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7.7.12 $\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$

$$= a\begin{vmatrix} a & b \\ c & a \end{vmatrix} - b\begin{vmatrix} c & b \\ b & a \end{vmatrix} + c\begin{vmatrix} c & a \\ b & c \end{vmatrix}$$

$$= a(a^2 - bc) - b(ca - b^2) + c(c^2 - ab)$$

$$= a^3 - abc - abc + b^3 + c^3 - abc$$

$$= a^3 + b^3 + c^3 - 3abc.$$

7.7.22 Cramer's rule:

$$A := \begin{bmatrix} 2 & -4 \\ 5 & 2 \end{bmatrix}, \quad u := \begin{bmatrix} x \\ y \end{bmatrix}, \quad b := \begin{bmatrix} -24 \\ 0 \end{bmatrix}.$$

$$D_1 := \begin{vmatrix} -24 & -4 \\ 0 & 2 \end{vmatrix} = -48$$

$$D_2 := \begin{vmatrix} 2 & -24 \\ 5 & 0 \end{vmatrix} = 120$$

$$D := |A| = 24$$

By Cramer's rule,

$$x = \frac{D_1}{D} = \frac{-48}{24} = -2 \qquad \text{and}$$

$$y = \frac{D_2}{D} = \frac{120}{24} = 5.$$

Gaussian elim. & back sub.:

$$\left[\begin{array}{cc|c} 2 & -4 & -24 \\ 5 & 2 & 0 \end{array}\right] \quad R_2 - 2.5R_1 \to R_2$$

$$\left[\begin{array}{cc|c} 2 & -4 & -24 \\ 0 & 12 & 60 \end{array}\right]$$

$$\Rightarrow \quad 2x = 4y - 24,$$
$$12y = 60.$$

$$\Rightarrow \quad x = 2y - 12,$$
$$y = 5.$$

$$\Rightarrow \boxed{\begin{array}{l} x = -2 \\ y = 5. \end{array}}$$

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7.8.5

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 5 & 4 & 1 & 0 & 0 & 1 \end{array}\right]$$

$R_2 - 2R_1 \to R_2$

$R_3 - 5R_1 \to R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 4 & 1 & -5 & 0 & 1 \end{array}\right]$$

$R_3 - 4R_2 \to R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 3 & -4 & 1 \end{array}\right]$$

$$\Rightarrow \boxed{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 4 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -4 & 1 \end{bmatrix}}$$

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7.9.3 $$\begin{bmatrix} -1 & 2 & 3 \\ -4 & 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Set is the nullspace of matrix

$\Rightarrow$ $\boxed{\text{set } \underline{\text{is}} \text{ a vector space.}}$

$$\begin{bmatrix} -1 & 2 & 3 \\ -4 & 1 & 1 \end{bmatrix} \quad R_2 - 4R_1 \to R_2$$

$$\begin{bmatrix} -1 & 2 & 3 \\ 0 & -7 & -11 \end{bmatrix}$$

$\therefore \quad -v_1 = -2v_2 - 3v_3 \ ,$

$-7v_2 = 11 v_3$

$\Rightarrow \ v_1 = 2\left(\frac{11}{-7} v_3\right) + 3v_3 = -\frac{v_3}{7}$

$v_2 = -\frac{11}{7} v_3$

$\therefore$ The set is

$$\left\{ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \;\middle|\; \text{where } v_1, v_2, v_3 \in \mathbb{R}, \; v_1 = -\frac{v_3}{7} \text{ and } v_2 = -\frac{11}{7} v_3 \right\}$$

$$= \left\{ \begin{bmatrix} -v_3/7 \\ -11v_3/7 \\ v_3 \end{bmatrix} \;\middle|\; v_3 \in \mathbb{R} \right\}$$

$\therefore$ A basis for the set is

$\left\{ \begin{bmatrix} -1/7 \\ -11/7 \\ 1 \end{bmatrix} \right\}$ and the set has

dimension 1.

7.9.4 Note:

Quicker methods exist for this problem (since we know that the set of 3×3 real matrices is a vector space), but since they don't appear to be used in this class, I won't use them here.

Let V denote the set of 3×3 real skew-symmetric matrices.

