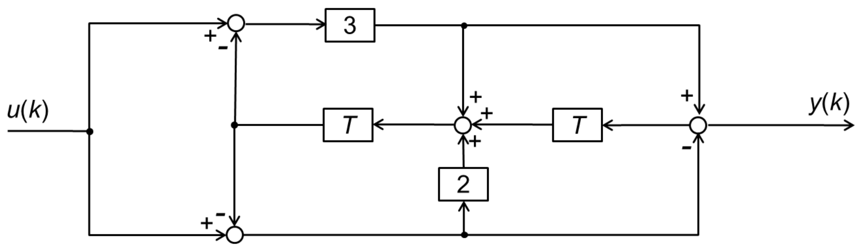
ECE 602 Midterm 1 Solution



Problem 1. (15 pts) A discrete-time system with the input $u[k]$ and output $y[k]$ is given above. The two blocks marked with "T" are time-delay units that delay their input signals by one time step (e.g., input $f[k]$ results in output $f[k-1]$). Find the transfer function relating $Y(z)$ and $U(z)$, where $Y(z)$ and $U(z)$ are the $\mathcal{Z}$-transforms of the output $y[k]$ and input $u[k]$, respectively.

Solution: Denote the outputs of the left and right time-delay units as $x_1[k]$ and $x_2[k]$. Then the state-space model is

$$\boldsymbol{x}[k+1] = \begin{bmatrix} -5 & 1 \\ -2 & 0 \end{bmatrix} \boldsymbol{x}[k] + \begin{bmatrix} 5 \\ 2 \end{bmatrix} u[k]$$
$$y[k] = \begin{bmatrix} -2 & 0 \end{bmatrix} \boldsymbol{x}[k] + 2u[k].$$

Or, with the roles of $x_1[k]$ and $x_2[k]$ switched, we have

$$\boldsymbol{x}[k+1] = \begin{bmatrix} 0 & -2 \\ 1 & -5 \end{bmatrix} \boldsymbol{x}[k] + \begin{bmatrix} 2 \\ 5 \end{bmatrix} u[k]$$
$$y[k] = \begin{bmatrix} 0 & -2 \end{bmatrix} \boldsymbol{x}[k] + 2u[k].$$

The transfer functions of the above state-space models are the same. We have

$$\begin{aligned} G(z) &= \boldsymbol{c}[z\boldsymbol{I}_2 - \boldsymbol{A}]^{-1}\boldsymbol{b} + d \\ &= \begin{bmatrix} -2 & 0 \end{bmatrix} \left[z\boldsymbol{I}_2 - \begin{bmatrix} -5 & 1 \\ -2 & 0 \end{bmatrix} \right]^{-1} \begin{bmatrix} 5 \\ 2 \end{bmatrix} + 2 \\ &= \begin{bmatrix} 0 & -2 \end{bmatrix} \left[z\boldsymbol{I}_2 - \begin{bmatrix} 0 & -2 \\ 1 & -5 \end{bmatrix} \right]^{-1} \begin{bmatrix} 2 \\ 5 \end{bmatrix} + 2 \\ &= \frac{2z^2}{z^2+5z+2}. \end{aligned}$$

Problem 2. (15 pts) Find the transfer function matrix relating $Y(s)$ and $U(s)$ of the following system:

$$\begin{aligned} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} &= \begin{bmatrix} -2 & 0 \\ -9 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \boldsymbol{u}(t), \\ y(t) &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}. \end{aligned}$$

Here, $Y(s)$ and $U(s)$ are the Laplace transforms of the output $y(t)$ and input $u(t)$, respectively.

Solution: We compute

$$\begin{aligned} G(s) &= \boldsymbol{c}\,[s\boldsymbol{I}_2 - \boldsymbol{A}]^{-1}\,\boldsymbol{B} \\ &= \frac{1}{(s-1)(s+2)} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s-1 & 0 \\ -9 & s+2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{s-7}{(s-1)(s+2)} & \frac{1}{s-1} \end{bmatrix}. \end{aligned}$$

Problem 3. (20 pts) Suppose the characteristic polynomial of an unknown matrix $\boldsymbol{A} \in \mathbb{R}^{3\times 3}$ is

$$\chi_{\boldsymbol{A}}(\lambda) = \det(\lambda \boldsymbol{I} - \boldsymbol{A}) = \lambda^3 - 4\lambda^2 + 5\lambda - 2.$$

(a) **(5 pts)** One of the zeros of the characteristic polynomial is at 2. What are the eigenvalues of $\boldsymbol{A}$? Is A one-to-one? onto?

