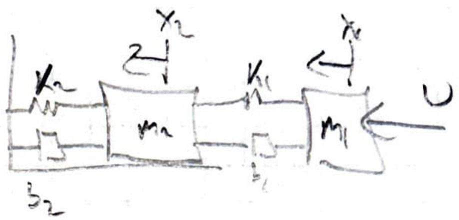
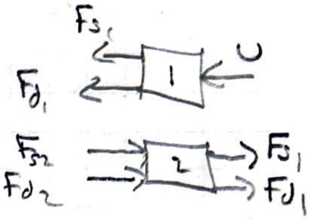
#1)





$$m\ddot{x}_1 = u + k_1(x_2-x_1) + b_1(\dot{x}_2-\dot{x}_1)$$

$$m\ddot{x}_2 = -k_2x_2 - b_2\dot{x}_2 - k_1(x_2-x_1) - b_1(\dot{x}_2-\dot{x}_1)$$

Let $z_1 = x_1$, $z_2 = x_2$, $z_3 = \dot{x}_1$, $z_4 = \dot{x}_2$

$$\dot{z}_3 = \frac{u}{m} + \frac{k_1}{m}z_2 - \frac{k_1}{m}z_1 + \frac{b_1 z_4}{m} - \frac{b_1 z_3}{m}$$

$$\dot{z}_4 = -\frac{k_2}{m}z_2 - \frac{b_2}{m}z_4 - \frac{k_1}{m}z_2 + \frac{k_1 z_1}{m} - \frac{b_1 z_4}{m} + \frac{b_1 z_3}{m}$$

$$\dot{x} = \begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{pmatrix} = Ax + Bu$$

$$\boxed{A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1}{m} & \frac{k_1}{m} & -\frac{b_1}{m} & \frac{b_1}{m} \\ \frac{k_1}{m} & -\frac{1}{m}(k_1+k_2) & \frac{b_1}{m} & -\frac{1}{m}(b_1+b_2) \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{m} \\ 0 \end{pmatrix}}$$

$m_1 = m_2 = 1, \quad b_1 = b_2 = 1, \quad K_2 = 1/4$

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -K_1 & K_1 & -1 & 1 \\ K_1 & (-K_1 - \frac{1}{4}) & 1 & -2 \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

If controllable : $\operatorname{rank}([B \ AB \ A^2B \ A^3B]) = 4$

let $CO = [B \ AB \ A^2B \ A^3B]$

$$AB = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} \qquad A^2B = (A)(AB) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -K_1 & K_1 & -1 & 1 \\ K_1 & (-K_1 - \frac{1}{4}) & 1 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

$$A^2B = \begin{pmatrix} -1 \\ 1 \\ -K_1 + 2 \\ K_1 - 3 \end{pmatrix} \qquad A^3B = (A)(A^2B)$$

$$A^3B = \begin{pmatrix} -K_1 + 2 \\ K_1 - 3 \\ 4K_1 - 5 \\ -K_1 - K_1 - \frac{1}{4} - K_1 + 2 - 2K_1 + 6 \end{pmatrix} = \begin{pmatrix} -K_1 + 2 \\ K_1 - 3 \\ 4K_1 - 5 \\ -5K_1 + \frac{31}{4} \end{pmatrix}$$

For CO to have full rank, $\det(CO) \neq 0$

$$CO = \begin{pmatrix} 0 & 1 & -1 & -K_1 + 2 \\ 0 & 0 & 1 & K_1 - 3 \\ 1 & -1 & -K_1 + 2 & 4K_1 - 5 \\ 0 & 1 & K_1 - 3 & -5K_1 + \frac{31}{4} \end{pmatrix}$$

$\det(CO) = a_{11} C_{11} - a_{12} C_{12} + a_{13} C_{13} - a_{14} C_{14}$

$a_{11} = 0 \ , \ a_{12} = 1 \ , \ a_{13} = -1 \ , \ a_{14} = -K_1 + 2$

$$- \begin{vmatrix} 0 & 1 & K_1 - 3 \\ 1 & -K_1 + 2 & 4K_1 - 5 \\ 0 & K_1 - 3 & -5K + \frac{31}{4} \end{vmatrix} - \begin{vmatrix} 0 & 0 & K_1 - 3 \\ 1 & -1 & 4K - 5 \\ 0 & 1 & -5K + \frac{31}{4} \end{vmatrix} - \begin{vmatrix} 0 & 0 & 1 \\ 1 & -1 & -K_1 + 2 \\ 0 & 1 & K_1 - 3 \end{vmatrix} (-K_1 + 2)$$

