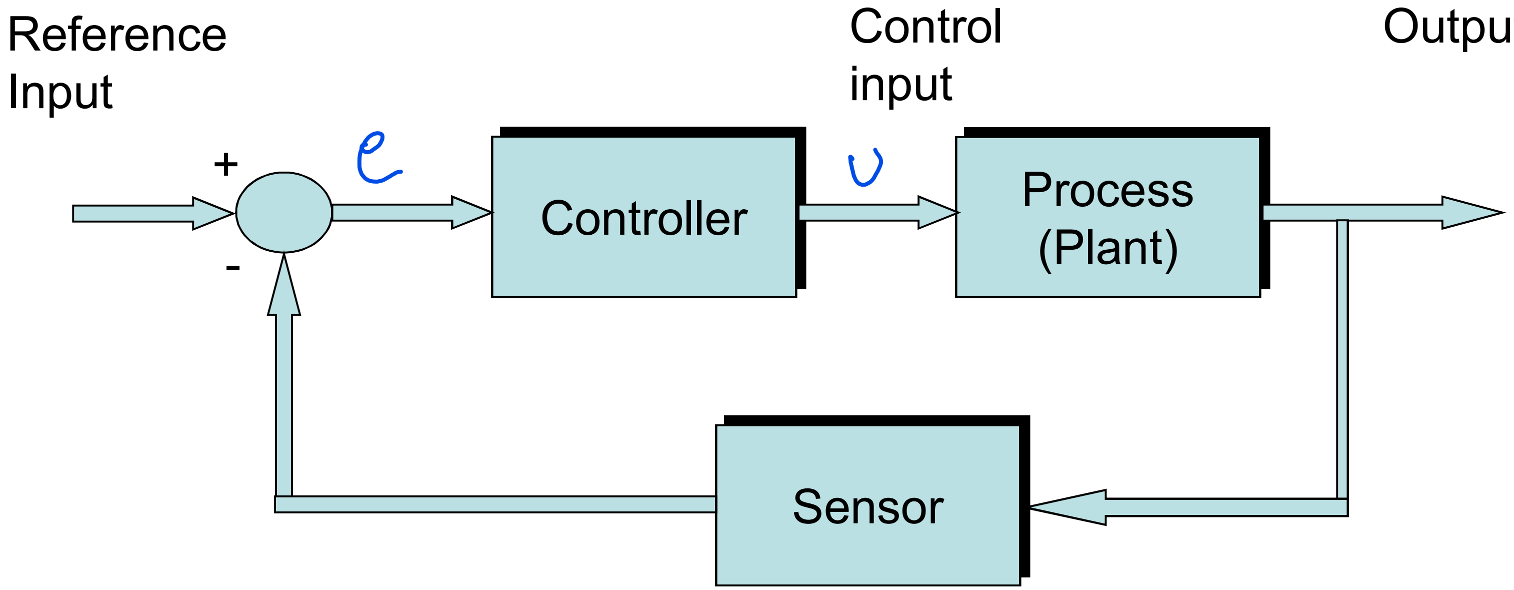
Continuous Control
MAE 443/543

Closed Loop Control system



Negative Feedback

Guilt-Tripping Radar Speed Sign

I see the display of 25 mph and slow down, if the display indicates I am below 10 mph, I speed up to try to reach and stay near the speed limit. This is the notion of negative feedback which endeavors to maintain stability.

Positive Feedback

Ethylene is a gas and is known as the “fruit-ripening hormone.” Every fruit has a certain level of ethylene production throughout its lifecycle. However, in some fruits, ethylene levels shoot up when the fruit starts ripening.

Ethylene gas is commercially used to ripen fruits after they have been picked. Fruits, such as tomato, banana, and pear are harvested just before ripening has started (typically in a hard, green, but mature stage). This allows time for the fruit to be stored and transported to distant places.

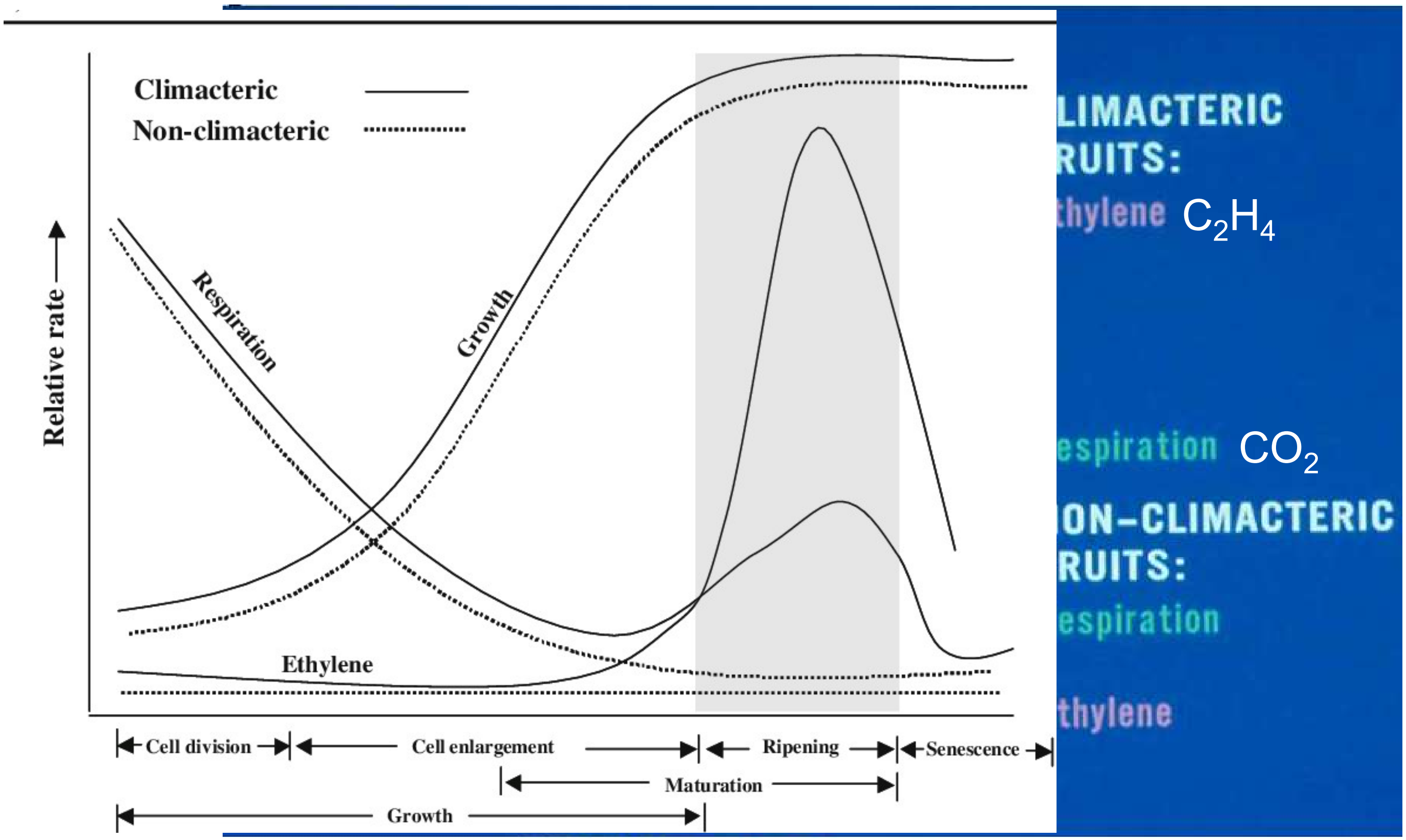
The saying, “*one bad apple can spoil the whole basket,”* is based upon the release of ethylene from rotting apples, which accelerates the ripening of other apples around the rotting one. (Positive Feedback)

Try at Home: If you have an unripe avocado or other fruits at home, try putting them in a paper bag with a ripening banana. This will speed up the ripening of the avocado Because of the ethylene emitted by the ripening banana.

Fruit Ripening Technology

Climacteric Fruits: are fruits that continue to ripen after harvest
Examples: Mango, Banana, Papaya, Kiwi, Apple etc.

Non-Climacteric Fruits: are fruits that once harvested do not ripen further
Examples: Orange, Grapefruit, Grapes, Strawberry, Cashew.



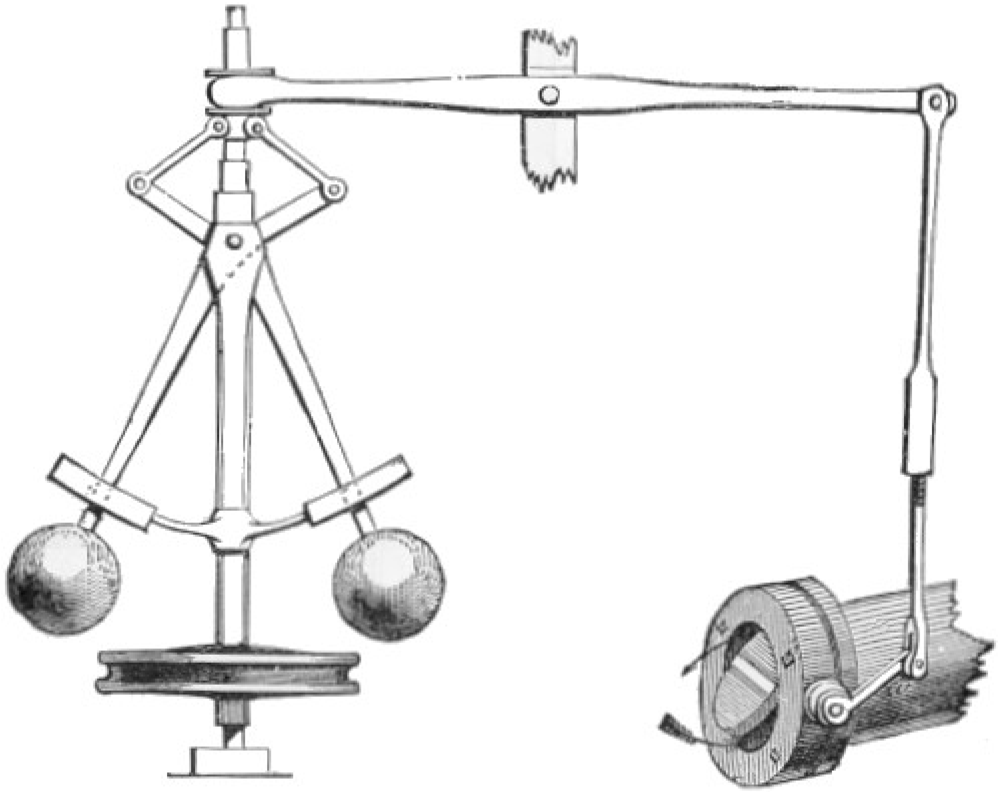
Positive vs Negative Feedback Story

"The distinction between the stabilizing and destabilizing character of negative and positive feedback loops is neatly captured in the story of the misconnected electric blanket. The newlyweds were given an electric blanket for their queen-size double bed. The blanket had separate temperature settings for the two sides of the bed, one for him and one for her. Properly connected, there should have been two separate negative feedback systems, each attempting to control the temperature of the blanket for the comfort of each individual. The story goes that the newlyweds misconnected the blanket so that his setting controlled her blanket temperature and hers controlled his. The result…was a nasty positive feedback system. She felt cold, turned up her setting, making his side too warm for him so he turned down his setting, making her even colder, so she raised her setting even further, and so on. How such a scenario would end is left up to the fertile imagination of the reader.

-Richardson, G. and Pugh, A. Introduction to System Dynamics Modeling with Dynamo, MIT Press, Cambridge, MA 1981, pp 11-12

Centrifugal Governor

http://en.wikipedia.org/wiki/Centrifugal_governor



Centrifugal governor, Boulton & Watt, 1798

http://en.wikipedia.org/wiki/James_Watt

Controlled Flight

The key innovation, which had no clear predecessor, was the use of wing warping to effect lateral control; that is, control for turning (Fig. 7). This innovation provided a full complement of movable aerodynamic surfaces to allow control over all three axes of rotational motion. This innovation was critical, since it made controlled flight possible.

