6.3.2

a) $A = \begin{pmatrix} 1 & 4 \\ 2 & 8 \end{pmatrix}$

$$AA^T = \begin{pmatrix} 1 & 4 \\ 2 & 8 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 4 & 8 \end{pmatrix} = \boxed{\begin{pmatrix} 17 & 34 \\ 34 & 68 \end{pmatrix}}$$

$$|A-\lambda I| = 0 \Rightarrow \lambda^2 - 85\lambda = 0$$

$$\boxed{\begin{aligned} \sigma_1^2 = \lambda_1 &= 85 \\ \lambda_2 &= 0 \end{aligned}}$$

$$(AA^T - \lambda_1 I)V_1 = 0$$

$$\begin{pmatrix} -68 & 34 \\ 34 & -17 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \vec{0}$$

$$-68x_1 + 34 = 0 \qquad x_1 = 1/2$$

$$x_2 = 1$$

$$V_1 = \begin{pmatrix} 1/2 \\ 1 \end{pmatrix} \qquad V_1^T V_1 = \frac{5}{4} \qquad \|V_1\| = \frac{\sqrt{5}}{2}$$

$$\boxed{U_1 = \begin{pmatrix} 1/\sqrt{5} \\ \frac{2}{\sqrt{5}} \end{pmatrix}}$$

$$(AA^T - \lambda_2 I)V_2 = \vec{0}$$

$$\begin{pmatrix} 17 & 34 \\ 34 & 68 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \vec{0}$$

6.3.2

$17x_1 + 34 = 0 \qquad x_2 = 1$

$x_1 = -2$

$v_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \qquad v_2^T v_2 = 5 \qquad \|v_2\| = \sqrt{5}$

$$\boxed{u_2 = \begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}}$$

b) From 1) $v_1 = \begin{pmatrix} 1/\sqrt{17} \\ 4/\sqrt{17} \end{pmatrix} \qquad v_2 = \begin{pmatrix} 4/\sqrt{17} \\ -1/\sqrt{17} \end{pmatrix}$

$$Av_1 = \begin{pmatrix} 1 & 4 \\ 2 & 8 \end{pmatrix} \begin{pmatrix} 1/\sqrt{17} \\ 4/\sqrt{17} \end{pmatrix} = \begin{pmatrix} 17/\sqrt{17} \\ 34/\sqrt{17} \end{pmatrix} = \underline{\begin{pmatrix} \sqrt{17} \\ 2\sqrt{17} \end{pmatrix}}$$

$\sigma_1 = \pm\sqrt{85} = \pm\sqrt{17}\sqrt{5}$

$$\sigma_1 u_1 = \sqrt{17} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \underline{\begin{pmatrix} \sqrt{17} \\ 2\sqrt{17} \end{pmatrix}}$$

$\therefore \boxed{Av_1 = \sigma_1 u_1}$

6.3.2

$$[u_1\ u_2] \begin{pmatrix} \sigma_1 & 0 \\ 0 & 0 \end{pmatrix} [v_1\ v_2]^T$$

$$= \begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix} \begin{pmatrix} \sqrt{85} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{17} & 4/\sqrt{17} \\ 4/\sqrt{17} & -1/\sqrt{17} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\sqrt{85}}{\sqrt{5}} & 0 \\ \frac{2\sqrt{85}}{\sqrt{5}} & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{17} & 4/\sqrt{17} \\ 4/\sqrt{17} & -1/\sqrt{17} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\sqrt{85}}{\sqrt{85}} & 4\frac{\sqrt{85}}{\sqrt{85}} \\ \frac{2\sqrt{85}}{\sqrt{85}} & \frac{8\sqrt{85}}{\sqrt{85}} \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} 1 & 4 \\ 2 & 8 \end{pmatrix} = A}$$

c) $A = \begin{pmatrix} 1 & 4 \\ 2 & 8 \end{pmatrix} \Rightarrow \begin{pmatrix} \boxed{1} & 4 \\ 0 & 0 \end{pmatrix}$

