AAE 532 - Orbit Mechanics

Purdue University

PS9 Solutions

Authors:
Mitchell Dominguez
Tyler Hook
Samantha Ramsey

Email:
doming18@purdue.edu
hook9@purdue.edu
ramsey87@purdue.edu

Instructor:
Kathleen Howell

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Useful Constants

	Axial Rotaional Period (Rev/Day)	Mean Equatorial Radius (km)	Gravitational Parameter $\mu = Gm(km^3/sec^2)$	Semi-major Axis of orbit (km)	Orbital Period (sec)	Eccentricity of Orbit	Inclination of Orbit to Ecliptic (deg)
☉ Sun	0.0394011	695990	132712440017.99	-	-	-	-
☾ Moon	0.0366004	1738.2	4902.8005821478	384400 (around Earth)	2360592 27.32 Earth Days	0.0554	5.16
☾ Mercury	0.0170514	2439.7	22032.080486418	57909101	7600537 87.97 Earth Days	0.20563661	7.00497902
♀ Venus	0.0041149 (Retrograde)	6051.9	324858.59882646	108207284	19413722 224.70 Earth Days	0.00676399	3.39465605
⊕ Earth	1.0027378	6378.1363	398600.4415	149597898	31558205 365.26 Earth Days	0.01673163	0.00001531
♂ Mars	0.9747000	3397	42828.314258067	227944135	59356281 686.99 Earth Days	0.09336511	1.84969142
♃ Jupiter	2.4181573	71492	126712767.8578	778279959	374479305 11.87 Years	0.04853590	1.30439695
♄ Saturn	2.2522053	60268	37940626.061137	1427387908	930115906 29.47 Years	0.05550825	2.48599187
⛢ Uranus	1.3921114 (Retrograde)	25559	5794549.0070719	2870480873	2652503938 84.05 Years	0.04685740	0.77263783
Ψ Neptune	1.4897579	25269	6836534.0638793	4498337290	5203578080 164.89 Years	0.00895439	1.77004347
♇ Pluto	-0.1565620 (Retrograde)	1162	981.600887707	5907150229	7830528509 248.13 Years	0.24885238	17.14001206

- First three columns of the body data are consistent with GMAT 2020a default values, which are mainly from JPL's ephemerides file de405.spk

- The rest of the data are from JPL website(`https://ssd.jpl.nasa.gov/?planet_pos`, retrieved at 09/01/2020)
based on E.M. Standish's "Keplerian Elements for Approximate Positions of the Major Planets"

Problem 1

Problem Statement

The due dates for the problem sets are known to be Fridays at 9:30 am ET. Is the due date listed above correct? What is YOUR age in JD days next Friday on the due date?

Solution

The listed Julian date JD 2460623.1041667 corresponds to November 8, 2024 at 9:30 AM, so the listed due date is correct.

I will be 10276.431 Julian days old when this homework is due.

Problem 2

Problem Statement

There is much discussion currently about a return to the vicinity of the Moon. Consider a Hohmann transfer to the Moon. Assume departure from a 175 km altitude circular Earth parking orbit; include the local gravity field at the Moon.

(a) Determine the Δv and TOF for a Hohmann transfer to the Moon if the spacecraft drops into a circular orbit at the Moon with radius of altitude 120 km. Assume arrival on the near (light) side with respect to Earth.

What are the transfer characteristics: r^-, v^-, γ^-, θ^* at Moon arrival, a, e, r_p, r_a, $\mathbb{P}$, ε?

Phase angle at departure from the parking orbit?

(b) If the spacecraft does not execute the capture maneuver, determine the orbit of the vehicle relative to the Earth, that is, the new orbit.

What are the orbital characteristics, relative the Earth, r^+, v^+, γ^+, θ^* after the lunar encounter?

Also, determine the characteristics a, e, r_p, r_a, $\mathbb{P}$, ε, $\Delta\omega$ in the new orbit

Will the new orbit come close to the Earth? If there is a crew onboard and the lunar capture maneuver cannot be performed, can the crew return to Earth?

(c) Plot these orbits in GMAT OR Matlab. What is the equivalent Δv_{equiv} produced via the lunar flyby in (b)?

Part (a)

A sketch of the Hohmann transfer appears in Figure 1a. The orbit characteristics of the transfer ellipse are

$$
\begin{aligned}
r_p &= R_\oplus + 175 && = \boxed{6553.1363\ km} \\
r_a &= a_☾ && = \boxed{384400\ km} \\
a_t &= \frac{1}{2}(r_p + r_a) && = \boxed{195476.568\ km} \\
e_t &= 1 - \frac{r_p}{a_t} && = \boxed{0.96648} \\
\mathbb{P} &= 2\pi\sqrt{\frac{a_t^3}{\mu_\oplus}} && = \boxed{9.95495\ days} \\
\varepsilon &= -\frac{\mu_\oplus}{2a_t} && = \boxed{-1.01956\ km^2/s^2}
\end{aligned}
$$

As the transfer ellipse is a geocentric orbit, all orbit parameters are computed using the Earth gravitational parameter $\mu_\oplus$. Given the transfer orbit characteristics, the arrival conditions are thus

$$
\begin{aligned}
r^- &= a_☾ && = \boxed{384400\ km} \\
v^- &= \sqrt{2\left(\varepsilon + \frac{\mu_\oplus}{r^-}\right)} && = \boxed{0.18645\ km/s} \\
\gamma^- &= \boxed{0°} \\
\theta^- &= \boxed{180°}
\end{aligned}
$$

Flight path angle and true anomaly at arrival are determined to be 0° and 180°, respectively, by the definition of the Hohmann transfer. The time of flight is half of the period of the transfer ellipse

$$TOF = \frac{\mathbb{P}}{2} = \boxed{4.9775\ days}$$

The phase angle at departure from the parking orbit is calculated assuming that the gravitational parameter of the Moon ($\mu_☾$) is negligible compared to that of the Earth. Thus,

$$\phi = 180° - \sqrt{\frac{\mu_\oplus}{a_☾^3}} = \boxed{114.7261°}$$

With the transfer ellipse is computed, the Δv cost is calculated. The vector diagram for the departure maneuver is sketched in Figure 1a. The geocentric orbital speeds of the circular parking orbit and the transfer ellipse at the maneuver location are

$$
\begin{aligned}
v_c &= \sqrt{\frac{\mu_\oplus}{r_p}} && = 7.79910\ km/s \\
v_{p/\oplus} &= \sqrt{2\left(\varepsilon + \frac{\mu_\oplus}{r_p}\right)} && = 10.9368\ km/s
\end{aligned}
$$

By definition, Δv is the magnitude of the difference between the final and initial velocity vectors. However, since v_c and $v_{p/\oplus}$ are aligned, Δv_1 is found by a simple scalar subtraction

$$\Delta v_1 = v_{p/\oplus} - v_c = 3.1377\ km/s$$

If the initial and final velocity vectors are not aligned, then Δv must be found by taking the magnitude of the ***vector*** *difference!*

The local gravity field of the Moon must be considered when calculating the arrival maneuver. Switching the central body from the Earth to the Moon requires determining the velocity of the spacecraft relative to the Moon at arrival, which is accomplished by leveraging the *vector* relationship

$$\bar{v}_{sc/☾} = \bar{v}_{sc/\oplus} + \bar{v}_{☾/\oplus} \tag{1}$$

where $\bar{v}_{j/i}$ denotes the velocity of body j with respect to body i. The vector diagram at the top of Figure 1a demonstrates this vector relationship, where $\bar{v}_{sc/\oplus} = \bar{v}^-$ and $\bar{v}_{sc/☾} = \bar{v}^-_{\infty/☾}$. In the frame of reference of the diagram, both the Moon and the spacecraft are moving directly left, but the Moon is traveling faster than the spacecraft. Thus, the hyperbolic velocity of the spacecraft with respect to the Moon is directed exactly rightwards. Once again, since the vectors are all aligned, vector operations reduce to scalar ones, and the relevant orbital speeds are found

$$v_{☾/\oplus} = \sqrt{\frac{\mu_\oplus}{a_☾}} \qquad = 1.01830\ km/s$$

$$v^-_{\infty/☾} = v^- - v_{☾/\oplus} \qquad = 0.83186\ km/s$$

As with the phase angle, the orbital speed of the Moon is calculated assuming the mass of the Moon is negligible compared to the Earth. This may seem like an incorrect assumption, but computing the orbital speed of the Moon using $\mu_\oplus + \mu_☾$ results in $v_{☾/\oplus} = 1.0245\ km/s$, which is a difference of roughly 6 m/s. Compared to the orbital maneuvers that are computed in this problem, 6 m/s is relatively small. Additionally, another large assumption is that the Moon's orbit is circular. According to the table of useful constants, the Moon's eccentricity is around 0.055. Considering an eccentric lunar orbit and using $\mu_\oplus + \mu_☾$, the orbital speeds at perigee and apogee are 1.0541 km/s and 0.9973 km/s, which is a difference of around 56 m/s. Thus, assuming negligible mass for the calculation of the Moon's orbit has a much smaller effect than the circular orbit assumption that has already been made.

