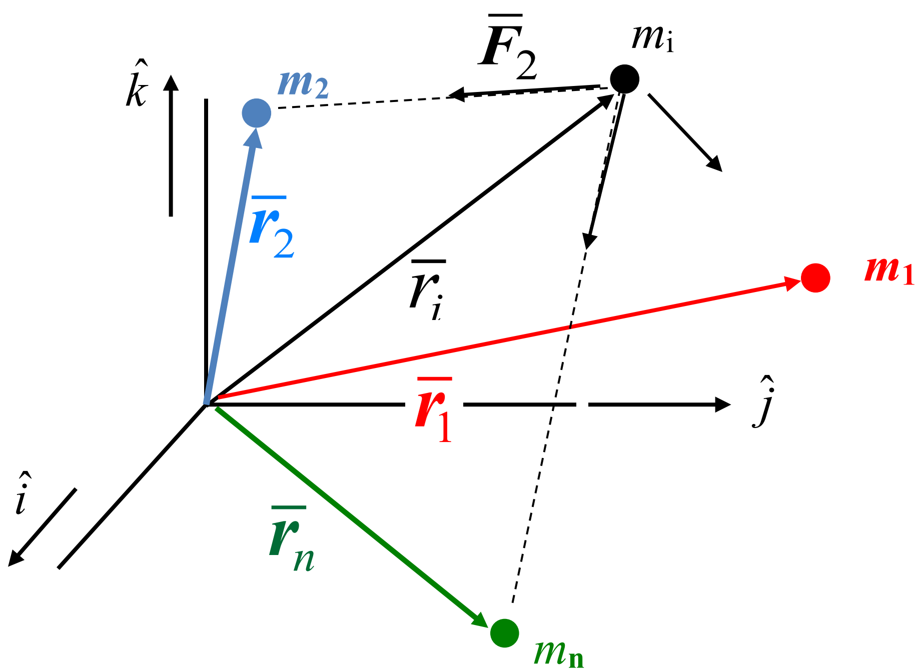
N – Body Problem

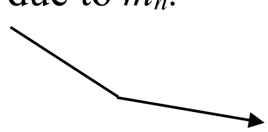
Write an expression for force acting on one body due to existence of multiple other bodies

Assume: Gravity is the only force acting
"System" of n – bodies (masses m_1, m_2, …, m_n)
All masses spherically symmetric

$$\left|\bar{f}_2\right| = \frac{G m_1 m_2}{r^2}$$



force on m_i due to m_n:



Sum all forces

$$\overline{F}_T = -\frac{Gm_i m_1}{r_{1i}^{\;3}}\overline{r}_{1i} - \frac{Gm_i m_2}{r_{2i}^{\;3}}\overline{r}_{2i} + \ldots - \frac{Gm_i m_n}{r_{ni}^{\;3}}\overline{r}_{ni}$$

$$\left(\text{does NOT include} \quad -\frac{Gm_i m_i}{r_{ii}^3}\overline{r}_{ii}\right)$$

