AAE 532 - Orbit Mechanics

Purdue University

PS5 Solutions

Authors:
Mitchell Dominguez
Tyler Hook
Samantha Ramsey

Email:
doming18@purdue.edu
hook9@purdue.edu
ramsey87@purdue.edu

Instructor:
Kathleen Howell

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Contents

Useful Constants

	Axial Rotaional Period (Rev/Day)	Mean Equatorial Radius (km)	Gravitational Parameter $\mu = Gm(km^3/sec^2)$	Semi-major Axis of orbit (km)	Orbital Period (sec)	Eccentricity of Orbit	Inclination of Orbit to Ecliptic (deg)
☉ Sun	0.0394011	695990	132712440017.99	-	-	-	-
☾ Moon	0.0366004	1738.2	4902.8005821478	384400 (around Earth)	2360592 27.32 Earth Days	0.0554	5.16
☾ Mercury	0.0170514	2439.7	22032.080486418	57909101	7600537 87.97 Earth Days	0.20563661	7.00497902
♀ Venus	0.0041149 (Retrograde)	6051.9	324858.59882646	108207284	19413722 224.70 Earth Days	0.00676399	3.39465605
⊕ Earth	1.0027378	6378.1363	398600.4415	149597898	31558205 365.26 Earth Days	0.01673163	0.00001531
♂ Mars	0.9747000	3397	42828.314258067	227944135	59356281 686.99 Earth Days	0.09336511	1.84969142
♃ Jupiter	2.4181573	71492	126712767.8578	778279959	374479305 11.87 Years	0.04853590	1.30439695
♄ Saturn	2.2522053	60268	37940626.061137	1427387908	930115906 29.47 Years	0.05550825	2.48599187
⛢ Uranus	1.3921114 (Retrograde)	25559	5794549.0070719	2870480873	2652503938 84.05 Years	0.04685740	0.77263783
Ψ Neptune	1.4897579	25269	6836534.0638793	4498337290	5203578080 164.89 Years	0.00895439	1.77004347
♇ Pluto	-0.1565620 (Retrograde)	1162	981.600887707	5907150229	7830528509 248.13 Years	0.24885238	17.14001206

- First three columns of the body data are consistent with GMAT 2020a default values, which are mainly from JPL's ephemerides file de405.spk

- The rest of the data are from JPL website(`https://ssd.jpl.nasa.gov/?planet_pos`, retrieved at 09/01/2020)
based on E.M. Standish's "Keplerian Elements for Approximate Positions of the Major Planets"

Problem 1

Problem Statement

A spacecraft is in orbit about Mars and assume that it is reasonable to model it as a relative two-body problem. The orbit is characterized such that $r_p = 1.8R_{♂}$ and $r_a = 8R_{♂}$. The vehicle is currently located such that $M_0 = -120°$. Note that this time is denoted as t_0.

(a) Determine the following orbit parameters and spacecraft state information:
a, e, p, period, $\mathcal{E}$; r_0, v_0, θ_0^*, E_0, γ_0, $(t_0 - t_p)$
All times are in hours and all angles in degrees.

(b) Write $\bar{\mathbf{r}}_0$ and $\bar{\mathbf{v}}_0$ in terms of components in the directions of both $\hat{\mathbf{r}}$ and $\hat{\theta}$; $\hat{\mathbf{e}}$ and $\hat{\mathbf{p}}$.

(c) Determine θ^* after a time equal to 75% of the period, i.e., $\Delta t = 0.75\mathcal{P}$. Determine the conditions at the new state: r_1, v_1, E_1, γ_1, $(t_1 - t_p)$. What is $\Delta\theta^*$ and ΔE between these two states?

(d) Use f and g relationships to write $\bar{\mathbf{r}}_1$, $\bar{\mathbf{v}}_1$ in terms of $\bar{\mathbf{r}}_0$, $\bar{\mathbf{v}}_0$. Use the eccentric anomaly form of the f and g relationships and demonstrate that:

$$\bar{\mathbf{r}}_1 = f\bar{\mathbf{r}}_0 + g\bar{\mathbf{v}}_0$$

$$\bar{\mathbf{v}}_1 = \dot{f}\bar{\mathbf{r}}_0 + \dot{g}\bar{\mathbf{v}}_0$$

(e) PLOT the orbit with your MATLAB scripts from last week. By hand, mark on the plot where the spacecraft is currently located by marking $\hat{\mathbf{r}}$, $\hat{\theta}$, $\bar{\mathbf{r}}_0$,θ_0^*; also sketch the local horizon, $\bar{\mathbf{v}}_0$, and γ_0. Do the same at the second location. Add the auxiliary circle; mark $\Delta\theta^*$ and ΔE. Identify the arc from t_0 to t_1.

Part (a)

Let's begin by solving for the semi-major axis of the spacecraft's orbit. Recalling that the major axis is equal to the apoapsis radius plus the periapsis radius, then:

$$\boxed{= \frac{r_p + r_a}{2} = 1.6645 \cdot 10^4 \text{ km}} \tag{1}$$

Next, one can use the periapsis or apoapsis radius equation to solve for the eccentricity:

$$\boxed{e = \frac{r_a}{a} - 1 = 0.6327} \tag{2}$$

Now, one can solve for the semi-latus rectum:

$$\boxed{p = a(1 - e^2) = 9.9830 \cdot 10^3 \text{ km}} \tag{3}$$

Since we just solved for semi-latus rectum, one can obtain the specific angular momentum magnitude through the following equation:

$$\boxed{h = \sqrt{\mu p} = 2.0677 \cdot 10^4 \text{ km}^2/\text{s}} \tag{4}$$

One can then solve for mean motion, which then can be used to solve for the orbit period of the spacecraft:

$$n = \sqrt{\frac{\mu}{a^3}} = 9.6367 \cdot 10^{-5} \text{ rad/s} \tag{5}$$

$$\boxed{\begin{aligned} \mathcal{P} = \frac{2\pi}{n} &= 6.5201 \cdot 10^4 \text{ s} \\ &= 18.1113 \text{ hrs} \end{aligned}} \tag{6}$$

Next, one can solve for energy, using the expression:

$$\boxed{\mathcal{E} = -\frac{\mu}{2a} = -1.2865 \text{ km}^2/\text{s}^2} \tag{7}$$

Next, one can solve for the time until periapsis $(t_0 - t_p)$. This means that our time value can be taken to be negative to indicate time that must elapse to get to the next periapsis passage. So, recalling Kepler's equation:

$$\boxed{\begin{aligned} (t_0 - t_p) = \frac{M_0}{n} &= -2.1734 \cdot 10^4 \text{ s} \\ &= -6.0371 \text{ hrs} \end{aligned}} \tag{8}$$

Now, using a Newton-Raphson method, one can solve iteratively to find the true value of eccentric anomaly when mean anomaly is $-120°$ and find:

$$\boxed{E_0 = -142.2112° = 217.7888°} \tag{9}$$

One can find the orbit radius of the spacecraft at this location, by using the conic equation or the following relationship:

$$\boxed{r_0 = a(1 - e\cos E) = 2.4967 \cdot 10^4 \text{ km}} \tag{10}$$

Then, one can solve for true anomaly using the following relationship with eccentric anomaly:

$$\boxed{\theta_0^* = 2 \cdot \arctan\left[\sqrt{\frac{1+e}{1-e}} \cdot \tan\frac{E}{2}\right] = -161.5568° = 198.4432°} \tag{11}$$

