#1) $x[k+1] = \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix} x[k] + \begin{pmatrix} 3 \\ 0 \end{pmatrix} u[k]$

$$\begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$C = [B \ AB] = \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}$$

$\text{rank}(C) = 1 \neq 2$ <u>Not reachable</u>

$$\begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix}$$

$$[B \ AB \ A^2] = \begin{bmatrix} \boxed{3} & 0 & 0 & 2 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix} \leftarrow \text{rank} = 2$$

$$\text{rank}(B \ AB) = 1 \neq \text{rank}([B \ AB \ A^2]) = 2$$

$\therefore$ System is neither reachable or controllable as the controllablity matrix isn't full rank & $[B \ AB \ A^2]$ doesn't have same rank as controllability matrix.

#2) $A = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$ $\quad C = [1 \;\; 1]$

$\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix}$ $\quad y = \begin{pmatrix} y[0] \\ y[1] \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$

$X[0] = \mathcal{O}^{-1} y$

$[1 \;\; 1] \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} = (-1 \;\; 1)$

$1\times2, 2\times2$

$\mathcal{O} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

$\mathcal{O}^{-1} = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

$X[0] = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 2 \\ 6 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$

$\boxed{X[0] = \begin{pmatrix} 2 \\ 4 \end{pmatrix}}$

check $\quad X[1] = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$y[1] = [1 \;\; 1] \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2$

#3) $A = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$q_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$q_2^T q_1 = 0 \quad \therefore \quad q_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad Q^{-1} = T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$\boxed{T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}$$

$\tilde{A} = T^{-1} A T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$= \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}$

$\tilde{B} = TB = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$C = CT = [1 \quad 1] \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = (1 \quad 1)$

$$\boxed{\tilde{A} = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C = (1 \quad 1), \quad D = -3}$$

#4) $G(s) = \begin{bmatrix} \frac{4s^2}{2s^2+6s+4} & \frac{1}{s+2} \end{bmatrix}$ $\quad U = 2\times 1$

$$\frac{4s^2}{2s^2+6s+4} = \frac{2s^2}{s^2+3s+2} \Rightarrow$$

$$\begin{array}{r|l} & 2 \\ \hline s^2+3s+2 & 2s^2 \\ & -(2s^2+6s+4) \\ \hline & -6s-4 \end{array}$$

$$\frac{4s^2}{2s^2+6s+4} = \frac{-6s-4}{s^2+3s+2} + 2$$

$$A_1 = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \quad B_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad C_1 = [-4 \;\; -6] \quad D_1 = 2$$

$$A_2 = -2 \quad B_2 = 1 \quad C_2 = 1 \quad D_2 = 0$$

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -2 & -3 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$\dot{X}_{3\times1} = A_{3\times3} X_{3\times1} + B_{3\times2} U_{2\times1}$$

$$y_{1\times1} = C_{1\times3} X_{3\times1} + D_{1\times2} U_{2\times1}$$

$$B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$C = [-4 \;\; -6 \;\; 1] = [C_1 \;\; C_2]$$

$$D = [2 \;\; 0] = [D_1 \;\; D_2]$$

#5) $P = \begin{pmatrix} -\frac{1}{2} & 0 \\ \gamma & -\frac{3}{2} \end{pmatrix}$

$$A = \frac{P+P^T}{2} = \frac{\begin{pmatrix} -\frac{1}{2} & 0 \\ \gamma & -\frac{3}{2} \end{pmatrix} + \begin{pmatrix} -\frac{1}{2} & \gamma \\ 0 & -\frac{3}{2} \end{pmatrix}}{2} = \begin{pmatrix} -\frac{1}{2} & \frac{\gamma}{2} \\ \frac{\gamma}{2} & -\frac{3}{2} \end{pmatrix}$$

A is negative semi-definite for $-A$ is positive semi-definite

$$-A = \begin{pmatrix} \frac{1}{2} & -\frac{\gamma}{2} \\ -\frac{\gamma}{2} & \frac{3}{2} \end{pmatrix}$$

For positive definite all principal minors > 0

$$\frac{1}{2} > 0, \quad \frac{3}{2} > 0, \quad \frac{3}{4} - \frac{\gamma^2}{4} > 0 \Rightarrow$$

$$\gamma^2 < 3 \Rightarrow \boxed{-\sqrt{3} < \gamma < \sqrt{3}}$$ $\leftarrow$ Range of γ so P is negative semi-definite

Check: If $\gamma = 0$ $\det(-A) > 0$, If $\gamma = 1$, $\det(-A) > 0$,
if $\gamma = -1$ $\det(-A) > 0$

