HW Sec 8.4 #24, #25.

#24 : $A_{n\times n}$ is symmetric

$Q(X) = X^T A X$ : a quadratic form.

(a) A is positive definite

iff $Q(X) > 0$ for any $X \neq 0 \in \mathbb{R}^n$

Show that A is positive definite

($\Leftrightarrow$) iff All eigenvalues of A are positive.

(idea). ($\Rightarrow$)

Assume that A is positive definite

Then $Q(X) > 0$ for any $X \neq 0$

$Q(X) = X^T A X > 0$

Since A is symmetric, A has an eigenbasis $\{V_1, V_2, \cdots, V_n\}$ : $AV_i = \lambda_i V_i$

$X = V_i$ : $Q(X) = Q(V_i) = V_i^T A V_i$

$= V_i^T \lambda_i V_i = \lambda_i |V_i|^2 > 0$

$\lambda_i > 0$, $i = 1, 2, \cdots, n$.

#25. $\underset{n\times n}{A}$ : symmetric, $A=[a_{ij}]$

A is positive definite

iff $A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \ddots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ :

$a_{11}>0$, $\det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}>0$, $\cdots$, $\det A>0$

#22 $4x_1^2+12x_1x_2+13x_2^2=$ ?

$A=\begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}$ : symmetric, $X=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$Q(X)=X^TAX=[x_1\ x_2]\begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$=[x_1\ x_2]\begin{bmatrix} a_{11}x_1+a_{12}x_2 \\ a_{12}x_1+a_{22}x_2 \end{bmatrix}$$

$$=a_{11}x_1^2+\underbrace{a_{12}x_1x_2+a_{12}x_1x_2}_{=2a_{12}x_1x_2}+a_{22}x_2^2$$

$$=4x_1^2+12x_1x_2+13x_2^2$$

$a_{11}=4$, $2a_{12}=12$, $a_{22}=13$

$a_{12}=6$ : $A=\begin{bmatrix} 4 & 6 \\ 6 & 13 \end{bmatrix}$

4.3. Case 2 & 3.

Case 2: $\begin{cases} Y' = AY \\ A \text{ has complex eigenvalues.} \end{cases}$

Goal: Derive real-valued solutions.

(Ex) $Y' = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{\text{let } = A} Y$

(1) $\lambda$: $\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix}$

$= \lambda^2 + 1 = 0$ ; $\lambda = i, -i$

(2) $V$: $\lambda = i$ $\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$-iv_1 + v_2 = 0$ ; $v_2 = iv_1$

$V_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$ $(v_1 = 1)$

$\lambda = -i$ : $V_2 = \overline{V_1} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$

$Y_1(t) = \begin{bmatrix} 1 \\ i \end{bmatrix} e^{it}$, $Y_2(t) = \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{-it}$

$\begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ i \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{= a} + i \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{= b}$

Euler formula: $e^{it} = \cos(t) + i\sin(t)$.

because of Taylor series.

$$y_1(t) = \left(\begin{bmatrix}1\\0\end{bmatrix} + i\begin{bmatrix}0\\1\end{bmatrix}\right)(\cos(t) + i\sin(t))$$

$$= \begin{bmatrix}1\\0\end{bmatrix}\cos(t) + \underbrace{i^2}_{-1}\begin{bmatrix}0\\1\end{bmatrix}\sin(t)$$

$$+ i\left(\begin{bmatrix}1\\0\end{bmatrix}\sin(t) + \begin{bmatrix}0\\1\end{bmatrix}\cos(t)\right)$$

$$y_2(t) = \left(\begin{bmatrix}1\\0\end{bmatrix} - i\begin{bmatrix}0\\1\end{bmatrix}\right)(\cos(t) - i\sin(t))$$

$$= \begin{bmatrix}1\\0\end{bmatrix}\cos(t) - \begin{bmatrix}0\\1\end{bmatrix}\sin(t)$$

$$- i\left(\begin{bmatrix}1\\0\end{bmatrix}\sin(t) + \begin{bmatrix}0\\1\end{bmatrix}\cos(t)\right)$$

(Superposition)

$y_1, y_2 \rightarrow c_1 y_1 + c_2 y_2$: a solution.

