HW3

8.5.1) $\begin{pmatrix} 6 & i \\ -i & 6 \end{pmatrix}$ $\bar{A}^T = \begin{pmatrix} 6 & i \\ -i & 6 \end{pmatrix} = A$

$\boxed{\text{A is Hermitian}}$ $A^{-1} = \frac{1}{35}\begin{pmatrix} 6 & -i \\ i & 6 \end{pmatrix}$

$|A - \lambda I| = \begin{vmatrix} 6-\lambda & i \\ -i & 6-\lambda \end{vmatrix} = (6-\lambda)^2 + 1 = \lambda^2 - 12\lambda + 35$

$= (\lambda - 7)(\lambda - 5) \Rightarrow \boxed{\begin{matrix} \lambda_2 = 7 \\ \lambda_1 = 5 \end{matrix}}$

$A v_1 = \lambda_1 v_1$

$\begin{pmatrix} 6 & i \\ -i & 6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 5\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, let $x_2 = 1$ $\therefore 6x_1 + i = 5x_1$

$\therefore x_1 = -i$ $\boxed{V_1 = \begin{pmatrix} -i \\ 1 \end{pmatrix}}$

$A V_2 = \lambda_2 V_2$

$\begin{pmatrix} 6 & i \\ -i & 6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 7x_1 \\ 7x_2 \end{pmatrix}$ Let $x_2 = 1$ $\therefore 6x_1 + i = 7x_1$

$\therefore x_1 = i$

$\boxed{V_2 = \begin{pmatrix} i \\ 1 \end{pmatrix}}$

8.5.2) $\begin{pmatrix} i & 1+i \\ -1+i & 0 \end{pmatrix}$ $\qquad \bar{A}^T = \begin{pmatrix} -i & -1-i \\ 1-i & 0 \end{pmatrix}$

$-A = \begin{pmatrix} -i & -1-i \\ 1-i & 0 \end{pmatrix}$ $\qquad \bar{A}^T = -A$

$A^{-1} = \frac{1}{2}\begin{pmatrix} 0 & -1-i \\ 1-i & i \end{pmatrix}$

$\boxed{A \text{ is skew - Hermitian}}$

$|A-\lambda I| = \begin{vmatrix} i-\lambda & 1+i \\ -1+i & -\lambda \end{vmatrix} = \lambda^2 - i\lambda + 2$

$\lambda_{1,2} = \frac{i \pm \sqrt{(-i)^2 - 8}}{2} = \frac{i}{2} \pm \frac{3i}{2} = -i, 2i$

$\boxed{\lambda_1 = -i, \quad \lambda_2 = 2i}$

$AV_1 = \lambda_1 V_1 = \begin{pmatrix} i & 1+i \\ -1+i & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -ix_1 \\ -ix_2 \end{pmatrix}$

Let $x_2 = 1$ $\therefore$ $ix_1 + i + 1 = -ix_1$ $\therefore$ $\frac{-(i+1)}{2i}$

$\boxed{V_1 = \begin{pmatrix} -\frac{1}{2} - \frac{1}{2i} \\ 1 \end{pmatrix}}$

$A V_2 = \lambda_2 V_2$

$$\begin{pmatrix} i & 1+i \\ 1+i & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2i x_1 \\ 2i x_2 \end{pmatrix}$$ let $x_2 = 1$ ∴

$x_1 i + 1+i = 2i x_1$ $x_1 = \frac{1+i}{i}$

$$\boxed{V_2 = \begin{pmatrix} 1 + \frac{1}{i} \\ 1 \end{pmatrix}}$$

8.5.5) $\begin{pmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix}$ $\bar{A}^T = \begin{pmatrix} -i & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{pmatrix} = -A = \begin{pmatrix} -i & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{pmatrix}$

