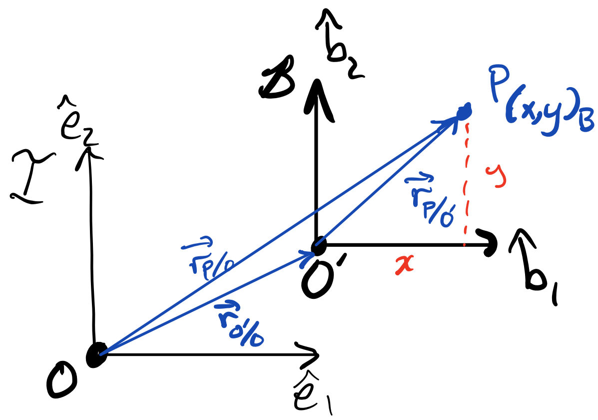
Introduction to Relative Motion

Section 3.6 of K+P



For now, consider the case of translation only (i.e., the $\mathcal{B}$ frame does not rotate).

What if we want the E.O.M.'s to be in terms of $(x,y)_B$ coordinates?

Body frame kinematics

$\mathcal{B}$

$$\vec{r}_{P/O'} = x\hat{b}_1 + y\hat{b}_2$$

$$^{B}\vec{v}_{P/O'} = \frac{^{B}d}{dt}(x\hat{b}_1 + y\hat{b}_2) = \dot{x}\hat{b}_1 + \dot{y}\hat{b}_2$$

$$^{B}\vec{a}_{P/O'} = \ddot{x}\hat{b}_1 + \ddot{y}\hat{b}_2$$

What is the inertial velocity vector for $\vec{r}_{P/O'}$?

$$^{\mathcal{I}}\vec{v}_{P/O'} = \frac{^{\mathcal{I}}d}{dt}(x\hat{b}_1 + y\hat{b}_2)$$

$$= \dot{x}\hat{b}_1 + x\frac{^{\mathcal{I}}d}{dt}(\hat{b}_1) + \dot{y}\hat{b}_2 + y\frac{^{\mathcal{I}}d}{dt}(\hat{b}_2)$$

But $\frac{^{\mathcal{I}}d}{dt}(\hat{b}_1) = \underbrace{^{\mathcal{I}}\vec{\omega}^{B}}_{\omega = 0} \times \hat{b}_1 = 0$ Similarly $\frac{^{\mathcal{I}}d}{dt}(\hat{b}_2) = 0$

So, ${}^{I}\vec{v}_{P/O'} = \dot{x}\hat{b}_1 + \dot{y}\hat{b}_2 = {}^{B}\vec{v}_{P/O'}$

EQUAL!

In the case of translation only, these expressions are equal.

Similarly,

$${}^{I}\vec{a}_{P/O'} = \ddot{x}\hat{b}_1 + \ddot{y}\hat{b}_2 = {}^{B}\vec{a}_{P/O'}$$

Let's reference the kinematics back to point O of the $\mathcal{I}$ frame to see what happens:

$$\vec{r}_{P/O} = \vec{r}_{P/O'} + \vec{r}_{O'/O}$$

$${}^{I}\vec{v}_{P/O} = {}^{I}\vec{v}_{P/O'} + {}^{I}\vec{v}_{O'/O}$$

$${}^{I}\vec{a}_{P/O} = {}^{I}\vec{a}_{P/O'} + {}^{I}\vec{a}_{O'/O}$$

Newton's 2nd law:

$$\vec{F}_P = m_P\,{}^{I}\vec{a}_{P/O}$$

$$= m_P\left({}^{I}\vec{a}_{P/O'} + {}^{I}\vec{a}_{O'/O}\right)$$

If the frame is translating only,

$$= m_P\left({}^{B}\vec{a}_{P/O'} + {}^{I}\vec{a}_{O'/O}\right)$$

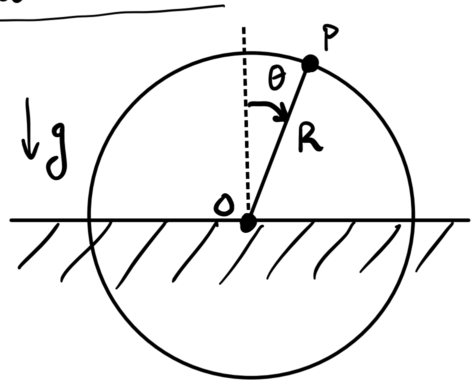
Re-arranging,

$$\underbrace{\vec{F}_P - m_P\,{}^{I}\vec{a}_{O'/O}}_{\text{Effect of frame acceleration.}} = \underbrace{m_P\,{}^{B}\vec{a}_{P/O'}}_{\text{Useful if you have sensor in the body frame}}$$

