HW6

11.1.12) $f(x) = |x|$

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x \geq 0 \end{cases}$$

$$a_0 = \frac{1}{2\pi}\left[\int_{-\pi}^{0} -x\,dx + \int_0^{\pi} x\,dx\right]$$

$$= \frac{1}{2\pi}\left[-\frac{x^2}{2}\Big|_{-\pi}^{0} + \frac{x^2}{2}\Big|_0^{\pi}\right] = \frac{1}{2\pi}\left[\frac{\pi^2}{2} + \frac{\pi^2}{2}\right] = \frac{\pi}{2}$$

$$\boxed{a_0 = \frac{\pi}{2}}$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -x\cos(nx)\,dx + \int_0^{\pi} x\cos(nx)\,dx\right]$$

Let $u = x$, $dv = \cos(nx)$
$du = dx$, $v = \frac{1}{n}\sin(nx)$

$\therefore \int x\cos(nx)\,dx = \frac{x}{n}\sin(nx) - \frac{1}{n}\int \sin(nx)\,dx$

$$= \left(\frac{x}{n}\sin(nx)\Big|_0^{\pi}\right) - \left(\frac{1}{n^2}\cos(nx)\Big|_0^{\pi}\right) = (0-0) - \left(\frac{\cos(n\pi)}{n^2} - \frac{1}{n^2}\right)$$

$$= \frac{1}{n^2} - \frac{\cos(n\pi)}{n^2}$$

$$-\int_{-\pi}^{0} x\cos(nx)\,dx$$

$$= -\left(\left[\frac{x}{n}\sin(nx)\Big|_{-\pi}^{0}\right] - \left[\frac{1}{n^2}\cos(nx)\Big|_{-\pi}^{0}\right]\right)$$

$$= -\left([0-0] - \left[\frac{1}{n^2} - \frac{1}{n^2}\cos(n\pi)\right]\right)$$

$$= \frac{1}{n^2} - \frac{1}{n^2}\cos(n\pi)$$

$$\therefore\ a_n = \frac{1}{\pi}\left[\frac{1}{n^2} - \frac{\cos(n\pi)}{n^2} + \frac{1}{n^2} - \frac{1}{n^2}\cos(n\pi)\right]$$

$$\boxed{a_n = \frac{2}{\pi n^2}\left[1 - \cos(n\pi)\right]}$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -x\sin(nx)\,dx + \int_{0}^{\pi} x\sin(nx)\,dx\right]$$

$U = x$, $dV = \sin(nx)$

$du = dx$, $V = \frac{-1}{n}\cos(nx)$

$$\therefore \int x\sin(nx)\,dx = -\frac{x}{n}\cos(nx) + \frac{1}{n}\int \cos(nx)\,dx$$

$$\left(-\frac{x}{n}\cos(nx)\Big|_{0}^{\pi}\right) - \frac{1}{n^2}\left(\sin(nx)\Big|_{0}^{\pi}\right)$$

$$= -\frac{\pi}{n}\cos(n\pi)$$

$$-\left(\frac{-x}{n}\cos(nx)\Big|_{-\pi}^{0} - \frac{1}{n^2}\left(\sin(nx)\Big|_{-\pi}^{0}\right)\right)$$

$$= -\left[0 - \frac{\pi}{n}\cos(n\pi)\right] = \frac{\pi}{n}\cos(n\pi)$$

$$\therefore b_n = \frac{1}{\pi}\left[-\frac{\pi}{n}\cos(n\pi) + \frac{\pi}{n}\cos(n\pi)\right] \therefore$$

$$\boxed{b_n = 0}$$

$$\boxed{f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2}(1 - \cos(n\pi))\cos(nx))}$$

11.14) $f(x) = x^2 \quad (-\pi < x < \pi)$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx = \left[\frac{x^3}{3}\right]\frac{1}{2\pi}\Big|_{-\pi}^{\pi} = \frac{1}{2\pi}\left[\frac{\pi^3}{3} + \frac{\pi^3}{3}\right] = \frac{\pi^2}{3}$$

$$\boxed{a_0 = \frac{\pi^2}{3}}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos(nx)\,dx$$

