ECE 602: LUMPED LINEAR SYSTEMS

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Observability tests for continuous-time (CT) linear time-invariant (LTI) systems

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- **Objective**: Discuss test for observability of CT linear time-invariant (LTI) systems modeled as

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0 \\ \boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t) \end{aligned}$$

  where $\boldsymbol{A} \in \mathbb{R}^{n\times n}$, $\boldsymbol{B} \in \mathbb{R}^{n\times m}$, $\boldsymbol{C} \in \mathbb{R}^{p\times n}$, $p < n$, and $\boldsymbol{D} \in \mathbb{R}^{p\times m}$

- Recall the solution of the state equation,

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}(0) + \int_0^t e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)d\tau$$

Observability definition

The system

$$\left.\begin{array}{rcl} \dot{\boldsymbol{x}} & = & \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} \\ \boldsymbol{y} & = & \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}\boldsymbol{u}, \end{array}\right\}$$

or equivalently the pair $(\boldsymbol{A}, \boldsymbol{C})$, is observable if there is a finite $t_1 > t_0$ such that for arbitrary $\boldsymbol{u}(\cdot)$ and resulting $\boldsymbol{y}(\cdot)$ over $[t_0, t_1]$, we can determine $\boldsymbol{x}(t_0)$ from the knowledge of the system input $\boldsymbol{u}$ and output $\boldsymbol{y}$.

- Note that once $\boldsymbol{x}(t_0)$ is known, we can determine $\boldsymbol{x}(t)$ from knowledge of $\boldsymbol{u}(\cdot)$ and $\boldsymbol{y}(\cdot)$ over any finite time interval $[t_0, t_1]$

Preliminary manipulations

- The solution $\boldsymbol{y}(t)$

$$\boldsymbol{y}(t) = \boldsymbol{C}e^{\boldsymbol{A}(t-t_0)}\boldsymbol{x}(t_0) + \int_{t_0}^{t} \boldsymbol{C}e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)d\tau + \boldsymbol{D}\boldsymbol{u}(t)$$

- Subtract $\int_{t_0}^{t} \boldsymbol{C}e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)d\tau + \boldsymbol{D}\boldsymbol{u}(t)$ from both sides
- Let

$$\boldsymbol{g}(t) = \boldsymbol{y}(t) - \int_{t_0}^{t} \boldsymbol{C}e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)d\tau - \boldsymbol{D}\boldsymbol{u}(t)$$

- Then we have

$$\boldsymbol{g}(t) = \boldsymbol{C}e^{\boldsymbol{A}(t-t_0)}\boldsymbol{x}(t_0),$$

where $\boldsymbol{g}$ is known to us

Observability tests

Theorem

The following statements are equivalent:

(1) *The pair* $(\boldsymbol{A}, \boldsymbol{C})$ *is observable;*

(2) *The matrix* $\boldsymbol{V}(t_0, t_1) = \int_{t_0}^{t_1} e^{\boldsymbol{A}^\top t} \boldsymbol{C}^\top \boldsymbol{C} e^{\boldsymbol{A} t} dt$ *is nonsingular for all* $t_1 > t_0$*;*

(3) *The* n *columns of* $\boldsymbol{C} e^{\boldsymbol{A} t}$ *are linearly independent for all* $t \in [0, \infty)$ *over the real numbers; is full column rank* n*.*

Observability tests—Contd

The following statements are equivalent:

(1) The pair $(\boldsymbol{A}, \boldsymbol{C})$ is observable;

(4) The observability matrix

$$\begin{bmatrix} \boldsymbol{C} \\ \boldsymbol{C}\boldsymbol{A} \\ \vdots \\ \boldsymbol{C}\boldsymbol{A}^{n-1} \end{bmatrix} \in \mathbb{R}^{pn \times n}$$

is full column rank n.

