Final Exam
Solutions

1. (**15 pts**) Determine if the following discrete-time system,

$$\begin{aligned} \boldsymbol{x}[k+1] &= \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} \boldsymbol{x}[k] + \begin{bmatrix} 3 \\ 0 \end{bmatrix} u[k] \\ &= \boldsymbol{A}\boldsymbol{x}[k] + \boldsymbol{b}u[k], \end{aligned}$$

is reachable, controllable or neither. Carefully justify your answer. ◇

The controllability matrix is

$$\begin{bmatrix} \boldsymbol{b} & \boldsymbol{A}\boldsymbol{b} \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}.$$

The system is **not reachable** because rank $\begin{bmatrix} \boldsymbol{b} & \boldsymbol{A}\boldsymbol{b} \end{bmatrix} = 1 < n = 2$.

The system is **not controllable** either because it does not satisfy the necessary and sufficient controllability condition for discrete-time linear time invariant systems. This controllability test is

$$\operatorname{rank} \begin{bmatrix} \boldsymbol{B} & \boldsymbol{A}\boldsymbol{B} & \cdots & \boldsymbol{A}^{n-1}\boldsymbol{B} \end{bmatrix} = \operatorname{rank} \begin{bmatrix} \boldsymbol{B} & \boldsymbol{A}\boldsymbol{B} & \cdots & \boldsymbol{A}^{n-1}\boldsymbol{B} & \boldsymbol{A}^n \end{bmatrix}.$$

We have, $n = 2$, and hence

$$\boldsymbol{A}^n = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}^2 = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}.$$

It is easy to see that the system fails the controllability test since

$$\operatorname{rank} \begin{bmatrix} \boldsymbol{b} & \boldsymbol{A}\boldsymbol{b} \end{bmatrix} = 1 \neq \operatorname{rank} \begin{bmatrix} \boldsymbol{b} & \boldsymbol{A}\boldsymbol{b} & \boldsymbol{A}^2 \end{bmatrix} = 2.$$

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2. (**15 pts**) For the discrete-time dynamical system,

$$\begin{aligned} \dot{\boldsymbol{x}}[k+1] &= \boldsymbol{A}\boldsymbol{x} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \boldsymbol{x}[k] \\ y[k] &= \boldsymbol{c}\boldsymbol{x} = \begin{bmatrix} 1 & 1 \end{bmatrix} \boldsymbol{x}[k], \end{aligned}$$

find the initial vector $\boldsymbol{x}[0]$ such that $y[0] = 2$ and $y[1] = 6$.

◇

We have

$$y[0] = 2 = \boldsymbol{c}\boldsymbol{x}[0] = \begin{bmatrix} 1 & 1 \end{bmatrix} \boldsymbol{x}[0]$$

and

$$y[1] = 6 = \boldsymbol{c}\boldsymbol{x}[1] = \boldsymbol{c}\boldsymbol{A}\boldsymbol{x}[0] = \begin{bmatrix} -1 & 1 \end{bmatrix} \boldsymbol{x}[0]$$

Combining the above gives

$$\begin{bmatrix} y[0] \\ y[1] \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix} = \begin{bmatrix} \boldsymbol{c} \\ \boldsymbol{c}\boldsymbol{A} \end{bmatrix} \boldsymbol{x}[0] = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \boldsymbol{x}[0].$$

Hence,

$$\boldsymbol{x}[0] = \begin{bmatrix} \boldsymbol{c} \\ \boldsymbol{c}\boldsymbol{A} \end{bmatrix}^{-1} \begin{bmatrix} y[0] \\ y[1] \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 6 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 \\ 6 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}.$$

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3. (**15 pts**) Let

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}u = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u, \\ y &= \boldsymbol{c}\boldsymbol{x} + du = \begin{bmatrix} 1 & 1 \end{bmatrix} \boldsymbol{x} - 3u. \end{aligned}$$

(a) (**5 pts**) Find the state transformation that transforms the pair $(\boldsymbol{A}, \boldsymbol{b})$ into the controller form.

(b) (**10 pts**) Find the representation of the dynamical system model in the new coordinates.

