ECE 68000: MODERN AUTOMATIC CONTROL

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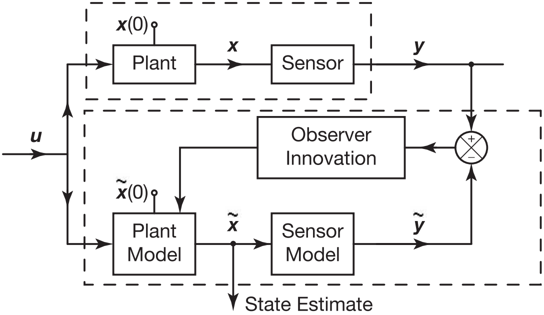
Observers for Systems With Unknown Inputs

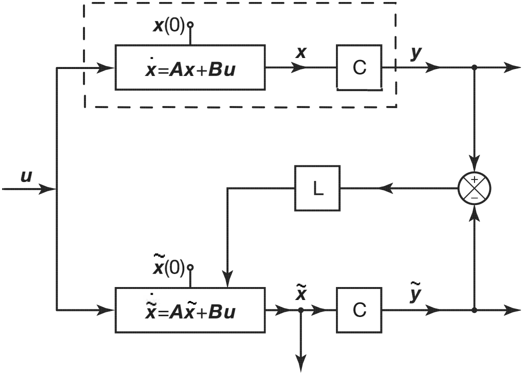
Closed-loop observer



- Luenberger's Innovation to obtain the closed-loop observer

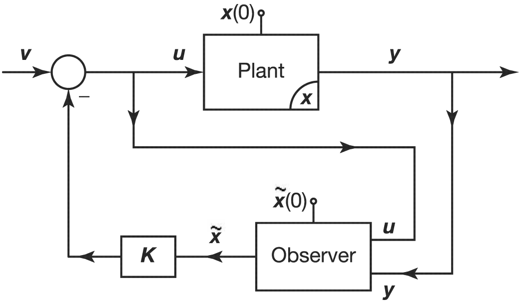
$$\dot{\tilde{\boldsymbol{x}}} = \boldsymbol{A}\tilde{\boldsymbol{x}} + \boldsymbol{B}\boldsymbol{u} + \boldsymbol{L}(\boldsymbol{y} - \tilde{\boldsymbol{y}})$$

Luenberger's closed-loop observer



- Luenberger's observer, $\boxed{\dot{\tilde{\boldsymbol{x}}} = \boldsymbol{A}\tilde{\boldsymbol{x}} + \boldsymbol{B}\boldsymbol{u} + \boldsymbol{L}(\boldsymbol{y} - \tilde{\boldsymbol{y}})}$
- Observation error dynamics, $\boxed{(\dot{\boldsymbol{x}} - \dot{\tilde{\boldsymbol{x}}}) = (\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C})(\boldsymbol{x} - \tilde{\boldsymbol{x}})}$

Combined observer-controller compensator



- Works well for systems without uncertainties
- What about systems with uncertainties?

Plant Model

Standard linear dynamical system model:

$$\begin{aligned}\dot{\boldsymbol{x}} &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} \\ \boldsymbol{y} &= \boldsymbol{C}\boldsymbol{x},\end{aligned}$$

where $\boldsymbol{B} \in \mathbb{R}^{n\times m}$, $\boldsymbol{C} \in \mathbb{R}^{p\times n}$

- Parameters $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ are known
- m_1 of inputs are known and $m_2 = m - m_1$ are unknown
- Re-arrange the order of the inputs if necessary, partition the input matrix $\boldsymbol{B}$ corresponding to the known, $\boldsymbol{u}_1$, and unknown inputs, $\boldsymbol{u}_2$, as

$$\boldsymbol{B} = \left[\begin{array}{cc} \boldsymbol{B}_1 & \boldsymbol{B}_2 \end{array}\right],$$

with $\boldsymbol{B}_1 \in \mathbb{R}^{n\times m_1}$ and $\boldsymbol{B}_2 \in \mathbb{R}^{n\times m_2}$ and

$$\boldsymbol{u} = \left[\begin{array}{c} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \end{array}\right]$$

System Model—Contd.