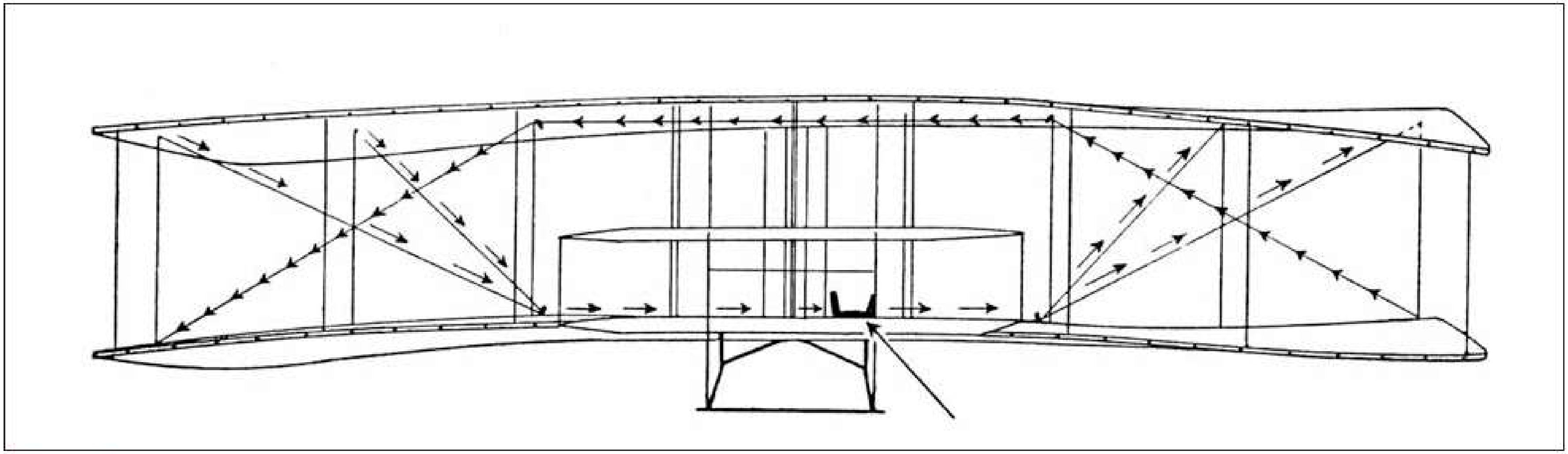


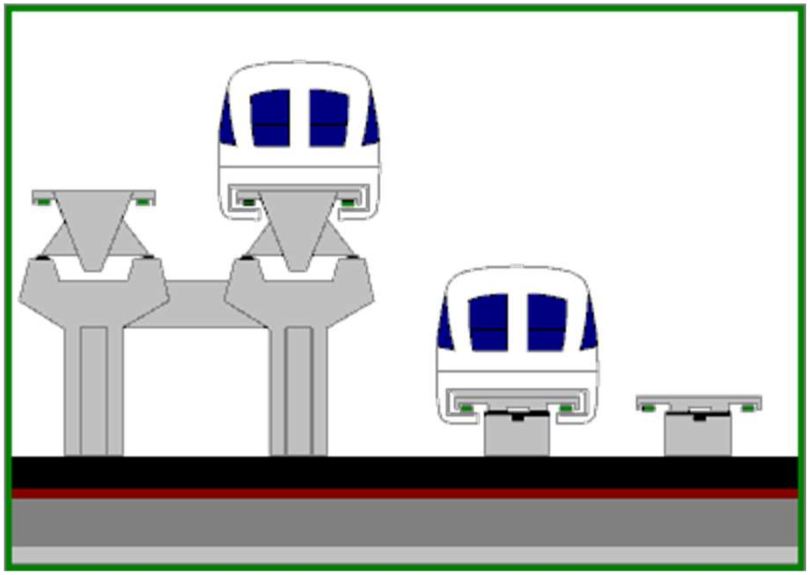
Figure 7. *The Wrights' key invention was the use of wing warping to effect lateral control. This innovation provided a full complement of movable aerodynamic surfaces to allow control over all three axes of rotational motion. ([Wright, 1953, p. 14], reproduced by permission.)*

Feedback Control: An Invisible Thread in the History of Technology, Dennis Bernstein

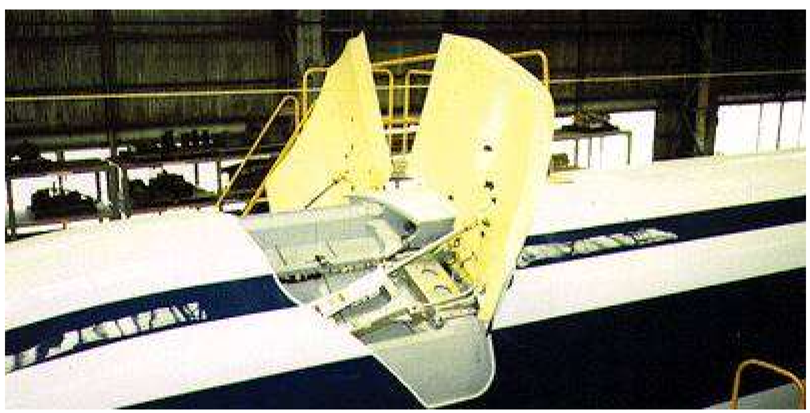
IEEE Control System Magazine, April 2002

MagLev Train

http://www.o-keating.com/hsr/mlx01.htm



The principal of a Magnet train is that floats on a magnetic field and is propelled by a linear induction motor. They follow guidance tracks with magnets. These trains are often referred to as *Magnetically Levitated* trains which is abbreviated to *MagLev.*

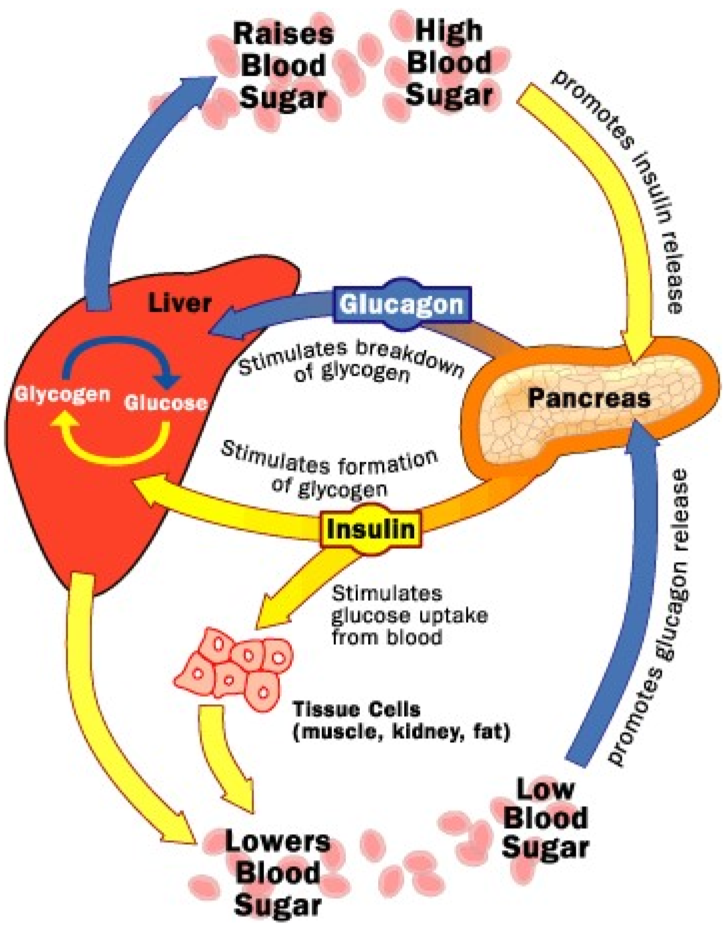


The Aerodynamic brakes on the MLX01

The MLX01
ML stands for maglev, and X for experimental.

Glucose Control (Diabetes)

http://health.howstuffworks.com/diseases-conditions/diabetes/diabetes1.htm



Type 1 Diabetes (Insulin Dependent, Juvenile Diabetes) *is the inability of the body* to regulate blood glucose by synthesizing the required insulin.

Type 2 Diabetes (Insulin Resistant, Adult onset Diabetes) *is the poor regulation of the* blood glucose due to reduction in insulin sensitivity.

Type 1 diabetes requires periodic Monitoring of blood glucose and Infusion of insulin.

Type 2 diabetes can be managed by diet and exercise.

Potential ramification of diabetes is retinopathy, neuropathy and nephropathy

Control of Epidemics

- I(t): infective class
- S(t): susceptive class
- R(t): removed class

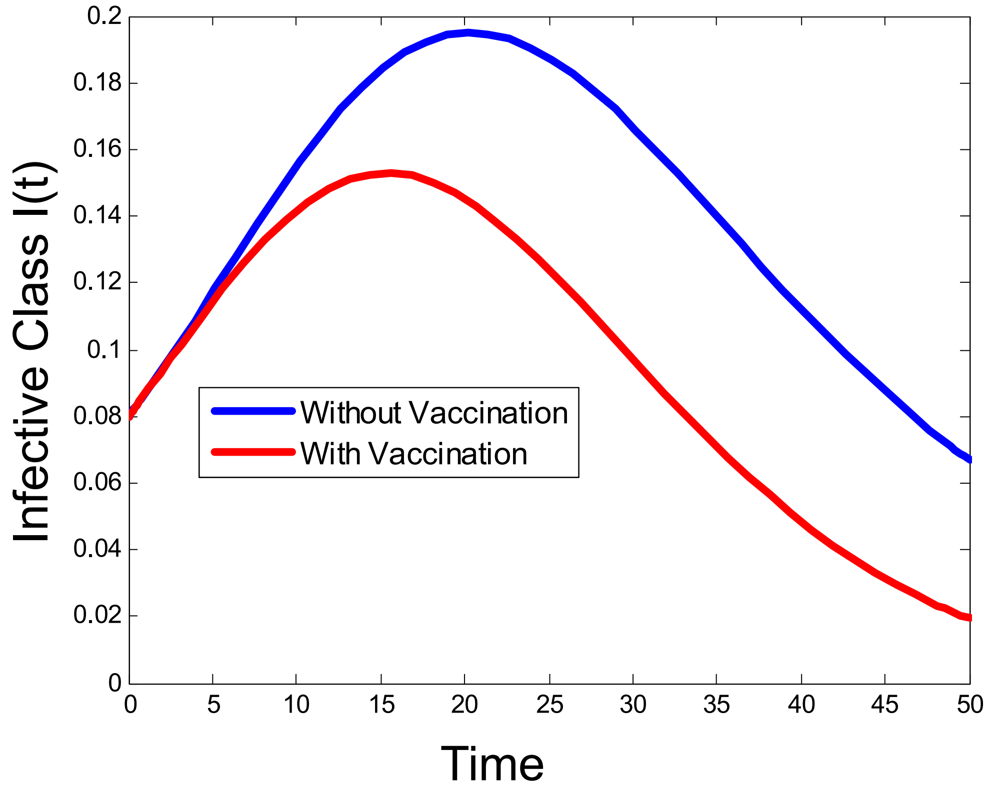
$$
\begin{aligned}
\dot{S}(t) &= -rS(t)I(t) & S(0) &= S_0 > 0 \\
\dot{I}(t) &= rS(t)I(t) - \gamma I(t) & I(0) &= I_0 > 0 \\
\dot{R}(t) &= \gamma I(t) & R(0) &= 0 \\
S_0 + I_0 &= 1, & S(t) + I(t) + R(t) &= 1
\end{aligned}
$$

- V(t): vaccinated individuals, α is the vaccination rate

$$
\begin{aligned}
\dot{S}(t) &= -rS(t)I(t) - \alpha & S(0) &= S_0 > 0 \\
\dot{I}(t) &= rS(t)I(t) - \gamma I(t) & I(0) &= I_0 > 0 \\
\dot{R}(t) &= \gamma I(t) & R(0) &= 0 \\
\dot{V}(t) &= \alpha & V(0) &= 0 \\
S_0 + I_0 &= 1, & S(t) + I(t) + R(t) + V(t) &= 1
\end{aligned}
$$

