#1) $\frac{d}{dt}\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} z_1(t-1) + |u(t)| \\ z_1(t) \end{bmatrix}$, $y = z_1(t+1) + z_2(t) \cdot u(t)$

$\boxed{\text{box}}$

<u>Non-linear System</u> - $|u(t)|$ is a non-linear operator

<u>Time-Invariant</u> - Time delay has no effect

<u>Non-Casual System</u> - Future dependence in $z_1$ for output & $\therefore$ Input.

<u>Distributed System</u> - Time delay in state $z_1$, $\therefore$ Infinite state variables.

<u>Continuous-Time</u> - $t \in (-\infty, \infty)$

#2) $\dddot{\theta} + \theta = \tau$

$u = \tau \quad , \quad y = \theta \qquad \underline{x} = \begin{pmatrix} \theta \\ \dot{\theta} \\ \ddot{\theta} \end{pmatrix} \qquad \dot{\underline{x}} = \begin{pmatrix} \dot{\theta} \\ \ddot{\theta} \\ \dddot{\theta} \end{pmatrix}$

$\dddot{\theta} = \tau - \theta = -x_1 + u$

$$\boxed{\begin{array}{ll} \dot{\underline{x}} = Ax + Bu & y = Cx + Du \\ A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} & B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} & D = 0 \end{array}}$$

$H(s) = C(sI - A)^{-1}B + D$

$sI - A = \begin{pmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 0 & s \end{pmatrix}$

$C_{11} = \begin{vmatrix} s & -1 \\ 0 & s \end{vmatrix} = s^2$

$C_{12} = (-1)\begin{vmatrix} 0 & -1 \\ 1 & s \end{vmatrix} = -1$

$C_{13} = \begin{vmatrix} 0 & s \\ 1 & 0 \end{vmatrix} = -s$

$C_{21} = (-1)\begin{vmatrix} -1 & 0 \\ 0 & s \end{vmatrix} = s$

$C_{22} = \begin{vmatrix} s & 0 \\ 1 & s \end{vmatrix} = s^2$

#2)

$C_{23} = (-1)\begin{vmatrix} s & -1 \\ 1 & 0 \end{vmatrix} = -1$

$C_{31} = \begin{vmatrix} -1 & 0 \\ s & -1 \end{vmatrix} = 1$

$C_{32} = (-1)\begin{vmatrix} s & 0 \\ 0 & -1 \end{vmatrix} = s$

$C_{33} = \begin{vmatrix} s & -1 \\ 0 & s \end{vmatrix} = s^2$

$C = \begin{pmatrix} s^2 & -1 & -s \\ s & s^2 & -1 \\ 1 & s & s^2 \end{pmatrix}$ $\quad C^T = \begin{pmatrix} s^2 & s & 1 \\ -1 & s^2 & s \\ -s & -1 & s^2 \end{pmatrix}$

$\det(sI-A) = (s)s^2 + (-1)(-1) = s^3+1$

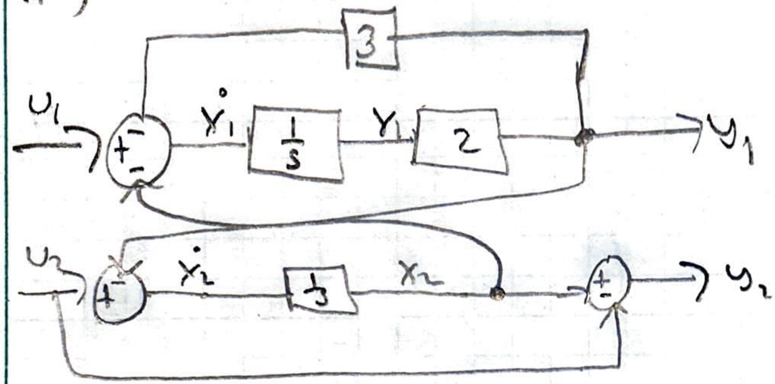
$(sI-A)^{-1} = \frac{1}{s^3+1}\begin{pmatrix} s^2 & s & 1 \\ -1 & s^2 & s \\ -s & -1 & s^2 \end{pmatrix}$

$(C)(sI-A)^{-1} = \begin{pmatrix} s^2 & s & 1 \end{pmatrix} \frac{1}{s^3+1}$

$(C)(sI-A^{-1})B = \frac{1}{s^3+1}$

$$\boxed{H(s) = \frac{1}{s^3+1}}$$

#3)




