#1)

$$\phi(y) = y^5$$

$$\phi(-y) = (-y)^5 = (-y)^4(-y) = -y^5 = -\phi(y) \quad \therefore \underline{y^5 \text{ is odd}}$$

<u>For odd functions</u>: $a_1 = 0$

$$b_1 = \frac{2}{\pi}\int_0^{\pi} \phi(a\sin\theta)\sin\theta \, d\theta$$

$\phi(a\sin\theta) = a^5\sin^5\theta$ - <u>compute fourier coefficients without integral</u>

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2j}$$

$$\sin^5\theta = (e^{i\theta} - e^{-i\theta})^5 \left(\frac{1}{2j}\right)^5 = \frac{(e^{i\theta} - e^{-i\theta})^5}{32j}$$

$$(e^{i\theta} - e^{-i\theta})(e^{i\theta} - e^{-i\theta}) = e^{i\theta}e^{i\theta} - 2e^{i\theta}e^{-i\theta} + e^{-i\theta}e^{-i\theta} = e^{2i\theta} - 2 + e^{-2i\theta}$$
$$= (e^{i\theta} - e^{-i\theta})^2$$

$$(e^{i\theta} - e^{-i\theta})^3 = (e^{i\theta} - e^{-i\theta})^2(e^{i\theta} - e^{-i\theta}) = (e^{2i\theta} + e^{-2i\theta} - 2)(e^{i\theta} - e^{-i\theta})$$

$$= e^{3i\theta} - e^{i\theta} + e^{-i\theta} - e^{-3i\theta} - 2e^{i\theta} + 2e^{-i\theta} = e^{3i\theta} - 3e^{i\theta} + 3e^{-i\theta} - e^{-3i\theta}$$

$$(e^{i\theta} - e^{-i\theta})^3(e^{i\theta} - e^{-i\theta})^2 = (e^{3i\theta} - 3e^{i\theta} + 3e^{-i\theta} - e^{-3i\theta})(e^{2i\theta} + e^{-2i\theta} - 2)$$
$$= e^{5i\theta} + e^{i\theta} - 2e^{3i\theta} - 3e^{3i\theta} - 3e^{-i\theta} + 6e^{i\theta} + 3e^{i\theta} + 3e^{-3i\theta} - 6e^{-i\theta} - e^{-i\theta} - e^{-5i\theta} + 2e^{-3i\theta}$$
$$= e^{5i\theta} - e^{-5i\theta} + 10e^{i\theta} - 5e^{3i\theta} - 10e^{-i\theta} + 5e^{-3i\theta} = (e^{i\theta} - e^{-i\theta})^5$$

$$\therefore \sin^5(\theta) = \frac{e^{5i\theta} - e^{-5i\theta} + 10e^{i\theta} - 10e^{-i\theta} + 5e^{-3i\theta} - 5e^{3i\theta}}{32j}$$

$$= \frac{10}{32}(e^{i\theta} - e^{i\theta}) + \frac{-5}{32}(e^{3i\theta} - e^{-3i\theta}) + \frac{1}{32}(e^{5i\theta} - e^{5i\theta})$$

$$= \frac{10}{16}\sin(\theta) - \frac{5}{16}\sin(3\theta) + \frac{1}{16}\sin(5\theta) = \sin^5(\theta)$$

$\therefore \phi(a\sin\theta) = a^5 \sin^5\theta = a^5(b_1 \sin\theta + b_3 \sin(3\theta) + b_5 \sin(5\theta))$

where $b_1 = 10/16$, $b_3 = -5/16$, & $b_5 = 1/16$

$\therefore b_1(a) = \dfrac{10a^5}{16}$

$$N(a) = \frac{b_1(a)}{a} = \frac{10a^5}{16a}$$

$\boxed{\text{Describing function } N(a) = \dfrac{10a^4}{16} \text{ for } \phi = y^5}$

