AAE 666

Homework Four: Solutions

Exercise 1

$$\begin{aligned} V(x) &= x_1^4 - x_1^2 x_2 + x_2^2 \\ &= \begin{pmatrix} x_1^2 \\ x_2 \end{pmatrix}' P \begin{pmatrix} x_1^2 \\ x_2 \end{pmatrix} \end{aligned}$$

where

$$P = \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix}$$

Since $P = P' > 0$, , the function is positive definite.

Exercise 2

With the following candidate Lyapunov function:

$$V(x) = \frac{1}{2}x^2$$

we have

$$\begin{aligned} DV(x)f(x) &= x(-x(1+\sin x)) \\ DV(x)f(x) &= -x^2(1+\sin x) \end{aligned}$$

Since $1 + \sin x > 0$ for $x \neq 0$ and $|x| < \pi/2$, we have $DV(x)f(x) < 0$ for $x \neq 0$ and $|x| < \pi/2$. Hence the system is AS about 0.

Exercise 3

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + x_1^3 - x_2 \end{aligned}$$

Consider the following candidate Lyapunov function:

$$V(x) = \frac{1}{2}x_1^2 - \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2 + \frac{1}{2}\lambda x_1^2 + \lambda x_1 x_2$$

where $0 < \lambda < 1$.

$$\begin{aligned}
DV(x) &= \begin{bmatrix} x_1 - x_1^3 + \lambda x_1 + \lambda x_2 & x_2 + \lambda x_1 \end{bmatrix} \\
DV(x^e) &= [0, 0] \\
D^2V(x) &= \begin{bmatrix} 1 - 3x_1^2 + \lambda & \lambda \\ \lambda & 1 \end{bmatrix} \\
D^2V(x^e) &= \begin{bmatrix} 1 + \lambda & \lambda \\ \lambda & 1 \end{bmatrix}
\end{aligned}$$

where $D^2V(x^e) > 0$ becase λ is positive. Therefore V is **lpd** about the origin and we have the following,

$$\begin{aligned}
DV(x)f(x) &= \begin{bmatrix} x_1 - x_1^3 + \lambda x_1 + \lambda x_2 & x_2 + \lambda x_1 \end{bmatrix} \begin{bmatrix} x_2 \\ -x_1 + x_1^3 - x_2 \end{bmatrix} \\
DV(x)f(x) &= (x_1x_2 - x_1^3x_2 + \lambda x_1x_2 + \lambda x_2^2) - (x_1x_2 + x_1^3x_2 - x_2^2 - \lambda x_1^2 + \lambda x_1^4 - \lambda x_1x_2) \\
DV(x)f(x) &= \lambda x_1^2(x_1^2 - 1) + x_2^2(\lambda - 1)
\end{aligned}$$

which $DV(x)f(x) < 0$ for $|x| < 1$ and $x \neq [0, 0]$. Therefore the system is asymptotically stable about the origin.

Exercise 4

Given the linear controller of the form $u = -k_1x_1 - k_2x_2$, we can write the close loop system as:

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_1 - x_1^3 - k_1x_1 - k_2x_2
\end{aligned}$$

Consider

$$V(x) = -\frac{1}{2}x_1^2 + \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2 + \frac{1}{2}\lambda k_2^2 x_1^2 + \lambda k_2 x_1 x_2 + \frac{1}{2}k_1 x_1^2$$

as a candidate Lyapunov function, where $0 < \lambda < 1$. We have:

$$\begin{aligned}
V(0) &= 0 \\
DV(x) &= \begin{pmatrix} -x_1 + x_1^3 + \lambda k_2^2 x_1 + \lambda k_2 x_2 + k_1 x_1 \\ x_2 + \lambda k_2 x_1 \end{pmatrix} \\
DV(0) &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
D^2V(x) &= \begin{pmatrix} -1 + 3x_1^2 + \lambda k_2^2 + k_1 & \lambda k_2 \\ \lambda k_2 & 1 \end{pmatrix} > \begin{bmatrix} k_1 - 1 + \lambda k_2^2 & \lambda k_2 \\ \lambda k_2 & 1 \end{bmatrix} = P
\end{aligned}$$

