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6.5.7 $t * e^t = \int_0^t \tau e^{t-\tau}\, d\tau$

$$= (\tau)(-e^{t-\tau})\Big|_0^t - \int_0^t (-e^{t-\tau})\, d\tau$$

integration by parts

$$= -t + \left[-e^{t-\tau}\right]_0^t$$

$$= -t - 1 + e^t$$

6.5.8 $\Rightarrow 2t = y(t) + 4\,(f*g)(t)$, where
$f(t) = y(t)$
$g(t) = t$

$$\Rightarrow 2\mathcal{L}(t) = \mathcal{L}(y) + 4\,\mathcal{L}(\underbrace{f}_{=y})\,\mathcal{L}(\underbrace{g}_{=t})$$

$$\Rightarrow \frac{2}{s^2} = \mathcal{L}(y)\left(1 + \frac{4}{s^2}\right)$$

$$\Rightarrow \mathcal{L}(y) = \frac{2}{s^2+4}$$

$$\Rightarrow y = \sin(2t)$$

6.5.23 $\mathcal{L}(f) = \dfrac{40.5}{s(s^2-9)} = \dfrac{81/2}{s(s^2-3^2)} = \underbrace{\dfrac{27}{2}\cdot\dfrac{1}{s}}_{P(s)}\cdot\underbrace{\dfrac{3}{s^2-3^2}}_{Q(s)}$

$$\Rightarrow f = \int_0^t \underbrace{\frac{27}{2}\cdot 1}_{p(t-\tau)}\ \underbrace{\sinh(3\tau)}_{q(t)}\, d\tau$$

$$= \frac{27}{2}\left[\frac{1}{3}\cosh(3\tau)\right]_0^t$$

$$= \frac{9}{2}\left(\cosh(3t) - 1\right)$$

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6.6.8 $f(t) = te^{-kt}\sin t$

$$\Rightarrow \mathcal{L}(f) = -\frac{d}{ds}\left(\mathcal{L}(e^{-kt}\sin t)\right)$$

$$= -\frac{d}{ds}\left(\mathcal{L}(\sin(t))\Big|_{s=s+k}\right)$$

$$= -\frac{d}{ds}\left(\frac{1}{(s+k)^2+1}\right)$$

$$= \frac{2(s+k)}{((s+k)^2+1)^2}$$

6.6.10 $\mathcal{L}(t^n e^{kt}) = (-1)^n \frac{d^n}{ds^n}\left(\mathcal{L}(e^{kt})\right)$

$$= (-1)^n \frac{d^n}{ds^n}\left((s-k)^{-1}\right)$$

$$= (-1)^n(-1)^n n!(s-k)^{-(n+1)}$$

$$= \frac{n!}{(s-k)^{n+1}}$$

6.6.16 $\mathcal{L}(f) = \dfrac{2s+6}{(s^2+6s+10)^2}$

$$= \frac{\frac{d}{ds}(s^2+6s+10)}{(s^2+6s+10)^2}$$

$$= \frac{d}{ds}\left(\frac{-1}{s^2+6s+10}\right) \qquad \text{(quotient rule)}$$

$$\Rightarrow f = -t\,\mathcal{L}^{-1}\left(\frac{-1}{s^2+6s+10}\right)$$

$$= t\,\mathcal{L}^{-1}\left(\frac{1}{(s+3)^2+1}\right)$$

$$= t\,e^{-3t}\sin(t)$$

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6.7.3 Taking L.T. on both sides of eqns,

$$s\mathcal{L}(y_1) - \underbrace{y_1(0)}_{=3} = -\mathcal{L}(y_1) + 4\mathcal{L}(y_2)$$

$$s\mathcal{L}(y_2) - \underbrace{y_2(0)}_{=4} = 3\mathcal{L}(y_1) - 2\mathcal{L}(y_2)$$

$$\Rightarrow \mathcal{L}(y_1)(s+1) + \mathcal{L}(y_2)(-4) = 3$$

$$\mathcal{L}(y_1)(-3) + \mathcal{L}(y_2)(s+2) = 4$$

$$\Rightarrow \underbrace{\begin{bmatrix} s+1 & -4 \\ -3 & s+2 \end{bmatrix}}_{:= A} \begin{bmatrix} \mathcal{L}(y_1) \\ \mathcal{L}(y_2) \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$\Rightarrow$ .... (find inverse of A) ..