#2)

$$\ddot{y} - y + y^3 = 0$$

$\phi(y) = y^3$, $u = -\phi(y)$

$\ddot{y} = y + u$

$x_1 = y$ $\quad x_2 = \dot{y}$ $\qquad y = x_1$

$\dot{x}_1 = x_2$
$\dot{x}_2 = x_1 + u$

$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $C = [1 \ \ 0]$, $D = 0$

$G(s) = C(sI - A)^{-1}B + D = (1 \ \ 0)\begin{pmatrix} s & -1 \\ -1 & s \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$= (1 \ \ 0)\frac{1}{s^2-1}\begin{pmatrix} s & 1 \\ 1 & s \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{s^2-1}(s \ \ 1)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{s^2-1}$

$G(s) = \frac{1}{s^2-1}$

$\phi(a\sin\theta) = a^3\sin^3\theta = a^3\left(\frac{e^{i\theta} - e^{-i\theta}}{2j}\right)^3 = a^3\left(\frac{e^{3i\theta} - 3e^{i\theta} + 3e^{-i\theta} - 3e^{-3i\theta}}{-8j}\right)$ From problem 1

$\phi(a\sin\theta) = a^3\left(\frac{3}{4}\sin\theta - \frac{\sin(3\theta)}{4}\right)$ $\quad \therefore b_1 = \frac{3a^3}{4}$, $b_3 = -a^3/4$

$N(a) = \frac{b_1(a)}{a} = \frac{3a^2}{4}$

<u>For periodic solution:</u> $1 + G(i\omega)N(a) = 0$

$G(i\omega) = \frac{1}{-\omega^2 - 1}$

$$1 + G(i\omega)\, N(a) = 1 + \frac{3a^2}{-4\omega^2 - 4} = 0$$

$$\Rightarrow \; 3a^2 = 4\omega^2 + 4$$

$$\therefore \; \omega = \sqrt{\frac{3a^2 - 4}{4}}$$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\frac{\sqrt{3a^2-4}}{2}} = \frac{4\pi}{\sqrt{3a^2-4}}$$

$$\boxed{\text{System has periodic solutions of all amplitudes } a \text{, with approximate period } T = \left(\frac{4\pi}{\sqrt{3a^2-4}}\right)}$$

#3) $\ddot{y} + \mu\left(\frac{\dot{y}^3}{3} - \dot{y}\right) + y = 0$ $\mu > 0$

$\phi(y) = \dot{y}^3$ $U = -\phi(y) = -\dot{y}^3$

$\ddot{y} = -y + \mu\dot{y} - \frac{\mu}{3}\dot{y}^3 = -y + \mu\dot{y} + \frac{\mu}{3}u$

$X_1 = y$ $X_2 = \dot{y}$ $y = X_2$ (output)

$\dot{X}_1 = X_2$
$\dot{X}_2 = -X_1 + \mu X_2 - \frac{\mu}{3}u$

$A = \begin{pmatrix} 0 & 1 \\ -1 & \mu \end{pmatrix}$ $B = \begin{pmatrix} 0 \\ \frac{\mu}{3} \end{pmatrix}$ $C = [0 \quad 1]$ $D = 0$

$G(s) = C(sI-A)^{-1}B + D = [0 \quad 1]\begin{pmatrix} s & -1 \\ 1 & s-\mu \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ \mu/3 \end{pmatrix}$

$= [0 \quad 1]\frac{1}{s^2-\mu s+1}\begin{pmatrix} s-\mu & 1 \\ -1 & s \end{pmatrix}\begin{pmatrix} 0 \\ \mu/3 \end{pmatrix}$

$= \frac{1}{s^2-\mu s+1}(-1 \quad s)\begin{pmatrix} 0 \\ \mu/3 \end{pmatrix} = \frac{\mu s}{3(s^2-\mu s+1)}$

$G(s) = \frac{\mu s}{3(s^2-\mu s+1)}$

From problem 2

$\phi(a\sin\theta) = \phi(a^3\sin^3\theta)$, $\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin(3\theta)$ $\therefore$

