Student Name Solutions

Orbit Mechanics
10/23/20

Exam 2

Please read the problems <u>carefully</u>.

Write clearly and use diagrams when necessary.

Use the following constant values when appropriate

Body	GM (km^3/s^2)	Radius (km)
Earth	4.0000×10^5	6400.0
Moon	5.000×10^3	1500.0
Earth-Moon Distance	4.0000×10^5 km	

Purdue Honor Pledge "As a Boilermaker pursuing academic excellence, I pledge to be honest and true in all that I do. Accountable together—We Are Purdue."

For this exam, I understand it is a take-home exam with the following requirements:

1. I can use my own class notes and my own previously completed assignments.
2. I am not allowed to search for any resources online.
3. I can use my own calculator. I cannot use Matlab or other commercial software.
4. I am expected to work the exam on my own. I am not allowed to work with another person. I am not allowed to contact another person for help while completing the exam.
5. If I have any questions during the exam period, I will email Prof Howell (howell@purdue.edu) AND/OR the TAs Mitch Dominguez (doming18@purdue.edu) or Samantha Ramsey (ramsey87@purdue.edu) or Tyler Hook (hook9@purdue.edu). Given this longer exam period, we will answer as soon as possible.

Signature ______________________________

(45 Points)

Problem 1: It is now 2050 and under the new Global Lunar Vision – an international commercial enterprise for development of lunar resources – facilities are based at various locations on the Moon's surface. The Lunar Transport Service uses vehicles (LTVs) in various orbits to facilitate operations between the bases and Earth. Note $\mu_{LTV} <<< \mu_{☾}$.

(a) Analysis for lunar activities requires a valid model. Complete a quick look by placing Earth—LTV—Moon along a line and in this order; assume that the LTV is at a radius distance $50R_{☾}$.

What equation might give you insight to decide if a relative two-body model is adequate at this distance?

Compare the various terms in an equation for motion of the LTV with respect to the Moon; is the relative two-body model reasonable at this distance? Why or why not?

(b) Assume a distance such that a relative 2B model is adequate. A Lunar Transport Service vehicle (LTV) is currently in a lunar orbit with $p = 4R_{☾}$ and $e = \frac{1}{\sqrt{3}}$. (See the figure on the next page.)

When the LTV is located at the end of the latus rectum and <u>ascending</u>, an in-plane maneuver will shift the vehicle into a new orbit for a run to Earth such that $\mathcal{E}_N = +\frac{1}{6}\ \text{km}^2/\text{s}^2$ and $h_N = 8200\ \text{km}^2/\text{s}$.

(i) What is the escape speed for this departure location?

(ii) Check the vector diagram. Two reasonable options exist to accomplish this task. Describe these options; which one offers the smallest $|\Delta\bar{v}|$? Why?

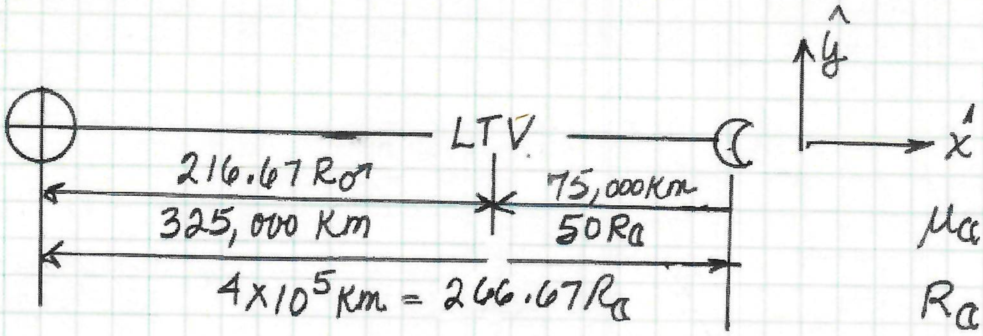
(iii) Determine the $\Delta\bar{v}$ (magnitude $\Delta\bar{v}$ and angle α) to accomplish the adjustment.

(iv) For the new orbit, determine e_N, θ_N^*, $\Delta\omega$.

Sketch the old and new orbits; mark the old and new line of apsides as well as $\Delta\omega$.

(c) Discuss: How are the results from (a) and (b) related? What are your conclusions and/or observations?