The system model

$$\boxed{\begin{array}{rcl} \dot{\boldsymbol{x}} & = & \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}_1\boldsymbol{u}_1 + \boldsymbol{B}_2\boldsymbol{u}_2 \\ \boldsymbol{y} & = & \boldsymbol{C}\boldsymbol{x} \end{array}}$$

- The vector function $\boldsymbol{u}_2$ may also model lumped uncertainties or nonlinearities in the plant
- Similar notation as in Basile and Marro, where $\boldsymbol{u}_2$ is called the disturbance vector
- The output matrix is $\boldsymbol{C} \in \mathbb{R}^{p\times n}$
- The pair $(\boldsymbol{A}, \boldsymbol{C})$ detectable

G. Basile and G. Marro, *On the observability of linear, time-invariant systems with unknown inputs*, Journal of Optimization Theory and Applications, Vol. 3, No. 6, pp. 410–415, Nov. 1969

Second page of the 1969 Basile and Marro's paper

We deal with a linear, purely dynamical, time-invariant system described by the equations

$$\dot{x} = Ax + B_1 u_1 + B_2 u_2 \tag{1}$$

$$y = Cx \tag{2}$$

where $x \in R^n$ is the *state vector*, $u_1 \in R^m$ is the *control vector*, $u_2 \in R^l$ is the *disturbance vector*, $y \in R^s$ is the *output vector*, and A, B_1, B_2, C are real, constant matrices of proper sizes. We call $\mathscr{F}_1 = \mathscr{R}(B_1)$ the subspace of control actions and $\mathscr{F}_2 = \mathscr{R}(B_2)$ the subspace of disturbance actions.

It is well known that, in the particular case where $B_1 \neq 0$, $B_2 = 0$, from the observation of input and output functions in a finite interval of time, it is possible to recognize the orthogonal projection of the state on the least subspace which is invariant under A^T and contains $\mathscr{R}(C^T)$. The word *least* is justified because the intersection of two invariants is an invariant. This subspace is sometimes called *observability subspace* and its orthogonal complement *unobservability subspace.*

In this particular case, when the input functions are completely known, the observation of the system (1)–(2) reduces to the observation of the corresponding autonomous system; that is, since

$$y(t) = C\Phi(t, 0)\, x_0 + C \int_0^t \Phi(t, \tau)\, B_1 u_1(\tau)\, d\tau \tag{3}$$

where $\Phi(t, \tau)$ is the state-transition matrix, it is possible to determine by a simple subtraction the output functions of the corresponding autonomous system, namely, the zero-input output functions.

By similar reasoning, the general case in which a part of the input is known and a part is unknown can be reduced to the case of completely unknown input. Thus, it is sufficient to consider only this last case. In the next section, we state a theorem that provides the observability subspace as the least conditioned invariant under the matrix A^T, with respect to the subspace $\mathscr{F}_2^{\perp}$, containing $\mathscr{R}(C^T)$, and which includes the previous results, corresponding to $B_2 = 0$.

2. Observability Subspace for Systems with Unknown Inputs

First, we recall some definitions and results given in a previous paper (Ref. 6) which provide a background for the analysis presented here. Consider

Projection Operator UIO—Idea

- Decompose the state $\boldsymbol{x}$ as

$$\begin{aligned}\boldsymbol{x} &= \boldsymbol{x} - \boldsymbol{M}\boldsymbol{y} + \boldsymbol{M}\boldsymbol{y} \\ &= (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})\boldsymbol{x} + \boldsymbol{M}\boldsymbol{C}\boldsymbol{x} \\ &= (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})\boldsymbol{x} + \boldsymbol{M}\boldsymbol{y},\end{aligned}$$

  where $\boldsymbol{M}$: $n \times p$ real matrix to be determined
- $\boldsymbol{q} = (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})\boldsymbol{x}$: Unknown part of the decomposition

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S. Hui and S. H. Żak, *Observer design for systems with unknown inputs*, International Journal of Applied Mathematics and Computer Science, Vol. 15, No. 4, pp. 431–446, 2005

