ECE 68000: MODERN AUTOMATIC CONTROL

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The Pontryagin's Minimum Principle (PMP)

Optimal Control With Constraints on Inputs

- Minimize the performance index

$$J = \Phi(\boldsymbol{x}(t_f)) + \int_{t_0}^{t_f} F(\boldsymbol{x}(t), \boldsymbol{u}(t))dt$$

subject to

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}), \quad \boldsymbol{x}(t_0) = \boldsymbol{x}_0, \quad \text{and} \quad \boldsymbol{x}(t_f) = \boldsymbol{x}_f$$

- To proceed we define the Hamiltonian function $H(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p})$ as

$$H(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p}) = H = F + \boldsymbol{p}^\top \boldsymbol{f}$$

where the co-state vector $\boldsymbol{p}$ will be determined in the analysis to follow

Analysis for fixed final time t_f

- "Adjoin" to J additional terms that sum up to zero
- Note that because the state trajectories must satisfy the equation, $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$, we have

$$\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) - \dot{\boldsymbol{x}} = \mathbf{0}$$

- Introduce the modified objective performance index,

$$\tilde{J} = J + \int_{t_0}^{t_f} \boldsymbol{p}(t)^\top (\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) - \dot{\boldsymbol{x}})\, dt$$

- Any state trajectory satisfies $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$
- For any choice of $\boldsymbol{p}(t)$ the value of $\tilde{J}$ is the same as that of J
- Express $\tilde{J}$ as

$$\begin{aligned}\tilde{J} \quad &= \quad \Phi(\boldsymbol{x}(t_f)) + \int_{t_0}^{t_f} F(\boldsymbol{x}(t), \boldsymbol{u}(t))\, dt \\ &\quad + \int_{t_0}^{t_f} \boldsymbol{p}(t)^\top (\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) - \dot{\boldsymbol{x}})\, dt\end{aligned}$$

Modified performance index manipulations

- We have

$$\tilde{J} = \Phi(\boldsymbol{x}(t_f)) + \int_{t_0}^{t_f} \left(H(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p}) - \boldsymbol{p}^\top \dot{\boldsymbol{x}}\right) dt$$

- Let $\boldsymbol{u}(t)$ be a nominal control strategy; it determines a corresponding state trajectory $\boldsymbol{x}(t)$
- If we apply another control strategy, say $\boldsymbol{v}(t)$, that is “close” to $\boldsymbol{u}(t)$, then $\boldsymbol{v}(t)$ will produce a state trajectory close to the nominal trajectory
- This new state trajectory is just a perturbed version of $\boldsymbol{x}(t)$ and we represent it as

$$\boldsymbol{x}(t) + \delta\boldsymbol{x}(t)$$

Change in modified performance index

- The change in the state trajectory, $\boldsymbol{x}(t) + \delta\boldsymbol{x}(t)$, yields a corresponding change in the modified performance index
- Represent this change, the variation $\delta\tilde{J}$ of $\tilde{J}$, as

$$
\begin{aligned}
\delta\tilde{J} &= \Phi(\boldsymbol{x}(t_f) + \delta\boldsymbol{x}(t_f)) - \Phi(\boldsymbol{x}(t_f)) \\
&\quad + \int_{t_0}^{t_f} \left(H(\boldsymbol{x} + \delta\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{p}) - H(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p}) - \boldsymbol{p}^\top \delta\dot{\boldsymbol{x}}\right) dt
\end{aligned}
$$

- Integrate by parts

$$
\int_{t_0}^{t_f} \boldsymbol{p}^\top \delta\dot{\boldsymbol{x}}\, dt = \boldsymbol{p}(t_f)^\top \delta\boldsymbol{x}(t_f) - \boldsymbol{p}(t_0)^\top \delta\boldsymbol{x}(t_0) - \int_{t_0}^{t_f} \dot{\boldsymbol{p}}^\top \delta\boldsymbol{x}\, dt
$$

- Note that $\delta\boldsymbol{x}(t_0) = \boldsymbol{0}$ because a change in the control strategy does not change the initial state

Computing the variation of $\tilde{J}$

- Hence

$$\begin{aligned}\delta\tilde{J} &= \Phi(\boldsymbol{x}(t_f)+\delta\boldsymbol{x}(t_f)) - \Phi(\boldsymbol{x}(t_f)) - \boldsymbol{p}(t_f)^\top\delta\boldsymbol{x}(t_f)\\ &\quad + \int_{t_0}^{t_f}\left(H(\boldsymbol{x}+\delta\boldsymbol{x},\boldsymbol{v},\boldsymbol{p}) - H(\boldsymbol{x},\boldsymbol{u},\boldsymbol{p}) + \dot{\boldsymbol{p}}^\top\delta\boldsymbol{x}\right)dt\end{aligned}$$

- Replace $\Phi(\boldsymbol{x}(t_f)+\delta\boldsymbol{x}(t_f)) - \Phi(\boldsymbol{x}(t_f))$ with its first-order approximation, and add and subtract the term $H(\boldsymbol{x},\boldsymbol{v},\boldsymbol{p})$ under the integral to obtain

$$\begin{aligned}\delta\tilde{J} &= \left(\nabla_{\boldsymbol{x}}\Phi\big|_{t=t_f} - \boldsymbol{p}(t_f)\right)^\top\delta\boldsymbol{x}(t_f)\\ &\quad + \int_{t_0}^{t_f}\Big(H(\boldsymbol{x}+\delta\boldsymbol{x},\boldsymbol{v},\boldsymbol{p}) - H(\boldsymbol{x},\boldsymbol{v},\boldsymbol{p}) + H(\boldsymbol{x},\boldsymbol{v},\boldsymbol{p})\\ &\qquad - H(\boldsymbol{x},\boldsymbol{u},\boldsymbol{p}) + \dot{\boldsymbol{p}}^\top\delta\boldsymbol{x}\Big)\,dt\end{aligned}$$

Computing the variation of $\tilde{J}$—contd.