$= -\left[-\left(-5K + \frac{31}{4}\right) + K_1 - 3(K_1 - 3) \right] - (K_1 - 3)(1) + K_1 - 2$

$= -5K + \frac{31}{4} - (K_1 - 3)^2 - K_1 + 3 + K_1 - 2$

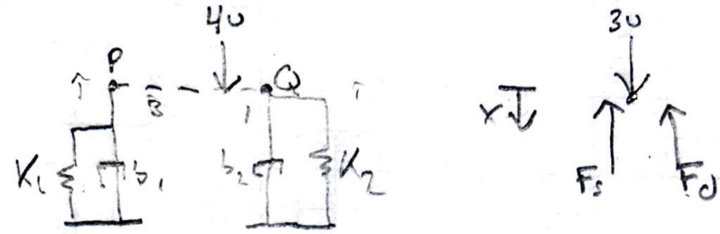
$= -5K_1 + \frac{31}{4} - (K_1^2 - 6K_1 + 9) + 1 = -K_1^2 + K_1 - \frac{1}{4} = -(K_1 - K_1 + \frac{1}{2})$

$= -(K_1 - \frac{1}{2})^2 = \det(CO)$

$\det(CO) = 0 \quad \text{if} \quad K_1 = \frac{1}{2}$

$\therefore$ $\boxed{\text{The system model is controllable if } K_1 \neq \frac{1}{2}}$

#2)



At equilibrium: $\Sigma F = 0 = F_P + F_Q - 4u = 0$

$\Sigma M_J = 0 = (F_P)(3) - (F_Q)(1)$ $\therefore$

$3F_P = F_Q \Rightarrow 4F_P = 4u \Rightarrow F_P = u$ & $F_Q = 3u$

$\Sigma F_P = m_P \ddot{x}_P = u - K_1 x_1 - b_1 \dot{x}_1$

$\Sigma F_Q = m_Q \ddot{x}_Q = 3u - K_2 x_2 - b_2 \dot{x}_2$

$m_P = m_Q = 0$, $K_1 = b_1 = b_2 = 1$, $K_2 = 2$

$\therefore \dot{x}_1 = \dfrac{u - K_1 x_1}{b_1}$, $\dot{x}_2 = \dfrac{3u - K_2 x_2}{b_2}$

$$\boxed{\dot{x} = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} x + \begin{pmatrix} 1 \\ 3 \end{pmatrix} u}$$

$x_1(0) = x_2(0) = 10$

$$CO = [B \;\; AB] = \begin{pmatrix} 1 & -1 \\ 3 & -6 \end{pmatrix}$$

$\det(CO) = -6+3 = 3 \neq 0 \quad \therefore \quad \operatorname{rank}(CO) = 2$ & system is Controllable

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)} B\, u(\tau)\, d\tau$$

$$x(1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$u(t) = -B^T e^{A^T(t_1-t)} W_c^{-1}(t_1) \left[e^{At} x(t_0) - x(t_1) \right]$$

Let $t_1 = 1$, $x(1) = 0 \;\therefore$

$$u(t) = -B^T e^{A^T(1-t)} W_c^{-1}(1)\, e^{A} x(0)$$

$$W_c(t) = \int_0^t e^{A(t-\tau)} BB^T e^{A^T(t-\tau)}\, d\tau$$

$$W_c(1) = \int_0^1 e^{A(1-\tau)} BB^T e^{A^T(1-\tau)}\, d\tau$$

$$W_c(1) = \int_0^1 \begin{pmatrix} e^{-(1-\tau)} & 0 \\ 0 & e^{-(2-\tau)} \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix} \begin{pmatrix} e^{-(1-\tau)} & 0 \\ 0 & e^{-(2-\tau)} \end{pmatrix} d\tau$$

