(Summary of Fourier series) Even/odd functions

1. $f(x)$ is periodic in $\mathbb{R}$ and even on $[-L, L]$

$b_n = 0$ for each n

$$F(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) = F_c(x)$$

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx, \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

2. $f(x)$ is periodic in $\mathbb{R}$ and odd on $[-L, L]$

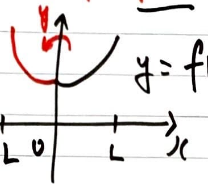
$$F(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) = F_s(x)$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

(Application) Missing data.

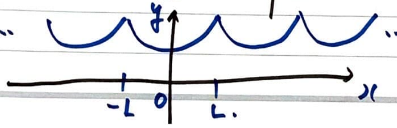
Q What if $f(x)$ is defined on $[0, L]$?

(idea)



1. Even extension: ① $f_E(x) = \begin{cases} f(x), & 0 < x < L \\ f(-x), & -L < x < 0 \end{cases}$

② Make $f_E(x)$ periodic



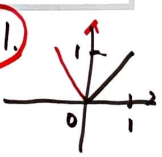
The Fourier series of $f_E(x)$:

$$F(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) = F_c(x)$$: the Fourier Cosine series of $f(x)$

$$a_0 = \frac{1}{L}\int_0^L f_E(x)\,dx = \frac{1}{L}\int_0^L f(x)\,dx$$

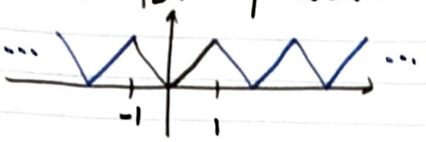
$$a_n = \frac{2}{L}\int_0^L f_E(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

(Ex) $f(x) = x, \quad 0 \le x < 1$: $L = 1$ Call half period.



(1) $f_E(x) = \begin{cases} x, & 0 \le x < 1 \\ -x, & -1 \le x < 0 \end{cases}$

(2) Make $f_E(x)$ periodic in $\mathbb{R}$



$L=1$

$$F(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x)$$

$$a_0 = \frac{1}{1}\int_0^1 x\,dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}$$

$$a_n = \frac{2}{1}\int_0^1 \underbrace{x}_{u}\,\underbrace{\cos(n\pi x)}_{v'}\,dx$$

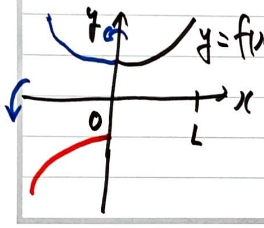
$$= 2\left(\cancel{\left[x\frac{\sin(n\pi x)}{n\pi}\right]_0^1} - \int_0^1 1\cdot\frac{\sin(n\pi x)}{n\pi}\,dx\right)$$

$$a_n = -2\frac{1}{n\pi}\left[\frac{-\cos(n\pi x)}{n\pi}\right]_0^1$$

$$= \frac{2}{(n\pi)^2}\left[\cos(n\pi) - 1\right]$$

$$F(x) = \frac{1}{2} + \sum_{n=1}^{\infty}\left[\frac{2(\cos(n\pi)-1)}{n^2\pi^2}\right]\cos(n\pi x) = F_c(x)$$

2. Odd extension:



(1) $f_o(x) = \begin{cases} f(x), & 0 \le x < L \\ -f(-x), & -L \le x < 0 \end{cases}$

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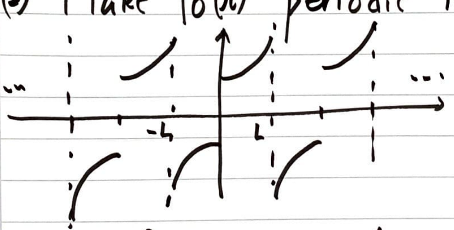
(2) Make $f_o(x)$ periodic in $\mathbb{R}$



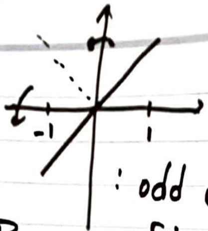
$f_o(x)$: periodic in $\mathbb{R}$ & odd on $[-L, L]$

$$F(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$b_n = \frac{2}{L}\int_0^L f_o(x)\sin\left(\frac{n\pi x}{L}\right)dx = \underline{\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx}$$

