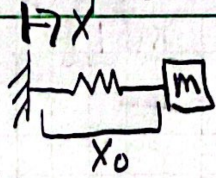
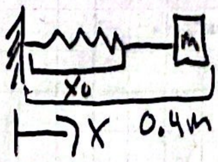
2.13)





a) Linear spring equation of motion

$$F_x = m\ddot{x} = -K(x - x_0) \Rightarrow \boxed{\ddot{x} = \frac{-K}{m}(x - x_0)}$$

Let $\underline{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} x \\ \dot{x} \end{pmatrix}$

This implies: $\boxed{\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= \frac{-K}{m}(z_1 - x_0) \end{aligned}}$

Non-linear spring equation of motion (scalar form)

$$F_x = m\ddot{x} = -K(x - x_0) - C(x - x_0)^3$$

$$\Rightarrow \boxed{\ddot{x} = \frac{-K}{m}(x - x_0) - \frac{C}{m}(x - x_0)^3}$$

Let $\underline{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} x \\ \dot{x} \end{pmatrix}$

This implies:

$$\boxed{\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= -\frac{K}{m}(z_1 - x_0) - \frac{C}{m}(z_1 - x_0)^3 \end{aligned}}$$

b) Integrate EoM from $0 \Rightarrow 20s$. Assume $\dot{x}(0) = z_2 = 0$

$x_0 = 0.25$ m $\quad x(0) = z_1(0) = 0.4$ m $\quad K = 1$ N/m

$C = 5$ N/m^3 $\quad m = 1$ Kg. <u>See MATLAB Code!!</u>

<u>Nonlinear spring has higher frequency of oscillation.</u>

2.14)

Equation of motion for simple damped harmonic oscillator:

$\ddot{z} + 2\zeta\omega_0\dot{z} + \omega_0^2 = 0$ (Eqn 2.24)

Assume solution of form $z = Ae^{\lambda t}$ $\therefore$ $\dot{z} = A\lambda e^{\lambda t}$, $\ddot{z} = A\lambda^2 e^{\lambda t}$

This gives: $Ae^{\lambda t}(\lambda^2 + 2\zeta\omega_0\lambda + \omega_0^2) = 0$

For unique solution $Ae^{\lambda t} \neq 0$, $\therefore$ $\lambda^2 + 2\zeta\omega_0\lambda + \omega_0^2 = 0$

For critically damped case, $\zeta = 1$ $\therefore$ $\underline{\lambda_{1,2} = -\zeta\omega_0}$

For repeated real roots, solution takes form of:

$\underline{z = Ae^{\lambda t} + Bte^{\lambda t}}$

Therefore for critically damped case:

$z(t) = Ae^{-\zeta\omega_0 t} + Bte^{-\zeta\omega_0 t}$

Apply intial condition $z(0)$:

$z(0) = Ae^0 + B(0)e^0 \Rightarrow \boxed{z(0) = A}$ (1)

Apply initial condition $\dot{z}(0)$:

$\dot{z}(t) = -\zeta\omega_0 Ae^{-\zeta\omega_0 t} + Be^{-\zeta\omega_0 t} + Bt\zeta\omega_0 e^{-\zeta\omega_0 t}$

$\dot{z}(0) = -\zeta\omega_0 A + B \Rightarrow B = \dot{z}(0) + \zeta\omega_0 A$ (2)

Substitute (1) => (2): $\boxed{B = \dot{z}(0) + \zeta\omega_0 z(0)}$ (3)

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Substitute (1) & (3) into $z(t) = Ae^{-\xi\omega_0 t} + Bte^{-\xi\omega_0 t}$

Gives

$$\boxed{z(t) = z(0)e^{-\xi\omega_0 t} + \left(\dot{z}(0) + z(0)\xi\omega_0\right)te^{-\xi\omega_0 t}}$$

Matches Eqn 2.26

2.18)

Atmospheric density: $\rho = \rho_0 e^{-y/h}$

$$f_y = \frac{-GMm}{(R_e+y)^2} + c\dot{y}^2 \quad \text{(Force acting on mass m)}$$

$$c = \tfrac{1}{2}\rho C_D A = \tfrac{1}{2}\rho_0 e^{-y/h} C_D A$$

__Knowns:__ $c(0) = 0.05\ \frac{Kg}{M}$ at sea level, $h = 7000\ m$,

$\rho_0 = 1.2\ Kg/m^3$, $M = 5.9742 \times 10^{24}\ Kg$,

$R_e = 6378100\ m$, $G = 6.673 \times 10^{-11}\ \frac{m^3}{Kg\,s^2}$, $m = 10\ Kg$

