ECE 68000: MODERN AUTOMATIC CONTROL

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Dynamic Programming for Discrete-Time Nonlinear Systems

Problem statement

- Nonlinear discrete-time plant

$$\boldsymbol{x}[k+1] = \boldsymbol{f}\left(k, \boldsymbol{x}[k], \boldsymbol{u}[k]\right),$$

  with a given initial condition $\boldsymbol{x}[k_0] = \boldsymbol{x}_0$
- The associated performance index

$$J = J_0 = \Phi(N, \boldsymbol{x}[N]) + \sum_{k=0}^{N-1} F(k, \boldsymbol{x}[k], \boldsymbol{u}[k])$$

- Let $J_k^*(\boldsymbol{x}[k])$ denote the minimum cost of transferring the system from $\boldsymbol{x}[k]$ to the terminal point $\boldsymbol{x}[N]$
- Because $\boldsymbol{x}[N]$ is the terminal point, therefore $J_N^*(\boldsymbol{x}[N]) = \Phi(N, \boldsymbol{x}[N])$

Use PO to evaluate cost

- Using the PO, obtain the cost of transfer from $\boldsymbol{x}[N-1]$ to the terminal point:

$$\begin{aligned} & J_{N-1}^*(\boldsymbol{x}[N-1]) \\ = & \min_{\boldsymbol{u}[N-1]} \left\{F(N-1, \boldsymbol{x}[N-1], \boldsymbol{u}[N-1]) + J_N^*(\boldsymbol{x}[N])\right\} \\ = & \min_{\boldsymbol{u}[N-1]} \left\{F(N-1, \boldsymbol{x}[N-1], \boldsymbol{u}[N-1]) + \Phi(N, \boldsymbol{x}[N])\right\} \end{aligned}$$

- When applying the PO to solving optimal control problems for discrete-time dynamical systems, perform two stage-by-stage passes through the time stages
- Begin with a backward pass
- First, eliminate $\boldsymbol{x}[N]$
- Next, carry out the minimization to find $\boldsymbol{u}^*[N-1]$
- Repeat the process

Typical step of backward pass

- Typical step of the backward pass

$$
\begin{aligned}
J_k^*(\boldsymbol{x}[k]) &= \min_{\boldsymbol{u}[k]} \left\{ F(k, \boldsymbol{x}[k], \boldsymbol{u}[k]) + J_{k+1}^*(\boldsymbol{x}[k]) \right\} \\
&= \min_{\boldsymbol{u}[k]} \left\{ F(k, \boldsymbol{x}[k], \boldsymbol{u}[k]) + J_{k+1}^* \left(\boldsymbol{f}(k, \boldsymbol{x}[k], \boldsymbol{u}[k]) \right) \right\}
\end{aligned}
$$

- The backward pass is completed when the initial time k_0 is reached

Forward pass

- After backward pass completed, because $\boldsymbol{x}_0 = \boldsymbol{x}[k_0]$ is known, find $\boldsymbol{u}_0 = \boldsymbol{u}[k_0]$ in terms of this state
- Proceed with the *forward pass*
- Use $\boldsymbol{x}_0, \boldsymbol{u}_0$ to compute $\boldsymbol{x}[k_0 + 1]$
- The state $\boldsymbol{x}[k_0 + 1]$ is then used to compute $\boldsymbol{u}[k_0 + 1]$ from $\boldsymbol{x}[k_0 + 2] = \boldsymbol{f}(k_0 + 1, \boldsymbol{x}[k_0 + 1], \boldsymbol{u}[k_0 + 1])$, and so on

Example

- Plant: scalar discrete-time dynamical system

$$x[k+1] = 2x[k] - 3u[k], \quad x[0] = 4,$$

  and the performance index

$$J = J_0 = (x[2] - 10)^2 + \frac{1}{2} \sum_{k=0}^{1} \left(x[k]^2 + u[k]^2 \right)$$

- The final state, in this example, free
- Use the PO to find optimal $u[0]$ and $u[1]$
- There are no constraints on $u[k]$
- Begin with the backward pass
- We have

$$J^*(x[2]) = (x[2] - 10)^2$$

Example—backward pass

- Evaluate

$$\begin{aligned} J^*(x[1]) &= \min_{u[1]} \left\{ \frac{1}{2}\left(x[1]^2 + u[1]^2\right) + J^*(x[2]) \right\} \\ &= \min_{u[1]} \left\{ \frac{1}{2}\left(x[1]^2 + u[1]^2\right) + (2x[1] - 3u[1] - 10)^2 \right\} \end{aligned}$$

- There are no constraints on $u[1]$; find optimal $u[1]$ as a function of $x[1]$ by applying the first-order necessary condition for unconstrained optimization

$$\frac{\partial}{\partial u[1]} \left\{ \frac{1}{2}\left(x[1]^2 + u[1]^2\right) + (2x[1] - 3u[1] - 10)^2 \right\} = 0$$

- Hence,

$$u[1] = \frac{1}{19}\left(12x[1] - 60\right)$$

Example—evaluate $J^*(x[1])$

- Therefore

$$\begin{aligned} J^*(x[1]) &= \frac{1}{2}\left(x[1]^2 + \left(\frac{12x[1]-60}{19}\right)^2\right. \\ &\quad \left. + \left(2x[1] - 3\frac{12x[1]-60}{19} - 10\right)^2\right) \end{aligned}$$

- Next, we compute $u[0]$. For this observe that

$$J^*(x[0]) = \min_{u[0]}\left\{\frac{1}{2}\left(x[0]^2 + u[0]^2\right) + J^*(x[1])\right\}$$

- Taking into account that

$$x[1] = 2x[0] - 3u[0], \quad x[0] = 4,$$

  and the fact that there are no constraints on $u[0]$, use the first-order necessary condition for unconstrained optimization to find optimal $u[0]$

Example—completing the backward pass

- Compute

$$\frac{\partial}{\partial u[0]}\left\{\frac{1}{2}\left(x[0]^2+u[0]^2\right)+J^*(x[1])\right\}=0$$

to get

$$\frac{\partial}{\partial u[0]}\left\{8+\frac{1}{2}u[0]^2+\frac{1}{2}(8-3u[0])^2 + \frac{1}{2}\left(\frac{12(8-3u[0])-60}{19}\right)^2\right\} + \frac{\partial}{\partial u[0]}\left\{2(8-3u[0])-3\frac{12(8-3u[0])-60}{19}-10\right\}^2 = 0$$

- Hence, $13.7895u[0]=27.7895$

Example—forward pass

- We obtain, $u[0] = 2.0153$
- The backward pass complete
- Ready for the forward pass
- Use the system difference equation, $x[0]$, and $u[0]$ to compute $x[1] = 1.9541$
- Hence
$$u[1] = \frac{12x[1] - 60}{19} = -1.9237$$
- Therefore,
$$x[2] = 2x[1] - 3u[1] = 9.6794,$$
which completes the forward pass