ECE 68000: MODERN AUTOMATIC CONTROL

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System Zeros and Unknown Input Observers

Zeros of SISO Systems

- For a SISO system model $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$, its zeros are defined to be the zeros of the polynomial

$$\boldsymbol{c} \operatorname{adj}(s\boldsymbol{I} - \boldsymbol{A})\, \boldsymbol{b}$$

  where adj denotes the classical adjoint
- For a MIMO system model $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})$, the product

$$\boldsymbol{C} \operatorname{adj}(s\boldsymbol{I} - \boldsymbol{A})\, \boldsymbol{B}$$

  is a matrix with polynomial entries
- The collection of zeros of the polynomials would seem to be a natural generalization of the definition of system zeros
- However, this possible definition does not lead to a generalization of the SISO theory

System Zeros of SISO Systems: A Different Look

- Note that

$$\begin{bmatrix} \boldsymbol{I}_n & -\boldsymbol{0} \\ -\boldsymbol{c}(s\boldsymbol{I}_n - \boldsymbol{A})^{-1} & 1 \end{bmatrix} \begin{bmatrix} s\boldsymbol{I}_n - \boldsymbol{A} & -\boldsymbol{b} \\ \boldsymbol{c} & 0 \end{bmatrix}$$
$$= \begin{bmatrix} s\boldsymbol{I}_n - \boldsymbol{A} & -\boldsymbol{b} \\ \boldsymbol{0} & \boldsymbol{c}(s\boldsymbol{I}_n - \boldsymbol{A})^{-1}\boldsymbol{b} \end{bmatrix}$$

- Hence

$$\begin{aligned} \det \begin{bmatrix} s\boldsymbol{I}_n - \boldsymbol{A} & -\boldsymbol{b} \\ \boldsymbol{c} & 0 \end{bmatrix} &= \det(s\boldsymbol{I}_n - \boldsymbol{A}) \det(\boldsymbol{c}(s\boldsymbol{I}_n - \boldsymbol{A})^{-1}\boldsymbol{b}) \\ &= \det(s\boldsymbol{I}_n - \boldsymbol{A}) \frac{\boldsymbol{c}\,\mathrm{adj}(s\boldsymbol{I}_n - \boldsymbol{A})\boldsymbol{b}}{\det(s\boldsymbol{I}_n - \boldsymbol{A})} \end{aligned}$$

- Thus, $\boldsymbol{c}\,\mathrm{adj}(s\boldsymbol{I}_n - \boldsymbol{A})\,\boldsymbol{b} = \det \begin{bmatrix} s\boldsymbol{I}_n - \boldsymbol{A} & -\boldsymbol{b} \\ \boldsymbol{c} & 0 \end{bmatrix}$

System Zeros of SISO Systems

- So for SISO systems, the system zeros are precisely the collection of s such that the matrix

$$\begin{bmatrix} s\boldsymbol{I}_n - \boldsymbol{A} & -\boldsymbol{b} \\ \boldsymbol{c} & 0 \end{bmatrix}$$

  does not have full rank
- Thus for an LTI SISO system, its transfer function can be written as

$$G(s) = \boldsymbol{c}(s\boldsymbol{I}_n - \boldsymbol{A})^{-1}\boldsymbol{b} = \frac{1}{\det(s\boldsymbol{I}_n - \boldsymbol{A})} \det \begin{bmatrix} s\boldsymbol{I}_n - \boldsymbol{A} & -\boldsymbol{b} \\ \boldsymbol{c} & 0 \end{bmatrix}$$

System Matrix and Normal Rank

- Rosenbrock's system matrix

$$\boldsymbol{P}(s) = \begin{bmatrix} s\boldsymbol{I}_n - \boldsymbol{A} & -\boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix}$$

- The normal rank of a matrix valued function $\boldsymbol{M}$ defined on the complex plane $\mathbb{C}$ is

$$\operatorname{normalrank} \boldsymbol{M} = \max \{\operatorname{rank} \boldsymbol{M}(s) : s \in \mathbb{C}\}$$

- In other words, the normal rank of a matrix function is the largest possible rank among the collection of matrices in the range $\{\boldsymbol{M}(s) : s \in \mathbb{C}\}$ of $\boldsymbol{M}$.
- The rank of a matrix valued function is defined to be its normal rank

System Matrix and the Laplace Transform

- Linear time invariant plant model:

$$\begin{aligned}\dot{\boldsymbol{x}} &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}, \quad \boldsymbol{x}(0) = \boldsymbol{x}_0 \\ \boldsymbol{y} &= \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}\boldsymbol{u}\end{aligned}$$