With the velocity of the spacecraft relative to the Moon determined, the local lunar hyperbolic orbit is considered, as illustrated in Figure 1b. The diagram is drawn in the same orientation as the geocentric view, so $v^-_{\infty/☾}$ is still directed rightwards. Because the Earth is located in the downwards direction relative to the perspective of the sketch, the *near side* arrival corresponds to a hyperbolic orbit where the spacecraft approaches the Moon on the bottom side of the sketch. Yet another vector diagram is required to solve for the arrival Δv. Once again, all of the vectors are aligned, so vector operations reduce to scalar ones. The lunar-centric orbital speed of the circular arrival orbit ($v_{c/☾}$) and hyperbolic speed at perilune ($v_{p/☾}$) are

$$v_{p/☾} = \sqrt{v^2_{\infty/☾} + \frac{2\mu_☾}{r_c}} \qquad = 2.44314\ km/s$$

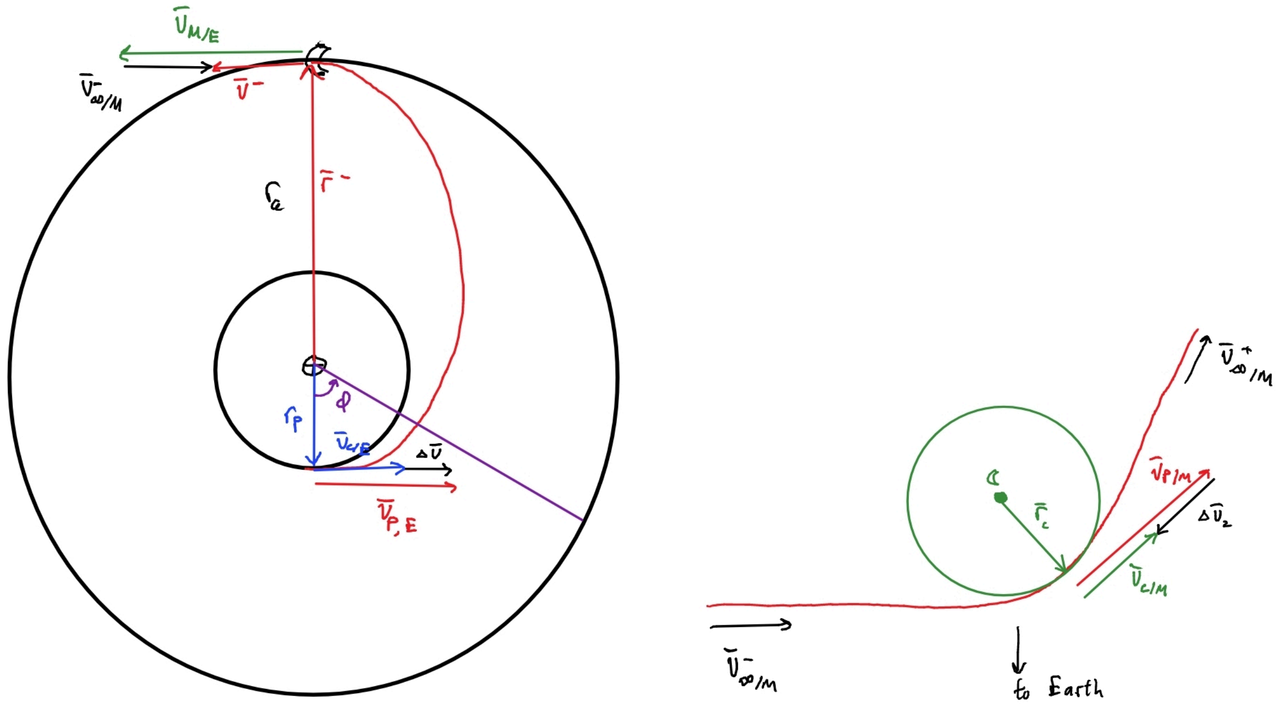
$$v_{c/☾} = \sqrt{\frac{\mu_☾}{r_c}} \qquad = 1.62434\ km/s$$

Finally, the arrival Δv is

$$\Delta v_2 = v_{c/☾} - v_{p/☾} = 0.81880\ km/s$$

and the total Δv expenditure is

$$\Delta v_{tot} = \Delta v_1 + \Delta v_2 = \boxed{3.95646\ km/s}$$



(a) Geocentric view **(b) Lunar-centric view**

Figure 1: Vector diagrams for Part (a)

Part (b)

If the spacecraft does not execute the capture maneuver, then it will fly past the Moon and exit the lunar vicinity on the other leg of the hyperbola. Determining the orbit post-lunar encounter requires the direction of $\bar{v}^+_{\infty/\rightmoon}$. A rough sketch of the scenario is illustrated in Figure 2a. The direction of the exit velocity is dependent on the turn angle δ, which is calculated using

$$\begin{aligned}
\varepsilon_h &= \frac{v_\infty^2}{2} && = 0.34599\ km^2/s^2 \\
|a_h| &= \frac{\mu}{2\varepsilon_h} && = 7085.1176\ km \\
e_h &= \frac{r_c}{|a_h|} + 1 && = 1.26227 \\
\delta &= 2\sin^{-1}\left(\frac{1}{e_h}\right) && = 104.7878°
\end{aligned}$$

The results from this calculation reveal that the original sketch in Figure 2a was not entirely accurate! Since δ is greater than 90°, $\bar{v}^+_{\infty/\rightmoon}$ is actually directed up and to the left. The next vector diagram remedies this situation, as demonstrated in Figure 2b. Now that the exit direction of the spacecraft from the lunar vicinity is determined, the motion is once again represented from a geocentric perspective. The *geocentric* post-lunar encounter velocity is determined by leveraging Equation (1), where $\bar{v}_{sc/\oplus} = \bar{v}^+$ and $\bar{v}_{sc/\rightmoon} = \bar{v}^+_{\infty/\rightmoon}$. This problem is planar, so direct use of vectors is avoided once more by applying trigonometry. The speed v^+ is found by leveraging the law of cosines, such that

$$v^+ = \sqrt{(v^+_{\infty/\rightmoon})^2 + v^2_{\rightmoon/\oplus} - 2v^+_{\infty/\rightmoon}v_{\rightmoon/\oplus}\cos\delta} = \boxed{1.47015\ km/s}$$

The flight path angle is found with the law of sines, where

$$\gamma^+ = \sin^{-1}\left(\frac{v^+_{\infty/\rightmoon}}{v^+}\sin\delta\right) = \boxed{33.16752°}$$

and the post-lunar encounter position is assumed to be identical to the arrival position, or

$$r^+ = r^- = \boxed{384400\ km}$$

Next, true anomaly is found to be

$$\theta^{*,+} = \tan^{-1}\left(\frac{\frac{r^+(v^+)^2}{\mu_\oplus}\cos\gamma^+\sin\gamma^+}{\frac{r^+(v^+)^2}{\mu_\oplus}\cos^2\gamma^+ - 1}\right) = \boxed{64.24632°}$$

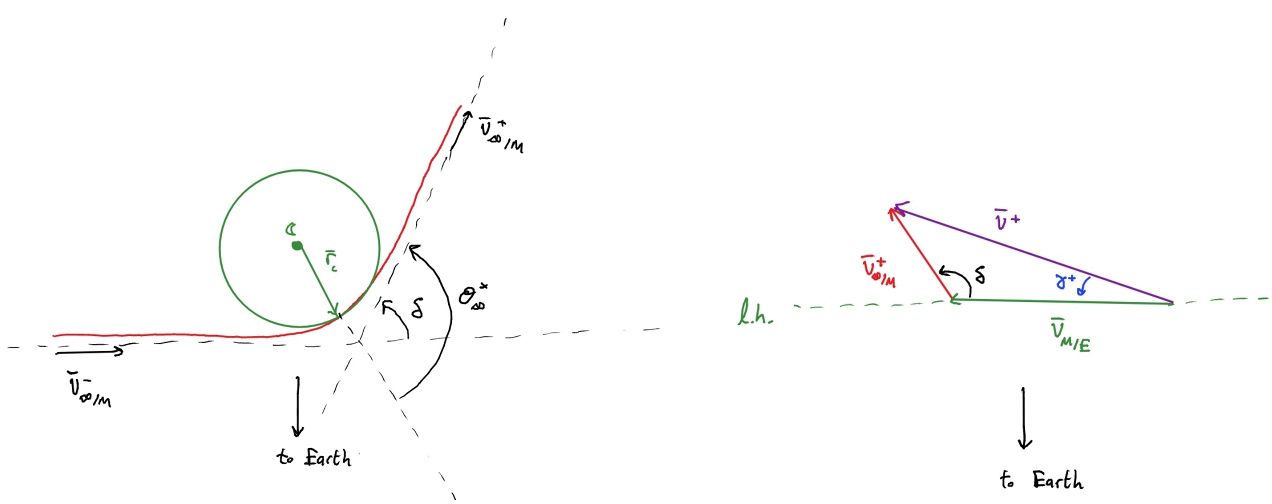
Thus, the change in argument of perigee is

$$\Delta\omega = \theta^{*,-} - \theta^{,+} = \boxed{115.7537°}$$

The remaining orbit characteristics are

$$\begin{aligned}
e^+ &= \sqrt{\left(\frac{r^+(v^+)^2}{\mu_\oplus} - 1\right)^2\cos^2\gamma + \sin^2\gamma} &&= \boxed{1.05981} \\
a^+ &= \frac{-\mu_\oplus}{2\left(\frac{(v^+)^2}{2} - \frac{\mu_\oplus}{r^+}\right)} &&= \boxed{-4557366.02\ km} \\
\varepsilon^+ &= \frac{(v^+)^2}{2} - \frac{\mu_\oplus}{r^+} &&= \boxed{0.04373\ km^2/s^2} \\
r_p &= a^+(1 - e^+) &&= \boxed{272555.778\ km}
\end{aligned}$$

The post-encounter geocentric orbit is hyperbolic! Thus, r_a and $\mathbb{P}$ are undefined. Additionally, it is clear that the spacecraft does **not** return to Earth after the lunar encounter. If the crew cannot perform the lunar capture maneuver, then they are never going to return to Earth ballistically.



