Control System Analysis and Design Via the "Second Method" of Lyapunov[1]

II Discrete-Time Systems

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The second method of Lyapunov is applied to the study of discrete-time (sampled-data) systems. With minor variations, the discussion parallels that of the companion paper on continuous-time systems. Theorems are stated in full but motivation, proofs, examples, and so on, are given only when they differ materially from their counterparts in the continuous-time case.

1—Introduction

We apply here the considerations of the companion paper,[3] "Control System Analysis and Design Via the 'Second Method' of Lyapunov. I Continuous-Time Systems," to the study of dynamic systems in which the time variable changes discretely, i.e., those governed by ordinary *difference* (rather than differential) equations.

In order to avoid needless duplication, all equations, theorems, examples marked by an asterisk will refer to corresponding items of the companion paper. Other equations, etc., will be numbered consecutively. All theorems will be stated in full, but examples, proofs, etc., will be given only if they differ in a nontrivial way from their continuous-time counterparts. A few results peculiar to the discrete-time case are included, however, with full discussion.

The need for a separate presentation is due to the increasing importance of sampled-data systems in connection with digital-computer control. The development of the study of continuous-time and discrete-time systems has tended to drift apart somewhat as a result of the popularity of the z-transform approach to the latter [1].[4] We shall endeavor to emphasize here the essential unity as well as some of the peculiar contrasts of the two subjects, from a point of view rather different from the z-transform.

2—Outline of Contents

In Section 4 we present a method of description of discrete-time dynamic systems which is analogous to the terminology now current in mathematics for the description of continuous-time dynamic systems. The important concepts are those of the state and state transition—these notions are indispensable not only for the application of the "second method" but even for a rigorous definition of stability. Section 5 recalls the customary definitions of stability; these are fully identical with the ones used in the continuous-time case.

Section 6 presents the main theorems on which the "second method" is based. Again, the analogy with the continuous-time case is complete and the reader is referred to the companion paper[2] for motivation and proofs.

Section 7 deals with applications to stability theory. Example 6* presents the derivation of the Routh-Hurwitz inequalities via the "second method." Since these inequalities are not well known in the discrete-time case, a full derivation is given. Theorem 4* and Example 2 illustrate the concept of a *contraction;* this simple notion leads to useful nonlinear stability criteria which are at present almost unknown in the sampled-data systems literature.

Sections 8, 9, and 10 are analogous with those of the companion paper, with minor variations in the details of the derivations.

3—Guide to Western Literature

There is very little literature on the stability theory of systems governed by difference equations, since such problems are usually trivial special cases of the stability theory of continuous-time systems. An early paper is that of Ta Li [2]. Recently, Hahn [3] has provided a self-contained summary of the application of the "second method" of Lyapunov to difference equations; this material is also included in his book [4].

An important prerequisite to the application of the "second method" is that system equations must be given from the "state" point of view. Often these equations cannot be obtained in a *natural* and convenient way from the z-transform [1] type of description of discrete-time systems, particularly when the system contains dynamic elements of two types—those governed by difference or by differential equations. A detailed treatment of the derivation of the discrete-time dynamic equations in cases of current practical interest has been given recently by the authors [5]. This paper provides a complete treatment of the linear case, including the analysis of sampling operations of complicated type (see also Reference [6]).

4—Description of Discrete-Time Dynamic Systems

Small boldface Roman or Greek letters will denote vectors. Capital boldface Roman or Greek letters will denote matrices. Unless otherwise specified, all scalars and all elements of vectors and matrices will be real numbers. The prime denotes the transpose of a vector or matrix; thus the inner (scalar) product is written as $\mathbf{x}'\mathbf{y}$ and a quadratic form as $\mathbf{x}'\mathbf{A}\mathbf{x}$. The norm is denoted by $\|\mathbf{x}\|$; in specific calculations this means always the Euclidean norm $(\mathbf{x}'\mathbf{x})^{1/2}$. The eigenvalues of a matrix $\mathbf{F}$ are $\lambda_i(\mathbf{F})$. The unit matrix is $\mathbf{I}$.

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[3] R. E. Kalman and J. E. Bertram, "Control System Analysis and Design Via the 'Second Method' of Lyapunov. I Continuous-Time Systems," published in this issue, pp. 371–393.

[4] Numbers in brackets designate References at end of paper.

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In this paper, we shall study systems governed by the *vector difference* equation.

$$\mathbf{x}(t_{k+1}) = \mathbf{h}(\mathbf{x}(t_k), \mathbf{u}(t_k), t_k) \tag{5*}$$

where the t_k indicate discrete values of time (k = integer)

$$-\infty < \ldots, t_{k-1} < t_k < t_{k+1}, \ldots < +\infty; \quad t_k \to \infty \text{ when } k \to \infty$$

at which the behavior of the system can be or is observed; t_k is regarded as an independent variable analogous to t in the continuous-time case. Of course, (5*) is equivalent to the set of n scalar difference equations

$$x_i(t_{k+1}) = h_i(x_1(t_k), \ldots, x_n(t_k), u_1(t_k), \ldots, u_m(t_k), t_k) \quad (5^*)$$

$$i = 1, \ldots, n$$

The vector $\mathbf{x}$ is the *state* of the system (5*), its components x_i are the *state variables*. The vector $\mathbf{u}(t_k)$ is the *control* function (or *forcing function* or *input*) of the system (5*); its components $u_i(t)$ are the *control variables*. The system is specified by the vector-valued function $\mathbf{h}$. It is always assumed that $\mathbf{h}$ is a 1-to-1 function for any fixed $\mathbf{u}$ and t_k, continuous in all of its arguments. The integer n is the *order* of the system. Usually, $m < n$. To assure that (5*) represents a physical system, we must require also that if $t_{k+1} - t_k \to 0$ as $k \to \infty$, then $\mathbf{h}(\mathbf{x}, \mathbf{u}, t_k) \to \mathbf{x}$.

If $\mathbf{u}(t_k) \equiv \mathbf{0}$ for all t_k, we say that (5*) is *free* (*unforced*):

$$\mathbf{x}(t_{k+1}) = \mathbf{h}(\mathbf{x}(t_k), t_k) \tag{5*-F}$$

Equation (5*) defines a *discrete-time dynamic system*. Unlike in the case of continuous-time dynamic systems which are usually specified by their infinitesimal state-transitions (i.e., by differential equations) there is no difficulty (i.e., no finite escape time) here to obtain from the primary description (5*) *all* possible state transitions.

