ECE 68000: MODERN AUTOMATIC CONTROL

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Stability of the Observation Error Dynamics

‘Practical Stability’

System:

$$\dot{e}(t) = \phi(t, e(t), w(t))$$

Performance output:

$$z(t) = \psi(t, e(t))$$

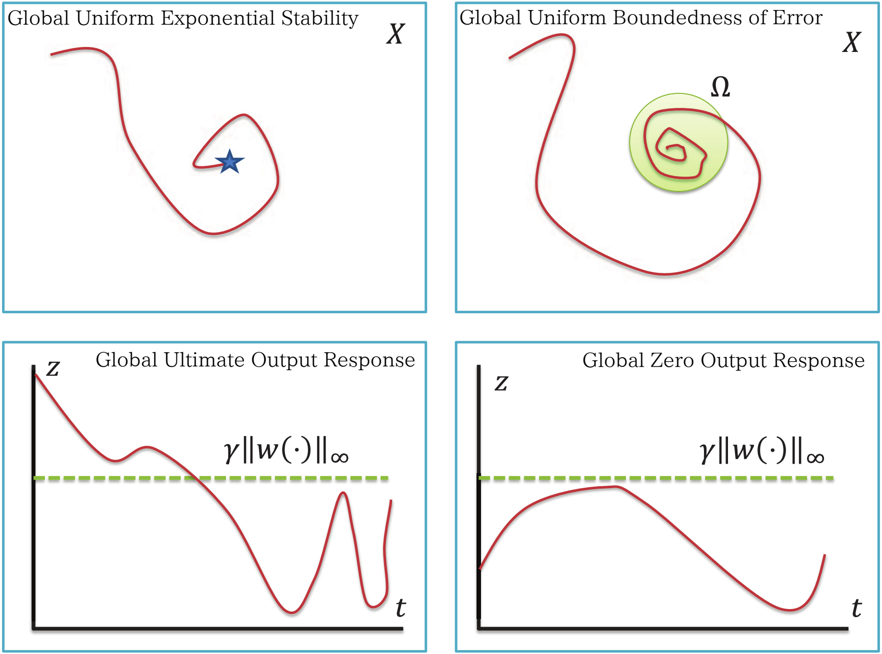
Definition ($\mathcal{L}_\infty$ stability with performance γ)

- Global uniform exponential stability ($w \equiv 0$)
- Global uniform boundedness of the error state ($w \neq 0$)
- Output response for zero initial error state

$$e(t_0) = 0 \implies \|z(t)\| \leq \gamma \|w(\cdot)\|_\infty, \;\; \forall t \geq t_0$$

- Global ultimate output response

$$\limsup_{t \to \infty} \|z(t)\| \leq \gamma \|w(\cdot)\|_\infty$$


Global Uniform Exponential Stability
X
Global Uniform Boundedness of Error
X
Ω
Global Ultimate Output Response
z
γ‖w(·)‖∞
t
Global Zero Output Response
z
γ‖w(·)‖∞
t

Sufficient Conditions for $\mathcal{L}_\infty$-stability with p.l. γ

$$\dot{e} = \phi(t, e, w), \qquad z = \psi(t, e), \qquad e \in \mathbb{R}^n$$

Theorem

There exists a differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ *and scalars* $\alpha, \beta_1, \beta_2 > 0$ *and* $\mu > 0$ *such that for all* $t \geq t_0$

$$\begin{aligned} \beta_1 \|e\|^2 \leq V(e) &\leq \beta_2 \|e\|^2 \\ \mathcal{D}V(e)\, \dot{e} &\leq -2\alpha \left(V(e) - \|w\|^2 \right) \\ \|z\|^2 &\leq \mu V(e) \end{aligned}$$

$\implies$ $\mathcal{L}_\infty$*-stability of general system with performance level* $\gamma = \sqrt{\mu}$

l_∞-stability with performance level (p.l.) γ

- Recall the observation error dynamics of the closed-loop UIO

$$\boldsymbol{e}[k+1] = (\boldsymbol{A}_1 - \boldsymbol{L}\boldsymbol{C})\boldsymbol{e}[k] - \boldsymbol{L}\boldsymbol{D}\boldsymbol{v}[k]$$

