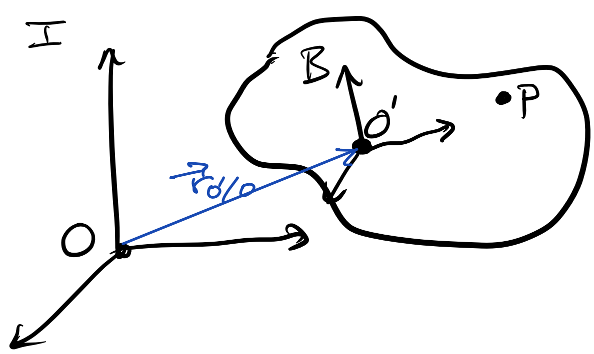
Rigid body motion in $\mathbb{R}^3$

We use RBT's to represent the instantaneous position and velocity of a body frame w.r.t. the inertial frame.

- We already discussed rotation
- Translation is trivial (just keep track of a point on the body)
- Let's put them together



The $\underline{\text{configuration}}$ of the body $(\vec{r}_{o'/o}, {}^{I}C^{B})$

The $\underline{\text{configuration space}}$ is $\mathbb{R}^3 \times SO(3) \triangleq SE(3)$

"Special Euclidean Group"

$$SE(3) \triangleq \left\{ (\vec{r}, R) \mid \vec{r} \in \mathbb{R}^3 \text{ and } R \in SO(3) \right\}$$

An element of SE(3) serves as both a specification of the configuration of the RB. and a transformation that takes coordinates of a point in one frame to another.

$$[\vec{r}_{P/O}]_I = [\vec{r}_{O'/O}]_I + {}^I C^B [\vec{r}_{P/O'}]_B$$

affine representation

Homogeneous representation

$$\begin{bmatrix} \vec{r}_{P/O} \\ 1 \end{bmatrix}_I = \underbrace{\begin{bmatrix} {}^I C^B & \vec{r}_{O'/O} \\ \underset{\sim}{0}^T & 1 \end{bmatrix}}_{SE(3)\ \text{element}} \begin{bmatrix} \vec{r}_{P/O'} \\ 1 \end{bmatrix}_B$$

$\underset{\sim}{0}^T \to [0\ 0\ 0]$

$y = Ax$

SE(3) element 4x4 ${}^I g^B \in SE(3)$

In general if $(\vec{r}, R) \in SE(3)$, then

$$g = \begin{bmatrix} R & \vec{r} \\ \underset{\sim}{0}^T & 1 \end{bmatrix}$$

$\underset{\sim}{0}^T \to 0_{1\times3}$

$\boxed{\text{Homogeneous representation of an element in SE(3)}}$

SE(3) as a Lie Group

$SE(3) = \{(r, R) \mid r \in \mathbb{R}^3 \text{ and } R \in SO(3)\}$

$$(r, R) \xleftrightarrow{\text{correspond}} g = \begin{bmatrix} R & r \\ \underline{0}^T & 1 \end{bmatrix}$$

Property SE(3) Transformations

$${}^{I}g^{C} = ({}^{I}g^{B})({}^{B}g^{C})$$

Proof:

① Closure Let $\begin{bmatrix} R_1 & \underline{a} \\ \underline{0}^T & 1 \end{bmatrix}$ and $\begin{bmatrix} R_2 & \underline{b} \\ \underline{0}^T & 1 \end{bmatrix} \in SE(3)$

$$\begin{bmatrix} R_1 & \underline{a} \\ \underline{0}^T & 1 \end{bmatrix}\begin{bmatrix} R_2 & \underline{b} \\ \underline{0}^T & 1 \end{bmatrix} = \begin{bmatrix} R_1 R_2 & R_1\underline{b} + \underline{a} \\ \underline{0}^T & 1 \end{bmatrix}$$

$R_1\underline{b} + \underline{a} \in \mathbb{R}^3$

$R_1 R_2 \in SO(3)$ since $R_1, R_2 \in SO(3)$ and $SO(3)$ is closed under matrix multiplication

$\Rightarrow$ closure

② Identity $\begin{bmatrix} I_{3\times3} & \underline{0}_{3\times1} \\ \underline{0}^T & 1 \end{bmatrix} \in SE(3)$ because

$$\begin{bmatrix} I_{3\times3} & \underline{0}_{3\times1} \\ 0_{1\times3} & 1 \end{bmatrix}\begin{bmatrix} R & \underline{b} \\ 0_{1\times3} & 1 \end{bmatrix} = \begin{bmatrix} R & b \\ 0_{1\times3} & 1 \end{bmatrix}\begin{bmatrix} I_{3\times3} & \underline{0}_{3\times1} \\ 0_{1\times3} & 1 \end{bmatrix} = \begin{bmatrix} R & b \\ \underset{\underset{\underline{0}^T}{\updownarrow}}{0_{1\times3}} & 1 \end{bmatrix}$$

