Final Exam
Solutions

1. (**20 pts**) Consider the following nonlinear dynamical system model:

$$\begin{aligned}\dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{G}(\boldsymbol{x})u \\ &= \begin{bmatrix} \frac{4}{1+x_2} \\ x_1 x_2 \end{bmatrix} + \begin{bmatrix} x_2 \\ 1 \end{bmatrix} u.\end{aligned}$$

(i) (**10 pts**) Find the equilibrium states corresponding to the constant input $u = -2$.

(ii) (**10 pts**) Derive Taylor linearized state-space models for small deviations about the obtained equilibria.

(i) We compute the equilibrium state corresponding to the constant input $u = -2$ by solving the following set of the algebraic equations, $\dot{\boldsymbol{x}} = \boldsymbol{0}$, that is,

$$\begin{aligned} 0 &= \frac{4}{1+x_2} - 2x_2 \\ 0 &= x_1 x_2 - 2. \end{aligned}$$

Manipulating the first of the above equations, we obtain

$$(x_2 - 1)(x_2 + 2) = 0$$

Hence, we have two equilibrium states corresponding to the constant input $u = -2$;

$$\boldsymbol{x}_e^{(1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{x}_e^{(2)} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}.$$

(ii) We denote the modeling equations as

$$\dot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x}, \boldsymbol{u}).$$

The linearized state-space model for small deviations about the equilibrium has the form:

$$\frac{d}{dt}\delta\boldsymbol{x} = \boldsymbol{A}\delta\boldsymbol{x} + \boldsymbol{B}\boldsymbol{\delta}u,$$

where

$$\boldsymbol{A} = \left.\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{x}}\right|_{\substack{\boldsymbol{x}=\boldsymbol{x}_e \\ u=u_e}} = \left.\begin{bmatrix} 0 & -\frac{4}{(1+x_2)^2} + u \\ x_2 & x_1 \end{bmatrix}\right|_{\substack{\boldsymbol{x}=\boldsymbol{x}_e \\ \boldsymbol{u}=\boldsymbol{u}_e}} \quad \text{and} \quad \boldsymbol{b} = \left.\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{u}}\right|_{\substack{\boldsymbol{x}=\boldsymbol{x}_e \\ u=u_e}} = \boldsymbol{G}(\boldsymbol{x}_e)$$

The Taylor linearized system about $\boldsymbol{x}_e^{(1)}$ has the form,

$$\frac{d}{dt}\delta\boldsymbol{x} = \begin{bmatrix} 0 & -3 \\ 1 & 2 \end{bmatrix} \delta\boldsymbol{x} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \delta u$$

The Taylor linearized system about $\boldsymbol{x}_e^{(2)}$ has the form,

$$\frac{d}{dt}\delta\boldsymbol{x} = \begin{bmatrix} 0 & -6 \\ -2 & -1 \end{bmatrix} \delta\boldsymbol{x} + \begin{bmatrix} -2 \\ 1 \end{bmatrix} \delta u$$

$\diamond$

2. **(20 pts)**

(i) (15 pts) Find the linear state-feedback control law that minimizes

$$J = \int_0^\infty \left(\frac{1}{4}x(t)^2 + 9u(t)^2\right) dt$$

subject to

$$\dot{x}(t) = \sqrt{2}x(t) + 3u(t), \quad x(0) = 1.$$

(ii) (5 pts) Find the value of the performance index for the closed-loop system driven by the optimal controller.

(i) (15 pts) To find the optimal controller for this problem, we first solve the algebraic Riccati equation (ARE),

$$\boldsymbol{A}^\top \boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} + \boldsymbol{Q} - \boldsymbol{P}\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^\top \boldsymbol{P} = \boldsymbol{O},$$

Then, we use its solution, $\boldsymbol{P} = \boldsymbol{P}^\top \succ 0$, to construct the optimal controller,

$$u^* = -\boldsymbol{K}\boldsymbol{x} = -\boldsymbol{R}^{-1}\boldsymbol{B}^\top \boldsymbol{P}\boldsymbol{x}.$$

The matrices that are needed when solving the ARE are

$$\boldsymbol{A} = \sqrt{2}, \quad \boldsymbol{B} = 3, \quad \boldsymbol{Q} = \frac{1}{4}, \quad \boldsymbol{R} = 9.$$