$$W_c(1) = \int_0^1 \begin{pmatrix} e^{2\tau-2} & 3e^{(\tau-1)}e^{(2\tau-2)} \\ 3e^{(\tau-1)(2\tau-2)}e & 9e^{(4\tau-4)} \end{pmatrix}$$

$$\int_0^1 e^{2\tau-2} = \frac{e^{2\tau-2}}{2}\Big|_0^1 = \frac{e^0}{2} - \frac{e^{-2}}{2} = .4323$$

$$\int_0^1 3e^{(\tau-1)}e^{(2\tau-2)} = \int_0^1 3e^{3\tau-3} = e^{3\tau-3}\Big|_0^1 = e^0 - e^{-3} = .9502$$

$$\int_0^1 9e^{4\tau-4} = \frac{9}{4}e^{4\tau-4}\Big|_0^1 = \frac{9}{4}\left(1-e^{-4}\right) = 2.2088$$

$$W_c = \begin{pmatrix} .4323 & .9502 \\ .9502 & 2.2088 \end{pmatrix} \qquad W_c^{-1} = \begin{pmatrix} 2.2088 & -.9502 \\ -.9502 & .4323 \end{pmatrix}\frac{1}{.052}$$

$$W_c^{-1} = \begin{pmatrix} 42.4549 & -18.2640 \\ -18.2640 & 8.3098 \end{pmatrix}$$

$$u(t) = -[1 \;\; 3]\begin{pmatrix} e^{-(1-t)} & 0 \\ 0 & e^{-2(1-t)} \end{pmatrix}\begin{pmatrix} 42.4549 & -18.2640 \\ -18.2640 & 8.3098 \end{pmatrix}\begin{pmatrix} e^{-1} & 0 \\ 0 & e^{-2} \end{pmatrix}\begin{pmatrix} 10 \\ 10 \end{pmatrix}$$

$$\boxed{u(t) = 167.83e^{(2t-2)} - 131.47e^{(t-1)}}$$

To prevent states from going to ∞, set $u(t) = 0$ if $t > 1$ second.

#3) $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\lambda_1 = -2,\ \lambda_2 = -3,\ U_e = 2$

$\dot{X} = Ax + bu = f(x, u)$

$\dot{x} = f(X_e + \delta x,\ U_e + \delta u)$ $x = X_e + \delta x,\ u = U_e + \delta U$

$\dot{X}_1 = X_2$

$\dot{X}_2 = x_1 + u$

At equilibrium, $f(X_e, U_e) = 0 \therefore \dot{X}_1 = \dot{X}_2 = 0,\ U = U_e = 2$

$\therefore X_2 = 0,\ X_1 = -U_e = -2$

$X_e = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$

$\frac{\partial f_1}{\partial x_1} = 0,\ \frac{\partial f_1}{\partial x_2} = 1,\ \frac{\partial f_2}{\partial x_1} = 1,\ \frac{\partial f_2}{\partial x_2} = 0$

$\frac{\partial f_1}{\partial u} = 0,\ \frac{\partial f_2}{\partial u} = 1$

$A = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $B = \begin{pmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\dot{\delta x} = A\delta x + B\delta u$

Let $\delta U = -K\delta x$ (State Feedback controller)

$\dot{\delta x} = A\delta x - BK\delta x = (A - BK)\delta x$

$K = [K_1 \; K_2]$

$$A - BK = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ K_1 & K_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1-K_1 & -K_2 \end{pmatrix}$$

$\lambda_1 + \lambda_2 = \text{trace}(A-BK) = -5 = -K_2$

$K_2 = 5$

$(\lambda_1)(\lambda_2) = 6 = \det(A-BK) = -1 + K_1 \quad \therefore \quad K_1 = 7$

$K = [7 \; 5]$

$\therefore \; \delta u = -[7 \; 5]\, \delta x$

$$\delta x = x - x_e = x - \begin{pmatrix} -2 \\ 0 \end{pmatrix}$$

$$\delta u = -[7 \; 5]\, x + [7 \; 5] \begin{pmatrix} -2 \\ 0 \end{pmatrix}$$

$\delta u = -[7 \; 5]\, x - 14$

$u = u_e + \delta u = 2 - [7 \; 5]\, x - 14$

$$\boxed{u = -[7 \; 5]\, x - 12}$$

Feedback control law to stabalize $\dot{x} = Ax + bu$ about (x_e, u_e)

