Sec 7.4.

(Ex) $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ -1 & -3 & -7 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$

(1) Find a basis for Row(A)

$\{ [1\ 2\ 3], [0\ 1\ 4] \}$.

(2) Find a basis for Col(A).

$\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} \right\}$.

(Subspaces)

(Ex) (1) $V = \mathbb{R}^2$

$W = \{(x, y) \in V : 2x - y = 0\}$ a subspace of V

Find a basis for W.

$(x, y) \in W : 2x - y = 0$ iff $2x = y$

$\| \quad (x, 2x) = x(1, 2)$

$W = \operatorname{span}\{(1, 2)\}.$

$\{(1, 2)\}$: a basis for W.

(2) $V = (\mathbb{R}^3, +, \cdot)$

$U = \left\{ \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \in V : u_1 - u_2 + 2u_3 = 0 \right\}$

① U is a subspace of V.

② Find a basis for U.

For $\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \in U$, $u_1 - u_2 + 2u_3 = 0$.

$u_1 = u_2 - 2u_3$: $X \overset{\text{let}}{=} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} u_2 - 2u_3 \\ u_2 \\ u_3 \end{bmatrix}$

u_2, u_3 : free variables

$$X = \begin{bmatrix} u_2 \\ u_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -2u_3 \\ 0 \\ u_3 \end{bmatrix} = u_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + u_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

因て $U = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$

lin. independent.

$\therefore \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$: a basis for U.

(Null space).

(Ex) $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}_{2\times 2}$ $(\mathbb{R}^2, +, \cdot)$: a v.s.

Let $W = \{ X \in \mathbb{R}^2 : \underline{AX = 0} \}$

(1) W is a subspace of $\mathbb{R}^2$? Yes

① any $X_1, X_2 \in W$: $\underline{AX_1 = 0}$, $\underline{AX_2 = 0}$

$A(X_1 + X_2) = AX_1 + AX_2 = 0$

$X_1 + X_2 \in W$.

② any $X_1 \in W$, $\beta \in \mathbb{R}$

<u>claim</u>: $\beta X_1 \in W$

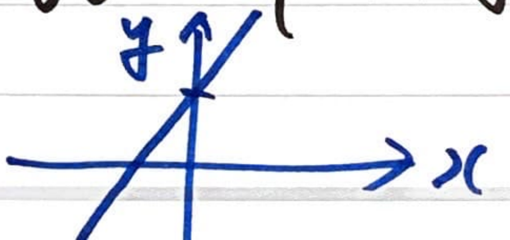
(Because $A(\beta X_1) = \beta A X_1 = \beta \cdot 0 = 0$)

W is a subspace of $\mathbb{R}^2$.

(2) Find a basis for W.

(Ex) $V = \mathbb{R}^2$

$W = \{(x, y) \in V : 2x - y + 5 = 0\}$.



$y = 2x + 5$.

$W = \{X \in \mathbb{R}^2 \mid AX = 0\}$, $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

① Solve $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\left[\begin{array}{cc:c} 1 & 2 & 0 \\ 2 & 4 & 0 \end{array}\right] \xrightarrow{r_2 - 2r_1} \left[\begin{array}{cc:c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $x + 2y = 0$

$x = -2y$: $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2y \\ y \end{bmatrix} = y \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$W = \text{span}\left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$.

$\therefore \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$: a basis for W.

$W = \text{Null}(A)$: the null space of A

$\dim \text{Null}(A) = 1$

<u>Remark</u>: $\#(\text{columns}) = 2.$

$\text{rank} A = 1$

Q $A_{m \times n}$:

$\underline{\text{rank} A + \dim \text{Null}(A) = n\,?}$

<u>Thm</u> $A_{m \times n}$

$\underline{\text{rank} A + \dim \text{Null}(A) = n.}$

(Ex) $A_{3\times 4}$: $AX = 0$: $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

$$[A \,\vdots\, 0] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 2 & 4 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(1) $\operatorname{rank} A = 2$

(2) $\dim \operatorname{Null}(A) = 4 - \operatorname{rank} A = 4 - 2 = 2$.

(3) $\operatorname{Null}(A) = ?$

$$\begin{cases} x_1 + 2x_3 + 4x_4 = 0 & \text{— ①} \\ \quad\quad\;\; x_3 - 2x_4 = 0 & \text{— ②} \end{cases}$$

② : $x_3 = 2x_4$

① : $x_1 = -2x_3 - 4x_4 = -2(2x_4) - 4x_4$

$= -8x_4$

$$X = \begin{bmatrix} -8x_4 \\ x_2 \\ 2x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -8x_4 \\ 0 \\ 2x_4 \\ x_4 \end{bmatrix}$$

$$= x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -8 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

$$\text{Null}(A) = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -8 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}$$

$\therefore \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -8 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}$: a basis for Null(A)

Q1

$AX = 0$: $\underline{X = 0}$.

$AX = 0$ has only one solution or infinitely many solutions.

Q2 $W = \{0\}$: $\dim W = 0$