For Positive definite: $A = \begin{pmatrix} -\frac{1}{2} & \frac{\gamma}{2} \\ \frac{\gamma}{2} & -\frac{3}{2} \end{pmatrix}$

Principal minors $-\frac{1}{2}$ & $-\frac{3}{2}$ are never > 0 $\therefore$ this Quadratic form is never positive semi-definite

#6) $A=\begin{pmatrix}0&0\\1&2\end{pmatrix}$ $B=\begin{pmatrix}1\\0\end{pmatrix}$

$\dot{x}=Ax+Bu=Ax-BKx+Br=(A-BK)x+Br$

$s_1=-1+2j$, $s_2=-1-2j$

$A-BK=\begin{pmatrix}0&0\\1&2\end{pmatrix}-\begin{pmatrix}1\\0\end{pmatrix}[K_1\ K_2]=\begin{pmatrix}0&0\\1&2\end{pmatrix}-\begin{pmatrix}K_1&K_2\\0&0\end{pmatrix}$

$=\begin{pmatrix}-K_1&-K_2\\1&2\end{pmatrix}$ $|\lambda I-(A-BK)|=\lambda^2-(2-K_1)\lambda+(-2K_1+K_2)$

$(\lambda-(-1+2j))(\lambda-(-1-2j))=(\lambda+1-2j)(\lambda+1+2j)$

$=\lambda^2+\lambda+\cancel{2j\lambda}+\lambda+1+\cancel{2j}-\cancel{2j\lambda}-\cancel{2j}-4j^2$

$=\lambda^2+2\lambda+5$

$2=-2+K_1 \Rightarrow K_1=4$

$5=-2K_1+K_2 \Rightarrow K_2=13$

$\boxed{u=-[4\ 13]x+r}$

$\lambda^2+2\lambda+5$

$\lambda_{1,2}=\frac{-2\pm\sqrt{4-(4)(5)}}{2}=-1\pm 2i$

$\dot{x}=(A-BK)x+Br=A_{CL}+B$

$y=Cx+Du=Cx-DKx+Dr=\underbrace{(C-DK)}_{C_{CL}}x+Dr$

$$sX(s) = (A-BK)X + BR(s)$$

$$(sI-(A-BK))X = BR(s)$$

$$X = (sI-(A-BK))^{-1} B R(s)$$

$$Y(s) = (C-DK)(sI-(A-BK))^{-1} B R(s) + DR(s)$$

$$\frac{Y(s)}{R(s)} = (C-DK)(sI-(A-BK))^{-1}B + D$$

$1\times2 \quad \times 2\times2$

$$C-DK = [1 \ 1]-(-3)[4 \ 13] = [13 \ 40]$$

$$sI-A+BK = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 4 & 13 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} s+4 & 13 \\ -1 & s-2 \end{pmatrix}$$

$$(sI-A+BK)^{-1} = \frac{1}{(s+4)(s-2)+13} \begin{pmatrix} s-2 & -13 \\ 1 & s+4 \end{pmatrix}$$

$$(C-DK)(sI-A+BK)^{-1} = \begin{pmatrix} \frac{13(s-2)+40}{(s+4)(s-2)+13} & \frac{(13)(-13)+40(s+4)}{(s+4)(s-2)+13} \end{pmatrix}$$

$$(C-DK)(sI-A+BK)^{-1}B+D = \frac{13(s-2)+40}{(s+4)(s-2)+13} - 3$$

$$\boxed{\frac{Y(s)}{R(s)} = \frac{13(s-2)+40}{(s+4)(s-2)+13} - 3}$$

#7)

$$G(s) = \begin{pmatrix} \dfrac{s^5 + 2s^4 + 7s^3 + s + 2}{s^6 - s^5 - 3s^4} + 1 \\[2em] \dfrac{s^5 + s^4 + 2s^3 + s^2 + 7}{s^6 - 2s^4} \end{pmatrix}$$

I had to BS this I had no time, It's not in CCF form

#8) $A = \begin{pmatrix} 0 & 6 \\ 0 & 0 \end{pmatrix}$

$\dot{x}_1 = x_2$

$\dot{x}_2 = 0$

$\frac{dx_2}{dt} = 0 \qquad \int_{x_2(0)}^{x_2} dx_2 = \int 0\, dt = 0 = x_2 - x_2(0)$

$x_2 = x_2(0)$

$\frac{dx_1}{dt} = x_2 = x_2(0) \qquad \int_{x_1(0)}^{x_1} dx_1 = x_2 \int_0^t dt$

$x_1 - x_1(0) = x_2(0)\, t$

$x_1 = x_2(0)\, t + x_1(0)$

$\phi(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \qquad x(t) = \phi(t)\, x(0)$