$\boxed{\bar{A}^T = A \therefore \text{ skew - Herretitian}}$

$$\left(\begin{array}{ccc|ccc} i & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & i & 0 & 1 & 0 \\ 0 & i & 0 & 0 & 0 & 1 \end{array}\right) \overset{\frac{1}{i} = -i}{\Rightarrow} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1-i & 0 & 0 \\ 0 & 0 & i & 0 & 1 & 0 \\ 0 & i & 0 & 0 & 0 & 1 \end{array}\right) \overset{\text{swap rows}}{\Rightarrow}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1-i & 0 & 0 \\ 0 & i & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & i & 1 & 0 & 1 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1-i & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & -i \\ 0 & 0 & 1 & 1 & 0 & -i & 0 \end{array}\right) \therefore A^{-1} = \begin{pmatrix} -i & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{pmatrix}$$

$\boxed{\bar{A}^T = A^{-1} \therefore A \text{ is also unitary}}$

$$|A - \lambda I| = \begin{vmatrix} i-\lambda & 0 & 0 \\ 0 & -\lambda & i \\ 0 & i & -\lambda \end{vmatrix} = 0 = (i-\lambda) \begin{vmatrix} -\lambda & i \\ i & -\lambda \end{vmatrix}$$

$(i-\lambda)[\lambda^2+1]=0$

$\therefore \boxed{\lambda_1 = i}$ $\qquad \lambda_{2,3} = \pm\sqrt{-1}$

$\boxed{\begin{matrix}\lambda_2 = i \\ \lambda_3 = -i\end{matrix}}$ $\qquad |\lambda_1| = |\lambda_2| = |\lambda_3| = 1$

$(A-\lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -i & i \\ 0 & i & -i \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \boxed{i} & -i \end{pmatrix}$$

Let $x_1 = 0$, $x_3 = 1$ $\therefore$ $\quad x_2 i - i = 0 \quad \therefore x_2 = 1$

$$\boxed{V_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}}$$

$(A-\lambda_2 I)V_2 = 0 \Rightarrow$

$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \boxed{i} & -i \end{pmatrix}$, Let $x_1 = 1$, $x_3 = 0$ $\therefore$ $\quad x_2 i = 0 \quad \therefore x_2 = 0$

$$\boxed{V_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}$$

$(A-\lambda_3 I)V_3 = 0$

$$\begin{pmatrix} 2i & 0 & 0 \\ 0 & i & i \\ 0 & i & i \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \vec{0}$$

$$\Rightarrow \begin{pmatrix} \boxed{2i} & 0 & 0 \\ 0 & \boxed{i} & i \\ 0 & 0 & 0 \end{pmatrix}$$ Let $x_3 = 1$ $\therefore$ $x_2 i + i = 0$, $x_2 = -1$

$2i x_1 = 0$, $x_1 = 0$

$$\boxed{V_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}}$$

8.5.13) $\overline{(ABC)}^T = -C^{-1}BA$

A - hermetian $(\bar{A}^T = A)$
B - skew hermatitian $(\bar{B}^T = -B)$
C - unitary $(\bar{C}^T = C^{-1})$

$\overline{(ABC)}^T = \bar{C}^T \bar{B}^T \bar{A}^T$

$\bar{A}^T = A$, $\bar{B}^T = -B$, $\bar{C}^T = C^{-1}$ $\therefore$ $\bar{C}^T\bar{B}^T\bar{A}^T = (C^{-1})(-B)(A)$

$= \bar{C}^T\bar{B}^T\bar{A}^T = \boxed{-C^{-1}BA} = \overline{(ABC)}^T$

8.4.1) $A = \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}$ $\lambda^2 - 25 = 0$ $\lambda_{1,2} = \pm 5$

$AV_1 = \lambda_1 V_1 = \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5x_1 \\ 5x_2 \end{pmatrix}$ $x_2 = 1$ $\therefore$ $3x_1 + 4 = 5x_1$

