(Practice test)

#1. $\left[\begin{array}{ccc|c} 1 & 2 & -3 & 2 \\ 2 & 7 & -11 & 5 \\ -1 & 1 & -2 & k \end{array}\right] \xrightarrow[r_3+r_1]{r_2-2\cdot r_1} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 2 \\ 0 & 3 & -5 & 1 \\ 0 & 3 & -5 & k+2 \end{array}\right]$

$\xrightarrow{r_3-r_2} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 2 \\ 0 & 3 & -5 & 1 \\ 0 & 0 & 0 & k+1 \end{array}\right]$

D

(1) No solution : $k+1 \neq 0$ iff $\underline{k \neq -1}$

(2) Inf. many solutions: $k+1=0$ iff $\underline{k=-1}$

A

(3) Only one solution : Impossible.
(unique)

E

#2 $A = \begin{bmatrix} 5 & 3 & 2 \\ 3 & 1 & 10 \\ 4 & 2 & 6 \end{bmatrix}$

(1) Null(A): Solve $AX=0$

$\left[\begin{array}{ccc|c} 5 & 3 & 2 & 0 \\ 3 & 1 & 10 & 0 \\ 4 & 2 & 6 & 0 \end{array}\right] \xrightarrow{\frac{1}{5}r_1} \left[\begin{array}{ccc|c} 1 & 3/5 & 2/5 & 0 \\ 3 & 1 & 10 & 0 \\ 4 & 2 & 6 & 0 \end{array}\right]$

$\xrightarrow[r_3-4r_1]{r_2-3r_1} \left[\begin{array}{ccc|c} 1 & 3/5 & 2/5 & 0 \\ 0 & -4/5 & 44/5 & 0 \\ 0 & -2/5 & 22/5 & 0 \end{array}\right]$

$1-\frac{9}{5}=-\frac{4}{5}$

$10-\frac{6}{5}=\frac{44}{5}$

$6-\frac{8}{5}=\frac{22}{5}$

$\xrightarrow[-\frac{5}{4}r_2]{r_3-\frac{1}{2}r_2} \left[\begin{array}{ccc|c} 1 & 3/5 & 2/5 & 0 \\ 0 & 1 & -11 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$x_1+\frac{3}{5}x_2+\frac{2}{5}x_3=0$

$x_2-11x_3=0$

$x_2=11x_3, \quad x_1+\frac{3}{5}(11x_3)+\frac{2}{5}x_3=0$

$x_1+\frac{35}{5}x_3=0 : \quad x_1=-7x_3$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -7x_3 \\ 11x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -7 \\ 11 \\ 1 \end{bmatrix} : \left\{ \begin{bmatrix} -7 \\ 11 \\ 1 \end{bmatrix} \right\}$

a basis for Null(A)

(2) rank(A) = 2

B

(3) A basis of Col(A):

$$\left\{ \begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \right\} :$$

(4) A^{-1} exists? $\quad$ rank $A = 2 < 3$

$\det A = 0$

(5) $W = \{ w \in \mathbb{R}^3 \mid w \perp \mathrm{Col}(A) \}$.

Let $w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$: $\quad w \cdot \begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix} = 5w_1 + 3w_2 + 4w_3 = 0$

$$w \cdot \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = 3w_1 + w_2 + 2w_3 = 0$$

$5w_1 + 3w_2 + 4w_3 = 0 \quad$ —①

$3w_1 + w_2 + 2w_3 = 0 \quad$ —②

① − 2·②

$$\begin{array}{rl} & 5w_1 + 3w_2 + 4w_3 = 0 \\ -) & 6w_1 + 2w_2 + 4w_3 = 0 \\ \hline & -w_1 + w_2 \qquad\quad = 0 \end{array} \qquad w_2 = w_1$$

①: $\underbrace{5w_1 + 3 \cdot w_1}_{= 8w_1} + 4w_3 = 0 \quad \therefore w_3 = -2w_1$

$$w = \begin{bmatrix} w_1 \\ w_1 \\ -2w_1 \end{bmatrix} = w_1 \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} : \left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \right\} : \text{a basis for } W.$$

Q $A_{3\times3}$: rank $A = 3$

Ⓐ $\dim \mathrm{Null}(A) = 3 - \mathrm{rank}A = 3 - 3 = 0$

$\mathrm{Null}(A) = \{0\}$

Remark $A_{m\times n}$

$$\mathrm{rank}A + \dim \mathrm{Null}(A) = n.$$

(Ex) $A_{3\times3}$: $\quad AX = 0 \quad$ (rank $A = 3$)

$\dim \mathrm{Null}(A) = 0$.

