ECE 68000: MODERN AUTOMATIC CONTROL

Professor Stan Żak

Model-Based Predictive Control (MPC) With Constraints

Summary of the Simple Discrete-Time MPC

- The matrix $\left(\boldsymbol{R} + \boldsymbol{Z}^\top \boldsymbol{Q}\boldsymbol{Z}\right)$ is invertible, and in fact, positive definite because $\boldsymbol{R} = \boldsymbol{R}^\top \succ 0$ and $\boldsymbol{Z}^\top \boldsymbol{Q}\boldsymbol{Z}$ is also symmetric and at least positive semi-definite
- Hence, $\Delta \boldsymbol{U}$ that satisfies the FONC is

$$\Delta \boldsymbol{U}^* = \left(\boldsymbol{R} + \boldsymbol{Z}^\top \boldsymbol{Q}\boldsymbol{Z}\right)^{-1} \boldsymbol{Z}^\top \boldsymbol{Q}\left(\boldsymbol{r}_p - \boldsymbol{W}\boldsymbol{x}_a\right)$$

- Apply the second derivative test to $J(\Delta \boldsymbol{U})$, which we refer to as the second-order sufficiency condition (SOSC)

$$\begin{aligned} \frac{\partial^2 J}{\partial \Delta \boldsymbol{U}^2} &= \boldsymbol{R} + \boldsymbol{Z}^\top \boldsymbol{Q}\boldsymbol{Z} \\ &\succ 0, \end{aligned}$$

which implies that $\Delta \boldsymbol{U}^*$ is a strict minimizer of J

Controller implementation details

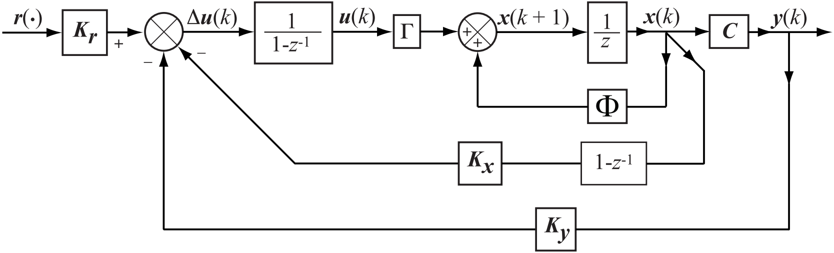
- Compute $\Delta\boldsymbol{u}[k]$,

$$
\begin{aligned}
\Delta\boldsymbol{u}[k] &= \overbrace{\left[\begin{array}{cccc} \boldsymbol{I}_m & \boldsymbol{O} & \cdots & \boldsymbol{O} \end{array}\right]}^{N_p \text{ block matrices}} \left(\boldsymbol{R}+\boldsymbol{Z}^\top\boldsymbol{Q}\boldsymbol{Z}\right)^{-1}\boldsymbol{Z}^\top\boldsymbol{Q}\left(\boldsymbol{r}_p\right. \\
&\qquad\left.-\boldsymbol{W}\boldsymbol{x}_a\right) \\
&= \boldsymbol{K}_{\boldsymbol{r}}\boldsymbol{r}_p - \boldsymbol{K}_{\boldsymbol{x}}\Delta\boldsymbol{x}[k] - \boldsymbol{K}_{\boldsymbol{y}}\boldsymbol{y}[k],
\end{aligned}
$$

where

$$
\begin{aligned}
\boldsymbol{K}_{\boldsymbol{r}} &= \left[\begin{array}{cccc} \boldsymbol{I}_m & \boldsymbol{O} & \cdots & \boldsymbol{O} \end{array}\right]\left(\boldsymbol{R}+\boldsymbol{Z}^\top\boldsymbol{Q}\boldsymbol{Z}\right)^{-1}\boldsymbol{Z}^\top\boldsymbol{Q}, \\
\boldsymbol{K}_{\boldsymbol{x}} &= \boldsymbol{K}_{\boldsymbol{r}}\boldsymbol{W}\left[\begin{array}{c} \boldsymbol{I}_n \\ \boldsymbol{O} \end{array}\right] \\
\boldsymbol{K}_{\boldsymbol{y}} &= \boldsymbol{K}_{\boldsymbol{r}}\boldsymbol{W}\left[\begin{array}{c} \boldsymbol{O} \\ \boldsymbol{I}_p \end{array}\right]
\end{aligned}
$$

