ECE 68000: MODERN AUTOMATIC CONTROL

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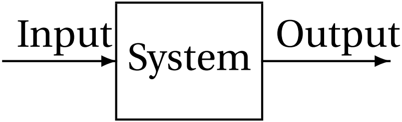
Linear system overview

Outline

- Some remarks on signals and systems
- Solving the state equation
- Solution of the controlled system
- Reachability and controllability
- Observability
- State-feedback controller design problem
- Pole placement

Some Remarks on Signals and Systems



- Signals are functions of time, which can be scalar-valued or vector-valued
- A system is any part of the real world surrounded by a well defined boundary
- The system is influenced by its environment via input signal, $\boldsymbol{u}(t)$ and acts on its environment via output signal $\boldsymbol{y}(t)$

State of the System

- The state of the system contains all past information of the system up to the initial time t_0
- If we wish to compute the system output for $t > t_0$, we only need $\boldsymbol{u}(t)$ for $t > t_0$ and the initial state $\boldsymbol{x}(t_0)$

Solving Uncontrolled State Equation

- Time-invariant linear model

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t)$$

  subject to an initial condition

$$\boldsymbol{x}(0) = \boldsymbol{x}_0$$

- Solution,

$$\boxed{\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}_0}$$

Solving Uncontrolled State Equation—More General Case

- $$\boxed{\boldsymbol{x}(t) = e^{\boldsymbol{A}(t-t_0)}\boldsymbol{x}(t_0)}$$
- $e^{\boldsymbol{A}(t-t_0)} = \Phi(t, t_0)$
  State transition matrix—it relates the state at any instant of time t_0 to the state at any other time t

State Equation Solution of the Controlled System

- Linear Time-Invariant (LTI) controlled dynamical system

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t)$$

- Premultiply by $e^{-\boldsymbol{A}t}$

$$e^{-\boldsymbol{A}t}\dot{\boldsymbol{x}}(t) = e^{-\boldsymbol{A}t}\boldsymbol{A}\boldsymbol{x}(t) + e^{-\boldsymbol{A}t}\boldsymbol{B}\boldsymbol{u}(t)$$

Solution of the Controlled System

- Re-arrange

$$e^{-\boldsymbol{A}t}\dot{\boldsymbol{x}}(t) - e^{-\boldsymbol{A}t}\boldsymbol{A}\boldsymbol{x}(t) = e^{-\boldsymbol{A}t}\boldsymbol{B}\boldsymbol{u}(t)$$

- Note that

$$\frac{d}{dt}\left(e^{-\boldsymbol{A}t}\boldsymbol{x}(t)\right) = -\boldsymbol{A}e^{-\boldsymbol{A}t}\boldsymbol{x}(t) + e^{-\boldsymbol{A}t}\dot{\boldsymbol{x}}(t)$$

- Hence, $\frac{d}{dt}\left(e^{-\boldsymbol{A}t}\boldsymbol{x}(t)\right) = e^{-\boldsymbol{A}t}\boldsymbol{B}\boldsymbol{u}(t)$

Controlled System Model Solution

- Integrate

$$e^{-\boldsymbol{A}t}\boldsymbol{x}(t) - \boldsymbol{x}(0) = \int_0^t e^{-\boldsymbol{A}\tau}\boldsymbol{B}\boldsymbol{u}(\tau)d\tau$$

- Manipulate to obtain

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}(0) + \int_0^t e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)d\tau$$

Controlled System—General Case

Important Solution Formula

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}(t-t_0)}\boldsymbol{x}(t_0) + \int_{t_0}^{t} e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)\,d\tau$$

Reachability Definition

We say that the system $\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t)$ is reachable if for any $\boldsymbol{x}_f$ there is $t_1 > 0$ and a control law, $\boldsymbol{u}(t)$, that transfers $\boldsymbol{x}(t_0) = \boldsymbol{0}$ to $\boldsymbol{x}(t_1) = \boldsymbol{x}_f$

Controllability Definition

We say that the system $\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t)$ is controllable if there is a control law $\boldsymbol{u}(t)$ that transfers any initial state $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$ to the origin at some time $t_1 > t_0$

- For **continuous** LTI systems controllability and reachability are equivalent

Some Controllability Tests

The following are equivalent:

- The system $\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t)$ is reachable
- $\operatorname{rank} \begin{bmatrix} \boldsymbol{B} & \boldsymbol{A}\boldsymbol{B} & \cdots & \boldsymbol{A}^{n-1}\boldsymbol{B} \end{bmatrix} = n$
- The matrix

$$\boldsymbol{W}(t_0, t_1) = \int_{t_0}^{t_1} e^{-\boldsymbol{A}t} \boldsymbol{B}\boldsymbol{B}^\top e^{-\boldsymbol{A}^\top t} dt$$

is nonsingular for all $t_1 > t_0$

Observability

- Suppose the system state is not directly accessible
- Instead, we have the output of the system

$$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}\boldsymbol{u}$$

- We still wish to know the behavior of the entire state

Observability Definition

The system

$$\left.\begin{array}{rcl} \dot{\boldsymbol{x}} & = & \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} \\ \boldsymbol{y} & = & \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}\boldsymbol{u} \end{array}\right\}$$

or equivalently the pair $(\boldsymbol{A}, \boldsymbol{C})$, is observable if there is a finite $t_1 > t_0$ such that for arbitrary $\boldsymbol{u}(t)$ and resulting $\boldsymbol{y}(t)$ over $[t_0, t_1]$, we can determine $\boldsymbol{x}(t_0)$ from complete knowledge of the system input $\boldsymbol{u}$ and output $\boldsymbol{y}$

Remark on Observability

Note that once $\boldsymbol{x}(t_0)$ is known, we can determine $\boldsymbol{x}(t)$ from knowledge of $\boldsymbol{u}(t)$ and $\boldsymbol{y}(t)$ over any finite time interval $[t_0, t_1]$

Observability Test

The following are equivalent:

- The pair $(\boldsymbol{A}, \boldsymbol{C})$ is observable
- The observability matrix

$$\begin{bmatrix} \boldsymbol{C} \\ \boldsymbol{C}\boldsymbol{A} \\ \vdots \\ \boldsymbol{C}\boldsymbol{A}^{n-1} \end{bmatrix} \in \mathbb{R}^{pn \times n}$$

  is of full rank n

State-Feedback Controller

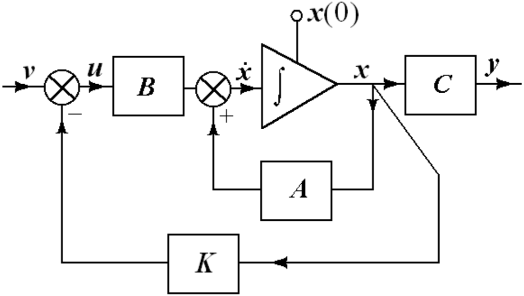
- Plant (System to be controlled)

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} \\ \boldsymbol{y} &= \boldsymbol{C}\boldsymbol{x} \end{aligned}$$

- Controller—linear state-feedback

$$\boldsymbol{u} = -\boldsymbol{K}\boldsymbol{x} + \boldsymbol{v}$$

Closed-Loop System



$$\dot{\boldsymbol{x}} = (\boldsymbol{A} - \boldsymbol{BK})\,\boldsymbol{x} + \boldsymbol{Bv}$$