③ Inverse

We want to solve for this to see what it is and if it is in SE(3)

$$\begin{bmatrix} R_1 & \underline{a} \\ \underline{0}^T & 1 \end{bmatrix}\overbrace{\begin{bmatrix} R_2 & \underline{b} \\ \underline{0}^T & 1 \end{bmatrix}} = \begin{bmatrix} I & \underline{0} \\ \underline{0}^T & 1 \end{bmatrix}$$

$$\begin{bmatrix} R_1R_2 & R_1\underline{b}+\underline{a} \\ \underline{0}^T & 1 \end{bmatrix} = \begin{bmatrix} I & \underline{0} \\ \underline{0}^T & 1 \end{bmatrix} \Rightarrow$$

$R_1\underline{b} + \underline{a} = \underline{0}$

$\underline{b} = -R_1^{-1}\underline{a}$

and

$R_1R_2 = I$

$\Rightarrow R_2 = R_1^{-1}$

So, the inverse element is

$$\begin{bmatrix} R_1^{-1} & -R_1^{-1}\underline{a} \\ \underline{0}^T & 1 \end{bmatrix} \in SE(3)$$

④ Associativity $g_1(g_2g_3) = (g_1g_2)g_3$

under matrix multiplication

(check yourself)

SE(3) elements are rigid body transformations

① ${}^{I}g^{B}$ preserves distance

$$\left\| {}^{I}g^{B}\begin{bmatrix}\vec{r}_{P/O}\\1\end{bmatrix}_B - {}^{I}g^{B}\begin{bmatrix}\vec{r}_{Q/O}\\1\end{bmatrix}_B \right\| \stackrel{?}{=} \left\| \vec{r}_{P/O} - \vec{r}_{Q/O} \right\|$$

$$\left\| \begin{bmatrix}R & \underline{a}\\ \underline{0}^T & 1\end{bmatrix}\begin{bmatrix}\vec{r}_{P/O}\\1\end{bmatrix}_B - \begin{bmatrix}R & \underline{a}\\ \underline{0}^T & 1\end{bmatrix}\begin{bmatrix}\vec{r}_{Q/O}\\1\end{bmatrix}_B \right\| \stackrel{?}{=}$$

$$\left\| \begin{bmatrix}R\vec{r}_{P/O} + \underline{a}\\1\end{bmatrix} - \begin{bmatrix}R\vec{r}_{Q/O} + \underline{a}\\1\end{bmatrix} \right\| \stackrel{?}{=}$$

$$\left\| \begin{bmatrix}R\overbrace{(\vec{r}_{P/O} - \vec{r}_{Q/O})}^{\vec{r}_{P/Q}}\\0\end{bmatrix} \right\| \stackrel{?}{=}$$

$$\|R\vec{r}_{P/Q}\| = \sqrt{\vec{r}_{P/Q}^{\,T}\underbrace{R^TR}_{I}\vec{r}_{P/Q}} = \|\vec{r}_{P/Q}\|$$

$$\|\vec{a}\| = \sqrt{\vec{a}\cdot\vec{a}} = \sqrt{\vec{a}^T\vec{a}}$$

$$\|R\vec{a}\| = \sqrt{(R\vec{a})^T(R\vec{a})}$$

$$\sqrt{\vec{a}^TR^TR\vec{a}}$$

② ${}^{I}g^{B}$ preserves orientation

$${}^{I}g^{B}\begin{bmatrix}\vec{a}\times\vec{b}\\1\end{bmatrix}_B \stackrel{?}{=} \left({}^{I}g^{B}\begin{bmatrix}\vec{a}\\1\end{bmatrix}_B\right)\times\left({}^{I}g^{B}\begin{bmatrix}\vec{b}\\1\end{bmatrix}_B\right)$$

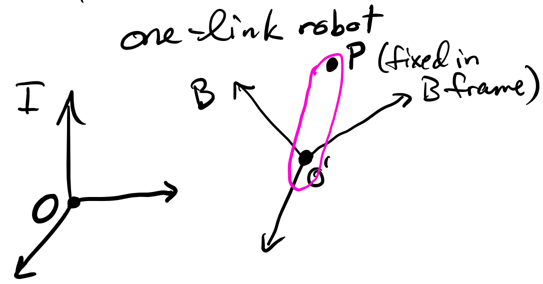
(check yourself)

③ ${}^{I}g^{B}$ preserves dot product

$${}^{I}g^{B}\left(\begin{bmatrix}\vec{a}\\1\end{bmatrix}_B\cdot\begin{bmatrix}\vec{b}\\1\end{bmatrix}_B\right) \stackrel{?}{=} \left({}^{I}g^{B}\begin{bmatrix}\vec{a}\\1\end{bmatrix}_B\right)\cdot\left({}^{I}g^{B}\begin{bmatrix}\vec{b}\\1\end{bmatrix}_B\right)$$

(check yourself)

