ECE 68000: MODERN AUTOMATIC CONTROL

Professor Stan Żak

State Estimation in Linear Systems

State Estimation in Linear Systems

- **Objective**: Construct asymptotic state observers to estimate state variables of linear lumped continuous-time (CT) or discrete-time (DT) systems
- We consider linear time-varying (LTI) system

$$
\begin{aligned}
\dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) \\
\boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t)
\end{aligned}
$$

or

$$
\begin{aligned}
\boldsymbol{x}[k+1] &= \boldsymbol{A}\boldsymbol{x}[k] + \boldsymbol{b}\boldsymbol{u}[k] \\
\boldsymbol{y}[k] &= \boldsymbol{C}\boldsymbol{x}[k] + \boldsymbol{D}\boldsymbol{u}[k]
\end{aligned}
$$

- We assume that the system at hand is both reachable and observable

Need for state estimators to implement state-feedback (SF) controllers

- To implement a SF controller, all the states need to be available
- Often, this requirement is not met, either because measuring of all the state variables would require excessive number of sensors, or because the state variables are not accessible for direct measurement
- Instead, only a subset of state variables or their combination may be available
- Use state estimate to implement a SF controller

Estimator or observer

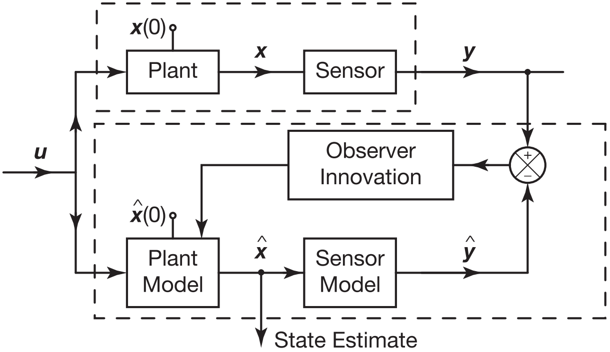
- Use state estimate to implement a SF controller
- Much of the literature refers to observers as “state estimators”
- One can argue that estimator is much more descriptive in its function because observer implies a direct measurement
- On the other hand, estimator implies non-deterministic approach
- Here we use a deterministic approach
- We follow the original terminology of the observer’s inventor

My dear Watson, you see but you do not observe*

*Sir Arthur Conan Doyle, *Scandal in Bohemia*, 1891

What is an observer?

- The first observer proposed by Luenberger in the early nineteen sixties for the purpose of estimating the state of a plant, based on limited measurements of that system
- An observer—a deterministic dynamical system that generates an estimate of the plant's state using that plant's input and output signals



Observers as virtual sensors

- The plant's state estimate are used in place of the true state to close the control loop
- Observers can be used as "software" or "virtual" sensors as opposed to hardware sensing devices directly measuring physical variables
- Observers augment or replace sensors in a control system
- Observers have been applied in secure communication using chaotic synchronization, machine vision, wind energy systems, speed-sensorless control of induction motors, or in model-based predictive controllers

Beginnings of the observer

- Observer—a dynamical system that estimates the system state based on the system inputs and outputs
- The observer provides a solution to the problem of incomplete state vector information
- D. G. Luenberger initiated the theory of observers in 1963 in his Ph.D. thesis, *Determining the State of a Linear System with Observers of Low Dynamic Order*, at Stanford

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† D. G. Luenberger, *Observing the state of a linear system*, IEEE Transactions on Military Electronics, Vol. 8, Issue 2, pp. 74–80, April 1964

First page of the 1964 Luenberger's paper

Observing the State of a Linear System

DAVID G. LUENBERGER, STUDENT MEMBER, IEEE

Summary—**In much of modern control theory designs are based on the assumption that the state vector of the system to be controlled is available for measurement. In many practical situations only a few output quantities are available. Application of theories which assume that the state vector is known is severely limited in these cases. In this paper it is shown that the state vector of a linear system can be reconstructed from observations of the system inputs and outputs.**

It is shown that the observer, which reconstructs the state vector, is itself a linear system whose complexity decreases as the number of output quantities available increases. The observer may be incorporated in the control of a system which does not have its state vector available for measurement. The observer supplies the state vector, but at the expense of adding poles to the over-all system.

