ECE 68000: MODERN AUTOMATIC CONTROL

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Discrete algebraic Riccati equation (DARE)

Linear Quadratic Regulator Problem for Discrete-Time Linear Systems

- The plant

$$\boldsymbol{x}[k+1] = \boldsymbol{A}\boldsymbol{x}[k] + \boldsymbol{B}\boldsymbol{u}[k], \quad k = 0, 1, 2, \ldots,$$

  with a specified initial condition $\boldsymbol{x}(0) = \boldsymbol{x}_0$, where $\boldsymbol{x}[k] \in \mathbb{R}^n$ and $\boldsymbol{u}[k] \in \mathbb{R}^m$
- Assumption: the pair $(\boldsymbol{A}, \boldsymbol{B})$ is reachable
- Objective: construct a stabilizing linear state-feedback

$$\boldsymbol{u}[k] = -\boldsymbol{K}\boldsymbol{x}[k]$$

  that minimizes the quadratic performance index

$$J(\boldsymbol{u}) = \sum_{k=0}^{\infty} \left\{ \boldsymbol{x}[k]^\top \boldsymbol{Q}\boldsymbol{x}[k] + \boldsymbol{u}[k]^\top \boldsymbol{R}\boldsymbol{u}[k] \right\},$$

  where $\boldsymbol{Q} = \boldsymbol{Q}^\top \succeq 0$, and $\boldsymbol{R} = \boldsymbol{R}^\top \succ 0$

Assumptions and objectives

- Assumption: the components of the control vector are unconstrained
- An optimal control law denoted $\boldsymbol{u}^*$
- The controller must stabilize the plant
- The closed-loop system,

$$\boldsymbol{x}[k+1] = (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K})\,\boldsymbol{x}[k],$$

  asymptotically stable, that is, the eigenvalues of the matrix $\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}$ are in the open unit disk
- Thus there exists a Lyapunov function, $V(\boldsymbol{x}[k]) = \boldsymbol{x}[k]^\top \boldsymbol{P}\boldsymbol{x}[k]$
- Therefore the first forward difference, $\Delta V(\boldsymbol{x}[k]) = V(\boldsymbol{x}[k+1]) - V(\boldsymbol{x}[k])$ evaluated on the trajectories of the closed-loop system negative definite
- Goal: sufficient condition for $\boldsymbol{u}[k] = -\boldsymbol{K}\boldsymbol{x}[k]$ to be optimal

Sufficient condition for $\boldsymbol{u}[k] = -\boldsymbol{K}\boldsymbol{x}[k]$ to be optimal

Theorem

If the state feedback controller, $\boldsymbol{u}[k] = -\boldsymbol{K}\boldsymbol{x}[k]$, is such that

$$\min_{\boldsymbol{u}} \left(\Delta V(\boldsymbol{x}[k]) + \boldsymbol{x}[k]^\top \boldsymbol{Q}\boldsymbol{x}[k] + \boldsymbol{u}[k]^\top \boldsymbol{R}\boldsymbol{u}[k]\right) = 0,$$

then it is optimal

Theorem's proof

- Represent the first difference as

$$\Delta V(\boldsymbol{x}[k])|_{\boldsymbol{u}=\boldsymbol{u}^*} + \boldsymbol{x}[k]^\top \boldsymbol{Q}\boldsymbol{x}[k] + \boldsymbol{u}^*[k]^\top \boldsymbol{R}\boldsymbol{u}^*[k] = 0$$

- Hence,

$$\begin{aligned} \Delta V(\boldsymbol{x}[k])|_{\boldsymbol{u}=\boldsymbol{u}^*} &= (V(\boldsymbol{x}[k+1]) - V(\boldsymbol{x}[k])|_{\boldsymbol{u}=\boldsymbol{u}^*} \\ &= -\boldsymbol{x}[k]^\top \boldsymbol{Q}\boldsymbol{x}[k] - \boldsymbol{u}^*[k]^\top \boldsymbol{R}\boldsymbol{u}^*[k] \end{aligned}$$

