Problem 1

The necessary and sufficient conditions for the existence of unknown input observer (UIO) for a plant model of the following form will be derived.

$$\begin{aligned} \dot{x} &= Ax + B_1 u_1 + B_2 u_2 \\ y &= Cx + D_1 u_1 + D_2 u_3 \end{aligned} \tag{1}$$

In equation 1, u_2 is the unknown input and u_3 is the output disturbance. The dimensions of the states, outputs, and inputs are as follows. There are n states, m_1 known inputs, m_2 unknown inputs, p outputs, and r output disturbances. To begin the derivation, we can first represent x as:

$$x = x - M(Cx + D_1 u_1) + M(Cx + D_1 u_1) \tag{2}$$

Substituting the output equation from equation 1 into 2 yields:

$$\begin{aligned} x &= x - M(Cx + D_1 u_1) + M(y - D_2 u_3) \\ x &= (I - MC)x - MD_1 u_1 + My - MD_2 u_3 \end{aligned} \tag{3}$$

The first condition for the existence of the UIO can be found by selecting M such that the effect of u_3 is the zero matrix. Therefore the first necessary and sufficient condition is:

$$MD_2 = 0_{n\times r} \tag{4}$$

Applying the above condition to equation 3 gives the following:

$$x = (I - MC)x - MD_1 u_1 + My \tag{5}$$

Next, the variable z will be defined as, $z = (I - MC)x$. Then equation 5 becomes:

$$x = z - MD_1 u_1 + My \tag{6}$$

It should be noted that the state estimate, $\hat{x}$, can be found by replacing x with $\hat{x}$ in equation 6. Using the definition of z, the dynamics of $\dot{z}$ are given below.

$$\dot{z} = (I - MC)\dot{x} = (I - MC)(Ax + B_1 u_1 + B_2 u_2) \tag{7}$$

Combining equations 6 and 7 gives the following dynamics:

$$\begin{aligned}\dot{z} &= (I - MC)(Az - AMD_1u_1 + AMy + B_1u_1 + B_2u_2)\\ \dot{z} &= (I - MC)[Az + AMy + (B_1 - AMD_1)u_1] + (I - MC)B_2u_2\end{aligned} \tag{8}$$

The second condition for the existence of the UIO can be found by selecting M such that the effect of u_2 is the zero matrix in equation 8. Therefore the second necessary and sufficient condition is:

$$(I - MC)B_2 = 0_{n\times m_2} \tag{9}$$

Applying this second condition to equation 8 and adding an innovation term to close the loop for the UIO, the dynamics for z becomes

$$\dot{z} = (I - MC)[Az + AMy + (B_1 - AMD_1)u_1] + L(y - \hat{y}) \tag{10}$$

The state estimate can then be determined with

$$\hat{x} = z - MD_1u_1 + My \tag{11}$$

If M is set to 0, then the Luenberger observer is obtained, as shown below:

$$\dot{z} = Az + B_1u_1 + L(y - \hat{y}) \tag{12}$$

Combining the two conditions for the existence of the UIO, M can be found by solving the following:

$$M\begin{bmatrix} CB_2 & D\end{bmatrix} = \begin{bmatrix} B_2 & 0_{n\times r}\end{bmatrix} \tag{13}$$

$$M = \begin{bmatrix} B_2 & 0_{n\times r}\end{bmatrix}\begin{bmatrix} CB_2 & D\end{bmatrix}^{\dagger} \tag{14}$$

The matrix M exists if and only if $\begin{bmatrix} CB_2 & D\end{bmatrix}$ is full column rank, and is therefore left invertable. The observation error is defined as $e = x - \hat{x}$. Using equation 11, this is expressed as

$$e = x - (z - MD_1u_1 + My) = x - z + MD_1u_1 - M(Cx + D_1u_1 + D_2u_3) \tag{15}$$

Applying the first condition (equation 4) eliminates the u_3 term in equation 15. Then the observation error can be simplified to $e = (I - MC)x - z$. The error dynamics can be expressed as:

$$\dot{e} = (I - MC)(Ax + B_1 u_1 + B_2 u_2) - [(I - MC)(Az + AMy + (B_1 - AMD_1)u_1) + L(y - \hat{y})] \tag{16}$$

The estimated output is given by $\hat{y} = C\hat{x} + D_1 u_1$. Therefore the innovation term, $L(y - \hat{y})$, can be written as:

$$L(y - \hat{y}) = L[(Cx + D_1 u_1 + D_2 u_3) - (C(z - MD_1 u_1 + My) + D_1 u_1)] \tag{17}$$

Substituting in the output equation for y into the above equation, then simplifying yields:

$$L(y - \hat{y}) = L[Cx + D_2 u_3 - Cz - CMCx - CMD_2 u_3] \tag{18}$$

Again applying the first condition in equation 4 (eliminating the MD_2 term), and using the definition of the error, equation 18 can be simplified to

$$L(y - \hat{y}) = LC(x - z - MCx) + LD_2 u_3 = LCe + LD_2 u_3 \tag{19}$$

Applying the second condition (equation 9) to equation 16, it can be simplified to:

$$\dot{e} = (I - MC)(Ax) - [(I - MC)(Az + AM(Cx + D_1 u_1 + D_2 u_3) - AMD_1 u_1) + L(y - \hat{y})] \tag{20}$$

Again applying the first condition, the u_3 term can be eliminated. Then simplifying yields:

$$\begin{aligned} \dot{e} &= (I - MC)(Ax) - [(I - MC)(Az + AMCx) + LCe + LD_2 u_3] \\ \dot{e} &= (I - MC)(Ax - Az - AMCx) - LCe - LD_2 u_3 \end{aligned} \tag{21}$$

Applying the error definition to equation 21, the error dynamics can be simplified to the final following form.

$$\dot{e} = ((I - MC)A - LC)e - LD_2 u_3 \tag{22}$$

It can then be shown that the state observation error is l_∞ stable if there exists a matrix $P = P^T \succ 0$ and an observer gain matrix L that satisfies the following matrix inequality

$$\begin{bmatrix} E^T PE - (1-\alpha)P & E^T PN \\ N^T PE & N^T PN - \alpha I \end{bmatrix} \preceq 0 \tag{23}$$

where $E = ((I - MC)A - LC)$, $N = -LD$, and $0 < \alpha < 1$. If the pair $((I - MC)A, C)$ is detectable (all observable states are stable), then we can find an observer gain matrix L such that E is Schur stable