center of mass is considered inertially fixed, i.e., the center of mass does not accelerate. Applying Newton's second law and dividing by the mass of the spacecraft, the vector equation of motion for the spacecraft is

$$\ddot{\bar{r}}_{s/c} = \frac{\mu_{\oplus}}{\|\bar{r}_{S\oplus}\|^3}\bar{r}_{S\oplus} + \frac{\mu_{\leftmoon}}{\|\bar{r}_{\leftmoon}\|^3}\bar{r}_{S\leftmoon} + \frac{\mu_{\odot}}{\|\bar{r}_{S\odot}\|^3}\bar{r}_{S\odot} + \frac{\mu_{\jupiter}}{\|\bar{r}_{S\jupiter}\|^3}\bar{r}_{S\jupiter}$$

where the positions of each body relative to the spacecraft are defined in Figure 2 and have values

$$\begin{aligned}
\bar{r}_{S\oplus} &= -461900\hat{x}\ km \\
\bar{r}_{S\leftmoon} &= -77500\hat{x}\ km \\
\bar{r}_{S\odot} &= 149135998\hat{x}\ km \\
\bar{r}_{S\jupiter} &= 927415957\hat{x}\ km
\end{aligned}$$

The resulting accelerations on the spacecraft due to the gravity of each of the other bodies are, in descending order

$$\begin{aligned}
\ddot{\bar{r}}_{S\odot} &= 5.967 \times 10^{-6}\hat{x}\ km/s^2 \\
\ddot{\bar{r}}_{S\oplus} &= -1.868 \times 10^{-6}\hat{x}\ km/s^2 \\
\ddot{\bar{r}}_{S\leftmoon} &= -8.163 \times 10^{-7}\hat{x}\ km/s^2 \\
\ddot{\bar{r}}_{S\jupiter} &= 1.473 \times 10^{-10}\hat{x}\ km/s^2
\end{aligned}$$

The Sun produces the largest acceleration on the spacecraft, followed by the Earth, Moon, and Jupiter. The total acceleration imparted on the spacecraft is

$$\ddot{\bar{r}}_{s/c} = 3.282 \times 10^{-6}\hat{x}\ km/s^2$$

Part (c)

The magnitude of each acceleration term $\|\ddot{\vec{r}}_{S,i}\|$ is compared to the magnitude of the net acceleration $\left\|\ddot{\vec{r}}_{s/c}\right\|$ using

$$\%_{acc} = \frac{\|\ddot{\vec{r}}_{S,i}\|}{\left\|\ddot{\vec{r}}_{s/c}\right\|} \times 100$$

By this metric, the Sun accounts for 182%, the Earth accounts for 56.9%, the Moon accounts for 24.9%, and Jupiter accounts for 0.004% of the net acceleration. The Sun dominates, as it accounts for greater than 100% of the acceleration, but this is reasonable considering that the Earth and Moon both oppose this acceleration. Jupiter and the Sun both cause an acceleration in the positive $\hat{x}$ direction, though the acceleration due to Jupiter is negligible. Meanwhile, the Earth and Moon both pull in the negative $\hat{x}$ direction. Overall, the acceleration magnitudes seem reasonable and fit with expectations. The spacecraft is closest to the Moon, but the Moon is by far the smallest body considered here, so the acceleration due to its gravity is still relatively small. Next, closest is the Earth, but the Earth is also not a large body relative to the Sun and Jupiter, so it also does not dominate, although the acceleration from the Earth is an order of magnitude greater than from the Moon. While the Sun is much farther away, it is also much larger, so it ends up having the largest effect on the resulting motion. Finally, while Jupiter is very massive, it is also incredibly far away from the spacecraft, so its effect is three orders of magnitude smaller than that of the Moon.

Problem 2

Problem Statement

Continue with problem 1.

(a) Now write the vector differential equation for the **relative** acceleration of the s/c with respect to the Moon, ${}^I\ddot{\bar{r}}_{☾\to s/c}$. At the instant when the bodies are assumed to be in the same configuration as in problem 1, that is,

Earth - Moon - s/c - Sun - Jupiter

compute the dominant, direct, indirect, and total perturbing accelerations.