Control of Epidemics

$$\alpha = 0.01$$
$$\gamma = 0.1$$
$$r = 0.2$$



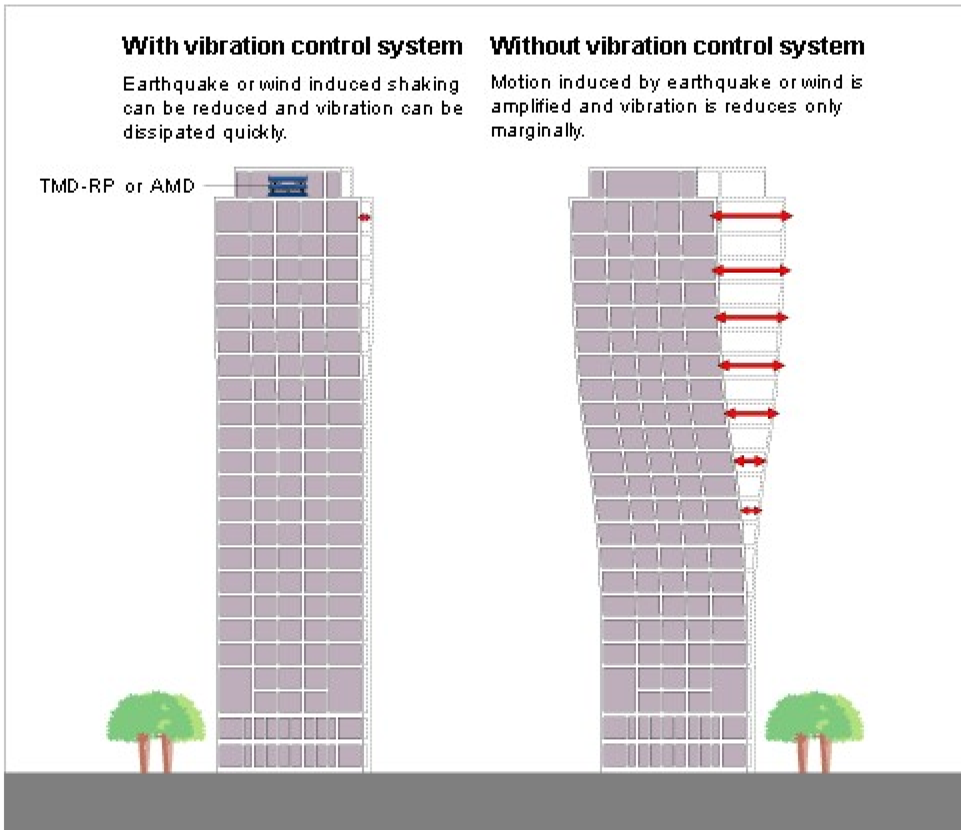
Introduction to Optimal Control, (pp 17-18)

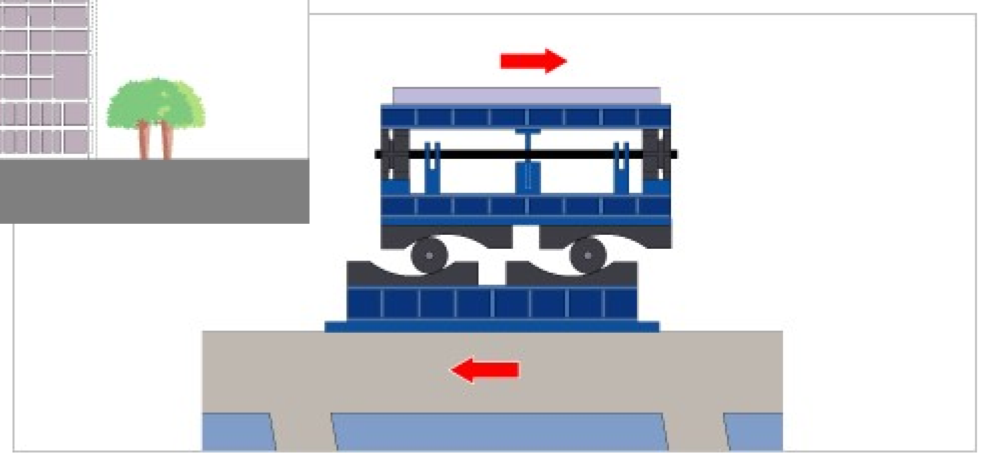
Jack Macki and Aaron Strauss

Springer-Verlag, 1982

Tuned Mass Damped (TMD)

http://www.oiles.co.jp/en/menshin/building/control/amd.html





Moniac (Money eNIAC)

Hydraulic simulator for economic modeling

ENIAC (**Electronic Numerical Integrator And Computer**, was the first large-scale, electronic, digital computer)

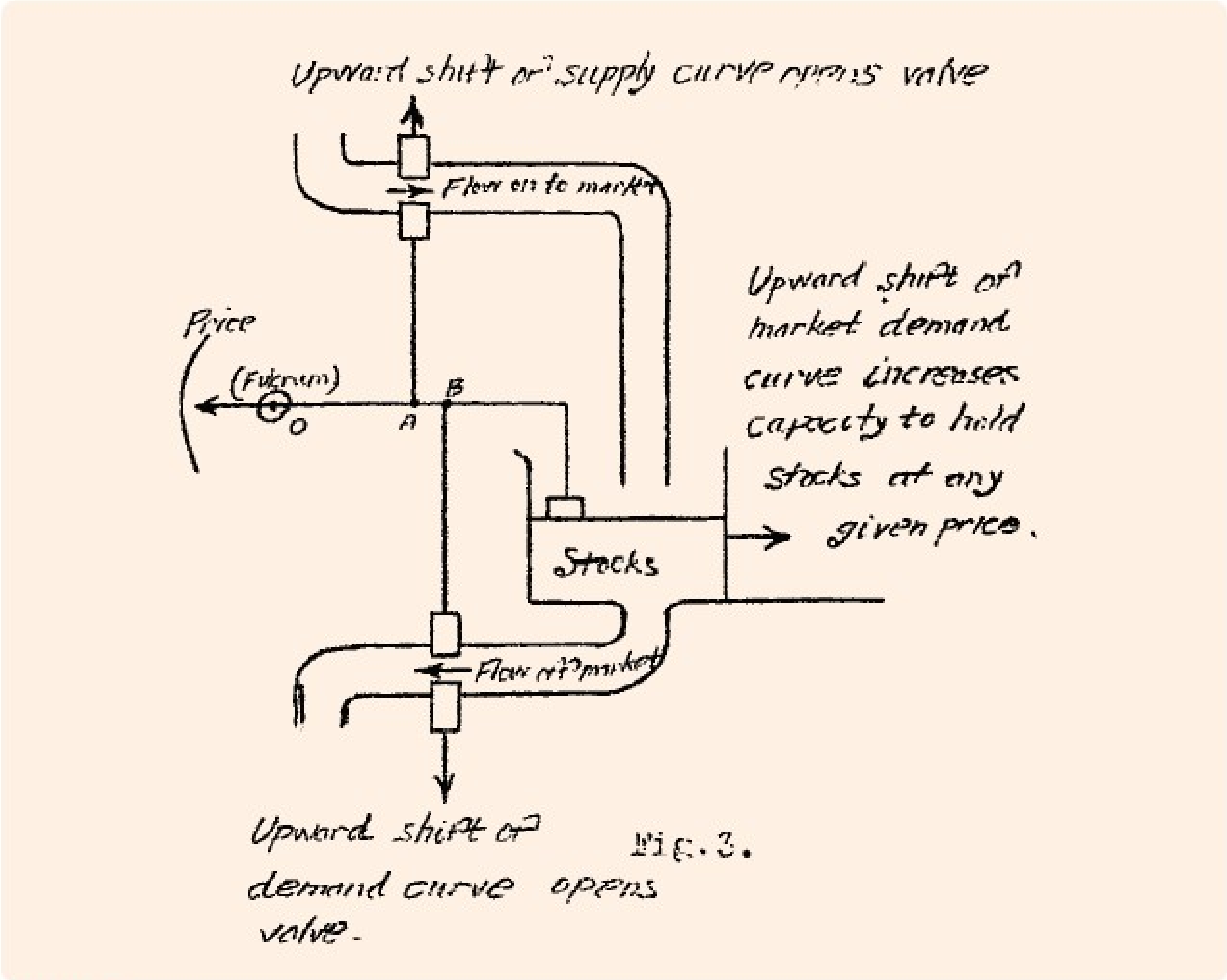
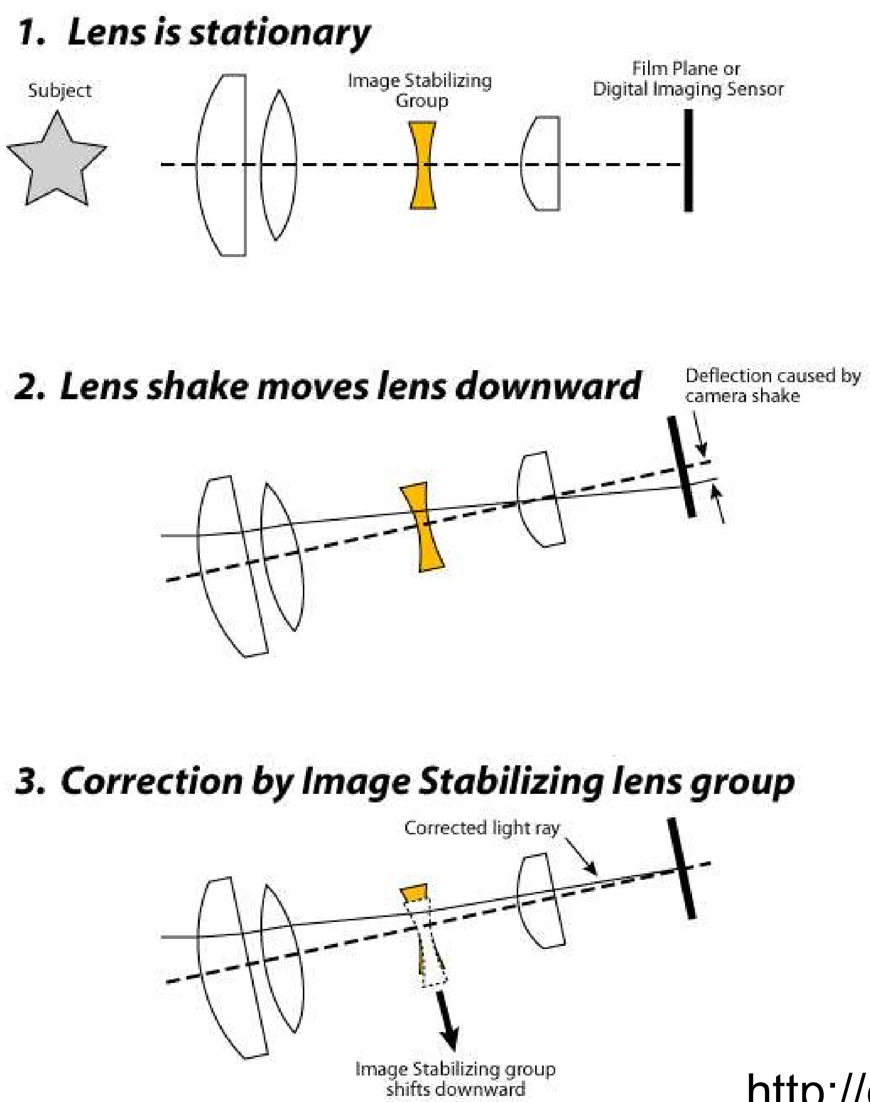
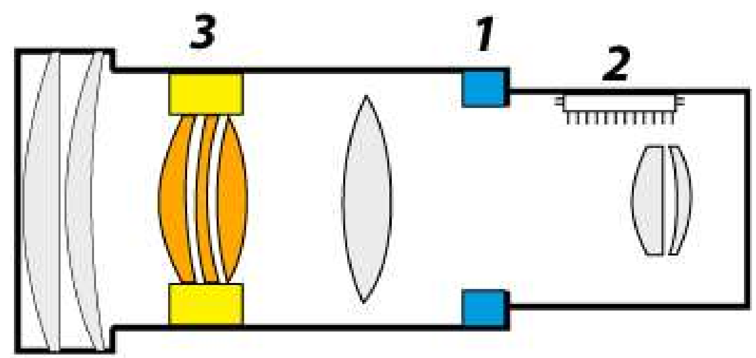


FIGURE 3 Phillips's sketch of the fundamental component of a hydraulic simulator of an economy, from an unpublished paper now in the possession of Martin Slater, Fellow of St. Edmund Hall, Oxford. This paper was the stimulus for building the prototype machine.