If the frame is not accelerating,

$$\vec{F}_P = m_P\,{}^{B}\vec{a}_{P/O'} \Rightarrow$$ Newton's 2nd law holds in this frame. The frame is inertial.

Tutorial 3.3 Find the E.O.M. for a particle sliding on a hemisphere.



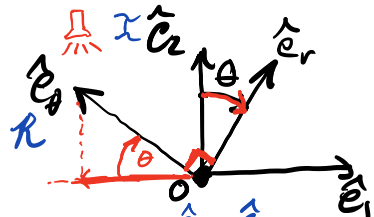
DOF?

$M = 2N - K$

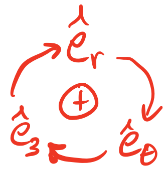
$N = 1, \quad K = 1$

$M = 1$

Reference frames



	$\hat{e}_r$	$\hat{e}_\theta$
$\hat{e}_1$	$\sin\theta$	$-\cos\theta$
$\hat{e}_2$	$\cos\theta$	$\sin\theta$



Kinematics

$\vec{r}_{P/O} = R\hat{e}_r$

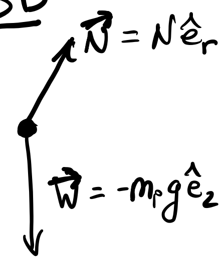
${}^{I}\vec{v}_{P/O} = -R\dot{\theta}\hat{e}_\theta$

Note: $\frac{{}^{I}d}{dt}(\hat{e}_r) = {}^{I}\vec{\omega}^{R} \times \hat{e}_r$

$= (-\dot{\theta}\hat{e}_3) \times \hat{e}_r$

$= -\dot{\theta}\hat{e}_\theta$

FBD



$${}^{I}\vec{a}_{P/O} = -R\ddot{\theta}\hat{e}_\theta - R\dot{\theta}\underbrace{\frac{{}^{I}d}{dt}(\hat{e}_\theta)}_{= -\dot{\theta}\hat{e}_3 \times \hat{e}_\theta = \dot{\theta}\hat{e}_r}$$

$${}^{I}\vec{a}_{P/O} = -R\ddot{\theta}\hat{e}_\theta - R\dot{\theta}^2\hat{e}_r$$

Newton's 2nd Law

$$\vec{F}_P = m_P\,{}^{I}\vec{a}_{P/O}$$

$$N\hat{e}_r - m_P g\hat{e}_2 = m_P(-R\ddot{\theta}\hat{e}_\theta - R\dot{\theta}^2\hat{e}_r)$$

$\hat{e}_2 \rightarrow \cos\theta\,\hat{e}_r + \sin\theta\,\hat{e}_\theta$

E.O.M.'s

$$\hat{e}_r: \quad N - m_p g\cos\theta = -m_p R\dot{\theta}^2 \Rightarrow N = m_p\left(g\cos\theta - R\dot{\theta}^2\right)$$

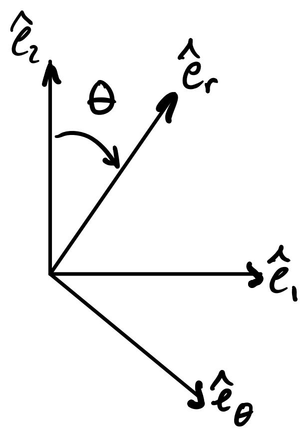
$$\hat{e}_\theta: \quad -\cancel{m_p} g\sin\theta = -\cancel{m_p} R\ddot{\theta}$$

$$\Rightarrow \boxed{\ddot{\theta} = \frac{g}{R}\sin\theta}$$

Note: 1 DOF, since 1 constraint. Therefore, 1 E.O.M.

See the remainder of the tutorial to solve for t^* and θ^* when the particle falls off.

What if we had defined our reference frames differently?



