ECE 68000: MODERN AUTOMATIC CONTROL

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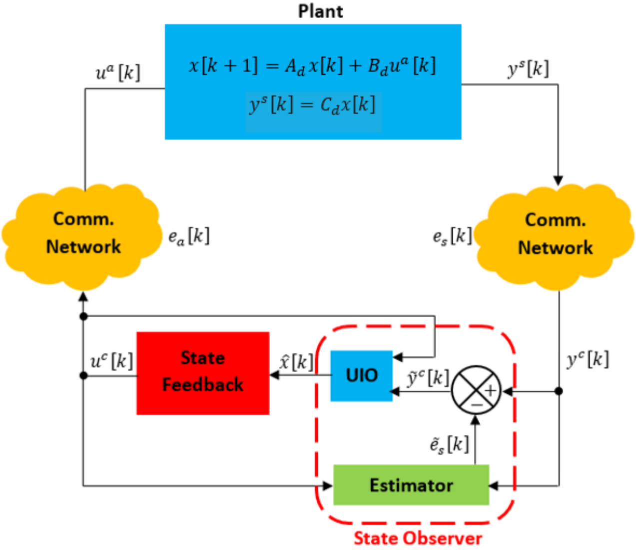
Recovering Corrupting Errors Using a Combined Observer Controller Compensator

State Observer = UIO + Output Sensor Error Estimator



Combined output sensor error estimator and UIO

- Construct an estimator of output sensor error $\boldsymbol{e}_s[k]$ to obtain its estimate denoted $\tilde{\boldsymbol{e}}_s[k]$
- Subtract $\tilde{\boldsymbol{e}}_s[k]$ from $\boldsymbol{y}^c[k]$ to obtain

$$\begin{aligned} \tilde{\boldsymbol{y}}^c[k] &= \boldsymbol{y}^c[k] - \tilde{\boldsymbol{e}}_s[k] \\ &= \boldsymbol{y}^s[k] + \boldsymbol{e}_s[k] - \tilde{\boldsymbol{e}}_s[k] \end{aligned}$$

- To proceed, assume $\tilde{\boldsymbol{y}}^c[k] = \boldsymbol{y}^s[k]$
- We obtain

$$\left.\begin{aligned} \boldsymbol{x}[k+1] &= \boldsymbol{A}\boldsymbol{x}[k] + \boldsymbol{B}(\boldsymbol{u}^c[k] + \boldsymbol{e}_a[k]) \\ \tilde{\boldsymbol{y}}^c[k] &= \boldsymbol{C}\boldsymbol{x}[k] \end{aligned}\right\}$$

Constructing UIO

- First decompose the state $\boldsymbol{x}[k]$ as

$$\begin{aligned}\boldsymbol{x}[k] &= \boldsymbol{x}[k] - \boldsymbol{M}\tilde{\boldsymbol{y}}^c[k] + \boldsymbol{M}\tilde{\boldsymbol{y}}^c[k] \\ &= \boldsymbol{x}[k] - \boldsymbol{M}\boldsymbol{C}\boldsymbol{x}[k] + \boldsymbol{M}\boldsymbol{C}\boldsymbol{x}[k] \\ &= (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})\boldsymbol{x}[k] + \boldsymbol{M}\tilde{\boldsymbol{y}}^c[k]\end{aligned}$$

  where $\boldsymbol{M} \in \mathbb{R}^{n\times p}$ is a parameter matrix to be constructed
- Let $\boldsymbol{z}[k] = (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})\boldsymbol{x}[k]$
- Then

$$\begin{aligned}\boldsymbol{z}[k+1] &= (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})\boldsymbol{x}[k+1] \\ &= (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})(\boldsymbol{A}\boldsymbol{x}[k] + \boldsymbol{B}\boldsymbol{u}^c[k] + \boldsymbol{B}\boldsymbol{e}_a[k]) \\ &= (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})(\boldsymbol{A}\boldsymbol{x}[k] + \boldsymbol{B}\boldsymbol{u}^c[k]) + (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})\boldsymbol{B}\boldsymbol{e}_a[k]\end{aligned}$$

