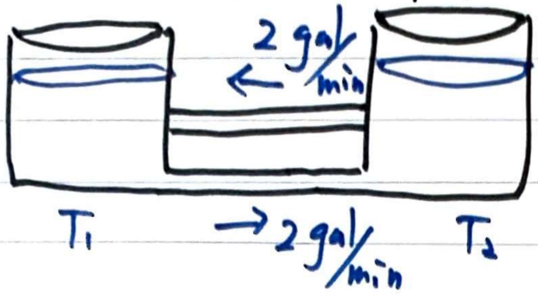
(Tank $T_1$ & $T_2$)




unknowns:
the amount of fertilizer

$y_1(t)_{(lb)}$ = the amount of the fertilizer in $T_1$

$y_2(t)_{(lb)}$ = " " $T_2$

rule: mass conservation.

$T_1$: $\underset{(lb/min)}{\dfrac{dy_1}{dt}}$ = in-flow − out flow

$$\underset{(lb/min)}{\frac{dy_1}{dt}} = 2\left(\frac{gal}{min}\right)\frac{y_2}{200}\left(\frac{lb}{gal}\right) - 2\cdot\frac{y_1}{200}$$

$$y_1' = \frac{1}{100}y_2 - \frac{y_1}{100}$$

$$\frac{dy_2}{dt} = 2\cdot\frac{y_1}{200} - 2\cdot\frac{y_2}{200} = \frac{y_1}{100} - \frac{y_2}{100}$$

$$\begin{cases} y_1' = -0.01y_1 + 0.01y_2 & y_1(0) = 0 \\ y_2' = 0.01y_1 - 0.01y_2, & y_2(0) = 100 \end{cases}$$

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} : \quad y' = \underbrace{\begin{bmatrix} -0.01 & 0.01 \\ 0.01 & -0.01 \end{bmatrix}}_{\text{let } A} y$$

$$y(0) = \begin{bmatrix} 0 \\ 100 \end{bmatrix}$$

(1) $\lambda$: $\det(A - \lambda I) = \begin{vmatrix} -0.01-\lambda & 0.01 \\ 0.01 & -0.01-\lambda \end{vmatrix}$

$= (-0.01-\lambda)^2 - 0.01^2 \overset{\text{let}}{=} 0$

$\lambda^2 + 2 \cdot 0.01\lambda + \cancel{0.01^2} - \cancel{0.01^2} = 0$

$\lambda(\lambda + 0.02) = 0$ : $\lambda = 0, -0.02$

(Case 1)

(2) $\lambda = 0$: $\begin{bmatrix} -0.01 & 0.01 \\ 0.01 & -0.01 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$-0.01 v_1 + 0.01 v_2 = 0$ : $-v_1 + v_2 = 0$

$v_2 = v_1$ : $\underline{V_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$ (let $v_1 = 1$)

$\lambda = -0.02$: $\begin{bmatrix} 0.01 & 0.01 \\ 0.01 & 0.01 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$0.01 v_1 + 0.01 v_2 = 0$ : $v_1 + v_2 = 0$

$v_2 = -v_1$

$\underline{V_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}}$

$$y(t) = c_1\begin{bmatrix}1\\1\end{bmatrix}e^{0} + c_2\begin{bmatrix}1\\-1\end{bmatrix}e^{-0.02t}$$

$$\text{IC}: \; y(0) = \begin{bmatrix}0\\100\end{bmatrix}$$

$$y(0) = c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}0\\100\end{bmatrix}$$

$$\begin{array}{rl} & c_1 + c_2 = 0 \\ + & c_1 - c_2 = 100 \\ \hline & 2c_1 \quad = 100 \end{array} \qquad \begin{cases} c_1 = 50 \\ c_2 = -c_1 = -50 \end{cases}$$

$$\underline{y(t) = 50\begin{bmatrix}1\\1\end{bmatrix} + (-50)\begin{bmatrix}1\\-1\end{bmatrix}e^{-0.02t}}$$