$\phi(a\sin\theta) = a^3\frac{3}{4}\sin\theta - \frac{1}{4}\sin(3\theta)$ $\therefore$ $b_1 = \frac{3a^3}{4}$ & $b_3 = -\frac{1}{4}a^3$

$\therefore$ $N(a) = \frac{b_1}{a} = \boxed{\frac{3a^2}{4}}$

For periodic solution: $1 + G(i\omega)\, N(a) = 0$

$$1 + G(i\omega)\, N(a) = 1 + \frac{\mu i\omega}{\cancel{3}(-\omega^2 - \mu i\omega + 1)} \left(\frac{\cancel{3}a^2}{4}\right) = 0 \quad \Rightarrow$$

$$(-\omega^2 - \mu i\omega + 1)4 + \mu a^2 i\omega = 0$$

$-\omega^2 + 1 = 0 \quad \Rightarrow \quad \omega = 1 \qquad T = 2\pi/\omega$

$-4\cancel{\mu}\omega + a^2 \cancel{\mu}\omega = 0 \quad \Rightarrow \quad a^2 = 4 \quad \Rightarrow \quad \underline{a = 2}$

System has a periodic solution with an amplitude, $a = 2$ & an approximate period, $T = 2\pi$.

#4)

$$\dot{x} = -x - 2\,\text{sgn}(x(t-h))$$

$U = -\text{sgn}(y) = -\phi(y)$ $\qquad y = x(t-h)$

$$\dot{x} = -x + 2u$$

$$\mathcal{L}[\dot{x}] = sX = \mathcal{L}[-x + 2u] = -X(s) + 2U(s)$$

$$\therefore (s+1)X(s) = 2U(s) \Rightarrow \frac{X(s)}{U(s)} = \frac{2}{s+1}$$

$$\mathcal{L}[y] = \mathcal{L}[x(t-h)]$$

$$Y(s) = e^{-sh} X(s) \qquad \therefore \frac{Y(s)}{X(s)} = e^{-sh}$$

$$\therefore \frac{Y(s)}{U(s)} = \boxed{G(s) = \frac{2e^{-sh}}{s+1}}$$

From Notes: $\text{sgn}(y) = \begin{cases} -1 & y < 0 \\ 0 & y = 0 \\ 1 & y > 0 \end{cases}$ (odd function)

$$N(a) = \frac{2}{\pi a}\int_0^{\pi} \phi(a\sin\theta)\sin\theta\, d\theta = \frac{2}{\pi a}\int_0^{\pi} \sin\theta\, d\theta = \boxed{\frac{4}{\pi a}}$$

For periodic solution: $1 + G(i\omega)N(a) = 0$

$$\therefore 1 + \left(\frac{2e^{-i\omega h}}{i\omega + 1}\right)\left(\frac{4}{\pi a}\right) = 0$$

$$e^{-i\omega h} = \cos(-\omega h) + i\sin(-\omega h) = \cos(\omega h) - i\sin(\omega h)$$

$$\therefore 1 + G(i\omega)N(a) = 0 = 1 + \frac{8(\cos(\omega h) - i\sin(\omega h))}{i\omega\pi a + \pi a} = 0$$

$$\frac{i\omega\pi a + \pi a + 8\cos(\omega h) - 8i\sin(\omega h)}{i\omega\pi a + \pi a} = 0 \implies$$

$$8\cos(\omega h) + \pi a = 0 \qquad \text{(Real part)}$$

$$\omega\pi a - 8\sin(\omega h) = 0 \qquad \text{(Imaginary part)}$$

$$\boxed{a = -\frac{8}{\pi}\cos(h\omega)} \quad \therefore$$

$$\omega\pi\left(\frac{-8}{\pi}\cos(h\omega)\right) - 8\sin(\omega h) = 0 = -\omega\cos(\omega h) - \sin(\omega h)$$

$$\therefore \; \underline{-\omega = \tan(\omega h)} \implies \boxed{\tan(\omega h) + \omega = 0}$$

