AAE 532 - Orbit Mechanics

Purdue University

PS10 Solutions

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Contents

Useful Constants

	Axial Rotaional Period (Rev/Day)	Mean Equatorial Radius (km)	Gravitational Parameter $\mu = Gm(km^3/sec^2)$	Semi-major Axis of orbit (km)	Orbital Period (sec)	Eccentricity of Orbit	Inclination of Orbit to Ecliptic (deg)
☉ Sun	0.0394011	695990	132712440017.99	-	-	-	-
☾ Moon	0.0366004	1738.2	4902.8005821478	384400 (around Earth)	2360592 27.32 Earth Days	0.0554	5.16
☾ Mercury	0.0170514	2439.7	22032.080486418	57909101	7600537 87.97 Earth Days	0.20563661	7.00497902
♀ Venus	0.0041149 (Retrograde)	6051.9	324858.59882646	108207284	19413722 224.70 Earth Days	0.00676399	3.39465605
⊕ Earth	1.0027378	6378.1363	398600.4415	149597898	31558205 365.26 Earth Days	0.01673163	0.00001531
♂ Mars	0.9747000	3397	42828.314258067	227944135	59356281 686.99 Earth Days	0.09336511	1.84969142
♃ Jupiter	2.4181573	71492	126712767.8578	778279959	374479305 11.87 Years	0.04853590	1.30439695
♄ Saturn	2.2522053	60268	37940626.061137	1427387908	930115906 29.47 Years	0.05550825	2.48599187
⛢ Uranus	1.3921114 (Retrograde)	25559	5794549.0070719	2870480873	2652503938 84.05 Years	0.04685740	0.77263783
Ψ Neptune	1.4897579	25269	6836534.0638793	4498337290	5203578080 164.89 Years	0.00895439	1.77004347
♇ Pluto	-0.1565620 (Retrograde)	1162	981.600887707	5907150229	7830528509 248.13 Years	0.24885238	17.14001206

- First three columns of the body data are consistent with GMAT 2020a default values, which are mainly from JPL's ephemerides file de405.spk

- The rest of the data are from JPL website(https://ssd.jpl.nasa.gov/?planet_pos, retrieved at 09/01/2020)
based on E.M. Standish's "Keplerian Elements for Approximate Positions of the Major Planets"

Problem 1

Problem Statement

Assume that a vehicle is in a circular Earth orbit and will transfer to a second orbit that is not in the same plane.

(a) The initial orbit possesses the following characteristics:

$$a = 4R_{\oplus}$$
$$e = 0$$
$$i = 28.5°$$
$$\Omega = 60°$$
$$\omega = 45°$$
$$M = 90°$$

Now assume that the departure occurs when the vehicle is at the ascending node. By hand, determine the vectors $\bar{r}_1, \bar{v}_1$ in the Earth Inertial coordinates $\hat{x}, \hat{y}, \hat{z}$ at this location. Insert this orbit into GMAT; use Earth Inertial coordinates. Check your answer in GMAT.

(b) The final orbit should possess the following parameters:

$$a = 6R_{\oplus}$$
$$e = 0$$
$$i = 45°$$
$$\Omega = 60°$$
$$\omega = 45°$$
$$M = 0°$$

Assume that the arrival state on the new arrival orbit occurs when $\theta = 120°$. (Note θ is latitude NOT true anomaly!) Define this vector as $\bar{r}_2$. Determine $\bar{r}_2$ and $\bar{v}_2$ in terms of Earth Inertial coordinates. Also enter this orbit into the same GMAT window. Check your $\bar{r}_2$ and $\bar{v}_2$ results in GMAT and confirm that they are correct. Plot the entire departure and arrival orbits. Observe that they do not intersect.

(c) Two vectors are now available and a transfer is to be constructed to connect them.

(i) Sketch the vectors in a 3D sketch.

(ii) The transfer orbit is defined such that the angular momentum vector has a positive z component, i.e., $\hat{h} \cdot \hat{z} > 0$. Add the $\hat{h}$ to your sketch. Determine i, Ω, TA for the transfer orbit.

(iii) Assume that you know that the transfer path is characterized by $p = 5R_{\oplus}$. Produce the additional information concerning the transfer orbit: $h, a, e, \theta_1^*, \theta_2^*, \bar{v}_1, \bar{v}_2, \gamma_1, \gamma_2$, TOF.

(d) In GMAT, plot only the transfer ellipse. If you can, add the Earth equator, the Earth inertial coordinate axes, the spacecraft orbit normal, the orbit line of nodes, and the transfer orbit periapsis. Plot a view down the orbit normal to see the orbit plane. Mark the departure and arrival location and the appropriate quantities at each ($\bar{r}_1, \bar{r}_2, l.h., \theta_1^*, \theta_2^*, \bar{v}_1, \bar{v}_2, \gamma_1, \gamma_2$). Mark the arc from the ellipse that comprises the transfer. What is the transfer type (1A, 2A, 1B, 2B)?

Part (a)

We know that at the ascending node, $\theta = 0°$. We can convert from the orbit-fixed frame to the inertial coordinate frame using the direction cosine matrix:

$$\begin{bmatrix} c_\Omega c_\theta - s_\Omega c_i s_\theta & -c_\Omega s_\theta - s_\Omega c_i c_\theta & s_\Omega s_i \\ s_\Omega c_\theta + c_\Omega c_i s_\theta & -s_\Omega s_\theta + c_\Omega c_i c_\theta & -c_\Omega s_i \\ s_i s_\theta & s_i c_\theta & c_i \end{bmatrix}$$

Therefore, the position vector $\bar{r}_1$ is expressed as:

$$\bar{r}_1 = 4R_\oplus \hat{r} = 4r_\oplus \left((c_\Omega c_\theta - s_{\Omega c_i s_{[}\theta})\hat{x} + (s_\Omega c_\theta + c_\Omega c_i s_\theta)\hat{y} + (s_i s_\theta)\hat{z}\right)$$

$$\boxed{\bar{r}_1 = 12756.28\hat{x} + 22094.53\hat{y} + 0\hat{z}\ km = 2\hat{x} + 3.4641\hat{y} + o\hat{z}\ R_\oplus}$$

We can also determine the velocity at this location using the equation for velocity of a circular orbit:

$$\bar{v}_{c,1} = \sqrt{\frac{\mu}{r_1}}\hat{\theta} = 3.9527\hat{\theta} = 3.9527\left((-c_\Omega s_\theta - s_\Omega c_i c_\theta)\hat{x} + (-s_\Omega s_\theta + c_\Omega c_i c_\theta)\hat{y} + (s_i c_\theta)\hat{z}\right)\ km/s$$

$$\boxed{\bar{v}_1 = -3.0083\ hatx + 1.7368\hat{y} + 1.8861\hat{z}\ km/s}$$

We can confirm these results with GMAT as well selecting an arbitrary epoch of Dec 1 2024:

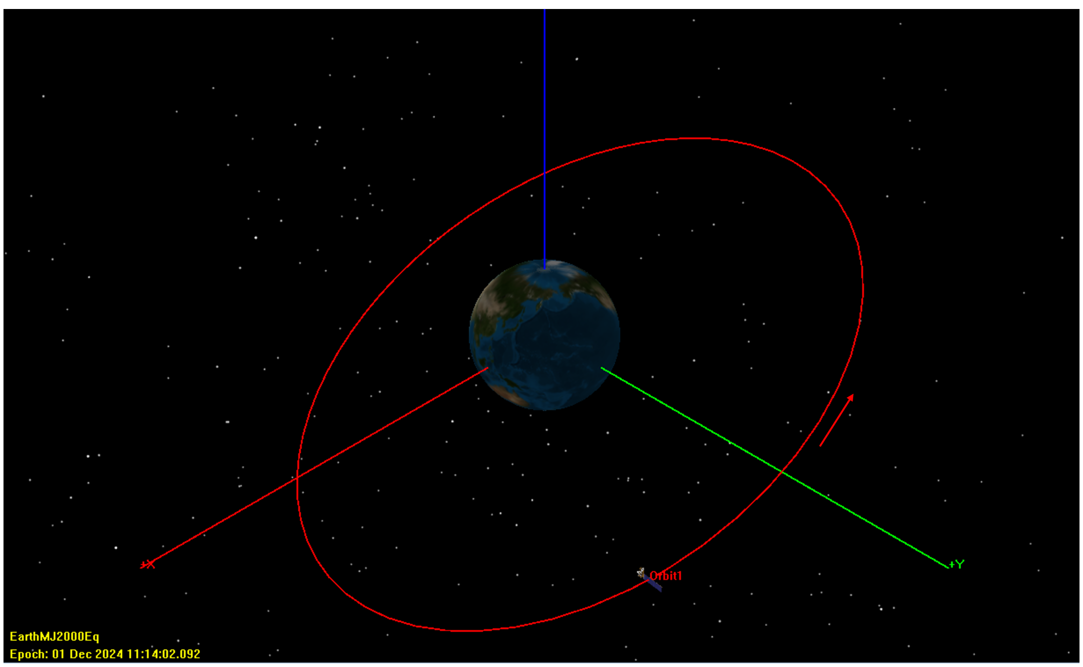


Figure 1: 3D orbit visualized in GMAT.

Orbit1.A1ModJulian	Orbit1.EarthMJ2000Eq.X	Orbit1.EarthMJ2000Eq.Y	Orbit1.EarthMJ2000Eq.Z	Orbit1.EarthMJ2000Eq.VX	Orbit1.EarthMJ2000Eq.VY	Orbit1.EarthMJ2000Eq.VZ
30645.50042863868	12756.2726	22094.51225839874	0	-3.008299911289018	1.736842763585842	1.886057355729414

Figure 2: GMAT report screenshot.

