ECE 68000: MODERN AUTOMATIC CONTROL

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Application of the discrete-time (DT) Lyapunov matrix equation to evaluate performance indices

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- Discrete-time (DT) Lyapunov matrix equation
- Discrete-time (DT) performance indices
- Evaluating Discrete-time (DT) performance indices

Discrete-time (DT) Lyapunov matrix equation

- Discrete-time (DT) system

$$\boldsymbol{x}[k+1] = \boldsymbol{A}\boldsymbol{x}[k], \quad \boldsymbol{x}[k_0] = \boldsymbol{x}_0$$

- By analogy with the continuous-time (CT) case, consider a quadratic positive definite form,

$$V(\boldsymbol{x}[k]) = \boldsymbol{x}[k]^\top \boldsymbol{P}\boldsymbol{x}[k]$$

- Evaluate the first difference of $V(\boldsymbol{x}[k])$ defined as $\Delta V(\boldsymbol{x}[k]) = V(\boldsymbol{x}[k+1]) - V(\boldsymbol{x}[k])$, on the trajectories of the system to obtain

$$\begin{aligned}
\Delta V(\boldsymbol{x}[k]) &= V(\boldsymbol{x}[k+1]) - V(\boldsymbol{x}[k]) \\
&= \boldsymbol{x}[k+1]^\top \boldsymbol{P}\boldsymbol{x}[k+1] - \boldsymbol{x}[k]^\top \boldsymbol{P}\boldsymbol{x}[k] \\
&= \boldsymbol{x}[k]^\top \boldsymbol{A}^\top \boldsymbol{P}\boldsymbol{A}\boldsymbol{x}[k] - \boldsymbol{x}[k]^\top \boldsymbol{P}\boldsymbol{x}[k]
\end{aligned}$$

Discrete-time (DT) Lyapunov's equation—contd

- We have

$$\begin{aligned}\Delta V(\boldsymbol{x}[k]) &= \boldsymbol{x}[k]^\top \boldsymbol{A}^\top \boldsymbol{P}\boldsymbol{A}\boldsymbol{x}[k] - \boldsymbol{x}[k]^\top \boldsymbol{P}\boldsymbol{x}[k] \\ &= \boldsymbol{x}[k]^\top \left(\boldsymbol{A}^\top \boldsymbol{P}\boldsymbol{A} - \boldsymbol{P}\right) \boldsymbol{x}[k],\end{aligned}$$

  where we replaced $\boldsymbol{x}[k+1]$ in the above with $\boldsymbol{A}\boldsymbol{x}[k]$
- For the solutions to decrease the "energy function" V, at each $k > k_0$, it is necessary and sufficient that $\Delta V(\boldsymbol{x}[k]) < 0$, or equivalently, that for some positive definite $\boldsymbol{Q}$

$$\Delta V(\boldsymbol{x}[k]) = -\boldsymbol{x}[k]^\top \boldsymbol{Q}\boldsymbol{x}[k]$$

- We obtain the so called *discrete-time Lyapunov matrix equation*

$$\boxed{\boldsymbol{A}^\top \boldsymbol{P}\boldsymbol{A} - \boldsymbol{P} = -\boldsymbol{Q}}$$

Discrete-time (DT) Lyapunov's theorem

Theorem

The matrix $\boldsymbol{A}$ has its eigenvalues in the open unit disk if and only if for any positive definite $\boldsymbol{Q}$ the solution $\boldsymbol{P}$ to

$$\boldsymbol{A}^\top \boldsymbol{P}\boldsymbol{A} - \boldsymbol{P} = -\boldsymbol{Q}$$

is positive definite.

Proof of the necessity condition ($\Rightarrow$)

- The solution $\boldsymbol{P}$ can be expressed as

$$\boldsymbol{P} = \sum_{k=0}^{\infty} \left(\boldsymbol{A}^\top\right)^k \boldsymbol{Q}\boldsymbol{A}^k$$

- The above expression is well defined because all eigenvalues of $\boldsymbol{A}$ have magnitude (strictly) less than unity
- Substituting gives

$$\begin{aligned}
\boldsymbol{A}^\top\boldsymbol{P}\boldsymbol{A} - \boldsymbol{P} &= \boldsymbol{A}^\top \left(\sum_{k=0}^{\infty} \left(\boldsymbol{A}^\top\right)^k \boldsymbol{Q}\boldsymbol{A}^k\right)\boldsymbol{A} - \sum_{k=0}^{\infty} \left(\boldsymbol{A}^\top\right)^k \boldsymbol{Q}\boldsymbol{A}^k \\
&= \sum_{k=1}^{\infty} \left(\boldsymbol{A}^\top\right)^k \boldsymbol{Q}\boldsymbol{A}^k - \sum_{k=1}^{\infty} \left(\boldsymbol{A}^\top\right)^k \boldsymbol{Q}\boldsymbol{A}^k - \boldsymbol{Q} \\
&= -\boldsymbol{Q}
\end{aligned}$$