Substituting the above matrices into the ARE gives

$$2\sqrt{2}\boldsymbol{P} + \frac{1}{4} - \boldsymbol{P}^2 = 0,$$

equivalently,

$$\boldsymbol{P}^2 - 2\sqrt{2}\boldsymbol{P} - \frac{1}{4} = 0$$

Solving the above gives

$$\boldsymbol{P} = \sqrt{2} \pm \frac{3}{2}.$$

We take positive definite $\boldsymbol{P}$. Hence, the optimal controller is

$$\boxed{u^* = -\boldsymbol{R}^{-1}\boldsymbol{B}^\top \boldsymbol{P}\boldsymbol{x} = -\frac{1}{3}\left(\sqrt{2} + \frac{3}{2}\right)x = -\frac{2\sqrt{2}+3}{6}x}$$

(ii) **(5 pts)** The value of the performance index on the trajectories of the closed-loop system driven by the optimal controller is

$$J = \boldsymbol{x}(0)^\top \boldsymbol{P}\boldsymbol{x}(0) = \left(\sqrt{2} + \frac{3}{2}\right) = \frac{2\sqrt{2}+3}{2}.$$

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3. (**20 pts**) Construct $u = u(t)$ that minimizes

$$J(u) = \frac{1}{2}\int_0^1 u(t)^2 dt$$

subject to

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\boldsymbol{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\boldsymbol{u}, \quad \boldsymbol{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{x}(1) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

We form the Hamiltonian,

$$H(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p}) = \frac{1}{2}\boldsymbol{u}^\top \boldsymbol{u} + \boldsymbol{p}^\top(\boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}).$$

Because there are no constraints on the control, we use the first-order-necessary condition for unconstrained minimum to obtain a candidate control law that minimizes the Hamiltonian,

$$\boldsymbol{0}^\top = \frac{\partial H}{\partial \boldsymbol{u}} = \boldsymbol{u}^\top + \boldsymbol{p}^\top \boldsymbol{B}.$$

Hence,

$$\boldsymbol{u}^* = -\boldsymbol{B}^\top \boldsymbol{p},$$

where $\boldsymbol{p}$ is the co-state vector obtained by solving the co-state equation

$$\dot{\boldsymbol{p}} = -\boldsymbol{A}^\top \boldsymbol{p}.$$

The solution to the above equation has the form,

$$\boldsymbol{p}(t) = e^{-\boldsymbol{A}^\top t}\boldsymbol{p}(0),$$

where

$$e^{-\boldsymbol{A}^\top t} = \begin{bmatrix} 1 & 0 \\ -t & 1 \end{bmatrix}.$$

Hence,

$$\begin{aligned}
\boldsymbol{u}^* &= -\boldsymbol{B}^\top e^{-\boldsymbol{A}^\top t}\boldsymbol{p}(0) \\
&= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -t & 1 \end{bmatrix} \boldsymbol{p}(0) \\
&= \begin{bmatrix} t & -1 \end{bmatrix} \boldsymbol{p}(0).
\end{aligned}$$

Recall the solution to $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$,

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}(0) + \int_0^t e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}u(\tau)d\tau.$$

Employing the above formula and taking into account the given boundary conditions, we obtain

$$\begin{aligned}
\boldsymbol{x}(1) &= e^{\boldsymbol{A}}\boldsymbol{x}(0) + \int_0^1 e^{\boldsymbol{A}(1-\tau)}\boldsymbol{B}u(\tau)d\tau \\
&= \int_0^1 \begin{bmatrix} 1-\tau \\ 1 \end{bmatrix} \begin{bmatrix} \tau & -1 \end{bmatrix} d\tau \boldsymbol{p}(0) \\
&= \int_0^1 \begin{bmatrix} \tau-\tau^2 & 1-\tau \\ \tau & -1 \end{bmatrix} d\tau \boldsymbol{p}(0) \\
&= \left. \begin{bmatrix} \frac{1}{2}\tau^2 - \frac{1}{3}\tau^3 & -\tau + \frac{1}{2}\tau^2 \\ \frac{1}{2}\tau^2 & -\tau \end{bmatrix} \right|_0^1 \boldsymbol{p}(0) \\
&= \begin{bmatrix} \frac{1}{6} & -\frac{1}{2} \\ \frac{1}{2} & -1 \end{bmatrix} \boldsymbol{p}(0).
\end{aligned}$$

