ECE 602: LUMPED LINEAR SYSTEMS

Professor Jianghai Hu

Solutions of Continuous-Time LTI Systems: General *A* Case

System Modes (General A Case)

Using the JCF: $A = TJT^{-1} = \begin{bmatrix} T_1 & \cdots & T_r \end{bmatrix} \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_r \end{bmatrix} \begin{bmatrix} S_1^T \\ \vdots \\ S_r^T \end{bmatrix}$

Solution to $\dot{x} = Ax$ with initial state $x(0)$:

$$x(t) = e^{At}x(0) = Te^{Jt}T^{-1}x(0) = \sum_{i=1}^{r} T_i e^{J_i t} \left(S_i^T x(0) \right)$$

- Columns of $T_i e^{J_i t} \in \mathbb{R}^{n \times n_i}$ are modes corresponding to eigenvalue λ_i, whose weights in $x(t)$ are given by entries of vector $S_i^T x(0) \in \mathbb{R}^{n_i}$

Recall:

$$J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix} \in \mathbb{C}^{n_i \times n_i} \quad \Rightarrow \quad e^{J_i t} = \begin{bmatrix} e^{\lambda_i t} & te^{\lambda_i t} & \cdots & \frac{1}{(n_i-1)!} t^{n_i-1} e^{\lambda_i t} \\ & e^{\lambda_i t} & \ddots & \vdots \\ & & \ddots & te^{\lambda_i t} \\ & & & e^{\lambda_i t} \end{bmatrix}$$

Example

$$\dot{x} = Ax \text{ with } A = \underbrace{\begin{bmatrix} -1 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{T} \underbrace{\begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}}_{J} \underbrace{\begin{bmatrix} 0 & -1 & 0 \\ 1 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix}}_{T^{-1}}$$

Decoupled Form (General A Case)

LTI system $\dot{x} = Ax$ with the JCF: $A = T \operatorname{diag}(J_1, \ldots, J_r) T^{-1}$

After a change coordinates $\tilde{x} = T^{-1}x$, the LTI system becomes:

$$\dot{\tilde{x}} = T^{-1}AT\tilde{x} = J\tilde{x} \quad \Rightarrow \quad \begin{cases} \dot{\tilde{x}}_1 = J_1\,\tilde{x}_1 \\ \quad\quad\vdots \\ \dot{\tilde{x}}_r = J_r\,\tilde{x}_r \end{cases} \quad \text{where } \tilde{x} = \begin{bmatrix} \tilde{x}_1 \\ \vdots \\ \tilde{x}_r \end{bmatrix}$$

- r groups of ODEs, each in "chain" form