◇

(a) We first construct the controllability matrix for the pair $(\boldsymbol{A}, \boldsymbol{b})$,

$$\begin{bmatrix} \boldsymbol{b} & \boldsymbol{A}\boldsymbol{b} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The pair $(\boldsymbol{A}, \boldsymbol{b})$ is controllable; therefore, we can proceed with constructing the transformation. The last row of the inverse of the controllability matrix is $\boldsymbol{q}_1 = \begin{bmatrix} 0 & 1 \end{bmatrix}$. Thus, the transformation, $\tilde{\boldsymbol{x}} = \boldsymbol{T}\boldsymbol{x}$, that brings the pair into the controller form is

$$\tilde{\boldsymbol{x}} = \boldsymbol{T}\boldsymbol{x} = \begin{bmatrix} \boldsymbol{q}_1 \\ \boldsymbol{q}_1 \boldsymbol{A}^\top \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \boldsymbol{x}.$$

(b) The system model in the new coordinates has the form

$$\dot{\tilde{\boldsymbol{x}}} = \tilde{\boldsymbol{A}}\tilde{\boldsymbol{x}} + \tilde{\boldsymbol{b}}u, \quad y = \tilde{\boldsymbol{c}}\tilde{\boldsymbol{x}} + \tilde{d}u,$$

where

$$\tilde{\boldsymbol{A}} = \boldsymbol{T}\boldsymbol{A}\boldsymbol{T}^{-1}, \quad \tilde{\boldsymbol{b}} = \boldsymbol{T}\boldsymbol{b}, \quad \tilde{\boldsymbol{c}} = \boldsymbol{c}\boldsymbol{T}^{-1}, \quad \tilde{d} = d,$$

that is,

$$\tilde{\boldsymbol{A}} = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}, \quad \tilde{\boldsymbol{b}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \tilde{\boldsymbol{c}} = \begin{bmatrix} -1 & 1 \end{bmatrix}, \quad \tilde{d} = -3.$$

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4. (**20 pts**) Construct a state-space model for the transfer function

$$\boldsymbol{G}(s) = \begin{bmatrix} \frac{4s^2}{2s^2+6s+4} & \frac{1}{s+2} \end{bmatrix}.$$

◇

This is a transfer function of a continuous-time (CT) system with one output and two inputs. The given transfer function is not strictly proper and its denominator of the first element is not monic. Therefore, we first divide the numerator and the denominator of the first element by the coefficient of the highest power of s. We next extract the strictly proper part of $\boldsymbol{G}(s)$ and represent $\boldsymbol{G}(s)$ as

$$\begin{aligned} \boldsymbol{G}(s) &= \begin{bmatrix} \frac{-6s-4}{s^2+3s+2} & \frac{1}{s+2} \end{bmatrix} + \begin{bmatrix} 2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{-6s-4}{s^2+3s+2} & \frac{s+1}{s^2+3s+2} \end{bmatrix} + \begin{bmatrix} 2 & 0 \end{bmatrix}. \end{aligned}$$

A possible state-space realization has the form,

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} -4 & 1 \\ -6 & 1 \end{bmatrix} \boldsymbol{u}(t) \\ y(t) &= \begin{bmatrix} 0 & 1 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 2 & 0 \end{bmatrix} \boldsymbol{u}(t). \end{aligned}$$

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5. (**20 pts**) For what range of the parameter γ the quadratic form

$$\boldsymbol{x}^\top \begin{bmatrix} -\frac{1}{2} & 0 \\ \gamma & -\frac{3}{2} \end{bmatrix} \boldsymbol{x}$$

(a) (**10 pts**) is negative semi-definite?

(b) (**10 pts**) For what range of the parameter γ this quadratic form is positive semi-definite?

