ECE 68000: MODERN AUTOMATIC CONTROL

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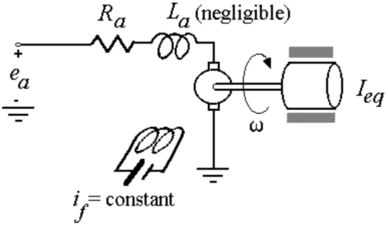
The Pontryagin's Minimum Principle (PMP): Examples

Pontryagin's minimum principle: Example 1



DC motor optimal control

- Plant: An armature-controlled DC motor
- Use the minimum principle to find the armature voltage $e_a(t)$ such that the motor angular velocity ω changes from $\omega(0) = 0$ rad/sec at $t = 0$ to $\omega(1) = 10$ rad/sec while minimizing the energy dissipated in the armature resistor, R_a
- No constraints on e_a
- The energy dissipated in the armature resistor is

$$J = \int_0^1 R_a i_a(t)^2 dt,$$

  where i_a is the armature current.
- Represent the expression for the energy dissipated in the armature resistor in terms of ω and e_a
- Apply Ohm's law to the armature circuit to get

$$e_a - K_b\omega = R_a i_a$$

The Hamiltonian function

- Hence

$$J = \int_0^1 R_a i_a(t)^2 dt = \int_0^1 \frac{(e_a - K_b\omega)^2}{R_a} dt.$$

- The Hamiltonian function

$$H = \frac{(e_a - K_b\omega)^2}{R_a} + p\left(-\frac{K_i K_b}{R_a I_{eq}}\omega + \frac{K_i}{I_{eq} R_a} e_a\right)$$

- No constraints on e_a, we can find the optimal control by solving the equation

$$0 = \frac{\partial H}{\partial e_a} = \frac{2\,(e_a - K_b\omega)}{R_a} + \frac{K_i}{I_{eq} R_a} p$$

- Hence,

$$e_a = 2\omega - p$$

Costate equation

- The costate equation

$$\dot{p} = -\frac{\partial H}{\partial \omega} = \frac{2K_b\left(e_a - K_b\omega\right)}{R_a} + \frac{K_i K_b}{R_a I_{eq}} p = 2(e_a - 2\omega) + 2p$$

- Substituting gives

$$\dot{p} = 0$$

- Hence,

$$p(t) = \text{constant} = c$$

- Therefore,

$$e_a = 2\omega - c$$

- Hence

$$\dot{\omega} = -c$$

Optimal control

- Solving $\dot{\omega} = -c$ yields

$$\omega(t) = -ct + \omega(0) = -ct$$

  because $\omega(0) = 0$
- Use the boundary condition $\omega(1) = 10$ to find $c = -10$
- Hence, the optimal armature voltage

$$\boxed{e_a^*(t) = 20t + 10}$$

Example 2

- The plant

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}u$$

- The performance index

$$J = \int_0^{t_f} dt$$

- The control is required to satisfy

$$|u(t)| \le 1$$

  for all $t \in [0, t_f]$
- This constraint means that the control must have magnitude no greater than 1
- Objective: find admissible control minimizing J that transfers the system from a given initial $\boldsymbol{x}_0$ to the origin

Solving the costate equation

- The Hamiltonian function

$$
\begin{aligned}
H &= 1 + \boldsymbol{p}^\top(\boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}u) \\
&= 1 + \begin{bmatrix} p_1 & p_2 \end{bmatrix} \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \right) \\
&= 1 + p_1 x_2 + p_2 u
\end{aligned}
$$

- The costate equations are

$$
\begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} = - \begin{bmatrix} \frac{\partial H}{\partial x_1} \\ \frac{\partial H}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 \\ -p_1 \end{bmatrix}
$$

- Solving the costate equations

$$
p_1 = d_1 \quad \text{and} \quad p_2 = -d_1 t + d_2,
$$

where d_1 and d_2 are integration constants

Admissible control

- Find an admissible control minimizing the Hamiltonian

$$
\begin{aligned}
\arg_u \min H &= \arg_u \min(1 + p_1 x_2 + p_2 u) \\
&= \arg_u \min(p_2 u) \\
&= \begin{cases} u(t) = 1 & \text{if} \quad p_2 < 0 \\ u(t) = ? & \text{if} \quad p_2 = 0 \\ u(t) = -1 & \text{if} \quad p_2 > 0 \end{cases}
\end{aligned}
$$

