ECE 68000: MODERN AUTOMATIC CONTROL

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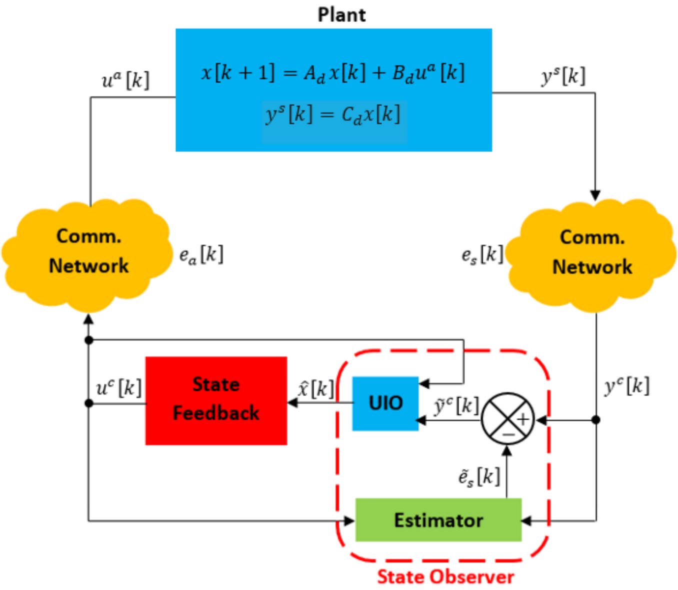
State Estimation of Networked Control Systems Corrupted by Unknown Input and Output Sparse Errors

UIO in State Estimation of NCS



State observer construction

- First, construct an estimator of network communication errors, $\boldsymbol{e}_s[k]$, in the signal flow from the sensor to the controller
- Use the estimation $\tilde{\boldsymbol{e}}_s[k]$ of $\boldsymbol{e}_s[k]$ to cancel out its effects
- Then, build **unknown input observer (UIO)** to estimate the plant state

Recovering error vector $\boldsymbol{e}_s[k]$

- Substitute $\boldsymbol{u}^a[k] = \boldsymbol{\Lambda}(k)\boldsymbol{u}^c[k]$ and $\boldsymbol{y}^c[k] = \boldsymbol{\Gamma}(k)\boldsymbol{y}^s[k]$ into

$$\left.\begin{array}{rcl} \boldsymbol{x}[k+1] & = & \boldsymbol{A}\boldsymbol{x}[k] + \boldsymbol{B}\boldsymbol{u}^a[k] \\ \boldsymbol{y}^s[k] & = & \boldsymbol{C}\boldsymbol{x}[k] \end{array}\right\}$$

  to obtain

$$\left.\begin{array}{rcl} \boldsymbol{x}[k+1] & = & \boldsymbol{A}\boldsymbol{x}[k] + \boldsymbol{B}\boldsymbol{\Lambda}(k)\boldsymbol{u}^c[k] \\ \boldsymbol{y}^c[k] & = & \boldsymbol{\Gamma}(k)\boldsymbol{C}\boldsymbol{x}[k] \end{array}\right\}$$

- The input to the controller, $\boldsymbol{y}^c[k]$, and the signal generated by the controller, $\boldsymbol{u}^c[k]$, are available to us
- Collect τ observations for the system

Notation

- For a given a vector $\boldsymbol{y}[k] \in \mathbb{R}^p$, $\tau \in \mathbb{N}$, the vector

$$\boldsymbol{y}^c|_{[k-\tau+1,k]}$$

  denotes the collection of τ samples of $\boldsymbol{y}[k]$
- That is,

$$\boldsymbol{y}^c|_{[k-\tau+1,k]} = \begin{bmatrix} \boldsymbol{y}^c[k-\tau+1] \\ \boldsymbol{y}^c[k-\tau+2] \\ \vdots \\ \boldsymbol{y}^c[k-1] \\ \boldsymbol{y}^c[k] \end{bmatrix}$$

Collecting observations

- Recall that $\overline{\boldsymbol{\Gamma}}(k) = \boldsymbol{\Gamma}(k) - \boldsymbol{I}_p \in \mathbb{R}^{p\times p}$
- One observation

$$\boldsymbol{y}^c|_{[k,k]} = \boldsymbol{\Gamma}(k)\boldsymbol{C}\boldsymbol{x}[k] = \boldsymbol{C}\boldsymbol{x}[k] + \overline{\boldsymbol{\Gamma}}(k)\boldsymbol{C}\boldsymbol{x}[k]$$

