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Ch 3 in K+P Particle Dynamics

Constraints and degrees of freedom (DOF)

Collection of particles: The number of DOF is the number of independent coordinates needed to describe the position of every particle.

$(x,y)_I$

$\underbrace{x^2+y^2=r^2}_{\text{constraint}}$

2-D: $\underbrace{2N}_{\text{Particle}} - \underbrace{K}_{\text{Constraints}} = \underbrace{M}_{\text{DOF}}$

3-D: $M = 3N - K$

Collection of rigid bodies: The number of DOF is the number of coordinates needed to describe the position and orientation of every body

(x, y, α)

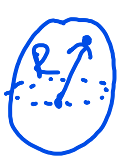
2-D: $M = 3N - K$

3-D: $M = \underbrace{6}_{\text{DOF per particle}}N - K$

A constraint force reduces the number of DOF and often acts orthogonal to direction of motion. In which case it is called a normal force

Constraints can often be expressed algebraically.

Ex. Particle on sphere

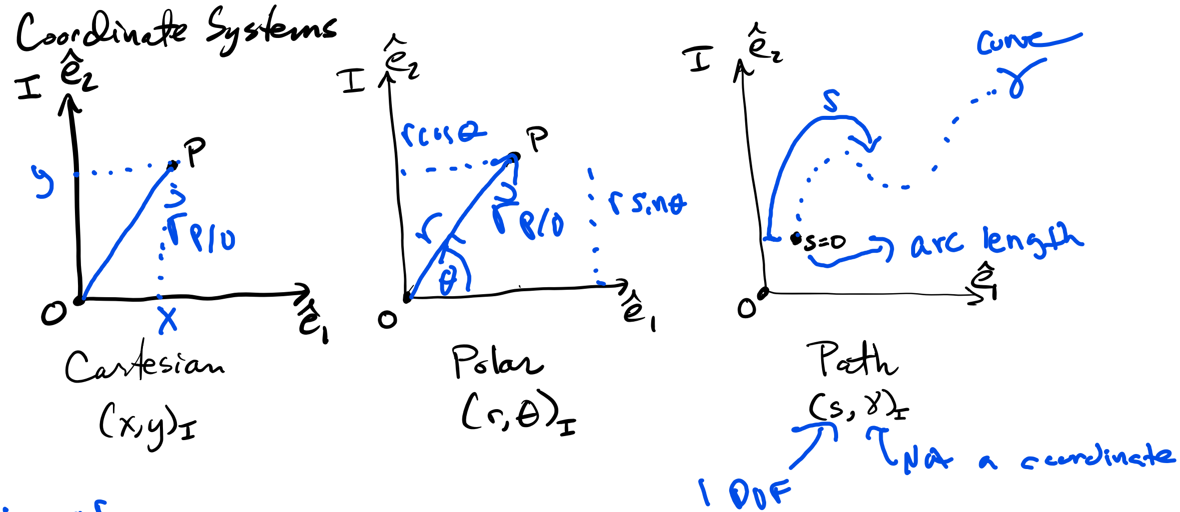


$\|\vec{r}_{P/O}\| = \sqrt{x^2+y^2+z^2} = R$

We will re-visit constraints later in the course.

Inertial Particle Kinematics in the Plane

Coordinate Systems



Description of systems motion based on geometry

in reference frames only

Solving for the Kinematics of a particle (or rigid body) involves calculating the velocity and acceleration by differentiating the position vector w.r.t. time.

