## 4.5. Nonlinear systems of DE.

(Ex) $\begin{cases} y_1' = y_2 \\ y_2' = y_1 - \frac{1}{2}y_1^2 \end{cases}$

(1) Equilibrium solutions: Let $y_1' = 0$ & $y_2' = 0$

$y_2 = 0$ & $\underline{y_1 - \frac{1}{2}y_1^2 = 0}$ : $y_1(1 - \frac{1}{2}y_1) = 0$

$y_1 = 0$ or $1 - \frac{1}{2}y_1 = 0$ : $y_1 = 2$

$\underline{(0,0), \ (2,0) : \text{critical points}}$

$\mathbf{y} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \ \mathbf{y} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

Want classification of critical points and their stabilities.

Q classify critical points & stabilities

(A) Linearization.

1. $(0, 0)$ : Analyze solution curves around $(0,0)$.

$(y_1, y_2) \approx (0, 0)$ : $|y_1| \approx 0$, $|y_2| \approx 0$

$\underline{|y_1| > |y_1|^2}$, $|y_2| > y_2^2$

| | |
|---|---|
| 0.1 | 0.01 |
| 0.01 | 0.0001 |

: Drop "$-\frac{1}{2}y_1^2$"

$$\Rightarrow \begin{cases} y_1' = y_2 \\ y_2' = y_1 \end{cases} : \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \qquad \underline{\mathbf{y}' = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{\text{let } A} \mathbf{y}.}$$

a linearized system.

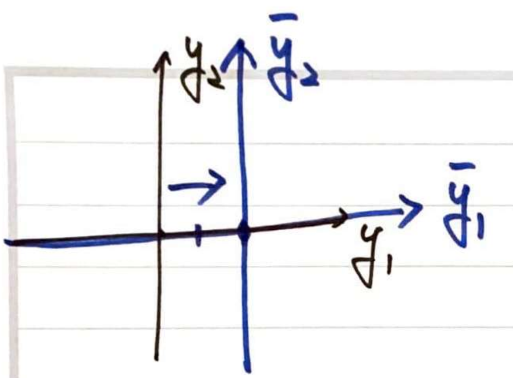
$$\det(A - \lambda I) = (-\lambda)^2 - 1 = \lambda^2 - 1 = 0 : \lambda = 1, -1.$$

$(0, 0)$ : a unstable saddle point.

$(2, 0)$ : $(y_1, y_2) \approx (2, 0)$

$y_1^2 \approx 4$ : <u>We cannot "$-\frac{1}{2} y_1^2$"</u>

(Idea) Translate $(y_1, y_2)$-coordinate system.




$$\begin{cases} \bar{y}_1 = y_1 - 2 \\ \bar{y}_2 = y_2 \end{cases}$$

$y_1 = \bar{y}_1 + 2, \quad y_2 = \bar{y}_2$

Ⓛ $y_1' = \bar{y}_1', \quad y_2' = \bar{y}_2'$

Ⓡ ① $y_2 = \bar{y}_2$, ② $y_1 - \frac{1}{2} y_1^2 = \bar{y}_1 + 2 - \frac{1}{2}(\bar{y}_1 + 2)^2$

$= \bar{y}_1 + \cancel{2} - \frac{1}{2}(\bar{y}_1^2 + 4\bar{y}_1 + \cancel{4})$

$y_1 - \frac{1}{2} y_1^2 = -\bar{y}_1 - \frac{1}{2} \bar{y}_1^2$

$$\begin{cases}\bar{y}_1' = \bar{y}_2 \\ \bar{y}_2' = -\bar{y}_1 - \frac{1}{2}\bar{y}_1^2\end{cases}$$ : $(0,0)$ in $(\bar{y}_1, \bar{y}_2)$ plane

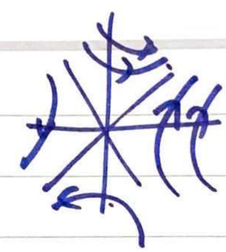
$(\bar{y}_1, \bar{y}_2) \approx (0,0)$: Drop "$-\frac{1}{2}\bar{y}_1^2$"

$$\Rightarrow \begin{cases}\bar{y}_1' = \bar{y}_2 \\ \bar{y}_2' = -\bar{y}_1\end{cases} : \begin{bmatrix}\bar{y}_1 \\ \bar{y}_2\end{bmatrix}' = \begin{bmatrix}0 & 1 \\ -1 & 0\end{bmatrix}\underbrace{\begin{bmatrix}\bar{y}_1 \\ \bar{y}_2\end{bmatrix}}_{= \bar{\mathbf{y}}}$$

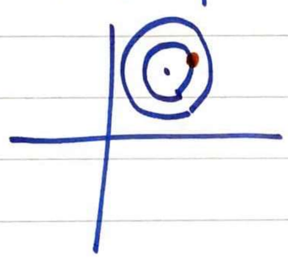
$$\bar{\mathbf{y}}' = \underbrace{\begin{bmatrix}0 & 1 \\ -1 & 0\end{bmatrix}}_{= A}\bar{\mathbf{y}} \qquad \underline{A: \lambda = i, -i}$$

$(0,0)$ : a stable center

$(2,0)$ in $(y_1, y_2)$ plane.

