3.1.2

$V_1 = \begin{pmatrix}1\\2\end{pmatrix}$, $V_2 = \begin{pmatrix}2\\5\end{pmatrix}$

$\langle V_1, V_2 \rangle = V_1^T V_2 = (1 \quad 2)\begin{pmatrix}2\\5\end{pmatrix} = 2+10 = 12$

$\langle V_1, V_2 \rangle \neq 0 \quad \therefore$ not orthogonal

$C_1 V_1 + C_2 V_2 = 0$

$C_1 \begin{pmatrix}1\\2\end{pmatrix} + C_2 \begin{pmatrix}2\\5\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$

$C_1 + 2C_2 = 0$

$2C_1 + 5C_2 = 0$

$C_1 = -\frac{5C_2}{2}$

$-\frac{5C_2}{2} + 2C_2 = 0 \Rightarrow C_2 = 0 \quad \therefore \quad C_1 = 0$

V_1 & V_2 linearly Independent

$\boxed{V_1 \text{ \& } V_2 \text{ are non-orthogonal and linearly independent}}$

$V_1 = \begin{pmatrix}0\\0\end{pmatrix}$, $V_2 = \begin{pmatrix}1\\1\end{pmatrix}$

$\langle V_1, V_2 \rangle = (0 \quad 0)\begin{pmatrix}1\\1\end{pmatrix} = 0$

$C_1 V_1 + C_2 V_2 = 0$

3.1.2

$$c_1\begin{pmatrix}0\\0\end{pmatrix} + c_2\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

c_1 doesn't have to be 0 $\therefore$ the vectors $\begin{pmatrix}0\\0\end{pmatrix}$ & $\begin{pmatrix}1\\1\end{pmatrix}$ are orthogonal but linearly dependent

3.1.5

$$V_1^T V_2 = [1\ 2\ {-2}\ 1] \begin{pmatrix} 4 \\ 0 \\ 4 \\ 0 \end{pmatrix} = 4 - 8 = -4 \neq 0$$

$$V_1^T V_3 = [1\ 2\ {-2}\ 1] \begin{pmatrix} 1 \\ -1 \\ -1 \\ -1 \end{pmatrix} = 1 - 2 + 2 - 1 = 0$$

$$V_1^T V_4 = [1\ 2\ {-2}\ 1] \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = 1 + 2 - 2 + 1 \neq 0$$

$$V_2^T V_4 = [4\ 0\ 4\ 0] \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = 8 \neq 0$$

$$V_2^T V_3 = [4\ 0\ 4\ 0] \begin{pmatrix} 1 \\ -1 \\ -1 \\ -1 \end{pmatrix} = 4 - 4 = 0$$

$$V_3^T V_4 = [1\ {-1}\ {-1}\ {-1}] \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = -2 \neq 0$$

The pairs V_2 & V_3 are orthogonal, so is the pair V_1 & V_3

3.1.12

$A = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \end{pmatrix}$

$C(A^T) = N(A)^\perp$

$\xrightarrow{E_{21}(-1)} \begin{pmatrix} \boxed{1} & 0 & 2 \\ 0 & \boxed{1} & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$x_1 + 2x_3 = 0$

$x_2 + 2x_3 = 0$

x_1 & x_2 are pivots $\therefore$ x_3 is free, let $x_3 = 1$

$x_2 = x_1 = -2$

$\boxed{N(A) = \text{span}\left\{ \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix} \right\}}$ Basis for Null space, which is $\perp$ to row space

$C(A^T) = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\}$

$\mathbb{R}^3 = N(A) + C(A^T) = \begin{pmatrix} -2 & 1 & 0 \\ -2 & 0 & 1 \\ 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$

$-2x_1 + x_2 = 3$ $\qquad$ $x_3 = 3 + 2x_1 = x_2$

$-2x_1 + x_3 = 3$

3.1.12

$X_1 + 2X_2 + 2X_3 = 3$

$X_1 + 4X_2 = 3$

$X_2 = 3 + 2X_1$

$X_1 + 12 + 8X_1 = 3$

$9X_1 = -9$

$X_1 = -1 \Leftarrow$ multiply $\begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}$ by -1

