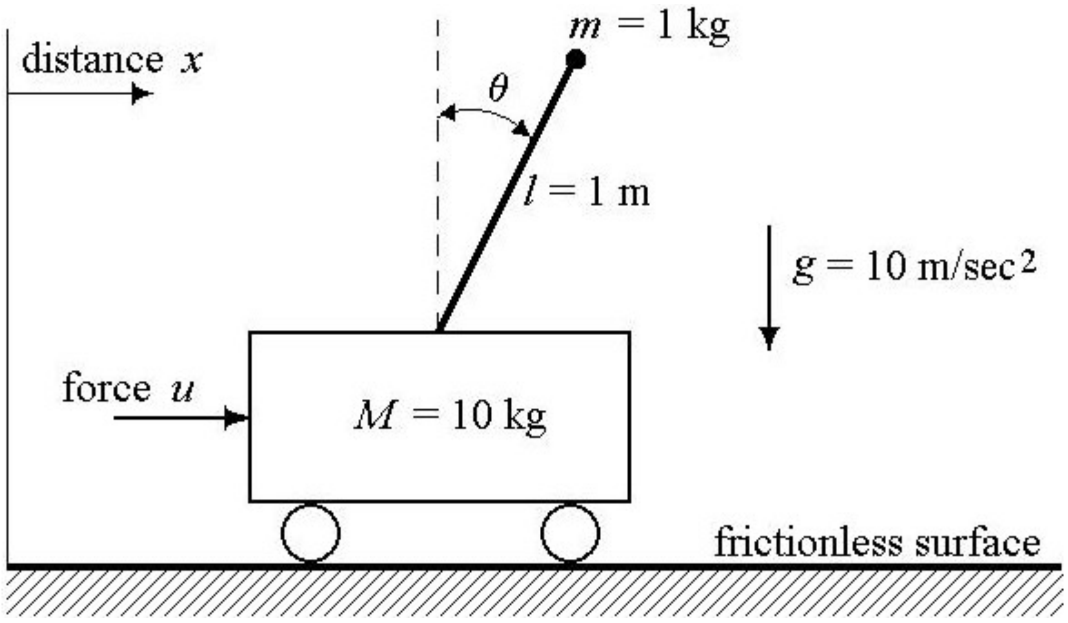
Case Study

We will construct a combined controller-observer compensator for a nonlinear system using a linearized model. The nonlinear dynamical system that we consider consists of a cart with an inverted pendulum (point mass on a mass-less shaft) attached to it as depicted in the figure below. We use a nonlinear system model derived in Module 4.3, linearize it, and use the linearized model as design model to construct a combined observer-controller compensator. In our compensator synthesis, we use linear matrix inequalities.



We use the following state variables:

$$x_1 = x, \quad x_2 = \dot{x}, \quad x_3 = \theta, \quad x_4 = \dot{\theta}.$$

We designate x_1 and x_3 as the system outputs.

The state-space model of the inverted pendulum moving on a cart has the form,

$$\begin{cases} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & \frac{mlx_4^2 \sin x_3 - mg \cos x_3 \sin x_3 + u}{M + m - m\cos^2 x_3} \\ \dot{x}_3 & = & x_4 \\ \dot{x}_4 & = & \frac{-mlx_4^2 \cos x_3 \sin x_3 + gM \sin x_3 + gm \sin x_3 - \cos x_3 u}{l(M + m - m\cos^2 x_3)} \end{cases}$$

We proceed to linearize the model about the equilibrium $x = 0, u = 0$. The linearized model is,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 11 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0.1 \\ 0 \\ -0.1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

We can use the following MATLAB's commands to obtain the linearized model;

```
clear all
clc
x0=[0 0 0 0]';
syms x1 x2 x3 x4 u m M g l
F=[x2;
  (m*l*x4^2*sin(x3)-m*g*cos(x3)*sin(x3)+u)/(M+m-m*cos(x3)^2);
   x4;
  (-m*l*x4^2*cos(x3)*sin(x3)+g*M*sin(x3)+g*m*sin(x3)-cos(x3)*u)/...
  (l*(M+m-m*cos(x3)^2))];
DFx=jacobian(F,[x1 x2 x3 x4]);
A=subs(DFx,[x1 x2 x3 x4 u],[x0(1) x0(2) x0(3) x0(4) 0]);
A=subs(A,[m M g l],[1 10 10 1]);
A=double(A)
DFu=jacobian(F,[u]);
B=subs(DFu,[x1 x2 x3 x4],[x0(1) x0(2) x0(3) x0(4)]);
B=subs(B,[m M g l],[1 10 10 1]);
B=double(B)
C=[1 0 0 0;0 0 1 0]
```

