Part II of course

Methods of Analytical Mechanics
Goldstein, Classical Mechanics, 3rd Ed.
Hand & Finch, Analytical Mechanics

- Newtonian formulation can be thought of as "vectorial mechanics"
- Although Newtonian approaches sometimes look at aggregated quantities (e.g. COM), bodies are usually treated separately and linked by constraint forces, which are not always of interest.
- We will focus on two scalar properties of motion
  ↳ kinetic and potential energies, instead of forces to analyze the motion.

Constraint Types (Ch 1 Goldstein)

Assume that our system has $q_1, \ldots, q_{N_c}$ generalized coordinates

N_c "Number of coordinates"

	Time-dependent "Rheonomous"	Time-independent "Scleronomous"
Holonomic	$f(q_1, \ldots, q_{N_c}, t) = 0$	$f(q_1, \ldots, q_{N_c}) = 0$
Non-holonomic: INEQUALITIES		
Semi-holonomic	$f(q_1, \ldots, q_{N_c}, \dot{q}_1, \ldots, \dot{q}_{N_c}, t) = 0$	$f(q_1, \ldots, q_{N_c}, \dot{q}_1, \ldots, \dot{q}_{N_c}) = 0$

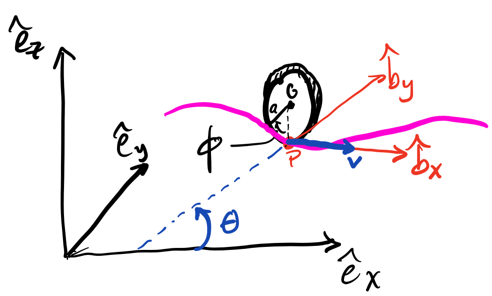
"Holonomic" constraints $\quad f(\vec{r}_{1/0}, \vec{r}_{2/0}, \ldots \vec{r}_{N/0}, t) = 0$

Ex. Rigid body $\quad \|\vec{r}_{i/j}\|^2 - c_{ij} = 0 \quad \forall\, i,j$

Other constraints are "Non-holonomic" if you cannot write simple equations connecting the #DOF with the location of all parts of the system.

Warning: Some constraints contain time rates of change and appear to be non-holonomic at first, but if they are integrable, then they are actually holonomic. Non-holonomic constraints resist integration.

Ex. Non-holonomic constraint: Rolling motion w/o slipping



coordinates

position: x, y, z

orientation: ϕ, θ, ψ

DOF

1 Rigid body

$M = 6(1) - \underset{2}{K_H} - \underset{2}{K_S} = 2 \qquad N_c = 4$

Constraints

1) Upright coin: $\hat{e}_z \cdot \hat{b}_y = 0 \Rightarrow \psi = 0$

2) In contact with the table: $\vec{r}_{G/0} \cdot \hat{e}_z = a$
$\Rightarrow z = a$

3+4) Cannot move sideways instantaneously

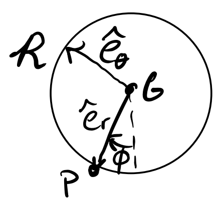
$${}^{I}\vec{V}_{G/0} \cdot \hat{b}_y = 0$$

(Leads to two scalar constraints that are non-holonomic)

Aside: Revisit rolling w/o slipping

(See 8.2, 13.3 in K+P)

Rolling w/o slipping ${}^I\vec{v}_{P/O} = 0$



$$\underbrace{\cancel{{}^I\vec{v}_{P/O}}}_{\substack{=0 \\ \text{By assumption}}} = {}^I\vec{v}_{G/O} + {}^I\vec{v}_{P/G}$$

$$= {}^I\vec{v}_{G/O} + \underbrace{\cancel{{}^R\vec{v}_{P/G}}}_{=0 \text{ (Body is rigid)}} + {}^I\vec{\omega}^R \times \vec{r}_{P/G}$$

$$0 = {}^I\vec{v}_{G/O} + (\dot{\phi}\hat{e}_3) \times (a\hat{e}_r)$$

$\Rightarrow {}^I\vec{v}_{G/O} = -a\dot{\phi}\hat{e}_\theta$

But ${}^I\vec{v}_{G/O} = v\hat{b}_x$ in the $\mathcal{B}$ frame

At the instant P contacts the ground

$\Rightarrow \boxed{v = a\dot{\phi}}$

Note: Not a constraint in this problem for the purposes of counting constraints, because v has been introduced here

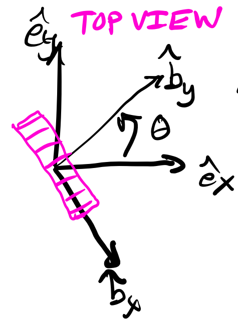
Q: How do we obtain coordinate representations of the nonholonomic constraints in the 3D coin problem?