(b) Compare the magnitudes of each term. Are they the same as the magnitude in Prob 1 for the generic acceleration? Should they be? Why or why not?

(c) Which body produces the largest/smallest accelerations on the s/c?
Is the descending order the same as in problem 1? Why might it be different? Will the spacecraft location influence the order of impact of the bodies?
Does the direction of the net perturbing acceleration seem reasonable for this configuration? Why or why not?

(d) Is it reasonable to model the spacecraft motion using only the gravity of the Moon (a two-body problem)? Why or why not?
Does Jupiter have significant influence?
Does the solar gravity have significant influence?
If you could leave one body out of the model, which one would you delete and why?
If you want to leave out two bodies, which would they be at this instant?

Part (a)

Now we seek to describe the dynamic evolution of the **relative** position vector $\ddot{\bar{r}}_{\leftmoon \to s/c}$. Since the basepoint of this vector (namely the Moon) is moving and accelerating, we cannot apply Newton's second law directly and must employ the relative formulation. First apply Newton's second law to both the spacecraft and the Moon:

$$\ddot{\bar{r}}_{s/c} = \frac{\mu_{\oplus}}{r^3_{s/c\to\oplus}}\bar{r}_{s/c\to\oplus} + \frac{\mu_{\leftmoon}}{r^3_{s/c\to\leftmoon}}\bar{r}_{s/c\to\leftmoon} + \frac{\mu_{\odot}}{r^3_{s/c\to\odot}}\bar{r}_{s/c\to\odot} + \frac{\mu_{\jupiter}}{r^3_{s/c\to\jupiter}}\bar{r}_{s/c\to\jupiter} \tag{1}$$

$$\ddot{\bar{r}}_{\leftmoon} = \frac{\mu_{\oplus}}{r^3_{\leftmoon\to\oplus}}\bar{r}_{\leftmoon\to\oplus} + \frac{\mu_{s/c}}{r^3_{\leftmoon\to s/c}}\bar{r}_{\leftmoon\to s/c} + \frac{\mu_{\odot}}{r^3_{\leftmoon\to\odot}}\bar{r}_{\leftmoon\to\odot} + \frac{\mu_{\jupiter}}{r^3_{\leftmoon\to\jupiter}}\bar{r}_{\leftmoon\to\jupiter} \tag{2}$$

So, if one were to subtract equation 2 from 1, one would obtain:

$$\ddot{\bar{r}}_{\leftmoon\to s/c} + \frac{G(m_{s/c}+m_{\leftmoon})}{r^3_{\leftmoon\to s/c}}\bar{r}_{\leftmoon\to s/c} = G\sum_{\substack{j=1\\ j\neq s/c,\leftmoon}}^{N} m_j\left(\frac{\bar{r}_{s/c\to j}}{r^3_{s/c\to j}} - \frac{\bar{r}_{\leftmoon\to j}}{r^3_{\leftmoon\to j}}\right) \tag{3}$$

where the orange term is referred to as the dominant acceleration, red is the direct perturbing acceleration, and blue is the indirect perturbing acceleration. Since, we seek the differential equation of the relative vector, we can update our free-body diagram so that the forces on the Moon (initial point of position vector) and the forces on the spacecraft (terminating point of position vector) are present.

Recognize that, for the following problem, we will rearrange equation 3 to isolate the total acceleration term on the left side such that:

$$\ddot{\bar{r}}_{\leftmoon\to s/c} = -\frac{G(m_{s/c}+m_{\leftmoon})}{r^3_{\leftmoon\to s/c}}\bar{r}_{\leftmoon\to s/c} + G\sum_{\substack{j=1\\ j\neq s/c,\leftmoon}}^{N} m_j\left(\frac{\bar{r}_{s/c\to j}}{r^3_{s/c\to j}} - \frac{\bar{r}_{\leftmoon\to j}}{r^3_{\leftmoon\to j}}\right) \tag{4}$$

