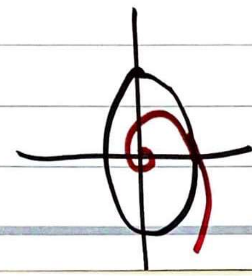
(Ex) $\mathbb{y}' = \begin{bmatrix} -2 & -1 \\ 4 & -2 \end{bmatrix} \mathbb{y}.$

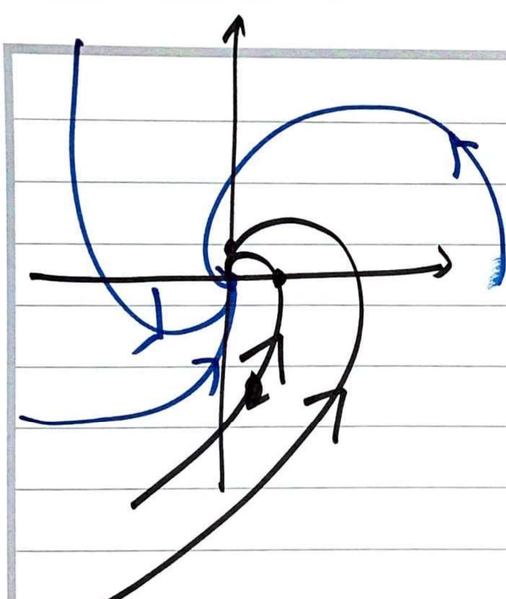
$$\mathbb{y}(t) = c_1 \underbrace{e^{-2t}\begin{bmatrix} \cos(2t) \\ 2\sin(2t) \end{bmatrix}}_{=\,\mathbb{y}_3(t)} + c_2 \underbrace{e^{-2t}\begin{bmatrix} \sin(2t) \\ -2\cos(2t) \end{bmatrix}}_{=\,\mathbb{y}_4(t)}$$

$$X_1(t) = \begin{bmatrix} \cos(2t) \\ 2\sin(2t) \end{bmatrix} \begin{matrix} = x_1 \\ = x_2 \end{matrix}$$

$$x_1^2 + \left(\frac{x_2}{2}\right)^2 = \cos^2(2t) + \sin^2(2t) = 1$$

$$\lim_{t\to\infty} e^{-2t} = 0$$





$\mathbb{y}_3(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \mathbb{y}_3(\frac{\pi}{4}) = e^{-\frac{\pi}{2}}\begin{bmatrix} 0 \\ 2 \end{bmatrix}$

(0, 0) : an asymptotically stable spiral point
(attractive)

Case 3: $\mathbb{Y}' = A\mathbb{Y}$

A is not diagonalizable.

Topics: 1. general solution, 2. Solution Curves

(Ex) $\mathbb{Y}' = \underbrace{\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}}_{\text{let } A} \mathbb{Y}$: Case 3.

(1) λ: $\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 \\ 2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 = 0$

$\lambda = 1, 1.$

(2) $\lambda = 1$: $\begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$: $2v_1 = 0$

$v_1 = 0$: $V = \begin{bmatrix} 0 \\ v_2 \end{bmatrix}$: Let $V_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\mathbb{Y}_1(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^t.$

Q $\mathbb{Y}_2(t) = ?$

(idea) $y'' - 2y' + y = 0$: Let $y(t) = e^{rt}$

$r^2 - 2r + 1 = 0$: $(r-1)^2 = 0$; $r = 1$

$y_1(t) = e^t$, $y_2(t) = e^t u(t)$

$\Rightarrow$ $u(t) = t$

$y(t) = C_1 e^t + C_2 t e^t$

? $Y_2(t) = \begin{bmatrix}0\\1\end{bmatrix} te^t$

L $Y_2'(t) = \begin{bmatrix}0\\1\end{bmatrix}(e^t + te^t)$

R $AY_2 = A\begin{bmatrix}0\\1\end{bmatrix}te^t = \begin{bmatrix}0\\1\end{bmatrix}te^t$

