ECE 602: LUMPED LINEAR SYSTEMS

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Reachability of Discrete-Time (DT) Linear Time-Invariant (LTI) Systems

Reachability of discrete-time (DT) linear time-invariant (LTI) systems

- **Objective**: Introduce notion of reachability of DT LTI systems modeled as

$$\boldsymbol{x}[k+1] = \boldsymbol{A}\boldsymbol{x}[k] + \boldsymbol{B}\boldsymbol{u}[k], \quad \boldsymbol{x}[0] = \boldsymbol{x}_0,$$

  where $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ and $\boldsymbol{B} \in \mathbb{R}^{n\times m}$
- First, obtain a solution of the system
- Note that

$$\boldsymbol{x}[1] = \boldsymbol{A}\boldsymbol{x}[0] + \boldsymbol{B}\boldsymbol{u}[0]$$

  and

$$\begin{aligned}\boldsymbol{x}[2] &= \boldsymbol{A}\boldsymbol{x}[1] + \boldsymbol{B}\boldsymbol{u}[1]\\ &= \boldsymbol{A}\left(\boldsymbol{A}\boldsymbol{x}[0] + \boldsymbol{B}\boldsymbol{u}[0]\right) + \boldsymbol{B}\boldsymbol{u}[1]\\ &= \boldsymbol{A}^2\boldsymbol{x}[0] + \boldsymbol{A}\boldsymbol{B}\boldsymbol{u}[0] + \boldsymbol{B}\boldsymbol{u}[1]\end{aligned}$$

Solving DT LTI system modeling equation

- We have

$$\boldsymbol{x}[2] = \boldsymbol{A}^2\boldsymbol{x}[0] + \boldsymbol{A}\boldsymbol{B}\boldsymbol{u}[0] + \boldsymbol{B}\boldsymbol{u}[1]$$

- Iterate to obtain

$$\begin{aligned}
\boldsymbol{x}[3] &= \boldsymbol{A}\boldsymbol{x}[2] + \boldsymbol{B}\boldsymbol{u}[2] \\
&= \boldsymbol{A}\left(\boldsymbol{A}^2\boldsymbol{x}[0] + \boldsymbol{A}\boldsymbol{B}\boldsymbol{u}[0] + \boldsymbol{B}\boldsymbol{u}[1]\right) + \boldsymbol{B}\boldsymbol{u}[2] \\
&= \boldsymbol{A}^3\boldsymbol{x}[0] + \boldsymbol{A}^2\boldsymbol{B}\boldsymbol{u}[0] + \boldsymbol{A}\boldsymbol{B}\boldsymbol{u}[1] + \boldsymbol{B}\boldsymbol{u}[2] \\
&= \boldsymbol{A}^3\boldsymbol{x}[0] + \begin{bmatrix} \boldsymbol{B} & \boldsymbol{A}\boldsymbol{B} & \boldsymbol{A}^2\boldsymbol{B} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}[2] \\ \boldsymbol{u}[1] \\ \boldsymbol{u}[0] \end{bmatrix}
\end{aligned}$$

Solving DT LTI system modeling equation—Contd

- We have

$$\boldsymbol{x}[3] = \boldsymbol{A}^3\boldsymbol{x}[0] + \begin{bmatrix} \boldsymbol{B} & \boldsymbol{AB} & \boldsymbol{A}^2\boldsymbol{B} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}[2] \\ \boldsymbol{u}[1] \\ \boldsymbol{u}[0] \end{bmatrix}$$

- In general

$$\boldsymbol{x}[i] = \boldsymbol{A}^i\boldsymbol{x}[0] + \begin{bmatrix} \boldsymbol{B} & \boldsymbol{AB} & \cdots & \boldsymbol{A}^{i-1}\boldsymbol{B} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}[i-1] \\ \vdots \\ \boldsymbol{u}[1] \\ \boldsymbol{u}[0] \end{bmatrix}$$

