#1) $\ddot{y} + 0.1\dot{y} - y + y^3 = 0$

At equilibrium: $\ddot{y} = \dot{y} = 0$

$\therefore\ -y + y^3 = 0$

$y(-1 + y^2) = 0 \qquad y^2 = 1 \qquad y = \pm\sqrt{1}$

$\therefore\ \boxed{y_e^{(1)} = 0 \qquad y_e^{(2)} = 1 \qquad y_e^{(3)} = -1}$

$X_e = \begin{pmatrix} y_e \\ 0 \end{pmatrix}, \quad X = \begin{pmatrix} y \\ \dot{y} \end{pmatrix}$

$X_1 = y \qquad X_2 = \dot{y} \qquad \ddot{y} = -0.1\dot{y} + y - y^3$

$\dot{X}_1 = X_2$

$\dot{X}_2 = -0.1X_2 + X_1 - X_1^3$

$A = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}\Bigg|_{X_e} = \begin{pmatrix} 0 & 1 \\ 1-3x_1^2 & -0.1 \end{pmatrix}$

For $X_e^{(1)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$: $\quad A = \begin{pmatrix} 0 & 1 \\ 1 & -0.1 \end{pmatrix}$

From matlab: $\lambda_1 = -1.0512, \quad \lambda_2 = 0.9512$

$\lambda_1 < 0 < \lambda_2$, $\boxed{\text{therefore } X_e^{(1)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ is a saddle}}$

For $X_e^{(2)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$: $\quad A = \begin{pmatrix} 0 & 1 \\ -2 & -0.1 \end{pmatrix}$

From matlab: $\lambda_1 = -0.05 + 1.4133i$ & $\lambda_2 = -0.05 - 1.4133i$

$\text{Real}(\lambda_1) < 0 \therefore$ $\boxed{X_e^{(2)} \text{ is a stable focus}}$

For $x_e^{(3)} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$: $A = \begin{pmatrix} 0 & 1 \\ -2 & -0.1 \end{pmatrix}$

therefore $\lambda_1 = -0.05 + 1.4133i$ & $\lambda_2 = -0.05 - 1.4133i$

$\text{Real}(\lambda_1) < 0 \therefore$ $\boxed{X_e^{(3)} \text{ is a stable focus}}$

#2) $\ddot{y} + \sin(y) = 0$

At equilibrium : $\ddot{y} = \dot{y} = 0$

$\sin(y_e) = 0$, $x = \begin{pmatrix} y \\ \dot{y} \end{pmatrix}$, $x_e = \begin{pmatrix} y_e \\ 0 \end{pmatrix}$

$$\boxed{y_e^{(1)} = 0 \qquad y_e^{(2)} = \pi}$$

$X_1 = y \qquad X_2 = \dot{y}$

$\dot{X}_1 = X_2$

$\dot{X}_2 = -\sin(X_1)$

$$A = \left.\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}\right|_{X_e} = \left.\begin{pmatrix} 0 & 1 \\ -\cos(X_1) & 0 \end{pmatrix}\right|_{X_e}$$

For $X_e^{(1)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$: $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $\therefore$ $\lambda^2 + 1 = 0$, $\lambda_{1,2} = \pm\sqrt{-1}$

$\lambda_1 = i$, $\lambda_2 = -i$

$\boxed{\text{Real}(\lambda_1) = 0 \text{ , } \therefore \text{ the behavior of the non-linear system about } X_e^{(1)} \text{ cannot be determined.}}$

For $X_e^{(1)} = \begin{pmatrix} \pi \\ 0 \end{pmatrix}$: $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\therefore$ $\lambda^2 - 1 = 0$, $\lambda_{1,2} = \pm\sqrt{1}$

$\lambda_1 = -1$, $\lambda_2 = 1$ $\qquad \lambda_1 < 0 < \lambda_2$ $\therefore$

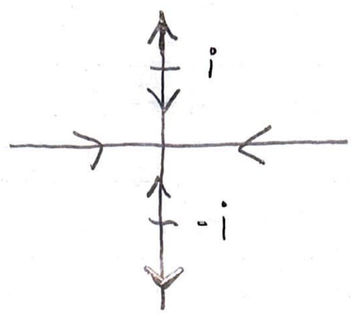
$\boxed{X_e^{(2)} = \begin{pmatrix} \pi \\ 0 \end{pmatrix} \text{ is a saddle}}$

#3)

$\dot{x} = -x - x^3$ at equilibrium: $0 = -x(1 + x^2)$

$x_e = 0, \pm i$

$x < 0$	$x > 0$
$\dot{x}(-1) = 2$	$\dot{x}(1) = -2$
$x > i$	$x < i$
$\dot{x}(2i) = 6i$	$\dot{x}(\frac{i}{2}) = -.375i$
$x > -i$	$x < -i$
$\dot{x}(-2i) =$	$\dot{x}(-\frac{i}{2}) = .375i$