$x_1 = 2$

$$\boxed{V_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}}$$

$AV_2 = -5V_2 = \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -5x_1 \\ -5x_2 \end{pmatrix}$ $x_2 = 1$ $\therefore$ $3x_1 + 4 = -5x_1$ $\therefore$ $x_1 = -\frac{1}{2}$

$$\boxed{V_2 = \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix}}$$

$$P = \begin{pmatrix} -4 & 2 \\ 3 & -1 \end{pmatrix} \qquad P^{-1} = \begin{pmatrix} \frac{1}{2} & 1 \\ \frac{3}{2} & 2 \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} 1/2 & 1 \\ 3/2 & 2 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} -4 & 2 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} 11/2 & -1 \\ 25/2 & 0 \end{pmatrix} \begin{pmatrix} -4 & 2 \\ 3 & -1 \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} -25 & 12 \\ -50 & 25 \end{pmatrix} = A_2$$

$$|A_2 - \lambda I| = \lambda^2 - 25 = 0 \qquad \boxed{\lambda_{1,2} = \pm 5}$$

$$A_2 y_1 = \lambda_1 y_1 = \begin{pmatrix} -25 & 12 \\ -50 & 25 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5x_1 \\ 5x_2 \end{pmatrix}, \quad x_2 = 1$$

$$\therefore -25x_1 + 12 = 5x_1 \qquad \therefore x_1 = 12/30$$

$$\boxed{y_1 = \begin{pmatrix} 12/30 \\ 1 \end{pmatrix}}$$

$$A_2 y_2 = \lambda_2 y_2 = \begin{pmatrix} -25 & 12 \\ -50 & 25 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -5x_1 \\ -5x_2 \end{pmatrix}, \quad x_2 = 1$$

$$\therefore -25x_1 + 12 = -5x_1 \qquad \therefore x_1 = 12/20$$

$$\boxed{y_2 = \begin{pmatrix} 12/20 \\ 1 \end{pmatrix}}$$

$$x_1 = \begin{pmatrix} -4 & 2 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 13/30 \\ 1 \end{pmatrix} = \begin{pmatrix} 2/5 \\ 1/5 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \checkmark$$

$$x_2 = \begin{pmatrix} -4 & 2 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 13/20 \\ 1 \end{pmatrix} = \begin{pmatrix} -2/5 \\ 4/5 \end{pmatrix} = \begin{pmatrix} -1/2 \\ 1 \end{pmatrix} \checkmark$$

8.4.9) $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ $\quad |A - \lambda I| = \lambda^2 - 5\lambda$

$\lambda_1 = 0 \quad , \lambda_2 = 5$

$$AV_1 = \lambda_1 V_1 = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad x_2 = 1 \therefore x_1 = -2$$

$$\boxed{V_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}}$$

$2 = 4x_1$

$$AV_2 = \lambda_2 V_2 = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5x_1 \\ 5x_2 \end{pmatrix} \qquad x_2 = 1 \therefore x_1 = 1/2$$

$$\boxed{V_2 = \begin{pmatrix} 1/2 \\ 1 \end{pmatrix}}$$

$$P = (V_1 \ V_2) = \begin{pmatrix} -2 & 1/2 \\ 1 & 1 \end{pmatrix}, \therefore P^{-1} = -\frac{2}{5}\begin{pmatrix} 1 & -1/2 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} -2/5 & 1/5 \\ 2/5 & 4/5 \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} -2/5 & 1/5 \\ 2/5 & 4/5 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} -2 & 1/2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} -2 & 1/2 \\ 1 & 1 \end{pmatrix}$$

$$\boxed{P^{-1}AP = \begin{pmatrix} 0 & 0 \\ 0 & 5 \end{pmatrix}}$$

8.4.10) $A = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$ $\lambda_1 = 1$, $\lambda_2 = -1$ (Triangular matrix)

$AV_1 = \lambda_1 V_1 = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $x_2 = 1$ $\therefore x_1 = 1$

$2x_1 - 1 = 1$

$\boxed{V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$

$AV_2 = \lambda_1 V_2 = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_1 \\ -x_2 \end{pmatrix}$, $x_2 = 1$ $\therefore x_1 = 0$

$2x_1 - 1 = -1$

$\boxed{V_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$ $\therefore P = (V_1 \ V_2) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

$P^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$, $P^{-1}AP = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

$P^{-1}AP = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \boxed{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} = P^{-1}AP$

8.4.25) $4x_1^2 + 12x_1x_2 + 13x_2^2$

$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ $A = \begin{pmatrix} 4 & 6 \\ 6 & 13 \end{pmatrix}$ $Q = x^T A x$

