ECE 68000: MODERN AUTOMATIC CONTROL

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Quadratic Lyapunov's Functions

Quadratic Forms

- $V = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}$, where $\boldsymbol{P} = \boldsymbol{P}^{\mathrm{T}}$
- If $\boldsymbol{P}$ not symmetric, need to symmetrize it
- First observe that because the transposition of a scalar equals itself, we have

$$\left(x^T P x\right)^T = x^T P^T x = x^T P x$$

Symmetrizing Quadratic Form

- Perform manipulations

$$\begin{aligned} \boldsymbol{x}^T \boldsymbol{P} \boldsymbol{x} &= \frac{1}{2} \boldsymbol{x}^T \boldsymbol{P} \boldsymbol{x} + \frac{1}{2} \boldsymbol{x}^T \boldsymbol{P} \boldsymbol{x} \\ &= \frac{1}{2} \boldsymbol{x}^T \boldsymbol{P} \boldsymbol{x} + \frac{1}{2} \boldsymbol{x}^T \boldsymbol{P}^T \boldsymbol{x} \\ &= \boldsymbol{x}^T \left(\frac{\boldsymbol{P} + \boldsymbol{P}^T}{2} \right) \boldsymbol{x} \end{aligned}$$

- Note that

$$\left(\frac{\boldsymbol{P} + \boldsymbol{P}^T}{2} \right)^T = \frac{\boldsymbol{P} + \boldsymbol{P}^T}{2}$$

Tests for Positive and Positive Semi-Definiteness of Quadratic Form

- $V=\boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}$, where $\boldsymbol{P}=\boldsymbol{P}^{\mathrm{T}}$, is positive definite if and only if all eigenvalues of $\boldsymbol{P}$ are positive
- $V=\boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}$, where $\boldsymbol{P}=\boldsymbol{P}^{\mathrm{T}}$, is positive semi-definite if and only if all eigenvalues of $\boldsymbol{P}$ are non-negative

Comments on the Eigenvalue Tests

- These tests are only good for the case when $\boldsymbol{P}=\boldsymbol{P}^{\mathrm{T}}$. You must symmetrize $\boldsymbol{P}$ before applying the above tests
- Other tests, the Sylvester's criteria, involve checking the signs of principal minors of $\boldsymbol{P}$

Negative Definite Quadratic Form

$V = \boldsymbol{x}^{\top}\boldsymbol{P}\boldsymbol{x}$ is negative definite if and only if

$$-\boldsymbol{x}^{\top}\boldsymbol{P}\boldsymbol{x} = \boldsymbol{x}^{\top}(-\boldsymbol{P})\boldsymbol{x}$$

is positive definite

Negative Semi-Definite Quadratic Form

$V=\boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}$ is negative semi-definite if and only if

$$-\boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}=\boldsymbol{x}^{\mathrm{T}}(-\boldsymbol{P})\boldsymbol{x}$$

is positive semi-definite

Example: Checking the Sign Definiteness of a Quadratic Form

- Is ***P***, equivalently, is the associated quadratic form, $V = \boldsymbol{x}^{\mathsf{T}}\boldsymbol{P}\boldsymbol{x}$, pd, psd, nd, nsd, or neither?

$$\boldsymbol{P} = \begin{bmatrix} 2 & -6 \\ 0 & 2 \end{bmatrix}$$

- The associated quadratic form

$$V = x^T \boldsymbol{P} x = 2x_1^2 - 6x_1 x_2 + 2x^2$$

Example: Symmetrizing the Underlying Matrix of the Quadratic Form

- Applying the eigenvalue test to the given quadratic form would seem to indicate that the quadratic form is pd, which turns out to be false
- Need to symmetrize the underlying matrix first and then can apply the eigenvalue test

Example: Symetrized Matrix

- Symmetrizing manipulations

$$\frac{1}{2}\left(P + P^T\right) = \frac{1}{2}\left(\begin{bmatrix} 2 & -6 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ -6 & 2 \end{bmatrix}\right) = \begin{bmatrix} 2 & -3 \\ -3 & 2 \end{bmatrix}$$

- The eigenvalues of the symmetrized matrix are: 5 and -1
- The quadratic form is indefinite!