For any initial state $\mathbf{x}_0$, any initial time t_0, and any time t_k, we define by induction the function

$$\boldsymbol{\phi}(t_k;\ \mathbf{x}_0, t_0) = \mathbf{x}(t_k)$$

satisfying

$$\boldsymbol{\phi}(t_0;\ \mathbf{x}_0, t_0) = \mathbf{x}_0 \quad \text{for all} \quad \mathbf{x}_0, t_0 \tag{7*}$$

$$\boldsymbol{\phi}(t_{k+1};\ \mathbf{x}(t_k), t_k) = \mathbf{h}(\mathbf{x}(t_k), \mathbf{u}, t_k) \quad \text{for all} \quad \mathbf{x}(t_k),\ t_k,$$

and some *fixed* sequence of values $\mathbf{u}(t_0), \ldots, \mathbf{u}(t_k), \ldots$ of the forcing function

We call $\boldsymbol{\phi}(t_k;\ \mathbf{x}_0, t_0)$ (where t_k takes on all possible discrete values) the *motion* (or *trajectory*) of the system (5*) going through state $\mathbf{x}_0$ at time t_0. Since the function $\mathbf{h}$ in (5*) is unique and one-to-one, it follows that for any $\mathbf{x}_0$ and any discrete values t_c, t_b, t_a of time

$$\boldsymbol{\phi}(t_c;\ \mathbf{x}_0, t_a) = \boldsymbol{\phi}(t_c;\ \boldsymbol{\phi}(t_b;\ \mathbf{x}_0, t_a), t_b) \tag{8*}$$

If $\mathbf{h}$ is not one-to-one, (8*) holds if $t_a \leqq t_b \leqq t_c$. It is usually not necessary to indicate explicitly the dependence of the motion on the forcing function.

A state $\mathbf{x}_e$ of a free dynamic system (5*-F) is an *equilibrium state* if

$$\mathbf{h}(\mathbf{x}_e, t_k) = \mathbf{x}_e \quad \text{for all} \quad t_k \tag{10*}$$

A dynamic system (5*) is *stationary* if $\mathbf{h}(\mathbf{x}, \mathbf{u}, t_k) \equiv \mathbf{h}(\mathbf{x}, \mathbf{u})$, i.e., $\mathbf{h}$ is not an explicit function of t_k. Then we take for simplicity $t_{k+1} - t_k$ = constant for all k. If a system is both free and stationary, it is *autonomous*.

A dynamic system (5*) is *linear* if $\mathbf{h}$ is a linear function of $\mathbf{x}$ and $\mathbf{u}$.

If $\mathbf{x}_e = \mathbf{0}$ is an equilibrium state and if (5*) is linear, then we can write

$$\mathbf{x}(t_{k+1}) = \mathbf{H}(t_k)\mathbf{x}(t_k) + \boldsymbol{\Delta}(t_k)\mathbf{u}(t_k) \tag{5*-L}$$

where

$$\mathbf{H}(t_k) = [\partial h_i(t_k)/\partial x_j]$$

$$\boldsymbol{\Delta}(t_k) = [\partial h_i(t_k)/\partial u_j]$$

Let us define

$$\begin{aligned} \boldsymbol{\Phi}(t_k, t_k) &= \mathbf{I} \quad \text{for all} \quad t_k \\ \boldsymbol{\Phi}(t_{k+1}, t_l) &= \mathbf{H}(t_k) \ldots \mathbf{H}(t_l) \quad \text{for all} \quad t_k \geqq t_l \end{aligned} \tag{7*-L}$$

As in the continuous case, the matrix $\boldsymbol{\Phi}(t_k, t_0)$ is called the *transition matrix* of (5*-L); but now its arguments t_k, t_0 are defined for discrete values of time only.

Now we can write a general expression for motions of (5*-L), valid for all $t_k \geqq t_0$

$$\boldsymbol{\phi}(t_k;\ \mathbf{x}_0, t_0) = \boldsymbol{\Phi}(t_k, t_0)\mathbf{x}_0 + \sum_{t_0 \leqq t_i < t_k} \boldsymbol{\Phi}(t_k, t_i)\boldsymbol{\Delta}(t_i)\mathbf{u}(t_i) \tag{6*-L}$$

If $\mathbf{H}(t_k)$ is nonsingular for every t_k, then its inverse is

$$\mathbf{H}^{-1}(t_k) = \boldsymbol{\Phi}^{-1}(t_{k+1}, t_k) = \boldsymbol{\Phi}(t_k, t_{k+1})$$

Then the transition matrix $\boldsymbol{\Phi}(t_k, t_l)$ is defined also for $t_l > t_k$ and we can write for any discrete t_a, t_b, t_c,

$$\boldsymbol{\Phi}(t_c, t_a) = \boldsymbol{\Phi}(t_c, t_b)\boldsymbol{\Phi}(t_b, t_a) \tag{8*-L}$$

and also

$$\boldsymbol{\Phi}^{-1}(t_a, t_b) = \boldsymbol{\Phi}(t_b, t_a) \tag{9*}$$

If $\mathbf{H}(t_k)$ is singular, then (8*-L) holds if we require that $t_a \leqq t_b \leqq t_c$.

If the linear system (5*-L) is stationary, then $\mathbf{H}(t_k)$, $\boldsymbol{\Delta}(t_k)$ are constants and $\boldsymbol{\Phi}(t_k, t_0)$ depends only on the difference $t_k - t_0$.

Example 1—Discretization of Continuous-Time Systems. A discrete-time dynamic system may arise from a continuous-time dynamic system by simply considering the latter at discrete instants of time only. Then our present notation coincides exactly with that of the companion paper.[3]

But not every discrete-time system can be regarded arising in a *natural* way from a continuous-time system. For instance, if $\boldsymbol{\Phi}(t_k, t_0)$ is the transition matrix of a linear, stationary, discrete-time system, obtained from a linear, stationary, continuous-time system, then we would have

$$\boldsymbol{\Phi}(t_k, t_0) \equiv \boldsymbol{\Phi}(t_k - t_0) = \exp\,(t_k - t_0)\mathbf{F} \tag{1}$$

But it is well known that the foregoing equation cannot be satisfied with any real, constant matrix $\mathbf{F}$ if $\boldsymbol{\Phi}(t_k - t_0)$ has any real, nonpositive eigenvalues. A simple instance of this is the equation

$$x(t_{k+1}) = -x(t_k)/2$$

Note also that the discrete version of a continuous-time stationary system is stationary only if $t_{k+1} - t_k$ = constant for all t_k.

