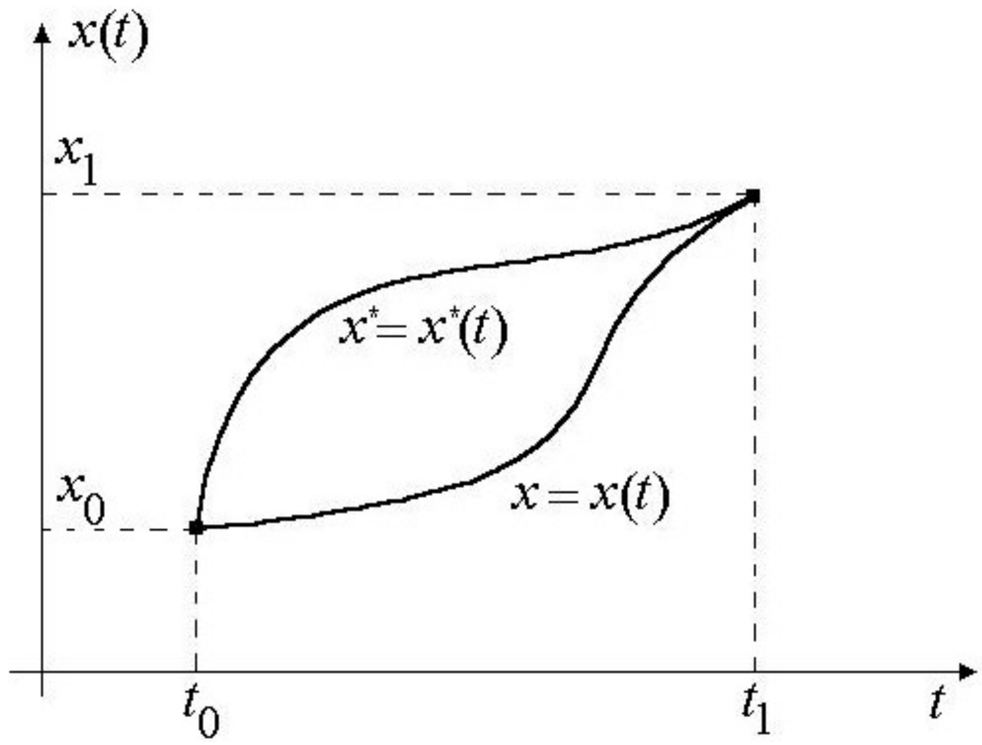
Case Study

The objective of this case study is present a condition that a curve has to satisfy to be a maximizer or a minimizer, that is, to be an extremal of a given functional.

Suppose that we are given two points $\left(t_0, x_0\right)$ and $\left(t_1, x_1\right)$ in the $\left(t, x\right)$-plane. We wish to find a curve, a trajectory, joining the given points such that the functional

$$v\left(x\right) = \int_{t_0}^{t_1} F\left(t, x, \dot{x}\right) dt$$

along this trajectory can achieve its extremal—that is, maximal or minimal—value. The problem is illustrated in the figure below.



We assume that there is an optimal trajectory $x = x(t)$ joining the points (t_0, x_0) and (t_1, x_1) such that $\delta v(x(t)) = 0$. Let us then consider an arbitrary acceptable curve $x^* = x^*(t)$ that is close to x and from a family of curves

$$x(t, \alpha) = x(t) + \alpha\left(x^*(t) - x(t)\right) = x(t) + \alpha \delta x(t).$$

Note that for $\alpha = 0$ we get $x(t)$, and for $\alpha = 1$ we obtain $x^*(t)$. Furthermore,

$$\frac{d}{dt}(\delta x(t)) = \frac{d}{dt}\left(x^*(t) - x(t)\right) = \delta x',$$

and analogously

$$\frac{d^2}{dt^2}(\delta x(t)) = \frac{d^2}{dt^2}\left(x^*(t) - x(t)\right) = \delta x''.$$

