AAE 666 Homework 7 Solution

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Exercise 1

Using Schur complement directly, we have

$$PA + A^T P + C^T C + 2\alpha P + \gamma^2 PBB^T P \leq 0$$

Let $V(x) = x^T P x$, then

$$\begin{aligned}\dot{V}(x) &= 2x^T P\dot{x} = 2x^T P(Ax + B\varphi(Cx)) \\ \dot{V}(x) &= x^T(PA + A^T P) + 2x^T PB\varphi(Cx)\end{aligned}$$

Since,

$$\begin{aligned}2x^T PB\varphi(Cx) &\leq 2|B^T Px||\varphi(Cx)| \\ &= 2|B^T Px||\gamma Cx|\end{aligned}$$

Thus, by $2ab \leq a^2 + b^2$

$$\begin{aligned}2x^T PB\varphi(Cx) &\leq \gamma^2|B^T Px|^2 + |Cx|^2 \\ &= x^T(PA + A^T P + \gamma^2 PBB^T P + C^T C)x\end{aligned}$$

Using the first equation, we end up with $\dot{V} \leq -2\alpha x^T Px = -2\alpha V(x)$, and the system is GEO about the origin with covergence rate α.

Exercise 2

Let $x_1 = \theta_1, x_2 = \theta_2, x_3 = \dot{\theta}_1, x_4 = \dot{\theta}_2$, then we have the system as:

$$\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4\end{bmatrix} = \begin{bmatrix}0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2k & k & -2 & 1 \\ k & -k & 1 & -1\end{bmatrix}\begin{bmatrix}x_1 \\ x_2 \\ x_3 \\ x_4\end{bmatrix} + \begin{bmatrix}0 \\ 0 \\ \sin x_1 \\ \sin x_2\end{bmatrix}$$

Also have $|\varphi_i(z_i)| = |\gamma sin z_i| \leq \gamma|z_i|$, where $\gamma = 1$. Then, $B_1 = [0,0,1,0]^T$, $B_2 = [0,0,0,1]^T$,$C_1 = [1,0,0,0]$, $C_2 = [0,0,1,0]$. Plug in matrix on pg.133 using LMI and obtain that $k \geq 18.1$.

Exercise 3

$$PA + A^T P + 2\alpha P \leq 0$$
$$B^T P = C$$

Let $V(x) = x^T P x$, then,

$$\dot{V}(x) = 2x^T P \dot{x} = 2x^T P(Ax - B\varphi(Cx)) = 2x^T PAx - 2(B^T Px)^T \varphi(Vx)$$

Using $B^T P = C$,

$$\dot{V}(x) = x^T (PA + A^T P)x - 2(Cx)^T \varphi(x)$$

Since $(Cx)^T \varphi(x) \geq 0$,

$$\dot{V}(x) \leq x^T (PA + A^T P)x$$

Thus,

$$\dot{V}(x) \leq -2\alpha x^T Px = -2\alpha V(x)$$

Thus, the system is GEI about the origin with convergence rate α.

Exercise 4

$$\hat{g}(s) = \frac{\beta s + 1}{s^2 + s + 2}$$

$$\dot{V}(x) \leq -2\alpha x^T Px = -2\alpha V(x)$$

(a) Stability: Poles: $s_{1,2} = \frac{1}{2} \pm \frac{\sqrt{7}j}{2}$. Stable with negative real part. **(b) Dissipativity**:

$$\hat{g}(j\omega) = \frac{\beta j\omega + 1}{(j\omega)^2 + j\omega + 2} = \frac{1 + \beta j\omega}{2 - \omega^2 + j\omega}$$
$$\hat{g}(j\omega) + \hat{g}(j\omega)' = \frac{1 + \beta j\omega}{2 - \omega^2 + j\omega} + \frac{1 - \beta j\omega}{2 - \omega^2 - j\omega}$$
$$\hat{g}(j\omega) + \hat{g}(j\omega)' = \frac{(1 + \beta j\omega)(2 - \omega^2 - j\omega) + (1 - \beta j\omega)(2 - \omega^2 + j\omega)}{(2 - \omega^2)^2 + \omega^2}$$
$$\hat{g}(j\omega) + \hat{g}(j\omega)' = \frac{2(2 + (\beta + 1)\omega^2)}{(2 - \omega^2)^2 + \omega^2} \geq 0 \quad if \quad \beta \geq 1$$

(c) Asymptotic side condition

$$D = \hat{g}(\infty) = \lim_{|\omega| \to \infty} \hat{g}(j\omega) = 0, \quad \rho = 1$$

$$\lim_{|\omega| \to \infty} \omega^2 (\hat{g}(j\omega) + \hat{g}(j\omega)') = \lim_{|\omega| \to \infty} \frac{2(\frac{2}{\omega^2} + \beta - 1)}{(\frac{2 - \omega^2}{\omega^2})^2 + \frac{1}{\omega^2}} = 2(\beta - 1) \neq 0, \quad if \quad \beta \neq 1$$

Therefore, the transfer function is SPR if $\beta > 1$.
Alternatively, by KYPSPR Lemma,

$$\begin{aligned}\hat{g}(s) &= \frac{\beta s + 1}{s^2 + s + 2} \\ A &= \begin{bmatrix} -1 & -2 \\ 1 & 0 \end{bmatrix} \\ B &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ C &= \begin{bmatrix} \beta & 1 \end{bmatrix} \\ D &= 0\end{aligned}$$

Using theorem 21 and solving LMI in MATLAB iteratively gives $\beta > 1$ by KYPSPR.