Image Stabilization

http://www.usa.canon.com/dlc/controller?act=GetArticleAct&articleID=706&fromTips=1



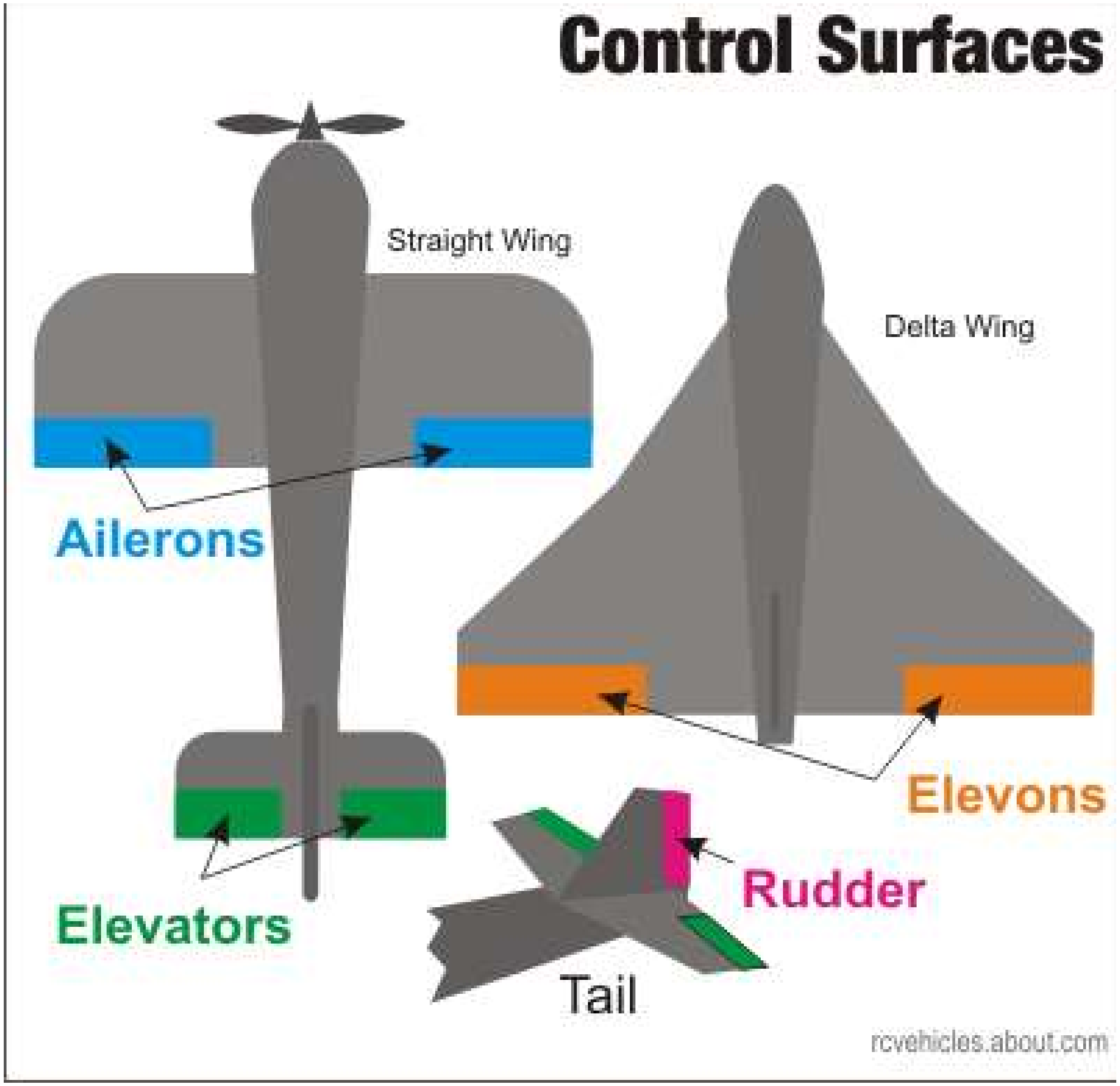


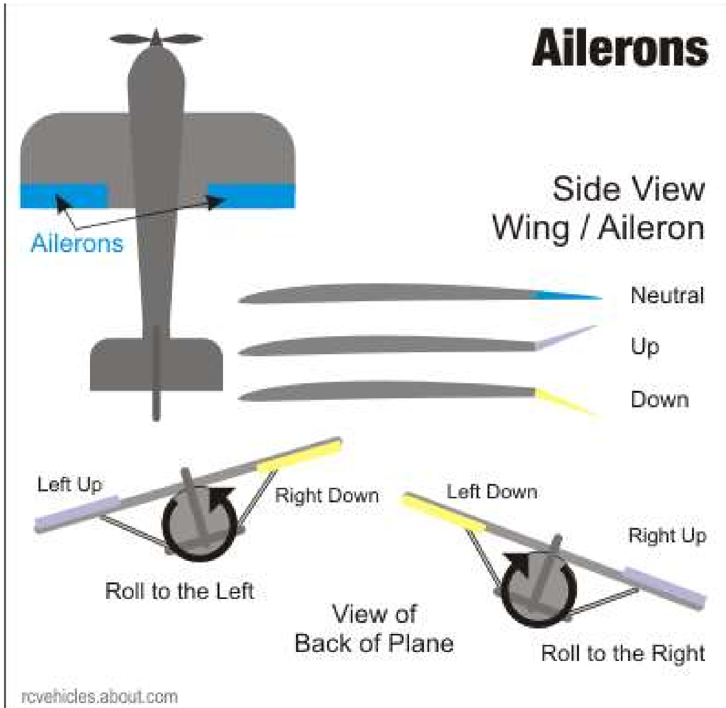
1. A pair of gyro sensors inside the lens detect any camera or lens movement. One sensor detects horizontal shake, the other vertical.

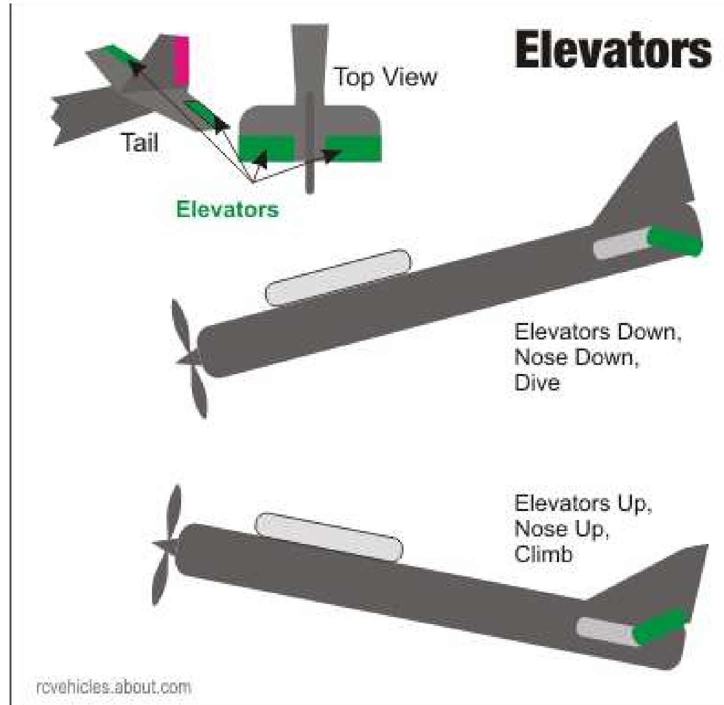
2. Signals from the pair of motion detectors are sent to a microprocessor, which instantly calculates the frequency and amplitude of the lens shake.

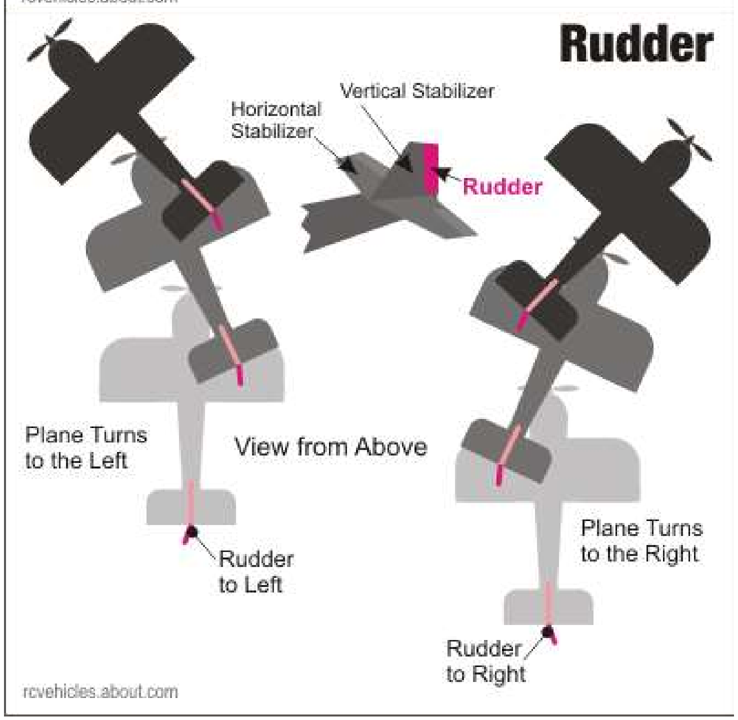
3. The group of optical elements which perform Image Stabilization are held in place by a device called a "coil". It's not a coil spring, as the name suggests — it's a round object which surrounds the Image Stabilizing element group. It can move the optics horizontally and vertically, and (when stabilization is off) lock and center the group. The coil receives signals from the microprocessor (step 2), which precisely tell the coil in what direction to move the Image Stabilizing optical elements, and at what rate.