#4) $A = \begin{pmatrix} 6 & 2 \\ -3 & 1 \end{pmatrix}$ $b = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$n = 2$

$CO = [B \;\; AB] = \begin{pmatrix} 1 & 4 \\ -1 & -4 \end{pmatrix}$

$CO \Rightarrow \begin{pmatrix} \boxed{1} & 4 \\ 0 & 0 \end{pmatrix}$ CO has 1 pivot

$\text{rank}(CO) = 1 \neq n \therefore$

$\boxed{\text{System is not reachable or Controllable}}$

#5) $A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ $B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

$CO = \begin{bmatrix} b_1 & \lambda_1 b_1 \\ b_2 & \lambda_2 b_2 \end{bmatrix}$

$\det(CO) = \lambda_2 b_1 b_2 - \lambda_1 b_1 b_2 = b_1 b_2 (\lambda_2 - \lambda_1)$

CO has full rank n if $\det(CO) \neq 0$.

$\det(CO) \neq 0$ if b_1 & $b_2 \neq 0$ & $\lambda_1 \neq \lambda_2$

$\therefore$ System is reachable & controllable if b_1 & $b_2 \neq 0$ & $\lambda_1 \neq \lambda_2$.

#6) $\begin{pmatrix} -3 & -8 \\ -1 & 4 \end{pmatrix} = A$ $\quad b = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\quad C = [1 \ -8]$, $n = 2$

$CO = [B \ AB] = \begin{pmatrix} 1 & 5 \\ -1 & -5 \end{pmatrix} \Rightarrow \begin{pmatrix} \boxed{1} & 5 \\ 0 & 0 \end{pmatrix}$

$\text{rank}(CO) = 1 \neq n$

$\boxed{\text{System is not Controllable}}$

QR decomposition of CO: (Gram-Schmidt to get Q)

$q_1 = \dfrac{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}{\left\| \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\|} = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$

$q_2' = \begin{pmatrix} 5 \\ -5 \end{pmatrix} - \left(q_1^T \begin{pmatrix} 5 \\ -5 \end{pmatrix}\right) q_1 = \begin{pmatrix} 5 \\ -5 \end{pmatrix} - \left([1/\sqrt{2} \ -1/\sqrt{2}] \begin{pmatrix} 5 \\ -5 \end{pmatrix}\right) \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$

$q_2' = \begin{pmatrix} 5 \\ -5 \end{pmatrix} - \dfrac{10}{\sqrt{2}} \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ (Not valid)

q_2 must be orthonormal to q_1 ($q_2^T q_1 = 0$)

If $q_2 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$ $\quad q_2^T q_1 = 0$ ✓

$Q = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$ $\quad A = QR$ $\quad R = Q^{-1}A = Q^T A$

$A = CO$

$Q^{-1} = Q^T$ (orthonormal matrix)

$\tilde{A} = Q^{-1} A Q$, $\tilde{B} = Q^{-1} B$

$$Q^{-1} = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$\tilde{A} = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} -3 & -8 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 5 & -7 \\ 0 & -4 \end{pmatrix}$$

$$\tilde{A} = \begin{pmatrix} A_1 & A_2 \\ 0 & A_4 \end{pmatrix}$$

A_4 is unreachable part of $\tilde{A}$. The eigenvalues of $A_4 = -4$. Eigenvalue of uncontrollable part of A is in open left hand plane. Therefore system is $\boxed{\text{stabalizable.}}$

$$\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{pmatrix} 1 & -8 \\ 5 & -40 \end{pmatrix} \Rightarrow \begin{pmatrix} \boxed{1} & -8 \\ 0 & 0 \end{pmatrix}$$

$\boxed{\text{rank}(\mathcal{O}) = 1 \neq n \;\therefore\; \text{system isn't observable}}$

$$A = \begin{pmatrix} -3 & -8 \\ -1 & 4 \end{pmatrix} \quad \chi_A = \lambda^2 - \lambda - 20 \Rightarrow (\lambda - 5)(\lambda + 4)$$