Part (b)

Using the same direction cosine matrix as the previous part, we can determine the second position vector using $\theta = 120°$ for the new orbit:

$$\bar{r}_2 = 6R_{\oplus}\hat{r} = 6R_{\oplus}\left((c_{\Omega}c_{\theta} - s_{\Omega}c_i s_{\theta})\hat{x} + (s_{\Omega}c_{\theta} + c_{\Omega}c_i s_{\theta})\hat{y} + (s_i s_{\theta})\hat{z}\right)$$

$$\boxed{\bar{r}_2 = -29862.31\hat{x} - 4853.50\hat{y} + 23434.77\hat{z}\ km = -4.6820\hat{x} - 0.7610\hat{y} + 3.6742\hat{z}\ R_{\oplus}}$$

Once again we have a circular orbit and can find the velocity vector as:

$$\bar{v}_{c,2} = \sqrt{\frac{\mu}{r_2}}\hat{\theta} = 3.2274\hat{\theta} = 3.2274\left((-c_{\Omega}s_{\theta} - s_{\Omega}c_i c_{\theta})\hat{x} + (-s_{\Omega}s_{\theta} + c_{\Omega}c_i c_{\theta})\hat{y} + (s_i c_{\theta})\hat{z}\right)\ km/s$$

$$\boxed{\bar{v}_1 = -0.4093\hat{x} - 2.9910\hat{y} - 1.1410\hat{z}\ km/s}$$

We can once again confirm these results using GMAT, noticing that the two orbits do not intersect:

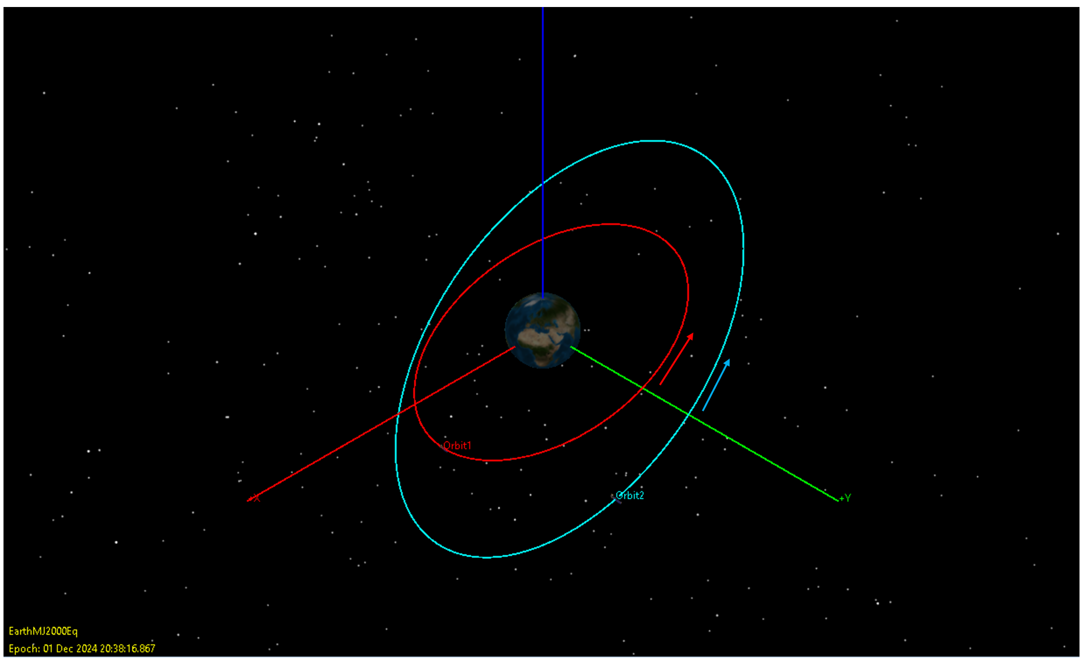
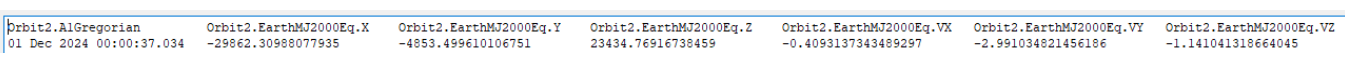


Figure 3: 3D orbits visualized in GMAT.



Orbit2.A1Gregorian	Orbit2.EarthMJ2000Eq.X	Orbit2.EarthMJ2000Eq.Y	Orbit2.EarthMJ2000Eq.Z	Orbit2.EarthMJ2000Eq.VX	Orbit2.EarthMJ2000Eq.VY	Orbit2.EarthMJ2000Eq.VZ
01 Dec 2024 00:00:37.034	-29862.30988077935	-4853.499610106751	23434.76916738459	-0.4093137343489297	-2.991034821456186	-1.141041318664045

Figure 4: GMAT report screenshot.

Part (c)

i

First we can sketch the position vectors:

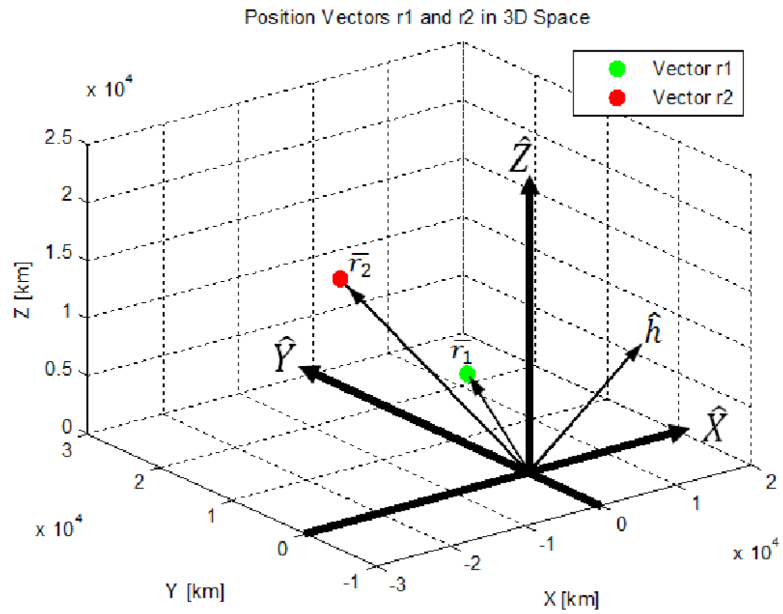


Figure 5: 3D sketch of $\bar{r}_1$ and $\bar{r}_2$.

ii

We can determine the transfer angle between the two by taking the dot product:

$$\bar{r}_1 \cdot \bar{r}_2 = |\bar{r}_1||\bar{r}_2|\cos(TA)$$

Rearranging and recognizing the sign ambiguity for the double valued inverse cosine:

$$TA = \pm 120°$$

We know that

$$\bar{h} \cdot \hat{z} > 0$$

, which can help us determine which sign is appropriate for the transfer angle. The h vector has been added to the figure in part i, and from looking at it we can see the direction of $\hat{h}$. The transfer orbit must go from $\bar{r}_1$ to $\bar{r}_2$ and lie in a plane perpendicular to $\hat{h}$. Thus, we can use the right hand rule to determine the direction the transfer orbit goes. Then looking at the projection onto the $x - y$ plane:

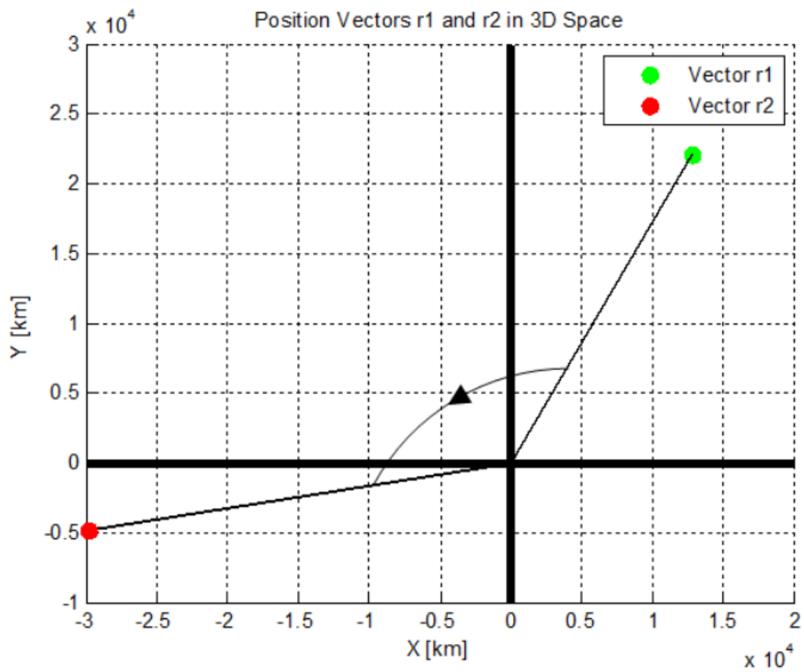


Figure 6: 2D projection of $\bar{r}_1$ and $\bar{r}_2$.