The equilibrium state $x_e = 0$ is asymptotically stable with Region of attraction $(-\infty, \infty)$

$\dot{x} = -x + x^3$ $\dot{x} = 0 = x(-1 + x^2)$

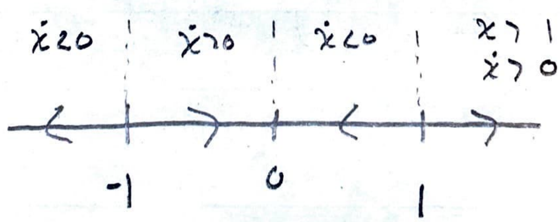
$x_e = 0, -1, 1$

$\dot{x}(2) = 6$

$\dot{x}(\frac{1}{2}) = -.375$

$\dot{x}(-\frac{1}{2}) = .375$

$\dot{x}(2) = -6$



Equilibrium states ± 1 are unbounded or unstable, $x = 0$ is an asymptotically stable equilibrium state with region of attraction $(-1, 1)$

$$\dot{x} = x - 2x^2 + x^3$$

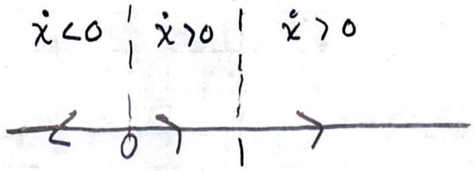
$$0 = x(1 - 2x + x^2) = x(x-1)(x-1)$$

$x_e = 0$, $x_e = 1$

$\dot{x}(-1) = -4$

$\dot{x}(\frac{1}{2}) = .125$

$\dot{x}(2) = 2$



The equilibrium state $x_e = 0$ is unstable, $x_e = 1$ is also unstable. However solutions are bounded if $0 < x_0 < 1$.

#4)

$$\dot{x} = x^3$$

$$\frac{dx}{dt} = x^3 \quad \Rightarrow \quad x^{-3}dx = dt$$

$$\int_{x_0}^{x} n^{-3}\,dn = \int_0^t dt = t$$

$$\frac{-n^{-2}}{2}\Big|_{x_0}^{x} = -\frac{1}{2x^2} + \frac{1}{2x_0^2} = t$$

$$\frac{-1}{2x^2} = t - \frac{1}{2x_0^2} = \frac{2x_0^2 t - 1}{2x_0^2}$$

$$\frac{1}{x^2} = \frac{-2x_0^2 t + 1}{x_0^2}$$

$$x(t) = \frac{x_0}{\sqrt{-2x_0^2 t + 1}}$$

$$\lim_{t \to T_e} x(t) = \infty \text{ if: } -2x_0^2 T_e + 1 = 0 \quad \Rightarrow \quad x(t) = \frac{x_0}{0}$$

$$\therefore \boxed{T_e = \frac{1}{2x_0^2}}$$

T_e is finite escape/blow up time

#5) $\dot{x} = \frac{x}{1+x^2} + \sin(x)$

For indefinitely extended solutions with no finite escape/blow up time:

$\|f(x)\| \leq \alpha\|x\| + \beta$

$\|f(x)\| = \|\frac{x}{1+x^2} + \sin(x)\| \leq \|\frac{x}{1+x^2}\| + \|\sin(x)\|$

$\leq \|\frac{1}{1+x^2}\| \|x\| + \|\sin(x)\|$

$0 \leq \frac{1}{1+x^2} \leq 1 \quad \therefore \quad \|\frac{1}{1+x^2}\| \leq 1$

$-1 \leq \sin(x) \leq 1 \quad \therefore \quad \|\sin(x)\| \leq 1$

$\therefore \|f(x)\| \leq \|\frac{1}{1+x^2}\| \|x\| + \|\sin(x)\| \leq \|x\| + 1$

$\|f(x)\| \leq \alpha \|x\| + \beta$

$\therefore \alpha = 1 \quad \& \quad \beta = 1$

$\|f(x)\| \leq \alpha \|x\| + \beta$, where $\alpha = 1$ & $\beta = 1$ $\therefore$ the solutions don't have a finite blow up time.