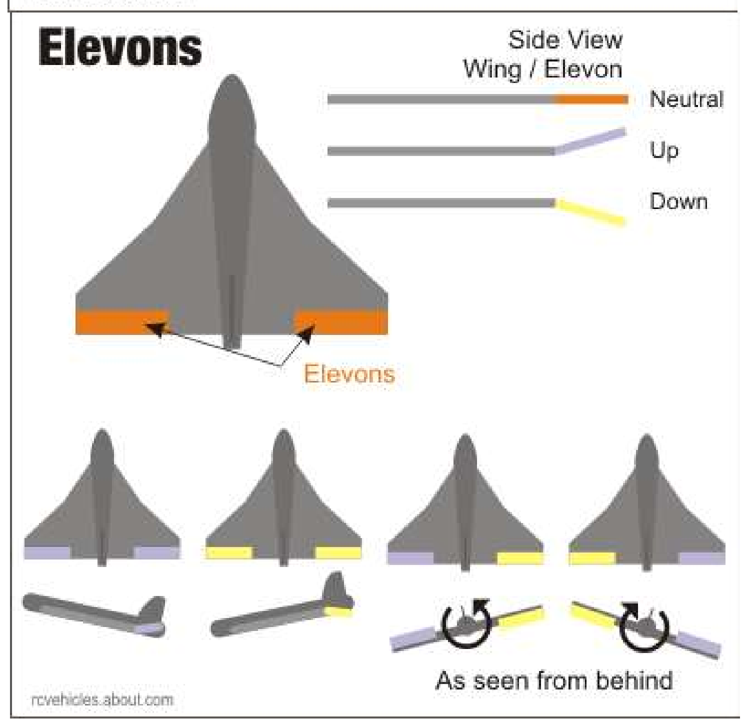
http://en.wikipedia.org/wiki/Vestibulo-ocular_reflex


Control Surfaces
Straight Wing
Delta Wing
Ailerons
Elevons
Rudder
Elevators
Tail
rcvehicles.about.com


Ailerons
Side View
Wing / Aileron
Ailerons
Neutral
Up
Down
Left Up
Right Down
Left Down
Right Up
Roll to the Left
View of
Back of Plane
Roll to the Right
rcvehicles.about.com


Elevators
Top View
Tail
Elevators
Elevators Down,
Nose Down,
Dive
Elevators Up,
Nose Up,
Climb
rcvehicles.about.com

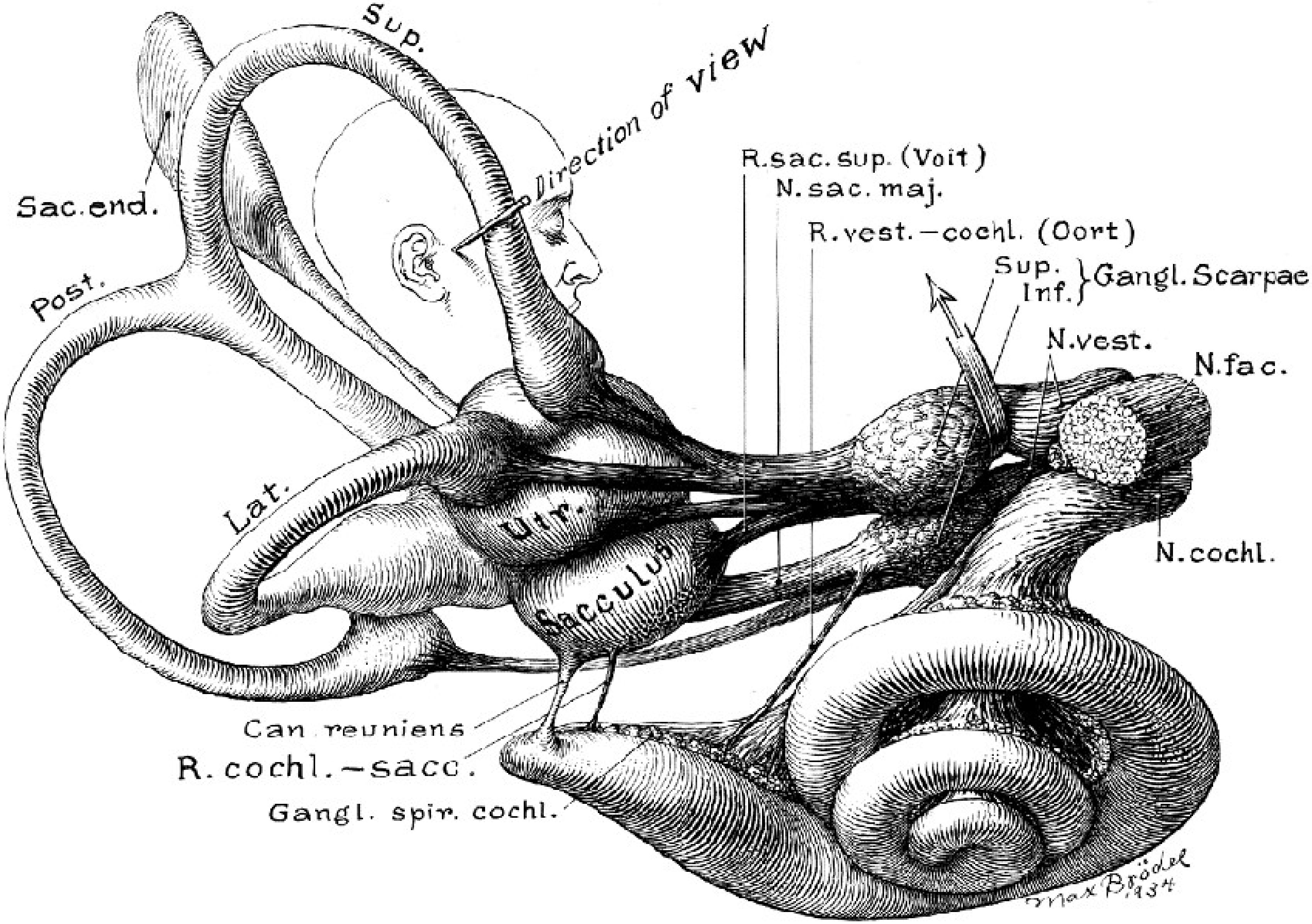

Rudder
Vertical Stabilizer
Horizontal
Stabilizer
Rudder
Plane Turns
to the Left
View from Above
Rudder
to Left
Plane Turns
to the Right
Rudder
to Right
rcvehicles.about.com

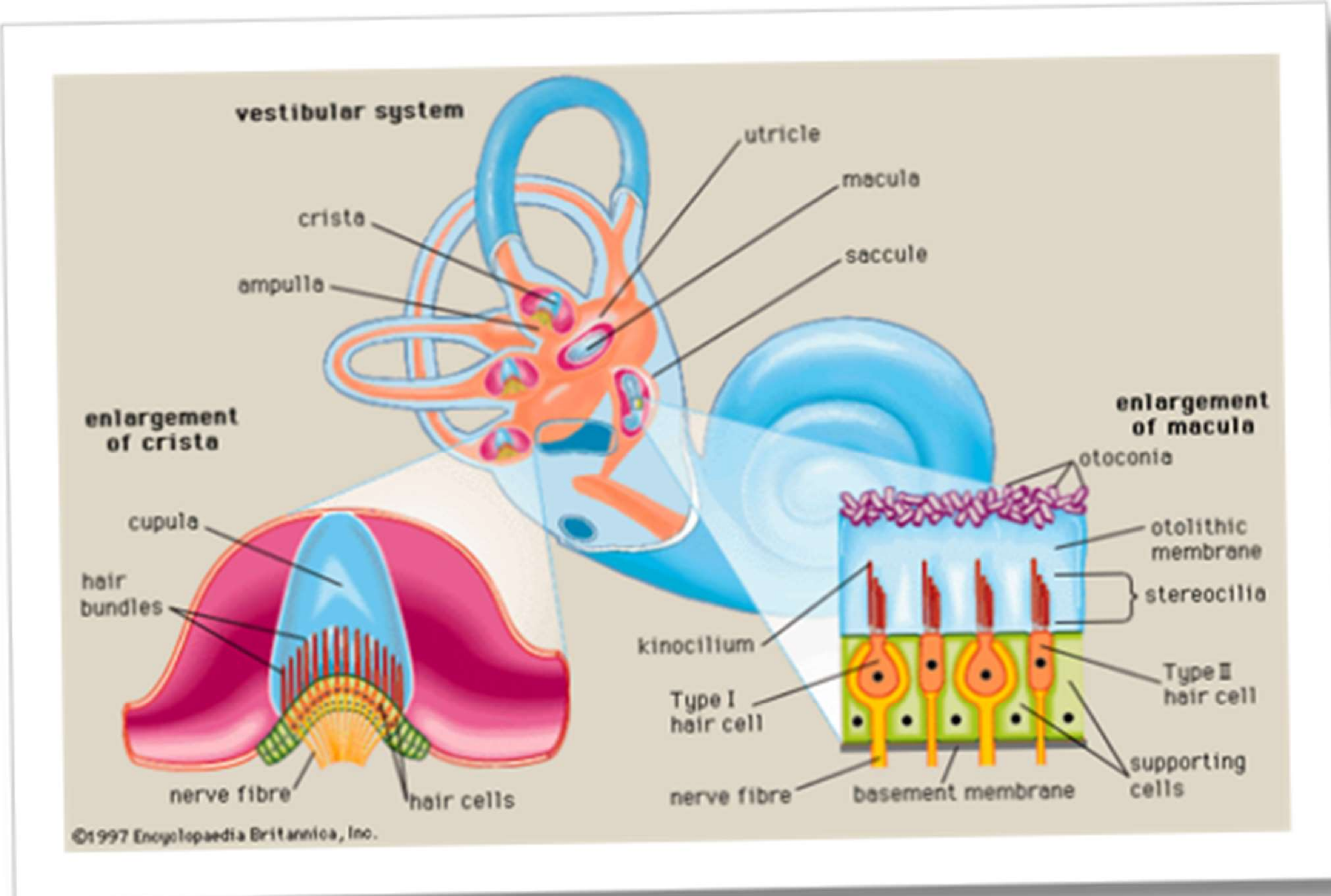

Elevons
Side View
Wing / Elevon
Neutral
Up
Down
Elevons
As seen from behind
rcvehicles.about.com

Pilot Induced Oscillations

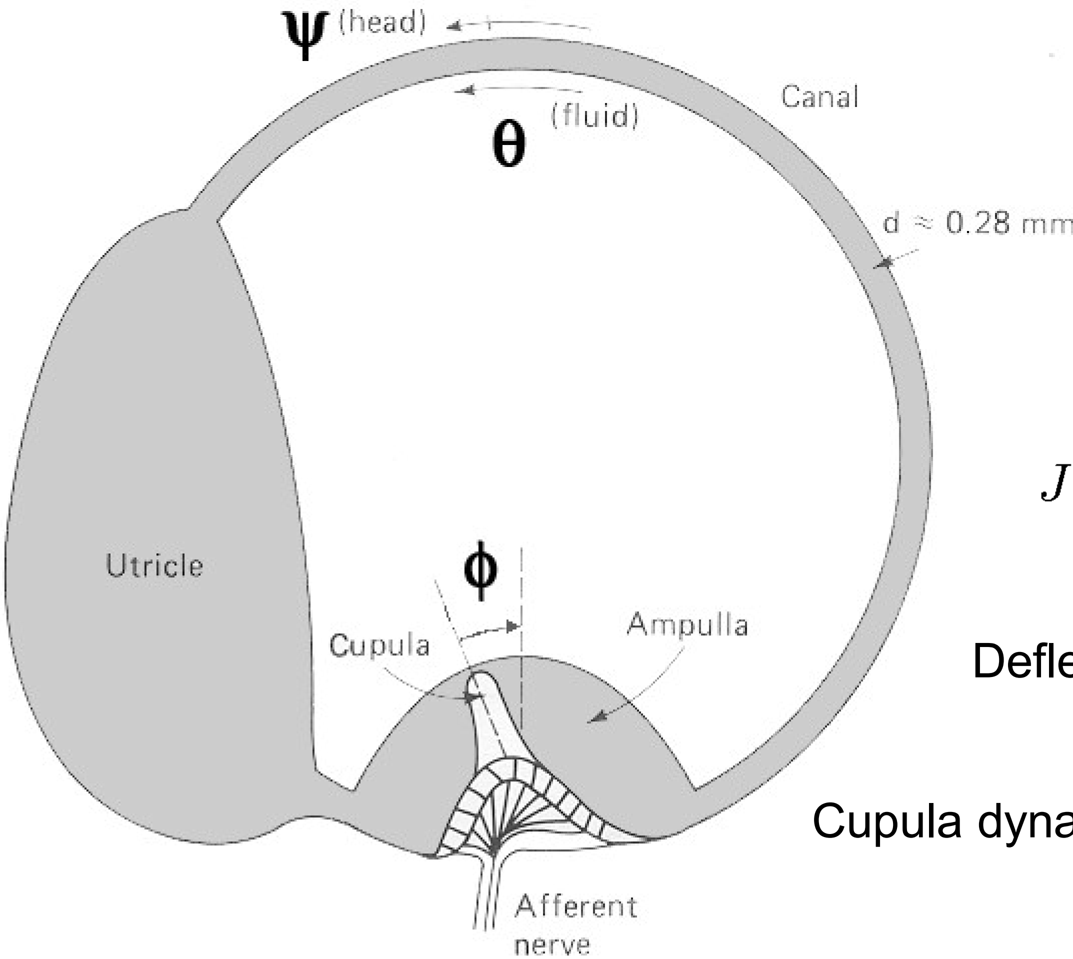
A pilot aiming for a 500-foot per minute descent, for example, may find himself descending too rapidly. He begins to apply up elevator until the vertical speed indicator shows 500 feet per minute. However, because the vertical speed indicator lags the actual vertical speed, the pilot is actually descending at much less than 500 feet per minute. The pilot then begins applying down elevator until the vertical speed indicator reads 500 feet per minute, starting the cycle over. It's harder than it might seem to stabilize the vertical speed because the airspeed also constantly changes.
https://en.wikipedia.org/wiki/Pilot-induced_oscillation

https://www.youtube.com/watch?v=XHPv0qt03aA


Sup.
Direction of view
Sac. end.
Post.
Lat.
Utr.
Sacculus
R.sac.sup. (Voit)
N.sac.maj.
R.vest.–cochl. (Oort)
Sup.
Inf.
Gangl. Scarpae
N.vest.
N.fac.
N.cochl.
Can. reuniens
R. cochl.–sacc.
Gangl. spir. cochl.
Max Brödel 1934


vestibular system
utricle
macula
crista
saccule
ampulla
enlargement of crista
enlargement of macula
otoconia
cupula
otolithic membrane
hair bundles
stereocilia
kinocilium
Type I hair cell
Type II hair cell
supporting cells
nerve fibre
hair cells
nerve fibre
basement membrane
©1997 Encyclopaedia Britannica, Inc.

