ECE 602: LUMPED LINEAR SYSTEMS

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Separating the reachable part from the non-reachable part for linear time-invariant (LTI) systems

Separating the reachable part from the non-reachable for an LTI system

- **Objective**: Separate the reachable part from the non-reachable for a CT or DT linear time-invariant (LTI) system

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0,$$

  where $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ and $\boldsymbol{B} \in \mathbb{R}^{n \times m}$

Theorem

The pair $(\boldsymbol{A}, \boldsymbol{B})$ is non-reachable if and only if there is a similarity transformation $\boldsymbol{z} = \boldsymbol{T}\boldsymbol{x}$ such that

$$\tilde{\boldsymbol{A}} = \boldsymbol{T}\boldsymbol{A}\boldsymbol{T}^{-1} = \begin{bmatrix} \boldsymbol{A}_1 & \boldsymbol{A}_2 \\ \boldsymbol{O} & \boldsymbol{A}_4 \end{bmatrix}, \quad \tilde{\boldsymbol{B}} = \boldsymbol{T}\boldsymbol{B} = \begin{bmatrix} \boldsymbol{B}_1 \\ \boldsymbol{O} \end{bmatrix},$$

where the pair $(\boldsymbol{A}_1, \boldsymbol{B}_1)$ is reachable, $\boldsymbol{A}_1 \in \mathbb{R}^{r \times r}$, $\boldsymbol{B}_1 \in \mathbb{R}^{r \times m}$, and the rank of the controllability matrix of the pair $(\boldsymbol{A}, \boldsymbol{B})$ equals r.

Separating the reachable part from the nonreachable—Preliminaries

- We first prove necessity ($\Rightarrow$)
- Form the controllability matrix of the pair $(\boldsymbol{A}, \boldsymbol{B})$
- Use row elementary operations to obtain
- Verify that

$$\boldsymbol{T}\boldsymbol{A}^{l}\boldsymbol{B} = \left(\boldsymbol{T}\boldsymbol{A}\boldsymbol{T}^{-1}\right)^{l}\boldsymbol{T}\boldsymbol{B}$$

- Hence

$$\boldsymbol{T}\left[\begin{array}{ccc} \boldsymbol{B} & \cdots & \boldsymbol{A}^{n-1}\boldsymbol{B} \end{array}\right] = \left[\begin{array}{ccc} \boldsymbol{T}\boldsymbol{B} & \cdots & \left(\boldsymbol{T}\boldsymbol{A}\boldsymbol{T}^{-1}\right)^{n-1}(\boldsymbol{T}\boldsymbol{B}) \end{array}\right]$$

Separating the reachable part from the non-reachable—Constructive proof

- We obtain

$$
\begin{aligned}
\boldsymbol{T} \begin{bmatrix} \boldsymbol{B} & \cdots & \boldsymbol{A}^{n-1}\boldsymbol{B} \end{bmatrix} &= \begin{bmatrix} x & x & x & \cdots & x & x & x \\ 0 & x & x & \cdots & x & x & x \\ \vdots & & & \vdots & & & \vdots \\ 0 & 0 & 0 & \cdots & x & x & x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & & & \vdots & & & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} \boldsymbol{B}_1 & \boldsymbol{A}_1\boldsymbol{B}_1 & \cdots & \boldsymbol{A}_1^{n-1}\boldsymbol{B}_1 \\ \boldsymbol{O} & \boldsymbol{O} & \cdots & \boldsymbol{O} \end{bmatrix},
\end{aligned}
$$

where the symbol x denotes a "don't care", that is, an unspecified scalar

Separating the reachable part from the non-reachable—Manipulations

- $$\tilde{A} = TAT^{-1} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, \quad \tilde{B} = TB = \begin{bmatrix} B_1 \\ O \end{bmatrix},$$

  where $A_3 = O$. It is clear that TB must have the above form
- If A_3 were not a zero matrix, then we would have