$$y_1 + y_2 = 2\left(\begin{bmatrix}1\\0\end{bmatrix}\cos(t) - \begin{bmatrix}0\\1\end{bmatrix}\sin(t)\right)$$

$$\frac{1}{2}y_1 + \frac{1}{2}y_2 = \underbrace{\begin{bmatrix}1\\0\end{bmatrix}\cos(t) - \begin{bmatrix}0\\1\end{bmatrix}\sin(t)}_{\text{let" } y_3}$$

$$y_1 - y_2 = 2i\left(\begin{bmatrix}1\\0\end{bmatrix}\sin(t) + \begin{bmatrix}0\\1\end{bmatrix}\cos(t)\right)$$

$$\frac{1}{2i}y_1 - \frac{1}{2i}y_2 = \underbrace{\begin{bmatrix}1\\0\end{bmatrix}\sin(t) + \begin{bmatrix}0\\1\end{bmatrix}\cos(t)}_{\text{let" } y_4}$$

Euler formula: $\underline{e^{it} = \cos(t) + i\sin(t).}$

because of Taylor series.

$$y_1(t) = \left(\begin{bmatrix}1\\0\end{bmatrix} + i\begin{bmatrix}0\\1\end{bmatrix}\right)\left(\cos(t) + i\sin(t)\right)$$

$$= \begin{bmatrix}1\\0\end{bmatrix}\cos(t) + \underbrace{i^2}_{-1}\begin{bmatrix}0\\1\end{bmatrix}\sin(t)$$

$$+ i\left(\begin{bmatrix}1\\0\end{bmatrix}\sin(t) + \begin{bmatrix}0\\1\end{bmatrix}\cos(t)\right)$$

$$y_2(t) = \left(\begin{bmatrix}1\\0\end{bmatrix} - i\begin{bmatrix}0\\1\end{bmatrix}\right)\left(\cos(t) - i\sin(t)\right)$$

$$= \begin{bmatrix}1\\0\end{bmatrix}\cos(t) - \begin{bmatrix}0\\1\end{bmatrix}\sin(t)$$

$$- i\left(\begin{bmatrix}1\\0\end{bmatrix}\sin(t) + \begin{bmatrix}0\\1\end{bmatrix}\cos(t)\right)$$

(Superposition)

$y_1, y_2 \rightarrow c_1 y_1 + c_2 y_2$ : a solution.

$$y_1 + y_2 = 2\left(\begin{bmatrix}1\\0\end{bmatrix}\cos(t) - \begin{bmatrix}0\\1\end{bmatrix}\sin(t)\right)$$

$$\frac{1}{2}y_1 + \frac{1}{2}y_2 = \underbrace{\begin{bmatrix}1\\0\end{bmatrix}\cos(t) - \begin{bmatrix}0\\1\end{bmatrix}\sin(t)}_{\text{let}'' \ y_3}$$

$$y_1 - y_2 = 2i\left(\begin{bmatrix}1\\0\end{bmatrix}\sin(t) + \begin{bmatrix}0\\1\end{bmatrix}\cos(t)\right)$$

$$\frac{1}{2i}y_1 - \frac{1}{2i}y_2 = \underbrace{\begin{bmatrix}1\\0\end{bmatrix}\sin(t) + \begin{bmatrix}0\\1\end{bmatrix}\cos(t)}_{\text{let}'' \ y_4}$$

$$Y_3(t) = \begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix}, \quad Y_4(t) = \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}$$

$$W(Y_3, Y_4) = \det[Y_3 \; Y_4] = \det\begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$$

$$= \cos^2(t) + \sin^2(t) = 1 \neq 0$$

Wronskian

$\therefore$ $Y_3(t)$, $Y_4(t)$: lin. independent.

$$Y(t) = C_1\left(\begin{bmatrix}1\\0\end{bmatrix}\cos(t) - \begin{bmatrix}0\\1\end{bmatrix}\sin(t)\right)$$
$$+ C_2\left(\begin{bmatrix}1\\0\end{bmatrix}\sin(t) + \begin{bmatrix}0\\1\end{bmatrix}\cos(t)\right)$$ : a general solution.

