AAE 666 Homework 8: Solution

Exercise 1

(a): $x(t) = \sin(t^2)$, oscillates within the interval [-1,1]. Its positive limit set is $[-1, 1]$.
(b): $x(t) = e^t \sin(t)$: For $t_k = 0, \pi, 2\pi, 3\pi, ...$ we have $x(t_k) = 0$. The positive limit set is $\{0\}$.

Exercise 2

Consider the candidate Lyapunov function:

$$V(x) = \frac{1}{2}(x_1^2 + x_2^2) - x_1$$

This function is radially unbounded. Now

$$\dot{V}(x) = \begin{bmatrix} x_1 - 1 & x_2 \end{bmatrix} \begin{bmatrix} x_2^2 \\ -x_1 x_2 \end{bmatrix} = -x_2^2 \leq 0$$

Therefore, all solutions are bounded and converge to the largest invariant set $\mathcal{M}$ contained in the set $S = \{x \in \mathbb{R}^n | x_2 = 0\}$. Thus all solutions must approach the x_1 axis.

Exercise 3

Let $x_1 = q$, and $x_2 = \dot{q}$, then:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -c(x_2) - k(x_1) \end{aligned}$$

Consider the candidate Lyapunov function:

$$V(x) = P(x_1) + \frac{1}{2}x_2^2 = \int_0^{x_1} k(\eta) d\eta + \frac{1}{2}x_2^2$$

If $\lim\limits_{x_1 \to \infty} P(x_1) = \infty$, then V is radially unbounded.

$$\begin{aligned} DV(x)f(x) &= \begin{bmatrix} k(x_1) & x_2 \end{bmatrix} \begin{bmatrix} x_2 \\ -c(x_2) - k(x_1) \end{bmatrix} \\ &= k(x_1)x_2 - c(x_2)x_2 - k(x_1)x_2 = -c(x_2)x_2 \leq 0 \end{aligned}$$

Hence all solutions are bounded. Since $c(x_2)x_2 > 0$ for all $x_2 \neq 0$, all solutions converge to the largest invariant set $\mathcal{M}$ in $\mathcal{S} = \{x \in \mathbb{R}^2 | x_2 = 0\}$. If x is in $\mathcal{M}$ then so is $-c(x_2) - k(x_1) = \dot{x}_2 = 0$. Hence $k(x_1) = 0$, that is, x is an equilibrium state.

Exercise 4

(a) Let $x_1 = q$, and $x_2 = \dot{q}$, then

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{c}{m}x_2 - \frac{k(x_1)}{m} \end{bmatrix}$$

(b)
(i) **La Salle Type.** Consider the candidate Lyapunov function:

$$V(x) \int_0^{x_1} k(\eta)d\eta + \frac{1}{2}mx_2^2$$

Clearly

$$V(0) = 0 \qquad \text{and} \qquad V(x) > 0 for x \neq 0$$

Since

$$\int_0^{x_1} k(\eta)d\eta = \infty$$

Together with the x_2^2 term, we have $\lim_{x\to\infty} V(x) = \infty$ and V is positive definite.

$$\begin{aligned} \dot{V} = DV(x)f(x) &= \begin{bmatrix} k(x_1) & mx_2 \end{bmatrix} \begin{bmatrix} x_2 \\ -\frac{c}{m}x_2 - \frac{k(x_1)}{m} \end{bmatrix} \\ &= k(x_1)x_2 - c(x_2)x_2 - k(x_1)x_2 = -c(x_2)x_2 \leq 0 \end{aligned}$$

If $\dot{V} \equiv 0$ for a solution, we must have $x_2 \equiv 0$; hence $\dot{x}_2 \equiv 0$ and $k(x_1) \equiv 0$. Since $k(x_1)x_1 > 0$ for all $x_1 \neq 0$, x_1 must be 0. Hence the only solution for which $\dot{V} \equiv 0$ is the zero solution. Therefore, the system is GAS about zero.