$$\Rightarrow \begin{bmatrix} \mathcal{L}(y_1) \\ \mathcal{L}(y_2) \end{bmatrix} = \underbrace{\frac{1}{(s+1)(s+2)-12} \begin{bmatrix} s+2 & 4 \\ 3 & s+1 \end{bmatrix}}_{= A^{-1}} \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$\Rightarrow \mathcal{L}(y_1) = \frac{3s+22}{(s+1)(s+2)-12} = \frac{3s+22}{(s+5)(s-2)},$$

$$\mathcal{L}(y_2) = \frac{4s+13}{(s+1)(s+2)-12} = \frac{4s+13}{(s+5)(s-2)}$$

$\Rightarrow$ ... (partial fraction decomp) ..

$$\Rightarrow \mathcal{L}(y_1) = \frac{-1}{s+5} + \frac{4}{s-2},$$

$$\mathcal{L}(y_2) = \frac{1}{s+5} + \frac{3}{s-2}$$

$$\Rightarrow y_1 = -e^{-5t} + 4e^{2t},$$

$$y_2 = e^{-5t} + 3e^{2t}$$

6.7.12 Taking L.T. on both sides of eqns,

$$s^2\mathcal{L}(y_1) - s\underbrace{y_1(0)}_{=1} - \underbrace{y_1'(0)}_{=0} = -2\mathcal{L}(y_1) + 2\mathcal{L}(y_2),$$

$$s^2\mathcal{L}(y_2) - s\underbrace{y_2(0)}_{=3} - \underbrace{y_2'(0)}_{=0} = 2\mathcal{L}(y_1) - 5\mathcal{L}(y_2)$$

$$\Rightarrow \mathcal{L}(y_1)(s^2+2) + \mathcal{L}(y_2)(-2) = s$$

$$\mathcal{L}(y_1)(-2) + \mathcal{L}(y_2)(s^2+5) = 3s$$

$$\Rightarrow \underbrace{\begin{bmatrix} s^2+2 & -2 \\ -2 & s^2+5 \end{bmatrix}}_{:=A} \begin{bmatrix} \mathcal{L}(y_1) \\ \mathcal{L}(y_2) \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} s$$

$$\Rightarrow \begin{bmatrix} \mathcal{L}(y_1) \\ \mathcal{L}(y_2) \end{bmatrix} = \frac{1}{\underbrace{(s^2+2)(s^2+5)-4}_{=(s^2-6)(s^2+1)}} \begin{bmatrix} s^2+5 & 2 \\ 2 & s^2+2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} s$$

obtained 6 as a root using "rational roots theorem"!
1 obtained via quadratic roots formula (after polynomial division)

$$\Rightarrow \mathcal{L}(y_1) = \frac{((s^2+5)+6)s}{(s^2+6)(s^2+1)}$$

$$\mathcal{L}(y_2) = \frac{(2+(s^2+2)(3))s}{(s^2+6)(s^2+1)}$$

$\Rightarrow$ ... (partial fractions) ...

$$\Rightarrow \mathcal{L}(y_1) = s\left(\frac{-1}{s^2+6} + \frac{2}{s^2+1}\right)$$

$$\mathcal{L}(y_2) = s\left(\frac{2}{s^2+6} + \frac{1}{s^2+1}\right)$$

$$\Rightarrow y_1 = -\cos(\sqrt{6}\,t) + 2\cos(t)$$

$$y_2 = 2\cos(\sqrt{6}\,t) + \cos(t)$$