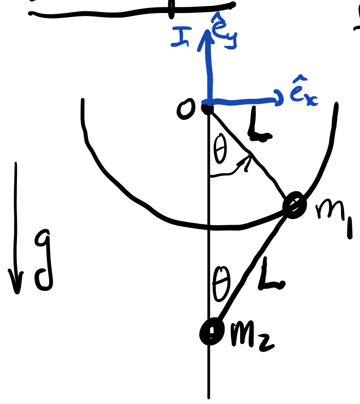
Example: Use the 1st form of D'Alembert's Principle to find the EOM(s) for a crank mechanism



DOF?

(x_1, y_1) and (x_2, y_2)

$$M = 2N - K = 2(2) - 3 = 1$$

$$x_1^2 + y_1^2 = L^2$$
$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = L^2$$
$$x_2 = 0$$

Kinematics

$$\vec{r}_{1/0} = L\sin\theta\,\hat{e}_x - L\cos\theta\,\hat{e}_y$$
$${}^I\vec{V}_{1/0} = L\dot{\theta}\cos\theta\,\hat{e}_x + L\dot{\theta}\sin\theta\,\hat{e}_y$$
$${}^I\vec{a}_{1/0} = L(\ddot{\theta}c\theta - \dot{\theta}^2 s\theta)\hat{e}_x + L(\ddot{\theta}s\theta + \dot{\theta}^2 c\theta)\hat{e}_y$$

$$\vec{r}_{2/0} = -2L\cos\theta\,\hat{e}_y$$
$${}^I\vec{V}_{2/0} = 2L\dot{\theta}\,s\theta\,\hat{e}_y$$
$${}^I\vec{a}_{2/0} = 2L(\ddot{\theta}s\theta + \dot{\theta}^2 c\theta)\hat{e}_y$$

D'Alembert's Principle: $\displaystyle\sum_{i=1}^{N}\left(\vec{F}_i^{(a)} - m_i\,{}^I\vec{a}_{i/0}\right)\cdot{}^I\delta\vec{r}_{i/0} = 0$

Applied forces:
$$\vec{W}_1 = -m_1 g\,\hat{e}_y$$
$$\vec{W}_2 = -m_2 g\,\hat{e}_y$$

How do we calculate ${}^{I}\delta\vec{r}_{1/0}$?

${}^{I}d\vec{r}_{1/0}$ real infinitesimal displacement

$${}^{I}\vec{v}_{1/0} = \frac{{}^{I}d\vec{r}_{1/0}}{dt}$$ ← Multiply this by dt to get ${}^{I}d\vec{r}_{1/0}$

$${}^{I}d\vec{r}_{1/0} = L\left(\frac{d\theta}{\cancel{dt}} c\theta\,\hat{e}_x + \frac{d\theta}{\cancel{dt}} s\theta\,\hat{e}_y\right)\cancel{dt}$$

$$= L\,d\theta\, c\theta\,\hat{e}_x + L\,d\theta\, s\theta\,\hat{e}_y$$

To get ${}^{I}\delta\vec{r}_{1/0}$, exchange all the differentials with δ's.

$$\delta\vec{r}_{1/0} = L\,\delta\theta\, c\theta\,\hat{e}_x + L\,\delta\theta\, s\theta\,\hat{e}_y$$

Similarly,

$${}^{I}\delta\vec{r}_{2/0} = 2L\,\delta\theta\, s\theta\,\hat{e}_y$$

Plugging in,

$$(m_1 g\,\hat{e}_y - m_1\,{}^{I}\vec{a}_{1/0})\cdot{}^{I}\delta\vec{r}_{1/0} + (m_2 g\,\hat{e}_y - m_2\,{}^{I}\vec{a}_{2/0})\cdot{}^{I}\delta\vec{r}_{2/0} = 0$$

$$\underbrace{m_1 g\,\hat{e}_y\cdot{}^{I}\delta\vec{r}_{1/0}}_{\text{Term \#1}} + \underbrace{m_1\,{}^{I}\vec{a}_{1/0}\cdot{}^{I}\delta\vec{r}_{1/0}}_{\text{Term \#2}} + \underbrace{m_2 g\,\hat{e}_y\cdot{}^{I}\delta\vec{r}_{2/0}}_{\text{Term \#3}} + \underbrace{m_2\,{}^{I}\vec{a}_{2/0}\cdot{}^{I}\delta\vec{r}_{2/0}}_{\text{Term \#4}} = 0$$