(i) **Non La Salle Type.** Consider the candidate Lyapunov function:

$$V(x) = \frac{1}{2}\lambda\frac{c^2}{m^2}x_1^2 + \lambda\frac{c}{m}x_1x_2 + \frac{1}{2}x_2^2 + \frac{1}{m}\int_0^{x_1} k(\eta)d\eta$$

Rewrite $V(x)$ with $E(x_1) = \int_0^{x_1} k(\eta)d\eta$ into:

$$\begin{aligned} V(x) &= \frac{1}{2}\lambda(\frac{c^2}{m^2}x_1^2 + 2\frac{c}{m}x_1x_2 + x_2^2) - \frac{1}{2}\lambda x_2^2 + \frac{1}{2}x_2^2 + \frac{1}{m}E(x_1) \\ &= \frac{1}{2}\lambda(\frac{c}{m}x_1 + x_2)^2 + \frac{1}{2}(1-\lambda)x_2^2 + \frac{1}{m}E(x_1) \\ &= \frac{1}{2}x' \begin{bmatrix} \lambda\frac{c^2}{m^2} & \lambda\frac{c}{m} \\ \lambda\frac{c}{m} & 1 \end{bmatrix} x + \frac{1}{m}E(x_1) \\ &= \frac{1}{2}x'Px + \frac{1}{m}E(x_1) \end{aligned}$$

Therefore, we see that $V(x)$ is radially unbounded,

$$\begin{aligned} V(x) &> 0, x \neq 0 \\ DV(x)f(x) &= -(1-\lambda)\frac{c}{m}x_2^2 - \lambda\frac{c}{m^2}x_1k(x_1) < 0, \forall x \neq 0 \end{aligned}$$

Therefore, the origin is GAS.

Exercise 5

Let $x_1 = q$ and $x_2 = \dot{q}$, then

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= a\sin x_1 + bu = a\sin x_1 - k_d b x_2 - k_p b x_1\end{aligned}$$

Consider the candidate Lyapunov function:

$$\begin{aligned}V(x) &= \int_0^{x_1} -a\sin\eta + k_p b\eta \, d\eta + \frac{1}{2}x_2^2 \\ &= E(x_1) + \frac{1}{2}x_2^2\end{aligned}$$

where

$$E(x_1) = a(\cos x_1 - 1) + \frac{1}{2}k_p b x_1^2$$

Since,

$$\begin{aligned}E(0) &= 0 \\ E'(0) &= 0 \\ E''(x_1) &= -a\cos x_1 + k_p b \geq -a + k_p b\end{aligned}$$

So if we choose

$$\boxed{k_p > \tfrac{a}{b}}$$

we have $E''(x_1) > 0$ for all x_1. Hence E and V are positive definite.

$$\begin{aligned}DV(x)f(x) &= \begin{bmatrix}-a\sin x_1 + k_p b x_1 & x_2\end{bmatrix}\begin{bmatrix}x_2 \\ a\sin x_1 - k_d b x_2 - k_p b x_1\end{bmatrix} \\ &= -a x_2 \sin x_1 + k_p b x_1 x_2 + a x_2 \sin x_1 - k_p b x_1 x_2 - k_d b x_2^2 \\ &= -k_d b x_2^2\end{aligned}$$

If

$$\boxed{k_d > 0}$$

we have

$$\begin{aligned}DV(x)f(x) &\leq 0 \text{ for all } x \\ DV(x)f(x) &= 0 \Rightarrow x_2 = 0\end{aligned}$$

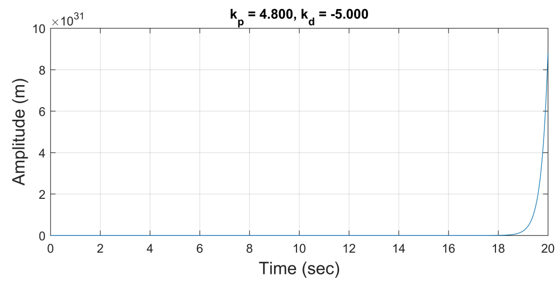
If $DV(x)f(x) \equiv 0$ for a solution, we must have $x_2 \equiv 0$; hence $\dot{x}_2 \equiv 0$ and

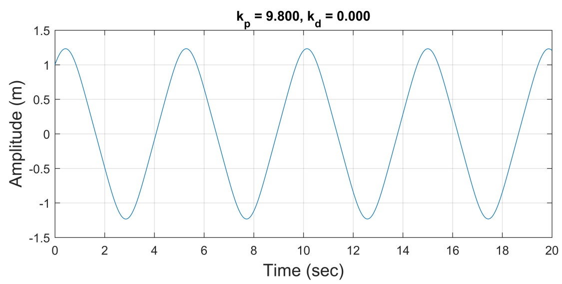
$$g(x_1) = a\sin x_1 - k_p b x_1 \equiv 0$$