(5)

$$\operatorname{rank} \begin{bmatrix} s\boldsymbol{I}_n - \boldsymbol{A} \\ \boldsymbol{C} \end{bmatrix} = n \quad \text{for all} \quad s \in \operatorname{eig}(\boldsymbol{A})$$

Relation between reachability and observability

- The pair (A, C) is observable if and only if the pair $\left(A^\top, C^\top\right)$ is reachable, and the other way around;
- The pair (A, B) is reachable if and only if the pair $\left(A^\top, B^\top\right)$ is observable

In other words

- The pair (A, C) is observable $\iff$ the pair $\left(A^\top, C^\top\right)$ is reachable
- The pair (A, B) is reachable $\iff$ if the pair $\left(A^\top, B^\top\right)$ is observable

(1) $\implies$ (2) by contraposition

- Recall
$$\boldsymbol{g}(t) = \boldsymbol{C} e^{\boldsymbol{A}(t-t_0)} \boldsymbol{x}(t_0)$$
- If $\boldsymbol{V}(t_0, t_1)$ is singular, there exists a nonzero constant vector, say $\boldsymbol{x}_a$, such that
$$\boldsymbol{V}(t_0, t_1)\boldsymbol{x}_a = \int_{t_0}^{t_1} e^{\boldsymbol{A}^\top t} \boldsymbol{C}^\top \boldsymbol{C} e^{\boldsymbol{A}t} dt \, \boldsymbol{x}_a = \mathbf{0}$$
- This implies that $\boldsymbol{C} e^{\boldsymbol{A}t} \boldsymbol{x}_a = \mathbf{0}$
- Therefore
$$\boldsymbol{g}(t) = \boldsymbol{C} e^{\boldsymbol{A}t} \boldsymbol{x}_0 = \boldsymbol{C} e^{\boldsymbol{A}t} \left(\boldsymbol{x}_0 + \boldsymbol{x}_a\right)$$
- Thus, $\boldsymbol{x}(0) = \boldsymbol{x}_0 + \boldsymbol{x}_a$ yields the same response as $\boldsymbol{x}(0) = \boldsymbol{x}_0$, which means that we cannot determine the system state
- In other words, the state fails to be observable if the observability Gramian is singular

Example

- For the dynamical system model,

$$\left.\begin{array}{rcl} \dot{x} & = & \boldsymbol{A}\boldsymbol{x} = \begin{bmatrix} 2 & 0 \\ 3 & -1 \end{bmatrix} \boldsymbol{x} \\ y & = & \boldsymbol{c}\boldsymbol{x} = \begin{bmatrix} -1 & 1 \end{bmatrix} \boldsymbol{x}, \end{array}\right\}$$

  find a non-zero initial vector $\boldsymbol{x}(0) = \boldsymbol{x}_0$ such that $y(t) = 0$ for all $t \geq 0$

- We have, $\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}_0$, where

$$e^{\boldsymbol{A}t} = \mathcal{L}^{-1}\left((s\boldsymbol{I} - \boldsymbol{A})^{-1}\right) = \begin{bmatrix} e^{2t} & 0 \\ e^{2t} - e^{-t} & e^{-t} \end{bmatrix}$$

Example—Contd

- Our objective is to find $\boldsymbol{x}_0$ such that

$$\begin{bmatrix} -1 & 1 \end{bmatrix} e^{\boldsymbol{A}t}\boldsymbol{x}_0 = 0 = y(t), \quad \text{for all} \quad t \geq .$$

- For example, if

$$\boldsymbol{x}_0 = \begin{bmatrix} 1 & 1 \end{bmatrix}^\top,$$

  then $y(t) = 0$ for all $t \geq 0$
- Note that we were able to find $\boldsymbol{x}_0 \neq \mathbf{0}$ such that $y(t) = 0$ for all $t \geq 0$, because for all $t \geq 0$, the columns of $\boldsymbol{c}e^{\boldsymbol{A}t}$ are linearly dependent over the real numbers
- This is because the pair $(\boldsymbol{A}, \boldsymbol{c})$ is nonobservable
- The vector $\boldsymbol{x}_0$ is in the null-space of the observability matrix

$$\begin{bmatrix} \boldsymbol{c} \\ \boldsymbol{c}\boldsymbol{A} \end{bmatrix}$$