Open-Loop UIO

- We have

$$\boldsymbol{z}[k+1] = (\boldsymbol{I} - \boldsymbol{MC})(\boldsymbol{Ax}[k] + \boldsymbol{Bu}^c[k]) + (\boldsymbol{I} - \boldsymbol{MC})\boldsymbol{Be}_a[k]$$

- Select $\boldsymbol{M}$ such that $(\boldsymbol{I} - \boldsymbol{MC})\boldsymbol{B} = \boldsymbol{O}$
- Then, $\boldsymbol{z}[k+1] = (\boldsymbol{I} - \boldsymbol{MC})(\boldsymbol{Ax}[k] + \boldsymbol{Bu}^c[k])$
- Substituting $\boldsymbol{x}[k] = \boldsymbol{z}[k] + \boldsymbol{M}\tilde{\boldsymbol{y}}^c[k]$ gives

$$\boldsymbol{z}[k+1] = (\boldsymbol{I} - \boldsymbol{MC})(\boldsymbol{Az}[k] + \boldsymbol{AM}\tilde{\boldsymbol{y}}^c[k] + \boldsymbol{Bu}^c[k])$$

- State estimate: $\hat{\boldsymbol{x}}[k] = \boldsymbol{z}[k] + \boldsymbol{M}\tilde{\boldsymbol{y}}^c[k]$
- State observation error:

$$\boldsymbol{e}[k+1] = \boldsymbol{x}[k+1] - \hat{\boldsymbol{x}}[k+1] = (\boldsymbol{I} - \boldsymbol{MC})\boldsymbol{Ae}[k]$$

Open-Loop Observer Error Analysis

- UIO observer

$$
\begin{aligned}
\boldsymbol{z}[k+1] &= (\boldsymbol{I}-\boldsymbol{M}\boldsymbol{C})(\boldsymbol{A}\boldsymbol{z}[k]+\boldsymbol{A}\boldsymbol{M}\tilde{\boldsymbol{y}}^c[k]+\boldsymbol{B}\boldsymbol{u}^c[k]) \\
\hat{\boldsymbol{x}}[k] &= \boldsymbol{z}[k]+\boldsymbol{M}\tilde{\boldsymbol{y}}^c[k]
\end{aligned}
$$

- Observation error dynamics

$$
\begin{aligned}
\boldsymbol{e}[k+1] &= \boldsymbol{x}[k+1]-\hat{\boldsymbol{x}}[k+1] \\
&= \boldsymbol{A}\boldsymbol{x}[k]+\boldsymbol{B}(\boldsymbol{u}^c[k]+\boldsymbol{e}_a[k])-\boldsymbol{z}[k+1] \\
&\quad -\boldsymbol{M}\tilde{\boldsymbol{y}}^c[k+1] \\
&= \boldsymbol{A}\boldsymbol{x}[k]+\boldsymbol{B}(\boldsymbol{u}^c[k]+\boldsymbol{e}_a[k])-\boldsymbol{z}[k+1] \\
&\quad -\boldsymbol{M}\boldsymbol{C}\boldsymbol{x}[k+1] \\
&= \boldsymbol{A}\boldsymbol{x}[k]+\boldsymbol{B}(\boldsymbol{u}^c[k]+\boldsymbol{e}_a[k])-\boldsymbol{z}[k+1] \\
&\quad -\boldsymbol{M}\boldsymbol{C}\boldsymbol{A}\boldsymbol{x}[k]-\boldsymbol{M}\boldsymbol{C}\boldsymbol{B}\boldsymbol{u}^c[k]-\boldsymbol{M}\boldsymbol{C}\boldsymbol{B}\boldsymbol{e}_a[k])
\end{aligned}
$$

