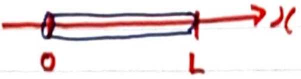
12.6 Heat equation

(IBVP)



$$\begin{cases} u_t - k u_{xx} = 0, & 0 < x < L, \ t > 0 \\ u(0,t) = 0, & u(L,t) = 0 \\ u(x,0) = f(x), & 0 < x < L \end{cases}$$

(Separation of variables)

Assume $f(x) \neq 0 \Rightarrow u(x,t) \neq 0$

Let $u(x,t) = F(x)\,G(t)$.

$u_t = F(x)\,G'(t), \quad u_{xx} = F''(x)\,G(t)$

$u_t = k u_{xx}$ iff $F(x)G'(t) = kF''(x)G(t)$

$$\frac{\cancel{F(x)}\,G'(t)}{k\,\cancel{F(x)}\,G(t)} = \frac{\cancel{k}\,F''(x)\,\cancel{G(t)}}{F(x)\,\cancel{G(t)}}$$

$$\frac{G'(t)}{kG(t)} = \frac{F''(x)}{F(x)} = m : \text{constant}$$

(1) $F'' = mF$ (2) $G' = mkG$

BC: $u(0,t) = F(0)G(t) = 0$ BC. for any $t>0$

$F(0) = 0$.

$u(L,t) = F(L)G(t) = 0$ ″

$F(L) = 0$

(1) $F'' - mF = 0, \quad F(0) = 0, \quad F(L) = 0$

⊛ $r^2 - m = 0 : \quad r = \pm\sqrt{m}$

① $m > 0 : \quad F(x) \equiv 0$

② $m = 0 : \quad F(x) \equiv 0$

③ $m < 0 :$

$$m = m_n = -\left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, \cdots$$

$$F_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$

(2) $G' - mkG = 0$ iff $G' + \left(\frac{n\pi}{L}\right)^2 kG = 0$

$\rho = e^{\left(\frac{n\pi}{L}\right)^2 kt}$: the integrating factor

$$e^{(\frac{n\pi}{L})^2 kt} G' + \underbrace{e^{(\frac{n\pi}{L})^2 kt} \left(\frac{n\pi}{L}\right)^2 k G}_{\frac{dP}{dt}} = 0$$

$$\frac{d}{dt}\left(e^{(\frac{n\pi}{L})^2 kt} G(t)\right) = 0$$

$$e^{(\frac{n\pi}{L})^2 kt} G(t) = A \text{ : constant.}$$

$$G_n(t) = A_n e^{-(\frac{n\pi}{L})^2 kt}, \quad n = 1, 2, \cdots$$

$$u_n(x,t) = F_n(x)\, G_n(t)$$

$$= A_n e^{-(\frac{n\pi}{L})^2 kt} \sin\left(\frac{n\pi x}{L}\right),$$

$$(n = 1, 2, \cdots)$$

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-(\frac{n\pi}{L})^2 kt} \sin\left(\frac{n\pi x}{L}\right)$$

IC

IC: $u(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) = f(x)$

The Fourier sine series of $f(x)$

$$A_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$

(1-dim)

(Steady-state heat flow) $u_t - k u_{xx} = 0$

$\frac{\partial u}{\partial t} = 0$: $-k\, u_{xx} = 0$

2-dim: $\frac{\partial u}{\partial t} - k \nabla^2 u = 0$

Def 1. $\Delta u = \nabla^2 u = \nabla \cdot (\nabla u) = u_{xx} + u_{yy}$

: the Laplacian of u.

2. 3-dim: $\Delta u = \nabla^2 u = u_{xx} + u_{yy} + u_{zz}$

Remark: $\Delta^s u$: fractional Laplacian

$\Delta^s u = f$: hot research topic

steady state heat equation: Laplace equation.

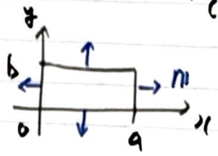
$0 = \frac{\partial u}{\partial t} - k \nabla^2 u = 0$: $\nabla^2 u = 0$

$\nabla^2 u = 0$ on $D \subset \mathbb{R}^2$.

① $D = \square$ rectangular

② $D = \bigcirc$: polar coordinates

$D = (0, a)(0, b)$



Boundary conditions.

the boundary of D.

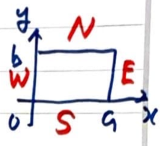
1. $u(x, y) = \phi(x, y)$. $(x, y) \in \partial D$

Dirichlet BC.

2. $\frac{\partial u}{\partial n} = \nabla u \cdot n = \psi(x, y)$, ∂D

Neumann BC.

3. $\alpha u(x, y) + \beta \frac{\partial u}{\partial n} = \gamma(x, y)$, ∂D.