Using equation 4:

$$
\begin{aligned}
-\frac{G(m_{s/c}(m_{s/c}+m_{☾})}{r^3_{☾\to s/c}}\bar{r}_{☾\to s/c} &= -8.1628\cdot 10^{-7}\ \hat{x}\ km/s^2 \ \ \text{dominant acceleration}\\
Gm_{\oplus}\frac{\bar{r}_{s/c\to\oplus}}{r^3_{s/c\to\oplus}} &= -1.8683\cdot 10^{-6}\ \hat{x}\ km/s^2 \ \ \text{Earth direct perturbing}\\
Gm_{\oplus}\frac{\bar{r}_{☾\to\oplus}}{r^3_{☾\to\oplus}} &= -2.6976\cdot 10^{-6}\ \hat{x}\ km/s^2 \ \ \text{Earth indirect perturbing}\\
Gm_{\oplus}\frac{\bar{r}_{s/c\to\oplus}}{r^3_{s/c\to\oplus}} - Gm_{\oplus}\frac{\bar{r}_{☾\to\oplus}}{r^3_{☾\to\oplus}} &= 8.2928\cdot 10^{-7}\ \hat{x}\ km/s^2 \ \ \text{Earth net perturbing}\\
Gm_{\odot}\frac{\bar{r}_{s/c\to\odot}}{r^3_{s/c\to\odot}} &= 5.9669\cdot 10^{-6}\ \hat{x}\ km/s^2 \ \ \text{Sun direct perturbing}\\
Gm_{\odot}\frac{\bar{r}_{☾\to\odot}}{r^3_{☾\to\odot}} &= 5.9607\cdot 10^{-6}\ \hat{x}\ km/s^2 \ \ \text{Sun indirect perturbing}\\
Gm_{\odot}\frac{\bar{r}_{s/c\to\odot}}{r^3_{s/c\to\odot}} - Gm_{\odot}\frac{\bar{r}_{☾\to\odot}}{r^3_{☾\to\odot}} &= 6.1967\cdot 10^{-9}\ \hat{x}\ km/s^2 \ \ \text{Sun net perturbing}\\
Gm_{♃}\frac{\bar{r}_{s/c\to ♃}}{r^3_{s/c\to ♃}} &= 1.4732\cdot 10^{-10}\ \hat{x}\ km/s^2 \ \ \text{Jupiter direct perturbing}\\
Gm_{♃}\frac{\bar{r}_{☾\to ♃}}{r^3_{☾\to ♃}} &= 1.4730\cdot 10^{-10}\ \hat{x}\ km/s^2 \ \ \text{Jupiter indirect perturbing}\\
Gm_{♃}\frac{\bar{r}_{s/c\to ♃}}{r^3_{s/c\to ♃}} - Gm_{♃}\frac{\bar{r}_{☾\to ♃}}{r^3_{☾\to ♃}} &= 2.4619\cdot 10^{-14}\ \hat{x}\ km/s^2 \ \ \text{Jupiter net perturbing}\\
\ddot{\bar{r}}_{☾\ \to s/c(Pert)} &= 8.3548\cdot 10^{-7}\ \hat{x}\ km/s^2 \ \ \text{total perturbing acceleration}\\
\ddot{\bar{r}}_{☾\ \to s/c} &= 1.9193\cdot 10^{-8}\ \hat{x}\ km/s^2 \ \ \text{total acceleration}
\end{aligned}
$$

Note that the positive $\hat{x}$ direction is to the right, and the negative $\hat{x}$ is to the left, as depicted in Figure 2.

Part (b)

The direct and indirect perturbing terms of the Sun are the two largest magnitude terms, respectively. This is then followed by the Earth's indirect perturbing term, the Earth's direct perturbing term, and then the dominant acceleration term of the Moon. The last two magnitude terms of Jupiter are the smallest, with Jupiter's direct perturbing term being ever so slightly larger than its indirect perturbing term.

Note that the net perturbing terms are not the same as the magnitude of the generic acceleration terms from problem 1 above. Instead, the direct perturbing terms and dominant acceleration term **are** the same as the generic acceleration terms. This should make sense as, by definition, the direct perturbing terms are the acceleration effects on the spacecraft due to that particular body.

