HW5

6.5.7) $t * e^t$

$$h(t) = \int_0^t \tau e^{(t-\tau)} d\tau = \int_0^t \tau e^t e^{-\tau} d\tau = e^t \int_0^t \tau e^{-\tau} d\tau$$

$u = \tau \quad dv = e^{-\tau}$

$du = d\tau \quad v = -e^{-\tau}$

$$uv - \int v\,du = -\tau e^{-\tau}\Big|_0^t + \int_0^t e^{-\tau} d\tau$$

$$= -te^{-t} + [-e^{-\tau}\Big|_0^t] = -te^{-t} - e^{-t} + 1$$

$$h(t) = e^t(-te^{-t} - e^{-t} + 1) = -t\cancel{e^{t-t}}^{1} - \cancel{e^{t-t}}^{1} + e^t$$

$$\boxed{h(t) = e^t - t - 1}$$

6.5.8) $e^{at} * e^{bt}$

$$h(t) = \int_0^t e^{a\tau} e^{b(t-\tau)} d\tau = e^{bt} \int_0^t e^{a\tau} e^{-b\tau} = e^{bt} \int_0^t e^{(a-b)\tau} d\tau$$

$$= e^{bt}\left[\frac{1}{a-b} e^{(a-b)\tau}\Big|_0^t\right] = e^{bt}\left[\frac{1}{a-b} e^{(a-b)t} - \frac{1}{a-b}\right]$$

$$= \frac{1}{a-b}\left(e^{bt} e^{(a-b)t} - e^{bt}\right)$$

$$\boxed{h(t) = \frac{1}{a-b}\left(e^{at} - e^{bt}\right)}$$

6.5.23) $\frac{40.5}{s(s^2-9)} = \frac{A}{s} + \frac{Bs+C}{s^2-9}$

$40.5 = As^2 - 9A + Bs^2 + Cs$

$0 = A + B \qquad A = -B$

$0 = C$

$40.5 = -9A \qquad A = -4.5 \qquad B = 4.5$

$\frac{40.5}{s(s^2-9)} = \frac{-4.5}{s} + \frac{4.5s}{s^2-9}$

$\mathcal{L}^{-1}\left[\frac{-4.5}{s}\right] = -4.5 \ , \ \mathcal{L}^{-1}\left[\frac{4.5s}{s^2-9}\right] = 4.5\cosh(3t)$

$f = -4.5 \ , \quad g = 4.5\cosh(3t)$

$f * g = \int_0^t (-4.5)(4.5)\cosh(3(t-\tau))\,d\tau$

$\cosh(x) = \frac{e^x + e^{-x}}{2}$

$\therefore \ f * g = -10.125 \int_0^t e^{3(t-\tau)} + e^{-3(t-\tau)}\,d\tau$

$= -10.125\left[e^{3t}\int_0^t e^{-3\tau}\,d\tau + e^{-3t}\int_0^t e^{3\tau}\,d\tau\right]$

$= -10.125\left[\frac{e^{3t}}{-3}\left[e^{-3t} - 1\right] + \frac{e^{-3t}}{3}\left[e^{3t} - 1\right]\right]$

$= -\frac{10.125}{3}\left[-1 + e^{3t} + 1 - e^{-3t}\right] = -\frac{10.125}{3}\left(e^{3t} - e^{-3t}\right)$

$-e^{-3t} + e^{3t} = 2\sinh(3t)$ ∴

$$\boxed{f(t) = -6.75 \sinh(3t)}$$

6.6.3) $\frac{1}{2} t e^{-3t}$

$\mathcal{L}[t f(t)] = (-1) \frac{d}{ds} L(f(t))$

Let $f(t) = e^{-3t}$, $\mathcal{L}[f(t)] = \frac{1}{s+3}$

$\frac{d}{ds}(s+3)^{-1} = (-1)(s+3)^{-2} = \frac{-1}{(s+3)^2}$ ∴

$\mathcal{L}[t f(t)] = (-1)(-1)\left(\frac{1}{(s+3)^2}\right) = \frac{1}{(s+3)^2}$

