ECE 680 Purdue University

Test #1
Solutions

1. **(10 pts)** Find the equilibrium pair $(\boldsymbol{x}_e, u_e)$ corresponding to $u_e = 3$ for the following nonlinear model,

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 3 + x_1x_2 \\ -6 + 5x_1x_2 \end{bmatrix} + \begin{bmatrix} -1 \\ x_2 \end{bmatrix} u \\ y &= x_1^2 + x_2 u. \end{aligned}$$

◇

Setting $\dot{x}_1 = \dot{x}_2 = 0$ and substituting $u = 3$, we obtain the following algebraic equations,

$$\begin{aligned} x_1x_2 &= 0 \\ -6 + 5x_1x_2 + 3x_2 &= 0. \end{aligned}$$

Taking into account that $x_1x_2 = 0$ in the second of the above equation gives

$$-6 + 3x_2 = 0.$$

We have the following equilibrium pair,

$$(\boldsymbol{x}_e, u_e) = \left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}, 3 \right).$$

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2. **(10 pts)** Linearize the nonlinear model,

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 3 + x_1x_2 \\ -6 + 5x_1x_2 \end{bmatrix} + \begin{bmatrix} -1 \\ x_2 \end{bmatrix} u \\ y &= x_1^2 + x_2 u, \end{aligned}$$

about the equilibrium found in the previous problem. ◇

Let $\dot{x}_1 = f_1$ and $\dot{x}_2 = f_2$. Then the linearized model has the form,

$$\frac{d}{dt}\delta\boldsymbol{x} = \boldsymbol{A}\,\delta\boldsymbol{x} + \boldsymbol{b}\,\delta u,$$

where

$$\boldsymbol{A} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} x_2 & x_1 \\ 5x_2 & 5x_1 + u \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{bmatrix} = \begin{bmatrix} -1 \\ x_2 \end{bmatrix}$$

evaluated at the equilibrium pair about which we linearize the nonlinear system. We have,

$$\frac{d}{dt}\delta \boldsymbol{x} = \begin{bmatrix} 2 & 0 \\ 10 & 3 \end{bmatrix} \delta \boldsymbol{x} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} \delta u.$$

The linearized output map has the form,

$$\delta y = \begin{bmatrix} 0 & 3 \end{bmatrix} \delta \boldsymbol{x} + \begin{bmatrix} 2 \end{bmatrix} \delta u,$$

where $\delta y = y - y_e = y - 6$.

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3. (**10 pts**) For the system modeled by

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}u \\ &= \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u, \end{aligned}$$

construct a state-feedback control law, $u = -\boldsymbol{k}\boldsymbol{x} + r$, such that the closed-loop system poles are located at -1 and -2.

$\diamond$

We can use Ackermann's formula to the pair $(\boldsymbol{A}, \boldsymbol{b})$ to obtain the feedback gain $\boldsymbol{k}$. We form the controllability matrix of the pair $(\boldsymbol{A}, \boldsymbol{b})$, then find the last row of its inverse and call it $\boldsymbol{q}_1$. We have

$$\begin{bmatrix} \boldsymbol{b} & \boldsymbol{A}\boldsymbol{b} \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

Hence, $\boldsymbol{q}_1 = \begin{bmatrix} 0 & 1 \end{bmatrix}$. The desired closed-loop characteristic polynomial (CLCP) is

$$\text{CLCP} = (s+1)(s+2) = s^2 + 3s + 2.$$

The feedback gain $\boldsymbol{k}$ then is

$$\boldsymbol{k} = \boldsymbol{q}_1 \left(\boldsymbol{A}^2 + 3\boldsymbol{A} + 2\boldsymbol{I}_2 \right)$$

$$
\begin{aligned}
&= \begin{bmatrix} 0 & 1 \end{bmatrix} \left\{ \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}^2 + 3 \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} + 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\} \\
&= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 6 & 0 \\ 6 & 12 \end{bmatrix} \\
&= \begin{bmatrix} 6 & 12 \end{bmatrix}.
\end{aligned}
$$

Hence,

$$u = -\begin{bmatrix} 6 & 12 \end{bmatrix} \boldsymbol{x} + r,$$

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4. **(15 pts)** Design an asymptotic observer for the plant,

$$
\begin{aligned}
\dot{\boldsymbol{x}} &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}u = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u, \\
y &= \boldsymbol{c}\boldsymbol{x} + du = \begin{bmatrix} 0 & 1 \end{bmatrix} \boldsymbol{x} + 3u.
\end{aligned}
$$

The observer poles are to be located at -3 and -4. Write down the equations of your observer, both symbolic and numeric.

