ECE 68000: MODERN AUTOMATIC CONTROL

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Functional and the Functional's Variation

What is a functional?

- A function can be viewed as a rule, or a mapping, that assigns to each element of some set a unique element of a possibly different set
- In particular, a function $x : \mathbb{R} \to \mathbb{R}$ of a real variable t, assigns to each real number a unique real number
- An increment of the argument of a function of one variable t is $\Delta t = t - t_1$
- Similarly, a functional is a mapping that assigns to each function, from some class of functions, a unique number
- We can say that a functional is a "function of a function"

Continuous functional

- Let $x : \mathbb{R} \to \mathbb{R}$ be an argument of a functional
- By a variation $\delta x(t)$ of an argument $x(t)$ of a functional v we mean the difference of two functions

$$\delta x(t) = x(t) - x_1(t)$$

- Assume that $x(t)$ can change in an arbitrary way in some class of functions
- Recall, that a function $x : \mathbb{R} \to \mathbb{R}$ is continuous if a small change of its argument t corresponds to a small change of the value $x(t)$ of the function
- Similarly, a functional v is said to be continuous if a "small" change of its argument $x = x(t)$ corresponds to a small change of the value of the functional

Linear functional

- A functional v is called linear if

$$v\left(ax_1 + x_2\right) = av\left(x_1\right) + v\left(x_2\right),$$

  where a is a constant.
- The variation of a functional is analogous to the notion of a function differential
- To connect the two, consider a function of one variable
- Let x be a differentiable function defined on an open interval U, and let $t \in U$
- The derivative of x at t is defined as

$$x'(t) = \lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t}$$

- Let

$$\varphi(\Delta t) = \frac{x(t + \Delta t) - x(t)}{\Delta t} - x'(t)$$

Differentiability of a function

- The function φ is not defined at $\Delta t = 0$, however

$$\lim_{\Delta t \to 0} \varphi(\Delta t) = 0$$

- Re-write

$$x(t + \Delta t) - x(t) = x'(t)\Delta t + \varphi(\Delta t)\Delta t$$

- The above has meaning only when $\Delta t \neq 0$
- To make it hold at $\Delta t = 0$, define

$$\varphi(\Delta t)\big|_{\Delta t = 0} = 0$$

- To proceed, let

$$\begin{array}{ll} \beta(\Delta t) = \varphi(\Delta t) & \text{if} \quad \Delta t > 0 \\ \beta(\Delta t) = -\varphi(\Delta t) & \text{if} \quad \Delta t < 0 \end{array}$$

Function linear in Δx

- If x is differentiable, there exists a function β such that

$$\begin{aligned} x(t+\Delta t) - x(t) = \Delta x &= x'(t)\Delta t + \beta(\Delta t)|\Delta t| \\ &= L(t, \Delta t) + \beta(\Delta t)|\Delta t|, \end{aligned}$$

  where $\lim_{\Delta t \to 0} \beta(\Delta t) = 0$ and $L(t, \Delta t) = x'(t)\Delta t$ is a linear function in Δt

- For a real-valued function $f = f(\boldsymbol{x}) : \mathbb{R}^n \to \mathbb{R}$

$$\begin{aligned} f(\boldsymbol{x}+\Delta \boldsymbol{x}) - f(\boldsymbol{x}) = \Delta f &= Df(\boldsymbol{x})\Delta \boldsymbol{x} + \beta(\Delta \boldsymbol{x})\|\Delta \boldsymbol{x}\| \\ &= L(\boldsymbol{x}, \Delta \boldsymbol{x}) + \beta(\Delta \boldsymbol{x})\|\Delta \boldsymbol{x}\|, \end{aligned}$$

  where $\lim_{\Delta \boldsymbol{x} \to \boldsymbol{0}} \beta(\Delta \boldsymbol{x}) = 0$, and

$$L(\boldsymbol{x}, \Delta \boldsymbol{x}) = Df(\boldsymbol{x})\Delta \boldsymbol{x} = \nabla f(\boldsymbol{x})^\top \Delta \boldsymbol{x}$$

  is a linear function in $\Delta \boldsymbol{x}$

The variation of a functional

- The functional is an operator over a Banach space of continuous functions, $\mathcal{C}([t_0, t_1])$
- If an increment $\Delta v = v(x + \delta x) - v(x)$ of a functional v can be represented as