Part (c)

As suggested by part (b) above, if we consider only direct perturbing accelerations, then the descending order and direction of the impact of each body acting on the spacecraft are the same as laid out in problem 1, which are:

$$\boxed{\text{Sun(perturbing, }+\hat{x}\text{)/Earth(perturbing, }-\hat{x}\text{)/Moon(dominant, }-\hat{x}\text{)/Jupiter(perturbing, }+\hat{x}\text{)}}$$

However, if we consider the net perturbing acceleration terms of each body, then the descending order and direction of the impact of each body are:

$$\boxed{\text{Earth(perturbing, }+\hat{x}\text{)/Moon(dominant, }-\hat{x}\text{)/Sun(perturbing, }+\hat{x}\text{)/Jupiter(perturbing, }+\hat{x}\text{)}}$$

The reason why the net perturbing terms might be different is that the net perturbing term now accounts for the indirect perturbing acceleration of the 'dominant' body. Given that this 'dominant' body is located close to the spacecraft and has significant mass, its indirect influence may shift the impact of the bodies within the model.

The spacecraft's location with respect to the other planetary objects certainly influences the order of the impact of each body. One can recall that these acceleration terms are position-dependent functions. For instance, if the spacecraft was, instead, placed in between the Sun and Jupiter with it being only 700000 km away from Jupiter, then it stands to reason that the influence of Jupiter would would be larger than it is in the current configuration.

For the most part, the direction of the net perturbing terms for each body seem reasonable for this configuration, all except Earth. Intuitively, one would think that the direction of the net perturbing acceleration of Earth would be pointed in the negative direction given its position 'left' of the spacecraft. However, one would then be confusing the net perturbing term for the direct perturbing term. The direct perturbing term of Earth does point in the negative direction. But again, for the net perturbing term, one must consider the impact of the 'dominant' body on the reference object (Earth). Because the indirect perturbing term of the Earth also points in the negative direction and is larger than the direct perturbing term, this, in a way, 'cancels' the direct perturbing acceleration of the Earth, creating a net perturbing acceleration in the positive $\hat{x}$ direction, which seems reasonable.

Part (d)

It may not be obvious, but 77,500 km is a fairly long distance from the Moon to the spacecraft to approximate it as a two-body problem. To put that distance into perspective, that is roughly one-fifth the distance from the Earth to the Moon! Looking at the individual acceleration terms, the net perturbing acceleration from the Earth is greater than the 'dominant' acceleration. So, it seems two-body is not adequate for this problem.

Jupiter's gravitational influence on the motion of the spacecraft is minimal at best. So, while including the net perturbing acceleration due to Jupiter would develop a more accurate result, it might not be noticeable in a short amount of time since the acceleration from Jupiter is relatively small compared to the rest of the bodies.

In contrast, the gravitational influence of the Sun has direct and indirect perturbing acceleration magnitudes comparable to that from Earth. In fact, its direct perturbing acceleration is the largest direct perturbing acceleration and is even larger than the dominant acceleration from the Moon. And while the net perturbing acceleration due to the Sun is not as large as Earth's net perturbing acceleration, being only two magnitudes smaller will still create a noticeable change in the motion of the spacecraft over time, meaning that its influence is, to a certain extent, significant.

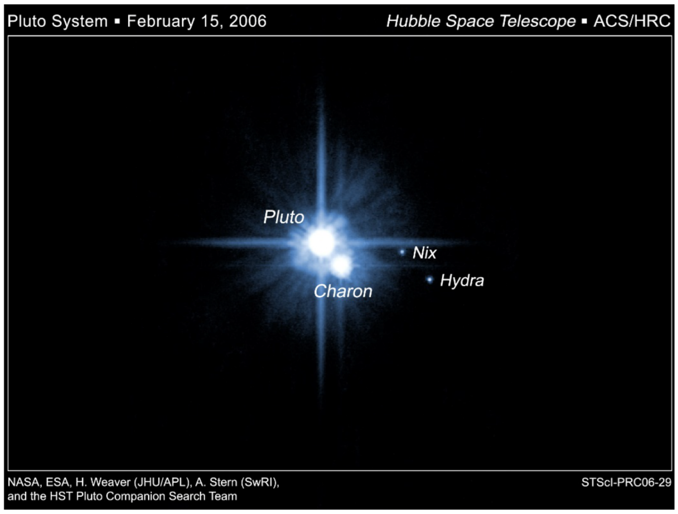
If one could leave one body out of the model, that body would be Jupiter as it was suggested above that, while the inclusion of Jupiter in the dynamic model adds accuracy, given the net perturbing acceleration is several orders of magnitude smaller than that of the Earth and the Sun means that its influence would be minimal and therefore, less impactful if removed. Thus, the four-body model chosen would be the Earth-Moon-s/c-Sun.