$y_2 = x_2 - u_2$

$\dot{x}_1 = u_1 - 6x_1 - x_2$

$\dot{x}_2 = u_2 - 2x_1$

$y_1 = 2x_1$

$$\boxed{\begin{aligned} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} &= \underbrace{\begin{pmatrix} -6 & -1 \\ -2 & 0 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{B} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \\ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &= \underbrace{\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}}_{C} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}}_{D} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \end{aligned}}$$

$$sI - A = \begin{pmatrix} s+6 & 1 \\ 2 & s \end{pmatrix}$$

#3)

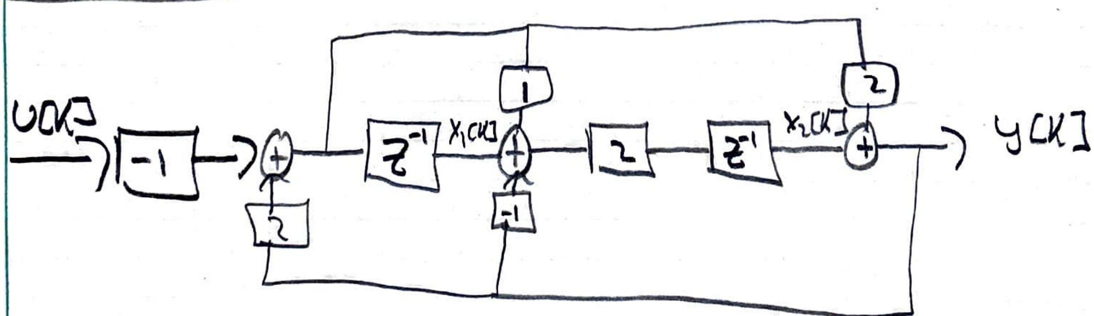
$$(SI-A)^{-1} = \frac{1}{s^2+6s-2}\begin{pmatrix} s & -1 \\ -2 & s+6 \end{pmatrix}$$

$$C(SI-A)^{-1} = \frac{1}{s^2+6s-2}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} s & -1 \\ -2 & s+6 \end{pmatrix} = \frac{1}{s^2+6s-2}\begin{pmatrix} 2s & -2 \\ -2 & s+6 \end{pmatrix}$$

$$C(SI-A)^{-1}B = \frac{1}{s^2+6s-2}\begin{pmatrix} 2s & -2 \\ -2 & s+6 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$C(SI-A)^{-1}B + D = \begin{pmatrix} \frac{2s}{s^2+6s-2} & \frac{-2}{s^2+6s-2} \\ \frac{-2}{s^2+6s-2} & \frac{s+6}{s^2+6s-2} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\boxed{\begin{aligned} \frac{Y_1(s)}{U_1(s)} &= \frac{2s}{s^2+6s-2} & \frac{Y_1(s)}{U_2(s)} &= \frac{-2}{s^2+6s-2} \\ \frac{Y_2(s)}{U_1(s)} &= \frac{-2}{s^2+6s-2} & \frac{Y_2(s)}{U_2(s)} &= \frac{-s^2-5s+8}{s^2+6s-2} \end{aligned}}$$




$$X_1[K+1] = -U[K] + 2y[K]$$

$$y[K] = 2X_1[K+1] + X_2[K]$$

$$X_2[K+1] = 2(X_1[K] + X_1[K+1] - y[K])$$

#3)

$$X_1[K+1] = -U[K] + 4X[K+1] + 2X_2[K]$$

$$X_1[K+1] = \frac{1}{3}U[K] - \frac{2}{3}X_2[K] \qquad \underline{1}$$

$$y[K] = \frac{2}{3}U[K] - \frac{4}{3}X_2[K] + X_2[K]$$

$$y[K] = \frac{2}{3}U[K] - \frac{1}{3}X_2[K] \qquad \underline{2}$$

$$X_2[K+1] = 2X_1[K] + \frac{2}{3}U[K] - \frac{4}{3}X_2[K] - \frac{4}{3}U[K] + \frac{2}{3}X_2[K]$$

$$X_2[K+1] = 2X_1[K] - \frac{2}{3}U[K] - \frac{2}{3}X_2[K] \qquad \underline{3}$$