Example: Further Analysis

- Direct check that the quadratic form is indefinite
- Take $\boldsymbol{x}$=[1 0]^T. Then

$$x^T P x = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -6 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2 > 0$$

- Take $\boldsymbol{x}$=[1 1]^T. Then

$$x^T P x = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -6 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = -2 < 0$$

Stability Test for $\boldsymbol{x}_e = \mathbf{0}$ of d$\boldsymbol{x}$/dt=$\boldsymbol{A}\boldsymbol{x}$

- Let $V = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}$ where $\boldsymbol{P} = \boldsymbol{P}^{\mathrm{T}} > 0$
- For V to be a Lyapunov function, that is, for $\boldsymbol{x}_e = \mathbf{0}$ to be a.s.,

$$\dot{V}(x(t)) < 0$$

- Evaluate the time derivative of V on the solution of the system d$\boldsymbol{x}$/dt=$\boldsymbol{A}\boldsymbol{x}$---Lyapunov derivative

Lyapunov Derivative for d$\boldsymbol{x}$/dt=$\boldsymbol{Ax}$

- Note that $V(\boldsymbol{x}(t))=\boldsymbol{x}(t)^{\mathsf{T}}\boldsymbol{P}\boldsymbol{x}(t)$
- Use the chain rule

$$\begin{aligned}\dot{V}(x(t)) &= \dot{x}^T \boldsymbol{P} x + x^T \boldsymbol{P} \dot{x} \\ &= x^T \boldsymbol{A}^T \boldsymbol{P} x + x^T \boldsymbol{P} \boldsymbol{A} x \\ &= x^T \left(\boldsymbol{A}^T \boldsymbol{P} + \boldsymbol{P} \boldsymbol{A} \right) x\end{aligned}$$

- We used

$$\dot{x}^T = x^T \boldsymbol{A}^T$$

Lyapunov Matrix Equation

- Denote

$$A^T P + P A = -Q$$

- Then the Lyapunov derivative can be represented as

$$\dot{V} = \frac{d}{dt} V = -x^T Q x$$

where

$$Q = Q^T > 0$$

Terms to Our Vocabulary

- Theorem---a major result of independent interest
- Lemma---an auxiliary result that is used as a stepping stone toward a theorem
- Corollary---a direct consequence of a theorem, or even a lemma

Lyapunov's Theorem

The real matrix ***A*** is a.s., that is, all eigenvalues of ***A*** have negative real parts if and only if for any $Q = Q^T > 0$ the solution P of the continuous matrix Lyapunov equation

$$A^T P + PA = -Q$$

is (symmetric) positive definite

How Do We Use the Lyapunov Theorem?

- Select an arbitrary symmetric positive definite $\boldsymbol{Q}$, for example, an identity matrix, $\boldsymbol{I}_n$
- Solve the Lyapunov equation for $\boldsymbol{P}=\boldsymbol{P}^T$
- If $\boldsymbol{P}$ is positive definite, the matrix $\boldsymbol{A}$ is a.s. If $\boldsymbol{P}$ is not p.d. then $\boldsymbol{A}$ is not a.s.

How NOT to Use the Lyapunov Theorem

- It would be no use choosing $\boldsymbol{P}$ to be positive definite and then calculating $\boldsymbol{Q}$
- For unless $\boldsymbol{Q}$ turns out to be positive definite, nothing can be said about a.s. of $\boldsymbol{A}$ from the Lyapunov equation

Example: How NOT to Use the Lyapunov Theorem

- Consider an a.s. matrix

$$A = \begin{bmatrix} -1 & 3 \\ 0 & -1 \end{bmatrix}$$

- Try

$$\boldsymbol{P} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- Compute $\boldsymbol{Q} = -\left(\boldsymbol{A}^T\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}\right)$

Example: Computing Q

$$
\begin{aligned}
\boldsymbol{Q} &= -\left(\boldsymbol{A}^T\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}\right) \\
&= -\left(\begin{bmatrix} -1 & 0 \\ 3 & -1 \end{bmatrix} + \begin{bmatrix} -1 & 3 \\ 0 & -1 \end{bmatrix}\right) \\
&= \begin{bmatrix} 2 & -3 \\ -3 & 2 \end{bmatrix}
\end{aligned}
$$

The matrix $\boldsymbol{Q}$ is indefinite!---
recall the previous example

Solving the Continuous Matrix Lyapunov Equation Using MATLAB

- Use the MATLAB's command lyap
- Example:

$$A = \begin{bmatrix} -1 & 3 \\ 0 & -1 \end{bmatrix}$$

- $\boldsymbol{Q}=\boldsymbol{I}_2$
- P=lyap(A',Q)

$$\boldsymbol{P}= \begin{bmatrix} 0.50 & 0.75 \\ 0.75 & 2.75 \end{bmatrix}$$

- Eigenvalues of ***P*** are positive: 0.2729 and 2.9771; ***P*** is positive definite

Benefits of the Lyapunov Theory

- Solution to differential equation are not needed to infer about stability properties of equilibrium state of interest

☞ **Lyapunov functions** are useful in designing robust and adaptive controllers