2.25

T^{-1}: $u+v+w \to u$

$u+v \to v$

$u \to w$

Therefore : $x = u+v+w$

$y = u+v$

$z = u$

$y = z+v$

$v = y-z$

$x = u+v+w = \cancel{z} + (y-\cancel{z}) + w$

$w = x-y$

$\boxed{T^{-1} \text{ transforms } (x,y,z) \text{ into } (z,\ y-z,\ x-y)}$

2.27

$$A_1 = \begin{pmatrix} \boxed{1} & 2 & 0 & 3 \\ 0 & \boxed{2} & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{4} \end{pmatrix}$$

$$\boxed{\begin{aligned} C(A_1) &= \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 0 \\ 4 \end{pmatrix} \right\} \\ C(A_1^T) &= \text{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 4 \end{pmatrix} \right\} \end{aligned}}$$

$x_1 + 2x_2 + 3x_4 = 0$

$2x_2 + 2x_3 + 2x_4 = 0$ $\qquad x_2 = -x_3$ ← Free, let $x_3 = 1$

$x_4 = 0$ $\qquad \therefore x_2 = -1$

$x_1 - 2 = 0 \qquad x_1 = 2$

$$\boxed{N(A_1) = \text{span}\left\{ \begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \end{pmatrix} \right\}}$$

$$A_1^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 3 & 2 & 0 & 4 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 3 & 2 & 0 & 4 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 2 & 0 & 4 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 4 \end{pmatrix}$$

2.27

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

$x_1 = 0$

$2x_2 = 0$

$4x_4 = 0$ $\quad x_3 = 1$, free

$$\boxed{N(A_1^T) = \text{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}}$$

$$A_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 1 & 4 \\ 1 & 4 \end{pmatrix}$$

$$A_2 \Rightarrow \begin{pmatrix} \boxed{1} & 4 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\boxed{\begin{aligned} C(A_2) &= \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\} \\ C(A_2^T) &= \text{span}\left\{ \begin{pmatrix} 1 \\ 4 \end{pmatrix} \right\} \end{aligned}}$$

$x_1 + 4x_2 = 0$ $\quad x_2 = 1$, free $\quad \therefore x_1 = -4$

$$\boxed{N(A_2) = \text{span}\left\{ \begin{pmatrix} -4 \\ 1 \end{pmatrix} \right\}}$$

2.27

$$A_2^T = \begin{pmatrix} 1 & 1 & 1 \\ 4 & 4 & 4 \end{pmatrix}$$

$$A_2^T \Rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow x_1 + x_2 + x_3 = 0$$

Free

$x_1 = -x_2$ or $x_1 = -x_3$

$$\boxed{N(A_2^T) = \text{span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}}$$

3.3.4

$$E^2 = \|Ax-b\|^2$$

$$Ax-b = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} u-1 \\ v-3 \\ u+v-4 \end{pmatrix}$$

$$\|Ax-b\|^2 = (Ax-b)^T(Ax-b)$$

$$= (u-1,\ v-3,\ u+v-4)\begin{pmatrix} u-1 \\ v-3 \\ u+v-4 \end{pmatrix} =$$

$$(u^2-2u+1) + (v^2-6v+9) + (u^2+v^2+2vu-8u-8v+16)$$

$$= E^2 = 2u^2+2v^2-10u-14v+2vu+26$$

$$\frac{\partial E^2}{\partial u} = 4u-10+2v = 0$$

$$\frac{\partial E^2}{\partial v} = 4v-14+2u = 0$$

$$\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 10 \\ 14 \end{pmatrix}$$

$$(A^TA) \quad \hat{x} = A^Tb$$

3.3.4

$$A^TA = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$A^Tb = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \end{pmatrix}$$

The calculus gives the same (A^TA) & (A^Tb) as the geometry but is scaled by a factor of 2. To make them match, E^2 should be defined as $E^2 = \frac{1}{2}(Ax-b)^T(Ax-b)$.

