AAE 666

Homework Two: Solutions

Exercise 1

$$\ddot{y} + 0.1\dot{y} - y + y^3 = 0$$

Define the state variable as $x_1 = y$ and $x_2 = \dot{y}$, and the system can be described as:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -0.1x_2 + x_1 - x_1^3 \end{aligned}$$

At equilibrium, we have $\dot{x}_1 = 0$, and $\dot{x}_2 = 0$. Thus:

$$\begin{aligned} x_2^e &= 0 \\ -0.1x_2^e + x_1^e - (x_1^e)^3 &= 0 \end{aligned}$$

Therefore, we have the second equation as:

$$\begin{aligned} 0 + x_1^e - (x_1^e)^3 &= 0 \\ x_1^e(1 - (x_1^e)^2) &= 0 \end{aligned}$$

and we have three equilibirum solutions, $x_1^e = 0, -1, 1$ and $x_2^e = 0$.
By linearizing about (x_1^e,x_2^e):

$$\begin{bmatrix} \delta\dot{x}_1 \\ \delta\dot{x}_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 - 3x_1^2 & -0.1 \end{bmatrix}\Bigg|_{(x_1^e, x_2^e)} \begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix}$$

1. At (x_1^e,x_2^e) = (0,0):

$$\begin{bmatrix} \delta\dot{x}_1 \\ \delta\dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -0.1 \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix}$$

The eigenvalues can be computed: λ_1 = -1.0512, and $\lambda_2 = 0.9512$, which means a **saddle point**.
2. At (x_1^e,x_2^e) = ($\pm$1,0):

$$\begin{bmatrix} \delta\dot{x}_1 \\ \delta\dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -0.1 \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix}$$

The eigenvalues can be computed: $\lambda_{1,2} = -0.05 \pm 1.4133i$, which means a **stable focus**. We can first numerically integrate the nonlinear system with multiple initial states so that we see the behavior of the system and plot the phase portrait

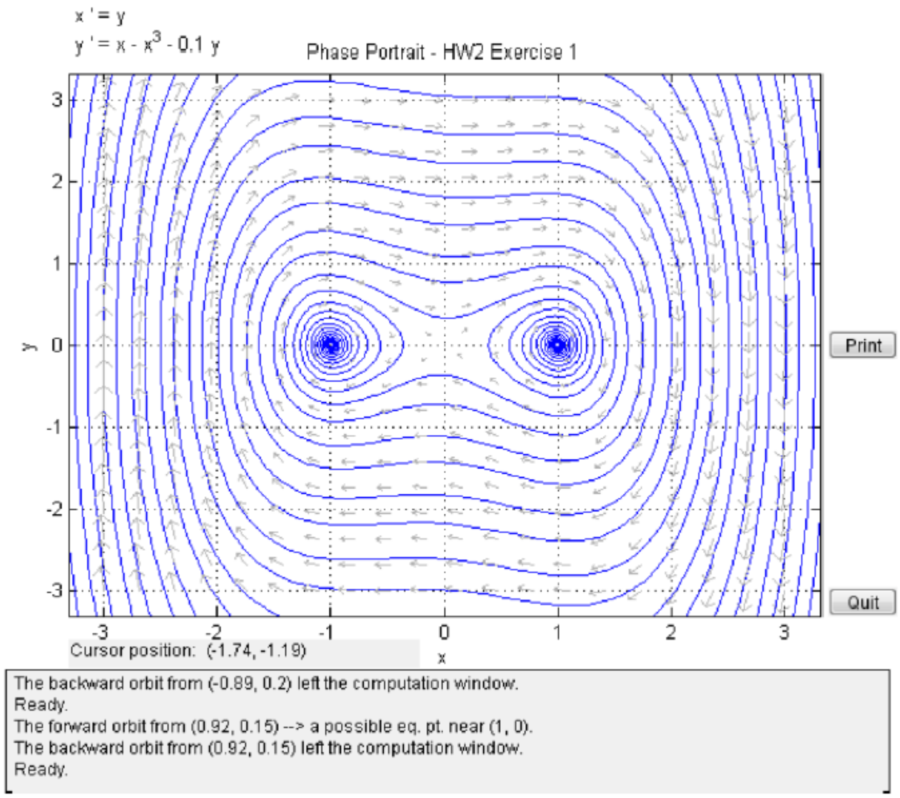

Figure 1: Exercise 1 Plot

Exercise 2

First define the state variables, $x_1 = y$, $x_2 = \dot{y}$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -sinx_1$$

We identify two kinds of equilibrium solutions from the above equations. 1. When $x_2 = 0$, $x_1^e = 2n\pi$

$$\delta\dot{x}_1 = \delta x_2$$
$$\delta\dot{x}_2 = -cosx_1^{\delta}x_1 = -\delta x_1$$
$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

The eigenvalues can be computed: $\lambda_{1,2} = \pm i$, which means a **center**.
2. When $x_2 = 0$, $x_1^e = 2n + 1\pi$

$$\begin{aligned} \delta\dot{x}_1 &= \delta x_2 \\ \delta\dot{x}_2 &= -cosx_1^{\delta}x_1 = \delta x_1 \\ A &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{aligned}$$

The eigenvalues can be computed: $\lambda_{1,2} = \pm 1$, which means a **saddle point**.