We proceed to design a stabilizing state-feedback control law $u = -kx$ for the linearized system. We use the following LMI to obtain the above results:

```
cvx_begin sdp quiet
% Variables definition
variable S(n,n) symmetric
variable Y(m,n)
% LMI
S*A'-Y'*B'+A*S-B*Y+2*S <=0
S >= eps*eye(n)
cvx_end
k=Y/S
```

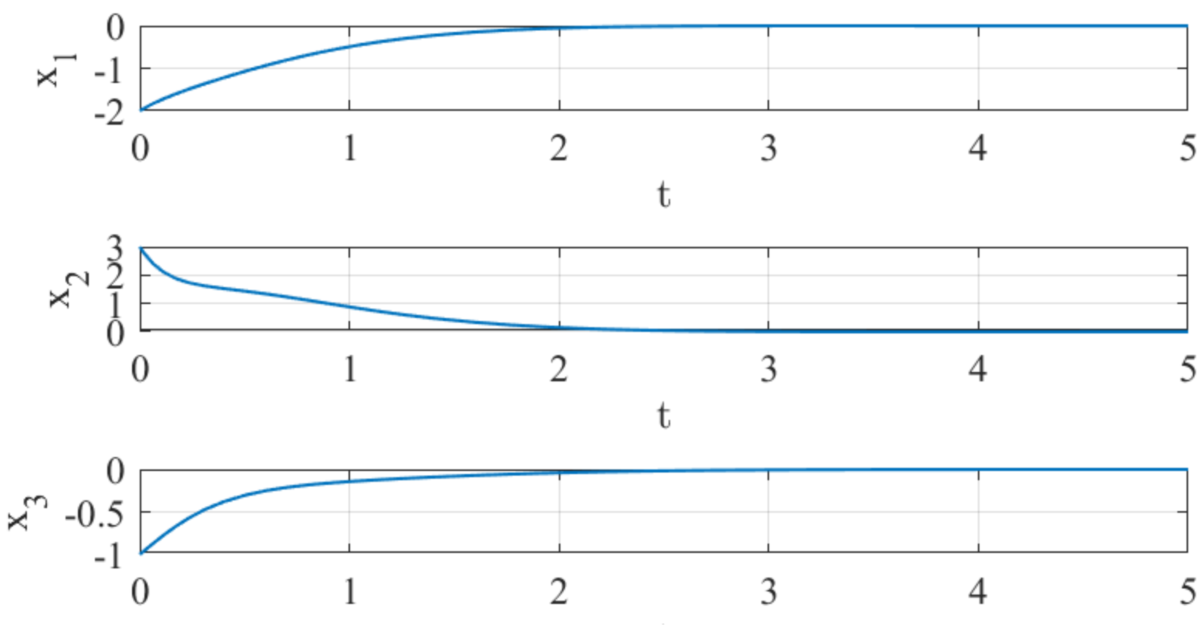
After a few iterations, we obtained the following feedback gain:

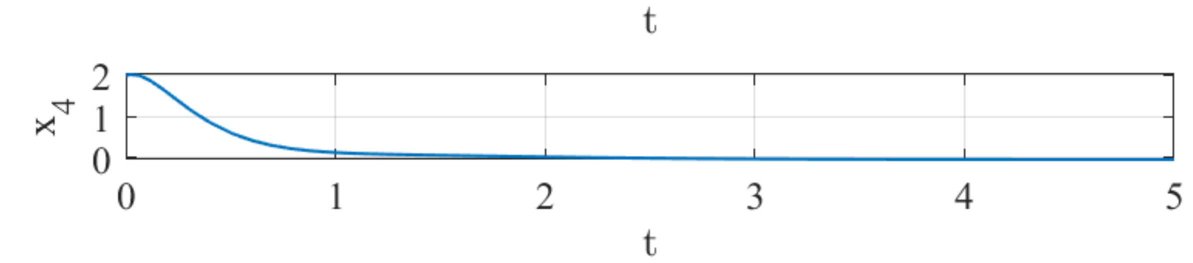
$$k = \begin{bmatrix} -88.1191 & -118.7902 & -809.3007 & -249.9562 \end{bmatrix}.$$