Afterthat, one can solve for the orbit velocity of the spacecraft, using the vis-viva equation:

$$\boxed{v_0 = \sqrt{2\left(\mathcal{E} + \frac{\mu}{r}\right)} = 0.9261 \text{ km/s}} \tag{12}$$

Lastly, one can also solve for the flight path angle γ. Realize that, since the spacecraft is descending in its orbit, the flight path angle will be negative (-). Thus, using the following relation, the flight path angle is:

$$\boxed{\gamma_0 = -\arccos\left(\frac{h}{rv}\right) = -26.5912°} \tag{13}$$

Part (b)

Now, let's write $\bar{\mathbf{r}}_0$, $\bar{\mathbf{v}}_0$ in terms of components in the directions of $\hat{\mathbf{r}}$ and $\hat{\theta}$ as well as $\hat{\mathbf{e}}$ and $\hat{\mathbf{p}}$. Note that, in the Radial-Transverse-Normal reference frame, writing the radius vector is easy as it is the orbit radius magnitude in the radial direction:

$$\boxed{\bar{\mathbf{r}}_0 = 2.4967 \cdot 10^4\, \hat{\mathbf{r}}\ [\text{km}]} \tag{14}$$

Recall that the velocity vector, in the RTN frame, can be expressed as:

$$\bar{\mathbf{v}} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\theta} \tag{15}$$

Which means we need to define $\dot{r}$ and $\dot{\theta}$. Recall that, for $\dot{\theta}$:

$$\dot{\theta} = \frac{h}{r^2} = 3.3170 \cdot 10^{-5}\ \text{rad/s} \tag{16}$$

Then, remembering that the spacecraft is still descending in its orbit, then:

$$\dot{r} = +\sqrt{v^2 - r^2\dot{\theta}^2} = -0.4146\ \text{km/s} \tag{17}$$

Which means that the velocity vector is then:

$$\boxed{\bar{\mathbf{v}}_0 = -0.4146\, \hat{\mathbf{r}} + 0.8282\, \hat{\theta}\ [\text{km/s}]} \tag{18}$$

Now, to write $\bar{\mathbf{r}}_0$, $\bar{\mathbf{v}}_0$ in terms of components in the directions of $\hat{\mathbf{e}}$ and $\hat{\mathbf{p}}$, one must note the transformation from the RTN frame to the Perifocal Coordinate System. This transformation simply involves the angle true anomaly and can be expressed as:

$$\begin{bmatrix} \hat{\mathbf{e}} \\ \hat{\mathbf{p}} \\ \hat{\mathbf{h}} \end{bmatrix} = \begin{bmatrix} \cos\theta^* & -\sin\theta^* & 0 \\ \sin\theta^* & \cos\theta^* & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{r}} \\ \hat{\theta} \\ \hat{\mathbf{h}} \end{bmatrix} \tag{19}$$

Thus, plugging in $\theta^* = -161.5568°$ from above, the position and velocity vectors in the PQW frame are:

$$\boxed{\bar{\mathbf{r}}_0 = -2.3685 \cdot 10^4\, \hat{\mathbf{e}} + -7.8988 \cdot 10^3\, \hat{\mathbf{p}}\ [\text{km}]} \tag{20}$$

$$\boxed{\bar{\mathbf{v}}_0 = 0.6553\, \hat{\mathbf{e}} - 0.6545\, \hat{\mathbf{p}}\ [\text{km/s}]} \tag{21}$$

Part (c)

To determine θ^* after a time equal to 75% of the period, one must first solve for $(t_1 - t_p)$, which is:

$$\boxed{\begin{aligned}\Delta t = 0.75\mathcal{P} = &= 4.8900 \cdot 10^4 \text{ s} \\ &= 13.5835 \text{ hrs}\end{aligned}} \tag{22}$$

$$\boxed{\begin{aligned}(t_1 - t_p) = \Delta t + (t_0 - t_p) = &= 2.7167 \cdot 10^4 \text{ s} \\ &= 7.5464 \text{ hrs}\end{aligned}} \tag{23}$$

Which means that the mean anomaly of the spacecraft at this location in its orbit is:

$$M_1 = n(t_1 - t_p) = 150° \tag{24}$$

Again, using a Newton-Raphson method, one can solve iteratively to find the true value of eccentric anomaly when mean anomaly is $-120°$ and find:

$$\boxed{E_1 = 161.5011°} \tag{25}$$

One can find the orbit radius of the spacecraft at this location, by using the conic equation or the following relationship:

$$\boxed{r_1 = a(1 - e\cos E) = 2.6632 \cdot 10^4 \text{ km}} \tag{26}$$

Then, one can solve for true anomaly using the following relationship with eccentric anomaly:

$$\boxed{\theta_1^* = 2 \cdot \arctan\left[\sqrt{\frac{1+e}{1-e}} \cdot \tan\frac{E}{2}\right] = 171.1657°} \tag{27}$$

One can then solve for the orbit velocity of the spacecraft, using the vis-viva equation:

$$\boxed{v_1 = \sqrt{2\left(\mathcal{E} + \frac{\mu}{r}\right)} = 0.8021 \text{ km/s}} \tag{28}$$

One can also solve for the flight path angle γ. Realize that, since the spacecraft is now <u>ascending</u> in its orbit, the flight path angle will be positive (+). Thus, using the following relation, the flight path angle is:

$$\boxed{\gamma_1 = +\arccos\left(\frac{h}{rv}\right) = 14.5312°} \tag{29}$$

Lastly, one can solve for $\Delta\theta^*$ and ΔE between these two states:

$$\boxed{\Delta\theta^* = \theta_1^* - \theta_0^* = 332.7225°} \tag{30}$$

$$\boxed{\Delta E = E_1 - E_0 = 303.7124°} \tag{31}$$

Part (d)

Now, let's use f and g relationships to write $\bar{\mathbf{r}}_1$ and $\bar{\mathbf{v}}_1$ in terms of $\bar{\mathbf{r}}_0$ and $\bar{\mathbf{v}}_0$. Recall that:

$$\bar{\mathbf{r}}_1 = f\bar{\mathbf{r}}_0 + g\bar{\mathbf{v}}_0 \tag{32}$$

$$\bar{\mathbf{v}}_1 = \dot{f}\bar{\mathbf{r}}_0 + \dot{g}\bar{\mathbf{v}}_0 \tag{33}$$

If we want to write f and g in terms of eccentric anomaly, then one must note the form of the f and g functions, where:

$$f = 1 - \frac{a}{r_0}\Big[1 - \cos(E_1 - E_0)\Big] \tag{34}$$

$$g = (t - t_0) - \sqrt{\frac{a^3}{\mu}}\Big[(E_1 - E_0) - \sin(E_1 - E_0)\Big] \tag{35}$$

$$\dot{f} = \frac{\sqrt{\mu a}}{r r_0}\sin(E_1 - E_0) \tag{36}$$

$$\dot{g} = 1 - \frac{a}{r}\Big[1 - \cos(E_1 - E_0)\Big] \tag{37}$$

So, given that $\Delta E = 303.7124°$, the values for f and g in terms of eccentric anomaly are:

$$\boxed{\begin{aligned} f(\Delta E) &= 0.7033 \text{ [unitless]} \\ g(\Delta E) &= -1.4738 \cdot 10^4 \text{ [s]} \\ \dot{f}(\Delta E) &= 3.3402 \cdot 10^{-5} \text{ [1/s]} \\ \dot{g}(\Delta E) &= 0.7219 \text{ [unitless]} \end{aligned}} \tag{38}$$