$$\hat{x} = (A^TA)^{-1}A^Tb$$

$$(A^TA)^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{pmatrix}$$

$$\hat{x} = \begin{pmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{pmatrix} \begin{pmatrix} 5 \\ 7 \end{pmatrix} = \boxed{\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \hat{x}}$$

$$P = A\hat{x} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$\boxed{P = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}}$$

3.3.4

$$C(A) = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

$$b = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + 3 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$$

$\boxed{b \text{ exists is } C(A) \text{ so } P = b.}$

3.3.13

$$A = \begin{pmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 2 \end{pmatrix} \qquad b = \begin{pmatrix} 4 \\ 3 \\ 1 \\ 0 \end{pmatrix}$$

$$A^TA = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 4 & -1 \\ -1 & 9 \end{pmatrix}$$

$$(A^TA)^{-1} = \frac{1}{35} \begin{pmatrix} 9 & 1 \\ 1 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 9/35 & 1/35 \\ 1/35 & 4/35 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 7/35 & 8/35 & 9/35 & 11/35 \\ -7/35 & -3/35 & 1/35 & 9/35 \end{pmatrix}$$

$$(A^TA)^{-1}A^T b = \begin{pmatrix} 7/35 & 8/35 & 9/35 & 11/35 \\ -7/35 & -3/35 & 1/35 & 9/35 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \\ 1 \\ 0 \end{pmatrix}$$

$$= \hat{x} = \begin{pmatrix} \frac{28+24+9}{35} \\ \frac{-28-9+1}{35} \end{pmatrix} = \boxed{\begin{pmatrix} 61/35 \\ -\frac{36}{35} \end{pmatrix} = \frac{61}{35} - \frac{36}{35}t}$$

$$P = A(A^TA)^{-1}A^T = \begin{pmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 7/35 & 8/35 & 9/35 & 11/35 \\ -7/35 & -3/35 & 1/35 & 9/35 \end{pmatrix}$$

$$P = \begin{pmatrix} 21/35 & 14/35 & 7/35 & -7/35 \\ 14/35 & 11/35 & 8/35 & 2/35 \\ 7/35 & 8/35 & 9/35 & 11/35 \\ -7/35 & 2/35 & 11/35 & 29/35 \end{pmatrix}$$

3.3.13

$$Pb = \begin{pmatrix} 21/35 & 14/35 & 7/35 & -7/35 \\ 14/35 & 11/35 & 8/35 & 2/35 \\ 7/35 & 8/35 & 9/35 & 11/35 \\ -7/35 & 2/35 & 11/35 & 29/35 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \\ 1 \\ 0 \end{pmatrix}$$

$$Pb = \begin{pmatrix} 84 + 42 + 7 \\ 56 + 33 + 8 \\ 28 + 24 + 9 \\ -28 + 6 + 11 \end{pmatrix} \frac{1}{35}$$

$$\boxed{Pb = \begin{pmatrix} 133/35 \\ 97/35 \\ 61/35 \\ -11/35 \end{pmatrix}}$$

3.4.9

q_1, q_2, q_3 are orthonormal $\therefore$ $q_1 \perp q_2$, $q_1 \perp q_3$, $q_2 \perp q_3$. The closest combination of q_1 & q_2 to q_3 is $\boxed{0q_1 + 0q_2}$

3.4.23

$$a_0 = \frac{\langle y, 1 \rangle}{\langle 1, 1 \rangle}$$

$$\langle y, 1 \rangle = \int_0^{2\pi} y\,dx = \int_0^{\pi} dx + \cancel{\int_{\pi}^{2\pi} 0\,dx}^{\,0}$$

$$\langle y, 1 \rangle = \pi$$

$$\langle 1, 1 \rangle = \int_0^{2\pi} dx = 2\pi$$

$$a_0 = \frac{\pi}{2\pi} = \frac{1}{2}$$

$$a_1 = \frac{\langle y, \cos x \rangle}{\langle \cos x, \cos x \rangle}$$

$$\langle y, \cos x \rangle = \int_0^{2\pi} y \cos x\,dx = \int_0^{\pi} \cos x\,dx + \cancel{\int_{\pi}^{2\pi} 0\,dx}^{\,0}$$

$$\langle y, \cos x \rangle = \sin(\pi) - \sin(0) = 0$$

$$a_1 = 0$$

$$b_1 = \frac{\langle y, \sin x \rangle}{\langle \sin x, \sin x \rangle}$$

$$\langle y, \sin x \rangle = \int_0^{2\pi} y \sin x = \int_0^{\pi} \sin x\,dx + \cancel{\int_{\pi}^{2\pi} 0\,dx}^{\,0}$$