Their $\hat{e}_3$ vectors are not aligned if they are both right-handed frames.

$$\begin{array}{c|cc} & \hat{e}_r & \hat{e}_\theta \\ \hline \hat{e}_1 & \sin\theta & \cos\theta \\ \hat{e}_2 & \cos\theta & -\sin\theta \end{array}$$

$\underbrace{\qquad}_{{}^I C^R}$ Transformation matrix, but not rotation matrix

Note:

$$\det({}^I C^R) = (-\sin\theta)(\sin\theta) - (\cos\theta)(\cos\theta)$$
$$= -\sin^2\theta - \cos^2\theta$$
$$= -1(\sin^2\theta + \cos^2\theta)$$
$$= -1$$

Read Ch 4 of K+P, skip impulse stuff

Ch 4: Linear and Angular Momentum of a Particle

Momentum is a tool for solving problems that can sometimes make find trajectories or properties of trajectories easier. → Can also make finding the E.O.M. easier in some cases. For example, see Ex 4.4 on the pendulum.

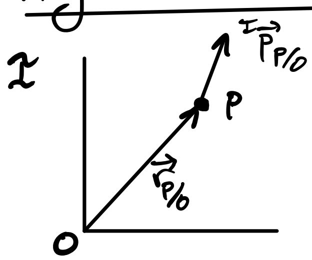
$\boxed{\text{Linear Momentum}}$ $\quad {}^{I}\vec{p}_{P/O} = m_P \, {}^{I}\vec{v}_{P/O}$

N2L: $\vec{F}_P = m_P \, {}^{I}\vec{a}_{P/O} = m_P \frac{{}^{I}d}{dt}\left({}^{I}\vec{v}_{P/O}\right) = \frac{{}^{I}d}{dt}\left({}^{I}\vec{p}_{P/O}\right)$

Conservation?

If $\vec{F}_P = 0$, then ${}^{I}\vec{p}_{P/O}$ is conserved

Angular momentum relative to an inertially fixed point O



$${}^{I}\vec{h}_{P/O} \triangleq \vec{r}_{P/O} \times {}^{I}\vec{p}_{P/O}$$
$$= \vec{r}_{P/O} \times m_P \, {}^{I}\vec{v}_{P/O}$$

How does angular momentum change with time?

$$\frac{{}^{I}d}{dt}\left({}^{I}\vec{h}_{P/O}\right) = \left(\frac{{}^{I}d}{dt}\left(\vec{r}_{P/O}\right) \times {}^{I}\vec{p}_{P/O}\right) + \left(\vec{r}_{P/O} \times \frac{{}^{I}d}{dt}\left(m_P \, {}^{I}\vec{v}_{P/O}\right)\right)$$
$$= \underbrace{\left({}^{I}\vec{v}_{P/O} \times m_P \, {}^{I}\vec{v}_{P/O}\right)}_{=0} + \left(\vec{r}_{P/O} \times \underbrace{m_P \, {}^{I}\vec{a}_{P/O}}_{\vec{F}_P}\right)$$

$$\frac{{}^{I}d}{dt}\left({}^{I}\vec{h}_{P/O}\right) = \underbrace{\vec{r}_{P/O} \times \vec{F}_P}_{\triangleq \vec{M}_{P/O}}$$

