8.4. Exercise

#1. $B = P^{-1}AP$

Y: an eigenvector of B $(Y = P^{-1}X)$

$X = PY.$

Compute (1) $B = P^{-1}AP$

(2) Y_1, Y_2

(3) $X_i = PY_i, \quad i = 1, 2$

8.5. Complex matrices

(Ex) Let $A = \begin{bmatrix} 1+2i & 2i \\ 5 & 3-4i \end{bmatrix}$

1. $\bar{A} = \begin{bmatrix} 1-2i & -2i \\ 5 & 3+4i \end{bmatrix}$: the complex conjugate of A

2. $\bar{A}^T = \begin{bmatrix} 1-2i & 5 \\ -2i & 3+4i \end{bmatrix}$: the conjugate transpose of A.

<u>Def</u> $A_{n\times n} = [a_{ij}]$

(1) If $\bar{A}^T = A$, A is called Hermitian

(2) If $\bar{A}^T = -A$, A is called <u>skew - Hermitian</u>

(3) If $\bar{A}^T = A^{-1}$, A is called <u>unitary.</u>

(Ex) (1) $A = \begin{bmatrix} 1 & 1+i \\ 1-i & 4 \end{bmatrix}$? Hermitian.

$\bar{A} = \begin{bmatrix} 1 & 1-i \\ 1+i & 4 \end{bmatrix}$, $\bar{A}^T = \begin{bmatrix} 1 & 1+i \\ 1-i & 4 \end{bmatrix} = A$.

(2) $A = \begin{bmatrix} 0 & i & 2 \\ i & 0 & 1+i \\ -2 & -1+i & 0 \end{bmatrix}$? skew - Hermitian

$\bar{A} = \begin{bmatrix} 0 & -i & 2 \\ -i & 0 & 1-i \\ -2 & -1-i & 0 \end{bmatrix}$, $\bar{A}^T = \begin{bmatrix} 0 & -i & -2 \\ -i & 0 & -1-i \\ 2 & 1-i & 0 \end{bmatrix}$

$\bar{A}^T = -\begin{bmatrix} 0 & i & 2 \\ i & 0 & 1+i \\ -2 & -1+i & 0 \end{bmatrix} = -A$

(3) $A = \begin{bmatrix} \frac{1}{2} & i\sqrt{\frac{3}{4}} \\ i\sqrt{\frac{3}{4}} & \frac{1}{2} \end{bmatrix}$? unitary.

$\bar{A} = \begin{bmatrix} \frac{1}{2} & -i\sqrt{\frac{3}{4}} \\ -i\sqrt{\frac{3}{4}} & \frac{1}{2} \end{bmatrix}$, $\bar{A}^T = \begin{bmatrix} \frac{1}{2} & -i\sqrt{\frac{3}{4}} \\ -i\sqrt{\frac{3}{4}} & \frac{1}{2} \end{bmatrix}$

$$A^{-1} = \frac{1}{\frac{1}{4} - i^2\frac{3}{4}} \begin{bmatrix} \frac{1}{2} & -i\sqrt{\frac{3}{4}} \\ -i\sqrt{\frac{3}{4}} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -i\sqrt{\frac{3}{4}} \\ -i\sqrt{\frac{3}{4}} & \frac{1}{2} \end{bmatrix}$$

$$\bar{A}^T = A^{-1} \checkmark$$

<u>Thm 1</u>

1. Every eigenvalue of a Hermitian matrix $A_{n \times n}$ is <u>real</u>
2. Every eigenvalue of a skew-Hermitian matrix $A_{n \times n}$ is <u>pure-imaginary or zero</u>
3. If $A_{n \times n}$ is unitary, then each eigenvalue λ of A has length 1 : $\underline{|\lambda| = 1}$

4 Systems of DE.

(Review)

(Ex) 1. $y' - ay = 0$: (Sec 1.5) $\rho = e^{\int -a\,dt} = e^{-at}$

$y(t) = Ce^{at}$

2. $y'' + 3y' + 2y = 0$: linear (chapter 2)

Assume $y(t) = e^{rt}$: $y' = re^{rt}$, $y'' = r^2 e^{rt}$

$r^2 + 3r + 2 = 0$

$(r+1)(r+2) = 0$: $r = -1, -2$

$y(t) = C_1 e^{-t} + C_2 e^{-2t}$: a general sol.

Q Derive a system of DE?

Let $y_1 = y$ & $\underline{y_2 = y'}$

$$\begin{cases} y_2' = y'' = -3y' + (-2)y = \underline{-2y_1 - 3y_2} \\ y_1' = y' = y_2 \end{cases}$$

$$\begin{cases} y_1' = y_2 \\ y_2' = -2y_1 - 3y_2 \end{cases} : \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

(Ex) $X' = \underbrace{\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}}_{\text{let" } A} X.$

A: $\lambda = -1,\ 5$

$\lambda = -1$: $V_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\lambda = 5$: $V_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Let $P = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$, $P^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$

$D = P^{-1}AP = \text{diag}(-1, 5) = \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix}$

$X' = AX \rightarrow P^{-1}X' = P^{-1}APP^{-1}X$

$(P^{-1}X)' = D(P^{-1}X)$: Let $Y = P^{-1}X$

(1) Solve $Y' = DY$

(2) $X = PY$.