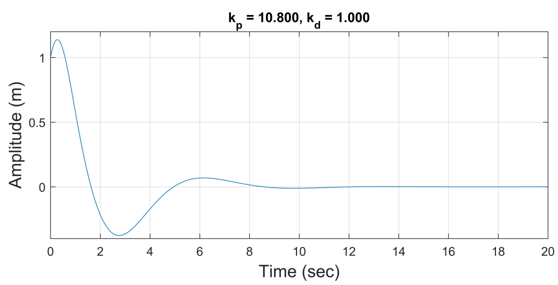
Note that

$$x_1 g(x_1) = a x_1 \sin x_1 - k_p b x_1^2 \leq (a - k_p b)x_1^2 < 0 \text{ for } x_1 \neq 0$$

Hence $g(x_1) = 0$ implies that $x_1 = 0$. Hence the only solution for which $DV(x)f(x) \equiv 0$ is the zero solution. Therefore, the system is GAS about zero.








```
clear all
close all
clc

% AAE 666
% Homework 8
% Problem 5
% Coder: Siwei Fan

tmat = linspace(0,20,2^13);

m = 1;
g = 9.8;
l = 1;
I = 1;
a = m*g*l/I;
```

```
b = 1/I;
kp_mat = [a/b-5 a/b a/b+1];
kd_mat = [-5 0 1]
for i = 1:3;
kp = kp_mat(i);
kd = kd_mat(i);
[t,x] = ode45(@(t,x)Func_Hw8_Prob5(t,x,kp,kd,a,b),tmat,[1 1]);
figure(i);
plot(t,x(:,1))
grid on;
set(gcf, 'Position',[180,100,2100/3,900/3])
temp=sprintf('k_p = %5.3f, k_d = %5.3f',kp,kd);
title(temp);
xlabel('Time (sec)','fontsize',14);
ylabel('Amplitude (m)','fontsize',14);
fig_name = sprintf('Fig_hw8_p5_%d',i);
print('-r600','-djpeg',fig_name)
end

% function [xdot]=Func_Hw8_Prob5(t,x,kp,kd,a,b)
% n = length(x);
% xdot = zeros(n,1);
% xdot(1) = x(2);
% xdot(2) = a*sin(x(1))-b*kp*x(1)-b*kd*x(2);
% end
```

Exercise 6

Let $x_1 = q$ and $x_2 = \dot{q}$. Then

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - x_2 + \theta \sin x_1 + u \end{aligned}$$

Consider

$$\boxed{u = -\hat{\theta} \sin x_1}$$

where $\hat{\theta}$ is an estimate of the constant parameter θ. This results in the closed loop system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - x_2 - \Delta\theta \sin x_1 \\ \Delta\dot{\theta} &= \dot{\hat{\theta}} \end{aligned}$$

where $\Delta\theta = \hat{\theta} - \theta$. For any $\alpha > 0$, consider the candidate Lyapunov function:

$$V(x, \Delta\theta) = \frac{3}{2}x_1^2 + x_1 x_2 + x_2^2 + \frac{1}{2}\alpha(\Delta\theta)^2$$

The function V is positive definite and

$$\dot{V} = DV(x, \Delta\theta)f(x, \Delta\theta) = \begin{bmatrix} 3x_1 + x_2 & x_1 + 2x_2 & \alpha\Delta\theta \end{bmatrix} \begin{bmatrix} x_2 \\ -x_1 - x_2 - \Delta\theta sin x_1 \\ \dot{\hat{\theta}} \end{bmatrix}$$

$$= -x_1^2 - x_2^2 - (x_1 + 2x_2)(\Delta\theta)\sin x_1 + \alpha(\Delta\theta)\dot{\hat{\theta}}$$

If we let

$$\boxed{\dot{\hat{\theta}} = \tfrac{1}{\alpha}(x_1 + 2x_2)\sin x_1}$$

we obtain that

$$\dot{V} = -x_1^2 - x_2^2 \leq 0$$

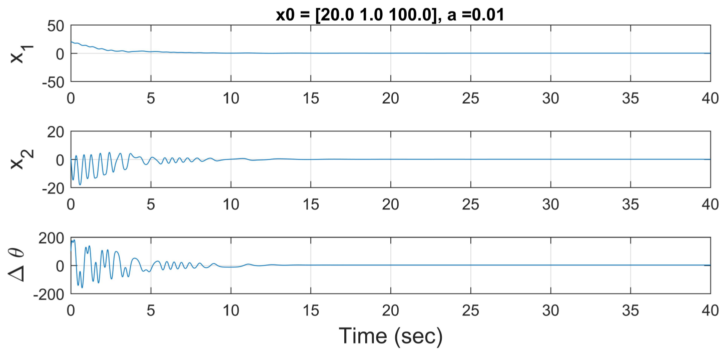
It now follows that all solutions are bounded and converge to the largest invariant set in $\{(x_1, x_2, \Delta\theta) \in \mathbb{R}^3 | x_1 = 0, x_2 = 0\}$. Hence