Considering

$$k_1 > 1, \qquad k_2 > 0$$

and $0 < \lambda < 1$, P is a positive definite symmetric matrix. Therefore $V(x)$ is pd. Then we proceed to calculate $DV(x)f(x)$:

$$
\begin{aligned}
DV(x)f(x) &= \begin{pmatrix} -x_1 + x_1^3 + \lambda k_2^2 x_1 + \lambda k_2 x_2 + k_1 x_1 & x_2 + \lambda k_2 x_1 \end{pmatrix} \begin{pmatrix} x_2 \\ x_1 - x_1^3 - k_1 x_1 - k_2 x_2 \end{pmatrix} \\
&= -x_1 x_2 + x_1^3 x_2 + \lambda k_2^2 x_1 x_2 + \lambda k_2 x_2^2 + k_1 x_1 x_2 + (x_2 + \lambda k_2 x_1)(x_1 - x_1^3 - k_1 x_1 - k_2 x_2) \\
&= -\lambda k_2 x_1^2 (k_1 - 1 + x_1^2) - k_2 x_2^2 (1 - \lambda)
\end{aligned}
$$

Since $0 < \lambda < 1$ and $1 - k_1 < 0$, $k_2 > 0$, $DV(x)f(x) < 0$ for $x \neq (0, 0)$ Then, by **Theorem 1**, we conclude that the system is **GAS** with the linear controller $u = -k_1 x_1 - k_2 x_2$ with $k_1 > 1$ and $k_2 > 0$. Phase portrait for the open-loop and a sample closed-loop system are shown.

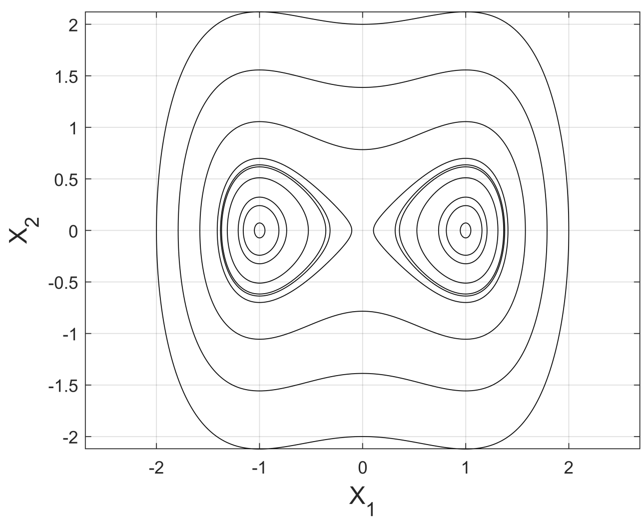

Figure 1: Open loop Duffing system

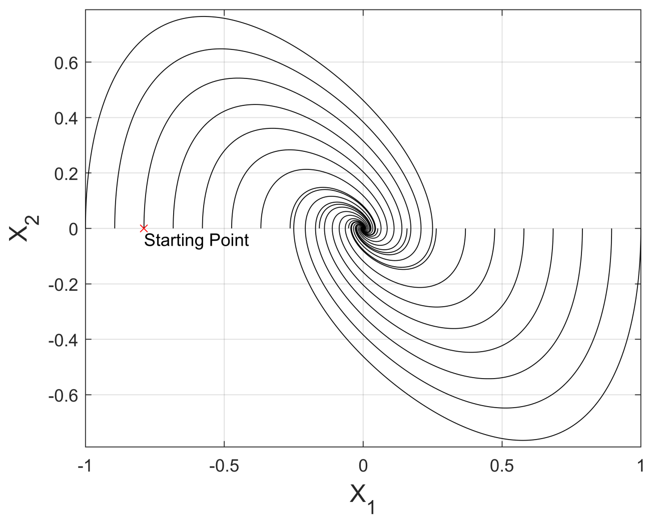

Figure 2: Closed loop Duffing system, $k_1 = 2$,$k_2 = 1$

Exercise 5

$$\begin{aligned} V(x) &= x_1 - x_1^3 + x_1^4 - x_2^2 + x_2^4 \\ &= V_1(x_1) + V_2(x_2) \end{aligned}$$

where $V_1 = x_1 - x_1^3 + x_1^4$ and $V_2 = -x_2^2 + x_2^4$. Since

$$\lim_{x_1 \to \infty} V_1(x_1) = \infty$$
$$\lim_{x_2 \to \infty} V_2(x_2) = \infty$$

V is radially unbounded.

