"A mathematical introduction to robotic manipulation"

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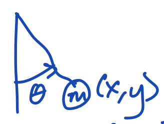
Goal: To provide a coordinate free description of RB motion using tools from linear algebra + Matrix groups and geometric mechanics which does not suffer from singularities

→ the study of how the structure of the configuration space can be used to gain deeper insight into the behavior of the system in physical space

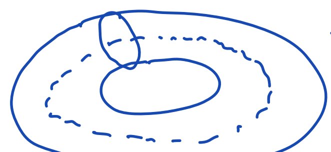
↳ By choosing the right geometry, constraints are intrinsically satisfied.

Ex. Pendulum lives on S^1 ← unit circle



$x^2+y^2=l^2$

Ex. Double pendulum lives on $\mathbb{T}^2 = S^1 \times S^1$



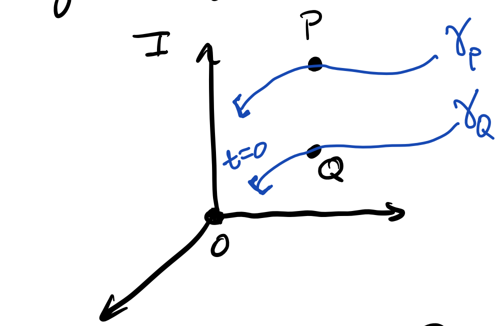
Torus

The geometric approach to kinematics ulitizes Screw theory. (Chasles and Poinsot in early 1800's and Robert Ball in 1900)

Dfn: Screw motion – a movement consisting of rotation about an axis and translation parallel to the axis.

Screws and twists are central to the geometric formulation of R.B. Kinematics. Wrenches provide the forces.

Rigid Body Transformations (Ch 2)



Parametric vector curves

$\vec{r}_{P/O}(t)$ $\gamma_P(t) = (x_P(t), y_P(t), z_P(t)) \in \mathbb{R}^3$

$\gamma_Q(t) = (x_Q(t), y_Q(t), z_Q(t)) \in \mathbb{R}^3$

If P and Q are on the same <u>R.B.</u>,

$$\| \gamma_P(t) - \gamma_Q(t) \| = \| \gamma_P(0) - \gamma_Q(0) \| = \text{constant}$$

In general, a rigid body transformation (RBT) involves both translation and rotation

An <u>R.B.T.</u> describes how points on a body move as a function of time relative to a fixed coordinate system.

An <u>RBT</u> is a mapping $g: \mathbb{R}^3 \to \mathbb{R}^3$ with the properties

1. length is preserved

$$\| g(\vec{r}_{P/O}) - g(\vec{r}_{Q/O}) \| = \| \vec{r}_{P/O} - \vec{r}_{Q/O} \|$$

2. Cross product is preserved (ensures internal reflections are avoided).

$$g(\vec{a} \times \vec{b}) = g(\vec{a}) \times g(\vec{b})$$

3. Inner product is preserved (conseq. of 1 & 2)

$$g(\vec{a} \cdot \vec{b}) = g(\vec{a}) \cdot g(\vec{b})$$

Implication: RBT's map orthonormal frames to orthonormal frames.

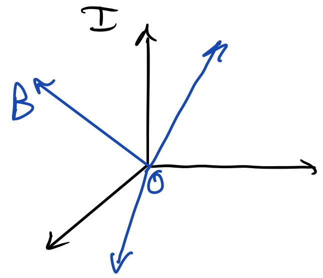
Dfn: The <u>configuration</u> of a R.B. is the <u>position</u> and <u>orientation</u> of the body frame wrt an inertial frame.

Under RBT, $B = (O', \hat{b}_1, \hat{b}_2, \hat{b}_3)$ at $\vec{r}_{O'/O}$ becomes

$$g(B) = \left(g(O'), g(\hat{b}_1), g(\hat{b}_2), g(\hat{b}_3)\right) \text{ at } g(\vec{r}_{O'/O})$$

$$\begin{array}{c|ccc} & \hat{b}_1 & \hat{b}_2 & \hat{b}_3 \\ \hline \hat{e}_1 & \langle \hat{e}_1, \hat{b}_1 \rangle & & \ddots \\ \hat{e}_2 & \langle \hat{e}_2, \hat{b}_1 \rangle & & \\ \hat{e}_3 & \langle \hat{e}_3, \hat{b}_1 \rangle & & \end{array}$$

Rotational motion in $\mathbb{R}^3$



Rotation matrix

$$[{}^{I}C^{B}] = \left[[\hat{b}_1]_I \ [\hat{b}_2]_I \ [\hat{b}_3]_I \right]$$

This serves as an RBT from B to I.