Based on the figure and discussion above:

$$\boxed{TA = +120^\circ}$$

From the direction cosine matrix:

$$\hat{h}_z = \cos i$$

And:

$$\hat{h} = \frac{\bar{r}_1 \times \bar{r}_2}{|\bar{r}_1 \times \bar{r}_2|} = 0.6124\hat{x} - 0.3536\hat{y} + 0.7071\hat{z}$$

Solving for i:

$$i = \cos^{-1}(\hat{h}_z) = \pm 45^\circ$$

Once again the inverse cosine is double valued, but we know that inclination is always between 0° and 180°, therefore:

$$\boxed{i = 45^\circ}$$

Then to find RAAN we return to the direction cosine matrix:

$$\hat{h}_x = \sin\Omega \sin i$$

$$\hat{h}_y = -\cos\Omega \sin i$$

We can substitute values for $\hat{h}_x$ and $\hat{h}_y$ along with the known inclination and the above equations become:

$$0.6124 = \sin\Omega\frac{\sqrt{2}}{2}$$

$$-0.3536 = -\cos\Omega\frac{\sqrt{2}}{2}$$

All inverse trig functions are doubled valued, so solving yields:

$$\Omega = 60°, 120°$$

$$\Omega = \pm 60°$$

We can see that one value is repeated in both equations and therefore that is the value of RAAN that we must choose:

$$\boxed{\Omega = 60°}$$

iii

Earlier we found the angle between the two position vectors, $TA = 120°$. This angle is also equal to the difference of true anomaly values between the two position vectors:

$$\Delta\theta^* = \theta_2^* - \theta_1^* = 120°$$

We already know the position vectors associated with the two points r_1 and r_2 on the transfer orbit, but we can find the velocity vectors associated with the two points using f and g relationships:

$$\bar{r}_2 = f\bar{r}_1 + g\bar{v}_1$$

$$\bar{v}_2 = \dot{f}\bar{r}_1 + \dot{g}\bar{v}_1$$

$$f = 1 - \frac{r_1}{p}\left(1 - \cos(\theta_2^* - \theta_1^*)\right)$$

$$g = \frac{r_2 r_1}{\sqrt{\mu p}}\sin(\theta_2^* - \theta_1^*)$$

$$\dot{f} = \frac{\bar{r}_1 \cdot \bar{v}_1}{p r_1}(1 - \cos(\theta_2^* - \theta_1^*)) - \frac{1}{r_1}\sqrt{\frac{\mu}{p}}\sin(\theta_2^* - \theta_1^*)$$

$$\dot{g} = 1 - \frac{r_1}{p}(1 - \cos(\theta_2^* - \theta_1^*))$$

Which we can solve to get:

$$f = -0.8$$

$$g = 7499\ s$$

$$\dot{f} = -0.0001\ 1/s$$

$$\dot{g} = -0.2$$

From these equations, we can solve to the velocity vectors:

$$\bar{v}_1 = \frac{\bar{r}_2 + f\bar{r}_1}{g}$$

$$\boxed{\bar{v}_1 = -2.6212\hat{x} + 1.7097\hat{y} + 3.1249\hat{z}\ km/s}$$

$$\bar{v}_2 = \dot{f}\bar{r}_1 + \dot{g}\bar{v}_1$$

$$\boxed{\bar{v}_2 = -0.9046\hat{x} - 2.8167\hat{y} - 0.6250\hat{z}\ km/s}$$

Next, we can use the energy equation to solve for semi-major axis:

$$\frac{v_1^2}{2} - \frac{\mu}{r_1} = \frac{-\mu}{2a}$$

$$\boxed{a = 34100.93\ km = 5.36R_{\oplus}}$$

Then, h:

$$\boxed{h = \sqrt{\mu p} = 112745.91\ km^2/s}$$

We now have enough information to solve for eccentricity:

$$\frac{p}{a} = 1 - e^2$$

$$\boxed{e = 0.2546}$$

Note that since $e < 1$ we know our transfer orbit is an ellipse. To solve for true anomaly we can use the conic equation:

$$\theta_1^* = \pm\cos^{-1}\left(\frac{\frac{p}{r_1} - 1}{e}\right)$$

$$\theta_1^* = \pm 10.89°$$

To determine which sign to choose we need to know if the spacecraft is ascending or descending at each point along the transfer orbit:

$$\bar{v}_1 \cdot \hat{r}_1 = 0.1701\ km/s > 0$$

Since this value is positive, the spacecraft is ascending at the r_1 location and we choose the positive true anomaly:

$$\boxed{\theta_1^* = 10.89°}$$

To solve for the true anomaly at the second position vector, we use the transfer angle as follows:

$$\theta_2^* = \theta_1^* + \Delta\theta^*$$

$$\boxed{\theta_2^* = 130.89°}$$

To find the flight path angle we use:

$$\gamma = \cos^{-1}\left(\frac{\sqrt{\mu p}}{r_1 v_1}\right)$$

$$\gamma_1 = \pm 2.2042°$$

$$\gamma_2 = \pm 13.0039°$$

Based on the true anomaly values, we know whether the spacecraft s ascending or descending which allows us to choose the correct sign for the flight path angle:

$$\boxed{\gamma_1 = 2.2042°}$$

$$\boxed{\gamma_2 = 13.0039°}$$

To find the time of flight of the transfer, we use Kepler's equation. Since neither of the two points are located at periapsis, we must perform Kepler's equation twice:

$$t_2 - t_1 = (t_2 - t_p) - (t_1 - t_p) = \frac{E_2 - e\sin E_2}{\sqrt{\frac{\mu}{a^3}}} - \frac{E_1 - e\sin E_1}{\sqrt{\frac{\mu}{a^3}}}$$

Where:

$$E = \pm\cos\left(\frac{\frac{r}{a} - 1}{-e}\right)$$

Thus,

$$E_1 = 8.41°$$

$$E_2 = 118.69°$$

Plugging in the known values into the double Kepler's equation we get:

$$\boxed{TOF = t_2 - t_1 = 4.8173\ hrs}$$

d

Once again we can produce these results in GMAT. Note that this is a type 1A transfer since the transfer angle is less than 180 degrees and the empty focus us not between the chord and the transfer arc.

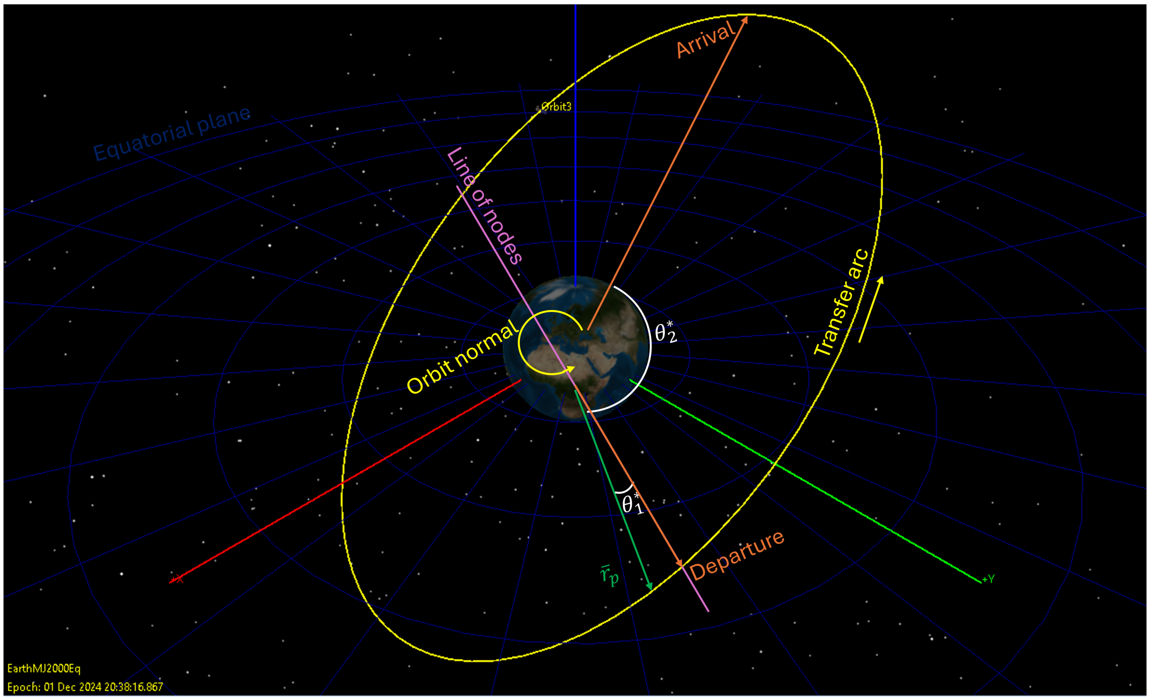


Figure 7: Full transfer ellipse visualized in GMAT.

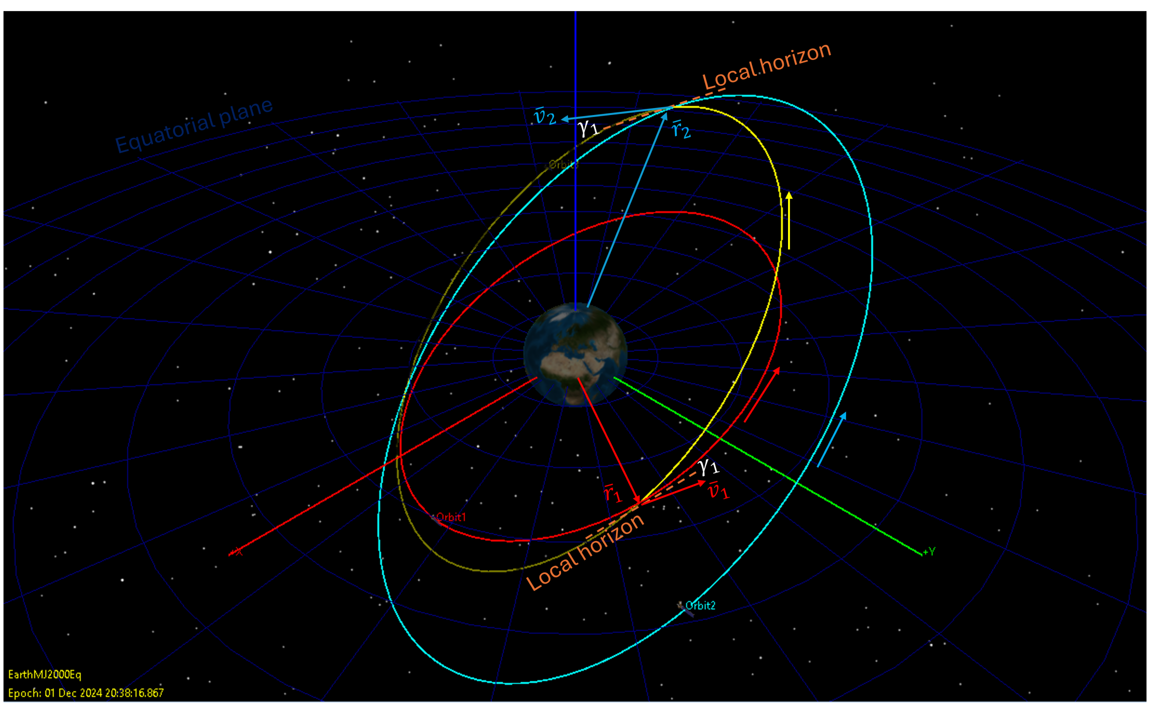


Figure 8: Initial and final orbits with transfer arc highlighted.

