A1) $\begin{pmatrix} PA + A^TP + C^TC + 2\alpha P & PB \\ B^TP & -\gamma^{-2} I \end{pmatrix} \leq 0$

$\dot{x} = Ax + B\phi \qquad z = Cx$

$\|\phi\| \leq \gamma \|z\|$

Choose canidate Lyapunov function: $V = x^TPx$

$\therefore \dot{V} = \dot{x}^TPx + x^TP\dot{x} = 2x^TP\dot{x} = 2x^TP(Ax + B\phi)$

$\dot{V} = x^T(PA + A^TP)x + 2x^TPB\phi$, $x^T(PA + A^TP)x = 2x^TPAx$

Scalar function

$2x^TPB\phi \leq 2|B^TPx||\phi|$

$\phi \leq \gamma|z| \leq \gamma\|Cx\|$

$\therefore 2x^TPB\phi \leq 2|B^TPx|\gamma|Cx|$

$2ab \leq a^2 + b^2 \quad \therefore \quad 2x^TPB\phi \leq \gamma^2|B^TPx|^2 + |Cx|^2$

$\gamma^2|B^TPx|^2 + |Cx|^2 = \gamma^2(B^TPx)^T(B^TPx) + (Cx)^T(Cx)$

$= \gamma^2 x^TPBB^TPx + x^TC^TCx$

$\therefore 2x^TPB\phi \leq \gamma^2 x^TPBB^TPx + x^TC^TCx$

& $\dot{V} = 2x^TP\dot{x} \leq x^T(PA + A^TP)x + \gamma^2 x^TPBB^TPx + x^TC^TCx$

<u>For G.E.S :</u> $\dot{V} \leq -2\alpha V$

<u>From Schur Complement :</u> IF $\begin{pmatrix} PA + A^TP + C^TC + 2\alpha P & PB \\ B^TP & -\gamma^2 I \end{pmatrix} \leq 0$

Then $PA + A^TP + C^TC + 2\alpha P \leq 0$

$\therefore \boxed{\dot{V} \leq -2\alpha x^TPx}$

#2) $\dot{x} = Ax + B_1\phi_1 + B_2\phi_2 \qquad \|\phi_i\| \le \gamma \|z_i\|$

$\ddot{\theta}_1 + 2\dot{\theta}_1 - \dot{\theta}_2 + 2K\theta_1 - K\theta_2 - \sin\theta_1 = 0$

$\ddot{\theta}_2 - \dot{\theta}_1 + \dot{\theta}_2 - K\theta_1 + K\theta_2 - \sin\theta_2 = 0$

$x = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix} \quad \therefore \quad A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2K & K & -2 & 1 \\ K & -K & 1 & -1 \end{pmatrix}$

$z_1 = C_1 x = \theta_1 = (1\ 0\ 0\ 0)x \quad \therefore \quad C_1 = (1\ 0\ 0\ 0)$

$z_2 = C_2 x = \theta_2 = (0\ 1\ 0\ 0)x \quad \therefore \quad C_2 = (0\ 1\ 0\ 0)$

$\therefore \phi_1 = \sin(z_1), \ \phi_2 = \sin(z_2) \quad \therefore B_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \ \& \ B_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

$\|\phi_1\| = \|\sin(z_1)\| \le \|z_1\|$

$\|\phi_2\| = \|\sin(z_2)\| \le \|z_2\|$

$\therefore \gamma = 1 \qquad \tilde{B} = (\lambda_1^{-1}B_1 \quad \lambda_2^{-1}B_2)$

$\tilde{C} = \begin{pmatrix} \lambda_1 C_1 \\ \lambda_2 C_2 \end{pmatrix}$

Let $\lambda_1 = 2$, $\lambda_2 = 3$ $\therefore$

$\tilde{B} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1/2 & 0 \\ 0 & 1/3 \end{pmatrix} \quad \& \quad \tilde{C} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{pmatrix}$

$\mu_1 = \lambda_1^2 = 4 \qquad \mu_2 = \lambda_2^2 = 9$

<u>QMI</u>: $PA + A^TP + \gamma^2 P\tilde{B}\tilde{B}^TP + \tilde{C}^T\tilde{C} < 0$

<u>LMI</u>: $\begin{pmatrix} PA + A^TP + \mu_1 C_1^T C + \mu_2 C_2^T C_2 & \gamma PB_1 & \gamma PB_2 \\ \gamma B_1^T P & -\mu_1 I & 0 \\ \gamma B_2^T P & 0 & -\mu_2 I \end{pmatrix} < 0$

#2)

$\boxed{\begin{array}{l}\text{From MATLAB, it was found that a spring constant of} \\ K = 20.9 \text{ gurantees that the system is G.E.S.} \\ \text{about the zero solution as } P = P^T > 0 \text{ exists} \\ \& \ PA + A^TP + \gamma^2 P\tilde{B}\tilde{B}^TP + \tilde{C}^T\tilde{C} < 0 \text{, where } \gamma = 1.\end{array}}$

P has positive eigenvalues & $PA + A^TP + \gamma^2 P\tilde{B}\tilde{B}^TP + \tilde{C}^TC$ has negative eigenvalues.