If we were to consider only a three-body model, the second body that would be left out would be the Sun. As suggested above, while the Sun's direct and indirect acceleration terms are noticeably large, for where the spacecraft is located, its net perturbing acceleration is two orders of magnitude smaller than that of the Earth and therefore, would have a smaller impact on the dynamics (by comparison with Earth) if removed. And so, the three-body model chosen would be the Earth-Moon-s/c, leaving out the Sun and Jupiter.

Problem 3

Problem Statement

The dwarf planet Pluto has 5 known moons. The largest moon, Charon, is nearly half the size of Pluto and is the largest known moon in comparison to its parent body. Assume the Pluto-Charon system is modeled as an isolated two-body problem for the motion of Charon relative to Pluto due to the mutual gravity. Ignore all other forces.



	Pluto	Charon*
Gm	$981.601\ km^3/s^2$	$119.480\ km^3/s^2$
Diameter	$1162\ km$	$606\ km$

Semi-major axis of orbit for Charon relative to Pluto = $19596\ km$

*https://nssdc.gsfc.nasa.gov/planetary/factsheet/plutofact.html

(a) Sketch the system and define appropriate unit vectors; let $\hat{x}, \hat{y}, \hat{z}$ be an inertial set of unit vectors such that $\hat{x}$ is parallel to $\bar{r}_{PC}$ at the initial time. Locate the center of mass and define position vectors for each object with respect to the fixed center of mass. Is the cm outside the radius of Pluto?

(b) Let $\bar{r}_{PC} = \bar{r}$ be the relative position of Charon with respect to Pluto. Write the kinematic expressions for the relative position and velocity for the motion of Charon relative to Pluto, that is, $\bar{r}, \bar{v}$ in terms of rotating unit vectors $\hat{r}, \hat{\theta}$. At $t = 0$, the inertial velocities are known such that $\dot{\bar{r}}_C = 0.211319 km/s\ \hat{y}$ and $\dot{\bar{r}}_P = -0.025717 km/s\ \hat{y}$. Determine angular velocity for the motion of Charon relative to Pluto.

(c) Determine the system linear momentum; use this result to compute the velocity of the system center of mass. Does this result make sense?

(d) Determine the constant $\bar{C}_3$ for this system. What are the correct units?

(e) Evaluate the energy constant C_4; of course, include the units!

Part (a)

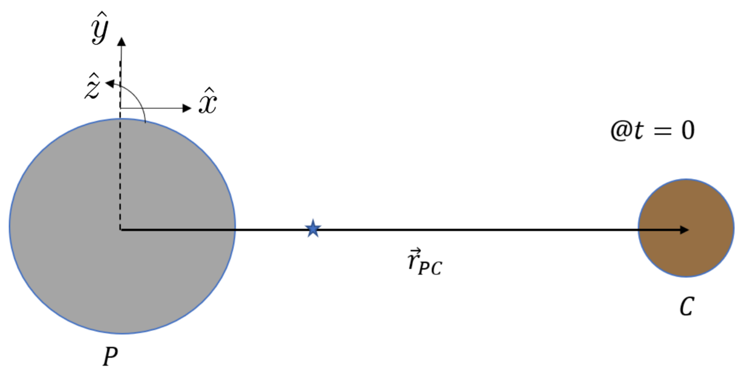


Figure 3: Pluto Charon Diagram.

First, the center of mass with respect to Pluto can be computed by:

$$\bar{r}_{PP} = 0\hat{x}$$

$$\bar{r}_{PC} = 19596\ km\ \hat{x}$$

$$\bar{r}_{Pcm} = \frac{\sum_{i=1}^{2} \mu_i \bar{r}_{Pi}}{\sum_{i=1}^{2} \mu_i} = 2126.392\ km\ \hat{x}$$

The position vectors with respect to the center of mass are:

$$\bar{r}_{cmP} = \bar{r}_{PP} - \bar{r}_{Pcm} = -2126.392\ km\ \hat{x}$$

$$\bar{r}_{cmC} = \bar{r}_{PC} - \bar{r}_{Pcm} = 17469.608\ km\ \hat{x}$$

The center of mass is outside the radius of Pluto.