1. (a)



$\mu_☾ = 5000 \frac{Km^3}{s^2}$

$R_☾ = 1500$ Km

$\mu_\oplus = 4\times10^5$

$\oplus☾ = 4\times10^5$

Relative EOM for LTV wrt ☾

$$\ddot{\bar{r}}_{☾L} + \frac{\mu_☾ + \overset{\sim 0}{\cancel{\mu_{LTV}}}}{r_{☾L}^2}\hat{r}_{☾L} = \mu_\oplus\left(\frac{\hat{r}_{L\oplus}}{r_{L\oplus}^2} - \frac{\hat{r}_{☾\oplus}}{r_{☾\oplus}^2}\right)$$

20 $R_☾$
(30,000)

Dom Accel $= -\frac{\mu_☾}{r_{☾L}^2}\hat{r}_{☾L} = 8.889\times10^{-7}\ \frac{Km}{s^2}\ \hat{x}$ $\quad 5.55\times10^{-6}$

Direct Accel $= \frac{\mu_\oplus}{r_{L\oplus}^2}\hat{r}_{L\oplus} = 3.787\times10^{-6}\ \frac{Km}{s^2}\ (-\hat{x})$ $\quad 2.92\times10^{-6}$

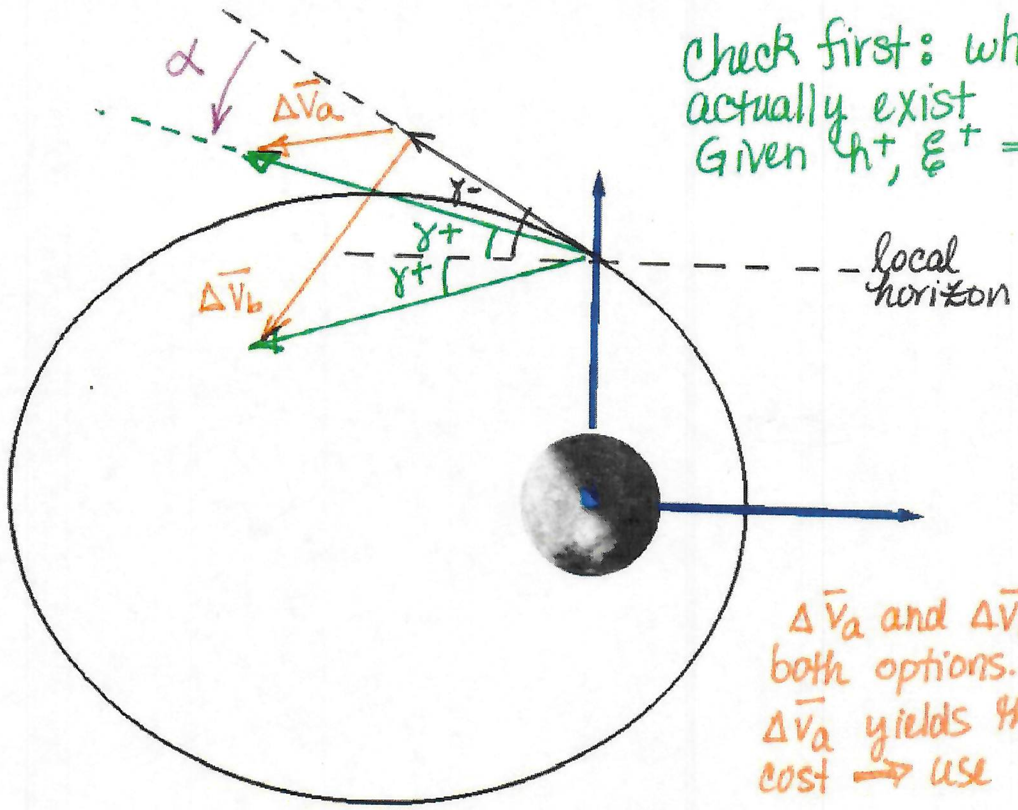
Indirect Accel $= \frac{\mu_\oplus}{r_{☾\oplus}^2}\hat{r}_{☾\oplus} = 2.5\times10^{-6}\ (-\hat{x})\ \frac{Km}{s^2}$ $\quad 4.2\times10^{-7}$

Net Perturbing $= -1.287\times10^{-6}\ \hat{x}\ \frac{Km}{s^2}$

At this distance from the Moon (75,000 Km), the relative 2B model is not sufficient. The perturbing effect of the Earth is significant and of larger magnitude than the dominant ☾ gravity.