$\diamond$

(a) We first symmetrize the given quadratic form to obtain

$$f = \frac{1}{2}\left(\boldsymbol{x}^\top \begin{bmatrix} -\frac{1}{2} & 0 \\ \gamma & -\frac{3}{2} \end{bmatrix} \boldsymbol{x} + \boldsymbol{x}^\top \begin{bmatrix} -\frac{1}{2} & \gamma \\ 0 & -\frac{3}{2} \end{bmatrix} \boldsymbol{x}\right) = \frac{1}{2}\boldsymbol{x}^\top \begin{bmatrix} -1 & \gamma \\ \gamma & -3 \end{bmatrix} \boldsymbol{x}.$$

This quadratic form is negative semi-definite if and only if its negative is positive semi-definite. We therefore check

$$g = -f = \frac{1}{2}\boldsymbol{x}^\top \begin{bmatrix} 1 & -\gamma \\ -\gamma & 3 \end{bmatrix} \boldsymbol{x}$$

for positive definiteness. The quadratic for g is positive semi-definite if and only if its principal minors of underlying symmetric matrix are non-negative. The two first-order principal minors are non-negative, and in fact, positive. We can ignore the $\frac{1}{2}$ when computing the second-order principal minor since we need to check if it non-negative. We have

$$\det \begin{bmatrix} 1 & -\gamma \\ -\gamma & 3 \end{bmatrix} = 3 - \gamma^2 \geq 0.$$

Therefore, the quadratic form $g = -f$ is positive semi-definite if and only if $\gamma \in [-\sqrt{3},\ \sqrt{3}]$. Hence the original quadratic form f is negative definite if and only if

$$\gamma \in [-\sqrt{3},\ \sqrt{3}].$$

(b) The quadratic form is positive semi-definite if and only if all principal minors of underlying symmetric matrix are non-negative. Already the first-order principal minors are negative. Therefore, the range of the parameter γ for which the quadratic form f is positive semi-definite is the empty set, that is, there is no value of γ for which $f \leq 0$ for $\boldsymbol{x} \neq 0$.

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6. (**15 pts**) For the system modeled by

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}u \\ &= \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u, \end{aligned}$$

(a) (**10 pts**) construct a state-feedback control law, $u = -\boldsymbol{k}\boldsymbol{x} + r$, such that the closed-loop system poles are located at $-1 \pm j2$;

(b) (**5 pts**) Let $y = \boldsymbol{c}\boldsymbol{x} + du = \begin{bmatrix} 1 & 1 \end{bmatrix} \boldsymbol{x} - 3u$. Find the transfer function, $\frac{Y(s)}{R(s)}$, of the closed-loop system.

$\diamond$

(a) We can use Ackermann's formula to the pair $(\boldsymbol{A}, \boldsymbol{b})$ to obtain the feedback gain $\boldsymbol{k}$. We form the controllability matrix of the pair $(\boldsymbol{A}, \boldsymbol{b})$, then find the last row of its inverse and call it $\boldsymbol{q}_1$. We have

$$\begin{bmatrix} \boldsymbol{b} & \boldsymbol{A}\boldsymbol{b} \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Hence, $\boldsymbol{q}_1 = \begin{bmatrix} 0 & 1 \end{bmatrix}$. The desired closed-loop characteristic polynomial (CLCP) is

$$\text{CLCP} = (s + 1 - j2)(s + 1 + j2) = s^2 + 2s + 5.$$

The feedback gain $\boldsymbol{k}$ then is

$$\begin{aligned} \boldsymbol{k} &= \boldsymbol{q}_1 \left(\boldsymbol{A}^2 + 2\boldsymbol{A} + 5\boldsymbol{I}_2 \right) \\ &= \begin{bmatrix} 0 & 1 \end{bmatrix} \left\{ \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}^2 + 2 \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} + 5 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\} \\ &= \begin{bmatrix} 4 & 13 \end{bmatrix}. \end{aligned}$$

Hence,

$$u = -\boldsymbol{k}\boldsymbol{x} + r = -\begin{bmatrix} 4 & 13 \end{bmatrix} \boldsymbol{x} + r.$$

