$\boxed{\text{Ch 7}}$: Angular momentum for a multiparticle system

Recall $\quad {}^{I}\vec{h}_{i/o} \triangleq \vec{r}_{i/o} \times m_i \, {}^{I}\vec{v}_{i/o} = \vec{r}_{i/o} \times {}^{I}\vec{p}_{i/o}$

For each particle, the angular momentum version (rotational version) of N2L holds:

$$\frac{{}^{I}d}{dt}\left({}^{I}\vec{h}_{i/o}\right) = \vec{M}_{i/o} = \vec{r}_{i/o} \times \underset{\substack{\uparrow \\ \text{Total force} \\ \text{on particle } i}}{\vec{F}_i}$$

Can we work towards a multiparticle version of the angular momentum version of N2L?

We can breakup the forces into internal and external contributions,

$$\left.\begin{aligned} \frac{{}^{I}d}{dt}\left({}^{I}\vec{h}_{i/o}\right) &= \vec{r}_{i/o} \times \left(\vec{F}_i^{(ext)} + \sum_{j=1}^{N} \vec{F}_{ij}\right) \\ &= \vec{M}_{i/o}^{(ext)} + \vec{M}_{i/o}^{(int)} \end{aligned}\right\} \text{For each particle}$$

Let's introduce <u>total angular momentum</u>

$${}^{I}\vec{h}_o \triangleq \sum_{i=1}^{N} {}^{I}\vec{h}_{i/o} = \sum_{i=1}^{N} \vec{r}_{i/o} \times m_i \, {}^{I}\vec{v}_{i/o}$$

Then look at the change in the total angular momentum,

$$\frac{{}^{I}d}{dt}\left({}^{I}\vec{h}_O\right) = \sum_{i=1}^{N} m_i \underbrace{\left({}^{I}\vec{v}_{i/O} \times {}^{I}\vec{v}_{i/O}\right)}_{=0} + \sum_{i=1}^{N} \vec{r}_{i/O} \times \underbrace{m_i \frac{{}^{I}d}{dt}\left({}^{I}\vec{v}_{i/O}\right)}_{=\vec{F}_i \text{ By N2L}} \quad \text{(Product rule)}$$

So, we have,

$$\frac{{}^{I}d}{dt}\left({}^{I}\vec{h}_O\right) = \sum_{i=1}^{N} \vec{r}_{i/O} \times \vec{F}_i$$

Let's breakup $\vec{F}_i$ into internal and external forces

$$\frac{{}^{I}d}{dt}\left({}^{I}\vec{h}_O\right) = \sum_{i=1}^{N} \vec{r}_{i/O} \times \left(\vec{F}_i^{(ext)} + \sum_{j=1}^{N} \vec{F}_{i,j}\right)$$

$$= \underbrace{\sum_{i=1}^{N} \vec{r}_{i/O} \times \vec{F}_i^{(ext)}}_{\triangleq \vec{M}_O^{(ext)}} + \sum_{i=1}^{N} \sum_{j=1}^{N} \vec{r}_{i/O} \times \vec{F}_{i,j}$$

Manipulate as shown in Eqn. 7.3 of K+P to get

$$\frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \vec{r}_{i/j} \times \vec{F}_{i,j}$$

We would like this to be zero, but it is not always for every multiparticle system. We will the "internal moment assumption" that this term is zero. It is very hard to find mechanical examples for which this is not true.

→ Prob 7.13 is a counter-example based on a magnetic dipole.

Then we obtain

$$\boxed{\frac{{}^{I}d}{dt}\left({}^{I}\vec{h}_O\right) = \vec{M}_O^{(ext)}}$$

(Under internal moment assumption)

Conservation?

If $\vec{M}_O^{(ext)} = 0$, the ${}^{I}\vec{h}_O$ is conserved.

(However, this isn't particularly useful, because ${}^{I}\vec{h}_O$ is usually hard to measure).

Useful concept: Separation principle for ang. mom.