Note: by the time any vehicle reaches 50 $R_☾$ (from either direction) assess model carefully

(b)



Check first: what options actually exist
Given $h^+, \mathcal{E}^+ \Longrightarrow \underbrace{v^+, \gamma^+}_{\text{options}}$ (next page)

$\Delta\bar{V}_a$ and $\Delta\bar{V}_b$ are both options. But $\Delta\bar{V}_a$ yields the smaller cost $\rightarrow$ use $\Delta\bar{V}_a$

$p = 4R_ℂ \quad e = \frac{1}{\sqrt{3}} \Longrightarrow p = a(1-e^2) \Longrightarrow a = \overset{9000\text{ km}}{6R_ℂ}$

$v_c = .91287$ Km/s $\quad v^- = v_c\sqrt{1+e^2} = 1.05409$ Km/s $\quad 1.05409$

$h = \sqrt{\mu_ℂ p} = 5477.226$ Km^2/s

$h = \underset{\uparrow r = p}{r} v c_\gamma \Longrightarrow \gamma^- = +30°$ ($\theta^* > 0$; ascending)

new orbit departing lunar vicinity on hyperbola

$\mathcal{E}_N = \frac{+1}{6} = \frac{\mu_ℂ}{2|a|} \longrightarrow a_N = -15000$ Km $= -10R_ℂ$

$h_N = 8200$ km^2/s $\qquad h^2 = \mu p \longrightarrow p_N = 13448$ Km $= 8.965R_ℂ$

(i) $\boxed{\text{Esape speed} = \sqrt{2}\,V_c = 1.29099 \text{ Km/s}}$

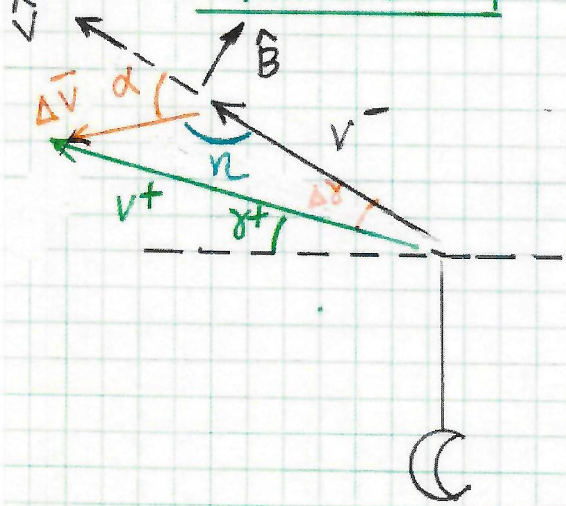
(ii) $r^+ = r^- = 4R_a$

$h^+ = 8200\ km^2/s = rv c_\gamma$

$\varepsilon^+ = \frac{v^{+2}}{2} - \frac{\mu_a}{r^+} \Longrightarrow \underbrace{v^+ = \sqrt{2}\ Km/s}_{v^+ > v_{esc}}$ ✓

$v^+ > v^-$

$\boxed{\gamma^+ = \pm 14.8993°}$



$\Delta\gamma = 15.1°$

cos law

$\Delta v^2 = v^{-2} + v^{+2} - 2v^- v^+ \cos\Delta\gamma$

$\boxed{\Delta v = .48232\ Km/s}$ 2.8784763

sine law

$\frac{v^+}{S_\eta} = \frac{\Delta v}{S_{\Delta\gamma}} \Longrightarrow \eta = 130.197°$

$\Downarrow$

$\alpha = 180° - \eta$

$\boxed{\alpha = -49.803°}$

velocity vector shifts toward Moon

$\boxed{\Delta\bar{v} = .3113\ \hat{V} \quad .3684\ \hat{B}\ Km/s}$

$$\tan\theta^* = \frac{\left(\frac{rv^2}{\mu}\right)\cos\gamma\sin\gamma}{\left(\frac{rv^2}{\mu}\right)\cos^2\gamma - 1}$$