(b) Note that

$$\begin{aligned} y &= \boldsymbol{c}\boldsymbol{x} + du \\ &= \boldsymbol{c}\boldsymbol{x} + d(-\boldsymbol{k}\boldsymbol{x} + r) \\ &= (\boldsymbol{c} - d\boldsymbol{k})\boldsymbol{x} + dr \\ &= \begin{bmatrix} 13 & 40 \end{bmatrix} \boldsymbol{x} - 3r, \end{aligned}$$

and

$$\begin{aligned} [s\boldsymbol{I}_2 - \boldsymbol{A} + \boldsymbol{b}\boldsymbol{k}]^{-1} &= \begin{bmatrix} s+4 & 13 \\ -1 & s-2 \end{bmatrix}^{-1} \\ &= \frac{1}{s^2 + 2s + 5} \begin{bmatrix} s-2 & -13 \\ 1 & s+4 \end{bmatrix}. \end{aligned}$$

The closed-loop transfer function is

$$\begin{aligned}
\frac{Y(s)}{R(s)} &= [\boldsymbol{c} - d\boldsymbol{k}]\left[s\boldsymbol{I}_2 - \boldsymbol{A} + \boldsymbol{b}\boldsymbol{k}\right]^{-1}\boldsymbol{b} + d \\
&= \frac{1}{s^2+2s+5}\begin{bmatrix} 13 & 40 \end{bmatrix}\begin{bmatrix} s-2 & -13 \\ 1 & s+4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + d \\
&= \frac{13s+14}{s^2+2s+5} - 3 \\
&= \frac{-3s^2+7s-1}{s^2+2s+5}.
\end{aligned}$$

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7. (**20 pts**) Find the transfer function of the system described by the following state-space equations,

$$\begin{aligned}
\dot{\boldsymbol{x}}(t) &= \left[\begin{array}{cccc|cc} 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 3 & 1 \\ \hline 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 2 & 0 \end{array}\right]\boldsymbol{x}(t) + \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 1 \\ \hline 0 \\ 0 \end{array}\right]u(t) \\
\boldsymbol{y}(t) &= \left[\begin{array}{cccc|cc} 2 & 1 & 0 & 7 & 2 & 1 \\ 7 & 0 & 1 & 2 & 1 & 1 \end{array}\right]\boldsymbol{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix}u(t).
\end{aligned}$$

◇

The transfer function contains only controllable and observable part of the state-space model. Therefore,

$$\boldsymbol{G}(s) = \begin{bmatrix} 2 & 1 & 0 & 7 \\ 7 & 0 & 1 & 2 \end{bmatrix}\left(s\boldsymbol{I}_4 - \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right)^{-1}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

We use the fact fact that the controllable pair $(A, \boldsymbol{b})$ is in the controller form. Hence

$$\boldsymbol{G}(s) = \frac{1}{s^4}\begin{bmatrix} 2 & 1 & 0 & 7 \\ 7 & 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ s \\ s^2 \\ s^3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Manipulating, we obtain

$$\boldsymbol{G}(s) = \begin{bmatrix} \frac{7s^3+s+2}{s^4} \\ \frac{2s^3+s^2+7}{s^4} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{s^4+7s^3+s+2}{s^4} \\ \frac{2s^3+s^2+7}{s^4} \end{bmatrix}.$$

8. (**15 pts**) For the following linear time-varying (LTV) system model,

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix} \boldsymbol{x}(t);$$

(a) (**10 pts**) Find the state transition matrix $\boldsymbol{\Phi}(t, \tau)$;

(b) (**5 pts**) Let $\boldsymbol{x}(2) = \begin{bmatrix} 0 & 4 \end{bmatrix}^\top$. Find $\boldsymbol{x}(1)$.

$\diamond$

(a) We present two methods for finding the state transition matrix. The first method uses the fact that $\boldsymbol{A}(t)\boldsymbol{A}(s) = \boldsymbol{A}(s)\boldsymbol{A}(t)$, where $\boldsymbol{A}(t) = \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix}$. Hence

$$\boldsymbol{\Phi}(t, \tau) = e^{\int_\tau^t \boldsymbol{A}(s) ds} = e^{\int_\tau^t \begin{bmatrix} 0 & s \\ 0 & 0 \end{bmatrix} ds} = e^{\begin{bmatrix} 0 & \frac{1}{2}(t^2 - \tau^2) \\ 0 & 0 \end{bmatrix}} = \begin{bmatrix} 1 & \frac{1}{2}(t^2 - \tau^2) \\ 0 & 1 \end{bmatrix}.$$

