ECE 68000: MODERN AUTOMATIC CONTROL

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Observer-Based Feedback Control Design: Further Analysis

Observer-Based Feedback Control Design—Further Analysis

- **Objective**: Construct Combined Observer-Controller Compensator to control linear continuous-time (CT) or discrete-time (DT) system using only input-output info
- We consider linear time-varying (LTI) systems

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t)\\ \boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t)\end{aligned}$$

  or

$$\begin{aligned}\boldsymbol{x}[k+1] &= \boldsymbol{A}\boldsymbol{x}[k] + \boldsymbol{B}\boldsymbol{u}[k]\\ \boldsymbol{y}[k] &= \boldsymbol{C}\boldsymbol{x}[k] + \boldsymbol{D}\boldsymbol{u}[k]\end{aligned}$$

- Assumption: the systems are reachable and observable

Combined Observer-Controller Compensator

- The observer,

$$\dot{\tilde{\boldsymbol{x}}}(t) = \boldsymbol{A}\tilde{\boldsymbol{x}}(t) + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{L}\left(\boldsymbol{y}(t) - \tilde{\boldsymbol{y}}(t)\right),$$

- In our analysis of the closed-loop system driven by the combined observer-controller compensator, take into account that $\tilde{\boldsymbol{y}}(t) = \boldsymbol{C}\tilde{\boldsymbol{x}}(t) + \boldsymbol{D}\boldsymbol{u}(t)$
- Substituting the above expressions for $\tilde{\boldsymbol{y}}(t)$ gives

$$\begin{aligned}\dot{\tilde{\boldsymbol{x}}}(t) &= \boldsymbol{A}\tilde{\boldsymbol{x}}(t) + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{L}\left(\boldsymbol{y}(t) - \tilde{\boldsymbol{y}}(t)\right) \\ &= (\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C})\tilde{\boldsymbol{x}}(t) + \boldsymbol{L}\boldsymbol{y}(t) + (\boldsymbol{B} - \boldsymbol{L}\boldsymbol{D})\boldsymbol{u}(t)\end{aligned}$$

Combined Observer-Controller Compensator Analysis

- Taking into account that $\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t)$ and $\tilde{\boldsymbol{y}}(t) = \boldsymbol{C}\tilde{\boldsymbol{x}}(t) + \boldsymbol{D}\boldsymbol{u}(t)$, we obtain

$$\begin{aligned}\dot{\tilde{\boldsymbol{x}}}(t) &= \boldsymbol{A}\tilde{\boldsymbol{x}}(t) + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{L}\left(\boldsymbol{y}(t) - \tilde{\boldsymbol{y}}(t)\right) \\ &= (\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C})\tilde{\boldsymbol{x}}(t) + \boldsymbol{L}\boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t)\end{aligned}$$

- Closed-loop system

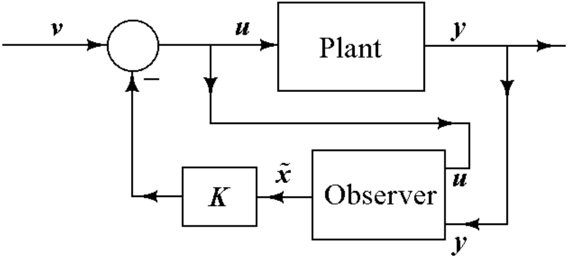
$$\left.\begin{aligned}\begin{bmatrix}\dot{\boldsymbol{x}}(t) \\ \dot{\tilde{\boldsymbol{x}}}(t)\end{bmatrix} &= \begin{bmatrix}\boldsymbol{A} & \boldsymbol{O} \\ \boldsymbol{L}\boldsymbol{C} & \boldsymbol{A} - \boldsymbol{L}\boldsymbol{C}\end{bmatrix}\begin{bmatrix}\boldsymbol{x}(t) \\ \tilde{\boldsymbol{x}}(t)\end{bmatrix} + \begin{bmatrix}\boldsymbol{B} \\ \boldsymbol{B}\end{bmatrix}\boldsymbol{u}(t) \\ \boldsymbol{y}(t) &= \begin{bmatrix}\boldsymbol{C} & \boldsymbol{O}\end{bmatrix}\begin{bmatrix}\boldsymbol{x}(t) \\ \tilde{\boldsymbol{x}}(t)\end{bmatrix} + \boldsymbol{D}\boldsymbol{u}(t)\end{aligned}\right\}$$