$C(A) = \text{span}\left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right\}$

$$\boxed{C(A) = u_1 = \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}}$$

$C(A^T) = \text{span}\left\{\begin{pmatrix} 1 \\ 4 \end{pmatrix}\right\}$

6.3.2

$$\boxed{C(A^T) = \begin{pmatrix} 1/\sqrt{17} \\ 4/\sqrt{17} \end{pmatrix} = v_1}$$

$$\begin{pmatrix} 1 & 4 \\ 2 & 8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \vec{0}$$

$x_1 + 4x_2 = 0$ $\quad\quad$ $x_2 = 1, \; x_1 = -4$

$$N(A) = \text{span}\left\{ \begin{pmatrix} 1 \\ -4 \end{pmatrix} \right\}$$

$$\boxed{N(A) = \begin{pmatrix} -4/\sqrt{17} \\ 1/\sqrt{17} \end{pmatrix} = v_2}$$

$$A^T = \begin{pmatrix} 1 & 2 \\ 4 & 8 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$$

$x_1 + 2x_2 = 0$ $\quad\quad$ $x_2 = 1, \; x_1 = -2$

$$N(A^T) = \text{span}\left\{ \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\}$$

$$\boxed{N(A^T) = u_2 = \begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}}$$

6.3.15

$A_{1\times 4} \therefore V_{4\times 4},\ U_{1\times 1},\ \Sigma_{1\times 4}$

$A = [1\ 1\ 1\ 1]$

$A^T A = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}[1\ 1\ 1\ 1] = \begin{pmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{pmatrix}$

$\text{rank}(A^TA) = 1$ as $A^TA \Rightarrow \begin{pmatrix}1&1&1&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$

$\text{trace} = 4 \therefore \Sigma\lambda_i = 4$

A^TA is symmetric, $\therefore$ $\text{rank}(A^TA) = \#$ of non-zero e-vals

$\text{rank} = 1,\ m = 4 \therefore \lambda_1 = \lambda_2 = \lambda_3 = 0.$

$\Sigma\lambda_i = 4 \therefore \lambda_4 = 4.$

$(A^TA - \lambda_1)V_1 = \vec{0}$

$\begin{pmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} = \vec{0} \Rightarrow x_1 + x_2 + x_3 + x_4 = 0$

Prefferred solution let $x_2 = 1,\ x_3 = x_4 = 0 \therefore x_1 = -1$

$V_1 = \begin{pmatrix}-1\\1\\0\\0\end{pmatrix}$

let $x_3 = 1,\ x_2 = x_4 = 0 \therefore x_1 = -1$

$V_2 = \begin{pmatrix}-1\\0\\1\\0\end{pmatrix}$

6.3.15

let $X_4 = 1$, $X_2 = X_3 = 0$ $\therefore X_1 = -1$

$$V_3 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$(A^TA - 4I)V_4 = \vec{0}$$

$$\begin{pmatrix} -3 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -3 \end{pmatrix} \begin{pmatrix} X_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$$\begin{pmatrix} -3 & 1 & 1 & 1 \\ 0 & -8/3 & 4/3 & 4/3 \\ 1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -3 \end{pmatrix} \Rightarrow \begin{pmatrix} -3 & 1 & 1 & 1 \\ 0 & -8/3 & 4/3 & 4/3 \\ 0 & 4/3 & -8/3 & 4/3 \\ 0 & 4/3 & 4/3 & -8/3 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -3 & 1 & 1 & 1 \\ 0 & -8/3 & 4/3 & 4/3 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 2 & -2 \end{pmatrix} \Rightarrow \begin{pmatrix} -3 & 1 & 1 & 1 \\ 0 & -8/3 & 4/3 & 4/3 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix}$$

$X_4 = 1$ $\therefore$ $-2X_3 + 2 = 0$ $X_3 = 1$

$-\frac{8}{3}X_2 + 4/3 + 4/3 = 0$ $X_2 = 1$

$-3X_1 + 3 = 0$ $X_1 = 1$

$$V_4 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \overset{\text{Normalize}}{\Rightarrow} V_4 = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}$$