MPC implementation



MPC With Constraints: Rate of change $\Delta \boldsymbol{u}$

- An attractive feature of the model-based predictive control approach is that a control engineer can incorporate different types of constraints on the control action
- Consider three types of such constraints.
- Hard constraints on the rate of change of the control signal

$$\Delta u_i^{\min} \leq \Delta u_i[k] \leq \Delta u_i^{\max}, \quad i = 1, 2, \ldots, m$$

- Let

$$\begin{aligned} \Delta \boldsymbol{u}^{\min} &= \begin{bmatrix} \Delta u_1^{\min} & \cdots & \Delta u_m^{\min} \end{bmatrix}^\top \quad \text{and} \\ \Delta \boldsymbol{u}^{\max} &= \begin{bmatrix} \Delta u_1^{\max} & \cdots & \Delta u_m^{\max} \end{bmatrix}^\top \end{aligned}$$

Implementing constraints on the rate of change $\Delta \boldsymbol{u}$

- We have
$$\Delta \boldsymbol{u}^{\min} \leq \Delta \boldsymbol{u}[k] \leq \Delta \boldsymbol{u}^{\max}$$
- Equivalently
$$\begin{bmatrix} -\boldsymbol{I}_m \\ \boldsymbol{I}_m \end{bmatrix} \Delta \boldsymbol{u}[k] \leq \begin{bmatrix} -\Delta \boldsymbol{u}^{\min} \\ \Delta \boldsymbol{u}^{\max} \end{bmatrix}.$$
- Can represent constraints on the rate of change of the control action over the whole prediction horizon, N_p, in terms of $\Delta \boldsymbol{U}$, by augmenting the above inequality to incorporate constraints for the remaining sampling times

Implementing constraints on the rate of change of the control action for all sampling times within the prediction horizon

$$
\begin{bmatrix}
-\boldsymbol{I}_m & \boldsymbol{O} & \cdots & \boldsymbol{O} & \boldsymbol{O} \\
\boldsymbol{I}_m & \boldsymbol{O} & \cdots & \boldsymbol{O} & \boldsymbol{O} \\
\boldsymbol{O} & -\boldsymbol{I}_m & \cdots & \boldsymbol{O} & \boldsymbol{O} \\
\boldsymbol{O} & \boldsymbol{I}_m & \cdots & \boldsymbol{O} & \boldsymbol{O} \\
\vdots & & & & \vdots \\
\boldsymbol{O} & \boldsymbol{O} & \cdots & -\boldsymbol{I}_m & \boldsymbol{O} \\
\boldsymbol{O} & \boldsymbol{O} & \cdots & \boldsymbol{I}_m & \boldsymbol{O} \\
\boldsymbol{O} & \boldsymbol{O} & \cdots & \boldsymbol{O} & -\boldsymbol{I}_m \\
\boldsymbol{O} & \boldsymbol{O} & \cdots & \boldsymbol{O} & \boldsymbol{I}_m
\end{bmatrix}
\begin{bmatrix}
\Delta\boldsymbol{u}[k] \\
\Delta\boldsymbol{u}[k+1] \\
\vdots \\
\Delta\boldsymbol{u}(k+N_p-2) \\
\Delta\boldsymbol{u}[k+N_p-1]
\end{bmatrix}
\leq
\begin{bmatrix}
-\Delta\boldsymbol{u}^{\min} \\
\Delta\boldsymbol{u}^{\max} \\
-\Delta\boldsymbol{u}^{\min} \\
\Delta\boldsymbol{u}^{\max} \\
\vdots \\
-\Delta\boldsymbol{u}^{\min} \\
\Delta\boldsymbol{u}^{\max} \\
-\Delta\boldsymbol{u}^{\min} \\
\Delta\boldsymbol{u}^{\max}
\end{bmatrix}
$$

