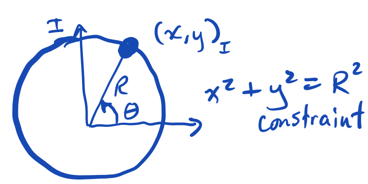
Ch 3 in K+P Particle Dynamics

Constraints and degrees of freedom (DOF)

Collection of particles: The number of DOF is the number of independent coordinates needed to describe the position of every particle.

In 2D $\quad M = 2(1) - 1 = 1$



2-D: $\quad \underbrace{M}_{\#\text{DOF}} = 2\underbrace{N}_{\#\text{particles}} - \underbrace{K}_{\#\text{constraints}}$

3-D: $\quad M = 3N - K$

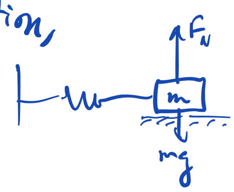
Collection of rigid bodies: The number of DOF is the number of coordinates needed to describe the position and orientation of every body

airfoil in 2D

2-D: $\quad M = 3N - K$

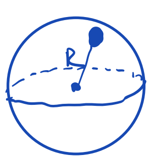
3-D: $\quad M = \underbrace{6}_{\text{Multiplier: DOF per particle}}N - K$

A constraint force reduces the number of DOF and often acts orthogonal to the direction of motion, in which case it is called a normal force.



Constraints can often be expressed algebraically.

Ex. Particle on sphere

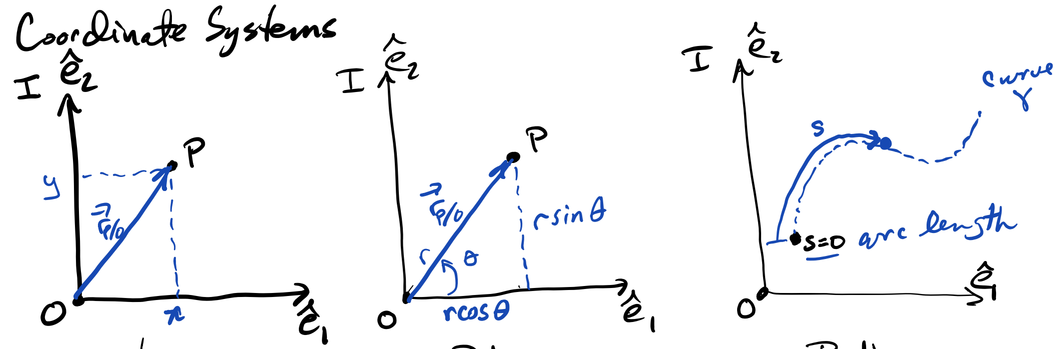


$$\|\vec{r}_{P/O}\| = \sqrt{x^2 + y^2 + z^2} = R$$

We will re-visit constraints later in the course.

Inertial Particle Kinematics in the Plane

Coordinate Systems



Cartesian $(x,y)_I$

Polar $(r,\theta)_I$

Path $(s,\gamma)_I$

Note: 1 DOF

Not a coordinate, but the curve γ

The description of a system's motion based on geometry and reference frames only; Does not include forces.

Solving for the **Kinematics** of a particle (or rigid body) involves calculating the velocity and acceleration by differentiating the position vector w.r.t. time.

Cartesian coord:

$$\vec{r}_{P/O} = x\,\hat{e}_1 + y\,\hat{e}_2$$

$${}^I\vec{v}_{P/O} = \frac{{}^I d}{dt}(\vec{r}_{P/O}) = \dot{x}\,\hat{e}_1 + \dot{y}\,\hat{e}_2$$

$${}^I\vec{a}_{P/O} = \frac{{}^I d}{dt}({}^I\vec{v}_{P/O}) = \ddot{x}\,\hat{e}_1 + \ddot{y}\,\hat{e}_2$$

Polar Coord.:

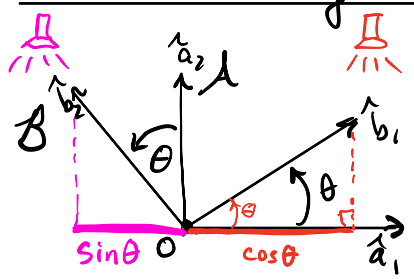
$$\vec{r}_{P/O} = r\cos\theta\,\hat{e}_1 + r\sin\theta\,\hat{e}_2$$

$${}^I\vec{v}_{P/O} = \frac{{}^I d}{dt}(\vec{r}_{P/O}) = (\dot{r}\cos\theta - r\sin\theta\,\dot{\theta})\hat{e}_1 + (\dot{r}\sin\theta + r\cos\theta\,\dot{\theta})\hat{e}_2$$

$${}^I\vec{a}_{P/O} = \frac{{}^I d}{dt}({}^I\vec{v}_{P/O}) = (\ddot{r}\cos\theta - 2\dot{r}\dot{\theta}\sin\theta - r\ddot{\theta}\sin\theta - r\dot{\theta}^2\cos\theta)\hat{e}_1$$
$$+ (\ddot{r}\sin\theta + 2\dot{r}\dot{\theta}\cos\theta + r\ddot{\theta}\cos\theta - r\dot{\theta}^2\sin\theta)\hat{e}_2$$

Sometimes problems are easily solved with making use of different types of reference frames, including moving and noninertial frames.
But first...

Transforming between reference frames



SOHCAHTOA

$$\mathcal{A} = (O, \hat{a}_1, \hat{a}_2, \hat{a}_3)$$
$$\mathcal{B} = (O, \hat{b}_1, \hat{b}_2, \hat{b}_3)$$

dot product

$\hat{a}_1 \cdot \hat{b}_2$ or $\hat{a}_1^T \hat{b}_2$

Transformation Table

$$\begin{array}{c|cc} & \hat{b}_1 & \hat{b}_2 \\ \hline \hat{a}_1 & \langle \hat{a}_1, \hat{b}_1 \rangle & \langle \hat{a}_1, \hat{b}_2 \rangle \\ \hat{a}_2 & \langle \hat{a}_2, \hat{b}_1 \rangle & \langle \hat{a}_2, \hat{b}_2 \rangle \end{array}$$

The $(i,j)^{th}$ entry represents the dot product between the i^{th} row and j^{th} column

$$\begin{array}{c|cc} & \hat{b}_1 & \hat{b}_2 \\ \hline \hat{a}_1 & \cos\theta & -\sin\theta \\ \hat{a}_2 & \sin\theta & \cos\theta \end{array}$$

To From

$${}^{\mathcal{A}}C^{\mathcal{B}} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$[\vec{r}_{P/O}]_{\mathcal{A}} = {}^{\mathcal{A}}C^{\mathcal{B}} [\vec{r}_{P/O}]_{\mathcal{B}}$$

This transformation table process results in <u>Transformation matrices</u>.

Often, but not always, these transformation matrices are also <u>rotation matrices</u>. (If there is a reflection involved, the resulting matrix will not be a rotation matrix.)

An important property for rotation matrices:

$$\left({}^{B}C^{A}\right)^{T} = \left({}^{B}C^{A}\right)^{-1} \quad \text{orthogonal matrices}$$

Now, if I know the ${}^{A}C^{B}$ is a rotation matrix, I can find ${}^{B}C^{A}$ easily:

$$[\vec{r}_{P/O}]_A = {}^{A}C^{B}[\vec{r}_{P/O}]_B$$

$$\left({}^{A}C^{B}\right)^{-1}[\vec{r}_{P/O}]_A = [\vec{r}_{P/O}]_B$$

or

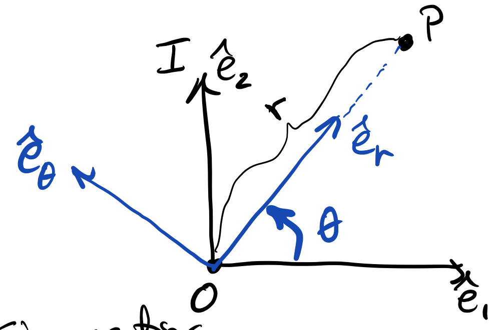
$$[\vec{r}_{P/O}]_B = \left({}^{A}C^{B}\right)^{-1}[\vec{r}_{P/O}]_A$$