$$\boxed{\begin{aligned} \begin{pmatrix} X_1[K+1] \\ X_2[K+1] \end{pmatrix} &= \underbrace{\begin{pmatrix} 0 & -2/3 \\ 2 & -2/3 \end{pmatrix}}_{A} \begin{pmatrix} X_1[K] \\ X_2[K] \end{pmatrix} + \underbrace{\begin{pmatrix} \frac{1}{3} \\ -2/3 \end{pmatrix}}_{B} U[K] \\ y[K] &= \underbrace{\begin{pmatrix} 0 & -1/3 \end{pmatrix}}_{C} \begin{pmatrix} X_1[K] \\ X_2[K] \end{pmatrix} + \underbrace{\left(2/3\right)}_{D} U[K] \end{aligned}}$$

$$\underline{X}[K+1] = A\underline{X}[K] + BU[K]$$

$$y[K] = C\underline{X}[K] + DU[K]$$

$$H(z) = C(zI-A)^{-1}B + D$$

$$zI - A = \begin{pmatrix} z & 2/3 \\ -2 & z+2/3 \end{pmatrix}$$

$$(zI-A)^{-1} = \frac{1}{z^2 + \frac{2}{3}z + \frac{4}{3}} \begin{pmatrix} z+2/3 & -2/3 \\ 2 & z \end{pmatrix}$$

#3)

$$C(zI-A)^{-1} = \frac{1}{z^2+\frac{2}{3}z+\frac{4}{3}} \begin{pmatrix} -\frac{2}{3} & -\frac{z}{3} \end{pmatrix}$$

$$C(zI-A)^{-1}B = \frac{1}{z^2+\frac{2}{3}z+\frac{4}{3}} \begin{pmatrix} -\frac{2}{3} & -\frac{z}{3} \end{pmatrix} \begin{pmatrix} \frac{1}{3} \\ \frac{-2}{3} \end{pmatrix}$$

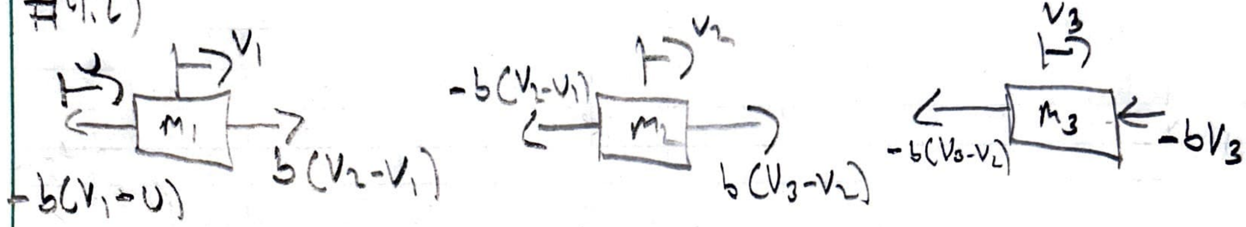
$$= \frac{\frac{2z-2}{9}}{z^2+\frac{2}{3}z+\frac{4}{3}}$$

$$C(zI-A)^{-1}B + D = \frac{2z-2}{9(z^2+\frac{2}{3}z+\frac{4}{3})} + \frac{6(z^2+\frac{2}{3}z+\frac{4}{3})}{9(z^2+\frac{2}{3}z+\frac{4}{3})}$$

$$H(z) = \frac{6z^2+6z+6}{9(z^2+\frac{2}{3}z+\frac{4}{3})} = \frac{6(z^2+z+1)}{9z^2+6z+12}$$

$$\boxed{H(z) = \frac{2(z^2+z+1)}{3z^2+2z+4}}$$

#4.2)




$$\Sigma F_1 = m\dot{v}_1 = -2bv_1 + bu + bv_2$$

$$\Sigma F_2 = m\dot{v}_2 = -2bv_2 + bv_1 + bv_3$$

$$\Sigma F_3 = m\dot{v}_3 = -2bv_3 + bv_2$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \qquad y = m\dot{v}_2 = m\dot{x}_2$$

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$$\boxed{\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{-2b}{m} & \frac{b}{m} & 0 \\ \frac{b}{m} & \frac{-2b}{m} & \frac{b}{m} \\ 0 & \frac{b}{m} & \frac{-2b}{m} \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \underbrace{\begin{pmatrix} \frac{b}{m} \\ 0 \\ 0 \end{pmatrix}}_{B} u}$$

$$\boxed{y = \underbrace{\begin{pmatrix} b & -2b & b \end{pmatrix}}_{C} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \underbrace{0}_{D} u}$$

