1.5.4

$$A = \begin{pmatrix} 2 & 1 \\ 8 & 7 \end{pmatrix}$$

$$E_{21}(-4)\,A = \begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 8 & 7 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} = U$$

$$L^{-1} = E_{21}(-4) \quad \therefore \quad L = E_{21}(4) = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$$

$$\boxed{L = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} \qquad U = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}}$$

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}$$

$$E_{21}(-1/3)\,A = \begin{pmatrix} 1 & 0 & 0 \\ -1/3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 1 \\ 0 & 8/3 & 2/3 \\ 1 & 1 & 3 \end{pmatrix}$$

$$E_{31}(-1/3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1/3 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1/3 & 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 1 & 1 \\ 0 & 8/3 & 2/3 \\ 1 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 1 \\ 0 & 8/3 & 2/3 \\ 0 & \frac{2}{3} & 8/3 \end{pmatrix}$$

$$E_{32}(-1/4) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1/4 & 1 \end{pmatrix} \qquad -\left(\frac{2}{3}\right)\left(\frac{1}{4}\right) = -\frac{2}{12} + \frac{32}{12} = \frac{30}{12} = \frac{10}{4} = \frac{5}{2}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1/4 & 1 \end{pmatrix}\begin{pmatrix} 3 & 1 & 1 \\ 0 & 8/3 & 2/3 \\ 0 & 2/3 & 8/3 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 1 \\ 0 & 8/3 & 2/3 \\ 0 & 0 & \frac{5}{2} \end{pmatrix} = U$$

1.5.4

$L^{-1} = E_{32}(-1/4)\, E_{31}(-1/3)\, E_{21}(-1/3)$ $\therefore$

$L = E_{21}(1/3)\, E_{31}(1/3)\, E_{32}(1/4)$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ 1/3 & 1/4 & 1 \end{pmatrix}$$

$$\boxed{L = \begin{pmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ 1/3 & 1/4 & 1 \end{pmatrix} \qquad U = \begin{pmatrix} 3 & 1 & 1 \\ 0 & 8/3 & 2/3 \\ 0 & 0 & 5/2 \end{pmatrix}}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 4 & 4 \\ 1 & 4 & 8 \end{pmatrix} = A$$

$$E_{21}(-1) = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$E_{21}(-1)A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 4 & 4 \\ 1 & 4 & 8 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & 3 \\ 1 & 4 & 8 \end{pmatrix}$$

$$E_{31}(-1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$E_{31}(-1) \begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & 3 \\ 1 & 4 & 8 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & 3 \\ 0 & 3 & 7 \end{pmatrix}$$

$$E_{32}(-1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

1.5.4

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & 3 \\ 0 & 3 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & 3 \\ 0 & 0 & 4 \end{pmatrix} = U$$

$L^{-1} = E_{32}(-1)\, E_{31}(-1)\, E_{21}(-1) \quad \therefore$

$$L = E_{21}(1)\, E_{31}(1)\, E_{32}(1) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\boxed{L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \qquad U = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & 3 \\ 0 & 0 & 4 \end{pmatrix}}$$

1.5.15

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{pmatrix}$$

$$\underset{n=3}{P_{13}} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$P_{13} A = \begin{pmatrix} 2 & 3 & 4 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$E_{21}(-\tfrac{1}{2}) P_{13} A = \begin{pmatrix} 1 & 0 & 0 \\ -\tfrac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & 4 \\ 0 & -\tfrac{3}{2} & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$E_{32}(\tfrac{2}{3}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \tfrac{2}{3} & 1 \end{pmatrix}$$

$$E_{32}(\tfrac{2}{3}) \begin{pmatrix} 2 & 3 & 4 \\ 0 & -\tfrac{3}{2} & -1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 4 \\ 0 & -\tfrac{3}{2} & -1 \\ 0 & 0 & \tfrac{1}{3} \end{pmatrix} = \hat{U}$$

$$L^{-1} = E_{32}(\tfrac{2}{3}) E_{21}(-\tfrac{1}{2}) \quad \therefore \quad L = E_{21}(\tfrac{1}{2}) E_{32}(-\tfrac{2}{3})$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \tfrac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\tfrac{2}{3} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \tfrac{1}{2} & 1 & 0 \\ 0 & -\tfrac{2}{3} & 1 \end{pmatrix}$$