Force Model

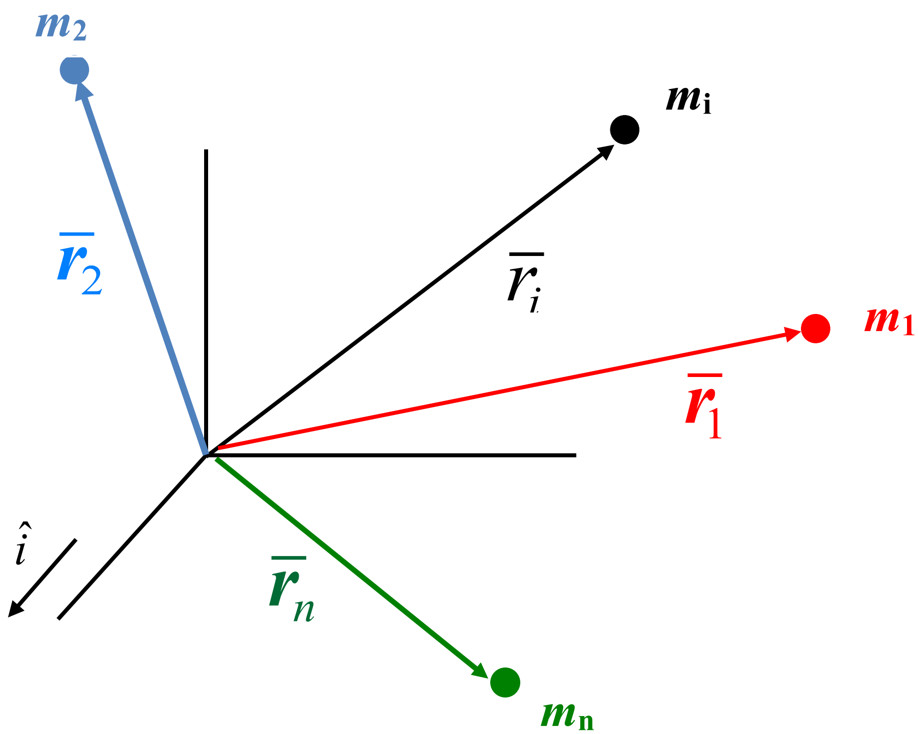
Using this force model, write EOM from Newton II

$$\frac{d}{dt}(m_i\,\overline{v}_i) = \overline{F}_{Total}$$ Note: only true if derivative wrt an inertial frame

$$m_i \frac{d\,\overline{v}_i}{dt} + \overline{v}_i \cancel{\frac{dm_i}{dt}} = \overline{F}_T$$

Acceleration as seen in the inertial frame

Assume m_i constant



$$m_i \ddot{\bar{r}}_i = -G \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{m_i m_j}{r_{ji}^3} \bar{r}_{ji}$$

Let $m_i \rightarrow$ s/c, $m_1 \rightarrow$ Sun, $m_2 \rightarrow$ Mars, $m_n \rightarrow$ Earth, Jupiter, Mercury, Uranus, ….

$$m_i \ddot{\bar{r}}_i = \underbrace{-\frac{G m_i m_1}{r_{1i}^3} \bar{r}_{1i}}\; \underbrace{-\frac{G m_i m_2}{r_{2i}^3} \bar{r}_{2i}}\; \underbrace{-\sum_{\substack{j=3 \\ j\neq i}}^{n} \frac{G m_i m_i}{r_{ji}^3} \bar{r}_{ji}}$$

Alternative formulation using potential function, U:

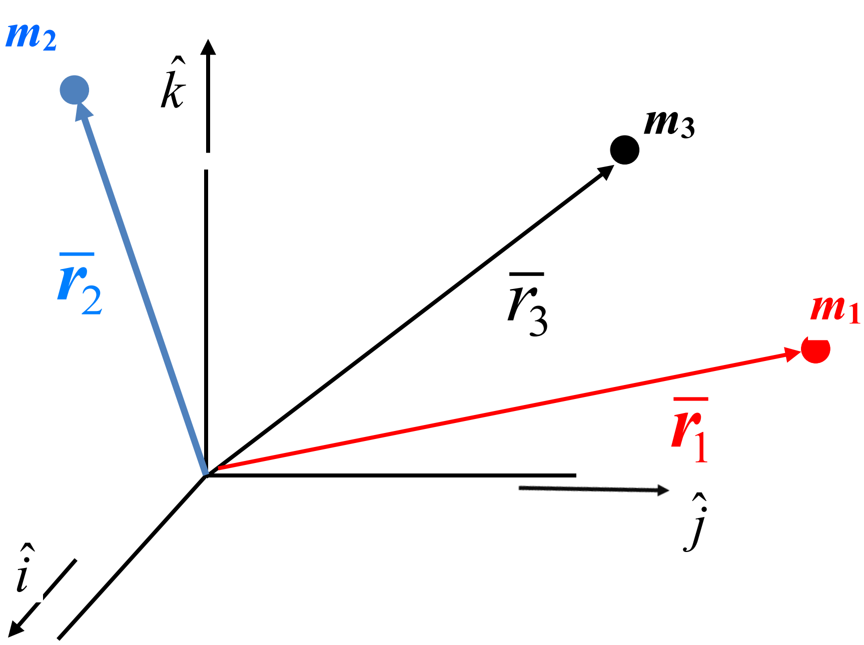
$$\boxed{m_i \frac{d^2 \bar{r}_i}{dt^2} = \bar{\nabla}_i U}$$

