ECE 602: LUMPED LINEAR SYSTEMS

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From a Transfer Function to a State-Space Description: Single-Input Single-Output Case

From a Transfer Function to a State-Space Model: Single-Input Single-Output Case

- **Objective**: Convert a transfer function into a state-space representation of linear lumped systems both discrete and continuous time-invariant single-input single output (SISO) systems
- A transfer function $\boldsymbol{G}(s)$ is realizable if there exists a quadruple of constant matrices $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D})$ such that $\boldsymbol{G}(s) = \boldsymbol{C}\left(s\boldsymbol{I}_n - \boldsymbol{A}\right)^{-1}\boldsymbol{B} + \boldsymbol{D}$. We call such a quadruple $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D})$ a realization of $\boldsymbol{G}(s)$

Definition

The dimension of a realization is the size of the matrix $\boldsymbol{A}$, that is, if $\boldsymbol{A}$ is an n-by-n matrix then we say that the dimension of the corresponding realization is n

Transfer function of a single-input single-output (SISO) system

- Consider a system modeled by a transfer function,

$$\begin{aligned} \frac{Y(s)}{U(s)} &= G(s) = \frac{N(s)}{D(s)} \\ &= \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} \end{aligned}$$

- Assume $G(s)$ is a proper rational function, that is,

$$\deg_s N(s) \leq \deg_s D(s)$$

- Note that the highest coefficient of the denominator polynomial is unity. If this was not the case, we would divide the numerator and the denominator by the highest coefficient of the denominator thus forcing $a_n = 1$

Proper, strictly proper, improper transfer functions

- $G(s)$ is strictly proper if $\deg_s N(s) < \deg_s D(s)$, equivalently, $m < n$
- $G(s)$ is improper if $\deg_s N(s) > \deg_s D(s)$, equivalently, $m > n$
- If $G(s)$ is proper but not strictly, that is, $\deg_s N(s) = \deg_s D(s)$, equivalently, $m = n$, then we divide the numerator $N(s)$ by the denominator $D(s)$ to obtain

$$G(s) = G(s)_{\mathrm{sp}} + G(\infty),$$

where $G(s)_{\mathrm{sp}}$ denotes the strictly proper part of $G(s)$ and $G(\infty) = b_m$

Improper transfer functions

- $G(s)$ is improper if $\deg_s N(s) > \deg_s D(s)$, equivalently, $m > n$
- If $G(s)$ is improper and we divide the numerator $N(s)$ by the denominator $D(s)$, we obtain

$$G(s) = G(s)_{\mathrm{sp}} + G_{\mathrm{rm}}(s),$$

  where $G(s)_{\mathrm{sp}}$ denotes the strictly proper part and the remainder $G_{\mathrm{rm}}(s)$ is a polynomial

Stating our problem more precisely

- Our goal is to find a realization of a proper transfer function $G(s)$, that is, a quadruple, $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}, d)$ such that

$$G(s) = \boldsymbol{c}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{b} + d.$$

- Note that $\boldsymbol{c}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{b}$ is a strictly proper rational function and d is a scalar
- In our construction of a realization of $G(s)$, we first extract the strictly proper part
- We then find a triple $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$ such that

$$G(s)_{\mathrm{sp}} = \boldsymbol{c}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{b}$$

- A realization of $G(s)$ will have the form $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}, G(\infty))$, that is, $d = G(\infty)$

First step in constructing a state-space realization

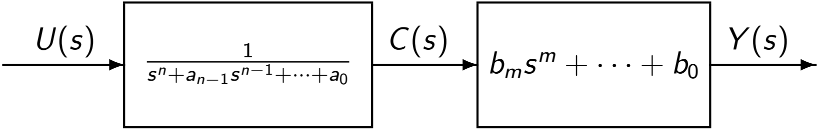
- To proceed, we assume that $G(s) = G(s)_{\mathrm{sp}}$
- We split our procedure of finding a triple $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$ into two steps
- In the first step, introduce an intermediate Laplace variable, $C(s)$, such that

$$\frac{Y(s)}{U(s)} = \frac{Y(s)}{C(s)} \frac{C(s)}{U(s)}.$$

- Thus

$$\begin{aligned} \frac{Y(s)}{U(s)} &= \frac{1}{D(s)} N(s) \\ &= \frac{1}{s^n + a_{n-1}s^{n-1} + \cdots + a_0} \left(b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0 \right) \\ &= \frac{Y(s)}{C(s)} \frac{C(s)}{U(s)} \end{aligned}$$

