# Optimal Estimation Methods – Homework 1

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# Exercise 1.7

Using the simple model

$$y = x_1 + x_2 \sin 10t + x_3 e^{2t^2}$$

with $x_1 = x_2 = x_3 = 1.0$, generate four sets of "synthetic data" at the instants $t = 0, 0.1, 0.2, 0.3, \ldots, 1.0$ by truncating each $y$ value after 6, 4, 2, and 1 significant figures, respectively, to simulate (crudely) measurement errors. Use the normal equations (1.26) to process the measurements and derive $\hat{x}_i$ estimates for each of the four cases. Compare the estimates with the true values (1, 1, 1) in each case.

```
% Exercise 1.7

% Truth
t=[0:0.1:1]';m=length(t);
x=[1;1;1];
y=x(1)+x(2)*sin(10*t)+x(3)*exp(2*t.^2);

% Truncation Values
r6=1e6;r4=1e4;r2=1e2;r1=1e1;

% Get Truncated Measurements
y1=ceil(y*r1)/r1;
y2=ceil(y*r2)/r2;
y4=ceil(y*r4)/r4;
y6=ceil(y*r6)/r6;

% Least Squares Solutions for All Sets
h=[ones(m,1) sin(10*t) exp(2*t.^2)];
z=inv(h'*h)*h';
x1=z*y1
x2=z*y2
x4=z*y4
x6=z*y6
```

```
>> problem1_7

x1 =

    1.0315
    0.9942
    1.0024


x2 =

    1.0046
    1.0015
    1.0001


x4 =

    1.0000
    1.0000
    1.0000
```

```
x6 =

    1.0000
    1.0000
    1.0000
```

# Exercise 1.9

Consider the following partitioned matrix (assume that $|A_{11}| \neq 0$ and $|A_{22}| \neq 0$):

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

Prove that the following matrices are all valid inverses:

$$A^{-1} = \begin{bmatrix} A_{11}^{-1} + A_{11}^{-1} A_{12} B_{22}^{-1} A_{21} A_{11}^{-1} & -A_{11}^{-1} A_{12} B_{22}^{-1} \\ -B_{22}^{-1} A_{21} A_{11}^{-1} & B_{22}^{-1} \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} B_{11}^{-1} & -B_{11}^{-1} A_{12} A_{22}^{-1} \\ -A_{22}^{-1} A_{21} B_{11}^{-1} & A_{22}^{-1} + A_{22}^{-1} A_{21} B_{11}^{-1} A_{12} A_{22}^{-1} \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} B_{11}^{-1} & -A_{11}^{-1} A_{12} B_{22}^{-1} \\ -A_{22}^{-1} A_{21} B_{11}^{-1} & B_{22}^{-1} \end{bmatrix}$$

where $B_{ii}$ is the *Schur complement* of $A_{ii}$, given by

$$B_{11} = A_{11} - A_{12} A_{22}^{-1} A_{21}, \quad B_{22} = A_{22} - A_{21} A_{11}^{-1} A_{12}$$

Also, prove the matrix inversion lemma from these matrix inverses.

# Exercise 1.9

- Brute Force (first form)

$$
\begin{aligned}
A\,A^{-1} &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} A_{11}^{-1} + A_{11}^{-1} A_{12} B_{22}^{-1} A_{21} A_{11}^{-1} & -A_{11}^{-1} A_{12} B_{22}^{-1} \\ -B_{22}^{-1} A_{21} A_{11}^{-1} & B_{22}^{-1} \end{bmatrix} \\
&= \begin{bmatrix} C & D \\ E & F \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
C &\equiv I + A_{12} B_{22}^{-1} A_{21} A_{11}^{-1} - A_{12} B_{22}^{-1} A_{21} A_{11}^{-1} = I \ \checkmark \\
D &\equiv A_{12} B_{22}^{-1} - A_{12} B_{22}^{-1} = 0 \ \checkmark \\
E &\equiv A_{21} A_{11}^{-1} + A_{21} A_{11}^{-1} A_{12} B_{22}^{-1} A_{21} A_{11}^{-1} - A_{22} B_{22}^{-1} A_{21} A_{11}^{-1} \\
F &\equiv A_{22} B_{22}^{-1} - A_{21} A_{11}^{-1} A_{12} B_{22}^{-1} \\
&= \underbrace{\left(A_{22} - A_{21} A_{11}^{-1} A_{12}\right)}_{=B_{22}} B_{22}^{-1} = I \ \checkmark
\end{aligned}
$$