where $\bar{\nabla}_i \rightarrow$ vector gradient operator

$$\bar{\nabla}_i(.) = \hat{i}\frac{\partial}{\partial x_i}(.) + \hat{j}\frac{\partial}{\partial y_i}(.) + \hat{k}\frac{\partial}{\partial z_i}(.)$$

$U \rightarrow$ gravitational potential (scalar)

Example: System of 3 particles



$$\bar{r}_{12} = \bar{r}_2 - \bar{r}_1$$
$$\bar{r}_{13} = \bar{r}_3 - \bar{r}_1$$

$$\bar{r}_1 = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$$
$$\bar{r}_2 = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$$
$$\bar{r}_3 = x_3\hat{i} + y_3\hat{j} + z_3\hat{k}$$

Force (total) on m_1 ➡ $\bar{F}_T = -G\sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{m_i\, m_j}{r_{ji}^3}\bar{r}_{ji}$ general expression

$$\overline{F}_1 = G\left(\frac{m_1 m_2}{r_{12}^3}\overline{r}_{12} + \frac{m_1 m_3}{r_{13}^3}\overline{r}_{13} \right) = m_1 \frac{d^2 \overline{r}_1}{dt^2}$$

Alternate expression

$$\overline{F}_1 = \overline{\nabla}_1 U \qquad \text{where} \qquad U = \frac{1}{2} G \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{m_i m_j}{r_{ij}}$$

$$U = \frac{1}{2} G\left(\frac{m_1 m_2}{r_{12}} + \frac{m_1 m_3}{r_{13}} + \frac{m_2 m_1}{r_{21}} + \frac{m_2 m_3}{r_{23}} + \frac{m_3 m_1}{r_{31}} + \frac{m_3 m_2}{r_{32}} \right)$$

$$U = G\left(\frac{m_1 m_2}{r_{12}} + \frac{m_1 m_3}{r_{13}} + \frac{m_2 m_3}{r_{23}} \right)$$

DOF? Coordinates used to describe configuration?

$\overline{r}_1, \quad \overline{r}_2, \quad \overline{r}_3$

→ All quantities in U must be written in terms of the independent variables

$$\overline{r}_{12} = \overline{r}_2 - \overline{r}_1 = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$
$$\overline{r}_{13} = \overline{r}_3 - \overline{r}_1 = (x_3 - x_1)\hat{i} + (y_3 - y_1)\hat{j} + (z_3 - z_1)\hat{k}$$
$$\overline{r}_{23} = \overline{r}_3 - \overline{r}_2 = (x_3 - x_2)\hat{i} + (y_3 - y_2)\hat{j} + (z_3 - z_2)\hat{k}$$

where
$$r_{12} = \left|\overline{r}_{12}\right| = \left[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \right]^{1/2}$$
$$r_{13} = \left|\overline{r}_{13}\right| = \left[(x_3 - x_1)^2 + (y_3 - y_1)^2 + (z_3 - z_1)^2 \right]^{1/2}$$