Term #1: $= m_1 g\,\hat{e}_y\cdot(L\,\delta\theta\, c\theta\,\hat{e}_x + L\,\delta\theta\, s\theta\,\hat{e}_y)$

$$= m_1 g L\,\delta\theta\, s\theta$$

Term #2:

$$= m_1 \, {}^{I}\vec{a}_{1/0} \cdot {}^{I}\delta\vec{r}_{1/0}$$

$$= m_1\left(L(\ddot{\theta}c\theta - \dot{\theta}^2 s\theta)\hat{e}_x + L(\ddot{\theta}s\theta + \dot{\theta}^2 c\theta)\hat{e}_y\right)\cdot\left(L\delta\theta c\theta\,\hat{e}_x + L\delta\theta s\theta\,\hat{e}_y\right)$$

$\vdots$ (Uses a trig identity)

$$= m_1 L^2 \ddot{\theta}\,\delta\theta$$

Similarly, Term #3 $= 2m_2 g L \delta\theta s\theta$

$$\text{Term \#4} = 4m_2 L^2(\ddot{\theta}s^2\theta + \dot{\theta}^2 s\theta c\theta)\delta\theta$$

Plug in all terms,

$$0 = m_1 L^2\ddot{\theta}\,\delta\theta + 4m_2 L^2(\ddot{\theta}s^2\theta + \dot{\theta}^2 s\theta c\theta)\delta\theta + m_1 g L\delta\theta s\theta + 2m_2 g L\delta\theta s\theta$$

Factor out $\delta\theta$, $(\ldots\text{stuff}\ldots)\delta\theta = 0 \quad \forall\,\delta\theta \Rightarrow \ldots\text{stuff}\ldots = 0$

$$\boxed{(m_1 + 4m_2 s^2\theta)L\ddot{\theta} + 4m_2 L\dot{\theta}^2 s\theta c\theta + (m_1 + 2m_2)g\,s\theta = 0}$$

E.O.M.

Notes:

- D'Alembert's Principle can in fact be used to solve problems
- However, it is a bit tedious because many of the calculations are similar to N2L (e.g. we calculated accelerations)
- We didn't need constraint forces!

D'Alembert's Principle (cont'd)

Let's derive another form of D'Alembert's Principle that is easier to work with.

↳ Let's put it in terms of kinetic energy

$$\underbrace{\sum_{i=1}^{N} \vec{F}_i^{(a)} \cdot \delta \vec{r}_{i/0}}_{\text{Applied force contribution}} - \underbrace{\sum_{i=1}^{N} {}^{I}\frac{d}{dt}\left({}^{I}\vec{P}_{i/0}\right) \cdot \delta \vec{r}_{i/0}}_{\text{inertial force contribution to virtual work}} = 0$$

We can write virtual displacements in terms of generalized coordinates:

$$\delta \vec{r}_{i/0} = \sum_{j=1}^{N_c} \frac{\partial \vec{r}_{i/0}}{\partial q_j} \delta q_j$$

(Recall: We froze time for virtual displacements so there is no $\frac{\partial \vec{r}_{i/0}}{\partial t}$ term).

Rewriting the applied force contribution of virtual work,

$$\sum_{i=1}^{N} \vec{F}_i^{(a)} \cdot \delta \vec{r}_{i/0} = \sum_{j=1}^{N_c} \boxed{\sum_{i=1}^{N} \vec{F}^{(a)} \cdot \frac{\partial \vec{r}_{i/0}}{\partial q_j}} \delta q_j$$

↳ $\triangleq Q_j$

Generalized force (Projects the particle's forces onto a single degree of freedom of the system).

$$\boxed{Q_j \triangleq \sum_{i=1}^{N} \vec{F}^{(a)} \cdot \frac{\partial \vec{r}_{i/0}}{\partial q_j}}$$

Let's look at the inertial force contribution:

$$\sum_{i=1}^{N} \frac{{}^{I}d}{dt}\left({}^{I}\vec{p}_{i/o}\right) \cdot {}^{I}\delta\vec{r}_{i/o} = \sum_{i=1}^{N} m_i\, {}^{I}\vec{a}_{i/o} \cdot {}^{I}\delta\vec{r}_{i/o} = \sum_{i=1}^{N}\sum_{j=1}^{N_c} \boxed{m_i\, {}^{I}\vec{a}_{i/o} \cdot \frac{\partial \vec{r}_{i/o}}{\partial q_j}}\, \delta q_j$$