Can we break up the total angular momentum?

$$\begin{aligned}{}^{I}\vec{h}_O &= \sum_{i=1}^{N} \vec{r}_{i/O} \times m_i\, {}^{I}\vec{v}_{i/O} \\ &= \sum_{i=1}^{N} \left(\vec{r}_{G/O} + \vec{r}_{i/G}\right) \times m_i \left({}^{I}\vec{v}_{G/O} + {}^{I}\vec{v}_{i/G}\right)\end{aligned}$$

$\Rightarrow$ Using the C.O.M. corollary twice,

$${}^{I}\vec{h}_O = \underbrace{m_G\, \vec{r}_{G/O} \times {}^{I}\vec{v}_{G/O}}_{{}^{I}\vec{h}_{G/O}} + \underbrace{\sum_{i=1}^{N} \vec{r}_{i/G} \times m_i\, {}^{I}\vec{v}_{i/G}}_{{}^{I}\vec{h}_G}$$

Ang. Mom. of multiparticle system about G.

Separation principle for angular momentum

$$\boxed{{}^{I}\vec{h}_O = {}^{I}\vec{h}_{G/O} + {}^{I}\vec{h}_G} \quad \bigstar$$

Note: 1) This contrasts linear momentum, because ${}^{I}\vec{h}_O \neq {}^{I}\vec{h}_{G/O}$
The reason is b/c we also have to account for the **angular momentum about G.**

2) The separation principle is extremely important because it allows us to solve for the **translational** and **rotational** motion separately.

With some additional manipulation, we can also get versions of N2L for ${}^{I}\vec{h}_{G/O}$ and ${}^{I}\vec{h}_G$.

Summary:

$$\frac{{}^{I}d}{dt}\left({}^{I}\vec{h}_O\right) = \vec{M}_O^{(ext)}$$ (req'd internal moment assumption)

(translational character)

$$\frac{{}^{I}d}{dt}\left({}^{I}\vec{h}_{G/O}\right) = \vec{M}_{G/O}^{(ext)}$$

(rotational character about G)

$$\frac{{}^{I}d}{dt}\left({}^{I}\vec{h}_G\right) = \underbrace{\vec{M}_G^{(ext)}}$$ (req'd internal moment assumption)

$$\triangleq \sum_{i=1}^{N} \vec{M}_{i/G}^{(ext)} = \sum \vec{r}_{i/G} \times \vec{F}_i^{(ext)}$$

Ch 7: Work and Energy of a multiparticle system

Total Kinetic Energy

$$T_O \triangleq \sum_{i=1}^{N} T_{i/O} = \sum_{i=1}^{N} \frac{m_i}{2} \| {}^{I}\vec{v}_{i/O} \|^2$$

$$= \sum_{i=1}^{N} \frac{m_i}{2} \| {}^{I}\vec{v}_{G/O} + {}^{I}\vec{v}_{i/G} \|^2$$

Using the C.O.M. corollary,

$$T_O = \underbrace{\frac{1}{2} m_G \| {}^{I}\vec{v}_{G/O} \|^2}_{= T_{G/O}} + \underbrace{\frac{1}{2} \sum_{i=1}^{N} m_i \| {}^{I}\vec{v}_{i/G} \|^2}_{\triangleq T_G}$$

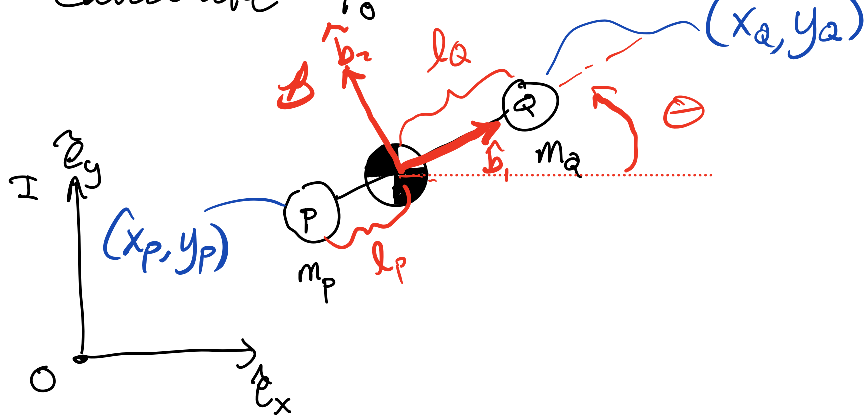
$T_{G/O}$: Translational kinetic energy

T_G: rotational kinetic energy

$$\boxed{T_O = T_{G/O} + T_G}$$

Separation principle for kinetic energy.

