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4.6.11 $\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 6 \\ -1 \end{bmatrix} e^{2t}$

$\underline{y}'$ $\quad A \quad$ $\underline{y}$ $\quad g$

Clearly (using ideas from previous hw),

$$\underline{y}_h = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} .$$

homogenous soln.

By the method of coefficients,

$$\underline{y}_p = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} e^{2t} .$$

u

particular soln.

$$\underline{y}_p' = A\underline{y}_p + g$$

$$\Rightarrow \begin{bmatrix} 2u_1 \\ 2u_2 \end{bmatrix} e^{2t} = \begin{bmatrix} u_2 \\ u_1 \end{bmatrix} e^{2t} + \begin{bmatrix} 6 \\ -1 \end{bmatrix} e^{2t}$$

$$\Rightarrow \begin{aligned} 2u_1 &= u_2 + 6 \\ 2u_2 &= u_1 - 1 \end{aligned}$$

$\Rightarrow 2u_1 - 6 = u_2,$

$2u_2 + 1 = u_1$

$\Rightarrow u_2 = 2(2u_2+1) - 6$

$u_1 = 2u_2 + 1$

$\Rightarrow u_2 = 4u_2 - 4, \quad u_1 = 2u_2 + 1$

$\Rightarrow u_2 = \frac{4}{3}, \quad u_1 = \frac{11}{3}$

$\therefore$ The general soln. is

$$\underline{y} = \underline{y}_h + \underline{y}_p = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + \begin{bmatrix} 11/3 \\ 4/3 \end{bmatrix} e^{2t}$$

Applying the initial conditions,

$1 = y_1(0) = c_1 + c_2 + 11/3,$

$0 = y_2(0) = c_1 - c_2 + 4/3.$

$\Rightarrow \ldots \Rightarrow c_1 = -2, \quad c_2 = -2/3.$

$\therefore$ $\boxed{\begin{array}{l} \text{The soln. is} \\ y_1 = -2e^{t} - \frac{2}{3}e^{-t} + \frac{11}{3}e^{2t}, \\ y_2 = -2e^{t} + \frac{2}{3}e^{-t} + 4/3\, e^{2t}. \end{array}}$

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6.1.13 $$F(s) = \int_0^1 e^{-st}\,dt + \int_1^2 -e^{-st}\,dt + \int_2^\infty 0\,e^{-st}\,dt$$

$$= \frac{e^{-s} - 1}{-s} - \frac{e^{-2s} - e^{-s}}{-s}$$

$$= \frac{-e^{-s} + 1 + e^{-2s} - e^{-s}}{s}$$

$$= \frac{e^{-2s} - 2e^{-s} + 1}{s}$$

$$= \frac{(1 - e^{-s})^2}{s}$$

6.1.30 $F(s) = \dfrac{4(s+8)}{s^2 - 4^2}$

$$= 4\left[\frac{s}{s^2-4^2} + 2\,\frac{4}{s^2-4^2} \right]$$

Now using Table 6.1,

$$\Rightarrow \mathcal{L}^{-1}(F(s)) = 4\left[\cosh(4t) + 2\sinh(4t) \right]$$

$$= 4\left[\frac{e^{4t}+e^{-4t}}{2} + 2\,\frac{e^{4t}-e^{-4t}}{2} \right]$$

$$= 2\left[3e^{4t} - e^{-4t} \right]$$

6.1.32 $F(s) = \dfrac{1}{(s+a)(s+b)}$

$$= \frac{\alpha}{s+a} + \frac{\beta}{s+b}, \text{ where } \alpha, \beta \text{ are const.}$$

(using partial fractions; I'm not sure if theres a better way).

$$= \frac{\alpha(s+b) + \beta(s+a)}{(s+a)(s+b)}$$

(i) If $a \neq b$:

$$\Rightarrow 1 = 0 + \beta(-b+a)$$

$$1 = \alpha(-a+b) + 0$$

$$\Rightarrow \beta = \frac{1}{a-b}, \quad \alpha = \frac{1}{b-a}$$

$$\Rightarrow F(s) = \frac{1}{b-a}\left(\frac{1}{s+a} - \frac{1}{s+b}\right)$$

$$\Rightarrow F(s) = \frac{1}{b-a}\left(\frac{1}{1-(-a)} - \frac{1}{1-(-b)}\right)$$

$$\Rightarrow f(t) = \frac{1}{b-a}\left(e^{-at} - e^{-bt}\right)$$

(ii) <u>If $a=b$</u>:

$$\Rightarrow F(s) = \frac{1}{(s+a)^2}$$

$$= \frac{1}{(s-(-a))^2}$$

$$\Rightarrow f(t) = e^{-at}t$$ (using 1st shifting theorem)

$$\therefore \quad f(t) = \begin{cases} \frac{1}{b-a}\left(e^{-at} - e^{-bt}\right), & \text{if } a \neq b \\ e^{-at}t & \text{if } a = b \end{cases}$$