6.3.15

$V_1^T V_2 = [-1\ 1\ 0\ 0]\begin{bmatrix}-1\\0\\1\\0\end{bmatrix} = 1$ $\therefore$ not normal use

gram - schmidt

$$q_1 = \frac{V_1}{\|V_1\|} = \begin{pmatrix}-1/\sqrt{2}\\1/\sqrt{2}\\0\\0\end{pmatrix}$$

$$q_2 = V_2 - (q_1^T V_2) q_1$$

$$= \begin{pmatrix}-1\\0\\1\\0\end{pmatrix} - \left[\begin{pmatrix}-1/\sqrt{2} & 1/\sqrt{2} & 0 & 0\end{pmatrix}\begin{pmatrix}-1\\0\\1\\0\end{pmatrix}\right]\begin{pmatrix}-1/\sqrt{2}\\1/\sqrt{2}\\0\\0\end{pmatrix}$$

$$= \begin{pmatrix}-1\\0\\1\\0\end{pmatrix} - \left[(1/\sqrt{2})\begin{pmatrix}-1/\sqrt{2}\\1/\sqrt{2}\\0\\0\end{pmatrix}\right] = \begin{pmatrix}-\frac{1}{2}\\-1/2\\1\\0\end{pmatrix}$$

$$\|q_2\| = \sqrt{\frac{1}{4} + \frac{1}{4} + 1} = \sqrt{6/4} = \frac{\sqrt{6}}{2}$$

$$V_2 = \begin{pmatrix}-1/\sqrt{6}\\-1/\sqrt{6}\\2/\sqrt{6}\\0\end{pmatrix}$$

$$q_3 = V_3 - (q_1^T V_3) q_1 - (V_2^T V_3) V_2$$

$$= \begin{pmatrix}-1\\0\\0\\1\end{pmatrix} - \begin{pmatrix}-1/2\\1/2\\0\\0\end{pmatrix} - \left[\begin{pmatrix}-1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} & 0\end{pmatrix}\begin{pmatrix}-1\\0\\0\\1\end{pmatrix}\right] V_2$$

$$= \begin{pmatrix}-1/2\\-1/2\\0\\1\end{pmatrix} - (1/\sqrt{6})\begin{pmatrix}-1/\sqrt{6}\\-1/\sqrt{6}\\2/\sqrt{6}\\0\end{pmatrix} = \begin{pmatrix}-1/2\\-1/2\\0\\1\end{pmatrix} - \begin{pmatrix}-1/6\\-1/6\\2/6\\1\end{pmatrix} = \begin{pmatrix}-2/6\\-2/6\\-2/6\\1\end{pmatrix}$$

6.3.15

$$q_3 = \begin{pmatrix} -1/3 \\ -1/3 \\ -1/3 \\ 1 \end{pmatrix}$$

$$\| q_3 \| = 2/\sqrt{3} = \sqrt{1 + 1/3} = \sqrt{4/3}$$

$$v_3 = \begin{pmatrix} -\sqrt{3}/6 \\ -\sqrt{3}/6 \\ -\sqrt{3}/6 \\ \sqrt{3}/2 \end{pmatrix}$$

$$\boxed{V = \begin{pmatrix} 1/2 & -1/\sqrt{2} & -1/\sqrt{6} & -\sqrt{3}/6 \\ 1/2 & 1/\sqrt{2} & -1/\sqrt{6} & -\sqrt{3}/6 \\ 1/2 & 0 & 2/\sqrt{6} & -\sqrt{3}/6 \\ 1/2 & 0 & 0 & \sqrt{3}/2 \end{pmatrix}}$$

$$\Sigma = [\sigma_1 \ 0 \ 0 \ 0] \qquad \sigma_1 = \sqrt{4}$$

$$\boxed{\Sigma = [2 \ 0 \ 0 \ 0]}$$

$$A v_4 = \sigma_1 u_1$$

$$[1 \ 1 \ 1 \ 1] \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix} = 2 = 2 u_1$$

$$\boxed{U = 1}$$

6.3.15

$B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ $\quad B_{2\times 3} \therefore V_{3\times 3}, \Sigma_{2\times 3}, U_{2\times 2}$

$B^T B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$|B^T B - \lambda I| = 0 \quad = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix}$

$= (1-\lambda) \begin{vmatrix} 1-\lambda & 0 \\ 0 & -\lambda \end{vmatrix} - 0 - 0$

