ECE 602: LUMPED LINEAR SYSTEMS

Professor Stan Żak

Minimal Realizations of Transfer Function Matrices

Minimal Realizations of Transfer Function Matrices

- A transfer function $\boldsymbol{G}(s)$ is realizable if there exists a quadruple of constant matrices $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D})$ such that $\boldsymbol{G}(s) = \boldsymbol{C}\left(s\boldsymbol{I}_n - \boldsymbol{A}\right)^{-1}\boldsymbol{B} + \boldsymbol{D}$. We call such a quadruple $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D})$ a realization of $\boldsymbol{G}(s)$

Definition

The dimension of a realization is the size of the matrix $\boldsymbol{A}$, that is, if $\boldsymbol{A}$ is an n-by-n matrix then we say that the dimension of the corresponding realization is n

- **Objective**: Find a minimal state-space realization a transfer function matrix of a linear lumped system either discrete or continuous time-invariant multi-input multi-output (MIMO) system

Realizations of a transfer function matrix of a multi-input multi-output (MIMO) system

- Different methods may yield realizations of different dimensions
- A proper rational transfer function matrix has infinitely many realizations of different dimensions
- A realization with the smallest possible dimension is called a minimal realization
- Necessary and sufficient conditions for a realization to be minimal was given by Kalman in 1963

Necessary and Sufficient Condition for a Realization to be Minimal

Theorem

A realization $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D})$ *of a given transfer function matrix* $\boldsymbol{G}(s)$ *is minimal if and only if the pair* $(\boldsymbol{A}, \boldsymbol{B})$ *is reachable and the pair* $(\boldsymbol{A}, \boldsymbol{C})$ *is observable*

Can always lower the dimension of a realization if it is not reachable/observable!

- If a realization $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D})$ is non-reachable, separate the reachable part from the non-reachable one
- That is, construct a similarity transformation $\boldsymbol{z} = \boldsymbol{T}\boldsymbol{x}$ such that

$$\tilde{\boldsymbol{A}} = \boldsymbol{T}\boldsymbol{A}\boldsymbol{T}^{-1} = \left[\begin{array}{c|c} \boldsymbol{A}_1 & \boldsymbol{A}_2 \\ \hline \boldsymbol{O} & \boldsymbol{A}_4 \end{array}\right], \quad \tilde{\boldsymbol{B}} = \boldsymbol{T}\boldsymbol{B} = \left[\begin{array}{c} \boldsymbol{B}_1 \\ \hline \boldsymbol{O} \end{array}\right],$$

  where the pair $(\boldsymbol{A}_1, \boldsymbol{B}_1)$ is reachable $\boldsymbol{A}_1 \in \mathbb{R}^{r \times r}$, $\boldsymbol{B}_1 \in \mathbb{R}^{r \times m}$
- The matrix $\boldsymbol{C}$ becomes

$$\tilde{\boldsymbol{C}} = \boldsymbol{C}\boldsymbol{T}^{-1} = \left[\begin{array}{c|c} \boldsymbol{C}_1 & \boldsymbol{C}_2 \end{array}\right],$$

  where $\boldsymbol{C}_1 \in \mathbb{R}^{p \times r}$, and $\tilde{\boldsymbol{D}} = \boldsymbol{D}$

Lowering the dimension of a realization

- The transfer functions of $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D})$ and $(\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{B}}, \tilde{\boldsymbol{C}}, \boldsymbol{D})$ are the same because the transfer function is invariant under similarity transformations
- We have

$$\begin{aligned}
\boldsymbol{G}(s) &= \boldsymbol{C}\left(s\boldsymbol{I}_n - \boldsymbol{A}\right)^{-1}\boldsymbol{B} + \boldsymbol{D} \\
&= \tilde{\boldsymbol{C}}\left(s\boldsymbol{I}_n - \tilde{\boldsymbol{A}}\right)^{-1}\tilde{\boldsymbol{B}} + \boldsymbol{D} \\
&= \left[\begin{array}{c|c} \boldsymbol{C}_1 & \boldsymbol{C}_2 \end{array}\right]\left[\begin{array}{c|c} s\boldsymbol{I}_r - \boldsymbol{A}_1 & -\boldsymbol{A}_2 \\ \hline \boldsymbol{O} & s\boldsymbol{I}_{n-r} - \boldsymbol{A}_4 \end{array}\right]^{-1}\left[\begin{array}{c} \boldsymbol{B}_1 \\ \hline \boldsymbol{O} \end{array}\right] + \boldsymbol{D} \\
&= \boldsymbol{C}_1[s\boldsymbol{I}_r - \boldsymbol{A}_1]^{-1}\boldsymbol{B}_1 + \boldsymbol{D}
\end{aligned}$$