<u>Rotation matrices</u>

Let $R \in \mathbb{R}^{3\times3}$ be a rotation matrix and $\vec{r}_1, \vec{r}_2, \vec{r}_3 \in \mathbb{R}^3$ be its columns.

$$\vec{r}_i \cdot \vec{r}_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

$$\Longrightarrow [\vec{r}_1 \ \vec{r}_2 \ \vec{r}_3]^T [\vec{r}_1 \ \vec{r}_2 \ \vec{r}_3] = I$$

$$R^T R = R R^T = I$$

$$\det(R) = \vec{r}_1 \cdot (\underbrace{\vec{r}_2 \times \vec{r}_3}) = \vec{r}_1 \cdot \underbrace{\vec{r}_1}_{w} = 1$$

Right-handed frame

The set of all 3x3 matrices satisfying these properties is denoted by SO(3) "special orthogonal" matrices of size 3x3.

In general,

SO(n) is a group of special orthogonal matrices of size n×n.

Lie Group $SO(3) \triangleq \{ R \in \mathbb{R}^{3\times3} \mid RR^T = I \text{ and } \det(R) = 1 \}$

SO(3) is a "group" under the operation of matrix multiplication. SO(3) is sometimes called the rotation group.

Dfn: A ~~set~~ set G together with a binary operation $\circ$ defined on elements of G is called a group if

1. Closure $g_1, g_2 \in G \Rightarrow g_1 \circ g_2 \in G$
2. identity $\exists\, e \in G$ s.t. $g \circ e = e \circ g = g \quad \forall g \in G$
3. Inverse $\exists\, g^{-1} \in G$ s.t. $g \circ g^{-1} = g^{-1} \circ g = e$
4. Associativity. $(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$

Show that SO(3) is a Lie group.

1. Closure $R_1, R_2 \in SO(3)$ is $R_1 R_2 \overset{?}{\in} SO(3)$

$$\underset{(3\times3)(3\times3)}{R_1 R_2} = (3\times3) \checkmark$$

$$(R_1 R_2)^T (R_1 R_2) = R_2^T (R_1^T R_1) R_2 = R_2^T (I) R_2 = I \checkmark$$

$$\det(R_1 R_2) = \det(R_1)\det(R_2) = 1 \checkmark$$

② Identity: Identity matrix I

③ Inverse: Matrix inverse $R^{-1}R = I$

Need to show $R^{-1} \in SO(3)$

$R \in SO(3) \Rightarrow RR^T = I \Rightarrow R^{-1} = R^T \in SO(3)$

④ Associativity

$$(R_1 R_2) R_3 = R_1 (R_2 R_3)$$

$\Rightarrow SO(3)$ is a <u>group</u>

The rotation group $SO(3)$ is the configuration space of a RB free to rotate but not translate $\Rightarrow$ The "trajectory" is a curve $R(t) \in SO(3)$

Rotations are R.B.T.

① ${}^{I}C^{B}$ preserves distance

$$\| \vec{r}_{P/Q} \| \stackrel{?}{=} \| {}^{I}C^{B} [\vec{r}_{P/O}]_B - {}^{I}C^{B} [\vec{r}_{Q/O}]_B \|$$

② ${}^{I}C^{B}$ preserve relative orientation

$${}^{I}C^{B} [\vec{a} \times \vec{b}]_B \stackrel{?}{=} {}^{I}C^{B} [\vec{a}]_B \times {}^{I}C^{B} [\vec{b}]_B$$