Ex. Two hockey pucks connected by a massless rod. Calculate T_O



Kinematics:

$$\vec{r}_{G/O} = \underbrace{\frac{m_P x_P + m_Q x_Q}{m_P + m_Q}}_{\triangleq x_G} \hat{e}_x + \underbrace{\frac{m_P y_P + m_Q y_Q}{m_P + m_Q}}_{\triangleq y_G} \hat{e}_y$$

$${}^I\vec{v}_{G/O} = \dot{x}_G \hat{e}_x + \dot{y}_G \hat{e}_y$$

$${}^I\vec{v}_{G/O} = \dot{x}_G \hat{e}_x + \dot{y}_G \hat{e}_y$$

$$\vec{r}_{P/G} = -l_P \hat{b}_1 \qquad \vec{r}_{Q/G} = l_Q \hat{b}_1$$

$${}^I\vec{v}_{P/G} = -l_P \dot{\theta} \hat{b}_2 \qquad {}^I\vec{v}_{Q/G} = l_Q \dot{\theta} \hat{b}_2$$

$$T_{G/O} = \frac{1}{2} m_G \|{}^I\vec{v}_{G/O}\|^2 = \frac{m_G}{2}(\dot{x}_G^2 + \dot{y}_G^2)$$

$$T_G = \frac{1}{2}\sum_{i=1}^{N} m_i \|{}^I\vec{v}_{i/G}\|^2 = \frac{1}{2} m_P (l_P \dot{\theta})^2 + \frac{1}{2} m_Q (l_Q \dot{\theta})^2$$

$$T_O = T_{G/O} + T_G$$

Work for multiparticle systems

$$W_i^{(tot)}(\vec{r}_{i/O}; \gamma_i) = \int_{\gamma_i} \vec{F}_i \cdot {}^I d\vec{r}_{i/O}$$

For each particle i

$$W^{(tot)} \triangleq \sum_{i=1}^{N} W_i^{(tot)}$$

Total work done on all particles.

Can we break this into internal and external contributions?

Break-up force: $\vec{F}_i = \vec{F}_i^{(ext)} + \sum_j \vec{F}_{i,j}$

Sum the first eqn over i, and break up the force into internal and external forces,

$$W^{(tot)} = \sum_{i=1}^{N} \int_{\gamma_i} \vec{F}_i^{(ext)} \cdot {}^{I}d\vec{r}_{i/0} + \underbrace{\sum_{i=1}^{N} \sum_{j=1}^{N} \int_{\gamma_i} \vec{F}_{ij} \cdot {}^{I}d\vec{r}_{i/0}}$$

Note: Dot product here is in contrast to the cross product that we previously saw in the internal moment assumption.

This work term $\neq 0$ in general $\triangleq W^{(int)}$

Since particles can do work on each other. This term represents work done against internal forces during relative motion of particles.

$$\boxed{W^{(tot)} = W^{(ext)} + W^{(int)}}$$

Work-energy formulas for multiparticle systems

① Total Work, Kinetic Energy

$$T_0(t_2) = T_0(t_1) + W^{(tot)}$$
$$= T_0(t_1) + W^{(ext)} + W^{(int)}$$

② Conservative Work, Potential Energy

Separation of Cons. forces cleanly breaks apart the conservative work and potential energies

$$U_0^{(ext)}(t_2) = U_0^{(ext)}(t_1) - W^{(c,ext)}$$
$$U_0^{(int)}(t_2) = U_0^{(int)}(t_1) - W^{(c,int)}$$

③ Nonconservative Work, Total energy

$$E_0(t_2) = E_0(t_1) + W^{(nc)}$$
$$= E_0(t_1) + W^{(nc,ext)} + W^{(nc,int)}$$

where,

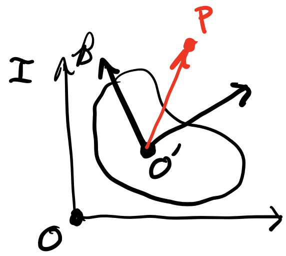
$$E_0 = T_0 + U_0^{(ext)} + U_0^{(int)}$$

Conservation? If $W^{(nc)} = 0$, the E_0 is conserved.