From kinematics: $\vec{r}_{G/O} = x\hat{e}_x + y\hat{e}_y + a\hat{e}_z$

$${}^I\vec{v}_{G/O} = \dot{x}\hat{e}_x + \dot{y}\hat{e}_y$$

In the B frame,

$$ {}^{I}\vec{v}_{G/O} = \dot{x}\left(-s\theta\,\hat{b}_x + c\theta\,\hat{b}_y\right) + \dot{y}\left(c\theta\,\hat{b}_x + s\theta\,\hat{b}_y\right) $$
$$ = \left(-\dot{x}s\theta + \dot{y}c\theta\right)\hat{b}_x + \left(\dot{x}c\theta + \dot{y}s\theta\right)\hat{b}_y $$



	$\hat{e}_x$	$\hat{e}_y$
$\hat{b}_x$	$-s\theta$	$c\theta$
$\hat{b}_y$	$c\theta$	$s\theta$

Then,

$$ {}^{I}\vec{v}_{G/O} \cdot \hat{b}_y = 0 $$ (Recall constraint of no sideways motion)

$$ \Rightarrow \dot{x}\cos\theta + \dot{y}\sin\theta = 0 $$

One form of the constraint, but let's try to isolate $\dot{x}$ and $\dot{y}$.

Instead, Look at ${}^{I}\vec{v}_{G/O} = v\,\hat{b}_x$

$$ = -vs\theta\,\hat{e}_x + vc\theta\,\hat{e}_y $$
$$ = \dot{x}\,\hat{e}_x + \dot{y}\,\hat{e}_y $$

$$ \begin{cases} \dot{x} = -v\sin\theta \\ \dot{y} = v\cos\theta \end{cases} $$

Plug in and eliminate v:

$$ \dot{x} + a\dot{\phi}\sin\theta = 0 $$
$$ \dot{y} - a\dot{\phi}\cos\theta = 0 $$

Eliminating dt,

$$ dx + a\,d\phi\,\sin\theta = 0 $$
$$ dy - a\,d\phi\,\cos\theta = 0 $$

Neither equation can be integrated without solving the entire problem.

You can't integrate to get $x(\theta,\phi)$ and $y(\theta,\phi)$

In theory, if you had this, you could plug these in to eliminate x and y from the problem.

However the values of x and y depend not only on θ and ϕ, but on the history of the system (Hand and Finch p37)

Question: How can we determine if a constraint is non-holonomic?

Let's look at a necessary condition to be holonomic.

If $f(x,\theta,\phi)=0$ is holonomic, then $df=0$, and we can write

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial \theta}d\theta + \frac{\partial f}{\partial \phi}d\phi = 0$$

Compare this to $dx + a\,d\phi \sin\theta = 0$,

$$\frac{\partial f}{\partial x} = 1 \qquad \frac{\partial f}{\partial \theta} = 0 \qquad \frac{\partial f}{\partial \phi} = a\sin\theta$$

Mixed partial derivatives trick:

$$\frac{\partial^2 f}{\partial\theta\,\partial\phi} = \frac{\partial^2 f}{\partial\phi\,\partial\theta}$$

$$\frac{\partial^2 f}{\partial\phi\,\partial\theta} = 0 \quad \text{and} \quad \frac{\partial^2 f}{\partial\theta\,\partial\phi} = a\cos\theta$$

$\neq$

$\Rightarrow$ Non-holonomic

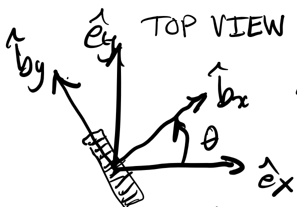
$\Rightarrow f$ that satisfies the form of an exact differential for our constraint doesn't exist.

$$\Rightarrow dx + a\,d\phi \sin\theta = 0$$

is a non-holonomic constraint.

Aside: The setup for this problem differs between texts and the equations vary slightly based on the reference frames used and definition of the θ angle.

