ECE 602: LUMPED LINEAR SYSTEMS

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Controllability of Discrete-Time (DT) Linear Time-Invariant (LTI) Systems

Controllability of discrete-time (DT) linear time-invariant (LTI) systems

- **Objective**: Introduce notion of controllability of DT LTI systems modeled as

$$\boldsymbol{x}[k+1] = \boldsymbol{A}\boldsymbol{x}[k] + \boldsymbol{B}\boldsymbol{u}[k], \quad \boldsymbol{x}[0] = \boldsymbol{x}_0,$$

  where $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ and $\boldsymbol{B} \in \mathbb{R}^{n\times m}$
- First, obtain a solution of the system
- Note that

$$\boldsymbol{x}[1] = \boldsymbol{A}\boldsymbol{x}[0] + \boldsymbol{B}\boldsymbol{u}[0]$$

  and

$$\begin{aligned} \boldsymbol{x}[2] &= \boldsymbol{A}\boldsymbol{x}[1] + \boldsymbol{B}\boldsymbol{u}[1] \\ &= \boldsymbol{A}\left(\boldsymbol{A}\boldsymbol{x}[0] + \boldsymbol{B}\boldsymbol{u}[0]\right) + \boldsymbol{B}\boldsymbol{u}[1] \\ &= \boldsymbol{A}^2\boldsymbol{x}[0] + \boldsymbol{A}\boldsymbol{B}\boldsymbol{u}[0] + \boldsymbol{B}\boldsymbol{u}[1] \end{aligned}$$

Solving DT LTI system modeling equation

- We have
$$\boldsymbol{x}[2] = \boldsymbol{A}^2\boldsymbol{x}[0] + \boldsymbol{A}\boldsymbol{B}\boldsymbol{u}[0] + \boldsymbol{B}\boldsymbol{u}[1]$$
- Iterate to obtain

$$\begin{aligned}
\boldsymbol{x}[3] &= \boldsymbol{A}\boldsymbol{x}[2] + \boldsymbol{B}\boldsymbol{u}[2] \\
&= \boldsymbol{A}\left(\boldsymbol{A}^2\boldsymbol{x}[0] + \boldsymbol{A}\boldsymbol{B}\boldsymbol{u}[0] + \boldsymbol{B}\boldsymbol{u}[1]\right) + \boldsymbol{B}\boldsymbol{u}[2] \\
&= \boldsymbol{A}^3\boldsymbol{x}[0] + \boldsymbol{A}^2\boldsymbol{B}\boldsymbol{u}[0] + \boldsymbol{A}\boldsymbol{B}\boldsymbol{u}[1] + \boldsymbol{B}\boldsymbol{u}[2] \\
&= \boldsymbol{A}^3\boldsymbol{x}[0] + \begin{bmatrix} \boldsymbol{B} & \boldsymbol{A}\boldsymbol{B} & \boldsymbol{A}^2\boldsymbol{B} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}[2] \\ \boldsymbol{u}[1] \\ \boldsymbol{u}[0] \end{bmatrix}
\end{aligned}$$

Solving DT LTI system modeling equation—Contd

- We have

$$\boldsymbol{x}[3] = \boldsymbol{A}^3\boldsymbol{x}[0] + \begin{bmatrix} \boldsymbol{B} & \boldsymbol{AB} & \boldsymbol{A}^2\boldsymbol{B} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}[2] \\ \boldsymbol{u}[1] \\ \boldsymbol{u}[0] \end{bmatrix}$$

- In general

$$\boldsymbol{x}[i] = \boldsymbol{A}^i\boldsymbol{x}[0] + \begin{bmatrix} \boldsymbol{B} & \boldsymbol{AB} & \cdots & \boldsymbol{A}^{i-1}\boldsymbol{B} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}[i-1] \\ \vdots \\ \boldsymbol{u}[1] \\ \boldsymbol{u}[0] \end{bmatrix}$$

Controllability

Definition

A DT system is controllable if it can be transferred from any given state to the origin $\mathbf{0}$ in finite number of steps

Theorem

The DT system $\boldsymbol{x}[k+1] = \boldsymbol{A}\boldsymbol{x}[k] + \boldsymbol{B}\boldsymbol{u}[k]$ is controllable if and only if

$$\textit{range}\,(\boldsymbol{A}^n) \subset \textit{range}\begin{bmatrix} \boldsymbol{B} & \boldsymbol{AB} & \cdots & \boldsymbol{A}^{n-1}\boldsymbol{B} \end{bmatrix}.$$