# Exercise 1.9

- Work on $E$

$$\begin{aligned}
E &\equiv A_{21}A_{11}^{-1} + A_{21}A_{11}^{-1}A_{12}B_{22}^{-1}A_{21}A_{11}^{-1} - A_{22}B_{22}^{-1}A_{21}A_{11}^{-1} \\
&= (I + A_{21}A_{11}^{-1}A_{12}B_{22}^{-1} - A_{22}B_{22}^{-1})A_{21}A_{11}^{-1} \\
&= (B_{22} \underbrace{+A_{21}A_{11}^{-1}A_{12} - A_{22})}_{=-B_{22}}B_{22}^{-1}A_{21}A_{11}^{-1} \\
&= (B_{22} - B_{22})B_{22}^{-1}A_{21}A_{11}^{-1} \\
&= 0 \ \checkmark
\end{aligned}$$

- Another Approach

$$X \equiv \begin{bmatrix} I & 0 \\ A_{21}A_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} A_{11} & 0 \\ 0 & \underbrace{A_{22} - A_{21}A_{11}^{-1}A_{12}}_{=B_{22}} \end{bmatrix} \begin{bmatrix} I & A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{bmatrix} \begin{bmatrix} I & A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = A$$

- Use the following matrix inverses

$$\begin{bmatrix} I & 0 \\ A_{21}A_{11}^{-1} & I \end{bmatrix}^{-1} = \begin{bmatrix} I & 0 \\ -A_{21}A_{11}^{-1} & I \end{bmatrix}$$

$$\begin{bmatrix} I & A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix}^{-1} = \begin{bmatrix} I & -A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix}$$

# Exercise 1.9

- Then $A^{-1}$ is given by

$$A^{-1} \equiv X^{-1} = \begin{bmatrix} I & -A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{11}^{-1} & 0 \\ 0 & B_{22}^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -A_{21}A_{11}^{-1} & I \end{bmatrix}$$

$$= \begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12}B_{22}^{-1} \\ 0 & B_{22}^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -A_{21}A_{11}^{-1} & I \end{bmatrix}$$

$$= \begin{bmatrix} A_{11}^{-1} + A_{11}^{-1}A_{12}B_{22}^{-1}A_{21}A_{11}^{-1} & -A_{11}^{-1}A_{12}B_{22}^{-1} \\ -B_{22}^{-1}A_{21}A_{11}^{-1} & B_{22}^{-1} \end{bmatrix} \quad \checkmark$$

# Exercise 1.9

- Brute Force (second form)

$$
\begin{aligned}
A\,A^{-1} &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11}^{-1} & -B_{11}^{-1} A_{12} A_{22}^{-1} \\ -A_{22}^{-1} A_{21} B_{11}^{-1} & A_{22}^{-1} + A_{22}^{-1} A_{21} B_{11}^{-1} A_{12} A_{22}^{-1} \end{bmatrix} \\
&= \begin{bmatrix} C & D \\ E & F \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
C &\equiv A_{11} B_{11}^{-1} - A_{12} A_{22}^{-1} A_{21} B_{11}^{-1} \\
&= \underbrace{\left(A_{11} - A_{12} A_{22}^{-1} A_{21}\right)}_{=B_{11}} B_{11}^{-1} = I \;\; \checkmark \\
D &\equiv A_{12} A_{22}^{-1} + A_{12} A_{22}^{-1} A_{21} B_{11}^{-1} A_{12} A_{22}^{-1} - A_{11} B_{11}^{-1} A_{12} A_{22}^{-1} \\
E &\equiv A_{21} B_{11}^{-1} - A_{21} B_{11}^{-1} = 0 \;\; \checkmark \\
F &\equiv I + A_{21} B_{11}^{-1} A_{12} A_{22}^{-1} - A_{21} B_{11}^{-1} A_{12} A_{22}^{-1} = I \;\; \checkmark
\end{aligned}
$$

# Exercise 1.9

- Work on $D$

$$
\begin{aligned}
D &\equiv A_{12}A_{22}^{-1} + A_{12}A_{22}^{-1}A_{21}B_{11}^{-1}A_{12}A_{22}^{-1} - A_{11}B_{11}^{-1}A_{12}A_{22}^{-1} \\
&= (I + A_{12}A_{22}^{-1}A_{21}B_{11}^{-1} - A_{11}B_{11}^{-1})A_{12}A_{22}^{-1} \\
&= (B_{11} \underbrace{+A_{12}A_{22}^{-1}A_{21} - A_{11}}_{=-B_{11}})B_{11}^{-1}A_{12}A_{22}^{-1} \\
&= (B_{11} - B_{11})B_{11}^{-1}A_{12}A_{22}^{-1} \\
&= 0 \ \checkmark
\end{aligned}
$$

# Exercise 1.9

- Another Approach

$$Y \equiv \begin{bmatrix} I & A_{12}A_{22}^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} \underbrace{A_{11} - A_{12}A_{22}^{-1}A_{21}}_{=B_{11}} & 0 \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ A_{22}^{-1}A_{21} & I \end{bmatrix}$$

$$= \begin{bmatrix} A_{11} - A_{12}A_{22}^{-1}A_{21} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ A_{22}^{-1}A_{21} & I \end{bmatrix}$$

$$= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = A$$

- Use the following matrix inverses

$$\begin{bmatrix} I & A_{12}A_{22}^{-1} \\ 0 & I \end{bmatrix}^{-1} = \begin{bmatrix} I & -A_{12}A_{22}^{-1} \\ 0 & I \end{bmatrix}$$

$$\begin{bmatrix} I & 0 \\ A_{22}^{-1}A_{21} & I \end{bmatrix}^{-1} = \begin{bmatrix} I & 0 \\ -A_{22}^{-1}A_{21} & I \end{bmatrix}$$

