HW7

11.7.1) $\int_0^\infty \frac{\cos(xw) + w\sin(xw)}{1+w^2}\,dx$

$f(x) = \pi e^{-x}$

$A(w) = \frac{2}{\pi}\int_0^\infty \pi e^{-v}\cos(wv)\,dv$

$= 2\int_0^\infty e^{-v}\cos(wv)\,dv$

$\int e^{av}\cos(bv)\,dv = \frac{e^{av}}{a^2+b^2}\left[a\cos(bv) + b\sin(bv)\right]$

$\therefore \int e^{-v}\cos(wv)\,dv = \frac{e^{-v}}{1+w^2}\left[-\cos(wv) + w\sin(wv)\right]\Big|_0^\infty$

$= 0 - \left[\frac{1}{1+w^2}\left[-1 - 0\right]\right] = \frac{1}{1+w^2}$

$\therefore \boxed{A(w) = \frac{2}{1+w^2}}$

$B = \frac{2}{\pi}\int_0^\infty \pi e^{-v}\sin(wv)\,dv$

$\int e^{av}\sin(bv)\,dv = \frac{e^{av}}{a^2+b^2}\left(a\sin(bv) - b\cos(bv)\right) \quad \therefore$

$\int e^{-v}\sin(wv)\,dv = \frac{e^{-v}}{1+w^2}\left(-\sin(wv) - w\cos(wv)\right)\Big|_0^\infty$

$$= 0 - \left[\frac{1}{1+\omega^2}(0-\omega)\right] = \frac{\omega}{1+\omega^2}$$

$$\boxed{B(\omega) = \frac{2\omega}{1+\omega^2}}$$

1.7.11) $f(x) = \begin{cases} \sin x & 0 < x < \pi \\ 0 & x > \pi \end{cases}$

$$f(x) = \int_0^\infty A(\omega)\cos(\omega x)\,d\omega$$

$$A(\omega) = \frac{2}{\pi}\int_0^\pi \sin(v)\cos(\omega v)\,dv$$

$$\int \sin(av)\cos(bv)\,dv = -\frac{\cos(a-b)v}{2(a-b)} - \frac{\cos(a+b)v}{2(a+b)}$$

$a = 1$
$b = \omega$

$$= \left. -\frac{\cos((1-\omega)v)}{2(1-\omega)} - \frac{\cos((1+\omega)v)}{2(1+\omega)} \right|_0^\pi$$

$$= -\frac{\cos((1-\omega)\pi)}{2(1-\omega)} - \frac{\cos((1+\omega)\pi)}{2(1+\omega)} + \frac{1}{2(1-\omega)} + \frac{1}{2(1+\omega)}$$

$$= \frac{\cos(\omega\pi) + 1}{1-\omega^2}$$

$$\therefore A(\omega) = \frac{2}{\pi} \frac{\cos(\omega\pi)+1}{1-\omega^2}$$

$$\boxed{f(x) = \int_0^\infty \frac{2}{\pi} \frac{\cos(\omega\pi)+1}{1-\omega^2} \cos(\omega x)\, d\omega}$$

11.7.18) $f(x) = \begin{cases} \cos(x) & 0 < x < \pi \\ 0 & x > \pi \end{cases}$

$$B(\omega) = \frac{2}{\pi} \int_0^\pi \cos(v) \sin(\omega v)\, dv$$

$$\int \sin(av)\cos(bv)\, dv = -\frac{\cos((a-b)v)}{2(a-b)} - \frac{\cos((a+b)v)}{2(a+b)}, \quad a=\omega,\ b=1$$

$$= -\frac{\cos((\omega-1)v)}{2(\omega-1)} - \frac{\cos((1+\omega)v)}{2(\omega+1)} \Bigg|_0^\pi$$

$$= -\frac{\cos(\pi\omega-\pi)}{2(\omega-1)} - \frac{\cos(\pi+\omega\pi)}{2(\omega+1)} + \frac{1}{2(\omega-1)} - \frac{1}{2(\omega+1)}$$

$$= \frac{\omega\cos(\pi\omega)+\omega}{\omega^2-1}$$

$$\therefore B(\omega) = \frac{2\omega}{\pi}\left(\frac{\cos(\pi\omega)+1}{\omega^2-1}\right)$$

$$\boxed{f(x) = \int_0^\infty \frac{2\omega}{\pi} \frac{(\cos(\pi\omega)+1)}{\omega^2-1} \sin(\omega x)\, d\omega}$$