- Sum both sides from $k = 0$ to $k = \infty$

$$V(\boldsymbol{x}[\infty]) - V(\boldsymbol{x}[0]) = -\sum_{k=0}^{\infty} \left(\boldsymbol{x}[k]^\top \boldsymbol{Q}\boldsymbol{x}[k] + \boldsymbol{u}^*[k]^\top \boldsymbol{R}\boldsymbol{u}^*[k]\right)$$

- By assumption the closed-loop system is asymptotically stable, we have $\boldsymbol{x}[\infty] = \boldsymbol{0}$

The controller is optimal

- Since the closed-loop system asymptotically stable,

$$V(\boldsymbol{x}[0]) = \boldsymbol{x}[0]^\top \boldsymbol{P}\boldsymbol{x}[0] = \sum_{k=0}^{\infty} \left(\boldsymbol{x}[k]^\top \boldsymbol{Q}\boldsymbol{x}[k] + \boldsymbol{u}^*[k]^\top \boldsymbol{R}\boldsymbol{u}^*[k]\right)$$

- Thus, if a linear stabilizing controller satisfies the hypothesis of the theorem, the value of the performance index resulting from applying this controller is

$$J = \boldsymbol{x}[0]^\top \boldsymbol{P}\boldsymbol{x}[0]$$

- Proof by contradiction that the controller is optimal
- Assume that the hypothesis of the theorem holds but the controller $\boldsymbol{u}^*$ that satisfies this hypothesis is not optimal, that is, there is a controller $\tilde{\boldsymbol{u}}$ such that

$$J\left(\tilde{\boldsymbol{u}}\right) < J\left(\boldsymbol{u}^*\right)$$

The controller is optimal—proof by contradiction

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- The hypothesis of the theorem implies that

$$\Delta V(\boldsymbol{x}[k])|_{\boldsymbol{u}=\tilde{\boldsymbol{u}}} + \boldsymbol{x}[k]^\top \boldsymbol{Q}\boldsymbol{x}[k] + \tilde{\boldsymbol{u}}[k]^\top \boldsymbol{R}\tilde{\boldsymbol{u}}[k] \geq 0$$

- Represent the above as

$$\Delta V(\boldsymbol{x}[k])|_{\boldsymbol{u}=\tilde{\boldsymbol{u}}} \geq -\boldsymbol{x}[k]^\top \boldsymbol{Q}\boldsymbol{x}[k] - \tilde{\boldsymbol{u}}[k]^\top \boldsymbol{R}\tilde{\boldsymbol{u}}[k]$$

- Sum from $k = 0$ to $k = \infty$ yields

$$V(\boldsymbol{x}[0]) = J(\boldsymbol{u}^*) \leq \sum_{k=0}^{\infty} \left(\boldsymbol{x}[k]^\top \boldsymbol{Q}\boldsymbol{x}[k] + \tilde{\boldsymbol{u}}[k]^\top \boldsymbol{R}\tilde{\boldsymbol{u}}[k]\right) = J(\tilde{\boldsymbol{u}}),$$

  that is,

$$J(\tilde{\boldsymbol{u}}) \geq J(\boldsymbol{u}^*),$$

  a contradiction $\square$

Finding an optimal controller

- Find an appropriate quadratic Lyapunov function $V(\boldsymbol{x}) = \boldsymbol{x}^\top \boldsymbol{P}\boldsymbol{x}$, to be used to construct the optimal controller
- First find $\boldsymbol{u}^*$ that minimizes the function

$$f = f(\boldsymbol{u}[k]) = \Delta V(\boldsymbol{x}[k]) + \boldsymbol{x}[k]^\top \boldsymbol{Q}\boldsymbol{x}[k] + \boldsymbol{u}[k]^\top \boldsymbol{R}\boldsymbol{u}[k].$$