Problem 2

Problem Statement

Assume that you are studying future space station concepts. You have been asked to provide information on transfer orbits to and from the space station by an OTV (Orbit Transfer Vehicle). The OTV is originally in a circular orbit at $2R_\oplus$. This future space station is in a coplanar, circular orbit at $6R_\oplus$. (All orbits are assumed to be in the same plane.)

(a) Determine the total $||\Delta\bar{v}||$ and TOF for a Hohmann transfer by the OTV to the space station. Determine the phase angle that is necessary at departure. How long is it until the geometry repeats?

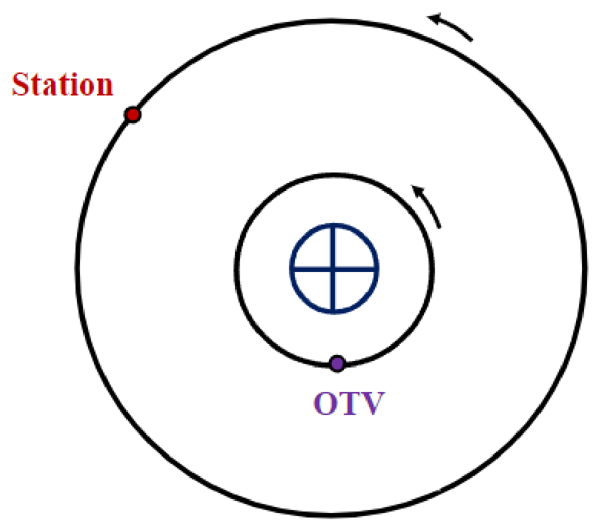
(b) For your scenario, the conditions for a Hohmann transfer do not fit the requirements. One set of conditions for departure and arrival are illustrated below with a transfer angle of 240°. Determine the following characteristics corresponding to the <u>minimum</u> energy transfer that will satisfy the conditions below:

$$a,\ e,\ r_p,\ TOF,\ r_D,\ v_D,\ \gamma_D,\ \theta^*_D,\ r_A,\ v_A,\ \gamma_A,\ \theta^*_A$$

(Note: D – departure; A – arrival.) What phase angle is required at departure?

(c) Determine the departure maneuver that is required and express it in terms of $||\Delta\bar{v}_D||$, α_D as well as $\hat{V}$, $\hat{N}$, $\hat{B}$ coordinates.
Also determine the arrival maneuver: $||\Delta\bar{v}_A||$, α_A and $\hat{V}$, $\hat{N}$, $\hat{B}$ components.
How does the total $||\Delta\bar{v}||$ and TOF compare to the Hohmann result?

(d) Use GMAT to plot these orbits (Use Earth Inertial coordinates for simplicity!) Include one rev in the initial orbit, add the maneuver, use TOF (or θ^*) for stopping conditions on the transfer arc, then a 2^{nd} maneuver, one rev in the final orbit. Mark the transfer arc and the appropriate $\bar{r}$, $\bar{v}$ vectors, local horizon, and other pertinent quantities.
Confirm all of your quantities in (b), (c). **Add the vacant focus F to your plot.**



Part (a)

Using a Hohmann transfer, our problem looks like the folowing: Then, the radii for the periapsis and apoapsis for

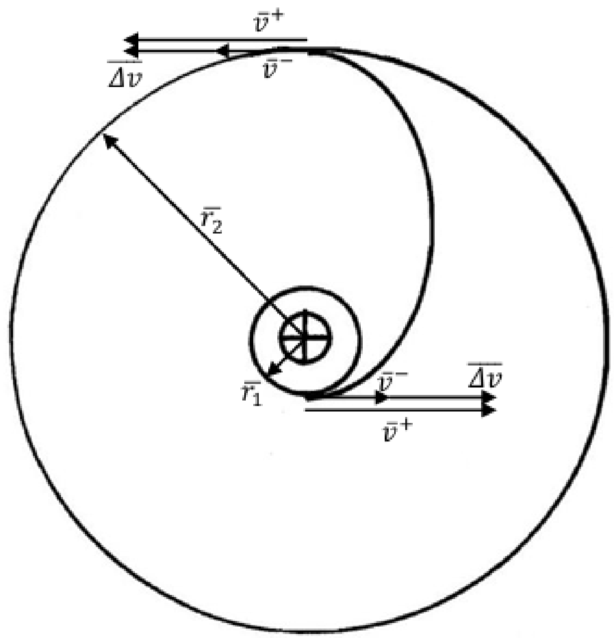


Figure 9: Geocentric View with velocity vector diagrams.

the transfer Hohmann are:

$$r_1 = r_p = 2R_{\oplus} = 1.2756 \cdot 10^4 \text{ km} \tag{1}$$

$$r_2 = r_a = 6R_{\oplus} = 3.8269 \cdot 10^4 \text{ km} \tag{2}$$

One can determine the maneuver at the departure location by first calculating the speed of the spacecraft in the initial circular orbit:

$$v_{c_1} = \sqrt{\frac{\mu_{\oplus}}{r_1}} = 5.5899 \text{ km/s} \tag{3}$$

Next, let's determine the semi-major axis for the transfer orbit:

$$a = \frac{r_1 + r_2}{2} = 2.5513 \cdot 10^4 \text{ km} \tag{4}$$

Then, the speed of the spacecraft at periapsis in the transfer orbit is:

$$v_p = \sqrt{\frac{2\mu_{\oplus}}{r_1} - \frac{\mu_{\oplus}}{a}} = 6.8462 \text{ km/s} \tag{5}$$

So, noting that the velocity vectors are along the same $\hat{\theta}$ direction at periapsis, the maneuver required is:

$$\Delta\bar{v}_1 = \bar{v}_p - \bar{v}_{c_1} = 1.2563\,\hat{\theta} \text{ km/s} \tag{6}$$

OR:

$$\boxed{||\Delta\bar{v}_1|| = 1.2563 \text{ km/s}} \tag{7}$$

Next, one can determine the maneuver at arrival by first calculating the speed of the spacecraft in the final circular orbit and then the speed of the spacecraft at apoapsis in the Hohmann transfer:

$$v_{c_2} = \sqrt{\frac{\mu_{\oplus}}{r_2}} = 3.2274 \text{ km/s} \tag{8}$$

$$v_a = \sqrt{\frac{2\mu_{\oplus}}{r_2} - \frac{\mu_{\oplus}}{a}} = 2.2821 \text{ km/s} \tag{9}$$

Then, the arrival maneuver is:

$$\Delta \bar{v}_2 = \bar{v}_{c_2} - \bar{v}_a = 0.9453\,\hat{\theta} \text{ km/s} \tag{10}$$

OR:

$$\boxed{||\Delta \bar{v}_2|| = 0.9453 \text{ km/s}} \tag{11}$$

This means that the total $||\Delta \bar{v}||$ require is:

$$\boxed{||\Delta \bar{v}||_{total} = ||\Delta \bar{v}_1|| + ||\Delta \bar{v}_2|| = 2.2016 \text{ km/s}} \tag{12}$$

Next, let's determine the period for the elliptic orbit of the transfer in order to then determine the time of flight (TOF):

$$\begin{aligned} \mathcal{P} &= 2\pi\sqrt{\frac{a^3}{\mu_\oplus}} \\ &= 4.0555 \cdot 10^4 \text{ s} \\ &= 11.2652 \text{ hrs} \end{aligned} \tag{13}$$

So, since the Hohmann transfer is half the period of the transfer ellipse, then:

$$\boxed{\begin{aligned} TOF &= \frac{1}{2}\mathcal{P} \\ &= 2.02774 \cdot 10^4 \text{ s} \\ &= 5.6326 \text{ hrs} \end{aligned}} \tag{14}$$

Lastly, let's determine the synodic period, which is how long it takes for the geometry to repeat. First, let's determine the mean motion for both the initial and final circular orbits:

$$n_1 = \sqrt{\frac{\mu_\oplus}{r_1^3}} = 4.3821 \cdot 10^{-4} \text{ 1/s} \tag{15}$$

$$n_2 = \sqrt{\frac{\mu_\oplus}{r_2^3}} = 8.4334 \cdot 10^{-5} \text{ 1/s} \tag{16}$$

Then, the synodic period is:

$$\boxed{\begin{aligned} s &= \frac{2\pi}{n_1 - n_2} \\ &= 1.7755 \cdot 10^4 \text{ s} \\ &= 4.9320 \text{ hrs} \end{aligned}} \tag{17}$$

Lastly, the phase angle necessary at departure is:

$$\boxed{\begin{aligned} \phi &= 180° - (n_2)(TOF) \\ &= 82.0204° \end{aligned}} \tag{18}$$

Part (b)

Now, let's determine the characteristics corresponding to the minimum energy transfer with a transfer angle of 240°. One can determine the semi-major axis first using cosine law:

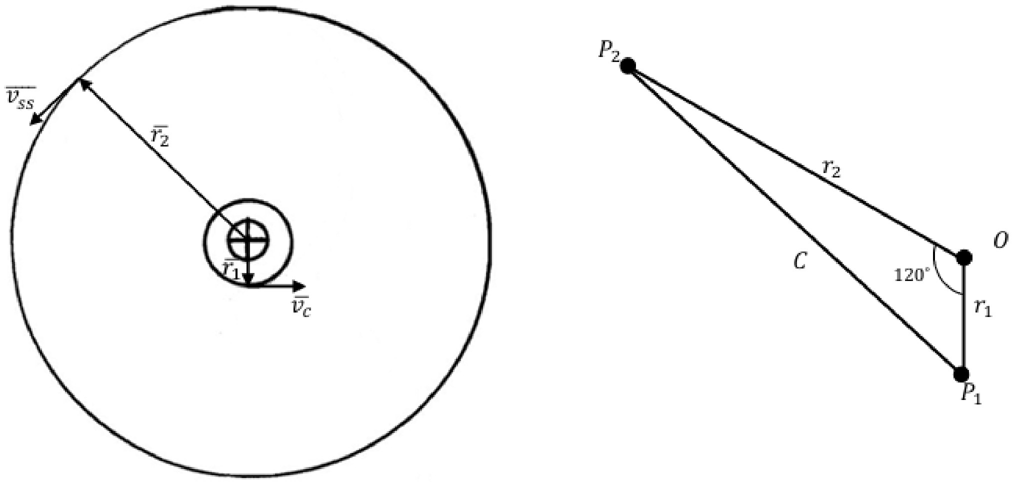


Figure 10: Geocentric View (left) and space triangle diagram (right).