$\therefore X_3 = X_2 = 1$

$$X_r = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \boxed{\begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}} \Leftarrow X_r$$

$$\boxed{X_n = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}}$$

3.2.3

$a = (1,1,1)$
$b = (2,4,4)$

$$p = \frac{a^T b}{a^T a} a$$

$a^T b = 2+4+4 = 10$

$a^T a = 3$

$$p = \frac{10}{3}(1,1,1)$$

$$= \boxed{(10/3,\ 10/3,\ 10/3)}$$

← multiple of a closest to b

$$p = \frac{b^T a}{b^T b} b$$

$b^T a = 10$

$b^T b = 4+16+16 = 36$

$$\frac{10}{36}(2,4,4) = \boxed{(5/9,\ 10/9,\ 10/9)}$$

3.2.5

$$\cos\theta = \frac{a^T b}{\|a\|\|b\|}$$

$b = (1, 0, \ldots, 0)$

$a = (1, 1, \ldots, 1)$

$a^T b = 1$

$\|a\| = \sqrt{(1)^2 + (1)^2 + \ldots + (1)^2} \quad = a^T a$

$\|a\| = \sqrt{(1)^2 n} \quad = \sqrt{n}$

$\|b\| = \sqrt{1^2 + 0^2 + \ldots} \quad = 1$

$$\cos\theta = \frac{1}{(1)(\sqrt{n})} = \frac{1}{\sqrt{n}}$$

$$\boxed{\theta = \cos^{-1}\left(1/\sqrt{n}\right)}$$

$$P = \frac{aa^T}{a^T a} = \frac{\underset{n \times 1}{\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}} \underset{1 \times n}{[1, 1, \ldots, 1]}}{[1, 1, \ldots 1] \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}} \qquad a^T a = \|a\|^2 = n$$

$$\boxed{P = \begin{pmatrix} 1/n & \cdots & 1/n \\ \vdots & \ddots \; 1/n \; \ddots & \vdots \\ 1/n & \cdots & 1/n \end{pmatrix}}$$

3.2.8

$$\cos\theta = \frac{a^T b}{\|a\|\,\|b\|}$$

$(0,0,0) - (\frac{1}{2},\frac{1}{2},\frac{1}{2}) = [-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}]^T$

$(1,1,0) - (\frac{1}{2},\frac{1}{2},\frac{1}{2}) = [\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}]^T$

$(1,0,1) - (\frac{1}{2},\frac{1}{2},\frac{1}{2}) = [\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}]^T$

$(0,1,1) - (\frac{1}{2},\frac{1}{2},\frac{1}{2}) = [-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}]^T$

$$\frac{[-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}]\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\\ -\frac{1}{2}\end{bmatrix}}{\left(\sqrt{(-\frac{1}{2})^2+(-\frac{1}{2})^2+(-\frac{1}{2})^2}\right)\left(\sqrt{(\frac{1}{2})^2+(\frac{1}{2})^2+(-\frac{1}{2})^2}\right)} = \frac{-\frac{1}{4} + -\frac{1}{4} + \frac{1}{4}}{\left(\sqrt{\frac{3}{4}}\right)\left(\sqrt{\frac{3}{4}}\right)}$$

$$= \frac{-\frac{1}{4}}{\frac{3}{4}} = -\frac{1}{3} = \cos\theta$$

Because tetrahedron is regular, the angles between the rays are all equal. This angle is $\cos\theta = -\frac{1}{3}$ or $\cos\theta = 109.5^\circ$

3.2.12

$x + 2y = 0$

$y = -\frac{x}{2}$, $x = 1$, $y = -\frac{1}{2}$

$P = \frac{aa^T}{a^Ta}$, $a = (1, -\frac{1}{2})$

$$P = \frac{\begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} \end{bmatrix}}{\begin{bmatrix} 1 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix}} = \frac{\begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{4} \end{bmatrix}}{\frac{5}{4}}$$

$$\boxed{P = \begin{pmatrix} \frac{4}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{pmatrix}}$$