- Perform preliminary manipulations on f

$$\begin{aligned}
f(\boldsymbol{u}[k]) &= \boldsymbol{x}[k+1]^\top \boldsymbol{P}\boldsymbol{x}[k+1] - \boldsymbol{x}[k]^\top \boldsymbol{P}\boldsymbol{x}[k] + \boldsymbol{x}[k]^\top \boldsymbol{Q}\boldsymbol{x}[k] \\
&\quad + \boldsymbol{u}[k]^\top \boldsymbol{R}\boldsymbol{u}[k] \\
&= \left(\boldsymbol{A}\boldsymbol{x}[k] + \boldsymbol{B}\boldsymbol{u}[k]\right)^\top \boldsymbol{P}\left(\boldsymbol{A}\boldsymbol{x}[k] + \boldsymbol{B}\boldsymbol{u}[k]\right) \\
&\quad - \boldsymbol{x}[k]^\top \boldsymbol{P}\boldsymbol{x}[k] + \boldsymbol{x}[k]^\top \boldsymbol{Q}\boldsymbol{x}[k] + \boldsymbol{u}[k]^\top \boldsymbol{R}\boldsymbol{u}[k]
\end{aligned}$$

First-order necessary condition for a relative minimizer

- Apply the first-order necessary condition for a relative minimizer

$$
\begin{aligned}
\frac{\partial f(\boldsymbol{u}[k])}{\partial \boldsymbol{u}[k]} &= 2\left(\boldsymbol{A}\boldsymbol{x}[k] + \boldsymbol{B}\boldsymbol{u}[k]\right)^\top \boldsymbol{P}\boldsymbol{B} + 2\boldsymbol{u}[k]^\top \boldsymbol{R} \\
&= 2\boldsymbol{x}[k]^\top \boldsymbol{A}^\top \boldsymbol{P}\boldsymbol{B} + 2\boldsymbol{u}[k]^\top \left(\boldsymbol{R} + \boldsymbol{B}^\top \boldsymbol{P}\boldsymbol{B}\right) \\
&= \boldsymbol{0}
\end{aligned}
$$

- The matrix $\boldsymbol{R} + \boldsymbol{B}^\top \boldsymbol{P}\boldsymbol{B}$ is positive definite because $\boldsymbol{R}$ is, and therefore the matrix $\boldsymbol{R} + \boldsymbol{B}^\top \boldsymbol{P}\boldsymbol{B}$ is invertible
- Hence,

$$
\boldsymbol{u}^*[k] = -\left(\boldsymbol{R} + \boldsymbol{B}^\top \boldsymbol{P}\boldsymbol{B}\right)^{-1} \boldsymbol{B}^\top \boldsymbol{P}\boldsymbol{A}\boldsymbol{x}[k] = -\boldsymbol{K}\boldsymbol{x}[k],
$$

where

$$
\boldsymbol{K} = \left(\boldsymbol{R} + \boldsymbol{B}^\top \boldsymbol{P}\boldsymbol{B}\right)^{-1} \boldsymbol{B}^\top \boldsymbol{P}\boldsymbol{A}
$$

Second-order necessary condition

- Let $\boldsymbol{S} = \boldsymbol{R} + \boldsymbol{B}^\top \boldsymbol{P} \boldsymbol{B}$
- Hence
$$\boldsymbol{u}^*[k] = -\boldsymbol{S}^{-1} \boldsymbol{B}^\top \boldsymbol{P} \boldsymbol{A} \boldsymbol{x}[k].$$
- Check if $\boldsymbol{u}^*$ satisfies the second-order sufficient condition
- We have
$$\begin{aligned}
&\frac{\partial^2}{\partial \boldsymbol{u}^2(k)} \left(\boldsymbol{x}[k+1]^\top \boldsymbol{P} \boldsymbol{x}[k+1] - \boldsymbol{x}[k]^\top \boldsymbol{P} \boldsymbol{x}[k] + \boldsymbol{x}[k]^\top \boldsymbol{Q} \boldsymbol{x}[k] \right.\\
&\quad \left. + \boldsymbol{u}[k]^\top \boldsymbol{R} \boldsymbol{u}[k] \right)\\
&\quad = \frac{\partial}{\partial \boldsymbol{u}[k]} \left(2\boldsymbol{x}[k]^\top \boldsymbol{A}^\top \boldsymbol{P} \boldsymbol{B} + 2\boldsymbol{u}[k]^\top \left(\boldsymbol{R} + \boldsymbol{B}^\top \boldsymbol{P} \boldsymbol{B} \right) \right)\\
&\quad = 2 \left(\boldsymbol{R} + \boldsymbol{B}^\top \boldsymbol{P} \boldsymbol{B} \right)\\
&\quad \succ 0
\end{aligned}$$
- $\boldsymbol{u}^*$ satisfies the second-order sufficient condition for a relative minimizer