http://www.swarthmore.edu/NatSci/echeeve1/Ref/InEar/InnerEar.html



ψ Position of Head

ϕ Deflection of Cupula

θ Displacement of Fluid

$$J\ddot{\theta} + b(\dot{\theta} - \dot{\psi}) + k(\theta - \psi) = 0$$

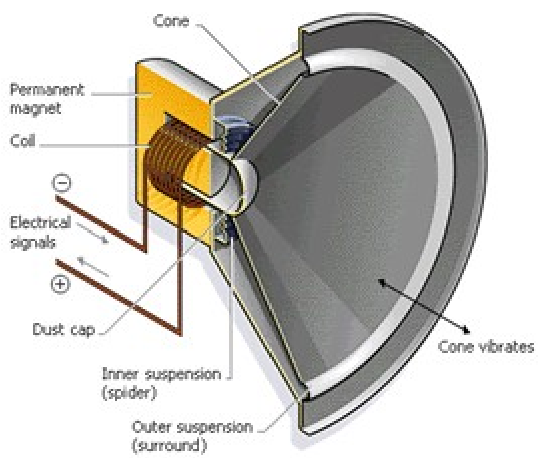
Deflection of Cupula: $\phi = \theta - \psi$

Cupula dynamics: $J\ddot{\phi} + b\dot{\phi} + k\phi = -J\ddot{\psi}$

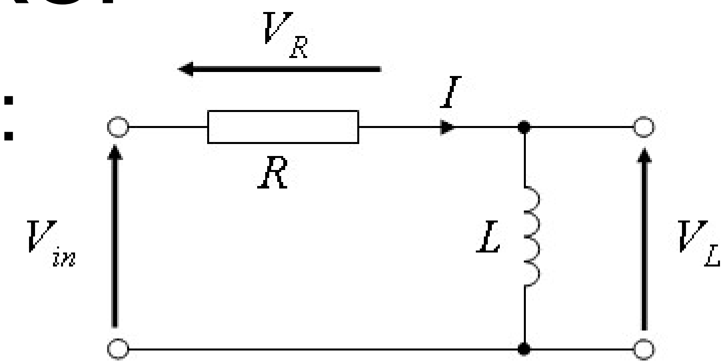
The spring constant k is very small, i.e., restoring force is very small. Therefore the System response if overdamped.

Why one gets dizzy: consider spinning for a long time. Spin for long enough that the small restoring force of the Cupula is enough to bring it back to its rest position. When one stops, the head acceleration goes to zero and the deflection of the Cupola in the opposite direction tells you that you are spinning in the opposite direction - you are dizzy

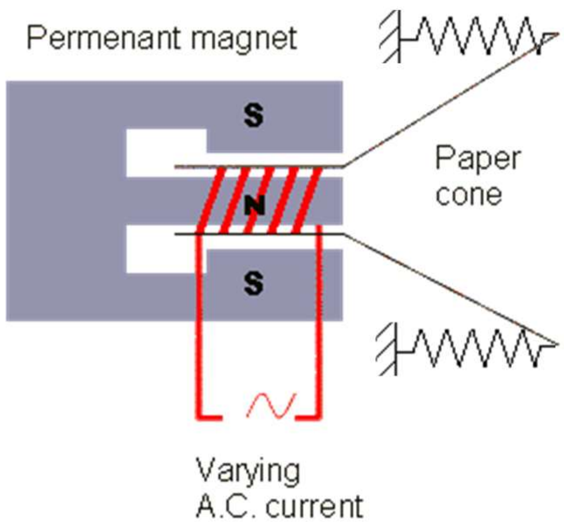
Loudspeaker



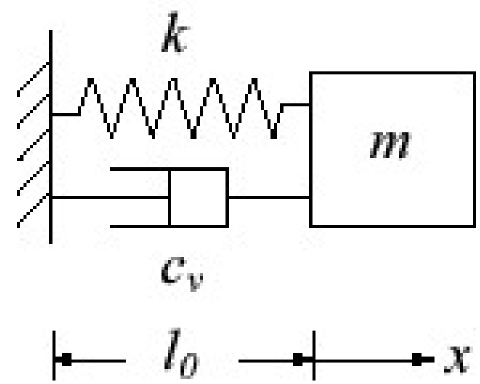
- RL circuit:



$$L\frac{dI}{dt} + RI + k_\tau \frac{dx}{dt} = V_{in}$$



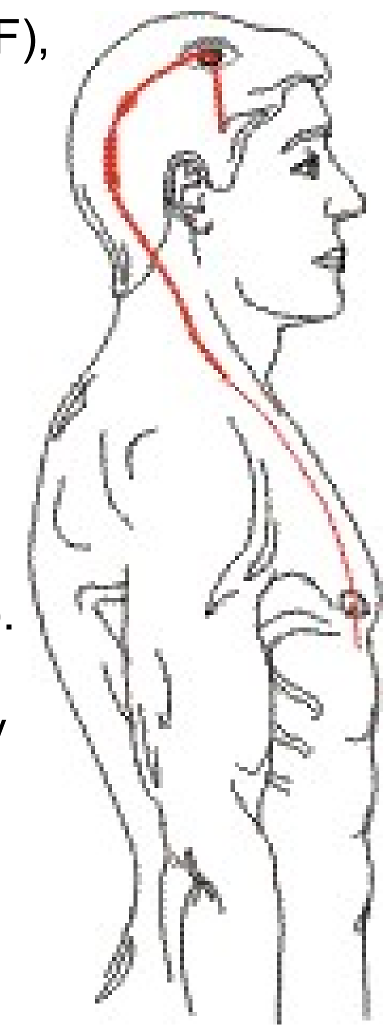
- Spring-Mass:



$$m\frac{d^2x}{dt^2} + c_v\frac{dx}{dt} + kx = k_\tau I$$

Hydrocephalus Therapy

http://www.medit.hia.rwth-aachen.de/en/research/hydrocephalus_implant/index.html



The human brain is immersed in a fluid (cerebrospinal fluid, CSF), which, among other things, protects the brain from mechanical stress (e.g. concussion) and helps support its weight through buoyancy. In normal situations, the production and reabsorption of this fluid are equal. However, a constant overproduction, blockage (i.e. tumor) or reabsorption difficulty can upset this natural balance, resulting in a build-up of fluid in the skull (Hydrocephalus). In adults, this excess fluid causes large pressures to develop rapidly in the skull, and impairs brain function. The most common solution today is the implantation of a passive pressure-control valve and catheter system (shunt). Once the pressure in the skull exceeds a certain critical value, the excess fluid is released through the open valve and typically drained into the stomach cavity (Fig. 1). Unfortunately, these passive valves encounter many problems, including over- and under- drainage, occlusions, and system failure. These problems may be avoided through the use of a mechatronic valve, which could monitor the patient's health and properly regulate the amount of fluid in the skull through pressure, flow and inclination sensors.

Need for Control Engineers

- “There are about 150,000 control engineers in the United States and a similar number in Japan and also in Europe. In the United States alone, the control industry does a business of over $50 billion per year! The theory, practice, and application of automatic control is a large, exciting, and extremely useful engineering discipline. One can readily understand the motivation for a study of modern control systems.”

Modern Control Systems, Dorf and Bishop, 10th Edition, Prentice Hall, 2005

Survey on Impact of Control Technologies

	Current Impact		Future Impact	
Control Technology	% High	Low/No	High	Low/No
PID control	91%	0%	78%	6%
System Identification	65%	5%	72%	5%
Estimation & filtering	64%	11%	63%	3%
Model-predictive control	62%	11%	85%	2%
Process data analytics	51%	15%	70%	8%
Fault detection & identification	48%	17%	8%	8%
Decentralized and/or coordinated control	29%	33%	54%	11%
Robust control	26%	35%	42%	23%
Intelligent control	24%	38%	5 9%	11%
Nonlinear control	21%	44%	42%	15%
Discrete-event systems	24%	45%	39%	27%
Adaptive control	18%	38%	44%	17%
Repetitive control	12%	74%	17%	51%
Other advanced control technology	11%	64%	25%	39%
Hybrid dynamical systems	11%	68%	33%	33%
Game theory	5%	76%	17%	52%

IFAC Newsletter April 2019

Stages in Control System Design

Modeling

- *Physics based model derivation*
- *Non-Physics based modeling*
- *System Identification*
  - *mode structure/class selection*
  - *parameter estimation*
  - *model validation*

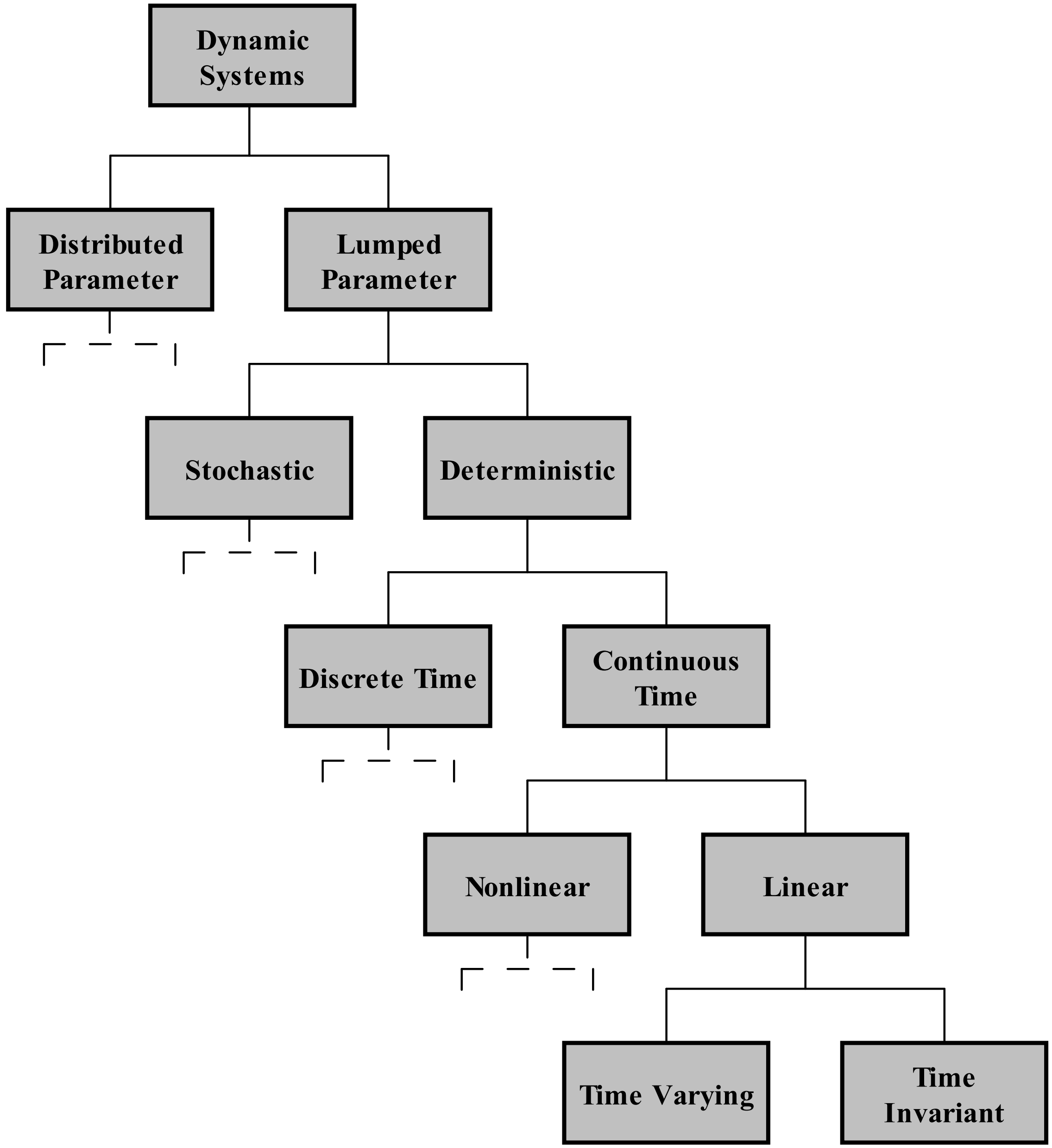
Analysis

- *Stability, Controllability, …*

Control System Design

- *Performance specification*
- *Synthesis of controller*
- *Simulation and testing*