Assuming C_D & A to be constant:

$$c(0) = .05 = \tfrac{1}{2}\rho_0 e^0 C_D A \Rightarrow \tfrac{1}{2}C_D A = \frac{.05}{\rho_0}$$

$$c = \frac{.05}{\rho_0}\rho_0 e^{-y/h} \Rightarrow \underline{c(y) = .05e^{-y/h}}$$

$$\therefore f_y = \frac{-GMm}{(R_e+y)^2} + .05e^{-y/h}\dot{y}^2 = m\ddot{y} \quad \text{(Scalar form)}$$

$$m\ddot{y} = \frac{-GMm}{(R_e+y)^2} + .05e^{-y/h}\dot{y}^2$$

$$\boxed{\ddot{y} = \frac{-GM}{(R_e+y)^2} + \frac{.05}{m}e^{-y/h}\dot{y}^2}$$

Let $\underline{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} y \\ \dot{y} \end{pmatrix}$

2.18)

$$\dot{\underline{z}} = \begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} z_2 \\ \frac{-GM}{(R_e+z_1)^2} + \frac{.05}{m} e^{-z_1/h} z_2^2 \end{pmatrix}}_{\text{Equation in 1st order form}}$$

See Mattab code for trajectories

$$\underline{z}(0) = \begin{pmatrix} 1000 \\ 0 \end{pmatrix}$$

The terminal velocity acheived in the simulation is ~ -46 m/s, similiar to that observed in figure 2.11 (~44 m/s). There is about a 3% difference in terminal velocity. However, unlike figure 2.11, the velocity acheived by the particle doesn't stay constant at the asymptote terminal velocity, our particle reaches terminal velocity then proceeds to slightly slow down as the final velocity at the end of the simulation is ~ -45 m/s.

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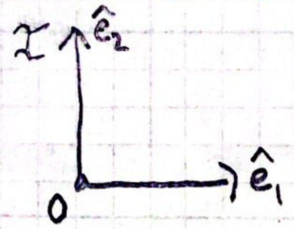
2.19)

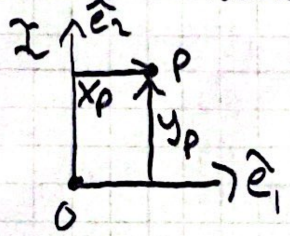
Define Reference Frame: (Origin O at ground)



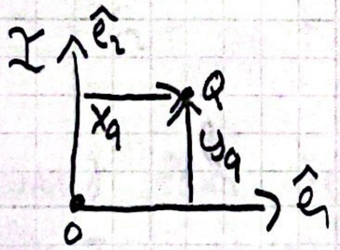
$\mathcal{I} = (O, \hat{e}_1, \hat{e}_2)$

Define Coordinate System: (2D planar motion)

↳ Constraint $z = 0$



$(x_p, y_p)_{\mathcal{I}}$



$(x_q, y_q)_{\mathcal{I}}$

FBD Mass P: (Neglect Drag)

$\underline{W} = -m_p g \hat{e}_2$

FBD Mass Q: (Neglect Drag)

$\underline{W} = -m_q g \hat{e}_2$

Newtons 2nd law for mass P (Scalar Form):

$$m_p \ddot{x}_p = 0 = f_{x_p}$$

$$m_p \ddot{y}_p = -m_p g = f_{y_p}$$

Equations of motion for mass P:

$$\ddot{x}_p = 0$$
$$\ddot{y}_p = -g$$

Solve: $\ddot{y}_p = \frac{d\dot{y}_p}{dt} = -g$

$$\int d\dot{y}_p = \int_0^t -g\,dt \Rightarrow \dot{y}_p + c_1 = -gt$$

$$\frac{dy_p}{dt} + c_1 = -gt$$

$$\int dy_p + \int_0^t c_1\,dt = \int_0^t -gt\,dt$$

$$c_2 + y_p + c_1 t = \frac{-gt^2}{2}$$

Dropped from height h at rest $\therefore$ $y_p(0) = h$ & $\dot{y}_p(0) = 0$

Apply IC: $y_p(0) = h$, $\dot{y}_p(0) = 0$

$$c_2 + h + (0)(t) = 0 \Rightarrow c_2 = -h$$
$$0 + c_1 = -g(0) \Rightarrow c_1 = 0$$
$$\therefore -h + y_p = \frac{-gt^2}{2} \Rightarrow$$