$\frac{rv^{+2}}{\mu_a} = 2.400$

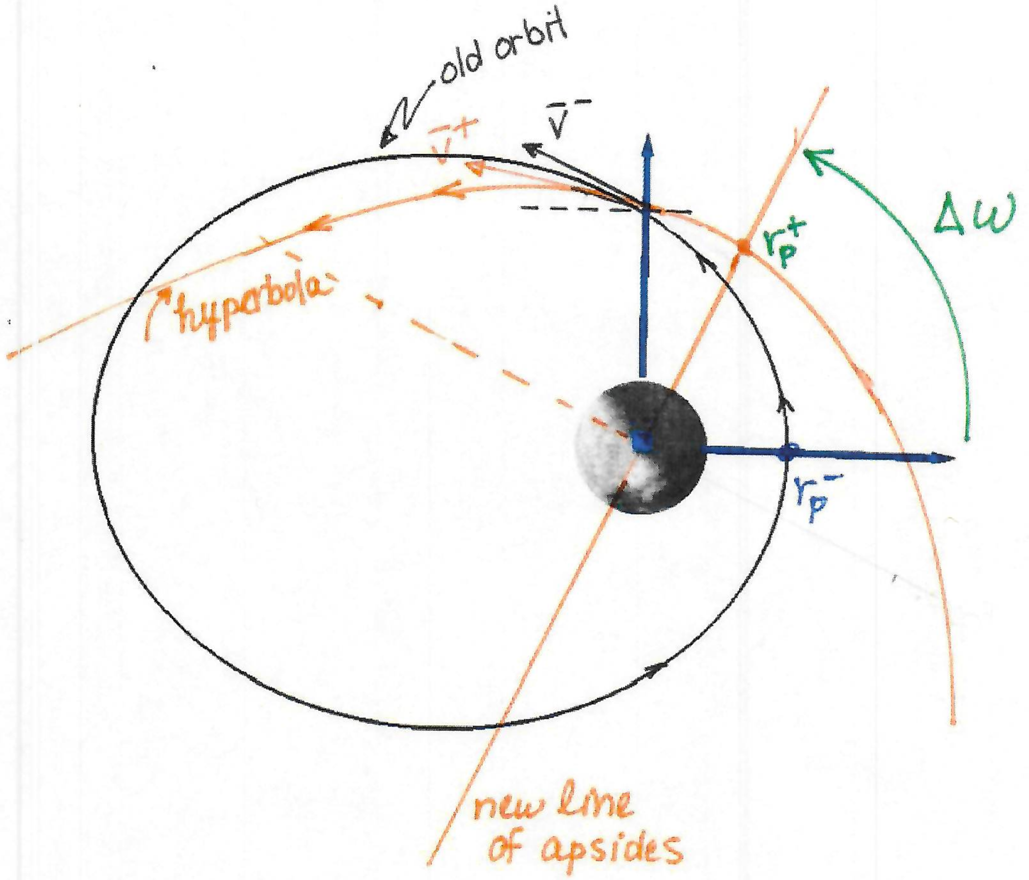
$\boxed{\theta^{*+} = \underbrace{25.660°}_{\gamma^+ > 0 \text{ ascending}}}$ OR -154.336

$\Delta\omega = 90° - 25.660°$

$\boxed{\Delta\omega = +64.34°}$

rotates about $+\hat{h}^-$

$\boxed{e_N^2 = 1 + \frac{p}{|a|} \rightarrow e_N = 1.3771}$



$p = 4R_a$

$r_p^- = 2.536\ R_a$

$r_p^+ = |a|(e-1) = 3.771\ R_a$

Orbit in (b) much closer to ☾. Similar analysis demonstrates that 2BP reasonable.

(c) In (a), conclude that by $50R_a$ away from Moon, two-body model (☾-s/c) not adequate. In (b), s/c is departing lunar vicinity along a hyperbolic path — which is a conic orbit in the ☾-s/c two-body problem. If we check the hyperbola and determine TOF till reaching $r = 50R_a$, it becomes apparent how long before this conic model is inadequate. Upon departure from lunar vicinity, influence of ⊕ grows rapidly.