A periodic solution exists where the freqency & anplitude are dependent on the time delay, h. The relationships for the amplitude and approximate period are

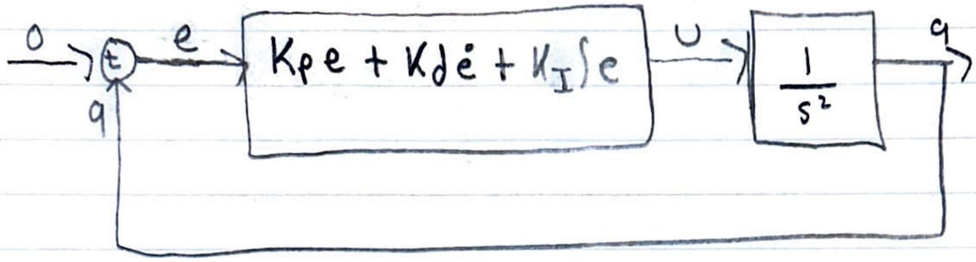
$a = -\frac{8}{\pi}\cos(h\omega)$ & $T = \frac{2\pi}{\omega}$, where $\tan(\omega h) + \omega = 0$ must be solved numerically for ω, based on h.

If $\underline{h = 0.25}$, then $\underline{a = 0.367,\ \omega = 6.862,\ \& \ T = 0.9156}$

#5)

a) $\ddot{q} = u = -K_p q - K_d \dot{q} - sat(K_I \int q)$

This system is in the form of: (Neglect saturation function)



where $e = -q$. $K_p = 1$ & $K_d = 2$. $\mathcal{L}[\ddot{q}] = s^2 Q(s)$,

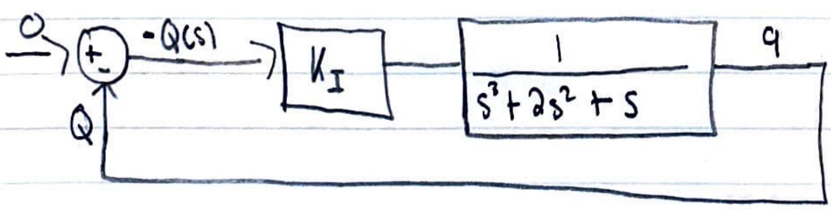
$\mathcal{L}[-\dot{q}] = -s\,Q(s)$, & $\mathcal{L}[\int q] = \frac{1}{s} Q(s)$ $\therefore$ $\ddot{q} + q + 2\dot{q} = -K_I \int q \Rightarrow$

$Q(s)(s^2 + 1 + 2s) = \frac{K_I}{s}(-Q(s))$ $\quad (-Q(s) = E(s))$

$\therefore \frac{Q(s)}{E(s)} = \frac{Q(s)}{-Q(s)} = K_I\left(\frac{1}{s(s^2+1+2s)}\right) = \boxed{K_I\left(\frac{1}{s^3+2s+s}\right)}$

↖ Nominal open loop system transfer function

Which takes the form of:



Can use root locus with system in this form:

$1 + (K_I)\left(\frac{1}{s^3+2s^2+s}\right)$

#5)

From MATLAB root locus plot, the largest $K_i \geq 0$

for which the system is asymptotically stable about $q = 0$

is $\boxed{K_i = 1.9\bar{9}}$. The closed loop system is asymptotically

stable for any $0 \leq K_i < 2$.

b) $U = -\phi(y) = -\text{sat}(K_I \int q)$ $\therefore$ $y = K_I \int q$

$\ddot{q} = -K_p q - K_d \dot{q} + U$ $\Rightarrow$

$\ddot{q} + K_p q + K_d \dot{q} = U$ $\quad (K_p = 1, K_d = 2)$

$\mathcal{L}[\ddot{q} + q + 2\dot{q}] = \mathcal{L}[U] = Q(s)(s^2 + 2s + 1) = U(s)$