(a) Lunar-centric view **(b) Geocentric view**

Figure 2: Vector diagrams for Part (b)

Part (c)

The final vector diagram of Problem 1 is drawn in Figure 3. There are multiple methods to calculate Δv_{eq}, given the information that has been obtained so far. One method is compute the vector components of $\bar{v}^+$ and $\bar{v}^-$ and perform the vector computation

$$\Delta v_{eq} = \left\| \bar{v}^+ - \bar{v}^- \right\|$$

The method used here, is to apply the law of cosines on the isosceles triangle formed by $\bar{v}^+_{\infty/☾}$, $\bar{v}^-_{\infty/☾}$, and $\Delta\bar{v}_{eq}$, such that

$$\Delta v_{eq} = \sqrt{(v^-_{\infty/☾})^2 + (v^+_{\infty/☾})^2 - 2v^-_{\infty/☾}v^+_{\infty/☾}\cos\delta} = \boxed{1.31804\ km}$$

This is a large Δv that is achieved for free! Figure 4 demonstrates the transfer trajectory and post-lunar encounter hyperbolic trajectory.

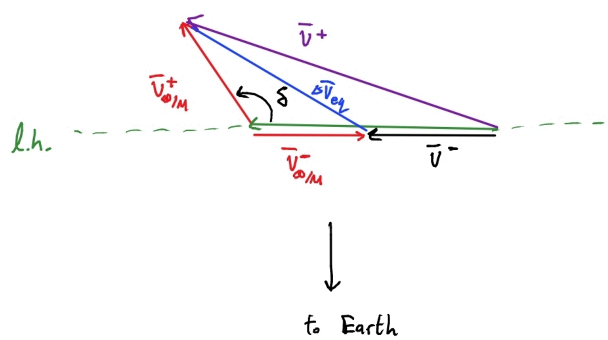


Figure 3: Vector diagram for computing Δv_{eq}

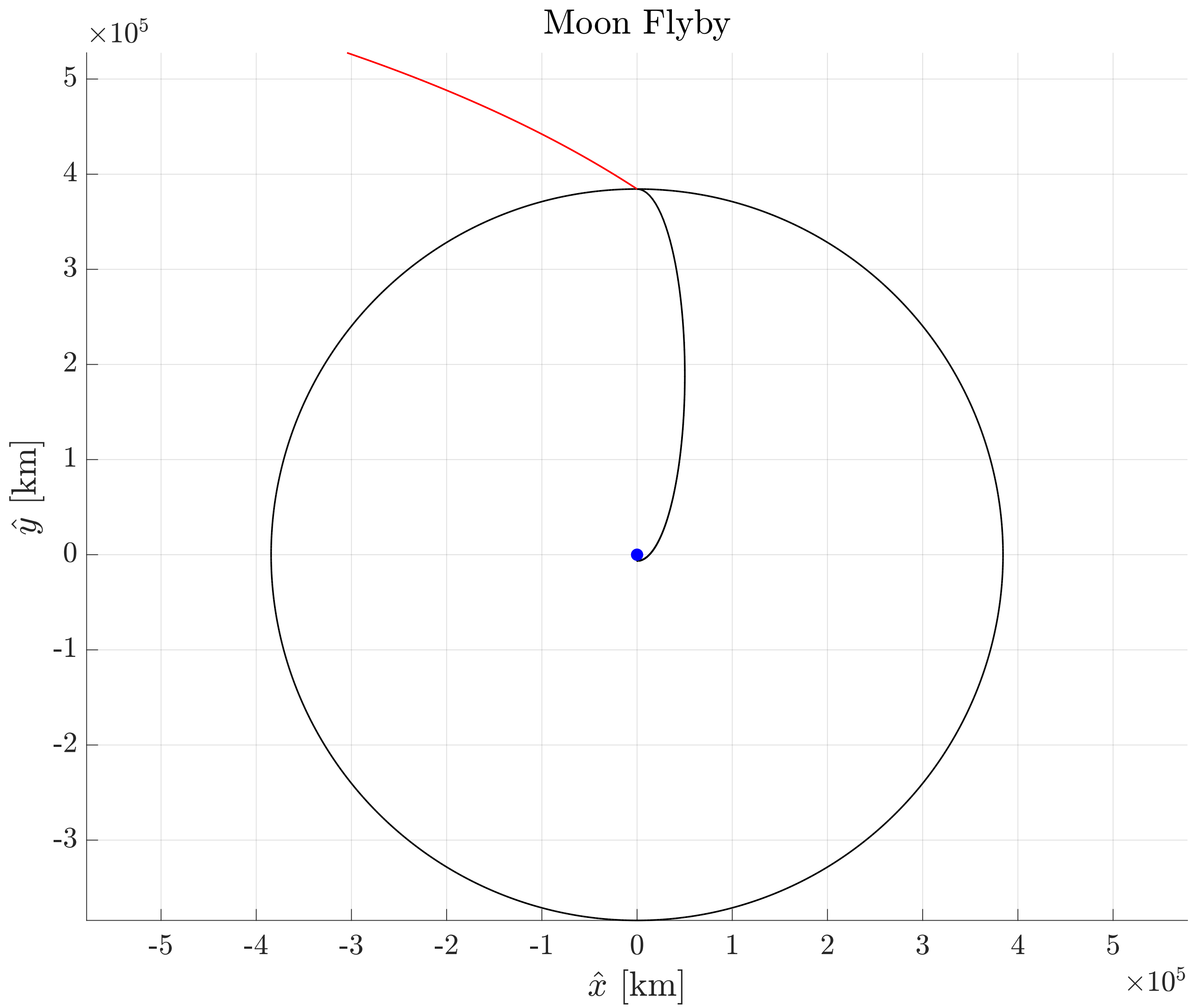


Figure 4: Plot of the Moon encounter trajectory

Problem 3

Problem Statement

Return to Problem 2. Recall that you (successfully) constructed a Hohmann transfer to the Moon assuming departure from a 175 km altitude circular Earth parking orbit; the local gravity field at the Moon was incorporated.

(a) The passage by the Moon modified the orbit relative to the Earth. Recall the conditions immediately after the lunar encounter, i.e., r^+, v^+, γ^+, θ^{*+}.
Assume that the goal is to return the Earth orbit. Should the vehicle pass on the light side or the dark side relative to Earth?
Determine the maneuver $||\Delta\bar{v}||$, α that would be required to immediately return the vehicle to the Hohmann transfer path for the return/inbound arc back to the Earth parking orbit.

(b) Rather than a Hohmann transfer path and a maneuver to return to the Earth, consider a free-return transfer. Assume that the vehicle departs the same 180-km Earth parking orbit. However, rather than a Hohmann transfer, design a free-return trajectory such that the transfer angle is 174°.
Determine the $\oplus \rightarrow$ ☾ time-of-flight. How does it compare to the Hohmann arc?
Determine the value of $r_{p/☾}$ that is required? Altitude?
Compute $||\Delta\bar{v}_{eq}||$, α that correspond to the free-return?

(c) For the free-return, plot the orbits: (i) in the Earth centered frame, plot the parking orbit, then the outbound and the return arcs only. On the plot, add $\bar{v}^-$, $\bar{v}^+$, local horizon, $\Delta\bar{v}_{eq}$, α. (ii) in the Moon centered frame, plot the hyperbola. Add the asymptotes, the vectors $\bar{v}_\infty^-$, $\bar{v}_\infty^+$ and the flyby angle δ. Use Matlab or GMAT.

Part (a)

Let's recall the conditions after the lunar encounter from Problem 2:

$$\boxed{\begin{aligned} r^+ &= 384400 \text{ km} \\ v^+ &= 1.4702 \text{ km/s} \\ \gamma^+ &= 33.1675° \\ \theta^{*+} &= 64.2463° \end{aligned}} \tag{2}$$

Then, let's examine both the light-side and dark-side passages. One can first notice the symmetry of the vector