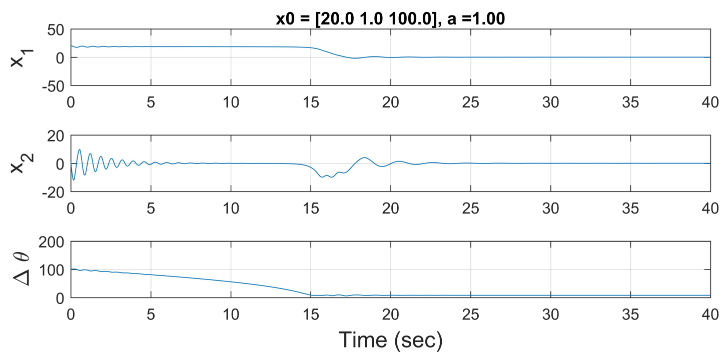
$$\lim_{t\to\infty} x(t) = 0 \text{ and } u(\cdot) \text{ is bounded}$$

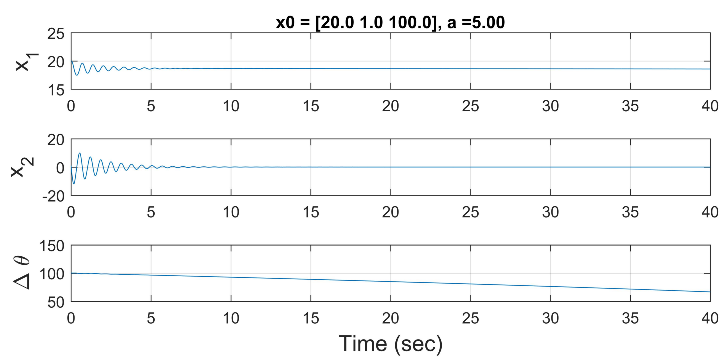
Numerical simulations are shown below to demonstrate the effectiveness of the controller regardless of the initial conditions (3-vector) and α value.

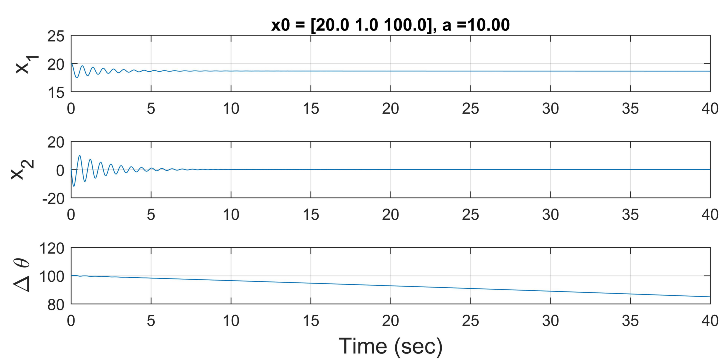
```
clear all
close all
clc
% AAE 666
% Homework 8
% Problem 6
% Coder: Siwei Fan

tmat = linspace(0,20,2^13);
alpha_mat = [0.01 1 5 10];
x0 = [20 100 100];
for i = 1:4;
alpha = alpha_mat(i);
[t,x] = ode45(@(t,x)Func_Hw8_Prob6(t,x,alpha),tmat,x0);
figure(i);
subplot(311);
plot(t,x(:,1))
subplot(312);
plot(t,x(:,2))
subplot(313);
plot(t,x(:,3))
```