- Two observations starting from $\boldsymbol{x}[k-1]$
  - First observation at time $k-1$

$$\begin{aligned} \boldsymbol{y}^c[k-1] &= \boldsymbol{\Gamma}(k-1)\boldsymbol{C}\boldsymbol{x}[k-1] \\ &= \boldsymbol{C}\boldsymbol{x}[k-1] + \overline{\boldsymbol{\Gamma}}(k-1)\boldsymbol{C}\boldsymbol{x}[k-1] \end{aligned}$$

  - Second observation at time k

$$\begin{aligned} \boldsymbol{y}^c[k] &= \boldsymbol{\Gamma}(k)\boldsymbol{C}\boldsymbol{x}[k] = \boldsymbol{C}\boldsymbol{x}[k] + \overline{\boldsymbol{\Gamma}}(k)\boldsymbol{C}\boldsymbol{x}[k] \\ &= \boldsymbol{C}(\boldsymbol{A}\boldsymbol{x}[k-1] + \boldsymbol{B}\boldsymbol{\Lambda}(k-1)\boldsymbol{u}^c[k-1]) \\ &\quad + \overline{\boldsymbol{\Gamma}}(k)\boldsymbol{C}\boldsymbol{x}[k] \\ &= \boldsymbol{C}\boldsymbol{A}\boldsymbol{x}[k-1] + \overline{\boldsymbol{\Gamma}}(k)\boldsymbol{C}\boldsymbol{x}[k] \\ &\quad + \boldsymbol{C}\boldsymbol{B}\boldsymbol{\Lambda}(k-1)\boldsymbol{u}^c[k-1] \end{aligned}$$

Collecting two observations together

$$
\begin{aligned}
\boldsymbol{y}^c|_{[k-1,k]} &= \begin{bmatrix} \boldsymbol{y}^c[k-1] \\ \boldsymbol{y}^c[k] \end{bmatrix} \\
&= \begin{bmatrix} \boldsymbol{C} \\ \boldsymbol{CA} \end{bmatrix} \boldsymbol{x}[k-1] + \begin{bmatrix} \overline{\boldsymbol{\Gamma}}(k-1)\boldsymbol{C}\boldsymbol{x}[k-1] \\ \overline{\boldsymbol{\Gamma}}(k)\boldsymbol{C}\boldsymbol{x}[k] \end{bmatrix} \\
&\quad + \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{CB\Lambda}(k-1)\boldsymbol{u}^c[k-1] \end{bmatrix}
\end{aligned}
$$

Collecting three observations together

$$
\begin{aligned}
\boldsymbol{y}^c|_{[k-2,k]} &= \begin{bmatrix} \boldsymbol{y}^c[k-2] \\ \boldsymbol{y}^c[k-1] \\ \boldsymbol{y}^c[k] \end{bmatrix} \\
&= \begin{bmatrix} \boldsymbol{C} \\ \boldsymbol{C}\boldsymbol{A} \\ \boldsymbol{C}\boldsymbol{A}^2 \end{bmatrix} \boldsymbol{x}[k-2] + \begin{bmatrix} \overline{\boldsymbol{\Gamma}}(k-2)\boldsymbol{C}\boldsymbol{x}[k-2] \\ \overline{\boldsymbol{\Gamma}}(k-1)\boldsymbol{C}\boldsymbol{x}[k-1] \\ \overline{\boldsymbol{\Gamma}}(k)\boldsymbol{C}\boldsymbol{x}[k] \end{bmatrix} \\
&\quad + \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{C}\boldsymbol{B}\boldsymbol{\Lambda}(k-2)\boldsymbol{u}^c[k-2] \\ \boldsymbol{C}\boldsymbol{B}\boldsymbol{\Lambda}(k-1)\boldsymbol{u}^c[k-1] + \boldsymbol{C}\boldsymbol{A}\boldsymbol{B}\boldsymbol{\Lambda}(k-2)\boldsymbol{u}^c[k-2] \end{bmatrix}
\end{aligned}
$$

Collecting τ observations

$$
\begin{aligned}
\boldsymbol{y}^c|_{[k-\tau+1,k]} &= \begin{bmatrix} \boldsymbol{C} \\ \boldsymbol{CA} \\ \vdots \\ \boldsymbol{CA}^{\tau-1} \end{bmatrix} \boldsymbol{x}[k-\tau+1] \\
&+ \begin{bmatrix} \overline{\boldsymbol{\Gamma}}(k-\tau+1)\boldsymbol{C}\boldsymbol{x}[k-\tau+1] \\ \overline{\boldsymbol{\Gamma}}(k-\tau+2)\boldsymbol{C}\boldsymbol{x}[k-\tau+2] \\ \vdots \\ \overline{\boldsymbol{\Gamma}}(k)\boldsymbol{C}\boldsymbol{x}[k] \end{bmatrix} \\
&+ \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{CB\Lambda}(k-\tau+1)\boldsymbol{u}^c[k-\tau+1] \\ \vdots \\ \Sigma_{i=1}^{\tau-1}\boldsymbol{CA}^{\tau-1-i}\boldsymbol{B\Lambda}(k-\tau+i)\boldsymbol{u}^c[k-\tau+i] \end{bmatrix}
\end{aligned}
$$

How many observations do we need?