$\mathcal{L}[Y] = \mathcal{L}[K_I \int q] = Y(s) = \frac{K_I}{s} Q(s)$

$$\therefore \frac{Y(s)}{U(s)} = \left(\frac{Y(s)}{Q(s)}\right)\left(\frac{Q(s)}{U(s)}\right) = \left(\frac{K_I}{s}\right)\left(\frac{1}{s^2+2s+1}\right) = \boxed{\frac{K_I}{s^3+2s^2+s} = G(s)}$$

From Slotine & Li, saturation describing function is:

$N(a) = \frac{2}{\pi}\left(\sin^{-1}\left(\frac{A}{a}\right) + \frac{A}{a}\sqrt{1 - \frac{A^2}{a^2}}\right)$ $\quad$ If $A = 1$, then

$N(a) = \frac{2}{\pi}\left(\sin^{-1}\left(\frac{1}{a}\right) + \frac{1}{a}\sqrt{\frac{a^2-1}{a^2}}\right)$

$$\text{sat}(y) = \begin{cases} -A & y < -A \\ y & -A \leq y \leq A \\ A & y > A \end{cases}$$

For periodic solutions: $1 + G(i\omega)N(a) = 0$

$$G(i\omega) = K_I \frac{1}{-i\omega^3 - 2\omega^2 + i\omega}$$

$$N(a)\,G(i\omega) = \left(\frac{K_I}{-i\omega^3 - 2\omega^2 + i\omega}\right)\left(\frac{2}{\pi}\sin^{-1}\left(\frac{A}{a}\right) + \frac{A}{a}\sqrt{1 - \frac{A^2}{a^2}}\right)$$

$$1 + N(a)\,G(i\omega) = -i\omega^3 - 2\omega^2 + i\omega + \frac{2}{\pi}\left[\sin^{-1}\left(\frac{A}{a}\right) + \frac{A}{a}\sqrt{1-\frac{A^2}{a^2}}\right]K_I = 0$$

$$-\omega^3 + \omega = 0 \quad \Rightarrow \quad \omega(-\omega^2 + 1) = 0 \quad \therefore \quad \omega = 1$$

$$-2\omega^2 + \frac{2K_I}{\pi}\left[\sin^{-1}\left(\frac{A}{a}\right) + \frac{A}{a}\sqrt{1-\frac{A^2}{a^2}}\right] = 0 \qquad \omega = 1$$

$$\Rightarrow \sin^{-1}\left(\frac{A}{a}\right) + \frac{A}{a}\sqrt{1-\frac{A^2}{a^2}} = \frac{\pi}{K_I}$$

$$\Rightarrow 1 - \frac{A^2}{a^2} = \frac{a^2}{A^2}\left(\frac{\pi}{K_I} - \sin^{-1}\left(\frac{A}{a}\right)\right)^2 = \frac{a^2 - A^2}{a^2}$$