$$[\vec{r}_{P/O}]_B = \underbrace{\left({}^{A}C^{B}\right)^{T}}_{{}^{B}C^{A}}[\vec{r}_{P/O}]_A$$

$$\Rightarrow \boxed{{}^{B}C^{A} = \left({}^{A}C^{B}\right)^{T}}$$

Polar Frame

↳ Reference frame that rotates to follow a point



$\mathcal{R} = (O, \hat{e}_r, \hat{e}_\theta, \hat{e}_3)$

$\mathcal{I} = (O, \hat{e}_1, \hat{e}_2, \hat{e}_3)$

Kinematics

$\vec{r}_{P/O} = r\hat{e}_r$

${}^{I}\vec{V}_{P/O} = \dot{r}\hat{e}_r + r\underbrace{\frac{{}^{I}d}{dt}(\hat{e}_r)}_{\text{what is this?}}$

Option #1

Convert all basis vectors to the I frame, then differentiate, and substitute back to get back to the polar frame

Transformation Table

	$\hat{e}_r$	$\hat{e}_\theta$
$\hat{e}_1$	$\cos\theta$	$-\sin\theta$
$\hat{e}_2$	$\sin\theta$	$\cos\theta$

$\} \; {}^{\mathcal{I}}C^{\mathcal{R}}$ (To ↓ $\mathcal{I}$, From ↓ $\mathcal{R}$)

$\vec{r}_{P/O} = r\hat{e}_r$

$= r(\cos\theta\hat{e}_1 + \sin\theta\hat{e}_2)$

${}^{I}\vec{V}_{P/O} = \dot{r}\cos\theta\hat{e}_1 - r\sin\theta\dot{\theta}\hat{e}_1 + \dot{r}\sin\theta\hat{e}_2 + r\cos\theta\dot{\theta}\hat{e}_2$

$= \dot{r}\underbrace{(\cos\theta\hat{e}_1 + \sin\theta\hat{e}_2)}_{\hat{e}_r} + r\dot{\theta}\underbrace{(-\sin\theta\hat{e}_1 + \cos\theta\hat{e}_2)}_{\hat{e}_\theta}$

$= \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$

Compare to

$= \dot{r}\hat{e}_r + r\frac{{}^{I}d}{dt}(\hat{e}_r)$

$\frac{{}^{I}d}{dt}(\hat{e}_r) = \dot{\theta}\hat{e}_\theta$

Caution: The $\hat{e}_3$ vectors must align between frames for this result.

Option #2: Differentiate unit vectors using the angular velocity vector

Angular velocity of $\mathcal{R}$ with respect to $\mathcal{I}$ $\left\{ {}^{\mathcal{I}}\vec{\omega}^{\mathcal{R}} = \dot{\theta}\,\hat{e}_3 \right.$

$$\frac{{}^{\mathcal{I}}d}{dt}(\hat{e}_r) = \frac{{}^{\mathcal{I}}d}{dt}\left(\cos\theta\,\hat{e}_1 + \sin\theta\,\hat{e}_2\right) =$$

$$= \dot{\theta}\underbrace{\left(-\sin\theta\,\hat{e}_1 + \cos\theta\,\hat{e}_2\right)}_{\hat{e}_\theta} = \dot{\theta}\,\hat{e}_\theta = \underbrace{\dot{\theta}\hat{e}_3}_{{}^{\mathcal{I}}\vec{\omega}^{\mathcal{R}}} \times \hat{e}_r$$

$$\boxed{\begin{aligned} \frac{{}^{\mathcal{I}}d}{dt}(\hat{e}_r) &= {}^{\mathcal{I}}\vec{\omega}^{\mathcal{R}} \times \hat{e}_r = \dot{\theta}\,\hat{e}_\theta \\ \frac{{}^{\mathcal{I}}d}{dt}(\hat{e}_\theta) &= {}^{\mathcal{I}}\vec{\omega}^{\mathcal{R}} \times \hat{e}_\theta = -\dot{\theta}\,\hat{e}_r \end{aligned}}$$

Inertial kinematics of point P in the Polar Frame

$$\vec{r}_{P/O} = r\,\hat{e}_r$$

$${}^{\mathcal{I}}\vec{v}_{P/O} = \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta$$

