ECE 68000: MODERN AUTOMATIC CONTROL

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Properties and Applications of the Kronecker Product

Some properties of the Kronecker product

- Recall the Sylvester matrix equation

$$\boldsymbol{A}\boldsymbol{X} + \boldsymbol{X}\boldsymbol{B} = \boldsymbol{C}$$

represented using the Kronecker product as

$$\left(\boldsymbol{I}_m \otimes \boldsymbol{A} + \boldsymbol{B}^\top \otimes \boldsymbol{I}_n\right) \operatorname{vec}(\boldsymbol{X}) = \operatorname{vec}(\boldsymbol{C})$$

- Example: let

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} -2 & 0 \\ -3 & 1 \end{bmatrix}, \quad \text{and} \quad \boldsymbol{C} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Example of the Sylvester equation

- We have

$$
\begin{aligned}
& \left(\boldsymbol{I}_m \otimes \boldsymbol{A} + \boldsymbol{B}^\top \otimes \boldsymbol{I}_n\right) \operatorname{vec}(\boldsymbol{X}) \\
= & \left(\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} -2 & 0 & -3 & 0 \\ 0 & -2 & 0 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x_{11} \\ x_{21} \\ x_{12} \\ x_{22} \end{bmatrix} \\
= & \operatorname{vec}(\boldsymbol{C}) = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}
\end{aligned}
$$

Useful identities

- We have

$$
\begin{aligned}
(\boldsymbol{A} \otimes \boldsymbol{B})(\boldsymbol{C} \otimes \boldsymbol{D}) &= \boldsymbol{AC} \otimes \boldsymbol{BD} \\
(\boldsymbol{A} \otimes \boldsymbol{B})^\top &= \boldsymbol{A}^\top \otimes \boldsymbol{B}^\top
\end{aligned}
$$

- We prove the first identity; indeed,

$$
\begin{aligned}
& (\boldsymbol{A} \otimes \boldsymbol{B})(\boldsymbol{C} \otimes \boldsymbol{D}) \\
= & \begin{bmatrix} a_{11}\boldsymbol{B} & \cdots & a_{1m}\boldsymbol{B} \\ \vdots & \ddots & \vdots \\ a_{m1}\boldsymbol{B} & \cdots & a_{mn}\boldsymbol{B} \end{bmatrix} \begin{bmatrix} c_{11}\boldsymbol{D} & \cdots & c_{1r}\boldsymbol{D} \\ \vdots & \ddots & \vdots \\ c_{n1}\boldsymbol{D} & \cdots & c_{nr}\boldsymbol{D} \end{bmatrix} \\
= & \begin{bmatrix} \sum_{k=1}^{n} a_{1k}c_{k1}\boldsymbol{BD} & \cdots & \sum_{k=1}^{n} a_{1k}c_{kr}\boldsymbol{BD} \\ \vdots & \ddots & \vdots \\ \sum_{k=1}^{n} a_{mk}c_{k1}\boldsymbol{BD} & \cdots & a_{mn}\sum_{k=1}^{n} a_{mk}c_{kr}\boldsymbol{BD} \end{bmatrix} \\
= & \boldsymbol{AC} \otimes \boldsymbol{BD}
\end{aligned}
$$

Analysis of the Sylvester's matrix equation

- Let λ_i, $\boldsymbol{v}_i$ be the eigenvalues and eigenvectors, respectively, of $\boldsymbol{A}$, and μ_j and $\boldsymbol{w}_j$ the eigenvalues and eigenvectors of $m \times m$ matrix $\boldsymbol{B}$
- Then,

$$\begin{aligned} (\boldsymbol{A} \otimes \boldsymbol{B}) \left(\boldsymbol{v}_i \otimes \boldsymbol{w}_j\right) &= \boldsymbol{A}\boldsymbol{v}_i \otimes \boldsymbol{B}\boldsymbol{w}_j \\ &= \lambda_i \boldsymbol{v}_i \otimes \mu_j \boldsymbol{w}_j \\ &= \lambda_i \mu_j \left(\boldsymbol{v}_i \otimes \boldsymbol{w}_j\right) \end{aligned}$$

- Thus, the eigenvalues of $\boldsymbol{A} \otimes \boldsymbol{B}$ are $\lambda_i \mu_j$, and their respective eigenvectors are $\boldsymbol{v}_i \otimes \boldsymbol{w}_j$ for $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$
- Represent $\boldsymbol{A}\boldsymbol{X} + \boldsymbol{X}\boldsymbol{B} = \boldsymbol{C}$ as

$$\boldsymbol{M} \operatorname{vec}(\boldsymbol{X}) = \operatorname{vec}(\boldsymbol{C}),$$

where

$$\boldsymbol{M} = \boldsymbol{I}_m \otimes \boldsymbol{A} + \boldsymbol{B}^\top \otimes \boldsymbol{I}_n$$

Sylvester's matrix equation analysis contd.

