PURDUE UNIVERSITY

ECE 68000: MODERN AUTOMATIC CONTROL

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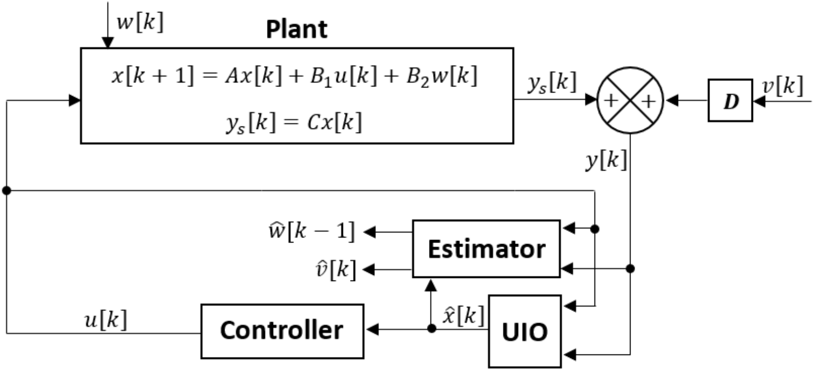
Secure state estimation of networked systems under arbitrary malicious attacks

Networked Control System (NCS) Security

- Networked Control Systems depend on wireless communication—a major challenge in the NCS design is their security
- Actuators and sensor measurements exposed to malicious attacks in communication networks
- Methods of detecting sparse malicious packet drop attacks in the communication networks proposed

Our Proposed Approach



B. Alenezi, M. Zhang, S. Hui, and S. H. Żak, Simultaneous Estimation of the State, Unknown Input, and Output Disturbance in Discrete-Time Linear Systems, *IEEE Transactions on Automatic Control*, Vol. 66, No. 12, December 2021, pp. 6115–6122

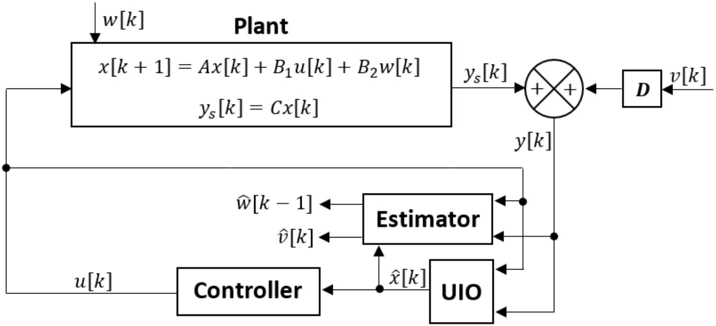
Plant Model

$$
\begin{aligned}
\boldsymbol{x}[k+1] &= \boldsymbol{A}\boldsymbol{x}[k] + \boldsymbol{B}_1\boldsymbol{u}[k] + \boldsymbol{B}_2\boldsymbol{w}[k] \\
\boldsymbol{y}[k] &= \boldsymbol{C}\boldsymbol{x}[k] + \boldsymbol{D}\boldsymbol{v}[k],
\end{aligned}
$$

where

- $\boldsymbol{A} \in \mathbb{R}^{n\times n}$, $\boldsymbol{B}_1 \in \mathbb{R}^{n\times m_1}$, $\boldsymbol{B}_2 \in \mathbb{R}^{n\times m_2}$, rank $\boldsymbol{B}_2 = m_2$, $\boldsymbol{C} \in \mathbb{R}^{p\times n}$, $\boldsymbol{D} \in \mathbb{R}^{p\times r}$, and rank $\boldsymbol{D} = r$
- Control input—$\boldsymbol{u}[k] \in \mathbb{R}^{m_1}$
- Unknown input—$\boldsymbol{w}[k] \in \mathbb{R}^{m_2}$
- Output disturbance—$\boldsymbol{v}[k] \in \mathbb{R}^{r}$
- $\boldsymbol{w}[k]$ and $\boldsymbol{v}[k]$ uniformly bounded as functions of k

Objectives



- Construct Unknown Input Observer (UIO) to estimate the plant state in the presence of unknown input $\boldsymbol{w}[k]$ and output disturbance $\boldsymbol{v}[k]$
- Estimate the unknown input and output disturbance

Combined UIO-controller compensator and an estimator of unknown input and output disturbance

