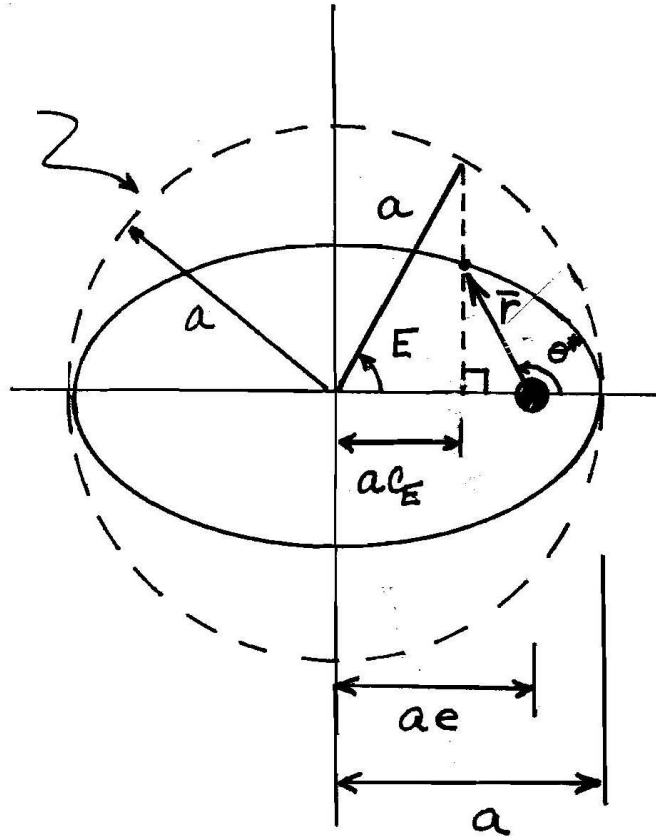


Eccentric Anomaly E

Additional variable defined for ellipse



$$a \cos E = ae - r \cos(180^\circ - \theta^*)$$

$$= ae + r \cos \theta^*$$

$$\cos E = \frac{ae + r \cos \theta^*}{a}$$

$$= e + \frac{r}{a} \cos \theta^*$$

$$r = \frac{p}{1 + e \cos \theta^*} \rightarrow \cos \theta^* = \frac{p}{re} - \frac{1}{e}$$

$$\cos E = \frac{a - r}{ae}$$

OR



E obviously related to θ^* . How??

Previously $r \cos \theta^* = a \cos E - ae$

Identity $\cos 2\alpha = 2 \cos^2 \alpha - 1$

$$r \left(2 \cos^2 \left(\frac{\theta^*}{2} \right) - 1 \right) = a \cos E - ae$$

$$\begin{aligned} 2r \cos^2 \left(\frac{\theta^*}{2} \right) &= a \cos E - ae + a(1 - e \cos E) \\ &= (a - ae) \cos E + (a - ae) \end{aligned}$$

$$2r \cos^2 \left(\frac{\theta^*}{2} \right) = a(1 - e)(1 + \cos E)$$


Identity $\cos 2\alpha = 1 - 2 \sin^2 \alpha$

$$r \left(1 - 2 \sin^2 \left(\frac{\theta^*}{2} \right) \right) = a \cos E - ae$$

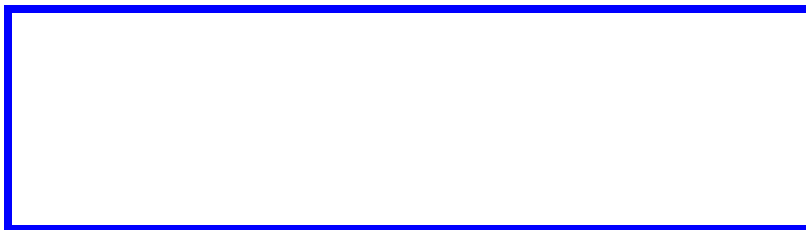
$$\begin{aligned} -2r \sin^2 \left(\frac{\theta^*}{2} \right) &= a \cos E - ae - a(1 - e \cos E) \\ &= (a + ae) \cos E - (a + ae) \end{aligned}$$