# Exercise 1.9

- Then $A^{-1}$ is given by

$$
\begin{aligned}
A^{-1} \equiv Y^{-1} &= \begin{bmatrix} I & 0 \\ -A_{22}^{-1}A_{21} & I \end{bmatrix} \begin{bmatrix} B_{11}^{-1} & 0 \\ 0 & A_{22}^{-1} \end{bmatrix} \begin{bmatrix} I & -A_{12}A_{22}^{-1} \\ 0 & I \end{bmatrix} \\
&= \begin{bmatrix} B_{11}^{-1} & 0 \\ -A_{22}^{-1}A_{21}B_{11}^{-1} & A_{22}^{-1} \end{bmatrix} \begin{bmatrix} I & -A_{12}A_{22}^{-1} \\ 0 & I \end{bmatrix} \\
&= \begin{bmatrix} B_{11}^{-1} & -B_{11}^{-1}A_{12}A_{22}^{-1} \\ -A_{22}^{-1}A_{21}B_{11}^{-1} & A_{22}^{-1} + A_{22}^{-1}A_{21}B_{11}^{-1}A_{12}A_{22}^{-1} \end{bmatrix} \quad \checkmark
\end{aligned}
$$

# Exercise 1.9

- Brute Force (third form)

$$
\begin{aligned}
A\,A^{-1} &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11}^{-1} & -A_{11}^{-1} A_{12} B_{22}^{-1} \\ -A_{22}^{-1} A_{21} B_{11}^{-1} & B_{22}^{-1} \end{bmatrix} \\
&= \begin{bmatrix} C & D \\ E & F \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
C &\equiv A_{11} B_{11}^{-1} - A_{12} A_{22}^{-1} A_{21} B_{11}^{-1} \\
&= \underbrace{\left(A_{11} - A_{12} A_{22}^{-1} A_{21}\right)}_{=B_{11}} B_{11}^{-1} = I \ \checkmark \\
D &\equiv A_{12} B_{22}^{-1} - A_{12} B_{22}^{-1} = 0 \ \checkmark \\
E &\equiv A_{21} B_{11}^{-1} - A_{21} B_{11}^{-1} = 0 \ \checkmark \\
F &\equiv A_{22} B_{22}^{-1} - A_{21} A_{11}^{-1} A_{12} B_{22}^{-1} \\
&= \underbrace{\left(A_{22} - A_{21} A_{11}^{-1} A_{12}\right)}_{=B_{22}} B_{22}^{-1} = I \ \checkmark
\end{aligned}
$$

# Exercise 1.9

- Matrix Inversion Lemma (look at first and second forms)

$$A^{-1} = \begin{bmatrix} A_{11}^{-1} + A_{11}^{-1}A_{12}B_{22}^{-1}A_{21}A_{11}^{-1} & -A_{11}^{-1}A_{12}B_{22}^{-1} \\ -B_{22}^{-1}A_{21}A_{11}^{-1} & B_{22}^{-1} \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} B_{11}^{-1} & -B_{11}^{-1}A_{12}A_{22}^{-1} \\ -A_{22}^{-1}A_{21}B_{11}^{-1} & A_{22}^{-1} + A_{22}^{-1}A_{21}B_{11}^{-1}A_{12}A_{22}^{-1} \end{bmatrix}$$

- Look at $1-1$ element to give the following identity

$$B_{11}^{-1} = A_{11}^{-1} + A_{11}^{-1}A_{12}B_{22}^{-1}A_{21}A_{11}^{-1}$$

- Change $A_{12}$ to $-A_{12}$, and substitute $B_{11}$ and $B_{22}$ to prove the matrix inversion lemma

$$(A_{11}+A_{12}A_{22}^{-1}A_{21})^{-1} = A_{11}^{-1}-A_{11}^{-1}A_{12}(A_{22}+A_{21}A_{11}^{-1}A_{12})^{-1}A_{21}A_{11}^{-1}$$

Now let $A \equiv A_{11}$, $B \equiv A_{12}$, $C \equiv A_{22}^{-1}$, and $D \equiv A_{21}$ to give

$$(A + BCD)^{-1} = A^{-1} - A^{-1}B\left(C^{-1} + DA^{-1}B\right)^{-1}DA^{-1} \quad \checkmark$$