11.8.1) $f(x) = \begin{cases} 1 & 0 < x < 1 \\ -1 & 1 < x < 2 \\ 0 & x > 2 \end{cases}$

$f_c = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos(\omega x)\, dx = f_{c_1} + f_{c_2} + \cancel{f_{c_3}}^{0}$

$f_{c_1} = \sqrt{2/\pi} \int_0^1 \cos(\omega x)\, dx = \sqrt{2/\pi}\, \frac{1}{\omega} \left[\sin(\omega x) \Big|_0^1 \right]$

$f_{c_1} = \sqrt{2/\pi}\, \frac{\sin(\omega)}{\omega}$

$f_{c_2} = \sqrt{2/\pi} \int_1^2 -\cos(\omega x)\, dx = -\sqrt{\frac{2}{\pi}}\, \frac{1}{\omega} \left[\sin(\omega x) \Big|_1^2 \right]$

$= -\sqrt{\frac{2}{\pi}}\, \frac{\sin(2\omega)}{\omega} + \sqrt{\frac{2}{\pi}}\, \frac{\sin(\omega)}{\omega}$

$$\boxed{f_c = 2\sqrt{\frac{2}{\pi}}\, \frac{\sin(\omega)}{\omega} - \sqrt{2/\pi}\, \frac{\sin(2\omega)}{\omega}}$$

11.8.2) $f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty f_c \cos(\omega x)\, d\omega$

$\int_0^\infty f_c \cos(\omega x)\, d\omega = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin(\omega)\cos(\omega x)}{\omega}\, d\omega - \sqrt{2/\pi} \int_0^\infty \frac{\sin(2\omega)\cos(\omega x)}{\omega}\, d\omega$

from 11.7, example 2:

$$\int_0^\infty \frac{\cos(\omega x)\sin(\omega)}{\omega} = \begin{cases} \pi/2 & 0 < x < 1 \\ \pi/4 & x = 1 \\ 0 & x > 1 \end{cases}$$

$$\int_0^\infty \frac{\sin(2\omega)\cos(\omega x)}{\omega}\,d\omega \qquad \begin{aligned} u &= 2\omega \\ du &= 2\,d\omega \end{aligned}$$

$$\frac{1}{2}\int_0^\infty \frac{\sin(u)\cos(\frac{u}{2}x)}{u}\,d\omega = \frac{1}{2}\begin{cases} \pi/2 & 0 < x < 2 \\ \pi/4 & x = 2 \\ 0 & x > 2 \end{cases}$$

$$\therefore \int_0^\infty f_c \cos(\omega x)\,d\omega = \sqrt{2/\pi}\left\{\begin{matrix} \pi/2 & 0 < x < 1 \\ \pi/4 & x = 1 \\ 0 & x > 1 \end{matrix}\right\} - \sqrt{2/\pi}\begin{cases} \pi/4 & 0 < x < 2 \\ \pi/8 & x = 2 \\ 0 & x > 2 \end{cases}$$

$$= \begin{matrix} \sqrt{2/\pi}\,(\pi/2 - \pi/4) & 0 < x < 1 \\ -\sqrt{2/\pi}\,(\pi/4) & 1 < x < 2 \\ 0 & x > 2 \end{matrix} = \boxed{\begin{cases} \sqrt{2/\pi}\,\pi/4 & 0 < x < 1 \\ -\sqrt{2/\pi}\,\pi/4 & 1 < x < 2 \\ 0 & x > 2 \end{cases}}$$

