MA 527

Lecture Notes (section 7.9)

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7.9. Vector spaces.

$(V, \oplus, \odot)$: a vector space.

(Ex) $\mathbb{P}_2 = \{ p(x) : p(x) = a_2x^2 + a_1x + a_0,$
$a_2, a_1, a_0 \in \mathbb{R} \}$

let $p(x) = a_2x^2 + a_1x + a_0, \quad q(x) = b_2x^2 + b_1x + b_0 \in \mathbb{P}_2$

$p(x) \oplus q(x) = (a_2+b_2)x^2 + (a_1+b_1)x + a_0 + b_0 \in \mathbb{P}_2.$

$\beta \in \mathbb{R}$

$\beta \odot p(x) = \beta a_2 x^2 + \beta a_1 x + \beta a_0 \in \mathbb{P}_2$

Q: $(\mathbb{P}_2, \oplus, \odot)$: a vector space?

(1) $p(x) + q(x) \overset{?}{=} q(x) + p(x)$

(Proof) $p(x) + q(x) = (a_2+b_2)x^2 + (a_1+b_1)x + a_0 + b_0$
$= (b_2+a_2)x^2 + (b_1+a_1)x + (b_0+a_0)$
$= q(x) + p(x).$

(2)

(3) $0 = ?$: $p(x) + ? = p(x).$

$0 = 0\cdot x^2 + 0\cdot x + 0 \in \mathbb{P}_2$. : the identity.

(4) $\exists\, -p(x)$: the inverse of $p(x)$.

(5). $k(p(x)+q(x)) \overset{?}{=} kp(x)+kq(x)$

⋮

(8) $L = k(p(x)+q(x)) = k((a_2+b_2)x^2+(a_1+b_1)x+a_0+b_0)$

$= (ka_2+kb_2)x^2+(ka_1+kb_1)x+(ka_0+kb_0)$

R RHS.

$= kp(x)+kq(x)$

$= ka_2x^2+ka_1x+ka_0+kb_2x^2+kb_1x+kb_0$

$= (ka_2+kb_2)x^2+(ka_1+kb_1)x+ka_0+kb_0$

Same.

(Ex) $(\mathbb{R}^2, +, \cdot)$: a vector space

Q: what if we define different operations?

$\oplus$: $[a\ b] \oplus [c\ d] = [a+c+1,\ b+d]$

$\odot$ $\alpha \in \mathbb{R}$: $\alpha \odot [a\ b] = [\alpha a, \alpha b]$.

$0 = ?$ the identity of $\oplus$:

$[a, b] \oplus \underset{[x,y]}{?} = [a, b]$

L $[a+x+1,\ b+y] = [a, b]$ R

$y=0$; $\cancel{a}+x+1 = \cancel{a}$: $x=-1$.

$0=[-1, 0]$.

(5) distribution.

$k=2$: Check $2\odot([a,b]\oplus[c,d])$

$\overset{?}{=} 2[a,b]\oplus 2[c,d]$. Ⓡ

Ⓛ: $2([a+c+1,\ b+d]) = [2a+2c+2,\ 2b+2d]$

Ⓡ $= [2a,\ 2b]\oplus[2c,\ 2d]$

$= [2a+2c+1,\ 2b+2d]$.

(Dot product & Inner product)

Def $\mathbb{a}=[a_1,\cdots,a_n]$, $\mathbb{b}=[b_1,\cdots,b_n]$ in $\mathbb{R}^n$

$\mathbb{a}\cdot\mathbb{b} = a_1b_1 + a_2b_2 + \cdots + a_nb_n. = \mathbb{a}\mathbb{b}^T$

(Properties)

(1) $(\alpha\mathbb{a}+\beta\mathbb{b})\cdot\mathbb{c} = \alpha\mathbb{a}\cdot\mathbb{c}+\beta\mathbb{b}\cdot\mathbb{c}$

(2) $\mathbb{a}\cdot\mathbb{b} = \mathbb{b}\cdot\mathbb{a}$

(3) $\mathbb{a}\cdot\mathbb{a} = |\mathbb{a}|^2 \geq 0$

If $\mathbb{a}\cdot\mathbb{a}=0$, $\mathbb{a}=0$

(4) $\mathbb{a} \cdot \mathbb{b} = |\mathbb{a}||\mathbb{b}| \cos\theta$

(5) $\cos\theta = \dfrac{\mathbb{a} \cdot \mathbb{b}}{|\mathbb{a}|\,|\mathbb{b}|}$: $\theta = \cos^{-1}\left(\dfrac{\mathbb{a}\cdot\mathbb{b}}{|\mathbb{a}||\mathbb{b}|}\right)$.

<u>Def</u> (Inner product).

$V = (V, \oplus, \odot)$: a vector space

$(\cdot, \cdot) : V \times V \to \mathbb{R}$ is called an <u>inner product</u>

if (1) for any $\alpha \in \mathbb{R}$, $\mathbb{a}, \mathbb{b}, \mathbb{c} \in V$

$(\alpha \mathbb{a} + \beta \mathbb{b}, \mathbb{c}) = \alpha(\mathbb{a}, \mathbb{c}) + \beta(\mathbb{b}, \mathbb{c})$

Linear property.