I. Introduction

IN THE PAST few years there has been an increasing percentage of control system literature written from the "state variable" point of view [1]–[8]. In the case of a continuous, time-invariant linear system the state variable representation of the system is of the form:

$$\dot{y}(t) = Ay(t) + Bx(t),$$

where

$y(t)$ is an $(n \times 1)$ state vector
$x(t)$ is an $(m \times 1)$ input vector
A is an $(n \times n)$ transition matrix
B is an $(n \times m)$ distribution matrix.

This state variable representation has some conceptual advantages over the more conventional transfer function representation. The state vector $y(t)$ contains enough information to completely summarize the past behavior of the system, and the future behavior is governed by a simple first-order differential equation. The properties of the system are determined by the constant matrices A and B. Thus the study of the system can be carried out in the field of matrix theory which is not only well developed, but has many notational and conceptual advantages over other methods.

When faced with the problem of controlling a system, some scheme must be devised to choose the input vector $x(t)$ so that the system behaves in an acceptable manner. Since the state vector $y(t)$ contains all the essential information about the system, it is reasonable to base the choice of $x(t)$ solely on the values of $y(t)$ and perhaps also t. In other words, x is determined by a relation of the form $x(t) = F[y(t), t]$.

This is, in fact, the approach taken in a large portion of present day control system literature. Several new techniques have been developed to find the function F for special classes of control problems. These techniques include dynamic programming [8]–[10], Pontryagin's maximum principle [11], and methods based on Lyapunov's theory [2], [12].

In most control situations, however, the state vector is not available for direct measurement. This means that it is not possible to evaluate the function $F[y(t), t]$. In these cases either the method must be abandoned or a reasonable substitute for the state vector must be found.

In this paper it is shown how the available system inputs and outputs may be used to construct an estimate of the system state vector. The device which reconstructs the state vector is called an observer. The observer itself as a time-invariant linear system driven by the inputs and outputs of the system it observes.

Kalman [3], [13], [14] has done some work on this problem, primarily for sampled-data systems. He has treated both the nonrandom problem and the problem of estimating the state when measurements of the outputs are corrupted by noise. In this paper only the nonstatistical problem is discussed but for that case a fairly complete theory is developed.

It is shown that the time constants of an observer can be chosen arbitrarily and that the number of dynamic elements required by the observer decreases as more output measurements become available. The novel point of view taken in this paper leads to a simple conceptual understanding of the observer process.

II. Observation of a Free Dynamic System

As a first step toward the construction of an observer it is useful to consider a slightly more general problem. Instead of requiring that the observer reconstruct the state vector itself, require only that it reconstruct some constant linear transformation of the state vector. This problem is simpler than the previous problem and its solution provides a great deal of insight into the theory of observers.

Assuming it were possible to build a system which reconstructs some constant linear transformation T of the state vector y, it is clear that it would then be possible to reconstruct the state vector itself, provided that the transformation T were invertible. This is the approach taken in this paper. It is first shown that it is relatively simple to build a system which will reconstruct some linear transformation of the state vector and then it is shown how to guarantee that the transformation obtained is invertible.

The first result concerns systems which have no inputs. (Such systems are called free systems.) The situa-

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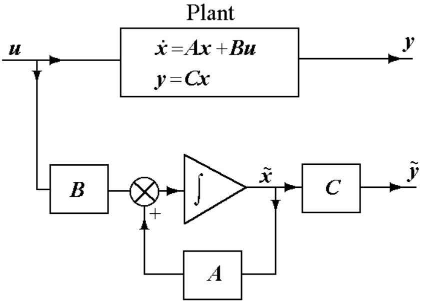
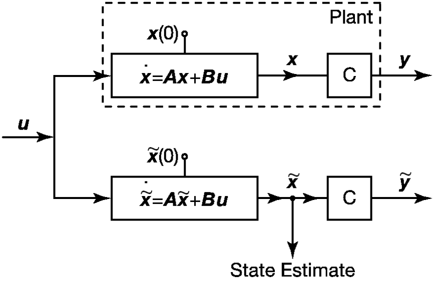
Trivial observer



- Trivial observer (open-loop observer)—the system model copy as an observer
- Observation error, $\boldsymbol{e} = \boldsymbol{x} - \tilde{\boldsymbol{x}}$

