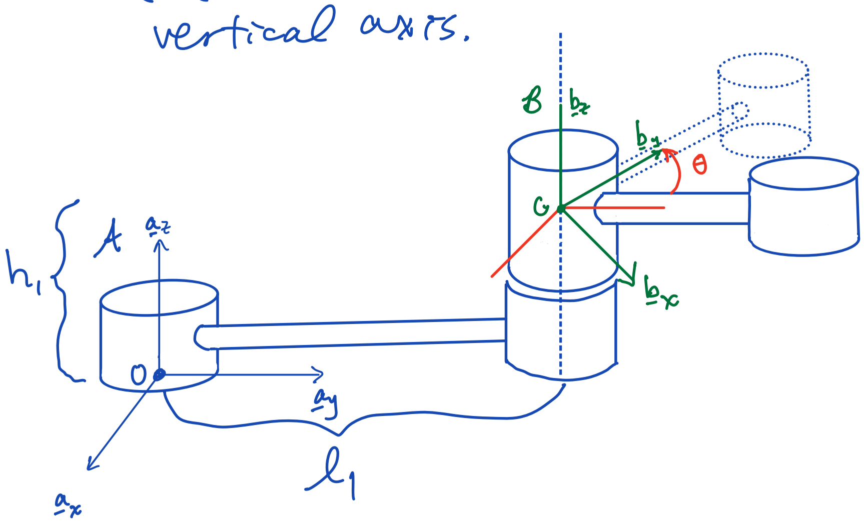
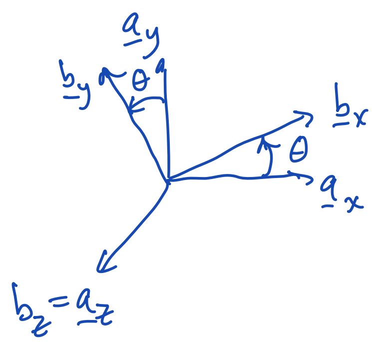
Ex. Find the configuration in SE(3) of a rigid body (the second link) that has been rotated about a vertical axis.





Position of the body frame:

$$\vec{r}_{G/O} = l_1 \underline{a}_y + h_1 \underline{a}_z \Rightarrow \left[\vec{r}_{G/O}\right]_A = \begin{bmatrix} 0 \\ l_1 \\ h_1 \end{bmatrix}_A$$

Orientation of the body frame after rotation through an angle of θ

$$\underbrace{\begin{array}{c|ccc} & \underline{b}_x & \underline{b}_y & \underline{b}_z \\ \hline \underline{a}_x & c\theta & -s\theta & 0 \\ \underline{a}_y & s\theta & c\theta & 0 \\ \underline{a}_z & 0 & 0 & 1 \end{array}}_{{}^A C^B}$$

→ orientation of B with respect to A

The homogeneous representation in SE(3) is:

$${}^A g^B = \begin{bmatrix} {}^A C^B & \vec{r}_{G/O} \\ 0_{1\times 3} & 1 \end{bmatrix} = \left[\begin{array}{ccc|c} c\theta & -s\theta & 0 & 0 \\ s\theta & c\theta & 0 & l_1 \\ 0 & 0 & 1 & h_1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

A set of <u>elementary homogeneous transformations</u> that can generate all of $SE(3)$ is given by:

$$\mathrm{Trans}_{x,a} = \left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \qquad \mathrm{Rot}_{x,\alpha} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c\alpha & -s\alpha & 0 \\ 0 & s\alpha & c\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{Trans}_{y,b} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathrm{Rot}_{y,\beta} = \begin{bmatrix} c\beta & 0 & s\beta & 0 \\ 0 & 1 & 0 & 0 \\ -s\beta & 0 & c\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{Trans}_{z,c} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathrm{Rot}_{z,\gamma} = \begin{bmatrix} c\gamma & -s\gamma & 0 & 0 \\ s\gamma & c\gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