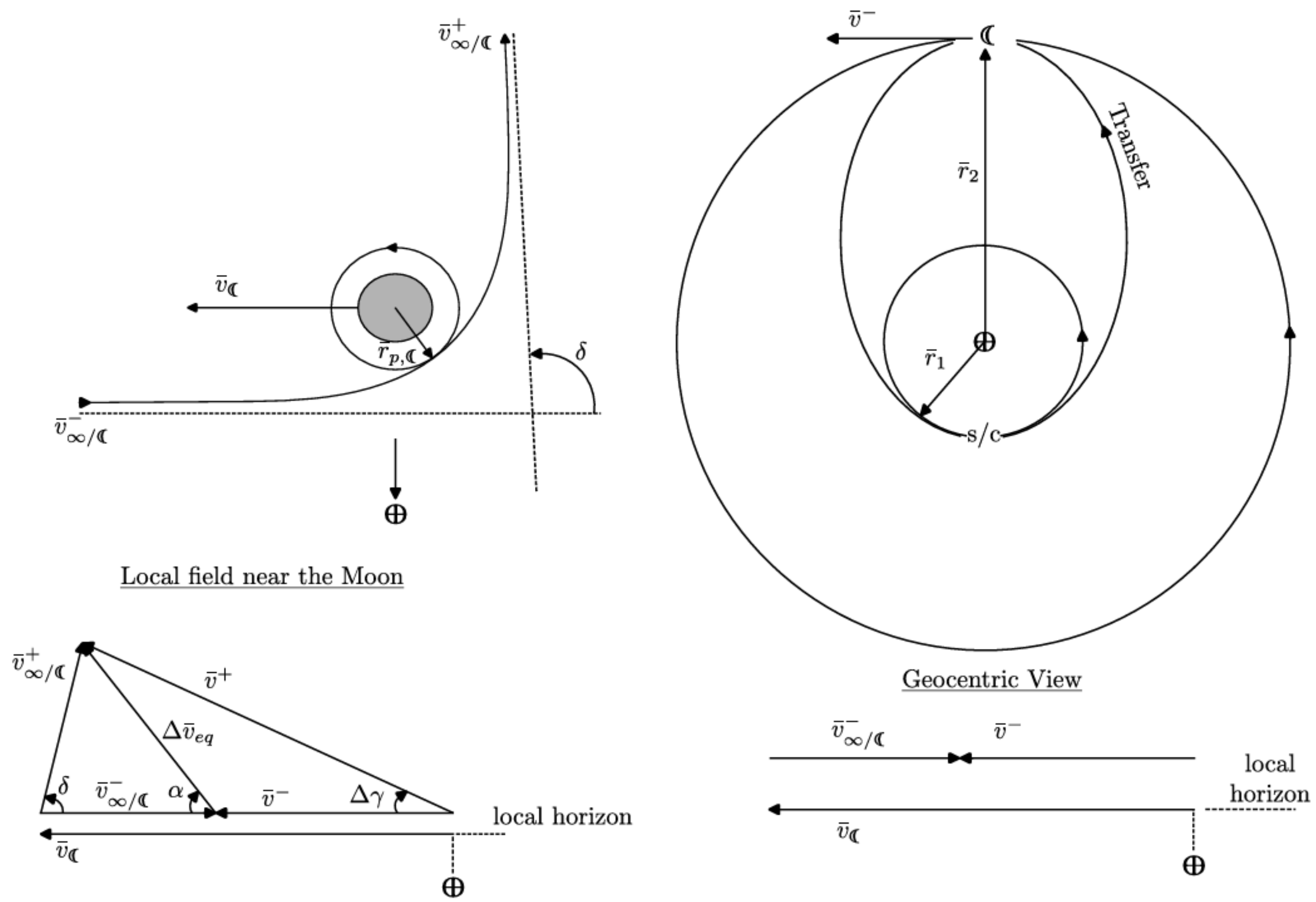


Figure 5: Moon-centered and Geocentric Views and vector diagrams at arrival at the Moon for light-side passage.

diagrams in figures 5 and 6. This behavior is due to the fact that the flyby occurs tangentially, so, from symmetry, one can note that the dark-side passage results in the same v^+ but a flight path angle γ^+ with an opposite sign.

If the goal is to return to Earth orbit, the vehicle should pass on the **far (dark) side** of the Moon. Previously, in Problem 2, the lunar passage was on the near (light) side. However, if the goal is to return to Earth, a pass on the dark side, in the patched-conic model, would provide an equivalent Δv that places the spacecraft on a trajectory descending towards the Earth. No matter which case, the spacecraft will be hyperbolic with respect to the Earth system in this scenario, but a passage on the dark side would enable a chance to save the crew, with the addition of a correction maneuver (since the periapsis radius will still be $2.7256 \cdot 10^5$ km).

If the goal is to offset the gravitational influence of the Moon by a return maneuver, one can construct velocity vector diagrams, such as the ones seen in figure 7. Note that both the light- and dark-side passages are depicted. Recognize that we want the velocity vector in the geocentric view after the maneuver $\bar{v}^+_{maneuver}$ to be the same as

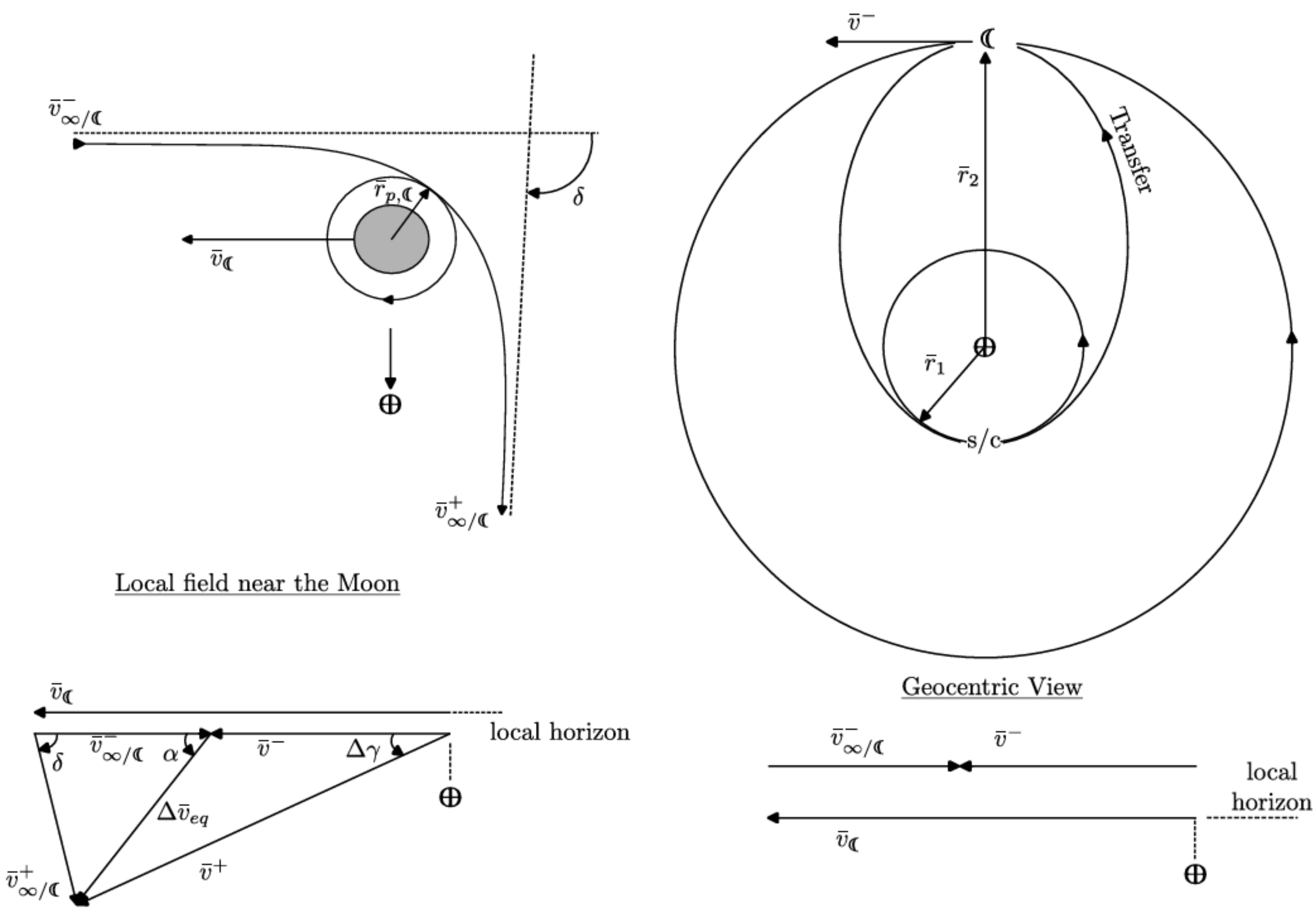


Figure 6: Moon-centered and Geocentric Views and vector diagrams at arrival at the Moon for dark-side passage.

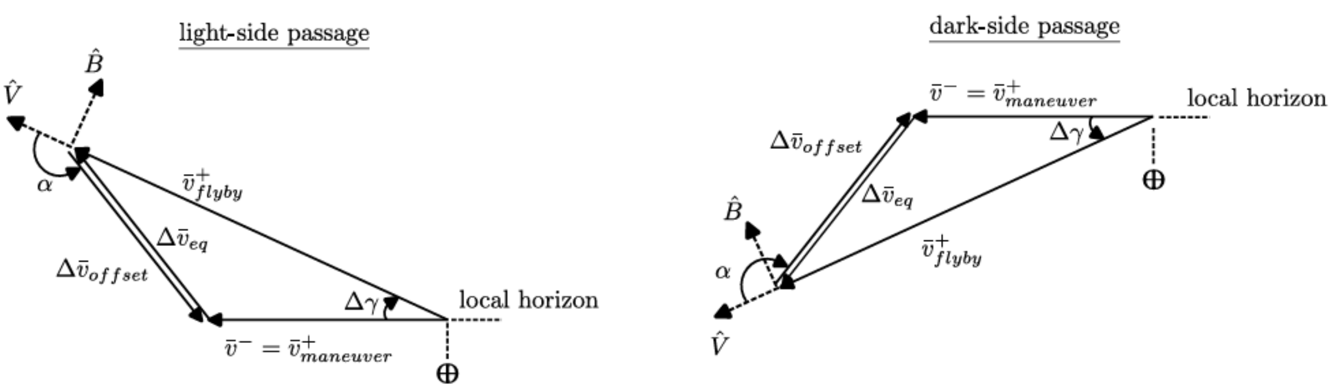


Figure 7: Velocity vector diagrams for light- and dark-side passages with $\Delta\bar{v}$ to offset the flyby.