Aside: Look at the product rule,

$$\frac{{}^{I}d}{dt}\left(m_i\, {}^{I}\vec{v}_{i/o} \cdot \frac{\partial \vec{r}_{i/o}}{\partial q_j}\right) = \boxed{m_i\, {}^{I}\vec{a}_{i/o} \cdot \frac{\partial \vec{r}_{i/o}}{\partial q_j}} + m_i\, {}^{I}\vec{v}_{i/o} \cdot \frac{{}^{I}d}{dt}\left(\frac{\partial \vec{r}_{i/o}}{\partial q_j}\right)$$

Re-arrange the product rule and sum over i,

$$\sum_{i=1}^{N} m_i\, {}^{I}\vec{a}_{i/o} \cdot \frac{\partial \vec{r}_{i/o}}{\partial q_j} = \sum_{i=1}^{N}\left(\frac{{}^{I}d}{dt}\left(m_i\, {}^{I}\vec{v}_{i/o} \cdot \underbrace{\frac{\partial \vec{r}_{i/o}}{\partial q_j}}_{\text{Term A}}\right) - m_i\, {}^{I}\vec{v}_{i/o} \cdot \underbrace{\frac{{}^{I}d}{dt}\left(\frac{\partial \vec{r}_{i/o}}{\partial q_j}\right)}_{\text{Term B}}\right)$$

Reminder: We are trying to work in kinetic energy terms, so we only want velocities to appear.

A useful identity for doing this is,

$$\frac{\partial \vec{r}_{i/o}}{\partial q_j} = \frac{\partial\left({}^{I}\vec{v}_{i/o}\right)}{\partial \dot{q}_j}$$

(See Hand & Finch, "How to cancel the dots")

recall $\vec{r}_{i/o} = \vec{r}_{i/o}(q_1, \ldots, q_{N_c}, t)$

For Term A:

$$\frac{\partial \vec{r}_{i/o}}{\partial q_j} = \frac{\partial}{\partial \dot{q}_j}\left(\underbrace{\sum_{k=1}^{N_c} \frac{\partial \vec{r}_{i/o}}{\partial q_k}\dot{q}_k} + \frac{\partial \vec{r}_{i/o}}{\partial t}\right) = \frac{\partial}{\partial \dot{q}_j}\left(\frac{{}^{I}d\vec{r}_{i/o}}{dt}\right)$$

Note: $\dfrac{\partial \dot{q}_k}{\partial \dot{q}_j} = \delta_{jk} = \begin{cases} 1 & \text{if } j=k \\ 0 & \text{otherwis} \end{cases}$

$$= \frac{\partial}{\partial \dot{q}_j}\left(\frac{{}^{I}d\,\vec{r}_{i/0}}{dt}\right) = \frac{\partial}{\partial \dot{q}_j}\left({}^{I}\vec{v}_{i/0}\right)$$

For Term B:

$$\frac{{}^{I}d}{dt}\left(\frac{\partial \vec{r}_{i/0}}{\partial q_j}\right) = \frac{\partial}{\partial q_j}\left(\frac{{}^{I}d}{dt}\vec{r}_{i/0}\right) = \frac{\partial}{\partial q_j}\left({}^{I}\vec{v}_{i/0}\right)$$

Plug in, (inertial force contribution to virtual work)

$$\sum_{i=1}^{N} m_i\, {}^{I}\vec{a}_{i/0} \cdot \frac{\partial \vec{r}_{i/0}}{\partial q_j} = \sum_{i=1}^{N}\left[\frac{{}^{I}d}{dt}\left(\boxed{m_i\, {}^{I}\vec{v}_{i/0} \cdot \frac{\partial\, {}^{I}\vec{v}_{i/0}}{\partial \dot{q}_j}}\right) - \boxed{m_i\, {}^{I}\vec{v}_{i/0} \cdot \frac{\partial\, {}^{I}\vec{v}_{i/0}}{\partial q_j}}\right]$$

Note that:

$$\frac{\partial}{\partial \dot{q}_j}\left(m_i \left\|{}^{I}\vec{v}_{i/0}\right\|^2\right) = \frac{\partial}{\partial \dot{q}_j}\left(m_i\, {}^{I}\vec{v}_{i/0} \cdot {}^{I}\vec{v}_{i/0}\right) = 2\, \boxed{m_i\, {}^{I}\vec{v}_{i/0} \cdot \frac{\partial\, {}^{I}\vec{v}_{i/0}}{\partial \dot{q}_j}}$$

Also,

$$\frac{\partial}{\partial q_j}\left(m_i \left\|{}^{I}\vec{v}_{i/0}\right\|^2\right) = 2\, \boxed{m_i\, {}^{I}\vec{v}_{i/0} \cdot \frac{\partial\, {}^{I}\vec{v}_{i/0}}{\partial q_j}}$$

