ECE 602: LUMPED LINEAR SYSTEMS

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State-Feedback Control of Single-Input Linear Systems

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- **Objective**: Construct state-feedback controllers for linear lumped both continuous-time (CT) and discrete-time (DT) systems
- We consider linear time-varying (LTI) system

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}u(t)$$

  or

$$\boldsymbol{x}[k+1] = \boldsymbol{A}\boldsymbol{x}[k] + \boldsymbol{b}u[k]$$

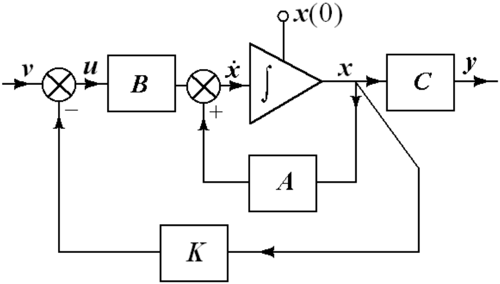
- We assume that the system at hand is reachable

Linear state-feedback (SF) controllers

- In general, the linear state-feedback control law, for a system modeled by $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$, is the feedback of a linear combination of all the state variables

$$\boldsymbol{u} = -\boldsymbol{K}\boldsymbol{x} + \boldsymbol{v},$$

- $\boldsymbol{K} \in \mathbb{R}^{m \times n}$ is a constant matrix and the vector $\boldsymbol{v}$ is an external input signal



Closed-loop system

- The closed-loop system

$$\dot{x} = (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K})\, x + \boldsymbol{B}\boldsymbol{v}$$

- The poles of the closed-loop system are the roots of the characteristic equation

$$\det\left(s\boldsymbol{I}_n - \boldsymbol{A} + \boldsymbol{B}\boldsymbol{K}\right) = 0$$

- The linear state-feedback control law design consists of selecting the gains

$$k_{ij}, \quad i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, n,$$

so that the roots of the closed-loop characteristic equation

$$\det\left(s\boldsymbol{I}_n - \boldsymbol{A} + \boldsymbol{B}\boldsymbol{K}\right) = 0$$

are in desirable locations in the complex plane

Preparing to select the controller gain matrix K

- The designer selects of the desired poles of the closed-loop system

$$s_1, s_2, \ldots, s_n$$

- The desired closed-loop poles can be real or complex
- If they are complex, then they must come in complex conjugate pairs
- This is because we use only real gains k_{ij}
- Having selected the desired closed-loop poles, form the desired closed-loop characteristic polynomial

$$\begin{aligned} \alpha_c(s) &= (s - s_1)(s - s_2)\cdots(s - s_n) \\ &= s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \alpha_0 \end{aligned}$$

Computing the controller gain k for single-input systems

- Our goal: Construct feedback matrix $\boldsymbol{K}$ such that

$$\det\left(s\boldsymbol{I}_n - \boldsymbol{A} + \boldsymbol{B}\boldsymbol{K}\right) = s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \alpha_0.$$

- This problem is also referred to as the pole placement problem
- For the single-input plants $\boldsymbol{K} = \boldsymbol{k} \in \mathbb{R}^{1\times n}$
- The solution to the problem is easily obtained if the pair $(\boldsymbol{A}, \boldsymbol{b})$ is already in the controller companion form

Computing k for the plant model in controller form

- If the plant model in the controller form, then

$$\boldsymbol{A} - \boldsymbol{bk} = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -a_0 - k_1 & -a_1 - k_2 & \cdots & -a_{n-2} - k_{n-1} & -a_{n-1} - k_n \end{bmatrix}$$

- Hence, the desired gains are

$$\begin{aligned} k_1 &= \alpha_0 - a_0, \\ k_2 &= \alpha_1 - a_1, \\ &\vdots \\ k_n &= \alpha_{n-1} - a_{n-1} \end{aligned}$$