#3 $L\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = \begin{bmatrix}0\\1\\4\end{bmatrix}$, $L\left(\begin{bmatrix}3\\2\end{bmatrix}\right) = \begin{bmatrix}-2\\0\\1\end{bmatrix}$

$L\left(\begin{bmatrix}-1\\2\end{bmatrix}\right) = ?$

$L: \mathbb{R}^2 \to \mathbb{R}^3$, $\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}3\\2\end{bmatrix}\right\}$: a basis for $\mathbb{R}^2$

$\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}3\\2\end{bmatrix}$: lin. independent

$a\begin{bmatrix}1\\2\end{bmatrix} + b\begin{bmatrix}3\\2\end{bmatrix} = \begin{bmatrix}-1\\2\end{bmatrix}$: $a + 3b = -1$ — ①, $2a + 2b = 2$ — ②

$2\cdot$① $-$ ②: $2a + 6b = -2$

$-)\ 2a + 2b = 2$

$4b = -4$: $b = -1$

①: $a - 3 = -1$, $a = 2$

$\begin{bmatrix}-1\\2\end{bmatrix} = 2\begin{bmatrix}1\\2\end{bmatrix} + (-1)\begin{bmatrix}3\\2\end{bmatrix}$

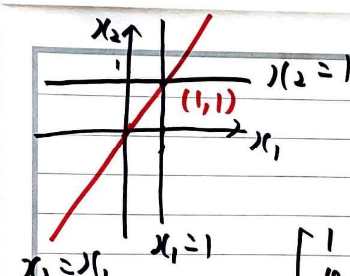
$L\left(\begin{bmatrix}-1\\2\end{bmatrix}\right) = 2L\left(\begin{bmatrix}1\\2\end{bmatrix}\right) - L\left(\begin{bmatrix}3\\2\end{bmatrix}\right)$

$= 2\begin{bmatrix}0\\1\\4\end{bmatrix} - \begin{bmatrix}-2\\0\\1\end{bmatrix} = \begin{bmatrix}2\\2\\7\end{bmatrix}$

#4. $AX = b$, $A_{3\times 2}$

(A) $AX = b$: consistent

$\begin{bmatrix}1 & 2\\2 & 4\\4 & 8\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}1\\2\\5\end{bmatrix}$ has no solutions.



$\begin{cases} x_1 = 1 \\ x_2 = 1 \\ x_1 - x_2 = 0 \end{cases}$:

$\begin{bmatrix}1 & 0\\0 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}1\\1\\0\end{bmatrix}$

$\begin{bmatrix}1 & 2\\2 & 4\\4 & 8\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}1\\2\\4\end{bmatrix}$ has infinitely many solutions

(D) $AX = b$ is inconsistent for at least one $\mathbb{b} \in \mathbb{R}^3$.

$A_{3\times 2} = [\mathbb{a}_1 \;\; \mathbb{a}_2]$

columns

$AX = b$ iff $[\mathbb{a}_1 \;\; \mathbb{a}_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = b$

$x_1 \mathbb{a}_1 + x_2 \mathbb{a}_2 = \mathbb{b}$. is consistent

iff $\mathbb{b} \in \mathrm{Col}(A)$.

rank $A \leq 2$. $\dim \mathrm{Col}(A) \leq 2$

$\mathrm{Col}(A) \subset \mathbb{R}^3$

Since $d \in \mathbb{R}^3 \setminus \mathrm{Col}(A)$ exists,

$AX = d$ is inconsistent.

#5. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\lambda^2 - 1 = 0$: $\lambda = 1, -1$

$\lambda = 1$: $V_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\lambda = -1$: $V_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

lin. independent.

A is diagonalizable

$P = [V_1 \;\; V_2] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ $D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$\sqrt{1^2+1^2} = \sqrt{2}$

$P = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$: $P^{-1}AP = D$. B