ECE 68000: MODERN AUTOMATIC CONTROL

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Solving LMIs with CVX

General Structure of CVX Code in MATLAB

```
cvx_begin sdp quiet
% sdp: semi-definite programming mode
% quiet: no display during computing
% include CVX [variables]
% very intuitive variable initialization
% for example: variable P(3,3) symmetric
% minimize([cost])  convex function
% subject to
% [affine constraints]
% preferably non-strict inequalities

cvx_end
disp(cvx_status) % solution status
```

Observer Design

Plant model:

$$\begin{aligned}\dot{\boldsymbol{x}} &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} \\ \boldsymbol{y} &= \boldsymbol{C}\boldsymbol{x}\end{aligned}$$

Linear observer:

$$\dot{\tilde{\boldsymbol{x}}} = \boldsymbol{A}\tilde{\boldsymbol{x}} + \boldsymbol{B}\boldsymbol{u} + \boldsymbol{L}(\boldsymbol{y} - \boldsymbol{C}\tilde{\boldsymbol{x}})$$

Goal: Design $\boldsymbol{L}$ to ensure asymptotic stability of the error dynamics

- Matrix inequality for observer design:

$$(\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C})^\top \boldsymbol{P} + \boldsymbol{P}(\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C}) \prec 0,\ \boldsymbol{P} = \boldsymbol{P}^\top \succ 0$$

Observer Design—Contd.

$$A^\top P + PA - C^\top L^\top P - PLC \prec 0, \; P \succ 0$$

- To-do: Find L, P
- Problem: Bi-linear matrix inequality in L and P
- **Technique #1**: Choose $Y = PL$
- LMIs:
$$\underbrace{A^\top P + PA}_{\text{linear in } P} - \underbrace{C^\top Y^\top - YC}_{\text{linear in } Y} \prec 0, \; P \succ 0$$
- For robustness of solution, rewrite as
$$A^\top P + PA - C^\top Y^\top - YC + 2\alpha P \preceq 0, \; P \succ 0$$
with fixed $\alpha > 0$
- Get back $L = P^{-1}Y$ ($P \succ 0$, hence invertible)

Snippet in CVX

```
cvx_begin sdp

% Variable definition
variable P(n, n) symmetric
variable Y(n, p)

% LMIs
P*sys.A + sys.A'*P - Y*sys.C - sys.C'*Y' + P <= 0
P >= eps*eye(n) % eps is a very small number in MATLAB

cvx_end
sys.L = P\Y; % compute L matrix
```

State/Output Feedback Control

LTI System with output feedback control:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \\ u &= -Ky \end{aligned}$$

Goal: Design K to ensure asymptotic stability of $(A - BKC)$

- Matrix inequality for output-feedback controller design:

$$(A - BKC)^\top P + P(A - BKC) \prec 0, \;\; P \succ 0$$

- Simpler case: state-feedback ($C = I$)

$$(A - BK)^\top P + P(A - BK) \prec 0, \;\; P \succ 0$$

Simpler Case: State-Feedback Control

$$(A - BK)^\top P + P(A - BK) \prec 0, \;\; P \succ 0$$

- To-do: Find K, P
- Problem: Bi-linear matrix inequality in K and P
- **Technique #2**: Congruence transformation with $S \triangleq P^{-1}$ and $Z \triangleq KS$
- New inequalities

$$SA^\top + AS - SK^\top B^\top - BKS \prec 0$$

- LMIs:

$$\underbrace{SA^\top + AS}_{\text{linear in } S} - \underbrace{Z^\top B^\top - BZ}_{\text{linear in } Z} \prec 0, \;\; P \succ 0$$

- Get back $P = S^{-1}$, $K = ZP$

Snippet in CVX

```
cvx_begin sdp

% Variable definition
variable S(n, n) symmetric
variable Z(m, n)

% LMIs
sys.A*S + S*sys.A' - sys.B*Z - Z'*sys.B' <= -eps*eye(n)
S >= eps*eye(n)

cvx_end
sys.K = Z/S; % compute K matrix
```

Output-Feedback Control

$$A^\top P + PA - C^\top K^\top B^\top P - PBKC \prec 0, \;\; P \succ 0$$

- To-do: Find K, P
- Problem: Bi-linear matrix inequality in K and P
- **Technique #3**: Choose M such that $BM = PB$ and $N \triangleq MK$
- New inequalities: $A^\top P + PA - C^\top K^\top MB^\top - BMKC \prec 0$
- Linear matrix (in)equalities:

$$\underbrace{A^\top P + PA}_{\text{linear in } P} - \underbrace{C^\top N^\top B^\top - BNC}_{\text{linear in } N} \prec 0, \;\; BM = PB, \;\; P \succ 0$$

- Get back $K = M^{-1}N$ (M is invertible if B has full column rank)

Snippet in CVX

Cool fact: CVX/YALMIP can handle *equality constraints*!

```
cvx_begin sdp quiet

% Variable definition
variable P(n, n) symmetric
variable N(m, p)
variable M(m, m)

% LMIs
P*sys.A + sys.A'*P - sys.B*N*sys.C ...
 - sys.C'*N'*sys.B' <= -eps*eye(n)
sys.B*M == P*sys.B
P >= eps*eye(n);

cvx_end
sys.K = M\N % compute K matrix
```