The second method solves two first-order ordinary differential equations, one by one. The given system consists of two first-order linear differential equations,

$$\dot{x}_1 = t x_2, \quad \text{and} \quad \dot{x}_2 = 0.$$

We can easily obtain the solution to the second of the above equations,

$$x_2(t) = x_2(0). \tag{1}$$

We next obtain the solution to $\dot{x}_1 = t x_2$ by integrating its both sides,

$$\begin{aligned} x_1(t) &= x_1(0) + \int_0^t \tau x_2(0) d\tau \\ &= x_1(0) + \frac{t^2 x_2(0)}{2}. \end{aligned} \tag{2}$$

We can now construct a fundamental matrix. Let

$$\boldsymbol{x}^{(1)}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{x}^{(2)}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Then using (2) and (1), we obtain the fundamental matrix corresponding to the above initial conditions is

$$\begin{aligned} \boldsymbol{X}(t) &= \begin{bmatrix} \boldsymbol{x}^{(1)}(t) & \boldsymbol{x}^{(2)}(t) \end{bmatrix} \\ &= \begin{bmatrix} 1 + \frac{t^2}{2} & \frac{t^2}{2} \\ 1 & 1 \end{bmatrix}. \end{aligned}$$

If we select the initial conditions

$$\boldsymbol{x}^{(1)}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{x}^{(2)}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

then we obtain a different fundamental matrix,

$$\boldsymbol{X}(t) = \begin{bmatrix} 1 & \frac{t^2}{2} \\ 0 & 1 \end{bmatrix}.$$

The state transition matrix, however, is unique and has the form

$$\begin{aligned}
\boldsymbol{\Phi}(t,\tau) &= \boldsymbol{X}(t)\boldsymbol{X}(\tau)^{-1} \\
&= \begin{bmatrix} 1+\frac{t^2}{2} & \frac{t^2}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1+\frac{\tau^2}{2} & \frac{\tau^2}{2} \\ 1 & 1 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} 1+\frac{t^2}{2} & \frac{t^2}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{\tau^2}{2} \\ -1 & 1+\frac{\tau^2}{2} \end{bmatrix} \\
&= \begin{bmatrix} 1 & -\frac{1}{2}(\tau^2-t^2) \\ 0 & 1 \end{bmatrix}.
\end{aligned}$$

(b) We have

$$\begin{aligned}
\boldsymbol{x}(1) &= \boldsymbol{\Phi}(1,2)\boldsymbol{x}(2) \\
&= \begin{bmatrix} 1 & -\frac{3}{2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 4 \end{bmatrix} \\
&= \begin{bmatrix} -6 \\ 4 \end{bmatrix}.
\end{aligned}$$

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9. (**15 pts**) Solve the equation

$$\dot{x}(t) = (\cos t)x(t), \quad x(0) = 5,$$

then find $x(\pi)$.

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We have

$$\begin{aligned}
x(t) &= e^{\int_0^t a(\tau)d\tau} x_0 \\
&= x_0 e^{\int_0^t \cos(\tau)d\tau} \\
&= 5e^{\sin t}.
\end{aligned}$$

Hence

$$x(\pi) = 5.$$

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10. (**15 pts**) Recall that an equilibrium state of a dynamical system is a state of rest, that is, the system starting from that state stays there thereafter. Determine the equilibrium state of the following continuous-time (CT) system,

$$\boldsymbol{x}(t) = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \boldsymbol{x}(t) - \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

◇

An equilibrium state, $\boldsymbol{x}_e$, of the given continuous-time (CT) system must satisfy the relation,

$$\mathbf{0} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \boldsymbol{x}_e - \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Performing manipulations yields,

$$\boldsymbol{x}_e = \frac{1}{5} \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

Hence,

$$\boldsymbol{x}_e = \begin{bmatrix} -\frac{1}{5} \\ \frac{2}{5} \end{bmatrix}.$$

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11. (**15 pts**) For the system described by the following state-space equations,

$$\dot{\boldsymbol{x}}(t) = \left[\begin{array}{ccc|cc} 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & a & 4 \\ 0 & 0 & 0 & -1 & 0 \end{array}\right] \boldsymbol{x}(t) + \left[\begin{array}{c} 0 \\ 0 \\ 1 \\ \hline 0 \\ 0 \end{array}\right] u(t)],$$

construct a state-feedback controller $u = -\boldsymbol{k}\boldsymbol{x}$ and determine the parameter a such that the closed-loop system poles are all at -2.

