MA 527
Fourier Series

Dr. Park

What can we do with Fouier series?

- **Application:** Image file compression (JPEG file)

Lossy
92% compression
6.9MB -> 0.5MB

Glossy
89% compression
6.9MB -> 0.8MB

Lossless
3% compression
6.9MB -> 6.7MB

Can you recognize the differences among the images of a cat?

How does JPEG file compression work?

Lossy	Glossy	Lossless
92% compression	89% compression	3% compression
6.9MB -> 0.5MB	6.9MB -> 0.8MB	6.9MB -> 6.7MB

- Human vision: Our eyes are **not sensitive to high frequency of images:**

  When high frequency parts in images are removed, we don’t see the difference.

  If we remove high frequency parts in images, we **can reduce the file size dramatically**.

(**Question**)

How do we represent images by well-known functions (filtering)? **sin(nx), cos(nx)**

JPEG and MPEG

- **The Joint Photographic Experts Group (JPEG)** released the JPEG standard for still image coding [Wallace, 1992]
  - JPEG remains the dominant format for still images and photographs, now.
  - JPEG uses Fourier series to compress image files.
  - In chapter 11 we will talk about Fourier series and Fourier transform.

    The Fourier transform will be also applied to solve PDE in chapter 12.

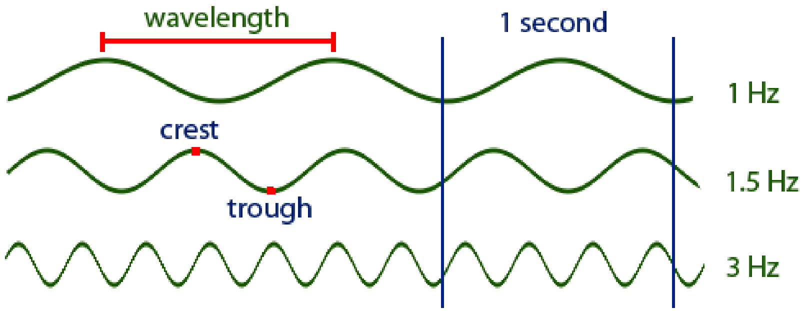
- **The Moving Picture Experts Group (MPEG)** published its first standard for coding of moving pictures and associated audio, MPEG-1 [Le Gall, 1991] and MPEG-2 [Haskell et al., 1996].

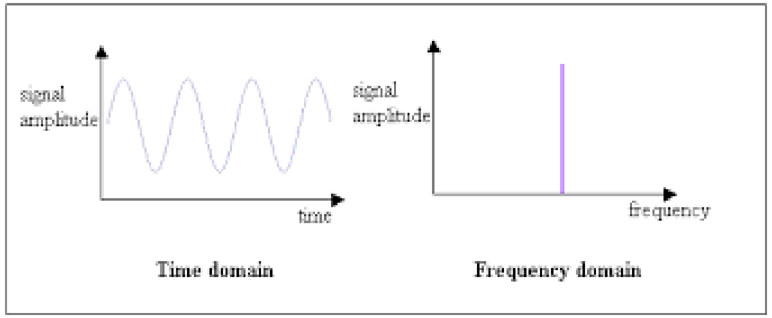
Waves and Signals

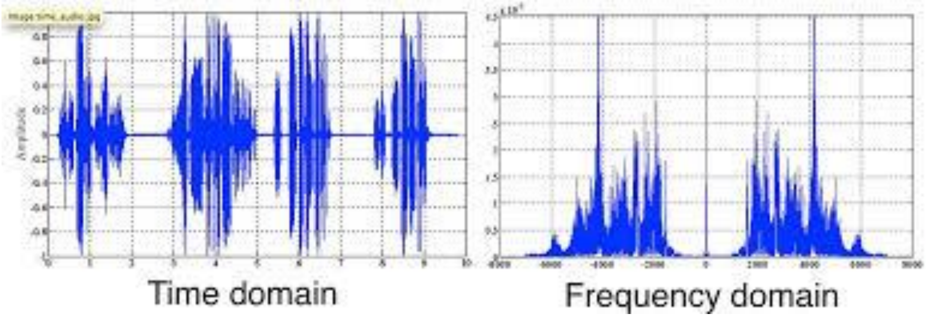
- The **Wavelength (λ)** of a wave = the distance between successive crests or high points of a wave (or between successive troughs or low points)
- The **Frequency (k)** of a wave = the number of waves that pass a given point in some unit of time (usually per second) -- unit: hertz
- The amplitude of a wave = the maximum height of a wave
- The speed of a wave (v): **V = λ k**
- **Questions: How do we measure or compute the frequencies and amplitudes?**

  Any easy way?