$$\boxed{\mathcal{L}[\tfrac{1}{2} t e^{-3t}] = \frac{1}{2(s+3)^2}}$$

6.6.8) $t e^{-Kt} \sin(t)$

$f(t) = e^{-Kt} \sin(t)$, $\mathcal{L}[f(t)] = \frac{1}{(s+K)^2+1}$

$\frac{d}{ds}\left([(s+K)^2+1]^{-1}\right) = \frac{(-1)(2)(s+K)}{((s+K)^2+1)^2} = \frac{(-2)(s+K)}{((s+K)^2+1)^2}$

$\mathcal{L}(t f(t)) = (-1) \frac{d}{ds}[L(f(t))]$ ∴

$$\boxed{\mathcal{L}[t e^{-Kt} \sin(t)] = \frac{2(s+K)}{[(s+K)^2+1]^2}}$$

6.6.10) $t^n e^{kt}$

$$L(t^n f(t)) = (-1)^n \frac{d^n}{ds^n} L(f(t))$$

$$f(t) = e^{kt} \qquad \mathcal{L}[e^{kt}] = \frac{1}{s-k}$$

$$\therefore \frac{d^n}{ds^n} L(f) = \frac{d^n}{ds^n} = \frac{1}{(s-k)}$$

$$\frac{d}{ds}(s-k)^{-1} = (-1)(s-k)^{-2}, \quad \frac{d^2}{ds^2}(-(s-k)^{-2}) = 2(s-k)^{-3}$$

$$\therefore \frac{d^n}{ds^n} = \frac{(-1)^n (n!)}{(s-k)^{n+1}}, \quad \frac{d^3}{ds^3} = -6(s-k)^{-4}$$

$$\boxed{\mathcal{L} = \frac{(-1)^n (-1)^n (n!)}{(s-k)^{n+1}}}$$

6.6.16) $\frac{2s+6}{(s^2+6s+10)^2}$

$$\int_s^\infty \frac{2s+6}{(s^2+6s+10)^2} ds, \quad u = s^2+6s+10$$
$$du = (2s+6)ds$$

$$\int_s^\infty \frac{1}{u^2} du = -u^{-1} = \frac{-1}{s^2+6s+10}\Bigg|_s^\infty$$

$$= 0 + \frac{1}{s^2+6s+10} = \frac{1}{s^2+6s+10} = \mathcal{L}\left[\frac{f(t)}{t}\right]$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^2+6s+10}\right] = \mathcal{L}^{-1}\left[\frac{1}{(s+3)^2+1}\right] = e^{-3t}\sin(t) = \frac{f(t)}{t}$$

$$\therefore \boxed{f(t) = t e^{-3t} \sin(t)}$$

6.7.3) $y_1' = -y_1 + 4y_2$ $\quad y_1(0) = 3$

$y_2' = 3y_1 - 2y_2$ $\quad y_2(0) = 4$

$$A = \begin{pmatrix} -1 & 4 \\ 3 & -2 \end{pmatrix} \qquad Y_0 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$(A - sI)Y = -Y(0)$$

$$(-2-s)(-1-s) = 2+2s+s+s^2$$

$$\underbrace{\begin{pmatrix} -1-s & 4 \\ 3 & -2-s \end{pmatrix}}_{M} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} -3 \\ -4 \end{pmatrix}$$

$$M^{-1} = \begin{pmatrix} -2-s & -4 \\ -3 & -1-s \end{pmatrix} \frac{1}{s^2+3s-10}$$

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} -2-s & -4 \\ -3 & -1-s \end{pmatrix} \begin{pmatrix} -3 \\ -4 \end{pmatrix} \frac{1}{s^2+3s-10} = \begin{pmatrix} 6+3s+16 \\ 9+4+4s \end{pmatrix} \frac{1}{s^2+3s-10}$$

