#1

$A = \begin{pmatrix} 7 & -4 \\ -2 & 5 \end{pmatrix}$

a) $\chi_A = \lambda^2 - 12\lambda + 27 = 0$

$(\lambda - 3)(\lambda - 9) = 0$ $\qquad \lambda^2 - 9\lambda - 3\lambda + 27 = 0$

$\boxed{\begin{matrix} \lambda_1 = 3 \\ \lambda_2 = 9 \end{matrix}}$

$(A - \lambda_1 I)v_1 = 0$

$\begin{pmatrix} 4 & -4 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$4v_1 - 4v_2 = 0$ $\quad$ Let $v_2 = 1 \therefore v_1 = 1$

$(A - \lambda_2 I)v_2 = 0$

$\begin{pmatrix} -2 & -4 \\ -2 & -4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$ $\qquad -2v_1 - 4v_2 = 0$

Let $v_2 = 1 \therefore v_1 = -2$

$$\boxed{\begin{matrix} \text{For } \lambda_1 = 3, \ v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ \text{For } \lambda_2 = 9, \ v_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \end{matrix}}$$

check

$\begin{pmatrix} 4 & -4 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4-4 \\ -2+2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} -2 & -4 \\ -2 & -4 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4-4 \\ -4+4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

#1)

b)

$A = S \Lambda S^{-1}$

$$S = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}$$

$$\Lambda = \begin{pmatrix} 3 & 0 \\ 0 & 9 \end{pmatrix}$$

$$S^{-1} = \frac{1}{1--2} \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1/3 & 2/3 \\ -1/3 & 1/3 \end{pmatrix}$$

$$e^{At} = S e^{\Lambda t} S^{-1}$$

$$= \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{9t} \end{pmatrix} \begin{pmatrix} 1/3 & 2/3 \\ -1/3 & 1/3 \end{pmatrix}$$

$$= \begin{pmatrix} e^{3t} & -2e^{9t} \\ e^{3t} & e^{9t} \end{pmatrix} \begin{pmatrix} 1/3 & 2/3 \\ -1/3 & 1/3 \end{pmatrix}$$

#1)

b)

$$\boxed{e^{At} = \begin{pmatrix} \frac{e^{3t}}{3} + \frac{2}{3}e^{9t} & \frac{2}{3}e^{3t} - \frac{2}{3}e^{9t} \\ \frac{e^{3t}}{3} - \frac{e^{9t}}{3} & \frac{2e^{3t}}{3} + \frac{1}{3}e^{9t} \end{pmatrix}}$$

c) $\frac{du}{dt} = Au$

$$u(t) = e^{At} u_0$$

$$u(t) = \begin{pmatrix} \frac{e^{3}}{3} + \frac{2}{3}e^{9t} & \frac{2}{3}e^{3t} - \frac{2}{3}e^{9t} \\ \frac{e^{3t}}{3} - \frac{e^{9t}}{3} & \frac{2}{3}e^{3t} + \frac{1}{3}e^{9t} \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2}{3}e^{3t} + \frac{4}{3}e^{9t} + \frac{2}{3}e^{3t} - \frac{2}{3}e^{9t} \\ \frac{2e^{3t}}{3} - \frac{2}{3}e^{9t} + \frac{2}{3}e^{3t} + \frac{1}{3}e^{9t} \end{pmatrix}$$

$$\boxed{u(t) = \begin{pmatrix} 4/3\, e^{3t} + 2/3\, e^{9t} \\ 4/3\, e^{3t} - 1/3\, e^{9t} \end{pmatrix}}$$

#1

d)

The differential equation is <u>unstable</u> because $Re(\lambda_1) > 0$ & $Re(\lambda_2) > 0$.