$$\tilde{A}\tilde{B} = \begin{bmatrix} A_1 B_1 \\ A_3 B_1 \end{bmatrix}.$$

- We conclude that $A_3 B_1 = O$, and therefore

$$\tilde{A}\tilde{B} = \begin{bmatrix} A_1 B_1 \\ O \end{bmatrix}.$$

More Analysis

- Next, we have
$$\tilde{\boldsymbol{A}}^2 \tilde{\boldsymbol{B}} = \begin{bmatrix} \boldsymbol{A}_1^2 \boldsymbol{B}_1 \\ \boldsymbol{A}_3 \boldsymbol{A}_1 \boldsymbol{B}_1 \end{bmatrix}.$$
- Hence
$$\boldsymbol{A}_3 \boldsymbol{A}_1 \boldsymbol{B}_1 = \boldsymbol{O},$$
and thus
$$\tilde{\boldsymbol{A}}^2 \tilde{\boldsymbol{B}} = \begin{bmatrix} \boldsymbol{A}_1^2 \boldsymbol{B}_1 \\ \boldsymbol{O} \end{bmatrix}.$$
- Continuing in this manner, we conclude that $\boldsymbol{A}_3 \boldsymbol{A}_1^{n-1} \boldsymbol{B}_1 = \boldsymbol{O}$
- It follows then that $\boldsymbol{A}_3 \begin{bmatrix} \boldsymbol{B}_1 & \boldsymbol{A}_1 \boldsymbol{B}_1 & \cdots & \boldsymbol{A}_1^{n-1} \boldsymbol{B}_1 \end{bmatrix} = \boldsymbol{O}$.
- Because $\operatorname{rank} \begin{bmatrix} \boldsymbol{B}_1 & \boldsymbol{A}_1 \boldsymbol{B}_1 & \cdots & \boldsymbol{A}_1^{n-1} \boldsymbol{B}_1 \end{bmatrix} = r$, that is, the controllability matrix of the pair $(\boldsymbol{A}_1, \boldsymbol{B}_1)$ is of full rank, we have to have $\boldsymbol{A}_3 = \boldsymbol{O}$, and the proof of necessity is complete.

Sufficiency $(\Leftarrow)$

- Note that if there is a similarity transformation such that

$$\tilde{A} = TAT^{-1} = \begin{bmatrix} A_1 & A_2 \\ O & A_4 \end{bmatrix}, \quad \tilde{B} = TB = \begin{bmatrix} B_1 \\ O \end{bmatrix},$$

then the controllability matrix of the pair $\left(\tilde{A}, \tilde{B}\right)$ has the form

$$\begin{bmatrix} B_1 & A_1 B_1 & A_1^2 B_1 & \cdots & A_1^{n-1} B_1 \\ O & O & O & \cdots & O \end{bmatrix}.$$

- Hence, the pair $\left(\tilde{A}, \tilde{B}\right)$ is non-reachable
- The similarity transformation preserves the reachability property
- Therefore, the pair (A, B) is non-reachable.

QED

Example

- For the system model,

$$\dot{\boldsymbol{x}} = \begin{bmatrix} -1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 1 & 1 & -1 & 2 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \boldsymbol{u},$$

  construct a state variable transformation so that in the new coordinates the reachable part is separated and distinct from the non-reachable part.

- Form the controllability matrix of the pair $(\boldsymbol{A}, \boldsymbol{B})$,

$$\begin{bmatrix} \boldsymbol{B} & \cdots & \boldsymbol{A}^3\boldsymbol{B} \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 & 0 & 3 & 4 & -1 & 4 \\ 0 & 0 & 1 & 1 & 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 3 & 2 & 7 & 7 & 19 \end{bmatrix}$$

Some manipulations

- Use the MATLAB function `CO=ctrb(A,B)` to compute the controllability matrix
- Use `[Q,R]=qr(CO)` to obtain

$$\boldsymbol{Q} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \ \boldsymbol{R} = \begin{bmatrix} -1 & -1 & 1 & 0 & -3 & -4 & 1 & -4 \\ 0 & -1 & -1 & -3 & -2 & -7 & -7 & -19 \\ 0 & 0 & -1 & -1 & 0 & -1 & -3 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

  where the matrices $\boldsymbol{Q}$ and $\boldsymbol{R}$ satisfy $\mathrm{CO} = \boldsymbol{Q}\boldsymbol{R}$
- $\boldsymbol{R}$ is upper row triangular such that $\operatorname{rank} \boldsymbol{R} = \operatorname{rank} \mathrm{CO}$
- Premultiply the controllability matrix, CO, by $\boldsymbol{Q}^{-1}$ to reduce this matrix to an upper row triangular matrix $\boldsymbol{R}$ because $\boldsymbol{Q}^{-1}\mathrm{CO} = \boldsymbol{R}$

Constructing the similarity transformation separating the reachable part from the non-reachable part

- Let $\boldsymbol{z} = \boldsymbol{Q}^{-1}\boldsymbol{x}$ be the state variable transformation
- In the new coordinates the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ take the form

$$\boldsymbol{Q}^{-1}\boldsymbol{A}\boldsymbol{Q} = \left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 1 & 2 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 4 \end{array}\right] \quad \text{and} \quad \boldsymbol{Q}^{-1}\boldsymbol{B} = \left[\begin{array}{cc} -1 & -1 \\ 0 & -1 \\ 0 & 0 \\ \hline 0 & 0 \end{array}\right]$$

- The dimension of the non-reachable part is 1