2.4.3

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{E_{31}(-1)} \begin{array}{c} \begin{matrix} p & p & & \end{matrix} \\ \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

$C(A)$: Basis = Span $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \right\}$

Dim = rank = 2

$x_1 + 2x_2 + x_4 = 0$

Pivot

$x_2 + x_3 = 0$

Pivot

$x_2 = -x_3$

$x_1 = 2x_3 - x_4$

Preffered solutions: $x_3 = 1, x_4 = 0$ & $x_3 = 0, x_4 = 1$

$N(A)$: Basis = span $\left\{ \begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

Dim = $n - r = 4 - 2 = 2$

$$A^T = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad E_{41}(-1) = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \xrightarrow{E_{21}(-2)} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{E_{32}(-1)} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

2.4.3

$C(A^T)$: Basis = span $\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}$

Dim = rank = 2

$$\overset{\swarrow \text{Pivot}}{x_1} + x_3 = 0$$
$$\underset{\uparrow \text{Pivot}}{x_2} = 0$$

Preferred solution: $x_3 = 1$ ∴

$N(A^T)$: Basis = span $\left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$

Dim = $m-r = 3-2 = 1$

$$U = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$ (Already in row echelon form)

$C(U)$: Basis = span $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \right\}$

Dim = rank = 2

$C(A^T) = C(U^T)$ &

$N(A) = N(U)$

Have same dimension & basis as U is the row echelon form of A.

2.4.3

$$U^T = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \xrightarrow{E_{41}(-1)} \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{E_{21}(-2)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{E_{32}(-1)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$x_1 = 0$

$x_2 = 0$

x_3 is free, set to 1 for preferred sol.

$$\boxed{\begin{array}{l} N(U^T): \text{Basis} = \text{Span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \\ \qquad \text{Dim} = m - r = 3 - 2 = 1 \end{array}}$$

2.4.7

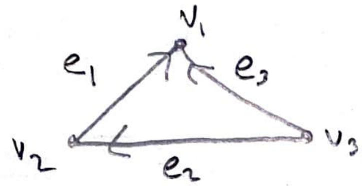
Because the row space must be orthogonal to the nullspace.

2.4.8

$A_{m\times n} X_{n\times 1} = 0_{n\times 1}$

The rank is $\boxed{n}$ as the solution $\vec{X} = \vec{0}$ can only occur if rank(A) equals n, as the columns of A are linearly independent.

2.5.1



$-x_2 + x_1$: e_1
$-x_3 + x_1$: e_3
$-x_3 + x_2$: e_2

$$\boxed{A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{pmatrix}}$$

$Ax = 0$

$$\xrightarrow{E_{31}(-1)} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right) \xrightarrow{E_{32}(-1)} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Free

$x_1 - x_2 = 0$

$x_2 - x_3 = 0$

pivot

Preferred solution let $x_3 = 1$ $\therefore$ $x_2 = 1$ & $x_1 = 1$

$$\boxed{\vec{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad N(A) = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}}$$

$A^T y = 0$

$$A^T = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & -1 \end{pmatrix} \xrightarrow{E_{32}(1)} \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ -1 & 0 & -1 \end{pmatrix}$$

$$\xrightarrow{E_{31}(1)} \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{E_{21}(1)} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

2.5.1

Pivot → $y_1 + y_3 = 0$

$y_2 + y_3 = 0$

Pivot　　Free

Preferred solution, $y_3 = 1$ $\therefore$ $y_2 = y_1 = -1$

$$\boxed{y = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \qquad N(A^T) = \text{span}\left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right\}}$$

2.5.2

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{pmatrix}$$

$Ax = b$

$$\begin{array}{ccc} P & P & F \end{array}$$

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$C(A) = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

← From 2.5.1
x_1 & x_2 are Pivots,
x_3 is free

$$= x_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$x_1 - x_2 = b_1$$
$$x_2 = b_2$$
$$x_1 = b_3$$

$$\therefore \quad b_3 - b_2 = b_1 \quad \Rightarrow \boxed{b_1 + b_2 - b_3 = 0}$$

$$x_1 - x_2 = b_1$$
$$x_2 - x_3 = b_2$$
$$x_1 - x_3 = b_3 \quad \Rightarrow \quad x_1 = b_3 + x_3$$

$$b_1 = b_3 + x_3 - x_2 \Rightarrow x_2 = b_3 - b_1 + x_3$$

$$b_3 - b_1 + \cancel{x_3} - \cancel{x_3} = b_2 \quad \Rightarrow \quad b_3 - b_1 = b_2$$

2.5.2

$\therefore$ $\boxed{b_1 + b_2 - b_3 = 0}$

The sum of potential differences around is zero

2.53

$$A^T = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & -1 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$C(A^T) = \text{span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}$$

$$y_1 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + y_2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$$

$y_1 = f_1$

$y_2 - y_1 = f_2$

$-y_2 = f_3$

$$\therefore\ -f_3 - f_1 = f_2 \Rightarrow \boxed{f_1 + f_2 + f_3 = 0}$$

$A^T y = f$

$$\begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$$

$y_1 + y_3 = f_1$

$y_2 - y_1 = f_2$

$-y_3 - y_2 = f_3 \Rightarrow y_3 = -f_3 - y_2$

$y_1 - f_3 - y_2 = f_1 \Rightarrow y_1 = f_1 + f_3 + y_2$

2.5.3

$\cancel{y/2} - f_1 - f_3 - \cancel{y/2} = f_2$

$-f_1 - f_3 = f_2 \Rightarrow \boxed{f_1 + f_2 + f_3 = 0}$

The sum of the current into a node is zero.

