ECE 602: LUMPED LINEAR SYSTEMS

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Linear Quadratic Regulation: Application Examples

Example Problem I

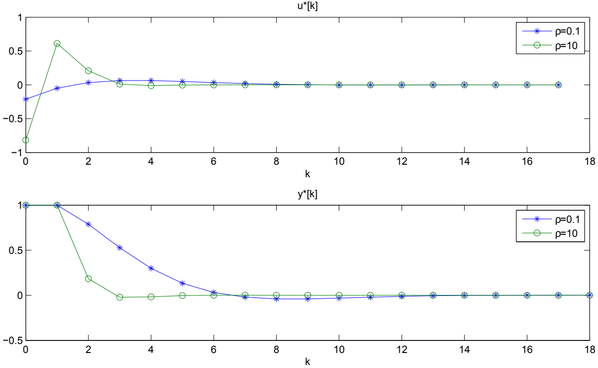
$$x[k+1] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x[k] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u[k], \quad x[0] = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$y[k] = \begin{bmatrix} 1 & 0 \end{bmatrix} x[k]$$

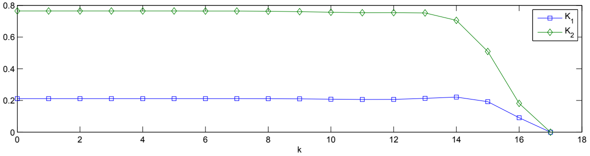
Cost function to be minimized: $J(U) = \sum_{k=0}^{N-1} \|u[k]\|^2 + \rho \sum_{k=0}^{N} \|y[k]\|^2$

- Time horizon $N = 18$
- State weights $Q = Q_f = \rho C^T C$, and control weight $R = 1$
- Optimal control is of the form $u^*[t] = \begin{bmatrix} K_1(t) & K_2(t) \end{bmatrix} x^*[t]$

Optimal Solutions of Example Problem I



Plot of the Kalman gain $K(t) = \begin{bmatrix} K_1(t) & K_2(t) \end{bmatrix}$ (assume $\rho = 0.1$):



Convergence of Riccati Recursion

Theorem

If (A, B) is stabilizable, then Riccati recusion starting from any P_N will converge to a solution $P_{\rm ss}$ of the **Algebraic Riccati Equation (ARE)***:*

$$P_{\rm ss} = Q + A^T P_{\rm ss} A - A^T P_{\rm ss} B (R + B^T P_{\rm ss} B)^{-1} B^T P_{\rm ss} A$$

If further $Q = C^T C$ for some C such that (C, A) is detectable, then the solution $P_{\rm ss} \succ 0$ to the ARE is also unique. In this case by applying the steady-state optimal control with gain

$$K_{\rm ss} = (R + B^T P_{\rm ss} B)^{-1} B^T P_{\rm ss} A,$$

the closed-loop system $A_{cl} = A - BK_{ss}$ is stable.

[0] "On the discrete time matrix Riccati equation of optimal control," P.E. Caines and D.Q. Mayne, Int. J. Control, vol. 12, no. 5, pp. 785-794, 1970.

Infinite Horizon LQR Problem

Problem: Find optimal $U = \{u[0], u[1], \ldots\}$ to minimize

$$J(U) = \sum_{k=0}^{\infty} \Big(x[k]^T Q x[k] + u[k]^T R u[k] \Big)$$

- Value function is independent of time, with Bellman equation:

$$V(x) = x^T Q x + \min_v \left[v^T R v + V(Ax + Bv) \right]$$

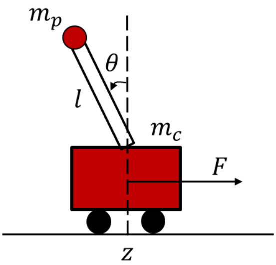
- Infinite value function possible

Theorem

If (A, B) is stabilizable and (C, A) is detectable where $Q = C^T C$, then the value function $V(x)$ of the infinite horizon problem is $V(x) = x^T P_{\mathrm{ss}} x$ where P_{ss} is the unique positive semidefinite solution to the discrete-time ARE; and the optimal control is stationary $u^*(t) = -K_{\mathrm{ss}} x^*(t)$.

Matlab command `lqr`

Example Problem II: Inverted Pendulum



States $x = \begin{bmatrix} z & \theta & \dot{z} & \dot{\theta} \end{bmatrix}^T$:

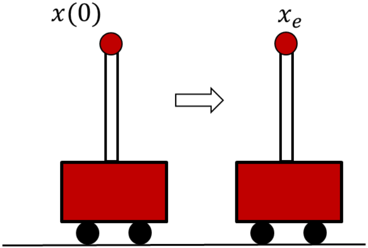
- θ: angle of pendulum
- $\dot{\theta}$: angular velocity of pendulum
- z: horizontal position of cart
- $\dot{z}$: velocity of cart

Input $u = F$

Linearized dynamics near $x_e = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T$ (sampled at $T = 0.5$ s):

$$x_{k+1} = \underbrace{\begin{bmatrix} 1 & 0.1054 & 0.06128 & 0.01688 \\ 0 & 5.753 & -1.859 & 1.154 \\ 0 & 0.5287 & -0.1617 & 0.1037 \\ 0 & 27.31 & -9.328 & 5.665 \end{bmatrix}}_{A} x_k + \underbrace{\begin{bmatrix} 0.05763 \\ 0.2442 \\ 0.1526 \\ 1.225 \end{bmatrix}}_{B} u_k$$

Inverted Pendulum: Stabilization



$$x_0 = \begin{bmatrix} -1 & 0 & 0 & 0 \end{bmatrix}^T$$

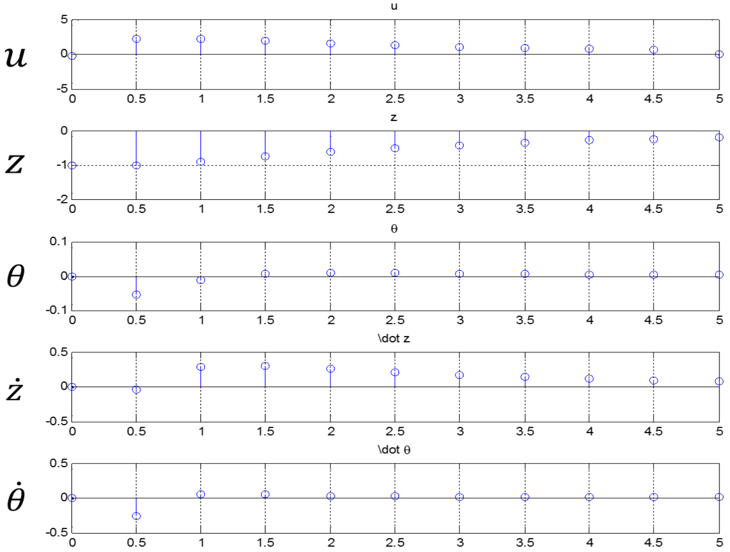
Goal: Find $u_0, \ldots, u_{N-1}$ to minimize

$$J = \alpha \sum_{k=1}^{N} \|x_k\|^2 + \beta \sum_{k=0}^{N-1} |u_k|^2$$

- LQR formulation: $Q = Q_f = \alpha I$, $R = \beta$
- Stabilization with energy consideration

Inverted Pendulum: Solutions ($\alpha = 10,\ \beta = 1$)

Optimal solution with horizon 10 ($\alpha = 10, \beta = 1$)



Inverted Pendulum: Solutions ($\alpha = 10^4$, $\beta = 1$)