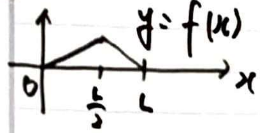
(Ex 1) P551 $\quad g(x) = 0, \quad 0 < x < L$

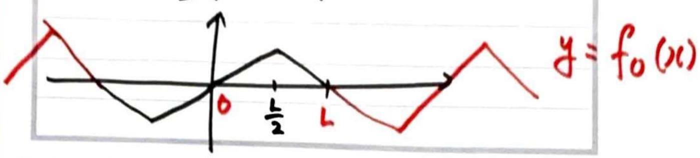


$$u(x,t) = \frac{f(x+ct) + f(x-ct)}{2}$$

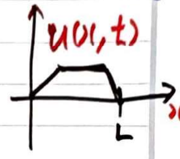
$$= \frac{1}{2}\left[f(x - \underline{(-ct)}) + f(x - \underline{ct})\right]$$

translation of $f_0(x)$

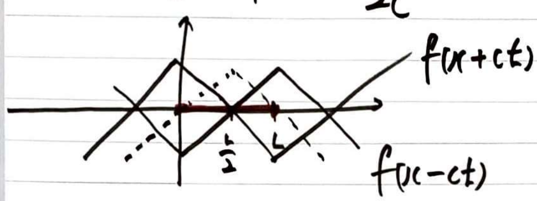
$$f(x) = \begin{cases} \frac{2}{L}x, & 0 < x < \frac{L}{2} \\ \frac{2}{L}(L-x), & \frac{L}{2} < x < L \end{cases}$$

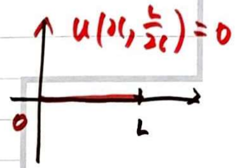


(1) $0 < ct < \frac{L}{2}$ $\qquad 0 < t < \frac{L}{2c}$

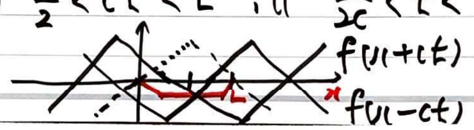


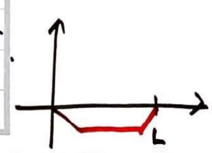
(2) $ct = \frac{L}{2}$ iff $t = \frac{L}{2c}$





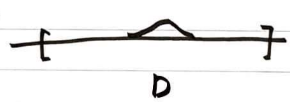
(3) $\frac{L}{2} < ct < L$ iff $\frac{L}{2c} < t < \frac{L}{c}$.





12.4. Initial value problem. (IVP)

(motivation).



Let $\underline{D = (-\infty, \infty)}$

(IVP)

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, & -\infty < x < \infty \\ u(x,0) = f(x), \quad u_t(x,0) = g(x), & x \in \mathbb{R} \end{cases}$$

(Method of characteristics)

Let $v = x + ct$ & $w = x - ct$.

$$u_t = \frac{\partial u}{\partial t} = \frac{\partial u}{\partial v}\underbrace{\frac{\partial v}{\partial t}}_{=c} + \frac{\partial u}{\partial w}\underbrace{\frac{\partial w}{\partial t}}_{=-c} = cu_v - cu_w$$

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial v}\underbrace{\frac{\partial v}{\partial x}}_{=1} + \frac{\partial u}{\partial w}\underbrace{\frac{\partial w}{\partial x}}_{=1} = u_v + u_w$$

$$u_{tt} = \frac{\partial}{\partial t}(cu_v - cu_w)$$

$$= \frac{\partial}{\partial v}(cu_v - cu_w)\underbrace{\frac{\partial v}{\partial t}}_{=c} + \frac{\partial}{\partial w}(cu_v - cu_w)\underbrace{\frac{\partial w}{\partial t}}_{=-c}$$

$$u_{tt} = c^2\frac{\partial^2 u}{\partial v^2} - 2c^2\frac{\partial^2 u}{\partial v\partial w} + c^2\frac{\partial^2 u}{\partial w^2}$$

$$u_{xx} = \frac{\partial^2 u}{\partial v^2} + 2\frac{\partial^2 u}{\partial v\partial w} + \frac{\partial^2 u}{\partial w^2}$$

$$u_{tt} - c^2u_{xx} = c^2\cancel{u_{vv}} - 2c^2u_{vw} + c^2\cancel{u_{ww}}$$

$$- c^2(\cancel{u_{vv}} + 2u_{vw} + \cancel{u_{ww}}) = -4c^2u_{vw} = 0$$

$\therefore \frac{\partial^2 u}{\partial v\partial w} = 0$: the normal equation

$\frac{\partial u}{\partial w} = h(w)$ (Because $\int \frac{\partial^2 u}{\partial v\partial w}\,dv = \int 0\,dv$)

$$u(v,w) = \underbrace{\int h(w)\,dw}_{=\text{ let } \psi(w)} + \phi(v)$$

$$u(v,w) = \phi(v) + \psi(w)$$

$$u(x,t) = u(v,w) = \phi(x+ct) + \psi(x-ct)$$

IC: $u(x,0) = \phi(x) + \psi(x) = f(x)$ ✓ IC. — ①

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\phi(x+ct) + \frac{\partial}{\partial t}\psi(x-ct)$$

$$= \phi'(x+ct)\underbrace{\frac{\partial}{\partial t}(x+ct)}_{=c} + \psi'(x-ct)(-c)$$

