#1)

$$A = \begin{pmatrix} 2 & 3 & 4 \\ 4 & 7 & 11 \\ 2 & 4 & 7 \end{pmatrix}$$

$$\xrightarrow{E_{21}(-2)} \begin{pmatrix} 2 & 3 & 4 \\ 0 & 1 & 3 \\ 2 & 4 & 7 \end{pmatrix} \xrightarrow{E_{31}(-1)} \begin{pmatrix} 2 & 3 & 4 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{pmatrix}$$

$$\xrightarrow{E_{32}(-1)} \begin{pmatrix} 2 & 3 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

$$L^{-1} = E_{32}(-1)\ E_{31}(-1)\ E_{21}(-2)$$

$$L = E_{21}(2)\ E_{31}(1)\ E_{32}(1)$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

#1)

$$\boxed{L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \qquad U = \begin{pmatrix} 2 & 3 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}}$$

$$A = LU = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 3 & 4 \\ 4 & 7 & 11 \\ 2 & 4 & 7 \end{pmatrix} \checkmark$$

#2)

a)

$$A = \begin{pmatrix} 1 & -2 & 3 \\ 2 & -2 & 5 \\ -3 & 4 & -8 \end{pmatrix}$$

$$\xrightarrow{E_{21}(-2)} \begin{pmatrix} 1 & -2 & 3 \\ 0 & 2 & -1 \\ -3 & 4 & -8 \end{pmatrix} \xrightarrow{E_{31}(3)} \begin{pmatrix} 1 & -2 & 3 \\ 0 & 2 & -1 \\ 0 & -2 & 1 \end{pmatrix}$$

$$\xrightarrow{E_{32}(1)} \begin{pmatrix} \boxed{1} & -2 & 3 \\ 0 & \boxed{2} & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

rank = 2

$$\boxed{\dim(N(A)) = 3-2 = 1} = n-r$$

$$\boxed{\dim(C(A)) = \dim(C(A^T)) = 2} = \text{rank}$$

$$\boxed{\dim(N(A^T)) = 3-2 = 1} = m-r$$

#2

b)

$$C(A^T) = \text{span}\left\{ \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \right\}$$

$$N(A) \perp C(A^T)$$

$$\boxed{x_1} - 2x_2 + 3x_3 = 0$$
$$2\boxed{x_2} - x_3 = 0$$

Pivot, x_3 is free. Preferred solution $x_3 = 1$

$\therefore$ $x_2 = \frac{1}{2}$

$x_1 = 2x_2 - 3x_3 = 1 - 3 = -2$

$$N(A) = \text{span}\left\{ \begin{pmatrix} -2 \\ \frac{1}{2} \\ 1 \end{pmatrix} \right\}$$

$$\boxed{x = \begin{pmatrix} -2 \\ \frac{1}{2} \\ 1 \end{pmatrix}}$$

#2)

c) $N(A^T) \perp C(A)$

$$A^T = \begin{pmatrix} 1 & 2 & -3 \\ -2 & -2 & 4 \\ 3 & 5 & -8 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 2 & -3 \\ 0 & 2 & -2 \\ 3 & 5 & -8 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & -3 \\ 0 & 2 & -2 \\ 0 & -1 & -1 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} \boxed{1} & 2 & -3 \\ 0 & \boxed{2} & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

$x_1 + 2x_2 - 3x_3 = 0$

$2x_2 - 2x_3 = 0$ $\quad$ x_3 - free, $x_3 = 1$

$x_2 = x_3 = 1$

$x_1 + 2 - 3 = 0$ $\quad$ $x_1 = 1$

$N(A^T) = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$

$$\boxed{y = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}$$

#2)

d) $N(A) \perp C(A^T)$

$$C(A^T) = \text{Span}\left\{ \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \right\}$$

$$\boxed{z = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}}$$

$$\langle x, z \rangle = (-2)(0) + (2)(1/2) + (1)(-1) = 0$$

#3)

Basis for P_3: $(1, t, t^2, t^3)$

Basis for P_2: $(1, t, t^2)$ $\qquad P_2 \xrightarrow{T} P_3$

$$(t)(1) + \frac{d(1)}{dt} = t$$

$$(t)(t) + \frac{d(t)}{dt} = t^2 + 1$$

$$(t)(t^2) + \frac{d(t^2)}{dt} = t^3 + 2t$$

$$\boxed{T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

#4)

$$A \begin{pmatrix} 1 & -1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \qquad y = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$\hat{x} = (A^T A)^{-1} A^T y$$

$$A^T A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 4 & 14 \end{pmatrix}$$

$$(A^T A)^{-1} = \frac{1}{42-16} \begin{pmatrix} 14 & -4 \\ -4 & 3 \end{pmatrix} = \begin{pmatrix} \frac{14}{26} & -4/26 \\ -4/26 & 3/26 \end{pmatrix}$$

$$\begin{pmatrix} 14/26 & -4/26 \\ -4/26 & 3/26 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ -1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 18/26 & 6/26 & 2/26 \\ -7/26 & 2/26 & 5/26 \end{pmatrix}$$

$$\hat{x} = \begin{pmatrix} 18/26 & 6/26 & 2/26 \\ -7/26 & 2/26 & 5/26 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{36}{26} \\ \frac{12}{26} \end{pmatrix}$$

$$\boxed{\begin{aligned} \hat{C} &= 36/26 \\ \hat{D} &= 12/26 \end{aligned}}$$

#5)

a) True, if $A = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_m \\ | & & | \end{pmatrix}$, the rank$(A) = m$, which $= \dim(C(A)) = \text{span}(S)$.

b) False, T is not closed under both addition & subtraction

c) False, the dot product would be non-zero

d) False, a unique solution can only be found if A is square $(m=n)$.

e) True, if the rows are linearly independent, the determinate of A is non-zero $\therefore$ A has an inverse.