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Find $t$ s.t. $\overline{y_1(t) \geq \frac{1}{2} y_2(t)}$

$$y(t) = \begin{bmatrix}y_1(t)\\y_2(t)\end{bmatrix} = \begin{bmatrix}50 - 50e^{-0.02t}\\50 + 50e^{-0.02t}\end{bmatrix}$$

$$y_1(t) = 50 - 50e^{-0.02t} \geq \frac{1}{2}\left(50 + 50e^{-0.02t}\right)$$

$$50 - 50e^{-0.02t} \geq 25 + 25e^{-0.02t}$$

$$-25 \qquad\qquad -25$$

$$25 \geq 75e^{-0.02t} \; : \; e^{-0.02t} \leq \frac{\cancel{25}}{\cancel{75}} = \frac{1}{3}$$

$$-0.02t \leq \ln\left(\frac{1}{3}\right)$$

$$t \geq \frac{\ln(\frac{1}{3})}{-0.02} \approx 54.93\cdots \text{(min)}$$

$$\therefore\ t \approx 55 \text{ (min)}$$

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(Equilibrium solutions)

<u>Remark</u> $Y' = \begin{bmatrix} -0.01 & 0.01 \\ 0.01 & -0.01 \end{bmatrix} Y$.

$Y_1(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ : a constant solution.

Ⓛ $Y_1'(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, Ⓡ $\begin{bmatrix} -0.01 & 0.01 \\ 0.01 & -0.01 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

<u>Def</u> $Y' = AY$ : $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

(1) $Y = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ is called an equilibrium solution if $A\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

i.e. let $Y' = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow AY = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

(2) A point $P_0(a, b)$ is called a critical point if $AY\Big|_{\substack{y_1 = a \\ y_2 = b}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

(Ex) 1. $\mathbb{y}' = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \mathbb{y}$

Find <u>equilibrium solutions.</u>

Let $\mathbb{y}' = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ : $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \mathbb{y} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Let $\mathbb{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ : $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$y_1 + 2y_2 = 0 \rightarrow y_1 = -2y_2$

$2y_1 + 4y_2 = 0 \qquad y_2 = -\frac{1}{2} y_1$


$\mathbb{y}(t) = \begin{bmatrix} y_1 \\ -\frac{1}{2} y_1 \end{bmatrix}$, $y_1 \in \mathbb{R}$ or

$\mathbb{y}(t) = \begin{bmatrix} \alpha \\ -\frac{1}{2} \alpha \end{bmatrix}$, $\alpha \in \mathbb{R}$ : $(\alpha, -\frac{1}{2}\alpha)$ critical points

(2) $\mathbb{y}' = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \mathbb{y}$

Find <u>critical points</u>

Let $\mathbb{y}' = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ : $\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

nonsingular.

$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ : $(0, 0)$ : a critical point

4.3 Constant-coefficient systems

Case1.

(Ex) $y' = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} y$

(1) critical points / Equilibrium solution

Let $y' = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$: $\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} y = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$y = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $(0, 0)$ : a critical point.

(2) a general solution.

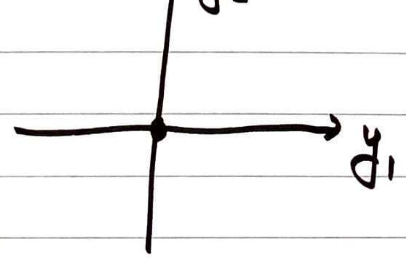
$A = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$ : $\lambda = 2, -2$ : Case1.

$V_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad V_2 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

$y(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-2t}$

(3) Solution Curves in the phase plane

$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ :




$-\infty < t < \infty$

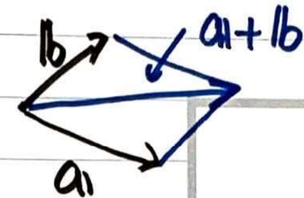
① Matlap : "pplane8.m"

## ② Sketch solution curves / Trajectories

**Remark**

1. $Y(t)$ = a linear combination of $V_1 = \begin{bmatrix}1\\1\end{bmatrix}$ & $V_2 = \begin{bmatrix}1\\-3\end{bmatrix}$