#3) $\dot{x} = Ax - B\phi$ $\quad$? $(\dot{x} = Ax + Bw)$

$z^T \phi \geq 0$ $\quad$ $(z = Cx, \; w = -\phi)$

$PA + A^T P + 2\alpha P \leq 0$

$B^T P = C$

Choose $V(x) = x^T P x$

$\therefore \dot{V} = 2x^T P \dot{x} = 2x^T P(Ax - B\phi) = 2x^T PAx - 2x^T PB\phi$

$= x^T(A^T P + PA)x - 2x^T PB\phi$, $x^T(A^T P + PA)x = 2x^T PAx$ for Scalar system

$\dot{V} = x^T(A^T P + PA)x - 2x^T C^T \phi$ $\quad$ $(B^T P = C, \; PB = C^T)$

$= x^T(A^T P + PA)x - 2(Cx)^T \phi$ $\quad$ $(z = Cx)$

$\therefore \dot{V} = x^T(A^T P + PA)x - 2z^T\phi$

$z^T\phi \geq 0$ $\therefore$ $-z^T\phi \leq 0$ $\therefore$ $-2z^T\phi \leq 0$

$\therefore \dot{V} \leq x^T(A^T P + PA)x$

$PA + A^T P + 2\alpha P \leq 0$ $\therefore$ $PA + A^T P \leq -2\alpha P$

$\therefore \dot{V} \leq x^T(-2\alpha P)x$

For G.E.S: $\dot{V} \leq -2\alpha V$

$\dot{V} \leq -2\alpha x^T P x = -2\alpha V$ $\quad$ $(V = x^T P x)$

$\therefore \dot{V} \leq -2\alpha V$ & the system is Globally exponentially stable about the origin with rate α for $P = P^T \succ 0$.

#4) $\hat{g}(s) = \dfrac{\beta s + 1}{s^2 + s + 2}$

For SPR: g is stable, $\hat{g} + \hat{g}^T > 0$ &

$\lim_{|\omega| \to \infty} \omega^{2P} \det[\hat{g} + \hat{g}^T] \neq 0$

Poles of $g(s)$: $s^2 + s + 2 = 0 \Rightarrow s_{1,2} = -\frac{1}{2} \pm \frac{\sqrt{1-8}}{2}$

$s_{1,2} = -\frac{1}{2} \pm \frac{\sqrt{7}}{2}$, Negative real part $\therefore$ $\hat{g}(s)$ is stable for any β.

$$g(i\omega) = \frac{\beta i\omega + 1}{(i\omega)^2 + i\omega + 2} = \frac{\beta i\omega + 1}{-\omega^2 + i\omega + 2}$$

$$g(i\omega)^T = \frac{-\beta i\omega + 1}{(-i\omega)^2 - i\omega + 2} = \frac{-\beta i\omega + 1}{-\omega^2 - i\omega + 2}$$

$$g(i\omega) + g(i\omega)^T = \frac{(\beta i\omega + 1)(-\omega^2 - i\omega + 2)}{(-\omega^2 + i\omega + 2)(-\omega^2 - i\omega + 2)} + \frac{(-\beta i\omega + 1)(-\omega^2 + i\omega + 2)}{(-\omega^2 - i\omega + 2)(-\omega^2 + i\omega + 2)}$$

$$= \frac{-\cancel{\beta i\omega^3} + \beta\omega^2 + 2\cancel{\beta} i\omega - \omega^2 - \cancel{i\omega} + 2 + \cancel{\beta i\omega^3} + \beta\omega^2 - 2\cancel{\beta} i\omega - \omega^2 + \cancel{i\omega} + 2}{(-\omega^2 + i\omega + 2)(-\omega^2 - i\omega + 2)}$$

$$= \frac{2\beta\omega^2 - 2\omega^2 + 4}{\omega^4 + \cancel{i\omega^3} - 2\omega^2 - \cancel{i\omega^3} + \omega^2 + \cancel{2i\omega} - 2\omega^2 - \cancel{2i\omega} + 4} = \frac{2\beta\omega^2 - 2\omega^2 + 4}{\omega^4 - 3\omega^2 + 4}$$

$$= \frac{2\omega^2(\beta - 1) + 4}{\omega^4 - 3\omega^2 + 4} > 0 \text{ for all } \omega \text{ if } \beta \geq 1$$