Which means that $\bar{\mathbf{r}}_1$ and $\bar{\mathbf{v}}_1$ can be expressed as:

$$\boxed{\bar{\mathbf{r}}_1 = 0.7033\,\bar{\mathbf{r}}_0 - 1.4738 \cdot 10^4 \text{ [s]}\,\bar{\mathbf{v}}_0 \text{ [km]}} \tag{39}$$

$$\boxed{\bar{\mathbf{v}}_1 = 3.3402 \cdot 10^{-5} \text{ [1/s]}\,\bar{\mathbf{r}}_0 + 0.7219\,\bar{\mathbf{v}}_0 \text{ [km/s]}} \tag{40}$$

Or, written in the PQW frame:

$$\boxed{\bar{\mathbf{r}}_1 = -2.6316 \cdot 10^4\,\hat{\mathbf{e}} + 4.0900 \cdot 10^3\,\hat{\mathbf{p}} \text{ [km]}} \tag{41}$$

$$\boxed{\bar{\mathbf{v}}_1 = -0.3181\,\hat{\mathbf{e}} - 0.7363\,\hat{\mathbf{p}} \text{ [km/s]}} \tag{42}$$

One can check this result through other means by, first, obtaining the position and velocity vector in the RTN frame, where:

$$\bar{\mathbf{r}}_1 = r_1\hat{\mathbf{r}} = 2.6632 \cdot 10^4\,\hat{\mathbf{r}} \text{ [km]} \tag{43}$$

And:

$$\bar{\mathbf{v}}_1 = v_1 \sin\gamma_1 \hat{\mathbf{r}} + v_1 \cos\gamma_1 \hat{\theta} = 0.2012\,\hat{\mathbf{r}} + 0.7764\,\hat{\theta} \text{ [km/s]} \tag{44}$$

So, recalling that the transformation from the RTN frame to the Perifocal Coordinate System simply involves the angle true anomaly and can be expressed as:

$$\begin{bmatrix} \hat{\mathbf{e}} \\ \hat{\mathbf{p}} \\ \hat{\mathbf{h}} \end{bmatrix} = \begin{bmatrix} \cos\theta^* & -\sin\theta^* & 0 \\ \sin\theta^* & \cos\theta^* & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{r}} \\ \hat{\theta} \\ \hat{\mathbf{h}} \end{bmatrix} \tag{45}$$

Thus, plugging in $\theta_1^* = 171.1657°$ from above, the position and velocity vectors in the PQW frame are:

$$\boxed{\bar{\mathbf{r}}_1 = -2.6316 \cdot 10^4\,\hat{\mathbf{e}} + 4.0900 \cdot 10^3\,\hat{\mathbf{p}} \text{ [km]}} \tag{46}$$

$$\boxed{\bar{\mathbf{v}}_1 = -0.3181\,\hat{\mathbf{e}} - 0.7363\,\hat{\mathbf{p}} \text{ [km/s]}} \tag{47}$$

Which matches the results obtained from the f and g functions above and concludes our check.

Part (e)

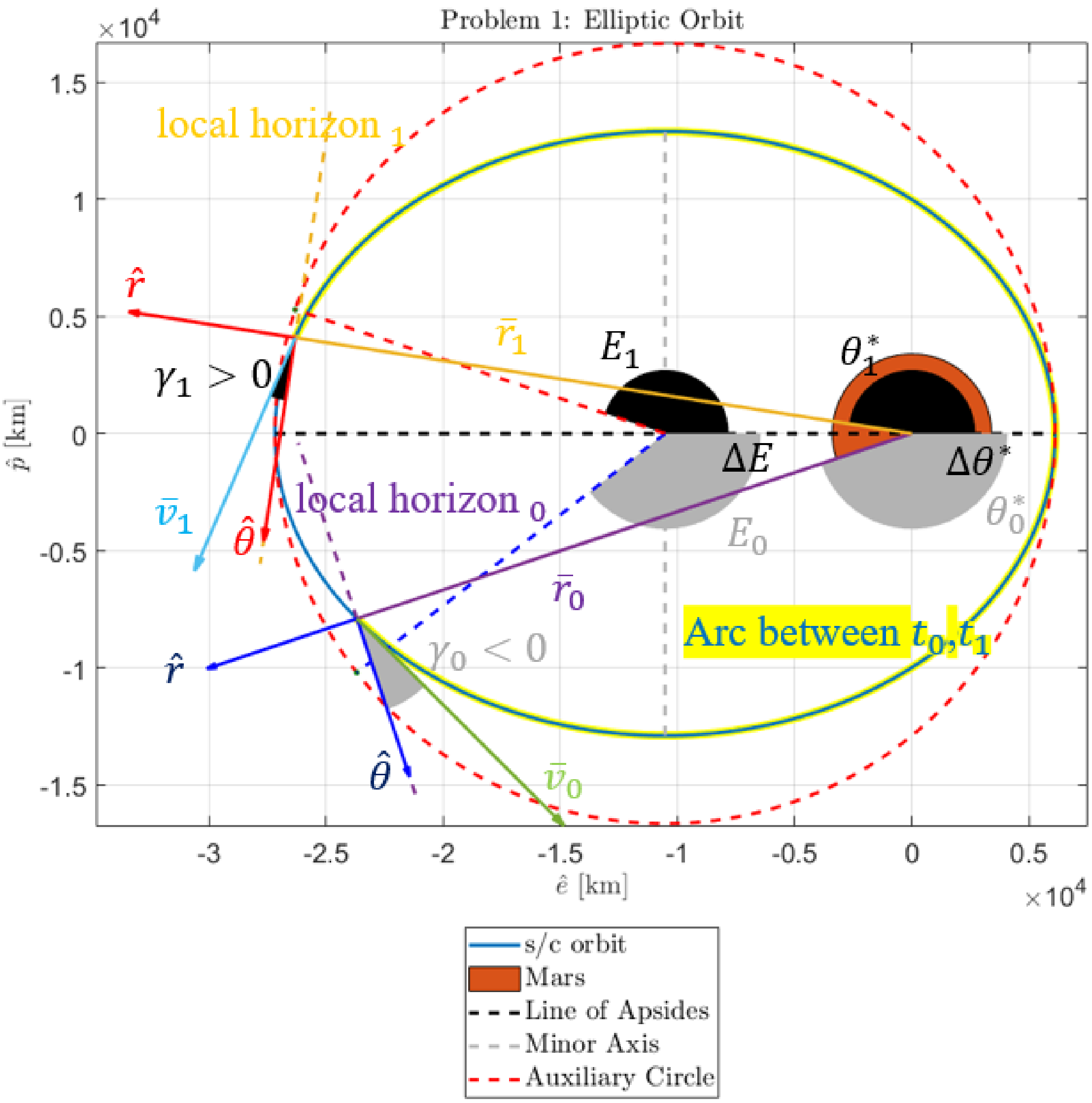

Figure 1: Plot of the s/c at its initial and final locations in its orbit about Mars.