<u>Goldstein appears to use:</u>



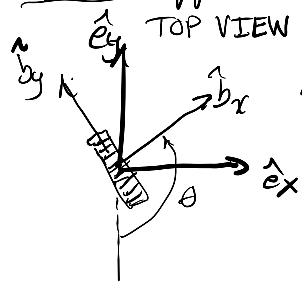
$$\begin{array}{c|cc} & \hat{e}_x & \hat{e}_y \\ \hline \hat{b}_x & c\theta & s\theta \\ \hat{b}_y & -s\theta & c\theta \end{array}$$

$${}^{I}\vec{v}_{G/O} = -v\,\hat{b}_y$$

$$\Rightarrow \begin{aligned} \dot{x} &= v\sin\theta \\ \dot{y} &= -v\cos\theta \end{aligned}$$

$$dx - a\sin\theta\, d\phi = 0$$
$$dy + a\cos\theta\, d\phi = 0$$

<u>H+F appear to use:</u>



$$\begin{array}{c|cc} & \hat{e}_x & \hat{e}_y \\ \hline \hat{b}_x & c(\theta-\frac{\pi}{2}) & s(\theta-\frac{\pi}{2}) \\ \hat{b}_y & -s(\theta-\frac{\pi}{2}) & c(\theta-\frac{\pi}{2}) \end{array}$$

$$\begin{array}{c|cc} & \hat{e}_x & \hat{e}_y \\ \hline \hat{b}_x & s\theta & -c\theta \\ \hat{b}_y & c\theta & s\theta \end{array}$$

$${}^{I}\vec{v}_{G/O} = -v\,\hat{b}_y$$

$$\Rightarrow \begin{aligned} \dot{x} &= -v\cos\theta \\ \dot{y} &= -v\sin\theta \end{aligned}$$

$$dx + a\cos\theta\, d\phi = 0$$
$$dy + a\sin\theta\, d\phi = 0$$

- These reference frames are not clearly defined in the text, but these interpretations appear to match.
- The convention isn't very important, because it does not impact the non-holonomy of the kinematic constraints.

Revisiting Degrees of freedom

In the presence of K_H holonomic constraints, we can introduce a set of $N_c = 3N - K_H$ independent scalars for the case of particles (or $N_c = 6N - K_H$ for rigid bodies) that are called <u>generalized cordinates</u> to describe the <u>configuration</u> of our system.

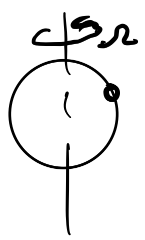
When non-holonomic constraints are present, they reduce the number of degrees of freedom, but we can't make use of them to reduce to configuration space.

3D particles
$$\begin{cases} \underset{\text{\#DOF}}{M} = 3N - K_H - \underset{\text{\# of non-holonomic constraints}}{K_s} \\ N_c = 3N - \underset{\text{\# of holonomic constraints}}{K_H} \end{cases}$$

Rigid bodies in 3D
$$\begin{cases} M = 6N - K_H - K_s \\ N_c = 6N - K_H \end{cases}$$

• Constraints can be further classified as time-dependent (rheonomous) or time-independent (scleronomous)

Ex. bead on a stationary hoop $\longrightarrow$ time-independent



bead on a moving hoop (with prescribed motion) $\longrightarrow$ time-dependent

bead on a moving wire that <u>freely</u> moves in response to the bead $\longrightarrow$ time-independent.

Difficulties in working with constraints

① Coordinates may not be independent when constraints are present.

② Forces that impose constraints are usually not provided "a priori" and depend upon the system's state

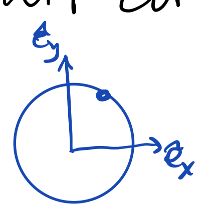
↳ They are among the unknowns of the system.

For holonomic constraints the positions $\vec{r}_{i/0}$ are not independent, however we can choose independent generalized coordinates to express them.

$$\vec{r}_{1/0} = \vec{r}_{1/0}(q_1, \dots, q_{N_c}, t)$$
$$\vdots$$
$$\vec{r}_{N/0} = \vec{r}_{N/0}(q_1, \dots, q_{N_c}, t)$$

Then, the constraint equations are contained in our formulation implicitly.

Ex. Particle in the plane constrained to the unit circle:



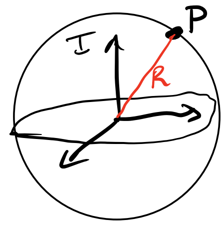
$\vec{r}_{P/0} = x\,\hat{e}_x + y\,\hat{e}_y$

with $x^2 + y^2 = 1$ (holonomic constraint; Note that x, y are dependent)

OR you could use polar coordinate θ:

$\vec{r}_{P/0} = \cos\theta\,\hat{e}_x + \sin\theta\,\hat{e}_y$ (constraint is implicit in our choice of coordinates)

Ex Particle constrained on a sphere



$\vec{r}_{P/O} = x\hat{e}_x + y\hat{e}_y + z\hat{e}_z$

with $x^2 + y^2 + z^2 = R^2$

OR You could use latitude and longitude as generalized cordinates. (Note: these angles are independent).