Projection Operator UIO—Decomposed Dynamics

- Some manipulations:

$$\begin{aligned}\dot{\boldsymbol{q}} &= (\boldsymbol{I}-\boldsymbol{MC})\dot{\boldsymbol{x}}\\ &= (\boldsymbol{I}-\boldsymbol{MC})(\boldsymbol{Ax}+\boldsymbol{B}_1\boldsymbol{u}_1+\boldsymbol{B}_2\boldsymbol{u}_2)\\ &= (\boldsymbol{I}-\boldsymbol{MC})(\boldsymbol{Ax}+\boldsymbol{B}_1\boldsymbol{u}_1)+(\boldsymbol{I}-\boldsymbol{MC})\boldsymbol{B}_2\boldsymbol{u}_2\end{aligned}$$

- Recall: $\boldsymbol{x}=\boldsymbol{q}+\boldsymbol{My}$

$$\dot{\boldsymbol{q}} = (\boldsymbol{I}-\boldsymbol{MC})(\boldsymbol{Aq}+\boldsymbol{AMy}+\boldsymbol{B}_1\boldsymbol{u}_1)+(\boldsymbol{I}-\boldsymbol{MC})\boldsymbol{B}_2\boldsymbol{u}_2$$

- Choose $\boldsymbol{M}$ to make $(\boldsymbol{I}-\boldsymbol{MC})\boldsymbol{B}_2=\boldsymbol{O}$
- Then

$$\boxed{\dot{\boldsymbol{q}} = (\boldsymbol{I}-\boldsymbol{MC})(\boldsymbol{Aq}+\boldsymbol{AMy}+\boldsymbol{B}_1\boldsymbol{u}_1)}$$

- Important: $\boldsymbol{u}_1$ and $\boldsymbol{y}$ are known

Projection Operator UIO—The Rank Condition

- Need:

$$(\boldsymbol{I} - \boldsymbol{MC})\boldsymbol{B}_2 = \boldsymbol{O}$$

- Linear Algebra:

$$\text{rank}\ (\boldsymbol{MCB}_2) \leq \text{rank}\ (\boldsymbol{CB}_2) \leq \text{rank}\ (\boldsymbol{B}_2)$$

- Necessary and Sufficient Condition:

$$\text{rank}\ (\boldsymbol{CB}_2) = \text{rank}(\boldsymbol{B}_2)$$

- Implication: At least as many independent outputs as unknown inputs for the method to work

Projection Operator UIO Dynamics

- If we know $\boldsymbol{q}$ and the initial condition

$$\boldsymbol{q}(0) = (\boldsymbol{I} - \boldsymbol{MC})\boldsymbol{x}(0),$$

  then

$$\boldsymbol{x} = \boldsymbol{q} + \boldsymbol{MCx} = \boldsymbol{q} + \boldsymbol{My}$$

  is known for all $t \geq 0$
- Indeed, integrate both sides of $\dot{\boldsymbol{q}} = (\boldsymbol{I} - \boldsymbol{MC})\dot{\boldsymbol{x}}$ to obtain

$$\boldsymbol{q}(t)\text{-}\boldsymbol{q}(0)= (\boldsymbol{I}\text{-}\boldsymbol{MC})(\boldsymbol{x}(t)\text{-}\boldsymbol{x}(0))$$

- Hence

$$\boldsymbol{q}(t) = (\boldsymbol{I} - \boldsymbol{MC})\boldsymbol{x}(t) - (\boldsymbol{I} - \boldsymbol{MC})\boldsymbol{x}(0) + \boldsymbol{q}(0)$$

- If $\boldsymbol{q}(0) = (\boldsymbol{I} - \boldsymbol{MC})\boldsymbol{x}(0)$, then

$$\boldsymbol{x} = \boldsymbol{q} + \boldsymbol{MCx} = \boldsymbol{q} + \boldsymbol{My}$$

  is known for all $t \geq 0$

Projection Operator UIO Dynamics—Contd.

- But we do not know $\boldsymbol{x}(0)$
- We have

$$\boldsymbol{q}(t) = (\boldsymbol{I} - \boldsymbol{MC})\boldsymbol{x}(t) - (\boldsymbol{I} - \boldsymbol{MC})\boldsymbol{x}(0) + \boldsymbol{q}(0)$$

- So we only get an approximation

$$\boxed{\tilde{\boldsymbol{x}} = \boldsymbol{q} + \boldsymbol{My}}$$

where $\boldsymbol{q}$ is obtained from

$$\boxed{\dot{\boldsymbol{q}} = (\boldsymbol{I} - \boldsymbol{MC})(\boldsymbol{Aq} + \boldsymbol{AMy} + \boldsymbol{B}_1\boldsymbol{u}_1)}$$