(35 points)
Problem 2: A vehicle is currently in Earth orbit. Assume that it is reasonable to employ a relative two-body model. At time t_1, the vehicle is located at position $\bar{r}_1$ and later, at t_2, it is observed at location $\bar{r}_2$. The orbit semi-major axis is given as $a = a_n$. In terms of inertial Earth equatorial coordinates, the locations are given as:

$$\bar{r}_1 = a_n\left(\cos 45^\circ\ \hat{x} + \sin 45^\circ\ \hat{z}\right)$$

$$\bar{r}_2 = a_n\left(\cos 45^\circ\ \hat{y} + \sin 45^\circ\ \hat{z}\right)$$

It is known that the spacecraft passes through apoapsis between t_1 and t_2.

(a) Make two sketches: (i) the general scenario in 3D; (ii) the scenario in the orbit plane. (Your sketches should be reasonably accurate!)

(b) Determine the angle between the vectors.

Is the specific angular momentum vector above or below the fundamental plane? Why?

(c) Determine the following orbital characteristics: Ω, i, ω, e, p

(d) Determine conditions at the given locations: $\theta_1^*, \theta_2^*, \gamma_1, \gamma_2, \Delta E$ or ΔH

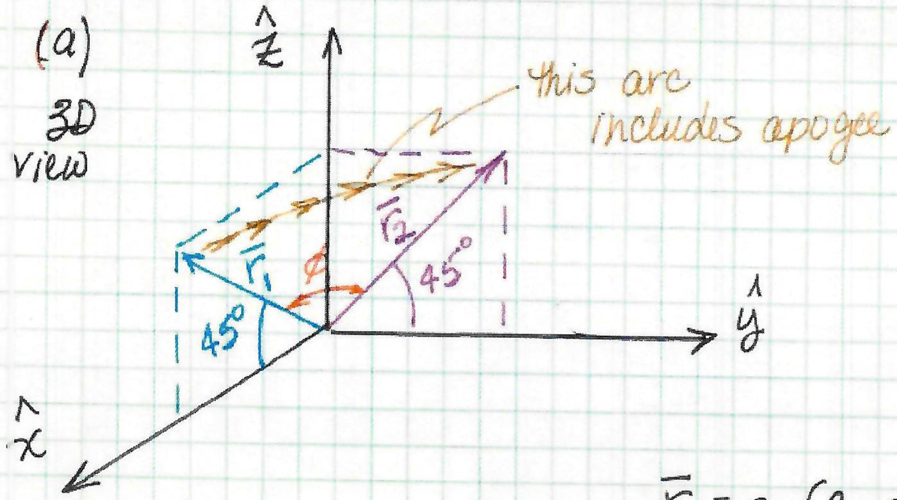
(e) Velocity $\bar{v}_1$ can be written $\bar{v}_1 = k_1 v_c \hat{v}_1$ where v_c is the corresponding circular speed. Determine k_1.

Use the f and g functions to determine the <u>velocity direction</u> in terms of the inertial coordinates $\hat{x}\,\hat{y}\,\hat{z}$. Also express the velocity direction in terms of $\hat{e}\,\hat{p}\,\hat{h}$ and $\hat{r}, \hat{\theta}, \hat{h}$.

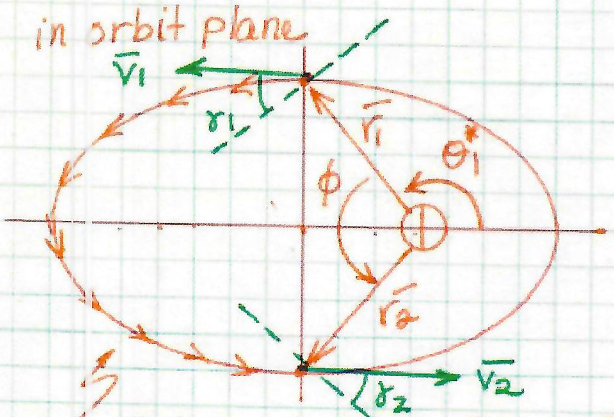
(f) Return to a sketch of the orbit plane. Mark $\bar{r}_1, \bar{r}_2, \bar{v}_1, \bar{v}_2$, local horizons, flight path angles. Also identify the arc between t_1 and t_2.