Example 2—Sampling Systems. A *sampling* (or *sampled-data*) system is one in which some variables are allowed to change only at discrete instants t_k of time. These instants may specify the time at which some physical measurement (say, radar pulse) is performed, the time at which the memory of a digital computer is read out, etc. There are several different types of sampling operations of practical significance:

(*a*) *Periodic* (*conventional*) *sampling:* $t_k = kT$, where T = constant > 0.

(*b*) *Multiple-order sampling:* The pattern of t_k's is repeated periodically, i.e., $t_{k+r} - t_k$ = constant for all k.

(*c*) *Multirate sampling:* In case of two concurrent sampling operations, $t_k = pT_1$ or qT_2 where T_1, T_2 are constants and p, q are integers.

(*d*) *Random sampling:* t_k is a random variable.

In case (*a*), the discrete dynamic system is stationary; in the remaining cases the system is nonstationary in general. In case (b), the system can be made stationary by considering only sampling points $t_k, t_{k+r}, t_{k+2r}, \ldots$ In case (c), the system can be made stationary by taking samples $L(T_1, T_2)$ units of time apart where $L(T_1, T_2)$ is the least common multiple of T_1, T_2; this fails, of course, if $L = \infty$; i.e., if T_1/T_2 is irrational.

The "second method" of Lyapunov, as usually stated, applies only to free dynamic systems with equilibrium state at $\mathbf{0}$. More generally, this means that the method is concerned only with deviations about some fixed motion (see the companion paper[3] for further details).

5—Concepts of Stability

The definitions of this section, as well as their motivation, are identical in every respect with those of the companion paper[3] except that the concept of finite escape time is missing.

(S_1) An equilibrium state $\mathbf{x}_e$ of a dynamic system is *stable* [(S_6): *uniformly stable*] if to any number $\epsilon > 0$ there corresponds a number $\delta(\epsilon, t_0) > 0 [\delta(\epsilon) > 0]$ such that if $\|\mathbf{x}_0 - \mathbf{x}_e\| \leqq \delta$ then $\|\boldsymbol{\phi}(t_k; \mathbf{x}_0, t_0) - \mathbf{x}_e\| \leqq \epsilon$ for all $t_k \geqq t_0$.

(S_3) An equilibrium state $\mathbf{x}_e$ of a dynamic system is *equiasymptotically stable* [(S_7): *uniformly asymptotically stable*] if

(a) it is stable [uniformly stable] and

(b) to any number $\mu > 0$ there corresponds a number $T(\mu, t_0)$ [$T(\mu)$] such that $\|\boldsymbol{\phi}(t_k; \mathbf{x}_0, t_0) - \mathbf{x}_e\| \leqq \mu$ for all $t_k \geqq t_0 + T$ whenever $\|\mathbf{x}_0 - \mathbf{x}_e\| \leqq r$; $r(t_0) > 0 [r > 0]$ being some fixed constant which does not depend on μ or $\mathbf{x}_0$.

(S_4) An equilibrium state $\mathbf{x}_e$ of a dynamic system is *equiasymptotically stable in the large* [(S_8): *uniformly asymptotically stable in the large*] if

(a) it is stable [uniformly stable];

(b) all motions are bounded [uniformly bounded];

(c) all motions $\boldsymbol{\phi}(t_k; \mathbf{x}_0, t_0)$ ($\mathbf{x}_0$, t_0 arbitrary) converge [converge uniformly in $\|\mathbf{x}_0\| \leqq r, t_0$; r being arbitrarily large] to $\mathbf{x}_e$ with increasing t_k.

The various implications between these definitions, as stated in the companion paper,[3] hold in the discrete case also.

6—The Main Theorems

We state here results corresponding to those of the companion paper.[3] After obvious modifications (replacing continuous t with discrete t_k, integrals by sums, etc.) the proofs are identical with those given in the companion paper and are therefore omitted (see Hahn [3] for a self-contained account).

Theorem 1*. *Consider the discrete-time, free dynamic system:*

$$\mathbf{x}(t_{k+1}) = \mathbf{h}(\mathbf{x}(t_k), t_k) \qquad (5^*\text{-F})$$

where $\mathbf{h}(\mathbf{0}, t_k) = \mathbf{0}$ *for all* t_k. SUPPOSE *there exists a scalar function* $V(\mathbf{x}, t_k)$ *such that* $V(\mathbf{0}, t_k) = 0$ *for all* t_k, *and*

(i) $V(\mathbf{x}, t_k)$ *is positive definite; i.e., there exists a continuous, nondecreasing scalar function* α *such that* $\alpha(0) = 0$ *and, for all* t_k *and all* $\mathbf{x} \neq \mathbf{0}$,

$$0 < \alpha(\|\mathbf{x}\|) \leqq V(\mathbf{x}, t_k);$$

(ii) *There exists a continuous scalar function* γ *such that* $\gamma(0) = 0$ *and, for all* t_k *and all* $\mathbf{x} \neq \mathbf{0}$,

$$\begin{aligned}\Delta V(\mathbf{x}, t_k) &= \text{rate of increase of } V \text{ along motion starting at } \mathbf{x}, t_k \\ &= [V(\boldsymbol{\phi}(t_{k+1}; \mathbf{x}, t_k), t_{k+1}) - V(\mathbf{x}, t_k)](t_{k+1} - t_k)^{-1} \\ &\leqq -\gamma(\|\mathbf{x}\|) < 0^5;\end{aligned}$$

(iii) *There exists a continuous, nondecreasing scalar function* β *such that* $\beta(0) = 0$ *and, for all* t_k *and all* $\mathbf{x} \neq \mathbf{0}$,

$$V(\mathbf{x}, t_k) \leqq \beta(\|\mathbf{x}\|);$$

(iv) $\alpha(\|\mathbf{x}\|) \to \infty$ *when* $\|\mathbf{x}\| \to \infty$.

THEN *the equilibrium state* $\mathbf{x}_e = \mathbf{0}$ *is uniformly asymptotically stable in the large and* $V(\mathbf{x}, t_k)$ *is a Lyapunov function of the system* (5*-F).

Corollary 1.1*. *The following conditions are sufficient for various weaker types of stability:*

1 *Uniform asymptotic stability:* (i–iii).
2 *Equiasymptotic stability in the large:* (i–ii), (iv).
3 *Equiasymptotic stability:* (i–ii).
4 *Uniform stability:* (i), (iii), and (ii$_1$): $\Delta V(\mathbf{x}, t_k) \leqq 0$ for all $\mathbf{x}$, t_k.
5 *Stability:* (i–ii$_1$).