the velocity vector in the geocentric view before the maneuver $\bar{v}^-$. It is apparent, from the figure, that the offset maneuver $\Delta\bar{v}_{offset}$ will have the same magnitude, but opposite direction as that of $\Delta\bar{v}_{eq}$:

$$\boxed{||\Delta\bar{v}_{offset}|| = ||\Delta\bar{v}_{eq}|| = 1.3180 \text{ km/s}} \tag{3}$$

Then, one can determine the maneuver angle α using the following equation and noting that $\Delta\gamma = \gamma^+$:

$$180° - \alpha = \arcsin\left(\frac{||\bar{v}^-|| \sin\Delta\gamma}{||\Delta\bar{v}_{eq}||}\right) = 4.4386° \tag{4}$$

Then, for a dark-side passage:

$$\boxed{\alpha = 175.5614°} \tag{5}$$

And for a light-side passage:

$$\boxed{\alpha = -175.5614°} \tag{6}$$

Part (b)

Now, we assume that $\theta^* = 174°$ when the flyby occurs, which is no longer a Hohmann transfer. One needs to determine eccentricity and eccentric anomaly in order to solve for time of flight (TOF). Recall that:

$$r_1 = 180 \text{ km} + R_\oplus = 6.558 \cdot 10^3 \text{ km} \tag{7}$$

$$r_2 = a_☾ = 3.8440 \cdot 10^5 \text{ km} \tag{8}$$

where r_1 is the location of the s/c leaving the Earth parking orbit and r_2 is the location of the spacecraft at the flyby. One can then solve for eccentricity by noting the conic equation:

$$r = \frac{p}{1 + e\cos\theta^*} \tag{9}$$

where p is the same for both r_1 and r_2, then:

$$r_1(1 = e\cos\theta_1^*) = r_2(1 = e\cos\theta_2^*) \tag{10}$$

Then, taking $\theta_1^* = 0°$ and $\theta_2^* = 174°$:

$$\boxed{e = \frac{(r_2 - r_1)}{r_1 - r_2\cos\theta_2^*} = 0.9717} \tag{11}$$

Now, one can solve for eccentric anomaly using the following equation:

$$E = 2\arctan\left[\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta^*}{2}\right)\right] = 132.7582° \tag{12}$$

which means that mean anomaly is then:

$$M = E - e\sin E = 91.8813° \tag{13}$$

One can then solve for TOF:

$$\boxed{\begin{aligned} TOF = t - t_p = M\sqrt{\frac{a^3}{\mu_\oplus}} &= 2.8312 \cdot 10^5 \text{ s} \\ &= 78.6449 \text{ hrs} \\ &= 3.2769 \text{ days} \end{aligned}} \tag{14}$$

Recall that, for the Hohmann arc, the TOF is half the period of the transfer ellipse, which is 4.9775 days. Thus, the TOF for the current transfer is over 1 day faster than the TOF for the Hohmann transfer.

Next, before considering the flyby encounter itself, one must calculate some orbital characteristics for the transfer orbit. One can begin with the semi-latus rectum:

$$p = r_1(1 - e) = 1.2931 \cdot 10^4 \text{ km} \tag{15}$$

Then, one can calculate the semi-major axis:

$$a = \frac{p}{1 - e^2} = 2.3161 \cdot 10^5 \text{ km} \tag{16}$$

This allows us to calculate the specific energy for the transfer orbit:

$$\mathcal{E} = -\frac{\mu_\oplus}{2a} = -0.8605 \text{ km}^2/\text{s}^2 \tag{17}$$

One can then calculate the speed of the spacecraft $v^- = v_2$ before the flyby:

$$v_2 = \sqrt{2\left(\mathcal{E} + \frac{\mu_\oplus}{r_2}\right)} = 0.5940 \text{ km/s} \tag{18}$$

We would like to know the flight path angle of the spacecraft before the flyby. One can calculate the flight path angle once one obtains the specific angular momentum. One can use the periapsis location within the transfer orbit to aid in the calculation for specific angular momentum or use the following relation:

$$h = \sqrt{\mu_{\oplus} p} = 7.1792 \cdot 10^4 \text{km}^2/\text{s} \tag{19}$$

Then, the flight path angle before the encounter is:

$$\gamma^- = \arccos\left(\frac{h}{r_2 v_2}\right) = \arccos\left(\frac{h}{r_2 v^-}\right) = 71.6757° \tag{20}$$

Now, one can consider the actual flyby encounter. Consider the following sketch with velocity vector diagrams for the free-transfer: Let's begin with calculating the speed of the Moon. Assuming that the Moon is in a circular orbit

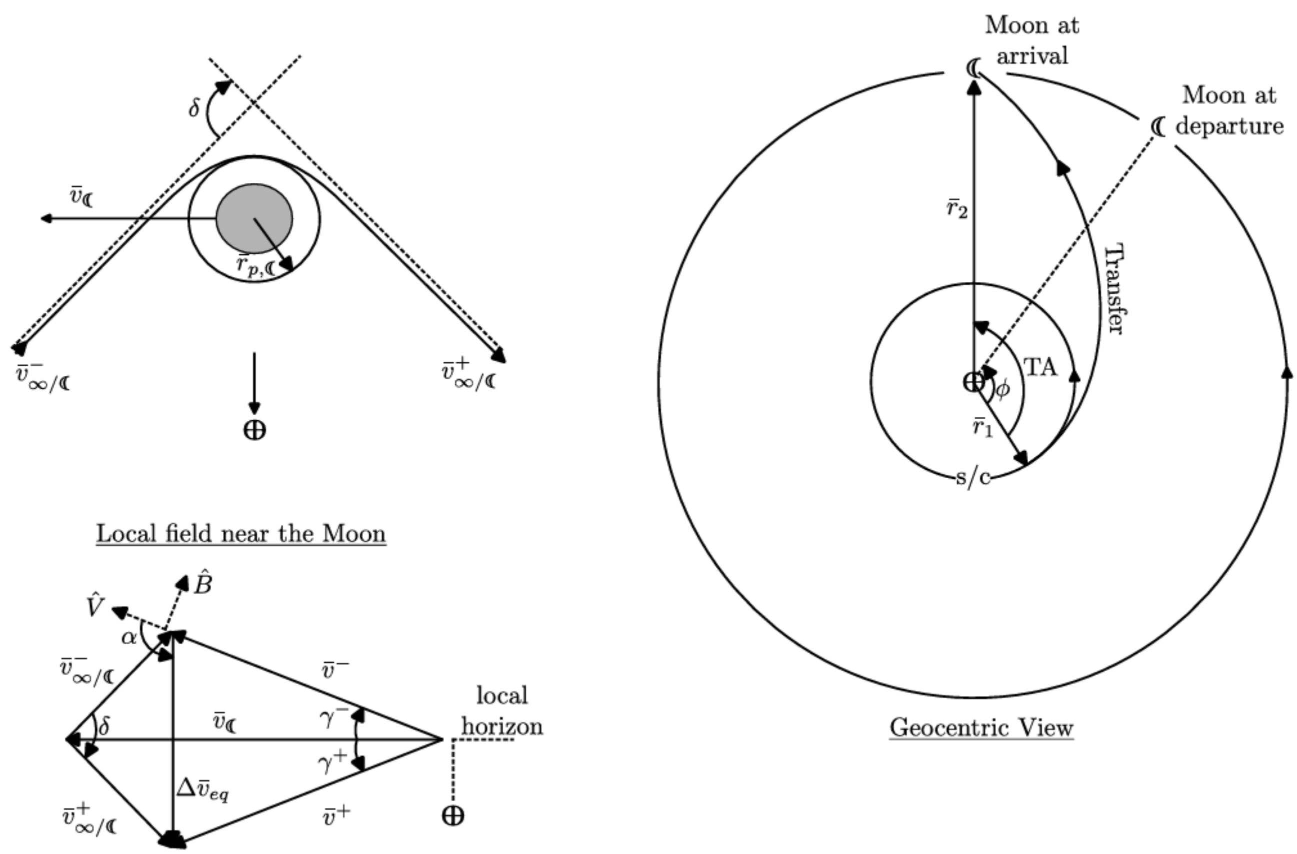


Figure 8: Moon-centered and Geocentric Views and vector diagrams at arrival at the Moon for dark-side passage for a free-return trajectory.

about the Earth and its mass is, essentially, negligible in comparison to the mass of the Earth:

$$v_{☾} = \sqrt{\frac{\mu/_{\oplus}}{a_{☾}}} = 1.0183 \text{ km/s} \tag{21}$$

Using cosine law, one can obtain the value for $||\bar{v}^-_{\infty/☾}||$ before the flyby:

$$||\bar{v}^-_{\infty/☾}|| = \sqrt{v_{☾}^2 + v_2^2 - 2 v_{☾} v_2 \cos\gamma^-} = 1.0047 \text{ km/s} \tag{22}$$

Before we determine the equivalent Δv from the flyby, one should first check that the flyby is feasible. To do this, one must check the altitude constraint, which is to say that the spacecraft cannot fly under the surface of the Moon at periapsis within its hyperbolic orbit within the Moon-centered view. First, let's calculate the turn angle for the flyby:

$$\delta = 2\arcsin\left(\frac{v_2 \sin\gamma^-}{||\bar{v}^-_{\infty/☾}||}\right) = 68.2877° \tag{23}$$

Since we have already defined $||\bar{v}^-_{\infty/☾}||$, one can calculate the specific energy for the hyperbolic orbit:

$$\mathcal{E}_H = \frac{||\bar{v}^-_{\infty/☾}||^2}{2} = 0.5047 \text{ km}^2/\text{s}^2 \tag{24}$$