UIO Synthesis: Preliminaries

- Begin by representing $\boldsymbol{x}[k]$ as

$$\begin{aligned}\boldsymbol{x}[k] &= \boldsymbol{x}[k] - \boldsymbol{MCx}[k] + \boldsymbol{MCx}[k] \\ &= (\boldsymbol{I}_n - \boldsymbol{MC})\boldsymbol{x}[k] + \boldsymbol{M}(\boldsymbol{y}[k] - \boldsymbol{Dv}[k]) \\ &= (\boldsymbol{I}_n - \boldsymbol{MC})\boldsymbol{x}[k] + \boldsymbol{My}[k] - \boldsymbol{MDv}[k])\end{aligned}$$

  where
- $\boldsymbol{M} \in \mathbb{R}^{n \times p}$ is to be determined
- Select $\boldsymbol{M}$ such that

$$\boldsymbol{MD} = \boldsymbol{O}_{n \times r}$$

  where $\boldsymbol{O}_{n \times r}$ is an n-by-r matrix of zeros
- We obtain:

$$\boldsymbol{x}[k] = (\boldsymbol{I}_n - \boldsymbol{MC})\boldsymbol{x}[k] + \boldsymbol{My}[k]$$

Manipulations

- We have: $\boldsymbol{x}[k] = (\boldsymbol{I}_n - \boldsymbol{MC})\boldsymbol{x}[k] + \boldsymbol{My}[k]$
- Let $\boldsymbol{z}[k] = (\boldsymbol{I}_n - \boldsymbol{MC})\boldsymbol{x}[k]$
- Hence

$$\boldsymbol{x}[k] = \boldsymbol{z}[k] + \boldsymbol{My}[k]$$

- We will now show that an estimate of the state $\boldsymbol{x}[k]$ can be obtained from

$$\hat{\boldsymbol{x}}[k] = \boldsymbol{z}[k] + \boldsymbol{My}[k]$$

- The signal $\boldsymbol{z}[k]$ is obtained from

$$\boldsymbol{z}[k+1] = (\boldsymbol{I}_n - \boldsymbol{MC})\boldsymbol{x}[k+1]$$

Manipulations—Contd.

- Substitute the state dynamics equation into $\boldsymbol{z}[k+1] = (\boldsymbol{I}_n - \boldsymbol{MC})\boldsymbol{x}[k+1]$ to obtain

$$\boldsymbol{z}[k+1] = (\boldsymbol{I}_n - \boldsymbol{MC})(\boldsymbol{Ax}[k] + \boldsymbol{B}_1\boldsymbol{u}[k] + \boldsymbol{B}_2\boldsymbol{w}[k])$$

- Substitute $\boldsymbol{x}[k] = \boldsymbol{z}[k] + \boldsymbol{My}[k]$ into the above

$$\begin{aligned}\boldsymbol{z}[k+1] &= (\boldsymbol{I}_n - \boldsymbol{MC})(\boldsymbol{Az}[k] + \boldsymbol{AMy}[k] + \boldsymbol{B}_1\boldsymbol{u}[k]) \\ &\quad + (\boldsymbol{I}_n - \boldsymbol{MC})\boldsymbol{B}_2\boldsymbol{w}[k]\end{aligned}$$

- Select $\boldsymbol{M}$ so that

$$(\boldsymbol{I}_n - \boldsymbol{MC})\boldsymbol{B}_2 = \boldsymbol{O}$$

Open-Loop UIO

$$\begin{aligned} \boldsymbol{z}[k+1] &= (\boldsymbol{I}_n - \boldsymbol{MC})(\boldsymbol{Az}[k] + \boldsymbol{AMy}[k] + \boldsymbol{B}_1\boldsymbol{u}[k]) \\ \hat{\boldsymbol{x}}[k] &= \boldsymbol{z}[k] + \boldsymbol{My}[k] \end{aligned}$$

- Observation error $\boldsymbol{e}[k] = \boldsymbol{x}[k] - \hat{\boldsymbol{x}}[k]$
- Observation error dynamics

$$\boldsymbol{e}[k+1] = (\boldsymbol{I}_n - \boldsymbol{MC})\boldsymbol{Ae}[k]$$

- Add innovation term—the closed-loop UIO

Synthesis of the Closed-Loop UIO

- Observation error dynamics of the open-loop UIO

$$\begin{aligned} \boldsymbol{e}[k+1] &= (\boldsymbol{I}_n - \boldsymbol{M}\boldsymbol{C})\boldsymbol{A}\boldsymbol{e}[k] \\ &= \boldsymbol{A}_1\boldsymbol{e}[k] \end{aligned}$$

- Add $\boldsymbol{L}(\boldsymbol{y}[k] - \hat{\boldsymbol{y}}[k])$, where $\boldsymbol{L} \in \mathbb{R}^{n\times p}$ and

$$\hat{\boldsymbol{y}}[k] = \boldsymbol{C}\hat{\boldsymbol{x}}[k] = \boldsymbol{C}(\boldsymbol{z}[k] + \boldsymbol{M}\boldsymbol{y}[k])$$