Cartesian coord:

$$\vec{r}_{P/O} = x\,\hat{e}_1 + y\,\hat{e}_2$$

$$^I\vec{v}_{P/O} = \frac{^I d}{dt}(\vec{r}_{P/O}) = \dot{x}\,\hat{e}_1 + \dot{y}\,\hat{e}_2$$

$$^I\vec{a}_{P/O} = \frac{^I d}{dt}(^I\vec{v}_{P/O}) = \ddot{x}\,\hat{e}_1 + \ddot{y}\,\hat{e}_2$$

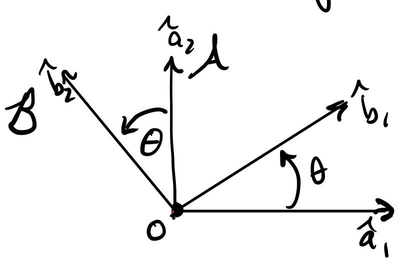
Polar Coord.: $\vec{r}_{P/O} = r\cos\theta\,\hat{e}_1 + r\sin\theta\,\hat{e}_2$

$$^I\vec{v}_{P/O} = \frac{^I d}{dt}(\vec{r}_{P/O}) = (\dot{r}\cos\theta - \dot{\theta} r\sin\theta)\hat{e}_1 + (\dot{r}\sin\theta + r\dot{\theta}\cos\theta)\hat{e}_2$$

$$^I\vec{a}_{P/O} = \frac{^I d}{dt}(^I\vec{v}_{P/O}) = (\ddot{r}\cos\theta - \dot{r}\dot{\theta}\sin\theta - \dot{r}\dot{\theta}\sin\theta - r\ddot{\theta}\sin\theta - r\dot{\theta}^2\cos\theta)\hat{e}_1 + (\ddot{r}\sin\theta + 2\dot{r}\dot{\theta}\cos\theta + r\ddot{\theta}\cos\theta - r\dot{\theta}^2\sin\theta)\hat{e}_2$$

Sometimes problems are easily solved when making use of different types of reference frames. Including moving & non-inertial frames.

Transforming between reference frames



$\mathcal{A} = (O, \hat{a}_1, \hat{a}_2, \hat{a}_3)$

$\mathcal{B} = (O, \hat{b}_1, \hat{b}_2, \hat{b}_3)$

Transformation Table

	$\hat{b}_1$	$\hat{b}_2$
$\hat{a}_1$	$\langle \hat{a}_1, \hat{b}_1 \rangle$	$\langle \hat{a}_1, \hat{b}_2 \rangle$
$\hat{a}_2$	$\langle \hat{a}_2, \hat{b}_1 \rangle$	$\langle \hat{a}_2, \hat{b}_2 \rangle$

$\hat{a}_1 \cdot \hat{b}_2$ or $\hat{a}_1^T \hat{b}_2$

The $(i,j)^{th}$ entry represents the dot product between the i^{th} row and j^{th} column

	$\hat{b}_1$	$\hat{b}_2$
$\hat{a}_1$	$\cos\theta$	$-\sin\theta$
$\hat{a}_2$	$\sin\theta$	$\cos\theta$

TO   From

$$\underbrace{{}^{\mathcal{A}}C^{\mathcal{B}}}_{\text{From } B \to A} = \begin{bmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{bmatrix}$$

$$[\vec{r}_{Q/O}]_A = {}^A C^B [\vec{r}_{Q/O}]_B$$

This transformation table process results in **Transformation matrices.**

Often, but not always, these transformation matrices are also rotation matrices. (If there is a reflection involved, the resulting matrix will not be a rotation matrix.

An important property for rotation matrices:

$$({}^{B}C^{A})^{T} = ({}^{B}C^{A})^{-1} \quad \text{orthogonal matrices}$$

Now, if I know the ${}^{A}C^{B}$ is a $\underbrace{\text{rotation matrix}}_{\det = +1}$, I can find ${}^{B}C^{A}$ easily:

$$[\vec{r}_{P/O}]_A = {}^{A}C^{B}[\vec{r}_{P/O}]_B$$

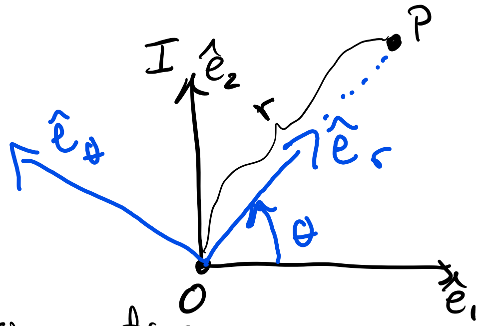
$$({}^{A}C^{B})^{-1}[\vec{r}_{P/O}]_A = [\vec{r}_{P/O}]_B$$

or

$$[\vec{r}_{P/O}]_B = ({}^{A}C^{B})^{-1}[\vec{r}_{P/O}]_A = \underbrace{({}^{A}C^{B})^{T}}_{{}^{B}C^{A}}[\vec{r}_{P/O}]_A$$