2. (a)

3D view



in orbit plane



path between $\bar{r}_1$ and $\bar{r}_2$ must look like this

$$\bar{r}_1 = a_n(c_{45°}\hat{x} + s_{45°}\hat{z})$$
$$\bar{r}_2 = a_n(c_{45}\hat{y} + s_{45}\hat{z})$$

observe: both vectors possess length = semimajor axis
$\Rightarrow$ both are at ends of minor axis

Note: $\boxed{\Delta E = 180°}$

$$\bar{r}_1 \cdot \bar{r}_2 = r_1 r_2 \cos\phi \Rightarrow c_\phi = s_{45}^2 \Rightarrow \boxed{\phi = 60°}$$

$$\hat{h} = \frac{\bar{r}_1 \times \bar{r}_2}{|\bar{r}_1 \times \bar{r}_2|} = \frac{a_n(c_{45}\hat{x} + s_{45}\hat{z}) \times a_n(c_{45}\hat{y} + s_{45}\hat{z})}{|\bar{r}_1 \times \bar{r}_2|}$$

$$= a_n^2 \tfrac{1}{2}\hat{z} - a_n^2 \tfrac{1}{2}\hat{y} - a_n^2 \tfrac{1}{2}\hat{x}$$

$$= \frac{\frac{a_n^2}{2}(\hat{z} - \hat{y} + \hat{x})}{\frac{a_n^2}{2}\sqrt{3}} = \underbrace{\frac{1}{\sqrt{3}}\hat{x} - \frac{1}{\sqrt{3}}\hat{y} + \frac{1}{\sqrt{3}}\hat{z}}_{\text{signs correct}}$$

$\hat{h}$ above fund plane because direction of motion MUST be as seen in sketch; opposite direction passes thru r_p

$$\hat{h}_z = \frac{1}{\sqrt{3}} = \cos i \qquad \boxed{i = 54.7356°} \quad (0 < i < \pi)$$

$$s_\Omega s_i = \hat{h}_x \Rightarrow \Omega = -45°, -135°$$
$$-c_\Omega s_i = \hat{h}_y \Rightarrow \Omega = \pm 45°$$

$$\boxed{\begin{array}{l}\Omega = -45° \\ \Omega = 315°\end{array}}$$

$$\hat{r}_1 = C_{45}\,\hat{x} + S_{45}\,\hat{z}$$

$$\hat{r}_{1z} = S_i S_\theta = S_{45} \Longrightarrow \theta_1 = 60°;\ 120°$$

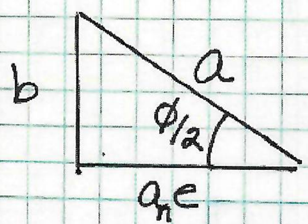
$$\hat{r}_{1y} = 0 = S_\Omega + C_\Omega C_i S_\theta \Longrightarrow \theta_1 = 60°$$

$$\omega = \theta_1 - \theta_1^* \Longrightarrow \boxed{\omega = -90°}$$

$$\theta_1^* = 180° - \phi/2 = \boxed{150°}$$

$$\boxed{\theta_2^* = 210°}$$

$$\theta_2^* - \theta_1^* = \phi = \Delta\theta^*$$



$$\cancel{a}\cos\phi/2 = \cancel{a}e \Longrightarrow \boxed{e = .866 = \frac{\sqrt{3}}{2}}$$

$$p = a_n(1-e^2) = \boxed{.25\,a_n}$$

end of minor axis $\quad v = v_c = \sqrt{\dfrac{\mu_\oplus}{a_n}} \Rightarrow \boxed{k_1 = 1}$

$$f = \left\{1 - \frac{r}{p}\left[1 - \cos\Delta\theta^*\right]\right\} = 1 - \frac{\overset{a_n}{r_2}}{\underset{a_n/4}{p}}(1 - C_\phi) = -1$$

$$= \left\{1 - \frac{a}{r_0}\left[1 - \cos\Delta E\right]\right\} = 1 - \frac{\cancel{a}}{\cancel{r_1}}(2) = -1$$

$$g = \left\{\frac{r r_0}{\sqrt{\mu p}}\sin\Delta\theta^*\right\} = \frac{r_1 r_2}{\sqrt{\mu p}} S_\phi = \frac{a_n^2(2)}{\sqrt{\mu_\oplus a_n}} S_{60} = \sqrt{\frac{a_n^3}{\mu_\oplus}}\sqrt{3}$$

$$= \left\{(t - t_0) - \sqrt{\frac{a^3}{\mu}}\left[\Delta E - \sin\Delta E\right]\right\} \qquad g = \frac{\sqrt{3}}{n}$$

