State Estimation of LTI Systems: Asymptotic Observers

- **Objective**: Construct asymptotic state observers to estimate state variables of linear lumped continuous-time (CT) or discrete-time (DT) systems
- We consider linear time-varying (LTI) system

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}u(t)\\ y(t) &= \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t)\end{aligned}$$

  or

$$\begin{aligned}\boldsymbol{x}[k+1] &= \boldsymbol{A}\boldsymbol{x}[k] + \boldsymbol{b}u[k]\\ \boldsymbol{y}[k] &= \boldsymbol{C}\boldsymbol{x}[k] + \boldsymbol{D}\boldsymbol{u}[k]\end{aligned}$$

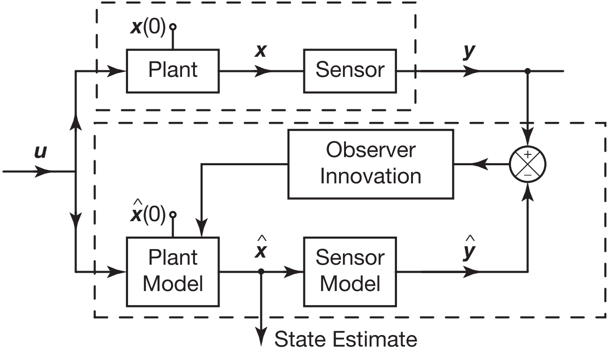
- We assume that the system at hand is both reachable and observable

Need for state estimators to implement state-feedback (SF) controllers

- To implement a SF controller, all the states need to be available
- Often, this requirement is not met, either because measuring of all the state variables would require excessive number of sensors, or because the state variables are not accessible for direct measurement
- Instead, only a subset of state variables or their combination may be available
- Use state estimate to implement a SF controller
- Much of the literature refers to observers as "state estimators"
- One can argue that estimator is much more descriptive in its function because observer implies a direct measurement
- On the other hand, estimator implies non-deterministic approach
- Here we use a deterministic approach
- We follow the original terminology of the observer's inventor

What is an observer?

- The first observer was proposed by Luenberger in the early nineteen sixties for the purpose of estimating the state of a plant, based on limited measurements of that system
- An observer—a deterministic dynamical system that generates an estimate of the plant's state using that plant's input and output signals



Observers as virtual sensors

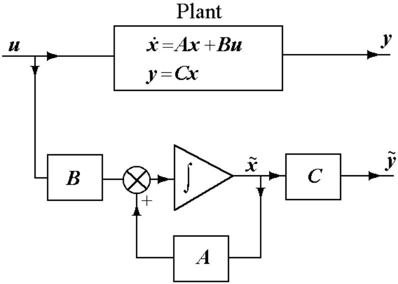
- The plant's state estimate are used in place of the true state to close the control loop
- Observers can be used as "software" or "virtual" sensors as opposed to hardware sensing devices directly measuring physical variables
- Observers augment or replace sensors in a control system
- Observers have been applied in secure communication using chaotic synchronization, machine vision, wind energy systems, speed-sensorless control of induction motors, or in model-based predictive among many other applications

Observer development

- Open-loop observer

$$\dot{\tilde{\boldsymbol{x}}}(t) = \boldsymbol{A}\tilde{\boldsymbol{x}}(t) + \boldsymbol{B}\boldsymbol{u}(t),$$

where $\tilde{\boldsymbol{x}}(t)$ is the estimate of $\boldsymbol{x}(t)$

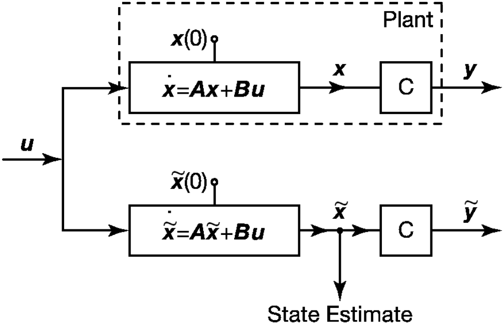


Open-loop observer estimation error

- Observation/estimation error, $\boldsymbol{e}(t) = \boldsymbol{x}(t) - \tilde{\boldsymbol{x}}(t)$
- Dynamics of the observation error

$$\dot{\boldsymbol{e}}(t) = \dot{\boldsymbol{x}}(t) - \dot{\tilde{\boldsymbol{x}}}(t) = \boldsymbol{A}\boldsymbol{e}(t),$$

with the initial observation error $\boldsymbol{e}(0) = \boldsymbol{x}(0) - \tilde{\boldsymbol{x}}(0)$