Constructing a minimal realization from a given realization

- The quadruple $(\boldsymbol{A}_1, \boldsymbol{B}_1, \boldsymbol{C}_1, \boldsymbol{D})$ is a realization of $\boldsymbol{G}(s)$ of lower dimension then the realization $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D})$
- Proceed to check if the pair $(\boldsymbol{A}_1, \boldsymbol{C}_1)$ is observable or not
- If it is, then $(\boldsymbol{A}_1, \boldsymbol{B}_1, \boldsymbol{C}_1, \boldsymbol{D})$ is a minimal realization of $\boldsymbol{G}(s)$
- If, on the other hand, the pair $(\boldsymbol{A}_1, \boldsymbol{C}_1)$ is not observable, extract from the pair $(\boldsymbol{A}_1, \boldsymbol{C}_1)$ an observable part
- This observable part along with corresponding input sub-matrix and the matrix $\boldsymbol{D}$ form a minimal realization of $\boldsymbol{G}(s)$

Example

- Can obtain a minimal realization of a given transfer function matrix using MATLAB's function ss
- Construct a minimal realization of the transfer function matrix

$$\boldsymbol{G}(s) = \begin{bmatrix} \frac{1}{s(s+2)} & \frac{2s-1}{s(s+2)} & \frac{s-1}{s(s+2)} \\ -\frac{1}{s+2} & \frac{1-s}{s^2+3s+2} & \frac{1}{s^2+3s+2} \end{bmatrix}.$$

- Represent the tf as

```
G=[tf([1],[1 2 0]) tf([2 -1],[1 2 0]) tf([1 -1],[1 2 0]);...
   tf([-1],[1 2]) tf([-1 1],[1 3 2])  tf([1],[1 3 2])]
```

- Then type in

```
sys = ss(G,'min')
```

Minimal realization

$$
\begin{aligned}
\boldsymbol{A} &= \begin{bmatrix} -0.2586 & -0.3622 & 0.1951 & -0.4402 \\ -0.3806 & -1.675 & 0.6493 & 0.3853 \\ -1.046 & 0.519 & -1.267 & 0.6196 \\ 1.082 & 0.1781 & 1.258 & -1.799 \end{bmatrix} \\
\boldsymbol{B} &= \begin{bmatrix} 0.4118 & -0.3924 & -0.4021 \\ 0.612 & 0.434 & -0.08897 \\ -0.5916 & -1.032 & -0.2202 \\ -0.395 & 1.429 & 0.9121 \end{bmatrix} \\
\boldsymbol{C} &= \begin{bmatrix} -0.4876 & -0.6661 & -1.355 & 0.4892 \\ -0.8541 & -1.105 & 0.2379 & -0.4269 \end{bmatrix} \\
\boldsymbol{D} &= \boldsymbol{O}
\end{aligned}
$$

Minimal realization

- Can check, using, for example, MATLAB's functions `ctrb` and `obsv` that the pair $(\boldsymbol{A}, \boldsymbol{B})$ is reachable and the pair $(\boldsymbol{A}, \boldsymbol{C})$ is observable
- Thus the above quadruple is indeed a minimal realization of the given transfer function matrix $\boldsymbol{G}(s)$.

Minimal realizations of the given $G(s)$ are equivalent

- There are infinitely many different minimal realizations of a given transfer function matrix
- They are related via a similarity transformation
- That is, minimal realizations of a given transfer function matrix are equivalent
- Indeed, suppose we have two different minimal realizations $(\boldsymbol{A}_1, \boldsymbol{B}_1, \boldsymbol{C}_1, \boldsymbol{D}_1)$ and $(\boldsymbol{A}_2, \boldsymbol{B}_2, \boldsymbol{C}_2, \boldsymbol{D}_2)$ of a given proper transfer function matrix $\boldsymbol{G}(s)$
- Then

$$\boldsymbol{G}(s) = \boldsymbol{C}_1(s\boldsymbol{I}_n - \boldsymbol{A}_1)^{-1}\boldsymbol{B}_1 + \boldsymbol{D}_1 = \boldsymbol{C}_2(s\boldsymbol{I}_n - \boldsymbol{A}_2)^{-1}\boldsymbol{B}_2 + \boldsymbol{D}_2$$

- Hence, we have to have

$$\boldsymbol{D}_1 = \boldsymbol{D}_2$$

Minimal realizations of the given $G(s)$ are equivalent—Contd

- We have

$$\boldsymbol{C}_1(s\boldsymbol{I}_n - \boldsymbol{A}_1)^{-1}\boldsymbol{B}_1 = \boldsymbol{C}_2(s\boldsymbol{I}_n - \boldsymbol{A}_2)^{-1}\boldsymbol{B}_2$$