As one can see in figure 1, the spacecraft is located at its initial and final positions, as specified in parts (a) and (c) above. At the initial location, the spacecraft is descending in its orbit, as evident by its negative flight path angle γ_0. However, at the final location, the spacecraft is ascending in its orbit, as evident by its positive flight path angle γ_1. The auxiliary circle, the line of apsides and the minor axis (although the last two are not necessary) are depicted for reference. Note that the angles ΔE and $\Delta\theta^*$ are depicted as the "difference" in the angles E_0 and E_1, and θ_0^* and θ_1^*, respectively.

Problem 2

Problem Statement

As part of the recent lunar initiative, an unmanned small probe is approaching the Moon on a hyperbola. (Maybe one of the small commercial payloads seeking a Moon landing in 2024!) The hyperbola is defined such that $a = 7050\ km$ and the passage altitude is $800km$ altitude. At the "current" time, the probe is located at $\theta^* = -90°$.

(a) Determine the following additional orbital characteristics: r_p, v_p, b, h, δ, v_∞, ε. Determine the following quantities at the current time: r, v, γ, H, time till perilune.

(b) What true anomaly value corresponds to the aim point? Determine the actual radius r at this location.

(c) Plot the hyperbola between $\pm\theta^* = 100°$. [You can use GMAT or Matlab.] Mark the probe at $\theta^* = -90°$ and label b, aim point, flyby angle, v, γ, r, θ^* (Always include the local horizon!)

Determine r, v, γ at $\theta^* = +100°$; add this information to the plot.

Part (a)

The desired orbital characteristics are:

$$r_p = R_{☽} + 800\ km = \boxed{2538.20\ km}$$

$$v_\infty = \sqrt{\frac{\mu}{|a|}} = \boxed{0.833926\ km/s}$$

$$\varepsilon = \frac{\mu}{2\,|a|} = \boxed{0.34772\ km^2/s^2}$$

$$e = 1 + \frac{r_p}{|a|} = 1.36003$$

$$v_p = \sqrt{\frac{2\mu}{r_p} + \frac{\mu}{|a|}} = \boxed{2.13510\ km/s}$$

$$p = |a|\left(e^2 - 1\right) = 5990.22\ km$$

$$h = \sqrt{\mu p} = \boxed{5419.31\ km^2/s}$$

$$\delta = 2\sin^{-1}\left(\frac{1}{e}\right) = \boxed{94.6616°}$$

$$b = |a|\sqrt{e^2 - 1} = \boxed{6498.54\ km}$$

At the current time, the following quantities are also calculated:

$$r = \frac{p}{1 + e\cos\theta^*} = \boxed{5990.22\ km}$$

$$v = \sqrt{2\left(\varepsilon + \frac{\mu}{r}\right)} = \boxed{1.52721\ km/s}$$

$$\gamma = \cos^{-1}\left(\frac{h}{rv}\right) = \boxed{-53.6737°}$$

$$H = \cosh^{-1}\left(\frac{a-r}{ea}\right) = \boxed{-0.824968}$$

$$t - t_p = \sqrt{\frac{|a|^3}{\mu}}\left(e\sinh H - H\right) = -1.00668\ hrs$$

Thus, the time till perilune is $\boxed{1.00668\ hrs}$.

Part (b)

Figure 2 illustrates the geometry of the hyperbolic probe orbit. The right triangle defined by the semiminor axis and the asymptotes of the hyerbola is used to define the true anomaly for the aim point as

$$\theta^*_{aim} = -\frac{\delta}{2} = \boxed{-47.3308^\circ}$$

The negative sign for θ^*_{aim} reflects that the spacecraft is descending when it arrives at the aim point. The corresponding orbital radius is

$$r_{aim} = \frac{p}{1 + e\cos\theta^*_{aim}} = \boxed{3117.02\ km}$$

Part (c)

When true anomaly is $\theta^* = 100\ km$, the position, orbital speed, and flight path angle are calculated using the expressions provided in part (a):

$$r = \boxed{7842.32\ km}$$

$$v = \boxed{1.39491\ km/s}$$

$$\gamma = \boxed{+60.3041°}$$

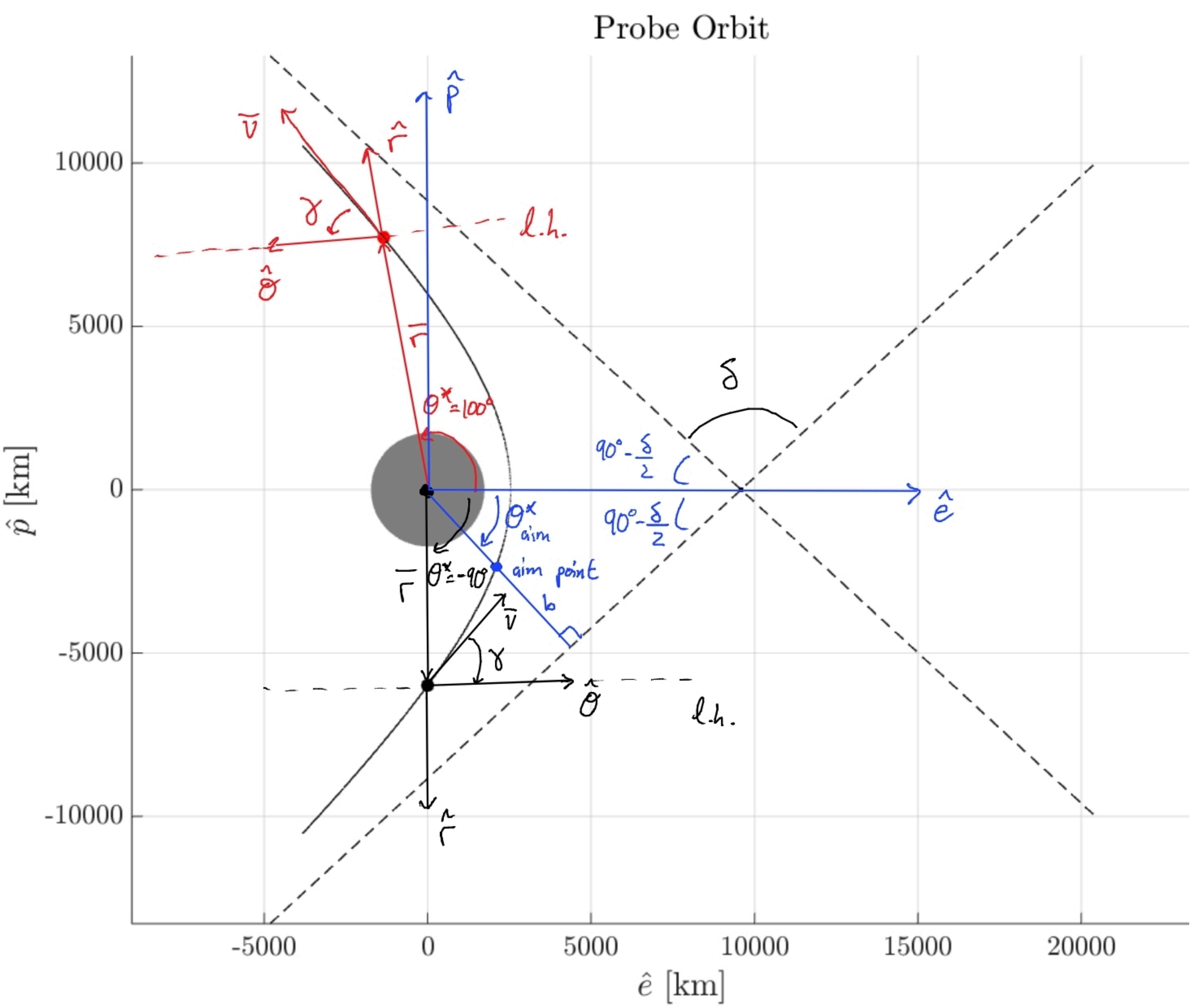

Figure 2: Lunar probe orbit, $-100° \leq \theta^* \leq +100°$

Problem 3

Problem Statement

Assume that a vehicle is currently moving in Earth orbit. Assume a two-body relative motion of the spacecraft with respect to Earth. The orbit is described with the following characteristics (relative to an Earth equatorial coordinate system):

$$
\begin{aligned}
a &= 66R_{\oplus} \\
e &= 0.9118 \\
i &= 34^\circ \\
\Omega &= 60^\circ \\
\omega &= 0^\circ \\
\theta &= 235^\circ
\end{aligned}
$$