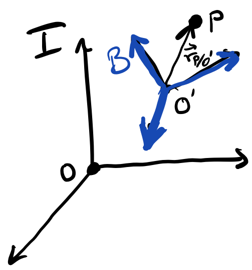
③ ${}^{I}C^{B}$ preserves the inner product

$$[\vec{a}]_B \cdot [\vec{b}]_B \stackrel{?}{=} ({}^{I}C^{B} [\vec{a}]_B) \cdot ({}^{I}C^{B} [\vec{b}]_B)$$

$$[\vec{a}]_B^T \underbrace{({}^{I}C^{B})^T ({}^{I}C^{B})}_{= I \text{ since } {}^{I}C^{B} \in SO(3)} [\vec{b}]_B \checkmark$$

Exponential coordinates for rotation

We want to describe a rotation $R \in SO(3)$ in terms of $\omega \in \mathbb{R}^3$ a unit vector for the direction of rotation and $\theta \in \mathbb{R}$ an angle of rotation



Transport eqn:

$$\frac{{}^{I}d}{dt}(\vec{r}_{P/O'}) = \underbrace{\frac{{}^{B}d}{dt}(\vec{r}_{P/O'})}_{=0} + {}^{I}\vec{\omega}^{B} \times \vec{r}_{P/O'}$$

$$\frac{{}^{I}d}{dt}(\vec{r}_{P/O'}) = {}^{I}\vec{\omega}^{B} \times \vec{r}_{P/O'}$$

$$\frac{{}^{I}d}{dt}\left([\vec{r}_{P/O'}]_B\right) = [{}^{I}\vec{\omega}^{B} \times]_B [\vec{r}_{P/O'}]_B$$

Change notation

$$[{}^{I}\vec{\omega}^{B} \times] = \hat{\omega}$$

$$\begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

Be careful of notation, since we have been using hat to denote unit vectors, but here it means

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MLS use the hat (or wedge) $\wedge$ notation to mean even more than cross product equivalent. In fact, they use it to mean "take array elements and form an element of the corresponding Lie algebra." We will use this later for twists.

Let ${}^{I}\vec{\omega}^{B} = \omega_1 \hat{b}_1 + \omega_2 \hat{b}_2 + \omega_3 \hat{b}_3$

$$[{}^{I}\vec{\omega}^{B} \times] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

and

$$\frac{d}{dt}[\vec{r}_{P/O'}]_B = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} [\vec{r}_{P/O'}]_B$$

From this, we can see that for a constant ${}^{I}\vec{\omega}^{B}$, we have a Linear Time-Invariant (LTI) differential equation.

$$\dot{\underline{x}} = A\underline{x} \quad \text{with} \quad \underline{x}(0) = x_0$$

Solution: $\underline{x}(t) = e^{At}\underline{x}_0$

Plugging in,

$$[\vec{r}_{P/O'}(t)]_B = \underbrace{e^{\hat{\omega}t}}_{\text{matrix exponential}} [\vec{r}_{P/O'}(0)]_B$$

exp

expm

$$e^{At} \triangleq I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \dots$$

Key idea: If we rotate with angular velocity ω for $t = \theta$ units of time, then the net rotation is $R(\omega, \theta)$

$$\underbrace{R(\omega,\theta)}_{\substack{\text{element of } SO(3) \\ \text{(Lie Group)}}} = e^{\hat{\omega}\theta}$$

element of little so(3) (Lie algebra) "Infinitesimal Generator"

The vector space of all 3x3 skew-symmetric matrices is called "little so(3)" and denoted so(3).

↳ Subcase of

$$so(n) = \{ S \in \mathbb{R}^{n \times n} \mid S = -S^T \} \qquad S + S^T = 0$$

- Little so(3) is a Lie Algebra that corresponds to the Lie Group "Big SO(3)" or $SO(3)$.
- The map between the Lie Algebra and the Lie Group is the matrix exponential

"subset of"

- $so(3) \subset \mathbb{R}^{3\times 3}$ is a vector space b/c the sum of 2 elements of so(3) is in so(3) and a scalar multiple of an element of so(3) is also in so(3).

