ECE 602: LUMPED LINEAR SYSTEMS

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Observability of continuous-time (CT) linear time-invariant (LTI) systems

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- **Objective**: Introduce the notion of observability of CT linear time-invariant (LTI) systems modeled as

$$\begin{aligned} \dot{x}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0 \\ \boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t) \end{aligned}$$

  where $\boldsymbol{A} \in \mathbb{R}^{n\times n}$, $\boldsymbol{B} \in \mathbb{R}^{n\times m}$, $\boldsymbol{C} \in \mathbb{R}^{p\times n}$, $p < n$, and $\boldsymbol{D} \in \mathbb{R}^{p\times m}$

- Recall the solution of the state equation,

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}(0) + \int_0^t e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)d\tau$$

Observability definition

The system

$$\left.\begin{array}{rcl} \dot{\boldsymbol{x}} & = & \boldsymbol{A}\boldsymbol{x}+\boldsymbol{B}\boldsymbol{u} \\ \boldsymbol{y} & = & \boldsymbol{C}\boldsymbol{x}+\boldsymbol{D}\boldsymbol{u}, \end{array}\right\}$$

or equivalently the pair $(\boldsymbol{A}, \boldsymbol{C})$, is observable if there is a finite $t_1 > t_0$ such that for arbitrary $\boldsymbol{u}(\cdot)$ and resulting $\boldsymbol{y}(\cdot)$ over $[t_0, t_1]$, we can determine $\boldsymbol{x}(t_0)$ from the knowledge of the system input $\boldsymbol{u}$ and output $\boldsymbol{y}$.

- Note that once $\boldsymbol{x}(t_0)$ is known, we can determine $\boldsymbol{x}(t)$ from knowledge of $\boldsymbol{u}(\cdot)$ and $\boldsymbol{y}(\cdot)$ over any finite time interval $[t_0, t_1]$
- **Objective**: Determine $\boldsymbol{x}(t_0)$, given $\boldsymbol{u}(\cdot)$ and $\boldsymbol{y}(\cdot)$

Preliminary manipulations

- The solution $\boldsymbol{y}(t)$

$$\boldsymbol{y}(t) = \boldsymbol{C}e^{\boldsymbol{A}(t-t_0)}\boldsymbol{x}(t_0) + \int_{t_0}^{t} \boldsymbol{C}e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)d\tau + \boldsymbol{D}\boldsymbol{u}(t)$$

- Subtract $\int_{t_0}^{t} \boldsymbol{C}e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)d\tau + \boldsymbol{D}\boldsymbol{u}(t)$ from both sides
- Let

$$\boldsymbol{g}(t) = \boldsymbol{y}(t) - \int_{t_0}^{t} \boldsymbol{C}e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)d\tau - \boldsymbol{D}\boldsymbol{u}(t)$$

- Then we have

$$\boldsymbol{g}(t) = \boldsymbol{C}e^{\boldsymbol{A}(t-t_0)}\boldsymbol{x}(t_0),$$

  where $\boldsymbol{g}$ is known to us

More manipulations

- Our goal now is to determine $\boldsymbol{x}(t_0)$
- Having $\boldsymbol{x}(t_0)$, we can determine the entire $\boldsymbol{x}(t)$ for all $t \in [t_0, t_1]$ from the formula

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}(t-t_0)}\boldsymbol{x}(t_0) + \int_{t_0}^{t} e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)d\tau.$$

- Premultiplying both sides of $\boldsymbol{g}(t)$ by $e^{\boldsymbol{A}^\top(t-t_0)}\boldsymbol{C}^\top$ and integrating between the limits t_0 and t_1,

$$\int_{t_0}^{t_1} e^{\boldsymbol{A}^\top(t-t_0)}\boldsymbol{C}^\top\boldsymbol{C}e^{\boldsymbol{A}(t-t_0)}dt\,\boldsymbol{x}(t_0) = \int_{t_0}^{t_1} e^{\boldsymbol{A}^\top(t-t_0)}\boldsymbol{C}^\top\boldsymbol{g}(t)dt$$