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6.2.4 $s^2 \mathcal{L}(y) - s\underbrace{y(0)}_{=0} - \underbrace{y'(0)}_{=0} + 9\mathcal{L}(y) = \frac{10}{s+1}$

$$\Rightarrow \mathcal{L}(y)(s^2+9) = \frac{10}{s+1}$$

$$\Rightarrow \mathcal{L}(y) = \frac{10}{(s+1)(s^2+9)}$$

$$= 10\left[\frac{\alpha}{s+1} + \frac{\beta s+\gamma}{s^2+9}\right]$$

partial fractions

$$\alpha(s^2+9) + (\beta s+\gamma)(s+1) = 1$$

$$\Rightarrow \alpha((-1)^2+9) = 1 \quad \Rightarrow \boxed{\alpha = 1/10}$$

$$\Rightarrow (3i\beta+\gamma)(3i+1) = 1$$

$$\Rightarrow (-9\beta+\gamma) + i(3\beta+3\gamma)) = 1$$

$$\Rightarrow -9\beta+\gamma = 1, \quad 3\beta+3\gamma = 0$$

$$\Rightarrow \beta = -(1+9\beta), \quad \beta = -\gamma$$

$$\Rightarrow \beta = \frac{-1}{10}, \quad \gamma = \frac{1}{10}$$

$$\therefore \quad \mathcal{L}(y) = \frac{1}{s+1} + \frac{-s+1}{s^2+9}$$

$\left(\text{cancelling: } 10 \cdot \frac{1}{10} = 1\right)$

$$= \frac{1}{s+1} - \frac{s}{s^2+3^2} + \frac{1}{3}\frac{3}{s^2+3^2}$$

$$\Rightarrow y = e^{-t} - \cos(3t) + \frac{1}{3}\sin(3t)$$

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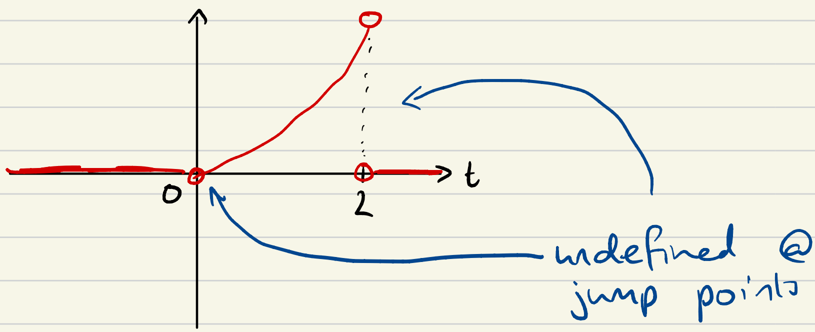
6.3.10 Here, $f(t) := \sinh(t)[u(t) - u(t-2)]$

Sketch:



$$\mathcal{L}(f(t)) = \mathcal{L}(\sinh(t)\,u(t)) - \mathcal{L}(\sinh(t)u(t-2))$$

$$= \mathcal{L}(\mathcal{L}^{-1}(e^{-0}\mathcal{L}(\sinh(t)))) - \mathcal{L}(\mathcal{L}^{-1}(e^{-2s}\mathcal{L}(\sinh(t+2))))$$

(Using 2nd shifting thm. In 2nd term, we did a similar substitution to that in (4**) on pg 220)

$$= \mathcal{L}(\sinh(t)) - e^{-2s}\mathcal{L}(\sinh(t+2))$$

$$= \frac{1}{s^2-1} - e^{-2s} \underbrace{\mathcal{L}\left(\frac{e^{t+2}}{2} - \frac{e^{-t-2}}{2} \right)}$$

$$= \mathcal{L}\left(\frac{e^2}{2} e^{t} - \frac{e^{-2}}{2} e^{-t} \right)$$

$$= \frac{e^2}{2} \frac{1}{s-1} - \frac{e^{-2}}{2} \frac{1}{s+1}$$

$$= \frac{1}{s^2-1} - \frac{e^{-2s+2}}{2(s-1)} + \frac{e^{-2s-2}}{2(s+1)}$$

6.3.25 $y'' + y = \begin{cases} t, & t \in (0,1) \\ 0; & t \in (1,\infty) \end{cases}$

$$\mathcal{L}(y'') = s^2 \mathcal{L}(y) - s\underbrace{y(0)}_{=0} - \underbrace{y'(0)}_{=0}$$

$$= s^2 \mathcal{L}(y).$$

(*)