Controller implementation

- The control law

$$\boldsymbol{u}(t) = -\boldsymbol{K}\tilde{\boldsymbol{x}}(t) + \boldsymbol{v}(t)$$

instead of the actual state-feedback control law



Closed-loop system

- Closing the loop

$$\left.\begin{aligned} \begin{bmatrix} \dot{\boldsymbol{x}}(t) \\ \dot{\tilde{\boldsymbol{x}}}(t) \end{bmatrix} &= \begin{bmatrix} \boldsymbol{A} & \boldsymbol{O} \\ \boldsymbol{LC} & \boldsymbol{A}-\boldsymbol{LC} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(t) \\ \tilde{\boldsymbol{x}}(t) \end{bmatrix} + \begin{bmatrix} \boldsymbol{B} \\ \boldsymbol{B} \end{bmatrix} (-\boldsymbol{K}\tilde{\boldsymbol{x}}(t) + \boldsymbol{v}(t)) \\ \boldsymbol{y}(t) &= \begin{bmatrix} \boldsymbol{C} & \boldsymbol{O} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(t) \\ \tilde{\boldsymbol{x}}(t) \end{bmatrix} + \boldsymbol{D}(-\boldsymbol{K}\tilde{\boldsymbol{x}}(t) + \boldsymbol{v}(t)) \end{aligned}\right\}$$

- Perform manipulations,

$$\begin{aligned} \begin{bmatrix} \dot{\boldsymbol{x}}(t) \\ \dot{\tilde{\boldsymbol{x}}}(t) \end{bmatrix} &= \begin{bmatrix} \boldsymbol{A} & -\boldsymbol{BK} \\ \boldsymbol{LC} & \boldsymbol{A}-\boldsymbol{LC}-\boldsymbol{BK} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(t) \\ \tilde{\boldsymbol{x}}(t) \end{bmatrix} + \begin{bmatrix} \boldsymbol{B} \\ \boldsymbol{B} \end{bmatrix} \boldsymbol{v}(t) \\ \boldsymbol{y}(t) &= \begin{bmatrix} \boldsymbol{C} & -\boldsymbol{DK} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(t) \\ \tilde{\boldsymbol{x}}(t) \end{bmatrix} + \boldsymbol{D}\boldsymbol{v}(t) \end{aligned}$$

- To analyze the above closed-loop system, it is convenient to perform a change of coordinates

Closed-loop system analysis

- Use the transformation

$$\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{x} - \tilde{\boldsymbol{x}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{I}_n & \boldsymbol{O} \\ \boldsymbol{I}_n & -\boldsymbol{I}_n \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \tilde{\boldsymbol{x}} \end{bmatrix}$$

- Note that

$$\begin{bmatrix} \boldsymbol{I}_n & \boldsymbol{O} \\ \boldsymbol{I}_n & -\boldsymbol{I}_n \end{bmatrix}^{-1} = \begin{bmatrix} \boldsymbol{I}_n & \boldsymbol{O} \\ \boldsymbol{I}_n & -\boldsymbol{I}_n \end{bmatrix}$$

Closed-loop system in the new coordinates

- The closed-loop system in the new coordinates

$$
\begin{aligned}
\begin{bmatrix} \dot{\boldsymbol{x}}(t) \\ \dot{\boldsymbol{x}}(t) - \dot{\tilde{\boldsymbol{x}}}(t) \end{bmatrix} &= \begin{bmatrix} \boldsymbol{A} - \boldsymbol{B}\boldsymbol{K} & \boldsymbol{B}\boldsymbol{K} \\ \boldsymbol{O} & \boldsymbol{A} - \boldsymbol{L}\boldsymbol{C} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(t) \\ \boldsymbol{x}(t) - \tilde{\boldsymbol{x}}(t) \end{bmatrix} \\
&\quad + \begin{bmatrix} \boldsymbol{B} \\ \boldsymbol{O} \end{bmatrix} \boldsymbol{v}(t) \\
\boldsymbol{y}(t) &= \begin{bmatrix} \boldsymbol{C} - \boldsymbol{D}\boldsymbol{K} & \boldsymbol{D}\boldsymbol{K} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(t) \\ \boldsymbol{x}(t) - \tilde{\boldsymbol{x}}(t) \end{bmatrix} + \boldsymbol{D}\boldsymbol{v}(t)
\end{aligned}
$$

