(Ex) $L^{-1}\left(\frac{2s}{(s-5)^2+4}\right) = 2L^{-1}\left(\frac{s-5+5}{(s-5)^2+2^2}\right)$

$= 2\left(L^{-1}\left(\frac{s-5}{(s-5)^2+2^2}\right) + \frac{5}{2}L^{-1}\left(\frac{1\cdot 2}{(s-5)^2+2^2}\right)\right)$

$L^{-1}(F(s-a)) = e^{at}L^{-1}(F(s)) = e^{at}f(t)$

$= 2\left(e^{5t}\cos(2t) + \frac{5}{2}e^{5t}\sin(2t)\right)$

6.2. Derivatives.

Q $y''+4y=0$: $L(y''+4y) = L(0)$

$L(y'')+4L(y) = 0$: $L(y'') = ?$

$$\underline{L(f'(t)) = sL(f(t)) - f(0)}$$

(Proof) $L(f'(t)) = \int_0^\infty \underbrace{e^{-st}}_{u} \underbrace{f'(t)}_{v'} dt$

$\int_a^b u(t)v'(t)\,dt = [u(t)v(t)]_a^b - \int_a^b u'(t)v(t)\,dt$

$= [e^{-st}f(t)]_0^\infty + \int_0^\infty (+s)e^{-st}f(t)\,dt$

$f(t)$ is of exponential order:
$|f(t)| \leq Me^{\beta t}$ $(t \geq t_0)$

$|e^{-st}f(t)| = e^{-st}|f(t)| \leq Me^{-st}e^{\beta t}$

$= \underline{Me^{(\beta-s)t}} \to 0$ $\quad (s > \beta)$ $\quad (\beta - s < 0)$

$\lim_{t\to\infty} e^{(\beta-s)t} = 0$; $\lim_{t\to\infty} e^{-st}f(t) = 0$

$L(f'(t)) = \underbrace{\lim_{t\to\infty} e^{-st}f(t)}_{=0} - \underline{f(0) + sL(f)}.$

$\therefore$ $\underline{L(f') = sL(f) - f(0).}$

$L(f'') = L((f')') = sL(f') - f'(0)$

$= s(sL(f) - f(0)) - f'(0)$

$$L(f'') = s^2 L(f) - s f(0) - f'(0)$$

$$L(f^{(n)}) = s^n L(f) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$$

(Ex) $y'' + 4y = 0, \quad y(0) = 0, \quad y'(0) = 1$

$L:$ $L(y'') + 4L(y) = 0$

$$s^2 L(y) - s\underbrace{y(0)}_{=0} - \underbrace{y'(0)}_{=1} + 4L(y) = 0$$

$(s^2+4)L(y) = 1 \; : \; L(y) = \dfrac{1}{s^2+4}.$

$$\begin{cases} L(\cos(kt)) = \dfrac{s}{s^2+k^2}, & s>0 \\ L(\sin(kt)) = \dfrac{k}{s^2+k^2}, & s>0 \end{cases}$$

$$y(t) = L^{-1}\left(\frac{1\cdot 2}{s^2+2^2}\right)\frac{1}{2} = \frac{1}{2}\cdot \sin(2t).$$

Q $L\left(\int_0^t f(\tau)\,d\tau\right) = ?$

(formula) $\underline{L\left(\int_0^t f(\tau)\,d\tau\right) = \dfrac{1}{s} L(f(t))}$

(Ex) $L^{-1}\left(\dfrac{5}{s^3+9s}\right) = L^{-1}\left(\dfrac{5}{s(s^2+9)}\right)$

① Partial fraction :

$$\frac{5}{s(s^2+9)} = \frac{A}{s} + \frac{Bs+C}{s^2+9}$$

② $L^{-1}\left(\dfrac{1}{s}F(s)\right) = \int_0^t L^{-1}(F(s))\,d\tau.$

$$L^{-1}\left(\frac{5}{s(s^2+9)}\right) = 5L^{-1}\left(\frac{1}{s}\frac{1}{s^2+9}\right)$$

$$= \frac{5}{3}\int_0^t L^{-1}\left(\frac{1\cdot 3}{s^2+3^2}\right)d\tau = \frac{5}{3}\int_0^t \sin(3\tau)\,d\tau$$

$\begin{pmatrix} z = 3\tau \\ dz = 3d\tau \end{pmatrix}$

$$= \frac{5}{3}\int_0^{3t} \sin(z)\,\frac{1}{3}\,dz$$

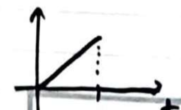
$$= \frac{5}{9}\int_0^{3t} \sin(z)\,dz = \frac{5}{9}\Big[-\cos(z)\Big]_0^{3t}$$

$= -\frac{5}{9}\left[\cos(3t) - 1\right]$

6.3 Unit step functions.

Topic : Discontinuous functions.