Then, the semi-major axis is:

$$a_H = -\frac{\mu_☾}{2\mathcal{E}_H} = -4.8568 \cdot 10^3 \text{ km} \tag{25}$$

Using the turn angle, the eccentricity for the hyperbolic trajectory is:

$$e_H = \frac{1}{\sin\delta/2} = 1.7817 \tag{26}$$

Thus, perilune is:

$$\boxed{r_{p,H} = a_H(1 - e_H) = 3.7964 \cdot 10^3 \text{ km}} \tag{27}$$

with an altitude:

$$\boxed{Altitude = r_{p,H} - R_☾ = 2.0582 \cdot 10^3 \text{ km}} \tag{28}$$

Lastly, one can determine the equivalent Δv from the flyby is the "maneuver" angle for the free-return. Using cosine law or an isosceles triangle relation:

$$\boxed{\begin{aligned} ||\Delta\bar{v}_{eq}|| &= \sqrt{2v_2^2 - 2v_2^2\cos(2|\gamma^-|)} \\ &= 2||\bar{v}^-_{\infty/☾}||\sin\left(\frac{\delta}{2}\right) \\ &= 1.1278 \text{ km/s} \end{aligned}} \tag{29}$$

One can determine the maneuver angle α using the following equation:

$$\begin{aligned} 180° - \alpha &= \arcsin\left(\frac{||\bar{v}^-||\sin\Delta\gamma}{||\Delta\bar{v}_{eq}||}\right) \\ &= \arcsin\left(\frac{||\bar{v}^-||\sin 2\gamma^-}{||\Delta\bar{v}_{eq}||}\right) \\ &= 18.3243° \end{aligned} \tag{30}$$

and since $\Delta\bar{v}_{eq}$ is pointed towards $-\hat{B}$ direction <u>or</u> the fact that $\Delta\bar{v}_{eq}$ is rotated in a counter-clockwise direction towards the moon, then $\alpha < 0$ and:

$$\boxed{\alpha = -161.6757°} \tag{31}$$

Part (c)

Note that the vectors depicted in the plots below are exaggerated for viewing.

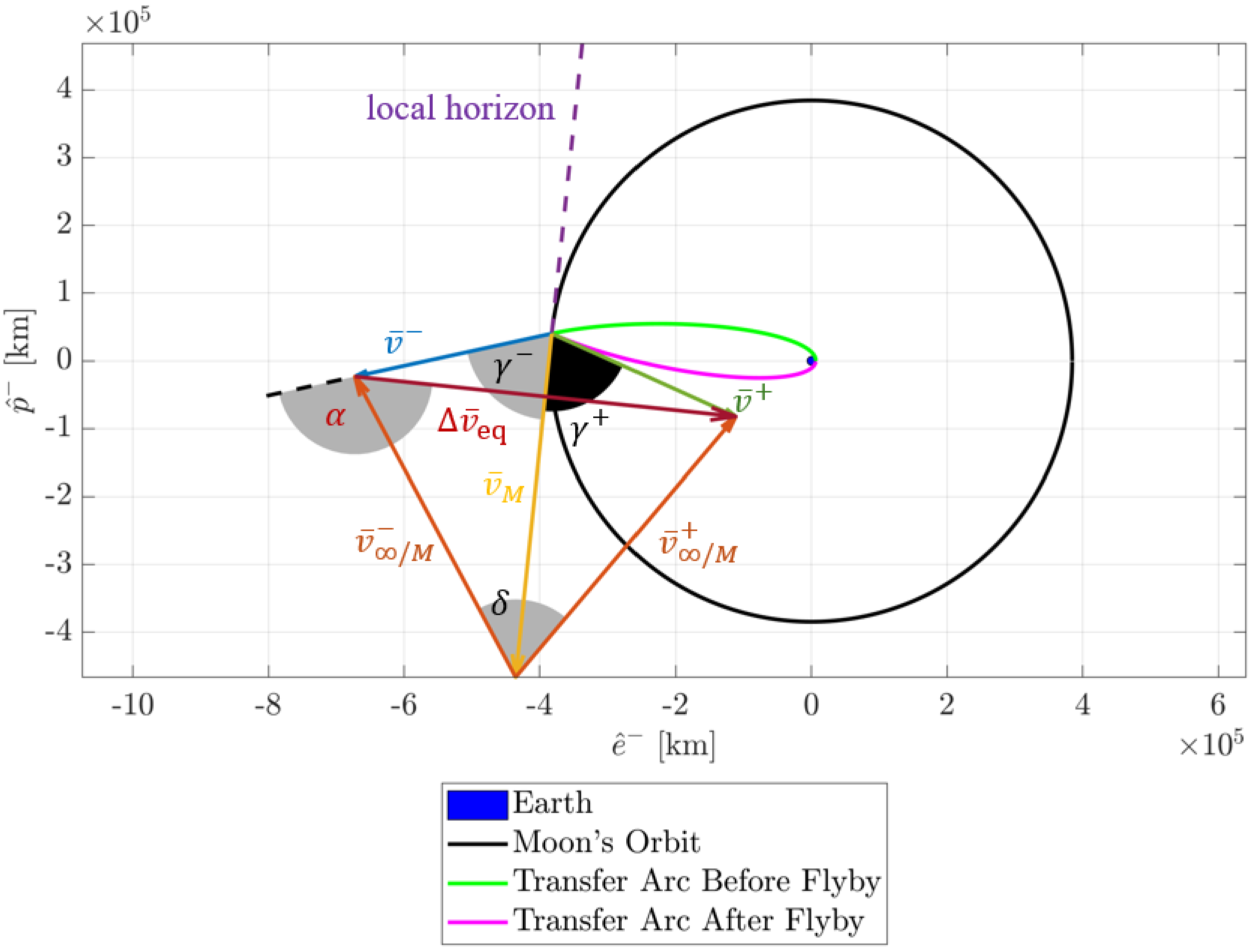


Figure 9: Geocentric view of the free-return trajectory.

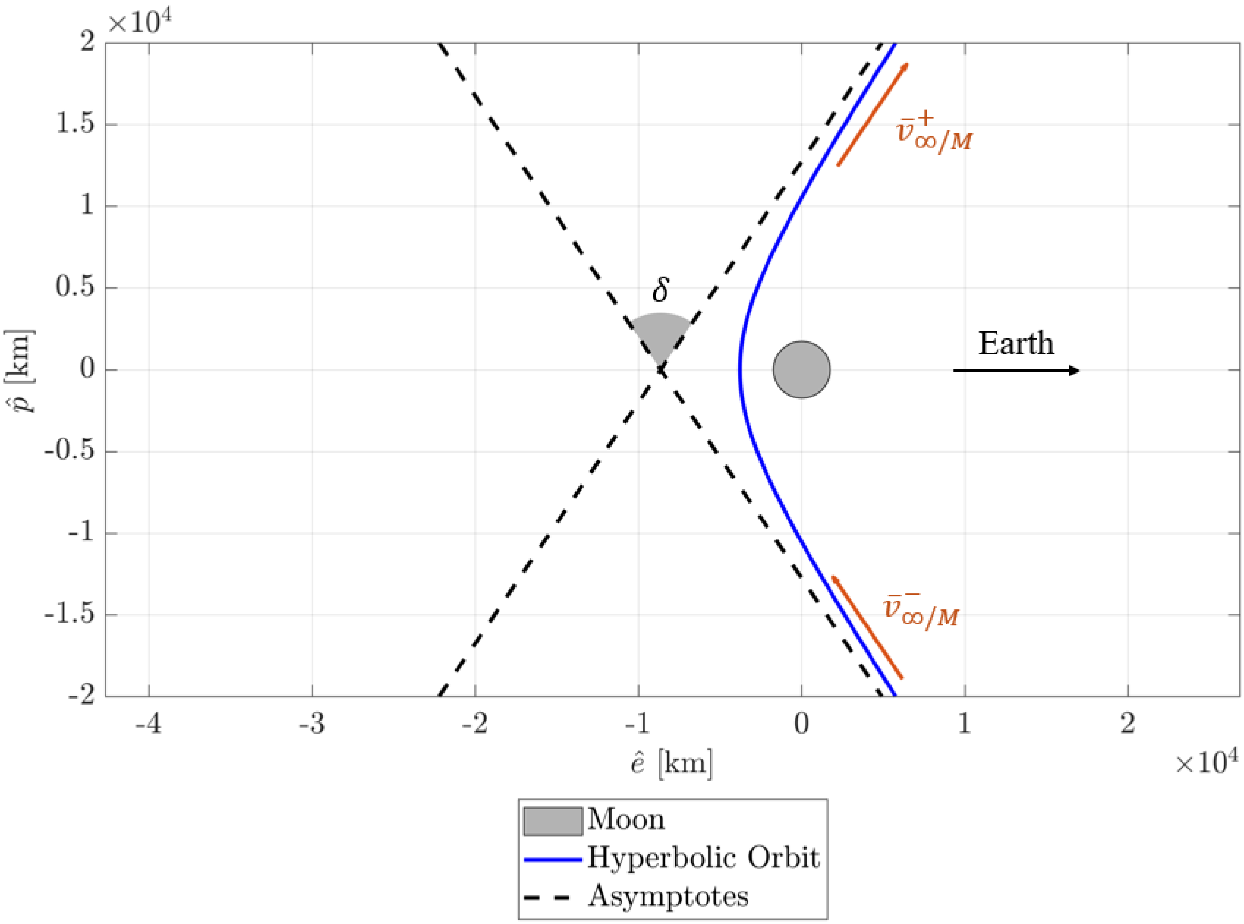


Figure 10: Local gravity near the Moon for the free-return trajectory.