#5.1)

$$H_s = H_2 H_1 \qquad H_2 = \frac{y}{y_1} \qquad H_1 = \frac{y_1}{u}$$

$$\dot{x}_1 = A_1 x_1 + B_1 u \qquad H_s = \frac{y}{u}$$

$$y_1 = C_1 x_1 + D_1 u$$

$$\dot{x}_2 = A_2 x_2 + B_2 y_1$$

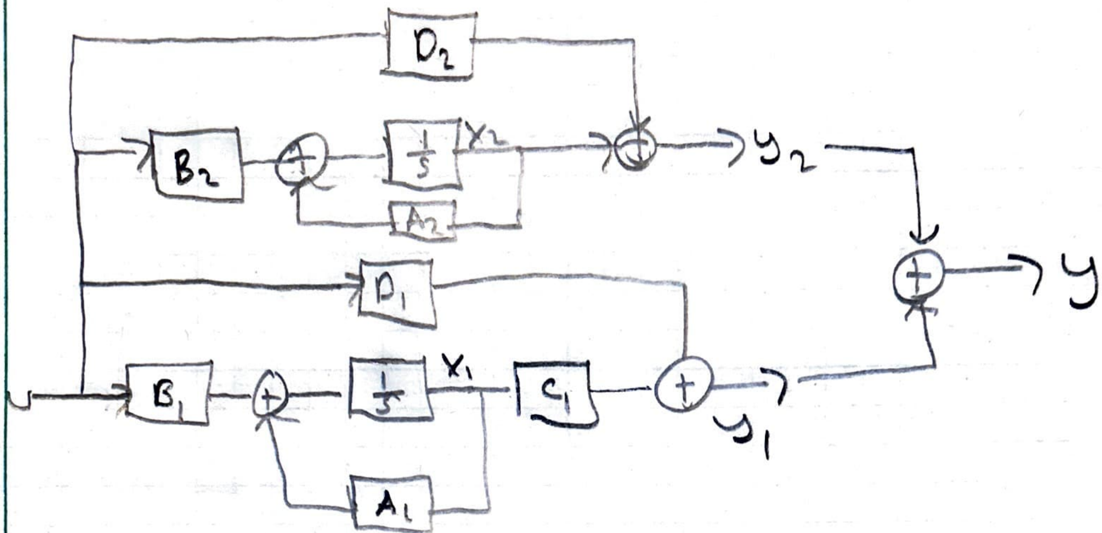
$$y = C_2 x_2 + D_2 y_1$$

$$\dot{x}_2 = A_2 x_2 + B_2 C_1 x_1 + B_2 D_1 u$$

$$y = C_2 x_2 + D_2 C_1 x_1 + D_2 D_1 u$$

$$\boxed{\begin{array}{l} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} A_1 & 0 \\ B_2 C_1 & A_2 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{pmatrix} B_1 \\ B_2 D_1 \end{pmatrix}}_{B} u \\ \\ y = \underbrace{\begin{pmatrix} D_2 C_1 & C_2 \end{pmatrix}}_{C} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{pmatrix} D_2 D_1 \end{pmatrix}}_{D} u \end{array}}$$