Set $Y_2(t) = \begin{bmatrix}0\\1\end{bmatrix}te^t + Ue^t \to Y' = AY$

L $= Y_2'(t) = \begin{bmatrix}0\\1\end{bmatrix}(e^t + te^t) + Ue^t$

R $= AY_2 = A\begin{bmatrix}0\\1\end{bmatrix}te^t + AUe^t$

$=$

$\cancel{\begin{bmatrix}0\\1\end{bmatrix}te^t} + \begin{bmatrix}0\\1\end{bmatrix}\cancel{e^t} + U\cancel{e^t} = \cancel{\begin{bmatrix}0\\1\end{bmatrix}te^t} + AU\cancel{e^t}$

$AU = U + \begin{bmatrix}0\\1\end{bmatrix}$

$AU - U = \begin{bmatrix}0\\1\end{bmatrix} \therefore (A - I)U = V_1 = \begin{bmatrix}0\\1\end{bmatrix}$

(formua) Solve $(A - \lambda I)U = V_1$

$\begin{bmatrix}0 & 0\\2 & 0\end{bmatrix}\begin{bmatrix}u_1\\u_2\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$ $\quad 2u_1 = 1 \therefore u_1 = \frac{1}{2}$

$U = \begin{bmatrix}\frac{1}{2}\\u_2\end{bmatrix} \therefore$ Let $U = \begin{bmatrix}\frac{1}{2}\\0\end{bmatrix}$

$\therefore Y_2(t) = \begin{bmatrix}0\\1\end{bmatrix}te^t + \begin{bmatrix}\frac{1}{2}\\0\end{bmatrix}e^t$

a general solution

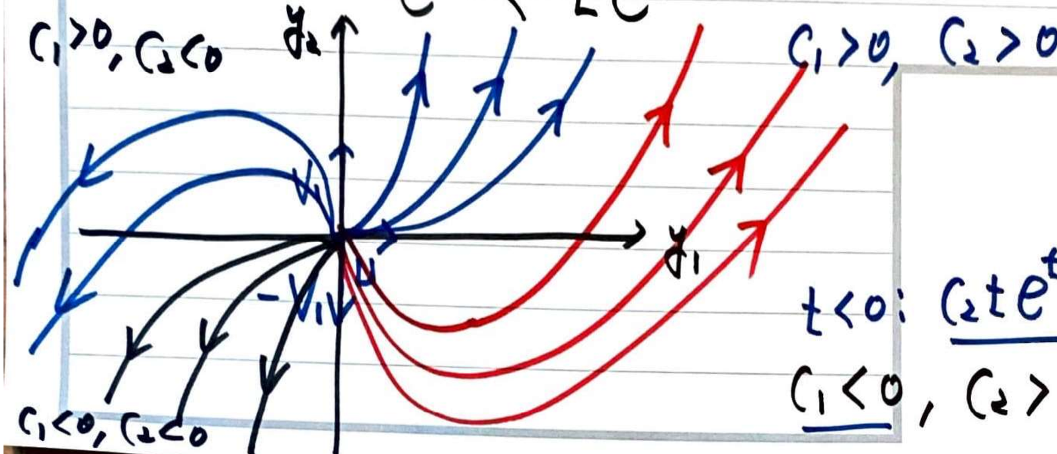
$\therefore Y(t) = C_1\begin{bmatrix}0\\1\end{bmatrix}e^t + C_2\left(\begin{bmatrix}0\\1\end{bmatrix}te^t + \begin{bmatrix}\frac{1}{2}\\0\end{bmatrix}e^t\right).$

(Trajectories) $Y' = \begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix} Y$.

$$Y(t) = (c_1 \underline{e^t} + c_2 \underline{t e^t})\begin{bmatrix}0\\ 1\end{bmatrix} + c_2 e^t \begin{bmatrix}\frac{1}{2}\\ 0\end{bmatrix}$$

1. $\lim_{t\to\infty} e^t = \infty$, $\lim_{t\to -\infty} e^t = 0$

2. $t > 1$: $e^t < t e^t$



$t<0$: $\underline{c_2 t e^t < 0}$

$\underline{c_1 < 0}$, $c_2 > 0$

$(0,0)$: a unstable degenerate node.