Open-Loop UIO Analysis

- Let $\boldsymbol{e}(t) = \boldsymbol{x}(t) - \tilde{\boldsymbol{x}}(t)$ be the estimation error
- Recall $(\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})\boldsymbol{B}_2 = \boldsymbol{O}$ and $\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x}$

Then we have

$$
\begin{aligned}
\frac{d\boldsymbol{e}}{dt} &= \frac{d}{dt}(\boldsymbol{x} - \tilde{\boldsymbol{x}}) \\
&= \frac{d}{dt}(\boldsymbol{x} - \boldsymbol{q} - \boldsymbol{M}\boldsymbol{C}\boldsymbol{x}) \\
&= \frac{d}{dt}((\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})\boldsymbol{x} - \boldsymbol{q}) \\
&= (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})(\boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}_1\boldsymbol{u}_1 + \boldsymbol{B}_2\boldsymbol{u}_2) \\
&\quad\; -(\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})(\boldsymbol{A}\boldsymbol{q} + \boldsymbol{A}\boldsymbol{M}\boldsymbol{y} + \boldsymbol{B}_1\boldsymbol{u}_1) \\
&= (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})(\boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}_1\boldsymbol{u}_1) + (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})\boldsymbol{B}_2\boldsymbol{u}_2 \\
&\quad\; -(\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})(\boldsymbol{A}\boldsymbol{q} + \boldsymbol{A}\boldsymbol{M}\boldsymbol{C}\boldsymbol{x} + \boldsymbol{B}_1\boldsymbol{u}_1) \\
&= (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})\boldsymbol{A}(\boldsymbol{x} - \boldsymbol{q} - \boldsymbol{M}\boldsymbol{C}\boldsymbol{x}) \\
&= (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})\boldsymbol{A}\boldsymbol{e}
\end{aligned}
$$

Closed-Loop UIO

- We add the innovation term to obtain the closed-loop UIO:

$$\begin{aligned}
\dot{\boldsymbol{q}} &= (\boldsymbol{I} - \boldsymbol{MC})((\boldsymbol{Aq} + \boldsymbol{AMy} + \boldsymbol{B}_1\boldsymbol{u}_1) + \boldsymbol{L}(\boldsymbol{y} - \tilde{\boldsymbol{y}})) \\
&= (\boldsymbol{I} - \boldsymbol{MC})((\boldsymbol{Aq} + \boldsymbol{AMy} + \boldsymbol{B}_1\boldsymbol{u}_1) \\
&\quad + \boldsymbol{L}(\boldsymbol{y} - \boldsymbol{Cq} - \boldsymbol{CMy})) \\
&= (\boldsymbol{I} - \boldsymbol{MC})((\boldsymbol{Aq} + \boldsymbol{AMy} + \boldsymbol{B}_1\boldsymbol{u}_1) \\
&\quad + \boldsymbol{LC}(\boldsymbol{x} - \boldsymbol{q} - \boldsymbol{My}))
\end{aligned}$$

- State estimate is

$$\tilde{\boldsymbol{x}} = \boldsymbol{q} + \boldsymbol{My}$$

Closed-Loop UIO Analysis

- Let $\boldsymbol{e} = \boldsymbol{x} - \tilde{\boldsymbol{x}}$
- We will show that

$$\dot{\boldsymbol{e}} = (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})(\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C})\boldsymbol{e}$$

  and $\boldsymbol{e}(t) \to \mathbf{0}$ as $t \to \infty$ under mild conditions
- Note that $(\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C})$ asymptotically stable does not guarantee that $(\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})(\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C})$ is asymptotically stable
- It is possible for a product of a projection matrix and an asymptotically stable matrix to be unstable

Example

- Let
$$\mathbf{\Pi} = \left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right] \text{ and } \boldsymbol{A} = \left[\begin{array}{cc} 1 & -3 \\ 3 & -2 \end{array} \right]$$
- $\boldsymbol{A}$ is asymptotically stable
- $\mathbf{\Pi}\boldsymbol{A}$ is unstable
- The system $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$ restricted to the range of $\mathbf{\Pi}$ is governed by $\dot{z} = z$, which is also unstable