Corollary 1.2*. *If the system* (5*) *is autonomous, it suffices tc take* V *as a function of* $\mathbf{x}$ *only, such that* $V(\mathbf{0}) = 0$ *and*

(i-A) $V(\mathbf{x}) > 0$ *when* $\mathbf{x} \neq \mathbf{0}$;
(ii-A) $\Delta V(\mathbf{x}) < 0$ *when* $\mathbf{x} \neq \mathbf{0}$;
(iii-A) $V(\mathbf{x})$ *is continuous in* $\mathbf{x}$;
(iv-A) $V(\mathbf{x}) \to \infty$ *when* $\|\mathbf{x}\| \to \infty$.

Corollary 1.3*. *In Corollary 1.2*, Condition* (ii-A) *may be replaced by*

(ii$_1$-A) $\Delta V(\mathbf{x}) \leqq 0$ *for all* $\mathbf{x}$;
(ii$_2$-A) $\Delta V(\boldsymbol{\phi}(t_k; \mathbf{x}_0, t_0))$ *does not vanish identically in* $t_k \geqq t_0$ *for any* t_0 *and any* $\mathbf{x}_0 \neq \mathbf{0}$.

The problem of the converse of Theorem 1* (i.e., does uniform asymptotic stability in the large imply the existence of a Lyapunov function with properties (i-iv)?) apparently has not been investigated so far. Theorem 2 of the companion paper[2] does not carry over trivially in this case because, as we have seen in Example 1, the class of all discrete-time dynamic systems is *larger* than the class of all continuous-time dynamic systems.

Theorem 3*. *Consider the discrete-time, linear, dynamic system*

$$\mathbf{x}(t_{k+1}) = \mathbf{H}(t_k)\mathbf{x}(t_k) + \boldsymbol{\Delta}(t_k)\mathbf{u}(t_k) \qquad (5^*\text{-L})$$

and ASSUME *that*

(i) $(t_k - t_{k-1})^{-1} \cdot \|\mathbf{H}(t_k) - \mathbf{I}\| \leqq c_1 < \infty$ *for all* t_k[6]

(ii) $0 < c_2 \leqq (t_{k+1} - t_k)^{-1} \cdot \|\boldsymbol{\Delta}(t_k)\mathbf{x}\| \leqq c_3 < \infty$ *for all* $\|\mathbf{x}\| = 1$, *all* t_k

THEN *the following propositions concerning this system are equivalent:*

(A) *Assuming* $\mathbf{x}_0 = \mathbf{0}$, *any uniformly bounded excitation*

$$\|\mathbf{u}(t_k)\| \leqq c_4 < \infty \text{ when } t_k \geqq t_0$$

gives rise to a uniformly bounded response for all $t_k \geqq t_0$, i.e.

$$\begin{aligned}\|\mathbf{x}(t_k)\| &= \Big\| \sum_{t_0 \leqq t_i < t_k} \boldsymbol{\Phi}(t_k, t_{i+1})\boldsymbol{\Delta}(t_i)\mathbf{u}(t_i) \Big\| \\ &\leqq c_5(c_4) < \infty;\end{aligned}$$

(B) *For all* $t_k > t_0$,

$$\sum_{t_0 < t_i \leqq t_k} (t_i - t_{i-1})\|\boldsymbol{\Phi}(t_k, t_i)\| \leqq c_6 < \infty;$$

(C) *The equilibrium state* $\mathbf{x}_e = \mathbf{0}$ *of the free system is uniformly asymptotically stable;*

(D) *There exist positive constants* c_7, c_8 *such that, whenever* $t_k \geqq t_0$,

[5] When the discrete-time system is given abstractly (i.e., not by discretization of a continuous-time system), there is no loss of generality in assuming that $t_{k+1} - t_k = 1$ for all k.

[6] In this theorem, $c_1, c_2, \ldots$ denote various positive constants.

$$\|\Phi(t_k, t_0)\| \leqq c_7 e^{-c_8(t_k - t_0)}; \tag{15*}$$

(E) Given any positive definite matrix $\mathbf{Q}(t_k)$ *satisfying, for all* $t_k \geqq t_0$,

$$0 < c_9\,\mathbf{I} \leqq \mathbf{Q}(t_k) \leqq c_{10}\,\mathbf{I} < \infty^{7}$$

the scalar function defined by

$$V(\mathbf{x}, t_k) = \sum_{t_k \leqq t_i \leqq \infty} (t_{i+1} - t_i)\|\Phi(t_i, t_k)\mathbf{x}\|^2_{\mathbf{Q}(t_i)}{}^{8} \tag{17*}$$

$$= \|\mathbf{x}\|^2_{\mathbf{P}(t_k)} \tag{18*}$$

exists[9] *and is a Lyapunov function for the free system, satisfying the requirements of Theorem 1* with*

$$\Delta V(\mathbf{x}, t_k) = -\|\mathbf{x}\|^2_{\mathbf{Q}(t_k)}$$

Corollary 3.1*. *A discrete-time, free, linear, stationary, dynamic system*

$$\mathbf{x}(t_{k+1}) = \mathbf{H}\mathbf{x}(t_k) \qquad (t_{k+1} - t_k = 1) \tag{5*-FLS}$$

is asymptotically stable if, and only if, given any symmetric, positive-definite matrix $\mathbf{Q}$ *there exists a symmetric, positive-definite matrix* $\mathbf{P}$ *which is the unique solution of the linear equation*

$$\mathbf{H}'\mathbf{P}\mathbf{H} - \mathbf{P} = -\mathbf{Q} \tag{19*}$$

and

$$V(\mathbf{x}) = \|\mathbf{x}\|^2_{\mathbf{P}}$$

is a Lyapunov function for (5*-FLS) *with*

$$\Delta V(\mathbf{x}) = -\|\mathbf{x}\|^2_{\mathbf{Q}}$$

Corollary 3.2.* *The eigenvalues of a constant matrix* $\mathbf{H}$ *are less than* ρ *in absolute value if, and only if, given any symmetric, positive definite matrix* $\mathbf{Q}$ *the linear equation*

$$\rho^{-2}\mathbf{H}'\mathbf{P}\mathbf{H} - \mathbf{P} = -\mathbf{Q} \tag{20*}$$

has a unique, symmetric, positive-definite solution $\mathbf{P}$.

The proof of Theorem 3 and its corollaries is entirely analogous to the proof in the continuous-time case.