$$-2r \sin^2 \left(\frac{\theta^*}{2} \right) = a(1 + e)(\cos E - 1)$$

Identity: $\tan^2 \frac{E}{2} = \frac{1 - \cos E}{1 + \cos E}$

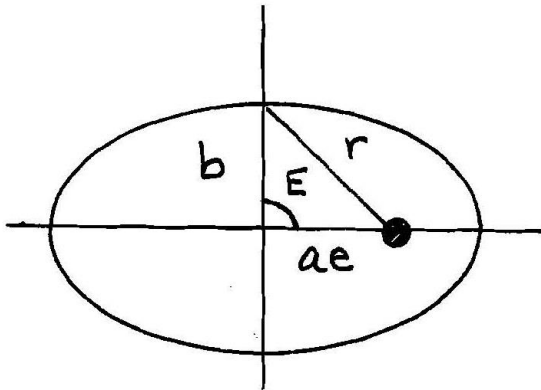


$$\frac{-2r \sin^2 \left(\frac{\theta^*}{2} \right)}{2r \cos^2 \left(\frac{\theta^*}{2} \right)} = \frac{a(1 + e)(\cos E - 1)}{a(1 - e)(1 + \cos E)}$$



Note:

At $E = 90^\circ$



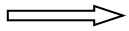
$$r = a(1 - e \cos 90^\circ) = a$$

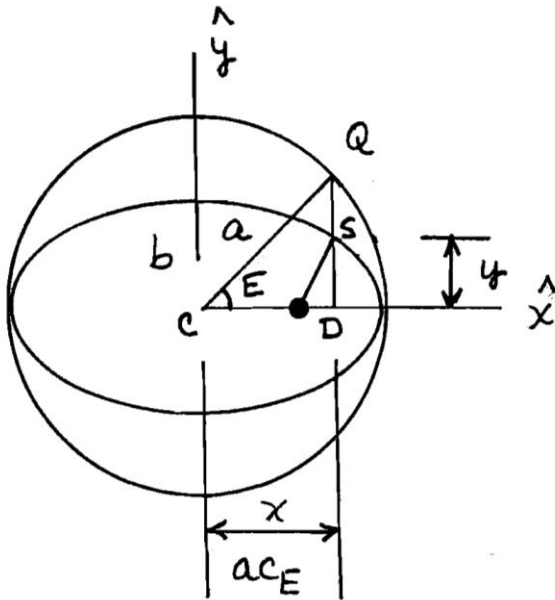
$$b^2 = r^2 - a^2 e^2$$

$$b^2 = a^2(1 - e^2)$$

$$b = a\sqrt{1 - e^2}$$

$$r = \frac{p}{1 + e \cos \theta^*} \rightarrow \cos \theta^* = \frac{p}{re} - \frac{1}{e} = \frac{a(1 - e^2)}{ae} - \frac{1}{e} = -e$$





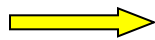
$$CD = a \cos E$$

Let x, y be point S on the ellipse

→ measured from center C

SD ?

Equation of an ellipse



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{(CD)^2}{a^2} + \frac{(SD)^2}{b^2} = 1$$

$$\frac{a^2 \cos^2 E}{a^2} + \frac{(SD)^2}{b^2} = 1$$

$$(SD)^2 = b^2 \left[1 - \frac{a^2 \cos^2 E}{a^2} \right]$$



All of these fundamental relationships are useful but one of the most important reasons to introduce E is to obtain a relation between position and time



How? Various approaches – consider one

Kepler's Equation (Relation between position and time)