#6) $\dot{x} = -\sqrt{(1-x)^2}$

For unique solution: f must be differentiable

$$f = -\sqrt{(1-x)^2} \qquad \|x\| = \sqrt{x^2}$$

$$\therefore f = -\|1-x\|$$

$$\frac{df}{dx} = -\frac{(1-x)}{\|(1-x)\|} \cdot \frac{d}{dx}(1-x) = \frac{-(1-x)}{\|(1-x)\|}(-1)$$

$$\frac{df}{dx} = \frac{(1-x)}{\|(1-x)\|}$$ ← Not differentiable at f=1

$\|(1-x)\| \neq 0$ for f to be differentiable $\therefore$ $x_0 \neq 1$

$\therefore$ $\boxed{\dot{x} = -\sqrt{(1-x)^2} \text{ has unique solutions for all } x_0 \text{ except when } x_0 = 1.}$

HW2 Gabriel Colangelo

```
close all
clear all
clc
```

Problem 1

```
% Problem 1 A Matrix
A_1                = @(x) [0 1;(1 - 3*x(1,1)^2) -0.1];

% Equilibrium States
xe1                = [0;0];
xe2                = [1;0];
xe3                = [-1;0];

% Eigenvalues
disp('Eigenvalues for x_e(1)')
```

```
Eigenvalues for x_e(1)
```

```
disp(eig(A_1(xe1)))
```

```
   -1.0512
    0.9512
```

```
disp('Eigenvalues for x_e(2)')
```

```
Eigenvalues for x_e(2)
```

```
disp(eig(A_1(xe2)))
```

```
  -0.0500 + 1.4133i
  -0.0500 - 1.4133i
```

```
disp('Eigenvalues for x_e(3)')
```

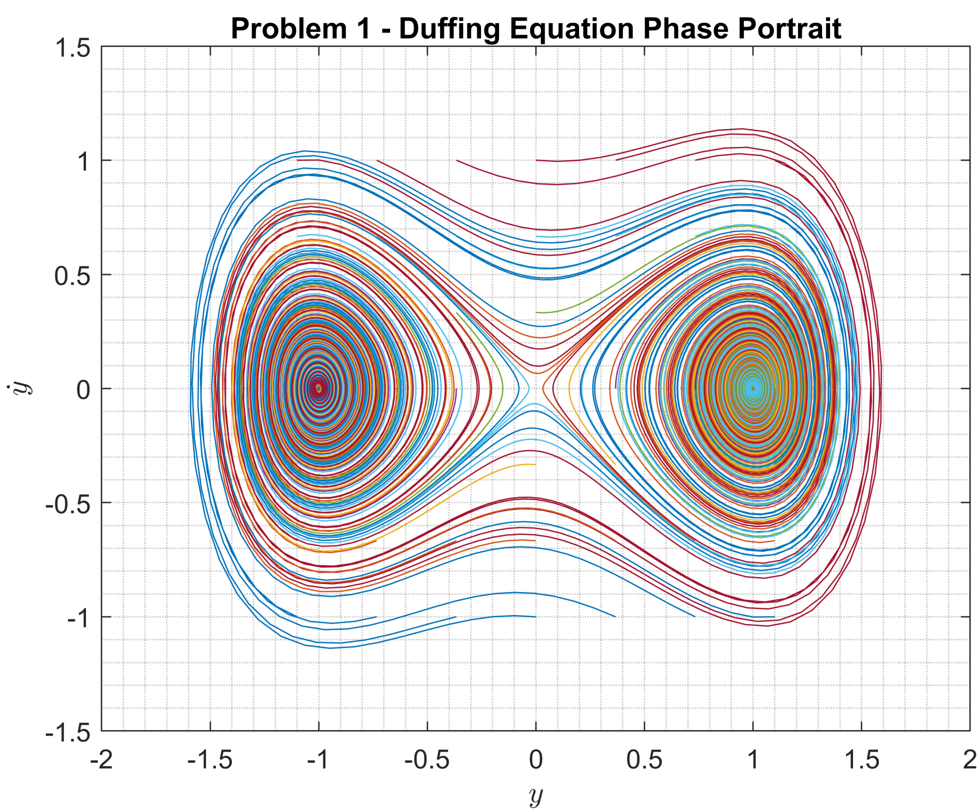
```
Eigenvalues for x_e(3)
```

```
disp(eig(A_1(xe3)))
```

```
  -0.0500 + 1.4133i
  -0.0500 - 1.4133i
```

```
% sim time
time               = (0:.1:80)';

% ODE45 solver options
options            = odeset('AbsTol',1e-8,'RelTol',1e-8);

% Initial Conditions to loop through
y_IC               = linspace(-1.1,1.1,7);
ydot_IC            = linspace(-1,1,7);

[IC_x,IC_y]        = meshgrid(y_IC,ydot_IC);
IC                 = [IC_x(:)';IC_y(:)'];
```