(b) **(10 pts)** For a given $t \geq 0$, express $e^{\boldsymbol{A}t}$ as a linear combination of $\boldsymbol{I}$, $\boldsymbol{A}$, and $\boldsymbol{A}^2$.

(c) **(5 pts)** From the given information, can you tell if the continuous-time LTI system $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$ is stable, marginally stable, or unstable? If so, what is the conclusion? If not, explain why. What about the discrete-time LTI system $\boldsymbol{x}[k+1] = \boldsymbol{A}\boldsymbol{x}[k]$?

Solution:

(a) Since $\chi_{\boldsymbol{A}}(\lambda) = (\lambda-1)^2(\lambda-2)$, $\boldsymbol{A}$ has eigenvalues $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = 2$. Since $\boldsymbol{A}$ has non-zero eigenvalues, it is non-singular. Hence it is one-to-one and onto.

(b) Let $f(\lambda) = e^{\lambda t}$ and $g(\lambda) = a_0 + a_1\lambda + a_2\lambda^2$. For them to agree on the spectrum $\{1, 1, 2\}$ of A, we need

$$\begin{aligned} f(1) = g(1) \quad &\Rightarrow \quad a_0 + a_1 + a_2 = e^t \\ f'(1) = g'(1) \quad &\Rightarrow \quad a_1 + 2a_2 = te^t \\ f(2) = g(2) \quad &\Rightarrow \quad a_0 + 2a_1 + 4a_2 = e^{2t}. \end{aligned}$$

Solving the above, we have

$$a_0 = -2te^t + e^{2t}, \quad a_1 = 3te^t + 2e^t - 2e^{2t}, \quad a_2 = e^{2t} - e^t - te^t.$$

As a result,

$$e^{\boldsymbol{A}t} = g(\boldsymbol{A}) = \left(-2te^t + e^{2t}\right)\boldsymbol{I} + \left(3te^t + 2e^t - 2e^{2t}\right)\boldsymbol{A} + \left(e^{2t} - e^t - te^t\right)\boldsymbol{A}^2.$$

(c) Since $\boldsymbol{A}$ has positive eigenvalues, the system $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$ is unstable. The discrete-time LTI system $\boldsymbol{x}[k+1] = \boldsymbol{A}\boldsymbol{x}[k]$ is also unstable since the eigenvaules are all outside of the unit disc.

Problem 4. (25 pts) Consider the following two problems whose solutions are related.

(a) **(15 pts)** Given a matrix $\boldsymbol{A} = \begin{bmatrix} -3 & 1 \\ 0 & -1 \end{bmatrix}$, find $e^{\boldsymbol{A}t}$ for $t \geq 0$.

(b) **(10 pts)** Find the state transition matrix $\boldsymbol{\Phi}(t,s)$, $s,t \geq 0$, for the following system:

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} -3e^{-t} & e^{-t} \\ 0 & -e^{-t} \end{bmatrix} \boldsymbol{x}(t).$$

Solution:

(a) Using the Laplace transform method:

$$e^{\boldsymbol{A}t} = \mathcal{L}^{-1} \begin{bmatrix} s+3 & -1 \\ 0 & s+1 \end{bmatrix}^{-1} = \mathcal{L}^{-1} \begin{bmatrix} \frac{1}{s+3} & \frac{1}{(s+1)(s+3)} \\ 0 & \frac{1}{s+1} \end{bmatrix} = \begin{bmatrix} e^{-3t} & \frac{1}{2}(e^{-t} - e^{-3t}) \\ 0 & e^{-t} \end{bmatrix}.$$