Begin with some relationships that are already known

$$p = \frac{h^2}{\mu}$$

$$r = \frac{p}{1 + e \cos \theta^*}$$

$$h = r^2 \dot{\theta} = r^2 \frac{d\theta}{dt}$$

Combine to eliminate r and h

$$h = r^2 \frac{d\theta}{dt}$$

$$\sqrt{\mu p} = \frac{p^2}{(1 + e \cos \theta^*)^2} \frac{d\theta}{dt}$$

Rearrange

$$\boxed{\sqrt{\frac{\mu}{p^3}} dt = \frac{d\theta}{(1 + e \cos \theta^*)^2}}$$

Need to integrate to get a useful relationship for time as a function of θ^* . Direct integration is nontrivial which is why t was always eliminated previously. However, it is possible to use eccentric anomaly E and the relationship

$$r = a (1 - e \cos E)$$

Note: introducing E implies that the result will only apply to elliptical orbits.

To use E to relate position and time:

- 1) Relate equation for r

$$\cos E = \frac{a-r}{ae} \quad \text{I}$$

- 2) Differentiate I and rearrange

$$\dot{r} = ae \dot{E} \sin E \quad \text{II}$$

- 3) Given

$$\mathcal{E} = \frac{\dot{r}^2 + r^2 \dot{\theta}^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a}$$

Multiply by $\frac{2r^2 a}{\mu}$

$$\frac{ar^2 \dot{r}^2}{\mu} = \frac{-ar^4 \dot{\theta}^2}{\mu} + 2ra - r^2$$

But

$$r^4 \dot{\theta}^2 = h^2 = \mu p = \mu a (1 - e^2)$$

$$\therefore \frac{ar^2 \dot{r}^2}{\mu} = a^2 e^2 - (a-r)^2 \quad \text{III}$$

- 4) Square I

$$a^2 e^2 \cos^2 E = (a-r)^2$$

Combine with II, III

$$\frac{ar^2}{\mu} [a^2 e^2 \dot{E}^2 \sin^2 E] = a^2 e^2 - a^2 e^2 \cos^2 E$$

$$\Rightarrow r \dot{E} = \sqrt{\frac{\mu}{a}}$$

- 5) Rewrite

$$r dE = \sqrt{\frac{\mu}{a}} dt$$

$$r = a(1 - e \cos E)$$

$$\sqrt{\frac{\mu}{a}} dt = a(1 - e \cos E) dE$$

6) Integrate (now easy)



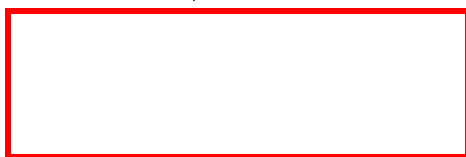
Corresponding relationship for a hyperbolic orbit



Can you derive this?

Define $\mathbf{n} = \sqrt{\frac{\mu}{a^3}}$

$$M = n(t - t_p)$$



Short equation but transcendental

Given time (M), cannot solve for E in closed form

Solution usually obtained iteratively



Solution of Kepler's Equation

Because it is frequently required, the solution of Kepler's equation is of great interest. Consider the equation as written

$$M = E - e \sin E$$

By differentiation

$$dM = (1 - e \cos E) dE$$

Integration between limits 0 and t

$$\int_0^{E_t} dE = \int_0^{M_t} \frac{dM}{1 - e \cos E}$$

An expansion using Fourier series and noting that the period of the function is 2π has coefficients

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} dE = 1$$

$$\begin{aligned} a_m &= \frac{1}{\pi} \int_0^{2\pi} \cos \{m(E - e \sin E)\} dE \\ &= 2 J_m(me) \end{aligned}$$

J_m is a Bessel function of the first kind of order m . For calculation,

$$J_m(me) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{me}{2}\right)^{2n+m}}{n!(n+m)!}$$

So that an explicit formula for eccentric anomaly is given by

$$E = M + 2 \sum_{m=1}^{\infty} \frac{1}{m} J_m(me) \sin(mM)$$

Numerically, at times a few terms can be used to start the solution and then with the first approximation, E_n , continue by a Newton procedure,

$$E_{n+1} = E_n - \frac{E_n - e \sin E_n - M}{1 - e \cos E_n}, \quad n = 1, 2, \dots, p$$

Until E_p no longer varies significantly.

