11.7.1 $$A(w) = \frac{1}{\pi}\int_0^\infty \pi e^{-v}\cos(wv)\,dv$$

$$= \underbrace{\frac{e^{-v}}{w}\sin(wv)\Big|_0^\infty}_{=0} + \int_0^\infty \frac{e^{-v}}{w}\sin(wv)\,dv$$

$$= -\frac{e^{-v}}{w^2}\cos(wv)\Big|_0^\infty - \int_0^\infty \frac{e^{-v}}{w^2}\cos(wv)\,dv$$

$$= \frac{1}{w^2} - \frac{1}{w^2}\int_0^\infty e^{-v}\cos(wv)\,dv$$

$\Rightarrow$

$$\int_0^\infty e^{-v}\cos(wv)\,dv = \frac{1}{w^2}\left(1 - \int_0^\infty e^{-v}\cos(wv)\,dv\right)$$

$$\Rightarrow \int_0^\infty e^{-v}\cos(wv)\,dv = \frac{1}{1+w^2}$$

$$B(\omega) = \frac{1}{\pi}\int_0^\infty \pi e^{-v} \sin(\omega v)\, dv$$

$$= \ldots \text{ (similarly to } A(\omega)) \ldots$$

$$= \frac{\omega}{1+w^2}$$

Since our function satisfies the hypothesis of Thm 1 and equals the average of its left and right hand limits @ $x=0$, the equality given is correct.

11.8.2 In part 1, we see that

$$\hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos(\omega x)\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left(\int_0^1 \cos(\omega x)\, dx + \int_1^2 -\cos(\omega x)\, dx \right)$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{\sin(\omega x)}{\omega}\Big|_0^1 - \frac{\sin(\omega x)}{\omega}\Big|_1^2 \right)$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{\omega} \left(\sin(\omega) - \sin(2\omega) + \sin(\omega) \right)$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{2\sin(\omega) - \sin(2\omega)}{\omega}.$$

Now,

$f(x)$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_c(\omega) \cos(\omega x)\, d\omega$$

$$= \frac{2}{\pi} \int_0^\infty \left(\frac{2\sin(\omega)\cos(\omega x)}{\omega} - \frac{\sin(2\omega)\cos(\omega x)}{\omega} \right) d\omega$$

Note: As we see from example 2 on page 5/4,

$$\int_0^\infty \frac{\sin(\omega)\cos(\omega x)}{\omega}\, d\omega = \begin{cases} 0, & x \in (-\infty, -1), \\ \pi/4, & x = -1, \\ \pi/2, & x \in (-1, 1), \\ \pi/4, & x = 1, \\ 0, & x \in (1, \infty). \end{cases}$$

Similarly, we could let $f(x) = \begin{cases} 1, & |x| < 2 \\ 0, & |x| > 1 \end{cases}$ and use the same idea to see

$$\int_0^\infty \frac{\sin(2\omega)\cos(\omega x)}{\omega}\, d\omega = \begin{cases} 0, & x \in (-\infty, -2), \\ \pi/4, & x = -2, \\ \pi/2, & x \in (-2, 2), \\ \pi/4, & x = 2, \\ 0, & x \in (2, \infty). \end{cases}$$

$$= \begin{cases} 0, & x \in (-\infty, -2), \\ -1/2, & x = -2, \\ -1, & x \in (-2, -1), \\ 0, & x = -1, \\ 1, & x \in (-1, 1), \\ 0, & x = 1, \\ -1, & x \in (1, 2), \\ -1/2, & x = 2, \\ 0, & x \in (2, \infty). \end{cases}$$

Without knowing what $f(x)$ was beforehand we don't know exactly what the value of f is at these points; we just know that jump discontinuities are so we should really exclude them when presenting f.

11.9.4 $\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{kx} e^{-i\omega x} \, dx$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{x(k-i\omega)}}{k-i\omega} \right]_{-\infty}^{0}$$

$$= \frac{1}{(k-i\omega)\sqrt{2\pi}}$$

$$\int uv' = uv - \int u'v$$

11.9.7 $$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_0^a x e^{-iwx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{x e^{-iwx}}{-iw} \Big|_0^a - \int_0^a \frac{e^{-iwx}}{-iw} \, dx \right)$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{a e^{-iwa}}{-iw} - \frac{e^{-iwa}}{-w^2} + \frac{1}{-w^2} \right)$$

$$= \frac{1}{w^2 \sqrt{2\pi}} \left(wiae^{-iwa} + e^{-iwa} - 1 \right)$$

$$= \frac{e^{-iwa}(1 + iaw) - 1}{w^2 \sqrt{2\pi}}$$

12.1.8 Heat Eqtn. : $u_t = c^2 u_{xx}$

$$u = e^{-9t}\sin(\omega x)$$

$$u_t = -9e^{-9t}\sin(\omega x)$$

$$u_x = \omega e^{-9t}\cos(\omega x)$$

$$u_{xx} = -\omega^2 e^{-9t}\sin(\omega x)$$

$$-9e^{-9t}\sin(\omega x) = \underbrace{\left(3/\omega\right)}_{c}^2\left(-\omega^2 e^{-9t}\sin(\omega x)\right)$$

$c = -3/\omega$ would also work.