3.4.23

$$\int_0^{\pi} \sin x \, dx = -\cos(\pi) + \cos(0) = 2$$

$$\langle \sin x, \sin x \rangle = \int_0^{2\pi} \sin^2 x \, dx = \pi \quad \text{(Equation 17 in 3.4)}$$

$$b_1 = \frac{2}{\pi}$$

$$\therefore \boxed{a_0 = \frac{1}{2}, \; a_1 = 0, \; b_1 = \frac{2}{\pi}}$$

$$y(x) = a_0 + b_1 \sin x + \ldots$$

3.5.1

$$F_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega_n & \omega_n^2 & \omega_n^3 \\ 1 & \omega_n^2 & \omega_n^4 & \omega_n^6 \\ 1 & \omega_n^3 & \omega_n^6 & \omega_n^9 \end{pmatrix}$$

$$\omega_n = e^{2\pi i/n} = e^{\pi i/2} = \cancel{\cos\left(\tfrac{\pi}{2}\right)}^{0} + i\left(\cancel{\sin\left(\tfrac{\pi}{2}\right)}^{1}\right) = i$$

$$F_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}$$

$$F_4^2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}$$

$$\boxed{F_4^2 = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 4 & 0 \\ 0 & 4 & 0 & 0 \end{pmatrix}}$$

$$F_4^4 = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 4 & 0 \\ 0 & 4 & 0 & 0 \end{pmatrix}\begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 4 & 0 \\ 0 & 4 & 0 & 0 \end{pmatrix}$$

$$\boxed{F_4^4 = \begin{pmatrix} 16 & 0 & 0 & 0 \\ 0 & 16 & 0 & 0 \\ 0 & 0 & 16 & 0 \\ 0 & 0 & 0 & 16 \end{pmatrix}}$$

3.5.5

$e^{ix} = -1$

$e^{ix} = \cos(x) + i\sin(x) = -1$

$e^{ix} = -1$ when $x = \underbrace{\pi,\ 3\pi,\ 5\pi,}\ \ldots$

$k=0: (2k+1)\pi = \pi$
$k=1: (2k+1)\pi = 3\pi$
$k=2: (2k+1)\pi = 5\pi$

$$\boxed{x = (2k+1)\pi}$$

$e^{i\theta} = i = \cos(\theta) + i\sin(\theta) = i$

$\theta = \frac{\pi}{2}$ gives $e^{i\theta} = i$

$e^{i\theta} = i$ for $x = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}$

$k=0: (4k+1)\frac{\pi}{2} = \frac{\pi}{2}$
$k=1: (4k+1)\frac{\pi}{2} = \frac{5\pi}{2}$
$k=2: (4k+1)\frac{\pi}{2} = \frac{9\pi}{2}$

$$\boxed{x = (4k+1)\frac{\pi}{2}}$$

3.18

$$Q = \begin{pmatrix} 2/3 & -1/3 \\ 2/3 & 2/3 \\ -1/3 & 2/3 \end{pmatrix}$$

$$Q^T Q = \begin{pmatrix} 2/3 & 2/3 & -1/3 \\ -1/3 & 2/3 & 2/3 \end{pmatrix} \begin{pmatrix} 2/3 & -1/3 \\ 2/3 & 2/3 \\ -1/3 & 2/3 \end{pmatrix}$$

$$\boxed{Q^T Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$$

$$Q Q^T = \begin{pmatrix} 2/3 & -1/3 \\ 2/3 & 2/3 \\ -1/3 & 2/3 \end{pmatrix} \begin{pmatrix} 2/3 & 2/3 & -1/3 \\ -1/3 & 2/3 & 2/3 \end{pmatrix}$$

$$\boxed{Q Q^T = \begin{pmatrix} 5/9 & 2/9 & -4/9 \\ 2/9 & 8/9 & 2/9 \\ -4/9 & 2/9 & 5/9 \end{pmatrix}}$$