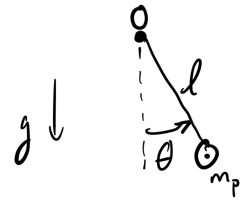
$${}^{\mathcal{I}}\vec{a}_{P/O} = \ddot{r}\,\hat{e}_r + 2\dot{r}\dot{\theta}\,\hat{e}_\theta + r\ddot{\theta}\,\hat{e}_\theta - r\dot{\theta}^2\,\hat{e}_r$$

$$= (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{e}_\theta$$

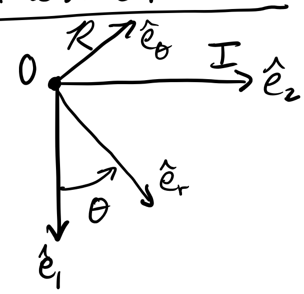
This is what you plug into N2L

$$\vec{F}_P = m_P\,{}^{\mathcal{I}}\vec{a}_{P/O}$$

Ex. Pendulum



Reference Frames



DOF

$M = 2N - K$

$M = 1$ DOF

Kinematics:

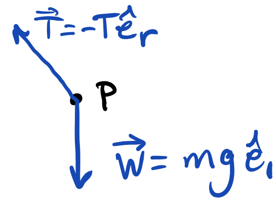
$\vec{r}_{P/O} = l\hat{e}_r$

${}^I\vec{v}_{P/O} = l\dot{\theta}\hat{e}_\theta$

$\boxed{{}^I\vec{a}_{P/O}} = l\ddot{\theta}\hat{e}_\theta - l\dot{\theta}^2\hat{e}_r$

$\boxed{\frac{{}^I d}{dt}(\hat{e}_\theta) = -\dot{\theta}\hat{e}_r}$

FBD



N2L:

$\vec{F}_P = m_P\,{}^I\vec{a}_{P/O}$

$-T\hat{e}_r + m_P g\hat{e}_1 = m(l\ddot{\theta}\hat{e}_\theta - l\dot{\theta}^2\hat{e}_r)$

To get the scalar eqn of motion, we will try to stay in the polar R frame; Let's convert the $\hat{e}_1$ vector

Rewriting $\hat{e}_1$

$-T\hat{e}_r + mg(\cos\theta\hat{e}_r - \sin\theta\hat{e}_\theta) = m_P(l\ddot{\theta}\hat{e}_\theta - l\dot{\theta}^2)\hat{e}_r$

Equate coefficients

$\hat{e}_r: \; -T + mg\cos\theta = -ml\dot{\theta}^2$

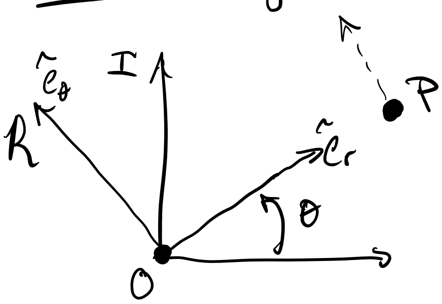
$\hat{e}_\theta: \; -mg\underbrace{s\theta}_{} = ml\ddot{\theta}$

Short-hand for sinθ

E.O.M. $\boxed{\ddot{\theta} = -\frac{g}{l}\sin\theta}$

Note: 1 DOF
Since 1
constraint

Ex. Straight line motion using a polar frame



$${}^{I}\vec{a}_{P/O} = (\ddot{r} - \underbrace{r\dot{\theta}^2}_{\text{centripetal acceleration term}})\hat{e}_r + (r\ddot{\theta} + \underbrace{2\dot{r}\dot{\theta}}_{\text{Coriolis acceleration}})\hat{e}_\theta$$

$$\underbrace{\cancel{\vec{F}_P}}_{0} = m_P\,{}^{I}\vec{a}_{P/O}$$

$$0 = m_P\left((\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta\right)$$

$$\Rightarrow \quad \hat{e}_r : \quad \ddot{r} = r\dot{\theta}^2$$

$$\hat{e}_\theta : \quad \ddot{\theta} = \frac{-2\dot{r}\dot{\theta}}{r}$$

These are kinematic terms that arise due to the rotating frame.