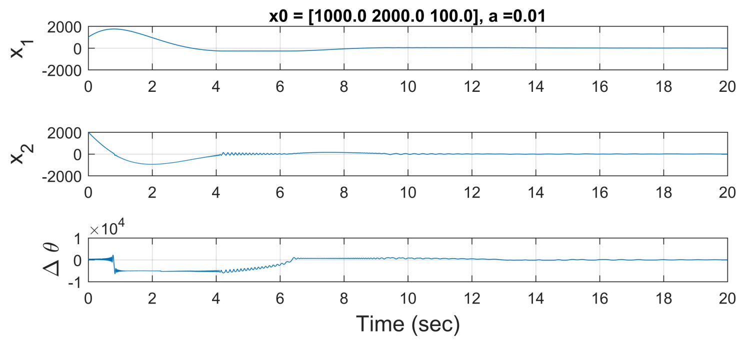
```
grid on;
set(gcf, 'Position',[180,100,2100/3,900/3])

subplot(311);grid on
ylabel('x_1','fontsize',14);
temp=[sprintf('x0 = [%3.1f %3.1f %3.1f]',x0) ...
sprintf(', a =%4.2f',alpha)];
title(temp);
subplot(312);grid on
ylabel('x_2','fontsize',14);
subplot(313);grid on
ylabel('\Delta \theta','fontsize',14);
xlabel('Time (sec)','fontsize',14);
gif_dir = sprintf('/Users/apple/Desktop/GIF/');
fig_name = sprintf('Fig_hw8_p6_%d',i);
print('-r600','-djpeg',fig_name)
end

% function [xdot]=Func_Hw8_Prob6(t,x,alpha)
% n = length(x);
% xdot = zeros(n,1);
% x1 = x(1);
% x2 = x(2);
% x3 = x(3);
% xdot(1) = x2;
% xdot(2) = -x1-x2-x3*sin(x1);
% xdot(3) = 1/alpha*(x1+2*x2)*sin(x1);
% end
```

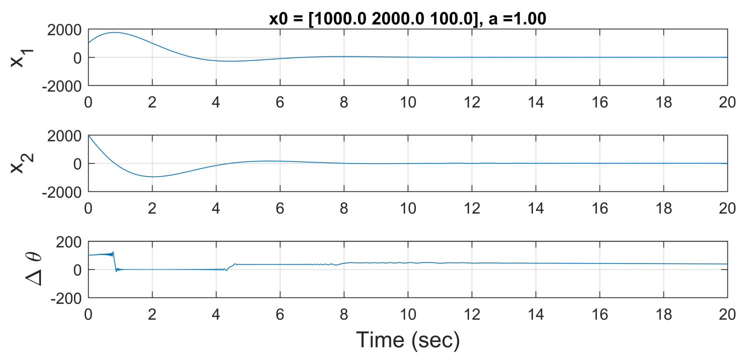

x0 = [20.0 1.0 100.0], a =0.01
x1
x2
Δ θ
Time (sec)

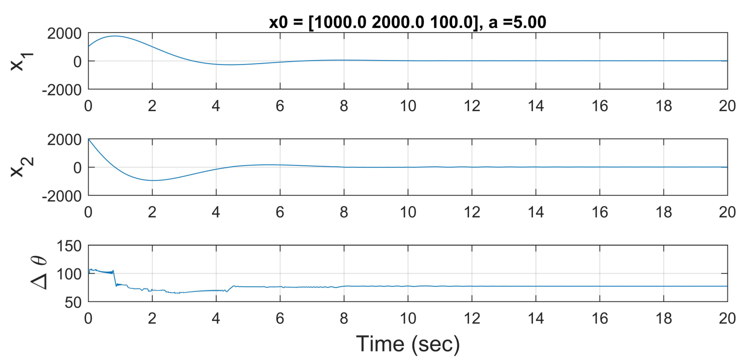

x0 = [20.0 1.0 100.0], a =1.00
x1
x2
Δ θ
Time (sec)

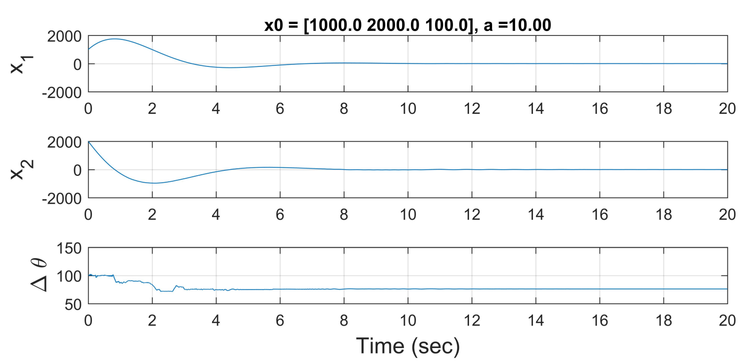

x0 = [20.0 1.0 100.0], a =5.00
x1
x2
Δ θ
Time (sec)


x0 = [20.0 1.0 100.0], a =10.00
x1
x2
Δ θ
Time (sec)


x0 = [1000.0 2000.0 100.0], a =0.01
x1
x2
Δ θ
×10^4
Time (sec)


x0 = [1000.0 2000.0 100.0], a =1.00
x1
x2
Δ θ
Time (sec)


x0 = [1000.0 2000.0 100.0], a =5.00
x1
x2
Δ θ
Time (sec)


x0 = [1000.0 2000.0 100.0], a =10.00
x1
x2
Δ θ
Time (sec)