Notation

- For any vector $\boldsymbol{v} \in \mathbb{R}^n$, denote $\|\boldsymbol{v}\| = \sqrt{\boldsymbol{v}^\top \boldsymbol{v}}$
- For a sequence of vectors $\boldsymbol{v}_{k=k_0}^{\infty}$, denote $\|\boldsymbol{v}\|_\infty \triangleq \sup_{k \geq k_0} \|\boldsymbol{v}_k\|$
- We say that a sequence $\{\boldsymbol{v}[k]\} \in l_\infty$ if $\|\boldsymbol{v}\|_\infty < \infty$

B. Alenezi, M. Zhang, S. Hui, and S. H. Żak, Simultaneous Estimation of the State, Unknown Input, and Output Disturbance in Discrete-Time Linear Systems, *IEEE Transactions on Automatic Control*, Vol. 66, No. 12, December 2021, pp. 6115–6122

l_∞-stability definition

The system $\boldsymbol{e}[k+1] = \boldsymbol{f}(k, \boldsymbol{e}[k], \boldsymbol{v}[k])$ is globally uniformly l_∞-stable with performance level γ if

1. $\boldsymbol{e}[k+1] = \boldsymbol{f}(k, \boldsymbol{e}[k], \boldsymbol{0})$ globally uniformly exponentially stable with respect to the origin
2. for $\boldsymbol{e}[k_0] = \boldsymbol{0}$, and every bounded unknown input $\boldsymbol{v}[k]$, $\|\boldsymbol{e}[k]\| \leq \gamma \|\boldsymbol{v}[k]\|_\infty \ \forall k \geq k_0$
3. for any $\boldsymbol{e}[k_0] = \boldsymbol{e}_0$ and $\boldsymbol{v}[\cdot]$,

$$\limsup_{k \to \infty} \|\boldsymbol{e}[k]\| \leq \gamma \|\boldsymbol{v}[k]\|_\infty$$

A. Chakrabarty, S. H. Żak, and S. Sundaram, *State and unknown input observers for discrete-time nonlinear systems*, 2016 IEEE 55th CDC, Las Vegas, Dec 12–14, 2016, pp. 7111–7116

Sufficient condition for l_∞-stability

Lemma

Suppose that for $\boldsymbol{e}[k+1] = \boldsymbol{f}(k, \boldsymbol{e}[k], \boldsymbol{v}[k])$, *there exists* $V : \mathbb{R}^n \to \mathbb{R}$ *and scalars* $\delta \in (0,1)$, $\beta_1, \beta_2 > 0$ *and* $\mu_1, \mu_2 \geq 0$ *such that*

$$\beta_1 \|\boldsymbol{e}[k]\|^2 \leq V(\boldsymbol{e}[k]) \leq \beta_2 \|\boldsymbol{e}[k]\|^2$$

and

$$\Delta V[k] \leq -\delta(V(\boldsymbol{e}[k]) - \mu_1 \|\boldsymbol{v}[k]\|^2)$$

for all $k \geq 0$, *where* $\Delta V[k] = V(\boldsymbol{e}[k+1]) - V(\boldsymbol{e}[k])$. *Then, the error system is globally uniformly* l_∞*-stable with performance level* $\gamma = \sqrt{\mu_1 \mu_2}$ *with respect to the disturbance input sequence* $\boldsymbol{v}[k]$

Note that $\|\boldsymbol{e}[k]\|^2 \leq \mu_2 V(\boldsymbol{e}[k])$ where $\mu_2 = 1/\beta_1$

Proof of the sufficient condition

- Expand $\Delta V[k] \leq -\delta(V(\boldsymbol{e}[k]) - \mu_1 \|\boldsymbol{v}[k]\|^2)$
- Use $\Delta V[k] = V(\boldsymbol{e}[k+1]) - V(\boldsymbol{e}[k])$ to obtain

$$V(\boldsymbol{e}[k+1]) \leq (1-\delta)V(\boldsymbol{e}[k]) + \delta\mu_1 \|\boldsymbol{v}[k]\|^2$$

- Hence, $V(\boldsymbol{e}[1]) = (1-\delta)V(\boldsymbol{e}[0]) + \delta\mu_1 \|\boldsymbol{v}[0]\|^2$
- Next

$$\begin{aligned} V(\boldsymbol{e}[2]) &= (1-\delta)V(\boldsymbol{e}[1]) + \delta\mu_1 \|\boldsymbol{v}[1]\|^2 \\ &= (1-\delta)^2 V(\boldsymbol{e}[0]) + (1-\delta)\delta\mu_1 \|\boldsymbol{v}[0]\|^2 + \delta\mu_1 \|\boldsymbol{v}[1]\|^2 \end{aligned}$$