(b) Note at $\boldsymbol{A}(t)$ and $\boldsymbol{A}(s)$ commute for all s,t. Thus,

$$\begin{aligned} \boldsymbol{\Phi}(t,s) = e^{\int_s^t \boldsymbol{A}(\tau)\,d\tau} &= \exp\left(\begin{bmatrix} -3(e^{-s} - e^{-t}) & e^{-s} - e^{-t} \\ 0 & -(e^{-s} - e^{-t}) \end{bmatrix} \right) \\ &= \begin{bmatrix} e^{-3(e^{-s}-e^{-t})} & \frac{1}{2}\left(e^{-(e^{-s}-e^{-t})} - e^{-3(e^{-s}-e^{-t})}\right) \\ 0 & e^{-(e^{-s}-e^{-t})} \end{bmatrix}, \end{aligned}$$

where in the last step we have used the result in (a) with t there replaced with $e^{-s} - e^{-t}$.

Problem 5. (25 pts) Consider a linear system $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$ with $A \in \mathbb{R}^{4\times 4}$ given by

$$\boldsymbol{A} = \underbrace{\begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 & \boldsymbol{v}_4 \end{bmatrix}}_{\boldsymbol{T}} \underbrace{\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\boldsymbol{J}} \underbrace{\begin{bmatrix} \boldsymbol{w}_1^T \\ \boldsymbol{w}_2^T \\ \boldsymbol{w}_3^T \\ \boldsymbol{w}_4^T \end{bmatrix}}_{\boldsymbol{T}^{-1}}.$$

(a) **(6 pts)** Find all the modes of the solutions to the system $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$.

(b) **(3 pts)** Given the initial condition $\boldsymbol{x}(0) = \boldsymbol{v}_1 - 2\boldsymbol{v}_3 + 3\boldsymbol{v}_4 \in \mathbb{R}^4$, find the corresponding solution $\boldsymbol{x}(t)$ and write it as a proper linear combination of the modes obtained in (a).

(c) **(6 pts)** Find three **nonzero** initial conditions $\boldsymbol{x}(0)$ from which the solution $\boldsymbol{x}(t)$ will

(i) converge to 0; (ii) remain constant; (iii) diverge to infinity, as $t \to \infty$.

Now consider the discrete-time system $\boldsymbol{x}[k+1] = \boldsymbol{A}\boldsymbol{x}[k]$ with $\boldsymbol{A}$ given above.

(d) **(6 pts)** Find all the modes of the solutions to the system $\boldsymbol{x}[k+1] = \boldsymbol{A}\boldsymbol{x}[k]$.

(e) **(4 pts)** Is the discrete-time system $\boldsymbol{x}[k+1] = \boldsymbol{A}\boldsymbol{x}[k]$ stable, marginally stable, or unstable?

Solution:

(a) We compute

$$\begin{aligned}
\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}(0) &= \begin{bmatrix}\boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 & \boldsymbol{v}_4\end{bmatrix} \begin{bmatrix} e^{-t} & 0 & 0 & 0 \\ 0 & 1 & t & \frac{1}{2}t^2 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{w}_1^T \\ \boldsymbol{w}_2^T \\ \boldsymbol{w}_3^T \\ \boldsymbol{w}_4^T \end{bmatrix} \boldsymbol{x}(0) \\
&= \begin{bmatrix} \underbrace{e^{-t}\boldsymbol{v}_1}_{\boldsymbol{x}^{(1)}(t)} & \underbrace{\boldsymbol{v}_2}_{\boldsymbol{x}^{(2)}(t)} & \underbrace{t\boldsymbol{v}_2 + \boldsymbol{v}_3}_{\boldsymbol{x}^{(3)}(t)} & \underbrace{\frac{1}{2}t^2\boldsymbol{v}_2 + t\boldsymbol{v}_3 + \boldsymbol{v}_4}_{\boldsymbol{x}^{(4)}(t)} \end{bmatrix} \begin{bmatrix} \boldsymbol{w}_1^\top\boldsymbol{x}(0) \\ \boldsymbol{w}_2^\top\boldsymbol{x}(0) \\ \boldsymbol{w}_3^\top\boldsymbol{x}(0) \\ \boldsymbol{w}_4^\top\boldsymbol{x}(0) \end{bmatrix} \\
&= [\boldsymbol{w}_1^\top\boldsymbol{x}(0)]\boldsymbol{x}^{(1)}(t) + [\boldsymbol{w}_2^\top\boldsymbol{x}(0)]\boldsymbol{x}^{(2)}(t) + [\boldsymbol{w}_3^\top\boldsymbol{x}(0)]\boldsymbol{x}^{(3)}(t) + [\boldsymbol{w}_4^\top\boldsymbol{x}(0)]\boldsymbol{x}^{(4)}(t),
\end{aligned}$$

where the four modes $\boldsymbol{x}^{(i)}(t)$, $i = 1, 2, 3, 4$, are labeled above.