- The solution of the above equation is unique if, and only if, the $mn \times mn$ matrix $\boldsymbol{M}$ is nonsingular
- To find the condition for this to hold, consider the following matrix

$$\left(\boldsymbol{I}_m + \varepsilon \boldsymbol{B}^\top\right) \otimes \left(\boldsymbol{I}_n + \varepsilon \boldsymbol{A}\right) = \boldsymbol{I}_m \otimes \boldsymbol{I}_n + \varepsilon \boldsymbol{M} + \varepsilon^2 \boldsymbol{B}^\top \otimes \boldsymbol{A}$$

whose eigenvalues are

$$\left(1 + \varepsilon \mu_j\right)\left(1 + \varepsilon \lambda_i\right) = 1 + \varepsilon \left(\mu_j + \lambda_i\right) + \varepsilon^2 \mu_j \lambda_i$$

because for a square matrix $\boldsymbol{Q}$,

$$\lambda_i \left(\boldsymbol{I}_n + \varepsilon \boldsymbol{Q}\right) = 1 + \varepsilon \lambda_i \left(\boldsymbol{Q}\right).$$

- Comparing terms in ε we conclude that the eigenvalues of $\boldsymbol{M}$ are $\lambda_i + \mu_j$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$

Conditions for the uniqueness of the solution of the Sylvester's equation

- Hence
$$\boldsymbol{M} = \boldsymbol{I}_m \otimes \boldsymbol{A} + \boldsymbol{B}^\top \otimes \boldsymbol{I}_n$$
is nonsingular if and only if
$$\lambda_i + \mu_j \neq 0$$
- The above is the necessary and sufficient condition for the solution $\boldsymbol{X}$ of the matrix equation $\boldsymbol{A}\boldsymbol{X} + \boldsymbol{X}\boldsymbol{B} = \boldsymbol{C}$ to be unique

Uniqueness of the solution of the continuous Lyapunov's matrix equation

- Represent the Lyapunov equation, $\boldsymbol{A}^\top \boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} = -\boldsymbol{Q}$, as

$$\left(\boldsymbol{I}_n \otimes \boldsymbol{A}^\top + \boldsymbol{A}^\top \otimes \boldsymbol{I}_n\right) \operatorname{vec}(\boldsymbol{P}) = -\operatorname{vec}(\boldsymbol{Q}).$$

- The condition for a solution $\operatorname{vec}(\boldsymbol{P})$ of the above equation to be unique is

$$\lambda_i(\boldsymbol{A}^\top) + \lambda_j(\boldsymbol{A}) \neq 0, \quad i, j = 1, 2, \ldots, n.$$

- Since $\lambda_i(\boldsymbol{A}^\top) = \lambda_i(\boldsymbol{A})$, and $\boldsymbol{A}$ is asymptotically stable by assumption, the condition $\lambda_i(\boldsymbol{A}^\top) + \lambda_j(\boldsymbol{A}) \neq 0$ is met, and hence the Lyapunov equation has a unique solution

Example

- Determine the stability of the system

$$\dot{\boldsymbol{x}} = \begin{bmatrix} -2 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & -2 & -2 \end{bmatrix} \boldsymbol{x}$$

  in the sense of Lyapunov by solving the Lyapunov matrix equation using the Kronecker product
- Take $\boldsymbol{Q} = \boldsymbol{I}$.
- Represent the continuous Lyapunov matrix equation,

$$\boldsymbol{A}^\top \boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} = -\boldsymbol{Q},$$

  using the Kronecker product as

$$\left(\boldsymbol{I}_n \otimes \boldsymbol{A}^\top + \boldsymbol{A}^\top \otimes \boldsymbol{I}_n\right) \mathrm{vec}(\boldsymbol{P}) = -\mathrm{vec}(\boldsymbol{Q})$$

Example contd

- Solve the above equation using the following MATLAB commands,

```
A=[-2 0 0;1 0 1;0 -2 -2];
nQ=-eye(3);
P_vec=(kron(eye(3),A')+kron(A',eye(3)))\nQ(:);
P=[P_vec(1:3) P_vec(4:6) P_vec(7:9)]
eig(P)
```

- We obtain,

$$\boldsymbol{P} = \begin{bmatrix} 0.4750 & 0.4500 & 0.1750 \\ 0.4500 & 1.2500 & 0.2500 \\ 0.1750 & 0.2500 & 0.3750 \end{bmatrix}.$$

- The eigenvalues of $\boldsymbol{P}$ are $\{0.2279, 0.3379, 1.5342\}$
- They are all positive; therefore $\boldsymbol{P}$ is positive definite and hence, by the Lyapunov theorem, the system is asymptotically stable

Checking if $\boldsymbol{P} = \boldsymbol{P}^\top \succ 0$

- The positive definiteness of $\boldsymbol{P}$ can also be checked using leading principal minors
- The leading principal minors are:

$$\Delta_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0.4750$$

$$\Delta_2 \begin{pmatrix} 1, & 2 \\ 1, & 2 \end{pmatrix} = \det \begin{bmatrix} 0.4750 & 0.4500 \\ 0.4500 & 1.2500 \end{bmatrix} = 0.3912,$$

$$\text{and} \quad \Delta_3 \begin{pmatrix} 1, & 2, & 3 \\ 1, & 2, & 3 \end{pmatrix} = \det \boldsymbol{P} = 0.1181.$$

  The leading principal minors are all positive; therefore $\boldsymbol{P}$ is positive definite

- The same $\boldsymbol{P}$ when using the MATLAB function, `lyap(A',eye(3))`