Open-Loop Observation Error

- Select $\boldsymbol{M}$ so that $(\boldsymbol{I} - \boldsymbol{MC})\boldsymbol{B} = \boldsymbol{O}$ gives

$$
\begin{aligned}
\boldsymbol{e}[k+1] &= \boldsymbol{Ax}[k] + \boldsymbol{B}(\boldsymbol{u}^c[k] + \boldsymbol{e}_a[k]) - \boldsymbol{z}[k+1] \\
&\quad -\boldsymbol{MCAx}[k] - \boldsymbol{MCBu}^c[k] - \boldsymbol{MCBe}_a[k]) \\
&= (\boldsymbol{I} - \boldsymbol{MC})(\boldsymbol{Ax}[k] + \boldsymbol{Bu}^c[k]) \\
&\quad +(\boldsymbol{I} - \boldsymbol{MC})\boldsymbol{Be}_a[k] \\
&\quad -\boldsymbol{z}[k+1] \\
&= (\boldsymbol{I} - \boldsymbol{MC})(\boldsymbol{Ax}[k] + \boldsymbol{Bu}^c[k]) \\
&\quad -(\boldsymbol{I} - \boldsymbol{MC})(\boldsymbol{A}\hat{\boldsymbol{x}}[k] + \boldsymbol{Bu}^c[k]) \\
&= (\boldsymbol{I} - \boldsymbol{MC})\boldsymbol{Ae}[k]
\end{aligned}
$$

- Open-loop UIO impractical—no control of the observation error dynamics

Closed-Loop UIO Error Dynamics

- Close the loop by adding the term $\boldsymbol{L}(\tilde{\boldsymbol{y}}^c[k] - \hat{\boldsymbol{y}})$ to the UIO, where $\hat{\boldsymbol{y}} = \boldsymbol{C}\hat{\boldsymbol{x}}$ is the UIO output
- Closed-loop UIO observation error

$$
\begin{aligned}
\boldsymbol{e}[k+1] &= (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})\boldsymbol{A}\boldsymbol{e}[k] - \boldsymbol{L}(\tilde{\boldsymbol{y}}^c[k] - \hat{\boldsymbol{y}}) \\
&= (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})\boldsymbol{A}\boldsymbol{e}[k] - \boldsymbol{L}(\boldsymbol{C}\boldsymbol{x}[k] - \boldsymbol{C}\hat{\boldsymbol{x}}) \\
&= (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})\boldsymbol{A}\boldsymbol{e}[k] - \boldsymbol{L}\boldsymbol{C}\boldsymbol{e}[k]
\end{aligned}
$$

- Let $\boldsymbol{A}_1 = (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})\boldsymbol{A}$, then we have

$$
\boldsymbol{e}[k+1] = (\boldsymbol{A}_1 - \boldsymbol{L}\boldsymbol{C})\boldsymbol{e}[k]
$$

- Closed-loop UIO—use the observer gain $\boldsymbol{L}$ to control the observation error dynamics

Closed-Loop UIO

- Compute $\boldsymbol{M}$ such that

$$(\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})\boldsymbol{B} = \boldsymbol{O}$$

- The UIO

$$\begin{aligned}
\boldsymbol{z}[k+1] &= (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})\left(\boldsymbol{A}\boldsymbol{z}[k] + \boldsymbol{A}\boldsymbol{M}\tilde{\boldsymbol{y}}^c[k] + \boldsymbol{B}\boldsymbol{u}^c[k]\right) \\
&\quad + \boldsymbol{L}(\tilde{\boldsymbol{y}}^c[k] - \hat{\boldsymbol{y}}[k]) \\
\hat{\boldsymbol{x}}[k] &= \boldsymbol{z}[k] + \boldsymbol{M}\tilde{\boldsymbol{y}}^c[k]
\end{aligned}$$

where

$$\hat{\boldsymbol{y}}[k] = \boldsymbol{C}\hat{\boldsymbol{x}}[k]$$

Solving $(\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})\boldsymbol{B} = \boldsymbol{O}$ for $\boldsymbol{M}$

- $(\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})\boldsymbol{B} = \boldsymbol{O} \iff \boldsymbol{M}\boldsymbol{C}\boldsymbol{B} = \boldsymbol{B}$
- $\boldsymbol{M}\boldsymbol{C}\boldsymbol{B} = \boldsymbol{B}$ implies

$$\operatorname{rank}(\boldsymbol{M}\boldsymbol{C}\boldsymbol{B}) = \operatorname{rank}\boldsymbol{B}$$