Exponential coordinates for Rigid Body Transformations



Kinematics for point P on rigid body:

$$ {}^I\frac{d}{dt}\left(\vec{r}_{P/O}\right) = \underbrace{{}^I\frac{d}{dt}\left(\vec{r}_{O'/O}\right)}_{{}^I\vec{v}_{O'/O}} + {}^I\vec{\omega}^B \times \left(\vec{r}_{P/O} - \vec{r}_{O'/O}\right) $$

Let's try to re-arrange this and put it into a homogeneous representation. (Because we want to work towards the matrix exp.)

$$ {}^I\frac{d}{dt}\left(\begin{bmatrix} \vec{r}_{P/O} \\ 1 \end{bmatrix}_I\right) = \underbrace{\left[\begin{array}{c|c} \left[\widehat{{}^I\vec{\omega}^B}\right] & {}^I\vec{v}_{O'/O} - {}^I\vec{\omega}^B \times \vec{r}_{O'/O} \\ \hline \underline{0}^T & 0 \end{array}\right]}_{} \begin{bmatrix} \vec{r}_{P/O} \\ 1 \end{bmatrix}_I $$

$\triangleq \left[\widehat{{}^I\xi^B}\right]$ in K&P style — **Twist**

$\triangleq \hat{\xi}$ in MLS style — a member of "little se(3)"

Notes on Notation

- Recall $\widehat{{}^I\omega^B}$ means cross product equivalent matrix
- We are now going to generalize the hat notation (also called a wedge operation).
- The wedge operator "$\wedge$" maps a vector of entries to a corresponding Lie algebra element. You will need to determine the correct Lie algebra from context
- There is a vee "$\vee$" operation that maps from the Lie algebra element to a vector of entries:

(will see in two pages what vee is)

The homogeneous representation gave us an LTI differential equation:

$$\underline{\dot{x}} = A\,\underline{x} \longleftrightarrow \underline{x}(t) = e^{At}\,\underline{x}(0)$$

$$\begin{bmatrix} \vec{r}_{P/O}(t) \\ 1 \end{bmatrix}_I = \underbrace{e^{\hat{\xi} t}}_{\text{Matrix exponential}} \begin{bmatrix} \vec{r}_{P/O}(0) \\ 1 \end{bmatrix}_I$$

$e^{\hat{\xi} t}$ is an element of $SE(3)$

$\hat{\xi}$ is an element of little $se(3)$, the associated Lie algebra.

Analogous to little $so(3)$, we have little $se(3)$:

$$se(3) \triangleq \{ (\vec{v}, \hat{\omega}) \mid \vec{v} \in \mathbb{R}^3 \text{ and } \hat{\omega} \in so(3) \}$$

The homogeneous representation is:

$$(\vec{v}, \hat{\omega}) \xleftrightarrow{\text{corresponds to}} \underbrace{\hat{\xi} = \begin{bmatrix} \hat{\omega} & \vec{v} \\ \underline{0}^T & 0 \end{bmatrix}}_{\text{Twist}}$$

Again we see that the matrix exponential is a mapping that takes elements of a Lie Algebra and maps them to elements of a Lie Group.

$$so(3) \xrightarrow{\text{expm}} SO(3)$$

$$se(3) \xrightarrow{\text{expm}} SE(3)$$

Twist $\quad \hat{\xi} = \begin{bmatrix} \hat{\omega} & \vec{v} \\ \underline{0}^T & 0 \end{bmatrix} \in se(3)$

We can use the vee "$\vee$" notation to extract <u>twist cordinates</u>

$$\begin{bmatrix} \hat{\omega} & \vec{v} \\ \underline{0}^T & 0 \end{bmatrix}^\vee = \begin{bmatrix} \vec{v} \\ \vec{\omega} \end{bmatrix} = \begin{bmatrix} \xi \end{bmatrix}$$

twist cordinates $\in \mathbb{R}^6$

MLS: ξ twist cordinates

Example: In the <u>plane</u>, the wedge operator would map

$$\mathbb{R}^3 \longrightarrow se(2)$$

$\begin{bmatrix} \vec{v} \\ \omega \end{bmatrix} \begin{matrix} \in \mathbb{R}^2 \\ \in \mathbb{R} \end{matrix}$

homogeneous representation

$\begin{bmatrix} \hat{\omega} & \vec{v} \\ \underline{0}^T & 0 \end{bmatrix} \in \mathbb{R}^{3\times 3}$

Exponential map of a twist creates the relative motion of a rigid body

$$\hat{\xi} \in se(3) \text{ and } \theta \in \mathbb{R}$$

Then $e^{\hat{\xi}\theta} \in SE(3)$

In turns out that (see MLS)

Every rigid body transformation $g \in SE(3)$ can be written as an exponential of some twist $\hat{\xi}\theta$ where $\xi\theta \in \mathbb{R}^6$ are the <u>exponential cordinates</u>

<u>Chasles' Thm:</u> "Every rigid body motion can be realized by a rotation about an axis, combined with a translation parallel to that axis"

↳ This is called screw motion

Note: The axis does not have to go through the body.