$u = x^2 \qquad dv = \cos(nx)\,dx$

$du = 2x\,dx \qquad v = \frac{1}{n}\sin(nx)$

$$\int x^2\cos(nx)\,dx = \frac{x^2}{n}\sin(nx) - \frac{2}{n}\int x\sin(nx)\,dx$$

$$\int x\sin(nx) = \frac{-x}{n}\cos(nx) + \frac{1}{n}\int\cos(nx)\,dx$$

$$\int_{-\pi}^{\pi} x^2 \cos(nx)\,dx = \cancel{\frac{x^2}{n}\sin(nx)}\Big|_{-\pi}^{\pi} - \frac{2}{n}\left[\frac{-x}{n}\cos(nx)\Big|_{-\pi}^{\pi} + \frac{1}{n}\int_{-\pi}^{\pi}\cos(nx)\,dx\right]$$

$$= -\frac{2}{n}\left[-\frac{\pi}{n}\cos(n\pi) - \frac{\pi}{n}\cos(n\pi)\right] + \frac{1}{n^2}\cancel{\sin(nx)}\Big|_{-\pi}^{\pi}\Big]$$

$$= -\frac{2}{n}\left[-\frac{2\pi}{n}\cos(n\pi)\right] = \frac{4\pi}{n^2}\cos(n\pi)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos(nx)\,dx$$

$$\boxed{a_n = \frac{4}{n^2}\cos(n\pi)}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin(nx)\,dx \quad , \quad x^2\sin(nx) \text{ is odd}$$

$$\therefore \int_{-\pi}^{\pi} x^2\sin(nx)\,dx = 0$$

$$\therefore \boxed{b_n = 0}$$

$$\boxed{f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}\cos(n\pi)\cos(nx)}$$

11.18) $f(x) = \begin{cases} 0 & -\pi < x < 0 \\ 1 & 0 < x < \pi \end{cases}$

$$a_0 = \frac{1}{2\pi}\int_0^{\pi} 1\,dx = \frac{\pi}{2\pi}$$

$$\boxed{a_0 = \frac{1}{2}}$$

$$a_n = \frac{1}{\pi} \int_0^{\pi} \cos(nx)\,dx$$

$$= \frac{1}{n\pi} \left[\sin(nx) \Big|_0^{\pi} \right] = 0$$

$$a_n = 0$$

$$b_n = \frac{1}{\pi} \int_0^{\pi} \sin(nx)\,dx = \frac{1}{n\pi} \left[-\cos(nx) \Big|_0^{\pi} \right]$$

$$= \frac{1}{n\pi} \left[-\cos(n\pi) - (-1) \right]$$

$$\boxed{b_n = \frac{1 - \cos(n\pi)}{n\pi}}$$

$$\boxed{f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(1 - \cos(n\pi))}{n\pi} \sin(nx)}$$

11.2.11) $f(x) = x^2 \quad (-1 < x < 1), \; P = 2$

$\boxed{\text{Even}}$ $\quad P = 2L = 2, \; \boxed{L = 1}$

$$a_0 = \frac{1}{2L} \int_{-L}^{L} x^2\,dx = \frac{1}{2} \left[\frac{x^3}{3} \Big|_{-1}^{1} \right] = \frac{1}{2} \left[\frac{1}{3} + \frac{1}{3} \right] = \frac{1}{3}$$

$$\boxed{a_0 = \frac{1}{3}}$$

$$a_n = \frac{1}{L}\int_{-1}^{1} x^2 \cos(n\pi x)\,dx$$

$$\int x^2 \cos(n\pi x)\,dx = \cancelto{0}{\frac{x^2}{n\pi}\sin(n\pi x)\Big|_{-1}^{1}} - \frac{2}{n\pi}\left[\frac{-x}{n\pi}\cos(n\pi x)\Big|_{-1}^{1} + \cancelto{0}{\frac{1}{n\pi}\int_{-1}^{1}\cos(n\pi x)}\right]$$