$$\begin{aligned} C^2 &= r_1^2 + r_2^2 - 2r_1r_2\cos(120°) \\ C &= 4.5993 \cdot 10^4 \text{ km} \end{aligned} \tag{19}$$

which means that the semi-major axis is:

$$\boxed{\begin{aligned} a &= \frac{s}{2} = \frac{1}{4}(r_1 + r_2 + C) \\ &= 2.4255 \cdot 10^4 \text{ km} \end{aligned}} \tag{20}$$

Recall that, for the minimum energy case, the empty focus is located on the chord of the space triangle: One can

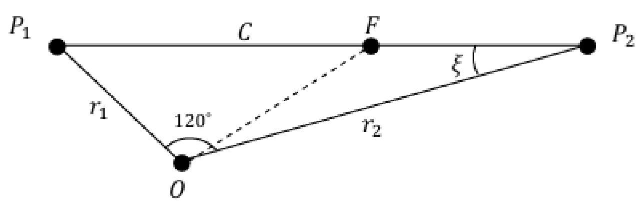


Figure 11: and space triangle diagram with empty focus.

solve for angle ξ using sine law:

$$\begin{aligned} \frac{C}{\sin(120°)} &= \frac{r_1}{\sin\xi} \\ \xi &= 13.8979°,\ 166.1021° \end{aligned} \tag{21}$$

where, since sine is doubled valued, we select $\boxed{\xi = 13.8979°}$ as this angle makes sense for our space triangle (the sum of the interior angle measures of a triangle always adds up to 180°). We then note the following to be true:

$$\begin{aligned} P_2F &= 2a - r_2 \\ &= 1.0240 \cdot 10^4 \text{ km} \end{aligned} \tag{22}$$

In addition, we also now that:

$$OF = 2ae \tag{23}$$

One can solve for OF, using cosine law:

$$\begin{aligned} (OF)^2 &= r_2^2 + (P_2F)^2 - 2r_2(P_2F)\cos\xi \\ OF &= 2.8435 \cdot 10^4 \text{ km} \end{aligned} \tag{24}$$

which means that eccentricity is then:

$$\boxed{e = \frac{OF}{2a} = 0.5862} \tag{25}$$

Then, the periapsis radius for the transfer orbit is:

$$\boxed{\begin{aligned} r_p &= a(1-e) \\ &= 1.0037 \cdot 10^4 \text{ km} \end{aligned}} \tag{26}$$

which verifies that the spacecraft will not crash into the Earth. Next, we should already know the departure and arrival radii for the spacecraft as we have solved for it previously in part (a):

$$\boxed{r_D = r_1 = 1.2756 \cdot 10^4 \text{ km}} \tag{27}$$

$$\boxed{r_A = r_2 = 3.8269 \cdot 10^4 \text{ km}} \tag{28}$$

In addition, one can use the vis-viva or energy equations to obtain the speed of the spacecraft at departure and arrival:

$$\boxed{\begin{aligned} v_D &= \sqrt{\frac{2\mu_\oplus}{r_D} - \frac{\mu_\oplus}{a}} \\ &= 6.7868 \text{ km/s} \end{aligned}} \tag{29}$$

$$\boxed{\begin{aligned} v_A &= \sqrt{\frac{2\mu_\oplus}{r_A} - \frac{\mu_\oplus}{a}} \\ &= 2.0970 \text{ km/s} \end{aligned}} \tag{30}$$

Next, one can solve for the departure and arrival true anomaly by rearranging the conic equation:

$$\theta^* = \arccos\left(\frac{p}{re} - \frac{1}{e}\right) \tag{31}$$

where:

$$\begin{aligned} p &= a(1-e^2) \\ &= 1.5921 \cdot 10^4 \text{ km} \end{aligned} \tag{32}$$

So, for departure:

$$\theta_D^* = \pm 64.9624° \tag{33}$$

and for arrival:

$$\theta_A^* = \pm 175.0376° \tag{34}$$

To figure out the sign ambiguity for the true anomaly for both departure and arrival, one can use the transfer angle and note that:

$$\theta_A^* = \theta_D^* + TA \tag{35}$$

where:

$$64.9624° + 240° = 304.9624° \tag{36}$$

OR:

$$-64.9624° + 240° = 175.0376° \tag{37}$$

This means that the true anomalies for departure and arrival are:

$$\boxed{\theta_D^* = -64.9624°} \tag{38}$$

$$\boxed{\theta^*_A = 175.0376°} \tag{39}$$

Now, one can determine the flight path angle for the spacecraft at departure and arrival. One can use the sign information from the true anomalies above to aid us in finding the proper flight path angles, using the following equation:

$$\gamma = \arccos\left(\frac{h}{rv}\right) = \arccos\left(\frac{\sqrt{\mu_\oplus p}}{rv}\right) \tag{40}$$

This means that the flight path angle at departure is:

$$\boxed{\gamma_D = -23.0511°} \tag{41}$$

and at arrival:

$$\boxed{\gamma_A = 6.9489°} \tag{42}$$

Lastly, to find TOF and the proper phase angle for departure, one can recognize that:

$$TOF = \frac{1}{n}[(E_A - E_D) - e\sin E_A + e\sin E_D] \tag{43}$$

Eccentric anomaly can be determined using the following equation:

$$E = \arccos\left(\frac{1}{e} - \frac{r}{ae}\right) \tag{44}$$

So, using the previous sign information, the eccentric anomalies for departure and arrival are:

$$E_D = -36.0257° \tag{45}$$

$$E_A = 170.3018° \tag{46}$$

Thus, plugging E_D and E_A into equation (43) provides:

$$\boxed{TOF = 1.8892 \cdot 10^4 \text{ s} = 5.2478 \text{ hrs}} \tag{47}$$

where:

$$n = \sqrt{\frac{\mu_\oplus}{a^3}} = 1.6714 \cdot 10^{-4} \text{ 1/s} \tag{48}$$

And, for the proper phasing:

$$\boxed{\begin{aligned}\phi &= 180° - (n_2)(TOF) \\ &= 88.7143°\end{aligned}} \tag{49}$$

Part (c)

Now, let's determine the departure and arrival maneuvers. Beginning with the departure location, one can find the departure velocity vector. In the rotating frame, recall that the general expression for the velocity vector is:

$$\bar{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} \tag{50}$$

where, for the $\hat{\theta}$ direction:

$$r\dot{\theta} = \frac{h}{r_D} = 6.2449 \text{ km/s} \tag{51}$$

For the $\hat{r}$ direction, note that:

$$\dot{r} = \pm\sqrt{v_D^2 - (r\dot{\theta})^2} \tag{52}$$

where the sign depends on where the spacecraft is ascending or descending. Recalling that the spacecraft is descending at departure, then:

$$\dot{r} = -2.6574 \text{ km/s} \tag{53}$$

So, the velocity at departure is:

$$\bar{v}_D = -2.6574\,\hat{r} + 6.2449\,\hat{\theta} \text{ km/s} \tag{54}$$

One can draw a basic sketch of the transfer as follows:

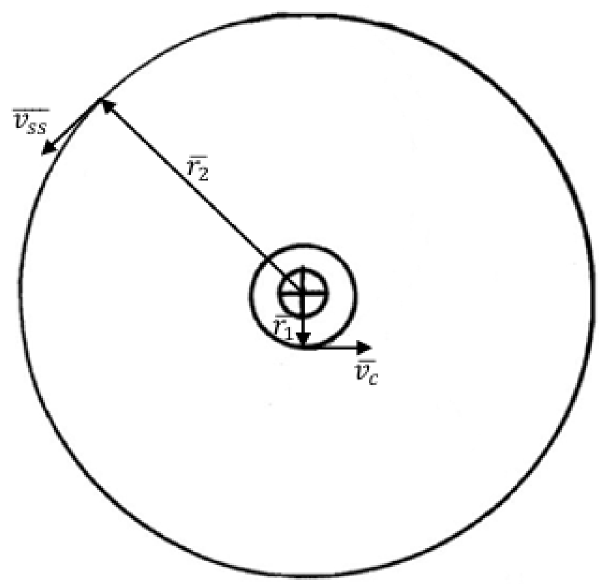


Figure 12: Geocentric View of transfer.

Since we know the circular velocity of the original orbit and the departure location velocity of the transfer ellipse, we can draw a vector diagram to help solve for $\Delta\bar{v}_D$:

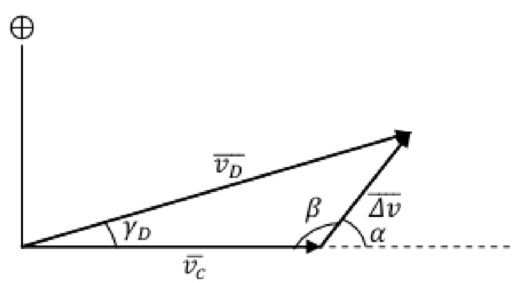


Figure 13: Velocity vector diagram at departure of transfer.