Q 1. saddle point  : unstable.

2. a center

 : stable.

3. Does linearization predict solution curves exact? No not exactly

(Quick way).

Use Jacobian matrix.

$$\begin{cases} y_1' = f_1(y_1, y_2) \\ y_2' = f_2(y_1, y_2) \end{cases} : \quad J = \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{bmatrix}$$

(Ex) $\begin{cases} y_1' = y_2 = f_1 \\ y_2' = \underline{y_1 - \frac{1}{2} y_1^2} = f_2 \end{cases}$ : $(0,0), (2,0)$

$$J = \begin{bmatrix} 0 & 1 \\ 1 - \frac{1}{2} 2y_1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 - y_1 & 0 \end{bmatrix}$$

$J(0,0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\lambda = 1, -1.$

$J(2,0) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $\lambda = i, -i.$

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(Ex) $y'' - 9y + y^3 = 0.$ ?

(idea) Use a system of DE.

(1) Let $y_1 = y$ & $y_2 = y'$

$$\begin{cases} y_1' = y_2 \\ y_2' = y'' = 9y - y^3 = 9y_1 - y_1^3. \end{cases}$$

$$\begin{cases} y_1' = y_2 \\ y_2' = 9y_1 - y_1^3 \end{cases}$$

(2) Critical points: Let $y_1' = 0$ & $\underline{y_2' = 0}$

$y_2 = 0$ & $9y_1 - y_1^3 = y_1(9 - y_1^2) = 0$

$y_1 = 0$, $\underline{9 - y_1^2 = 0}$ : $y_1 = 3, -3$

$(0, 0), (3, 0), (-3, 0)$

(3) $J = \begin{bmatrix} 0 & 1 \\ 9 - 3y_1^2 & 0 \end{bmatrix}$

a linearized system

$(0,0)$: $J(0,0) = \begin{bmatrix} 0 & 1 \\ 9 & 0 \end{bmatrix}$ : $\underline{y' = \begin{bmatrix} 0 & 1 \\ 9 & 0 \end{bmatrix} y}$

$(-\lambda)^2 - 9 = 0 \quad \lambda^2 - 9 = 0$ : $\lambda = 3, -3.$

$(0, 0)$: a unstable saddle point.

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$(3,0)$: $J(3,0) = \begin{bmatrix} 0 & 1 \\ 9 - 3\cdot 9 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -18 & 0 \end{bmatrix}$

$(-\lambda)^2 + 18 = 0$ : $\lambda = \pm\sqrt{18}\,i$

$(3,0)$: a stable center.

$(-3,0)$: $J(-3,0) = \begin{bmatrix} 0 & 1 \\ -18 & 0 \end{bmatrix}$ : $\lambda = \pm\sqrt{18}\,i$

$(-3,0)$: a stable center.

4.6 Nonhomogeneous systems of DE.

$y' = Ay + g(t)$

(Ex) $y' = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} y + \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t}$ ?

(idea) $y'' - 4y = e^{3t}$

(1) Solve $y'' - 4y = 0$ : $r^2 - 4 = 0$

$y_c(t) = c_1 e^{2t} + c_2 e^{-2t}$ $r = 2, -2$

(2) Find a particular solution $y_p(t)$

$y(t) = y_c(t) + y_p(t)$

1. Solve $y' = \underbrace{\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}}_{A} y \to y_c(t)$

2. Find ~~y~~ $y_p(t)$.

$y(t) = y_c(t) + y_p(t)$ : a general solution

(1)

$\det(A - \lambda I) = (1-\lambda)(-1-\lambda) - 3 = 0$

$\lambda^2 - 4 = 0$ $\lambda = 2, -2$

$V_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $V_2 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

$y_c(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-2t}$

(2) $y_p(t) = ?$

(idea) $\frac{d}{dt}$ (polynomial) = a polynomial

$\frac{d}{dt} e^{at} = a e^{at}$, $\frac{d}{dt} \sin(kt) = k \cos(kt)$

$\frac{d}{dt} \cos(kt) = -k \sin(kt)$

Set $y_p(t) = Ve^{3t}$