$= (1-\lambda)(-\lambda + \lambda^2) = -\lambda^3 + \lambda^2 - \lambda + \lambda^2 = -\lambda^3 + \lambda^2 - \lambda$

$= -\lambda(\lambda^2 - \lambda + 1) = 0$

$-\lambda(\lambda - 1)^2 \quad \lambda_1 = \lambda_2 = 1, \lambda_3 = 0$

$(B^T B - \lambda_1 I) V_1 = 0$

$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \qquad -x_3 = 0$

Let $x_1 = 1, x_2 = 0 \therefore x_3 = 0$

$V_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Let $x_1 = 0, x_2 = 1 \therefore x_3 = 0$

$V_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

V_1 & V_2 are orthonormal

$(B^T B - \lambda_3) V_1 = 0$

6.3.15

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \vec{0}$$

$x_1 = 0$
$x_2 = 0$

$x_3 = 1 \quad \therefore \quad x_2 = 0, \; x_1 = 0$

$$V_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\boxed{V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

$\sigma_1 = \sqrt{1} \qquad \sigma_2 = \sqrt{1}$

$$\boxed{\Sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}$$

$B v_1 = \sigma_1 u_1$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 1\, u_1 \Rightarrow u_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$B v_2 = \sigma_2 u_2$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 1\, u_2 \Rightarrow u_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\boxed{U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}$$

6.3.15

$C = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad C_{2\times2} : V_{2\times2}, \Sigma_{2\times2}, U_{2\times2}$

$C^T C = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

$\lambda^2 - 2\lambda = 0 \Rightarrow \lambda_1 = 2, \quad \lambda_2 = 0$

$(C^T C - \lambda_1 I) V_1 = 0$

$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \quad \Rightarrow V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \xrightarrow{\text{Normalize}} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$

$(C^T C - \lambda_2 I) V_2 = 0$

$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \quad \Rightarrow V_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \xrightarrow{\text{Normalize}} \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$

$\sigma_1 = \sqrt{2}$

$$\boxed{V = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \qquad \Sigma = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{pmatrix}}$$

$C V_1 = \sigma_1 U_1$

$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 2/\sqrt{2} \\ 0 \end{pmatrix} = \sqrt{2} \quad U_1$

$U_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

6.3.15

$U_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ as it's orthonormal to U_1

$\therefore$ $\boxed{U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$

$C^+ = V\Sigma^+ U^T$

$$= \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/2 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\boxed{C^+ = \begin{pmatrix} 1/2 & 0 \\ 1/2 & 0 \end{pmatrix}}$$

$$B^+ = \underset{3\times3}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \underset{3\times2}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}} \underset{2\times2}{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}$$

$$\boxed{B^+ = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}}$$

$$A^+ = \begin{pmatrix} 1/2 & -1/\sqrt{2} & -1/\sqrt{6} & -\sqrt{3}/6 \\ 1/2 & 1/\sqrt{2} & -1/\sqrt{6} & -\sqrt{3}/6 \\ 1/2 & 0 & 2/\sqrt{6} & -\sqrt{3}/6 \\ 1/2 & 0 & 0 & \sqrt{3}/2 \end{pmatrix} \begin{pmatrix} 1/2 \\ 0 \\ 0 \\ 0 \end{pmatrix} 1 = \boxed{\begin{pmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{pmatrix} = A^+}$$

6.3.19

$$A^+ = (A^T A)^{-1} A^T$$

$$A^T A x^+ = A^T b$$

$$x^+ = (A^T A)^{-1} A^T b$$

$$x^+ = A^+ b$$

$$\therefore \boxed{A^+ = (A^T A)^{-1} A^T}$$

b)

$$A^+ = A^T (A A^T)^{-1}$$

$$A^T A x^+ = A^T b$$

$$\underbrace{A^{-T} A^T}_{I} A x^+ = b$$

$$A x^+ = b$$

$$x^+ = A^+ b$$

$$\boxed{x^+ = A^T (A A^T)^{-1} b}$$

6.4.1

$P(x) = \frac{1}{2} x^T A x - x^T b$

$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 4 \\ 0 \\ 4 \end{pmatrix}$