- If $p_2 = 0$ is not sustained over an interval time, the control law is piecewise constant taking the values 1 or -1
- This control law has at most two intervals of constancy because the argument is a linear function, $-d_1 t + d_2$, that changes its sign at most once
- This type of control is called a *bang-bang control* because it switches back and forth between its extreme values

Bang-bang control

- Implement the bang-bang control law using a relay element, which is a signum function, as

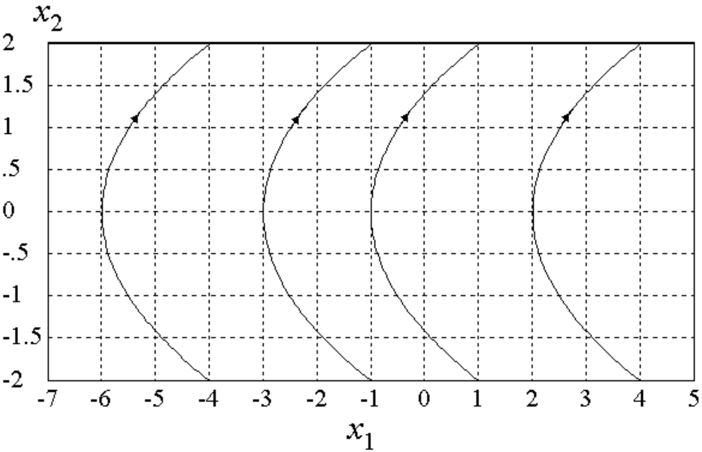
$$u^*(t) = -\mathrm{sgn}(p_2^*) = -\mathrm{sgn}(-d_1 t + d_2),$$

where "sgn" is the label for the signum function defined as

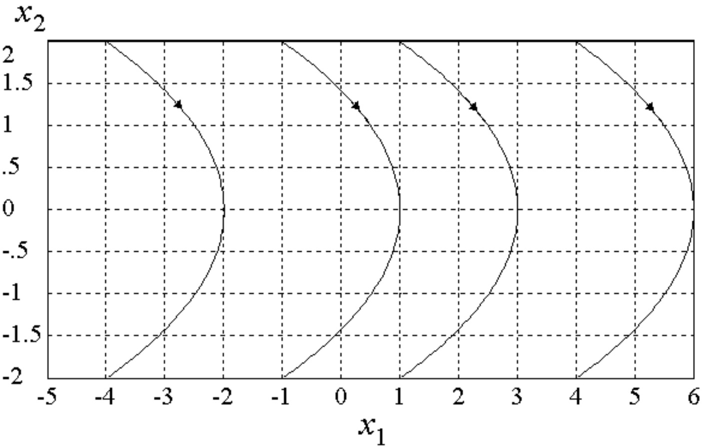
$$\mathrm{sgn}(z) = \begin{cases} \frac{z}{|z|} \quad \text{or} \quad \frac{|z|}{z} & \text{if} \quad z \neq 0 \\ 0 & \text{if} \quad z = 0 \end{cases} = \begin{cases} 1 & \text{if} \quad z > 0 \\ 0 & \text{if} \quad z = 0 \\ -1 & \text{if} \quad z < 0 \end{cases}$$

- System trajectories for $u = 1$ and $u = -1$ are families of parabolas
- Only one parabola from each family passes through the specified terminal point $\boldsymbol{x} = \begin{bmatrix} 0 & 0 \end{bmatrix}^\top$ in the state plane

Trajectories for $u = 1$



Trajectories for $u = -1$



Switching curve

- Segments of the two parabolas through the origin form the switching curve,

$$x_1 = -\frac{1}{2} x_2^2 \operatorname{sgn}(x_2)$$

- If an initial state is above the switching curve, then $u = -1$ is used until the switching curve is reached
- Then, $u = 1$ is used to reach the origin
- For an initial state below the switching curve, the control $u = 1$ is used first to reach the switching curve, and then the control is switched to $u = -1$
- Implement the above control action as $u = -\operatorname{sgn}(v)$, where

$$v = v(x_1, x_2) = x_1 + \frac{1}{2} x_2^2 \operatorname{sgn}(x_2)$$

  is the equation describing the switching curve

The closed-loop system

- The switching curve

$$v = v(x_1, x_2) = x_1 + \frac{1}{2}x_2^2 \mathrm{sgn}(x_2)$$

- Can use $u = -\mathrm{sgn}(v)$ to synthesize a closed-loop system such that starting at an arbitrary initial state in the state plane, the trajectory will always be moving in an optimal fashion towards the origin
- Once the origin is reached, the trajectory will stay there.

Phase portrait of the closed-loop system