↓

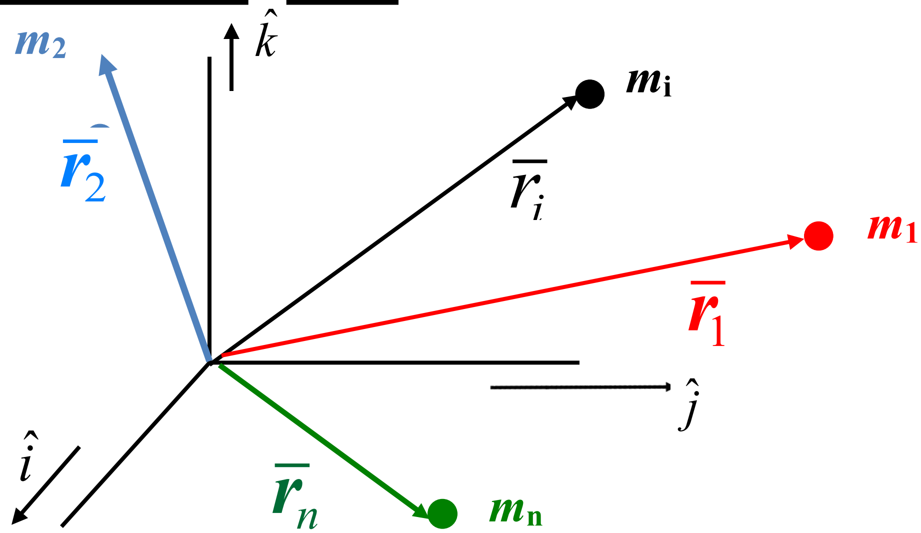
$$\frac{\partial U}{\partial x_1} = G m_1 m_2 \frac{\partial\left(r_{12}^{-1}\right)}{\partial x_1} + G m_1 m_3 \frac{\partial\left(r_{13}^{-1}\right)}{\partial x_1} + G m_2 m_3 \frac{\partial\left(r_{23}^{-1}\right)}{\partial x_1}$$
$$= \frac{G m_1 m_2}{r_{12}^3}(x_2 - x_1) + \frac{G m_1 m_3}{r_{13}^3}(x_3 - x_1)$$

$$\frac{\partial U}{\partial y_1} = \dots$$

$$\frac{\partial U}{\partial z_1} = \dots$$

$$\begin{aligned}
\bar{\bar{\nabla}}_1 U = &\left\{ \frac{Gm_1 m_2}{r_{12}^3}\left(x_2 - x_1\right) + \frac{Gm_1 m_3}{r_{13}^3}\left(x_3 - x_1\right) \right\} \hat{i} \\
&\left\{ \frac{Gm_1 m_2}{r_{12}^3}\left(y_2 - y_1\right) + \frac{Gm_1 m_3}{r_{13}^3}\left(y_3 - y_1\right) \right\} \hat{j} \\
&+\left\{ \frac{Gm_1 m_2}{r_{12}^3}\left(z_2 - z_1\right) + \frac{Gm_1 m_3}{r_{13}^3}\left(z_3 - z_1\right) \right\} \hat{k}
\end{aligned}$$

N – Body Problem



Vector Equation of Motion for m_i

$$\boxed{m_i \ddot{\bar{r}}_i = -G \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{m_i m_j}{r_{ji}^3} \bar{r}_{ji}}$$

→ 6 scalar first-order differential equations for m_i } solvable?

Observations concerning solution $\bar{r}_i(t)$:

1. Independent variable –

2. Time history $\bar{r}_j(t)$ NOT KNOWN

   m_j affected by m_i ; motion of m_i changes force on m_j → changes acceleration on m_j → changes position of m_j

   → scalar components of $\bar{r}_j$, $\dot{\bar{r}}_j$ are also unknown dependent variables

3. Add additional equations so no. of equations = no. of unknowns

   $\Downarrow$

   Need 6 scalar, first-order differential equations <u>for each particle in the system</u>

   $\Downarrow$

   first-order (scalar) differential equations are necessary
   equations nonlinear and coupled

4. For every first-order differential equation that appears, a <u>complete analytical</u> solution requires the ability to analytically integrate the DE

   If you can integrate a differential equation, you have an <u>integral of the motion</u> (note that a <u>constant</u> appears)

   Given a coupled set of differential equations, increasingly difficult to integrate (may try lots of approaches …)
   But must be accomplished for a complete solution.