$\phi(\tau) = \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix} \qquad \phi(\tau)^{-1} = \begin{pmatrix} 1 & -\tau \\ 0 & 1 \end{pmatrix}$

$\phi(t,\tau) = \phi(t)\, \phi(\tau)^{-1} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\tau \\ 0 & 1 \end{pmatrix}$

$$\boxed{\phi(t,\tau) = \begin{pmatrix} 1 & t \cdot \tau \\ 0 & 1 \end{pmatrix}}$$

$$\phi(2,1) = \begin{pmatrix} 1 & 2-1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$x(2) = \phi(2,1)\, x(1)$$

$$x(1) = \phi^{-1}(2,1)\, x(2) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 4 \end{pmatrix} = \begin{pmatrix} -4 \\ 4 \end{pmatrix}$$

$$\boxed{x(1) = \begin{pmatrix} -4 \\ 4 \end{pmatrix}}$$

#9)

$\dot{x} - (\cos t)x = 0$ $\qquad$ $\dot{x} + Px = Q$

$P = -\cos t$

$\rho = e^{\int P dt} = e^{\int -\cos t\, dt} = e^{-\sin t}$

$e^{-\sin t}\dot{x} - \cos t\, e^{-\sin t} x = 0 = \frac{d}{dt}(e^{-\sin t}x)$

$\int d(e^{-\sin t}x) = 0 + x(0) = e^{-\sin t}x$

$\boxed{x(t) = \frac{5}{e^{-\sin t}}}$

$x(\pi) = \frac{5}{e^{-\sin(\pi)}} = \frac{5}{e^{-0}}$

$\boxed{x(\pi) = 5}$

#10)

$$\dot{x} = \begin{pmatrix} 2x_1 + x_2 \\ x_1 + 3x_2 - 1 \end{pmatrix}$$

$0 = 2x_1 + x_2 \Rightarrow x_2 = -2x_1$

$0 = x_1 + 3x_2 - 1 \Rightarrow x_1 + 3(-2x_1) - 1 = 0$

$1 = -5x_1 \qquad x_1 = -\frac{1}{5}$

$\therefore x_2 = \frac{2}{5}$

$$\boxed{x_e = \begin{pmatrix} -1/5 \\ 2/5 \end{pmatrix}}$$

Check: $\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} -1/5 \\ 2/5 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

#11)

$$BK = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} (k_1 \; k_2 \; k_3 \; k_4 \; k_5) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ k_1 & k_2 & k_3 & k_4 & k_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A - BK = \begin{pmatrix} 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 & 0 \\ -k_1 & -k_2 & -k_3 & -k_4 & 1-k_5 \\ 0 & 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$s = (s+2)^5$$

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#12) $A = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}$ $B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$C = [1 \quad 1]$ $D = -3$

$(A - LC) = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix} - \begin{pmatrix} L_1 \\ L_2 \end{pmatrix} (1 \quad 1) = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix} - \begin{pmatrix} L_1 & L_1 \\ L_2 & L_2 \end{pmatrix}$

$= \begin{pmatrix} -L_1 & -L_1 \\ 1-L_2 & 2-L_2 \end{pmatrix}$

$|\lambda I - (A - LC)| = \lambda^2 - (2 - L_2 - L_1)\lambda + (-2L_1 + \cancel{L_2 L_1} - (-L_1 + \cancel{L_2 L_1}))$

$= \lambda^2 - (2 - L_2 - L_1)\lambda - L_1$

$(s+3)(s+4) = s^2 + 7\lambda + 12$, $s_1 = -3$, $s_2 = -4$

$L_1 = -12$

$-2 + L_2 + L_1 = 7$

$L_2 = 7 + 2 - L_1$

$\boxed{L = \begin{pmatrix} -12 \\ -3 \end{pmatrix}}$

$\boxed{\hat{x}[k+1] = A\hat{x}[k] + Bu[k] + L(y[k] - \hat{y}[k])}$

$\hat{y}[k] = C\hat{x}[k]$

$$(A-BK) = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} [K_1 \; K_2] = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix} - \begin{pmatrix} K_1 & K_2 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -K_1 & -K_2 \\ 1 & 2 \end{pmatrix}$$

$$z^2 + 3z + 2 = \lambda^2 - (2 - K_1)\lambda + (-2K_1 + K_2)$$

$$3 = -2 + K_1 \qquad K_1 = 5$$

$$2 = -2(5) + K_2 \qquad K_2 = 12$$

$$\boxed{K = [5 \;\; 12]}$$

$$\boxed{\frac{Y}{R} = \frac{13(z-2) + 40}{(z+4)(z-2) + 13} - 3}$$

Same As problem 6 TF