The constraints on the rate of change of the control are imposed only on the first component of $\Delta \boldsymbol{U}$

$$\begin{bmatrix} -\boldsymbol{I}_m & \boldsymbol{O} & \cdots & \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{I}_m & \boldsymbol{O} & \cdots & \boldsymbol{O} & \boldsymbol{O} \end{bmatrix} \Delta \boldsymbol{U} \leq \begin{bmatrix} -\Delta \boldsymbol{u}^{\min} \\ \Delta \boldsymbol{u}^{\max} \end{bmatrix}$$

Constraints on the Control Action Magnitude

- Hard constraints on the control action magnitude at the time sampling k

$$u_i^{\min} \leq u_i[k] \leq u_i^{\max}, \quad i = 1, 2 \ldots, m$$

- Let $\boldsymbol{u}^{\min} = \begin{bmatrix} u_1^{\min} & \cdots & u_m^{\min} \end{bmatrix}^\top$ and $\boldsymbol{u}^{\max} = \begin{bmatrix} u_1^{\max} & \cdots & u_m^{\max} \end{bmatrix}^\top$
- Then, we have

$$\boldsymbol{u}^{\min} \leq \boldsymbol{u}[k] \leq \boldsymbol{u}^{\max}$$

- Equivalently

$$\begin{bmatrix} -\boldsymbol{I}_m \\ \boldsymbol{I}_m \end{bmatrix} \boldsymbol{u}[k] \leq \begin{bmatrix} -\boldsymbol{u}^{\min} \\ \boldsymbol{u}^{\max} \end{bmatrix}$$

Constraints on the control action magnitude over the whole prediction horizon

- Note that

$$
\begin{aligned}
\boldsymbol{u}[k] &= \boldsymbol{u}[k-1] + \Delta\boldsymbol{u}[k] \\
&= \boldsymbol{u}[k-1] + \begin{bmatrix} \boldsymbol{I}_m & \boldsymbol{O} & \cdots & \boldsymbol{O} \end{bmatrix} \Delta\boldsymbol{U},
\end{aligned}
$$

  where, recall that
  $\Delta\boldsymbol{U} = \begin{bmatrix} \Delta\boldsymbol{u}[k] & \Delta\boldsymbol{u}[k+1] & \cdots & \Delta\boldsymbol{u}[k+N_p-1] \end{bmatrix}^\top$

- Similarly,

$$
\begin{aligned}
\boldsymbol{u}[k+1] &= \boldsymbol{u}[k] + \Delta\boldsymbol{u}[k+1] \\
&= \boldsymbol{u}[k-1] + \begin{bmatrix} \boldsymbol{I}_m & \boldsymbol{O} & \cdots & \boldsymbol{O} \end{bmatrix} \Delta\boldsymbol{U} + \Delta\boldsymbol{u}[k+1] \\
&= \boldsymbol{u}[k-1] + \begin{bmatrix} \boldsymbol{I}_m & \boldsymbol{I}_m & \cdots & \boldsymbol{O} \end{bmatrix} \Delta\boldsymbol{U}
\end{aligned}
$$

Constraints on the control action magnitude over the whole prediction horizon—contd.

- Collect into one equation

$$\begin{bmatrix} \boldsymbol{u}[k] \\ \boldsymbol{u}[k+1] \\ \vdots \\ \boldsymbol{u}[k+N_p-1] \end{bmatrix} = \begin{bmatrix} \boldsymbol{I}_m \\ \boldsymbol{I}_m \\ \vdots \\ \boldsymbol{I}_m \end{bmatrix} \boldsymbol{u}[k-1] + \begin{bmatrix} \boldsymbol{I}_m & \boldsymbol{O} & \cdots & \boldsymbol{O} \\ \boldsymbol{I}_m & \boldsymbol{I}_m & \cdots & \boldsymbol{O} \\ \vdots & & \ddots & \vdots \\ \boldsymbol{I}_m & \boldsymbol{I}_m & \cdots & \boldsymbol{I}_m \end{bmatrix} \begin{bmatrix} \Delta\boldsymbol{u}[k] \\ \Delta\boldsymbol{u}[k+1] \\ \vdots \\ \Delta\boldsymbol{u}[k+N_p-1] \end{bmatrix}$$