$\frac{1}{2}(x_1 \ x_2 \ x_3) \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} - (x_1 \ x_2 \ x_3) \begin{pmatrix} 4 \\ 0 \\ 4 \end{pmatrix}$

$\frac{1}{2} [(2x_1 - x_2), (-x_1 + 2x_2 - x_3), (-x_2 + 2x_3)] \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} - (4x_1 + 4x_3)$

$\frac{1}{2}(2x_1^2 - x_2 x_1 - x_1 x_2 + 2x_2^2 - x_3 x_2 - x_2 x_3 + 2x_3^2) - 4x_1 - 4x_3$

$\boxed{P(x) = x_1^2 - x_1 x_2 + x_2^2 - x_3 x_2 + x_3^2 - 4x_1 - 4x_3}$

$\frac{\partial P}{\partial x_1} = 2x_1 - x_2 - 4$

$\frac{\partial P}{\partial x_2} = -x_1 + 2x_2 - x_3$

$\frac{\partial P}{\partial x_3} = -x_2 + 2x_3 - 4$

$\frac{\partial P}{\partial x_1} = 0 \Rightarrow \underline{2x_1 - x_2 = 4}$

$\frac{\partial P}{\partial x_2} \underline{= 0 = -x_1 + 2x_2 - x_3}$

$\frac{\partial P}{\partial x_3} = 0 \Rightarrow \underline{-x_2 + 2x_3 = 4}$

6.4.11

$$R(x) = \frac{x_1^2 - x_1 x_2 + x_2^2}{x_1^2 + x_2^2} = \frac{x^T A x}{x^T x}$$

$$\underset{1\times 2 \times 2\times 2}{(x_1 \; x_2)} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x^T A x = (ax_1 + bx_2, \; bx_1 + cx_2) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$ax_1^2 + bx_2x_1 + bx_1x_2 + cx_2^2 = ax_1^2 + 2bx_2x_1 + cx_2^2$$

$\therefore$ $a = 1$, $b = -1/2$, $c = 1$

$$A = \begin{pmatrix} 1 & -1/2 \\ -1/2 & 1 \end{pmatrix}$$

$|A - \lambda I| = 0 \Rightarrow \lambda^2 - 2\lambda + 3/4$

$\lambda_1 = 1/2$, $\lambda_2 = 3/2$

$$\boxed{\min(R(x)) = 1/2 = \lambda_1}$$

$$R(x) = \frac{x_1^2 - x_1 x_2 + x_2^2}{2x_1^2 + x_2^2}$$

$x^T A x = x_1^2 - x_1 x_2 + x_2^2 \quad A = \begin{pmatrix} 1 & -1/2 \\ -1/2 & 1 \end{pmatrix}$

$x^T M x = 2x_1^2 + x_2^2$, $M = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$

$ax_1^2 + 2bx_2x_1 + cx_2^2 \quad \therefore \quad a = 2,\; b = 0,\; c = 1$

$$M = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

$\min(R(x)) = \lambda_1(M^{-1}A)$

6.4.11

$$M^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix}$$

$$M^{-1}A = \begin{pmatrix} 1/2 & -1/4 \\ -1/2 & 1 \end{pmatrix}$$

$$|M^{-1}A - \lambda I| = 0 = \lambda^2 - \frac{3}{2}\lambda + \frac{3}{8}$$

$$\lambda_{1,2} = \frac{\frac{3}{2} \pm \sqrt{\frac{9}{4} - \frac{12}{8}}}{2} = \frac{\frac{3}{2} \pm \frac{\sqrt{3}}{2}}{2}$$

$$\boxed{\lambda_1 = \frac{3 - \sqrt{3}}{4} = \min(R(x))}$$

6.4.16

$\sum \lambda_i = \text{trace}(A)$

$$A = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ \vdots & & 0 & 2 \\ & & & \\ 0 & 0 & 0 & \vdots \\ 1 & 2 & \cdots & n \end{pmatrix}$$