   $\downarrow$

   We have $6n$ equations in $6n$ dependent variables
   We need <u>$6n$ integrals of the motion</u> or <u>$6n$ constant</u>s to solve our system of differential equations

5. To date, we only know how to obtain 10 integrals
   $\Rightarrow$ The n-body problem is NOT completely solvable

Known since Euler's time (1707-1783)
Nothing new since

Ten Known Integrals

Can't integrate individual equations directly but collect equations into certain combinations -- allows integration of some of the new equations
Aided immensely by physical significance of the integrals

1. $\boxed{\text{Linear Momentum}}$

Conserved for system ← no external forces in FBD

$$m_i \ddot{\bar{r}}_i = -G \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{m_i m_j}{r_{ji}^3} \underbrace{(\bar{r}_i - \bar{r}_j)}_{\bar{r}_{ji}}$$

To get total $\bar{p}$, add up all equations

$$\sum_{i=1}^{n} m_i \ddot{\bar{r}}_i = -G \underbrace{\sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{m_i m_j}{r_{ji}^3} (\bar{r}_i - \bar{r}_j)}$$

Integrate twice

$$\underbrace{\sum_{i=1}^{n} m_i \bar{r}_i} = \bar{C}_1 t + \bar{C}_2 \quad \rightarrow$$

Note: $\bar{p} = \left(\sum_{i=1}^{n} m_i \dot{\bar{r}}_i \right) =$ constant $\bar{C}_1$

2. $\boxed{\text{Angular Momentum}}$

Conserved for <u>system</u> ⟵ no external forces (or moments) in FBD

$$m_i \ddot{\bar{r}}_i = -G \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{m_i m_j}{r_{ji}^3} \underbrace{(\bar{r}_i - \bar{r}_j)}_{\bar{r}_{ji}}$$

Vector cross with $\bar{r}_i$; add up all equations

$$\sum_{i=1}^{n} m_i \ddot{\bar{r}}_i \times \bar{r}_i = \sum_{i=1}^{n} G \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{m_i m_j}{r_{ij}^3} \left(\bar{r}_j - \bar{r}_i\right) \times \bar{r}_i$$

$$\left(\bar{r}_j \times \bar{r}_i\right) - \underbrace{\cancel{\left(\bar{r}_i \times \bar{r}_i\right)}}_{\text{zero}}$$

$$(\bar{r}_1 \times \bar{r}_2) + (\bar{r}_2 \times \bar{r}_1)$$

$$\sum_{i=1}^{n} m_i \ddot{\bar{r}}_i \times \bar{r}_i = \bar{0}$$

Equation we can integrate

Integrate once

<u>Total</u> angular momentum of a system of *n* particles → constant in magnitude AND direction

Can define significant surface: <u>**invariable plane**</u>

3. $\boxed{\text{Total Energy}}$

Conserved for <u>system</u> $\leftarrow$ internal forces derivable from potential so system conservative

$$m_i \ddot{\bar{r}}_i = \bar{\nabla}_i U$$

Scalar dot product with $\dot{\bar{r}}_i$; add up all equations

$$\sum_{i=1}^{n} m_i \ddot{\bar{r}}_i \bullet \dot{\bar{r}}_i = \underbrace{\sum_{i=1}^{n} \bar{\nabla}_i U \bullet \dot{\bar{r}}_i}$$

$$\left(\frac{\partial U}{\partial x}\hat{\imath} + \frac{\partial U}{\partial y}\hat{\jmath} + \frac{\partial U}{\partial z}\hat{k}\right)\left(\frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath} + \frac{dz}{dt}\hat{k}\right)$$

$$\sum_{i=1}^{n} m_i \frac{d}{dt}\left(\frac{1}{2}\dot{\bar{r}}_i \bullet \dot{\bar{r}}_i\right) = \frac{dU}{dt}$$

$$\frac{d}{dt}\left[\sum_{i=1}^{n} m_i \left(\frac{1}{2} m_i \dot{\bar{r}}_i \bullet \dot{\bar{r}}_i\right)\right] = \frac{dU}{dt}$$

$$\frac{d}{dt}T = \frac{d}{dt}U \quad \text{OR} \quad \frac{d}{dt}T - \frac{d}{dt}U = 0 \Leftarrow \text{ Equation we can integrate}$$

Integrate once