# Exercise 1.9

- Another Approach
- Look at $2-2$ element to give the following identity

$$B_{22}^{-1} = A_{22}^{-1} + A_{22}^{-1} A_{21} B_{11}^{-1} A_{12} A_{22}^{-1}$$

- Change $A_{21}$ to $-A_{21}$, and substitute $B_{11}$ and $B_{22}$ to prove the matrix inversion lemma

$$(A_{22}+A_{21}A_{11}^{-1}A_{12})^{-1} = A_{22}^{-1} - A_{22}^{-1}A_{21}(A_{11}+A_{12}A_{22}^{-1}A_{21})^{-1}A_{12}A_{22}^{-1}$$

Now let $A \equiv A_{22}$, $B \equiv A_{21}$, $C \equiv A_{11}^{-1}$, and $D \equiv A_{12}$ to give

$$(A+BCD)^{-1} = A^{-1} - A^{-1}B\left(C^{-1}+DA^{-1}B\right)^{-1}DA^{-1} \quad \checkmark$$

# Exercise 1.10

Create 101 synthetic measurements $\tilde{y}$ at 0.1 second intervals of the following:

$$\tilde{y}_j = a \sin t_j - b \cos t_j + v_j$$

where $a = b = 1$, and $v$ is a zero-mean Gaussian noise process with standard deviation given by 0.01. Determine the unweighted least squares estimates for $a$ and $b$. Using the same measurements, find a value of $\tilde{\mathbf{y}}$ that is near zero (near time $\pi/4$), and set that "measurement" value to 1. Compute the unweighted least squares solution, and compare it to the original solution. Then, use weighted least squares to "deweight" the measurement.

```
% Exercise 1.10

% Truth
dt=0.1;tf=10;
t=[0:dt:tf]';m=length(t);
x=[1;1];
ym=x(1)*sin(t)-x(2)*cos(t)+0.01*randn(m,1);

% Find Measurement Near Time pi/4
point_pi4=round((pi/4)/dt);

% Corrputed Measurements
ym_corrupt=ym;
ym_corrupt(point_pi4)=1;

% Least Squares Solution with Good Measurements
h=[sin(t) -cos(t)];
xe=inv(h'*h)*h'*ym

% Least Squares Solution with Corrupt Measurements
h=[sin(t) -cos(t)];
xe_corrupt=inv(h'*h)*h'*ym_corrupt
```

```
% Set Weight Matrix
w=eye(m);w(point_pi4,point_pi4)=1e-8;

% Weighted Least Squares Solution with Corrupt Measurements
h=[sin(t) -cos(t)];
xe_weighted=inv(h'*w*h)*h'*w*ym_corrupt
```

```
>> problem1_10

xe =

    1.0034
    0.9986


xe_corrupt =

    1.0178
    0.9829


xe_weighted =

    1.0034
    0.9986
```

# Exercise 1.12

Using the method of Lagrange multipliers, find all solutions **x** of the first necessary conditions for extremals of the function

$$J(\mathbf{x}) = (\mathbf{x} - \mathbf{a})^T W (\mathbf{x} - \mathbf{a})$$
$$\text{subject to } \mathbf{b}^T \mathbf{x} = c$$

where **a** and **b** are constant vectors, $c$ is a scalar, and $W$ is a symmetric, positive definite matrix.

# Exercise 1.12

- Appended Loss Function

$$J(\mathbf{x}) = (\mathbf{x} - \mathbf{a})^T W (\mathbf{x} - \mathbf{a}) + \lambda(\mathbf{b}^T \mathbf{x} - c)$$

- Necessary Conditions

$$\nabla_{\mathbf{x}} J = 2W\mathbf{x} - 2W\mathbf{a} + \lambda\mathbf{b} = \mathbf{0}$$

$$\mathbf{x} = \mathbf{a} - \frac{1}{2}\lambda W^{-1}\mathbf{b} \tag{1}$$

$$\nabla_{\lambda} J = \mathbf{b}^T \mathbf{x} - c = 0 \tag{2}$$

# Exercise 1.12

- Substitute Eq. (1) into Eq. (2)

$$\mathbf{b}^T\mathbf{a} - \frac{1}{2}\lambda\,\mathbf{b}^T W^{-1}\mathbf{b} - c = 0$$

$$\lambda = -2(\mathbf{b}^T W^{-1}\mathbf{b})^{-1}(c - \mathbf{b}^T\mathbf{a})$$

- Substitute $\lambda$ into Eq. (1)

$$\mathbf{x} = \mathbf{a} + (\mathbf{b}^T W^{-1}\mathbf{b})^{-1}(c - \mathbf{b}^T\mathbf{a})W^{-1}\mathbf{b}$$

# Exercise 1.13

Consider the following dynamic model:

$$y_k = \sum_{i=1}^{n} \phi_i y_{k-i} + \sum_{i=1}^{p} \gamma_i u_{k-i}$$

where $u_i$ is a known input. This ARX (AutoRegressive model with eXogenous input) model extends the simple scalar model given in example 1.2. Given measurements of $y_i$ and the known inputs $u_i$ recast the above model into least squares form and determine estimates for $\phi_i$ and $\gamma_i$.