$$\hat{U} = D U$$

$$\begin{pmatrix} 2 & 3 & 4 \\ 0 & -\tfrac{3}{2} & -1 \\ 0 & 0 & \tfrac{1}{3} \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -\tfrac{3}{2} & 0 \\ 0 & 0 & \tfrac{1}{3} \end{pmatrix} \begin{pmatrix} 1 & \tfrac{3}{2} & 2 \\ 0 & 1 & \tfrac{2}{3} \\ 0 & 0 & 1 \end{pmatrix}$$

$$L=\begin{pmatrix}1&0&0\\ \frac{1}{2}&1&0\\ 0&-\frac{2}{3}&1\end{pmatrix} \quad D=\begin{pmatrix}2&0&0\\ 0&-\frac{3}{2}&0\\ 0&0&\frac{1}{3}\end{pmatrix} \quad U=\begin{pmatrix}1&\frac{3}{2}&2\\ 0&1&\frac{2}{3}\\ 0&0&1\end{pmatrix}$$

$$PA=\begin{pmatrix}0&0&1\\ 0&1&0\\ 1&0&0\end{pmatrix}\begin{pmatrix}0&1&1\\ 1&0&1\\ 2&3&4\end{pmatrix}=\begin{pmatrix}2&3&4\\ 1&0&1\\ 0&1&1\end{pmatrix}$$

$$LDU=\begin{pmatrix}1&0&0\\ \frac{1}{2}&1&0\\ 0&-\frac{2}{3}&1\end{pmatrix}\begin{pmatrix}2&0&0\\ 0&-\frac{3}{2}&0\\ 0&0&\frac{1}{3}\end{pmatrix}\begin{pmatrix}1&\frac{3}{2}&2\\ 0&1&\frac{2}{3}\\ 0&0&1\end{pmatrix}$$

$$LDU=\begin{pmatrix}1&0&0\\ \frac{1}{2}&1&0\\ 0&-\frac{2}{3}&1\end{pmatrix}\begin{pmatrix}2&3&4\\ 0&-\frac{3}{2}&-1\\ 0&0&\frac{1}{3}\end{pmatrix}=\begin{pmatrix}2&3&4\\ 1&0&1\\ 0&1&1\end{pmatrix}=PA$$

$\therefore$

$$\boxed{L=\begin{pmatrix}1&0&0\\ \frac{1}{2}&1&0\\ 0&-\frac{2}{3}&1\end{pmatrix} \quad D=\begin{pmatrix}2&0&0\\ 0&-\frac{3}{2}&0\\ 0&0&\frac{1}{3}\end{pmatrix} \quad U=\begin{pmatrix}1&\frac{3}{2}&2\\ 0&1&\frac{2}{3}\\ 0&0&1\end{pmatrix}}$$

1.5.15

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$ ← Note: row 2 = (2)(row 1), not indelendent

$$P_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$P_{13} A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{pmatrix}$$

$$E_{21}(-2) = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$E_{21}(-2)\, P_{13}\, A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 1 & 2 & 1 \end{pmatrix}$$

$$E_{31}(-1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$E_{31}(-1) \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$E_{32}(-\tfrac{1}{2}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\tfrac{1}{2} & 1 \end{pmatrix}$$

$$E_{32}(-\tfrac{1}{2}) \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \hat{U}$$

$$L^{-1} = E_{32}(-\tfrac{1}{2})\, E_{31}(-1)\, E_{21}(-2) \quad \therefore \quad L = E_{21}(2)\, E_{31}(1)\, E_{32}(\tfrac{1}{2})$$

1.5.15

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2} & 1 \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & \frac{1}{2} & 1 \end{pmatrix}$$

$$\hat{U} = DU = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$PA = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{pmatrix}$$

$$LDU = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$LDU = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{pmatrix} = P_{13} A$$

$\therefore$

$$\boxed{L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & \frac{1}{2} & 1 \end{pmatrix} \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad U = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}}$$

1.5.27

$$A = \begin{pmatrix} 2 & 4 & 8 \\ 0 & 3 & 9 \\ 0 & 0 & 7 \end{pmatrix}$$

$$U = \begin{pmatrix} 2 & 4 & 8 \\ 0 & 3 & 9 \\ 0 & 0 & 7 \end{pmatrix}$$ (A is already a upper Δ matrix)