Decomposing $G(s)$ to construct its state-space realization



$$
\begin{aligned}
Y(s) &= N(s)C(s) = N(s)\frac{1}{D(s)}U(s) \\
&= \left(b_m s^m + b_{m-1}s^{m-1} + \cdots + b_0\right)\frac{1}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}U(s)
\end{aligned}
$$

Manipulating $\frac{C(s)}{U(s)}$

- We first concern ourselves with the transfer function
$$\frac{C(s)}{U(s)} = \frac{1}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}$$
- Perform cross-multiplication
$$\left(s^n + a_{n-1}s^{n-1} + \cdots + a_0\right) C(s) = U(s).$$
- Take the inverse Laplace transform with zero initial conditions
$c^{(n)} + a_{n-1}c^{(n-1)} + \cdots + a_1\dot{c} + a_0 c = u.$
- Define the state variables:
$$\left.\begin{array}{lcl} x_1 & = & c \\ & \vdots & \\ x_{n-1} & = & c^{(n-2)} = \dot{x}_{n-2} \\ x_n & = & c^{(n-1)} = \dot{x}_{n-1} \end{array}\right\}$$

Selecting state variables for a state-space realization of $\frac{C(s)}{U(s)}$

- Define the state variables:

$$\left.\begin{array}{lcl} x_1 & = & c \\ & \vdots & \\ x_{n-1} & = & c^{(n-2)} = \dot{x}_{n-2} \\ x_n & = & c^{(n-1)} = \dot{x}_{n-1} \end{array}\right\}$$

- Note that

$$\begin{array}{rcl} \dot{x}_n & = & c^{(n)}r \\ & = & -a_0 c - a_1 \dot{c} - \cdots - a_{n-1} c^{(n-1)} + u \\ & = & -a_0 x_1 - a_1 x_2 - \cdots - a_{n-1} x_n + u \end{array}$$

State-space model of $\frac{C(s)}{U(s)}$

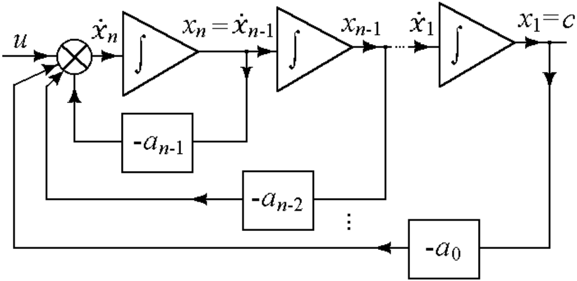
- Matrix-vector format representation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u,$$

where

$$c(t) = x_1$$

Block diagram of the state-space realization of $\frac{C(s)}{U(s)}$



Including the transfer function $\frac{Y(s)}{C(s)}$

- Need to include the transfer function

$$\frac{Y(s)}{C(s)} = b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0$$

- Represent it as

$$Y(s) = b_m s^m C(s) + b_{m-1} s^{m-1} C(s) + \cdots + b_1 s C(s) + b_0 C(s)$$

- The inverse Laplace transform with zero initial conditions, $y = b_m c^{(m)} + b_{m-1} c^{(m-1)} + \cdots + b_1 \dot{c} + b_0 c$
- Note that

$$\begin{aligned} c &= x_1 \\ \dot{c} &= \dot{x}_1 = x_2 \\ &\vdots \\ c^{(m)} &= \dot{x}_m = x_{m+1} \end{aligned}$$

State-space realization of $\frac{Y(s)}{C(s)}$

- Represent y in terms of the state variables,

$$y = b_m x_{m+1} + b_{m-1} x_m + \cdots + b_1 x_2 + b_0 x_1$$

- Equivalently as

$$y = \begin{bmatrix} b_0 & b_1 & \cdots & b_{m-1} & b_m & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \\ x_{m+1} \\ \vdots \\ x_n \end{bmatrix}$$

Combining all together ...