All rotations about current axis

Ex. Find a homogeneous transformation that represents rotation by α about current x-axis, followed by translation of b units along the current x-axis, followed by translation of d units along the current z-axis, followed by rotation by θ about the current z-axis.

$$g = (Rot_{x,\alpha})(Trans_{x,b})(Trans_{z,d})(Rot_{z,\theta})$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c\alpha & -s\alpha & 0 \\ 0 & s\alpha & c\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \left[\begin{array}{ccc|c} 1 & 0 & 0 & b \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} c\theta & -s\theta & 0 & 0 \\ s\theta & c\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

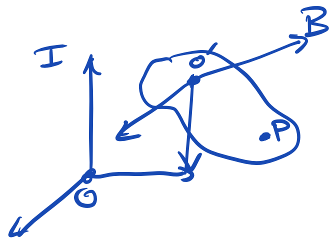
$$= \begin{bmatrix} 1 & 0 & 0 & b \\ 0 & c\alpha & -s\alpha & 0 \\ 0 & s\alpha & c\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\theta & -s\theta & 0 & 0 \\ s\theta & c\theta & 0 & 0 \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c\theta & -s\theta & 0 & b \\ c\alpha s\theta & c\alpha c\theta & -s\alpha & -ds\alpha \\ s\alpha s\theta & s\alpha c\theta & c\alpha & dc\alpha \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rigid body velocity

${}^Ig^B(t)$ is a curve in $SE(3)$
What is its velocity?

→ Rotational Velocity of a R.B.



$$\underbrace{\left({}^I\dot{C}^B\right)}_{\text{what is this?}} \notin SO(3),\ so(3)$$

$$\left[\vec{r}_{P/O'}(t)\right]_I = {}^IC^B(t)\left[\vec{r}_{P/O'}(0)\right]_B$$

Differentiate,

$$\frac{{}^Id}{dt}\left[\vec{r}_{P/O'}(t)\right]_I = \frac{{}^Id}{dt}\left({}^IC^B(t)\left[\vec{r}_{P/O'}(0)\right]_B\right)$$

$$= \underbrace{\frac{{}^Id}{dt}\left({}^IC^B(t)\right)}\left[\vec{r}_{P/O'}(0)\right]_B$$

This quantity maps body coordinates of P to the inertial velocity of P w.r.t. O'

Note: ${}^I\dot{C}^B$ has 9 entries and it would be convenient to have a more compact representation of the rotational velocity.

Rewrite:

(insert the identity here

$$\frac{{}^Id}{dt}\left[\vec{r}_{P/O'}(t)\right]_I = \boxed{\frac{{}^Id}{dt}\left({}^IC^B(t)\right)\left({}^IC^B\right)^T}\underbrace{\left({}^IC^B\right)\left[\vec{r}_{P/O'}(0)\right]_B}$$

Look at this term.

Intuition: trying to get everything in the equation in inertial coordinates

Want to compare to

$$\frac{{}^Id}{dt}\left[\vec{r}_{P/O'}(t)\right] = \left[{}^I\vec{\omega}^B \times\right]_I\left[\vec{r}_{P/O'}\right]_I$$

$\boxed{\text{Lemma}}$: $\frac{{}^{I}d}{dt}({}^{I}C^{B})({}^{I}C^{B})^{T} = \frac{{}^{I}d}{dt}({}^{I}C^{B})({}^{I}C^{B})^{-1} \in so(3)$

i.e. it's a skew symmetric matrix

Proof: Let $R \in SO(3)$

$$RR^{T} = I$$

$$\frac{{}^{I}d}{dt}(RR^{T}) = 0$$

$$\dot{R}R^{T} + R\dot{R}^{T} = 0 \Rightarrow \dot{R}R^{T} = -R\dot{R}^{T}$$

$$= -(\dot{R}R^{T})^{T}$$

$$= \text{skew symmetric} \checkmark$$

Recall A skew sym $\Rightarrow A = -A^{T}$

In fact, $\frac{{}^{I}d}{dt}({}^{I}C^{B})({}^{I}C^{B})^{T} \triangleq \widehat{[{}^{I}\vec{\omega}^{B}]_{I}}$

instantaneous angular velocity in inertial coordinates

${}^{I}\dot{C}^{B}$ doesn't have alot of meaning alone, but ${}^{I}\dot{C}^{B}({}^{I}C^{B})^{T}$ does.