Trivial observer—A different look



- Recall: Trivial observer (open-loop observer)—the plant model copy as an observer
- Observation error, $\boldsymbol{e} = \boldsymbol{x} - \tilde{\boldsymbol{x}}$

Problems with the open-loop observer

- Observation error dynamics,

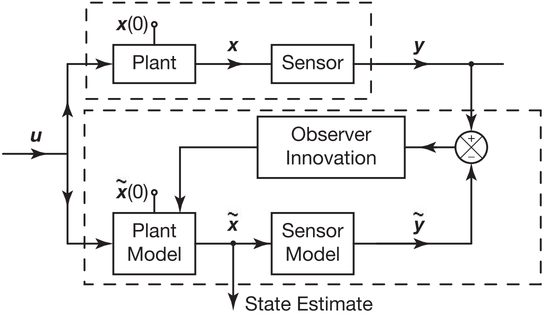
$$(\dot{\boldsymbol{x}} - \dot{\tilde{\boldsymbol{x}}}) = \boldsymbol{A}(\boldsymbol{x} - \tilde{\boldsymbol{x}})$$

- The observation error tends to zero only if the observed system is stable
- There is no control over the observation error dynamics
- There is a fix—add observer innovation to get the closed-loop observer

Open-loop observer analysis

- If the eigenvalues of the matrix $\boldsymbol{A}$ are in the open left-hand plane, then the error converges to zero
- However, we have no control over the convergence rate
- Furthermore, the matrix $\boldsymbol{A}$ does not have to have all its eigenvalues in the open left-hand plane
- Thus, the open-loop observer is impractical
- Modify this observer by adding a feedback term to it
- The resulting structure is called the closed-loop observer, or the Luenberger observer, or the asymptotic full-order observer

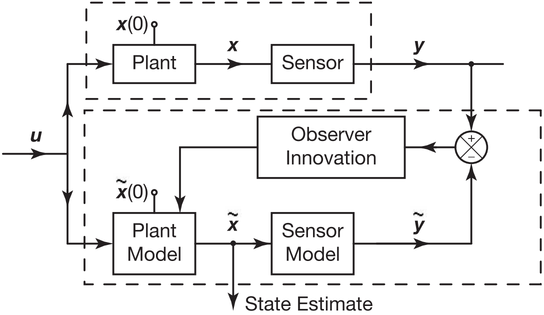
Closed-loop observer



- Luenberger's Innovation to obtain the closed-loop observer

$$\dot{\tilde{\boldsymbol{x}}} = \boldsymbol{A}\tilde{\boldsymbol{x}} + \boldsymbol{B}\boldsymbol{u} + \boldsymbol{L}(\boldsymbol{y} - \tilde{\boldsymbol{y}})$$

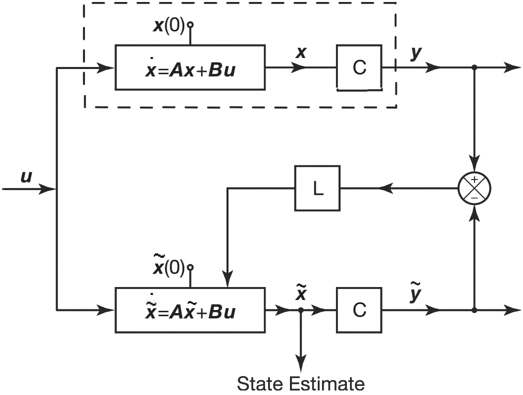
Closed-loop observer



- Luenberger's Innovation to obtain the closed-loop observer

$$\dot{\tilde{\boldsymbol{x}}} = \boldsymbol{A}\tilde{\boldsymbol{x}} + \boldsymbol{B}\boldsymbol{u} + \boldsymbol{L}(\boldsymbol{y} - \tilde{\boldsymbol{y}})$$