Exercise 6

Consider the following candidate Lyapunov function :

$$V(x) = -\frac{1}{2}x_1^2 + \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2 + \frac{1}{2}\lambda c^2 x_1^2 + \lambda c x_1 x_2$$

where $0 < \lambda < 1$. Let

$$P = \frac{1}{2}\begin{bmatrix} \lambda c^2 & \lambda c \\ \lambda c & 1 \end{bmatrix}$$

Since $c > 0$, $P = P' > 0$ and we see that

$$\begin{aligned} V(x) &= x^T P x - \frac{1}{2}x_1^2 + \frac{1}{4}x_1^4 \\ &= x^T P x + (\frac{1}{2}x_1^2 - \frac{1}{2})^2 - \frac{1}{4} \\ &\geq x^T P x - \frac{1}{4} \\ &\geq \lambda_{\min}(P)||x||^2 - \frac{1}{4} \end{aligned}$$

Since $\lambda_{\min}(P) > 0$, V is radially unbounded.

$$DV(x) = \begin{pmatrix} \lambda c^2 x_1 + x_1^3 + \lambda c x_2 - x_1 & x_2 + \lambda c x_1 \end{pmatrix}$$

and

$$\begin{aligned} DV(x)f(x) &= (\lambda c^2 x_1 + x_1^3 + \lambda c x_2 - x_1)x_2 + (x_2 + \lambda c x_1)(x_1 - x_1^3 - c x_2 + 1) \\ &= -\lambda c(x_1^4 - x_1^2 - x_1) - (1-\lambda)c x_2^2 + x_2 \\ &= -L_1(x_1) - L_2(x_2) \end{aligned}$$

where

$$L_1(x_1) = \lambda c(x_1^4 - x_1^2 - x_1), \qquad L_2(x_2) = (1-\lambda)c x_2^2 + x_2$$

Since

$$\lim_{x_1 \to \infty} L_1(x_1) = \infty, \qquad \lim_{x_2 \to \infty} L_2(x_2) = \infty$$

It follows that

$$\lim_{x \to \infty} DV(x)f(x) = -\infty$$

Hence $DV(x)f(x) \leq 0$ when $\|x\| \geq R$ and R is sufficiently large. It now follows tha tall solutions are bounded.

Exercise 7

With $V(x) = x^2$ as a candidate Lyapunov function, we have

$$DV(x)f(x) = 2xx\cos x - 2x^4 + 200x$$

The term $-x^4$ term dominates when $|x| > R$ and R is sufficiently large. Therefore, all solutions of the system are bounded.

Exercise 8

As a candidate Lyapunov function, consider

$$V(x) = \frac{1}{4}x_1^4 - \sin x_1 - 100x_1 + \frac{1}{2}x_2^2$$

This function is radially unbounded and

$$DV(x)f(x) = 0$$

Hence all solutions of this system are bounded.

Exercise 9

With $V(x) = x^2$ as a candidate Lyapunov function, we have

$$\begin{aligned} DV(x)f(x) &= -2x(2 + \sin x)x \\ &= -2(2 + \sin x)x^2 \end{aligned}$$

since $2 + \sin x \geq 1$, we obtain that

$$\begin{aligned} DV(x)f(x) &\leq -2x^2 \\ &= -2V(x) \end{aligned}$$

Considering the inequality form in **Theorem 5** to conclude the rate of convergence: $DV(x)f(x) \leq -2\alpha V(x)$. We have the rate of convergence of 1.

Exercise 10

Letting$y = x - 1$, we can rewrite the original equation as:

$$\dot{y} = f(y) := -(2 + \sin(y + 1))y$$

With $V(y) = y^2$ as a candidate Lyapunov function, we have

$$\begin{aligned} DV(y)f(y) &= -2y(2 + \sin(y + 1)y \\ &= -2(2 + \sin(y + 1)y^2 \end{aligned}$$

Since $2 + \sin(y + 1) \geq 1$, we obtain that

$$\begin{aligned} DV(y)f(y) &\leq -2y^2 \\ &= -2V(y) \end{aligned}$$

Considering the inequality in **Theorem 5** to conclude the rate of convergence: $DV(y)f(y) \leq -2\alpha V(y)$. We have the rate of convergence of 1.

Exercise 11

Considering as $V(x) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2)$ as a candidate Lyapunov function we have

$$\begin{aligned} DV(x)f(x) &= -x_1^2 + (I_2 - I_3)x_1x_2x_3 - 2x_2^2 + (I_3 - I_1)x_1x_2x_3 - 3x_3^2 + (I_1 - I_2)x_1x_2x_3 \\ &= -x_1^2 - 2x_2^2 - 3x_3^2 \\ &\leq -x_1^2 - x_2^2 - x_3^2 \\ &= -2V(x) \end{aligned}$$

Considering the inequality in**Theorem 5** to conclude the rate of convergence: $DV(y)f(y) \leq -2\alpha V(y)$. We have the rate of convergence of 1.