$u_t(x,0) = c\phi'(x) - c\psi'(x) = g(x)$ ✓ IC — ②

①$'$: $\phi'(x) + \psi'(x) = f'(x)$

$\frac{1}{c}$② : $+ \big)\ \phi'(x) - \psi'(x) = \frac{1}{c}g(x)$

$$2\phi'(x) = f'(x) + \frac{1}{c}g(x)$$

$$\phi'(x) = \frac{1}{2} f'(x) + \frac{1}{2c} g(x)$$

$$\int_{x_0}^{x} \phi'(z)\,dz = \frac{1}{2}\int_{x_0}^{x} f'(z)\,dz + \frac{1}{2c}\int_{x_0}^{x} g(z)\,dz$$

$$\phi(x) - \phi(x_0) = \frac{1}{2}\left(f(x) - f(x_0)\right) + \frac{1}{2c}\int_{x_0}^{x} g(z)\,dz$$

$$\phi(x) = \phi(x_0) + \frac{1}{2} f(x) - \frac{1}{2} f(x_0) + \frac{1}{2c}\int_{x_0}^{x} g(z)\,dz$$

$$\psi(x) = f(x) - \phi(x)$$

$$u(x,t) = \phi(x_0) + \frac{1}{2} f(x+ct) - \frac{1}{2} f(x_0) + \frac{1}{2c}\int_{x_0}^{x+ct} g(z)\,dz + f(x-ct) - \phi(x-ct)$$

$$u(x,t) = \cancel{\phi(x_0)} + \frac{1}{2} f(x+ct) - \cancel{\frac{1}{2} f(x_0)}$$

$$+ \frac{1}{2c}\int_{x_0}^{x+ct} g(z)\,dz + f(x-ct) = \frac{1}{2} f(x-ct)$$

$$+ \left(\cancel{\phi(x_0)} + \frac{-1}{2} f(x-ct) + \cancel{\frac{1}{2} f(x_0)} + \frac{-1}{2c}\int_{x_0}^{x-ct} g(z)\,dz\right)$$

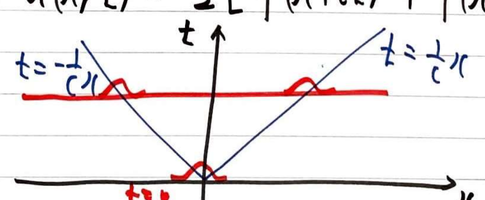
$$= \frac{1}{2}\left(f(x+ct) + f(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(z)\,dz$$

(d'Alembert formula)

$$\therefore\ u(x,t) = \frac{1}{2}\left(f(x+ct) + f(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(z)\,dz$$

Remark Let $g(x) \equiv 0$

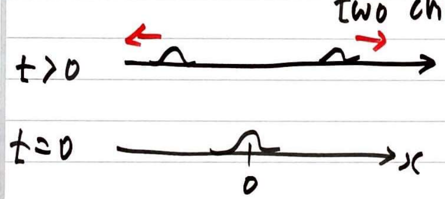
$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right]$$



$$\begin{cases} x + ct = 0 \\ x - ct = 0 \end{cases}$$

$$\begin{cases} t = -\frac{1}{c}x \\ t = \frac{1}{c}x \end{cases}$$

two characteristic lines.



(Types of 2nd order PDEs)

$A u_{xx} + 2B u_{xy} + C u_{yy} + F(x, y, u, u_x, u_y) = 0$ — ①

Def $\Delta = B^2 - AC$: the discriminant of ①

(1) $\Delta > 0$: ① is called a hyperbolic PDE

ex $u_{tt} - c^2 u_{xx} = 0$ $(y = t)$

$\Delta = 0 - (-c^2)\cdot 1 = c^2 > 0$

① has two characteristic lines.

(2) $\Delta = 0$: ① is called a parabolic PDE

ex heat equation $u_t - k u_{xx} = 0$

$\Delta = 0 - (-k)\cdot 0 = 0$

① has one characteristic line

(3) $\Delta < 0$: ① is called an elliptic PDE

ex $u_{xx} + u_{yy} = 0$

$\Delta = 0 - 1\cdot 1 = -1 < 0$

① has No characteristic lines

Remark : $\frac{\partial u}{\partial t} + \mathbf{v}\cdot\nabla u - \nabla^2 u = f(x, y)$

12.5.

12.6. Heat equation.

$q = -k\nabla u$: Fourier law

$\frac{\partial u}{\partial t} + \nabla\cdot q = 0$: energy conservation

$\frac{\partial u}{\partial t} + \nabla\cdot(-k\nabla u) = 0$ k: constant

$\frac{\partial u}{\partial t} - k\nabla^2 u = 0$:

$\frac{\partial u}{\partial t} - k u_{xx} = 0$: heat equation.