- Observation error dynamics of the closed-loop UIO

$$\boldsymbol{e}[k+1] = (\boldsymbol{A}_1 - \boldsymbol{L}\boldsymbol{C})\boldsymbol{e}[k] - \boldsymbol{L}\boldsymbol{D}\boldsymbol{v}[k]$$

Closed-Loop UIO

- Observation error dynamics of the closed-loop UIO

$$\boldsymbol{e}[k+1] = (\boldsymbol{A}_1 - \boldsymbol{L}\boldsymbol{C})\boldsymbol{e}[k] - \boldsymbol{L}\boldsymbol{D}\boldsymbol{v}[k]$$

- The closed-loop UIO

$$\begin{aligned}
\boldsymbol{z}[k+1] &= (\boldsymbol{I}_n - \boldsymbol{M}\boldsymbol{C})(\boldsymbol{A}\boldsymbol{z}[k] + \boldsymbol{A}\boldsymbol{M}\boldsymbol{y}[k] + \boldsymbol{B}_1\boldsymbol{u}[k]) \\
&\quad + \boldsymbol{L}(\boldsymbol{y}[k] - \hat{\boldsymbol{y}}[k]) \\
\hat{\boldsymbol{x}}[k] &= \boldsymbol{z}[k] + \boldsymbol{M}\boldsymbol{y}[k]
\end{aligned}$$

UIO Synthesis: Solving for $\boldsymbol{M}$

Theorem

There exists a solution $\boldsymbol{M}$ *to*

$$\begin{aligned} (\boldsymbol{I}_n - \boldsymbol{M}\boldsymbol{C})\boldsymbol{B}_2 &= \boldsymbol{O}_{n \times m_2} \\ \boldsymbol{M}\boldsymbol{D} &= \boldsymbol{O}_{n \times r} \end{aligned}$$

if and only if

$$rank \begin{bmatrix} \boldsymbol{C}\boldsymbol{B}_2 & \boldsymbol{D} \\ \boldsymbol{B}_2 & \boldsymbol{O}_{n \times r} \end{bmatrix} = rank \begin{bmatrix} \boldsymbol{C}\boldsymbol{B}_2 & \boldsymbol{D} \end{bmatrix}$$

Solving for $\boldsymbol{M}$—Proof of Theorem

- Represent

$$\begin{aligned}(\boldsymbol{I}_n - \boldsymbol{M}\boldsymbol{C})\boldsymbol{B}_2 &= \boldsymbol{O}_{n\times m_2}\\ \boldsymbol{M}\boldsymbol{D} &= \boldsymbol{O}_{n\times r}\end{aligned}$$

  as

$$\boldsymbol{M}\begin{bmatrix}\boldsymbol{C}\boldsymbol{B}_2 & \boldsymbol{D}\end{bmatrix} = \begin{bmatrix}\boldsymbol{B}_2 & \boldsymbol{O}_{n\times r}\end{bmatrix}$$

- A necessary and sufficient condition (NASC) for $\boldsymbol{M}$ to solve the above matrix equation is that the space spanned by the rows of the matrix $\begin{bmatrix}\boldsymbol{B}_2 & \boldsymbol{O}_{n\times r}\end{bmatrix}$ is in the range of the space spanned by the rows of the matrix $\begin{bmatrix}\boldsymbol{C}\boldsymbol{B}_2 & \boldsymbol{D}\end{bmatrix}$
- This is equivalent to

$$\operatorname{rank}\begin{bmatrix}\boldsymbol{C}\boldsymbol{B}_2 & \boldsymbol{D}\\ \boldsymbol{B}_2 & \boldsymbol{O}_{n\times r}\end{bmatrix} = \operatorname{rank}\begin{bmatrix}\boldsymbol{C}\boldsymbol{B}_2 & \boldsymbol{D}\end{bmatrix}$$

Solving for $\boldsymbol{M}$—Another NASC

Theorem

There exists a solution $\boldsymbol{M}$ *to*

$$\boldsymbol{M}\left[\begin{array}{cc}\boldsymbol{C}\boldsymbol{B}_2 & \boldsymbol{D}\end{array}\right]=\left[\begin{array}{cc}\boldsymbol{B}_2 & \boldsymbol{O}_{n\times r}\end{array}\right]$$

if and only if

$$\mathit{rank}\left[\begin{array}{cc}\boldsymbol{C}\boldsymbol{B}_2 & \boldsymbol{D}\end{array}\right]=\mathit{rank}(\boldsymbol{B}_2)+\mathit{rank}(\boldsymbol{D})$$