$\underbrace{\lambda_1 = 5}_{\text{unstable } \lambda}$, $\lambda_2 = -4$

If detectable $\text{rank}\begin{pmatrix} \lambda I - A \\ C \end{pmatrix} = n$

$$\begin{pmatrix} \lambda I - A \\ C \end{pmatrix} = \begin{pmatrix} 8 & 8 \\ 1 & 1 \\ 1 & -8 \end{pmatrix} = \begin{pmatrix} \boxed{1} & 0 \\ 0 & \boxed{1} \\ 0 & 0 \end{pmatrix}$$

2 pivots $\therefore$ rank $= 2 = n$

$$\boxed{\text{System is detectable as } \operatorname{rank}\begin{pmatrix} 5I - A \\ C \end{pmatrix} = n}$$

For unstable eigenvalue, 5

#7) $\dot{x} = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix}$ $b > 0,\ a > 0$

$A^T P + PA = -Q$

Let $Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_{2\times 2}$

$$\begin{pmatrix} 0 & -b \\ 1 & -a \end{pmatrix}\begin{pmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{pmatrix} + \begin{pmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} -bP_{12} & -bP_{22} \\ P_{11} - aP_{12} & P_{12} - aP_{22} \end{pmatrix} + \begin{pmatrix} -bP_{12} & P_{11} - P_{12}a \\ -bP_{22} & P_{12} - aP_{22} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} -2bP_{12} & -bP_{22} + P_{11} - P_{12}a \\ -bP_{22} + P_{11} - aP_{12} & 2P_{12} - 2aP_{22} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$-2bP_{12} = -1$

$-bP_{22} + P_{11} - P_{12}a = 0$

$2P_{12} - 2aP_{22} = -1$

$P_{12} = \dfrac{1}{2b}$

$P_{22} = \dfrac{P_{11}}{b} - \dfrac{P_{12}a}{b} = \dfrac{1}{2a} + \dfrac{P_{12}}{a}$

$P_{22} = \dfrac{P_{11}}{b} - \dfrac{a}{2b^2} = \dfrac{1}{2a} + \dfrac{1}{2ba}$

$$P_{11} = \frac{b}{2a} + \frac{1}{2a} + \frac{a}{2b} = \frac{(b+1)}{2a} + \frac{a}{2b}$$

$$P_{11} = \frac{b^2+b+a^2}{2ab}$$

$$P_{22} = \frac{P_{11}}{b} - \frac{a}{2b^2} = \frac{b^2+b+a^2}{2ab^2} - \frac{a^2}{2ab^2}$$

$$P_{22} = \frac{b+1}{2ab}$$

$$\therefore \quad P = \begin{pmatrix} \frac{b^2+b+a^2}{2ab} & \frac{1}{2b} \\ \frac{1}{2b} & \frac{b+1}{2ab} \end{pmatrix}$$

P is positive definite if all leading principal minors are positive

$$\left| \frac{b^2+b+a^2}{2ab} \right| > 0 \quad \checkmark$$

$$\begin{vmatrix} \frac{b^2+b+a^2}{2ab} & \frac{1}{2b} \\ \frac{1}{2b} & \frac{b+1}{2ab} \end{vmatrix} = \frac{(b^2+b+a^2)(b+1)}{(2ab)^2} - \frac{1}{4b^2}$$

$$= \frac{b^3+b^2+a^2b+b^2+b+a^2}{4a^2b^2} - \frac{a^2}{4a^2b^2} = \frac{b^3+2b^2+b(a^2+1)}{4a^2b^2} > 0$$