We are now ready to investigate the behavior of the functional

$$v = v\left(x\right) = \int_{t_0}^{t_1} F\left(t, x, \dot{x}\right) dt$$

on the curves from the family $x(t, \alpha)$. Note that the functional v can be considered as a function of α, that is, $v\left(x(t, \alpha)\right) = \Phi(\alpha)$. It follows from the first order-necessary condition for a point to be an extremizer of a function of one variable that $\alpha = 0$ is a candidate to be an extremizer of Φ if

$$\left.\frac{d\Phi(\alpha)}{d\alpha}\right|_{\alpha=0} = 0,$$

where

$$\Phi(\alpha) = \int_{t_0}^{t_1} F\left(t, x(t, \alpha), x'(t, \alpha)\right) dt,$$

and $x'(t, \alpha) = \frac{d}{dt} x(t, \alpha)$. We evaluate $\frac{d\Phi(\alpha)}{d\alpha}$ to obtain

$$\frac{d\Phi(\alpha)}{d\alpha} = \int_{t_0}^{t_1} \left(F_x \frac{d}{d\alpha} x(t, \alpha) + F_{x'} \frac{d}{d\alpha} x'(t, \alpha)\right) dt,$$

where $F_x = \frac{\partial}{\partial x} F\left(t, x(t, \alpha), x'(t, \alpha)\right)$, and $F_{x'} = \frac{\partial}{\partial x'} F\left(t, x(t, \alpha), x'(t, \alpha)\right)$. Because

$$\frac{d}{d\alpha} x(t, \alpha) = \frac{d}{d\alpha}\left(x(t) + \alpha \delta x(t)\right) = \delta x(t),$$

and

$$\frac{d}{d\alpha}x'(t,\alpha) = \frac{d}{d\alpha}\left(x'(t) + \alpha\delta x'(t)\right) = \delta x'(t),$$

we can write

$$\frac{d}{d\alpha}\Phi(\alpha) = \int_{t_0}^{t_1}\left(F_x(t, x(t,\alpha), x'(t,\alpha))\delta x(t) + F_{x'}(t, x(t,\alpha), x'(t,\alpha))\delta x'(t)\right)dt.$$

We have,

$$\frac{d}{d\alpha}\Phi(0) = \delta v.$$

Therefore,

$$\delta v = \int_{t_0}^{t_1}\left(F_x\delta x + F_{x'}\delta x'\right)dt,$$

where for convenience we dropped the argument t. Integrating by parts the last term, and taking into account that $\delta x' = (\delta x)'$ gives

$$\delta v = (F_{x'}\delta x)\big|_{t_0}^{t_1} + \int_{t_0}^{t_1}\left(F_x - \frac{d}{dt}F_{x'}\right)\delta x dt.$$

The end points (t_0, x_0) and (t_1, x_1), in this study, are fixed. This implies that

$$\delta x\big|_{t=t_0} = x^*(t_0) - x(t_0) = 0,$$

and

$$\delta x\big|_{t=t_1} = x^*(t_1) - x(t_1) = 0.$$

Hence

$$\delta v = \int_{t_0}^{t_1}\left(F_x - \frac{d}{dt}F_{x'}\right)\delta x dt.$$

Therefore,

$$\delta v = 0 \quad \text{if and only if} \quad F_x - \frac{d}{dt}F_{x'} = 0.$$

The equation $F_x - \frac{d}{dt} F_{x'} = 0$ can also be represented as

$$F_x - F_{x't} - F_{x'x} x' - F_{x'x'} x'' = 0.$$

The equation,

$$\boxed{F_x - \frac{d}{dt} F_{x'} = 0}$$

is known as the **Euler-Lagrange equation**. The curves $x = x(t, C_1, C_2)$ that are solutions to the Euler-Lagrange equations are called **extremals**. The constants C_1 and C_2 are the integration constants. Thus, a functional can only achieve its extremum on the extremals.