$$\boxed{y_p(t) = \frac{-gt^2}{2} + h}$$

$$\ddot{x}_p = 0 = \frac{d\dot{x}_p}{dt} \qquad \int d\dot{x}_p = \int_0^t 0\,dt \Rightarrow \dot{x}_p + c_1 = 0$$

$$\frac{dx_p}{dt} + c_1 = 0 \qquad \int dx_p + \int_0^t c_1 t = \int_0^t 0\,dt \Rightarrow$$

$$x_p + c_2 + c_1 t = 0$$

Dropped from rest from $x_p(0)$, $\therefore$ $\dot{x}_p(0) = 0$

Apply IC $\dot{x}_p(0)$ & $x_p(0)=0$:

$$x_p(0) + c_2 + c_1(0) = 0 \Rightarrow c_2 = -x_p(0)$$

$$\dot{x}_p(0) + c_1 = 0 \Rightarrow 0 + c_1 = 0 \Rightarrow c_1 = 0 \quad \therefore$$

$$x_p + (-x_p(0)) = 0$$

$$\boxed{x_p(t) = x_p(0)}$$

<u>Newtons 2^{nd} law for mass Q in scalar form</u>:

$$m_q \ddot{x}_q = 0 \Rightarrow \ddot{x}_q = 0 \quad (\text{Equation of motion in } \hat{e}_1)$$

$$m_q \ddot{y}_q = -m_q g \Rightarrow \ddot{y}_q = -g \quad (\text{Equation of motion in } \hat{e}_2)$$

Solve: $\ddot{x}_q = \frac{d\dot{x}_q}{dt} = 0 \Rightarrow \int d\dot{x}_q = \int_0^t 0\,dt = \dot{x}_q + c_1 = 0$

$$\frac{dx_q}{dt} + c_1 = 0 \Rightarrow \int dx_q + \int_0^t c_1\,dt = \int_0^t 0\,dt \Rightarrow$$

$$x_q + c_2 + c_1 t = 0$$

Launched from $x_q(0)$ with speed V_0 $\therefore$ IC $\dot{x}_q(0) = V_0$

$$x_q(0) + c_2 + c_1(0) = 0 \Rightarrow c_2 = -x_q(0)$$

$$\dot{x}_q(0) + c_1 = 0 = V_0 + c_1 \Rightarrow c_1 = -V_0$$

$$x_q - x_q(0) - V_0 t = 0$$

$$\boxed{x_q(t) = x_q(0) + V_0 t}$$

$$\ddot{y}_q = \frac{d\dot{y}_q}{dt} = -g \Rightarrow \int d\dot{y}_q = \int_0^t -g\,dt = \dot{y}_q + c_1 = -gt$$

$$\Rightarrow \int y_q + \int_0^t c_1\,dt = \int_0^t -gt = y_q + c_2 + c_1 t = -\frac{gt^2}{2}$$

Dropped from $y_q = h$ at $t=0$ $\therefore$ Apply $y_q(0)=h$: $h + c_2 + c_1(0) = -\frac{g(0)^2}{2}$

$\therefore \quad c_2 = -h \quad (h + c_2 = 0)$

Apply IC $\dot{y}_q(0) = 0$: (Dropped from rest)

$C_1 + \dot{y}_q = -gt$

$\dot{y}_q(0) = 0 = C_1 \quad \Rightarrow C_1 = 0 \qquad y_q + \cancel{c_2}^{-h} + \cancel{c_1}^{0}t = -\frac{gt^2}{2}$

$$\therefore \boxed{y_q(t) = h - \frac{gt^2}{2}}$$

Solve $y_q(t)$ and $y_p(t)$ for time when ball hits ground (ie $y_q = y_p = 0$)