$a_{11} = 4 > 0$ ✓

$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} 4 & 6 \\ 6 & 13 \end{vmatrix}$

$$|A| = 52 - 36 = 16 > 0$$

$$\boxed{\text{Leading principal minors greater than } 0, \therefore A \text{ is positive definite}}$$

$$-11x_1^2 + 84x_1x_2 + 24x_2^2$$

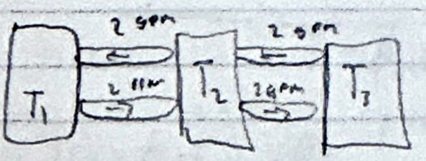
$$A = \begin{pmatrix} -11 & 42 \\ 42 & 24 \end{pmatrix}$$

$$a_{11} = -11 < 0$$

$$|A| = (11)(24) - 42^2 = -2028 < 0$$

$$\boxed{\text{Leading principal minors } < 0, \therefore A \text{ is indefinite}}$$

4.1.5)



$$y_1' = y_{2in} - y_{1out}$$
$$y_2' = y_{1in} + y_{3in} - 2y_{2out}$$
$$y_3' = y_{2in} - y_{3out}$$

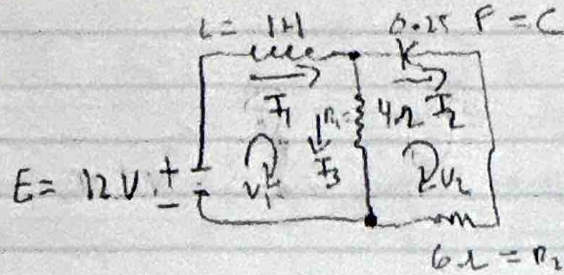
$$y_1' = \frac{2}{100}y_2 - \frac{2}{100}y_1$$

$$y_2' = \frac{2}{100}y_1 + \frac{2}{100}y_3 - \frac{4}{100}y_2$$

$$y_3' = \frac{2}{100}y_2 - \frac{2}{100}y_3$$

$$\therefore \boxed{y' = \begin{pmatrix} -.02 & .02 & 0 \\ .02 & -.04 & .02 \\ 0 & .02 & -.02 \end{pmatrix} y}$$

4.1.7)



$I_1 = I_2 + I_3$

$i = C \frac{dV}{dt}$

$C \int i\,dt = V$

Loop 1: $\Sigma V = 0 = E - \dot{I}_1 L - I_3 R_1$

Loop 2: $\Sigma V = 0 = R_1 I_3 - I_2 R_2 - \frac{1}{C}\int I_2\,dt$

$\dot{I}_1 = \frac{1}{L}\left(E + R_1(I_2 - I_1)\right) = \underline{12 + 4I_2 - 4I_1}$

$R_1(I_1 - I_2) - I_2 R_2 = \frac{1}{C}\int I_2\,dt \quad \xRightarrow{\frac{d}{dt}}$

$R_1(\dot{I}_1 - \dot{I}_2) - \dot{I}_2 R_2 = \frac{1}{C} I_2 = \dot{I}_2(-R_1 - R_2) + R_1 \dot{I}_1$

$\dot{I}_2 = \dfrac{-R_1 \dot{I}_1 + \frac{I_2}{C}}{(-R_1 - R_2)} = \dfrac{-\frac{R_1}{L}\left(E + R_1(I_2 - I_1)\right) + \frac{I_2}{C}}{(-R_1 - R_2)}$

$\dot{I}_2 = \dfrac{-4(12 + 4(I_2 - I_1)) + 4I_2}{-4 - 6} = \dfrac{-48 - 12I_2 + 16I_1}{-10}$

$\dot{I}_2 = \underline{4.8 + 1.2I_2 - 1.6I_1}$

$\dot{I} = \begin{pmatrix} -4 & 4 \\ -1.6 & 1.2 \end{pmatrix} I + \begin{pmatrix} 12 \\ 4.8 \end{pmatrix}$