(Ex) $g(t) = \begin{cases} t, & 0 \le t < 5 \\ 0, & t \ge 5. \end{cases}$



$L(g) = ?$ Want a fast way.

What if $g(t) = \begin{cases} 0, & 0 \le t < 3 \\ -2, & 3 \le t < 6 \\ t-8, & 6 \le t < 12 \\ 2, & t \ge 12 \end{cases}$

Def $u(t-a) = u_a(t) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \ge a \end{cases}$

: the unit step function.

$a > 0$: $L(u(t-a)) = \frac{1}{s} e^{-as}$, $s > 0$

(pf) $L(u(t-a)) = \int_0^\infty e^{-st} u(t-a)\, dt$

$= \int_0^a 0\, dt + \int_a^\infty e^{-st} dt$ $\left(\begin{array}{l} z = -st \\ dz = -s\,dt \end{array}\right.$ $(s>0)$

$= \int_{-as}^{-\infty} e^{z} \frac{1}{-s} dz = -\frac{1}{s}\left[e^{z}\right]_{-as}^{-\infty}$

$= -\frac{1}{s}(0 - e^{-as}) = \frac{1}{s} e^{-as}$, $s > 0$.

Thm 1 (Shifting Theorem) #27.

Ⓗ $F(s) = L(f(t))$

Ⓒ $L(u(t-a) f(t-a)) = e^{-as} F(s)$.

(pf) $L(u(t-a) f(t-a)) = \int_0^\infty e^{-st} u(t-a) f(t-a)\, dt$

$= \underbrace{\int_0^a 0\, dt}_{=0} + \int_a^\infty e^{-st} f(t-a)\, dt$ $\quad r = t-a$, $dr = dt$, $t = r + a$

$= \int_0^\infty e^{-s(r+a)} f(r)\, dr = \int_0^\infty e^{-sr} e^{-as} f(r)\, dr$

$= e^{-as} L(f(t))$

$$L(u(t-a)) = \frac{1}{s}e^{-as}, \quad s>0$$

(Ex) $g(t) = \begin{cases} t, & 0 \le t < 5 \\ 0, & t \geq 5 \end{cases}$

(1) Introduce $u(t-a)$:

$$g(t) = \underline{t + (0-t)\,u(t-5)}$$

① $0 \le t < 5$: RHS $= t = g(t)$

② $t \geq 5$: RHS $= t - t = 0 = g(t)$

(2) $L(g) = L(t) - L(t\,u(t-5))$.

(Ex) 1. $L(u(t-a)(t-a)) = e^{-as}L(t)$

$$= e^{-as}\frac{1}{s^2}, \quad s>0$$

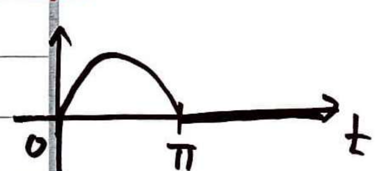
2. $L(t\,u(t-5)) = L((\underline{t-5}+\underline{5})\,u(t-5))$

$$= L((t-5)u(t-5)) + 5L(u(t-5))$$

$$= e^{-5s}\frac{1}{s^2} + 5\frac{1}{s}e^{-5s}, \quad s>0.$$

Q "$s>0$" is required? Yes.

(Ex) $g(t) = \begin{cases} \sin(t), & t < \pi \\ 0, & t \geq \pi \end{cases}$



1. $g(t) = \sin(t) + (0 - \sin(t))\,u(t-\pi)$

2. $L(g) = L(\sin(t)) - L(\sin(t)\,u(t-\pi))$

#27. $\sin(t) = \sin(t-\pi+\pi) = -\sin(t-\pi)$.

$\sin(\theta+\pi) = -\sin\theta$ ✓

$$L(g) = \frac{1}{s^2+1} + L\left((+1)\sin(t-\pi)\,u(t-\pi)\right)$$

$$= \frac{1}{s^2+1} + e^{-\pi s}\frac{1}{s^2+1}, \quad s>0$$

(Ex) $g(t) = \begin{cases} 0, & t<3 \\ -2, & 3 \le t < 6 \\ t-8, & 6 \le t < 12 \\ 2, & t \geq 12 \end{cases}$

$$g(t) = 0 + (-2-0)\,u(t-3) + \cancel{(t-8-(-2))}$$
$$+ (t-8-(-2))\,u(t-6)$$
$$+ (2-(t-8))\,u(t-12).$$