Luenberger's closed-loop observer



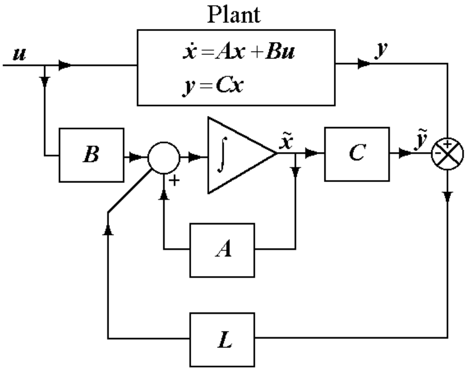
- Luenberger's observer, $\dot{\tilde{\boldsymbol{x}}} = \boldsymbol{A}\tilde{\boldsymbol{x}} + \boldsymbol{B}\boldsymbol{u} + \boldsymbol{L}(\boldsymbol{y} - \tilde{\boldsymbol{y}})$
- Observation error dynamics, $(\dot{\boldsymbol{x}} - \dot{\tilde{\boldsymbol{x}}}) = (\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C})(\boldsymbol{x} - \tilde{\boldsymbol{x}})$

Closed-loop observer

- The closed-loop observer

$$\dot{\tilde{\boldsymbol{x}}}(t) = \boldsymbol{A}\tilde{\boldsymbol{x}}(t) + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{L}\left(\boldsymbol{y}(t) - \tilde{\boldsymbol{y}}(t)\right)$$

where $\tilde{\boldsymbol{y}}(t) = \boldsymbol{C}\tilde{\boldsymbol{x}}(t)$



Closed-loop observer analysis

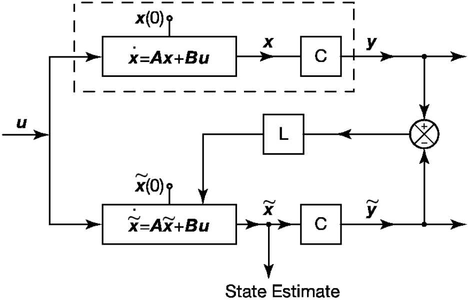
- The dynamics of the observation error

$$\begin{aligned}\dot{\boldsymbol{e}}(t) &= \dot{\boldsymbol{x}}(t) - \dot{\tilde{\boldsymbol{x}}}(t) \\ &= (\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C})\,\boldsymbol{e}(t), \quad \boldsymbol{e}(0) = \boldsymbol{x}(0) - \tilde{\boldsymbol{x}}(0).\end{aligned}$$

- The pair $(\boldsymbol{A}, \boldsymbol{C})$ is observable, if and only if the dual pair $(\boldsymbol{A}^\top, \boldsymbol{C}^\top)$ is reachable
- By assumption, the pair $(\boldsymbol{A}, \boldsymbol{C})$ is observable, and therefore the pair $(\boldsymbol{A}^\top, \boldsymbol{C}^\top)$ is reachable
- Thus, we can solve the pole placement problem for the dual pair $(\boldsymbol{A}^\top, \boldsymbol{C}^\top)$
- That is, for any set of prespecified n complex numbers, symmetric with respect to the real axis, there is a matrix, call it $\boldsymbol{L}^\top$, such that the eigenvalues of $\boldsymbol{A}^\top - \boldsymbol{C}^\top\boldsymbol{L}^\top$ and hence of $\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C}$ are in the prespecified locations

On the design of the closed-loop observer

- If the pair (A, C) is observable, then in addition to forcing the observation error to converge to zero, we can also control its rate of convergence by appropriately selecting the eigenvalues of the matrix $A - LC$
- Computing the closed-loop observer gain matrix L—can use the same tools as for the construction of the gain matrix K in the linear SF controller design



Example

- Given an observable pair

$$\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & -21 & 5 \end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Construct $\boldsymbol{L} \in \mathbb{R}^{4\times 2}$ so that the eigenvalues of $\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C}$ are located at

$$\{-2, -3+j, -3-j, -4\}.$$

- Our goal then is to construct $\boldsymbol{L}$ so that the characteristic polynomial of $\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C}$ is

$$\det(s\boldsymbol{I}_4 - \boldsymbol{A} + \boldsymbol{L}\boldsymbol{C}) = s^4 + 12s^3 + 54s^2 + 108s + 80$$

Closed-loop observer

- One possible choice

$$\boldsymbol{L} = \begin{bmatrix} 3.5029 & -0.4044 \\ 0.6545 & -2.2111 \\ -0.4807 & -2.9293 \\ -0.7599 & 16.4971 \end{bmatrix}$$

- Other possible gain matrices $\boldsymbol{L}$ could be used to allocate the eigenvalues of $\boldsymbol{A} - \boldsymbol{LC}$ into desired locations for multi-output systems