Properties of $\begin{bmatrix} B & AB & \cdots & A^{i-1}B \end{bmatrix}$

- Let
$$\boldsymbol{U}_i = \begin{bmatrix} \boldsymbol{B} & \boldsymbol{A}\boldsymbol{B} & \cdots & \boldsymbol{A}^{i-1}\boldsymbol{B} \end{bmatrix}$$
- Note that
$$\boldsymbol{U}_i \in \mathbb{R}^{n \times mi}$$
- If
$$\operatorname{rank} \boldsymbol{U}_l = \operatorname{rank} \boldsymbol{U}_{l+1}$$
then all the columns of $\boldsymbol{A}^l\boldsymbol{B}$ are linearly dependent on those of $\boldsymbol{U}_l$
- This, in turn, implies that all the columns of $\boldsymbol{A}^{l+1}\boldsymbol{B}$, $\boldsymbol{A}^{l+2}\boldsymbol{B}, \ldots$ must also be linearly dependent on those of $\boldsymbol{U}_l$
- Hence,
$$\operatorname{rank} \boldsymbol{U}_l = \operatorname{rank} \boldsymbol{U}_{l+1} = \operatorname{rank} \boldsymbol{U}_{l+2} = \ldots$$

Properties of $\left[\begin{array}{cccc} B & AB & \cdots & A^{i-1}B \end{array}\right]$—Contd

- If
$$\operatorname{rank} \boldsymbol{U}_l = \operatorname{rank} \boldsymbol{U}_{l+1}$$
then
$$\operatorname{rank} \boldsymbol{U}_l = \operatorname{rank} \boldsymbol{U}_{l+1} = \operatorname{rank} \boldsymbol{U}_{l+2} = \ldots$$
- Indeed, if $\operatorname{rank} \boldsymbol{U}_l = \operatorname{rank} \boldsymbol{U}_{l+1}$, then all the columns of $\boldsymbol{A}^l \boldsymbol{B}$ are linearly dependent on those of $\boldsymbol{U}_l$
- Hence,
$\boldsymbol{A}^{l+1}\boldsymbol{B} = \boldsymbol{A}(\boldsymbol{A}^l\boldsymbol{B}) = \boldsymbol{A} * (\text{linear combo of columns of } \boldsymbol{U}_l)$
- But since $\operatorname{rank} \boldsymbol{U}_l = \operatorname{rank} \boldsymbol{U}_{l+1}$, every column of $\boldsymbol{A}^{l+1}\boldsymbol{B}$ must be a linear combo of the columns of $\boldsymbol{U}_l$
- Therefore, $\operatorname{rank} \boldsymbol{U}_l = \operatorname{rank} \boldsymbol{U}_{l+1} = \operatorname{rank} \boldsymbol{U}_{l+2} = \ldots$

More Analysis of U_i

- The rank of $\boldsymbol{U}_i$ increases by at least one when i is increased by one, until the maximum value of rank $\boldsymbol{U}_i$ is attained
- The maximum value of the rank of $\boldsymbol{U}_i$ is guaranteed to be achieved for $i = n$
- Indeed, by the Cayley-Hamilton theorem the matrix $\boldsymbol{A}$ satisfies its own characteristic equation
- Hence

$$\boldsymbol{A}^n = -a_{n-1}\boldsymbol{A}^{n-1} - \cdots - a_1\boldsymbol{A} - a_0\boldsymbol{I}_n.$$

- Therefore

$$\text{rank}\begin{bmatrix} \boldsymbol{B} & \cdots & \boldsymbol{A}^{n-1}\boldsymbol{B} \end{bmatrix} = \text{rank}\begin{bmatrix} \boldsymbol{B} & \cdots & \boldsymbol{A}^{n-1}\boldsymbol{B} & \boldsymbol{A}^n\boldsymbol{B} \end{bmatrix}$$

The controllability matrix

- By the Cayley-Hamilton (C-H) theorem, the maximum value of the rank of $\boldsymbol{U}_i$ is guaranteed to be achieved for $i = n$
- Indeed, by C-H, $\boldsymbol{A}^n = -a_{n-1}\boldsymbol{A}^{n-1} - \cdots - a_1\boldsymbol{A} - a_0\boldsymbol{I}_n$
- Hence

$$\begin{aligned}\boldsymbol{A}^{n+1} &= \boldsymbol{A}\left(\boldsymbol{A}^n\right) = \boldsymbol{A}\left(-a_{n-1}\boldsymbol{A}^{n-1} - \cdots - a_1\boldsymbol{A} - a_0\boldsymbol{I}_n\right)\\ &= -a_{n-1}\boldsymbol{A}^n - \cdots - a_1\boldsymbol{A}^2 - a_0\boldsymbol{A}\\ &= -a_{n-1}\left(-a_{n-1}\boldsymbol{A}^{n-1} - \cdots - a_0\boldsymbol{I}_n\right) - \cdots - a_0\boldsymbol{A}\end{aligned}$$