7—Applications to Stability Theory

7.1 Routh-Hurwitz Conditions for Linear Stationary Systems. As in the companion paper,[3] Corollary 3.1* gives a purely algebraic procedure for checking the asymptotic stability of a free, linear, stationary system. (As is well known, a necessary and sufficient condition for this is that all eigenvalues of the matrix $\mathbf{H}$ in (5* FLS) be less than unity in absolute value.) Take any symmetric, positive-definite $\mathbf{Q}$ (say, the unit matrix); invert (19*) to find $\mathbf{P}$, which is always possible in case of asymptotic stability; test $\mathbf{P}$ for positive-definiteness, in other words, require the leading principal minors of $\mathbf{P}$ to be positive—these n inequalities are analogous to the Routh-Hurwitz conditions.

This procedure is illustrated in Example 6*. The procedure is most useful for machine computation when the number of nonzero elements of $\mathbf{H}$ is larger than about 4 or 5; in such cases, a stability test may require very extensive numerical computations by conventional methods. The analog of the canonic matrix (33) (Example 7) of the companion paper is so far not yet available for discrete systems.

Example 6*. Routh-Hurwitz Conditions for General Second-Order Case. Consider the case when $\mathbf{H}$ is a 2×2 matrix. Direct substitution of the general expression for $\mathbf{H}$ into (19*) leads to very complicated algebra so that some simplification must be sought.

If $\mathbf{H} = \lambda\mathbf{I}$, then $\lambda^2 = \det \mathbf{H} < 1$ insures stability. If $\mathbf{H} \neq \lambda\mathbf{I}$, then it is easy to show that there is some nonsingular matrix $\mathbf{T}$ such that

$$\mathbf{T}^{-1}\mathbf{H}\mathbf{T} = \mathbf{C} = \begin{bmatrix} 0 & 1 \\ -a_1 & -a_2 \end{bmatrix}$$

Writing out (19*) in detail for $\mathbf{H} = \mathbf{C}$, with $\mathbf{Q} = \mathbf{I}$, we get

$$\begin{bmatrix} -1 & 0 & a_1^2 \\ 0 & -1 - a_1 & a_1 a_2 \\ 1 & -2a_2 & a_2^2 - 1 \end{bmatrix} \cdot \begin{bmatrix} p_{11} \\ p_{12} \\ p_{22} \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$$

This set of equations has a unique solution if the determinant of the matrix on the left does not vanish:

$$\det \mathfrak{C} = (a_1 - 1)[(1 + a_1)^2 - a_2^2] \neq 0 \tag{26*}$$

Assuming (26*), we find after elementary computations

$$\mathbf{P} = (\det \mathfrak{C})^{-1} \begin{bmatrix} \det \mathfrak{C} - 2a_1^2(1 + a_1) & -2a_1 a_2 \\ -2a_1 a_2 & -2(1 + a_1) \end{bmatrix} \tag{27*}$$

Now $\mathbf{P}$ is positive definite if, and only if,

$$\det \mathbf{P} = \frac{2(1 + a_1^2)}{(1 - a_1)^2[(1 + a_1)^2 - a_2^2]} > 0 \tag{28*}$$

$$p_{22} = \frac{2(1 + a_1)}{(1 - a_1)[(1 + a_1)^2 - a_2^2]} > 0 \tag{29*}$$

But (28*) implies that

$$(1 + a_1)^2 - a_2^2 > 0$$

and this fact, with (29*) implies that

$$1 - a_1^2 > 0$$

Now observe that $a_1 = \det \mathbf{C}$ and $a_2 = -\operatorname{tr} \mathbf{C}$. Since the determinant and the trace are invariant under similarity transformations, it follows that we can write equally well

$$|\operatorname{tr} \mathbf{H}| < |1 + \det \mathbf{H}| \tag{30*}$$

$$|\det \mathbf{H}| < 1 \tag{31*}$$

which is the general form of the Routh-Hurwitz inequalities in the second-order case. Finally, it follows from (30*–31*) that condition (26*) is always satisfied.

7.2 Stability in Nonlinear Systems. The analog of Krasovskii's theorem (Theorem 4) of the companion paper[2] is well known in mathematics [7] but not in engineering. It is based on the following concept: A function $\mathbf{f}(\mathbf{x})$ is said to be a *contraction* if

$$\|\mathbf{f}(\mathbf{x})\| < \|\mathbf{x}\| \qquad (\mathbf{f}(\mathbf{0}) = \mathbf{0})$$

for some set of values of $\mathbf{x} \neq \mathbf{0}$ and *some* norm. With this in mind, we can state

Theorem 4*. *Consider the discrete-time, free, stationary, dynamic system*

$$\mathbf{x}(t_{k+1}) = \mathbf{h}(\mathbf{x}(t_k)) \qquad (\mathbf{h}(\mathbf{0}) = \mathbf{0}) \tag{5*-FS}$$

ASSUME *that* $\mathbf{h}$ *is a contraction for all* $\mathbf{x}$ *and some norm.*

THEN *the system* (5*-FS) *is asymptotically stable in the large, and one of its Lyapunov functions is*

$$V(\mathbf{x}) = \|\mathbf{x}\| \tag{2}$$

[7] If $\mathbf{A}$, $\mathbf{B}$ are two matrices, we write $\mathbf{A} < \mathbf{B}$ [$\mathbf{A} \leqq \mathbf{B}$] to express the fact that $\mathbf{B} - \mathbf{A}$ is positive definite [semidefinite].

[8] We write $\mathbf{x}'\mathbf{A}\mathbf{x} = \|\mathbf{x}\|^2_{\mathbf{A}}$.

[9] I.e., the infinite sum (17*) converges.

PROOF: Computing ΔV and assuming $t_{k+1} - t_k = 1$ (see footnote 4), we find that

$$\Delta V = \|\mathbf{h}(\mathbf{x})\| - \|\mathbf{x}\| \tag{3}$$

which is negative definite by definition of a contraction. Q.E.D.

Example 2. Conditions for Contraction. Consider the system

$$\mathbf{x}(t_{k+1}) = \mathbf{H}(\mathbf{x}(t_k))\mathbf{x}(t_k) \tag{4}$$

Assume that there are positive constants $c_1, \ldots, c_n$ such that either

$$(a) \quad \underset{i}{\text{Max}} \left\{ \sum_{j=1}^{n} \frac{c_i}{c_j} |h_{ij}(\mathbf{x})| \right\} < 1 \quad \text{for all} \quad \mathbf{x}$$

or

$$(b) \quad \underset{j}{\text{Max}} \left\{ \sum_{i=1}^{n} \frac{c_i}{c_j} |h_{ij}(\mathbf{x})| \right\} < 1 \quad \text{for all} \quad \mathbf{x}$$

Then in either case $\mathbf{H}(\mathbf{x})\mathbf{x}$ is a contraction for all $\mathbf{x}$ and therefore the system (4) is asymptotically stable in the large.