The optimal controller architecture

- The optimal controller can be constructed if we have found an appropriate positive definite matrix $\boldsymbol{P}$
- Next task: devise a method that would allow us to compute the desired matrix $\boldsymbol{P}$
- The equation describing the closed-loop system driven by the optimal controller

$$\boldsymbol{x}[k+1] = \left(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{S}^{-1}\boldsymbol{B}^\top \boldsymbol{P}\boldsymbol{A}\right)\boldsymbol{x}[k]$$

- The controller satisfies the hypothesis of the theorem

$$\begin{aligned}
&\boldsymbol{x}[k+1]^\top \boldsymbol{P}\boldsymbol{x}[k+1] - \boldsymbol{x}[k]^\top \boldsymbol{P}\boldsymbol{x}[k] \\
&+\boldsymbol{x}[k]^\top \boldsymbol{Q}\boldsymbol{x}[k] + \boldsymbol{u}^*[k]^\top \boldsymbol{R}\boldsymbol{u}^*[k] \\
&= 0
\end{aligned}$$

Optimal controller—manipulations

- Substituting and performing manipulations

$$
\begin{aligned}
&\boldsymbol{x}[k]^\top \left(\boldsymbol{A}-\boldsymbol{B}\boldsymbol{S}^{-1}\boldsymbol{B}^\top\boldsymbol{P}\boldsymbol{A}\right)^\top \boldsymbol{P}\left(\boldsymbol{A}-\boldsymbol{B}\boldsymbol{S}^{-1}\boldsymbol{B}^\top\boldsymbol{P}\boldsymbol{A}\right)\boldsymbol{x}[k]\\
&\quad -\boldsymbol{x}[k]^\top\boldsymbol{P}\boldsymbol{x}[k]\\
&\quad +\boldsymbol{x}[k]^\top\boldsymbol{Q}\boldsymbol{x}[k]+\boldsymbol{x}[k]^\top\boldsymbol{A}^\top\boldsymbol{P}\boldsymbol{B}\boldsymbol{S}^{-1}\boldsymbol{R}\boldsymbol{S}^{-1}\boldsymbol{B}^\top\boldsymbol{P}\boldsymbol{A}\boldsymbol{x}[k]\\
&\quad =\boldsymbol{x}[k]^\top\boldsymbol{A}^\top\boldsymbol{P}\boldsymbol{A}\boldsymbol{x}[k]-\boldsymbol{x}[k]^\top\boldsymbol{A}^\top\boldsymbol{P}\boldsymbol{B}\boldsymbol{S}^{-1}\boldsymbol{B}^\top\boldsymbol{P}\boldsymbol{A}\boldsymbol{x}[k]\\
&\qquad -\boldsymbol{x}[k]^\top\boldsymbol{A}^\top\boldsymbol{P}\boldsymbol{B}\boldsymbol{S}^{-1}\boldsymbol{B}^\top\boldsymbol{P}\boldsymbol{A}\boldsymbol{x}[k]\\
&\quad +\boldsymbol{x}[k]^\top\boldsymbol{A}^\top\boldsymbol{P}\boldsymbol{B}\boldsymbol{S}^{-1}\boldsymbol{B}^\top\boldsymbol{P}\boldsymbol{B}\boldsymbol{S}^{-1}\boldsymbol{B}^\top\boldsymbol{P}\boldsymbol{A}\boldsymbol{x}[k]\\
&\qquad -\boldsymbol{x}[k]^\top\boldsymbol{P}\boldsymbol{x}[k]+\boldsymbol{x}[k]^\top\boldsymbol{Q}\boldsymbol{x}[k]\\
&\quad +\boldsymbol{x}[k]^\top\boldsymbol{A}^\top\boldsymbol{P}\boldsymbol{B}\boldsymbol{S}^{-1}\boldsymbol{R}\boldsymbol{S}^{-1}\boldsymbol{B}^\top\boldsymbol{P}\boldsymbol{A}\boldsymbol{x}[k]\\
&\quad =\boldsymbol{x}[k]^\top\boldsymbol{A}^\top\boldsymbol{P}\boldsymbol{A}\boldsymbol{x}[k]-\boldsymbol{x}[k]^\top\boldsymbol{P}\boldsymbol{x}[k]+\boldsymbol{x}[k]^\top\boldsymbol{Q}\boldsymbol{x}[k]\\
&\qquad -2\boldsymbol{x}[k]^\top\boldsymbol{A}^\top\boldsymbol{P}\boldsymbol{B}\boldsymbol{S}^{-1}\boldsymbol{B}^\top\boldsymbol{P}\boldsymbol{A}\boldsymbol{x}[k]\\
&\qquad +\boldsymbol{x}[k]^\top\boldsymbol{A}^\top\boldsymbol{P}\boldsymbol{B}\boldsymbol{S}^{-1}\left(\boldsymbol{R}+\boldsymbol{B}^\top\boldsymbol{P}\boldsymbol{B}\right)\boldsymbol{S}^{-1}\boldsymbol{B}^\top\boldsymbol{P}\boldsymbol{A}\boldsymbol{x}[k]
\end{aligned}
$$