- Taking the inverse Laplace transform of both sides yields

$$\boldsymbol{C}_1 e^{\boldsymbol{A}_1 t}\boldsymbol{B}_1 = \boldsymbol{C}_2 e^{\boldsymbol{A}_2 t}\boldsymbol{B}_2$$

- Taking into account that

$$e^{\boldsymbol{A}t} = \boldsymbol{I} + t\boldsymbol{A} + \frac{t^2}{2!}\boldsymbol{A}^2 + \cdots = \sum_{k=0}^{\infty} \frac{t^k}{k!}\boldsymbol{A}^k$$

we obtain

$$\sum_{k=0}^{\infty} \frac{t^k}{k!}\boldsymbol{C}_1\boldsymbol{A}_1^k\boldsymbol{B}_1 = \sum_{k=0}^{\infty} \frac{t^k}{k!}\boldsymbol{C}_2\boldsymbol{A}_2^k\boldsymbol{B}_2$$

Markov's parameters

- We obtain

$$\sum_{k=0}^{\infty} \frac{t^k}{k!} \boldsymbol{C}_1 \boldsymbol{A}_1^k \boldsymbol{B}_1 = \sum_{k=0}^{\infty} \frac{t^k}{k!} \boldsymbol{C}_2 \boldsymbol{A}_2^k \boldsymbol{B}_2$$

- That is,

$$\boldsymbol{C}_1 \boldsymbol{A}_1^k \boldsymbol{B}_1 = \boldsymbol{C}_2 \boldsymbol{A}_2^k \boldsymbol{B}_2, \quad k = 0, 1, 2, \ldots$$

- The matrices $\boldsymbol{C}_i \boldsymbol{A}_i^k \boldsymbol{B}_i$ are called Markov parameters
- Thus Markov parameters of different realizations of a given transfer function matrix are equal

Constructing a similarity transformation linking two different minimal realizations of the same tf

- Using Markov parametrs, we obtain

$$\begin{bmatrix} \boldsymbol{C}_1 \\ \boldsymbol{C}_1\boldsymbol{A}_1 \\ \vdots \\ \boldsymbol{C}_1\boldsymbol{A}_1^{n-1} \end{bmatrix} \begin{bmatrix} \boldsymbol{B}_1 & \cdots & \boldsymbol{A}_1^{n-1}\boldsymbol{B}_1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{C}_2 \\ \boldsymbol{C}_2\boldsymbol{A}_2 \\ \vdots \\ \boldsymbol{C}_2\boldsymbol{A}_2^{n-1} \end{bmatrix} \begin{bmatrix} \boldsymbol{B}_2 & \cdots & \boldsymbol{A}_2^{n-1}\boldsymbol{B}_2 \end{bmatrix}$$

- Let

$$\mathcal{U}_i = \begin{bmatrix} \boldsymbol{B}_i & \boldsymbol{A}_i\boldsymbol{B}_i & \cdots & \boldsymbol{A}_i^{n-1}\boldsymbol{B}_i \end{bmatrix}, \quad i = 1, 2$$

  and

$$\mathcal{V}_i = \begin{bmatrix} \boldsymbol{C}_i \\ \boldsymbol{C}_i\boldsymbol{A}_i \\ \vdots \\ \boldsymbol{C}_i\boldsymbol{A}_i^{n-1} \end{bmatrix}, \quad i = 1, 2.$$

Constructing a similarity transformation linking two different minimal realizations

- Then

$$\mathcal{V}_1\mathcal{U}_1 = \mathcal{V}_2\mathcal{U}_2$$

Pre-multiplying both sides by $\mathcal{V}_1^\top$ gives

$$\mathcal{V}_1^\top\mathcal{V}_1\mathcal{U}_1 = \mathcal{V}_1^\top\mathcal{V}_2\mathcal{U}_2$$

- Both realizations are minimal, therefore their controllability and observability matrices have full ranks, that is,

$$\text{rank}\,\mathcal{U}_i = \text{rank}\,\mathcal{V}_i = n, \quad i = 1, 2$$

Similarity transformation

- The $n \times n$ matrix
$$\mathcal{V}_1^\top \mathcal{V}_1$$
is invertible
- Pre-multiply both sides of $\mathcal{V}_1^\top \mathcal{V}_1 \mathcal{U}_1 = \mathcal{V}_1^\top \mathcal{V}_2 \mathcal{U}_2$ by
$$\left[\mathcal{V}_1^\top \mathcal{V}_1\right]^{-1}$$
- We obtain
$$\mathcal{U}_1 = \left[\mathcal{V}_1^\top \mathcal{V}_1\right]^{-1} \mathcal{V}_1^\top \mathcal{V}_2 \mathcal{U}_2$$
- Let
$$\boldsymbol{T}_1 = \left[\mathcal{V}_1^\top \mathcal{V}_1\right]^{-1} \mathcal{V}_1^\top \mathcal{V}_2$$
- Then
$$\mathcal{U}_1 = \boldsymbol{T}_1 \mathcal{U}_2$$