◇

The system uncontrollable part will have its two poles located at -1 for $a = -4$. We construct next the state-feedback controller such that all the poles of the controllable part are at -2 as well. The desired characteristic polynomial of the controllable part is $p(s) = (s+2)^3 = s^3 + 6s^2 + 12s + 8$. The controllable part is in the controller form, which makes it easy to detrmine the feedback gain k. We have

$$u = -\begin{bmatrix} 8 & 12 & 6 & 0 & 0 \end{bmatrix} \boldsymbol{x}.$$

It is easy to veriy that the closed-loop characteristic polynomial (CLCP) is

$$\text{CLCP} = \det \left[\begin{array}{ccc|cc} s & -1 & 0 & -1 & -2 \\ 0 & s & -1 & -2 & 0 \\ 8 & 12 & s+6 & 0 & -1 \\ \hline 0 & 0 & 0 & s+2 & -1 \\ 0 & 0 & 0 & 1 & s \end{array}\right] = (s^3+6s^2+12s+8)(s^2+4s+4) = (s+1)^5.$$

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12. (**20 pts**) Consider the following model of a dynamical system:

$$\begin{aligned} \boldsymbol{x}[k+1] &= \boldsymbol{A}\boldsymbol{x}[k] + \boldsymbol{b}u[k] = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} \boldsymbol{x}[k] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u[k] \\ y[k] &= \boldsymbol{c}\boldsymbol{x}[k] + du[k] = \begin{bmatrix} 1 & 1 \end{bmatrix} \boldsymbol{x}[k] - 3u[k]. \end{aligned}$$

(a) (**10 pts**) Design an asymptotic state observer for the above system with the observer poles located at -3 and -4. Write the equations of the observer dynamics.

(b) (**10 pts**) Denote the observer state by $\tilde{\boldsymbol{x}}$. Let $u = -\boldsymbol{k}\tilde{\boldsymbol{x}} + r$. Determine the feedback gain $\boldsymbol{k}$ such that the controller's poles are at -1 and -2, that is, $\det[z\boldsymbol{I}_2 - \boldsymbol{A} + \boldsymbol{b}\boldsymbol{k}] = (z+1)(z+2)$. Then find the transfer function, $\frac{Y[z]}{R[z]}$, of the closed-loop system driven by the combined observer controller compensator.

◇

(a) We first find the observer gain $\boldsymbol{l}$ such that the matrix $\boldsymbol{A} - \boldsymbol{l}\boldsymbol{c}$ has its poles at -3 and -4. The desired characteristic polynomial of $\boldsymbol{A} - \boldsymbol{l}\boldsymbol{c}$ is $z^2 + 7z + 12$. We use the Ackermann's formula. We first form the controllability matrix of dual pair, $(\boldsymbol{A}^\top, c^\top)$ and find the last row vector of the inverse of the controllability matrix of the dual pair, $\boldsymbol{q}_1 = \begin{bmatrix} -1 & 1 \end{bmatrix}$. Then the observer gain is

$$\begin{aligned} \boldsymbol{l}^\top &= \boldsymbol{q}_1 \left((\boldsymbol{A}^\top) \right)^2 + 7\boldsymbol{A}^\top + 12\boldsymbol{I}_2 \Big) = \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} 12 & 9 \\ 0 & 30 \end{bmatrix} \\ &= \begin{bmatrix} -12 & 21 \end{bmatrix} \end{aligned}$$