4.4. Stability. (Summary)

$Y' = AY$, $A = \begin{bmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}$

$$\det(A - \lambda I) = (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21}$$

$$= \lambda^2 - (\underbrace{a_{11} + a_{22}}_{= \mathrm{tr}A})\lambda + \underbrace{a_{11}a_{22} - a_{12}a_{21}}_{= \det A} = 0$$

<u>Def</u> $\mathrm{tr}A = a_{11} + a_{22}$: the trace of A

Remark: If A has eigenvalues λ_1, λ_2,

$$(\lambda-\lambda_1)(\lambda-\lambda_2)=0$$

iff $\lambda^2-(\lambda_1+\lambda_2)+\lambda_1\lambda_2=0$

$\mathrm{tr}A=\lambda_1+\lambda_2$ & $\det A=\lambda_1\lambda_2$

Set $p=\mathrm{tr}A=a_{11}+a_{22}$ & $q=\det A$.

Then $\det(A-\lambda I)=\lambda^2-p\lambda+q=0$

$$\lambda=\frac{1}{2}\left(p\pm\sqrt{p^2-4q}\right) : \Delta=p^2-4q$$

the discriminant.

(1) $\Delta>0$: two different real eigenvalues (Case 1)

① $p>0,\ q>0$: $p=\sqrt{p^2}$

$p>\sqrt{p^2-4q}$: $\lambda=\frac{1}{2}(p\pm\sqrt{\Delta})$

$\lambda_1=\frac{1}{2}(p+\sqrt{\Delta})>0,\ \lambda_2=\frac{1}{2}(p-\sqrt{\Delta})>0$

$(0,0)$: a unstable improper node.

Because $\lim_{t\to\infty}e^{\lambda_1 t}=\infty,\ \lim_{t\to\infty}e^{\lambda_2 t}=\infty$.

② $p>0,\ q<0$: $p<\sqrt{p^2-4q}$

$\lambda_1 = \frac{1}{2}(p+\sqrt{\Delta}) > 0, \ \lambda_2 = \frac{1}{2}(p-\sqrt{\Delta}) < 0$

$(0,0)$: a unstable saddle point

③ $p<0, \ q>0$: $\lambda = \frac{1}{2}(p \pm \sqrt{p^2-4q})$.

$|p| > \sqrt{p^2-4q}$ iff $|p| > \sqrt{\Delta}$

$\lambda_1 = \frac{1}{2}(p+\sqrt{\Delta}) < 0, \ \lambda_2 = \frac{1}{2}(p-\sqrt{\Delta}) < 0$

$(0,0)$: an asymptotically stable (attractive) improper node.

④ $p<0, \ q<0$: $|p| < \sqrt{p^2-4q}$

$\lambda_1 = \frac{1}{2}(p+\sqrt{\Delta}) > 0, \ \lambda_2 = \frac{1}{2}(p-\sqrt{\Delta}) < 0$

$(0,0)$: a unstable saddle point.

⑤ $p=0; \ q<0$: $\lambda = \frac{1}{2}(0 \pm \sqrt{0-4q})$

$(-4q>0)$

$\lambda_1 = \underset{\vee 0}{\sqrt{\Delta}}, \ \lambda_2 = -\sqrt{\Delta} < 0$

$(0,0)$: a unstable saddle point

(2) $\Delta < 0$: $p^2 - 4q < 0$: Case 2.
$(-\Delta > 0)$

① $p > 0$: $\lambda_1 = \frac{1}{2}(p + \sqrt{\Delta}) = \frac{1}{2}(p + i\sqrt{-\Delta})$

$\lambda_2 = \frac{1}{2}(p - \sqrt{\Delta})$: $\alpha = \frac{p}{2} > 0$

(0,0): a unstable spiral point

② $p < 0$: $\lambda = \frac{1}{2}(p \pm \sqrt{-\Delta}\, i)$

(0,0) : an asymptotically stable spiral point.

③ $p = 0$: $\lambda = \pm \frac{1}{2}\sqrt{-\Delta}\, i$

(0,0) : a stable center.

(3) $\Delta = 0$: Case 3 $\lambda = \frac{p}{2}$

① $p > 0$:

(0,0): a unstable degenerate node

② $p < 0$:

(0,0); an asymptotically stable degenerate node.

Q $\begin{cases} \text{improper node : Case 1} \\ \text{degenerate node : Case 3.} \end{cases}$