Unsure how to get minimum K_i from this relation

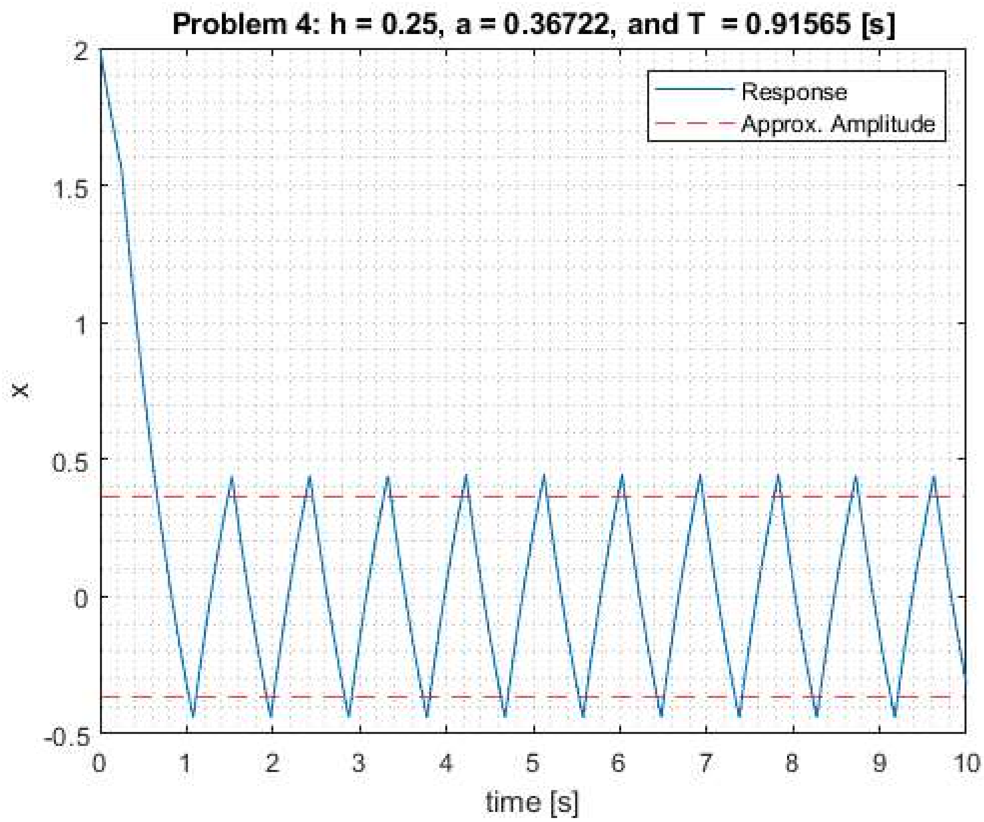
Contents


```
clear
close all
clc
```

Problem 4

```
% Time delay [s]
h           = 0.25;

% Get non-linear equation roots
w           = fzero(@(w) tan(w*h) + w, 10);

% Verify non-linear equation is near 0.
fprintf('For a time delay of h = %.2f, tan(wh) + w = %.4f if w = %.4f \n'...
        ,h, tan(w*h) + w, w)

% Period of periodic solution
T           = 2*pi/w;

% Amplitude of periodic solution
a           = -8*cos(w*h)/pi;

fprintf(['If h = %.2f, the system has an approximate period of %.3f seconds with an '...
         'amplitude of %.3f \n'],h,T,a)

% Run Model
simout      = sim("Model.slx");

% Plot
figure
plot(simout.tout, simout.logsout{1}.Values.Data)
hold on
yline([a -a],'--r')
grid minor
xlabel('time [s]')
legend('Response','Approx. Amplitude')
ylabel('x')
title_str = ['Problem 4: h = ', num2str(h), ', a = ', num2str(a),...
            ', and T  = ',num2str(T),' [s]'];
title(title_str)
```

```
For a time delay of h = 0.25, tan(wh) + w = 0.0000 if w = 6.8620
If h = 0.25, the system has an approximate period of 0.916 seconds with an amplitude of 0.367
```