Hence,

$$
\begin{aligned}
\boldsymbol{p}(0) &= \begin{bmatrix} \frac{1}{6} & -\frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}^{-1} \boldsymbol{x}(1) \\
&= \frac{12}{5} \begin{bmatrix} 1 & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{6} \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \\
&= \begin{bmatrix} 6 \\ 0 \end{bmatrix}
\end{aligned}
$$

and therefore

$$\boxed{u^* = 6t}$$

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4. **(20 pts)** Use dynamic programming to find $u[0]$ and $u[1]$ that minimize the performance index,

$$J = (x[2] - 1)^2 + 2 \sum_{k=0}^{1} u[k]^2$$

subject to

$$x[k+1] = bu[k], \quad x[0] = 10,$$

where $b \neq 0$. Note that there are no constraints on $u[k]$. Also, find the value of J^*.

Let $J^*(x[k])$ denote the minimal cost of transfer of $x[k]$ at time k to some final state $x[2]$. Then,

$$J^*(x[2]) = (x[2] - 1)^2 = (bu[1] - 1)^2,$$

and

$$J^*(x[1]) = \min_{u[1]} \left(2u[1]^2 + J^*(x[2]) \right) = \min_{u[1]} \left(2u[1]^2 + (bu[1] - 1)^2 \right).$$

There are no constraints on $u[1]$ so we can apply the first-derivative test. Differentiating the above with respect to $u[1]$ and equating the result to zero gives

$$\frac{\partial J(x[1])}{\partial u[1]} = 4u[1] + 2b(bu[1] - 1) = 0.$$

Hence,

$$u^*[1] = \frac{b}{2 + b^2}.$$

Substituting the above into $J^*(x[1])$ yields

$$J^*(x[1]) = \frac{2}{(2+b^2)^2}.$$

Continuing, we obtain

$$J^*(x[0]) = \min_{u[0]} \left(2u[0]^2 + J^*(x[1])\right) = \min_{u[0]} \left(2u[0]^2 + \frac{2}{(2+b^2)^2}\right).$$

Hence

$$u^*[0] = 0.$$

To obtain the optimal cost, we find

$$x^*[2] = bu^*[1] = \frac{b^2}{2+b^2}.$$

The optimal cost is

$$\begin{aligned} J^* &= (x^*[2]-1)^2 + 2\sum_{k=0}^{1} u^*[k]^2 \\ &= (bu^*[1]-1)^2 + 2u^*[0]^2 + 2u^*[1]^2 \\ &= \frac{2}{2+b^2}. \end{aligned}$$

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5. **(20 pts)** Minimize

$$J_0 = 3\sum_{k=0}^{\infty} \|\boldsymbol{x}[k]\|_2^2$$

subject to

$$\boldsymbol{x}[k+1] = \begin{bmatrix} -0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \boldsymbol{x}[k], \quad \boldsymbol{x}[0] = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

We have

$$J_0 = \boldsymbol{x}[0]^\top \boldsymbol{P}\boldsymbol{x}[0],$$

where $\boldsymbol{P} = \boldsymbol{P}^\top \succ 0$ is the solution to the discrete Lyapunov equation

$$\boldsymbol{A}^\top \boldsymbol{P}\boldsymbol{A} - \boldsymbol{P} = -\boldsymbol{Q} = -3\boldsymbol{I}_2.$$

We have

$$\boldsymbol{P} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}.$$

Hence,

$$J_0 = 4$$

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6. (**20 pts**) Given the following model of a dynamical system:

$$\dot{x} = 2u_1 + 2u_2, \qquad x(0) = 3,$$

and the associated performance index

$$J = \int_0^\infty \left(x^2 + ru_1^2 + ru_2^2\right) dt,$$

where $r > 0$ is a parameter.

(i) (**10 pts**) Find the solution to the algebraic Riccati equation corresponding to the linear state-feedback optimal controller.

(ii) (**5 pts**) Write the equation of the closed-loop system driven by the optimal controller.

(iii) (**5 pts**) Find the value of J for the optimal closed-loop system.