$y_q = 0 = h - \frac{gt^2}{2}$

$\boxed{t_q = \sqrt{\frac{2h}{g}}}$ - time for mass q to hit ground

$y_p = 0 = -\frac{gt^2}{2} + h$

$\boxed{t_p = \sqrt{\frac{2h}{g}}}$ - time for mass p to hit ground

It can be shown that by neglecting drag, both mass p and q will hit the ground at the same time if dropped from the same height h. The time for impact is given by $t = \sqrt{\frac{2h}{g}}$

Contents

MAE 562 HW1 Gabriel Colangelo 50223306

```
clear
close all
clc
```

Problem 2.13

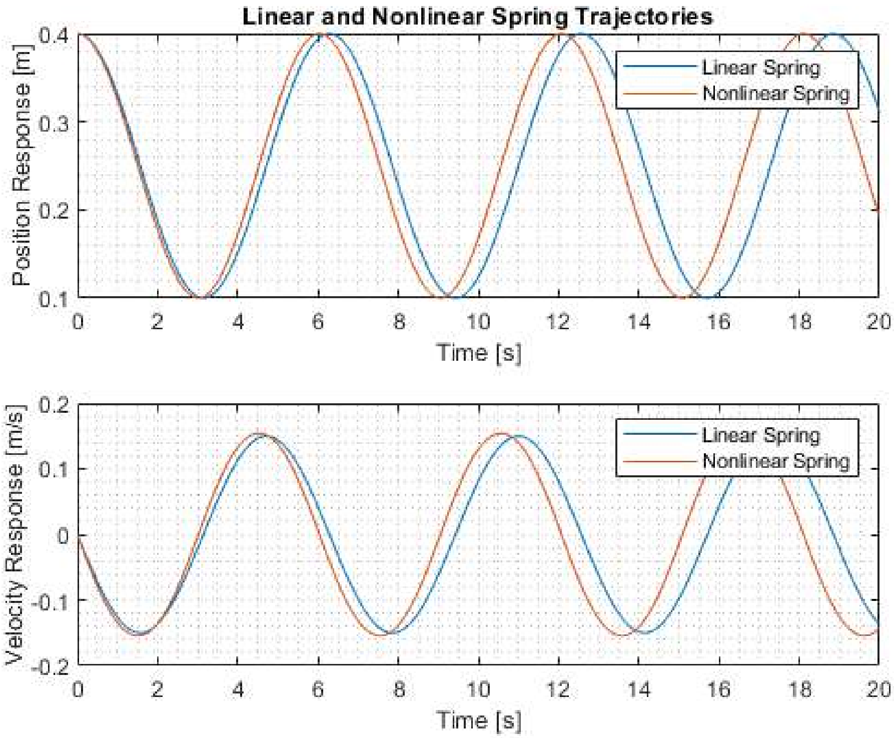
```
m       = 1                                                          ; % mass [kg]
x0      = 0.25                                                       ; % unstretched spring length [m]
k       = 1                                                          ; % Spring stiffness [N/m]
c       = 5                                                          ; % Spring constant [N/m^3]

IC      = [.4 0]'                                                    ; % Initial conditions of 0.4 [m] and 0 [m/s]
time    = (0:.01:20)'                                                ; % Time vector for 0-20 [s]
options = odeset('AbsTol',1e-8,'RelTol',1e-8)                        ; % ODE45 solver options

[T1,Z1] = ode45(@(t,z) LinearSpring(t,z,m,x0,k),time,IC,options)     ; % Linear Spring Simulation
[T2,Z2] = ode45(@(t,z) NonlinearSpring(t,z,m,x0,k,c),time,IC,options)   ; % NonLinear Spring Simulation

figure
ax1 = subplot(2,1,1);
plot(T1,Z1(:,1),T2,Z2(:,1))
xlabel('Time [s]')
ylabel('Position Response [m]')
grid minor
title('Linear and Nonlinear Spring Trajectories')
legend('Linear Spring','Nonlinear Spring')

ax2 = subplot(2,1,2);
plot(T1,Z1(:,2),T2,Z2(:,2))
xlabel('Time [s]')
ylabel('Velocity Response [m/s]')
grid minor
legend('Linear Spring','Nonlinear Spring')
linkaxes([ax1 ax2],'x')
```