① $Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$: $\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$y_1' = -y_1$, $y_2' = 5y_2$

$y_1(t) = c_1 e^{-t}$, $y_2 = c_2 e^{5t}$

② $X = PY = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} c_1 e^{-t} \\ c_2 e^{5t} \end{bmatrix}$

$$X(t) = \underline{\begin{bmatrix} 1 \\ -1 \end{bmatrix} c_1 e^{-t} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} c_2 e^{5t}}$$: a general solution.

Remark: $X_1(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$, $X_2(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{5t}$

eigenvectors

(Ex) $X' = \underbrace{\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}}_{\text{let " } A} X$

(1) λ: $A - \lambda I = \begin{bmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{bmatrix}$

$\det(A - \lambda I) = -\lambda(-3-\lambda) - (-2) = 0$

$\lambda^2 + 3\lambda + 2 = 0$ $(\lambda+1)(\lambda+2) = 0$

$\lambda = -1, -2$

(2) $\lambda = -1$: $\begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $v_1 + v_2 = 0$

$v_2 = -v_1$

$V = \begin{bmatrix} v_1 \\ -v_1 \end{bmatrix}$: Let $V_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

$\lambda = -2$: $\begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad 2v_1 + v_2 = 0$, $v_2 = -2v_1$

$V = \begin{bmatrix} v_1 \\ -2v_1 \end{bmatrix}$: $v_1 = 1$: $V_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$$X(t) = \underline{C_1\begin{bmatrix} 1 \\ -1 \end{bmatrix}e^{-t} + C_2\begin{bmatrix} 1 \\ -2 \end{bmatrix}e^{-2t}}$$

a general solution

Remark

Since $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ are linearly independent.

$X_1(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}e^{-t}$ & $X_2(t) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}e^{-2t}$ are

linearly independent.

Thm 1. (Case 1) $Y' = AY$

If $A_{n\times n}$ has an eigenbasis $\{X_1, \cdots, X_n\}$ and the corresponding eigenvalues $\lambda_1, \cdots, \lambda_n$, then $Y(t) = \underline{C_1X_1e^{\lambda_1 t} + C_2X_2e^{\lambda_2 t} + \cdots + C_nX_ne^{\lambda_n t}}$ is a general solution of $Y' = AY$.

(Ex) $Y' = \underbrace{\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}}_{A}Y$. Find a general solution.

(1) $\lambda = ?$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ 3 & -1-\lambda \end{vmatrix} = (1-\lambda)(-1-\lambda) - 3 = 0$$

$\lambda^2 - 1 - 3 = 0$: $\lambda^2 - 4 = 0$, $\lambda = 2, -2$

(2) $\lambda = 2$: $\begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad -v_1 + v_2 = 0$

$v_2 = v_1$

Let $V_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = -2$: $\begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad 3v_1 + v_2 = 0$

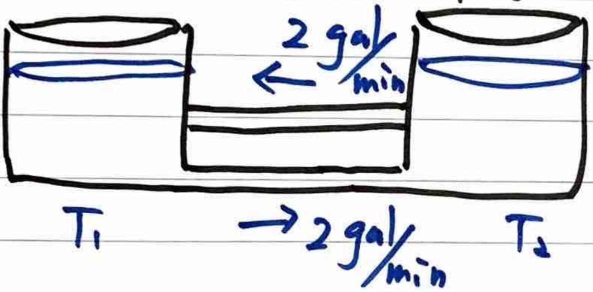
$v_2 = -3v_1$

Let $V_2 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$: lin. indep. (Case 1)

$$Y(t) = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + C_2 \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-2t}$$

: a general solution.

(Tank T_1 & T_2)



unknowns:
the amount of fertilizer

$y_1(t)_{(lb)}$ = the amount of the fertilizer in T_1

$y_2(t)_{(lb)}$ = " " T_2

rule: mass conservation.

T_1: $\dfrac{dy_1}{dt}_{(lb/min)}$ = in-flow − out flow

$$\frac{dy_1}{dt} = 2\left(\frac{gal}{min}\right) \frac{y_2}{200}\left(\frac{lb}{gal}\right) - 2 \cdot \frac{y_1}{200}$$

$\left(\frac{lb}{min}\right)$

$$y_1' = \frac{1}{100} y_2 - \frac{y_1}{100}$$

$$\frac{dy_2}{dt} = 2 \cdot \frac{y_1}{200} - 2 \cdot \frac{y_2}{200} = \frac{y_1}{100} - \frac{y_2}{100}$$

$$\begin{cases} y_1' = -0.01 y_1 + 0.01 y_2 & y_1(0) = 0 \\ y_2' = 0.01 y_1 - 0.01 y_2, & y_2(0) = 100 \end{cases}$$