$\therefore$ $g(i\omega) + g(i\omega)^T > 0$ for all ω if $\beta \geq 1$

#4) $|G| = \left|\dfrac{Bi\omega + 1}{-\omega^2 + i\omega + 2}\right| = \dfrac{\sqrt{1 + (B\omega)^2}}{\sqrt{(-\omega^2+2)^2 + (\omega)^2}}$

$\lim_{|\omega|\to\infty} |G| = \left.\dfrac{1}{\omega^4}\right|_{\omega\to\infty} = 0 \quad \therefore \; g(\infty) = 0 = D = \lim_{|\omega|\to\infty} g(i\omega)$

Scalar valued TF with $D = 0 \;\therefore\; \rho = 1$

Asymptotic side condition: $\lim_{\|\omega\|\to\infty} \omega^2 [g + g^T] \neq 0$

$$\lim_{\|\omega\|\to\infty} \left(\frac{2\omega^4(B-1) + 4}{\omega^4 - 3\omega^2 + 4}\right) = 2(B-1) \neq 0 \quad \text{if } B > 1$$

$\therefore \lim_{\|\omega\|\to\infty} \omega^2 [g + g^T] \neq 0$ if $B > 1$.

$\hat{g}(s)$ is strictly positive real only if $B > 1$, then the transfer function is stable, dissipative, & the $\lim_{|\omega|\to\infty} \omega^2 \det[\hat{g}(i\omega) + \hat{g}(i\omega)^T] \neq 0$.

For $\alpha = 0.001$, $B = 1.1$ gives a $P = P^T \geq 0$ for

$$\begin{pmatrix} PA + A^TP + 2\alpha P & PB - C^T \\ B^TP - C & -(D + D^T) \end{pmatrix} \leq 0$$

is satisfied & (A, B) is controllable & (A, C) is observable.

<u>From MATLAB</u>: $A = \begin{pmatrix} -1 & -2 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $C = \begin{pmatrix} 1.1 & 1 \end{pmatrix}$

& $D = 0$

#14)

Controllability Gramian, $W_c = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/4 \end{pmatrix}$ which is invertible $\therefore$ system is Controllable.

Observability Gramian, $W_o = \begin{pmatrix} 0.855 & 0.25 \\ 0.25 & 0.86 \end{pmatrix}$ is invertible $\therefore$ system is observable.

$P = \begin{pmatrix} 1.1 & 1 \\ 1 & 3.63 \end{pmatrix}$ with $\lambda_1 = 0.75$ & $\lambda_2 = 3.98$

$\therefore P = P^T \succ 0$.

$$\begin{pmatrix} PA + A^TP + 2\alpha P & PB - C^T \\ B^TP - C & -(D + D^T) \end{pmatrix} = \begin{pmatrix} -.1978 & .4365 & 0 \\ .4365 & -3.99 & 0 \\ 0 & 0 & 0 \end{pmatrix} = Q$$

with $\lambda_1 = -4.04$, $\lambda_2 = -0.148$, & $\lambda_3 = 0$ $\therefore$

$Q \preceq 0$ & LMI is satisfied.