Using the indexing on pg.s 309 and 310:

Let $A, B, C \in V$ and $\alpha, \beta \in \mathbb{R}$.

$$(A+B)^T = A^T + B^T$$
$$= -A - B$$
$$= -(A+B)$$
$$\in V.$$

i.e. $\boxed{V \text{ is closed under "+"}}$.

$\boxed{\text{I.1 and I.2 hold}}$ for the same reason that they hold for the set of 3×3 matrices.

Let $\underline{0} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Clearly, $\underline{0} \in V$ and

$A + \underline{0} = A$. (existence)

If there exists $Z \in V$ such that

$A + Z = A$, then

$-A + A + Z = -A + A$

$\Rightarrow Z = A - A = \underline{0}$ (uniqueness).

So, $\boxed{\text{I.3 is satisfied.}}$

Also, $(-A)^T = -(A^T)$

$= -(-A)$

$= A$,

so, $-A \in V$. (existence)

If there exists $N \in V$ such that

$A + N = \underline{0}$, then

$-A + A + N = -A + \underline{0}$

$\Rightarrow N = -A$ (uniqueness)

So, $\boxed{\text{I.4 is satisfied.}}$

$(\alpha A)^T = \alpha A^T$

$\Rightarrow$ $\boxed{\text{V is closed under scalar multiplication.}}$

$\boxed{\text{II.1 - 4 hold}}$ for the same reason that they hold for the set of 3×3 matrices.

$\therefore$ $\boxed{\text{V is a vector space}}$

If $A \in V$, then $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}$

where $a, b, c, d, e, f, g, h, k \in \mathbb{R}$.

But $A^T = -A$ implies that

$$\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & k \end{bmatrix} = \begin{bmatrix} -a & -b & -c \\ -d & -e & -f \\ -g & -h & -k \end{bmatrix}$$

$\Rightarrow a = e = k = 0,$

$b = -d,$

$c = -g$ and

$f = -h$

Claim:

$$S := \left\{ \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \right\}$$

is a basis for V.

Proof: From the above discussion, if $A \in V$, then

$$A = \begin{bmatrix} 0 & -d & -g \\ d & 0 & -h \\ g & h & 0 \end{bmatrix}$$ for some $d, g, h \in \mathbb{R}$.

and since $S \subseteq V$ and

$$A = d\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + g\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + h\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix},$$

we see that <u>S spans V.</u>

Suppose there exists $c_1, c_2, c_3 \in \mathbb{R}$ s.t.

$$c_1\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c_2\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + c_3\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \underline{0}$$

$$\Rightarrow \begin{bmatrix} 0 & -c_1 & -c_2 \\ c_1 & 0 & -c_3 \\ c_2 & c_3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow c_1 = c_2 = c_3 = 0$

$\therefore$ S is a <u>linearly independent</u> set.

the vector space,

$$\boxed{\begin{array}{l} S \text{ is a basis for } V. \\ \text{So, } V \text{ has dimension } 3. \end{array}}$$

$\therefore$

7.9.22 Let V be the set in question

i.e.

$$V = \left\{ \begin{bmatrix} x & y & z \end{bmatrix}^T \;\middle|\; \begin{array}{l} x, y, z \in \mathbb{R}, \\ \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 0 \end{bmatrix} \end{array} \right\}$$

$$= \left\{ \begin{bmatrix} x & y & z \end{bmatrix}^T \mid x, y, z \in \mathbb{R}, 2x + z = 0 \right\}$$

$$= \left\{ \begin{bmatrix} x \\ y \\ -2x \end{bmatrix} \;\middle|\; x, y \in \mathbb{R} \right\}$$

Similarly to the previously solved problem,

$$\boxed{\begin{array}{l} S := \left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \text{ is a basis} \\ \text{for the vector space } V. \\ \text{So, it has dim. } 2. \end{array}}$$

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8.1.12 Let $A := \begin{bmatrix} 3 & 5 & 3 \\ 0 & 4 & 6 \\ 0 & 0 & 1 \end{bmatrix}$.