(a) Determine the state in terms of $\bar{r},\ \bar{v},\ r,\ v,\ \gamma,\ \theta^*,\ M,\ E,\ (t-t_p)$; write $\bar{r},\ \bar{v}$ in terms of both rotating orbit unit vectors $(\hat{r},\ \hat{\theta},\ \hat{h})$ as well as inertial unit vectors $(\hat{e},\ \hat{p},\ \hat{h})$ and $(\hat{x},\ \hat{y},\ \hat{z})$.

(b) Confirm the results in GMAT with the conic propagator. (Include the Moon's orbit again in the GMAT image.) Use the epoch date 07 Oct 2024.

(c) A short document is posted under GMAT Tips, i.e., labelled 'Propagators with Different Force Models'. The propagator 'EarthPointMass' is already available (under the name you have selected for previous assignments). Produce the new propagators: 'EarthMoon', 'EarthSun', 'EarthMoonSun'. Add the different forces to create the new propagators as described in the GMAT Tips document. Add the Moon's orbit (Luna) to the output image. Plot the orbit in GMAT again. But, this time, add the other 3 vehicles with different propagators, i.e., the EarthMoonSun propagator plus the others. So, the GMAT image includes 4 orbits, one produced with each mode1. Print the image – be sure that each orbit is labelled (using color is most helpful.) Is the Earth-spacecraft conic two-body relative model actually a good representative model for this problem? Why or why not? Which additional mass (Moon, or Sun) seems to have a significant impact? Why? Why might this particular orbit more sensitive to the additional gravity fields?

Part (a)

First, we can write:

$$\theta^* = \theta - \omega = \boxed{235°}$$

$$p = a(1 - e^2)$$

$$r = \frac{p}{1 + e\cos\theta^*} = \boxed{1.4881 \times 10^5\ km}$$

$$v = \sqrt{2\left(-\frac{\mu}{2a} + \frac{\mu}{r}\right)} = \boxed{2.1001\ km/s}$$

$$h = \sqrt{\mu p}$$

$$\gamma = -\cos\left(\frac{h}{rv}\right) = \boxed{-57.4355°}$$

Note that the spacecraft is in the descending leg of the orbit because the radial velocity is negative.

$$E = 2\tan^{-1}\left[\left(\frac{1+e}{1-e}\right)^{-1/2}\tan\frac{\theta^*}{2}\right] = \boxed{315.157°}$$

$$M = E - e\sin E = \boxed{351.997°}$$

$$t - t_p = \frac{M}{n} = \boxed{738.246\ hr = 30.76\ days}$$

For the coordinate frames, we start with $\hat{r},\ \hat{\theta}$:

$$\bar{r} = r\hat{r} = \boxed{23.3305\ R_{\oplus}\hat{r}}$$

$$\bar{v} = v\sin\gamma\hat{r} + v\cos\gamma\hat{\theta} = \boxed{-1.7699\hat{r} + 1.13304\hat{\theta}\ km/s}$$

To go from polar coordinates to the $\hat{e}, \hat{p}$ orbit frame we rotate about $\hat{h}$ by θ using the DCM:

$$^{eph}C^{r\theta n} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{\bar{r} = -13.3818\hat{e} - 19.1113\hat{p}\ R_{\oplus}}$$

$$\boxed{\bar{v} = 1.9411\hat{e} + 0.8015\hat{p}\ km/s}$$

Now we can rotate to the $\hat{x}, \hat{y}, \hat{z}$ Cartesian coordinate frame:

$$
{}^{xyz}C^{eph} = \begin{bmatrix} \cos\Omega & -\sin\Omega & 0 \\ \sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos i & -\sin i \\ 0 & \sin i & \cos i \end{bmatrix} = \begin{bmatrix} \cos\Omega & -\cos i \sin\Omega & \sin i \sin\Omega \\ \sin\Omega & \cos i \cos\Omega & -\sin i \cos\Omega \\ 0 & \sin i & \cos i \end{bmatrix}
$$

Note that we could also use the full body 313 DCM provided in the notes to go straight from $\hat{r}, \hat{\theta}, \hat{n}$ to $\hat{x}, \hat{y}, \hat{z}$, but since we have already rotated from the $\hat{r}, \hat{\theta}, \hat{n}$ frame to the $\hat{e}, \hat{p}, \hat{h}$ frame, I have removed the third rotation from the matrix sequence.

$$
\boxed{\bar{r} = 7.0303\hat{x} - 19.5110\hat{y} - 10.6869\hat{z} \ R_{\oplus}}
$$

$$
\boxed{\bar{v} = 0.3951\hat{x} + 2.0133\hat{y} + 0.4482\hat{z} \ km/s}
$$

Part (b)

First, we set the spacecraft up by defining the semimajor axis, eccentricity, inclination, and RAAN. Note that the propagator is defined with Earth as a point mass and no perturbing forces are included. To confirm the results in part (a), we begin with $\theta = 235°$ and generate an output report to provide all of the requested parameters. Since we calculated the values at the initial time, the propagation length is arbitrary. To show the entire orbit, I chose true anomaly as the stopping condition with a value of $\theta = 234.9°$. We can check a few of the orbital parameters to verify our results:

Parameter	Calculated	GMAT
γ	$-57.4355°$	$147.4355°$
M	$351.997°$	$351.997°$
E	$315.157°$	$315.157°$

Similarly to the last problem set, we see that each of the values align perfectly aside from the offset in flight path angle which is expected. We can also verify the inertial $\hat{x}, \hat{y}, \hat{z}$ components of position and velocity. GMAT uses units of km for position, so if we divide those values by $R_\oplus$ or multiply our previously computed position by the same value, we will see that the position as well as velocity components match. Note that if we were to compare the values to more significant figures we would see some discrepancies due to limits in numerical precision.

