ECE 68000: MODERN AUTOMATIC CONTROL

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An Introduction to Dynamic Programming for Discrete-Time Systems

Bellman's Principle of Optimality

- Dynamic programming is based on the principle of optimality (PO) invented by Richard E. Bellman (1920–1984) in 1957

An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision

- Dynamic programming is used to synthesize optimal controllers for nonlinear, time-varying dynamical systems
- Objective: present dynamic programming for discrete-time systems and then discuss its continuous version

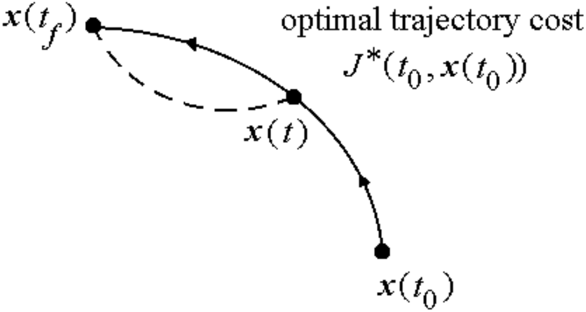
Some discussion of PO

- Alternative statement of PO

> An optimal trajectory has the property that at an intermediate point, no matter how it was reached, the rest of the trajectory must coincide with an optimal trajectory as computed from this intermediate point as the initial point

- PO seems to be already known to Johann Bernoulli who stated in 1706 that "… all curves, which should give a maximum, preserve also in all their sections the laws of the same maximum." Here, interpret maximum as optimum
- Bellman's principle of optimality serves to "limit the number of potentially optimal control strategies that must be investigated"
- The optimal control strategy is determined, using the PO, by working backward from the final stage

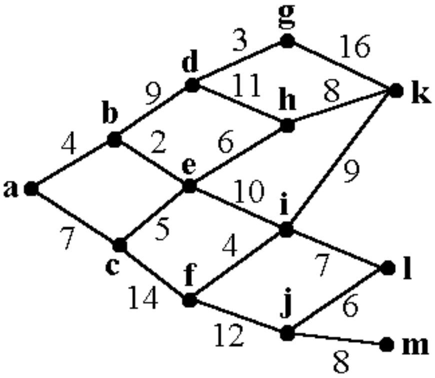
Illustration of the PO



Justification of the PO by contradiction

- Suppose that we are given an optimal trajectory starting from initial state $\boldsymbol{x}_0 = \boldsymbol{x}(t_0)$ at time t_0 and terminating at the final state $\boldsymbol{x}(t_f)$ at time t_f
- Let $\boldsymbol{x}(t)$ be an intermediate state on the optimal trajectory
- Let $J^*(t, \boldsymbol{x}(t))$ be the optimal cost of traveling from $\boldsymbol{x}(t)$ to the final state $\boldsymbol{x}(t_f)$
- Thus, $J^*(t_0, \boldsymbol{x}(t_0))$ is the optimal cost of traveling from $\boldsymbol{x}_0 = \boldsymbol{x}(t_0)$ to $\boldsymbol{x}(t_f)$
- It then follows that the portion of the optimal trajectory from $\boldsymbol{x}(t)$ to $\boldsymbol{x}(t_f)$ is also optimal
- For if some other path connecting $\boldsymbol{x}(t)$ and $\boldsymbol{x}(t_f)$ resulted in a smaller cost than the solid path, then this would provide a less expensive route from $\boldsymbol{x}_0$ to $\boldsymbol{x}(t_f)$, thus contradicting the optimality of the original trajectory

Example—a routing network



Finding optimal route

- Use the PO to find the minimum cost path that starts at the node **a** and ends at any one of the nodes **k**, **l**, **m** in the routing network
- The travel costs are shown beside each path segment
- The line segments can only be traversed from left to right
- Apply the PO to find the optimal path by performing a backward pass through the network
- Let J_q^* be the minimum cost from a generic node **q** to any one of the three possible final nodes
- Then, we have

$$J_g^* = 16, \quad J_h^* = 8, \quad J_i^* = 7, \quad J_j^* = 6$$

- The next stage, yields

$$J_d^* = \min\{3 + J_g^*, 11 + J_h^*\} = \min\{19, 19\} = 19$$

Finding optimal route—contd.

- Two optimal paths from node **d**
- Can take either **dgk** or **dhk** paths
- We then compute optimal paths from nodes **e** and **f**
- We have

$$J_e^* = \min\{6 + J_h^*, 10 + J_i^*\} = \min\{6 + 8, 10 + 7\} = 14$$

and

$$J_f^* = \min\{4 + J_i^*, 12 + J_j^*\} = \min\{4 + 7, 12 + 6\} = 11$$

- Thus, the optimal path emanating from node **e** is **ehk**, and from node **f** is **fil**
- The next stage yields

$$J_b^* = \min\{9 + J_d^*, 2 + J_e^*\} = \min\{9 + 19, 2 + 14\} = 16$$

and

$$J_c^* = \min\{5 + J_e^*, 14 + J_f^*\} = \min\{5 + 14, 14 + 11\} = 19$$

Finding optimal route—almost there

- Therefore, the optimal path starting from **b** is **behk**, and starting from **c** the optimal path is **cehk**
- Finally, the initial, and hence final, stage yields

$$J_a^* = \min\{4 + J_b^*, 7 + J_c^*\} = \min\{4 + 16, 7 + 19\} = 20$$

- Hence, the optimal, minimum total cost, route from **a** to any one of the three possible termination nodes **k**, **l**, or **m** is **abehk**
- The total cost of the optimal path is 20