$$= \frac{-2}{n\pi}\left[\frac{-1}{n\pi}\cos(n\pi) - \frac{1}{n\pi}\cos(-n\pi)\right] = \left(\frac{-2}{n\pi}\right)\left(\frac{-2}{n\pi}\cos(n\pi)\right)$$

$$\therefore \boxed{a_n = \frac{4}{n^2\pi^2}\cos(n\pi)}$$

$$b_n = \frac{1}{L}\int_{-1}^{1} x^2 \sin(n\pi x)\,dx \quad , \quad x^2 \sin(n\pi x) \text{ is odd}$$

$$\therefore \boxed{b_n = 0}$$

$$\boxed{f(x) = \frac{1}{3} + \sum_{K=1}^{\infty} \frac{4}{n^2\pi^2}\cos(n\pi)\cos(\pi n x)}$$

11.2.24) $f(x) = \begin{cases} 0 & 0 < x < 2 \\ 1 & 2 < x < 4 \end{cases}$

$L = 4$

$a_0 = \frac{1}{L} \int_0^L f(x)\,dx = \frac{1}{4} \int_2^4 dx = \frac{[4-2]}{4} = \frac{1}{2}$

$\boxed{a_0 = \frac{1}{2}}$

$f(x) = 1$ is even function $\therefore$ $\boxed{b_n = 0}$ ← Fourier sine series

$a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi}{L} x\right) dx$

$= \frac{1}{2} \int_2^4 \cos\left(\frac{n\pi}{4} x\right) dx$

$= \left(\frac{1}{2}\right)\left(\frac{4\pi}{n}\right) \left[\sin\left(\frac{n\pi}{4} x\right) \Big|_2^4 \right] = \frac{2\pi}{n} \left[\sin(n\pi) - \sin\left(\frac{n\pi}{2}\right) \right]$

$\boxed{a_n = \frac{2\pi}{n} \sin\left(\frac{n\pi}{2}\right)}$

11.2.29) $\sin(x)$ $(0 < x < \pi)$ $\underline{L = \pi}$

$\sin(x)$ is $\underline{\text{odd}}$ $\therefore$ $a_0 = a_n = 0$

$$b_n = \frac{2}{\pi} \int_0^{\pi} \sin(x)\sin(nx)\,dx$$

$$\sin(x)\sin(nx) = \frac{1}{2}\left[\cos(x - nx) - \cos(x + nx)\right] \therefore$$

$$b_n = \frac{1}{\pi}\left[\int_0^{\pi} \cos(x - nx)\,dx - \int_0^{\pi} \cos(x + nx)\,dx\right]$$

$$\frac{1}{\pi}\left[\frac{1}{1-n}\sin(x - nx)\Big|_0^{\pi} - \frac{1}{1+n}\sin(x + nx)\Big|_0^{\pi}\right]$$

$$= \left[\frac{1}{1-n}\sin(\pi - n\pi) - \frac{1}{1+n}\sin(\pi + n\pi)\right]$$

$$= \left(\frac{1}{\pi}\right)\left(\frac{-2\sin(n\pi)}{n^2 - 1}\right)$$

$$\boxed{b_n = \frac{-2\sin(n\pi)}{\pi(n^2 - 1)}}$$

$$\boxed{f(x) = \sum_{n=1}^{\infty} \frac{-2\sin(n\pi)}{\pi(n^2 - 1)} \sin(nx)}$$