2.61

Rotation of 90°: $\begin{pmatrix} \cos(90) & -\sin(90) \\ \sin(90) & \cos(90) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Projection onto x axis: $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \boxed{\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}}$$

Rotate by 90° then project to x

Projection onto y: $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \boxed{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}$$

project to x then y

2.6.7

Basis: $1, t, t^2, t^3$

$\frac{d^2}{dt^2}(1) = 0$

$\frac{d^2}{dt^2}(t) = 0$

$\frac{d^2}{dt^2}(t^2) = 2$

$\frac{d^2}{dt^2}(t^3) = 6t$

$$\boxed{A = \begin{array}{cc} \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{array}{c} 1 \\ t \\ t^2 \\ t^3 \end{array} \end{array}}$$

$$\boxed{C(A) = \text{span}\left\{ \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 6 \\ 0 \\ 0 \end{pmatrix} \right\}}$$

X_1 & X_2 are free, X_3 & X_4 are pivots

$2X_3 = 0$

$6X_4 = 0$

Preferred solutions: $X_1 = 1$, $X_2 = 0$ & $X_1 = 0$, $X_2 = 1$ $\therefore$

$$\boxed{N(A) = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}}$$

2.6.7

$\boxed{\text{The nullspace shows that taking } \frac{d}{dt^2} \text{ on a 1st or zero order polynomial gives zero.}}$

2.6.8

Basis: $(1, t, t^2, t^3, t^4)$

$(2+3t)(1) = 2+3t$

$(2+3t)(t) = 2t+3t^2$

$(2+3t)(t^2) = 2t^2+3t^3$

$(2+3t)(t^3) = 2t^3+3t^4$

$$\boxed{A = \begin{array}{c} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 3 \end{pmatrix} \end{array}} \begin{array}{c} 1 \\ t \\ t^2 \\ t^3 \\ t^4 \end{array}$$

2.6.18

$P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

$$\int_0^1 a_0 + a_1 x + a_2 x^2 + a_3 x^3 \, dx = 0$$

$$= a_0 x + \frac{a_1 x^2}{2} + \frac{a_2 x^3}{3} + \frac{a_3 x^4}{4} \Big|_0^1$$

$$= a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \frac{a_3}{4} = 0$$

IF $a_0 = a_1 = a_2 = a_3 = 0 \therefore \int_0^1 P(x)\,dx = 0$

$\therefore$ Zero vecter is in subspace.

Let $a_0 = 1$, $a_1 = -2$, $a_2 = 3$, $a_3 = -4$

$$\therefore \int_0^1 P(x)\,dx = 0 = 1 + (-1) + 1 + (-1) = 0 \checkmark$$

$$c \int_0^1 P(x)\,dx = c[0] = 0 \checkmark$$

$$\int_0^1 c\,P(x)\,dx = (c)(1)x + \frac{(c)(-2)x^2}{2} + \frac{3cx^3}{3} + \frac{(-4)cx^4}{4} \Big|_0^1 = 0$$

$$c - c + c - c = 0 \checkmark$$

Closed under multiplication.

$$\int_0^1 P_1(x)\,dx + \int_0^1 P_2(x)\,dx = \int_0^1 P_1(x) + P_2(x)\,dx$$

Let $b_0 = 1$, $b_1 = -4$, $b_2 = 6$, $b_3 = -4$

2.6.18

$a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \frac{a_3}{4} = 0 = \int_0^1 P_1(x)\,dx$

$b_0 + \frac{b_1}{2} + \frac{b_2}{3} + \frac{b_3}{4} = 0 = 1 - 2 + 2 - 1 = 0 = \int_0^1 P_2(x)\,dx$

$0 + 0 = 0$ ✓ $= \int_0^1 P_1(x)\,dx + \int_0^1 P_2(x)\,dx$

$\int a_0 + b_0 + (a_1+b_1)x + (a_2+b_2)x^2 + (a_3+b_3)x^3\,dx$

$= (a_0+b_0)x + \frac{(a_1+b_1)x^2}{2} + \frac{(a_2+b_2)x^3}{3} + \frac{(a_4+b_4)x^4}{4}\Big|_0^1$

$= a_0 + b_0 + \frac{a_1+b_1}{2} + \frac{a_2+b_2}{3} + \frac{a_3+b_3}{4}$

$= (1+1) + \frac{(-2+-4)}{2} + \frac{(3+6)}{3} + \frac{(-4+-4)}{4}$

$= 2 - 3 + 3 + -2 = 0$ ✓ $= \int_0^1 P_1 + P_2\,dx$

Therefore, closed under addition.

S is closed under addition & multipliaction, while including the zero vector. Therefore S is a subspace.

Basis: $\begin{pmatrix} 1 \\ -2 \\ 3 \\ -4 \end{pmatrix}$ $\begin{matrix} \\ t \\ t^2 \\ t^3 \end{matrix}$