(2) $(\mathbb{a}, \mathbb{b}) = (\mathbb{b}, \mathbb{a})$

(3) For any $\mathbb{a} \in V$, $(\mathbb{a}, \mathbb{a}) \geq 0$

If $(\mathbb{a}, \mathbb{a}) = 0$, $\mathbb{a} = 0$: the identity.

<u>Ex / Application</u>

1. Dot product : an inner product.

2. V : the set of square-integrable functions. on $[a, b]$

$f \in V$ iff $\int_a^b f(x)^2\, dx$ is finite.

$(f, g) \overset{\text{define}}{=} \int_a^b f(x)\, g(x)\, dx.$

: an inner product.

(Proof) (1) $(\alpha f + \beta g, h) = \alpha (f, h) + \beta (g, h)$

for any $f, g, h \in V$

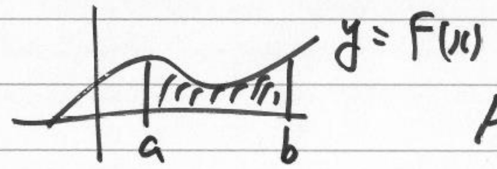
$(\because$ Ⓛ $= (\alpha f + \beta g, h) = \int_a^b (\alpha f(x) + \beta g(x)) h(x)\, dx$

$= \alpha \int_a^b f(x) h(x)\, dx + \beta \int_a^b g(x) h(x)\, dx$

$= \alpha (f, h) + \beta (g, h) =$ Ⓡ

(2) ✓

(3) $(f, f) = \int_a^b f(x)^2\, dx \geq 0$



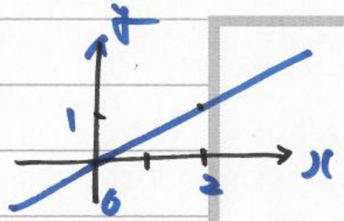
$\text{Area} = \int_a^b F(x)\, dx$

If $(f, f) = 0$, $f(x)^2 = 0$: $f(x) = 0$

(Linear transformation).

(Ex) $y = f(x) = \frac{1}{2} x$

a linear polynomial.



1. $f(x_1 + x_2) = \frac{1}{2}(x_1 + x_2) = \frac{1}{2}x_1 + \frac{1}{2}x_2$

$= f(x_1) + f(x_2)$

2. $f(\alpha x) = \frac{1}{2}\alpha x = \alpha(\frac{1}{2}x) = \alpha f(x)$.

<u>Def</u> X, Y: vector spaces

A function $F: X \to Y$ is called a <u>linear transformation</u> if for any $X_1, X_2 \in X$ and $c \in \mathbb{R}$

(1) $F(X_1 + X_2) = F(X_1) + F(X_2)$

(2) $F(cX_1) = cF(X_1)$.

<u>Remark</u>: (1), (2) means

$$F(\alpha_1 X_1 + \beta X_2) = \alpha F(X_1) + \beta F(X_2)$$

(Ex) Let $F: \mathbb{R}^2 \to \mathbb{R}^2$ be a function defined by $F\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 - 2x_2 \\ 3x_1 + 4x_2 \end{pmatrix}$.

$$F\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \underbrace{\begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}}_{=A}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

$F(X) = AX$. is a <u>linear transformation</u>

(Proof) 1. $F(X_1 + X_2) = A(X_1 + X_2) = AX_1 + AX_2$

$= F(X_1) + F(X_2)$

2. $F(cX_1) = AcX_1 = cAX_1 = cF(X_1)$

<u>Thm</u> If $F : X \to Y$ is a linear transformation, then $F(X)$ is represented by a matrix:

$$\underline{F(X) = AX.}$$

(Ex) $F\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \underbrace{\begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}}_{A}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. : $F^{-1} = ?$

$$A^{-1} = \frac{1}{4+6}\begin{bmatrix} 4 & 2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 0.4 & 0.2 \\ -0.3 & 0.1 \end{bmatrix}$$

$F^{-1}(X) = A^{-1}X$

Nonlinear transformation?

$F : \mathbb{R}^2 \to \mathbb{R}^2$: $\quad F\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1^2 + x_2 \\ x_1 x_2 + e^{x_2} \end{pmatrix}$

(Orthogonality).

(Ex) Find all vectors in $\mathbb{R}^2$ orthogonal to $u = [1, 2]$.

<u>Do they form a vector space?</u> Yes.

Find $v = [v_1, v_2]$ s.t. $u \cdot v = 0$

$v_1 + 2v_2 = 0$. iff $v_1 = -2v_2$

$v = [-2v_2,\ v_2] = v_2[-2,\ 1].$

Let $W = \{ v \in \mathbb{R}^2 \mid v \cdot u = 0 \}$

$W = \mathrm{span}\{[-2,\ 1]\}$: a subspace of $\mathbb{R}^2$.