From figure 13, we can obtain an equation for $\Delta\bar{v}_D$ as follows:

$$\Delta\bar{v}_D = \bar{v}_D - \bar{v}_{c_1} \tag{55}$$

where:

$$v_{c_1} = 5.5899\,\hat{\theta}\text{ km/s} \tag{56}$$

Then, because the transfer is within the same plane as the initial and final orbits, the required maneuver at departure can be expressed as:

$$\boxed{\Delta\bar{v}_D = -2.6574\,\hat{r} + 0.6550\,\hat{\theta}\text{ km/s}} \tag{57}$$

OR:

$$\boxed{||\Delta\bar{v}_D|| = 2.7369\text{ km/s}} \tag{58}$$

Now, one can find α_D by calculating its supplementary angle β using sine law:

$$\begin{gathered}\frac{||\Delta\bar{v}_D||}{\sin|\gamma_D|} = \frac{v_D}{\sin\beta} \\ \beta = 76.1538°,\ 103.8462°\end{gathered} \tag{59}$$

But, where $v_D^2 > v_{c_1}^2 + (\Delta v_D)^2$, then β is the larger angle:

$$\beta = 103.8462° \tag{60}$$

meaning that the maneuver angle is:

$$\boxed{\alpha_D = -(180° - \beta) = -76.1538°} \tag{61}$$

Note that the maneuver angle is negative as the maneuver is rotated counter-clockwise or the angle points towards the Earth. Now, one can write the maneuver in $\hat{V}$-$\hat{N}$-$\hat{C}$ components, recalling that:

$$\Delta\bar{v}_D = \Delta v_D(\cos\beta_{OP}\cos\alpha_D\hat{V} + \sin\beta_{OP}\hat{N} + \cos\beta_{OP}\sin\alpha_D\hat{C}) \tag{62}$$

where $\beta_{OP} = 0°$ as the maneuver has no out-of-plane (OP) component. So, the departure maneuver is then:

$$\boxed{\Delta\bar{v}_D = 0.6550\,\hat{V} - 2.6574\,\hat{C}\text{ km/s}} \tag{63}$$

Note that the $\hat{V}$-$\hat{N}$-$\hat{C}$ follow the convention $\hat{V}\times\hat{N} = \hat{C}$, where $\hat{V}$ lies along the velocity direction of the spacecraft, $\hat{N}$ lies along the angular momentum direction of the system, and $\hat{C}$ completes the orthonormal triad. However, if we keep the order of the $\hat{V}$ and $\hat{N}$ components the same and consider, instead, the frame where $\hat{N}\times\hat{V} = \hat{B}$, then the departure maneuver is then:

$$\boxed{\Delta\bar{v}_D = 0.6550\,\hat{V} + 2.6574\,\hat{B}\text{ km/s}} \tag{64}$$

Following the same procedure to obtain the corresponding values at arrival, one can again recall that the general expression for the velocity vector is:

$$\bar{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} \tag{65}$$

where, for the $\hat{\theta}$ direction:

$$r\dot{\theta} = \frac{h}{r_A} = 2.0816\text{ km/s} \tag{66}$$

For the $\hat{r}$ direction, note that:

$$\dot{r} = \pm\sqrt{v_A^2 - (r\dot{\theta})^2} \tag{67}$$

where the sign depends on where the spacecraft is ascending or descending. Recalling that the spacecraft is ascending at arrival, then:

$$\dot{r} = 0.2537\text{ km/s} \tag{68}$$

So, the velocity at arrival is:

$$\bar{v}_A = 0.2537\,\hat{r} + 2.0816\,\hat{\theta}\text{ km/s} \tag{69}$$

Since we know the circular velocity of the final orbit and the arrival location velocity of the transfer ellipse, we can draw a vector diagram to help solve for $\Delta\bar{v}_A$:

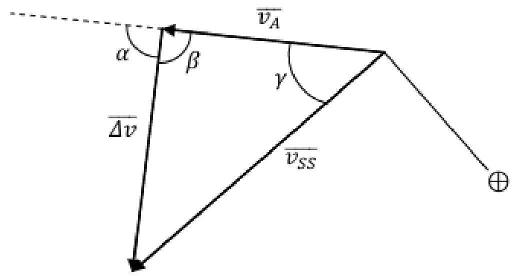


Figure 14: Velocity vector diagram at arrival of transfer.

From figure 14, we can obtain an equation for $\Delta\bar{v}_A$ as follows:

$$\Delta\bar{v}_A = \bar{v}_A - \bar{v}_{c_2} \tag{70}$$

where:

$$v_{c_2} = 3.2274\,\hat{\theta}\text{ km/s} \tag{71}$$

Then, because the transfer is within the same plane as the initial and final orbits, the required maneuver at arrival can be expressed as:

$$\boxed{\Delta\bar{v}_A = 0.2537\,\hat{r} - 1.1457\,\hat{\theta}\text{ km/s}} \tag{72}$$

OR:

$$\boxed{||\Delta\bar{v}_A|| = 1.1735\text{ km/s}} \tag{73}$$

Now, one can find α_A by calculating its supplementary angle β using sine law:

$$\begin{gathered}\frac{||\Delta\bar{v}_A||}{\sin|\gamma_A|} = \frac{v_{c_2}}{\sin\beta} \\ \beta = 19.4353°,\ 160.5647°\end{gathered} \tag{74}$$

But, where $v_{c_2}^2 > v_A^2 + (\Delta v_A)^2$, then β is the larger angle:

$$\beta = 160.5647° \tag{75}$$

meaning that the maneuver angle is:

$$\boxed{\alpha_A = -(180° - \beta) = -19.4353°} \tag{76}$$

Note that the maneuver angle is negative as the maneuver is rotated counter-clockwise or the angle points towards the Earth. Now, one can write the maneuver in $\hat{V}$-$\hat{N}$-$\hat{C}$ components, recalling that:

$$\Delta\bar{v}_A = \Delta v_A(\cos\beta_{OP}\cos\alpha_A\hat{V} + \sin\beta_{OP}\hat{N} + \cos\beta_{OP}\sin\alpha_A\hat{C}) \tag{77}$$

where $\beta_{OP} = 0°$ as the maneuver has no out-of-plane (OP) component. So, the arrival maneuver is then:

$$\boxed{\Delta\bar{v}_A = 1.1066\,\hat{V} - 0.3905\,\hat{C}\text{ km/s}} \tag{78}$$

Note that the $\hat{V}$-$\hat{N}$-$\hat{C}$ follow the convention $\hat{V}\times\hat{N} = \hat{C}$, where $\hat{V}$ lies along the velocity direction of the spacecraft, $\hat{N}$ lies along the angular momentum direction of the system, and $\hat{C}$ completes the orthonormal triad. However, if we keep the order of the $\hat{V}$ and $\hat{N}$ components the same and consider, instead, the frame where $\hat{N}\times\hat{V} = \hat{B}$, then the arrival maneuver is then:

$$\boxed{\Delta\bar{v}_A = 1.1066\,\hat{V} + 0.3905\,\hat{B}\text{ km/s}} \tag{79}$$

Thus, the total Δv for this transfer is:

$$\boxed{\Delta v_{total} = ||\Delta\bar{v}_D|| + ||\Delta\bar{v}_A|| = 3.9104\text{ km/s}} \tag{80}$$

We can recall that the total Δv for the Hohmann was 2.2016 km/s, which is way smaller than the total Δv for the minimum energy transfer. However, if one recalls the TOF from part (b) above was 5.2478 hrs, this TOF is smaller than the TOF for the Hohmann transfer, which was 5.6326 hrs.

Part (d)

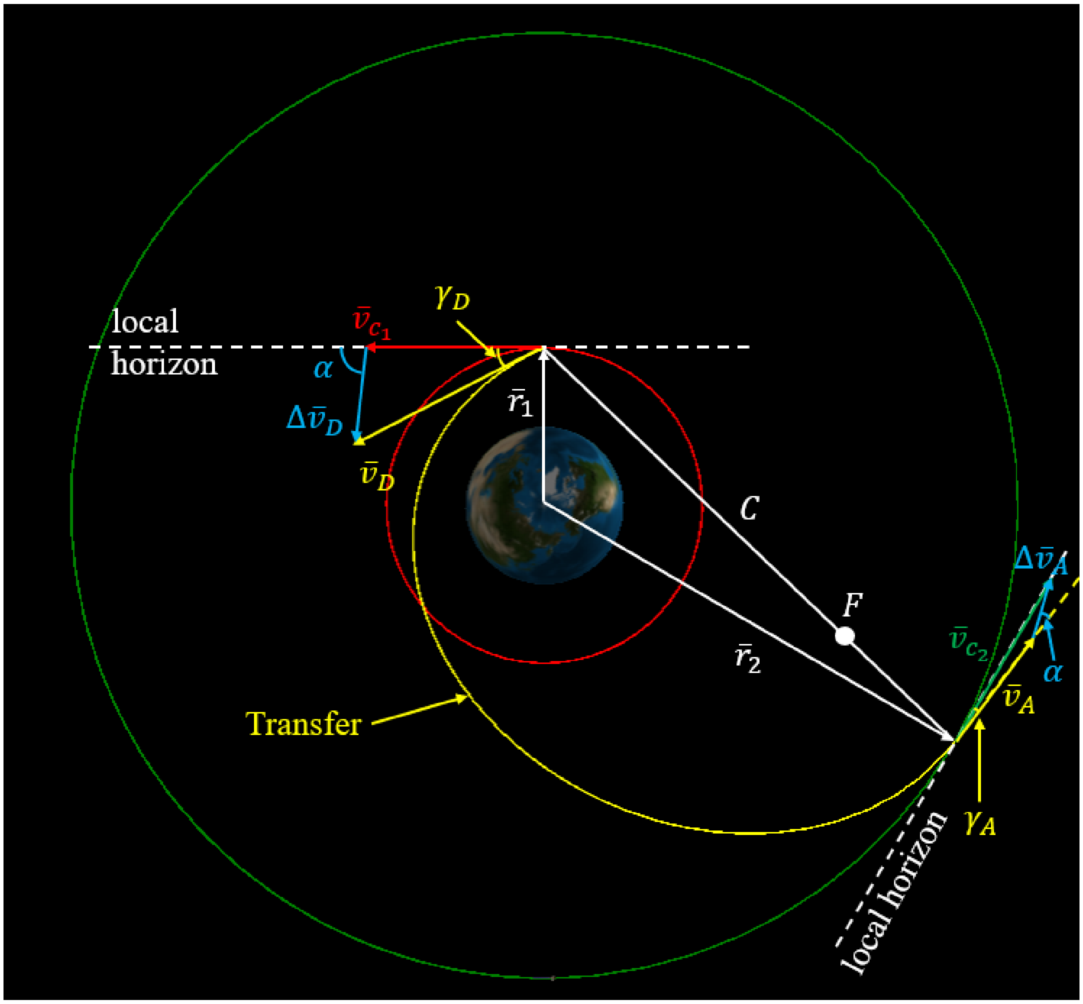


Figure 15: GMAT diagram of transfer. Note that velocity vectors are not to scale.