$\text{trace}(A) = n$

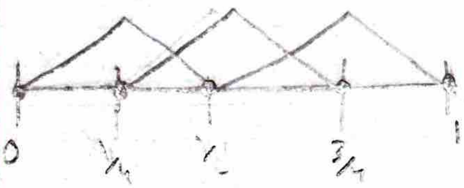
$\prod(\lambda_i) = \det(A) = 0$

$\text{Rank}(A) = 2$ = # of non-zero eigenvalues

$\therefore$ $\boxed{\text{all eigenvales are zero except for two positive ones}}$

6.5.1

$h = 1/4 \qquad u(0) = u(1) = 0 \qquad f = 2$



$A_{ii} = \frac{2}{h} \qquad A_{ij} = \frac{-1}{h}$

$$A = 4 \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

$$b_1 = \int V_1 f\,dx = \int_0^{1/4} \frac{(x-0)}{h} f\,dx + \int_{1/4}^{1/2} (1/2 - x)\frac{f}{h}\,dx$$

$$= \int_0^{1/4} 8x\,dx + \int_{1/4}^{1/2} 8(1/2 - x)\,dx$$

$$= \frac{8x^2}{2}\Big|_0^{1/4} + \left(4x - \frac{8x^2}{2}\right)\Big|_{1/4}^{1/2}$$

$$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$b_2 = \int V_2 f\,dx = \int_{1/4}^{1/2} \frac{2(x - 1/4)}{\frac{1}{4}}\,dx + \int_{1/2}^{3/4} \left(\frac{3}{4} - x\right)\frac{2}{1/4}\,dx$$

$$= \left(\frac{8x^2}{2} - 2x\right)\Big|_{1/4}^{1/2} + (6 - 8x)\Big|_{1/4} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$b_3 = \int V_3 f\,dx = \int_{1/2}^{3/4} 8(x - 1/2)\,dx + \int_{3/4}^{1} 8(1-x)\,dx$$

$$b_3 = \left(\frac{8x^2}{2} - 4x\right)\Big|_{1/2}^{3/4} + \left(8x - \frac{8x^2}{2}\right)\Big|_{3/4}^{1} = \frac{1}{4} + \frac{1}{4} = 1/2$$

6.5.1

$b = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}$

$y = A^{-1} b$

$C_{11} = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3$

$C_{12} = -\begin{vmatrix} -1 & -1 \\ 0 & 2 \end{vmatrix} = 2$

$C_{13} = \begin{vmatrix} -1 & 2 \\ 0 & -1 \end{vmatrix} = 1$

$C_{21} = -\begin{vmatrix} -1 & 0 \\ -1 & 2 \end{vmatrix} = 2$

$C_{22} = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4$

$C_{23} = -\begin{vmatrix} 2 & -1 \\ 0 & -1 \end{vmatrix} = 2$ $\qquad C_{33} = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3$

$C = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{pmatrix} \qquad C = C^T$

$\det(\underset{h}{A}) = 2C_{11} + -1\,C_{12} = 4$

$A^{-1} = \begin{pmatrix} 3/4 & 1/2 & 1/4 \\ 1/2 & 1 & 1/2 \\ 1/4 & 1/2 & 3/4 \end{pmatrix} \frac{1}{4} = \begin{pmatrix} 3/16 & 1/8 & 1/16 \\ 1/8 & 1/4 & 1/8 \\ 1/16 & 1/8 & 3/16 \end{pmatrix}$

$y = \begin{pmatrix} \frac{3}{32} + \frac{1}{16} + \frac{1}{32} \\ \frac{1}{16} + \frac{1}{8} + \frac{1}{16} \\ \frac{1}{32} + \frac{1}{16} + \frac{3}{32} \end{pmatrix} = \begin{pmatrix} 6/32 \\ 8/32 \\ 6/32 \end{pmatrix}$

$\boxed{y = \begin{pmatrix} 3/16 \\ 4/16 \\ 3/16 \end{pmatrix}}$

6.5.1

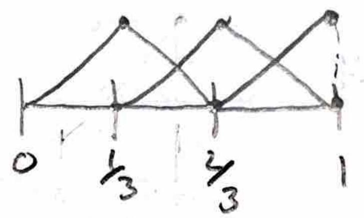
$U = X - X^2$

$U(1/4) = \frac{1}{4} - \left(\frac{1}{4}\right)^2 = \frac{4}{16} - \frac{1}{16} = 3/16$ ✓