Open-loop observer analysis

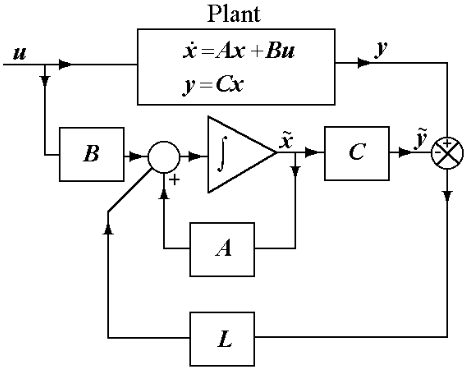
- If the eigenvalues of the matrix $\boldsymbol{A}$ are in the open left-hand plane, then the error converges to zero
- However, we have no control over the convergence rate
- Furthermore, the matrix $\boldsymbol{A}$ does not have to have all its eigenvalues in the open left-hand plane
- Thus, the open-loop observer is impractical
- Modify this observer by adding a feedback term to it
- The resulting structure is called the colorblue closed-loop observer or the Luenberger observer or the asymptotic full-order observer

Closed-loop observer

- The closed-loop observer

$$\dot{\tilde{\boldsymbol{x}}}(t) = \boldsymbol{A}\tilde{\boldsymbol{x}}(t) + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{L}\left(\boldsymbol{y}(t) - \tilde{\boldsymbol{y}}(t)\right)$$

where $\tilde{\boldsymbol{y}}(t) = \boldsymbol{C}\tilde{\boldsymbol{x}}(t)$



Closed-loop observe analysis

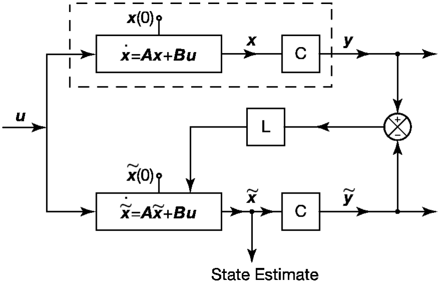
- The dynamics of the observation error

$$\begin{aligned}\dot{\boldsymbol{e}}(t) &= \dot{\boldsymbol{x}}(t) - \dot{\tilde{\boldsymbol{x}}}(t)\\ &= (\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C})\,\boldsymbol{e}(t), \quad \boldsymbol{e}(0) = \boldsymbol{x}(0) - \tilde{\boldsymbol{x}}(0).\end{aligned}$$

- The pair $(\boldsymbol{A}, \boldsymbol{C})$ is observable, if and only if the dual pair $(\boldsymbol{A}^\top, \boldsymbol{C}^\top)$ is reachable
- By assumption, the pair $(\boldsymbol{A}, \boldsymbol{C})$ is observable, and therefore the pair $(\boldsymbol{A}^\top, \boldsymbol{C}^\top)$ is reachable
- Thus, we can solve the pole placement problem for the dual pair $(\boldsymbol{A}^\top, \boldsymbol{C}^\top)$
- That is, for any set of prespecified n complex numbers, symmetric with respect to the real axis, there is a matrix, call it $\boldsymbol{L}^\top$, such that the eigenvalues of $\boldsymbol{A}^\top - \boldsymbol{C}^\top\boldsymbol{L}^\top$ and hence of $\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C}$ are in the prespecified locations

On the design of the closed-loop observer

- If the pair $(\boldsymbol{A}, \boldsymbol{C})$ is observable, then in addition to forcing the observation error to converge to zero, we can also control its rate of convergence by appropriately selecting the eigenvalues of the matrix $\boldsymbol{A} - \boldsymbol{LC}$
- Computing the closed-loop observer gain matrix $\boldsymbol{L}$ can be approached in exactly the same fashion as the construction of the gain matrix $\boldsymbol{K}$ in the linear SF controller design



Example

- Given an observable pair

$$\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & -21 & 5 \end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Construct $\boldsymbol{L} \in \mathbb{R}^{4\times 2}$ so that the eigenvalues of $\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C}$ are located at

$$\{-2, -3+j, -3-j, -4\}.$$

- Our goal then is to construct $\boldsymbol{L}$ so that the characteristic polynomial of $\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C}$ is

$$\det(s\boldsymbol{I}_4 - \boldsymbol{A} + \boldsymbol{L}\boldsymbol{C}) = s^4 + 12s^3 + 54s^2 + 108s + 80$$

Closed-loop observer

- One possible choice

$$\boldsymbol{L} = \begin{bmatrix} 1137 & 54 \\ 3955 & 188 \\ 5681 & 270 \\ -23626 & -1117 \end{bmatrix}$$

- Other possible gain matrices $\boldsymbol{L}$ could be used to allocate the eigenvalues of $\boldsymbol{A} - \boldsymbol{LC}$ into desired locations for multi-output systems