In case (a), define the norm by

$$\|\mathbf{x}\| = \underset{i}{\text{Max}} \{c_i|x_i|\}$$

Then

$$\|\mathbf{H}(\mathbf{x})\mathbf{x}\| = \underset{i}{\text{Max}} \left\{ c_i \left| \sum_{j=1}^{n} h_{ij}(\mathbf{x})x_j \right| \right\}$$

$$\leq \underset{i}{\text{Max}} \left\{ \sum_{j=1}^{n} \frac{c_i}{c_j} |h_{ij}(\mathbf{x})| \cdot c_j|x_j| \right\}$$

$$\leq \underset{i}{\text{Max}} \left\{ \sum_{j=1}^{n} \frac{c_i}{c_j} |h_{ij}(\mathbf{x})| \right\} \cdot \underset{j}{\text{Max}} \{c_j|x_j|\}$$

which verifies that $\mathbf{H}(\mathbf{x})\mathbf{x}$ is a contraction. Note that in this norm, the contours V = constant = c_0 are cubes with edges of length $2c_0c_i^{-1}$.

In case (b), define the norm by

$$\|\mathbf{x}\| = \sum_{i=1}^{n} c_i|x_i|$$

Then

$$\|\mathbf{H}(\mathbf{x})\mathbf{x}\| = \sum_{i=1}^{n} c_i \left| \sum_{j=1}^{n} h_{ij}(\mathbf{x})x_j \right|$$

$$\leq \sum_{i=1}^{n} \frac{c_i}{c_j} \sum_{j=1}^{n} |h_{ij}(\mathbf{x})| \cdot c_j|x_j|$$

$$\leq \underset{j}{\text{Max}} \left\{ \sum_{i=1}^{n} \frac{c_i}{c_j} |h_{ij}(\mathbf{x})| \right\} \cdot \sum_{j=1}^{n} c_j|x_j|$$

which yields the desired result.

It is clear that whether or not a given function is a contraction depends on the norm chosen. This is where ingenuity is needed.

8—Estimation of Transient Behavior

By regarding V as a measure of "distance" in state space, the estimation of transient behavior proceeds in a manner which is completely analogous to that given in the companion paper.[3] Only the details of some of the calculations are different. This is illustrated by the analog of Example 15:

Example 15*. Effect of Perturbations and Parameter Variations in the Linear Stationary Case. Consider the system $(t_{k+1} - t_k = 1)$

$$\mathbf{x}(t_{k+1}) = \mathbf{H}\mathbf{x}(t_k) + \mathbf{K}(\mathbf{x}(t_k))\mathbf{x}(t_k) + \mathbf{u}(t_k) \tag{60*}$$

where

$$\left. \begin{aligned} \|\mathbf{K}(\mathbf{x}(t_k))\| &\leq c_0 \\ \|\mathbf{u}(t_k)\| &\leq c_1 \end{aligned} \right\} \quad \text{for all} \quad t_k$$

Assuming $\mathbf{H}$ is asymptotically stable, take any symmetric, positive-definite $\mathbf{Q}$ and define a Lyapunov function

$$V(\mathbf{x}) = \|\mathbf{x}\|^2_{\mathbf{P}}$$

by means of Corollary 3.1*. Then, evidently,

$$\Delta V(\mathbf{x}) = -\|\mathbf{x}\|^2_{\mathbf{Q}} + 2[\mathbf{x}'(t_k)\mathbf{K}'(\mathbf{x}(t_k)) + \mathbf{u}'(t_k)]\mathbf{P}\mathbf{H}\mathbf{x}(t_k)$$
$$+ [\mathbf{x}'(t_k)\mathbf{K}'(\mathbf{x}(t_k)) + \mathbf{u}'(t_k)]\mathbf{P}[\mathbf{K}(\mathbf{x}(t_k))\mathbf{x}(t_k) + \mathbf{u}(t_k)] \tag{61*}$$

Employing the Schwarz inequality, (61*) may be written in the form:

$$\Delta V(\mathbf{x}) \leq \left[\frac{-\|\mathbf{x}\|^2_{\mathbf{Q}} + 2c_0\|\mathbf{x}\| \cdot \|\mathbf{P}\mathbf{H}\mathbf{x}\| + c_0^2\|\mathbf{P}\| \cdot \|\mathbf{x}\|^2}{\|\mathbf{x}\|^2_{\mathbf{P}}} \right] V(\mathbf{x})$$
$$+ 2c_1 \left[\frac{\|\mathbf{P}\mathbf{H}\mathbf{x}\| + c_0\|\mathbf{P}\| \cdot \|\mathbf{x}\|}{\|\mathbf{x}\|_{\mathbf{P}}} \right] V^{1/2}(\mathbf{x}) + c_1^2\|\mathbf{P}\| \tag{5}$$

Using repeatedly the relation ($\mathbf{A}$, $\mathbf{B}$ = symmetric, positive-definite)

$$\underset{\mathbf{x}}{\text{Max}} \{\|\mathbf{x}\|^2_{\mathbf{A}}/\|\mathbf{x}\|^2_{\mathbf{B}}\} = \lambda_{\max}(\mathbf{A}\mathbf{B}^{-1})$$

each one of the terms in (5) may be maximized separately, which leads to the following:

$$\Delta V(\mathbf{x}) \leq [-e^{2\eta T} + 2c_0c_3c_4 + c_0^2c_2^2c_3^2]V$$
$$+ 2c_1[c_4 + c_0c_2^2c_3]V^{1/2} + c_1^2c_2^2 \tag{6}$$

where

$$e^{2\eta T} = \lambda_{\min}(\mathbf{Q}\mathbf{P}^{-1}); \quad T = t_{k+1} - t_k$$

η being an estimate of the time constant of the system similar to that discussed in the companion paper;[3] and

$$c_2 = \lambda_{\max}{}^{1/2}(\mathbf{P})$$

$$c_3 = [\lambda_{\min}(\mathbf{P})]^{-1/2}$$

$$c_4 = \lambda_{\max}{}^{1/2}(\mathbf{H}'\mathbf{P}^2\mathbf{H}\mathbf{P}^{-1})$$

From (6) we see that $\Delta V(\mathbf{x})$ will be negative if

$$\|\mathbf{x}\| > \frac{c_1\left[c_0c_2^2c_3 + c_4 + \sqrt{e^{2\eta T}c_2^2 + c_4^2}\right]}{e^{2\eta T} - 2c_0c_3c_4 - c_0^2c_2^2c_3^2}$$

provided that the denominator of the foregoing expression is positive, which is just the condition for asymptotic stability when $c_1 = 0$.