Normalize by $\sqrt{2/\pi}\,\pi/4$

11.8.3) $\begin{cases} x & 0<x<2 \\ 0 & x>2 \end{cases}$

$$f_c = \int_0^2 x\cos(\omega x)\,dx \;\sqrt{2/\pi}$$

$u = \omega x$

$du = \omega\,dx$

$$f_c = \int_0^2 \frac{u}{\omega}\cos(u)\frac{du}{\omega} = \frac{1}{\omega^2}\int u\cos(u)\,du$$

$$\int u\cos u\,du = \cos(u) + u\sin(u)$$

$$\therefore\; f_c = \left(\frac{1}{\omega^2}\left[\cos(\omega x) + \omega x\sin(\omega x)\right]\Big|_0^2\right)\left(\sqrt{2/\pi}\right)$$

$$\boxed{f_c = \left(\sqrt{2/\pi}\right)\left(1/\omega^2\right)\left[\cos(2\omega) + 2\omega\sin(2\omega) - 1\right]}$$

11.8.5) $\begin{cases} x^2 & 0<x<1 \\ 0 & x>1 \end{cases}$

$$f_c = \sqrt{2/\pi}\int_0^1 x^2\cos(\omega x)\,dx$$

$u = \omega x \qquad du = \omega\,dx$

$$\int x^2 \cos(\omega x)\,dx = \int \left(\frac{u}{\omega}\right)^2 \cos(u)\frac{du}{\omega}$$

$$= \frac{1}{\omega^3}\int u^2 \cos(u)\,du$$

$$\int u^n \cos u\,du = u^n \sin(u) - n\int u^{n-1}\sin(u)\,du \quad , \quad n=2$$

$$\therefore \int u^2\cos(u)\,du = u^2 \sin(u) - 2\int u\sin(u)\,du$$

$$\int u\sin(u)\,du = \sin(u) - u\cos(u)$$

$$\int u^2 \cos(u)\,du = u^2\sin(u) - 2\sin(u) - 2u\cos(u)$$

$$\therefore \int_0^1 x^2\cos(\omega x)\,dx = \frac{1}{\omega^3}\left[(\omega x)^2\sin(\omega x) - 2\sin(\omega x) - 2\omega x\cos(\omega x)\right]\Big|_0^1$$

$$= \frac{1}{\omega^3}\left[\omega^2\sin(\omega) - 2\sin(\omega) - 2\omega\cos(\omega)\right]$$

$$\boxed{f_c = \left(\sqrt{2/\pi}\right)\left(\frac{1}{\omega^3}\right)\left(\omega^2\sin(\omega) - 2\sin(\omega) - 2\omega\cos(\omega)\right)}$$

11.9.3) $\begin{cases} 1 & a<x<b \\ 0 & \end{cases}$

$$f(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{-i\omega x}\,dx$$

$$f(\omega) = \frac{1}{\sqrt{2\pi}}\int_a^b e^{-i\omega x}\,dx$$

$$\int_a^b e^{-i\omega x}\,dx = \frac{-1}{i\omega}\left[e^{-i\omega b} - e^{-i\omega a}\right]$$

$$\boxed{f(\omega) = \frac{-1}{\sqrt{2\pi}\,i\omega}\left[e^{-i\omega b} - e^{-i\omega a}\right]}$$

11.9.4) $\begin{cases} e^{Kx} & x < 0 \\ 0 \end{cases}$

$$f(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} e^{Kx} e^{-i\omega x}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} e^{(K-i\omega)x}\,d\omega$$

$e^0 = 1, \; e^{-\infty} = 0$

$$= \left(\frac{1}{\sqrt{2\pi}}\right)\left(\frac{1}{K-i\omega}\right)(1-0)$$

$$= \left(\frac{1}{2\pi}\right)\left(\frac{1}{K-i\omega}\right)$$

$$\boxed{f(\omega) = \frac{1}{\sqrt{2\pi}\,(K-i\omega)}}$$

11.9.7) $\begin{cases} x & 0 < x < a \\ 0 \end{cases}$

$$f(\omega) = \frac{1}{\sqrt{2\pi}}\int_0^a x e^{-i\omega x}\,dx$$

$u = -i\omega x \qquad du = -i\omega\,dx$

$\therefore \int x e^{-i\omega x}\,dx = \frac{1}{(-i\omega)^2}\int u e^{u}\,du$

$\int u e^{u}\,du = (u-1)e^{u} = (-i\omega x - 1)e^{-i\omega x}\Big|_0^a$

$= (-i\omega a - 1)e^{-i\omega a} + 1$

$\therefore \int_0^a x e^{-i\omega x}\,dx = \frac{-1}{\omega^2}\left(-i\omega a e^{-i\omega a} - e^{-i\omega a} + 1\right)$

