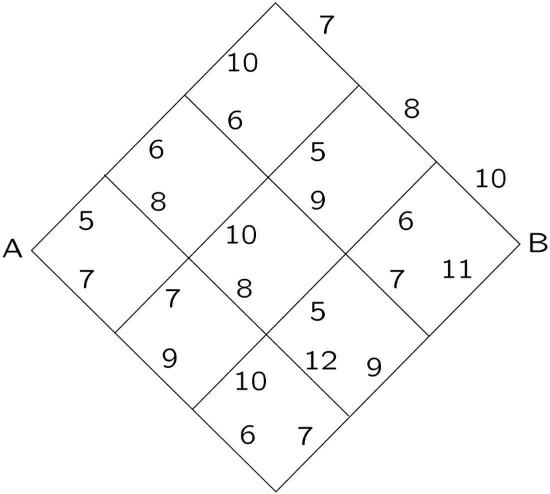
ECE 602: LUMPED LINEAR SYSTEMS

Professor Jianghai Hu

Dynamic Programming: Motivating Example

Movitating Example

- Start from point A
- Try to reach point B
- Each step only move right
- Cost labeled on each edge



Problem: Least costly path from A to B?

Formulated as an Optimal Control Problem

- $A = (0, 0),\ B = (3, 3)$
- State $x[k]$ with

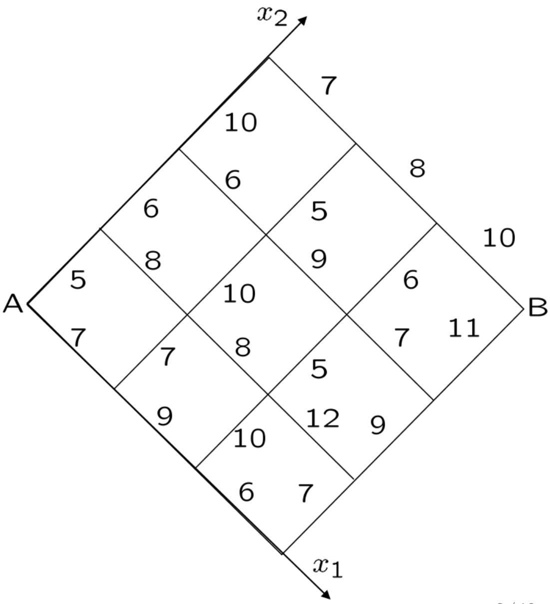
$$x[0] = A,\ x[6] = B$$

- Control $u[k] = \pm 1$
- Dynamics:

$$x[k+1] = \begin{cases} \mathrm{x[k]+(0,1)} & \mathrm{u[k]=1} \\ \mathrm{x[k]+(1,0)} & \mathrm{u[k]=-1} \end{cases}$$

- Cost to be minimized:

$$\sum_{k=0}^{5} \underbrace{w(x[k], u[k])}_{\text{edge weight}}$$



Direct Solution

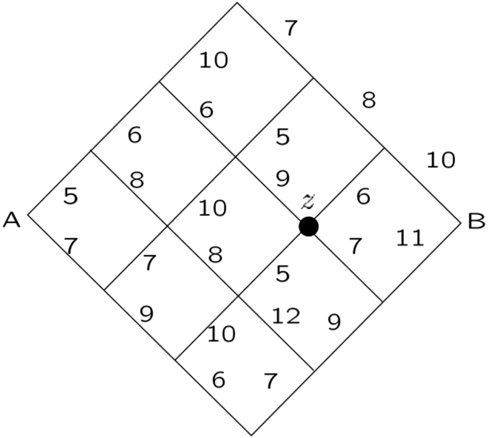
Enumerate all possible legal paths from A to B and compare their costs to find the one with the least cost.

- A total of 20 possible paths
- For ℓ-by-ℓ grid, the total number of legal paths is $\frac{(2\ell)!}{(\ell!)^2}$, impractical to enumerate for large ℓ

Value Function

At any point z, the value function (or cost-to-go) $V(z)$ is the least possible cost to reach B from z.

- Obtained by solving shorter time horizon problems
- Original problem is to find $V(A)$



Principle of Optimality

Principle of Optimality: Given a least-cost path from $x_0^* = A$ to $x_6^* = B$:

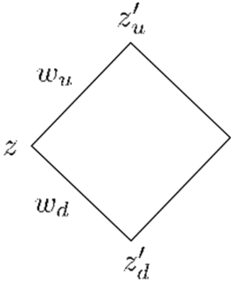
$$x_0^* \to x_1^* \to x_2^* \to \cdots \to x_6^*,$$

any truncation $x_t^* \to x_{t+1}^* \to \cdots \to x_6^*$ is a least-cost path from x_t^* to B