- so(3) can be identified with $\mathbb{R}^3$ using

$$\hat{a} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \qquad \text{Note: MLS use } [\vec{a}] = a$$

- Some properties of little so(3):

$$\hat{a}^2 = aa^T - \|a\|^2 I$$
$$\hat{a}^3 = -\|a\|^2 \hat{a}$$
$$\vdots$$

Suppose $\|\omega\| = 1$, and look at the matrix exponential:

$$e^{\hat{\omega}\theta} = I + \hat{\omega}\theta + \frac{\hat{\omega}^2}{2!}\theta^2 + \frac{\hat{\omega}^3}{3!}\theta^3 + \dots$$
$$= I + \left(\theta - \frac{\theta^3}{3!} + \dots\right)\hat{\omega} + \left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \dots\right)\hat{\omega}^2$$

Recall,
$$\sin x = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \dots$$
$$\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \dots$$

This results in Rodrigues' Formula:

$$\boxed{e^{\hat{\omega}\theta} = I + \sin\theta\,\hat{\omega} + (1 - \cos\theta)\hat{\omega}^2}$$

Notes:

- No infinite series present
- Use this when you want the matrix exponential of a skew symmetric matrix
- There is also a version for when $\|\omega\| \neq 1$

$$e^{\hat{\omega}\theta} = I + \frac{\hat{\omega}}{\|\omega\|}\sin(\|\omega\|\theta) + \frac{\hat{\omega}^2}{\|\omega\|^2}\left(1 - \cos(\|\omega\|\theta)\right)$$

Using Rodrigues' formula,

$$(e^{\hat{\omega}\theta})^{-1} = e^{-\hat{\omega}\theta} = e^{\hat{\omega}^T\theta} = (e^{\hat{\omega}\theta})^T$$

$\Rightarrow e^{\hat{\omega}\theta}$ is orthogonal.

A continuity argument can be used to show

$$\det(e^{\hat{\omega}\theta}) = +1$$

$$\Rightarrow e^{\hat{\omega}\theta} \in SO(3)$$

Geometrically, the skew sym. matrix $\hat{\omega}$ corresponds to an <u>axis of rotation</u>

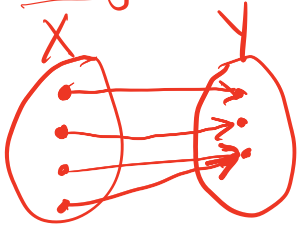
The exponential map creates rotation about the ω axis by an angle of θ.

Prop 2.5: The map $\exp: so(3) \longrightarrow SO(3)$ is surjective (onto). Given $R \in SO(3)$ $\exists\ \omega \in \mathbb{R}^3$ and $\theta \in \mathbb{R}$ s.t.

$$R = \exp(\hat{\omega}\theta).$$

Aside:

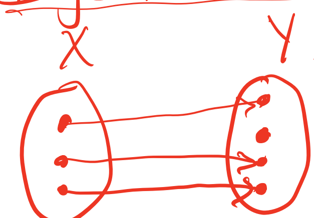
Surjective "Onto"



Surjective but not injective

Each element of codomain Y maps from at least one element of the domain X

Injective "one-to-one"



Injective but not surjective

Distinct elements of domain X map to distinct elements of codomain Y.

Key idea: The exp map transforms skew-sym matrices into special orthogonal ones and the result is that we can create exponential coordinates.

Let $R \in SO(3)$

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = e^{\hat{\omega}\theta}$$

We could use Rodrigues' formula,

trace

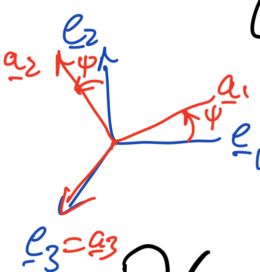
$$\theta = \cos^{-1}\left(\frac{tr(R) - 1}{2}\right) \text{ and } \omega = \frac{1}{2\sin\theta}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

The exponential coordinates are the components of the $\omega\theta$ vector

Recall: Euler's theorem says that any orientation $R \in SO(3)$ is equivalent to the rotation about a unit vector $\omega \in \mathbb{R}^3$ through an angle of θ.