9—Relations With System Optimization

Similarly to the continuous-time case, define a *performance index* in the stationary case by

$$V(\mathbf{x}) = \sum_{t_i=0}^{\infty} \rho(\boldsymbol{\phi}(t_i; \mathbf{x}, 0)) \tag{65*}$$

where ρ is the *error criterion*.

For exactly the same reasons as before, we are led to

Theorem 5*. (Kalman [8]) *Consider a free, linear, stationary dynamic system with an equilibrium state at the origin and* ASSUME:

(i) *The error criterion* $\rho(\mathbf{x})$ *is positive definite and* $\rho(\mathbf{0}) = 0$;

(ii) *The performance index* $V(\mathbf{x})$ *defined by* (65) *is finite in some neighborhood of the origin.*

THEN $\mathbf{x}_e = \mathbf{0}$ *is asymptotically stable.*

The entire discussion of this theorem in the companion paper[3] is applicable here too. We add only one remark:

In case $\rho(\mathbf{x}) = x_1^2$ (x_1 being the output of the system), the performance index (65*) is obviously minimized for arbitrary $\mathbf{x}(0)$ if $x_1(t_k) \equiv 0$ for all $t_k > 0$. As Bergen and Ragazzini have shown [9], this can indeed be accomplished by suitable digital compensation of single-input systems. As they also pointed out, however, their design procedure (which is unique) does not necessarily result in a *stable* system.

The situation is as follows: If $H(z)$ is the open-loop pulse transfer function of the plant to be controlled including the zero-order hold, then the transfer function $D(z)$ of the Bergen-Ragazzini controller must contain poles to cancel the zeros of $H(z)$ because of the condition $D(z)H(z) = z - 1$. This cancellation is never exact, and if $H(z)$ contains zeros outside the region $|z| \leq 1$, then elementary root-locus arguments show that the resulting system will have a pole outside the unit circle and hence be unstable, no matter how nearly the zero is canceled. This can happen quite readily even if the plant is open-loop stable; in such cases Bergen and Ragazzini recommend a modified procedure.

It is sometimes stated that the Bergen-Ragazzini design as just sketched fails because of the impossibility of perfect cancellation. This is misleading and erroneous, for *even with perfect cancellation the Bergen-Ragazzini design is not asymptotically stable when* $H(z)$ *contains zeros outside the unit circle.*

In fact, suppose the contrary; i.e., under perfect cancellation the system is asymptotically stable. An imperfect cancellation is readily seen to be equivalent to a perturbation of the equations of the system, as in (60*), But, by Example 15*, if c_0 is sufficiently small (i.e., good but not necessarily perfect cancellation) then the system remains asymptotically stable. *In other words, asymptotic stability is a system property which cannot be destroyed by small perturbations in the coefficients.* But this contradicts the foregoing conclusions about imperfect cancellation.

The fallacy in the conventional reasoning is that one cannot, in general, conclude that a stable response to a step input implies stability of motions starting at arbitrary initial states (see also condition (ii) of Theorem 3*).

We have here a particularly simple example of the fact that minimization of performance indexes based on semidefinite error criteria is not necessarily free of stability difficulties.

10—Application to Design

In the corresponding section of the companion paper,[3] we considered the problem of designing a regulating system for the plant governed by the equations

$$d\mathbf{x}/dt = \mathbf{F}\mathbf{x} + \mathbf{D}\mathbf{u}(t) \tag{7}$$

where the control variables are subject to the constraints

$$|u_i(t)| \leq c_i < \infty \quad (i = 1, \ldots, m) \tag{8}$$

If the controller is to be a sampled-data system, it is convenient to transform (7) first into the discrete-time equivalent form

$$\mathbf{x}(t_{k+1}) = \mathbf{H}\mathbf{x}(t_k) + \mathbf{\Delta}\mathbf{u}(t_k) \tag{70*}$$

where it was assumed that $t_{k+1} - t_k = T =$ constant for all k, and

$$\mathbf{H} = \exp T\mathbf{F}; \quad \mathbf{\Delta} = \int_0^T (\exp \tau\mathbf{F})\mathbf{D}\, d\tau$$

Assume that $\mathbf{F}$ (and therefore $\mathbf{H}$) is asymptotically stable and given any symmetric, positive-definite $\mathbf{Q}$ find the corresponding symmetric, positive-definite matrix $\mathbf{P}$ by means of Corollary 3.1*. Then

$$V(\mathbf{x}) = \|\mathbf{x}\|^2_{\mathbf{P}}$$

is clearly a Lyapunov function for the free part of system (70*). The difference of V along a motion of (70*) is then

$$\Delta V(\mathbf{x}(t_k), t_k) = T^{-1}[-\|\mathbf{x}(t_k)\|^2_{\mathbf{Q}} + 2\mathbf{u}'(t_k)\mathbf{\Delta}'\mathbf{P}\mathbf{H}\mathbf{x}(t_k) + \mathbf{u}'(t_k)\mathbf{\Delta}'\mathbf{P}\mathbf{\Delta}\mathbf{u}(t_k)] \tag{9}$$

An approximately optimal and practically useful design is obtained by selecting $\mathbf{u}(t_k)$ at each sampling instant t_k in such a way as to minimize ΔV.

It is clear that the right-hand side of (9) is a quadratic function in the components of $\mathbf{u}$. Disregarding at first the limits (8) on the components of $\mathbf{u}$, $\partial\Delta V/\partial\mathbf{u} = \mathbf{0}$ if $\mathbf{u}$ is given by

$$\mathbf{u}^\circ(t_k) = -(\mathbf{\Delta}'\mathbf{P}\mathbf{\Delta})^{-1}\mathbf{\Delta}'\mathbf{P}\mathbf{H}\mathbf{x}(t_k) \tag{10}$$

The matrix $\mathbf{\Delta}'\mathbf{P}\mathbf{\Delta}$ is obviously positive semidefinite since $\mathbf{P}$ is positive definite. If $\mathbf{\Delta}$ is 1-to-1 (i.e., $\mathbf{\Delta}\mathbf{u} = \mathbf{0}$ implies $\mathbf{u} = \mathbf{0}$ or, equivalently, the columns of $\mathbf{\Delta}$ are a linearly independent set), then $\mathbf{\Delta}'\mathbf{P}\mathbf{\Delta}$ is positive definite so that the inverse required in formula (10) actually exists. Moreover, positive definiteness of $\mathbf{\Delta}'\mathbf{P}\mathbf{\Delta}$ implies also that (9) has a true minimum at the point where the derivative with respect to $\mathbf{u}$ vanishes.