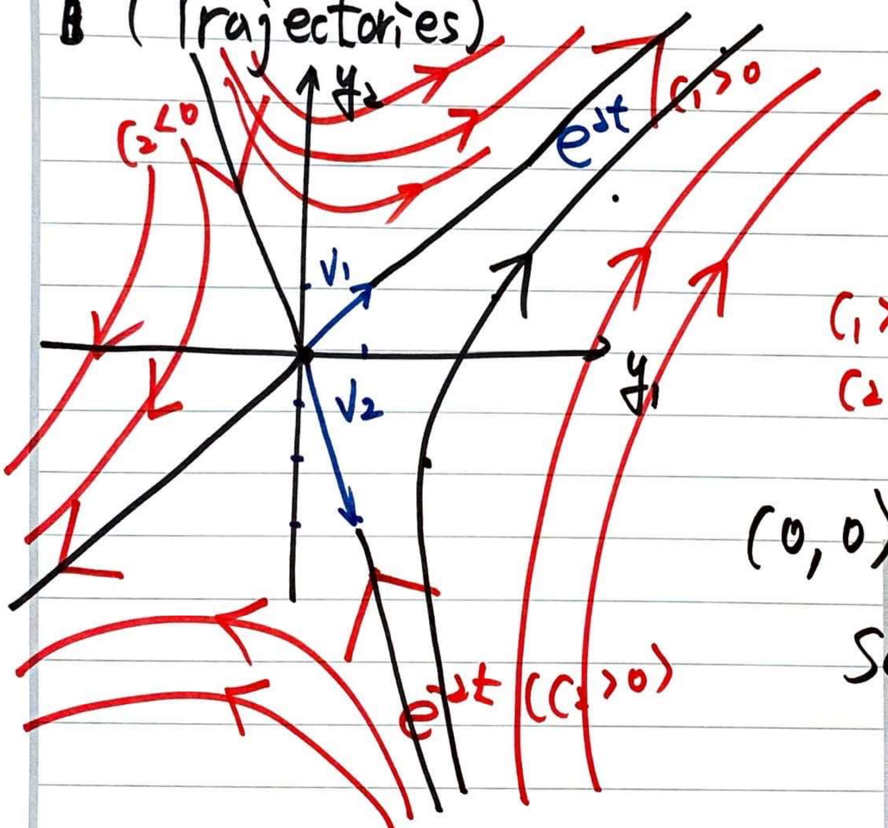
   Parallelogram law:




2. $\lim\limits_{t\to\infty} e^{2t} = \infty$, $\lim\limits_{t\to\infty} e^{-2t} = 0$

   $\lim\limits_{t\to-\infty} e^{2t} = 0$, $\lim\limits_{t\to-\infty} e^{-2t} = \infty$

(Trajectories)




$c_1 > 0$
$c_2 > 0$

$(0,0)$: a unstable saddle point

(Ex) $y' = \underset{A}{\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}} y$.

(1) Critical points : Let $y' = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} y = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ : $\underline{y = \begin{bmatrix} 0 \\ 0 \end{bmatrix}}$

$(0, 0)$ : a critical point.

(2) $\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ 3 & 4-\lambda \end{vmatrix} = (2-\lambda)(4-\lambda) - 3 = 0$

$\lambda^2 - 6\lambda + 5 = 0 \quad (\lambda - 1)(\lambda - 5) = 0$

$\lambda = 1, 5$.

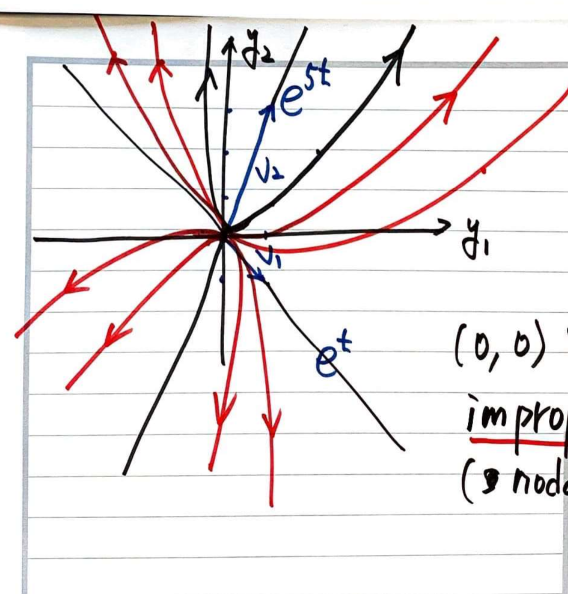
$\lambda = 1$: $v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\lambda = 5$: $v_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$\underline{y(t) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{5t}}$

(3) $\begin{cases} \lim_{t \to \infty} e^t = \infty, & \lim_{t \to \infty} e^{5t} = \infty \\ \lim_{t \to -\infty} e^t = 0, & \lim_{t \to -\infty} e^{5t} = 0 \end{cases}$

$e^t < e^{5t} \quad (t > 0)$

Let $y(t) = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$




$(0, 0)$ : a unstable <u>improper node</u>.

(nodal source).