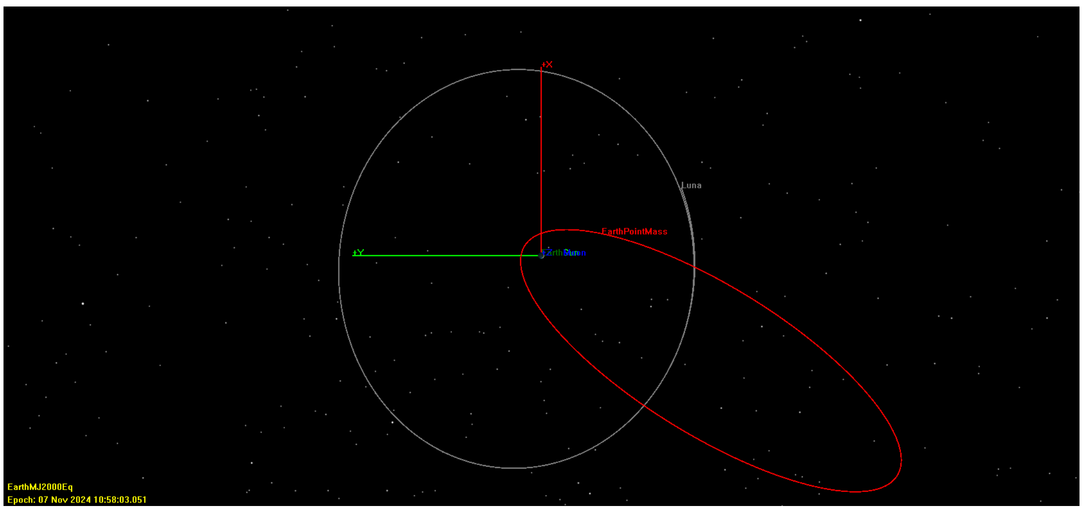

Figure 3: Orbit $\hat{x} - \hat{y}$ plane projection.

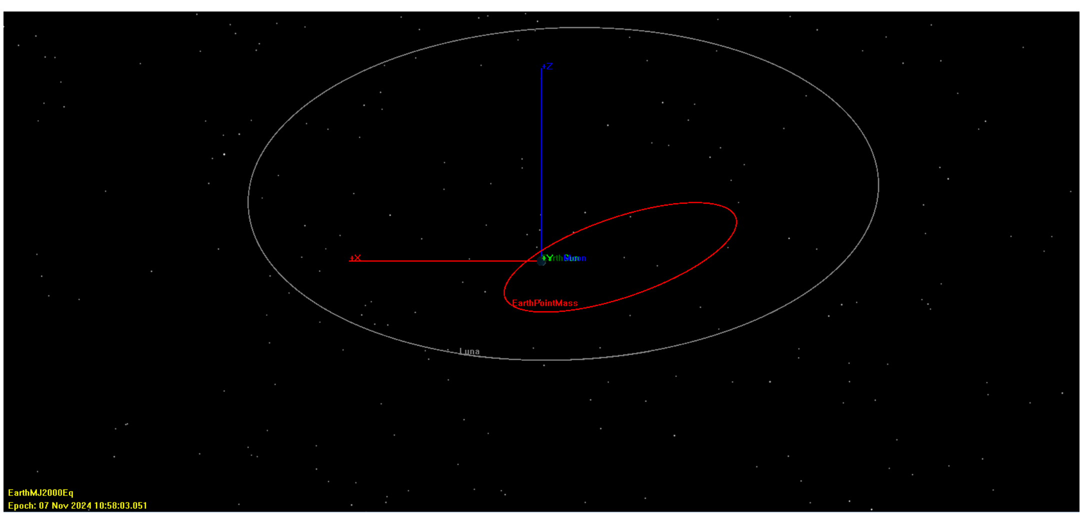

Figure 4: Orbit $\hat{x} - \hat{z}$ plane projection.

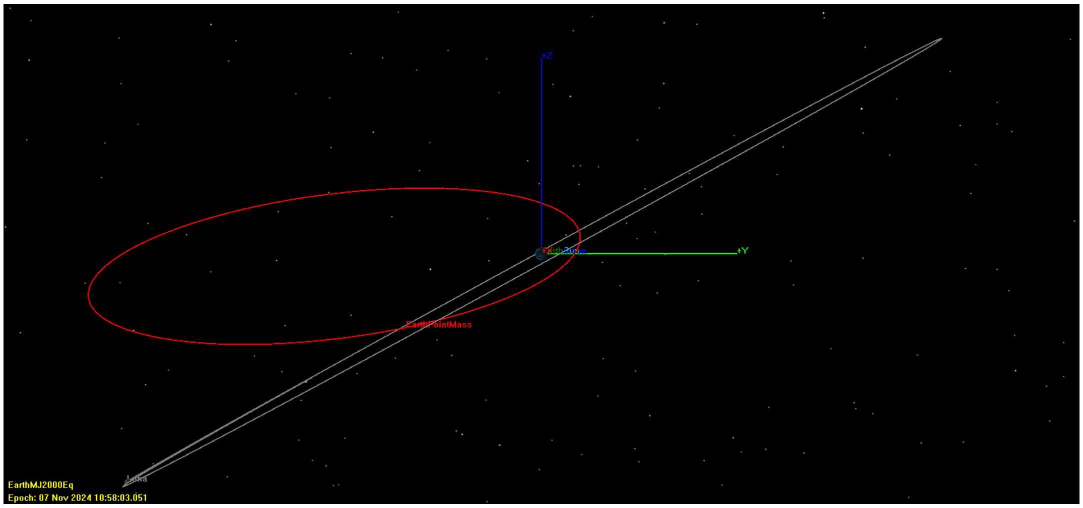

Figure 5: Orbit $\hat{y} - \hat{z}$ plane projection.

Part (c)

This time we want to run 4 separate spacecraft each with a different propagator. Each one will include a different force model to allow us to assess the effects of perturbations from different bodies. 'EarthPointMass' includes only the gravitational effects of the Earth, modeled as a point mass. This is the model we have used in previous assignments and effectively models the scenario as a relative two-body problem. 'EarthMoon,' 'EarthSun,' and 'EarthMoonSun' each include Earth modeled as a point mass along with the perturbing effects of either the Moon, Sun, or both. If we propagate for one period of the orbit we can see each of the planar projections as follows:

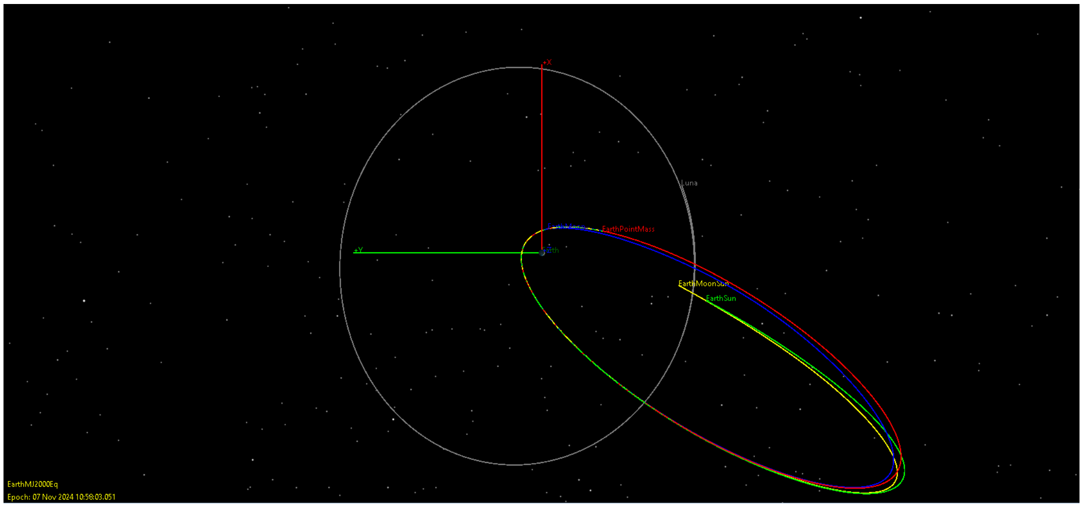

Figure 6: Orbit $\hat{x} - \hat{y}$ plane projection.