- Consider a system model

$$\left.\begin{array}{rcl} \boldsymbol{x}[k+1] & = & \boldsymbol{A}\boldsymbol{x}[k] \\ \boldsymbol{y}^c[k] & = & \boldsymbol{\Gamma}(k)\boldsymbol{C}\boldsymbol{x}[k] \end{array}\right\}$$

- Collect τ observations:

$$\begin{aligned} \boldsymbol{y}^c|_{[0,\tau-1]} &= \begin{bmatrix} \boldsymbol{\Gamma}(0)\boldsymbol{C} \\ \boldsymbol{\Gamma}(1)\boldsymbol{C}\boldsymbol{A} \\ \vdots \\ \boldsymbol{\Gamma}(\tau-1)\boldsymbol{C}\boldsymbol{A}^{\tau-1} \end{bmatrix} \boldsymbol{x}[0] \\ &= \mathrm{diag}\{\ \boldsymbol{\Gamma}(0) \quad \boldsymbol{\Gamma}(1) \quad \cdots \quad \boldsymbol{\Gamma}(\tau-1)\ \}\mathcal{O}^{\tau-1}\boldsymbol{x}[0] \end{aligned}$$

where $\mathcal{O}^{\tau-1}$ is the τ-step observability matrix

More Notation

- For a given matrix $\boldsymbol{M} \in \mathbb{R}^{n \times m}$ and a set $\Xi \subseteq \{1, \ldots, n\}$, denote
$$\boldsymbol{M}_{\bar{\Xi}} \in \mathbb{R}^{(n-|\Xi|) \times m}$$
the matrix obtained from $\boldsymbol{M}$ by removing the rows whose indices are contained in Ξ
- For a given matrix $\boldsymbol{M} \in \mathbb{R}^{n \times m}$ and a set $\Xi \subseteq \{1, \ldots, n\}$, denote
$$\boldsymbol{M}_{\Xi} \in \mathbb{R}^{|\Xi| \times m}$$
the matrix obtained from $\boldsymbol{M}$ by removing the rows whose indices are **not** contained in Ξ

$\mathcal{O}^{\tau-1}$—τ-step observability matrix

$$\mathcal{O}^{\tau-1} = \begin{bmatrix} \boldsymbol{C}\boldsymbol{A}^{\tau-1} \\ \vdots \\ \boldsymbol{C}\boldsymbol{A} \\ \boldsymbol{C} \end{bmatrix}$$

Resilient System Against Packet Drops

Definition

The linear system

$$\left.\begin{array}{rcl} \boldsymbol{x}[k+1] & = & \boldsymbol{A}\boldsymbol{x}[k] \\ \boldsymbol{y}^c[k] & = & \boldsymbol{\Gamma}(k)\boldsymbol{C}\boldsymbol{x}[k] \end{array}\right\}$$

is said to be resilient against d_s packet drops if there exists $\tau \in \mathbb{N}$ such that for any set $\Xi \subseteq \{1, \ldots, \xi\}$ with $|\Xi| \leq d_s$ the matrix $\mathcal{O}_{\bar{\Xi}}^{\tau-1}$ has full column rank

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G. Fiore, Y. H. Chang, Q. Hu, M. D. Di Benedetto, and C. J. Tomlin, *Secure state estimation for Cyber Physical Systems with sparse malicious packet drops*, 2017 ACC, Sheraton Seattle Hotel, Seattle, May 24–26, pp. 1898–1903

Some Manipulations

- Let $\boldsymbol{U}^c[k] \in \mathbb{R}^{m\times m}$ be a diagonal matrix whose components consist of $\boldsymbol{u}^c[k]$
- Let vec $(\boldsymbol{\Lambda}(k)) \in \mathbb{R}^m$ represents vectorization of diagonal components of $\boldsymbol{\Lambda}(k)$
- Then,

$$\boldsymbol{\Lambda}(k)\boldsymbol{u}^c[k] = \boldsymbol{U}^c[k]\text{vec}\ (\boldsymbol{\Lambda}(k))$$