# Exercise 1.13

- Expand the series with $m$ measurements, with $y_i = u_i = 0$ for $i < 0$

$$\begin{aligned}
y_1 &= \phi_1 y_0 + \gamma_1 u_0 \\
y_2 &= \phi_1 y_1 + \phi_2 y_0 + \gamma_1 u_1 + \gamma_2 u_0 \\
\vdots &= \qquad\qquad \vdots \\
y_m &= \phi_1 y_{m-1} + \phi_2 y_{m-2} + \cdots + \phi_n y_{m-n} \\
&\quad + \gamma_1 u_{m-1} + \gamma_2 u_{m-2} + \cdots + \gamma_p u_{m-p}
\end{aligned}$$

- Define the following vector $\mathbf{x}$

$$\mathbf{x} = \begin{bmatrix} \phi_1 & \phi_2 & \cdots & \phi_n & \gamma_1 & \gamma_2 & \cdots & \gamma_p \end{bmatrix}^T$$

- Define the following vector $\mathbf{y}$

$$\mathbf{y} = \begin{bmatrix} y_1 & y_2 & \cdots & y_m \end{bmatrix}^T$$

# Exercise 1.13

Now we have

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} y_0 & 0 & \cdots & 0 & u_0 & 0 & \cdots & 0 \\ y_1 & y_0 & \cdots & 0 & u_1 & u_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \cdots & \\ y_{m-1} & y_{m-2} & \cdots & y_{m-n} & u_{m-1} & u_{m-2} & \cdots & u_{m-p} \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_n \\ \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_p \end{bmatrix}$$

- This has the form $\mathbf{y} = H\mathbf{x}$
- $\mathbf{y}$ is $m \times 1$, $H$ is $m \times (n+p)$, and $\mathbf{x}$ is $(n+p) \times 1$
- We require $m \geq n + p$ to have a valid least squares solution

# Exercise 1.16

Consider the following dynamic model:

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}_{k+1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}_k$$

and measurement model

$$\tilde{y}_k = \begin{bmatrix} \sin(\omega_0 \,\Delta t\, k) & \cos(\omega_0 \,\Delta t\, k) \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}_k + v_k$$

where $\omega_0$ is the harmonic frequency, and $\Delta t$ is the sampling interval. Create synthetic measurements of the above process with $\omega_0 = 0.4\pi$ rad/sec and $\Delta t = 0.1$ seconds. Also, create different synthetic measurement sets using various values for the standard deviation of $v$ in the measurement errors. Use nonlinear least squares to find an estimate for $\omega_0$ for each synthetic measurement set.

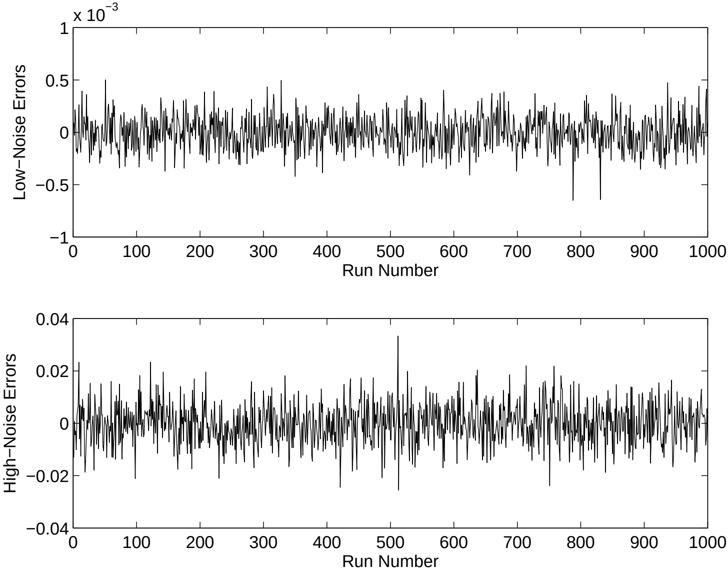
```
% Exercise 1.16

% Time
dt=0.1;tf=10;
t=[0:dt:tf]';m=length(t);

% Set k Vector
k=[0:1:m-1]';

% Pick Low-Noise and High-Noise Standard Deviation
sig_low=0.01;
sig_high=0.5;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Monte Carlo Runs
m_monte=1000;
w0_est_low=zeros(m_monte,1);
w0_est_high=zeros(m_monte,1);
for j=1:m_monte
```

```
% Generate Measurements
w0=0.4*pi;
ym_low=sin(w0*dt*k)+cos(w0*dt*k)+sig_low*randn(m,1);
ym_high=sin(w0*dt*k)+cos(w0*dt*k)+sig_high*randn(m,1);

% Low-Noise NLS Solution
w0_c=0.4*pi*1.1; % Try 0 (it diverges; very sensitive)
dx_low=1000;eps=1e-8;i_count=1;max_iter=1000;
while abs(dx_low) > eps
 h=dt*k.*(cos(w0_c*dt*k)-sin(w0_c*dt*k));
 ye_low=sin(w0_c*dt*k)+cos(w0_c*dt*k);
 dy_low=ym_low-ye_low;
 dx_low=inv(h'*h)*h'*dy_low;
 w0_c=w0_c+dx_low;
 i_count=i_count+1;
 if i_count==max_iter
  disp(' Max Iterations Reached')
  break
 end
end
w0_est_low(j)=w0_c;
```

```
% High-Noise NLS Solution
w0_c=0.4*pi*1.1; % Try 0 (it diverges; very sensitive)
dx_high=1000;eps=1e-8;i_count=1;max_iter=1000;
while abs(dx_high) > eps
 h=dt*k.*(cos(w0_c*dt*k)-sin(w0_c*dt*k));
 ye_high=sin(w0_c*dt*k)+cos(w0_c*dt*k);
 dy_high=ym_high-ye_high;
 dx_high=inv(h'*h)*h'*dy_high;
 w0_c=w0_c+dx_high;
 i_count=i_count+1;
 if i_count==max_iter
  disp(' Max Iterations Reached')
  break
 end
end
w0_est_high(j)=w0_c;

end

% Final Estimates
ye_low=sin(w0_est_low(j)*dt*k)+cos(w0_est_low(j)*dt*k);
ye_high=sin(w0_est_high(j)*dt*k)+cos(w0_est_high(j)*dt*k);
```

```
% Plot Error from Monte Carlo Runs
subplot(211)
plot([1:m_monte]',w0_est_low-w0)
set(gca,'fontsize',12)
ylabel('Low-Noise Errors')
xlabel('Run Number')
subplot(212)
plot([1:m_monte]',w0_est_high-w0)
set(gca,'fontsize',12)
ylabel('High-Noise Errors')
xlabel('Run Number')
```