$$\boxed{\frac{{}^{I}d}{dt}\left({}^{I}\vec{h}_{P/O}\right) = \vec{M}_{P/O}}$$

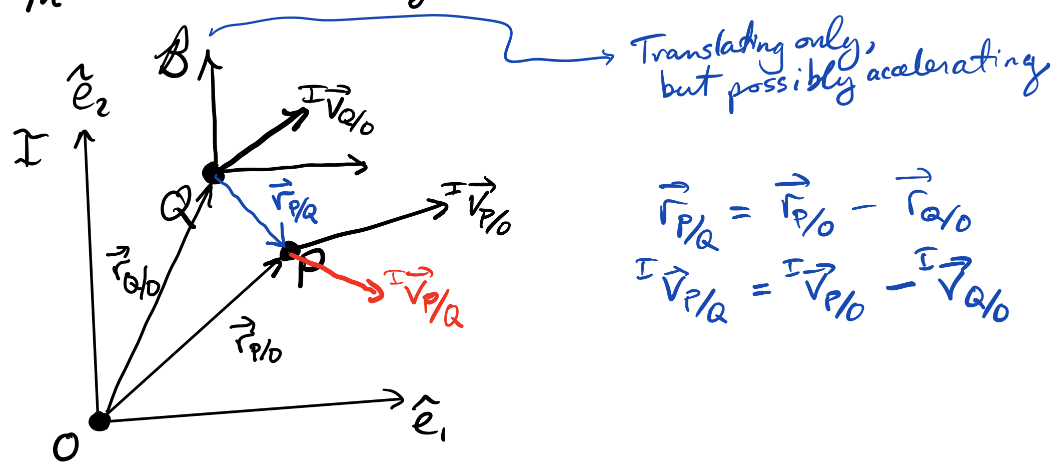
Rotational version of N2L
(Angular momentum form of N2L)

Conservation?

If $\vec{M}_{P/O} = 0$, ${}^{I}\vec{h}_{P/O}$ is conserved.

Angular momentum about an arbitrary point

Useful when you want to use angular momentum in an accelerating frame.



$$\vec{r}_{P/Q} = \vec{r}_{P/O} - \vec{r}_{Q/O}$$

$${}^{I}\vec{v}_{P/Q} = {}^{I}\vec{v}_{P/O} - {}^{I}\vec{v}_{Q/O}$$

$$\begin{aligned}
{}^{I}\vec{h}_{P/Q} &= \vec{r}_{P/Q} \times m_P\, {}^{I}\vec{v}_{P/Q} \\
&= \left(\vec{r}_{P/O} - \vec{r}_{Q/O}\right) \times m_P\left({}^{I}\vec{v}_{P/O} - {}^{I}\vec{v}_{Q/O}\right)
\end{aligned}$$

$$\begin{aligned}
\frac{{}^{I}d}{dt}\left({}^{I}\vec{h}_{P/Q}\right) &= \left({}^{I}\vec{v}_{P/O} - {}^{I}\vec{v}_{Q/O}\right) \times m_P\left({}^{I}\vec{v}_{P/O} - {}^{I}\vec{v}_{Q/O}\right) \\
&+ \underbrace{\left(\vec{r}_{P/O} - \vec{r}_{Q/O}\right)}_{\vec{r}_{P/Q}} \times \underbrace{m_P\left({}^{I}\vec{a}_{P/O}\right.}_{\vec{F}_P} \left. - {}^{I}\vec{a}_{Q/O}\right)
\end{aligned}$$

$$\frac{{}^{I}d}{dt}\left({}^{I}\vec{h}_{P/Q}\right) = \underbrace{\vec{r}_{P/Q} \times \vec{F}_P}_{\triangleq \vec{M}_{P/Q}} - \vec{r}_{P/Q} \times m_P\, {}^{I}\vec{a}_{Q/O}$$

$$\boxed{\frac{{}^{I}d}{dt}\left({}^{I}\vec{h}_{P/Q}\right) = \vec{M}_{P/Q} - \underbrace{\vec{r}_{P/Q} \times m_P\, {}^{I}\vec{a}_{Q/O}}_{\text{inertial moment}}}$$

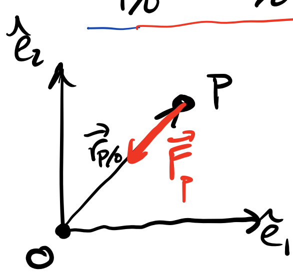
Rotational form of N2L about an arbitrary point Q.

Note: If ${}^{I}\vec{a}_{Q/O} = 0$, the rotational form of N2L holds.

Dfn: A <u>central force</u> is a nonzero force that ... That is, for $\vec{F}_P \neq 0$ a central force satisfies

$$\vec{M}_{P/O} = \vec{r}_{P/O} \times \vec{F}_P = 0$$

$$\frac{{}^{I}d}{dt}\left({}^{I}\vec{h}_{P/O}\right) = \cancel{\vec{M}_{P/O}} \; 0$$

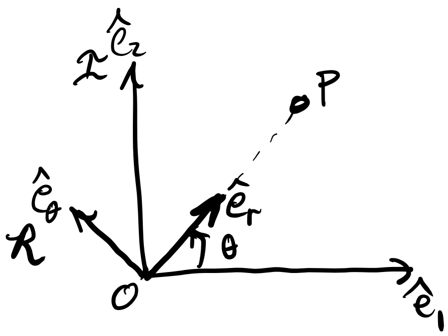


Note: for a central force, ${}^{I}\vec{h}_{P/O}$ is conserved.