12.3.11 $k = 0.01$, $L = 1$, $c^2 = 1$.

From eqtn (14), pg 548,

$$B_n = 2\left[\int_{1/4}^{1/2} (x - 1/4) \sin(n\pi x)\, dx + \int_{1/2}^{3/4} -(x - 3/4) \sin(n\pi x)\, dx\right]$$

$$= \ldots$$

$$= \frac{1}{2\pi^2 n^2}\left[-4\sin\left(\frac{\pi n}{4}\right) + 4\sin\left(\frac{\pi n}{2}\right) \cancel{- \pi n \cos\left(\frac{\pi n}{2}\right)} \right.$$
$$\left. 4\sin\left(\frac{\pi n}{2}\right) - 4\sin\left(\frac{3\pi n}{4}\right) \cancel{+ \pi n \cos\left(\frac{\pi n}{2}\right)}\right]$$

$$= \frac{1}{2\pi^2 n^2}\left[-8\sin\left(\frac{\pi n}{2}\right)\cos\left(-\frac{\pi n}{4}\right) + 8\sin\left(\frac{\pi n}{2}\right)\right]$$

$$= \frac{4}{\pi^2 n^2} \underbrace{\sin\left(\frac{\pi n}{2}\right)\left(1 - \cos\left(\frac{\pi n}{4}\right)\right)}$$

n	$\sin\left(\frac{\pi n}{2}\right)\left(1-\cos\left(\frac{\pi n}{4}\right)\right)$
1	$1 - 1/\sqrt{2}$
2	0
3	$-(1 + 1/\sqrt{2})$
4	0
5	$1 + 1/\sqrt{2}$
6	0
7	$-(1 - 1/\sqrt{2})$
8	0
9	$1 - 1/\sqrt{2}$
10	0
11	$-(1 + 1/\sqrt{2})$
12	0
13	$1 + 1/\sqrt{2}$
14	0
15	$-(1 - 1/\sqrt{2})$
16	0
17	$1 - 1/\sqrt{2}$
$\vdots$	$\vdots$

$$g(x) = 0 \implies B_n^* = 0$$

$\therefore$

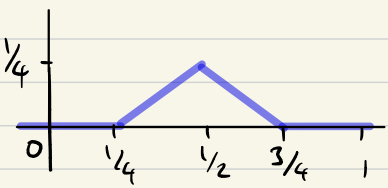
$$u(x,t)$$

$$= \sum_{n=1}^{\infty} \frac{4}{\pi^2 n^2} \sin\left(\frac{\pi n}{2}\right)\left(1 - \cos\left(\frac{\pi n}{4}\right)\right) \cos(n\pi t) \sin(n\pi x)$$

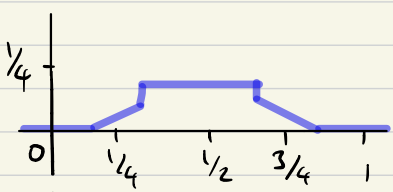
$\cos(-A) = \cos(A)$
so it doesn't
matter if $c=1$
or $c=-1$.

Sketch

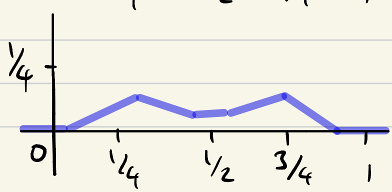
$t=0$



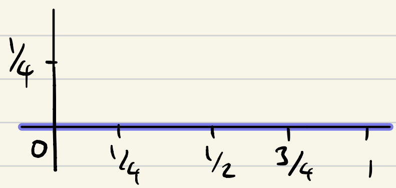
$t=0.1$



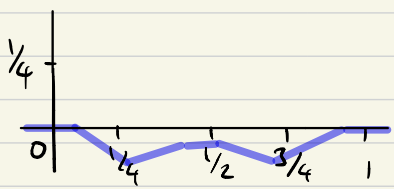
$t=0.2$



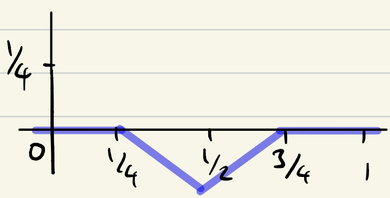
$t=0.5$



$t=0.8$



$t=1$



12.3.16 From previous problem,

$u = FG$,

$F = A\cos\beta x + B\sin\beta x + C\cosh\beta x + D\sinh\beta x$,

$G = a\cos c\beta^2 t + b\sin c\beta^2 t$.