Polar Frame

↳ Reference frame that rotates to follow a point



$$\mathcal{R} = (0, \hat{e}_r, \hat{e}_\theta, \hat{e}_3)$$

$$I = (0, \hat{e}_1, \hat{e}_2, \hat{e}_3)$$

Kinematics

$$\vec{r}_{P/O} = r\hat{e}_r$$

$$^I\vec{V}_{P/O} = \dot{r}\hat{e}_r + r\,\frac{^I d}{dt}(\hat{e}_r)$$

Option #1

Convert all basis vectors to I frame, differentiate, substitute back

Transformation Table

	$\hat{e}_r$	$\hat{e}_\theta$
$\hat{e}_1$	$\cos\theta$	$-\sin\theta$
$\hat{e}_2$	$\sin\theta$	$\cos\theta$

$= {}^I C^R$

$$\vec{r}_{P/O} = r\hat{e}_r$$

$$= r(\cos\theta\,\hat{e}_1 + \sin\theta\,\hat{e}_2)$$

$$^I\vec{V}_{P/O} = \dot{r}\cos\theta\,\hat{e}_1 - r\sin\theta\,\dot{\theta}\,\hat{e}_1 + \dot{r}\sin\theta\,\hat{e}_2 + r\cos\theta\,\dot{\theta}\,\hat{e}_2$$

$$^I\vec{a}_{P/O} = \dot{r}\underbrace{(\cos\theta\,\hat{e}_1 + \sin\theta\,\hat{e}_2)}_{\hat{e}_r} + r\dot{\theta}\underbrace{(-\sin\theta\,\hat{e}_1 + \cos\theta\,\hat{e}_2)}_{\hat{e}_\theta}$$

$$= \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$$

Compare to

$$= \dot{r}\hat{e}_r + r\,\frac{^I d}{dt}(\hat{e}_r)$$

$$\frac{^I d}{dt}(\hat{e}_r) = \dot{\theta}\hat{e}_\theta$$

Option #2: Differentiate unit vectors using the angular velocity vector

Angular velocity of R wrt I

$$\underbrace{{}^I\vec{\omega}^R}_{\text{Angular velocity of } R \text{ wrt } I} = \dot{\theta}\,\hat{e}_3$$

$$\frac{{}^I d}{dt}(\hat{e}_r) = \frac{{}^I d}{dt}(\cos\theta\,\hat{e}_1 + \sin\theta\,\hat{e}_2) =$$

$$= \dot{\theta}\underbrace{(-\sin\theta\,\hat{e}_1 + \cos\theta\,\hat{e}_2)}_{\hat{e}_\theta} = \dot{\theta}\,\hat{e}_\theta = \underbrace{\dot{\theta}\,(\hat{e}_3}_{{}^I\vec{\omega}^R} \times \hat{e}_r)$$

$$\boxed{\begin{aligned} \frac{{}^I d}{dt}(\hat{e}_r) &= {}^I\vec{\omega}^R \times \hat{e}_r = \dot{\theta}\,\hat{e}_\theta \\ \frac{{}^I d}{dt}(\hat{e}_\theta) &= {}^I\vec{\omega}^R \times \hat{e}_\theta = -\dot{\theta}\,\hat{e}_r \end{aligned}}$$