# Exercise 1.16



A measurement process used in three-axis magnetometers for low-Earth attitude determination involves the following measurement model:

$$\mathbf{b}_j = A_j \mathbf{r}_j + \mathbf{c} + \boldsymbol{\epsilon}_j$$

where $\mathbf{b}_j$ is the measurement of the magnetic field (more exactly, magnetic induction) by the magnetometer at time $t_j$, $\mathbf{r}_j$ is the corresponding value of the geomagnetic field with respect to some reference coordinate system, $A_j$ is the orthogonal attitude matrix (see §A.7.1), $\mathbf{c}$ is the magnetometer bias, and $\boldsymbol{\epsilon}_j$ is the measurement error. We can eliminate the dependence on the attitude by transposing terms and computing the square, and can define an effective measurement by

$$\tilde{y}_j = \mathbf{b}_j^T \mathbf{b}_j - \mathbf{r}_j^T \mathbf{r}_j$$

which can be rewritten to form the following measurement model:

$$\tilde{y}_j = 2\mathbf{b}_j^T\mathbf{c} - \mathbf{c}^T\mathbf{c} + v_j$$

where $v_j$ is the effective measurement error, whose closed-form expression is not required for this problem. For this exercise assume that

$$A\mathbf{r} = \begin{bmatrix} 10\sin(0.001t) \\ 5\sin(0.002t) \\ 10\cos(0.001t) \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 0.5 \\ 0.3 \\ 0.6 \end{bmatrix}$$

Also, assume that $\epsilon$ is given by a zero-mean Gaussian noise process with standard deviation given by 0.05 in each component. Using the above values create 1001 synthetic measurements of $\mathbf{b}$ and $\tilde{y}$ at 5-second intervals. The estimated output is computed from

$$\hat{y}_j = 2\mathbf{b}_j^T \hat{\mathbf{c}} - \hat{\mathbf{c}}^T \hat{\mathbf{c}}$$

where $\hat{\mathbf{c}}$ is the estimated solution from the nonlinear least square iterations. Use nonlinear least squares to determine $\hat{\mathbf{c}}$ for a starting value of $\mathbf{x}_c = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$. Also, try various starting values to check convergence. Note: $\mathbf{r}^T \mathbf{r} = \mathbf{r}^T A^T A \mathbf{r}$, since $A^T A = I$.

```
% Exercise 1.17

% Time
dt=5;tf=5000;
t=[0:dt:tf]';m=length(t);

% Measurements
x=[0.5;0.3;0.6];
ar=[10*sin(0.001*t) 5*sin(0.002*t) 10*cos(0.001*t)];
bm=ar+kron(ones(m,1),x')+0.05*randn(m,3);
ym=bm(:,1).*bm(:,1)+bm(:,2).*bm(:,2)+bm(:,3).*bm(:,3)...
  -ar(:,1).*ar(:,1)-ar(:,2).*ar(:,2)-ar(:,3).*ar(:,3);
```

```
% NLS Solution
x_c=[0;0;0];
dx=1000;eps=1e-8;i_count=1;max_iter=1000;
while norm(dx) > eps
 h=2*bm-2*kron(ones(m,1),x_c');
 ye=2*(x_c(1)*bm(:,1)+x_c(2)*bm(:,2)+x_c(3)*bm(:,3))...
   -x_c'*x_c;
 dy=ym-ye;
 dx=inv(h'*h)*h'*dy;
 x_c=x_c+dx;
 i_count=i_count+1;
 if i_count==max_iter
  disp(' Max Iterations Reached')
  break
 end
end
x,x_c
```

```
>> problem1_17

x =

    0.5000
    0.3000
    0.6000


x_c =

    0.5003
    0.2906
    0.5986
```