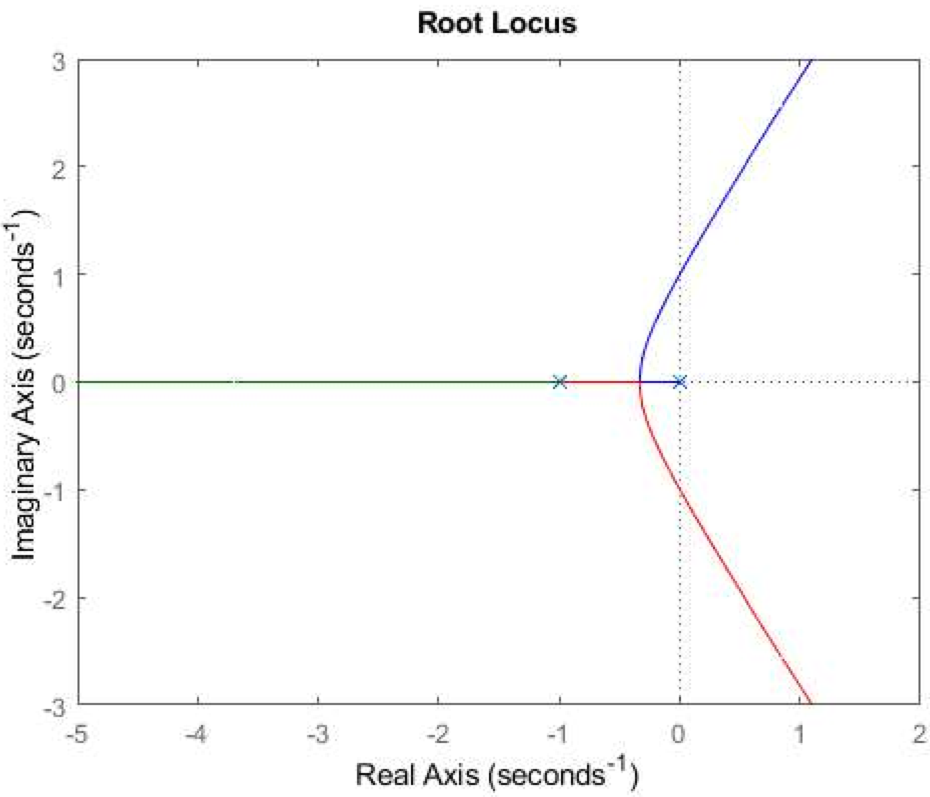


Problem 5

```
% PID Gains
Kp          = 1;
Kd          = 2;

% Sytem Transfer Function: Ki * G
G           = tf([1],[1 Kd Kp 0]);

% Root Locus plot
figure
rlocusplot(G)
[poles,Ki]   = rlocus(G,0:.001:5);

% Locate Largest Ki such that poles are in RHP
Ki_max       = max(Ki(max(real(poles)) < 0));

fprintf(['The largest Ki >=0, for which the closed loop system '...
         'is asymptotically stable about q = 0 is %.3f. \n'], Ki_max)

% Check poles of Closed Loop System
disp('The poles of the closed loop system with the max Ki are: ')
CL_poles    = pole(feedback(Ki_max*G,1))
```

```
The largest Ki >=0, for which the closed loop system is asymptotically stable about q = 0 is 1.999.
The poles of the closed loop system with the max Ki are:

CL_poles =

  -1.9998 + 0.0000i
  -0.0001 + 0.9998i
```

```
-0.0001 - 0.9998i
```



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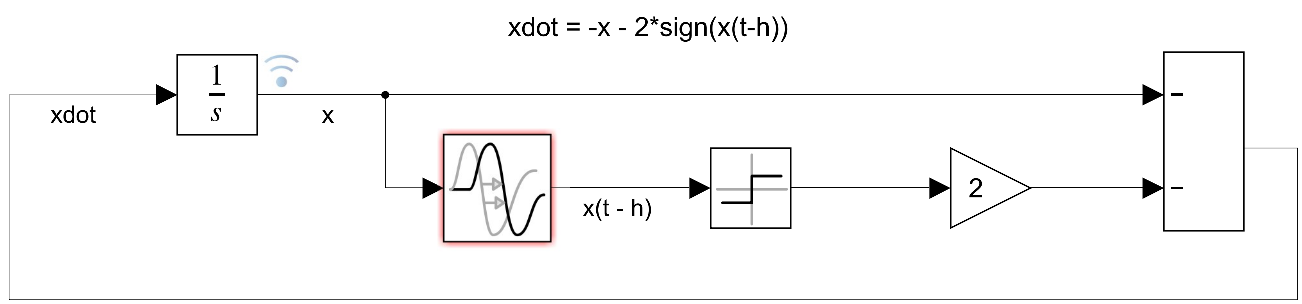

xdot = -x - 2*sign(x(t-h))
1
s
xdot
x
x(t - h)
2