#2

$x_n = .8x_{n-1} + .4y_{n-1}$

$y_n = .2x_{n-1} + .6y_{n-1}$

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} .8 & .4 \\ .2 & .6 \end{pmatrix} \begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix}$$

a) Yes, this is a Markov matrix as all colums add to 1 and each entry is greater than 0. Each entry is a probability.

b) Markov Matrix, so $\lambda_1 = 1$. Trace$(A) = 1.4$, therefore $\lambda_1 + \lambda_2 = 1.4$, $\therefore \lambda_2 = 0.4$

$$\boxed{\begin{aligned} \lambda_1 &= 1 \\ \lambda_2 &= 0.4 \end{aligned}}$$

Check: $(\lambda - 1)(\lambda - .4) = \lambda^2 \underbrace{-1.4}_{\text{trace}} + \underbrace{.4}_{\text{det}}$

$\det(A) = (.8)(.6) - (.4)(.2) = .48 - .08 = .4$

c) $(A - \lambda_1 I)v_1 = 0$

$$\begin{pmatrix} -0.2 & .4 \\ .2 & -.4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$-0.2v_1 + .4v_2 = 0$ Let $v_2 = 1 \therefore v_1 = 2$

$$v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

#2

c) $(A-\lambda_2 I)V_2 = 0$

$$\begin{pmatrix} .4 & .4 \\ .2 & .2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

Let $V_2 = 1$, $V_1 = -1$

$$V_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$S = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \qquad S^{-1} = \frac{1}{2-{-1}}\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1/3 & 1/3 \\ -1/3 & 2/3 \end{pmatrix}$$

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & .4 \end{pmatrix}$$

$$A^k = S\Lambda^k S^{-1} = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1^k & 0 \\ 0 & .4^k \end{pmatrix}\begin{pmatrix} 1/3 & 1/3 \\ -1/3 & 2/3 \end{pmatrix}$$

$$U_{ss} = \lim_{k\to\infty}(A^k U_0)$$

$$\lim_{k\to\infty}(A^k) = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1/3 & 1/3 \\ -1/3 & 2/3 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1/3 & 1/3 \\ -1/3 & 2/3 \end{pmatrix} = \begin{pmatrix} 2/3 & 2/3 \\ 1/3 & 1/3 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}$$

#2

c)

$$U_{ss} = \frac{1}{3}\begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$$

$$U_{ss} = \frac{1}{3}\begin{pmatrix} 1+1 \\ 1/2+1/2 \end{pmatrix} = 1/3\begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

$$\boxed{U_{ss} = \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}}$$

Steady state exists as A is diagonlizable, and for Markov matrices, the steady state projects to eigenvector associated with $\lambda=1$.

#3

a)

$\boxed{\det(C_1) = 0}$

$\det(C_2) = a_{11}C_{11} - a_{12}C_{12} = (0)(0) - (1)(1) =$

$\boxed{\det(C_2) = -1}$

$\det(C_3) = a_{11}C_{11} - a_{12}C_{12} + a_{13}C_{13} \quad = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

$= \cancel{0C_{11}} - (1)\begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} + \cancel{0C_{13}}$

$\boxed{\det(C_3) = 0}$

$\det(C_4) = \cancel{a_{11}C_{11}}^{0} - a_{12}C_{12} + \cancel{a_{13}C_{13}}^{0} + \cancel{a_{14}C_{14}}$

$= \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix} = (-1)\begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix}$

$= (-1)\left[(1)\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} - \cancel{(1)\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}}^{0}\right] = (-1)(-1) = 1$

$\boxed{\det(C_4) = 1}$

#3

b) $\det(C_n) = (-1)(C_{n-2})$

$C_2 = -1$, $C_4 = 1$, $C_6 = -1$, $C_8 = 1$, $C_{10} = -1$, $C_{12} = 1$, $C_{14} = -1$

$C_{16} = 1$, $C_{18} = -1$, $C_{20} = 1$

$\boxed{\det(C_{20}) = 1}$

#4

$A = \begin{pmatrix} -3 & 1 \\ -1 & -1 \end{pmatrix}$ $\begin{pmatrix} -3-\lambda & 1 \\ -1 & -1-\lambda \end{pmatrix} = (-3-\lambda)(-1-\lambda) = 3 + 4\lambda + \lambda^2 + 1$