$\therefore \boxed{f(\omega) = \left(\frac{1}{\sqrt{2\pi}\,\omega^2}\right)\left(e^{-i\omega a}(1 + i\omega a) - 1\right)}$

12.1.2) $u = x^2 + t^2$

$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

$\frac{\partial^2 u}{\partial t^2} = 2 \qquad \frac{\partial^2 u}{\partial x^2} = 2 \quad \therefore$

$2 = c^2\, 2 \quad \therefore \boxed{c = \pm 1,\ u \text{ is solution}}$

12.1.8) $u = e^{-4t}\sin(\omega x)$

$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$

$$\frac{\partial u}{\partial t} = -9e^{-9t}\sin(\omega x)$$

$$\frac{\partial^2 u}{\partial x^2} = -\omega^2 e^{-9t}\sin(\omega x)$$

$$\therefore -9e^{-9t}\sin(\omega x) = (c^2)(-\omega^2 e^{-9t}\sin(\omega x))$$

$$\Rightarrow \quad 9 = c^2\omega^2 \quad \therefore \quad \boxed{c = \pm\frac{3}{\omega}}$$

$$c^2 = \frac{9}{\omega^2}$$

↑ u is solution for this c

12.1.10) $u = e^x\cos(y)$ $\quad u = e^x\sin(y)$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{\partial^2 u}{\partial x^2} = \cos(y)e^x \qquad \frac{\partial^2}{\partial y^2} = -\cos(y)e^x$$

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^x\cos(y) - \cos(y)e^x = 0}$$ ✓

$$\frac{\partial^2 u}{\partial x^2} = \sin(y)e^x \qquad \frac{\partial^2 u}{\partial y^2} = -\sin(y)e^x$$

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^x\sin(y) - \sin(y)e^x = 0}$$ ✓

12.1.1a) $U_y + y^2 u = 0$

$$u' + y^2 u = 0$$

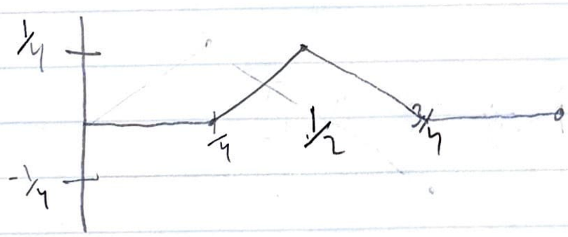
$$\frac{\partial u}{\partial y} = -y^2 u$$

$$\int \frac{\partial u}{u} = \int -y^2 \, \partial y$$

$$\ln|u| = C(x) - \frac{y^3}{3} \quad \therefore$$

$$\boxed{U = C(x)\, e^{-y^3/3}}$$

12.3.11)



$L = 1,\ c^2 = 1 \quad \therefore \quad \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$

$u(0,t) = 0 \qquad u(1,t) = 0$ (Boundary Condition)

$U(x,0) = f(x) \qquad U_t(x,0) = g(x)$

$$f(x) = \begin{cases} x - 1/4 & 1/4 < x < 1/2 \\ -x + 3/4 & 1/2 < x < 3/4 \end{cases}$$

$g(x) = 0$ (Zero Initial Velocity)