- Let

$$\boldsymbol{U} = \begin{bmatrix} \boldsymbol{u}[k] \\ \boldsymbol{u}[k+1] \\ \vdots \\ \boldsymbol{u}[k+N_p-1] \end{bmatrix} \quad \boldsymbol{E} = \begin{bmatrix} \boldsymbol{I}_m \\ \boldsymbol{I}_m \\ \vdots \\ \boldsymbol{I}_m \end{bmatrix}, \text{ and } \boldsymbol{H} = \begin{bmatrix} \boldsymbol{I}_m & \boldsymbol{O} & \cdots & \boldsymbol{O} \\ \boldsymbol{I}_m & \boldsymbol{I}_m & \cdots & \boldsymbol{O} \\ \vdots & & \ddots & \vdots \\ \boldsymbol{I}_m & \boldsymbol{I}_m & \cdots & \boldsymbol{I}_m \end{bmatrix}$$

Constraints over the whole prediction horizon

- Then

$$\boldsymbol{U} = \boldsymbol{E}\boldsymbol{u}[k-1] + \boldsymbol{H}\Delta\boldsymbol{U}$$

- Suppose we are faced with constructing a control action subject to the constraints,

$$\boldsymbol{U}^{\min} \leq \boldsymbol{U} \leq \boldsymbol{U}^{\max}$$

- The above constraints can be equivalently represented as

$$\begin{bmatrix} -\boldsymbol{U} \\ \boldsymbol{U} \end{bmatrix} \leq \begin{bmatrix} -\boldsymbol{U}^{\min} \\ \boldsymbol{U}^{\max} \end{bmatrix}$$

Implementing the constraints on the control action magnitude in terms of $\Delta \boldsymbol{U}$

- We obtain

$$\left[\begin{array}{c} -\left(\boldsymbol{E}\boldsymbol{u}[k-1]+\boldsymbol{H}\Delta\boldsymbol{U}\right) \\ \boldsymbol{E}\boldsymbol{u}[k-1]+\boldsymbol{H}\Delta\boldsymbol{U} \end{array}\right] \leq \left[\begin{array}{c} -\boldsymbol{U}^{\min} \\ \boldsymbol{U}^{\max} \end{array}\right]$$

- The above can be represented as

$$\left[\begin{array}{c} -\boldsymbol{H} \\ \boldsymbol{H} \end{array}\right] \Delta\boldsymbol{U} \leq \left[\begin{array}{c} -\boldsymbol{U}^{\min}+\boldsymbol{E}\boldsymbol{u}[k-1] \\ \boldsymbol{U}^{\max}-\boldsymbol{E}\boldsymbol{u}[k-1] \end{array}\right]$$

- Special case: control designer elects to impose constraints only on the first component of $\Delta\boldsymbol{U}$, that is, on $\Delta\boldsymbol{u}[k]$ only
- We have

$$\left[\begin{array}{cccc} -\boldsymbol{I}_m & \boldsymbol{O} & \cdots & \boldsymbol{O} \\ \boldsymbol{I}_m & \boldsymbol{O} & \cdots & \boldsymbol{O} \end{array}\right] \Delta\boldsymbol{U} \leq \left[\begin{array}{c} -\boldsymbol{u}^{\min}+\boldsymbol{u}[k-1] \\ \boldsymbol{u}^{\max}-\boldsymbol{u}[k-1] \end{array}\right],$$

where $\boldsymbol{u}^{\min}$ and $\boldsymbol{u}^{\max}$ are lower and upper bounds on $\boldsymbol{u}[k]$