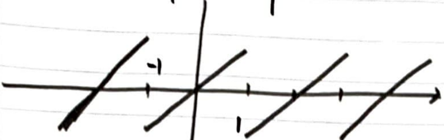
(Ex) $f(x) = x, \quad 0 \le x < 1.$

(1) $f_0(x) = \begin{cases} x, & 0 \le x < 1 \\ x, & -1 \le x < 0 \end{cases}$



: odd on $[-1, 1]$

(2) Make $f_0(x)$ periodic in $\mathbb{R}$



$$F(x) = \underline{\sum_{n=1}^{\infty} \left[\frac{-2\cos(n\pi)}{n\pi} \right] \sin(n\pi x)} = F_s(x)$$

$$b_n = \frac{2}{1} \int_0^1 x \sin(n\pi x)\, dx = \frac{-2\cos(n\pi)}{n\pi}$$

11.3. Forced Oscillations.

or

(Spring)

Topic : external force.

mass

$m y'' + c y' + k y = r(t)$: (c: damping constant, k: spring ", r(t): external force)

$y(t)$: the displacement. at t.

(Ex) $y'' + 9y = \cos(t)$

(1) $y'' + 9y = 0$: $y(t) = e^{\lambda t}$

$\lambda^2 + 9 = 0$ $\lambda = \pm 3i$

$y_c(t) = C_1 \cos(3t) + C_2 \sin(3t)$

(2) Find $y_p(t)$:

Set $y_p(t) = A\cos(t) + B\sin(t) \to$ DE

$y_p' = -A\sin(t) + B\cos(t)$

$y_p'' = -A\cos(t) - B\sin(t)$

(L) $y_p'' + 9y_p = -A\cos(t) - B\sin(t)$

$+ 9A\cos(t) + 9B\sin(t)$

$= \underbrace{8A}_{=1}\cos(t) + \underbrace{8B}_{=0}\sin(t)$

(R) $= \cos(t)$

$A = \frac{1}{8},\quad B = 0: \quad y_p(t) = \frac{1}{8}\cos(t)$

$y(t) = C_1\cos(3t) + C_2\sin(3t) + \frac{1}{8}\cos(t).$

(Ex) $m = 1\,(g),\quad c = 0.1\,(g/s),\quad k = 2\,(g/s^2)$

$y'' + 0.1y' + 2y = \sin(t)$

(1) $y'' + 0.1y' + 2y = 0: \quad \lambda^2 + 0.1\lambda + 2 = 0$

$\lambda = \frac{1}{2}\left(-0.1 \pm \underbrace{\sqrt{0.1^2 - 4\cdot 2}}_{=\sqrt{0.01-8} = \sqrt{7.99}i}\right) = -0.05 \pm \frac{\sqrt{7.99}i}{2}$

$y_c(t) = C_1 e^{-0.05t}\cos\left(\frac{\sqrt{7.99}}{2}t\right)$
$\quad + C_2 e^{-0.05t}\sin\left(\sqrt{7.99}t/2\right).$

(2) Set $y_p(t) = A\cos(t) + B\sin(t) \to$ DE

$y_p'(t) = -A\sin(t) + B\cos(t)$

$y_p'' = -A\cos(t) + (-B)\sin(t)$

Ⓛ $y_p'' + 0.1y_p' + 2y_p$

$= -A\cos(t) - B\sin(t) - 0.1A\sin(t) + 0.1B\cos(t)$
$\quad + 2A\cos(t) + 2B\sin(t)$

$= \underbrace{(A + 0.1B)}_{=0}\cos(t) + \underbrace{(-0.1A + B)}_{=1}\sin(t)$

Ⓡ $= \sin(t)$

$\begin{cases} A + 0.1B = 0 & \to A = -0.1B \\ -0.1A + B = 1 \end{cases}$

$\underbrace{-0.1(-0.1)B}_{0.01B} + B = 1$ iff $1.01B = 1$

$B = \frac{1}{1.01} = \frac{100}{101}$

$A = -\frac{1}{10}\frac{100}{101} = -\frac{10}{101}$

$y_p(t) = -\frac{10}{101}\cos(t) + \frac{100}{101}\sin(t)$

$y(t) = y_c(t) + y_p(t)$: a general solution.