- Note that if $\boldsymbol{Q}$ is symmetric so is $\boldsymbol{P}$
- If $\boldsymbol{Q}$ is positive definite, so is $\boldsymbol{P}$

Proof of the sufficiency condition ($\Leftarrow$)

- Let λ_i be an eigenvalue of $\boldsymbol{A}$ and $\boldsymbol{v}_i$ be a corresponding eigenvector, that is, $\boldsymbol{A}\boldsymbol{v}_i = \lambda_i \boldsymbol{v}_i$
- Observe that λ_i and $\boldsymbol{v}_i$ may be complex
- Denote by $\boldsymbol{v}_i^*$ the complex conjugate transpose of $\boldsymbol{v}_i$
- Note that $\boldsymbol{v}_i^* \boldsymbol{P} \boldsymbol{v}_i$ is a scalar
- We will show that its complex conjugate equals itself, which means that $\boldsymbol{v}_i^* \boldsymbol{P} \boldsymbol{v}_i$ is real

Proof of sufficiency ($\Leftarrow$)—contd.

- Indeed,
$$\overline{\boldsymbol{v}_i^* \boldsymbol{P} \boldsymbol{v}_i} = \boldsymbol{v}_i^* \boldsymbol{P}^* \boldsymbol{v}_i = \boldsymbol{v}_i^* \boldsymbol{P}^\top \boldsymbol{v}_i = \boldsymbol{v}_i^* \boldsymbol{P} \boldsymbol{v}_i$$
where we used the fact that the complex conjugate transpose of a real matrix reduces to its transpose
- Thus, $\boldsymbol{v}_i^* \boldsymbol{P} \boldsymbol{v}_i$ is real for any complex $\boldsymbol{v}_i$.
- Premultiplying by $\boldsymbol{v}_i^*$ and postmultiplying by $\boldsymbol{v}_i$ gives
$$\boldsymbol{v}_i^* \left(\boldsymbol{A}^\top \boldsymbol{P} \boldsymbol{A} - \boldsymbol{P} \right) \boldsymbol{v}_i = -\boldsymbol{v}_i^* \boldsymbol{Q} \boldsymbol{v}_i$$
- The left-hand side of the above equality evaluates to
$$\begin{aligned} \boldsymbol{v}_i^* \boldsymbol{A}^\top \boldsymbol{P} \boldsymbol{A} \boldsymbol{v}_i - \boldsymbol{v}_i^* \boldsymbol{P} \boldsymbol{v}_i &= \bar{\lambda}_i \boldsymbol{v}_i^* \boldsymbol{P} \boldsymbol{v}_i \lambda_i - \boldsymbol{v}_i^* \boldsymbol{P} \boldsymbol{v}_i \\ &= \left(|\lambda_i|^2 - 1 \right) \boldsymbol{v}_i^* \boldsymbol{P} \boldsymbol{v}_i \end{aligned}$$

End of proof of sufficiency ($\Leftarrow$)

- Comparing the above with the right-hand side yields

$$\left(1 - |\lambda_i|^2\right) \boldsymbol{v}_i^* \boldsymbol{P} \boldsymbol{v}_i = \boldsymbol{v}_i^* \boldsymbol{Q} \boldsymbol{v}_i > 0$$

  because $\boldsymbol{Q}$ is positive definite
- Since $\boldsymbol{P}$ is positive definite, we have to have $|\lambda_i| < 1$
- The proof is complete.