(b) If $\boldsymbol{x}(0) = \boldsymbol{v}_1 - 2\boldsymbol{v}_3 + 3\boldsymbol{v}_4$, then using the fact that $\boldsymbol{w}_i^\top\boldsymbol{v}_j = 0$ for $i \neq j$ and $\boldsymbol{w}_i^T\boldsymbol{v}_i = 1$, we have

$$\boldsymbol{x}(t) = \boldsymbol{x}^{(1)}(t) - 2\boldsymbol{x}^{(3)}(t) + 3\boldsymbol{x}^{(4)}(t).$$

(c) For (i) we can choose $\boldsymbol{x}(0) = \alpha\boldsymbol{v}_1$ for any $\alpha \neq 0$. For (ii) we can choose $\boldsymbol{x}(0) = \alpha\boldsymbol{v}_2$ for any $\alpha \neq 0$. For (iii) we can choose any $\boldsymbol{x}(0)$ such that either $\boldsymbol{w}_3^\top\boldsymbol{x}(0) \neq 0$ or $\boldsymbol{w}_4^\top\boldsymbol{x}(0) \neq 0$, e.g., $\boldsymbol{x}(0) = \boldsymbol{v}_3$ or $\boldsymbol{x}(0) = \boldsymbol{v}_4$.

(d) For the discrete-time system, the solution is

$$\begin{aligned}
\boldsymbol{x}[k] = \boldsymbol{A}^k\boldsymbol{x}[0] &= \begin{bmatrix}\boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 & \boldsymbol{v}_4\end{bmatrix} \begin{bmatrix} (-1)^k & 0 & 0 & 0 \\ 0 & 0^k & k \cdot 0^{k-1} & \frac{k(k-1)}{2} \cdot 0^{k-2} \\ 0 & 0 & 0^k & k \cdot 0^{k-1} \\ 0 & 0 & 0 & 0^k \end{bmatrix} \begin{bmatrix} \boldsymbol{w}_1^\top \\ \boldsymbol{w}_2^\top \\ \boldsymbol{w}_3^\top \\ \boldsymbol{w}_4^\top \end{bmatrix} \boldsymbol{x}[0] \\
&= \begin{bmatrix} \underbrace{(-1)^k\boldsymbol{v}_1}_{\boldsymbol{x}^{(1)}[k]} & \underbrace{0^k \cdot \boldsymbol{v}_2}_{\boldsymbol{x}^{(2)}[k]} & \underbrace{k \cdot 0^{k-1}\boldsymbol{v}_2 + 0^k \cdot \boldsymbol{v}_3}_{\boldsymbol{x}^{(3)}[k]} & \underbrace{\frac{k(k-1)}{2} \cdot 0^{k-2} \cdot \boldsymbol{v}_2 + k \cdot 0^{k-1} \cdot \boldsymbol{v}_3 + 0^k \cdot \boldsymbol{v}_4}_{\boldsymbol{x}^{(4)}[k]} \end{bmatrix} \begin{bmatrix} \boldsymbol{w}_1^\top\boldsymbol{x}[0] \\ \boldsymbol{w}_2^\top\boldsymbol{x}[0] \\ \boldsymbol{w}_3^\top\boldsymbol{x}[0] \\ \boldsymbol{w}_4^\top\boldsymbol{x}[0] \end{bmatrix}.
\end{aligned}$$

In the above, 0^k takes the value 1 for $k = 0$ and the value 0 for any other k. The four modes $\boldsymbol{x}^{(i)}[k]$ are labeled above.

(e) The discrete-time system is marginally stable due to the nondefective eigenvalue at -1.