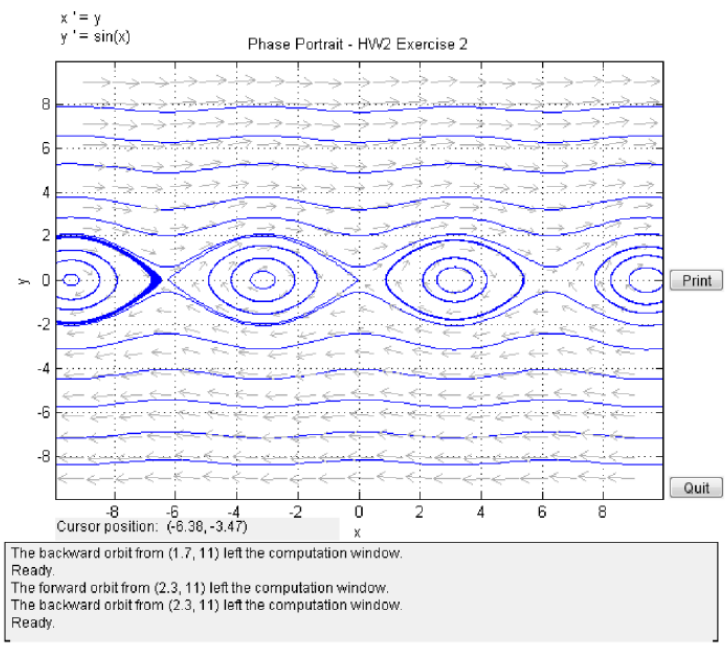

Figure 2: Exercise 2 Plot

Exercise 3

a. $\dot{x} = -x - x^3$

$$\begin{aligned} -x(x^2 + 1) &= 0 \\ x^e &= 0 \end{aligned}$$

The only real equilibrium is $\dot{x} = 0$, where the region of attraction is $(-inf, +inf)$.

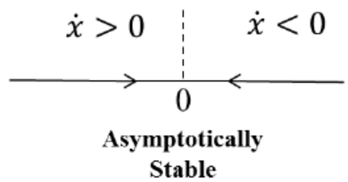

Figure 3: Exercise 3a Plot

b.$\dot{x} = -x + x^3$

$$-x(-x^2 + 1) = 0$$
$$x^e = 0, -1, +1$$

b.$\dot{x} = x - 2x^2 + x^3$

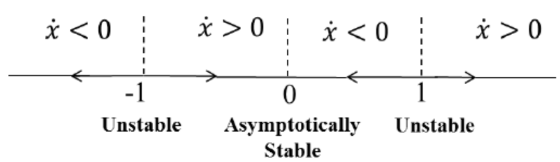

Figure 4: Exercise 3b Plot

$$x(1 - 2x + x^2) = 0$$
$$x^e = 0, +1$$

There is no region of attraction for these equilibrium solutions.

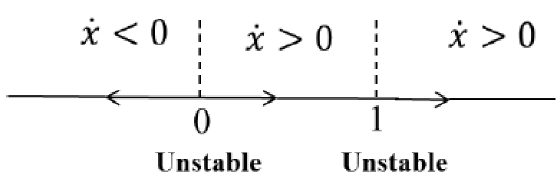

Figure 5: Exercise 3c Plot

Exercise 4

$$\dot{x} = x^3$$
$$\int_{x_0}^{x} \frac{1}{x^3} dx = \int_0^t dt$$
$$\left.\frac{-1}{2x^2}\right|_{x0}^{x} = t$$
$$\frac{-1}{2x^2} + \frac{1}{2x_0^2} = t$$
$$\frac{1 - 2x_0^2 t}{2x_0^2} = \frac{1}{2x^2}$$
$$x(t) = \frac{x_0}{\sqrt{1 - 2x_0^2 t}}$$
$$t_{x\,\mathrm{inf}} = \frac{1}{2x_0^2}$$

Exercise 5

$$\dot{x} = \frac{x}{1+x^2} + sin(x)$$
$$\mid \frac{x}{1+x^2} \mid = \mid \frac{1}{1+x^2} \mid\mid x \mid \leq \mid x \mid$$
$$\mid sin(x) \mid \leq 1$$
$$\mid \frac{x}{1+x^2} + sin(x) \mid \leq \mid x \mid + 1$$

Exercise 6

$$\dot{x} = -\sqrt{(1-x)^2}$$

If $x < 1$, then $\dot{x} = x - 1$
If $x > 1$, then $\dot{x} = -x + 1$
Differentiability of function f guarantees uniqueness, f is not differentiable at x=1. Therefore, $x_0 < 1$ or $x_0 > 1$ guarantees unique solutions.

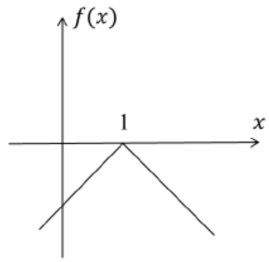

Figure 6: Exercise 6 Plot