Example

- Determine the stability of the system

$$\begin{aligned} \boldsymbol{x}[k+1] &= \begin{bmatrix} -0.8 & 0 & 0 \\ 0.4 & 0 & 0.4 \\ 0 & -0.8 & -0.8 \end{bmatrix} \boldsymbol{x}[k] \\ &= \boldsymbol{A}\boldsymbol{x}[k] \end{aligned}$$

  in the sense of Lyapunov by solving the Lyapunov matrix equation using the MATLAB function dlyap
- Use P=dlyap(A',eye(3))

Evaluating Discrete-time (DT) performance indices

- Evaluate

$$K_r = \sum_{k=0}^{\infty} k^r \boldsymbol{x}[k]^\top \boldsymbol{Q}\boldsymbol{x}[k], \quad r = 0, 1, \ldots$$

where $\boldsymbol{Q} = \boldsymbol{Q}^\top \succ 0$ subject to

$$\boldsymbol{x}[k+1] = \boldsymbol{A}\boldsymbol{x}[k], \quad \boldsymbol{x}[k_0] = \boldsymbol{x}_0$$

- $\boldsymbol{A}$ has its eigenvalues in the open unit disk, that is, the matrix $\boldsymbol{A}$ is convergent
- First consider the case when $r = 0$

$$K_0 = \sum_{k=0}^{\infty} \boldsymbol{x}[k]^\top \boldsymbol{Q}\boldsymbol{x}[k]$$

Evaluating K_0

- Since $\boldsymbol{A}$ is convergent, $\boldsymbol{P}$ is positive definite
- We write

$$\begin{aligned}\boldsymbol{x}[k]^\top \boldsymbol{Q}\boldsymbol{x}[k] &= -\boldsymbol{x}[k]^\top \left(\boldsymbol{A}^\top \boldsymbol{P}\boldsymbol{A} - \boldsymbol{P}\right)\boldsymbol{x}[k] \\ &= \boldsymbol{x}[k]^\top \boldsymbol{P}\boldsymbol{x}[k] - \boldsymbol{x}[k+1]^\top \boldsymbol{P}\boldsymbol{x}[k+1]\end{aligned}$$

- Summing both sides form $k = 0$ to $k = \infty$ and using the fact that $\boldsymbol{x}[\infty] = \mathbf{0}$ because the matrix $\boldsymbol{A}$ is convergent, we obtain

$$K_0 = \boldsymbol{x}[0]^\top \boldsymbol{P}\boldsymbol{x}[0]$$

Evaluating K_1

- Evaluate
$$K_1 = \sum_{k=0}^{\infty} k\boldsymbol{x}[k]^\top \boldsymbol{Q}\boldsymbol{x}[k]$$
subject to
$$\boldsymbol{x}[k+1] = \boldsymbol{A}\boldsymbol{x}[k], \quad \boldsymbol{x}[k_0] = \boldsymbol{x}_0$$
- Multiply both sides by k to get
$$k\boldsymbol{x}[k]^\top \boldsymbol{Q}\boldsymbol{x}[k] = k\boldsymbol{x}[k]^\top \boldsymbol{P}\boldsymbol{x}[k] - k\boldsymbol{x}[k+1]^\top \boldsymbol{P}\boldsymbol{x}[k+1]$$
- Summing the above from $k = 0$ to $k = \infty$ gives
$$\begin{aligned} K_1 &= \boldsymbol{x}[1]^\top \boldsymbol{P}\boldsymbol{x}[1] - \boldsymbol{x}[2]^\top \boldsymbol{P}\boldsymbol{x}[2] + 2\boldsymbol{x}[2]^\top \boldsymbol{P}\boldsymbol{x}[2] - 2\boldsymbol{x}[3]^\top \boldsymbol{P}\boldsymbol{x}[3] + \\ &= \sum_{k=0}^{\infty} \boldsymbol{x}[k]^\top \boldsymbol{P}\boldsymbol{x}[k] - \boldsymbol{x}[0]^\top \boldsymbol{P}\boldsymbol{x}[0] \end{aligned}$$

$K_1 = \boldsymbol{x}[0]^\top (\boldsymbol{P}_1 - \boldsymbol{P}) \boldsymbol{x}[0]$

- Evaluate the first term on the right-hand side to obtain

$$\sum_{k=0}^{\infty} \boldsymbol{x}[k]^\top \boldsymbol{P} \boldsymbol{x}[k] = \boldsymbol{x}[0]^\top \boldsymbol{P}_1 \boldsymbol{x}[0]$$

  where $\boldsymbol{P}_1$ satisfies

$$\boldsymbol{A}^\top \boldsymbol{P}_1 \boldsymbol{A} - \boldsymbol{P}_1 = -\boldsymbol{P}$$

- Hence

$$K_1 = \boldsymbol{x}[0]^\top (\boldsymbol{P}_1 - \boldsymbol{P}) \boldsymbol{x}[0]$$

- Proceeding in a similar manner, we can find expressions for K_r, where $r > 1$