$$\Delta v = L(x, \delta x) + \beta(\delta x) \|\delta x\|,$$

  where $L(x, \delta x)$ is a linear functional with respect to δx, the term $\|\delta x\| = \max_{t \in [t_0, t_1]} |\delta x|$ denotes the maximal value of $|\delta x|$, and $\beta(\delta x) \to 0$ if $\|\delta x\| \to 0$, then the linear part L of Δv is called the variation of the functional and is denoted δv, that is,

$$\delta v = L(x, \delta x)$$

Example

- Find the variation of the functional

$$v = \int_0^1 \left(2x^2(t) + x(t)\right) dt$$

- For this we first calculate its increment to get

$$\begin{aligned} \Delta v &= v\left(x + \delta x\right) - v\left(x\right) \\ &= \int_0^1 \left(2(x + \delta x)^2 + (x + \delta x)\right) dt - \int_0^1 \left(2x^2 + x\right) dt \\ &= \int_0^1 \left(2x^2 + 4x\delta x + 2(\delta x)^2 + x + \delta x - 2x^2 - x\right) dt \\ &= \int_0^1 (4x + 1)\delta x dt + 2\int_0^1 (\delta x)^2 dt. \end{aligned}$$

- The linear part of Δv is $\delta v = \int_0^1 (4x + 1)\delta x dt$, which is the variation of the given functional

Different way to obtain δv

- The linear part of Δv can be computed as

$$Df(\boldsymbol{x})\Delta\boldsymbol{x} = L(\boldsymbol{x}, \Delta\boldsymbol{x}) = \frac{d}{d\alpha} f(\boldsymbol{x} + \alpha\Delta\boldsymbol{x})\Big|_{\alpha=0}$$

Lemma

$$\delta v = \frac{d}{d\alpha} v(x + \alpha\delta x)\Big|_{\alpha=0}$$

Lemma's proof

- Suppose that for a given functional v there exists its variation
- This means that we can represent Δv as

$$\Delta v = v\left(x + \alpha \delta x\right) - v\left(x\right) = L(x, \alpha \delta x) + \beta(\alpha \delta x)|\alpha| \, \|\delta x\|$$

- Then, the derivative of $v\left(x + \alpha \delta x\right)$ with respect to α evaluated at $\alpha = 0$ is equal to

$$\begin{aligned}
\lim_{\alpha \to 0} \frac{\Delta v}{\alpha} &= \lim_{\alpha \to 0} \frac{L(x, \alpha \delta x) + \beta(\alpha \delta x)|\alpha| \, \|\delta x\|}{\alpha} \\
&= \lim_{\alpha \to 0} \frac{L(x, \alpha \delta x)}{\alpha} + \lim_{\alpha \to 0} \frac{\beta(\alpha \delta x)|\alpha| \, \|\delta x\|}{\alpha} \\
&= L(x, \delta x),
\end{aligned}$$

since $L(\cdot, \cdot)$ is linear with respect to the second argument

Lemma's proof—contd.

- Hence

$$L(x, \alpha\delta x) = \alpha L(x, \delta x)$$

- Furthermore,

$$\lim_{\alpha\to 0} \frac{\beta(\alpha\delta x)|\alpha|\,\|\delta x\|}{\alpha} = \lim_{\alpha\to 0} \beta(\alpha\delta x)\|\delta x\| = 0$$

This completes the proof $\square$

Example

- Find the variation of the functional

$$v = \int_0^1 \left(2x^2(t) + x(t)\right) dt$$

- Use the lemma to get

$$\begin{aligned}
\delta v &= \frac{d}{d\alpha} v\left(x + \alpha\delta x\right)|_{\alpha=0} \\
&= \frac{d}{d\alpha}\left(\int_0^1 \left(2(x+\alpha\delta x)^2 + (x+\alpha\delta x)\right) dt\right)\Big|_{\alpha=0} \\
&= \int_0^1 \left(4(x+\alpha\delta x)\delta x + \delta x\right) dt\,\Big|_{\alpha=0} \\
&= \int_0^1 (4x+1)\delta x dt
\end{aligned}$$