AAE 666

$\boxed{\text{Homework Three: Solutions}}$

Exercise 1

a.

$$V(x) = x_1^2 - x_1^4 + x_2^2$$

For $x^e = (x_1, x_2) = (0, 0)$

$$\begin{aligned} DV(x) &= (x_1 - 4x_1^3, 2x_2) \\ DV(x^e) &= 0 \\ D^2V(x) &= \begin{bmatrix} 1 - 12x_1^2 & 0 \\ 0 & 2 \end{bmatrix} \\ D^2V(x^e) &= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \end{aligned}$$

The above function is *lpd* about one of its equilirbium at the origin.

b.

$$V(x) = x_1 + x_2^2$$

For $x^e = (x_1, x_2) = (0, 0)$

$$\begin{aligned} DV(x) &= (x_1, 2x_2) \\ DV(x^e) &= 0 \\ D^2V(x) &= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \\ D^2V(x^e) &= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \end{aligned}$$

The above function is *lpd* about one of its equilirbium at the origin.

c.

$$V(x) = 2x_1^2 - x_1^3 + x_1x_2 + x_2^2$$

For $x^e = (x_1, x_2) = (0, 0)$

$$\begin{aligned} DV(x) &= (4x_1 - 3x_1^2 + x_2, x_1 + 2x_2) \\ DV(x^e) &= 0 \\ D^2V(x) &= \begin{bmatrix} 4 - 3x_1 & 1 \\ 1 & 2 \end{bmatrix} \\ D^2V(x^e) &= \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix} \end{aligned}$$

The eigenvalues for this matrix is $\lambda_1 = 1.5858$, $\lambda_2 = 4.4152$. Thus, the function is *lpd* about one of its equilirbium at the origin.

Exercise 2

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1^3$$

Consider $V(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$ as a candidate Lyapunov function. We have:

$$DV(x) = \{x_1^3, x_2\}$$
$$DV(0) = \{0, 0\}$$
$$D^2V(x) = \begin{bmatrix} 3x_1^2 & 0 \\ 0 & 1 \end{bmatrix}$$
$$DV(0) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

The V is LPD and we have

$$DV(x)f(x) = x_2 x_1^3 - x_2 x_1^3 = 0$$

By **Theorem 1**, we conclude that the system is stable about zero state.

Exercise 3

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + x_1^3$$

Consider $V(x) = \frac{1}{2}x_1^2 - \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$ as a candidate Lyapunov function. We have:

$$DV(x) = \{x_1 - x_1^3, x_2\}$$
$$DV(0) = \{0, 0\}$$
$$D^2V(x) = \begin{bmatrix} 1 + 3x_1^2 & 0 \\ 0 & 1 \end{bmatrix}$$
$$DV(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The V is LPD and we have

$$DV(x)f(x) = x_2(x_1 - x_1^3) + x_2(-x_1 + x_1^3) = 0 \tag{1}$$

By **Theorem 1**, we conclude that the system is stable about zero state.

Exercise 4

$$\dot{x}_1 = x_2^3$$
$$\dot{x}_2 = -x_2^2 x_1$$

Consider $V(x) = x_1^2 + x_2^2$ as a candidate Lyapunov function. We have:

$$DV(x) = \{2x_1, 2x_2\}$$
$$DV(0) = \{0, 0\}$$
$$D^2V(x) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$
$$DV(0) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

The $V(x)$ is LPD and we have

$$DV(x)f(x) = 2x_1(x_2^3) + 2x_2(-x_1 x_2^2) = 0 \tag{2}$$

By **Theorem 1**, we conclude that the system is stable about zero state.

Exercise 5

$$\dot{x} = -(2 + cosx)x$$

Consider $V(x) = \frac{1}{2}x^2$ as a cnadidate Lyapunov function. We have:

$$DV(x)f(x) = -(2 + cosx)x^2$$
$$(2 + cosx) \geq 1$$

Thus $DV(x)f(x) < 0$ for all $x \neq 0$, by **Theorem 4**, the system is GAS about zero.

Exercise 6

$$\dot{x} = -(2 + cosx)(x - 1)$$

Consider $V(x) = \frac{1}{2}(x-1)^2$ as a cnadidate Lyapunov function. We have:

$$DV(x)f(x) = -(2 + cosx)(x - 1)^2$$

Because $2 + cosx > 1$, and $(x-1)^2 > 0$ for all $x \neq 1$. Thus $DV(x)f(x) < 0$ for all $x \neq 1$, by **Theorem 4**, the system is GAS about 1.