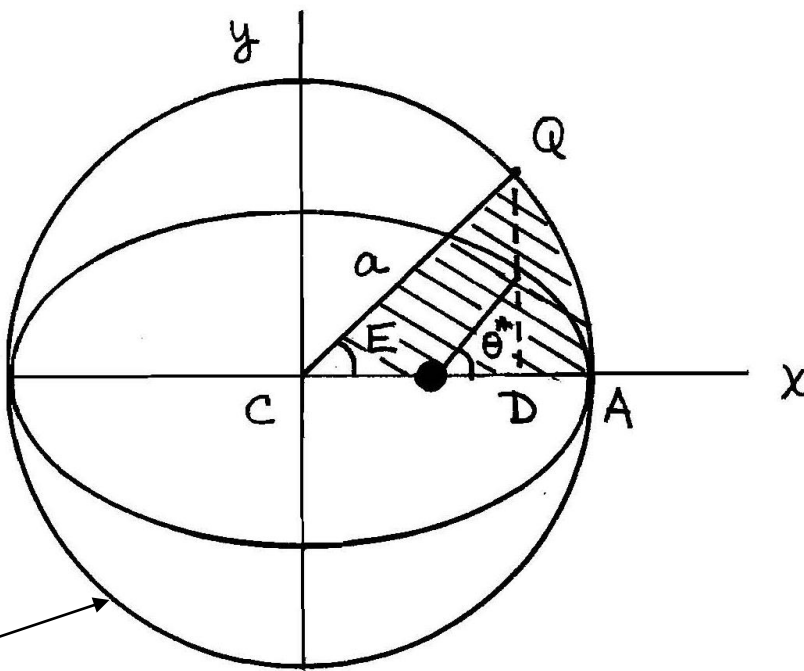
Hyperbolic Anomaly H

Need analog to E for hyperbola ← NOT an angle

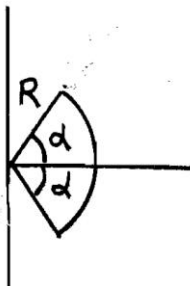
To accomplish, note that E actually represents an area