3.3.3

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \qquad b = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\hat{x} = (A^T A)^{-1} A^T b$$

$$A^T A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$(A^T A)^{-1} = \frac{1}{3} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{pmatrix}$$

$$\hat{x} = \begin{pmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\hat{x} = \begin{pmatrix} 2/3 & -1/3 & 1/3 \\ -1/3 & 2/3 & 1/3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1/3 \end{pmatrix}$$

$$\boxed{\hat{x} = \begin{pmatrix} 1/3 \\ 1/3 \end{pmatrix}}$$

$$P = A\hat{x} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1/3 \\ 1/3 \end{pmatrix}$$

$$\boxed{P = \begin{pmatrix} 1/3 \\ 1/3 \\ 2/3 \end{pmatrix}}$$

3.3.3

$$b-p = \begin{pmatrix} 2/3 \\ 2/3 \\ -2/3 \end{pmatrix}$$

$$[1\ 0\ 1]\begin{bmatrix} 2/3 \\ 2/3 \\ -2/3 \end{bmatrix} = \frac{2}{3} - \frac{2}{3} = 0 \checkmark$$

$$[0\ 1\ 1]\begin{bmatrix} 2/3 \\ 2/3 \\ -2/3 \end{bmatrix} = \frac{2}{3} - \frac{2}{3} = 0 \checkmark$$

$b-p$ is $\perp$ to columns of A as their dot products are 0.

3.3.12

$\text{span}\left\{\begin{pmatrix}1\\1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}\right\} = V$

Let V be $C(A^T)$ such that $V^\perp$ is $N(A)$

$A = \begin{pmatrix}1 & 1 & 0 & 1\\ 0 & 0 & 1 & 0\end{pmatrix} \begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}$

$N(A)$: $x_1 + x_2 + x_4 = 0$

$x_3 = 0$

x_1 & x_3 are pivots, x_2 & x_4 are free

Preferred solutions: $x_2 = 1$, $x_4 = 0$ & $x_2 = 0$, $x_4 = 1$

gives basis for $N(A)$ & $V^\perp$: $\boxed{\text{Span}\left\{\begin{pmatrix}-1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\0\\1\end{pmatrix}\right\}}$

$\tilde{A} = A^T \quad P = \tilde{A}(\tilde{A}^T\tilde{A})^{-1}\tilde{A}^T$

$\tilde{A}^T\tilde{A} = \begin{pmatrix}1 & 1 & 0 & 1\\ 0 & 0 & 1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 1 & 0\\ 0 & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}3 & 0\\ 0 & 1\end{pmatrix}$

$\text{inv}(\tilde{A}^T\tilde{A}) = \frac{1}{3}\begin{pmatrix}1 & 0\\ 0 & 3\end{pmatrix} = \begin{pmatrix}1/3 & 0\\ 0 & 1\end{pmatrix}$

3.3.12

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1/3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1/3 & 0 \\ 1/3 & 0 \\ 0 & 1 \\ 1/3 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\boxed{P = \begin{pmatrix} 1/3 & 1/3 & 0 & 1/3 \\ 1/3 & 1/3 & 0 & 1/3 \\ 0 & 0 & 1 & 0 \\ 1/3 & 1/3 & 0 & 1/3 \end{pmatrix}}$$

$$Pb = \begin{pmatrix} 1/3 & 1/3 & 0 & 1/3 \\ 1/3 & 1/3 & 0 & 1/3 \\ 0 & 0 & 1 & 0 \\ 1/3 & 1/3 & 0 & 1/3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\boxed{Pb = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}}$$

3.3.24

$C + Dt = b$

$$\begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -3 \\ -5 \end{pmatrix}$$

$$\hat{x} = (A^T A)^{-1} A^T b$$

$$A^T A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 2 & 6 \end{pmatrix}$$