(i) We have

$$\boldsymbol{A} = 0, \quad \boldsymbol{B} = \begin{bmatrix} 2 & 2 \end{bmatrix}, \quad \boldsymbol{Q} = 1, \quad \boldsymbol{R} = r\boldsymbol{I}_2.$$

The algebraic Riccati equation (ARE) for this problem has the form

$$\begin{aligned} \mathbf{0} &= \boldsymbol{A}^\top \boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} + \boldsymbol{Q} - \boldsymbol{P}\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^\top \boldsymbol{P} \\ &= 1 - \frac{8}{r}p^2. \end{aligned}$$

Hence, the solution to the ARE is

$$\boldsymbol{P} = \sqrt{\frac{r}{8}}.$$

(ii) The optimal controller has the form

$$\begin{aligned} \boldsymbol{u} &= -\boldsymbol{R}^{-1}\boldsymbol{B}^\top \boldsymbol{P}\boldsymbol{x} \\ &= -\frac{1}{\sqrt{2r}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} x. \end{aligned}$$

Hence the closed-loop optimal system is described by

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 2 & 2 \end{bmatrix} \boldsymbol{u} \\ &= -\frac{4}{\sqrt{2r}} x. \end{aligned}$$

(iii)

$$\begin{aligned} \min J(\boldsymbol{u}) &= \int_0^\infty \left(x^2 + r u_1^2 + r u_2^2\right) dt \\ &= 9\int_0^\infty \exp\left(-\frac{8t}{\sqrt{2r}}\right)(1+1)\, dt \\ &= 9\sqrt{\frac{r}{8}}. \end{aligned}$$

Note that

$$\min J(\boldsymbol{u}) = \boldsymbol{x}(0)^\top \boldsymbol{P}\boldsymbol{x}(0).$$

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7. (**20 pts**) Given the following model of a dynamical system:

$$\begin{aligned} \dot{x}_1 &= x_2 - 2 \\ \dot{x}_2 &= u, \end{aligned}$$

where

$$|u| \leq 1.$$

The performance index to be minimized is

$$J = \int_0^{t_f} dt.$$

Find the state-feedback control law $u = u(x_1, x_2)$ that minimizes J and drives the system from a given initial condition $\boldsymbol{x}(0) = [x_1(0), x_2(0)]^\top$ to the final state $\boldsymbol{x}(t_f) = \mathbf{0}$. Proceed as indicated below.

(i) (**5 pts**) Derive the equations of the optimal trajectories.

(ii) (**5 pts**) Derive the equation of the switching curve.

(iii) (**10 pts**) Write the expression for the optimal state-feedback controller.

(i) The Hamiltonian is

$$H = 1 + \psi_1 (x_2 - 2) + \psi_2 u.$$

By the Minimum Principle, the optimal control has the form

$$u = \text{sign}(-\psi_2).$$

When $u = 1$, then

$$\begin{aligned} x_2 &= t + c_1 \\ x_1 &= \frac{1}{2}(t + c_1 - 2)^2 + c_2 \\ &= \frac{1}{2}(x_2 - 2)^2 + c_2. \end{aligned}$$

Similarly, when $u = -1$, then

$$\begin{aligned} x_2 &= -t + c_3 \\ x_1 &= -\frac{1}{2}(-t + c_3 - 2)^2 + c_4 \\ &= -\frac{1}{2}(x_2 - 2)^2 + c_4. \end{aligned}$$

Hence the optimal trajectories have the form of parabolas.

(ii) The switching curve γ is obtained by combining parabolas passing through the origin of the state plane. For $u = 1$ the parabola that passes through the origin is

$$x_1 = \frac{1}{2}(x_2 - 2)^2 - 2,$$

while the parabola that passes through the origin corresponding to $u = -1$ is

$$x_1 = -\frac{1}{2}(x_2 - 2)^2 + 2.$$

Thus the switching curve γ can be described as

$$x_1 = (\text{sign}(x_2)) \left(-\frac{1}{2}(x_2 - 2)^2 + 2 \right).$$

(iii) The optimal state-feedback control law is

$$u(x_1, x_2) = -\text{sign} \left(x_1 - (\text{sign} x_2) \left(-\frac{1}{2}(x_2 - 2)^2 + 2 \right) \right).$$