Recall → the definition for E was based on concept of auxiliary circle

→ now define an area as a sector of that circle



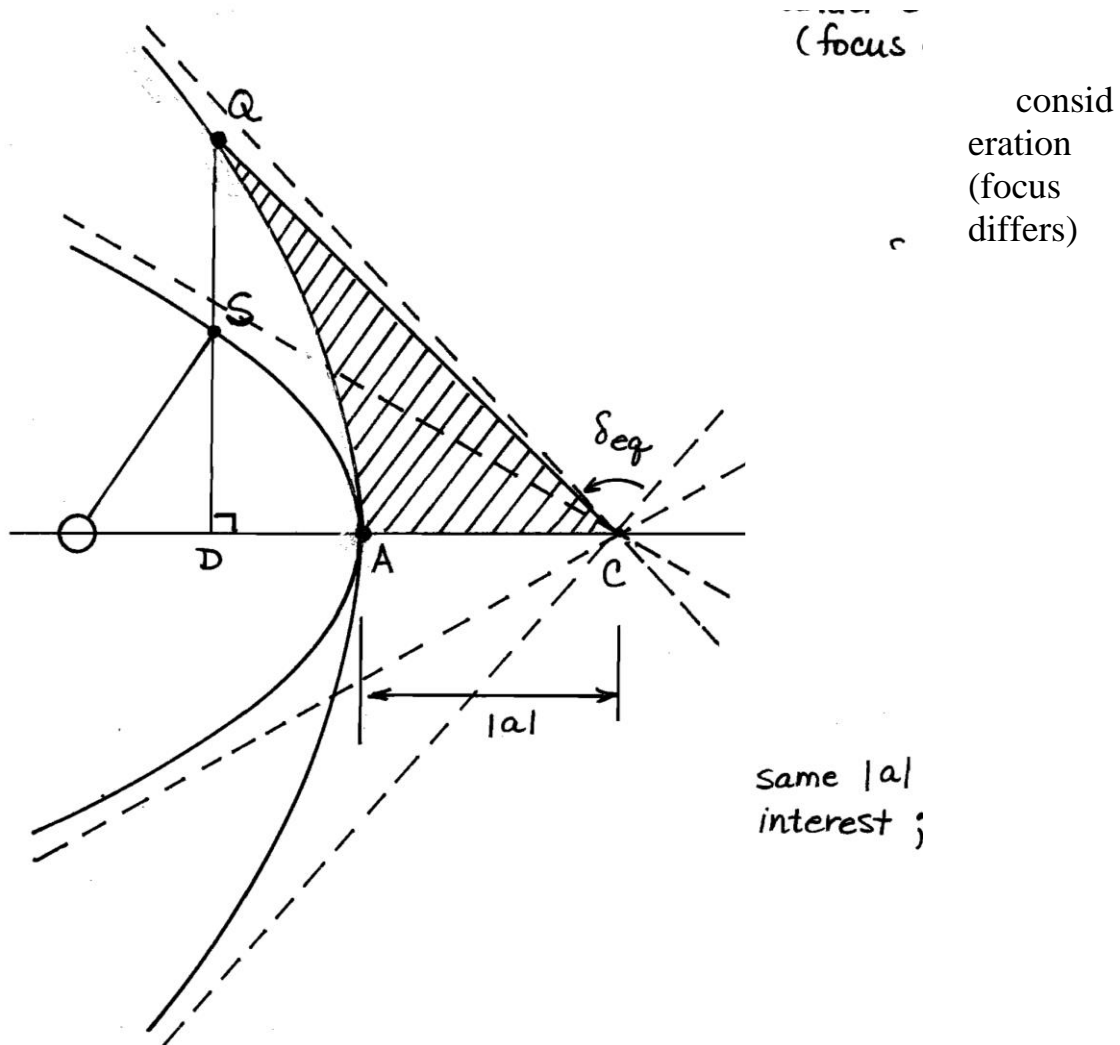
Reference geometric shape is circle



Area of a sector of a circle = $R^2 \alpha$

$$\text{Area QCA} = R^2 \left(\frac{E}{2} \right) = a^2 \frac{E}{2}$$

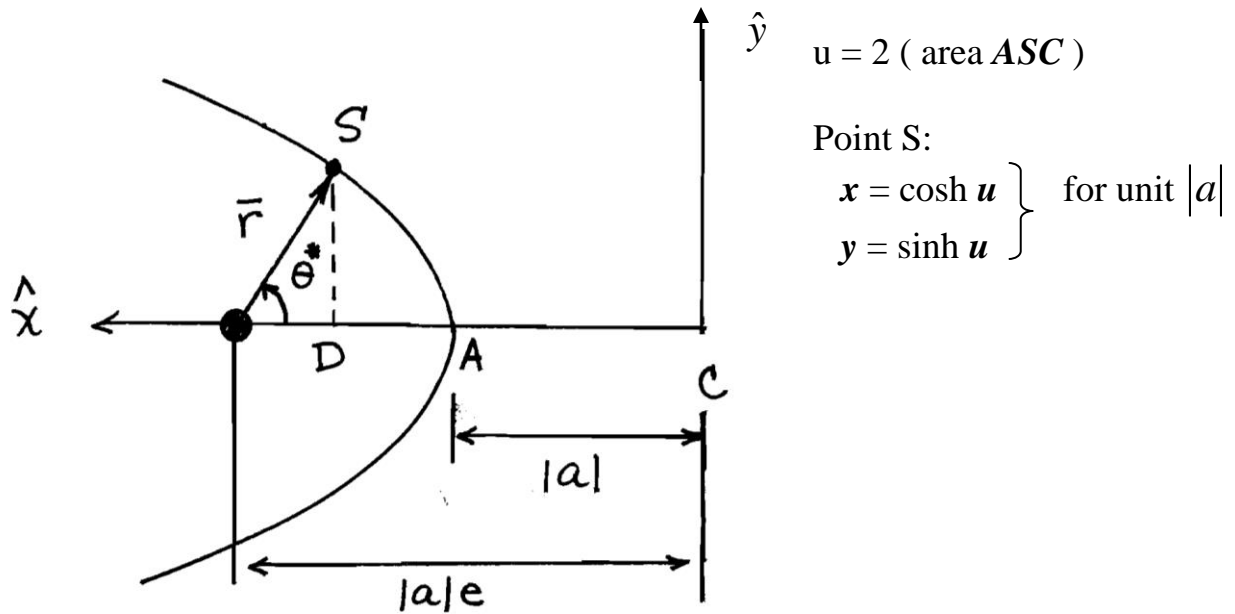
Corresponding situation for hyperbola:
Based on reference geometric shape –



Same $|a|$ as orbit of interest; same pericenter; actually different focus

Define hyperbolic anomaly H in terms of relationship to area CQA

Useful relationship in terms of H



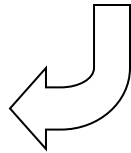
Equation for point on hyperbola measured from the center

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\frac{(CD)^2}{a^2} - \frac{(SD)^2}{b^2} = 1$$

$$CD = |a| \cosh H$$

$$SD = |b| \sinh H$$



Useful to obtain other relationships

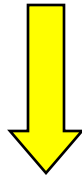
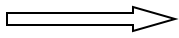
Note: $|a|e - r \cos \theta^* = CD$

$$|a|e - r \cos \theta^* = |a| \cosh H$$

$$|a| e^{-r \left(\frac{p}{re} - \frac{1}{e} \right)} = |a| \cosh H$$