$\lambda^2 + 2.8 + 1.6 \Rightarrow \lambda_1 = -2, \ \lambda_2 = -0.8$

$$AV_1 = \lambda_1 V_1 = \begin{pmatrix} -4 & 4 \\ -1.6 & 1.2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2x_1 \\ -2x_2 \end{pmatrix}$$

$$V_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$AV_2 = \lambda_2 V_2 = \begin{pmatrix} -4 & 4 \\ 1.6 & 1.2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -.8x_1 \\ -.8x_2 \end{pmatrix}$$

$$V_2 = \begin{pmatrix} 1 \\ 0.8 \end{pmatrix}$$

$$I_h = c_1 V_1 e^{\lambda_1 t} + c_2 V_2 e^{\lambda_2 t}$$

$$b = \begin{pmatrix} 12 \\ 4.8 \end{pmatrix} \qquad I_p = c \ , \ I_p' = 0$$

$$I' = AI + b = I' - AI = b = I_p' - A I_p = b$$

$$\begin{pmatrix} 4 & -4 \\ 1.6 & -1.2 \end{pmatrix} \begin{pmatrix} y_{p_1} \\ y_{p_2} \end{pmatrix} = \begin{pmatrix} 12 \\ 4.8 \end{pmatrix} \Rightarrow \begin{pmatrix} y_{p_1} \\ y_{p_2} \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

$$I = I_h + I_p = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ 0.8 \end{pmatrix} e^{-0.8t} + \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

$$I(0) = \begin{pmatrix} 0 \\ -3 \end{pmatrix} = \begin{pmatrix} 2c_1 + c_2 + 3 \\ c_1 + 0.8c_2 \end{pmatrix}$$

$$\begin{pmatrix} -3 \\ -3 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 0.8 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$A^{-1} = \frac{1}{0.6}\begin{pmatrix} 0.8 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1.33 & -1.67 \\ -1.67 & 3.33 \end{pmatrix}$$

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 4/3 & -5/3 \\ -5/3 & 10/3 \end{pmatrix}\begin{pmatrix} -3 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \end{pmatrix}$$

$$\therefore \boxed{\begin{aligned} I_1 &= 2e^{-2t} - 5e^{-.8t} + 3 \\ I_2 &= e^{-2t} - 4e^{-.8t} \end{aligned}}$$

4.1.12) $y''' + 2y'' - y' - 2y = 0$

$y_1 = y \quad , \quad y_2 = y' \quad , \quad y_3 = y''$

$y_1' = y_2 \quad , \quad y_2' = y_3 \quad , \quad y_3' = y''' = -2y_3 + y_2 + 2y_1$

$$y' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & -2 \end{pmatrix} y$$

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 2 & 1 & -2-\lambda \end{vmatrix} = -\lambda \begin{vmatrix} -\lambda & 1 \\ 1 & -2-\lambda \end{vmatrix} - \begin{vmatrix} 0 & 1 \\ 2 & -2-\lambda \end{vmatrix} = 0$$

$$= (-\lambda)[\lambda^2 + 2\lambda - 1] + 2 = -\lambda^3 - 2\lambda^2 + \lambda + 2 = -\lambda^2(\lambda+2) + (\lambda+2) = 0$$

$(\lambda+2)(-\lambda^2 + 1) = 0 \qquad \therefore \lambda_1 = -2 \, , \; \lambda_2 = 1 , \; \lambda_3 = -1$

$(A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 2 & 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \boxed{2} & 1 & 0 \\ 0 & \boxed{2} & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$x_3 = 1$, $\therefore$ $x_2 = -1/2$ & $x_1 = 1/4$

$$V_1 = \begin{pmatrix} 1/4 \\ -1/2 \\ 1 \end{pmatrix}$$

$(A - \lambda_2 I) V_2 = 0$

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 1 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 3 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \Rightarrow \begin{pmatrix} \boxed{-1} & 1 & 0 \\ 0 & \boxed{-1} & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$x_3 = 1$ $\therefore$ $x_2 = 1$ & $x_1 = 1$

$$V_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$(A - \lambda_3 I) V_3 = 0$