Now, taking $\mathcal{L}$ on both sides of the ODE

$$\mathcal{L}(y'' + y) = \mathcal{L}(t[u(t) - u(t-1)])$$

$$= \mathcal{L}(tu(t)) - \mathcal{L}(tu(t-1))$$

$$= e^0 \frac{1}{s^2} - e^{-s}\mathcal{L}(t+1)$$

$$= \frac{1}{s^2} - e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right)$$

(**)

again, as on pg 220

$$\Rightarrow \mathcal{L}(y) = \frac{1}{s^2+1}\left(\frac{1}{s^2} - e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right)\right)$$

(using (*) & (**))

$\Rightarrow \ldots$ (using partial fractions) ...

$$\Rightarrow \mathcal{L}(y) = \frac{1}{s^2} - \frac{1}{s^2+1} - \frac{e^{-s}(-s-1)}{s^2+1} - \frac{e^{-s}}{s^2} - \frac{e^{-s}}{s}$$

$$\Rightarrow y = t - \sin(t) + \mathcal{L}^{-1}\left(\frac{e^{-s}s}{s^2+1^2}\right) + \mathcal{L}^{-1}\left(\frac{e^{-s}}{s^2+1^2}\right)$$

$$- \mathcal{L}^{-1}\left(\frac{e^{-s}}{s^2}\right) - \mathcal{L}\left(\frac{e^{-s}}{s}\right)$$

$$= t - \sin(t) + \cos(t-1)u(t-1)$$

$$+ \sin(t-1)u(t-1) - (t-1)u(t-1)$$

$$- (1)u(t-1)$$

$$= t - \sin(t) + u(t-1)\left[\cos(t-1) + \sin(t-1) - t\right]$$

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6.4.10 $\mathcal{L}(y'') = s^2\mathcal{L}(y) - \underbrace{s\, y(0) - y'(0)}_{=0},$

$$\mathcal{L}(y') = s\mathcal{L}(y) - \underbrace{y(0)}_{=0}$$

$$\mathcal{L}\left(\delta\left(t-\tfrac{1}{2}\pi\right) + u(t-\pi)\cos(t)\right)$$

$$= e^{-\frac{\pi}{2}s} + \mathcal{L}\left(u(t-\pi)\cos(t-\pi)\underbrace{(-1)}\right)$$

using properties of cosine.

If we didn't have this nice property, we may have had to evaluate an integral; We don't have a theorem for dealing with L.T.s of arbitrarily shifted fn.s. i.e. relating $\mathcal{L}(f(t-a))$ to $\mathcal{L}(f(t))$.

$$= e^{-\frac{\pi}{2}s} - e^{-\pi s}\frac{s}{s^2+1}$$

$\therefore$ Using the O.D.E.,

$$s^2\mathcal{L}(y)+5s\,\mathcal{L}(y)+6\mathcal{L}y=e^{-\frac{\pi}{2}s}-e^{-\pi s}\frac{s}{s^2+1}$$

$$\Rightarrow \mathcal{L}(y)=\frac{1}{\underbrace{s^2+5s+6}_{=(s+2)(s+3)}}\left(e^{-\frac{\pi}{2}s}-e^{-\pi s}\frac{s}{s^2+1}\right)$$

$$=e^{-\frac{\pi}{2}s}\frac{1}{(s+2)(s+3)}-e^{-\pi s}\frac{s}{(s^2+1)(s+2)(s+3)}$$

$$=e^{-\frac{\pi}{2}s}\left(\frac{1}{s+2}-\frac{1}{s+3}\right)-e^{-\pi s}\left(\frac{s+1}{10(s^2+1)}-\frac{2}{5(s+2)}+\frac{3}{10(s+3)}\right)$$

(partial fractions)

$$\Rightarrow y=u\left(t-\tfrac{\pi}{2}\right)\left[e^{-2\left(t-\frac{\pi}{2}\right)}-e^{-3\left(t-\frac{\pi}{2}\right)}\right]$$

$$-u(t-\pi)\left[\frac{1}{10}\left(\underbrace{\cos(t-\pi)}_{=-\cos(t)}+\underbrace{\sin(t-\pi)}_{=-\sin(t)}\right)\right.$$

$$\left.-\frac{2}{5}e^{-2(t-\pi)}+\frac{3}{10}e^{-3(t-\pi)}\right]$$