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Case 2: $Y' = AY$

$\lambda = \alpha + \beta i$

A: $\lambda = \alpha \pm \beta i$, $V_1 = \mathbb{a} + \mathbb{b} i$

$$Y_1(t) = V_1 e^{(\alpha+\beta i)t} = V_1 e^{\alpha t} e^{i\beta t}$$
$$= e^{\alpha t}(\mathbb{a} + \mathbb{b} i)(\cos(\beta t) + i\sin(\beta t))$$

$$Y_2(t) = \overline{V_1} e^{(\alpha-\beta i)t}$$
$$= e^{\alpha t}(\mathbb{a} - \mathbb{b} i)(\cos(\beta t) - i\sin(\beta t))$$

$$Y(t) = C_1 e^{\alpha t}(\mathbb{a}\cos(\beta t) - \mathbb{b}\sin(\beta t))$$
$$+ C_2 e^{\alpha t}(\mathbb{a}\sin(\beta t) + \mathbb{b}\cos(\beta t))$$

(Ex) $y' = \begin{bmatrix} -2 & -1 \\ 4 & -2 \end{bmatrix} y$ let $\overset{\text{let}}{=} A$

(1) critical points: Let $y' = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} -2 & -1 \\ 4 & -2 \end{bmatrix} y = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ : $\underline{y = \begin{bmatrix} 0 \\ 0 \end{bmatrix}}$

$(0, 0)$ : a critical point.

(2) General solution

$\det(A - \lambda I) = \begin{vmatrix} -2-\lambda & -1 \\ 4 & -2-\lambda \end{vmatrix} = (-2-\lambda)^2 + 4 = 0$

$\lambda = -2 \pm 2i$ : Case 2. $(\alpha = -2, \beta = 2)$

$\lambda = -2 + 2i$ : $\begin{bmatrix} -2i & -1 \\ 4 & -2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$-2i v_1 - v_2 = 0$ : $v_2 = -2i v_1$

$V_1 = \begin{bmatrix} 1 \\ -2i \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{= a} + i \underbrace{\begin{bmatrix} 0 \\ -2 \end{bmatrix}}_{= b}$

$y(t) = c_1 e^{-2t} \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cos(2t) - \begin{bmatrix} 0 \\ -2 \end{bmatrix} \sin(2t) \right)$

$+ c_2 e^{-2t} \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \sin(2t) + \begin{bmatrix} 0 \\ -2 \end{bmatrix} \cos(2t) \right)$

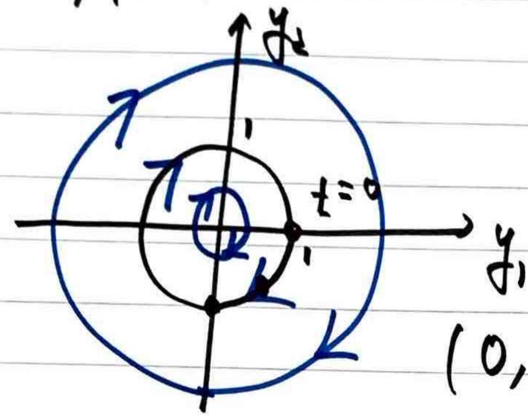
(Solution curves)

(Ex) $y' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} y$.

$$\mathbf{y}(t) = c_1 \underbrace{\begin{bmatrix}\cos(t)\\ -\sin(t)\end{bmatrix}}_{=\,\mathbf{y}_3(t)} + c_2 \underbrace{\begin{bmatrix}\sin(t)\\ \cos(t)\end{bmatrix}}_{=\,\mathbf{y}_4(t)}$$

$$|\mathbf{y}_3(t)| = \sqrt{\cos^2 t + (-\sin t)^2} = 1$$

$$|\mathbf{y}_4(t)| = 1$$




$\mathbf{y}_3(0) = \begin{bmatrix}1\\0\end{bmatrix}$, $\mathbf{y}_3(\frac{\pi}{2}) = \begin{bmatrix}0\\-1\end{bmatrix}$

$\mathbf{y}_3(\frac{\pi}{4}) = \begin{bmatrix}\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\end{bmatrix}$

$(0,0)$: a stable center

(Ex) $\mathbf{y}' = \begin{bmatrix}-2 & -1\\ 4 & -2\end{bmatrix}\mathbf{y}$.

$$\mathbf{y}(t) = c_1 \underbrace{e^{-2t}\begin{bmatrix}\cos(2t)\\ 2\sin(2t)\end{bmatrix}}_{=\,\mathbf{y}_3(t)} + c_2 \underbrace{e^{-2t}\begin{bmatrix}\sin(2t)\\ -2\cos(2t)\end{bmatrix}}_{=\,\mathbf{y}_4(t)}$$

$$X_1(t) = \begin{bmatrix}\cos(2t)\\ 2\sin(2t)\end{bmatrix}\begin{matrix}=x_1\\ =x_2\end{matrix}$$

$$x_1^2 + \left(\frac{x_2}{2}\right)^2 = \cos^2(2t) + \sin^2(2t) = 1$$

$$\lim_{t\to\infty} e^{-2t} = 0$$