Inertial kinematics of point P in the Polar Frame

$$\vec{r}_{P/O} = r\,\hat{e}_r$$

$${}^I\vec{v}_{P/O} = \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta$$

$${}^I\vec{a}_{P/O} = \ddot{r}\,\hat{e}_r + 2\dot{r}\dot{\theta}\,\hat{e}_\theta + r\ddot{\theta}\,\hat{e}_\theta - r\dot{\theta}^2\,\hat{e}_r$$

$${}^I\vec{a}_{P/O} = (\ddot{r} - r\dot{\theta}^2)\,\hat{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\,\hat{e}_\theta$$

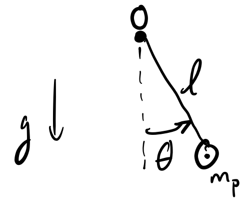
$$\vec{F}_P = m_P\,{}^I\vec{a}_{P/O}$$

DoF: $M = 2N - n$

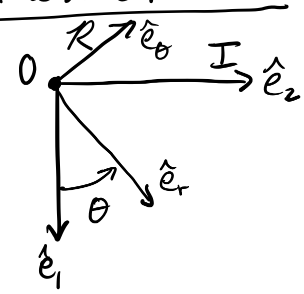
$2(1) - 1$

$r = l$

Ex. Pendulum



Reference Frames



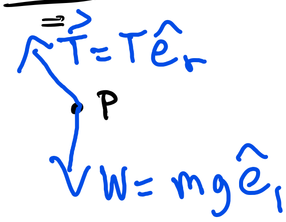
Kinematics:

$$\vec{r}_{P/O} = l\hat{e}_r$$

$$^I\vec{v}_{P/O} = l\dot{\theta}\hat{e}_\theta$$

$$^I\vec{a}_{P/O} = l\ddot{\theta}\hat{e}_\theta - l\dot{\theta}^2\hat{e}_r$$

FBD



N2L:

$$\vec{F}_P = m_P\, {}^I\vec{a}_{P/O}$$

$$-T\hat{e}_r + m_P g\hat{e}_1 = m_p(l\ddot{\theta}\hat{e}_\theta - l\dot{\theta}^2\hat{e}_r)$$

To get the scalar eqn of motion, we will try to stay in the polar $\mathcal{R}$ frame; Let's convert the $\hat{e}_1$ vector

$$-T\hat{e}_r + mg(\cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta) = m_P(l\ddot{\theta}\hat{e}_\theta - l\dot{\theta}^2)\hat{e}_r$$

Equate coefficients

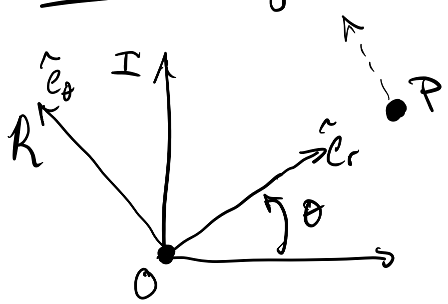
$\hat{e}_r$: $-T + mg\cos\theta = -m_P l\dot{\theta}^2$ (Constraint Force)

$\hat{e}_\theta$: $-mg\sin\theta = m_P l\ddot{\theta}$

E.O.M.

$$\boxed{\ddot{\theta} = -\frac{g}{l}\sin\theta}$$ — 1 DoF

Ex. Straight line motion using a polar frame



$${}^I\vec{a}_{P/O} = (\ddot{r} - \underbrace{r\dot{\theta}^2}_{\text{Centripetal acceleration}})\hat{e}_r + (r\ddot{\theta} + \underbrace{2\dot{r}\dot{\theta}}_{\text{Coriolis acceleration}})\hat{e}_\theta$$

$$\vec{F}_P = m_P\,{}^I\vec{a}_{P/O}$$

$$0 = m_P\left((\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta\right)$$

$$\Rightarrow \left.\begin{aligned} \hat{e}_r &: \ddot{r} = r\dot{\theta}^2 \\ \hat{e}_\theta &: \ddot{\theta} = \frac{-2\dot{r}\dot{\theta}}{r} \end{aligned}\right\}$$ Arise due to rotating frame effects