$$\bar{r}_2 = f\bar{r}_1 + g\bar{v}_1 \Longrightarrow \bar{v}_1 = \frac{\bar{r}_2 - f\bar{r}_1}{g}$$

$$\bar{v}_1 = \frac{\bar{r}_2 - f\,\bar{r}_1}{g} = \frac{1}{g}\left\{ a_n (c_{45}\hat{y} + s_{45}\hat{z}) - (-1)\, a_n (c_{45}\hat{x} + s_{45}\hat{z}) \right\}$$

$$\bar{v}_1 = \frac{n}{\sqrt{3}}\, a_n \left\{ \frac{1}{\sqrt{2}}\hat{x} + \frac{1}{\sqrt{2}}\hat{y} + \frac{2}{\sqrt{2}}\hat{z} \right\} \qquad n\,a_n = \sqrt{\frac{\mu}{a_N^3}}\sqrt{a_N^2} = V_c$$

$$\boxed{\bar{v}_1 = v_c \underbrace{\left\{ \frac{1}{\sqrt{6}}\hat{x} + \frac{1}{\sqrt{6}}\hat{y} + \frac{2}{\sqrt{6}}\hat{z} \right\}}_{-\hat{e}}}$$

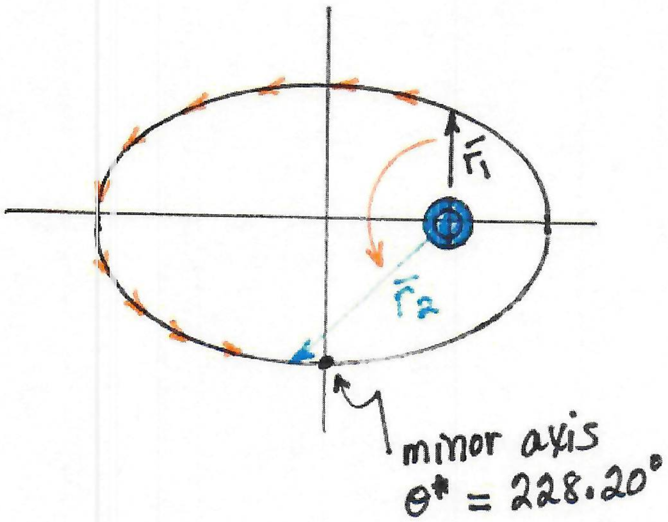
(20 points)

Problem 3: Assume a relative two-body model is appropriate. In an elliptical Earth orbit, a spacecraft is located such that the true anomaly is $\theta_1^* = 90°$; at a later time $\theta_2^* = 225°$. If $r_p = 2R_\oplus$ and $r_a = 10R_\oplus$, determine the time $(t_2 - t_1)$.

Sketch the orbit and identify the arc of interest.

$p = \frac{10}{3} R_\oplus$

$\left.\begin{array}{l} r_p = 2R_\oplus \\ r_a = 10 R_\oplus \end{array}\right\} a = 6R_\oplus$



$r_p = a(1-e) \Rightarrow e = \frac{2}{3}$

$r_1 = \frac{p}{1+e\cos\theta_1^*}$

$r_1 = \frac{10}{3} R_\oplus \quad r_2 = 6.304 R_\oplus$

$= 3.33 R_\oplus$

$n = \sqrt{\frac{\mu}{a^3}} = 8.4049 \times 10^{-5} \frac{rad}{s}$

$n(t_2 - t_p) = E_2 - e\sin E_2$

$-\left[n(t_1 - t_p) = E_1 - e\sin E_1\right]$

$n(t_2 - t_1) = E_2 - E_1 - e s_{E_2} + e s_{E_1}$

$t_1 = 4094.8$ sec

$t_2 = 63043.9$ sec

$$\tan\frac{E}{2} = \underbrace{\left(\frac{1-e}{1+e}\right)^{1/2}}_{\left(\frac{1}{5}\right)^{1/2}} \tan\frac{\theta^*}{2} \Longrightarrow \boxed{\begin{array}{l} E_1 = 48.1896° \\ E_2 = 265.6123 \end{array}}$$

ck $r = a(1 - e c_E)$ ✓

$$\boxed{\begin{array}{l} t_2 - t_1 = 58949.1 \text{ sec} \\ = 16.375 \text{ hr} \end{array}}$$