Ch 6

Linear momentum of multiparticle systems

N2L for particle j:

$$\frac{{}^{I}d}{dt}\Big(\underbrace{m_j \, {}^{I}\vec{v}_{j/O}}\Big) = \vec{F}_j^{(ext)} + \sum_{i=1}^{N} \vec{F}_{j,i}$$

linear momentum for particle j, ${}^{I}\vec{P}_{j/O}$

Summing over all particles j = 1 to N in the system,

$$\underbrace{\sum_{j=1}^{N} \frac{{}^{I}d}{dt}\left({}^{I}\vec{P}_{j/O}\right)} = \sum_{j=1}^{N} \vec{F}_j^{(ext)} + \sum_{j=1}^{N}\sum_{i=1}^{N} \vec{F}_{j,i}$$

The differentiation and summation can be swapped here

$$\sum_{j=1}^{N} \frac{{}^{I}d}{dt}\left({}^{I}\vec{P}_{j/O}\right) = \frac{{}^{I}d}{dt}\Big(\underbrace{\sum_{j=1}^{N} {}^{I}\vec{P}_{j/O}}\Big)$$

Let's introduce a new quantity:

$${}^{I}\vec{P}_{O} \triangleq \sum_{j=1}^{N} {}^{I}\vec{P}_{j/O}$$ Total linear momentum

Now, the summed version of N2L can be rewritten:

$$\frac{{}^{I}d}{dt}\left({}^{I}\vec{P}_0\right) = \underbrace{\sum_{j=1}^{N} \vec{F}_j^{(ext)}}_{\triangleq \vec{F}^{(ext)}} + \underbrace{\sum_{j=1}^{N}\sum_{i=1}^{N} \vec{F}_{j,i}}_{=0}$$

Since $\vec{F}_{j,i} = -\vec{F}_{i,j}$

Finally,

$$\boxed{\frac{{}^{I}d}{dt}\left({}^{I}\vec{P}_0\right) = \vec{F}^{(ext)}}$$

One form of N2L for a multiparticle system.

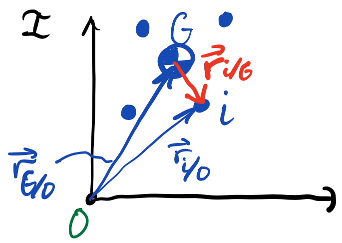
Conservation?

$\Rightarrow$ ${}^{I}\vec{P}_0$ is conserved if $\vec{F}^{(ext)} = 0$

Notes:

1. This shows the Law of conservation of linear momentum for a multiparticle system:
$${}^{I}\vec{P}_0(t_1) = {}^{I}\vec{P}_0(t_2) \quad \text{if} \quad \vec{F}^{(ext)} = 0$$
2. It's also possible to get conservation laws on individual <u>components</u> since this a vector equation.
3. The total change in linear momentum has nothing to do with <u>internal forces</u>
4. The book's derivation in Section 6.1.2 uses linear Impulse

Useful concept:
Center of Mass of a multiparticle system



$$m_G \triangleq \sum_{i=1}^{N} m_i$$

$$\vec{r}_{G/O} \triangleq \frac{1}{m_G} \sum_{i=1}^{N} m_i \vec{r}_{i/O}$$

But $\vec{r}_{i/O} = \vec{r}_{G/O} + \vec{r}_{i/G}$

Then plugging in, (into the definition of $\vec{r}_{G/O}$

$$m_G \vec{r}_{G/O} = \sum_{i=1}^{N} m_i (\vec{r}_{G/O} + \vec{r}_{i/G})$$

$$= \underbrace{\left(\sum_{i=1}^{N} m_i\right)}_{m_G} \vec{r}_{G/O} + \sum_{i=1}^{N} m_i \vec{r}_{i/G}$$

$$\cancel{m_G \vec{r}_{G/O}} = \cancel{m_G \vec{r}_{G/O}} + \sum_{i=1}^{N} m_i \vec{r}_{i/G}$$

$$\Rightarrow \boxed{\sum_{i=1}^{N} m_i \vec{r}_{i/G} = 0}$$

"Center of mass corollary"

What can we say about the motion of the center of mass?

Let's start with N2L for each particle j

$$\underbrace{m_j \, {}^{I}\vec{a}_{j/O}}_{} = \underbrace{\vec{F}_j}_{}$$

and sum over j

$$\text{RHS: } \sum_{j=1}^{N} \left(\vec{F}_j^{(ext)} + \sum_{i=1}^{N} \vec{F}_{j,i} \right) = \underbrace{\sum_{j=1}^{N} \vec{F}_j^{(ext)}}_{\triangleq \vec{F}_G^{(ext)}} + \underbrace{\sum_{j=1}^{N} \sum_{i=1}^{N} \vec{F}_{j,i}}_{=0 \text{ by N3L}}$$

LHS: $\sum_{j=1}^{N} m_j \frac{{}^{I}d}{dt}\left({}^{I}\vec{v}_{j/O}\right) = \frac{{}^{I}d}{dt}\sum_{j=1}^{N} m_j \left({}^{I}\vec{v}_{j/G} + {}^{I}\vec{v}_{G/O}\right)$

$$= \frac{{}^{I}d}{dt}\left(\underbrace{\sum_{j=1}^{N} m_j\, {}^{I}\vec{v}_{j/G}} + \underbrace{m_G\, {}^{I}\vec{v}_{G/O}}_{= {}^{I}\vec{p}_{G/O}}\right)$$

$$\sum_{j=1}^{N} m_j \frac{{}^{I}d}{dt}\vec{r}_{j/G} = \frac{{}^{I}d}{dt}\left(\sum_{j=1}^{N} m_j \vec{r}_{j/G}\right) = 0$$ (By center of mass corollary)

In Summary,

$$\boxed{\frac{{}^{I}d}{dt}\left({}^{I}\vec{p}_{G/O}\right) = \vec{F}_G^{(ext)}}$$

N2L for a multiparticle system.

To study the translational motion of a multiparticle system, treat the C.O.M. as a particle and ignore **internal forces.**

Conservation?

$\Rightarrow$ ${}^{I}\vec{p}_{G/O}$ is conserved if $\vec{F}_G^{(ext)} = \sum_{j=1}^{N} \vec{F}_j^{(ext)} = 0$

How does ${}^{I}\vec{p}_O$ relate to ${}^{I}\vec{p}_{G/O}$?

$${}^{I}\vec{p}_O = \sum_{j=1}^{N} {}^{I}\vec{p}_{j/O} = \sum_{j=1}^{N} m_j\, {}^{I}\vec{v}_{j/O} = \sum_{j=1}^{N} m_j \left({}^{I}\vec{v}_{j/G} + {}^{I}\vec{v}_{G/O}\right)$$

$$= 0 + m_G\, {}^{I}\vec{v}_{G/O} = {}^{I}\vec{p}_{G/O}$$

$$\Rightarrow \boxed{{}^{I}\vec{p}_O = {}^{I}\vec{p}_{G/O}}$$

Note: This will not be true for angular momentum.