Note: The relative motion of particles in a multiparticle system can cause internal work.
(Not true for a rigid body).

Ch8 : Relative motion in a rotating frame

(Fixed in frame B for now)



Planar rigid body

Frame B is rotating and translating

Kinematics

$$\vec{r}_{P/O} = \vec{r}_{O'/O} + \vec{r}_{P/O'}$$

$$^I\vec{v}_{P/O} = \underbrace{\frac{^I d}{dt}(\vec{r}_{O'/O})}_{^I\vec{v}_{O'/O}} + \underbrace{\frac{^I d}{dt}(\vec{r}_{P/O'})}_{\text{what is this?}}$$

Although P is fixed in B, it is moving b/c frame B is moving.

Using cartesian coordinates,

$$\frac{^I d}{dt}(\vec{r}_{P/O'}) = \frac{^I d}{dt}(x\hat{b}_1 + y\hat{b}_2) = \dot{x}\hat{b}_1 + x\frac{^I d}{dt}(\hat{b}_1) + \dot{y}\hat{b}_2 + y\frac{^I d}{dt}(\hat{b}_2)$$

$$= \underbrace{\dot{x}\hat{b}_1 + \dot{y}\hat{b}_2}_{\frac{^B d}{dt}(\vec{r}_{P/O'})} + \underbrace{{}^I\vec{\omega}^B \times (x\hat{b}_1 + y\hat{b}_2)}_{{}^I\vec{\omega}^B \times \vec{r}_{P/O'}}$$

Now we can extrapolate how to do this calculation in general,

$$\frac{^I d}{dt}(\vec{r}_{P/O'}) = \frac{^B d}{dt}(\vec{r}_{P/O'}) + {}^I\vec{\omega}^B \times \vec{r}_{P/O'}$$

Angular velocity of Frame B with respect to I

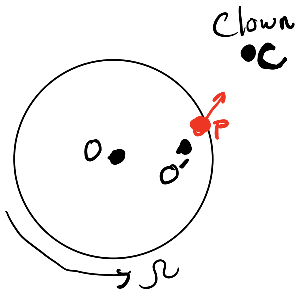
In general, this is the Transport equation:

$$\boxed{\frac{^I d}{dt}(\vec{r}) = \frac{^B d}{dt}(\vec{r}) + {}^I\vec{\omega}^B \times \vec{r}}$$

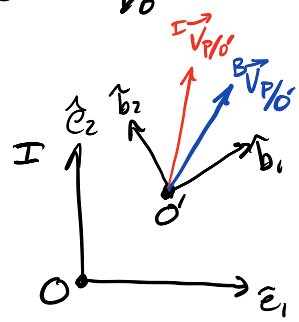
Applies to any vector.

Ex. Hit the clown while riding a carousel.

Goal is to find the velocity needed to project P into the clown's mouth.



We need to find ${}^B\vec{v}_{P/O'}$



Kinematics

$${}^I\vec{v}_{P/O} = \underbrace{{}^I\vec{v}_{P/O'}} + {}^I\vec{v}_{O'/O}$$

$$= \underbrace{{}^B\vec{v}_{P/O'}}_{\text{We want this}} + \underbrace{{}^I\vec{\omega}^B}_{\Omega\hat{e}_3} \times \vec{r}_{P/O'}$$

Re-arrange,

$${}^B\vec{v}_{P/O'} = \underbrace{{}^I\vec{v}_{P/O}}_{s\,\hat{r}_{C/O'}} - \underbrace{{}^I\vec{v}_{O'/O}} - \underbrace{{}^I\vec{\omega}^B \times \vec{r}_{P/O'}}_{\text{At instant of release } \vec{r}_{P/O'} = 0}$$

$${}^I\vec{v}_{O'/O} = \cancelto{0}{{}^B\vec{v}_{O'/O}} + {}^I\vec{\omega}^B \times \vec{r}_{O'/O}$$

$${}^B\vec{v}_{P/O'} = s\,\hat{r}_{C/O'} - {}^I\vec{\omega}^B \times \vec{r}_{O'/O} = \boxed{s\,\hat{r}_{C/O'} - \Omega\hat{e}_3 \times \vec{r}_{O'/O}}$$