- Proceeding, we obtain

$$\begin{aligned} V(\boldsymbol{e}[k]) &= (1-\delta)^k V(\boldsymbol{e}[0]) + (1-\delta)^{k-1}\delta\mu_1 \|\boldsymbol{v}[0]\|^2 \\ &\quad \cdots + (1-\delta)\delta\mu_1 \|\boldsymbol{v}[k-2]\|^2 + \delta\mu_1 \|\boldsymbol{v}[k-1]\|^2 \\ &\leq (1-\delta)^k V(\boldsymbol{e}[0]) + \delta\mu_1 \left((1-\delta)^{k-1} + \cdots + 1\right) \|\boldsymbol{v}\|_\infty^2 \\ &= (1-\delta)^k V(\boldsymbol{e}[0]) + \delta\mu_1 \left(\frac{1-(1-\delta)^k}{1-(1-\delta)}\right) \|\boldsymbol{v}\|_\infty^2 \end{aligned}$$

Proof of the sufficient condition contd.

- We have

$$V(\boldsymbol{e}[k]) \leq (1-\delta)^k V(\boldsymbol{e}[0]) + \delta\mu_1 \left(\frac{1-(1-\delta)^k}{1-(1-\delta)}\right) \|\boldsymbol{v}\|_\infty^2$$

- Hence

$$V(\boldsymbol{e}[k]) \leq (1-\delta)^k V(\boldsymbol{e}[0]) + \mu_1 \|\boldsymbol{v}\|_\infty^2$$

  for any $k \geq 0$ since $0 < \delta < 1$

Proof of the sufficient condition—Conclusion

- We have

$$V(\boldsymbol{e}[k]) \leq (1-\delta)^k V(\boldsymbol{e}[0]) + \mu_1 \|\boldsymbol{v}\|_\infty^2$$

- Hence

$$\begin{aligned} \|\boldsymbol{e}[k]\|^2 &\leq \mu_2 V(\boldsymbol{e}[k]) \\ &\leq \mu_2 (1-\delta)^k V(\boldsymbol{e}[0]) + \mu_1 \mu_2 \|\boldsymbol{v}\|_\infty^2 \end{aligned}$$

- This implies

$$\limsup_{k\to\infty} \|\boldsymbol{e}[k]\|^2 \leq \mu_1 \mu_2 \|\boldsymbol{v}\|_\infty^2$$

- In sum, the error dynamics are l_∞-stable with performance level $\gamma = \sqrt{\mu_1 \mu_2}$

Stability of the error dynamics

Recall the observation error dynamics of the closed-loop UIO

$$\boldsymbol{e}[k+1] = (\boldsymbol{A}_1 - \boldsymbol{LC})\boldsymbol{e}[k] - \boldsymbol{LDv}[k]$$

Theorem

The observation error dynamics are globally uniformly l_∞-stable with performance level γ if $(\boldsymbol{A}_1 - \boldsymbol{LC})$ is Schur stable and either of the conditions of the definition of the l_∞-stability is satisfied

In sum: We proved stability of the error dynamics lemma that we will use next

Observation error: $\boldsymbol{e}[k+1] = (\boldsymbol{A}_1 - \boldsymbol{LC})\boldsymbol{e}[k] - \boldsymbol{LDv}[k]$

Lemma

Suppose there exists a function $V : \mathbb{R}^n \to \mathbb{R}$ *and scalars* $\delta \in (0,1)$, $\beta_1, \beta_2 > 0$ *and* $\mu_1, \mu_2 \geq 0$ *such that*

$$\beta_1 \|\boldsymbol{e}[k]\|^2 \leq V(\boldsymbol{e}[k]) \leq \beta_2 \|\boldsymbol{e}[k]\|^2,$$

$$\Delta V[k] \leq -\delta(V(\boldsymbol{e}[k]) - \mu_1 \|\boldsymbol{v}[k]\|^2)$$

for all $k \geq 0$. *Then, the observation error is globally uniformly* l_∞*-stable with performance level* $\gamma = \sqrt{\mu_1 \mu_2}$ *with respect to the output disturbance* $\boldsymbol{v}[k]$, *where* $\|\boldsymbol{e}[k]\|^2 \leq \mu_2 V(\boldsymbol{e}[k])$ *with* $\mu_2 = 1/\beta_1$