- Replace $(H(\boldsymbol{x}+\delta\boldsymbol{x},\boldsymbol{v},\boldsymbol{p})-H(\boldsymbol{x},\boldsymbol{v},\boldsymbol{p}))$ with its first-order approximation,

$$H(\boldsymbol{x}+\delta\boldsymbol{x},\boldsymbol{v},\boldsymbol{p})-H(\boldsymbol{x},\boldsymbol{v},\boldsymbol{p})=\frac{\partial H}{\partial \boldsymbol{x}}\delta\boldsymbol{x}$$

- Substituting gives

$$\begin{aligned}\delta\tilde{J} &= \left(\nabla_{\boldsymbol{x}}\Phi|_{t=t_f}-\boldsymbol{p}(t_f)\right)^\top \delta\boldsymbol{x}(t_f)\\ &\quad+\int_{t_0}^{t_f}\left(\left(\frac{\partial H}{\partial \boldsymbol{x}}+\dot{\boldsymbol{p}}^\top\right)\delta\boldsymbol{x}+H(\boldsymbol{x},\boldsymbol{v},\boldsymbol{p})-H(\boldsymbol{x},\boldsymbol{u},\boldsymbol{p})\right)\,dt\end{aligned}$$

Apply the fundamental lemma of calculus of variations

- By the fundamental lemma of calculus of variation necessary condition for the trajectory to be optimal $\delta\tilde{J} = 0$
- We have to have

$$\frac{\partial H}{\partial \boldsymbol{x}} + \dot{\boldsymbol{p}}^\top = \boldsymbol{0}^\top$$

  with the final condition

$$\boldsymbol{p}(t_f) = \nabla_{\boldsymbol{x}} \Phi\big|_{t=t_f}$$

- We are left with

$$\delta\tilde{J} = \int_{t_0}^{t_f} \left(H(\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{p}) - H(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p}) \right) \, dt$$

- Represent the control strategy, $\boldsymbol{v} = \boldsymbol{v}(t)$, that is “close” to $\boldsymbol{u} = \boldsymbol{u}(t)$, as $\boldsymbol{v} = \boldsymbol{u} + \delta\boldsymbol{u}$

Apply the fundamental lemma of calculus of variations—contd.

- Replace

$$H(\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{p}) - H(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p}) = H(\boldsymbol{x}, \boldsymbol{u} + \delta\boldsymbol{u}, \boldsymbol{p}) - H(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p})$$

  with its first-order approximation

$$H(\boldsymbol{x}, \boldsymbol{u} + \delta\boldsymbol{u}, \boldsymbol{p}) - H(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p}) = \frac{\partial H}{\partial \boldsymbol{u}}\delta\boldsymbol{u}$$

- Substitute it into $\delta\tilde{J}$

$$\begin{aligned} \delta\tilde{J} &= \int_{t_0}^{t_f} \left(H(\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{p}) - H(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p})\right) dt \\ &= \int_{t_0}^{t_f} \frac{\partial H}{\partial \boldsymbol{u}}\delta\boldsymbol{u}\, dt \end{aligned}$$

Finishing calculating $\delta\tilde{J}$

- By the fundamental lemma of calculus of variation necessary condition for the trajectory to be optimal is $\delta\tilde{J} = 0$
- Hence
$$\frac{\partial H}{\partial \boldsymbol{u}} = \mathbf{0}^\top$$
- That is, since $\boldsymbol{u}$ is optimal it must satisfy the first-order necessary condition to be a minimizer of H

Pontryagin's minimum principle (PMP)

Theorem

Necessary conditions for $\boldsymbol{u} \in U$ to minimize J are:

$$\dot{\boldsymbol{p}} = -\left(\frac{\partial H}{\partial \boldsymbol{x}}\right)^{\top},$$

where $H = H(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p}) = F(\boldsymbol{x}, \boldsymbol{u}) + \boldsymbol{p}^{\top} \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$,

$$H(\boldsymbol{x}^*, \boldsymbol{u}^*, \boldsymbol{p}^*) = \min_{\boldsymbol{u} \in U} H(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p}).$$

If the final state, $\boldsymbol{x}(t_f)$, is free, then in addition to the above conditions it is required that the following end-point condition is satisfied,

$$\boldsymbol{p}(t_f) = \nabla_{\boldsymbol{x}} \Phi\big|_{t=t_f}$$

Costate equation

- The equation,

$$\dot{\boldsymbol{p}} = -\left(\frac{\partial H}{\partial \boldsymbol{x}}\right)^{\top} = -\nabla_{\boldsymbol{x}} H$$

is called in the literature the *adjoint* or *costate* equation