ECE 68000: MODERN AUTOMATIC CONTROL

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Fuzzy Modeling

Combining mathematical description of the plant with its linguistic description

- In many situations there may be human experts who can provide a linguistic description in terms of IF-THEN rules of the plant dynamical behavior
- Combining available mathematical description of the plant with its linguistic description results in a fuzzy system model
- Can construct a fuzzy model if local description of the plant is available in terms of local models,

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}_i \boldsymbol{x}(t) + \boldsymbol{B}_i \boldsymbol{u}(t), \quad i = 1, 2, \ldots, r,$$

where the state vector $\boldsymbol{x}(t) \in \mathbb{R}^n$, the control input $\boldsymbol{u}(t) \in \mathbb{R}^m$

Combine local models with IF-THEN rules

- The i-th rule can have the form:

$$\text{Rule i} \;:\; \text{IF } x_1(t) \text{ is } F_1^i \text{ AND} \cdots \text{ AND } x_n(t) \text{ is } F_n^i,$$

$$\text{THEN } \dot{\boldsymbol{x}}(t) = \boldsymbol{A}_i \boldsymbol{x}(t) + \boldsymbol{B}_i \boldsymbol{u}(t),$$

  where F_j^i, $j = 1, 2, \ldots, n$, is the j-th fuzzy set of the i-th rule
- Let $\mu_j^i(x_j)$ be the membership function of the fuzzy set F_j^i
- Let

$$w^i = w^i(\boldsymbol{x}) = \prod_{j=1}^{n} \mu_j^i(x_j)$$

- The resulting fuzzy system model is the weighted average of the local models

$$\dot{\boldsymbol{x}} = \frac{\sum_{i=1}^{r} w^i \left(\boldsymbol{A}_i \boldsymbol{x} + \boldsymbol{B}_i \boldsymbol{u}\right)}{\sum_{i=1}^{r} w^i}$$

Fuzzy model

- Fuzzy system model

$$
\begin{aligned}
\dot{\boldsymbol{x}} &= \frac{\sum_{i=1}^{r} w^i \left(\boldsymbol{A}_i \boldsymbol{x} + \boldsymbol{B}_i \boldsymbol{u}\right)}{\sum_{i=1}^{r} w^i} \\
&= \sum_{i=1}^{r} \alpha_i \left(\boldsymbol{A}_i \boldsymbol{x} + \boldsymbol{B}_i \boldsymbol{u}\right) \\
&= \left(\sum_{i=1}^{r} \alpha_i \boldsymbol{A}_i\right) \boldsymbol{x} + \left(\sum_{i=1}^{r} \alpha_i \boldsymbol{B}_i\right) \boldsymbol{u} \\
&= \boldsymbol{A}(\boldsymbol{\alpha}) \boldsymbol{x} + \boldsymbol{B}(\boldsymbol{\alpha}) \boldsymbol{u},
\end{aligned}
$$

where for $i = 1, 2, \ldots, r$,

$$
\alpha_i = \frac{w^i}{\sum_{i=1}^{r} w^i}
$$

Fuzzy model also known as polytopic model

- Note that, for $i = 1, 2, \ldots, r$,

$$\alpha_i \geq 0, \quad \sum_{i=1}^{r} \alpha_i = 1$$

and

$$\boldsymbol{\alpha} = \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_r \end{bmatrix}^\top \in [0, 1]^r$$

- For the reason above, fuzzy model is also referred to as in the literature as the *polytopic model*
- Fuzzy models are also called the Takagi-Sugeno (T-S) fuzzy models or Takagi-Sugeno-Kang (TSK) fuzzy models

Stabilizing fuzzy models

- Stabilization problem of a nonlinear plant: construct a controller so that starting from an arbitrary point, in some neighborhood of the operating point, the controller forces the closed-loop system trajectory to converge to this operating point
- If the starting point coincides with the operating point, the closed-loop system trajectory is expected to stay at this point for all subsequent time
- Asymptotic stability of continuous-time fuzzy model

$$\dot{\boldsymbol{x}} = \sum_{i=1}^{r} \alpha_i \boldsymbol{A}_i \boldsymbol{x},$$

where $\alpha_i = \alpha_i(\boldsymbol{x}(t)) \geq 0$ for $i = 1, 2, \ldots, r$, and $\sum_{i=1}^{r} \alpha_i = 1$