Projection matrix: $P^2 = P$ and P is symmetric

QQ^T is Symmetric

3.18

$$(QQ^T)(QQ^T) = \begin{pmatrix} 5/9 & 2/9 & -4/9 \\ 2/9 & 8/9 & 2/9 \\ -4/9 & 2/9 & 5/9 \end{pmatrix} \begin{pmatrix} 5/9 & 2/9 & -4/9 \\ 2/9 & 8/9 & 2/9 \\ -4/9 & 2/9 & 5/9 \end{pmatrix}$$

$$= (QQ^T)^2 = \begin{pmatrix} 45/81 & 18/81 & -36/81 \\ 18/81 & 72/81 & 18/81 \\ -36/81 & 18/81 & 45/81 \end{pmatrix}$$

$$= \begin{pmatrix} 5/9 & 2/9 & -4/9 \\ 2/9 & 8/9 & 2/9 \\ -4/9 & 2/9 & 5/9 \end{pmatrix}$$

Because $(QQ^T)^2 = QQ^T$ & (QQ^T) is symmetric, QQ^T is a projection

3.28

$$a_1 = \begin{pmatrix} 4 \\ 5 \\ 2 \\ 2 \end{pmatrix}$$

$$q_1 = \frac{a_1}{\|a_1\|} \qquad \|a_1\|^2 = a_1^T a_1$$

$$\|a_1\|^2 = (4\ 5\ 2\ 2) \begin{pmatrix} 4 \\ 5 \\ 2 \\ 2 \end{pmatrix} = 49$$

$$\|a_1\| = \sqrt{\|a_1\|^2} = 7$$

$$\boxed{q_1 = \begin{pmatrix} 4/7 \\ 5/7 \\ 2/7 \\ 2/7 \end{pmatrix}}$$

$$a_2' = a_2 - (q_1^T a_2)\, q_1$$

$$= \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix} - \left[(4/7,\ 5/7,\ 2/7,\ 2/7) \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix} \right] \begin{pmatrix} 4/7 \\ 5/7 \\ 2/7 \\ 2/7 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 8/7 \\ 10/7 \\ 4/7 \\ 4/7 \end{pmatrix}$$

$$a_2' = \begin{pmatrix} -1/7 \\ 4/7 \\ -4/7 \\ -4/7 \end{pmatrix}$$

$$q_2 = \frac{a_2'}{\|a_2'\|}$$

3.28

$$\|a_2'\|^2 = a_2'^T a_2' = \begin{pmatrix} -1/7 & 4/7 & -4/7 & -4/7 \end{pmatrix} \begin{pmatrix} -1/7 \\ 4/7 \\ -4/7 \\ -4/7 \end{pmatrix}$$

$$\|a_2'\|^2 = 1$$

$$\|a_2\| = 1$$

$$\boxed{q_2 = \begin{pmatrix} -1/7 \\ 4/7 \\ -4/7 \\ -4/7 \end{pmatrix}}$$

$$\boxed{\begin{aligned} a_1 &= 7q_1 \\ a_2 &= 2q_1 + q_2 \end{aligned}}$$

$$Q = \begin{pmatrix} 4/7 & -1/7 \\ 5/7 & 4/7 \\ 2/7 & -4/7 \\ 2/7 & -4/7 \end{pmatrix}$$

$$R = Q^T A = \begin{pmatrix} 4/7 & 5/7 & 2/7 & 2/7 \\ -1/7 & 4/7 & -4/7 & -4/7 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 5 & 3 \\ 2 & 0 \\ 2 & 0 \end{pmatrix}$$

$$\boxed{R = \begin{pmatrix} 7 & 2 \\ 0 & 1 \end{pmatrix}}$$