Case Study

Consider an overdetermined system of linear equations of the form,

$$Ax = b,$$

where $A \in \mathbb{R}^{m\times n}, m \geq n, \operatorname{rank}(A) = n$, and x and b are of appropriate dimensions. Consider also associated objective function

$$J(x) = (b - Ax)^\top (b - Ax)\,.$$

Write down an iterative steepest descent algorithm, in terms of A and b, for minimizing the above objective function. Carefully derive an expression for the step size α_k in terms of $\nabla J\left(x^{(k)}\right)$ and A only.

Please take a moment to work through this problem on your own. When you are ready, scroll down the page to view the solution.

Explanation: The steepest descent algorithm for minimizing J has the form

$$x^{(k+1)} = x^{(k)} - \alpha_k \nabla J\left(x^{(k)}\right), \quad k = 0, 1 \ldots$$

where

$$\nabla J\left(x^{(k)}\right) = A^\top A x^{(k)} - A^\top b = A^\top \left(Ax^{(k)} - b\right).$$

Substituting the above into the iterative algorithm gives

$$\boxed{x^{(k+1)} = x^{(k)} + \alpha_k A^\top \left(b - Ax^{(k)}\right), \quad k = 0, 1, \ldots}$$

The step size is calculated form the formula,

$$\begin{aligned}
\alpha_k &= \arg\min_\alpha J\left(x^{(k)} - \alpha \nabla J\left(x^{(k)}\right)\right) \\
&= \frac{\nabla J(x^{(k)})^\top \nabla J(x^{(k)})}{\nabla J(x^{(k)})^\top A^\top A \nabla J(x^{(k)})} \\
&= \frac{\left\|\nabla J(x^{(k)})\right\|^2}{\left\|A\nabla J(x^{(k)})\right\|^2}.
\end{aligned}$$

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