Hence,

$$\boldsymbol{l} = \begin{bmatrix} -12 \\ 21 \end{bmatrix}.$$

The observer's output is $\tilde{y} = \boldsymbol{c}\tilde{\boldsymbol{x}} + du$. Then, the dynamics of the observer are

$$\begin{aligned} \tilde{\boldsymbol{x}}[k+1] &= \boldsymbol{A}\tilde{\boldsymbol{x}}[k] + \boldsymbol{b}u[k] + \boldsymbol{l}(y[k] - \tilde{y}[k]) \\ &= (\boldsymbol{A} - \boldsymbol{l}\boldsymbol{c})\tilde{\boldsymbol{x}}[k] + \boldsymbol{l}y[k] + \boldsymbol{b}u[k] - \boldsymbol{l}du[k] \\ &= (\boldsymbol{A} - \boldsymbol{l}\boldsymbol{c})\tilde{\boldsymbol{x}}[k] + \boldsymbol{l}y[k] + (\boldsymbol{b} - \boldsymbol{l}d)u[k] \\ &= \begin{bmatrix} 12 & 12 \\ -20 & 19 \end{bmatrix} \tilde{\boldsymbol{x}}[k] + \begin{bmatrix} -12 \\ 21 \end{bmatrix} y[k] + \begin{bmatrix} -35 \\ 63 \end{bmatrix} u[k]. \end{aligned}$$

(b) The control law

$$u = -\boldsymbol{k}\tilde{\boldsymbol{x}} + r = -\begin{bmatrix} 5 & 12 \end{bmatrix} \tilde{\boldsymbol{x}} + r.$$

The closed loop system with the combined observer-controller compensator is

$$\begin{aligned}
\boldsymbol{x}[k+1] &= \boldsymbol{A}\boldsymbol{x}[k] - \boldsymbol{b}\boldsymbol{k}\tilde{\boldsymbol{x}}[k] + \boldsymbol{b}r[k] \\
\tilde{\boldsymbol{x}}[k+1] &= (\boldsymbol{A} - \boldsymbol{l}\boldsymbol{c})\tilde{\boldsymbol{x}}[k] + \boldsymbol{l}\boldsymbol{c}\boldsymbol{x}[k] - \boldsymbol{b}\boldsymbol{k}\tilde{\boldsymbol{x}}[k] + \boldsymbol{b}r[k] \\
y[k] &= \boldsymbol{c}\boldsymbol{x}[k] - d\boldsymbol{k}\tilde{\boldsymbol{x}}[k] + dr[k].
\end{aligned}$$

In matrix form, the above equations become

$$\begin{aligned}
\begin{bmatrix} \boldsymbol{x}[k+1] \\ \tilde{\boldsymbol{x}}[k+1] \end{bmatrix} &= \begin{bmatrix} \boldsymbol{A} & -\boldsymbol{b}\boldsymbol{k} \\ \boldsymbol{l}\boldsymbol{c} & \boldsymbol{A} - \boldsymbol{l}\boldsymbol{c} - \boldsymbol{b}\boldsymbol{k} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}[k] \\ \tilde{\boldsymbol{x}}[k] \end{bmatrix} + \begin{bmatrix} \boldsymbol{b} \\ \boldsymbol{b} \end{bmatrix} r[k] \\
y[k] &= \begin{bmatrix} \boldsymbol{c} & -d\boldsymbol{k} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}[k] \\ \tilde{\boldsymbol{x}}[k] \end{bmatrix} + dr[k].
\end{aligned}$$

The transfer function of the closed loop system is

$$\begin{aligned}
\frac{Y[z]}{R[z]} &= (\boldsymbol{c} - d\boldsymbol{k})(z\boldsymbol{I} - \boldsymbol{A} + \boldsymbol{b}\boldsymbol{k})^{-1}\boldsymbol{b} + d \\
&= \begin{bmatrix} 16 & 37 \end{bmatrix} \begin{bmatrix} z+5 & 12 \\ -1 & z-2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} - 3 \\
&= \frac{16z+5}{z^2 + +3z + 2} - 3 \\
&= \frac{-3z^2 + 7z - 1}{z^2 + 3z + 2}.
\end{aligned}$$