System Models



Online examples

http://www.engin.umich.edu/group/ctm/

http://www.mathworks.com/applications/controldesign/

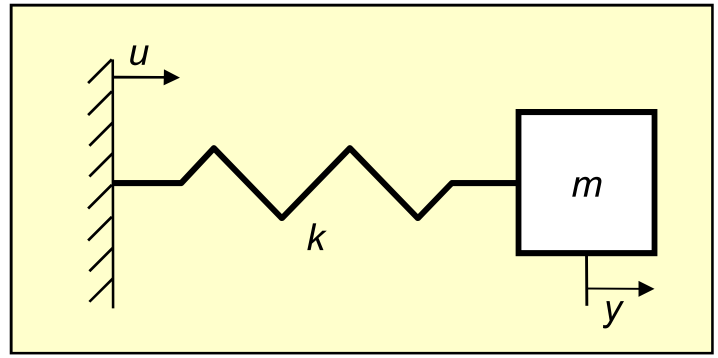
http://lorien.ncl.ac.uk/ming/Dept/Swot/connotes.htm

http://www-control.eng.cam.ac.uk/extras/Virtual_Library/Control_VL.html

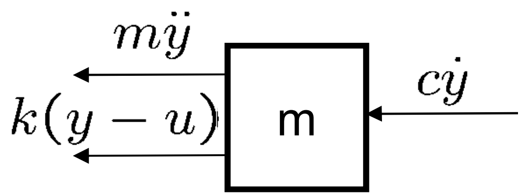
http://www.jhu.edu/~signals/explore/index.html

Example

For the spring-mass dashpot system



Force balance leads to the free body diagram



$$m\ddot{y} + c\dot{y} + ky = ku$$

At initial time (t=0), we need the values of y *and* $\dot{y}$ *to solve the differential equation which are referred to as initial conditions.*

Overview of Control Approaches

- *Classical Control: employs primarily frequency domain tools to achieve control objectives*
    - *MAE 443/543 (Continuous Control)*
    - *MAE 444/544 (Digital Control)*
- *Modern Control: employs primarily time-domain tools*
    - *MAE 571 (System Analysis)*
    - *MAE 672 (Optimal Control)*
    - *MAE 670 (Nonlinear Control)*

•*Post-Modern Control: integrate time-domain and frequency domain tools.*

Software Tools

• *MATLAB (MATrix LABoratory)*

powerful numerical software package with toolboxes for control, optimization, system identification, etc.

http://www.engin.umich.edu/group/ctm/

•*MAPLE (MAniPulator LanguagE)*

symbolic manipulator for analytically solving algebraic, differential equation and for linear algebra besides other functionality

http://www.mapleapps.com/categories/whatsnew/html/SCCCmapletutorial.shtml

•*MATLAB is now bundled with MAPLE permitting the user to exploit the strengths of both packages*

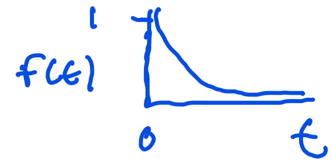
Laplace Transform

Let f(t) be a function of time t, such that f(t) = 0, for t < 0, then

$$\mathcal{L}[f(t)] = F(s) = \int_0^{\infty} f(t) e^{-st} dt$$

e^{-at} $a > 0$

is the Laplace transform of f(t).



s is a complex variable

Example:

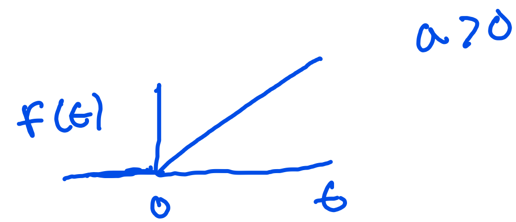
$$f(t) = e^{-at}$$

$$F(s) = \int_0^{\infty} e^{-at} e^{-st} dt = \int_0^{\infty} e^{-(a+s)t} dt$$

$$= -\left. \frac{e^{-(a+s)t}}{s+a} \right|_0^{\infty}$$

$$= -\left[\frac{e^{-(a+s)\infty}}{s+a} - \frac{1}{s+a} \right] = \frac{1}{s+a}$$

Laplace Transform



Example:

$$f(t) = at$$

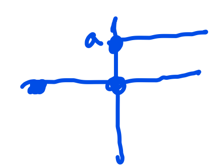
$$F(s) = \int_0^\infty ate^{-st}dt$$

$$= -\left.\frac{ate^{-st}}{s}\right|_0^\infty + \int_0^\infty \frac{ae^{-st}}{s}dt$$

$$= -\left.\frac{ae^{-st}}{s^2}\right|_0^\infty = -\left[\frac{ae^{-\infty}}{s^2} - \frac{a}{s^2}\right] = \frac{a}{s^2}$$

Example (Step function):

Unit step: $f(t) = \begin{cases} a \text{ for } t>0 \\ 0 \text{ for } t<0 \end{cases}$



Derivative of ramp is step.

$$\underbrace{\left(\frac{a}{s^2}\right)}_{\text{ramp}}\underbrace{(s)}_{\text{derivative}} = \frac{a}{s}$$

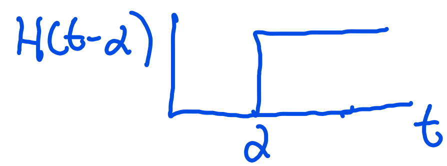
$$f(t) = \begin{cases} 0 & t < 0 \\ A & t > 0 \end{cases}$$

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty Ae^{-st}dt = \frac{A}{s}$$

Laplace Transform (Impulse Function)

Example (Heaviside Step function):

$$\mathcal{H}(t-\alpha) = \begin{cases} 0 & t < \alpha \\ A & t > \alpha \end{cases} \qquad \mathcal{L}[\mathcal{H}(t-\alpha)] = \int_0^\infty \mathcal{H}(t-\alpha) e^{-st} dt$$



Define $\tau = t - \alpha \Rightarrow t = \tau + \alpha$

$dt = d\tau$

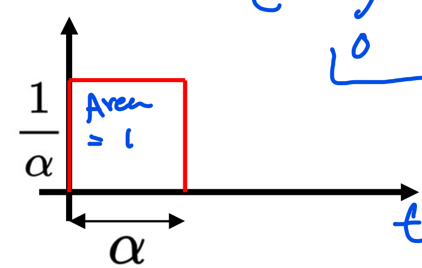
$$\mathcal{L}[\mathcal{H}(t-\alpha)] = \int_0^\infty \mathcal{H}(t-\alpha) e^{-st} dt \longrightarrow \mathcal{L}[\mathcal{H}(\tau)] = \int_{-\alpha}^\infty \mathcal{H}(\tau) e^{-s(\tau+\alpha)} d\tau$$

$\int_{-\delta}^{\infty} H e^{-s(\tau+\alpha)} = \int_{-\infty}^{0} H e^{-s(\tau+\alpha)} d\tau + \int_0^\infty H e^{-s(\tau+\alpha)} d\tau$

$\Rightarrow e^{-s\alpha} \underbrace{\int_0^\infty H(\tau) e^{-s\tau} dt}_{\text{pure delay}} = \frac{e^{-s\alpha}}{s}$

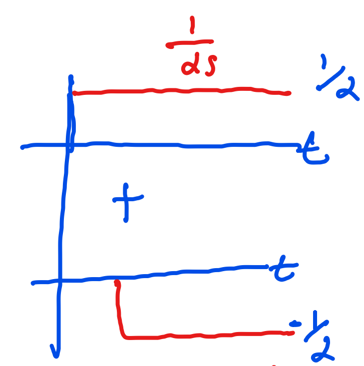
$$\mathcal{L}[\mathcal{H}(t-\alpha)] = \frac{e^{-s\alpha}}{s}$$

Define a pulse function as:



Addition of Heaviside and Step function

$$f(t) = \begin{cases} 0 & t < 0 \\ \frac{1}{\alpha} & t < \alpha \\ 0 & t > \alpha \end{cases} \qquad \mathcal{L}[f(t)] = \frac{1}{\alpha}\left(\frac{1}{s} - \frac{e^{-s\alpha}}{s}\right)$$



$= \frac{-e^{-s\alpha}}{\alpha s}$

In the limit as α goes to zero, the Laplace transform goes to unity.

Unit pulse: $\lim_{\alpha \to 0} \left(\frac{1 - e^{-s\alpha}}{\alpha s}\right) = \infty \Rightarrow$ l'Hopital's $\lim_{\alpha \to 0} \left(\frac{s e^{-s\alpha}}{s}\right) = 1$

Derivative of Step is unit impulse. $\left(\frac{1}{s}\right)(s) = 1$
(step) (derivative)

Laplace Transform of the Derivative of a Function

$$\mathcal{L}\left[\frac{df(t)}{dt}\right] = ?$$

Consider $$\int_0^\infty f(t)e^{-st}dt = f(t)\frac{e^{-st}}{-s}\Big|_0^\infty - \int_0^\infty \left[\frac{df(t)}{dt}\right]\frac{e^{-st}}{-s}dt$$

$$\Rightarrow F(s) = \frac{f(0)}{s} + \frac{1}{s}\mathcal{L}\left[\frac{df(t)}{dt}\right]$$

$$\Rightarrow \mathcal{L}\left[\frac{df(t)}{dt}\right] = sF(s) - f(0)$$

$$\frac{df(t)}{dt} = g(t)$$

$$\mathcal{L}\left[\frac{d^2f(t)}{dt^2}\right] = \mathcal{L}\left[\frac{dg(t)}{dt}\right] = s\mathcal{L}[g(t)] - g(0) = s\left(\mathcal{L}\left[\frac{df(t)}{dt}\right]\right) - \dot{f}(0)$$

$$= s^2F(s) - sf(0) - \dot{f}(0)$$

Laplace Transform of the Derivative of a Function

Similarly

$$\mathcal{L}\left[\frac{d^n f(t)}{dt^n}\right] = s^n F(s) - s^{n-1} f(0) - s^{n-2} \dot{f}(0) - \ldots - \frac{d^{n-1} f}{dt^{n-1}}(0)$$

$s^3 F - s^2 f(0) - s \dot{f}(0) - \ddot{f}(0)$

Laplace Transform of a Function multiplied by e^{-at}

Let f(t) be Laplace transformable, its Laplace transform being F(s), then the Laplace transform of $e^{-at}f(t)$