The resulting closed-loop poles are located at

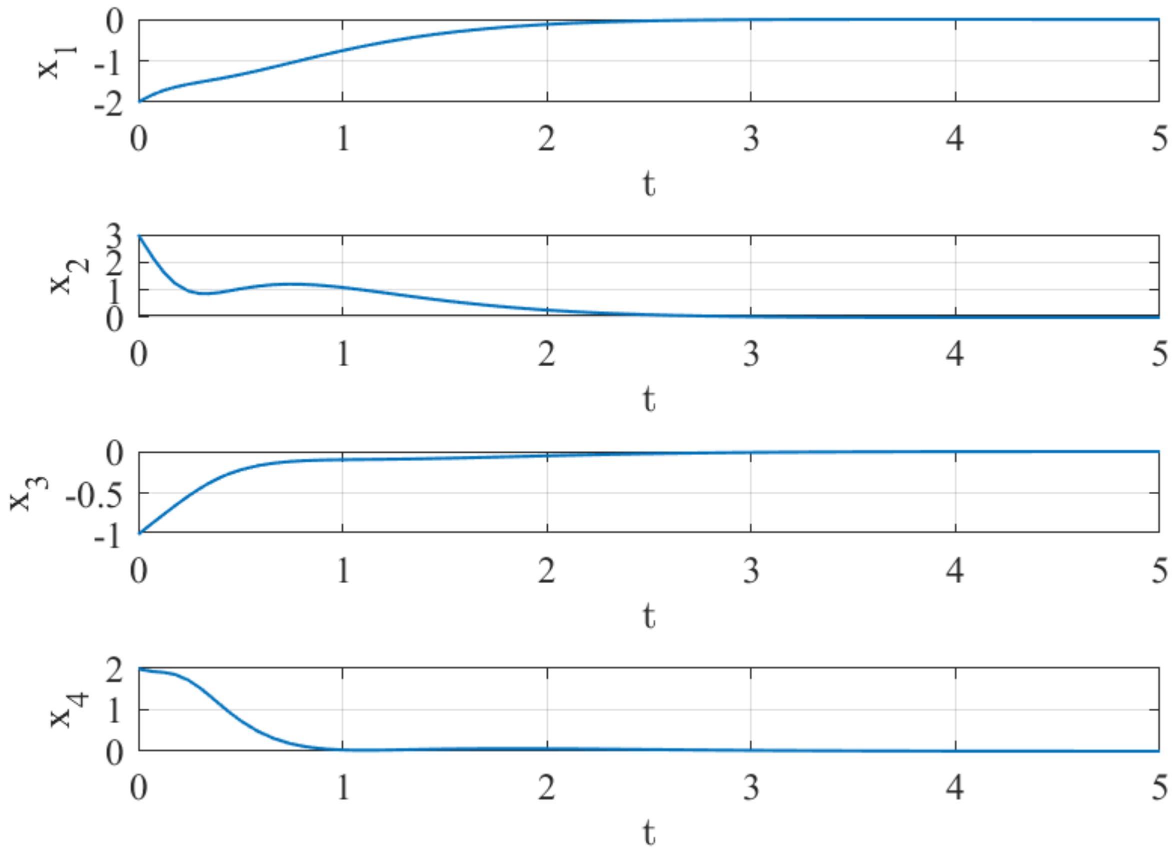
$$\{ -4.8096 + 0.7977i,\ -4.8096 - 0.7977i,\ -1.7487 + 0.8059i,\\ -1.7487 - 0.8059i\}$$

Next, we generated plots of state variables versus time on the interval $[0,\ 5\,\mathrm{sec}]$ for the initial condition, $x(0) = \begin{bmatrix} -2 & 3 & -1 & 2 \end{bmatrix}^\top$. Plots are shown in the following figure.





We tested the above state-feedback controller on the nonlinear system. We generated plots of state variables versus time on the interval $[0,\ 5\,\mathrm{sec}]$ for the initial condition, $x(0) = \begin{bmatrix} -2 & 3 & -1 & 2 \end{bmatrix}^\top$. Plots of state variables of the closed-loop system are shown in the figure below.



The state feedback controller stabilizes both linearized model and the nonlinear model of the inverted pendulum moving on a cart. Plots of the state variables of the linearized and nonlinear models are very similar for the the same initial conditions. For larger deviations from the equilibrium, the linear state-feedback fails to stabilize the nonlinear system. The linearized model is globally stabilizable by the linear feedback.