$\therefore B_n^* = 0$

$$B_n = 2\int_0^1 f(x)\sin(n\pi x)$$

$$= 2\left[\int_{1/4}^{1/2} (x - 1/4)\sin(n\pi x)\,dx + \int_{1/2}^{3/4} (-x + 3/4)\sin(n\pi x)\,dx\right]$$

$$= 2\left[-\tfrac{1}{4}\int_{1/4}^{1/2} \sin(n\pi x)\,dx + \int_{1/4}^{1/2} x\sin(n\pi x)\,dx - \int_{1/2}^{3/4} x\sin(n\pi x)\,dx + \tfrac{3}{4}\int_{1/2}^{3/4}\sin(n\pi x)\,dx\right]$$

$\int x\sin(n\pi x)\,dx, \quad u = n\pi x \quad du = n\pi\,dx$

$$\frac{1}{(n\pi)^2}\int u\sin(u)\,du = \frac{1}{(n\pi)^2}\left[\sin(n\pi x) - n\pi x\cos(n\pi x)\right]\Big|_a^b$$

$$-\frac{1}{4}\int_{1/4}^{1/2}\sin(n\pi x)\,dx = \frac{1}{4n\pi}\left[\cos\left(\frac{n\pi}{2}\right) - \cos\left(\frac{n\pi}{4}\right)\right]$$

$$\frac{3}{4}\int_{1/2}^{3/4}\sin(n\pi x)\,dx = \frac{-3}{4n\pi}\left[\cos\left(\frac{3n\pi}{4}\right) - \cos\left(\frac{n\pi}{2}\right)\right]$$

$$\int_{1/4}^{1/2} x\sin(n\pi x)\,dx = \frac{1}{(n\pi)^2}\left[\sin\left(\frac{n\pi}{2}\right) - \frac{n\pi}{2}\cos\left(\frac{n\pi}{2}\right) - \sin\left(\frac{n\pi}{4}\right) + \cos\left(\frac{n\pi}{4}\right)\frac{n\pi}{4}\right]$$

$$-\int_{1/2}^{3/4} x\sin(n\pi x)\,dx = \frac{-1}{(n\pi)^2}\left[\sin\left(\frac{3\pi}{4}n\right) - \frac{3\pi n}{4}\cos\left(\frac{3n\pi}{4}\right) - \sin\left(\frac{n\pi}{2}\right) + \frac{n\pi}{2}\cos\left(\frac{n\pi}{2}\right)\right]$$

$$\int f(x)\sin(n\pi x)\,dx = \frac{1}{(n\pi)^2}\left(2\sin\left(\frac{n\pi}{2}\right) - \sin\left(\frac{3\pi n}{4}\right) - \sin\left(\frac{n\pi}{4}\right)\right)$$

$\therefore B_n = \frac{2}{n^2\pi^2}\left(2\sin\left(\frac{n\pi}{2}\right) - \sin\left(\frac{3\pi n}{4}\right) - \sin\left(\frac{n\pi}{4}\right)\right)$

$$\therefore \boxed{U(x,t) = \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}\left(2\sin\left(\frac{n\pi}{2}\right) - \sin\left(\frac{3\pi n}{4}\right) - \sin\left(\frac{n\pi}{4}\right)\right)\cos(n\pi t)\sin(n\pi x)}$$

$U(x,t) = \sum\left(B_n\cos(\lambda_n t) + \cancel{B_n^* \sin(\lambda_n t)}^{\,0}\right)\sin\left(\frac{n\pi x}{L}\right)$

$\lambda_n = \frac{cn\pi}{L} = n\pi$

12.3.15) $\frac{\partial^2 u}{\partial t^2} = -c^2\frac{\partial^4 u}{\partial x^4}$

$U = F(x)\,G(t)$

$\frac{\partial^2 u}{\partial t^2} = F\ddot{G} \qquad \frac{\partial^4 u}{\partial x^4} = F^{(4)}G$

$\therefore F\ddot{G} = -c^2F^{(4)}G \Rightarrow \frac{\ddot{G}}{G} = -c^2\frac{F^{(4)}}{F}$

$$\therefore \boxed{\frac{-\ddot{G}}{c^2G} = \frac{F^{(4)}}{F} = K}$$

$F^{(4)} - KF = 0$

$\lambda^4 - K = 0 \Rightarrow \lambda = \pm\sqrt[4]{K} = B \therefore \quad \boxed{K = B^4}$