- Take the Laplace transforms

$$\begin{aligned}s\boldsymbol{X}(s) - \boldsymbol{x}(0) &= \boldsymbol{A}\boldsymbol{X}(s) + \boldsymbol{B}\boldsymbol{U}(s) \\ \boldsymbol{Y}(s) &= \boldsymbol{C}\boldsymbol{X}(s) + \boldsymbol{D}\boldsymbol{U}(s)\end{aligned}$$

- Equivalent representation

$$\begin{bmatrix} s\boldsymbol{I}_n - \boldsymbol{A} & -\boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix} \begin{bmatrix} \boldsymbol{X}(s) \\ \boldsymbol{U}(s) \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}(0) \\ \boldsymbol{Y}(s) \end{bmatrix}$$

General Definition of System Zeros

A complex number z_0 is a system zero of the system $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D})$ if

$$\operatorname{rank} \begin{bmatrix} z_0 \boldsymbol{I}_n - \boldsymbol{A} & -\boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix} < \operatorname{normalrank} \begin{bmatrix} s \boldsymbol{I}_n - \boldsymbol{A} & -\boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix}$$

- The system zeros are also referred to as the invariant zeros of the system
- Note that the system matrix may not be square and so the determinant is not always defined for a system matrix

The Meaning of System Zeros—SISO Case

- Consider the second order input-output system model

$$\ddot{y} + y = \dot{u} - z_0 u \quad \Rightarrow \quad \text{Transfer function } = G(s) = \frac{s - z_0}{s^2 + 1}$$

- Only one system zero at $s = z_0$
- If $u(t) = e^{z_0 t}$, then $\dot{u} - z_0 u = 0$ and so this input has the same effect as the zero input.
- In other words, the system does not see the input $u(t) = e^{z_0 t}$
- If $z_0 < 0$, $u(t) = e^{z_0 t} \to 0$ and so $u(t)$ is asymptotically the same as 0
- If $z_0 \geq 0$, $u(t) = e^{z_0 t} \to \infty$ or is equal to 1 for all t and this $u(t)$ is far from 0 but the output cannot provide any information about this input

The Meaning of System Zeros—MIMO Case

- Consider the case when $m \le p$ and

$$\boldsymbol{P}(s) = \begin{bmatrix} s\boldsymbol{I}_n - \boldsymbol{A} & -\boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix}$$

  has rank less than $n + m$ for $s = z_0$

- Then there exists $\left[\boldsymbol{x}_0^\top \ \boldsymbol{u}_0^\top\right]^\top \neq \boldsymbol{0}$ such that

$$\boldsymbol{P}(z_0) \begin{bmatrix} \boldsymbol{x}_0 \\ \boldsymbol{u}_0 \end{bmatrix} = \begin{bmatrix} z_0\boldsymbol{I}_n - \boldsymbol{A} & -\boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_0 \\ \boldsymbol{u}_0 \end{bmatrix} = \boldsymbol{0}$$

- We have

$$\boldsymbol{P}(s) - \boldsymbol{P}(z_0) = (s - z_0) \begin{bmatrix} \boldsymbol{I}_n & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} \end{bmatrix}$$

M. L. J. Hautus, *Strong detectability and observers*, Linear Algebra and Its Applications, Vol. 50, pp. 353–368, 1983; see page 366

The Meaning of System Zeros—MIMO Case Contd.

- Let

$$\boldsymbol{X}(s) = \frac{1}{s - z_0}\boldsymbol{x}_0 \text{ and } \boldsymbol{U}(s) = \frac{1}{s - z_0}\boldsymbol{u}_0$$

  Then

$$\begin{bmatrix} s\boldsymbol{I}_n - \boldsymbol{A} & -\boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix} \begin{bmatrix} \boldsymbol{X}(s) \\ \boldsymbol{U}(s) \end{bmatrix} = \begin{bmatrix} \boldsymbol{I}_n & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_0 \\ \boldsymbol{0} \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}_0 \\ \boldsymbol{0} \end{bmatrix}$$

- Hence

$$\boldsymbol{x}(t) = \mathcal{L}^{-1}(\boldsymbol{X}(s)) = e^{z_0 t}\boldsymbol{x}_0 \text{ and } \boldsymbol{u}(t) = \mathcal{L}^{-1}(\boldsymbol{U}(s)) = e^{z_0 t}\boldsymbol{u}_0$$

  satisfy

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}, \quad \boldsymbol{x}(0) = \boldsymbol{x}_0 \\ \boldsymbol{y} &= \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}\boldsymbol{u} \end{aligned}$$

  and the corresponding output equals 0.

The Importance of System Zeros in the UIO Synthesis

- System $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D})$ with a system zero not in the open LHP will ignore certain unbounded or persistent inputs
- Conclusion: It is impossible to design a general unknown input estimator if there are system zeros not in the open LHP