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{pmatrix} \Rightarrow \begin{pmatrix} \boxed{1} & 1 & 0 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$x_3 = 1$ $\therefore$ $x_2 = -1$, $x_1 = 1$

$$V_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$\boxed{y(t) = c_1 \begin{pmatrix} 1/4 \\ -1/2 \\ 1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{t} + c_3 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} e^{-t}}$$

$y''' + 2y'' - y' - 2y = 0$

$r^3 + 2r^2 - r - 2 = 0$

$r^2(r+2) - (r+2) = 0$

$(r+2)(r^2 - 1) = 0 \quad , \; r_1 = -2 , \; r_2 = 1 , \; r_3 = -1$

$y = c_1 e^{r_1 t} + c_2 e^{r_2 t} + c_3 e^{r_3 t}$

$$\boxed{y = c_1 e^{-2t} + c_2 e^{t} + c_3 e^{-t}}$$

4.3.3) $y_1' = y_1 + 2y_2$
$y_2' = \frac{1}{2} y_1 + y_2$

$y' = \begin{pmatrix} 1 & 2 \\ 1/2 & 1 \end{pmatrix} y \qquad |A - \lambda I| = \lambda^2 - 2\lambda = 0 \quad \therefore \; \lambda_1 = 0, \; \lambda_2 = 2$

$(A - \lambda_1 I) V_1 = 0$

$\begin{pmatrix} 1 & 2 \\ 1/2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad x_2 = 1 \quad \therefore \quad x_1 = -2$

$V_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$(A - \lambda_2 I) V_2 = 0 \qquad x_2 = 1 \quad \therefore \quad x_1 = 2$

$\begin{pmatrix} -1 & 2 \\ 1/2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad V_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$$\boxed{y = \begin{pmatrix} -2 \\ 1 \end{pmatrix} c_1 + \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t} c_2}$$

4.3.13) $y_1' = y_2$ $\quad$ $y_1(0) = 0$

$y_2' = y_1$ $\quad$ $y_2(0) = 2$

$$y' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} y \qquad |A - \lambda I| = \lambda^2 - 1 = 0 \qquad \therefore \lambda_1 = 1,\ \lambda_2 = -1$$

$$(A - \lambda_1 I) V_1 = \vec{0}$$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad x_2 = 1 \ \therefore\ x_1 = 1 \qquad V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$(A - \lambda_2 I) V_2 = 0$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \vec{0} \qquad x_2 = 1 \ \therefore\ x_1 = -1 \qquad V_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$y = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t}$$

$$y(0) = \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\boxed{y = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t}}$$

$$\sinh(t) = \frac{e^{t} - e^{-t}}{2}$$

$$\cosh(t) = \frac{e^{t} + e^{-t}}{2}$$

$$\boxed{\begin{aligned} y_1(t) &= e^{t} - e^{-t} = 2\sinh(t) \\ y_2(t) &= e^{t} + e^{-t} = 2\cosh(t) \end{aligned}}$$

4.3.18)

Pure water

$$y_1' = \cancel{(12)(0)} + \frac{4y_2}{200} - \frac{16y_1}{200}$$

$$y_2' = \frac{16y_1}{200} - \frac{4y_2}{200} - \frac{12y_2}{200}$$

$$y' = \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} -16/200 & 4/200 \\ 16/200 & -16/200 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$|A - \lambda I| = \lambda^2 + .16\lambda + .0048$$

$$\lambda_{1,2} = \frac{-.16 \pm \sqrt{.16^2 - (4)(.0048)}}{2} = -.08 \pm .04$$

$$\lambda_1 = -.04 \quad , \quad \lambda_2 = -.12$$

$$A V_1 = \lambda_1 V_1$$

$$\begin{pmatrix} -.08 & .02 \\ .08 & -.08 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -.04x_1 \\ -.04x_2 \end{pmatrix}$$

$x_2 = 1 \quad \therefore \quad .02 = .04x_1$

$x_1 = 1/2$

$$V_1 = \begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$$

$$A V_2 = \lambda_2 V_2$$

$$\begin{pmatrix} -.08 & .02 \\ .08 & -.08 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -.12x_1 \\ -.12x_2 \end{pmatrix}$$