- The transfer function matrix of the closed-loop system

$$
\boldsymbol{Y}(s) = \Big((\boldsymbol{C} - \boldsymbol{D}\boldsymbol{K}) \left(s\boldsymbol{I}_n - \boldsymbol{A} + \boldsymbol{B}\boldsymbol{K} \right)^{-1} \boldsymbol{B} + \boldsymbol{D} \Big) \boldsymbol{V}(s)
$$

- The closed-loop system driven by the combined observer-controller compensator has is the same transfer function as the system driven by the state-feedback control law $\boldsymbol{u}(t) = -\boldsymbol{K}\boldsymbol{x}(t) + \boldsymbol{v}(t)$

Implementing the combined observer-controller compensator for nonlinear systems

- Non-linear system $\dot{\boldsymbol{x}} = f(\boldsymbol{x}, \boldsymbol{u})$, $\boldsymbol{y} = \boldsymbol{h}(\boldsymbol{x}, \boldsymbol{u})$ linearized about $(\boldsymbol{x}_e, \boldsymbol{u}_e)$
- Linearized model

$$
\begin{aligned}
\frac{d}{dt}\delta\boldsymbol{x} &= \boldsymbol{A}\delta\boldsymbol{x} + \boldsymbol{B}\delta\boldsymbol{u} \\
\delta\boldsymbol{y} &= \boldsymbol{C}\delta\boldsymbol{x} + \boldsymbol{D}\delta\boldsymbol{u}
\end{aligned}
$$

  where

$$
\delta\boldsymbol{x} = \boldsymbol{x} - \boldsymbol{x}_e, \quad \delta\boldsymbol{u} = \boldsymbol{u} - \boldsymbol{u}_e, \quad \text{and} \quad \delta\boldsymbol{y} = \boldsymbol{y} - \boldsymbol{y}_e
$$

- Control law designed for the linearized system

$$
\delta\boldsymbol{u} = -\boldsymbol{K}\delta\boldsymbol{x} + \boldsymbol{v}
$$

Observer for nonlinear systems

- Observer constructed for the linearized model

$$
\begin{aligned}
\frac{d}{dt}\delta\boldsymbol{x} &= \boldsymbol{A}\delta\boldsymbol{x} + \boldsymbol{B}\delta\boldsymbol{u} \\
\delta\boldsymbol{y} &= \boldsymbol{C}\delta\boldsymbol{x} + \boldsymbol{D}\delta\boldsymbol{u}
\end{aligned}
$$

- Observer

$$
\begin{aligned}
\frac{d}{dt}\boldsymbol{z} &= \boldsymbol{A}\boldsymbol{z} + \boldsymbol{B}\delta\boldsymbol{u} + \boldsymbol{L}(\delta\boldsymbol{y} - \delta\tilde{\boldsymbol{y}}) \\
\delta\tilde{\boldsymbol{y}} &= \boldsymbol{C}\boldsymbol{z} + \boldsymbol{D}\delta\boldsymbol{u}
\end{aligned}
$$

- Note that, for simplicity, I used $\boldsymbol{z}$ instead of $\delta\tilde{\boldsymbol{x}}$
- Observer's estimate $\boldsymbol{z}$ estimates $\delta\boldsymbol{x}$, that is, $\boldsymbol{z} \to \delta\boldsymbol{x}$
- Control law, $\delta\boldsymbol{u} = -\boldsymbol{K}\boldsymbol{z} + \boldsymbol{v}$, designed for the linearized system

Combined observer-controller compensator for nonlinear systems

- Observer constructed for the linearized model
- Observer's estimate $\boldsymbol{z} = \delta\tilde{\boldsymbol{x}}$ estimates $\delta\boldsymbol{x}$
- Control law, $\delta\boldsymbol{u} = -\boldsymbol{K}\delta\boldsymbol{x} + \boldsymbol{v}$, designed for the linearized system
- Use $\boldsymbol{z} = \delta\tilde{\boldsymbol{x}}$ instead of $\delta\boldsymbol{x}$

$$\begin{aligned}
\delta\boldsymbol{u} &= \boldsymbol{u} - \boldsymbol{u}_e \\
&= -\boldsymbol{K}\delta\tilde{\boldsymbol{x}} + \boldsymbol{v} \\
&= -\boldsymbol{K}\boldsymbol{z} + \boldsymbol{v}
\end{aligned}$$

- Controller applied to the nonlinear system

$$\boldsymbol{u} = -\boldsymbol{K}\boldsymbol{z} + \boldsymbol{u}_e + \boldsymbol{v}$$