$U(1/2) = \frac{1}{2} - \frac{1}{4} = 1/4 = 4/16$ ✓

$U(3/4) = \frac{3}{4} - \left(\frac{3}{4}\right)^2 = \frac{12}{16} - \frac{9}{16} = 3/16$ ✓

C.5.3



$$V_3 = \frac{(x - 2/3)}{\frac{1}{3}} = \frac{(x - x_{j-1})}{h}$$

$$V_3' = 3 = \frac{1}{h} \qquad A_{33} = \int (V_3')^2 \, dx$$

$$A_{33} = \int_{2/3}^{1} 9 \, dx = 9x \Big|_{2/3}^{1} = 3$$

$$\boxed{A_{33} = 3} \quad \text{or} \quad \frac{1}{h}$$

$$f_3 = \int_{2/3}^{1} 2 V_3 \, dx$$

$$f_3 = \int_{2/3}^{1} 6(x - 2/3) = \frac{6x^2}{2} - 4x \Big|_{2/3}^{1}$$

$$\boxed{f_3 = \frac{1}{3}}$$

6.5.3

$$A = 3\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \qquad f_i = \int h f v_i \, dx = \int 6 v_i \, dx$$

$$f_1 = \int_0^{1/3} 6x \, dx + \int_{1/3}^{2/3} 6\left(\tfrac{2}{3} - x\right) dx$$

$$f_1 = \frac{6x^2}{2}\Big|_0^{1/3} + 4x - \frac{6x^2}{2}\Big|_{1/3}^{2/3} = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

$$f_2 = \int_{1/3}^{2/3} 6\left(x - \tfrac{1}{3}\right) dx + \int_{2/3}^{1} 6(1-x)\, dx = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

$$f = \begin{pmatrix} 2/3 \\ 2/3 \\ 1/3 \end{pmatrix}$$

$$y = A^{-1} f$$

$$C_{11} = \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} = 1$$

$$C_{12} = -\begin{vmatrix} -1 & -1 \\ 0 & 1 \end{vmatrix} = 1$$

$$C_{13} = \begin{vmatrix} -1 & 2 \\ 0 & -1 \end{vmatrix} = 1$$

$$C_{21} = -\begin{vmatrix} -1 & 0 \\ -1 & 1 \end{vmatrix} = 1 \qquad C = C^T = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

$$C_{22} = \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} = 2$$

$$C_{23} = -\begin{vmatrix} 2 & -1 \\ 0 & -1 \end{vmatrix} = 2 \qquad \det\left(\tfrac{1}{h}A\right) = 2 - 1 = 1$$

$$C_{33} = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3$$

6.5.3

$$A^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

$$y = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 2/3 & 2/3 \\ 1/3 & 2/3 & 1 \end{pmatrix}\begin{pmatrix} 2/3 \\ 2/3 \\ 1/3 \end{pmatrix}$$

$$\boxed{y = \begin{pmatrix} 5/9 \\ 8/9 \\ 1 \end{pmatrix}}$$

6.5.5

$$\int f(x)\, v_j(x)\, dx = \int -y_1 v_1'' v_j \, dx + \int -y_2 v_2'' dx + \ldots$$

$$-y_i \int v_i'' v_j \, dx$$

Let $u = v_j \qquad dv = v_i''$

$\qquad u' = v_j' \qquad v = v_i'$

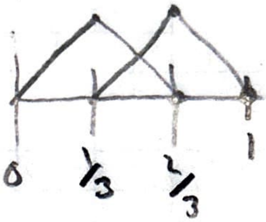
$$-y_i \int u\, dv = uv\Big|_0^1 - \int_0^1 v\, du = -\left(v_j v_i' - \int v_i' v_j' \, dx \right) y_i$$

$$y_i \left(-\cancel{v_j v_i'}^{\,0} + \int_0^1 v_i' v_j' \, dx \right)$$

$$A_{ij} = \int_0^1 v_i' v_j' \, dx$$

$$\therefore \boxed{A_{ij}\, y_i = f_i} \Rightarrow Ay = f$$

6.5.8



$h = \frac{1}{3}$

$$M_{11} = \int_0^1 V_1 V_1\,dx = \int_0^{1/3} V_1 V_1\,dx + \int_{1/3}^{2/3} V_1 V_1\,dx$$