$\diamond$

8. **(20 pts)** Let $\boldsymbol{A} \in \mathbb{R}^{m\times m}$ and $\boldsymbol{B} \in \mathbb{R}^{n\times n}$. Express $\det(\boldsymbol{A}\otimes\boldsymbol{B})$ in terms of $\det \boldsymbol{A}$ and $\det \boldsymbol{B}$, where the symbol $\otimes$ denotes the Kronecker product. (You may find the identities $(\boldsymbol{A}\otimes\boldsymbol{C})(\boldsymbol{D}\otimes\boldsymbol{B}) = \boldsymbol{AD}\otimes\boldsymbol{CB}$ and $\det(\boldsymbol{A}\otimes\boldsymbol{I}_r) = (\det\boldsymbol{A})^r$ to be useful in your derivation.) Then employ the obtained formula to evaluate $\det(\boldsymbol{A}\otimes\boldsymbol{B})$ for the case when

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 2 & -2 \end{bmatrix} \quad \text{and} \quad \boldsymbol{B} = \begin{bmatrix} 2 & 6 & -8 \\ 0 & -1 & 0 \\ 0 & 2 & 3 \end{bmatrix}.$$

Consider the identity, $(\boldsymbol{A}\otimes\boldsymbol{C})(\boldsymbol{D}\otimes\boldsymbol{B}) = \boldsymbol{AD}\otimes\boldsymbol{CB}$. Let $\boldsymbol{C} = \boldsymbol{I}_n$ and $\boldsymbol{D} = \boldsymbol{I}_m$. Then, we obtain

$$(\boldsymbol{A}\otimes\boldsymbol{C})(\boldsymbol{D}\otimes\boldsymbol{B}) = (\boldsymbol{A}\otimes\boldsymbol{I}_n)(\boldsymbol{I}_m\otimes\boldsymbol{B}) = \boldsymbol{AI}_m\otimes\boldsymbol{BI}_n = \boldsymbol{A}\otimes\boldsymbol{B}.$$

Note that

$$\det(\boldsymbol{I}_m\otimes\boldsymbol{B}) = \det\begin{bmatrix} \boldsymbol{B} & \boldsymbol{O} & \cdots & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{B} & \cdots & \boldsymbol{O} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{O} & \boldsymbol{O} & \cdots & \boldsymbol{B} \end{bmatrix} = (\det\boldsymbol{B})^m.$$

Taking into account that $\det(\boldsymbol{A}\otimes\boldsymbol{I}_n) = (\det\boldsymbol{A})^n$, we obtain
$\det(\boldsymbol{A}\otimes\boldsymbol{B}) = (\det\boldsymbol{A})^n(\det\boldsymbol{B})^m$. For the case when

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 2 & -2 \end{bmatrix} \quad \text{and} \quad \boldsymbol{B} = \begin{bmatrix} 2 & 6 & -8 \\ 0 & -1 & 0 \\ 0 & 2 & 3 \end{bmatrix},$$

we obtain

$$\det(\boldsymbol{A}\otimes\boldsymbol{B}) = (\det\boldsymbol{A})^3(\det\boldsymbol{B})^2 = (-2)^3(-6)^2 = (-8)(36) = -288.$$

$\diamond$

9. **(20 pts)** For the nonlinear system model of Problem 1, that is, the model

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \frac{4}{1+x_2} + x_2 u \\ x_1 x_2 + u \end{bmatrix},$$

find the equilibrium state $\boldsymbol{x}_e$ corresponding to $u_e = -2$ and such that $x_{e1} = -1$. Then, construct a **linear** in $\boldsymbol{x}$ and $\boldsymbol{u}$ model describing the system operation about $(\boldsymbol{x}_e, u_e)$.

The equilibrium pair $(\boldsymbol{x}_e, u_e)$ satisfies the algebraic equations,

$$\dot{x}_1 = \dot{x}_2 = 0,$$

that is,

$$\frac{4}{1+x_{2e}} + x_{2e}u_e = 0 \quad \text{and} \quad x_{1e}x_{2e} + u_e = 0.$$

From the second of the above equations, we obtain

$$x_{e2} = -2.$$

Hence,

$$\boldsymbol{x}_e = \begin{bmatrix} -1 \\ -2 \end{bmatrix}, \quad u_e = -2.$$

The system model is linear in the control variable and can be represented in the following form:

$$\dot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x}, u) = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{G}(\boldsymbol{x})u = \begin{bmatrix} \frac{4}{1+x_2} \\ x_1x_2 \end{bmatrix} + \begin{bmatrix} x_2 \\ 1 \end{bmatrix} u.$$

