MA 527

Lecture Notes (section 8.1 & 8.2)

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8.1 Eigenvalue problem

(Motivation) DE.

(1) $y' = ay$: $y(t) = Ce^{at}$

(2) ~~see~~ Chapter 4.

$\underline{X' = AX}$: Assume $X(t) = \mathbf{v}e^{\lambda t}$

($\mathbf{v}$: constant vector)

$$\begin{cases} X' = \lambda \mathbf{v} e^{\lambda t} \\ AX = A\mathbf{v}e^{\lambda t} \end{cases}$$

$$A\mathbf{v}\cancel{e^{\lambda t}} = \lambda \mathbf{v} \cancel{e^{\lambda t}}$$

$\therefore$ $A\mathbf{v} = \lambda \mathbf{v}$

Def A : an $n \times n$ matrix

If $\underline{A\mathbf{v} = \lambda \mathbf{v}}$, then

(1) λ is called an eigenvalue of A

(2) $\mathbf{v}_{\neq 0}$ 〃 an eigenvector of A associated with λ.

Q How do we find λ?

$A\mathbf{v} - \underset{I\mathbf{v}}{\underset{\parallel}{\lambda \mathbf{v}}} = 0$: $\underset{\mathbf{v} \neq 0}{\underline{(A - \lambda I)\mathbf{v} = 0}}$

$(A - \lambda I)$ is singular

$\therefore$ $\det(A - \lambda I) = 0$: characteristic equation

(Ex) (1) $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$: $A - \lambda I = \begin{bmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{bmatrix}$

① $\det(A - \lambda I) = (1-\lambda)(3-\lambda) - 8 = \lambda^2 - 4\lambda + 3 - 8 = 0$

$\lambda^2 - 4\lambda - 5 = 0$: $(\lambda+1)(\lambda-5) = 0$

$\lambda = -1,\ 5.$

② $\lambda = -1$:

Solve $(A - (-1)I)V = 0$

$$\begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 2 & 2 & 0 \\ 4 & 4 & 0 \end{array}\right] \xrightarrow{\frac{1}{2}r_1} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 4 & 4 & 0 \end{array}\right] \xrightarrow{r_2 - 4r_1} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$v_1 + v_2 = 0$: $v_2 = -v_1$

$$V = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ -v_1 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad (v_1 \neq 0)$$

$V_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\left(\text{or } \begin{bmatrix} 2 \\ -2 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix}, \cdots \right)$

$\lambda = 5$: $(A - 5I) = \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix}$

$$\left[\begin{array}{cc|c} -4 & 2 & 0 \\ 4 & -2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} -4 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$-4v_1 + 2v_2 = 0$ $\quad 2v_2 = 4v_1$

$v_2 = 2v_1$

$V = \begin{bmatrix} v_1 \\ 2v_1 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$: $V_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

(EX) $A = \begin{bmatrix} 2 & 0 & -1 \\ 0 & \frac{1}{2} & 0 \\ 1 & 0 & 4 \end{bmatrix}$ 3x3.

(1) $\lambda = ?$ $\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 0 & -1 \\ 0 & \frac{1}{2}-\lambda & 0 \\ 1 & 0 & 4-\lambda \end{vmatrix}$ ←

$= \left(\frac{1}{2} - \lambda\right)(-1)^4 \begin{vmatrix} 2-\lambda & -1 \\ 1 & 4-\lambda \end{vmatrix} = \left(\frac{1}{2} - \lambda\right)\left(\underline{(2-\lambda)(4-\lambda)+1}\right)$

$\lambda^2 - 6\lambda + 9$

$= \left(\frac{1}{2} - \lambda\right)(\lambda - 3)^2 = 0$

multiplicity: 2

$\lambda = \frac{1}{2}$, $\underline{\lambda = 3}$.

(2) $\lambda = \frac{1}{2}$: Solve $\begin{bmatrix} \frac{3}{2} & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & \frac{7}{2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\begin{cases} \frac{3}{2}v_1 - v_3 = 0 \\ v_1 + \frac{7}{2}v_3 = 0 \end{cases}$ $\underline{v_1 = 0, \quad v_3 = 0}$

$\underbrace{\begin{bmatrix} \frac{3}{2} & -1 \\ 1 & \frac{7}{2} \end{bmatrix}}_{B} \begin{bmatrix} v_1 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\det B = \frac{21}{4} + 1 \neq 0$

$V = \begin{bmatrix} 0 \\ v_2 \\ 0 \end{bmatrix} = v_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$: $\underline{V_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}$

$\lambda = 3$: $A - 3I = \begin{bmatrix} -1 & 0 & -1 \\ 0 & -\frac{5}{2} & 0 \\ 1 & 0 & 1 \end{bmatrix}$

Solve $(A - 3I)V = 0$.

$$\begin{cases} -v_1 + v_3 = 0 \\ -\frac{5}{2} v_2 = 0 \\ v_1 + v_3 = 0 \end{cases} \longrightarrow \begin{array}{l} v_1 + v_3 = 0 : \ v_3 = -v_1 \\ v_2 = 0 \end{array}$$

$$V = \begin{bmatrix} v_1 \\ 0 \\ -v_1 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} : \ V_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

(Eigenvalue: complex number).

$AX = \lambda X. : \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \ a, b, c, d \in \mathbb{R}$

(Ex) $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

(1) $\lambda = ?$ $\quad \det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix}$

$= \lambda^2 - (-1) = \lambda^2 + 1 = 0. \quad \lambda = i, -i.$

(Ex) $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

(1) $\lambda = ? : \quad \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix}$

$= (1-\lambda)^2 + 1 = 0 \quad : \ \lambda = 1 \pm i.$

(2) $\underline{\lambda = 1 + i}$:

Solve $(A - (1+i)I) X = 0$

$$\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} : \begin{cases} i^2 = -1 \\ i^4 = 1. \end{cases}$$

$$\left[\begin{array}{cc|c} -i & -1 & 0 \\ 1 & -i & 0 \end{array}\right] \xrightarrow{i r_1} \left[\begin{array}{cc|c} 1 & -i & 0 \\ 1 & -i & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_1 - i x_2 = 0 \quad : \quad x_1 = i x_2$

$$X = \begin{bmatrix} i x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} i \\ 1 \end{bmatrix} \quad : \quad X_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

$$\lambda = 1 - i : \quad \underline{X_2 = \overline{X_1}} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

$*$ $\overline{a+bi} = a - bi$: the conjugate of $a+bi$
$(a, b \in \mathbb{R})$

$$X_2 = \overline{X_1}$$

(Proof) $AX_1 = (1+i)X_1$

$$\overline{AX_1} = \overline{(1+i)X_1}$$