◇

We can use Ackermann's formula to to the pair $(\boldsymbol{A}^\top, \boldsymbol{c}^\top)$ to obtain the estimator gain vector $\boldsymbol{l}$. We form the controllability matrix of the dual pair $(\boldsymbol{A}^\top, \boldsymbol{c}^\top)$, then find the last row of its inverse and call it $\boldsymbol{q}_1$. We have

$$\begin{bmatrix} \boldsymbol{c}^\top & \boldsymbol{A}^\top \boldsymbol{c}^\top \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}.$$

Hence, $\boldsymbol{q}_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}$. The desired characteristic polynomial of $\boldsymbol{A} - \boldsymbol{l}\boldsymbol{c}$ is

$$\det(s\boldsymbol{I}_2 - \boldsymbol{A} + \boldsymbol{l}\boldsymbol{c}) = (s+3)(s+4) = s^2 + 7s + 12.$$

Therefore, the estimator gain $\boldsymbol{l}$ is

$$
\begin{aligned}
\boldsymbol{l}^\top &= \boldsymbol{q}_1 \left(\left(\boldsymbol{A}^\top \right)^2 + 7\boldsymbol{A}^\top + 12\boldsymbol{I}_2 \right) \\
&= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 20 & 10 \\ 0 & 30 \end{bmatrix} \\
&= \begin{bmatrix} 20 & 10 \end{bmatrix}.
\end{aligned}
$$

Hence,

$$\boldsymbol{l} = \begin{bmatrix} 20 \\ 10 \end{bmatrix}.$$

The observer dynamics are:

$$\begin{aligned} \dot{\tilde{\boldsymbol{x}}} &= \boldsymbol{A}\tilde{\boldsymbol{x}} + \boldsymbol{b}u + \boldsymbol{l}\left(y - \tilde{y}\right) \\ &= \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \tilde{\boldsymbol{x}} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 20 \\ 10 \end{bmatrix} \left(y - \tilde{y}\right), \end{aligned}$$

where

$$\tilde{y} = c\tilde{\boldsymbol{x}} + du.$$

Equivalently,

$$\begin{aligned} \dot{\tilde{\boldsymbol{x}}} &= (\boldsymbol{A} - \boldsymbol{l}\boldsymbol{c})\tilde{\boldsymbol{x}} + \boldsymbol{b}u + \boldsymbol{l}(y - du) \\ &= \begin{bmatrix} 1 & -20 \\ 1 & -8 \end{bmatrix} \tilde{\boldsymbol{x}} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 20 \\ 10 \end{bmatrix} (y - 3u) \\ &= \begin{bmatrix} 1 & -20 \\ 1 & -8 \end{bmatrix} \tilde{\boldsymbol{x}} + \begin{bmatrix} -59 \\ -30 \end{bmatrix} u + \begin{bmatrix} 20 \\ 10 \end{bmatrix} y. \end{aligned}$$

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5. (**15 pts**) Is the following quadratic form,

$$f = \boldsymbol{x}^\top \boldsymbol{Q} \boldsymbol{x} = \boldsymbol{x}^\top \begin{bmatrix} 1 & 2 & 6 & 0 \\ 0 & 2 & 0 & 6 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \boldsymbol{x},$$

positive definite, positive semi-definite, negative definite, negative semi-definite, or indefinite? Carefully justify your answer.

◇

We first symmetrize the quadratic form to obtain

$$\begin{aligned} f &= \frac{1}{2}\boldsymbol{x}^\top \left(\boldsymbol{Q} + \boldsymbol{Q}^\top\right) \boldsymbol{x} \\ &= \boldsymbol{x}^\top \begin{bmatrix} 1 & 1 & 3 & 0 \\ 1 & 2 & 0 & 3 \\ 3 & 0 & 3 & 0 \\ 0 & 3 & 0 & 4 \end{bmatrix} \boldsymbol{x}. \end{aligned}$$

The first-order leading principal minor is $\Delta_1 = 1 > 0$, the second-order leading principal minor $\Delta_2 = 1 > 0$, while the third-order leading principal minor $\Delta_3 = -15$, which mean tha the quadratic form is indefinite.