$x_2 = 1 \quad \therefore \quad .02 = -.04x_1$

$x_1 = -1/2$

$$V_2 = \begin{pmatrix} -1/2 \\ 1 \end{pmatrix}$$

$$\boxed{y(t) = c_1 \begin{pmatrix} 1/2 \\ 1 \end{pmatrix} e^{-0.04t} + c_2 \begin{pmatrix} -1/2 \\ 1 \end{pmatrix} e^{-0.12t}}$$

4.4.3)

$$y_1' = y_2$$
$$y_2' = -9y_1$$

$$y' = \begin{pmatrix} 0 & 1 \\ -9 & 0 \end{pmatrix} y \qquad |A - \lambda I| = \lambda^2 + 9 = 0$$

$$\lambda_{1,2} = \pm 3i \quad \therefore \quad \boxed{\text{stable , center}}$$

$$AV_1 = \lambda_1 V_1 \Rightarrow \begin{pmatrix} 0 & 1 \\ -9 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3ix_1 \\ 3ix_2 \end{pmatrix}$$

$$x_1 = 1 \therefore x_2 = 3i, \quad V_1 = \begin{pmatrix} 1 \\ 3i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 3 \end{pmatrix} i$$

$$y = c_1 \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(3t) - \begin{pmatrix} 0 \\ 3 \end{pmatrix} \sin(3t) \right] + c_2 \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin(3t) + \begin{pmatrix} 0 \\ 3 \end{pmatrix} \cos(3t) \right]$$

$$\boxed{\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} c_1 \cos(3t) + c_2 \sin(3t) \\ -3c_1 \sin(3t) + 3c_2 \cos(3t) \end{pmatrix}}$$

4.4.5) $y' = \begin{pmatrix} -2 & 2 \\ -2 & -2 \end{pmatrix} \Rightarrow \lambda^2 + 4\lambda + 8$

$\lambda_{1,2} = \frac{-4 \pm \sqrt{16-32}}{2} = -2 \pm 2i$

$\therefore$ $\boxed{\text{stable, attractive, spiral point}}$

$A V_1 = \lambda_1 V_1$

$\begin{pmatrix} -2 & 2 \\ -2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2+2i \; x_1 \\ -2+2i \; x_2 \end{pmatrix}$

$x_1 = 1 \quad \therefore \quad -2 + 2x_2 = -2 + 2i \quad \therefore \quad x_2 = i$

$V_1 = \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} i$

$y = c_1 e^{-2t}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(2t) - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(2t)\right]$
$+ c_2 e^{-2t}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin(2t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos(2t)\right]$

$$\boxed{\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} c_1 e^{-2t} \cos(2t) + c_2 e^{-2t} \sin(2t) \\ -c_1 e^{-2t} \sin(2t) + c_2 e^{-2t} \cos(2t) \end{pmatrix}}$$

4.4.7) $y' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \Rightarrow \lambda^2 - 2\lambda - 3 \Rightarrow (\lambda-3)(\lambda+1)$

$\lambda_1 = 3$, $\lambda_2 = -1$

$\therefore$ $\boxed{\text{Unstable saddle point}}$

$AV_1 = \lambda_1 V_1 \Rightarrow \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3x_1 \\ 3x_2 \end{pmatrix}$ $\quad x_2 = 1 \therefore x_1 = 1$

$V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$AV_2 = \lambda_2 V_2 \Rightarrow \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_1 \\ -x_2 \end{pmatrix}$ $\quad x_2 = 1 \therefore x_1 = -1$

$V_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$y = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t}$

$$\boxed{\begin{aligned} y_1 &= c_1 e^{3t} - c_2 e^{-t} \\ y_2 &= c_1 e^{3t} + c_2 e^{-t} \end{aligned}}$$

4.4.11) $y'' + 2y' + 2y = 0$

$\lambda^2 + 2\lambda + 2 = 0$ $\quad \lambda_{1,2} = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$