All leading principal minors are positive, $\therefore$

$\boxed{\text{P is positive definite \& the system is asymptotically stable}}$

#8) $A = \begin{pmatrix} c & 1 \\ 0 & d \end{pmatrix}$

$A^T P A - P = -Q$

Let $Q = I_{2\times 2}$

$$\begin{pmatrix} c & 0 \\ 1 & d \end{pmatrix}\begin{pmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{pmatrix}\begin{pmatrix} c & 1 \\ 0 & d \end{pmatrix} - \begin{pmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} P_{11}c & P_{12}c \\ P_{11}+P_{12}d & P_{12}+P_{22}d \end{pmatrix}\begin{pmatrix} c & 1 \\ 0 & d \end{pmatrix} = \begin{pmatrix} P_{11}c^2 & P_{11}c + P_{12}cd \\ P_{11}c+P_{12}cd & P_{11}+P_{12}d + P_{12}d + P_{22}d^2 \end{pmatrix}$$

$P_{11}c^2 - P_{11} = -1$

$P_{11}c + P_{12}cd - P_{12} = 0$

$P_{11} + 2P_{12}d + P_{22}d^2 - P_{22} = -1$

$$P_{11} = \frac{-1}{c^2-1} = \frac{1}{1-c^2}$$

$$P_{12} = \frac{-P_{11}c}{cd-1} = \left(\frac{c}{1-c^2}\right)\left(\frac{1}{cd-1}\right)$$

$$P_{22} = \frac{-1-2P_{12}d - P_{11}}{(d^2-1)} = \frac{-1}{d^2-1} - \frac{2dc}{(1-c^2)(cd-1)(d^2-1)} - \frac{1}{(1-c^2)(d^2-1)}$$

$$P_{22} = \frac{(-1)(1-c^2)(cd-1) - 2dc - cd+1}{(d^2-1)(1-c^2)(cd-1)}$$

$$P_{22} = \frac{-(cd-1-c^3d+c^2) - 2dc - cd + 1}{(d^2-1)(1-c^2)(cd-1)}$$

$$P_{22} = \frac{2+c^3d-c^2}{(d^2-1)(1-c^2)(cd-1)}$$

$$P = \begin{pmatrix} \frac{1}{1-c^2} & \frac{c}{(1-c^2)(cd-1)} \\ \frac{c}{(1-c^2)(cd-1)} & \frac{2+c^3d-c^2}{(d^2-1)(1-c^2)(cd-1)} \end{pmatrix} \qquad \begin{matrix} |c|<1 \\ |d|<1 \end{matrix}$$

$$\frac{1}{1-c^2} > 0 \quad \checkmark$$

$$|P| = \frac{2+c^3d-c^2}{(d^2-1)(1-c^2)^2(cd-1)} - \frac{c^2}{(1-c^2)(cd-1)}$$

$$= \frac{2+c^3d-c^2-c^2(1-c^2)}{\underbrace{(d^2-1)}_{\text{negatie}}\underbrace{(1-c^2)^2}_{\text{positive}}\underbrace{(cd-1)}_{\text{negative}}} \Rightarrow \frac{c^2d^2-2cd+2}{\underbrace{(c^2-1)}_{\text{negative}}\underbrace{(d^2-1)}_{\text{negative}}\underbrace{(cd-1)^2}_{\text{positive}}}$$

$$= \frac{1+(1-cd)^2}{(c^2-1)(d^2-1)(cd-1)^2} > 0$$ (Numerator & denominator always positive)

P is positive definite by Sylvester criteria, ∴ the system is asymptotically stable.

#9) $g[k] = 10k(.7)^k$

Ratio test: $\lim_{k\to\infty} \left| \frac{g[k+1]}{g[k]} \right| = \frac{10(k+1)(.7)^{k+1}}{10k(.7)^k} = r$

$= \lim_{k\to\infty} = \frac{(k+1)(.7)}{k} = .7 \lim_{k\to\infty} \left(1 + \frac{1}{k}\right) = (.7)(1)$

$r = .7 < 1$, series is convergent

The series is convergent, therefore

$\boxed{\text{g[k] is absolutely summable \& the system is BIBO stable}}$

The series converges to $\frac{700}{9}$, i.e. $M < \infty$

$\sum_0^\infty g[k] = 10 \sum_0^\infty k(.7)^k$

$\sum_0^\infty k r^k = \frac{r}{(1-r)^2}$ $\therefore$ $r = .7$ & $\sum_0^\infty k(.7)^k = \frac{.7}{.09} = \frac{70}{9}$