Part (b)

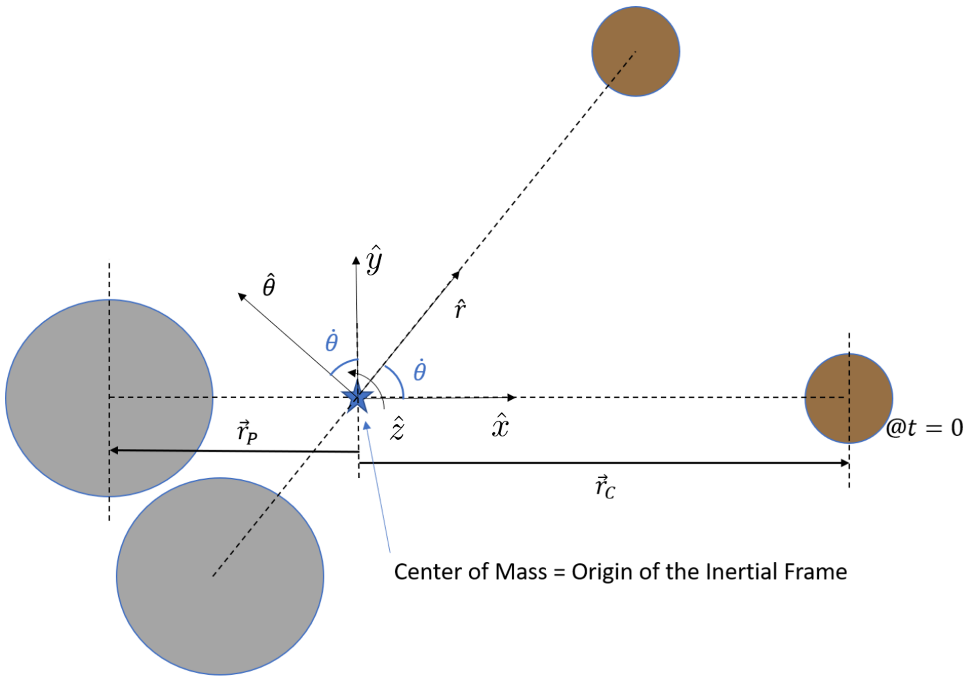


Figure 4: Inertial and Rotating Frame Diagram.

The inertial velocity is related to the rotating velocity by the transport theorem (basic kinematic equation):

$$\frac{{}^{I}d\bar{r}}{dt} = \frac{{}^{R}d\bar{r}}{dt} +^{I} \bar{\omega}^{R} \times \bar{r} = \dot{r}\hat{r} + (\dot{\theta}\hat{z} + r\hat{r}) = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

A number of observations are in order:

1 Firstly, the vectors employed in the expression above are relative vectors, indicating the dynamical behavior of the relative position vector $\bar{r}_{PS}$ over time.

2 Secondly, the velocity components in the rotating frame are expressed in terms of the radial velocity $\dot{r}$ and transverse velocity $r\dot{\theta}$. The magnitude of the velocity expressed in the rotating frame must be equal to the magnitude of the velocity in the standard frame $\hat{x} - \hat{y} - \hat{z}$.

3 Thirdly, the inertial velocities are initially give to be in the $\hat{y}$ frame, indicating that there is no radial velocity at that given time (thus $\dot{r} = 0$). We can use this fact to solve directly for angular velocity $\dot{\theta}$ which are the same for both primaries relative to the center of mass.