Contents

Homework 7 Gabriel Colangelo

```
clear
close all
clc
```

Problem 2

```
% Scaling parameters
lambda1 = 2;
lambda2 = 3;
mu1     = lambda1^2;
mu2     = lambda2^2;

% xdot  = Ax + B1*phi1 + B2*phi2
% zi    = Ci*X
C1      = [1 0 0 0];
C2      = [0 1 0 0];
B1      = [0 0 1 0]';
B2      = [0 0 0 1]';

% Initialize spring constant
K       = 0.1;

% Initialize counter
count   = 0;

% Initialize while loop logic
tfeas   = 1;

% Options for feasp - silent
opts    = [0;0;0;0;1];

% Create iterative loop
while tfeas > -.01

    % Increase counter
    count   = count + 1;

    % Create counter break
    if count > 1000
        disp('Stable spring constant not found')
        fprintf('\n')
        break
    end

    % LMI toolbox setup
    setlmis([]);

    % Constant matrix
    A       = [0 0 1 0; 0 0 0 1; -2*K K -2 1; K -K 1 -1];

    % Positive definite matrix
    P       = lmivar(1, [4,1]);

    % Create LMI -[PA+A'P+C1'C1+C2'C2 PB1 PB2; B1'P -I 0; B2'P 0 -I];
    lmi1    = newlmi;
```

```
    lmiterm([lmi1,1,1,P],1,A,'s');      % PA + A'P
    lmiterm([lmi1 1 1 0],mu1*(C1'*C1)); %mu1*C1'C1
    lmiterm([lmi1 1 1 0],mu2*(C2'*C2)); %mu2*C2'C2
    lmiterm([lmi1 1 2 P],1,B1);         %PB1
    lmiterm([lmi1 1 3 P],1,B2);         %PB2
    lmiterm([lmi1 2 2 0],-mu1);         %-mu1*I
    lmiterm([lmi1 3 3 0],-mu2);         %-mu2*I

    Plmi    = newlmi;
    lmiterm([-Plmi,1,1,P],1,1);
    lmiterm([Plmi,1,1,0],1);
    lmis    = getlmis;

    % Solve LMIS
    [tfeas, xfeas] = feasp(lmis,opts);

    % Create P matrix
    P       = dec2mat(lmis,xfeas,P);

    % If not feasible, increase K
    if tfeas > -.01
       K       = K + .1;
    end

end

% Check P
fprintf('\n')
disp('Final P is')
disp(P)

disp('Eigenvalues of P are')
disp(eig(P))

fprintf(['A spring constant value that guarantees' ...
        ' the system is globally exponentially stable about' ...
        ' the zero solution is K = %.1f \n'],K)

% Check QMI
Ctilde = [lambda1*C1;lambda2*C2];
Btilde = [lambda1^-1*B1, lambda2^-1*B2];
Q      = P*A + A'*P + P*Btilde*Btilde'*P + Ctilde'*Ctilde;

fprintf('\n')
disp('The QMI from Equation 14.37 is:')
disp(Q)
disp('With eigenvalues:')
disp(eig(Q))
```

```
Final P is
  639.3024 -384.3185    3.8190   -4.9162
 -384.3185  258.6842   -1.7025    3.3037
    3.8190   -1.7025   12.2526   -6.0652
   -4.9162    3.3037   -6.0652    6.3122

Eigenvalues of P are
    2.5066
   15.9522
   20.1819
  877.9106

A spring constant value that guarantees the system is globally exponentially stable about the zero solution is K = 20.9

The QMI from Equation 14.37 is:
 -514.4370  319.3492    2.8361    0.6291
  319.3492 -198.3229   -2.2082   -0.1117
```

```
   2.8361   -2.2082  -11.8837    7.3092
   0.6291   -0.1117    7.3092   -4.5236

With eigenvalues:
 -712.7215
  -16.3885
   -0.0450
   -0.0123
```

Problem 4

```
% Laplace variable
s       = tf('s');

% Initial beta
beta    = 0;

% Initialize counter
count   = 0;

% Initialize Loop logic
logic   = 0;

% Define small posiitve alpha
alpha   = 1e-3;

while logic == 0

    % Increase counter
    count   = count + 1;

    % Create counter break
    if count > 1000
        disp('SPR beta not found')
        fprintf('\n')
        break
    end

    % Transfer Funtion
    g       = (beta*s + 1)/(s^2 + s + 2);

    % Transfer function to state space
    [A,B,C,D] = tf2ss(g.Numerator{1},g.Denominator{1});

    % Get size of A
    n        = length(A);

    cvx_begin sdp quiet

    % Variable definition
    variable P(n, n) symmetric

    % LMIs
    [(P*A + A'*P + 2*alpha*P), (P*B - C');...
     (B'*P - C), -(D + D')] <= -eps*eye(n+1);
    P >= eps*eye(n);
    cvx_end

    % Check in P = P' > 0
    logic = all(eig(P) > 0);

    % If logic is true, increase beta
    if logic == 0
       beta  = beta + .1;
    end
```

```
end

% Create controllability and observability gramian
Wc      = gram(ss(A,B,C,D),'c');
Wo      = gram(ss(A,B,C,D),'o');

% Check if gramian is invertible and A is Hurwitz
if det(Wc) ~= 0 && all(eig(A) < 0)
    disp('System is controllable')
end

if det(Wo)~= 0 && all(eig(A) < 0)
    disp('System is observable')
end

fprintf('The minimum beta needed for P = P'' > 0 is %.1f \n',beta)

LMI     = [(P*A + A'*P + 2*alpha*P), (P*B - C');...
           (B'*P - C), -(D + D')]

disp('The eigenvalues of the LMI are: ')
disp(eig(LMI))
```

```
System is controllable
System is observable
The minimum beta needed for P = P' > 0 is 1.1

LMI =

   -0.1978    0.4365    0.0000
    0.4365   -3.9927   -0.0000
    0.0000   -0.0000         0

The eigenvalues of the LMI are:
   -4.0423
   -0.1482
    0.0000
```

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