- Combining all together, we obtain a state-space realization

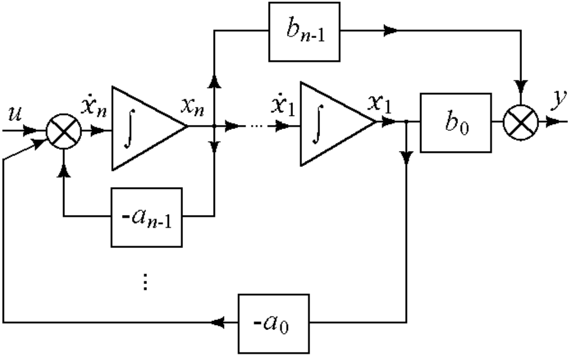
$$
\begin{aligned}
\dot{x} &= \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix} u \\
y &= \begin{bmatrix} b_0 & b_1 & \cdots & b_m & 0 & \cdots & 0 \end{bmatrix} x
\end{aligned}
$$

A very nice property of the realization with a pair (A, b) in the controller form

- Let $D(s) = \det[s\boldsymbol{I}_n - \boldsymbol{A}]$
- Then

$$
\begin{aligned}
& [s\boldsymbol{I}_n - \boldsymbol{A}]^{-1}\boldsymbol{b} \\
= \;& \begin{bmatrix} s & -1 & \cdots & 0 \\ 0 & s & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 \\ a_0 & a_1 & \cdots & s + a_{n-1} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = \frac{1}{D(s)} \begin{bmatrix} 1 \\ s \\ s^2 \\ \vdots \\ s^{n-2} \\ s^{n-1} \end{bmatrix}
\end{aligned}
$$

Block diagram of the state-space realization of $\frac{Y(s)}{U(s)} = G(s)$ where $m = n - 1$



Example

- Construct a realization of the transfer function

$$G(s) = \frac{4s^3 - 2s^2 + 3s + 1}{s^3 + 3s^2 - 5s + 7}$$

- First, represent $G(s)$ as

$$\begin{aligned} G(s) &= G(s)_{\mathrm{sp}} + G(\infty) \\ &= \frac{-14s^2 + 23s - 27}{s^3 + 3s^2 - 5s + 7} + 4. \end{aligned}$$

- Using the results of the above discussion, we obtain

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -7 & 5 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} -27 & 23 & -14 \end{bmatrix} x + 4u \end{aligned}$$

How many state-space realizations of a given tf are there?

- Infinitely many possible realizations of a given $G(s)$
- Can easily obtain another realization of $G(s)$
- Indeed, note that $G(s)$ can be viewed as an 1×1 matrix; therefore, its transpose equals itself, that is,

$$G(s)^\top = G(s)$$

- Transpose the previous realization

$$\begin{aligned} G(s) &= G(s)^\top \\ &= \left(\boldsymbol{c}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{b} + d\right)^\top \\ &= \boldsymbol{b}^\top \left((s\boldsymbol{I} - \boldsymbol{A})^{-1}\right)^\top \boldsymbol{c}^\top + d^\top \\ &= \boldsymbol{b}^\top \left(s\boldsymbol{I} - \boldsymbol{A}^\top\right)^{-1} \boldsymbol{c}^\top + d \\ &= \tilde{\boldsymbol{c}} \left(s\boldsymbol{I} - \tilde{\boldsymbol{A}}\right)^{-1} \tilde{\boldsymbol{b}} + d. \end{aligned}$$

Another state-space realization of the same tf

- Another realization of $G(s)$,

$$
\begin{aligned}
\dot{\tilde{x}} &= \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -a_{n-2} \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix} \tilde{x} + \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-2} \\ b_{n-1} \end{bmatrix} u \\
y &= \begin{bmatrix} 0 & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} \tilde{x} + G(\infty)
\end{aligned}
$$

- Can generalize to multi-input multi-output (MIMO) transfer function matrices?