Stability of $\dot{\boldsymbol{x}} = \sum_{i=1}^{r} \alpha_i \boldsymbol{A}_i \boldsymbol{x}$

- A sufficient condition for asymptotic stability in the large of the equilibrium state $\boldsymbol{x} = \boldsymbol{0}$ of system $\dot{\boldsymbol{x}} = \sum_{i=1}^{r} \alpha_i \boldsymbol{A}_i \boldsymbol{x}$ is that there exists a symmetric positive definite matrix $\boldsymbol{P}$ such that for $i = 1, 2, \ldots, r$,

$$\boldsymbol{A}_i^\top \boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}_i \prec 0$$

- It is obvious that a necessary condition for the existence of a common symmetric positive definite $\boldsymbol{P}$ is that each $\boldsymbol{A}_i$ be asymptotically stable, that is, the eigenvalues of each $\boldsymbol{A}_i$ be in the open left-hand complex plane

Necessary condition for the existence of a common $\boldsymbol{P}$

Theorem

Suppose that each $\boldsymbol{A}_i$*,* $i = 1, 2, \ldots, r$*, is asymptotically stable and there exists a symmetric positive definite* $\boldsymbol{P}$*. Then, the matrices*

$$\sum_{k=1}^{s} \boldsymbol{A}_{i_k},$$

where $i_k \in \{1, 2, \ldots, r\}$ *and* $s = 1, 2, 3, \ldots, r$ *are asymptotically stable*

Proof of theorem

- Let
$$\boldsymbol{A}_{i_k}^\top \boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}_{i_k} = -\boldsymbol{Q}_{i_k},$$
where $\boldsymbol{Q}_{i_k} = \boldsymbol{Q}_{i_k}^\top \succ 0$ for $i_k \in \{1, 2, \dots, r\}$
- Summing both sides yields
$$\left(\sum_{k=1}^{s} \boldsymbol{A}_{i_k}^\top\right) \boldsymbol{P} + \boldsymbol{P}\left(\sum_{k=1}^{s} \boldsymbol{A}_{i_k}\right) = -\sum_{k=1}^{s} \boldsymbol{Q}_{i_k}.$$
- By assumption $\boldsymbol{P} = \boldsymbol{P}^\top \succ 0$, and because $\boldsymbol{Q}_{i_k} = \boldsymbol{Q}_{i_k}^\top \succ 0$ for $i_k \in \{1, 2, \dots, r\}$, the symmetric matrices
$$\sum_{k=1}^{s} \boldsymbol{Q}_{i_k}, \quad s = 2, 3, \dots, r,$$
are also positive definite
- By the Lyapunov theorem, the matrices $\sum_{k=1}^{s} \boldsymbol{A}_{i_k}$, $s = 2, 3, \dots, r$, are asymptotically stable $\square$

Some corollaries

Corollary

Suppose that each $\boldsymbol{A}_i$, $i = 1, 2, \ldots, r$ is asymptotically stable and there exists common symmetric positive definite $\boldsymbol{P}$. Then, the matrices

$$\boldsymbol{A}_i + \boldsymbol{A}_j, \quad i, j = 1, 2, \ldots, r,$$

are asymptotically stable

Equivalently

Corollary

If there exist i and j such that a matrix $\boldsymbol{A}_i + \boldsymbol{A}_j$ is not asymptotically stable, then there is no positive definite matrix $\boldsymbol{P}$ such that $\boldsymbol{A}_i^\top \boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}_i \prec 0$ for $i = 1, 2, \ldots, r$

Example

- Let
$$A_1 = \begin{bmatrix} -1 & 4 \\ 0 & -2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 0 \\ 4 & -2 \end{bmatrix}$$
- Obviously each A_i, $i = 1, 2$, is asymptotically stable
- However, there is no common positive definite P such that $A_i^\top P + PA_i \prec 0$ for $i = 1, 2$ because the matrix
$$A_1 + A_2 = \begin{bmatrix} -1 & 4 \\ 0 & -2 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 4 & -2 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ 4 & -4 \end{bmatrix}$$
is unstable since its eigenvalues are
$$\{1.1231, -7.1231\}$$