# Exercise 1.18

An approximate linear solution to exercise 1.17 is possible. The original loss function is quartic in $\hat{\mathbf{c}}$. But this can be approximated by a quadratic loss function using a process known as *centering*. The linearized solution proceeds as follows. First, compute the following averaged values:

$$\bar{y} = \frac{1}{m}\sum_{j=1}^{m} \tilde{y}_j$$

$$\bar{\mathbf{b}} = \frac{1}{m}\sum_{j=1}^{m} \mathbf{b}_j$$

where $m$ is the total number of measurements, which is equal to 1001 from exercise 1.17. Next, define the following variables:

$$\breve{y}_j = \tilde{y}_j - \bar{y}$$

$$\breve{\mathbf{b}}_j = \mathbf{b}_j - \bar{\mathbf{b}}$$

# Exercise 1.18

The centered estimate now minimizes the following loss function:

$$\bar{J}(\hat{\mathbf{c}}) = \frac{1}{2}\sum_{j=1}^{m}\left(\breve{y}_j - 2\breve{\mathbf{b}}_j^T\,\hat{\mathbf{c}}\right)^2$$

Minimizing this function yields

$$\hat{\mathbf{c}} = P\sum_{j=1}^{m} 2\breve{y}_j\breve{\mathbf{b}}_j$$

where

$$P \equiv \left[\sum_{i=1}^{m} 4\breve{\mathbf{b}}_j\breve{\mathbf{b}}_j^T\right]^{-1}$$

# Exercise 1.18

Using the parameters described in exercise 1.17, compare the linear solution described here to the solution obtained by nonlinear least squares. Furthermore, find solutions for $\hat{\mathbf{c}}$ using both approaches with the following trajectory for $A\mathbf{r}$:

$$A\mathbf{r} = \begin{bmatrix} 10\sin(0.001t) \\ 5 \\ 10\cos(0.001t) \end{bmatrix}$$

Discuss the performance of the linear solution using this assumed trajectory for $A\mathbf{r}$.

```
% Exercise 1.18

% Time
dt=5;tf=5000;
t=[0:dt:tf]';m=length(t);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Monte Carlo Runs
m_monte=1000;
x_nls=zeros(m_monte,3);x_cen=zeros(m_monte,3);

% First Set of Basis Functions
for j = 1:m_monte

% Measurements
x=[0.5;0.3;0.6];
ar=[10*sin(0.001*t) 5*sin(0.002*t) 10*cos(0.001*t)];
bm=ar+kron(ones(m,1),x')+0.05*randn(m,3);
ym=bm(:,1).*bm(:,1)+bm(:,2).*bm(:,2)+bm(:,3).*bm(:,3)...
  -ar(:,1).*ar(:,1)-ar(:,2).*ar(:,2)-ar(:,3).*ar(:,3);

% NLS Solution
x_c=[0;0;0];
```

```
dx=1000;eps=1e-8;i_count=1;max_iter=1000;
while norm(dx) > eps
 h=2*bm-2*kron(ones(m,1),x_c');
 ye=2*(x_c(1)*bm(:,1)+x_c(2)*bm(:,2)+x_c(3)*bm(:,3))-x_c'*x_c
 dy=ym-ye;
 dx=inv(h'*h)*h'*dy;
 x_c=x_c+dx;
 i_count=i_count+1;
 if i_count==max_iter, disp(' Max Iterations Reached'), brea
end
x_nls(j,:)=x_c';

% Get Average Values
ym_bar=mean(ym);
bm_bar=mean(bm);

% Get Centered Measurements
y_cen=ym-ym_bar;
bm_cen=bm-kron(ones(m,1),bm_bar);

% Centered Estimate
p=1/4*inv(bm_cen'*bm_cen);
xe_cen=2*p*sum(kron(y_cen,[1 1 1]).*bm_cen)';
```

```
x_cen(j,:)=xe_cen';

end

disp('First Set of Basis Functions')
mean_nls=mean(x_nls),mean_cen=mean(x_cen)
std_nls=std(x_nls),std_cen=std(x_cen)

% Plot Estimates
subplot(211)
plot([1:m_monte]',x_nls)
axis([0 m_monte 0.2 0.7])
set(gca,'fontsize',12)
set(gca,'ytick',[0.2 0.3 0.4 0.5 0.6 0.7])
ylabel('NLS Solution')
xlabel('Run Number')
subplot(212)
plot([1:m_monte]',x_cen)
axis([0 m_monte 0.2 0.7])
set(gca,'fontsize',12)
set(gca,'ytick',[0.2 0.3 0.4 0.5 0.6 0.7])
ylabel('Centered Solution')
xlabel('Run Number')
```