$+1$

$\chi_A = \lambda^2 + 4\lambda + 4$

$\chi_A = 0 = (\lambda+2)(\lambda+2)$

$\lambda_1 = \lambda_2 = -2$

$(A - \lambda I) v_1 = 0$

$\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$

Let $v_2 = 1 \therefore v_1 = 1$

$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

a) $u_1 = \dfrac{v_1}{\|v_1\|} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$ $\|v_1\| = \sqrt{(1\ 1)\begin{pmatrix} 1 \\ 1 \end{pmatrix}} = \sqrt{2}$

$= \sqrt{v_1^T v_1}$

Let $u_2 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$

$u_2^T u_2 = (1/\sqrt{2} \ \ -1/\sqrt{2}) \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} = \left(\frac{1}{2} + \frac{1}{2}\right) = 1$

$u_2^T u_1 = (1/\sqrt{2} \ \ -1/\sqrt{2}) \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \left(\frac{1}{2} - \frac{1}{2}\right) = 0$

$\therefore u_2 \perp u_1$

#4)

$$\boxed{U = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}}$$

$$U^{-1} = \frac{1}{\left(\frac{1}{\sqrt{2}}\right)\left(\frac{-1}{\sqrt{2}}\right) - \left(1/\sqrt{2}\right)\left(1/\sqrt{2}\right)} \begin{pmatrix} -1/\sqrt{2} & -1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$U^{-1} = -\tfrac{1}{2} - \tfrac{1}{2} = -1$$

$$U^{-1} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$

Check $U^{H} = U = U^{-1}$ ✓

$$T = U^{-1} A U = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} -3 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{-3}{\sqrt{2}} - \frac{1}{\sqrt{2}} & 1/\sqrt{2} - 1/\sqrt{2} \\ \frac{-3}{\sqrt{2}} + \frac{1}{\sqrt{2}} & 1/\sqrt{2} + 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} -4/\sqrt{2} & 0 \\ -2/\sqrt{2} & 2/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} -4/2 & -4/2 \\ 0 & \frac{-2}{2} - \frac{2}{2} \end{pmatrix}$$

#4)

a) $\boxed{T = \begin{pmatrix} -2 & -2 \\ 0 & -2 \end{pmatrix}}$

b) $m^a(\lambda) = 2$

$m^g(\lambda) = 1$

$\therefore \boxed{J = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix}}$ ← 1 Jordan block $J_{-2,2}$

$(A - \lambda I) V_2 = V_1$

$\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$x = 0$
$y = 1$ $\therefore V_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\therefore \boxed{M = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}}$

#4

$$M^{-1} = \frac{1}{1}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

$$\overset{M^{-1}}{\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}} \overset{A}{\begin{pmatrix} -3 & 1 \\ -1 & -1 \end{pmatrix}} = \begin{pmatrix} -3 & 1 \\ 2 & -2 \end{pmatrix}$$

$$\underset{M^{-1}A}{\begin{pmatrix} -3 & 1 \\ 2 & -2 \end{pmatrix}} \underset{M}{\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}} = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix} \checkmark \quad M^{-1}AM = J$$

#5

a) $(1-i)(1+i) = 1^2 \cancel{+i-i} + 1^2$

$\boxed{\text{False, all eigenvalues of } U \text{ have } |\lambda| = 1}$

$|1+i| \neq 1$

b) $\boxed{\text{False, similior matrices have same eigenvalues}}$

c) $\boxed{\text{False, they have same eigenvalues but can have different } \chi_A.}$

d) $\boxed{\text{True}}$

e) $\boxed{\text{True}}$

$A^H = iA \quad \therefore \quad e^{A^H} = e^{iA} =$