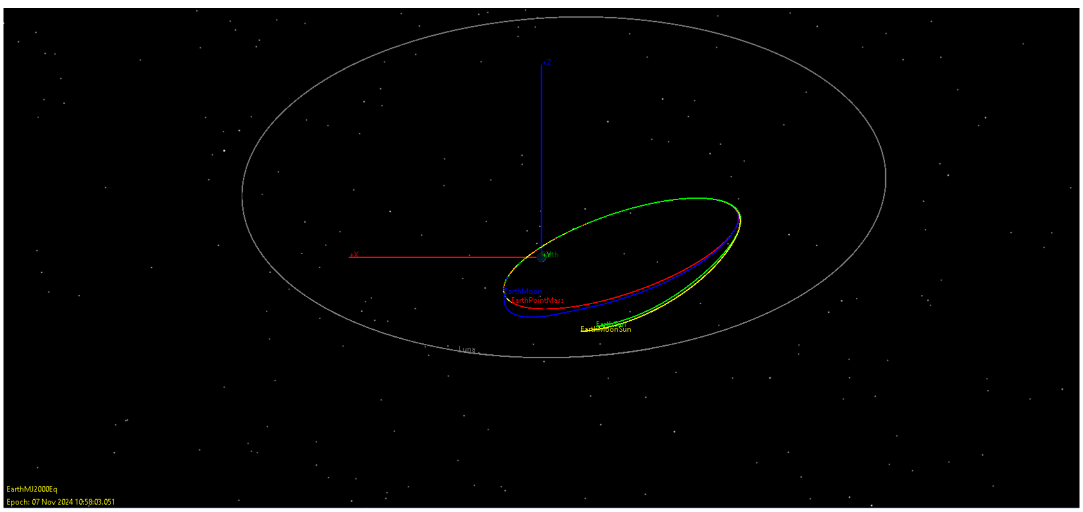

Figure 7: Orbit $\hat{x} - \hat{z}$ plane projection.

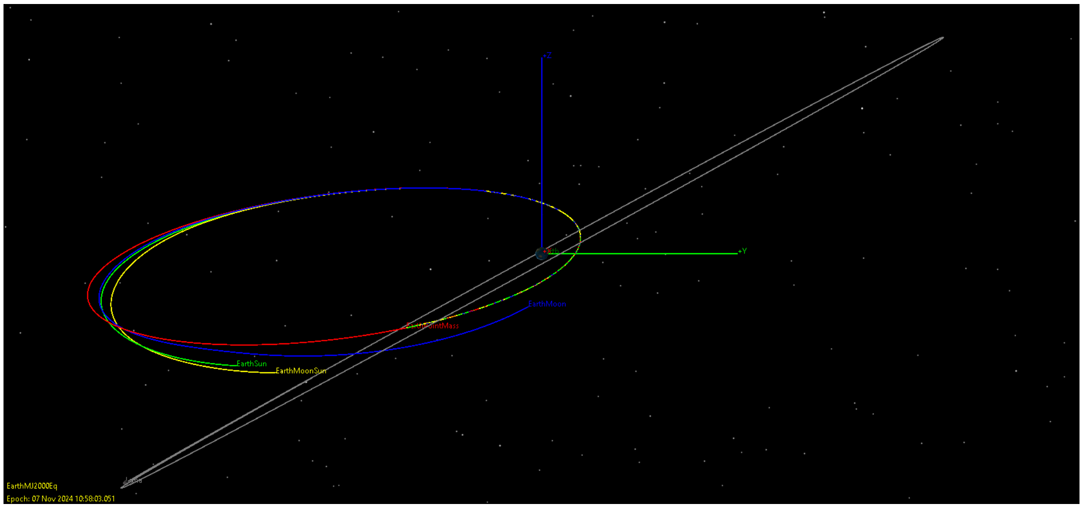

Figure 8: Orbit $\hat{y} - \hat{z}$ plane projection.

From these figures it is clear that the additional forces have a significant impact on the orbit even over the course of a single period. The 'EarthPointMass' spacecraft returns back to its initial position, while those affected by perturbations do not. Thus, we can conclude that the two-body relative model is not a good representation for this orbit. To determine which additional mass has the most significant impact long term, we can propagate the orbits for a longer time period as well as look at some GMAT generated plots for a couple of the orbital parameters.

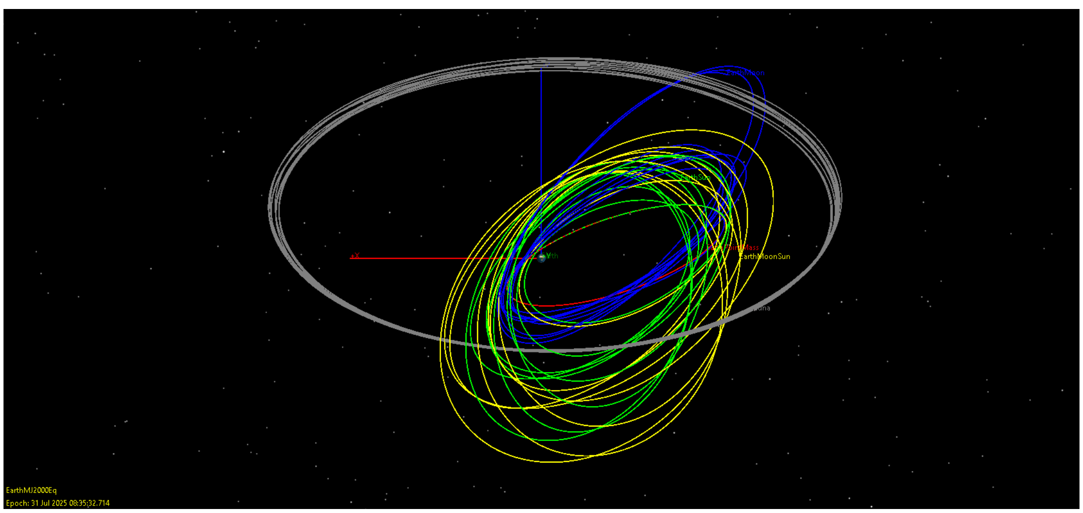

Figure 9: 300 day orbit propagation.

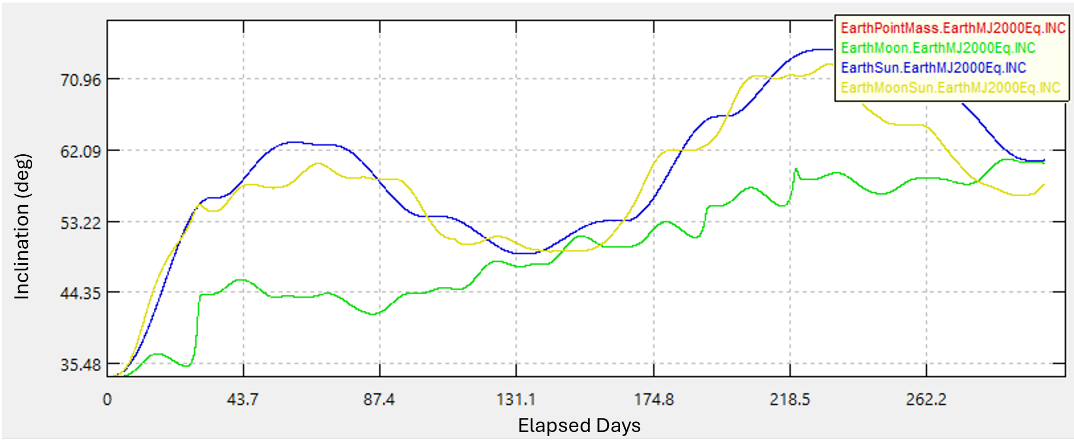

Figure 10: Inclination vs time.