$$|a| e^{-\frac{r|a|(e^2-1)}{re}} + \frac{r}{e} = |a| \cosh H$$

$$\frac{|a|+r}{e} = |a| \cosh H$$



Again, solution iterative!

Parabolic Orbits and Barker's Equation

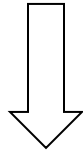
$$r = \frac{p}{1 + \cos \theta^*} = \frac{p}{2} \left(1 + \tan^2 \frac{\theta^*}{2} \right)$$

$$h = r^2 \dot{\theta} = r^2 \frac{d\theta}{dt} = \sqrt{\mu p}$$

$$\sqrt{\frac{\mu}{p^3}} dt = \frac{d\theta}{\left(1 + \cos \theta^*\right)^2} = \frac{1}{4} \left(1 + \tan^2 \frac{\theta^*}{2} \right)^2 d\theta$$

$$4 \sqrt{\frac{\mu}{p^3}} dt = \left(1 + \tan^2 \frac{\theta^*}{2} \right)^2 d\theta$$

$$4 \sqrt{\frac{\mu}{p^3}} dt = \sec^4 \frac{\theta^*}{2} d\theta$$



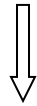
Integrate



← Barker's Eqn

(Barker prepared extensive tables of solutions in the 18th century.)

Define $B = 3 \sqrt{\frac{\mu}{p^3}} (t - t_p)$



$$\tan \frac{\theta^*}{2} = \left(B + \sqrt{1 + B^2} \right)^{\frac{1}{3}} - \left(B + \sqrt{1 + B^2} \right)^{-\frac{1}{3}}$$

(from Jerome Cardan method of solving cubic equations 1545 AD)

Now better methods!