(BVP)
$$\begin{cases} \nabla^2 u = 0, \quad D = (0, a)(0, b) \\ u(x, y) = 0, \quad E \cup W \cup S \\ u(x, b) = f(x), \quad y = b \ (:N) \end{cases}$$

(Separation of Variables)

$f(x) \neq 0 \Rightarrow u(x, y) \neq 0$

Set $u(x, y) = F(x) G(y)$

$u_{xx} + u_{yy} = 0$ iff $u_{xx} = -u_{yy}$

$$\frac{F''(x) \cancel{G(y)}}{F(x) \cancel{G(y)}} = -\frac{\cancel{F(x)} G''(y)}{\cancel{F(x)} G(y)}$$

$$\frac{F''(x)}{F(x)} = -\frac{G''(y)}{G(y)} = k : \text{constant}$$

(1) $F'' = kF$ (2) $G'' = -kG$

BC.

BC.: $x = 0$: $u(0, y) = F(0) G(y) = 0$ for any $y \in (0, b)$
(W)

$\therefore F(0) = 0$

$x = a$: $u(a, y) = F(a) G(y) = 0$ for $\forall y \in (0, b)$

$F(a) = 0$

$F'' - kF = 0, \quad F(0) = 0, \quad F(a) = 0$

$\underline{k_n = -\left(\frac{n\pi}{a}\right)^2}, \quad \underline{F_n(x) = \sin\left(\frac{n\pi}{a}x\right)}$

$(n = 1, 2, \cdots)$

(2) $G'' + kG = 0 : \quad \underline{G'' - \left(\frac{n\pi}{a}\right)^2 G = 0}$

BC: $u(x, 0) = F(x)G(0) = 0, \quad \forall x \in (0, a)$

$\underline{G(0) = 0}$

$r^2 - \left(\frac{n\pi}{a}\right)^2 = 0 : \quad r = \pm\left(\frac{n\pi}{a}\right), \quad n = 1, 2, \cdots$

$G(y) = Ae^{\frac{n\pi}{a}y} + \underbrace{B}_{=-A}e^{-\frac{n\pi}{a}y}$

$G(0) = A + B = 0 : \quad B = -A$

Let $G_n(y) = A_n\left(e^{\frac{n\pi}{a}y} - e^{-\frac{n\pi}{a}y}\right), \quad n = 1, 2, \cdots$

$u_n(x, y) = F_n(x)\, G_n(y)$

$u(x, y) = \sum_{n=1}^{\infty} u_n(x, y)$

$u(x, y) = \sum_{n=1}^{\infty} A_n\left(e^{\frac{n\pi y}{a}} - e^{-\frac{n\pi y}{a}}\right)\sin\left(\frac{n\pi x}{a}\right)$

Remark: $\sinh(z) = \frac{1}{2}\left(e^{z} - e^{-z}\right)$

$$G_n(y) = 2A_n \frac{e^{\frac{n\pi y}{a}} - e^{-\frac{n\pi y}{a}}}{2}$$

$$\overset{\alpha_n \text{ let}}{=} 2A_n \sinh\left(\frac{n\pi y}{a}\right).$$

$$u(x,y) = \sum_{n=1}^{\infty} \alpha_n \sinh\left(\frac{n\pi y}{a}\right) \sin\left(\frac{n\pi x}{a}\right)$$

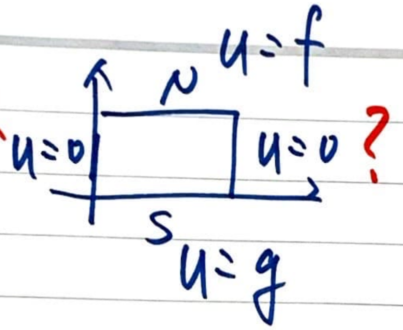
$y=b$: $$u(x,b) = \sum_{n=1}^{\infty} \alpha_n \sinh\left(\frac{n\pi b}{a}\right) \sin\left(\frac{n\pi x}{a}\right)$$

$$= f(x) \leftarrow \text{BC on } N$$

$$\alpha_n \sinh\left(\frac{n\pi b}{a}\right) = \frac{2}{a}\int_0^a f(x) \sin\left(\frac{n\pi x}{a}\right) dx$$

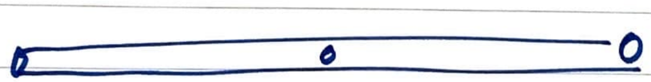
$$\alpha_n = \frac{2}{a\sinh(n\pi b/a)} \int_0^a f(x) \sin\left(\frac{n\pi x}{a}\right) dx$$

Q



? : $u(x,y) = \underline{u_1(x,y)} + \underline{u_2(x,y)}$

12.7. Heat equation on very long bars.



No boundary effect/condition.

(IVP)

$$\begin{cases} u_t - k u_{xx} = 0, & -\infty < x < \infty \\ u(x,0) = f(x) \end{cases}$$

Fourier integral / Fourier transform.