We proceed to design a state observer using the linearized model. We denote the observer state vector by z. The observer is

$$\dot{z} = Az + bu + L(y - \tilde{y}),$$

where $\tilde{y} = Cz$. We selected the following observer gain,

$$L = \begin{bmatrix} 11.3151 & -1.0830 \\ 39.1666 & -4.6530 \\ 1.0830 & 11.3151 \\ 3.6530 & 50.1666 \end{bmatrix}$$

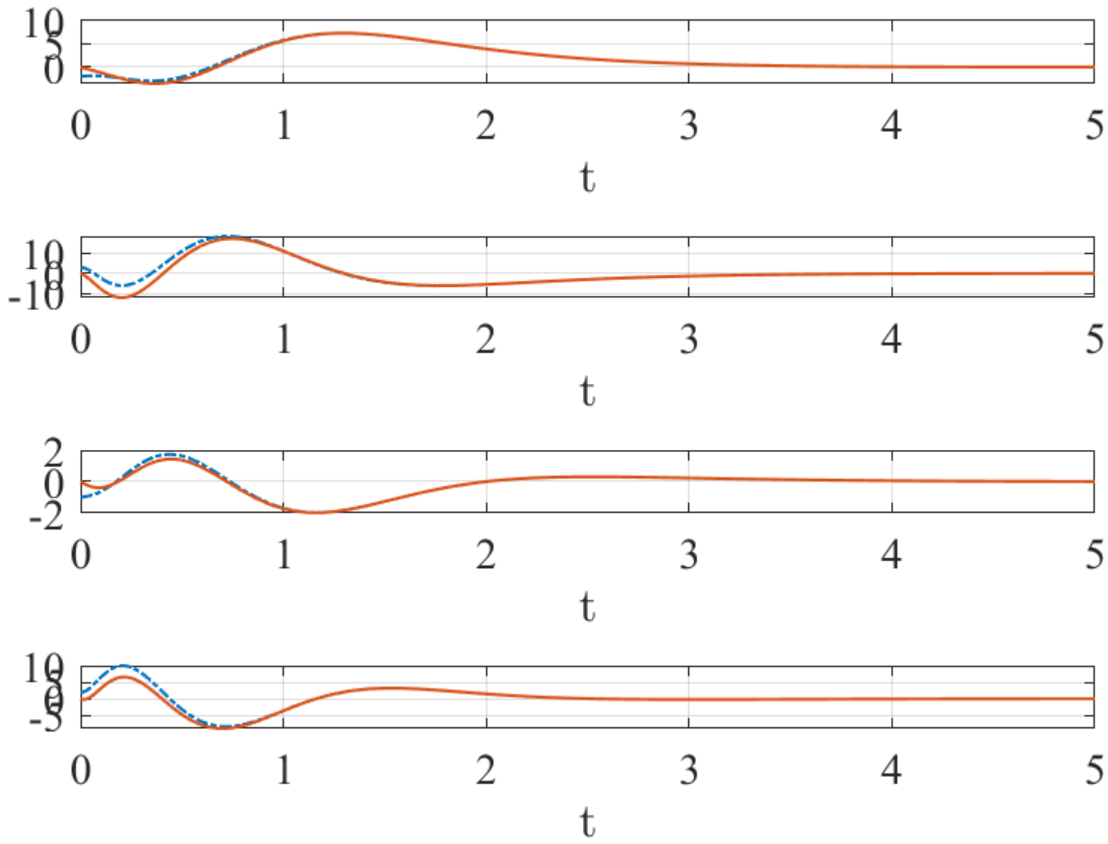
The resulting observer's poles are located at

$$\{ -6.1047 + 3.3077i,\ -6.1047 - 3.3077i,\ -5.2104 + 2.2248i, \\ -5.2104 - 2.2248i\}$$

We obtained the above using the following LMI,

```
cvx_begin sdp quiet
% Variable definition
variable P(n, n) symmetric
variable Z(n, p)
% LMI
A'*P - C'*Z' +P*A - Z*C +4*P <= 0
P >= eps*eye(n)
cvx_end
L=P\Z
```

We used the previously constructed state-feedback to synthesize a combined observer-controller compensator. We first tested our design on the linearized model. In our simulations, we set the initial condition $x(0) = \begin{bmatrix} -2 & 3 & -1 & 2 \end{bmatrix}^\top$ for the plant and $z(0) = 0$ for the observer. Plots of state variables of the closed-loop linear system are shown in the figure below.