We wondered what ${}^{I}\dot{C}^{B}$ was, and we see that,

$$\boxed{\frac{{}^{I}d}{dt}({}^{I}C^{B}) = \widehat{[{}^{I}\vec{\omega}^{B}]_{I}}\,{}^{I}C^{B}}$$

Matrix differential equation

Returning to the kinematics,

$$\frac{{}^{I}d}{dt}[\vec{r}_{P/O'}]_{I} = \widehat{[{}^{I}\vec{\omega}^{B}]_{I}}[\vec{r}_{P/O'}]_{I}$$

This is the more compact way to write the kinematics requiring only ${}^{I}\vec{\omega}^{B}$ because we made use of the structure of $SO(3)$ group.

Useful formulas for working with angular velocities:

$$\boxed{\begin{aligned} \left[{}^{I}\widehat{\vec{\omega}}^{B}\right]_I &= \frac{{}^{I}d}{dt}\left({}^{I}C^{B}\right)\left({}^{I}C^{B}\right)^T \\ \left[{}^{I}\widehat{\vec{\omega}}^{B}\right]_B &= \left[{}^{I}C^{B}\right]^T \frac{{}^{I}d}{dt}\left({}^{I}C^{B}\right) \end{aligned}}$$

instantaneous inertial angular velocity

instantaneous body angular velocity.

The rotational velocity of a rigid body can be described using these formulas. The orientation lives in $SO(3)$ but we need $\omega \in so(3)$ to describe its velocity.

Rigid Body Velocity (General case)

- We know the ${}^{I}g^{B}(t) \in SE(3)$ is the pose of the R.B.
- What is the velocity, in a geometric sense?

Look at homogeneous rep. $\quad \left({}^{I}C^{B}, \vec{r}_{0'/0}\right) \longleftrightarrow {}^{I}g^{B}$

$${}^{I}g^{B}(t) = \begin{bmatrix} {}^{I}C^{B}(t) & \vec{r}_{0'/0}(t) \\ \underline{0}^T & 1 \end{bmatrix}_I \in SE(3)$$

$$\left({}^{I}g^{B}\right)^{-1} = \begin{bmatrix} \left({}^{I}C^{B}\right)^T & -\left({}^{I}C^{B}\right)^T \vec{r}_{0'/0} \\ \underline{0}^T & 1 \end{bmatrix}_I \in SE(3)$$

Take the time derivative,

$$\frac{{}^{I}d}{dt}\left({}^{I}g^{B}\right) = \begin{bmatrix} \frac{{}^{I}d}{dt}\left({}^{I}C^{B}\right) & {}^{I}\vec{v}_{0'/0} \\ \underline{0}^T & 0 \end{bmatrix}_I \notin SE(3), se(3)$$

So what is $\left({}^{I}\dot{g}^{B}\right) = ?$

Let's consider what we did in the rotational case:

$$\frac{{}^{I}d}{dt}({}^{I}C^{B})({}^{I}C^{B})^{-1} \in so(3)$$

Look at, $({}^{I}\dot{g}^{B}) = (\,?\,)({}^{I}g^{B})$

Look at $({}^{I}\dot{g}^{B})({}^{I}g^{B})^{-1}$

$$\frac{{}^{I}d}{dt}({}^{I}g^{B})({}^{I}g^{B})^{-1} = \begin{bmatrix} \frac{{}^{I}d}{dt}({}^{I}C^{B}) & \vec{v}_{0'/0}(t) \\ 0_{1\times3} & 0 \end{bmatrix}\begin{bmatrix} ({}^{I}C^{B})^{T} & -({}^{I}C^{B})^{T}\vec{r}_{0'/0} \\ 0_{1\times3} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{{}^{I}d}{dt}({}^{I}C^{B})({}^{I}C^{B})^{T} & -\frac{{}^{I}d}{dt}({}^{I}C^{B})({}^{I}C^{B})^{T}\vec{r}_{0'/0} + \vec{v}_{0'/0} \\ 0_{1\times3} & 0 \end{bmatrix}$$