11.3.2) $y_n = A_n \cos(nt) + B_n \sin(nt)$

$y_n' = -A_n n \sin(nt) + B_n n \cos(nt)$

$y_n'' = -A_n n^2 \cos(nt) - B_n n^2 \sin(nt)$

$\therefore$

$-n^2(A_n \cos(nt) + B_n \sin(nt)) + .05(-A_n n \sin(nt) + B_n n \cos(nt))$

$+ 49 A_n \cos(nt) + 49 B_n \sin(nt) = \frac{4}{n^2 \pi} \cos(nt)$

$-n^2 A_n + .05 B_n n + 49 A_n = \frac{4}{n^2 \pi}$

$-n^2 B_n - .05 A_n n + 49 B_n = 0$

$B_n = \frac{(.05 A_n) n}{(49 - n^2)}$

$-n^2 A_n + \frac{(.05n)^2 A_n}{(49 - n^2)} + 49 A_n = \frac{4}{n^2 \pi}$

$A_n = \frac{4(49-n^2)}{n^2 \pi [(49-n^2)^2 + (.05n^2)]}$

$B_n = \frac{0.2}{n \pi [(49-n^2)^2 - (.05n)^2]}$

$$C_n = \sqrt{A_n^2 + B_n^2} = \frac{4}{n^2\pi\sqrt{[(49-n^2)^2 + (0.05n)^2]}}$$

Denominator increases, $\therefore$ the amplitudes will decrease. Similarly, if the damping increases, the amplitudes will decrease.

11.3.6) $y'' + \omega^2 y = r(t)$

$r(t) = \sin(\alpha t) + \sin(\beta t)$

$\lambda^2 + \omega^2 = 0$

$\lambda = \pm \omega i$

$\therefore y_c(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t)$

$y_p = a\sin(\alpha t) + b\sin(\beta t)$

$y_p' = a\alpha\cos(\alpha t) + b\beta\cos(\beta t)$

$y_p'' = -a\alpha^2\sin(\alpha t) - b\beta^2\sin(\beta t)$

$\therefore -a\alpha^2\sin(\alpha t) - b\beta^2\sin(\beta t) + a\alpha\cos(\alpha t) + b\beta\cos(\beta t)$
$+ a\sin(\alpha t) + b\sin(\beta t) = \sin(\alpha t) + \sin(\beta t)$

$-a\alpha^2 + a = 1 \quad \Rightarrow \quad a(1-\alpha^2) = 1$

$-b\beta^2 + b = 1 \quad \Rightarrow \quad b(1-\beta^2) = 1$

$\therefore \; y_p = \frac{1}{1-\alpha^2}\sin(\alpha t) + \frac{1}{1-\beta^2}\sin(\beta t)$

$$\boxed{y(t) = C_1\cos(\omega t) + C_2\sin(\omega t) + \frac{\sin(\alpha t)}{1-\alpha^2} + \frac{\sin(\beta t)}{1-\beta^2}}$$

11.5.7) $y'' + \lambda y = 0 \quad y(0) = 0, \; y(10) = 0$

$[p y']' + [q + \lambda r]y = 0$

$p = 1, \; q = 0, \; r = 1$

$s^2 + \lambda = 0 \quad \therefore \quad s_{1,2} = \pm\sqrt{-\lambda}$

$\lambda \le 0: \quad s_{1,2} = \pm\sqrt{-\lambda} \quad \therefore \quad y(x) = C_1 e^{\sqrt{\lambda}x} + C_2 e^{-\sqrt{\lambda}x}$

$\lambda > 0: \; s_{1,2} = \pm\sqrt{\lambda}\, i \quad \therefore \; y(x) = C_1\cos(\sqrt{\lambda}x) + C_2\sin(\sqrt{\lambda}x)$

$\lambda \le 0: \; y(0) = C_1 + C_2 = 0 \quad \Rightarrow \quad C_1 = -C_2$

$y(10) = C_1 e^{\sqrt{\lambda}10} - C_1 e^{-\sqrt{\lambda}10} = 0 \Rightarrow \; C_1 = 0 \; \therefore \; C_2 = 0$