Therefore the particle's motion is confined to a plane perpendicular to the ${}^{I}\vec{h}_{P/O}$ vector.

↳ The 3D problem becomes 2D.

Ex. 4.9 Simple Satellite
Tutorial 4.2 Orbit Equation



FBD

$\vec{F}_p = \dfrac{-G m_o m_p}{r^2}\hat{e}_r$ gravitational force

Kinematics

$$\vec{r}_{P/O} = r\hat{e}_r$$
$$^{\mathcal{I}}\vec{v}_{P/O} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$$
$$^{\mathcal{I}}\vec{a}_{P/O} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta$$

N2L

$$\vec{F}_p = m_p\,{}^{\mathcal{I}}\vec{a}_{P/O}$$
$$-\frac{G m_o m_p}{r^2}\hat{e}_r = m_p\left[(\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta\right]$$

Equating coefficients:

$$\left.\begin{aligned} \ddot{r} &= r\dot{\theta}^2 - \frac{G m_o}{r^2} \\ \ddot{\theta} &= \frac{-2\dot{r}\dot{\theta}}{r} \end{aligned}\right\}$$

Appears to be 2 D.O.F. But we have 1 conservation law (angular momentum) so we should be able to reduce this to 1 D.O.F.

$\vec{F}_p$ is a central force in this problem

$$\vec{M}_{P/O} = \vec{r}_{P/O} \times \vec{F}_p = 0 \Rightarrow {}^{\mathcal{I}}\vec{h}_{P/O} \text{ is conserved}$$

$$
\begin{aligned}
{}^{I}\vec{h}_{P/O} &= \vec{r}_{P/O} \times m_P\, {}^{I}\vec{v}_{P/O} \\
&= r\hat{e}_r \times m_P(\dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta) \\
&= m_P \underbrace{r^2\dot{\theta}}_{h_0}\, \hat{e}_3
\end{aligned}
$$

conserved

h_0 "specific angular momentum"

Let $\mu = Gm_0$

$$\Rightarrow \ddot{r} = \frac{h_0^2}{r^3} - \frac{\mu}{r^2} \qquad \ddot{\theta} = \frac{-2\dot{r}h_0}{r^3}$$

Solving for $r(t)$ and $\theta(t)$ in this problem is known as "Kepler's Problem" (see Goldstein)

Instead, let's focus on a parametric solution $r(\theta)$

Changing variables $y = \frac{1}{r} \Rightarrow h_0 = \frac{\dot{\theta}}{y^2}$

$$\dot{r} = \frac{d}{dt}\left(\frac{1}{y}\right) = \frac{-1}{y^2}\dot{y} = \frac{-1}{y^2}\frac{dy}{d\theta}\frac{d\theta}{dt} = -h_0\frac{dy}{d\theta}$$

$$\ddot{r} = -h_0\frac{d^2y}{d\theta^2}\dot{\theta} = -h_0^2 y^2 \frac{d^2y}{d\theta^2} \quad \text{(sub-ing in for } \dot{\theta}\text{)}$$

Equating with the other $\ddot{r}$ expression, and substituting

$$-h_0^2 y^2 \frac{d^2y}{d\theta^2} = h_0^2 y^3 - \mu y^2$$

$$\boxed{\frac{d^2y}{d\theta^2} = \underbrace{-y}_{\substack{\text{Simple}\\\text{Harmonic}\\\text{Motion}}} + \underbrace{\frac{\mu}{h_0^2}}_{\substack{\text{Forcing}\\\text{term}}}}$$ Parametric equation of motion

(Ch 12 on Simple Harmonic Motion)

eccentricity term

Solution:

$$y(\theta) = \frac{\mu}{h_0^2}\left(1 + e\cos(\theta - \theta_0)\right)$$

where e, θ_0 are constants

Putting back in terms of r

$$r(\theta) = \frac{h_0^2/\mu}{1 + e\cos(\theta - \theta_0)}$$

Conic section with eccentricity e