ECE 68000: MODERN AUTOMATIC CONTROL

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Constructing performance indices

Optimal control problem

- Motivation
- Constructing a performance index
- Combining different performance indices
- General performance indices
- Linear quadratic regulator (LQR) problem
- Index for optimal tracking a desired state trajectory

Motivation

- One of the essential elements of the control problem is a means of testing the performance of any proposed control law
- Whenever we use the term “best” or “optimal” to describe the effectiveness of a given control strategy, we do so with respect to some numerical index of performance called the performance index, or cost function, or penalty function
- We assume that the value of the performance index decreases as the quality of the given admissible control law increases
- The admissible controller that ensures the completion of the system objective and at the same time minimizes the performance index is called an *optimal controller* for the system

Constructing a performance index

- Constructing a performance index, that is, choosing a means to measure the system performance, can be considered as a part of the system modeling
- Suppose that the objective is to control a linear dynamical system model,

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t), \quad \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \\ \boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{x}(t)\end{aligned}$$

  on a fixed interval $[t_0, t_f]$ so that the components of the state vector are "small"
- A suitable performance index to be minimized would be

$$J_1 = \int_{t_0}^{t_f} \boldsymbol{x}(t)^\top \boldsymbol{x}(t) dt = \int_{t_0}^{t_f} \|\boldsymbol{x}(t)\|^2 dt$$

- Obviously, if J_1 is small, then the state vector norm, $\|\boldsymbol{x}(t)\|$, is small in the sense of the above performance index

Examples of performance indices

- If the objective is to control the system so that the components of the output, $\boldsymbol{y}(t)$, are to be small, then we could use the performance index

$$
\begin{aligned}
J_2 &= \int_{t_0}^{t_f} \boldsymbol{y}(t)^\top \boldsymbol{y}(t)\,dt \\
&= \int_{t_0}^{t_f} \boldsymbol{x}(t)^\top \boldsymbol{C}^\top \boldsymbol{C}\boldsymbol{x}(t)\,dt \\
&= \int_{t_0}^{t_f} \boldsymbol{x}(t)^\top \boldsymbol{Q}\boldsymbol{x}(t)\,dt
\end{aligned}
$$

  where the weight matrix $\boldsymbol{Q} = \boldsymbol{C}^\top \boldsymbol{C}$ is symmetric positive semi-definite

Inputs “not too large”

- If we wish to control the system in such a manner that the components of the input, $\boldsymbol{u}(t)$, are “not too large,” a suitable performance index to be minimized is

$$J_3 = \int_{t_0}^{t_f} \boldsymbol{u}(t)^\top \boldsymbol{u}(t) dt$$

or

$$J_4 = \int_{t_0}^{t_f} \boldsymbol{u}(t)^\top \boldsymbol{R}\boldsymbol{u}(t) dt$$

where the weight matrix $\boldsymbol{R}$ is symmetric positive definite

Inputs “not too large”—contd.

- There is no loss of generality in assuming the weight matrix $\boldsymbol{R}$ to be symmetric in $J_4 = \int_{t_0}^{t_f} \boldsymbol{u}(t)^\top \boldsymbol{R}\boldsymbol{u}(t)dt$
- For if $\boldsymbol{R}$ was not symmetric, we could represent the quadratic term $\boldsymbol{u}^\top \boldsymbol{R}\boldsymbol{u}$ equivalently as

$$\boldsymbol{u}^\top \boldsymbol{R}\boldsymbol{u} = \boldsymbol{u}^\top \left(\frac{\boldsymbol{R} + \boldsymbol{R}^\top}{2} \right) \boldsymbol{u}$$

where the matrix $\frac{1}{2}\left(\boldsymbol{R} + \boldsymbol{R}^\top\right)$ is symmetric

Combining different performance indices

- Cannot simultaneously minimize the performance indices because minimization of J_1 requires large control signals, while minimization of J_3 requires small control signals
- To solve the dilemma, could compromise between the two conflicting objectives by minimizing the performance index that is a convex combination of J_1 and J_3

$$\begin{aligned} J &= \lambda J_1 + (1-\lambda) J_3 \\ &= \int_{t_0}^{t_f} \left(\lambda \boldsymbol{x}(t)^\top \boldsymbol{x}(t) + (1-\lambda) \boldsymbol{u}(t)^\top \boldsymbol{u}(t) \right) dt, \end{aligned}$$

  where λ is a parameter in the range $[0, 1]$
- If $\lambda = 1$, then $J = J_1$
- If $\lambda = 0$, then $J = J_3$
- By trial and error, select λ from the interval $[0, 1]$ to compromise between the two extremes

General performance indices

- A generalization of the performance index

$$J = \int_{t_0}^{t_f} \left(\boldsymbol{x}(t)^\top \boldsymbol{Q} \boldsymbol{x}(t) + \boldsymbol{u}(t)^\top \boldsymbol{R} \boldsymbol{u}(t) \right) dt$$

- In certain applications, may wish the final state $\boldsymbol{x}(t_f)$ to be as close as possible to $\mathbf{0}$
- A possible performance measure to be minimized is

$$\boldsymbol{x}(t_f)^\top \boldsymbol{F} \boldsymbol{x}(t_f)$$

  where $\boldsymbol{F}$ is a symmetric positive definite matrix
- Combine the performance measures when our control aim is to keep the state “small,” the control “not too large,” and the final state as near to $\mathbf{0}$ as possible

The linear quadratic regulator (LQR) problem

- The resulting performance index

$$J = \frac{1}{2}\boldsymbol{x}(t_f)^\top \boldsymbol{F}\boldsymbol{x}(t_f) + \frac{1}{2}\int_{t_0}^{t_f} \left(\boldsymbol{x}(t)^\top \boldsymbol{Q}\boldsymbol{x}(t) + \boldsymbol{u}(t)^\top \boldsymbol{R}\boldsymbol{u}(t)\right) dt$$

  where the factor $1/2$ is to simplify subsequent algebraic manipulations
- Minimizing J is called the *linear quadratic regulator*, or the LQR problem for short

Index for optimal tracking a desired state trajectory

- In some cases, the controller's goal is to force the system state to track a desired state trajectory, $\boldsymbol{x}_d(t)$, throughout the interval $[t_0, t_f]$ while maintaining the deviations of the actual state $\boldsymbol{x}(t)$ "small" from the desired trajectory with the control effort $\boldsymbol{u}(t)$ "not too large" and the final state $\boldsymbol{x}(t_f)$ being as near as possible to some desired state $\boldsymbol{x}_d(t_f)$
- A suitable performance index to be minimized

$$
\begin{aligned}
J \;=\; & \frac{1}{2}\left(\boldsymbol{x}(t_f) - \boldsymbol{x}_d(t_f)\right)^\top \boldsymbol{F}\left(\boldsymbol{x}(t_f) - \boldsymbol{x}_d(t_f)\right) \\
& + \frac{1}{2}\int_{t_0}^{t_f} \Big(\left(\boldsymbol{x}(t) - \boldsymbol{x}_d(t)\right)^\top \boldsymbol{Q}\left(\boldsymbol{x}(t) - \boldsymbol{x}_d(t)\right) \\
& \quad + \boldsymbol{u}(t)^\top \boldsymbol{R}\boldsymbol{u}(t)\Big)\, dt
\end{aligned}
$$