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6. **(20 pts)** Evaluate

$$J_0 = \int_0^\infty y(t)^2 dt$$

subject to

$$\begin{aligned} \dot{\boldsymbol{x}} &= \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \boldsymbol{x}, \quad \boldsymbol{x}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ y &= \begin{bmatrix} \sqrt{2} & 0 \end{bmatrix} \boldsymbol{x}. \end{aligned}$$

$\diamond$

We first represent the performance index as

$$J_0 = \int_0^\infty \boldsymbol{x}^\top \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \boldsymbol{x} dt.$$

Note that pair $(\boldsymbol{A}, \begin{bmatrix} \sqrt{2} & 0 \end{bmatrix})$ is observable. The value of the performance index J_0 is $J_0 = \boldsymbol{x}(0)^\top \boldsymbol{P} \boldsymbol{x}(0)$, where $\boldsymbol{P} = \boldsymbol{P}^\top \succ 0$ is the solution to the Lyapunov equation,

$$\boldsymbol{A}^\top \boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} = -\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix},$$

that is,

$$\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}.$$

Performing simple manipulations gives

$$\begin{bmatrix} -p_2 & -p_3 \\ p_1 - p_2 & p_2 - p_3 \end{bmatrix} + \begin{bmatrix} -p_2 & p_1 - p_2 \\ -p_3 & p_2 - p_3 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix},$$

that is,

$$\begin{bmatrix} -2p_2 & p_1 - p_2 - p_3 \\ p_1 - p_2 - p_3 & 2(p_2 - p_3) \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}.$$

Solving the above, we obtain

$$\boldsymbol{P} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

Therefore,

$$J_0 = \boldsymbol{x}(0)^\top \boldsymbol{P} \boldsymbol{x}(0) = 1$$

7. (**10 pts**) Evaluate

$$J_1 = 4\int_0^\infty t\|\boldsymbol{x}(t)\|_2^2\,dt$$

subject to

$$\dot{\boldsymbol{x}} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}\boldsymbol{x}, \quad \boldsymbol{x}(0) = \begin{bmatrix} 1 \\ 8 \end{bmatrix}.$$

We first represent the performance index as

$$J_1 = \int_0^\infty \boldsymbol{x}^\top \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \boldsymbol{x} dt.$$

The value of the performance index J_1 is $J_1 = \boldsymbol{x}(0)^\top \boldsymbol{P}_1 \boldsymbol{x}(0)$, where $\boldsymbol{P}_1 = \boldsymbol{P}_1^\top > 0$ is obtained by solving two Lyapunov equations,

$$\boldsymbol{A}^\top \boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} = -4\boldsymbol{I}_2,$$

and

$$\boldsymbol{A}^\top \boldsymbol{P}_1 + \boldsymbol{P}_1\boldsymbol{A} = -\boldsymbol{P}.$$

Solving the first Lyapunov equation yields

$$\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix}.$$

Performing simple manipulations gives

$$\boldsymbol{P} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

We next solve the second Lyapunov equation,

$$\boldsymbol{A}^\top \boldsymbol{P}_1 + \boldsymbol{P}_1\boldsymbol{A} = -\boldsymbol{P}$$

to obtain

$$\boldsymbol{P}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1/4 \end{bmatrix}.$$

Therefore,

$$J_1 = \boldsymbol{x}(0)^\top \boldsymbol{P}_1 \boldsymbol{x}(0) = 17$$

$\diamond$

8. (**10 pts**) Determine the optimal state-feedback controller, $u = -kx$, that minimizes

$$J = \int_0^\infty u(t)^2 dt$$

subject to

$$\dot{x}(t) = x(t) + 2u(t), \quad x(0) = 1,$$

and determine the optimal value of J.

The optimal controller has the form

$$\boldsymbol{u}^* = -\boldsymbol{R}^{-1}\boldsymbol{B}^\top \boldsymbol{P}\boldsymbol{x} = -2px,$$

where $\boldsymbol{P} = p > 0$ is the solution to the ARE

$$\boldsymbol{A}^\top \boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} + \boldsymbol{Q} - \boldsymbol{P}\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^\top \boldsymbol{P} = \boldsymbol{O}.$$

In our problem, the ARE takes the form

$$2p - 4p^2 = 0.$$

Solving the above yields two solutions::

$$p_1 = \frac{1}{2} \quad \text{and} \quad p_2 = 0.$$

We take positive definite solution of the ARE. Hence,

$$u^* = -2p_1 x = -x.$$

The optimal value of J is

$$J = \boldsymbol{x}(0)^\top \boldsymbol{P}\boldsymbol{x}(0) = p_1 x(0)^2 = \frac{1}{2}.$$