Equivalently,

$$\textit{rank}\begin{bmatrix} \boldsymbol{B} & \boldsymbol{AB} & \cdots & \boldsymbol{A}^{n-1}\boldsymbol{B} \end{bmatrix} = \textit{rank}\begin{bmatrix} \boldsymbol{B} & \boldsymbol{AB} & \cdots & \boldsymbol{A}^{n-1}\boldsymbol{B} & \boldsymbol{A}^n \end{bmatrix}$$

Necessary and Sufficient Condition for Controllability

- For $\boldsymbol{x}[0] = \boldsymbol{x}_0$ and $\boldsymbol{x}_f = \boldsymbol{x}[q] = \boldsymbol{0}$,

$$
\begin{aligned}
\boldsymbol{A}^q \boldsymbol{x}_0 &= -\sum_{k=0}^{q-1} \boldsymbol{A}^{q-k-1} \boldsymbol{B} \boldsymbol{u}[k] \\
&= -\begin{bmatrix} \boldsymbol{B} & \boldsymbol{AB} & \cdots & \boldsymbol{A}^{q-1}\boldsymbol{B} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}[q-1] \\ \vdots \\ \boldsymbol{u}[0] \end{bmatrix}
\end{aligned}
$$

- The DT system is controllable if and only if for some $q > 0$ and arbitrary initial condition $\boldsymbol{x}_0$, the vector $\boldsymbol{A}^q \boldsymbol{x}_0$ is in the range of $\begin{bmatrix} \boldsymbol{B} & \boldsymbol{AB} & \cdots & \boldsymbol{A}^{q-1}\boldsymbol{B} \end{bmatrix} = \boldsymbol{U}_q$
- The maximal range of $\boldsymbol{U}_q$ is guaranteed to be attained for $q = n$

Nonreachable Yet Controllable

- The DT system

$$\begin{aligned} \boldsymbol{x}[k+1] &= \boldsymbol{A}\boldsymbol{x}[k] + \boldsymbol{b}u[k] \\ &= \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \boldsymbol{x}[k] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u[k], \quad a \in \mathbb{R} \end{aligned}$$

- The solution for arbitrary $\boldsymbol{x}[0]$ and $\boldsymbol{x}_f = \boldsymbol{x}[2] = \boldsymbol{0}$

$$\begin{aligned} \boldsymbol{x}[2] &= \boldsymbol{0} \\ &= \boldsymbol{A}\boldsymbol{x}[1] + \boldsymbol{b}u[1] \\ &= \boldsymbol{A}\left(\boldsymbol{A}\boldsymbol{x}[0] + \boldsymbol{b}u[0]\right) + \boldsymbol{b}u[1] \\ &= \boldsymbol{A}^2\boldsymbol{x}[0] + \begin{bmatrix} \boldsymbol{b} & \boldsymbol{A}\boldsymbol{b} \end{bmatrix} \begin{bmatrix} u[1] \\ u[0] \end{bmatrix} \end{aligned}$$

- We have $\boldsymbol{A}^2 = \boldsymbol{O}$, hence $\boldsymbol{A}^2\boldsymbol{x}[0] = \boldsymbol{0}$ for any $\boldsymbol{x}[0] \in \mathbb{R}^2$

Nonreachable Yet Controllable—Contd

- We have
$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u[1] \\ u[0] \end{bmatrix}$$
- Therefore, the system is controllable because an arbitrary initial state $\boldsymbol{x}[0] \in \mathbb{R}^2$ can be transferred to the origin of $\mathbb{R}^2$ using, for example, the zero control sequence, $u[0] = u[1] = 0$
- The DT system passes the controllability test,

$$\begin{aligned} \text{rank} \begin{bmatrix} \boldsymbol{b} & \boldsymbol{Ab} \end{bmatrix} &= \text{rank} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \text{rank} \begin{bmatrix} \boldsymbol{b} & \boldsymbol{Ab} & \boldsymbol{A}^2 \end{bmatrix} \\ &= \text{rank} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

- That is, range $\left(\boldsymbol{A}^2\right) = \text{range}\,(\boldsymbol{O}) \subset \text{range} \begin{bmatrix} \boldsymbol{b} & \boldsymbol{Ab} \end{bmatrix}$

Summary

- For discrete-time linear systems, reachability implies controllability
- The two notions are equivalent if the matrix $\boldsymbol{A}$ of the given discrete-time system is nonsingular