(Lucas V. Barbosa)







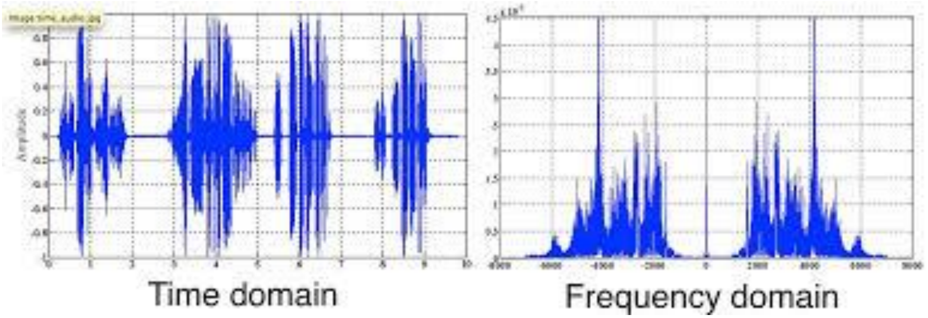
Decomposing a signal

- The red signal is a sum of six sine functions of different amplitudes and harmonically related frequencies. Their summation is called a **Fourier series**.

- The **Fourier transform** of the signal depicts amplitude vs frequency. The Fourier transform reveals the six frequencies and their amplitudes.

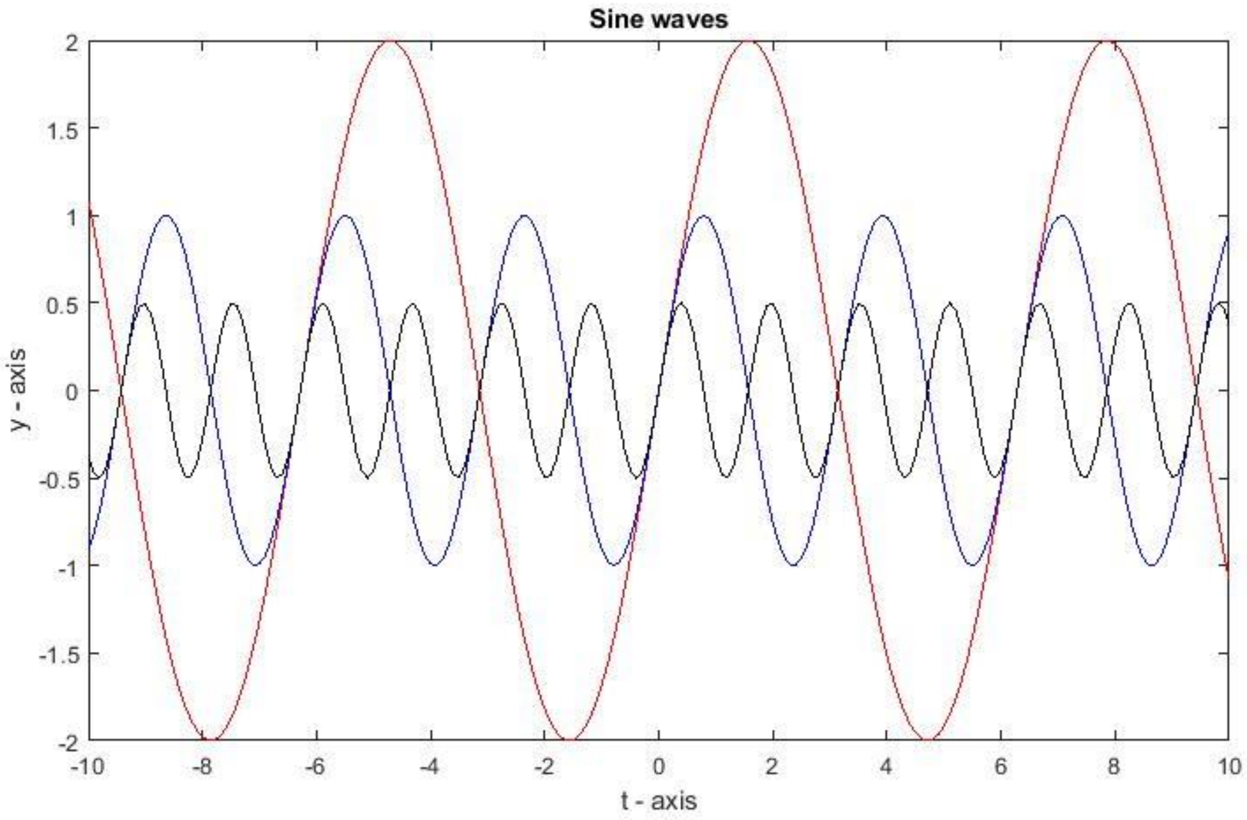
- A time-domain graph shows how a signal changes over time.