- Let $\boldsymbol{v}[k] = [\boldsymbol{0}^\top \ \cdots \ \Sigma_{i=1}^{\tau-1}(\boldsymbol{C}\boldsymbol{A}^{\tau-1-i}\boldsymbol{B}\boldsymbol{u}^c(k-\tau+i))^\top]^\top$
- Note that $\boldsymbol{v}[k]$ is known for all k and τ

M. Zhang, S. Hui, M. R. Bell, and S. H. Żak, Vector Recovery for a Linear System Corrupted by Unknown Sparse Error Vectors With Applications to Secure State Estimation, *IEEE Control Systems Letters*, Vol. 3, No. 4, pp. 895–900, October 2019

More Manipulations

- Let $\hat{\boldsymbol{y}}^c|_{[k-\tau+1,k]} = \boldsymbol{y}^c|_{[k-\tau+1,k]} - \boldsymbol{v}[k]$
- Then,

$$
\begin{aligned}
\boldsymbol{Y}[k] &\triangleq \begin{bmatrix} \hat{\boldsymbol{y}}^c[k] \\ \hat{\boldsymbol{y}}^c[k-1] \\ \vdots \\ \hat{\boldsymbol{y}}^c[k-\tau+1] \end{bmatrix} = \begin{bmatrix} \boldsymbol{C}\boldsymbol{A}^{\tau-1} \\ \boldsymbol{C}\boldsymbol{A}^{\tau-2} \\ \vdots \\ \boldsymbol{C} \end{bmatrix} \boldsymbol{x}[k-\tau+1] \\
&+ \boldsymbol{I}_{\tau p} \begin{bmatrix} \overline{\boldsymbol{\Gamma}}(k)\boldsymbol{C}\boldsymbol{x}[k] \\ \overline{\boldsymbol{\Gamma}}(k-1)\boldsymbol{C}\boldsymbol{x}[k-1] \\ \vdots \\ \overline{\boldsymbol{\Gamma}}(k-\tau+1)\boldsymbol{C}\boldsymbol{x}[k-\tau+1] \end{bmatrix} + \boldsymbol{F}[k] \begin{bmatrix} \mathrm{vec}(\overline{\boldsymbol{\Lambda}}(k-1)) \\ \mathrm{vec}(\overline{\boldsymbol{\Lambda}}(k-2)) \\ \vdots \\ \mathrm{vec}(\overline{\boldsymbol{\Lambda}}(k-\tau+1)) \end{bmatrix} \\
&\triangleq \mathcal{O}^{\tau-1}\boldsymbol{x}[k-\tau+1] + \boldsymbol{I}_{\tau p}\boldsymbol{E}_s[k] + \boldsymbol{F}[k]\mathcal{V}[k]
\end{aligned}
$$

Organizing Output Observations for Further Processing

- We have

$$\boldsymbol{Y}[k] = \mathcal{O}^{\tau-1}\boldsymbol{x}[k-\tau+1] + \boldsymbol{I}_{\tau p}\boldsymbol{E}_s[k] + \boldsymbol{F}[k]\mathcal{V}[k]$$

  where
  - $\mathcal{O}^{\tau-1} \in \mathbb{R}^{\tau p \times n}$
  - $\boldsymbol{Y}[k] \in \mathbb{R}^{\tau p}$
  - $\boldsymbol{F}[k] \in \mathbb{R}^{\tau p \times (\tau-1)m}$
  - $\boldsymbol{F}[k] = \begin{bmatrix} \boldsymbol{CBU}^c[k-1] & \cdots & \boldsymbol{CA}^{\tau-2}\boldsymbol{BU}^c[k-\tau+1] \\ \vdots & \ddots & \vdots \\ \boldsymbol{O}_{p\times m} & \cdots & \boldsymbol{CBU}^c[k-\tau+1] \\ \boldsymbol{O}_{p\times m} & \cdots & \boldsymbol{O}_{p\times m} \end{bmatrix}$

Organizing the Observations for Further Processing—Final Form

- Let $\boldsymbol{\Omega}[k] = [\boldsymbol{I}_{\tau p} \quad \boldsymbol{F}[k]]$
- Let $\boldsymbol{E}[k] = [\boldsymbol{E}_s^\top[k] \quad \mathcal{V}^\top[k]]^\top$
- Then
$$\boldsymbol{Y}[k] = \mathcal{O}^{\tau-1}\boldsymbol{x}[k-\tau+1] + \boldsymbol{\Omega}[k]\boldsymbol{E}[k]$$
  where $\boldsymbol{\Omega} \in \mathbb{R}^{\tau p \times [\tau p + (\tau-1)m]}$ and $\boldsymbol{E} \in \mathbb{R}^{\tau p + (\tau-1)m}$
- Objective: Recover $\boldsymbol{E}[k]$, a sparse vector