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A = LU = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & 8 \\ 0 & 3 & 9 \\ 0 & 0 & 7 \end{pmatrix}$$

$$U = D\,U_{new} = \begin{pmatrix} 2 & 4 & 8 \\ 0 & 3 & 9 \\ 0 & 0 & 7 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 7 \end{pmatrix} \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\boxed{L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad U = \begin{pmatrix} 2 & 4 & 8 \\ 0 & 3 & 9 \\ 0 & 0 & 7 \end{pmatrix} \quad D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 7 \end{pmatrix} \quad U_{new} = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}}$$

1.6.4

a) $AB = AC$

$A^{-1}AB = A^{-1}AC$ (Left multiply by A^{-1}, which exists)

$A^{-1}A = I$

$IB = IC$ $IB = B$, $IC = C$

$\boxed{B = C}$

b) $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$AB = AC$ $B \neq C$

$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix}$

$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} e & f \\ 0 & 0 \end{pmatrix}$

$\therefore$ $a = e$
$b = f$

Let $a = e = b = f = 1$. Let $g = h = 2$, Let $c = d = 3$

$$\boxed{B = \begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}}$$

$AB = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = AC$

1.6.6

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{E_{21}(-1)} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{E_{23}(-1)} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\boxed{A_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}}$$

$$A_2 = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right) \xrightarrow{E_{21}(\frac{1}{2})} \left(\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & \frac{1}{2} & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{E_{32}(\frac{2}{3})} \left(\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{array}\right) \xrightarrow{E_3(\frac{3}{4})} \left(\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{array}\right)$$

1.6.6

$$\xrightarrow{E_{23}(1)} \left(\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & 0 & \frac{3}{4} & \frac{3}{2} & \frac{3}{4} \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{array}\right)$$

$$\xrightarrow{E_2(\frac{2}{3})} \left(\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{array}\right)$$

$$\xrightarrow{E_{12}(1)} \left(\begin{array}{ccc|ccc} 2 & 0 & 0 & \frac{3}{2} & 1 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{array}\right)$$

$$\xrightarrow{E_1(\frac{1}{2})} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{array}\right)$$

$$\boxed{A_2^{-1} = \begin{pmatrix} \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{pmatrix}}$$

$$A_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{P_{13}} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right)$$

1.6.6

$$\xrightarrow{E_{23}(-1)} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{E_{12}(-1)} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{E_{13}(-1)} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right)$$

$$\boxed{A_3^{-1} = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}}$$

1.6.13

$A = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad B = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

$A^T B = [3 \quad 1] \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 6 + 2$

$\boxed{A^T B = 8}$

$B^T A = [2 \quad 2] \begin{bmatrix} 3 \\ 1 \end{bmatrix} = 6 +$

$\boxed{B^T A = 8}$

$AB^T = \begin{pmatrix} 3 \\ 1 \end{pmatrix} [2 \quad 2] = \begin{pmatrix} (3)(2) & (3)(2) \\ (1)(2) & (1)(2) \end{pmatrix}$

$\boxed{AB^T = \begin{pmatrix} 6 & 6 \\ 2 & 2 \end{pmatrix}}$

$BA^T = \begin{pmatrix} 2 \\ 2 \end{pmatrix} [3 \quad 1] = \begin{pmatrix} (2)(3) & (2)(1) \\ (2)(3) & (2)(1) \end{pmatrix}$

$\boxed{BA^T = \begin{pmatrix} 6 & 2 \\ 6 & 2 \end{pmatrix}}$

1.6.14

Let $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\therefore B^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$

$A = B + B^T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 2a & b+c \\ b+c & 2d \end{pmatrix}$

$\therefore$ OFF diagonal entries are equal

$K = B - B^T = \begin{pmatrix} 0 & b-c \\ c-b & 0 \end{pmatrix}$

$K^T = \begin{pmatrix} 0 & c-b \\ b-c & 0 \end{pmatrix}$ $\quad -K = \begin{pmatrix} 0 & -b+c \\ -c+b & 0 \end{pmatrix} = \begin{pmatrix} 0 & c-b \\ b-c & 0 \end{pmatrix}$