Observability Gramian

- Perform simple manipulations

$$
\begin{aligned}
&\int_{t_0}^{t_1} e^{\boldsymbol{A}^\top (t-t_0)} \boldsymbol{C}^\top \boldsymbol{C} e^{\boldsymbol{A}(t-t_0)} dt\, \boldsymbol{x}(t_0) \\
&= e^{-\boldsymbol{A}^\top t_0} \int_{t_0}^{t_1} e^{\boldsymbol{A}^\top t} \boldsymbol{C}^\top \boldsymbol{C} e^{\boldsymbol{A}t} dt\, e^{-\boldsymbol{A}t_0} \boldsymbol{x}(t_0) \\
&= \int_{t_0}^{t_1} e^{\boldsymbol{A}^\top (t-t_0)} \boldsymbol{C}^\top \boldsymbol{g}(t) dt
\end{aligned}
$$

- Let

$$
\boldsymbol{V}(t_0, t_1) = \int_{t_0}^{t_1} e^{\boldsymbol{A}^\top t} \boldsymbol{C}^\top \boldsymbol{C} e^{\boldsymbol{A}t} dt
$$

- $\boldsymbol{V}(t_0, t_1)$ is called the observability Gramian

Reconstructing the state

- We have

$$e^{-\boldsymbol{A}^\top t_0}\boldsymbol{V}(t_0,t_1)\,e^{-\boldsymbol{A}t_0}\,\boldsymbol{x}(t_0)=\int_{t_0}^{t_1}e^{\boldsymbol{A}^\top(t-t_0)}\boldsymbol{C}^\top\boldsymbol{g}(t)dt$$

- After some manipulations and assuming that the matrix $\boldsymbol{V}(t_0,t_1)$ is invertible, we obtain

$$\boldsymbol{x}(t_0)=e^{\boldsymbol{A}t_0}\boldsymbol{V}^{-1}(t_0,t_1)e^{\boldsymbol{A}^\top t_0}\int_{t_0}^{t_1}e^{\boldsymbol{A}^\top(t-t_0)}\boldsymbol{C}^\top\boldsymbol{g}(t)dt$$

- Knowledge of $\boldsymbol{x}(t_0)$ allows us to reconstruct the entire state $\boldsymbol{x}(\cdot)$ over the interval $[t_0,t_1]$
- In sum, if the matrix $\boldsymbol{V}(t_0,t_1)$ is invertible, then the system is observable

Example

- For a dynamical system modeled by

$$\left.\begin{array}{rcl}\dot{\boldsymbol{x}} & = & \begin{bmatrix} -1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}\boldsymbol{x} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u \\ y & = & \begin{bmatrix} 1 & 1 \end{bmatrix}\boldsymbol{x}, \end{array}\right\}$$

  determine $\boldsymbol{x}(\cdot)$ over the time interval $[0, 10]$ knowing that $u(t) = 1$ for $t \geq 0$ and $y(t) = t^2 + 2t$ for $t \geq 0$
- First find $\boldsymbol{x}(0)$

Reconstructing the initial state

- Knowledge of $\boldsymbol{x}(0)$ will allow to find the entire state vector for all $t \in [0, 10]$
- We have $t_0 = 0$ and $t_1 = 10$
- Hence,

$$\begin{aligned} \boldsymbol{x}(t_0) = \boldsymbol{x}(0) &= e^{\boldsymbol{A} t_0} \boldsymbol{V}^{-1}(t_0, t_1) e^{\boldsymbol{A}^\top t_0} \int_{t_0}^{t_1} e^{\boldsymbol{A}^\top (t - t_0)} \boldsymbol{c}^\top g(t) dt \\ &= \boldsymbol{V}^{-1}(0, 10) \int_0^{10} e^{\boldsymbol{A}^\top t} \boldsymbol{c}^\top g(t) dt \end{aligned}$$

where

$$g(t) = y(t) - \boldsymbol{c} \int_0^t e^{\boldsymbol{A}(t-\tau)} \boldsymbol{b} u(\tau) d\tau$$