Using the given inertial velocity at the initial time where $t = 0$, $\hat{x} = \hat{r}$, $\hat{y} = \hat{\theta}$:

$$^{I}\dot{\bar{r}} =^{I} \dot{\bar{r}}_C -^{I} \dot{\bar{r}}_P = 0.211319\ km/s\ \hat{y} + 0.025717\ km/s\ \hat{y}$$

$$= \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} = \dot{r}\hat{x} + r\dot{\theta}\hat{y}$$

$$r\dot{\theta} = 0.211319\ km/s + 0.025717\ km/s = 0.237036\ km/s$$

$$\dot{\theta} = \frac{0.237036\ km/s}{19596\ km} = 1.2096e-5\ rad/s$$

Part (c)

Note that the the origin of the inertial frame coincides with the center of mass of the two-particle system. This means that since we do not have any external forces, the linear momentum of the system should be zero. However when we compute the linear momentum, it is a vector with a "small" non-zero number in $\hat{y}$ direction. Linear momentum, $\bar{C}_2$ is:

$$G\bar{C}_2 = \sum_{i=1}^{2} Gm_i\dot{\bar{r}}_i = \sum_{i=1}^{2} \mu_i \ {}^I\dot{\bar{r}}_P + \mu_p \ {}^I\dot{\bar{r}}_P + \mu_C \ {}^I\dot{\bar{r}}_C = 0.00456 \ km^4/s^3 \ \hat{y}$$

Then divide by G to get linear momentum:

$$\bar{C}_2 = 6.8345e + 16 \ kg \ km/s \ \hat{y}$$

And the linear momentum of a system is linked to the velocity of the center of mass by the following equation:

$$\sum_{i=1}^{2} m_i \ {}^I\dot{\bar{r}}_{cm} = \sum_{i=1}^{2} m_i\dot{\bar{r}}_i$$

$${}^I\dot{\bar{r}}_{cm} = \frac{\sum_{i=1}^{2} m_i\dot{\bar{r}}_i}{\sum_{i=1}^{2} m_i} = 4e - 6 \ km/s \ \hat{y} \approx \bar{0}$$

Without the external force, we know the center of mass should be stationary. But the computed velocity is a non-zero number, which we could interpret as a numerical error. We notice that the magnitude of this value is 4e-6, the order of which is the same to the last digit of the numbers given for the inertial velocities. We can thus deduce that this non-zero number is coming from the numerical error (limits on the significant digits that we have) associated with given numbers. If they were represented with simple ratios, the linear momentum $\bar{C}_2$ as well as the velocity of the center of mass would be equal to zero vectors. In short:

$$\bar{C}_2 = \bar{0}$$
$${}^I\dot{\bar{r}}_{cm} = {}^I\bar{v}_{cm} = \bar{0}$$

Part (d)

$$\bar{C}_3 = \sum_{i=1}^{2} m_i(\bar{r}_i \times \ {}^I\dot{\bar{r}}_i) = \sum_{i=1}^{2} \frac{\mu_i}{G}(\bar{r}_i \times \ {}^I\dot{\bar{r}}_i)$$

$$= \frac{\mu_P}{G}(\bar{r}_P \times \ {}^I\dot{\bar{r}}_P) + \frac{\mu_C}{G}(\bar{r}_C \times \ {}^I\dot{\bar{r}}_C)$$

$$= \frac{\mu_P}{G}(-2126.392\ km\ \hat{x} \times (-0.025721\ km/s\ \hat{y})) + \frac{\mu_C}{G}(17469.608\ km\ \hat{x} \times (0.211314\ km/s\ \hat{y}))$$

$$= 7.4134e + 24\ kg\ km^2/s\ \hat{z}$$

The units should be $kg\ km^2/s$. Note that $\bar{C}_3$ is a vector, so we should include the direction as well.

Part (e)

$$C_4 = T - U = \sum_{i=1}^{N} m_i \left(\frac{1}{2} \ {}^{I}\dot{\bar{r}}_i \cdot {}^{I}\dot{\bar{r}}_i \right) - \frac{1}{2} G \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{m_i m_j}{r_{ij}}$$

$$= \frac{1}{2} m_P ({}^{I}\dot{\bar{r}}_P \cdot {}^{I}\dot{\bar{r}}_P) + \frac{1}{2} m_C ({}^{I}\dot{\bar{r}}_C \cdot {}^{I}\dot{\bar{r}}_C) - G \frac{m_P m_C}{r_{PC}}$$

$$= -4.4841e + 19 \ kg \ km^2/s^2$$

We can interpret this negative energy value such that Charon will continue to orbit around Pluto in this configuration and won't escape without additional forces. More insight into orbital energy will follow in later classes!