$\therefore K^T = -K$ as both matrices are equal

$B = \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}$ $\quad B^T = \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix}$

$A = B + B^T = \boxed{\begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix}}$

$K = B - B^T = \boxed{\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}}$

1.6.14

$A = B + B^T \qquad K = B - B^T$

$A - B = B^T = B - K$

$A + K = 2B$

$B = \frac{1}{2}A + \frac{1}{2}K$

$$B = \frac{1}{2}\begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$$

$$\boxed{B = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}$$

1.6.17

$A = L_1 D_1 U_1 = L_2 D_2 U_2$

a) Left multiply by L_1^{-1}:

$L_1^{-1} L_1 D_1 U_1 = L_1^{-1} L_2 D_2 U_2$

$L_1^{-1} L_1 = I$ $\therefore$ $D_1 U_1 = L_1^{-1} L_2 D_2 U_2$

Right Multiply by U_2^{-1}

$D_1 U_1 U_2^{-1} = L_1^{-1} L_2 D_2 U_2 U_2^{-1}$

$U_2 U_2^{-1} = I$ $\therefore$

$$\boxed{D_1 U_1 U_2^{-1} = L_1^{-1} L_2 D_2}$$

The inverse of an upper Δ matrix is also an upper Δ matrix. The product of two upper Δ matrices is also an upper Δ matrix. Similiar logic applies for lower Δ matrices. Multiplication by a diagonal matrix maintains the lower / upper Δ matrix.

b) $L_1^{-1} L_2 D_2 = \begin{pmatrix} L_1 & 0 & 0 \\ L_2 & L_4 & 0 \\ L_3 & L_5 & L_6 \end{pmatrix}$ $\quad D_1 U_1 U_2^{-1} = \begin{pmatrix} U_1 & U_2 & U_3 \\ 0 & U_4 & U_5 \\ 0 & 0 & U_6 \end{pmatrix}$

If $L_1^{-1} L_2 D_2 = D_1 U_1 U_2^{-1}$ then via inspection:
$U_2 = U_3 = U_5 = L_2 = L_3 = L_5 = 0$ & $L_1 = U_1$, $U_4 = L_4$, & $L_6 = U_6$.

1.6.17

$\therefore$ $L_1^{-1} L_2 D_2$ & $D_1 U_1 U_2^{-1}$ are both diagonal matrices with 0 on off diagonals and equivalent entries on the diagonal. Because of this, the following is true:

$$L_1 = L_2$$
$$D_1 = D_2$$
$$U_1 = U_2$$

1.12

1) If A is invertible then B is also invertible as the set of equations remains the same, just with swapped rows. Therefore the statement is $\boxed{\text{true}}$.

2) $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = A \qquad B = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$

$$AB = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 5 \\ 6 & 8 \end{pmatrix}$$

Not symmetric

$\therefore$ $\boxed{\text{False}}$

3) The inverse of BA is given by the identity $(BA)^{-1} = A^{-1}B^{-1}$. Therefore if A^{-1} & B^{-1} both exist then so must $(BA)^{-1}$. The statement is $\boxed{\text{true}}$

4) $\boxed{\text{True}}$ a permutation can be used to move a zero out of a pivot spot, allowing for the LU decomposition.

1.19

$u+v+w=0$

$u+2v+3w=0$

$3u+5v+7w=0$

Subtract ① from ②

$u+v+w=0$

$v+2w=0$

$3u+5v+7w=1$

Subtract 3 times ① from ③

$u+v+w=0$

$v+2w=0$

$2v+4w=1$

Subtract 2 times ② from ③:

$u+v+w=0$

$v+2w=0$

$0=1$

$\boxed{\text{Singular, i.e. No solution}}$

$u+v+w=0$

$u+v+3w=0$

$3u+5v+7w=1$

1.19

Subtract ① from ②

$u + v + w = 0$

$2w = 0$

$3u + 5v + 7w = 1$

Subtract 3 times ① from ③

$u + v + w = 0$

$2w = 0$

$2v + 4w = 1$

$w = 0 \therefore v = \frac{1}{2} \therefore u = -v = -\frac{1}{2}$

$$\therefore \boxed{\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}}$$