PURDUE UNIVERSITY

ECE 68000: MODERN AUTOMATIC CONTROL

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Introduction to linear matrix inequalities

Outline

- Motivation
- Definitions of convex set and convex function
- Linear matrix inequality (LMI)
- Canonical LMI
- Example of LMIs

The Lyapunov theorem

Lyapunov's thm:

A constant square matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ has its eigenvalues in the open left half-complex plane if and only if for any real, symmetric, positive definite $\boldsymbol{Q} \in \mathbb{R}^{n\times n}$, the solution $\boldsymbol{P} = \boldsymbol{P}^\top$ to the Lyapunov matrix equation

$$\boldsymbol{A}^\top \boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} = -\boldsymbol{Q}$$

is positive definite

The Lyapunov thm re-stated

Lyapunov's thm:

The real parts of the eigenvalues of $\boldsymbol{A}$ are all negative if and only if there exists a real symmetric positive definite matrix $\boldsymbol{P}$ such that

$$\boldsymbol{A}^\top \boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} \prec 0$$

Equivalently,

$$-\boldsymbol{A}^\top \boldsymbol{P} - \boldsymbol{P}\boldsymbol{A} \succ 0$$

Background results

- Let

$$\boldsymbol{P} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ x_2 & x_{n+1} & \cdots & x_{2n-1} \\ \vdots & & & \vdots \\ x_n & x_{2n-1} & \cdots & x_q \end{bmatrix}$$

- Define

$$\boldsymbol{P}_1 = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \ \boldsymbol{P}_2 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

Defining P_is

- $$\boldsymbol{P}_q = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$
- Note that each $\boldsymbol{P}_i$ has only non-zero elements corresponding to x_i in $\boldsymbol{P}$
- We have
$$\boldsymbol{P} = x_1\boldsymbol{P}_1 + x_2\boldsymbol{P}_2 + \cdots + x_q\boldsymbol{P}_q \succ 0$$
- Problem: Find x_i, $i = 1, 2, \ldots, q$, such that $\boldsymbol{P} \succ 0$

$\boldsymbol{A}^\top \boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} \prec 0$ re-stated

- Let as before

$$\boldsymbol{P} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ x_2 & x_{n+1} & \cdots & x_{2n-1} \\ \vdots & & & \vdots \\ x_n & x_{2n-1} & \cdots & x_q \end{bmatrix}$$

- Define as before

$$\boldsymbol{P}_1 = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \ \boldsymbol{P}_2 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

Use P_is in the Lyapunov's equation

- $$\boldsymbol{P}_q = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$
- Note that each $\boldsymbol{P}_i$ has only non-zero elements corresponding to x_i in $\boldsymbol{P}$
- Let
$$\boldsymbol{F}_i = -\boldsymbol{A}^\top \boldsymbol{P}_i - \boldsymbol{P}_i \boldsymbol{A}, \quad i = 1, 2, \ldots, q$$

Manipulate

- $$\begin{aligned} \boldsymbol{A}^\top \boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} &= x_1 \left(\boldsymbol{A}^\top \boldsymbol{P}_1 + \boldsymbol{P}_1 \boldsymbol{A} \right) \\ &\quad + x_2 \left(\boldsymbol{A}^\top \boldsymbol{P}_2 + \boldsymbol{P}_2 \boldsymbol{A} \right) \\ &\quad + \cdots + x_q \left(\boldsymbol{A}^\top \boldsymbol{P}_q + \boldsymbol{P}_q \boldsymbol{A} \right) \\ &= -x_1 \boldsymbol{F}_1 - x_2 \boldsymbol{F}_2 - \cdots - x_q \boldsymbol{F}_q \\ &\prec 0 \end{aligned}$$
- Let $\boldsymbol{F}(\boldsymbol{x}) = x_1 \boldsymbol{F}_1 + x_2 \boldsymbol{F}_2 + \cdots + x_q \boldsymbol{F}_q$

Lyapunov's Inequality restated

- $$P = P^\top \succ 0 \quad \text{and} \quad A^\top P + PA \prec 0$$

  if and only if

  $$F(x) \succ 0$$

- Equivalently,

  $$\begin{bmatrix} P & O \\ O & -A^\top P - PA \end{bmatrix} \succ 0$$

Linear Matrix Inequality

- Consider $n+1$ real symmetric matrices

$$\boldsymbol{F}_i = \boldsymbol{F}_i^\top \in \mathbb{R}^{m\times m}, \quad \boldsymbol{i} = 0, 1, \ldots, \boldsymbol{n}$$

  and a vector $\boldsymbol{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^\top$
- Construct an affine function

$$\begin{aligned} \boldsymbol{F}(\boldsymbol{x}) &= \boldsymbol{F}_0 + x_1\boldsymbol{F}_1 + \ldots + x_n\boldsymbol{F}_n \\ &= \boldsymbol{F}_0 + \sum_{i=1}^{n} x_i\boldsymbol{F}_i \end{aligned}$$

