ECE 68000: MODERN AUTOMATIC CONTROL

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Solving algebraic Riccati equation's using MATLAB's functions and LMIs

Solving continuous algebraic Riccati equation (CARE) using MATLAB's functions

- Continuous LQR problem: minimize the performance index
$$J = \int_0^{\infty} \left(\boldsymbol{x}^\top \boldsymbol{Q} \boldsymbol{x} + \boldsymbol{u}^\top \boldsymbol{R} \boldsymbol{u} + 2 \boldsymbol{x}^\top \boldsymbol{N} \boldsymbol{u} \right) dt$$
subject to
$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}, \quad \boldsymbol{x}(0) = \boldsymbol{x}_0$$
- MATLAB's function lqr computes the optimal state-feedback control $\boldsymbol{u} = -\boldsymbol{K}\boldsymbol{x}$
- lqr also returns the solution $\boldsymbol{P}$ of the associated algebraic Riccati equation and the closed-loop poles $\mathrm{CLP} = \mathrm{eig}(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K})$
- The matrix N is set to zero when omitted

Solving discrete algebraic Riccati equation (DARE) using MATLAB's functions

- Discrete LQR problem: minimize the performance index

$$J(\boldsymbol{u}) = \sum_{k=0}^{\infty} \left\{ \boldsymbol{x}[k]^\top \boldsymbol{Q}\boldsymbol{x}[k] + \boldsymbol{u}[k]^\top \boldsymbol{R}\boldsymbol{u}[k] + 2\boldsymbol{x}[k]^\top \boldsymbol{N}\boldsymbol{u}[k] \right\}$$

subject to

$$\boldsymbol{x}[k+1] = \boldsymbol{A}\boldsymbol{x}[k] + \boldsymbol{B}\boldsymbol{u}[k], \quad k = 0, 1, 2, \ldots$$

with a specified initial condition $\boldsymbol{x}(0) = \boldsymbol{x}_0$

- $[K, P, CLP] = \text{dlqr}(A, B, Q, R, N)$ calculates optimal $\boldsymbol{K}$
- dlqr also returns the solution $\boldsymbol{P}$ of the associated algebraic Riccati equation and the closed-loop poles $\text{CLP} = \text{eig}(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K})$
- The matrix N is set to zero when omitted

The Schur Complement Lemma

- Very common trick used in control systems
- Block symmetric matrix

$$\begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{B}^\top & \mathcal{C} \end{bmatrix}$$

Theorem

$$\begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{B}^\top & \mathcal{C} \end{bmatrix} \prec 0 \iff \mathcal{A} \prec 0,\ \mathcal{C} - \mathcal{B}^\top \mathcal{A}^{-1} \mathcal{B} \prec 0$$

$$\begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{B}^\top & \mathcal{C} \end{bmatrix} \prec 0 \iff \mathcal{C} \prec 0,\ \mathcal{A} - \mathcal{B}\mathcal{C}^{-1} \mathcal{B}^\top \prec 0$$

Solving CARE—Background

- CARE: $A^\top P + PA + Q - PBR^{-1}B^\top P = 0$
- Matrices $Q = Q^\top \succeq 0$, $R = R^\top \succ 0$
- Riccati inequality: $A^\top P + PA + Q - PBR^{-1}B^\top P \preceq 0$
- Recall the condition of theorem

Theorem

If the state-feedback controller $\boldsymbol{u}^* = -\boldsymbol{K}\boldsymbol{x}$ *is such that*

$$\min_{\boldsymbol{u}} \left(\frac{dV}{dt} + \boldsymbol{x}^\top \boldsymbol{Q}\boldsymbol{x} + \boldsymbol{u}^\top \boldsymbol{R}\boldsymbol{u} \right) = 0,$$

for some $V = \boldsymbol{x}^\top \boldsymbol{P}\boldsymbol{x}$, *then the controller is optimal.*

Theorem's sufficiency condition

- Let

$$f(\boldsymbol{u}) = \frac{dV}{dt} + \boldsymbol{x}^\top \boldsymbol{Q}\boldsymbol{x} + \boldsymbol{u}^\top \boldsymbol{R}\boldsymbol{u}$$

- Then

$$\begin{aligned} f(\boldsymbol{u}) &= 2\boldsymbol{x}^\top \boldsymbol{P}\dot{\boldsymbol{x}} + \boldsymbol{x}^\top \boldsymbol{Q}\boldsymbol{x} + \boldsymbol{u}^\top \boldsymbol{R}\boldsymbol{u} \\ &= \boldsymbol{x}^\top \left(\boldsymbol{A}^\top \boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} - \boldsymbol{P}\boldsymbol{B}\boldsymbol{K} - \boldsymbol{K}^\top \boldsymbol{B}^\top \boldsymbol{P} + \boldsymbol{Q} + \boldsymbol{K}^\top \boldsymbol{R}\boldsymbol{K}\right) \boldsymbol{x} \\ &\leq 0 \end{aligned}$$

- Equivalently

$$\boldsymbol{A}^\top \boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} - \boldsymbol{P}\boldsymbol{B}\boldsymbol{K} - \boldsymbol{K}^\top \boldsymbol{B}^\top \boldsymbol{P} + \boldsymbol{Q} + \boldsymbol{K}^\top \boldsymbol{R}\boldsymbol{K} \preceq 0$$

Solving CARE

- Let $S = P^{-1}$ and $Z = KS$
- Pre-multiply and post-multiply the Riccati inequality by S

$$SA^\top + AS - Z^\top B^\top - BZ + SQS + Z^\top RZ \preceq 0$$

- Taking Schur complements:

$$\begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{B}^\top & \mathcal{C} \end{bmatrix} \equiv \begin{bmatrix} SA^\top + AS - Z^\top B^\top - BZ & S & Z^\top \\ S & -Q^{-1} & 0 \\ Z & 0 & -R^{-1} \end{bmatrix} \preceq 0$$

- Note that controller gain $K = ZS^{-1}$

Snippet in CVX

```
% Specify your system and weight matrices

cvx_begin sdp quiet
% Variable definition
variable S(n, n) symmetric
variable Z(m,n)
% LMIs
[S*sys.A' + sys.A*S - sys.B*Z - Z'*sys.B, S, Z';...
S, -inv(sys.Q), zeros(n,m);...
Z, zeros(m,n),  -inv(sys.R)] <= 0
S >= eps*eye(n)
cvx_end
sys.K = Z/S % compute K matrix
```