6.6. DE with variable coefficients

$L(t f(t)) = -\frac{d}{ds} L(f(t))$

∘ $L^{-1}(F'(s)) = -t L^{-1}(F)$: $L^{-1}(F) = \frac{1}{-t} L^{-1}(F'(s))$

(Ex) $G(s) = \frac{s}{(s^2+16)^2}$: $L^{-1}(G) = ?$

(1) $*$: $L^{-1}(G) = L^{-1}\left(\frac{s}{s^2+16} \cdot \frac{1}{s^2+16}\right)$

$= L^{-1}\left(\frac{s}{s^2+4^2}\right) * L^{-1}\left(\frac{1 \cdot 4}{s^2+4^2}\right)\frac{1}{4}$

$= \cos(4t) * \left(\frac{1}{4}\sin(4t)\right)$

(2) Let $G(s) = H'(s)$

$H(s) = \int G(s)\,ds = \int \frac{s}{(s^2+16)^2}\,ds$ $\quad z = s^2+16$, $dz = 2s\,ds$

$= \int \underbrace{\frac{1}{z^2}}_{z^{-2}} \cdot \frac{1}{2}\,dz = \frac{1}{2}\frac{z^{-1}}{-1} + C = -\frac{1}{2z} + C.$

Let $C = 0$: $H(s) = -\frac{1}{2(s^2+16)}$

$L^{-1}(G) = L^{-1}(H'(s)) = -t L^{-1}(H)$

$= (-t)\left(-\frac{1}{2}\right) L^{-1}\left(\frac{1 \cdot 4}{s^2+4^2}\right)\frac{1}{4} = \frac{t}{8}\sin(4t).$

(1) $L^{-1}(G) = \frac{1}{4}\int_0^t \cos(4\tau)\sin(4(t-\tau))\,d\tau.$

$= \cdots = \frac{t}{8}\sin(4t).$

Q $L^{-1}(\ln(s^2+1)) = ?$

$\frac{d}{ds}\ln(s^2+1) = \frac{1}{s^2+1} \cdot \frac{d}{ds}(s^2+1) = \frac{2s}{s^2+1}$

$L^{-1}(\ln(s^2+1)) = \frac{1}{-t} L^{-1}\left(\frac{d}{ds}\ln(s^2+1)\right)$

$= -\frac{1}{t} L^{-1}\left(\frac{2s}{s^2+1}\right) = -\frac{2}{t} L^{-1}\left(\frac{s}{s^2+1}\right)$

$= -\frac{2}{t}\cos(t).$

6.7 System of DEs.

Q $\begin{cases} y_1'' = y_1 + 3y_2, & y_1(0)=0,\ y_1'(0)=3 \\ y_2'' = 4y_1, & y_2(0)=0,\ y_2'(0)=-4 \end{cases}$

1. Let $z_1 = y_1,\ z_2 = y_1',\ z_3 = y_2,\ z_4 = y_2'$

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}' = \begin{bmatrix} z_2 \\ z_1 + 3z_3 \\ z_4 \\ 4z_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 4 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}$$

Eigenvalues / Eigenvectors.

2: Laplace transform.

(Ex) $\begin{cases} y_1' = -y_2, & y_1(0)=1 \\ y_2' = -y_1, & y_2(0)=0 \end{cases}$

L:

$L(y_1') = -L(y_2):\quad sL(y_1) - \underbrace{y_1(0)}_{=1} = -L(y_2)$

$L(y_2') = -L(y_1):\quad sL(y_2) - \underbrace{y_2(0)}_{=0} = -L(y_1)$

$sL(y_1) + L(y_2) = 1 \quad -①$

$L(y_1) + sL(y_2) = 0 \quad -②$

$s① - ②:$

$$\begin{aligned} s^2L(y_1) + sL(y_2) &= s \\ -\ \big(L(y_1) + sL(y_2) &= 0\big) \\ \hline (s^2-1)L(y_1) &= s \end{aligned}$$

$L(y_1) = \dfrac{s}{s^2-1}:\quad \underline{y_1(t) = \cosh(t)}$

$L(y_2) = ?\quad ②:\ sL(y_2) = -L(y_1) = -\dfrac{s}{s^2-1}$

$L(y_2) = -\dfrac{1}{s^2-1}:\quad \underline{y_2(t) = -\sinh(t)}$

(Ex) $\begin{cases} y_1'' = y_1 + 3y_2, & y_1(0)=0,\ y_1'(0)=3 \\ y_2'' = 4y_1 & y_2(0)=0,\ y_2'(0)=-4 \end{cases}$

L:

$L(y_1'') = L(y_1) + 3L(y_2)$

$s^2L(y_1) - s\underbrace{y_1(0)}_{=0} - \underbrace{y_1'(0)}_{=3} = L(y_1) + 3L(y_2)$

$$(s^2-1)L(y_1) - 3L(y_2) = 3 \quad -①$$

$$L(y_2'') = 4L(y_1) : \quad \underline{s^2 L(y_2)} - s\,\underbrace{y_2(0)}_{=0} - \underbrace{y_2'(0)}_{=4}$$
$$= 4L(y_1)$$

$$4L(y_1) - s^2 L(y_2) = 4 \quad -②$$

$s^2 ① - 3 ②$:

$$s^2(s^2-1)L(y_1) - 3s^2 L(y_2) = 3s^2$$
$$-)\ 12L(y_1) \quad - 3s^2 L(y_2) = 12$$
$$(s^4 - s^2 - 12)L(y_1) = 3s^2 - 12$$

$$L(y_1) = \frac{3(s^2-4)}{s^4-s^2-12} = \frac{3\cancel{(s^2-4)}}{\cancel{(s^2-4)}(s^2+3)}$$

$$L(y_1) = \frac{3}{s^2+3} = \frac{\sqrt{3}\cdot\sqrt{3}}{s^2+\sqrt{3}^2} \quad -③ \quad : y_1(t) = \sqrt{3}\sin(\sqrt{3}t)$$

$③ \to ②$

$$②: \quad 4\cdot\frac{3}{s^2+3} - s^2 L(y_2) = 4$$

$$-s^2 L(y_2) = 4 - \frac{12}{s^2+3}.$$

$$L(y_2) = -\frac{4}{s^2} + \frac{12}{s^2(s^2+3)}$$

$$\frac{12}{s^2(s^2+3)} = \frac{As+B}{s^2} + \frac{Cs+D}{s^2+3}$$: No "s"

$A=0,\ C=0$

$$12 = (As+B)(s^2+3) + (Cs+D)s^2$$

$$12 = As^3 + Bs^2 + 3As + 3B + Cs^3 + Ds^2$$

$$12 = \underbrace{(A+C)}_{=0}s^3 + \underbrace{(B+D)}_{=0}s^2 + \underbrace{3A}_{=0}s + \underbrace{3B}_{=12}$$

$A=0,\ C=0,\ B=4,\ D=-B=-4.$

$$y_2(t) = L^{-1}\left(-\cancel{\frac{4}{s^2}} + \cancel{\frac{4}{s^2}} + \frac{-4}{s^2+3}\right) = -4L^{-1}\left(\frac{1}{s^2+3}\right)$$

$$= -\frac{4}{\sqrt{3}}L^{-1}\left(\frac{\sqrt{3}}{s^2+\sqrt{3}^2}\right) = -\frac{4}{\sqrt{3}}\sin(\sqrt{3}t).$$

Q: No IC : Laplace transform is not applicable.