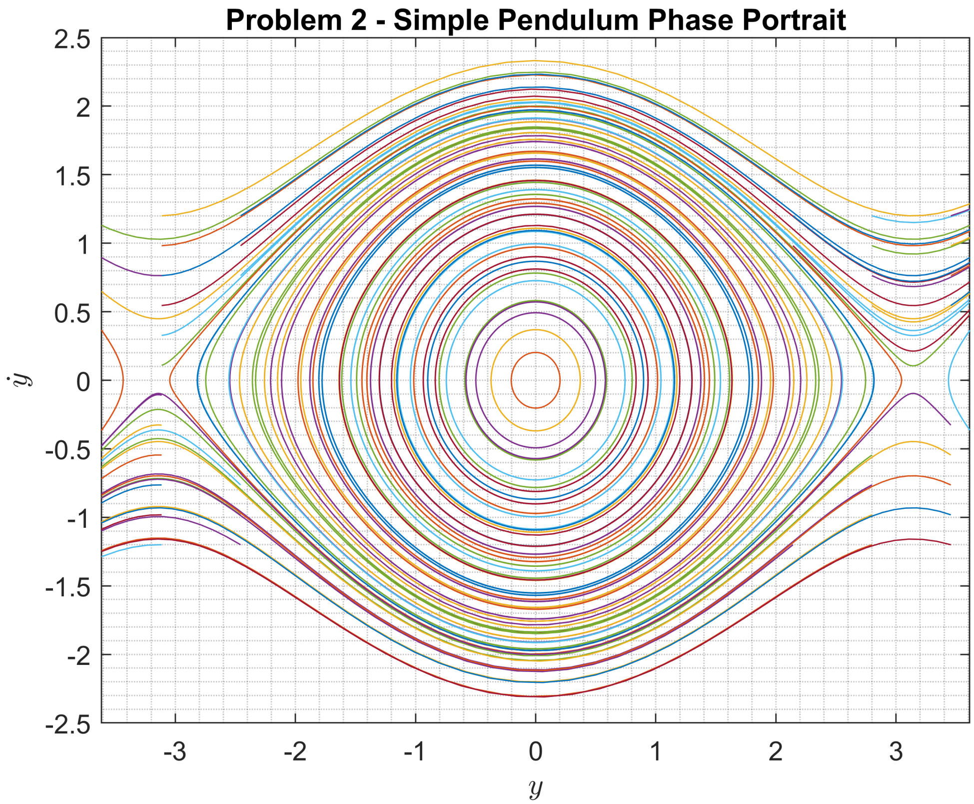
```
% Initialize vectors
y               = zeros(length(time),length(IC));
ydot            = y;

for i = 1:length(IC)
    % ODE45 Function call
    [~, X]      = ode45(@(t,x) Duffing(t,x), time, IC(:,i), options);

    % Extract and Store States
    y(:,i)      = X(:,1);
    ydot(:,i)   = X(:,2);
end

figure
plot(y,ydot)
grid minor
ylabel('$\dot{y}$','Interpreter','latex')
xlabel('$y$','Interpreter','latex')
title('Problem 1 - Duffing Equation Phase Portrait')
```



Problem 2

```
% Sim time
pend_t                  = 0:.1:50;

% Pendulum Equilibrium States
pend_xe1                = [0;0];
```

```
pend_xe2                = [pi;0];

% Initial Conditions to loop through
pend_y_IC               = linspace(-1.2*pi,1.1*pi,12);
pend_ydot_IC            = linspace(-1.2,1.2,12);

[pend_IC_x,pend_IC_y]   = meshgrid(pend_y_IC,pend_ydot_IC);
pend_IC                 = [pend_IC_x(:)';pend_IC_y(:)'];

% Initialize vectors
pend_y                  = zeros(length(pend_t),length(pend_IC));
pend_ydot               = pend_y;

for i = 1:length(pend_IC)
    % ODE45 Function call
    [~, X]          = ode45(@(t,x) SimplePendulum(t,x),...
                            pend_t, pend_IC(:,i), options);

    % Extract and Store States
    pend_y(:,i)     = X(:,1);
    pend_ydot(:,i)  = X(:,2);
end

figure
plot(pend_y,pend_ydot)
xlim([-1.15*pi 1.15*pi])
grid minor
ylabel('$\dot{y}$','Interpreter','latex')
xlabel('$y$','Interpreter','latex')
title('Problem 2 - Simple Pendulum Phase Portrait')
```



Functions

```
function xdot = Duffing(t,x)
xdot(1,1)    = x(2,1);
xdot(2,1)    = -0.1*x(2,1) + x(1,1) - x(1,1)^3;

end

function xdot = SimplePendulum(t,x)
xdot(1,1)    = x(2,1);
xdot(2,1)    = -sin(x(1,1));
end
```