Case Study

The purpose of this case study is to show that if s_i is an eigenvalue of the matrix

$$A - BR^{-1}B^{\top}P,$$

and $v_i \neq 0$ is the corresponding eigenvector, where $P = P^{\top} \succ 0$ is the symmetric positive definite solution of the algebraic Riccati equation, then

$$\begin{bmatrix} v_i \\ Pv_i \end{bmatrix}$$

is an eigenvector of the associated Hamiltonian matrix corresponding to the eigenvalue s_i.

Indeed, we have

$$\begin{aligned}\begin{bmatrix} A & -BR^{-1}B^{\top} \\ -Q & -A^{\top} \end{bmatrix}\begin{bmatrix} v_i \\ Pv_i \end{bmatrix} &= \begin{bmatrix} Av_i - BR^{-1}B^{\top}Pv_i \\ -Qv_i - A^{\top}Pv_i \end{bmatrix} \\ &= \begin{bmatrix} \left(A - BR^{-1}B^{\top}P\right)v_i \\ P\left(A - BR^{-1}B^{\top}P\right)v_i \end{bmatrix}\end{aligned}$$

because P is the solution of the algebraic Riccati equation

$$A^{\top}P + PA + Q - PBR^{-1}B^{\top}P = O,$$

and hence

$$-Q - A^{\top}P = PA - PBR^{-1}B^{\top}P.$$

Since s_i is an eigenvalue of the matrix $A - BR^{-1}B^{\top}P$ and $v_i \neq 0$ is the corresponding eigenvector, we have

$$\begin{bmatrix} A & -BR^{-1}B^{\top} \\ -Q & -A^{\top} \end{bmatrix}\begin{bmatrix} v_i \\ Pv_i \end{bmatrix} = \begin{bmatrix} \left(A - BR^{-1}B^{\top}P\right)v_i \\ P\left(A - BR^{-1}B^{\top}P\right)v_i \end{bmatrix} = s_i\begin{bmatrix} v_i \\ Pv_i \end{bmatrix}.$$

Because by assumption $v_i \neq 0$, the above means that s_i is an eigenvalue of the Hamiltonian matrix and

$$\begin{bmatrix} v_i \\ Pv_i \end{bmatrix}$$

is the corresponding eigenvector.

PURDUE
UNIVERSITY.