Problem 4

Problem Statement

The Messenger spacecraft offered exploration of the planet Mercury after launch in 2004 with Mercury orbit insertion in March 2011. The transfer to Mercury employed more than one Venus gravity assist.

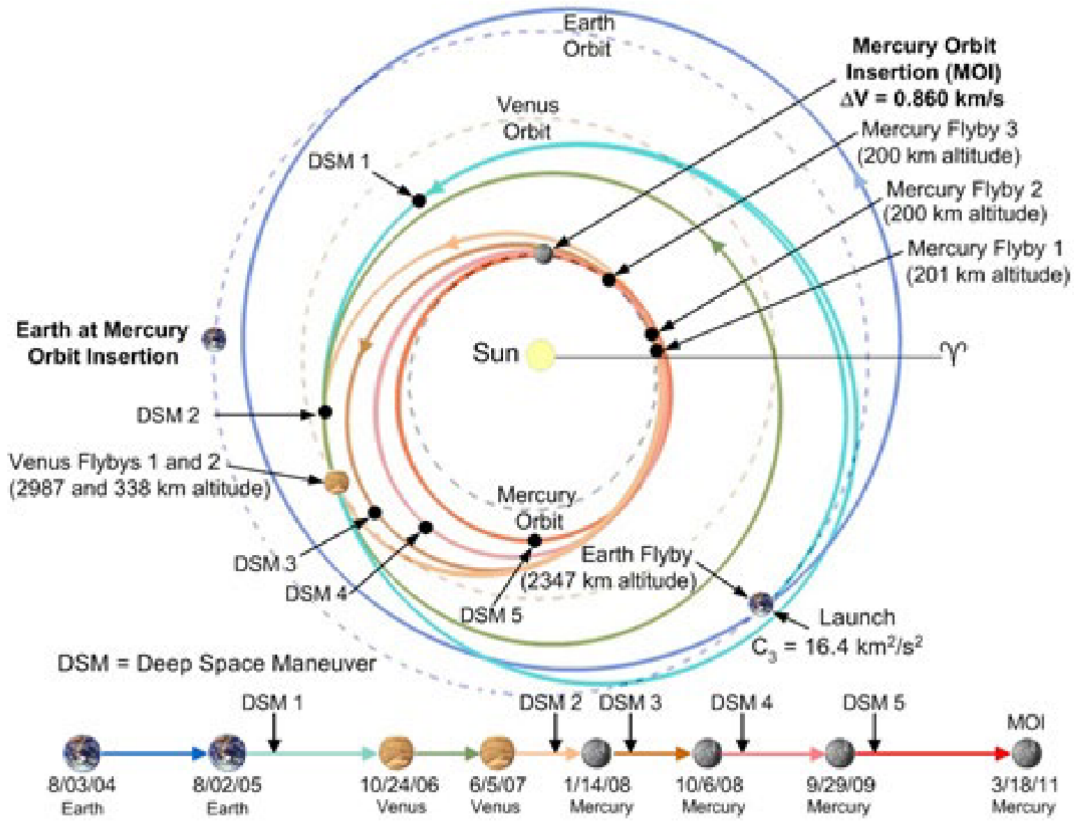


Figure 11: Messenger trajectory.

(a) Determine the actual values for TOF for the following:

(i) Earth launch to the first Venus flyby.

(ii) Earth launch to first Mercury flyby.

(iii) Earth launch to MOI (Mercury Orbit Insertion).

(b) Assume that you are completing a preliminary analysis for such a Mercury mission but assume that all planetary orbits are co-planar and circular and use patched conics. Assume a Hohmann transfer from Earth-to-Mercury. Let all planetary orbits be circular and coplanar. Along the path to Mercury, an encounter with Venus occurs. Assume that the closest approach to Venus is $4R_{♀}$. TOF from Earth to Venus encounter along your Hohmann arc? (For the actual Venus Flyby 1, the flyby distance was 2990 km altitude. How many Venus radii was the actual encounter?)

(c) To continue down to Mercury most efficiently, is it desirable to gain or lose energy? Should the spacecraft pass 'ahead' or 'behind' Venus? Determine the following quantities for the post-encounter heliocentric orbit: $a, e, r, v, \gamma, \theta^*, r_a, r_p, P, \Delta\omega$.

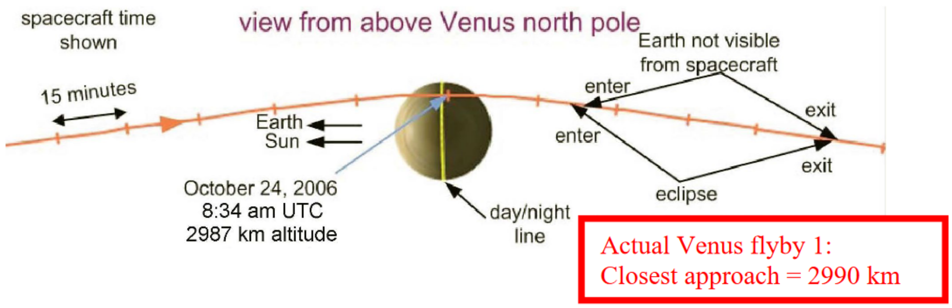


Figure 12: Venus flyby diagram

(d) Determine the equivalent $\Delta\bar{v}_{eq}$. What is the magnitude and direction, i.e. $|\Delta\bar{v}_{eq}|$ and α?

(e) Plot the old and the new orbit. (Use either Matlab or GMAT.) Identify $\bar{v}^-, \bar{v}^+, \Delta\bar{v}_{eq}, \alpha$, line of apsides, $\Delta\omega$. Does the s/c still cross the orbit of Mercury? Compare your TOF for the legs Earth-to-Venus; Venus-to-Mercury with the actual TOF for various events. Are they reasonable?

(f) If you want to reach Mercury along the path that you constructed, determine the phase angles between Earth and Venus and Earth and Mercury that are independently required.

Part (a)

To determine the TOF of each event, we first convert them to Julian dates. This way the TOF can be found through simple subtraction rather than having to count calendar days.

Event	Calendar Date	Julian Date
Earth Launch	8/03/04	2453220.5
Earth Flyby	8/02/05	2453584.5
Venus Flyby 1	10/24/06	2454032.5
Venus Flyby 2	6/05/07	2454256.5
Mercury Flyby 1	1/14/08	2454479.5
Mercury Flyby 2	10/06/08	2454745.5
Mercury Flyby 3	9/29/09	2455103.5
Mercury Orbit Insertion	3/18/11	2455638.5

i

Earth launch to first Venus flyby:

$$TOF = 2453220.5 - 2454032.5 \boxed{= 812 \ days = 2.23 \ yrs}$$

ii

Earth launch to first Mercury flyby:

$$TOF = 2453220.5 - 2454479.5 \boxed{= 1259 \ days = 3.45 \ yrs}$$

iii

Earth launch to Mercury orbit insertion:

$$TOF = 2453220.5 - 2455638.5 \boxed{= 2418 \ days = 6.62 \ yrs}$$

JPL date converter: https://ssd.jpl.nasa.gov/tools/jdc/#/cd

Part (b)

Now we evaluate a Hohmann transfer arc from Earth to Mercury assuming that at some point along that path the spacecraft will have an encounter with Venus. We can determine the characteristics of the transfer ellipse first, then the location of the Venus encounter will be determined based on where the transfer ellipse intersects Venus's orbital distance from the Sun.

First we determine the parameters of the Hohmann transfer arc:

$$r_{a,t} = a_{\oplus} = 149597898\ km$$

$$r_{p,t} = a_{☿} = 57909101\ km$$

$$a_t = \frac{r_a + r_p}{2} = 103753499.5\ km$$

$$e_t = \frac{r_{a,t}}{a_t} - 1 = 0.4419$$

$$p_t = a_t(1 - e_t^2) = 83496747.83\ km$$

$$\theta^*_{t,♀} = \cos^{-1}\left(\left(\frac{p_t}{a_{♀}} - 1\right)\frac{1}{e_t}\right) = -121.12°$$

$$E_{t,♀} = 2\tan^{-1}\left(\tan\frac{\theta^*_{t,♀}}{2}\left(\frac{1+e_t}{1-e_t}\right)^{-1/2}\right) = -95.58°$$

$$(t - t_p)_{t,♀} = \frac{M_{t,♀}}{\sqrt{\mu_{\odot}/a_t^3}} = 169.72\ days$$

$$P_t = 2\pi\sqrt{\frac{a_t^3}{\mu_{\odot}}}$$

$$TOF = \frac{P_t}{2} - (t - t_p)_{t,♀}\ \boxed{= 64.24\ days}$$

The actual Venus flyby had an altitude of 2990 km which we can convert to Venus radii:

$$\frac{6051.9\ km + 2990\ km}{6051.9\ km}\ \boxed{= 1.49 R_{♀}}$$

Part (c)

Since the final destination is Mercury which orbits closer to the Sun, our objective is to lose energy after the Venus encounter. This suggests that we need to pass $\boxed{ahead}$ of Venus.