$\boxed{y = e^{-t}(c_1 \cos(t) + c_2 \sin(t)), \text{ stable \& attractive spiral point}}$

4.5.4) $y_1' = 4y_1 - y_1^2$

$y_2' = y_2$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 4y_1 - y_1^2 \\ y_2 \end{pmatrix} \quad \therefore \quad y_2 = 0, \; y_1(4-y_1) = 0, \; y_1 = 0, 4$$

$$\boxed{\text{Critical points: } \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \end{pmatrix}}$$

$$A = \left. \begin{pmatrix} \frac{\partial y_1'}{\partial y_1} & \frac{\partial y_1'}{\partial y_2} \\ \frac{\partial y_2'}{\partial y_1} & \frac{\partial y_2'}{\partial y_2} \end{pmatrix} \right|_{x_e} = \begin{pmatrix} 4 - 2y_1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A_1 = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -4 & 0 \\ 0 & 1 \end{pmatrix}$$

$\lambda_1 = 4, \lambda_2 = 1$ (diagonal matrix) $\qquad \lambda_1 = -4, \lambda_2 = 1$ (diagonal matrix)

$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ - Unstable Node

$\begin{pmatrix} 4 \\ 0 \end{pmatrix}$ - Unstable saddle point

4.5.7

$y_1' = -y_1 + y_2 - y_2^2$
$y_2' = -y_1 - y_2$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -y_1 + y_2 - y_2^2 \\ -y_1 - y_2 \end{pmatrix} \quad \therefore \quad y_1 = -y_2$$

$\therefore \quad 2y_2 - y_2^2 = 0 \qquad y_2(2 - y_2) = 0 \qquad y_2 = 0, 2$

$\therefore \quad y_1 = 0, -2$

$$\boxed{\text{Critical points: } \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \end{pmatrix}}$$

$$A = \left.\begin{pmatrix} -1 & 1-2y_2 \\ -1 & -1 \end{pmatrix}\right|_{x_c}$$

$$A_1 = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} \qquad A_2 = \begin{pmatrix} -1 & -3 \\ -1 & -1 \end{pmatrix}$$

$|A_1 - \lambda I| = 0 = \lambda^2 + 2\lambda + 2 \qquad \lambda_{1,2} = \dfrac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$

$$\boxed{\begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ is a stable \& attractive spical point}}$$

$|A_2 - \lambda I| = 0 = \lambda^2 + 2\lambda - 2 = \dfrac{-2 \pm \sqrt{4+8}}{2} \quad \therefore \lambda_{1,2} = -1 \pm \sqrt{3}$

$$\boxed{\begin{pmatrix} -2 \\ 2 \end{pmatrix} \text{ is unstable saddle point}}$$

4.5.11) $y'' + \cos(y) = 0$

$y_1 = y \qquad y_2 = y'$

$y_1' = y_2 \qquad y_2' = -\cos(y_1)$

$$y' = \begin{pmatrix} y_2 \\ -\cos(y_1) \end{pmatrix}$$

$$y' = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} y_2 \\ -\cos(y_1) \end{pmatrix} \quad \therefore\ y_2 = 0,\ y_1 = \frac{\pi}{2}, -\frac{\pi}{2}$$

$$\boxed{\text{Critical points: } \begin{pmatrix} \frac{\pi}{2} \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{\pi}{2} \\ 0 \end{pmatrix}}$$

$$A = \begin{pmatrix} 0 & 1 \\ \sin(y_1) & 0 \end{pmatrix}\Bigg|_{x_e} \quad \therefore\ A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ A_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$|A_1 - \lambda I| = 0 = \lambda^2 - 1 \Rightarrow \lambda_{1,2} = \pm 1$$

$$\therefore\ \boxed{\begin{pmatrix} \pi/2 \\ 0 \end{pmatrix} \text{ is an unstable saddle point}}$$

$$|A_2 - \lambda I| = 0 = \lambda^2 + 1 \Rightarrow \lambda_{1,2} = \pm i$$

$$\boxed{\begin{pmatrix} -\pi/2 \\ 0 \end{pmatrix} \text{ is a stable center}}$$