$F(x) = A\cos(Bx) + B\sin(Bx) + C_1e^{Bx} + C_2e^{-Bx}$

$$C\cosh(Bx) = \frac{Ce^{Bx} + Ce^{-Bx}}{2}$$

$$D\sinh(Bx) = \frac{De^{Bx} - De^{-Bx}}{2}$$

$$C\cosh(Bx) + D\sinh(Bx) = \frac{(C+D)e^{Bx}}{2} + \frac{(C-D)e^{-Bx}}{2}$$

Let $\frac{(C+D)}{2} = C_1$ & $\frac{(C-D)}{2}e^{-Bx} = C_2$

$\therefore C_1 e^{Bx} + C_2 e^{-Bx} = C\cosh(Bx) + D\sinh(Bx)$

$$\boxed{F(x) = A\cos(Bx) + B\sin(Bx) + C\cosh(Bx) + D\sinh(Bx)}$$

$\ddot{G} + c^2 K G = 0$

$K = B^4$

$\lambda^2 + c^2 K = 0 \Rightarrow \lambda_{1,2} = \pm\sqrt{-c^2K} = \pm(cB^2)i$

$$\therefore \boxed{G(t) = a\cos(cB^2t) + b\sin(cB^2t)}$$

12.3.16) Zero initial velocity $(\dot{G}^* = 0)$

$u(0,t) = 0$, $u(L,t) = 0$

$u_{xx}(0,t) = 0$, $u_{xx}(L,t) = 0$

$U_n = F_n(x)\, G_n(t)$

$F(0) = 0, \quad F(L) = 0$

$F''(0) = 0, \quad F_{xx}(L) = 0$

$F(0) = A + C = 0$

$F'' = -\beta^2 A\cos(\beta x) - \beta^2 B\sin(\beta x) + C\beta^2\cosh(\beta x) + D\beta^2\sinh(\beta x)$

$F''(0) = -A\beta^2 + C\beta^2 = 0$

$$\begin{pmatrix} 1 & 1 \\ -\beta^2 & \beta^2 \end{pmatrix}\begin{pmatrix} A \\ C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \quad A = C = 0$$

$F(L) = B\sin(\beta L) + D\sinh(\beta L) = 0 \quad \Rightarrow \sin(\beta L) = -\frac{D}{B}\sinh(\beta L)$

$F''(L) = -\beta^2 B\sin(\beta L) + D\beta^2\sinh(\beta L) = 0$

$\therefore \quad \beta^2 D\sinh(\beta L) + D\beta^2\sinh(\beta L) = 0 \Rightarrow 2\beta^2 D\sinh(\beta L) = 0 \quad \therefore$

$D = 0 \quad \therefore \quad \sin(\beta L) = 0 \quad \therefore \quad \beta = \frac{n\pi}{L} \quad \therefore \quad B = 1$

$\boxed{F(x) = \sinh\left(\frac{n\pi}{L}x\right)}$

$B_n^* = 0$ (Zero Velocity)

$\dot{G}(0) = 0 = b\,c\beta^2\cos(0) \Rightarrow \quad b = 0 \quad \therefore$

$\boxed{G(t) = a\cos(c\beta^2 t)}$ $\quad - \; U = F(x)\, G(t)$

12.4.7) $f(x) = K\sin(2\pi x)$, Zero initial velocity, $L = c = 1$

$f(x+ct) = K\sin(2\pi x + 2\pi ct)$
$f(x-ct) = K\sin(2\pi x - 2\pi ct)$

Zero initial velocity $\therefore$ $U(x,t) =$

$U = \frac{1}{2}K\sin(2\pi x + 2\pi t) + \frac{1}{2}K\sin(2\pi x - 2\pi t)$

$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$
$\sin(A-B) = \sin(A)\cos(B) - \sin(B)\cos(A)$

$\therefore \sin(A+B) + \sin(A-B) = 2\sin(A)\cos(B)$

$A = 2\pi x$, $B = 2\pi t$ $\therefore$

$$\boxed{U = K\sin(2\pi x)\cos(2\pi t)}$$