$0 = u(0,t) = F(0)G(t)$

$\Rightarrow 0 = F(0) = A + C \Rightarrow \boxed{A = -C}$

$0 = u_{xx}(0,t) = F''(0)G(t)$

$\Rightarrow 0 = F''(0) = \beta^2(-A\cos 0 - B\sin 0$

$-A\cosh 0 + D\sinh 0)$

$\Rightarrow 0 = -A - A \Rightarrow \boxed{A = C = 0}$

$0 = u(L,t) = F(L)G(t)$

$\Rightarrow 0 = F(L) = B\sin\beta L + D\sinh\beta L$ (*)

$0 = u_{xx}(L,t) = 0 = F''(L)G(t)$

$\Rightarrow 0 = F''(L) = \beta^2(-B\sin\beta L + D\sinh\beta L)$

$\Rightarrow 0 = -B\sin\beta L + D\sinh\beta L$ (**)

$(*)$ & $(**)$ $\Rightarrow$ $2D\sinh \beta L = 0$,

$2B\sin \beta L = 0$

$\Rightarrow D = 0$ (since $\beta L \neq i\pi n$, $n \in \mathbb{Z}$)

and

$\beta L = \pi n$, $n \in \mathbb{Z}$

$\Rightarrow \beta = \frac{\pi n}{L}$, $n \in \mathbb{Z}$

$\therefore$ solutions for F are

$$\boxed{F_n(x) := B_n \sin\left(\frac{n\pi}{L} x\right), \quad n \in \mathbb{N}.}$$

avoiding repititions and trivial solns

zero initial velocity $\Rightarrow u_t(x,0) = 0$

$\Rightarrow G'(0) = 0 \Rightarrow b = 0$

$\Rightarrow$ Solutions for G are

$$\boxed{G_n(t) := a_n \cos\left(c\,\frac{\pi^2 n^2}{L^2} t\right), \quad n \in \mathbb{N}}$$

$\therefore$ Solutions for u are

$$\boxed{u_n := \sin\left(\frac{n\pi}{L}x\right)\cos\left(c\,\frac{\pi^2 n^2}{L^2}t\right), \quad n \in \mathbb{N}}$$

12.4.19 Like before assume $u(x,t) = F(x)\,G(t)$.

Substituting this into PDE gives:

$$FG'' = c^2 F'' G$$

$$\Rightarrow \frac{F''}{F} = \frac{G''}{c^2 G} = \beta, \qquad (\text{Assume } u \neq 0)$$

where β is const.

$\beta = 0 \Rightarrow u$ trivial

$\beta > 0 \Rightarrow F = a_1 e^{\sqrt{\beta}x} + a_2 e^{-\sqrt{\beta}x}$

and $u(0,t) = 0 \Rightarrow a_1 + a_2 = 0$

and $u_x(L,t) = 0 \Rightarrow L = \dfrac{i(2\pi n + \pi)}{2\sqrt{\beta}}$, $n \in \mathbb{Z}$

$\therefore$ The only possible solns are where $\beta < 0$

$\Rightarrow F = A\cos\sqrt{\beta}x + B\sin\sqrt{\beta}x$

$u(0,t) = 0 \Rightarrow A = 0$

$u_x(L,t) = 0$

$\Rightarrow B\sqrt{\beta}\cos(\sqrt{\beta}L) = 0$

$\Rightarrow \sqrt{\beta}L = \frac{(2n+1)\pi}{2}, \quad n \in \{0,1,2,\dots\}$

$\Rightarrow \sqrt{\beta} = \frac{(2n+1)}{2}\frac{\pi}{L}$

So solns for F are

$$\boxed{F_n := \sin\Big(\underbrace{\frac{2n+1}{2}\cdot\frac{\pi}{L}}_{p_n}\cdot x\Big), \quad n \in \{0,1,2,\dots\}}$$

$G'' = \beta c^2 G$

$\Rightarrow G = C\cos(c\sqrt{\beta}t) + D\sin(c\sqrt{\beta}t)$

$u_t(x,0) = 0$

$\Rightarrow Dc\sqrt{\beta} = 0 \quad \Rightarrow D = 0.$

So soln.s for G are

$$\boxed{G_n := \cos\Big(c\underbrace{\frac{2n+1}{2}\cdot\frac{\pi}{L}}_{p_n}t\Big), \quad n \in \{0,1,2,\dots\}}$$

$$u(x,0) = f(x)$$

$$\Rightarrow F(x)\underbrace{G(0)}_{=1} = f(x)$$

$$\Rightarrow F(x) = f(x)$$

Now for some superposition solns of F,

$$\sum_{n=0}^{\infty} A_n F_n(x) = f(x)$$

$$\Rightarrow \underbrace{\sum_{n=0}^{\infty} A_n \sin(p_n x)}_{\text{Fourier sine series}} = f(x)$$

$$\Rightarrow A_n = \frac{2}{L}\int_0^L f(x) \sin(p_n x)\, dx$$

(since in this model, we don't care about what happens when $x > L$)

$\therefore$ The general soln. is

$$u = \sum_{n=0}^{\infty} A_n \sin(p_n x)\cos(p_n c t).$$