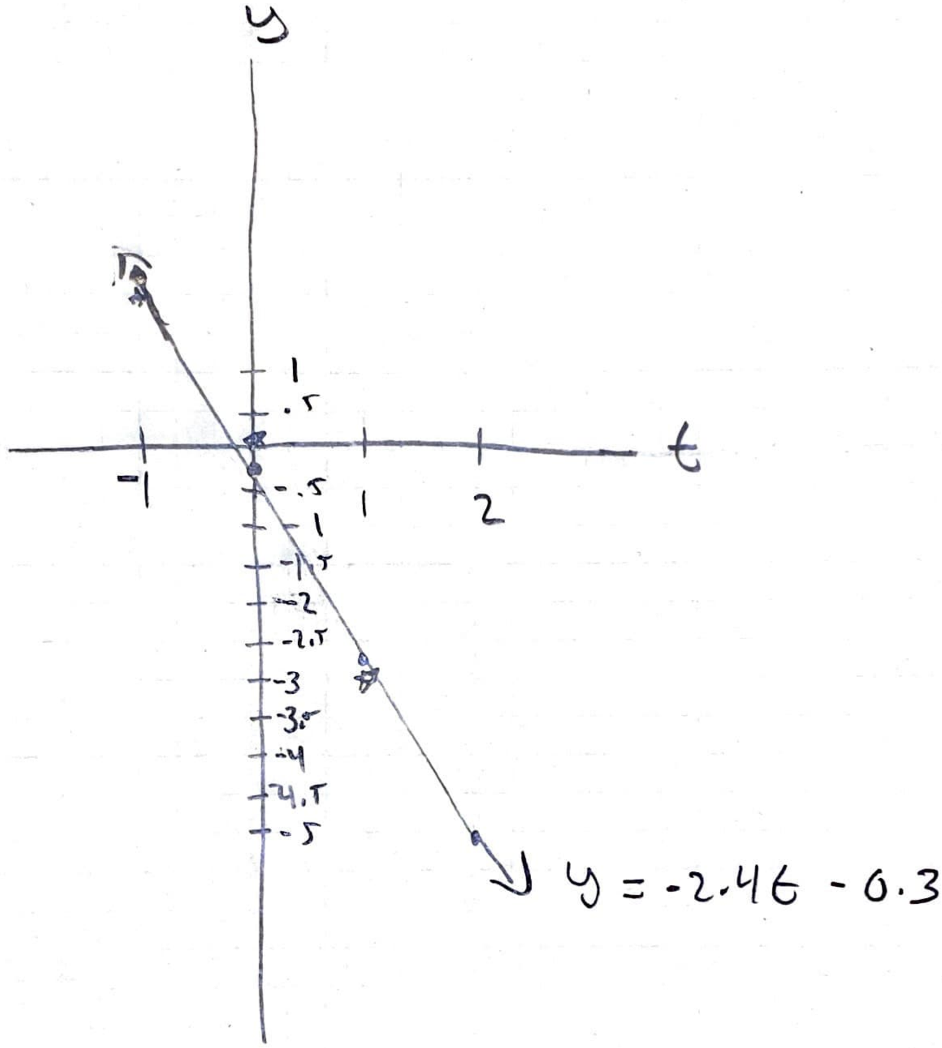
$$(A^T A)^{-1} = \frac{1}{20} \begin{pmatrix} 6 & -2 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} 3/10 & -1/10 \\ -1/10 & 1/5 \end{pmatrix}$$

$$\begin{pmatrix} 3/10 & -1/10 \\ -1/10 & 1/5 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 4/10 & 3/10 & 1/5 & 1/10 \\ -3/10 & -1/10 & 1/10 & 3/10 \end{pmatrix}$$

$$\begin{pmatrix} 4/10 & 3/10 & 1/5 & 1/10 \\ -3/10 & -1/10 & 1/10 & 3/10 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ -3 \\ -5 \end{pmatrix} = \begin{pmatrix} -3/10 \\ -24/10 \end{pmatrix}$$

$$\boxed{\hat{x} = \begin{pmatrix} -0.3 \\ -2.4 \end{pmatrix}}$$

3.3.24



$-2.4(-1) + -.3 = 2.1$

$-2.4(1) + -.3 = -2.7$

$-2.4(2) + -.3 = -5.1$