ECE 68000: MODERN AUTOMATIC CONTROL

Professor Stan Żak

Two-Point Boundary-Value Problem (TPBVP): Example

Example

- Plant
$$\dot{x} = ax + bu, \quad x(t_0) = x_0,$$
- The associated cost index
$$J(t_0) = \frac{1}{2} f x(t_f)^2 + \frac{1}{2} \int_{t_0}^{t_f} \left(q x(t)^2 + r u(t)^2 \right) dt$$
with the initial time t_0
- The final time t_f is fixed such that $t_0 < t_f < \infty$
- The final state $x(t_f)$ is free
- Objective: construct the control u^* that minimizes the cost index J

Optimal control

- The Hamiltonian function

$$H\left(t, x, u, \frac{\partial J^*}{\partial x}\right) = \frac{1}{2}qx^2 + \frac{1}{2}ru^2 + \frac{\partial J^*}{\partial x}(ax + bu).$$

- No constraints on the control u
- Can determine the form of the optimal control by applying the first order necessary condition for static optimization
- Compute

$$\frac{\partial H}{\partial u} = ru + b\frac{\partial J^*}{\partial x} = 0$$

- The optimal control candidate

$$u^* = -\frac{1}{r}b\frac{\partial J^*}{\partial x}$$

Apply the HJB equation

- This control indeed minimizes the Hamiltonian function H because
$$\frac{\partial^2 H}{\partial u^2} = r > 0,$$
that is, the second order sufficient condition for static minimization is satisfied
- Reduced the problem of constructing the optimal control to that of finding $\frac{\partial J^*}{\partial x}$
- Substituting u^* into the HJB equation yields

$$\begin{aligned}
&\frac{\partial J^*}{\partial t} + \frac{1}{2}qx^2 + \frac{b^2}{2r}\left(\frac{\partial J^*}{\partial x}\right)^2 + ax\frac{\partial J^*}{\partial x} - \frac{b^2}{r}\left(\frac{\partial J^*}{\partial x}\right)^2 \\
&= \frac{\partial J^*}{\partial t} + \frac{1}{2}qx^2 - \frac{b^2}{2r}\left(\frac{\partial J^*}{\partial x}\right)^2 + ax\frac{\partial J^*}{\partial x} \\
&= 0
\end{aligned}$$

Set the HJB equation

- Assume that J^* is a quadratic function of the state for all $t \leq t_f$, that is,

$$J^* = \frac{1}{2}p(t)x^2,$$

  where $p(t)$ is a scalar function of time to be determined
- Partial derivatives of J^*

$$\frac{\partial J^*}{\partial t} = \frac{1}{2}\dot{p}(t)x^2 \quad \text{and} \quad \frac{\partial J^*}{\partial x} = p(t)x.$$

- Substituting the above into the HJB equation and performing some manipulations

$$\left(\frac{1}{2}\dot{p}(t) + \frac{1}{2}q + ap(t) - \frac{b^2 p(t)^2}{2r}\right) x^2 = 0$$

Preparing to solve nonlinear ODE

- Equivalently

$$\frac{1}{2}\dot{p}(t) + \frac{1}{2}q + ap(t) - \frac{b^2 p(t)^2}{2r} = 0,$$

that is,

$$\dot{p}(t) + q + 2ap(t) - \frac{b^2 p(t)^2}{r} = 0$$

subject to the boundary condition $p(t_f) = f$ that we obtain by comparing $J^* = \frac{1}{2}p(t)x^2$ and $J(t_f) = \frac{1}{2}fx(t_f)^2$

- To solve the above equation assume that

$$p(t) = \rho\frac{\dot{w}(t)}{w(t)},$$

where the parameter ρ is to be determined

- Note that

$$\dot{p}(t) = \rho\frac{\ddot{w}(t)w(t) - \dot{w}(t)^2}{w(t)^2}$$

Solve nonlinear ODE

- Substituting

$$
\begin{aligned}
\dot{p} + q + 2ap - \frac{b^2 p^2}{r} \\
&= \rho \frac{\ddot{w} w - \dot{w}^2}{w^2} + q \frac{w^2}{w^2} + 2a\rho \frac{\dot{w}}{w} - \frac{b^2 \rho^2 \dot{w}^2 / r}{w^2} \\
&= \frac{\rho \ddot{w} w - \rho \dot{w}^2 + q w^2 + 2a\rho \dot{w} w - b^2 \rho^2 \dot{w}^2 / r}{w^2} \\
&= 0
\end{aligned}
$$

- Choose ρ so that the terms nonlinear in $\dot{w}$ cancel out
- For this it is enough that

$$
-\rho \dot{w}^2 - b^2 \rho^2 \dot{w}^2 / r = 0.
$$

Reducing nonlinear ODE to linear ODE

- Set
$$\rho = -\frac{r}{b^2},$$
then the equation to be solved takes the form
$$\frac{\left(-\frac{r}{b^2}\ddot{w} - \frac{2ar}{b^2}\dot{w} + qw\right) w}{w^2} = 0.$$
- The above is satisfied if
$$-\frac{r}{b^2}\ddot{w} - \frac{2ar}{b^2}\dot{w} + qw = 0,$$
or equivalently
$$\ddot{w} + 2a\dot{w} - \frac{qb^2}{r}w = 0,$$
which is just a second order linear differential equation

Solving linear ODE

- Its characteristic equation has two real roots:

$$s_{1,2} = -a \pm \sqrt{a^2 + \frac{qb^2}{r}}$$

- Hence

$$w(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t},$$

  and

$$p(t) = \rho \frac{\dot{w}(t)}{w(t)} = \rho \frac{s_1 C_1 e^{s_1 t} + s_2 C_2 e^{s_2 t}}{C_1 e^{s_1 t} + C_2 e^{s_2 t}}$$

- Using the boundary value $p(t_f) = f$, we express C_1 in terms of C_2 to obtain

$$C_1 = \frac{\rho s_2 C_2 e^{s_2 t_f} - f C_2 e^{s_2 t_f}}{f e^{s_1 t_f} - \rho s_1 e^{s_1 t_f}}$$

- Substitute the above expression for C_1 into $p(t)$ and cancel out C_2

Obtain the optimal controller

- Use the obtained result to construct the optimal state-feedback controller

$$\boxed{u^* = -\frac{1}{r} b \frac{\partial J^*}{\partial x} = -\frac{1}{r} b p(t) x}$$