Manipulating

- Compute,

$$
\begin{aligned}
e^{\boldsymbol{A}t} &= \mathcal{L}^{-1}\left([s\boldsymbol{I}_2 - \boldsymbol{A}]^{-1}\right) \\
&= \mathcal{L}^{-1}\left(\begin{bmatrix} \frac{1}{s} - \frac{1}{2}\frac{1}{s^2} & \frac{1}{2}\frac{1}{s^2} \\ -\frac{1}{2}\frac{1}{s^2} & \frac{1}{s} + \frac{1}{2}\frac{1}{s^2} \end{bmatrix}\right) = \begin{bmatrix} 1 - \frac{1}{2}t & \frac{1}{2}t \\ -\frac{1}{2}t & 1 + \frac{1}{2}t \end{bmatrix}
\end{aligned}
$$

- Next compute

$$
\begin{aligned}
\boldsymbol{V}(0,10) &= \int_0^{10} e^{\boldsymbol{A}^\top t}\boldsymbol{c}^\top \boldsymbol{c} e^{\boldsymbol{A}t} dt \\
&= \int_0^{10} \left(\begin{bmatrix} 1-t \\ 1+t \end{bmatrix}\begin{bmatrix} 1-t & 1+t \end{bmatrix}\right) dt
\end{aligned}
$$

More manipulations

- We have

$$
\begin{aligned}
\boldsymbol{V}(0,10) &= \int_0^{10} e^{\boldsymbol{A}^\top t}\boldsymbol{c}^\top \boldsymbol{c} e^{\boldsymbol{A}t} dt \\
&= \left[\begin{array}{cc} t - t^2 + \frac{t^3}{3} & t - \frac{t^3}{3} \\ t - \frac{t^3}{3} & t + t^2 + \frac{t^3}{3} \end{array}\right]\Bigg|_0^{10} \\
&= \left[\begin{array}{cc} 243.333 & -323.333 \\ -323.333 & 443.333 \end{array}\right]
\end{aligned}
$$

- The inverse of $\boldsymbol{V}(0,10)$

$$
\boldsymbol{V}^{-1}(0,10) = \left[\begin{array}{cc} 0.133 & 0.097 \\ 0.97 & 0.073 \end{array}\right]
$$

More calculating

- Compute $g(t)$ to get

$$
\begin{aligned}
g(t) &= y(t) - \boldsymbol{c} \int_0^t e^{\boldsymbol{A}(t-\tau)} \boldsymbol{b} u(\tau) d\tau \\
&= t^2 + 2t - \left[\begin{array}{cc} 1 & 1 \end{array} \right] \int_0^t \left[\begin{array}{c} -1 + t - \tau \\ 1 + t - \tau \end{array} \right] d\tau \\
&= t^2 + 2t - \int_0^t (2t - 2\tau) d\tau \\
&= t^2 + 2t - 2t\tau \Big|_0^t + \tau^2 \Big|_0^t = t^2 + 2t - t^2 \\
&= 2t
\end{aligned}
$$

Reconstructing the state

- Thus,

$$\int_0^{10} e^{\boldsymbol{A}^\top t}\boldsymbol{c}^\top g(t)dt = \int_0^{10} \begin{bmatrix} (1-t)2t \\ (1+t)2t \end{bmatrix} dt$$

$$= \begin{bmatrix} t^2 - \frac{2}{3}t^3 \\ t^2 + \frac{2}{3}t^3 \end{bmatrix} \Bigg|_0^{10} = \begin{bmatrix} -566.6667 \\ 766.6667 \end{bmatrix}$$

- Hence, $\boldsymbol{x}(0) = \boldsymbol{V}^{-1}(0,10)\int_0^{10} e^{\boldsymbol{A}^\top t}\boldsymbol{c}^\top g(t)dt = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$
- Knowing $\boldsymbol{x}(0)$, we can find $\boldsymbol{x}(t)$ for $t \geq 0$,

$$\begin{aligned} \boldsymbol{x}(t) &= e^{\boldsymbol{A}t}\boldsymbol{x}(0) + \int_0^t e^{\boldsymbol{A}(t-\tau)}\boldsymbol{b}u(\tau)d\tau \\ &= \begin{bmatrix} -1+t \\ 1+t \end{bmatrix} + \begin{bmatrix} (-1+t)t - \frac{t^2}{2} \\ (1+t)t - \frac{t^2}{2} \end{bmatrix} = \begin{bmatrix} \frac{t^2}{2} - 1 \\ \frac{t^2}{2} + 2t + 1 \end{bmatrix} \end{aligned}$$