As a result, value function at any point z satisfies

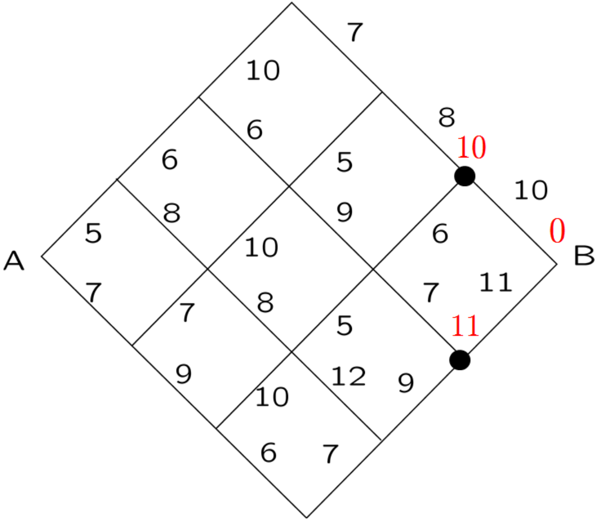
$$V(z) = \min\{w_u + V(z_u'), w_d + V(z_d')\}$$

- $V(z)$: cost-to-go from current position
- w_u, w_d: Running cost of current step
- $V(z')$: cost-to-go from next state position



Value Function Iteration

Idea: Use above to iteratively evaluate $V(z)$ from right to left



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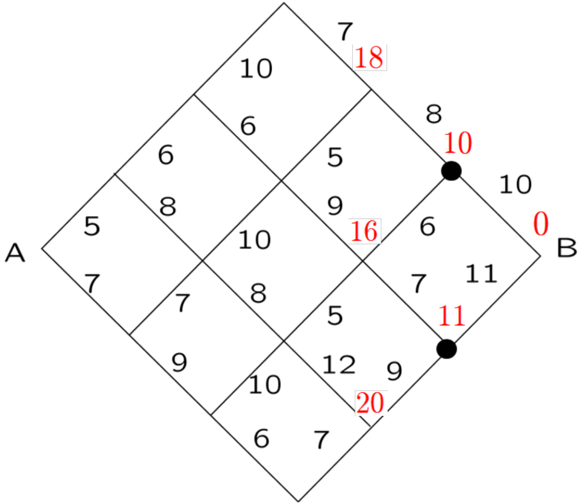
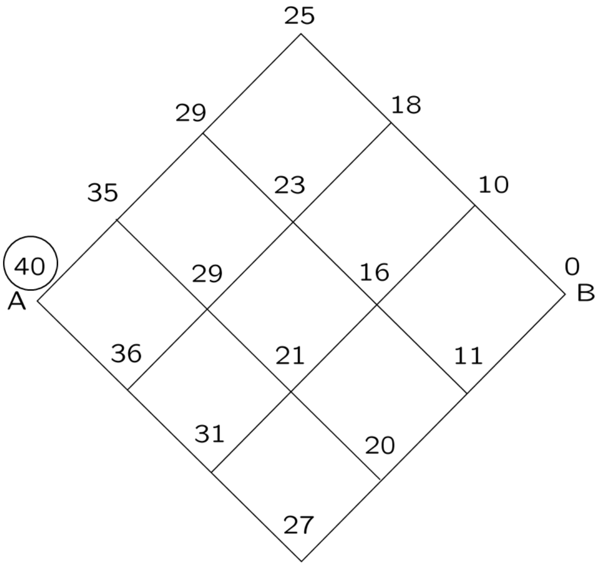


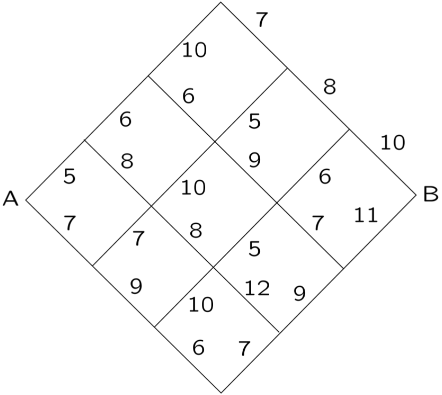
Value Function Iteration

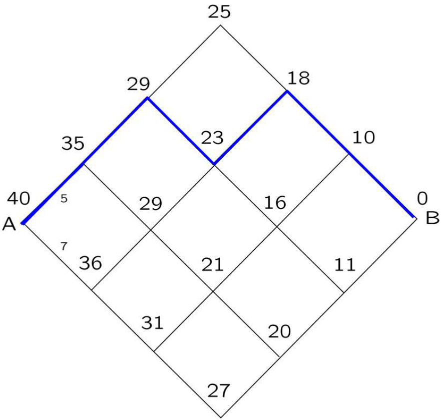
Conclusion: The least cost from A to B is 40



Recover the Optimal Control

Optimal control $u[0]$ is recovered from $V(A) = \min\{5 + 35, 7 + 36\}$





Advantages of Dynamic Programming

Reduced computational complexity: for ℓ-by-ℓ grid

- Only need to compute ℓ^2 value functions (no need to enumerate $\frac{(2\ell)!}{(\ell!)^2}$ paths)
- Solve an optimization problem of fixed size in each iteration

Provide solutions to all instances of the optimal control problem

- No need for re-computation if starting from a different initial position