$\lambda > 0: \; y(0) = C_1 = 0$

$y(10) = C_2\sin(\sqrt{\lambda}10) = 0$

$\sin(\sqrt{\lambda}10) = 0 \; , \quad \sqrt{\lambda}\,10 = n\pi \quad \therefore \quad \lambda = \left(\frac{n\pi}{10}\right)^2$

$$\therefore \; \boxed{\lambda = \left(\frac{n\pi}{10}\right)^2, \quad y(x) = \sin\left(\frac{n\pi x}{10}\right)}$$

$$(y_m, y_n) = \int_a^b r\, y_m y_n \, dx$$

$$\therefore (y_m, y_n) = \int_0^{10} \sin\left(\frac{m\pi x}{10}\right) \sin\left(\frac{n\pi x}{10}\right) dx$$

$$= \frac{1}{2} \int_0^{10} \cos\left(\frac{(m-n)\pi x}{10}\right) dx - \frac{1}{2} \int_0^{10} \cos\left(\frac{(m+n)\pi x}{10}\right) dx$$

$$|y_m|^2 = \int_0^{10} \sin^2\left(\frac{m\pi x}{10}\right) dx = 5$$

11.5.5) (Sorry I missed this one)

$$\int_0^{\pi} \cos(\theta) \sin(\theta)\, d\theta = \frac{1}{2} \int_0^{\pi} \sin(2\theta)\, d\theta$$

$$U = 2\theta \quad du = 2\, d\theta \quad \therefore \frac{1}{4} \int_0^{\pi} \sin(u)\, du = -\frac{1}{4} \left[\cos(2\theta) \Big|_0^{\pi} \right]$$

$$= -\frac{1}{4}[1 - 1] = 0$$

$$\therefore \boxed{\int_0^{\pi} \cos(\theta) \sin(\theta)\, d\theta = 0 \text{ , orthogonal}}$$

11.5.13) $y'' + 8y' + (\lambda + 16)y = 0 \quad y(0) = 0, \; y(\pi) = 0$

$p = e^{\int 8dx} = e^{8x}$

$q = (16)(e^{8x}) = 16e^{8x}$

$r = (1)(e^{8x}) = e^{8x}$

$\therefore \; [e^{8x} y']' + [16e^{8x} + \lambda e^{8x}]y = 0$

$s^2 + 8s + (\lambda + 16) = 0$

$s_{1,2} = \dfrac{-8 \pm \sqrt{64 - 4(\lambda+16)}}{2} = -4 \pm \sqrt{-\lambda}$

$\lambda > 0 : \; y(x) = e^{-4x}(C_1 \cos(\sqrt{\lambda} x) + C_2 \sin(\sqrt{\lambda} x))$

$y(0) = 0 = C_1$

$y(\pi) = 0 = e^{-4\pi} C_2 \sin(\sqrt{\lambda}\pi)$

$\sin(\sqrt{\lambda}\pi) = 0$ if $\sqrt{\lambda} = n = 1, 2, 3 \ldots$

$\therefore \; \boxed{\lambda = n^2}, \; \boxed{y(x) = e^{-4x} \sin(nx)}$

$\lambda < 0 :$ yields $C_1 = C_2 = 0$

$y(x) = C_1 e^{-4+\sqrt{\lambda}} + C_2 e^{-4-\sqrt{\lambda}}$

$y(0) = C_1 + C_2 = 0$

$y(\pi) = 0 = C_1 e^{-4+\sqrt{\lambda}} - C_1 e^{-4-\sqrt{\lambda}}$

11.6.1) $63x^5 - 90x^3 + 35x$

$$a_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)\, y_n(x)\, dx$$

$y_0 = 1$

$$\therefore a_0 = \frac{1}{2}\int_{-1}^{1} 63x^5 - 90x^3 + 35x = \frac{1}{2}\left[\frac{63x^6}{6} - \frac{90x^4}{4} + \frac{35x^2}{2}\right]\Big|_{-1}^{1}$$