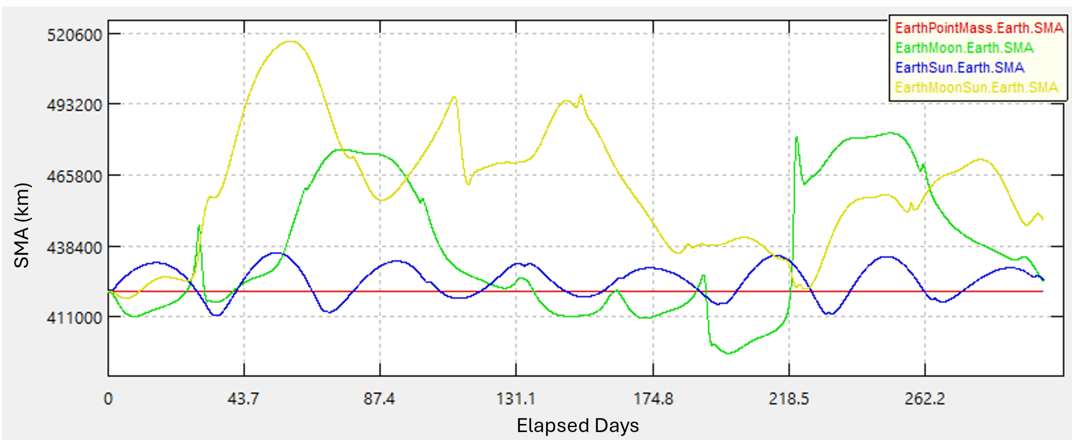

Figure 11: Semi-major axis vs time.

These figures make it clear that both masses have a significant impact on the spacecraft. Each additional mass affects various parameters differently though. For example, we can see that while the moon tends to have a more significant impact on the semi-major axis compared to the Sun, the Sun appears to have a greater effect on the inclination of the orbit. This orbit is likely more sensitive to additional gravity fields because of its large semi-major axis. As the spacecraft moves further from the Earth, the gravitational effects from the additional bodies have a greater effect. The magnitude of these effects depends on the phasing of the bodies as well. Since the semi-major axis is beyond the orbital distance of the moon, there is the potential for the spacecraft to have very close encounters with the lunar gravity field. When an event like this occurs, we see larger than normal spikes in the variation of the orbital parameters. Note that we can also exploit these dynamics during the mission design process by using the gravitational effects of the Moon and Sun to increase or decrease the energy of our orbit without requiring the use of a maneuver.

Problem 4

Problem Statement

The relationship between sets of unit vectors is clearly important! A vehicle is moving in some Earth orbit. At a certain time, the following information is given:

$$\bar{r}_1 = 2.12R_{\oplus}\hat{x} + 2.73R_{\oplus}\hat{y} - 0.6R_{\oplus}\hat{z}$$
$$\bar{v}_1 = -3.4\hat{x} + 1.62\hat{y} + 2.9\hat{z}\ km/s$$

Determine the following orbital characteristics: $a,\ e,\ p,\ i,\ \omega,\ \Omega,\ r,\ v,\ \gamma,\ \theta^*,\ M,\ E,\ (t-t_p)$. Be sure to include the appropriate quadrant checks. Is the orbit an ellipse? How do you know?

Solution

First, we can get the magnitudes of the position and velocity vectors:

$$r = |\bar{r}_1| = 22375.6\ km = \boxed{3.5082 R_\oplus}$$

$$v = |\bar{v}_1| = \boxed{4.7534\ km/s}$$

We will use these quantities along with specific energy and angular momentum for calculating the other orbital parameters.

$$h = |\bar{r} \times \bar{v}| = 1.0237\ km^2/s$$

$$\epsilon = \frac{1}{2}v^2 - \frac{\mu}{r} = 6.5168\ km^2/s^2$$

$$a = \frac{-\mu}{2\epsilon} = 30582.31\ km = \boxed{4.7949 R_\oplus}$$

$$p = \frac{h^2}{\mu} = 26289.98\ km = \boxed{4.1219 R_\oplus}$$

$$e = \sqrt{1 - \frac{p}{a}} = \boxed{0.3746}$$

Note that the spacecraft is on the descending leg of the orbit since the dot product between the position and velocity vectors is negative. Therefore:

$$\gamma = \cos^{-1}\left(\frac{h}{rv}\right) = \boxed{-15.7463°}$$

$$\theta^* = \cos^{-1}\left(\frac{1}{e}\left(\frac{h^2}{\mu r} - 1\right)\right) = \boxed{-62.1632°}$$

$$E = 2\tan^{-1}\left[\left(\frac{1+e}{1-e}\right)^{1/2} \tan\frac{\theta^*}{2}\right] = \boxed{315.7487°}$$

$$M = E - e\sin E = \boxed{330.7222°}$$

$$(t - t_p) = M\sqrt{\frac{a^3}{\mu}} = 48897\ s\ \boxed{13.5828\ hrs}$$

Then looking at the direction cosine matrix:

$$^{I}C^{R} = \begin{bmatrix} \cos\Omega\cos\theta - \sin\Omega\cos i\sin\theta & -\cos\Omega\sin\theta - \sin\Omega\cos i\cos\theta & \sin\Omega\sin i \\ \sin\Omega\cos\theta + \cos\Omega\cos i\sin\theta & -\sin\Omega\sin\theta + \cos\Omega\cos i\cos\theta & -\cos\Omega\sin i \\ \sin i\sin\theta & \sin i\cos\theta & \cos i \end{bmatrix}$$

And using the fact that:

$$\hat{h} = \frac{\bar{h}}{|\bar{h}|} = 0.5538\hat{x} - 0.2560\hat{y} + 0.7923\hat{z}$$

We know:

$$\cos i = 0.7923$$

$$i = \boxed{37.5983°}$$

To check quadrants, we need to analyze both expressions for Ω and θ and pick the result that appears as a solution to both.

$$-\cos\Omega \sin i = -0.2560$$

$$\sin\Omega \sin i = 0.5538$$

$$\Omega = \boxed{65.1914°}$$

$$\bar{\theta} = \bar{h} \times \bar{r} = 0.5728\hat{x} + 0.5735\hat{y} + 0.5857\hat{z}$$

$$\sin i \sin\theta = -0.1710$$

$$\sin i \cos\theta = 0.5857$$

$$\theta = 343.7207°$$

$$\omega = \theta - \theta^* = \boxed{45.8840°}$$

There are multiple ways to confirm that the spacecraft is in an elliptical orbit, including by looking at the eccentricity which is less than 1, and the energy which is less than 0. Additionally, without computing the orbital parameters we could compare the velocity of the spacecraft to the circular velocity at that radius and since:

$$v < \sqrt{2}v_c$$

We once again know that the spacecraft must be in an elliptical orbit.