Minimal realizations are equivalent

- Re-write $\mathcal{U}_1 = T_1 \mathcal{U}_2$ as

$$\begin{aligned} \begin{bmatrix} B_1 & \cdots & A_1^{n-1} B_1 \end{bmatrix} &= T_1 \begin{bmatrix} B_2 & \cdots & A_2^{n-1} B_2 \end{bmatrix} \\ &= \begin{bmatrix} T_1 B_2 & \cdots & T_1 A_2^{n-1} T_1^{-1} T_1 B_2 \end{bmatrix} \end{aligned}$$

- Recall

$$T_1 A_2^k T_1^{-1} = \left(T_1 A_2 T_1^{-1} \right)^k$$

- Therefore, the pairs (A_1, B_1) and (A_2, B_2) are equivalent
- That is, they are related via the similarity transformation
- In other words,

$$A_1 = T_1 A_2 T_1^{-1} \quad \text{and} \quad B_1 = T_1 B_2$$

- Express the similarity transformation matrix T_2 relating the observability matrices and show that $T_1 = T_2$

Example

- Construct a realization of the transfer function

$$G(s) = \frac{4s^3 - 2s^2 + 3s + 1}{s^3 + 3s^2 - 5s + 7}$$

- First, represent $G(s)$ as

$$G(s) = G(s)_{\mathrm{sp}} + G(\infty) = \frac{-14s^2 + 23s - 27}{s^3 + 3s^2 - 5s + 7} + 4$$

- Using the method we discussed previously

$$\begin{aligned}
\dot{\boldsymbol{x}} &= \boldsymbol{A}_1 \boldsymbol{x} + \boldsymbol{b}_1 u \\
&= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -7 & 5 & -3 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \\
y &= \boldsymbol{c}_1 \boldsymbol{x} + d_1 u \\
&= \begin{bmatrix} -27 & 23 & -14 \end{bmatrix} \boldsymbol{x} + 4u
\end{aligned}$$

Another minimal realization of $G(s)$

- Using the results of the above discussion, we obtain

$$
\begin{aligned}
\dot{\tilde{\boldsymbol{x}}} &= \boldsymbol{A}_2\tilde{\boldsymbol{x}} + \boldsymbol{b}_2 u \\
&= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -7 & 5 & -3 \end{bmatrix} \tilde{\boldsymbol{x}} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \\
y &= \boldsymbol{c}_2\tilde{\boldsymbol{x}} + d_2 u \\
&= \begin{bmatrix} -27 & 23 & -14 \end{bmatrix} \boldsymbol{x} + 4u
\end{aligned}
$$

- Note that

$$d_1 = d_2 = 4$$

- Construct

$$\boldsymbol{T}_1 = \left[\mathcal{V}_1^\top \mathcal{V}_1\right]^{-1} \mathcal{V}_1^\top \mathcal{V}_2$$

Verifying the equivalence of the minimal realizations

- Minimal realizations $(\boldsymbol{A}_1, \boldsymbol{b}_1, \boldsymbol{c}_1, d_1)$ and $(\boldsymbol{A}_2, \boldsymbol{b}_2, \boldsymbol{c}_2, d_2)$ of $G(s)$ are equivalent if

$$\boldsymbol{A}_1 = \boldsymbol{T}_1 \boldsymbol{A}_2 \boldsymbol{T}_1^{-1}, \quad \boldsymbol{b}_1 = \boldsymbol{T}_1 \boldsymbol{b}_2, \quad \boldsymbol{c}_1 = \boldsymbol{c}_2 \boldsymbol{T}_1^{-1}, \quad d_1 = d_2$$

where

$$\boldsymbol{T}_1 = \left[\boldsymbol{\mathcal{V}}_1^\top \boldsymbol{\mathcal{V}}_1\right]^{-1} \boldsymbol{\mathcal{V}}_1^\top \boldsymbol{\mathcal{V}}_2 = \begin{bmatrix} -0.0198 & -0.0555 & -0.0529 \\ -0.0555 & -0.0529 & 0.0201 \\ -0.0529 & 0.0201 & 0.0636 \end{bmatrix}$$

where, in our Example,

$$\boldsymbol{\mathcal{V}}_i = \begin{bmatrix} \boldsymbol{c}_i \\ \boldsymbol{c}_i \boldsymbol{A}_i \\ \boldsymbol{c}_i \boldsymbol{A}_i^2 \end{bmatrix} \quad i = 1, 2$$