Linear Matrix Inequality—Definition

$$\boldsymbol{F}(\boldsymbol{x}) = \boldsymbol{F}_0 + x_1\boldsymbol{F}_1 + \cdots + x_n\boldsymbol{F}_n \succ 0$$

Find a set of vectors $\boldsymbol{x}$ such that

$$\boldsymbol{z}^\top \boldsymbol{F}(\boldsymbol{x})\boldsymbol{z} > 0 \;\text{ for all }\; \boldsymbol{z} \in \mathbb{R}^m, \;\; \boldsymbol{z} \neq \boldsymbol{0},$$

that is, $\boldsymbol{F}(\boldsymbol{x})$ is positive definite

Linear Matrix Inequality—Another Definition

$$\boldsymbol{F}(\boldsymbol{x}) = \boldsymbol{F}_0 + x_1\boldsymbol{F}_1 + \cdots + x_n\boldsymbol{F}_n \succeq 0$$

Find a set of vectors $\boldsymbol{x}$ such that

$$\boldsymbol{z}^\top \boldsymbol{F}(\boldsymbol{x})\boldsymbol{z} \geq 0 \;\text{ for all }\; \boldsymbol{z} \in \mathbb{R}^m,$$

that is, $\boldsymbol{F}(\boldsymbol{x})$ is positive semidefinite

Convex Set

Definition

A set $\Omega \subseteq \mathbb{R}^n$ is convex if for any $\boldsymbol{x}$ and $\boldsymbol{y}$ in Ω, the line segment between $\boldsymbol{x}$ and $\boldsymbol{y}$ lies in Ω, that is,

$$\alpha \boldsymbol{x} + (1 - \alpha)\boldsymbol{y} \in \Omega \quad \text{for any} \quad \alpha \in (0, 1)$$

Convex Function

Definition

A real-valued function

$$f : \boldsymbol{\Omega} \to \mathbb{R}$$

defined on a convex set $\boldsymbol{\Omega} \subseteq \mathbb{R}^n$ is convex if for all $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{\Omega}$ and all $\alpha \in (0, 1)$,

$$f(\alpha\boldsymbol{x} + (1 - \alpha)\boldsymbol{y}) \leq \alpha f(\boldsymbol{x}) + (1 - \alpha)f(\boldsymbol{y})$$

Convex Optimization Problem

Definition

A convex optimization problem is the one where the objective function to be minimized is convex and the constraint set, over which we optimize the objective function, is a convex set.

Warning: If f is a convex function, then

$$\begin{aligned} &\max f(\boldsymbol{x}) \\ &\text{subject to } \boldsymbol{x} \in \boldsymbol{\Omega} \end{aligned}$$

is NOT a convex optimization problem!

Another LMI Example

A system of LMIs,

$$\boldsymbol{F}_1(\boldsymbol{x}) \succeq 0, \ \boldsymbol{F}_2(\boldsymbol{x}) \succeq 0, \ldots, \boldsymbol{F}_k(\boldsymbol{x}) \succeq 0$$

can be represented as one single LMI

$$\boldsymbol{F}(\boldsymbol{x}) = \begin{bmatrix} \boldsymbol{F}_1(\boldsymbol{x}) & & & \\ & \boldsymbol{F}_2(\boldsymbol{x}) & & \\ & & \ddots & \\ & & & \boldsymbol{F}_k(\boldsymbol{x}) \end{bmatrix} \succeq 0$$

Yet Another LMI Example

A linear matrix inequality, involving an m-by-n constant matrix $\boldsymbol{A}$, of the form,

$$\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$$

can be represented as m LMIs

$$b_i - \boldsymbol{a}_i^\top \boldsymbol{x} \geq 0, \quad i = 1, 2, \ldots, m,$$

where $\boldsymbol{a}_i^\top$ is the i-th row of the matrix $\boldsymbol{A}$

Example # 2 Contd

- View each scalar inequality as an LMI
- Represent m LMIs as one LMI,

$$\boldsymbol{F}(\boldsymbol{x}) = \begin{bmatrix} b_1 - \boldsymbol{a}_1^\top \boldsymbol{x} & & & \\ & b_2 - \boldsymbol{a}_2^\top \boldsymbol{x} & & \\ & & \ddots & \\ & & & b_m - \boldsymbol{a}_m^\top \boldsymbol{x} \end{bmatrix} \succeq 0$$

Notation $>$ versus $\geq$

- Most of the optimization solvers do not handle strict inequalities
- Therefore, the operator $>$ is the same as $\geq$, and so $>$ implements the non-strict inequality $\geq$

Solving LMIs

- $\boldsymbol{F}(\boldsymbol{x}) = \boldsymbol{F}_0 + x_1\boldsymbol{F}_1 + \cdots + x_n\boldsymbol{F}_n \succeq 0$ is called the *canonical representation* of an LMI
- The LMIs in the canonical form are very inefficient from a storage view-point as well as from the efficiency of the LMI solvers view-point
- The LMI solvers use a structured representation of LMIs