Computing the controller gain k for the system not in the controller form

- If the pair $(\boldsymbol{A}, \boldsymbol{b})$ is not in the controller form, transform it into the controller form, then compute the gain vector $\tilde{\boldsymbol{k}}$ such that

$$\det\left(s\boldsymbol{I}_n - \tilde{\boldsymbol{A}} + \tilde{\boldsymbol{b}}\tilde{\boldsymbol{k}}\right) = s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \alpha_0$$

- Thus,

$$\tilde{\boldsymbol{k}} = \begin{bmatrix} \alpha_0 - a_0 & \alpha_1 - a_1 & \cdots & \alpha_{n-1} - a_{n-1} \end{bmatrix}.$$

- Then,

$$\boldsymbol{k} = \tilde{\boldsymbol{k}}\boldsymbol{T},$$

where $\boldsymbol{T}$ is the transformation that brings the pair $(\boldsymbol{A}, \boldsymbol{b})$ into the controller form

Computing k in one shot

- Represent the formula for the gain matrix in an alternative way
- Note that

$$
\begin{aligned}
\tilde{\boldsymbol{k}}\boldsymbol{T} &= \begin{bmatrix} \alpha_0 - a_0 & \alpha_1 - a_1 & \cdots & \alpha_{n-1} - a_{n-1} \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_1 \\ \boldsymbol{q}_1\boldsymbol{A} \\ \vdots \\ \boldsymbol{q}_1\boldsymbol{A}^{n-1} \end{bmatrix} \\
&= \boldsymbol{q}_1\left(\alpha_0\boldsymbol{I}_n + \alpha_1\boldsymbol{A} + \cdots + \alpha_{n-1}\boldsymbol{A}^{n-1}\right) \\
&\quad -\boldsymbol{q}_1\left(a_0\boldsymbol{I}_n + a_1\boldsymbol{A} + \cdots + a_{n-1}\boldsymbol{A}^{n-1}\right)
\end{aligned}
$$

Computing k in one shot—Contd

- By the Cayley-Hamilton theorem,

$$\boldsymbol{A}^n = -\left(a_0\boldsymbol{I}_n + a_1\boldsymbol{A} + \cdots + a_{n-1}\boldsymbol{A}^{n-1}\right)$$

- Hence,

$$\boldsymbol{k} = \boldsymbol{q}_1\alpha_c\left(\boldsymbol{A}\right)$$

- The above expression for the gain row vector was proposed by Ackermann in 1972, and is now referred to as the Ackermann's formula for pole placement

Example

- Dynamical system

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u$$

- Use the Ackermann's formula to design a state-feedback controller, $u = -\boldsymbol{k}\boldsymbol{x}$, such that the closed-loop poles are located at $\{-1, -2\}$
- Form the controllability matrix of the pair $(\boldsymbol{A}, \boldsymbol{b})$ and then find the last row of its inverse denoted $\boldsymbol{q}_1$
- The controllability matrix is

$$\begin{bmatrix} \boldsymbol{b} & \boldsymbol{A}\boldsymbol{b} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$$

Computing k using Ackermann's formula

- The controllability matrix inverse is

$$\begin{bmatrix} \boldsymbol{b} & \boldsymbol{A}\boldsymbol{b} \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}.$$

- Hence, $\boldsymbol{q}_1 = \begin{bmatrix} 1 & -2 \end{bmatrix}$
- The desired closed-loop characteristic polynomial is

$$\alpha_c(s) = (s+1)(s+2) = s^2 + 3s + 2$$

- Therefore,

$$\begin{aligned} \boldsymbol{k} &= \boldsymbol{q}_1 \alpha_c(\boldsymbol{A}) \\ &= \boldsymbol{q}_1 \left(\boldsymbol{A}^2 + 3\boldsymbol{A} + 2\boldsymbol{I}_2 \right) \\ &= \boldsymbol{q}_1 \left(\begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix} + 3 \begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix} + 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \\ &= \boldsymbol{q}_1 \left(\begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix} + \begin{bmatrix} 3 & -3 \\ 3 & -6 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \right) \end{aligned}$$