Problem 2.18

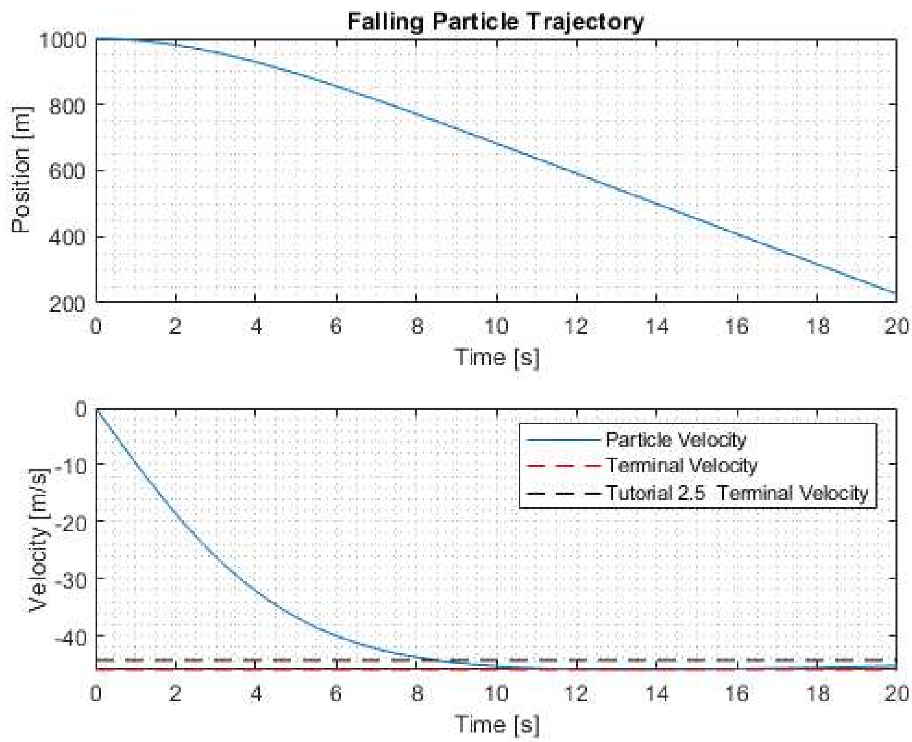
```
M       = 5.9742e24                                                    ; % Mass of Earth [kg]
Re      = 6378100                                                      ; % Radius of Earth [m]
G       = 6.673e-11                                                    ; % Gravitational Constant [m^3/kg-s^2]
h       = 7000                                                         ; % Scale height of atmoshere [m]
m       = 10                                                           ; % Particle mass [kg]
IC      = [1000 0]'                                                    ; % Initial condition of 1000 m and 0 m/s

[T3,Z3] = ode45(@(t,z) FallingParticle(t,z,M,G,Re,m,h),time,IC,options)     ; % Linear Spring Simulation

figure
ax1 = subplot(2,1,1);
plot(T3,Z3(:,1))
xlabel('Time [s]')
ylabel('Position [m]')
grid minor
title('Falling Particle Trajectory')

ax2 = subplot(2,1,2);
plot(T3,Z3(:,2))
line([0 20],[min(Z3(:,2)) min(Z3(:,2))],'Color','red','LineStyle','--')
line([0  20],[-44.2945  -44.2945],'Color','black','LineStyle','--')
xlabel('Time [s]')
legend('Particle Velocity','Terminal Velocity','Tutorial 2.5  Terminal Velocity')
ylabel('Velocity [m/s]')
grid minor

linkaxes([ax1 ax2],'x')
```



Function Definitions

```
function zdot = LinearSpring(t,z,m,x0,k)
z1          = z(1,1); % z1 = x
z2          = z(2,1); % z2 = xdot

% Equations of motion is first order form
zdot(1,1)   = z2;
zdot(2,1)   = -k/m*(z1 - x0);

end

function zdot = NonlinearSpring(t,z,m,x0,k,c)
z1          = z(1,1); % z1 = x
z2          = z(2,1); % z2 = xdot

% Equations of motion is first order form
zdot(1,1)   = z2;
zdot(2,1)   = (-k/m*(z1 - x0)) -(c/m*(z1 - x0)^3);

end

function zdot = FallingParticle(t,z,M,G,Re,m,h)
z1          = z(1,1); % z1 = y
z2          = z(2,1); % z2 = ydot

% Equations of motion is first order form
zdot(1,1)   = z2;
zdot(2,1)   = (-G*M/(Re + z1)^2) + (z2^2*(.05/m)*exp(-z1/h));
end
```

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