- On the other hand,

$$\operatorname{rank}(\boldsymbol{M}\boldsymbol{C}\boldsymbol{B}) \leq \operatorname{rank}(\boldsymbol{C}\boldsymbol{B}) \leq \operatorname{rank}(\boldsymbol{B})$$

- Hence, a necessary and sufficient condition for solvability of $(\boldsymbol{I} - \boldsymbol{M}\boldsymbol{C})\boldsymbol{B} = \boldsymbol{O}$ is

$$\operatorname{rank}(\boldsymbol{C}\boldsymbol{B}) = \operatorname{rank}(\boldsymbol{B})$$

Summary of UIO Construction

Theorem

Let $\boldsymbol{A}_1 = (\boldsymbol{I} - \boldsymbol{MC})\boldsymbol{A}$ *and* $\boldsymbol{T} = \boldsymbol{PL}$. *If*

1. $(\boldsymbol{I} - \boldsymbol{MC})\boldsymbol{B} = \boldsymbol{O}$,
2. *there exists* $\boldsymbol{P} = \boldsymbol{P}^\top \succ 0$ *such that*

$$\begin{bmatrix} -\boldsymbol{P} & \boldsymbol{A}_1^\top \boldsymbol{P} - \boldsymbol{C}^\top \boldsymbol{T}^\top \\ \boldsymbol{P}\boldsymbol{A}_1 - \boldsymbol{TC} & -\boldsymbol{P} \end{bmatrix} \prec 0,$$

then the UIO exists

Theorem Discussion

- $(\boldsymbol{A}_1 - \boldsymbol{L}\boldsymbol{C})$ Schur stable $\iff$ there exists $\boldsymbol{P} = \boldsymbol{P}^\top \succ 0$ such that
$$(\boldsymbol{A}_1 - \boldsymbol{L}\boldsymbol{C})^\top \boldsymbol{P}(\boldsymbol{A}_1 - \boldsymbol{L}\boldsymbol{C}) - \boldsymbol{P} \prec 0$$
- Substitute $\boldsymbol{P} = \boldsymbol{P}\boldsymbol{P}^{-1}\boldsymbol{P}$ into the Lyapunov matrix inequality to obtain
$$(\boldsymbol{A}_1 - \boldsymbol{L}\boldsymbol{C})^\top \boldsymbol{P}\boldsymbol{P}^{-1}\boldsymbol{P}(\boldsymbol{A}_1 - \boldsymbol{L}\boldsymbol{C}) - \boldsymbol{P} \prec 0$$
which is equivalent to the LMI condition of the Theorem by taking the Schur complement

Combined Estimator-UIO Design

1. Design an estimator $\tilde{\boldsymbol{e}}_s[k]$ of $\boldsymbol{e}_s[k]$
2. Design the UIO performing the following steps:
   - Check if $\text{rank}(\boldsymbol{CB}) = \text{rank}(\boldsymbol{B})$ is satisfied. If the condition is not satisfied, STOP
   - Solve $(\boldsymbol{I} - \boldsymbol{MC})\boldsymbol{B} = \boldsymbol{O}$ to obtain

     $$\boldsymbol{M} = \boldsymbol{B}\left((\boldsymbol{CB})^{\dagger} + \boldsymbol{H}_0(\boldsymbol{I}_p - (\boldsymbol{CB})(\boldsymbol{CB})^{\dagger})\right),$$

     where the superscript $\dagger$ denotes the Moore-Penrose pseudo-inverse and $\boldsymbol{H}_0$ is a design parameter matrix
   - Solve for $\boldsymbol{P}$ and $\boldsymbol{T}$
   - If $\boldsymbol{P} = \boldsymbol{P}^{\top} \succ 0$, UIO exists
   - Compute $\boldsymbol{L}_1 = \boldsymbol{P}^{-1}\boldsymbol{T}$