#5.2) $H_p = H_1 + H_2 = \frac{y_1}{U} + \frac{y_2}{U}$




$\dot{X}_1 = A_1 X_1 + B_1 U$

$y_1 = C_1 X_1 + D_1 U$

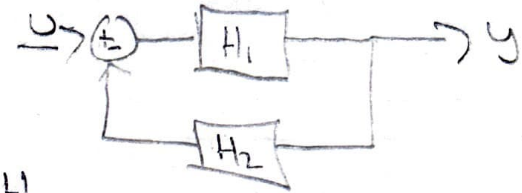
$\dot{X}_2 = A_2 X_1 + B_2 U$

$y_2 = C_2 X_2 + D_2 U$

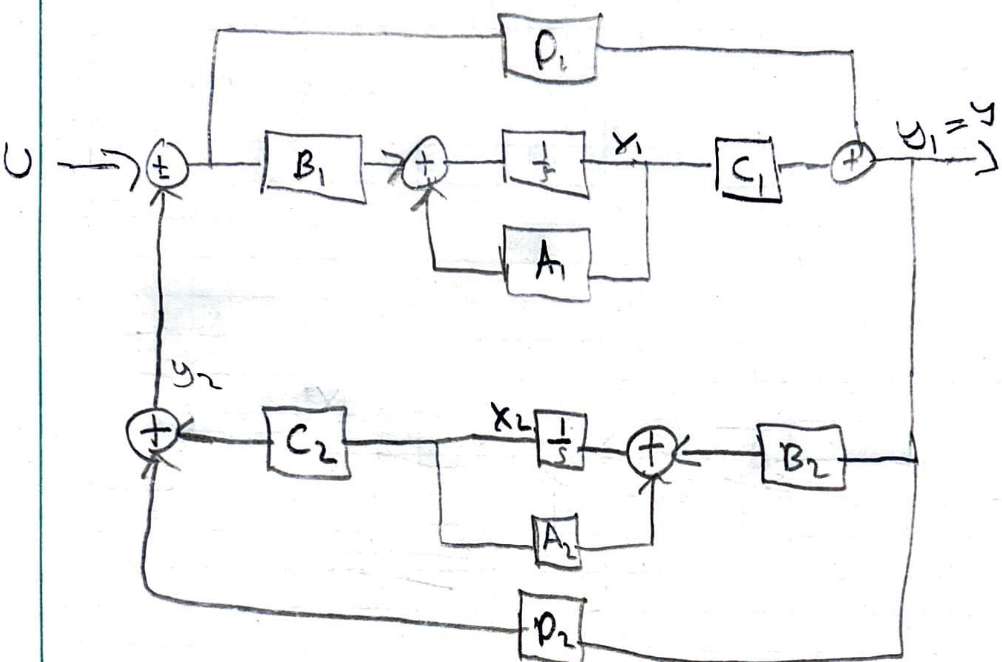
$y = y_1 + y_2 = C_1 X_1 + C_2 X_2 + D_1 U + D_2 U$

$$\boxed{\begin{pmatrix} \dot{X}_1 \\ \dot{X}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}}_{A} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} U \qquad y = \underbrace{\begin{pmatrix} C_1 & C_2 \end{pmatrix}}_{C} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} + \underbrace{(D_1 + D_2)}_{D} U}$$

#5.3)




$$\frac{y}{U} = H_F = \frac{H_1}{1 \pm H_1 H_2}$$




$$\dot{x}_1 = A_1 x_1 + B_1(U - y_2)$$

$$y = C_1 x_1 + D_1(U - y_2)$$

$$\dot{x}_2 = A_2 x_2 + B_2 y$$

$$y_2 = C_2 x_2 + D_2 y$$

$$\dot{x}_1 = A_1 x_1 + B_1 U - B_1 C_2 x_2 - B_1 D_2 y$$

$$y = C_1 x_1 + D_1 U - D_1 C_2 x_2 - D_1 D_2 y$$

$$y = (I + D_1 D_2)^{-1}(C_1 x_1 + D_1 U - D_1 C_2 x_2)$$

#53)

$$\dot{x}_1 = A_1 x_1 + B_1 u - B_1 C_2 x_2 - B_1 D_2 (I + D_1 D_2)^{-1} (C_1 x_1 + D_1 u - D_1 C_2 x_2)$$

$$\dot{x}_2 = A_2 x_2 + B_2 (I + D_1 D_2)^{-1} (C_1 x_1 + D_1 u - D_1 C_2 x_2)$$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} A_1 - B_1 D_2 (I + D_1 D_2)^{-1} C_1 & -B_1 C_2 + B_1 D_2 (I + D_1 D_2)^{-1} D_1 C_2 \\ B_2 (I + D_1 D_2)^{-1} C_1 & A_2 - B_2 (I + D_1 D_2)^{-1} D_1 C_2 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$+ \underbrace{\begin{pmatrix} B_1 - B_1 D_2 (I + D_1 D_2)^{-1} D_1 \\ B_2 (I + D_1 D_2)^{-1} D_1 \end{pmatrix}}_{B} (u)$$

$$y = \underbrace{[(I + D_1 D_2)^{-1} C_1 \quad -(I + D_1 D_2)^{-1} D_1 C_2]}_{C} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{[(I + D_1 D_2)^{-1} D_1]}_{D} u$$