Using GMAT we find the following values for the transfer arc:

		SMA	ECC	ElapsedSecs	RadPer	VMAG	FPA	TA
Departure	Before Maneuver	12756.3	0	0	12756.273	5.58994	90	0
	After Maneuver	24254.6	0.586172	0	10037.241	6.78681	113.05	295.04
Arrival	Before Maneuver	24254.6	0.586172	18892.04575	10037.241	2.09705	83.051	175.04
	After Maneuver	38268.8	8.28E-12	18892.04575	38268.818	3.22735	90	240

Figure 16: GMAT output before and after each maneuver.

Noting the difference in definitions of flight path angle, one can see that the values from the GMAT report above match those reported in part (b). Furthermore, the values after each maneuver were correctly obtained using an impulse maneuver in GMAT, with each maneuver defined using $\Delta\bar{v}_D$ and $\Delta\bar{v}_A$ in $\hat{V}$-$\hat{N}$-$\hat{C}$ coordinates from part (c). This then confirms our results in part (c) as well (Realize that GMAT calls the $\hat{V}$-$\hat{N}$-$\hat{C}$ frame 'VNB', but the definition from their manuals match that of $\hat{V}$-$\hat{N}$-$\hat{C}$.).

Problem 3

Problem Statement

Consider a transfer from Earth to Mars. Assume that the orbits of Earth and Mars are circular and coplanar.

(a) First, note the geometry for a Hohmann transfer from Earth to Mars. Determine the corresponding time-of-flight along the Hohmann transfer, i.e., TOF_{Hoh}. What is the synodic period?

(b) It has, however, been decided that the Hohmann transfer geometry is too restrictive. Convenient launch and arrival dates have been set such that the departure/arrival geometry can be modeled for a transfer angle equal to 130°.

Determine the minimum energy transfer for this geometry. Determine the following transfer characteristics: a, e, p, r_p, r_a, ε, θ^*_D, θ^*_A, v_d, v_a, γ_D, γ_A, TOF_{min}.

Plot the orbits – Earth orbit, Mars orbit, transfer orbit. Highlight the transfer arc; **add the vacant focus F to the plot!**

(c) For the minimum energy transfer, determine the phase angle required at departure. When will this geometry appear again?

[Note that if you miss the "launch", it is actually not necessary to wait for the entire synodic period. With a Lambert procedure, simply compute a new transfer for a slightly changed geometry!!]

(d) The s/c departs from a 250 km circular Earth parking orbit. For departure from Earth, compute $\left|\bar{v}^+_{\infty/\oplus}\right|$.

Determine the actual departure maneuver, i.e., $|\Delta\bar{v}_D|$. Be aware that Earth departure is no longer tangential (wrt the Sun)!!

Part (a)

The geometry for a Hohmann transfer is illustrated in Figure 17. In the Earth to Mars transfer scenario,

$$r_p = a_{\oplus} = 149597898\ km$$
$$r_a = a_{♂} = 227944135\ km$$

Thus, the transfer orbit possesses the characteristics

$$a_h = \frac{1}{2}(r_p + r_a) = 188771016.5\ km$$
$$\mathbb{P} = 2\pi\sqrt{\frac{a_h^3}{\mu_{\odot}}} = 517.742\ days$$

The transfer TOF is then

$$TOF_{Hoh} = \frac{\mathbb{P}}{2} = \boxed{258.8709\ days}$$

The mean motion values for the Earth and Mars are

$$n_{\oplus} = \sqrt{\frac{\mu_{\odot}}{a_{\oplus}^3}} = 1.99098 \times 10^{-7}\ rad/s$$
$$n_{♂} = \sqrt{\frac{\mu_{\odot}}{a_{♂}^3}} = 1.05855 \times 10^{-7}\ rad/$$

The synodic period for Earth and Mars is thus

$$T_{syn} = \frac{2\pi}{n_{\oplus} - n_{♂}} = \boxed{2.13531\ yrs}$$

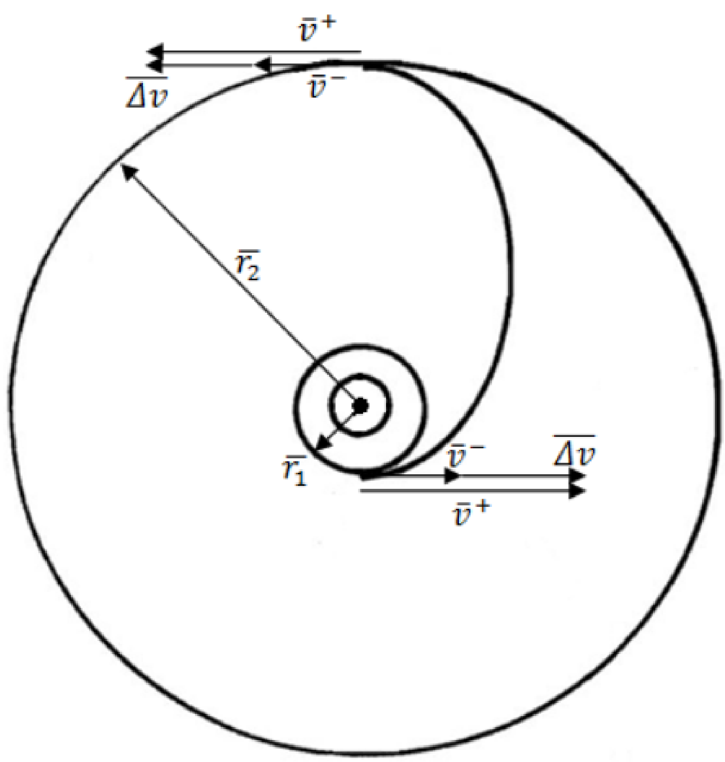


Figure 17: Hohmann transfer geometry

Part (b)

The space triangle is defined by sides r_1, r_2, and c, where the angle between r_1 and r_2, denoted δ, is the transfer angle. The geometry for the minimum energy transfer is shown in Figure 18. The length of the chord is found from the law of cosines,

$$c = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos\delta} = 2.29794\ AU$$

and the semimajor axis of the minimum energy transfer is

$$a = \frac{1}{4}(r_1 + r_2 + c) = \boxed{1.20541\ AU}$$

The energy of the transfer ellipse is then found

$$\varepsilon = -\frac{\mu_\odot}{2a} = \boxed{-367.97637\ km^2/s^2}$$

To find the remaining orbit parameters, consider the space triangle drawn in Figure 19. The vacant focus of the minimum energy ellipse lies on the chord, between P_1 and P_2, at a distance of $2ae$ from the central body. The angle ξ is computed by leveraging the law of cosines and triangle $O - P_1 - P_2$,

$$\xi = \sin^{-1}\left(\frac{r_1}{c}\sin\delta\right) = 19.47289°$$

The sign ambiguity is resolved by considering that the angle ξ must be acute to match the space triangle geometry. Now, considering triangle $O - F - P_2$, the law of cosines is applied to determine the length of $OF = 2ae$,

$$OF = 2ae = \sqrt{r_2^2 + (P_2F)^2 - 2r_2(P_2F)\cos\xi} = 0.748260\ AU$$

where $P_2F = 2a - r_2$. Thus, the eccentricity of the minimum energy transfer ellipse is

$$e = \frac{OF}{2a} = 0.31037$$

Remaining orbital characteristics are

$$\begin{aligned}
p &= a(1-e^2) &&= \boxed{1.08929\ AU} \\
r_p &= a(1-e) &&= \boxed{0.83128\ AU} \\
r_a &= a(1+e) &&= \boxed{1.57954\ AU} \\
r_D &= r_1 &&= \boxed{1\ AU} \\
r_A &= r_2 &&= \boxed{1.52371\ AU} \\
v_D &= \sqrt{\frac{2\mu_\odot}{r_D} - \frac{\mu_\odot}{a}} &&= \boxed{32.22270\ km/s} \\
v_A &= \sqrt{\frac{2\mu_\odot}{r_A} - \frac{\mu_\odot}{a}} &&= \boxed{20.69968\ km/s}
\end{aligned}$$

Determining the true anomaly at departure and arrival requires resolving another sign ambiguity. Initial solutions are obtained by rearranging the conic equation

$$\begin{aligned}
\theta_D^* &= \cos^{-1}\left[\frac{1}{e}\left(\frac{p}{r_D} - 1\right)\right] &&= \pm 73.27991° \\
\theta_A^* &= \cos^{-1}\left[\frac{1}{e}\left(\frac{p}{r_A} - 1\right)\right] &&= \pm 156.72009°
\end{aligned}$$

There are two possible solutions for each true anomaly. To resolve the ambiguity, the transfer angle is leveraged to determine the correct combination of θ_D^* and θ_A^* such that

$$\theta_D^* + \delta = \theta_A^*$$

Thus, the departure and arrival true anomalies are

$$\theta^*_D = \boxed{73.27991°}$$
$$\theta^*_A = \boxed{-156.72009°}$$

With the correct signs for true anomaly determined, the flight path angles at departure and arrival are

$$\gamma_D = \cos^{-1}\left(\frac{\sqrt{\mu_\odot p}}{r_D v_D}\right) = \boxed{15.26355°}$$
$$\gamma_A = \cos^{-1}\left(\frac{\sqrt{\mu_\odot p}}{r_A v_A}\right) = \boxed{-9.73645°}$$