If we now take into account (8), it follows that $\mathbf{u}$ should be set as close to $\mathbf{u}^\circ$ as allowed by the constraints. Hence

$$\left.\begin{aligned} u_i^*(t_k) &= -c_i \operatorname{sat}[c_i^{-1}(\mathbf{\Delta}'\mathbf{P}\mathbf{\Delta})^{-1}\mathbf{\Delta}'\mathbf{P}\mathbf{H}\mathbf{x}(t_k)]_i \quad (i = 1, \ldots, m) \\ \mathbf{u}^*(t) &= \mathbf{u}^*(t_k), \qquad t_k \leq t < t_{k+1} \end{aligned}\right. \tag{72*}$$

where

$$\operatorname{sat} x = \begin{cases} +1, & x > 1 \\ x, & |x| \leq 1 \\ -1, & x < -1 \end{cases}$$

The difficulties encountered in the companion paper[3] in proving asymptotic stability in the large when $\mathbf{F}$ and $\mathbf{D}$ are not known exactly and when some state variables are not measurable do not arise here because the function *sat* changes by small amounts for small perturbations in the argument.

Theorem 6*. *Let* $\mathbf{x}$ *be expressed uniquely as* $\mathbf{y} + \mathbf{z}$ *where* $\mathbf{y}$ *denotes the measurable and* $\mathbf{z}$ *the unmeasurable state variables;* $\hat{\mathbf{z}}$ *is the predicted value of* $\mathbf{z}$. *Let* $\hat{\mathbf{H}}$, $\hat{\mathbf{\Delta}}$ *denote the available estimates of* $\mathbf{H}$, $\mathbf{\Delta}$ *and* $\hat{\mathbf{P}} = \mathbf{P}(\hat{\mathbf{H}}, \mathbf{Q})$. IF

(i) $|\lambda_i(\mathbf{H}_{22})| < 1$ *for all i and*

(ii) $\mathbf{H} - \hat{\mathbf{H}}$, $\mathbf{\Delta} - \hat{\mathbf{\Delta}}$ *are sufficiently small*

THEN *the over-all system*

$$\mathbf{x}(t_{k+1}) = \mathbf{H}\mathbf{x}(t_k) + \mathbf{\Delta}\mathbf{u}^*(\hat{\mathbf{H}}, \hat{\mathbf{\Delta}}, \hat{\mathbf{P}}, \mathbf{y}(t_k) + \hat{\mathbf{z}}(t_k))$$

$$\hat{\mathbf{z}}(t_{k+1}) = \hat{\mathbf{H}}_{21}\mathbf{y}(t_k) + \hat{\mathbf{H}}_{22}\hat{\mathbf{z}}(t_k) + \hat{\mathbf{\Delta}}_2\mathbf{u}^*(\hat{\mathbf{H}}, \hat{\mathbf{\Delta}}, \hat{\mathbf{P}}, \mathbf{y}(t_k) + \hat{\mathbf{z}}(t_k))$$

is asymptotically stable in the large. (In the foregoing, $\mathbf{u}^*$ denotes the function defined by (72*); $\mathbf{H}_{21}$, $\mathbf{H}_{22}$ are the lower two submatrices obtained by partitioning $\mathbf{H}$ into four parts, in accordance with decomposing $\mathbf{x}$ as the direct sum of $\mathbf{y}$ and $\mathbf{z}$.)

PROOF. Assume first that $\mathbf{H}$, $\mathbf{\Delta}$ are known exactly. Using the notation $\tilde{\mathbf{z}} = \mathbf{z} - \hat{\mathbf{z}}$ for the error in predicting the values of the unmeasurable state variables, the over-all system equations become:

$$\left.\begin{aligned} \mathbf{x}(t_{k+1}) &= \mathbf{H}\mathbf{x}(t_k) + \mathbf{\Delta}\mathbf{u}^*(\mathbf{x}(t_k) + \tilde{\mathbf{z}}(t)) \\ \tilde{\mathbf{z}}(t_{k+1}) &= \mathbf{H}_{22}\tilde{\mathbf{z}}(t_k) \end{aligned}\right\} \tag{11}$$

Let $\mathbf{N}$ be a symmetric, positive-definite matrix. Using assumption (i), find the corresponding symmetric, positive-definite matrix $\mathbf{M}$ via Corollary 3.1*. Define the Lyapunov function

$$V(\mathbf{x}, \tilde{\mathbf{z}}) = \|\mathbf{x}\|^2_{\mathbf{P}} + \|\tilde{\mathbf{z}}\|^2_{\mathbf{M}} \tag{12}$$

Using the abbreviation $\mathbf{W} = \mathbf{H}'\mathbf{P}\boldsymbol{\Delta}(\boldsymbol{\Delta}'\mathbf{P}\boldsymbol{\Delta})^{-1}\boldsymbol{\Delta}'\mathbf{P}\mathbf{H}$, we find after simple calculations

$$\Delta V(\mathbf{x}, \tilde{\mathbf{z}}) \leqq T^{-1}\{-\lambda_{\min}(\mathbf{Q})\|\mathbf{x}\|^2 - \lambda_{\min}(\mathbf{N})\|\tilde{\mathbf{z}}\|^2 + \|\mathbf{W}\|[2\|\mathbf{x}\|.\|\tilde{\mathbf{z}}\| + \|\tilde{\mathbf{z}}\|^2]\}$$

This expression is negative definite if

$$-\lambda_{\min}(\mathbf{Q})[\lambda_{\min}(\mathbf{N}) - \|\mathbf{W}\|] + \|\mathbf{W}\|^2 < 0;$$

the foregoing condition is satisfied if $\lambda_{\min}(\mathbf{N})$ is sufficiently large, which can be brought about by making $\mathbf{N}$ large.

It is now clear that (11) is asymptotically stable in the large, with a Lyapunov function of the form (12) and

$$\Delta V(\mathbf{x}, \tilde{\mathbf{z}}) \leqq -k(\|\mathbf{x}\|^2 + \|\tilde{\mathbf{z}}\|^2)$$

where $0 < k < T^{-1}\lambda_{\min}(\mathbf{Q})$.

Inexact knowledge of $\mathbf{H}$ and $\boldsymbol{\Delta}$ adds perturbation terms to (11), analogous to the terms $\mathbf{K}(\mathbf{x}(t_k))\mathbf{x}(t_k)$ considered in Example 15*. By assumption (ii) and the results of Example 15*, the conclusion of Theorem 6* follows. Q. E. D.

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