Notes:

- Generalized cordinates do not need to be positional or angular $\longrightarrow$ they can be general.
- If constraints are nonholonomic, they do not help reduce the number of coordinates needed
- Proper choice of cordinates can cause the constraint force to disappear since the constraint will appear implicitly

D'Alembert's Principle

We want a formulation of mechanics that doesn't require **constraint** forces to be calculated. Only **applied forces** need to be known.
The trick: The constraint forces do no work.

Start with N2L

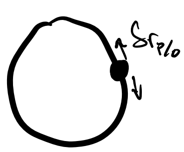
$$\vec{F}_i = {}^{I}\frac{d}{dt}\left({}^{I}\vec{p}_{i/O}\right)$$

Rewrite and sum over all particles

$$\sum_{i=1}^{N} \vec{F}_i - {}^{I}\frac{d}{dt}\left({}^{I}\vec{p}_{i/O}\right) = 0$$

Let's consider an infinitesimal displacement $\delta\vec{r}_{i/O}$ of particle i, which is called a **virtual displacement**, which is consistent with the instantaneous constraints and forces.

Bead on hoop



Let's look at the **virtual work** associated with the virtual displacement,

$$\sum_{i=1}^{N} \vec{F}_i \cdot \delta \vec{r}_{i/0}$$ virtual work

$$\vec{F}_i = \vec{F}_i^{(a)} + \vec{F}^{(c)}$$

"applied" or "active" forces

constraint forces

Dot both sides of N2L with the virtual displacement to create an expression that contains virtual work.

$$\sum_{i=1}^{N} \left(\vec{F}_i - \frac{^I d}{dt}\left({}^I\vec{p}_{i/0}\right)\right) \cdot \delta \vec{r}_{i/0} = 0$$

Sub in the force labeling we came up with:

$$\sum_{i=1}^{N} \left(\vec{F}_i^{(a)} + \vec{F}_i^{(c)} - \frac{^I d}{dt}\left({}^I\vec{p}_{i/0}\right)\right) \cdot \delta \vec{r}_{i/0} = 0$$

$$\sum_{i=1}^{N} \left(\vec{F}_i^{(a)} - \frac{^I d}{dt}\left({}^I\vec{p}_{i/0}\right)\right) \cdot \delta \vec{r}_{i/0} + \underbrace{\boxed{\sum_{i=1}^{N} \vec{F}_i^{(c)} \cdot \delta \vec{r}_{i/0}}}_{=0} = 0$$

By assumption, since we designed our virtual displacements to be consistent with the constraints

Virtual displacements for which this holds are called "reversible" (Meirovitch)

$$\boxed{\sum_{i=1}^{N}\left(\vec{F}^{(a)} - \frac{{}^{I}d}{dt}\left({}^{I}\vec{p}_{i/0}\right)\right)\cdot \delta\vec{r}_{i/0} = 0}$$

D'Alembert's Principle (Form #1 of 2)

The total virtual work from applied forces and "inertial forces" vanishes for a reversible virtual displacement.

Let's rewrite D'Alembert's principle to make it more useful for solving problems

Let's rewrite D'Alembert's principle using M generalized cordinates which are independent of each other.

$$\vec{r}_{i/0} = \vec{r}_{i/0}(q_1, \ldots, q_M, t)$$

<u>Rewrite velocity</u>:

$${}^{I}\vec{V}_{i/0} = \frac{{}^{I}d}{dt}\left(\vec{r}_{i/0}\right) = \sum_{j=1}^{M}$$

(Note: that we drop the I superscript on partial derivatives for convenience).

Rewrite virtual displacement:

$$\delta\vec{r}_{i/0} = \sum_{j=1}^{M} \frac{\partial \vec{r}_{i/0}}{\partial q_j}\,\delta q_j$$

Note: there is no variation in time in a virtual displacement

Rewriting Virtual work:

$$\sum_{i=1}^{N} \vec{F}_i^{(a)} \cdot \delta \vec{r}_{i/o} = \sum_{j=1}^{M} \sum_{i=1}^{N} \vec{F}_i^{(a)} \cdot \frac{\partial \vec{r}_{i/o}}{\partial q_j} \delta q_j$$