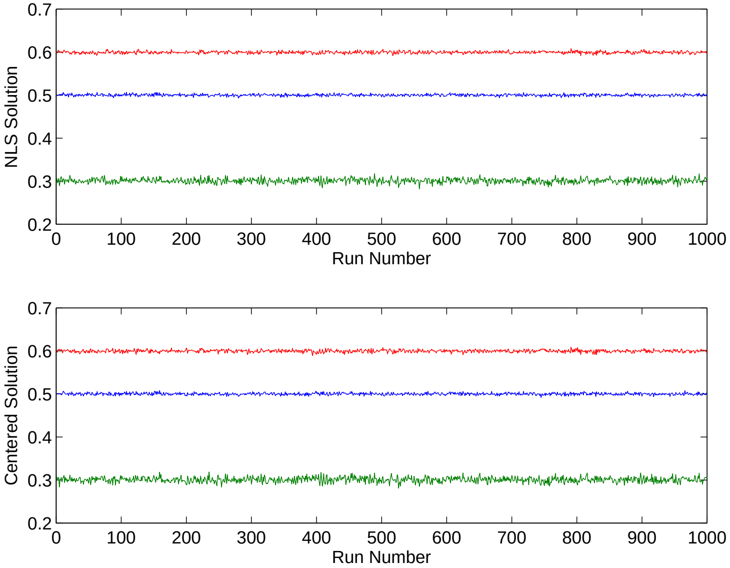
```
pause

% Second Set of Basis Functions
for j = 1:m_monte

% Measurements
x=[0.5;0.3;0.6];
ar=[10*sin(0.001*t) 5*ones(m,1) 10*cos(0.001*t)];
bm=ar+kron(ones(m,1),x')+0.05*randn(m,3);
ym=bm(:,1).*bm(:,1)+bm(:,2).*bm(:,2)+bm(:,3).*bm(:,3)...
  -ar(:,1).*ar(:,1)-ar(:,2).*ar(:,2)-ar(:,3).*ar(:,3);

% NLS Solution
x_c=[0;0;0];
dx=1000;eps=1e-8;i_count=1;max_iter=1000;
while norm(dx) > eps
 h=2*bm-2*kron(ones(m,1),x_c');
 ye=2*(x_c(1)*bm(:,1)+x_c(2)*bm(:,2)+x_c(3)*bm(:,3))-x_c'*x_c
 dy=ym-ye;
 dx=inv(h'*h)*h'*dy;
 x_c=x_c+dx;
 i_count=i_count+1;
```

```
 if i_count==max_iter, disp(' Max Iterations Reached'), brea
end
x_nls(j,:)=x_c';

% Get Average Values
ym_bar=mean(ym);
bm_bar=mean(bm);

% Get Centered Measurements
y_cen=ym-ym_bar;
bm_cen=bm-kron(ones(m,1),bm_bar);

% Centered Estimate
p=1/4*inv(bm_cen'*bm_cen);
xe_cen=2*p*sum(kron(y_cen,[1 1 1]).*bm_cen)';
x_cen(j,:)=xe_cen';

end

disp(' ')
disp('Second Set of Basis Functions')
mean_nls=mean(x_nls),mean_cen=mean(x_cen)
std_nls=std(x_nls),std_cen=std(x_cen)
```

```
% Plot Estimates
subplot(211)
plot([1:m_monte]',x_nls)
axis([0 m_monte 0.2 0.7])
set(gca,'fontsize',12)
set(gca,'ytick',[0.2 0.3 0.4 0.5 0.6 0.7])
ylabel('NLS Solution')
xlabel('Run Number')
subplot(212)
plot([1:m_monte]',x_cen)
set(gca,'fontsize',12)
ylabel('Centered Solution')
xlabel('Run Number')
```

```
>> problem1_18

 First Set of Basis Functions

mean_nls =

    0.5003    0.3011    0.5996


mean_cen =

    0.5001    0.3006    0.5998


std_nls =

    0.0024    0.0053    0.0026


std_cen =

    0.0026    0.0059    0.0030
```

# Exercise 1.18



Second Set of Basis Functions

```
mean_nls =

    0.5001    0.3013    0.5999


mean_cen =

    0.5001    5.3004    0.5998


std_nls =

    0.0026    0.0040    0.0028


std_cen =

    0.0023    0.3141    0.0024
```

# Exercise 1.18