MA 527

Lecture Notes (section 7.7 & 7.8)

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7.7. Determinants : $A_{n\times n}$.

<u>Def</u> $A = [a_{ij}]_{n\times n}$ (~~n ≥ 4~~)

$\det A = |A| = \sum_{p} \operatorname{sign}(p)\, a_{1p(1)} a_{2p(2)} \cdots a_{np(n)}$

$n=2$: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$: $\det A = ad - bc$

$n=3$:

$n \geq 1$: <u>Cofactor formula</u>

Q $A \to B$: row-equivalent.

$\det B \overset{?}{=} \det A$.

(row op 1). $A = \begin{bmatrix} A_1 \\ \vdots \\ A_n \end{bmatrix}_{n\times n}$ $A_i \leftrightarrow A_j$ $\longrightarrow B$

$\det B = (-1) \det A$.

(row op 2)

$A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \xrightarrow{r_2 - 3r_1} \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} = B$: $\det B = -1$

$\det A = -1$,

row op 2 : addition of a scalar multiple of a row and another row

$A \to B$: $\underline{\det B = \det A}$.

(row op 3)

$$A \longrightarrow B = \begin{bmatrix} A_1 \\ \vdots \\ kA_i \\ \vdots \\ A_n \end{bmatrix} : \underline{\det B = k \det A}.$$

(Cofactors)

$$A = [a_{ij}]_{n\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \cdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$\det A = a_{k1}C_{k1} + a_{k2}C_{k2} + \cdots + a_{kn}C_{kn}$$

$$= a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

: Cofactor formula.

$C_{ij} = (-1)^{i+j} \det A_{ij}$: $(i,j)^{th}$ cofactor.

A_{ij} : the submatrix obtained by removing the i^{th} row & the j^{th} column.

(Ex) $A = \begin{bmatrix} 0 & 3 & 5 \\ 3 & 5 & 0 \\ 5 & 0 & 10 \end{bmatrix}$

$$\det A = 0\cdot C_{11} + 3C_{12} + 5C_{13}$$

$$= 0 + 3\cdot(-1)^3 \det\begin{bmatrix} 3 & 0 \\ 5 & 10 \end{bmatrix} + 5(-1)^4 \det\begin{bmatrix} 3 & 5 \\ 5 & 0 \end{bmatrix}$$

$$= (-3)\cdot 30 + 5\cdot(0-25) = -215.$$

(Ex) (1) $\det \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} = A$: triangular.

$= 1 \cdot C_{11} + 0 + 0 = (-1)^2 \begin{vmatrix} 4 & 5 \\ 0 & 6 \end{vmatrix} = 24.$

$\det A = 1 \cdot 4 \cdot 6 = 24.$

(2) $\det \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 10 & 1 & 2 & 3 \\ 0 & 0 & 0 & 4 & 10 \\ 0 & 0 & 0 & 20 & 9 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} = 0$

7.8 Inverse matrix.

<u>Def</u> $A = [a_{ij}]_{n \times n}$

If $AB = I$ and $BA = I$ for a matrix B, B is called the inverse matrix of A.

Write $\underline{A^{-1} = B}$

Q Compute A^{-1}?

(Ex) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$: $\det A = ad - bc \neq 0$

(1) $A^{-1} = \dfrac{1}{\det A} \boxed{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}$: adjoint matrix of A.

$$A^{-1}A = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \frac{1}{\det A}\begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = I.$$

$n \geq 3.$: ?

<u>Thm 2</u> (Inverse matrix).

$A = [a_{ij}]_{n\times n}$: nonsingular. ($\det A \neq 0$)

(2) $n \geq 3$

$$A^{-1} = \frac{1}{\det A}\boxed{\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ & \cdots & & \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}^T}$$ = the adjoint matrix of A.

$$C_{ij} = (-1)^{i+j}\det A_{ij} :$$ the (i,j)th cofactor.

(Ex) $A = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 2 & 0 \end{bmatrix}$: $\det A = 8\begin{vmatrix} 0 & 4 \\ 2 & 0 \end{vmatrix} = -64$

$C_{11} = (-1)^2\begin{vmatrix} 0 & 4 \\ 2 & 0 \end{vmatrix} = -8,\quad C_{12} = (-1)^3\begin{vmatrix} 0 & 4 \\ 0 & 0 \end{vmatrix} = 0$

$C_{13} = (-1)^4\begin{vmatrix} 0 & 0 \\ 0 & 2 \end{vmatrix} = 0, \quad \cdots$

$$A^{-1} = \frac{1}{\cancel{-64}}\begin{bmatrix} \cancel{-8}\ {+\tfrac{1}{8}} & 0 & 0 \\ 0 & 0 & \cancel{-16}\ \tfrac{1}{4} \\ 0 & \cancel{-32}\ \tfrac{1}{2} & 0 \end{bmatrix}^T = \begin{bmatrix} +\frac{1}{8} & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{4} & 0 \end{bmatrix}$$

(Properties) A, B, C: nonsingular

(1) $(AB)^{-1} = B^{-1}A^{-1}$

(Proof) $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AA^{-1} = I$.

∘ $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

(2) $AX = b$: $\underline{X = A^{-1}b}$

(3) $\det(AB) = \det A \cdot \det B$.

<u>Remark</u>: $AB \neq BA$ in general.

(4) $(A^T)^{-1} = (A^{-1})^T$

(Proof) $A^T(A^{-1})^T = (A^{-1}A)^T = I$

(Gauss-Jordan method) $A_{n \times n}$.

A^{-1} : expensive.

(Ex) $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$: $A^{-1} = \frac{1}{5-6}\begin{bmatrix} 5 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$

(idea)

Use GE. : Compute A^{-1}. : $\underline{AA^{-1} = I}$

Solve "$AX = I$"

(Motivation)

$$\underline{GE: \quad \#(+/\cdot) \approx \frac{n^3}{3}.} \quad : \quad \underline{AX = I}$$

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 5 & 0 & 1 \end{array}\right] \xrightarrow{r_2 - 3r_1} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -1 & -3 & 1 \end{array}\right]$$

$$\xrightarrow{(-1)r_2} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 3 & -1 \end{array}\right] \xrightarrow{r_1 - 2r_2} \left[\begin{array}{cc|cc} 1 & 0 & -5 & 2 \\ 0 & 1 & 3 & -1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}.$$