Applying Ackermann's formula to find the controller gain k

- Continuing

$$\begin{aligned} \boldsymbol{k} &= \boldsymbol{q}_1 \alpha_c(\boldsymbol{A}) \\ &= \begin{bmatrix} 1 & -2 \end{bmatrix} \begin{bmatrix} 5 & -2 \\ 2 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \end{aligned}$$

The Pole Placement Theorem

Theorem

The pole placement problem is solvable for all choices of n desired closed-loop poles, symmetric with respect to the real axis, if and only if the given pair $(\boldsymbol{A}, \boldsymbol{B})$ is reachable.

- The proof of necessity ($\Leftarrow$) follows immediately from the discussion preceding the example
- Indeed, the pair $(\boldsymbol{A}, \boldsymbol{B})$ is reachable, if and only if it can be transformed into the controller form
- Once the transformation is performed, solve the pole placement problem for the given set of desired closed-loop poles, symmetric with respect to the real axis
- Transform the pair $(\boldsymbol{A}, \boldsymbol{B})$ and the gain matrix back into the original coordinates

Proof of Sufficiency Part of the Pole Placement Theorem

- The similarity transformation does not affect neither reachability nor the characteristic polynomial
- Thus the eigenvalues of $\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}$ are precisely the desired prespecified closed-loop poles.
- This completes the proof of the necessity part
- Use a proof by contraposition to prove the sufficiency part $(\Rightarrow)$
- Assume that the pair $(\boldsymbol{A}, \boldsymbol{B})$ is nonreachable
- There is a similarity transformation $\boldsymbol{z} = \boldsymbol{T}\boldsymbol{x}$ such that the pair $(\boldsymbol{A}, \boldsymbol{B})$ in the new coordinates has the form

$$\tilde{\boldsymbol{A}} = \boldsymbol{T}\boldsymbol{A}\boldsymbol{T}^{-1} = \begin{bmatrix} \boldsymbol{A}_1 & \boldsymbol{A}_2 \\ \boldsymbol{O} & \boldsymbol{A}_4 \end{bmatrix}, \quad \tilde{\boldsymbol{B}} = \boldsymbol{T}\boldsymbol{B} = \begin{bmatrix} \boldsymbol{B}_1 \\ \boldsymbol{O} \end{bmatrix},$$

where the pair $(\boldsymbol{A}_1, \boldsymbol{B}_1)$ is reachable, $\boldsymbol{A}_1 \in \mathbb{R}^{r\times r}$, and $\boldsymbol{B}_1 \in \mathbb{R}^{r\times m}$

Sufficiency for Pole Placement

- Let

$$\boldsymbol{u} = -\tilde{\boldsymbol{K}}\boldsymbol{z} = -\begin{bmatrix} \boldsymbol{K}_1 & \boldsymbol{K}_2 \end{bmatrix}\boldsymbol{z},$$

  where $\boldsymbol{K}_1 \in \mathbb{R}^{m\times r}$ and $\boldsymbol{K}_2 \in \mathbb{R}^{m\times(n-r)}$

- Then,

$$\tilde{\boldsymbol{A}} - \tilde{\boldsymbol{B}}\tilde{\boldsymbol{K}} = \begin{bmatrix} \boldsymbol{A}_1 - \boldsymbol{B}_1\boldsymbol{K}_1 & \boldsymbol{A}_2 - \boldsymbol{B}_1\boldsymbol{K}_2 \\ \boldsymbol{O} & \boldsymbol{A}_4 \end{bmatrix}$$

- The nonreachable portion of the system not affected by the state-feedback
- That is, the eigenvalues of $\boldsymbol{A}_4$ cannot be allocated
- Hence, the pole placement problem cannot be solved if the pair $(\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{B}})$ is nonreachable
- We call the system stabilizable if the nonreachable part is stable