$$V_1 = \begin{cases} \frac{1}{h}(x) & 0 \le x \le 1/3 \\ \frac{1}{h}(\frac{2}{3}-x) & \frac{1}{3} \le x \le \frac{2}{3} \\ 0 & x > \frac{2}{3} \end{cases}$$

$$M_{11} = \int_0^{1/3} (3x)^2 dx + \int_{1/3}^{2/3} (3(\tfrac{2}{3}-x))^2\,dx$$

$$= \int_0^{1/3} 9x^2 dx + \int_{1/3}^{2/3} 9(\tfrac{2}{3}-x)^2\,dx = \frac{1}{9} + \frac{1}{9} = \frac{2}{9}$$

$$M_{12} = M_{21} = \int_0^1 V_1 V_2\,dx = \int_0^{1/3} V_1 V_2\,dx + \int_{1/3}^{2/3} V_1 V_2\,dx$$
$$+ \int_{2/3}^{1} V_1 V_2\,dx$$

$$V_2 = \begin{cases} 0 & x < 1/3 \\ \frac{1}{h}(x-\frac{1}{3}) & \frac{1}{3} \le x \le 2/3 \\ \frac{1}{h}(1-x) & \frac{2}{3} \le x \le 1 \end{cases}$$

$\therefore \int_0^{1/3} V_1 V_2\,dx = 0$ & $\int_{2/3}^{1} V_1 V_2\,dx = 0$

6.5.8

$$M_{12} = \int_{1/3}^{2/3} 3(x-1/3)\,3\left(\tfrac{2}{3}-x\right) dx = \int_{1/3}^{2/3} 9\,(x-1/3)(2/3-x)\,dx$$

$$M_{12} = \frac{1}{18} = M_{21}$$

$$M_{22} = \int_0^1 V_2 V_2\,dx = \cancel{\int_0^{1/3} V_2 V_2\,dx}^{\,0} + \int_{1/3}^{2/3} V_2 V_2\,dx + \int_{2/3}^{1} V_2 V_2\,dx$$

$$= \int_{1/3}^{2/3} 9\left(x-\tfrac{1}{3}\right)^2 dx + \int_{2/3}^{1} 9(1-x)^2\,dx = \frac{1}{9} + \frac{1}{9} = \frac{2}{9}$$

$$\therefore \boxed{M = \begin{pmatrix} 2/9 & 1/18 \\ 1/18 & 2/9 \end{pmatrix}}$$

$$Ax = \lambda M x$$

$$(A - \lambda M)\,x = 0$$

$$A = 3\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 6 & -3 \\ -3 & 6 \end{pmatrix}$$

$$|A - \lambda M| = \begin{vmatrix} 6 - \frac{2\lambda}{9} & -3 - \lambda/18 \\ -3 - \frac{\lambda}{18} & 6 - \frac{2\lambda}{9} \end{vmatrix}$$

6.5.8

$$\left(6-\frac{2\lambda}{9}\right)\left(6-\frac{2\lambda}{9}\right) = 36 - \frac{24\lambda}{9} + \frac{4\lambda^2}{81}$$

$$\left(-3-\frac{\lambda}{18}\right)\left(-3-\frac{\lambda}{18}\right) = 9 + \frac{6\lambda}{18} + \frac{\lambda^2}{324}$$

$$36 - \frac{24\lambda}{9} + \frac{4\lambda^2}{81} - 9 - \frac{6\lambda}{18} - \frac{\lambda^2}{324} = 0$$

$$(36)(324) - (24)(36)\lambda + 16\lambda^2 - (324)(9) - (6)(18)\lambda - \lambda^2 = 0$$

$$15\lambda^2 - 972\lambda + 8748 = 0$$

$$\lambda_{1,2} = \frac{972 \pm \sqrt{(972)^2 - (4)(8748)(15)}}{30}$$

$\lambda_1 = 10.8$, $\lambda_2 = 54$

Minimum value is $\lambda_1 = 10.8$