We proceed with constructing a linear approximation. We first obtain the input matrix

$$\boldsymbol{B} = \boldsymbol{G}(\boldsymbol{x}_e) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

We next proceed with computing the matrix $\boldsymbol{A}$. We first compute the Jacobian matrix of $\boldsymbol{f}$,

$$D\boldsymbol{f}(\boldsymbol{x}_e) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}\Bigg|_{\boldsymbol{x}=\boldsymbol{x}_e} = \begin{bmatrix} 0 & -\frac{4}{(1+x_2)^2} \\ x_2 & x_1 \end{bmatrix}\Bigg|_{\boldsymbol{x}=\boldsymbol{x}_e} = \begin{bmatrix} 0 & -4 \\ -2 & -1 \end{bmatrix}.$$

Then,

$$\begin{aligned} \boldsymbol{A} &= D\boldsymbol{f}(\boldsymbol{x}_e) + \frac{\boldsymbol{f}(\boldsymbol{x}_e) - D\boldsymbol{f}(\boldsymbol{x}_e)\boldsymbol{x}_e}{\|\boldsymbol{x}_e\|^2}\boldsymbol{x}_e^\top \\ &= \begin{bmatrix} 0 & -4 \\ -2 & -1 \end{bmatrix} + \frac{1}{5}\left(\begin{bmatrix} -4 \\ 2 \end{bmatrix} - \begin{bmatrix} 0 & -4 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} -1 \\ -2 \end{bmatrix}\right)\begin{bmatrix} -1 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -4 \\ -2 & -1 \end{bmatrix} + \frac{1}{5}\begin{bmatrix} -12 \\ -2 \end{bmatrix}\begin{bmatrix} -1 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -4 \\ -2 & -1 \end{bmatrix} + \frac{1}{5}\begin{bmatrix} 12 & 24 \\ 2 & 4 \end{bmatrix} \\ &= \begin{bmatrix} \frac{12}{5} & \frac{4}{5} \\ -\frac{8}{5} & -\frac{1}{5} \end{bmatrix}. \end{aligned}$$

The resulting linear model has the form,

$$\dot{\boldsymbol{x}} = \begin{bmatrix} \frac{12}{5} & \frac{4}{5} \\ -\frac{8}{5} & -\frac{1}{5} \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} -2 \\ 1 \end{bmatrix} u.$$

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10. (**20 pts**) Consider the following continuous-time fuzzy model,

$$\begin{aligned} \dot{\boldsymbol{x}} &= \left(\alpha_1 \boldsymbol{A}_1 + \alpha_2 \boldsymbol{A}_2\right) \boldsymbol{x} \\ &= \left(\alpha_1 \begin{bmatrix} -1 & 0 \\ 2 & -1 \end{bmatrix} + \alpha_2 \begin{bmatrix} -2 & 4 \\ 0 & -1 \end{bmatrix} \right) \boldsymbol{x}, \end{aligned}$$

where $\alpha_i = \alpha_i(\boldsymbol{x}) \geq 0$ for $i = 1, 2$ and $\alpha_1 + \alpha_2 = 1$. Does there exist a quadratic Lyapunov function for this system? If yes, find one, if not explain why not.

Both local systems are asymptotically stable. Let us check if the matrix $(\boldsymbol{A}_1 + \boldsymbol{A}_2)$ is Hurwitz. We have

$$\begin{aligned} \boldsymbol{A}_1 + \boldsymbol{A}_2 &= \begin{bmatrix} -1 & 0 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} -2 & 4 \\ 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} -3 & 4 \\ 2 & -2 \end{bmatrix}. \end{aligned}$$

The characteristic polynomial of $(\boldsymbol{A}_1 + \boldsymbol{A}_2)$ is

$$\det \begin{bmatrix} s+3 & -4 \\ -2 & s+2 \end{bmatrix} = s^2 + 6s - 2.$$

It is easy to check that the characteristic polynomial zeros are located at

$$-3 \pm \sqrt{11}.$$

Thus one of the zeros is positive while the other one is negative meaning that the matrix $(\boldsymbol{A}_1 + \boldsymbol{A}_2)$ is not Hurwitz and therefore the system does not have a quadratic Lyapunov function. ◇