Relation to Euler Angles

Consider 3-2-3 (ψ, θ, ϕ)



$${}^IC^A \qquad {}^AC^B \qquad {}^BC^C$$

This one is wedge

$$R(\underline{e}_3, \psi) = e^{\hat{e}_3 \psi}$$

$$[\underline{e}_3]_I = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}_I \qquad \hat{e}_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Using Rodrigues' formula

$$e^{\hat{\omega}\theta} = I + \sin\theta\, \hat{\omega} + (1-\cos\theta)\, \hat{\omega}^2$$

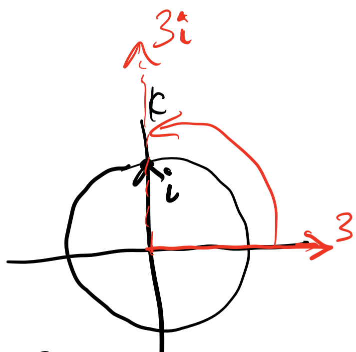
$$e^{\hat{e}_3 \psi} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \sin\psi \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + (1-\cos\psi) \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} c\psi & -s\psi & 0 \\ s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} = {}^IC^A$$

$$R(\underline{a}_2, \theta) = e^{\hat{a}_2 \theta} = \begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix} = {}^AC^B$$

$$R(\underline{b}_3, \phi) = e^{\hat{b}_3 \phi} = \begin{bmatrix} c\phi & -s\phi & 0 \\ s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} = {}^BC^C$$

Quaternions: Generalize complex numbers and can be used to represent 3D rotations just like complex numbers on the unit circle represent planar rotations.



Planar case

Unlike Euler angles, quaternions give a global parametrization of SO(3) at the cost of using 4 numbers (and 1 constraint) instead of 3.

$$Q = q_0 + q_1 \hat{i} + q_2 \hat{j} + q_3 \hat{k} \qquad q_i \in \mathbb{R}$$

q_0 is a scalar component

$\vec{q} = q_1 \hat{i} + q_2 \hat{j} + q_3 \hat{k}$ vector component

$Q = (q_0, \vec{q})$

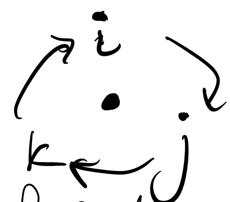
4 dimensional vector space, and a group w.r.t. quaternion multiplication.

Properties of Quat. multiplication

$a\hat{i} = \hat{i}a \qquad a\hat{j} = \hat{j}a \qquad a\hat{k} = \hat{k}a$

$\hat{i}\cdot\hat{i} = j\cdot j = k\cdot k = -1$

$\left\{ i\cdot j = -j\cdot i = k \quad j\cdot k = -k\cdot j = i \quad k\cdot i = -i\cdot k = j \right.$



For quaternion multiplication

Other Quaternion Properties

$\vec{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$

Conjugate: $Q^* = (q_0, -\vec{q})$

Magnitude: $\|Q\| = \sqrt{Q\cdot Q^*} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$

Inverse: $Q^{-1} = \dfrac{Q^*}{\|Q\|^2}$

Identity: $Q = (1, \vec{0})$

Product: $Q\cdot P = (q_0 p_0 - \vec{q}\cdot\vec{p},\ q_0\vec{p} + p_0\vec{q} + \vec{q}\times\vec{p})$

Vector form of Quaternions

Complex form: $Q\cdot P = (q_0 + q_1 i + q_2 j + q_3 k)\cdot(p_0 + p_1 i + p_2 j + p_3 k)$

$= q_0 p_0 + q_0(p_1 i + p_2 j + p_3 k) + p_0(q_1 i + q_2 j + q_3 k)$

$+ q_1 i(p_1 i + p_2 j + p_3 k) + \ldots$

We can convert between exponential coordinates and quaternions

Exp coords $\rightarrow$ Quat.

Given $R = e^{\hat{\omega}\theta}$

The unit quaternion is

$$Q = \left(\cos\frac{\theta}{2}, \omega \sin\left(\frac{\theta}{2}\right)\right)$$

Quat $\rightarrow$ Exp coords.

Given $Q = (q_0, \vec{q})$

$$\theta = 2\cos^{-1}(q_0)$$

$$\omega = \begin{cases} \dfrac{\vec{q}}{\sin\left(\frac{\theta}{2}\right)} & \text{if } \theta \neq 0 \\ 0 & \text{otherwise} \end{cases}$$