$\therefore \sum_{k=0}^\infty g[k]$ converges to $(10)\left(\frac{70}{9}\right) = \frac{700}{9}$

Contents


```
clear
close all
clc
```

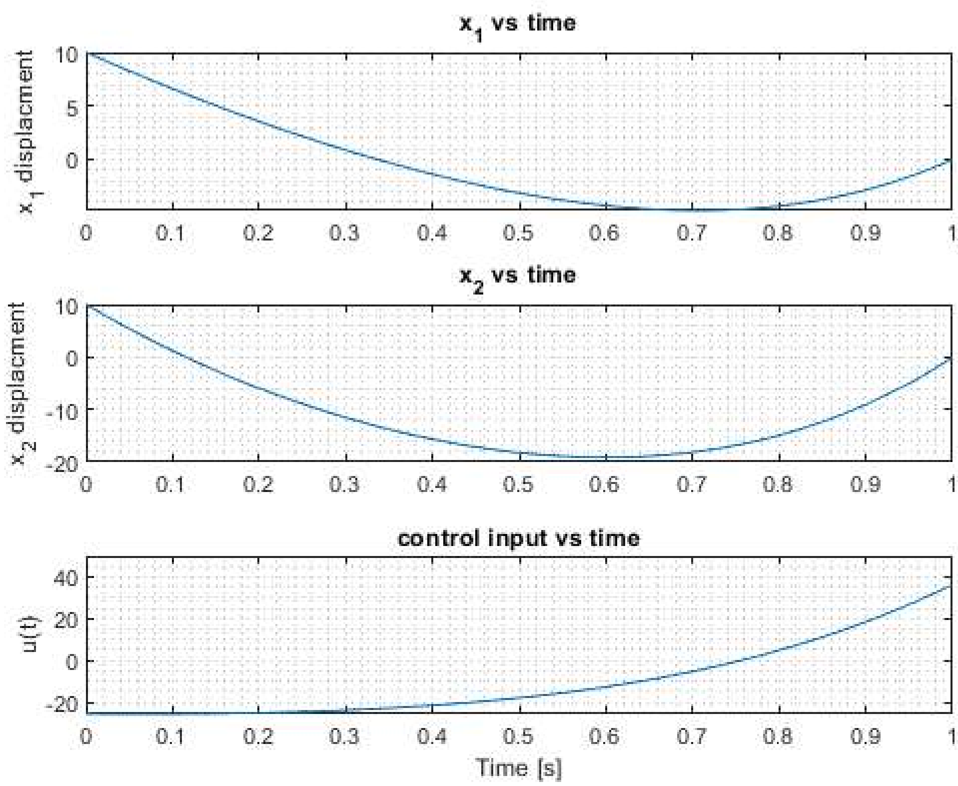
Run and Plot Simulation

```
time        = (0:.005:1)';                         % Time [s]
IC          = [10; 10];                            % 10 unit initial displacment at each end

% ODE45 solver options
options     = odeset('AbsTol',1e-8,'RelTol',1e-8);

% ODE45 Function call
[T, X]      = ode45(@(t,x) SuspensionSystem(t,x),time,IC,options);

% Get control input at each time
U           = zeros(size(T));
for i = 1:length(T)
    [~,U(i)]    = SuspensionSystem(T(i),X);
end

figure
subplot(3,1,1)
plot(T,X(:,1))
title('x_1 vs time')
grid minor
ylabel('x_1 displacment')
subplot(3,1,2)
plot(T,X(:,2))
title('x_2 vs time')
grid minor
ylabel('x_2 displacment')
subplot(3,1,3)
plot(T,U)
title('control input vs time')
grid minor
ylabel('u(t)')
xlabel('Time [s]')
```

State Space Function for Simple Suspension System

```
function [xdot, u]  = SuspensionSystem(t,x)

% State Space Matrices
A               = [-1 0; 0 -2];
B               = [1; 3];

% State Vector
x               = [x(1,1);x(2,1)];

% Control Law
```

```
if t <= 1
    u               = 167.83*exp(2*t - 2) - 131.47*exp(t - 1);
else
    u = 0;
end

% State Space Equation
xdot            = A*x + B*u;
end
```



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