The overall inertial force contribution to the virtual work can be rewritten:

$$\sum_{j=1}^{m}\left({}^{I}\frac{d}{dt}\left[\frac{\partial}{\partial\dot{q}_j}\underbrace{\left(\sum_{i=1}^{N}\frac{1}{2}m_i\|\vec{v}_{i/0}\|^2\right)}_{\sum_{i=1}^{N}T_{i/0}=T_0}\right]-\frac{\partial}{\partial q_j}\underbrace{\left(\sum_{i=1}^{N}\frac{1}{2}m_i\|{}^{I}\vec{v}_{i/0}\|^2\right)}_{T_0}\right)\delta q_j$$

Together, both contributions (applied force + inertial force) to virtual work give:

$$\boxed{\sum_{j=1}^{N_c}\left({}^{I}\frac{d}{dt}\left(\frac{\partial T_0}{\partial\dot{q}_j}\right)-\frac{\partial T_0}{\partial q_j}-Q_j\right)\delta q_j=0}$$

D'Alembert's Principle (#2 of 2)

This form of D'Alembert's Principle is a stepping stone to Lagrange's Equations.

Euler–Lagrange Equations

If the constraints are **holonomic**, then it is possible to find a set of independent q_j's. Since the δq_j's are independent, each individual coefficient must vanish.

(Imagine taking a virtual disp. such that $\delta q_k \neq 0$, but $\delta q_j = 0 \quad \forall j \neq k$)

$$\Rightarrow \boxed{\frac{{}^{I}d}{dt}\left(\frac{\partial T_0}{\partial \dot{q}_j}\right) - \frac{\partial T_0}{\partial q_j} = Q_j}$$

$$\forall j = 1, \ldots, M$$

Lagrange's Eqns
or
Euler–Lagrange equations
(Form #1 of 2)

Let's break up the generalized forces Q_j into conservative and nonconservative forces.

$$Q_j = Q_j^{(c)} + Q_j^{(nc)}$$

$$Q_j^{(c)} \triangleq \sum_{i=1}^{N} \vec{F}_i^{(c)} \cdot \frac{\partial \vec{r}_{i/0}}{\partial q_j} = \sum_{i=1}^{N} \left(-\nabla U_i \cdot \frac{\partial \vec{r}_{i/0}}{\partial q_j}\right)$$

$$Q_j^{(c)} = -\sum_{i=1}^{N} \frac{\partial u_i}{\partial q_j} \quad \text{(Note: chain rule used here)}$$

$$= \frac{-\partial U_0}{\partial q_j} \quad \text{where} \quad U_0 = \sum_{i=1}^{N} u_i \quad \text{and} \quad u_i = u_i(\vec{r}_{i/O}) \text{ only}$$

Recall the first form of Lagrange's equations:

$$ {}^{I}\frac{d}{dt}\left(\frac{\partial T_0}{\partial \dot{q}_j}\right) - \frac{\partial T_0}{\partial q_j} = Q_j^{(c)} + Q_j^{(nc)}$$

$$= \frac{-\partial U_0}{\partial q_j} + Q_j^{(nc)}$$

$$\Rightarrow \quad {}^{I}\frac{d}{dt}\left(\frac{\partial T_0}{\partial \dot{q}_j}\right) - \frac{\partial}{\partial q_j}\left(T_0 - U_0\right) = Q_j^{(nc)}$$

Define $L \triangleq T_0 - U_0$

And note $\frac{\partial T_0}{\partial \dot{q}_j} = \frac{\partial L_0}{\partial \dot{q}_j}$ Since $\frac{\partial U_0}{\partial \dot{q}_j} = 0$

$$\Rightarrow \quad {}^{I}\frac{d}{dt}\left(\frac{\partial L_0}{\partial \dot{q}_j}\right) - \frac{\partial L_0}{\partial q_j} = Q_j^{(nc,a)}$$

Let $L \triangleq T_0 - U_0$ be the Lagrangian

Then we can rewrite the 1st form of the E.L. equations:

$$\boxed{{}^{I}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j^{(nc,a)} \quad \forall j = 1, \ldots, N_c}$$

Lagrange's Equations or the Euler-Lagrange Eqns
(Form #2 of 2)

Notes:

1) For holonomic constraints
2) Notice that the conservative forces were moved into the $\frac{\partial L}{\partial q_j}$ term → Don't even have to handle forces like gravity or springs explicitly.