The eccentric anomalies corresponding to the departure and arrival points are also determined

$$E_D = \cos^{-1}\left[-\frac{1}{e}\left(\frac{r_D}{a} - 1\right)\right] = 56.69850°$$
$$E_A = \cos^{-1}\left[-\frac{1}{e}\left(\frac{r_A}{a} - 1\right)\right] = -148.29546° = 211.70454°$$

where the signs must match the corresponding true anomalies. Finally, time of flight is calculated by applying Kepler's Equation

$$TOF = \sqrt{\frac{a^3}{\mu_\odot}}\left[(E_A - E_D) - e\sin E_A + e\sin E_D\right] = \boxed{240.64357\ days}$$

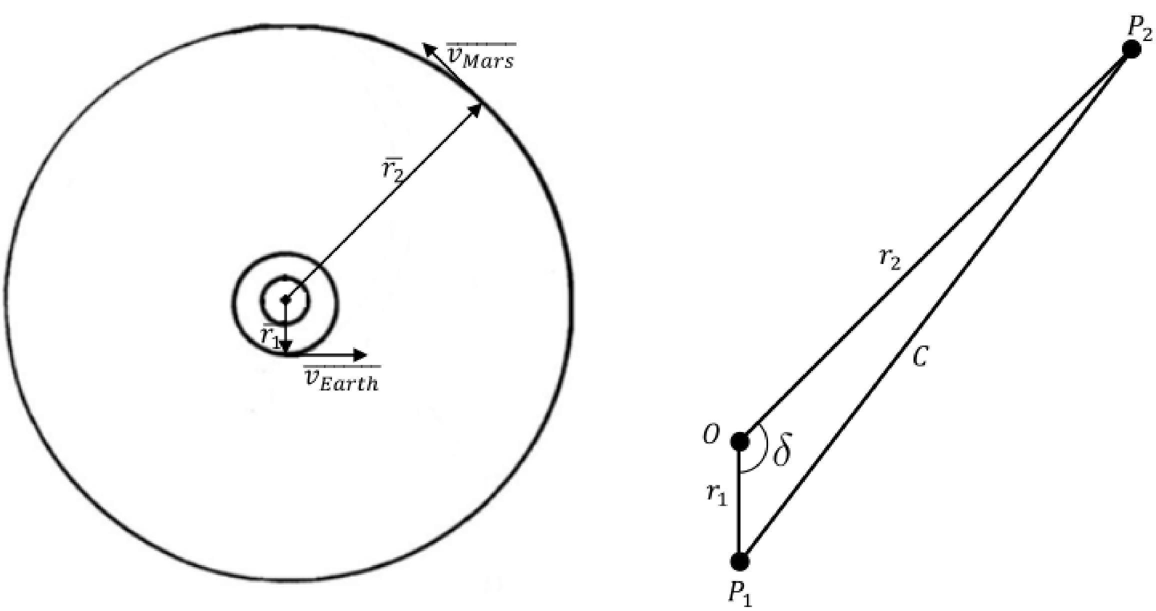


Figure 18: System and space triangle

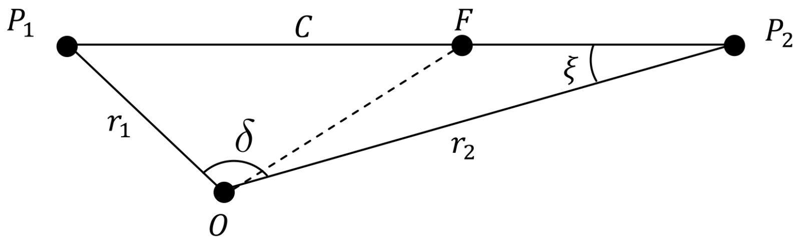


Figure 19: Space triangle with vacant focus

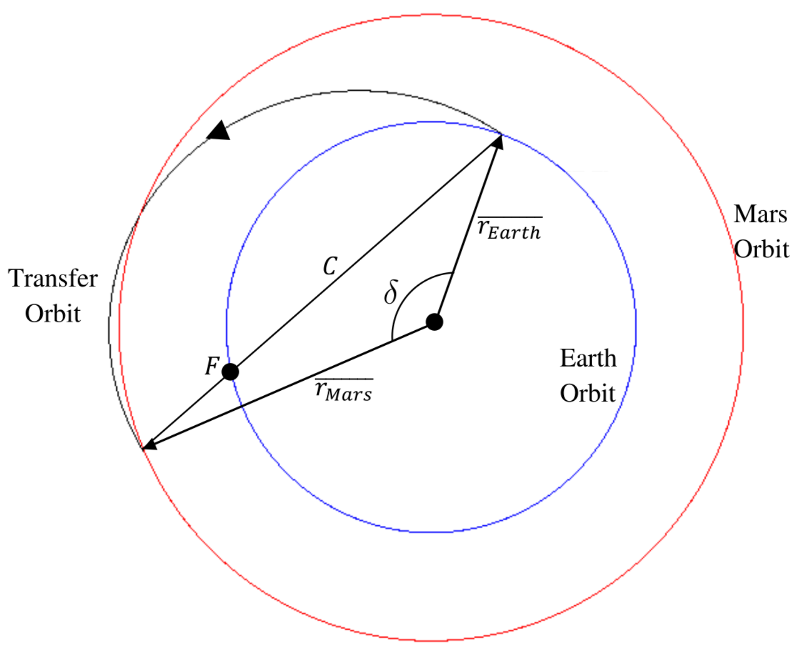


Figure 20: Minimum energy transfer

Part (c)

The phase angle at departure is

$$\phi = \delta - n_{♂}(TOF) = \boxed{3.89746°}$$

This exact geometry will reappear in one synodic period, which was determined in Part (a) to be $\boxed{T_{syn} = 2.1353\ yr}$

Part (d)

Figure 21 demonstrates the vector diagram to find the desired $\bar{v}^+_{\infty/\oplus}$. Applying the law of cosines,

$$v_{\infty/\oplus} = \sqrt{v_\oplus^2 + v_D^2 - 2v_\oplus v_D \cos\gamma_D} = \boxed{8.58217\ km/s}$$

The geocentric orbital speed of the hyperbola at the radius of the parking orbit is

$$v_{p/\oplus} = \sqrt{v^2_{\infty/\oplus} + \frac{2\mu_\oplus}{r_p}} = 13.92584\ km/s$$

and the geocentric orbital speed of the parking orbit is

$$v_{c/\oplus} = \sqrt{\frac{\mu_\oplus}{r_p}} = 7.75485\ km/s$$

As illustrated in Figure 22, the velocity vector of the hyperbola at perigee is tangent to the parking orbit. Thus, the departure maneuver is aligned with the velocities of both the parking orbit and the hyperbolic orbit, and the maneuver magnitude is

$$\Delta v_D = v_{p/\oplus} - v_{c/\oplus} = \boxed{6.17100\ km/s}$$

From the heliocentric perspective, the departure is no longer tangential! However, since this is a planar problem, the departure from the parking orbit is still a tangential maneuver from the geocentric perspective.

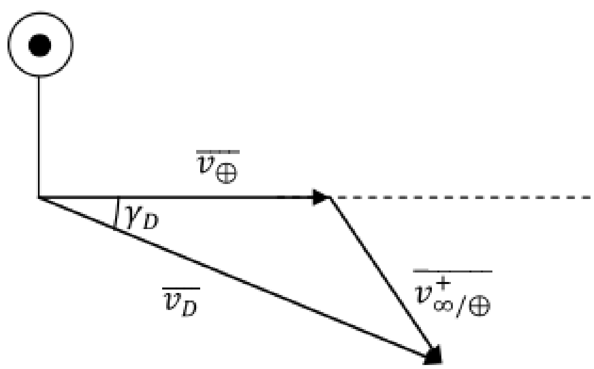


Figure 21: Vector diagram to determine departure geometry

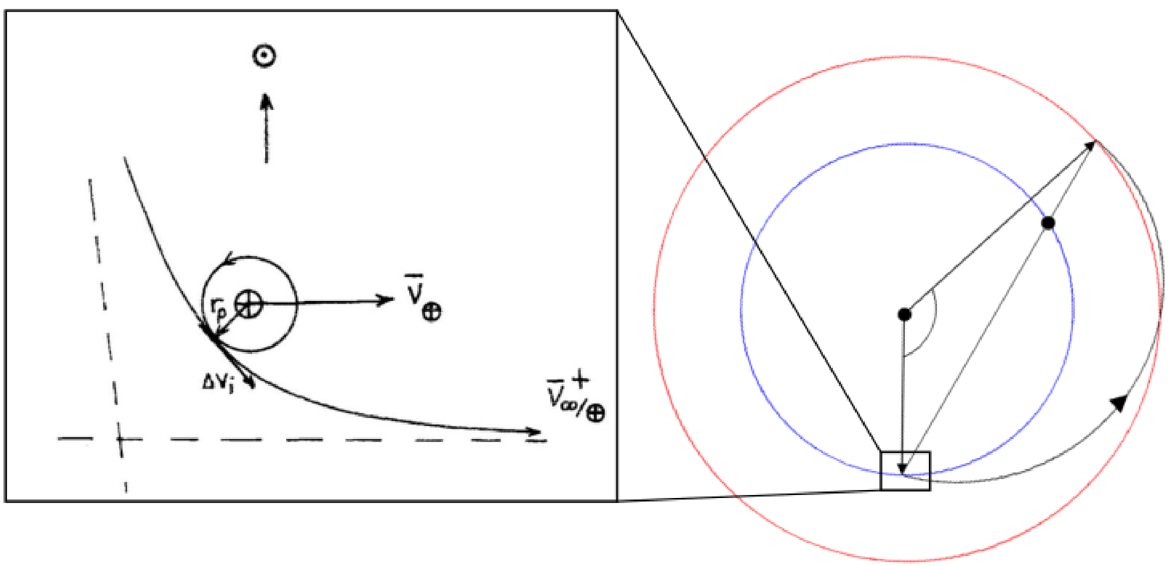


Figure 22: Hyperbolic departure from Earth