First we can evaluate parameters of the transfer orbit from Earth at the Venus encounter:

$$r^- = a_{♀} = 108207284\ km$$

$$v^- = \sqrt{\mu_{\odot}\left(-\frac{1}{a_t} + \frac{2}{r^-}\right)} = 34.26\ km/s$$

$$\gamma^- = \cos^{-1}\left(\frac{\sqrt{\mu_{\odot} p_t}}{r^- v^-}\right) - 26.12^{\circ}\ \text{(descending)}$$

$$v_{♀} = \sqrt{\frac{\mu_{\odot}}{r_{♀}}} = 35.02\ km/s$$

Now looking at the vector diagram for the Venus encounter:

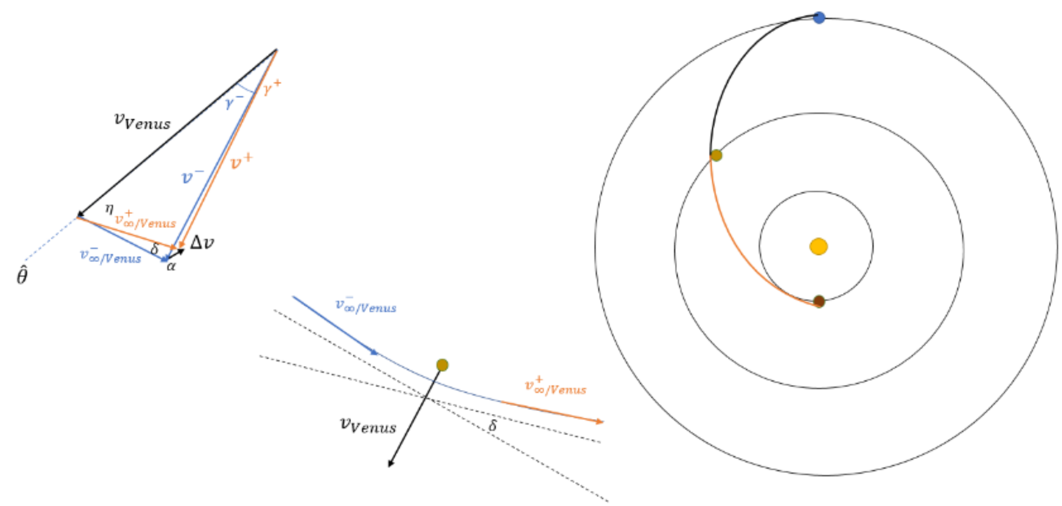


Figure 13: Vector diagram for Venus encounter.

From geometry:

$$v^-_{\infty,♀} = \sqrt{v^{-2} + v_{♀}^2 - 2v^- v_{♀}\cos\gamma} = 15.67\ km/s$$

The orbit relative to Venus is hyperbolic, therefore:

$$a_h = \frac{\mu_{♀}}{v_{\infty}^2} = -1322.94\ km$$

$$e_h = \frac{r_p}{|a_h|} + 1 = 19.23$$

$$\delta = 2\sin^{-1}\left(\frac{1}{e_h}\right) = 5.9406^{\circ}$$

$$\Delta v_{eq} = \sqrt{2v_\infty^2 - 2v_\infty^2 \cos\delta} \boxed{= 1.62\ km/s}$$

Now we can proceed with the computation of the angle characterizations in the vector diagram along with spacecraft parameters after the Venus encounter. If $\epsilon = \eta + \delta$:

$$v^{-2} = v_♀^2 + v_{\infty ♀}^{-2} - 2v_♀ v_{\infty ♀}^{-} \cos\epsilon \qquad \longrightarrow \qquad \epsilon = 74.2°$$

$$\eta = \epsilon - \delta = 68.3°$$

$$v^{+} = \sqrt{v_♀^2 + v_{\infty ♀}^{+2} - 2v_♀ v_{\infty ♀}^{+} \cos\eta} \boxed{= 32.65\ km/s}$$

$$v_{\infty ♀}^2 = v^{+2} + v_♀^2 - 2v^{+} v_♀ \cos\gamma^{+} \qquad \longrightarrow \qquad \gamma^{+} = -26.48°$$

$$r^{+} = r^{-} \boxed{= 108207284\ km}$$

$$h^{+} = r^{+} v^{+} \cos\gamma^{+} = 3.1624e9\ km^2/s^2$$

$$p^{+} = \frac{h^{+2}}{\mu_\odot} = 7.5357e7\ km$$

$$\theta^{*+} = \tan^{-1} \frac{\frac{r^{+} v^{+2}}{\mu_\odot} \cos\gamma^{+} \sin\gamma^{+}}{\frac{r^{+} v^{+2}}{\mu_\odot} \cos\gamma^{+2} - 1} \boxed{= 228.81°}$$

$$e^{+} = \frac{r^{+} v^{+2}}{\mu_\odot} \cos\gamma^{+2} + \sin\gamma^{+2} \boxed{= 0.4610}$$

$$a^{+} = \frac{p^{+}}{1 + e^{+2}} \boxed{= 95694545\ km}$$

$$r_a^{+} = a^{+}(1 + e^{+}) \boxed{139810284\ km}$$

$$r_p^{+} = a^{+}(1 - e^{+}) \boxed{51578806\ km}$$

$$P = 2\pi \sqrt{\frac{a^{+3}}{\mu_\odot}} \boxed{= 186.87\ days}$$

$$\Delta\omega = -\Delta\theta^{*} \boxed{= 10.07°}$$

Part (d)

Recall from the previous part:

$$\Delta v_{eq} = \sqrt{2v_\infty^2 - 2v_\infty^2 \cos\delta}\ \boxed{= 1.62\ km/s}$$

We can use the vector diagram to find the angle:

$$\alpha = 180 - \cos^{-1}\left(\frac{v^{+2} - v^{-2} - \Delta v_{eq}^2}{-2v^{-}\Delta v_{eq}}\right)\ \boxed{= -172.62^\circ}\ \text{(towards the Sun)}$$

Part (e)

Plotting both transfer ellipses we can see that the spacecraft still crosses the orbit of Mercury despite the Venus encounter changing the trajectory; however, the periapse of the transfer ellipse no longer aligns with Mercury's orbit. This has the benefit of there now being two crossing opportunities, one before periapse and one after, but the drawback that the Δv to insert into a final orbit at the same orbital radius as Mercury will no longer be tangential.

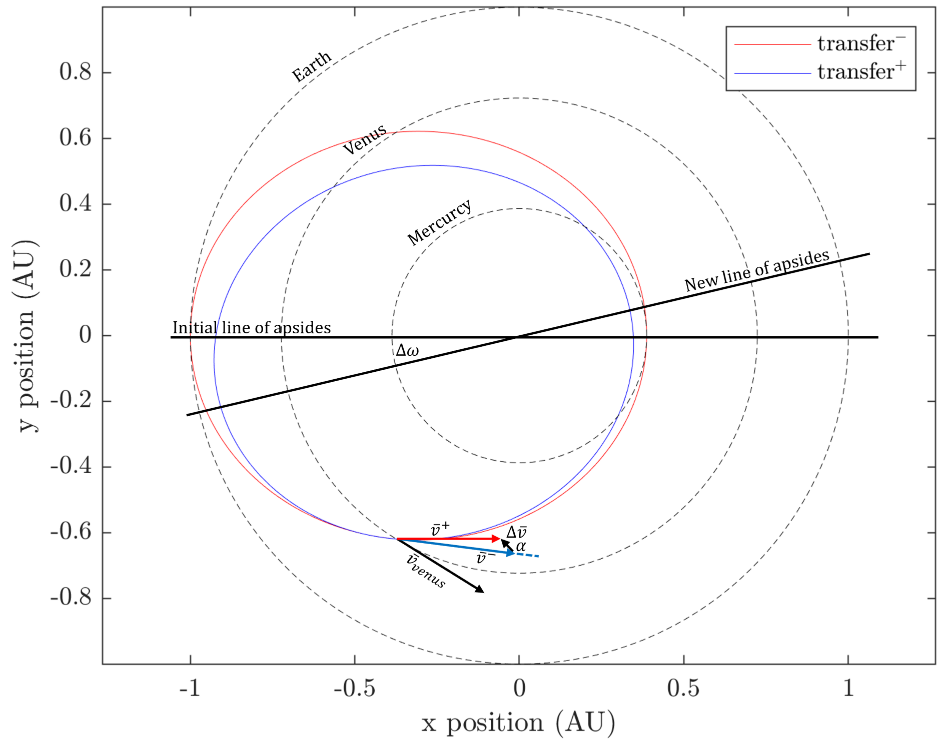


Figure 14: Transfer orbits pre and post Venus flyby.

We previously calculated that it took 64.24 days to go from Earth to Venus, and 33.07 days to go from Venus to Mercury. We can compare this to the TOFs from the Messenger mission (812 days and 447 days respectively). There is clearly a significant difference since our calculations are based on a simple Hohmann transfer that passes through Venus's gravitational field briefly on the way to Mercury; whereas the Messenger mission performed an Earth gravity assist followed by multiple Venus and Mercury gravity assists before the final Mercury orbit insertion.

Part (f)

The phase angle from Earth to Venus is straightforward:

$$TA_1 = \theta^*_{♀} - 180° = 58.88°$$

$$n_{♀} TOF_{\oplus \to ♀} = TA_1 - \phi_{\oplus \to ♀}$$

$$\boxed{\phi_{\oplus \to ♀} = -44.04°}$$

Measured with respect to the Line of Apsides through Earth. Next we evaluate the phase angle from Earth to Mercury:

$$TOF_{\oplus \to ☿} = TOF_{\oplus \to ♀} + TOF_{♀ \to ☿}$$

$$TA_2 = \theta^*_{☿} + \Delta\omega - 180 = 140.88°$$

$$n_{☿} TOF_{\oplus \to ☿} = TA_2 - \phi_{\oplus \to ☿}$$

$$\boxed{\phi_{\oplus \to ☿} = 102.65°}$$