$$|A-\lambda| = \begin{vmatrix} 3-\lambda & 5 & 3 \\ 0 & 4-\lambda & 6 \\ 0 & 0 & 1-\lambda \end{vmatrix}$$

$$= (3-\lambda)\begin{vmatrix} 4-\lambda & 6 \\ 0 & 1-\lambda \end{vmatrix} \quad \left(\begin{array}{c}\text{expand along} \\ \text{1st column}\end{array}\right)$$

$$= (3-\lambda)(4-\lambda)(1-\lambda)$$

$\therefore$ $\boxed{\text{Eigenvectors are } 1, 3 \text{ and } 4.}$

Let $v := \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

$$Av = \lambda v$$

$$\Updownarrow$$

$$\begin{bmatrix} 3x_1 & 5x_2 & 3x_3 \\ 0 & 4x_2 & 6x_3 \\ 0 & 0 & x_3 \end{bmatrix} = \begin{bmatrix} \lambda x_1 \\ \lambda x_2 \\ \lambda x_3 \end{bmatrix}$$

<u>Solve $Av = v$ for v</u> :

$$x_3 = x_3$$

$$4x_2 - x_2 = -6x_3 \Rightarrow x_2 = -2x_3$$

$$3x_1 - x_1 = -3x_3 - 5x_2$$

$$= -3x_3 - 5(-2x_3)$$

$$= 7x_3$$

$$\Rightarrow x_1 = \frac{7}{2}x_3$$

$$\therefore \boxed{v = \begin{bmatrix} \frac{7}{2} \\ -2 \\ 1 \end{bmatrix}}$$ works.

Solve $Av = 3v$ for v :

$x_3 = 3x_3 \Rightarrow x_3 = 0$

$4x_2 - 3x_2 = -6x_3 \Rightarrow x_2 = 0$

$3x_1 - 3x_1 = -5x_2 - 3x_3$

$\therefore$ $\boxed{v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{works.}}$

Solve $Av = 4v$ for v :

$x_3 = 4x_3 \Rightarrow x_3 = 0$

$4x_2 - 4x_2 = -6x_3 \Rightarrow x_2 = x_2$

$3x_1 - 4x_1 = -5x_2 - 3x_3$

$\Rightarrow -x_1 = -5x_2 \Rightarrow x_1 = 5x_2$

$\therefore$ $\boxed{v = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} \quad \text{works.}}$

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8.3.8 Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$.

$\boxed{\begin{array}{l} A \text{ is } \underline{\text{not}} \text{ symmetric.} \\ A \text{ is } \underline{\text{not}} \text{ skew-symmetric.} \end{array}}$

$$AA^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos^2\theta+\sin^2\theta & \cos\theta\sin\theta-\sin\theta\cos\theta \\ 0 & \sin\theta\cos\theta-\cos\theta\sin\theta & \sin^2\theta+\cos^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow$ $\boxed{A \text{ is orthogonal}}$

(Every "right inverse" of a square matrix is automatically a "left inverse". I'm not sure if this covered in this course. So we don't need to check that $A^TA = I$).

$$|A - \lambda I| = (1-\lambda)\left((\cos\theta - \lambda)^2 + \sin^2\theta\right)$$

$$= (1-\lambda)(\cos^2\theta + \lambda^2 - 2\lambda\cos\theta + \sin^2\theta)$$

$$= (1-\lambda)(\underbrace{\lambda - 2\lambda\cos\theta + 1}_{\text{roots are}})$$

$$\lambda = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$$

$$= \cos\theta \pm \sqrt{\cos^2\theta - 1}$$

$$= \cos\theta \pm \sqrt{-\sin^2\theta}$$

$$= \cos\theta \pm i|\sin\theta|$$

$$= \cos\theta \pm i\sin\theta$$

(for fixed θ)

$$= (1-\lambda)(\lambda - (\cos\theta + i\sin\theta))(\lambda - (\cos\theta - i\sin\theta))$$

$\therefore$ $\boxed{\text{The spectrum of } A \text{ is } \{1, \cos\theta + i\sin\theta, \cos\theta - i\sin\theta\}.}$

A is neither symmetric nor skew-symmetric so Thm 1 doesn't apply to it and is not illustrated.

A is orthogonal and its eigvals are:

$1, \longleftarrow$ real

$$\left.\begin{array}{l} \cos\theta + i\sin\theta, \\ \cos\theta - i\sin\theta. \end{array}\right\} \longleftarrow \text{pair of complex conjugates}$$

Also,

$$|1| = |\cos\theta + i\sin\theta| = |\cos\theta + i\sin\theta| = 1$$

$\therefore$ Thm 5 is illustrated here.