$s' = s+a$

$\int_0^{\infty} f(t) e^{-s't} dt$

$= \int_0^{\infty} e^{-(s+a)t} f(t)\, dt$

$$\mathcal{L}\left(e^{-at}f(t)\right) = \int_0^{\infty} e^{-at}f(t)e^{-st}dt = F(s+a)$$

We see that multiplication of f(t) by e^{-at} *has the effect of replacing s by (s+a) in the Laplace Transform*

Example:

$$\mathcal{L}\left[\ sin(\omega t)\right] = \frac{\omega}{s^2+\omega^2}$$

$$\mathcal{L}\left[\ e^{-at}sin(\omega t)\right] = \frac{\omega}{(s+a)^2+\omega^2}$$

If $f(t) = e^{-3t} \sin(5t)$

$\Rightarrow \frac{5}{(s+3)^2+25}$

$e^{t} \sin(5t) \Rightarrow \frac{5}{(s-1)^2+25}$

Complex-Differentiation Theorem

If f(t) is Laplace transformable, then

$$\mathcal{L}\left[tf(t)\right] = -\frac{dF(s)}{ds}$$

$$\mathcal{L}\left[t^2 f(t)\right] = \frac{d^2F(s)}{ds^2}$$

In general

$$\mathcal{L}\left[t^n f(t)\right] = (-1)^n \frac{d^nF(s)}{ds^n}$$

Example:

$$\mathcal{L}\left[\ sin(\omega t)\right] = \frac{\omega}{s^2+\omega^2}$$

$$\mathcal{L}\left[tsin(\omega t)\right] = -\frac{d}{ds}\left(\frac{\omega}{s^2+\omega^2}\right) = \frac{2s\omega}{(s^2+\omega^2)^2}$$

$(\omega)(s^2+\omega^2)^{-1}$

$\frac{(-1)(\omega)(2s)}{(s^2+\omega^2)^2}(-1)$

$= \frac{2s\omega}{(s^2+\omega^2)^2}$

Final and Initial Value Theorem

Final Value Theorem:

If f(t) and $\dfrac{df(t)}{dt}$ *are Laplace transformable, if F(s) is the Laplace transform of f(t), and if* $\lim_{t \to \infty} f(t)$ *exists, then*

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$$

Consider

$$\lim_{s \to 0} \int_0^\infty \left[\frac{df(t)}{dt}\right] e^{-st} dt = \lim_{s \to 0} \left[sF(s) - f(0)\right]$$

Since

$$\lim_{s \to 0} e^{-st} = 1,$$

Final and Initial Value Theorem

we have

$$\int_0^\infty \left[\frac{df(t)}{dt}\right] dt = f(t)|_0^\infty = f(\infty) - f(0) = \lim_{s \to 0} sF(s) - f(0)$$

Thus,

$$f(\infty) = \lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$$

Final Value theorem is applicable only if

1. *F(s) has no roots of the denominator (poles) in the complex right half plane*
2. *F(s) should have no poles on the imaginary axis, except at most one pole at s=0*

Initial Value Theorem:

If f(t) and $\frac{df(t)}{dt}$ *are Laplace transformable, if F(s) is the Laplace transform of f(t), and if* $\lim_{s \to \infty} sF(s)$ *exists, then*

$$f(0^+) = \lim_{s \to \infty} sF(s)$$

Inverse Laplace Transform

$$\mathcal{L}^{-1}[F(s)] = f(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(s)e^{st}ds$$

where $j = \sqrt{-1}$

Not easy. We use table to find the Inverse Laplace Transforms

Partial Fraction method for Inverse Laplace Transform

Let

$$F(s) = \frac{B(s)}{A(s)}$$

B(s), A(s) are polynomials in s with the order of A(s) > B(s). If F(s) can be represented as

$$F(s) = F_1(s) + F_2(s) + \ldots + F_n(s)$$

then $$\mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}[F_1(s)] + \mathcal{L}^{-1}[F_2(s)] + \ldots + \mathcal{L}^{-1}[F_n(s)]$$

$$\mathcal{L}^{-1}[F(s)] = f_1(t) + f_2(t) + \ldots + f_n(t)$$

Inverse Laplace Transform

Example: (Distinct Poles)

$$F(s) = \frac{B(s)}{A(s)} = \frac{k(s+z_1)(s+z_2)\ldots(s+z_m)}{(s+p_1)(s+p_2)\ldots(s+p_n)} \quad \text{where } m < n$$

We can write

$$F(s) = \frac{B(s)}{A(s)} = \frac{a_1}{s+p_1} + \frac{a_2}{s+p_2} + \ldots + \frac{a_n}{s+p_n}$$

where a_k – residue at the pole $s=p_k$.

$s+3 = a_1(s+2) + a_2(s+1)$

$\left(\frac{s+3}{s+2}\right) = a_1 + \frac{a_2(s+1)}{(s+2)}$

Example:

$$F(s) = \frac{(s+3)}{(s+1)(s+2)} = \frac{a_1}{(s+1)} + \frac{a_2}{(s+2)}$$

Let $s=-1$

$\frac{2}{1} = a_1 + 0$

$$a_1 = \left[(\cancel{s+1}) \frac{(s+3)}{(\cancel{s+1})(s+2)} \right]\Bigg|_{s=-1} = 2$$

$\frac{s+3}{s+1} = \frac{a_1(s+2)}{(s+1)} + a_2$

$s = -2$

$\frac{1}{-1} = a_2 + 0$

$$a_2 = \left[(\cancel{s+2}) \frac{(s+3)}{(s+1)(\cancel{s+2})} \right]\Bigg|_{s=-2} = -1$$

Inverse Laplace Transform

therefore,

$$f(t) = \mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}[\frac{2}{(s+1)}] - \mathcal{L}^{-1}[\frac{-1}{(s+2)}]$$

$$f(t) = 2e^{-t} - e^{-2t} \text{ for } t > 0$$

Example:

$$F(s) = \frac{s^3 + 5s^2 + 9s + 7}{(s+1)(s+2)}$$

This has to be written as

$$s^2 + 3s + 2\,)\,s^3 + 5s^2 + 9s + 7\,(\,s + 2$$

$$s^3 + 3s^2 + 2s$$

$$------$$

$$2s^2 + 7s + 7$$

$$2s^2 + 6s + 4$$

$$------$$

$$s + 3$$

Inverse Laplace Transform

therefore,

$$F(s) = (s+2) + \frac{(s+3)}{(s+1)(s+2)}$$

$$f(t) = \frac{d}{dt}(\delta(t)) + 2\delta(t) + 2e^{-t} - e^{-2t} \text{ for } t > 0$$

Example: (Multiple Poles) - Repeated Roots

$$F(s) = \frac{s^2+2s+3}{(s+1)^3}$$

$$F(s) = \frac{B(s)}{A(s)} = \frac{a_1}{(s+1)} + \frac{a_2}{(s+1)^2} + \frac{a_3}{(s+1)^3}$$

$$a_3 = \left[(s+1)^3 F(s)\right]\Big|_{s=-1}$$

$= s^2+2s+3 = a_1(s+1)^2 + \frac{a_2}{(s+1)} + a_3$

$$(s+1)^3\frac{B(s)}{A(s)} = a_1(s+1)^2 + a_2(s+1) + a_3$$

$$\therefore a_3 = \left[(s+1)^3\frac{B(s)}{A(s)}\right]\Bigg|_{s=-1}$$

$\Rightarrow a_3 = 2$

$a_1(s^2+2s+1) + a_2(s+1)$

$a_1 s^2 + (2a_1 + a_2)s + (a_1 + a_2)$

$a_1 = 1$

$2(1) + a_2 = 2 \quad a_2 = 0$

check

$3 = a_1 + a_2 + 2$

$3 = (1) + 0 + 2$ ✓

$s^2 + 2s + 3 = a_1(s+1)^2 + a_2(s+1) + 2$

$\frac{d}{ds} \Rightarrow \quad 2s+2 = 2a_1(s+1) + a_2$

$a_2 = 0$

$\frac{d}{ds} \Rightarrow \quad 2 = 2a_1$

$a_1 = 1$

Inverse Laplace Transform

Consider,

$$\frac{d}{ds}\left[(s+1)^3 \frac{B(s)}{A(s)}\right] = 2a_1(s+1) + a_2$$

$$\therefore a_2 = \frac{d}{ds}\left[(s+1)^3 \frac{B(s)}{A(s)}\right]\Bigg|_{s=-1}$$

$$\therefore a_2 = \frac{d}{ds}\left[(s^2+2s+3)\right]\Big|_{s=-1} = 0$$

Similiarly

$$2a_1 = \frac{d^2}{ds^2}\left[(s+1)^3 \frac{B(s)}{A(s)}\right]\Bigg|_{s=-1}$$

$$2a_1 = \frac{d^2}{ds^2}\left[(s^2+2s+3)\right]\Big|_{s=-1} \Rightarrow a_1 = 1$$

$F(s) = \frac{1}{s+1} + \frac{2}{(s+1)^3}$

OR Compare Coefficients

$\mathcal{L}^{-1}(F) = e^{-t} + t^2 e^{-t}$

Matlab function

Matlab command for determining residue for the transfer function G(s) = B/A:

[R,P,K] = RESIDUE(B,A)

```
[R,P,K] = residue([1 2 3],[1 3 3 1])
R =

    1.0000
    0.0000
    2.0000

P =

   -1.0000
   -1.0000
   -1.0000

K =

     []
```

Solving Differential Equations

Consider the differential equation

$$\ddot{x} + 3\dot{x} + 2x = 0, \quad x(0) = a, \dot{x}(0) = b$$

Define

$$X(s) = \mathcal{L}\left[x(t)\right]$$

$$\mathcal{L}\left[\dot{x}(t)\right] = sX(s) - x(0)$$

$$\mathcal{L}\left[\ddot{x}(t)\right] = s^2X(s) - sx(0) - \dot{x}(0)$$

Therefore

$$\left[s^2X(s) - sx(0) - \dot{x}(0)\right] + 3\left[sX(s) - x(0)\right] + 2X(s) = 0$$

$$(s^2 + 3s + 2)X(s) = sx(0) + \dot{x}(0) + 3x(0)$$

$a_1 = 2a+b$
$a_2 = -(a+b)$

$$X(s) = \frac{(s+3)a + b}{s^2 + 3s + 2} = \frac{2a+b}{s+1} - \frac{a+b}{s+2} = \frac{a_1}{s+1} + \frac{a_2}{s+2}$$

$$x(t) = (2a+b)e^{-t} - (a+b)e^{-2t} \quad \forall t > 0$$

Solving Differential Equations

Consider the differential equation

$$\ddot{x} + 2\dot{x} + 5x = 3, \quad x(0) = 0, \dot{x}(0) = 0$$

$$X(s) = \frac{a_1}{s} + \frac{a_2 s + a_3}{s^2+2s+5}$$

Laplace transformation leads to:

$$(s^2 + 2s + 5)X(s) = \frac{3}{s}$$

$$\frac{3}{s^2+2s+5} = a_1 + \frac{a_2 s^2 + a_3 s}{s^2+2s+5} \Big|_{s=0}$$

$$a_1 = \frac{3}{5}$$

Therefore

$$X(s) = \frac{3}{s(s^2 + 2s + 5)} = \frac{3}{5s} - \frac{3}{10((s+1)^2 + 2^2)} - \frac{3(s+1)}{5((s+1)^2 + 2^2)}$$

$$x(t) = \frac{3}{5} - \frac{3}{10}e^{-t}sin(2t) - \frac{3}{5}e^{-t}cos(2t) \quad \forall t > 0$$

$$3 = a_1(s^2+2s+5) + (a_2 s + a_3)s$$

$$3 = \left(\frac{3}{5} + a_2\right)s^2 + \left(\frac{6}{5} + a_3\right)s + 3$$

$$X = \frac{3}{5s} - \frac{\frac{3}{5}s + \frac{6}{5}}{(s+1)^2+4} = \frac{3}{5s} - \frac{\frac{3}{5}(s+1) + \frac{3}{5}}{(s+1)^2+4}$$