- So we always have $\operatorname{rank}\boldsymbol{U}_n = \operatorname{rank}\boldsymbol{U}_{n+1} = \operatorname{rank}\boldsymbol{U}_{n+2} = \ldots$
- The matrix

$$\boldsymbol{U}_n = \left[\begin{array}{cccc}\boldsymbol{B} & \boldsymbol{AB} & \cdots & \boldsymbol{A}^{n-1}\boldsymbol{B}\end{array}\right] \in \mathbb{R}^{n\times mn}$$

is called the controllability matrix

Reachability

Definition

We say that the system $\boldsymbol{x}[k+1] = \boldsymbol{A}\boldsymbol{x}[k] + \boldsymbol{B}\boldsymbol{u}[k]$ is reachable if for any $\boldsymbol{x}_f$ there exists a finite integer $q > 0$ and a control sequence, $\{\boldsymbol{u}[i] : i = 0, 1, \ldots, q-1\}$, that transfers $\boldsymbol{x}[0] = \mathbf{0}$ to $\boldsymbol{x}[q] = \boldsymbol{x}_f$.

Theorem

The system $\boldsymbol{x}[k+1] = \boldsymbol{A}\boldsymbol{x}[k] + \boldsymbol{B}\boldsymbol{u}[k]$ is reachable if and only if

$$rank \begin{bmatrix} \boldsymbol{B} & \boldsymbol{A}\boldsymbol{B} & \cdots & \boldsymbol{A}^{n-1}\boldsymbol{B} \end{bmatrix} = n.$$

Necessary and Sufficient Condition for Reachability

- For $\boldsymbol{x}[0] = \mathbf{0}$ and $\boldsymbol{x}[q] = \boldsymbol{x}_f$,

$$\begin{aligned} \boldsymbol{x}_f &= \sum_{k=0}^{q-1} \boldsymbol{A}^{q-k-1} \boldsymbol{B} \boldsymbol{u}[k] \\ &= \begin{bmatrix} \boldsymbol{B} & \boldsymbol{A}\boldsymbol{B} & \cdots & \boldsymbol{A}^{q-1}\boldsymbol{B} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}[q-1] \\ \vdots \\ \boldsymbol{u}[0] \end{bmatrix} \end{aligned}$$

- Thus for any $\boldsymbol{x}_f$ there exists a control sequence $\{\boldsymbol{u}[i] : i = 0, 1, \dots, q-1\}$ that transfers $\boldsymbol{x}[0] = \mathbf{0}$ to $\boldsymbol{x}[q] = \boldsymbol{x}_f$ if and only if

$$\operatorname{rank} \begin{bmatrix} \boldsymbol{B} & \boldsymbol{A}\boldsymbol{B} & \cdots & \boldsymbol{A}^{q-1}\boldsymbol{B} \end{bmatrix} = \operatorname{rank} \boldsymbol{U}_q = n.$$

Example of Nonreachable DT System

- A DT system,

$$\begin{aligned} \boldsymbol{x}[k+1] &= \boldsymbol{A}\boldsymbol{x}[k] + \boldsymbol{b}u[k] \\ &= \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \boldsymbol{x}[k] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u[k], \quad a \in \mathbb{R} \end{aligned}$$

- The solution for $\boldsymbol{x}[0] = \boldsymbol{0}$ and arbitrary $\boldsymbol{x}_f$

$$\boldsymbol{x}_f = \begin{bmatrix} \boldsymbol{b} & \boldsymbol{A}\boldsymbol{b} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}[1] \\ \boldsymbol{u}[0] \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{u}[1] \\ \boldsymbol{u}[0] \end{bmatrix}$$

- Take the state $\boldsymbol{x}_f = \begin{bmatrix} 0 & 1 \end{bmatrix}^\top$
- Then, this state is reachable $\iff$ there are $u[0]$ and $u[1]$ such that

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{u}[1] \\ \boldsymbol{u}[0] \end{bmatrix}$$

So Why Nonreachable?

- Tthe state $\boldsymbol{x}_f = \begin{bmatrix} 0 & 1 \end{bmatrix}^\top$ is not reachable because there are no $u[0]$, $u[1]$ such that

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{u}[1] \\ \boldsymbol{u}[0] \end{bmatrix}$$

- Equivalently, the necessary and sufficient condition for reachability is NOT satisfied,

$$\operatorname{rank} \begin{bmatrix} \boldsymbol{b} & \boldsymbol{Ab} \end{bmatrix} = \operatorname{rank} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 1 < 2.$$

- Note that nonreachable states are

$$\left\{ c \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; c \in \mathbb{R} \right\}$$