Recall $\begin{bmatrix} \hat{\omega} & \vec{v} \\ 0^{T} & 0 \end{bmatrix} \in se(3)$ "Twist"

$$= \underbrace{\begin{bmatrix} \widehat{[{}^{I}\vec{\omega}^{B}]}_{I} & -{}^{I}\vec{\omega}^{B} \times \vec{r}_{0'/0} + {}^{I}\vec{v}_{0'/0} \\ 0_{1\times3} & 0 \end{bmatrix}}_{\triangleq \underbrace{\widehat{[{}^{I}V^{B}]}_{I}}_{\text{"Twist"}}} \in se(3)$$

"Spatial velocity"

MLS notation: $\hat{V}^{s}_{IB}$

Finally,

$$\boxed{\frac{{}^{I}d}{dt}({}^{I}g^{B}) = \widehat{[{}^{I}V^{B}]}_{I}({}^{I}g^{B})}$$

Matrix differential equation

is the infinitesimal generator of SE(3)

Remove the "$\wedge$" operation,

$$\left[{}^{I}V^{B}\right]_I = \begin{bmatrix} {}^{B}\vec{v}_{O'/O} \\ {}^{I}\vec{\omega}^{B} \end{bmatrix}_I = \begin{bmatrix} -{}^{I}\vec{\omega}^{B} \times \vec{r}_{O'/O} + {}^{I}\vec{v}_{O'/O} \\ {}^{I}\vec{\omega}^{B} \end{bmatrix}_I$$

Twist coordinates

~~We are going to show that the spatial velocity can be used to find the inertial velocity of a point.~~

Recall $\begin{bmatrix} \vec{r}_{P/O} \\ 1 \end{bmatrix}_I = {}^{I}g^{B} \begin{bmatrix} \vec{r}_{P/O'} \\ 1 \end{bmatrix}_B$

$$\frac{{}^{I}d}{dt}\begin{bmatrix} \vec{r}_{P/O} \\ 1 \end{bmatrix}_I = \frac{{}^{I}d}{dt}({}^{I}g^{B})\begin{bmatrix} \vec{r}_{P/O'} \\ 1 \end{bmatrix}_B = \frac{{}^{I}d}{dt}({}^{I}g^{B})$$

$$= \begin{bmatrix} \vec{r}_{P/O} \\ 1 \end{bmatrix}_I$$

$$\Rightarrow \boxed{\begin{bmatrix} {}^{I}\vec{v}_{P/O} \\ 0 \end{bmatrix}_I = \begin{bmatrix} \vec{r}_{P/O} \\ 1 \end{bmatrix}_I}$$

Writing this out,

$$\begin{bmatrix} {}^I\vec{v}_{P/O} \\ 0 \end{bmatrix}_I = \begin{bmatrix} 0 & & 0 \end{bmatrix} \begin{bmatrix} \vec{r}_{P/O} \\ 1 \end{bmatrix}_I = \begin{bmatrix} & 0 & \end{bmatrix}_I$$

Note: Compare this to

$$\frac{{}^I d}{dt}\left(\vec{r}_{P/O}\right) =$$

Use T.E.,

$$= {}^B\vec{v}_{P/O'} + {}^I\vec{\omega}^B \times \vec{r}_{P/O'} + {}^B\vec{v}_{O'/O} + {}^I\vec{\omega}^B \times \vec{r}_{O'/O}$$

$$=$$