Discrete-time algebraic Riccati equation (DARE)

- Continuing

$$
\begin{aligned}
&\boldsymbol{x}[k]^\top \boldsymbol{A}^\top \boldsymbol{P}\boldsymbol{A}\boldsymbol{x}[k] - \boldsymbol{x}[k]^\top \boldsymbol{P}\boldsymbol{x}[k] + \boldsymbol{x}[k]^\top \boldsymbol{Q}\boldsymbol{x}[k] \\
&\quad -2\boldsymbol{x}[k]^\top \boldsymbol{A}^\top \boldsymbol{P}\boldsymbol{B}\boldsymbol{S}^{-1}\boldsymbol{B}^\top \boldsymbol{P}\boldsymbol{A}\boldsymbol{x}[k] \\
&\quad +\boldsymbol{x}[k]^\top \boldsymbol{A}^\top \boldsymbol{P}\boldsymbol{B}\boldsymbol{S}^{-1}\boldsymbol{B}^\top \boldsymbol{P}\boldsymbol{A}\boldsymbol{x}[k] \\
&= \boldsymbol{x}[k]^\top \left(\boldsymbol{A}^\top \boldsymbol{P}\boldsymbol{A} - \boldsymbol{P} + \boldsymbol{Q} - \boldsymbol{A}^\top \boldsymbol{P}\boldsymbol{B}\boldsymbol{S}^{-1}\boldsymbol{B}^\top \boldsymbol{P}\boldsymbol{A}\right) \boldsymbol{x}[k] \\
&= 0
\end{aligned}
$$

- The above is to hold for any $\boldsymbol{x}$
- Hence

$$
\boxed{\boldsymbol{A}^\top \boldsymbol{P}\boldsymbol{A} - \boldsymbol{P} + \boldsymbol{Q} - \boldsymbol{A}^\top \boldsymbol{P}\boldsymbol{B}\boldsymbol{S}^{-1}\boldsymbol{B}^\top \boldsymbol{P}\boldsymbol{A} = \boldsymbol{O}}
$$

- The *discrete-time algebraic Riccati equation* (DARE)