Let $u = -kz + r$. We find the transfer function, of the closed-loop system driven by the combined controller-observer compensator. The state-space equations of the closed-loop system driven by the combined observer-controller compensator are:

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} Ax - bkz + br \\ Az - bkz + br + L\left(y - \tilde{y}\right) \end{bmatrix}$$

$$y = \begin{bmatrix} C & O \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix}$$

To proceed with further analysis, we represent the above equations in the form,

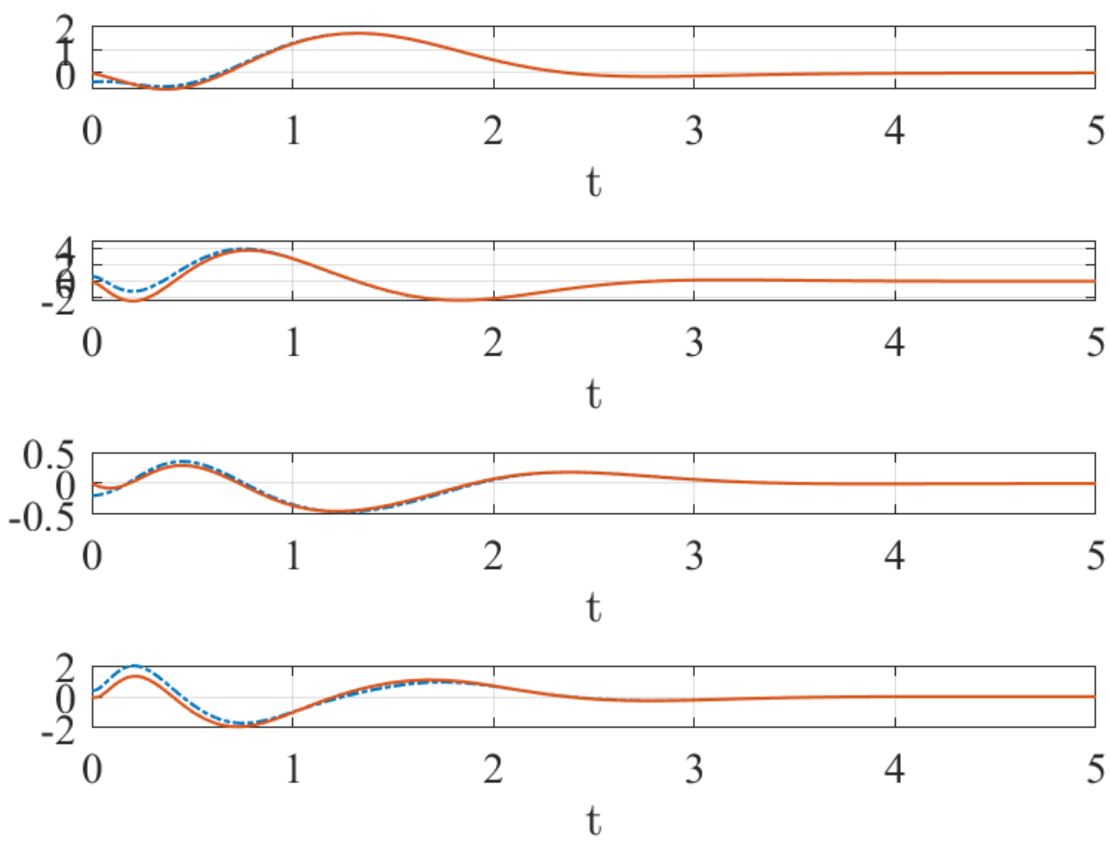
$$
\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A & -bk \\ LC & A-bk-LC \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} b \\ b \end{bmatrix} r
$$

$$
y = \begin{bmatrix} C & O \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix}.
$$

The transfer function of the closed-loop system driven by the combined observer-controller compensator is given by

$$
Y(s) = C(sI - A + bk)^{-1} R(s) = \begin{bmatrix} \frac{0.1s^2-1}{s^2(s^2-1)} \\ \frac{-0.1}{s^2-1} \end{bmatrix} R(s)
$$

Finally, we test our compensator on the nonlinear model. The initial conditions are: $x(0) = \begin{bmatrix} -0.2 & 0.3 & -0.1 & 0.2 \end{bmatrix}^\top$ for the nonlinear plant and $z(0) = 0$ for the observer. Plots of state variables of the closed-loop linear system are shown in the figure below.



Note that for large deviations from the equilibrium, the compensator is unable to force the pendulum into the desired vertical position. This is because the presence of the observer in the loop increases the error that the compensator has to reject. Recall that the compensator was constructed using the design model, which is a linearized model of the plant, while its performance is tested on the simulation model, that is, the nonlinear model of the plant. Thus the compensator is required to reject errors from two sources.

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