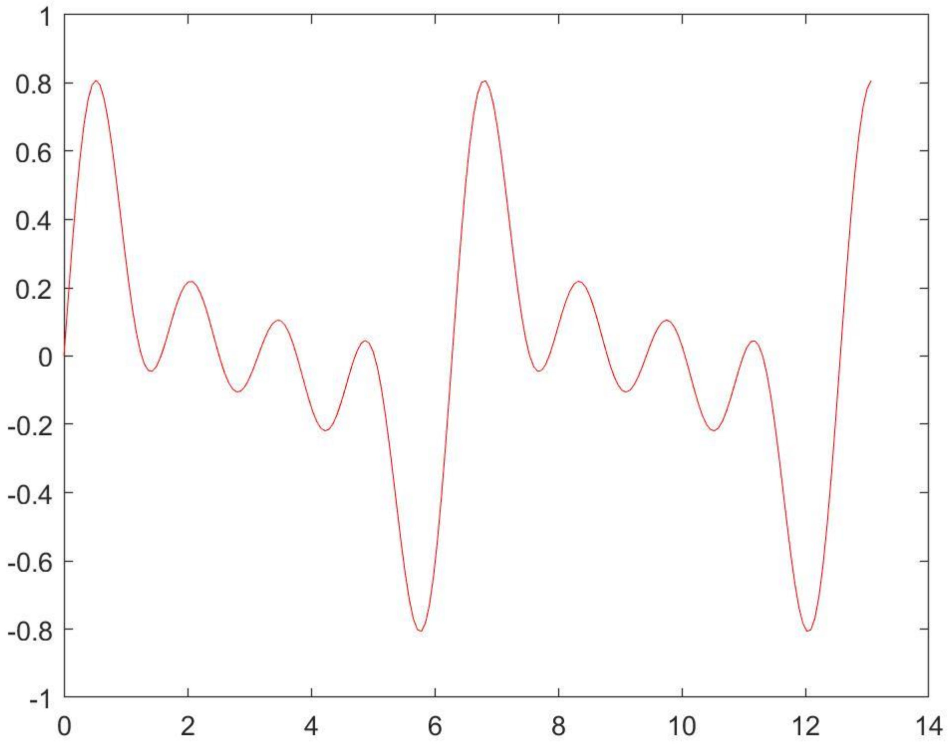
- A frequency-domain graph shows how much of the signal lies within each given frequency band over a range of frequencies.



Sine waves:

- y1 = 2sin(t)
- y2 = sin(2t)
- y3 = 0.5sin(4t)





: It is a sum of **four sine curves**.

Can you find the FOUR curves?

Fourier Series

(Def)

- A function f(x) is called a **periodic function** if there is a positive number, P, such that **f(x + P) = f(x)** for all real number x.
- P is called a period of f(x).

(Ex) **sin(x), cos(x), tan(x)**

(Question) Given a periodic function or signal f(x) of period P, can we approximate or represent f(x) by an infinite series of periodic functions?

Use cos(nx) and sin(nx):

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Use cos(nx) and sin(nx):

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

(Question) Does it converge?

Theorem 2 (Fourier series)

(H)

f(x) is a periodic function with period 2π and piecewise continuous in [-π, π].

f(x) has a left-had derivative and a right-hand derivative at the left and right end points, respectively.

(C)

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

(0) $$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

(a) $$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx \qquad n = 1, 2, \cdots$$

(b) $$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx \qquad n = 1, 2, \cdots .$$

Section 11.5 Sturm-Liouville problem

Orthogonality of Eigenfunctions of Sturm–Liouville Problems

Suppose that the functions p, q, r, and p' *in the Sturm–Liouville equation* (1) *are real-valued and continuous and* $r(x) > 0$ *on the interval* $a \leqq x \leqq b$. *Let* $y_m(x)$ *and* $y_n(x)$ *be eigenfunctions of the Sturm–Liouville problem* (1), (2) *that correspond to different eigenvalues* λ_m *and* λ_n, *respectively. Then* y_m, y_n *are orthogonal on that interval with respect to the weight function* r, *that is,*

$$(y_m, y_n) = \int_a^b r(x) y_m(x) y_n(x) \, dx = 0 \qquad (m \neq n). \tag{6}$$

If $p(a) = 0$, *then* (2a) *can be dropped from the problem. If* $p(b) = 0$, *then* (2b) can be dropped. [It is then required that y and y' remain bounded at such a point, and the problem is called **singular**, as opposed to a **regular problem** in which (2) is used.]

If $p(a) = p(b)$, *then* (2) *can be replaced by the* "**periodic boundary conditions**"

$$y(a) = y(b), \qquad y'(a) = y'(b). \tag{7}$$