Case Study

Consider the following optimization problem:

$$\min \frac{x^\top Q x}{x^\top P x},$$

where $Q = Q^\top \succ 0$ and $P = P^\top \succ 0$. The above could represent the optimization problem:

$$\min \frac{-\dot{V}(x)}{V(x)},$$

where $V(x) = x^\top P x$, $\dot{V}(x) = -x^\top Q x$, and $A^\top P + PA = -Q$.

Note that if a point $x = [x_1, \ldots, x_n]^\top$ is a solution to the problem, then so is any nonzero scalar multiple of it,

$$tx = [tx_1, \ldots, tx_n]^\top, \qquad t \neq 0.$$

Indeed,

$$\frac{(tx)^\top Q(tx)}{(tx)^\top P(tx)} = \frac{t^2 x^\top Q x}{t^2 x^\top P x} = \frac{x^\top Q x}{x^\top P x}.$$

Therefore, to avoid the multiplicity of solutions, we further impose the constraint

$$x^\top P x = 1.$$

The optimization problem becomes

$$\begin{aligned} &\min x^\top Q x \\ &\text{subject to} \quad x^\top P x = 1. \end{aligned}$$

Let us write

$$\begin{aligned} f(x) &= x^\top Q x, \\ h(x) &= 1 - x^\top P x. \end{aligned}$$

We now apply Lagrange's method. We first form the Lagrangian function

$$l(x, \lambda) = x^\top Q x + \lambda(1 - x^\top P x).$$

Applying the Lagrange condition yields

$$D_x l(x,\lambda) = 2x^\top Q - 2\lambda x^\top P = 0^\top,$$
$$D_\lambda l(x,\lambda) = 1 - x^\top P x = 0.$$

The first of the equations above can be represented as

$$Qx - \lambda P x = 0$$

or

$$(\lambda P - Q)x = 0.$$

This representation is possible because $P = P^\top$ and $Q = Q^\top$. By assumption $P \succ 0$, and hence, P^{-1} exists. Premultiplying $(\lambda P - Q)x = 0$ by P^{-1}, we obtain

$$(\lambda I_n - P^{-1}Q)x = 0$$

or, equivalently,

$$P^{-1}Qx = \lambda x.$$

Therefore, the solution, if it exists, is an eigenvector of $P^{-1}Q$, and the Lagrange multiplier is the corresponding eigenvalue. As usual, let x^* and λ^* be the optimal solution. Because $x^{*\top} P x^* = 1$ and $P^{-1}Qx^* = \lambda^* x^*$, we have

$$\lambda^* = x^{*\top} Q x^*.$$

Hence, λ^* is the minimum of the objective function, and therefore is, in fact, the minimal eigenvalue of $P^{-1}Q$. It is also called the minimal **generalized eigenvalue**.

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