$$a_0 = 0$$

$y_1 = x$

$$\therefore a_1 = \frac{3}{2}\int_{-1}^{1} 63x^6 - 90x^4 + 35x^2 = \frac{3}{2}\left[\frac{63x^7}{7} - \frac{90x^5}{5} + \frac{35x^3}{3}\Big|_{-1}^{1}\right]$$

$$a_1 = 8$$

$y_2 = \frac{1}{2}(3x^2 - 1)$

$$\therefore a_2 = \frac{5}{4}\int_{-1}^{1}(63x^5 - 90x^3 + 35x)(3x^2 - 1)\, dx$$

$$a_2 = \frac{5}{4}\int_{-1}^{1} 189x^7 - 270x^5 + 105x^3 - 63x^5 + 90x^3 - 35x\, dx$$

$$= \frac{5}{4}\left[\frac{189x^8}{8} - \frac{333x^6}{6} + \frac{195x^4}{4} - \frac{35x^2}{2}\Big|_{-1}^{1}\right]$$

$$a_2 = 0$$

$$y_3 = \frac{1}{2}(5x^3 - 3x)$$

$$a_3 = \frac{7}{4}\int_{-1}^{1}(63x^5 - 90x^3 + 35x)(5x^3 - 3x)\,dx$$

$$a_3 = \frac{7}{4}\int_{-1}^{1} 315x^8 - 450x^6 + 175x^4 - 189x^6 + 270x^4 - 105x^2\,dx$$

$$= \frac{7}{4}\int_{-1}^{1} 315x^8 - 639x^6 + 445x^4 - 105x^2\,dx$$

$$a_3 = -8$$

$$a_0 = a_2 = 0 \quad \therefore \quad a_4 = 0$$

$$y_5 = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

$$\therefore a_5 = \frac{11}{16}\int_{-1}^{1}(63x^5 - 90x^3 + 35x)(63x^5 - 70x^3 + 15x)\,dx$$

$$a_5 = \frac{11}{16}\int_{-1}^{1} 3969x^{10} - 4410x^8 + 945x^6 - 5670x^8 + 6300x^6 - 1350x^4$$
$$+ 2205x^6 - 2450x^4 + 525x^2$$

$$= \frac{11}{16}\int_{-1}^{1} 3969x^{10} - 10080x^8 + 9450x^6 - 3800x^4 + 525x^2$$

$$a_5 = 8$$

$$\therefore \boxed{f(x) = 8x - \frac{8}{2}(5x^3 - 3x) + (63x^5 - 70x^3 + 15x)} = 8y_1 - 8y_2 + 8y_3$$

11.6.3) $1-x^4$

$y_0 = 1$: $a_0 = \frac{1}{2}\int_{-1}^{1} 1-x^4 = \frac{8}{10}$

$y_1 = x$: $a_1 = \frac{3}{2}\int_{-1}^{1} x - x^5 = 0$

$y_2 = \frac{1}{2}(3x^2-1)$: $a_2 = \frac{5}{4}\int_{-1}^{1} 3x^2 - 1 - 3x^6 + x^4 \, dx = -4/7$

$y_3 = \frac{1}{2}(5x^3-3x)$: $a_3 = \frac{7}{4}\int_{-1}^{1} (1-x^4)(5x^3-3x) = 0$ (

$y_4 = \frac{1}{8}(35x^4 - 30x^2 + 3)$; $a_4 = \frac{9}{16}\int_{-1}^{1} (1-x^4)(35x^4 - 30x^2 + 3)\,dx$

$= 9/16 \int_{-1}^{1} 35x^4 - 30x^2 + 3 - 35x^8 + 30x^6 - 3x^4$

$= 9/16 \int_{-1}^{1} 32x^4 - 30x^2 + 3 - 35x^8 + 30x^6 = -8/35$

$$\boxed{\begin{aligned} \therefore F(x) &= 8/10 - 4/14(3x^2-1) - \tfrac{1}{35}(35x^4 - 30x^2 + 3) \\ &= 8/10\, y_0 - 4/7\, y_2 - 8/35\, y_4 \end{aligned}}$$