We have

$$\begin{aligned}
\operatorname{rank}\begin{bmatrix}\boldsymbol{C}\boldsymbol{B}_2 & \boldsymbol{D}\end{bmatrix} &= \operatorname{rank}\begin{bmatrix}\boldsymbol{C}\boldsymbol{B}_2 & \boldsymbol{D}\\ \boldsymbol{B}_2 & \boldsymbol{O}\end{bmatrix}\\
&= \operatorname{rank}\left(\begin{bmatrix}\boldsymbol{I}_p & -\boldsymbol{C}\\ \boldsymbol{O} & \boldsymbol{I}_n\end{bmatrix}\begin{bmatrix}\boldsymbol{C}\boldsymbol{B}_2 & \boldsymbol{D}\\ \boldsymbol{B}_2 & \boldsymbol{O}\end{bmatrix}\right)\\
&= \operatorname{rank}\begin{bmatrix}\boldsymbol{O} & \boldsymbol{D}\\ \boldsymbol{B}_2 & \boldsymbol{O}\end{bmatrix} = \operatorname{rank}(\boldsymbol{B}_2)+\operatorname{rank}(\boldsymbol{D})
\end{aligned}$$

A Formula to Compute $\boldsymbol{M}$

- Represent

$$\begin{aligned} (\boldsymbol{I}_n - \boldsymbol{M}\boldsymbol{C})\boldsymbol{B}_2 &= \boldsymbol{O}_{n\times m_2} \\ \boldsymbol{M}\boldsymbol{D} &= \boldsymbol{O}_{n\times r} \end{aligned}$$

as

$$\boldsymbol{M}\begin{bmatrix} \boldsymbol{C}\boldsymbol{B}_2 & \boldsymbol{D} \end{bmatrix} = \begin{bmatrix} \boldsymbol{B}_2 & \boldsymbol{O}_{n\times r} \end{bmatrix}$$

- If $\operatorname{rank}\begin{bmatrix} \boldsymbol{C}\boldsymbol{B}_2 & \boldsymbol{D} \end{bmatrix} = \operatorname{rank}(\boldsymbol{B}_2) + \operatorname{rank}(\boldsymbol{D})$ then

$$\begin{bmatrix} \boldsymbol{C}\boldsymbol{B}_2 & \boldsymbol{D} \end{bmatrix}$$

has full column rank and therefore it is left invertible

Computing $\boldsymbol{M}$—Contd.

- We are solving

$$\boldsymbol{M}\begin{bmatrix} \boldsymbol{C}\boldsymbol{B}_2 & \boldsymbol{D} \end{bmatrix} = \begin{bmatrix} \boldsymbol{B}_2 & \boldsymbol{O}_{n\times r} \end{bmatrix}$$

- We obtain

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{B}_2 & \boldsymbol{O}_{n\times r} \end{bmatrix}\begin{bmatrix} \boldsymbol{C}\boldsymbol{B}_2 & \boldsymbol{D} \end{bmatrix}^{\dagger}$$

- A general class of solutions

$$\begin{aligned} \boldsymbol{M} &= \begin{bmatrix} \boldsymbol{B}_2 & \boldsymbol{O}_{n\times r} \end{bmatrix}\Big(\begin{bmatrix} \boldsymbol{C}\boldsymbol{B}_2 & \boldsymbol{D} \end{bmatrix}^{\dagger} \\ &\quad + \boldsymbol{H}_0\left(\boldsymbol{I}_p - \begin{bmatrix} \boldsymbol{C}\boldsymbol{B}_2 & \boldsymbol{D} \end{bmatrix}\begin{bmatrix} \boldsymbol{C}\boldsymbol{B}_2 & \boldsymbol{D} \end{bmatrix}^{\dagger}\right)\Big) \end{aligned}$$

where $\boldsymbol{H}_0 \in \mathbb{R}^{(m_2+r)\times p}$ is a design parameter matrix

More on the Synthesis of the UIO

- Proposed UIO

$$
\begin{aligned}
\boldsymbol{z}[k+1] &= (\boldsymbol{I}_n - \boldsymbol{MC})(\boldsymbol{Az}[k] + \boldsymbol{AMy}[k] + \boldsymbol{B}_1\boldsymbol{u}[k]) \\
&\quad + \boldsymbol{L}(\boldsymbol{y}[k] - \hat{\boldsymbol{y}}[k]) \\
\hat{\boldsymbol{x}}[k] &= \boldsymbol{z}[k] + \boldsymbol{My}[k]
\end{aligned}
$$

- Observation error dynamics:

$$
\boldsymbol{e}[k+1] = (\boldsymbol{A}_1 - \boldsymbol{LC})\boldsymbol{e}[k] - \boldsymbol{LDv}[k]
$$

where $\boldsymbol{A}_1 = (\boldsymbol{I}_n - \boldsymbol{MC})\boldsymbol{A}$

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