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} \frac{3s+22}{s^2+3s-10} \\ \frac{4s+13}{s^2+3s-10} \end{pmatrix}$$

$$\frac{3s+22}{s^2+3s-10} = \frac{3s+22}{(s+5)(s-2)} = \frac{A}{s+5} + \frac{B}{s-2}$$

$3 = A + B$ $\qquad$ $A = 3 - B$

$22 = 5B - 2A$ $\qquad$ $22 = 5B - 6 + 2B \Rightarrow 28 = 7B$

$B = 4$

$A = -1$

$$\frac{3s+22}{s^2+3s-10} = \frac{-1}{s+5} + \frac{4}{s-2} = Y_1(s)$$

$$Y_2(s) = \frac{4s+13}{(s+5)(s-2)} = \frac{A}{(s+5)} + \frac{B}{(s-2)}$$

$4 = A + B$ $\qquad$ $A = 4 - B$

$13 = 5B - 2A = 5B - 8 + 2B \Rightarrow B = 3$

$A = 1$

$$Y_2(s) = \frac{1}{s+5} + \frac{3}{s-2}$$

$$\mathcal{L}^{-1}\left[\frac{-1}{s+5} + \frac{4}{s-2}\right] = -e^{-5t} + 4e^{2t}$$

$$\mathcal{L}^{-1}\left[\frac{1}{s+5} + \frac{3}{s-2}\right] = e^{-5t} + 3e^{2t}$$

$$\boxed{\begin{aligned} Y_1(t) &= -e^{-5t} + 4e^{2t} \\ Y_2(t) &= e^{-5t} + 3e^{2t} \end{aligned}}$$

6.7.12) $y_1'' = -2y_1 + 2y_2$
$y_2'' = 2y_1 - 5y_2$

$$s^2Y_1 - s\cancel{Y_1(0)}^{1} - \cancel{Y_1'(0)}^{0} = -2Y_1(s) + 2Y_2(s)$$

$$s^2Y_2 - s\cancel{Y_2(0)}^{3} - \cancel{Y_2'(0)}^{0} = 2Y_1(s) - 5Y_2(s)$$

$$s^2Y_1 - s = -2y_1 + 2y_2$$
$$s^2Y_2 - 3s = 2y_1 - 5y_2$$

$$\begin{pmatrix} s^2+2 & -2 \\ -2 & s^2+5 \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} s \\ 3s \end{pmatrix}$$

$$M^{-1} = \begin{pmatrix} s^2+5 & 2 \\ 2 & s^2+2 \end{pmatrix} \frac{1}{(s^2+5)(s^2+2)-4}$$

$$Y = \begin{pmatrix} s^2+5 & 2 \\ 2 & s^2+2 \end{pmatrix} \begin{pmatrix} s \\ 3s \end{pmatrix} \frac{1}{(s^2+5)(s^2+2)-4} = \begin{pmatrix} s^3+5s+6s \\ 2s+3s^3+6s \end{pmatrix} \frac{1}{(s^2+5)(s^2+2)-4}$$

$$= \begin{pmatrix} s(s^2+11) \\ s(3s^2+8) \end{pmatrix} \frac{1}{(s^2+5)(s^2+2)-4} = \begin{pmatrix} \frac{2s}{s^2+1} - \frac{s}{s^2+6} \\ \frac{s}{s^2+1} + \frac{2s}{s^2+6} \end{pmatrix}$$

$$\mathcal{L}^{-1}\left[\frac{2s}{s^2+6}\right] = 2\cos(\sqrt{6}\,t)$$

$$\mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right] = \cos(t)$$

$$\mathcal{L}^{-1}\left[\frac{-s}{s^2+6}\right] = -\cos(\sqrt{6}\,t)$$

$$\mathcal{L}^{-1}\left[\frac{2s}{s^2+1}\right] = 2\cos(t)$$

$$\therefore \boxed{\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} 2\cos(t) - \cos(\sqrt{6}\,t) \\ 2\cos(\sqrt{6}\,t) + \cos(t) \end{pmatrix}}$$