AAE 532 - Orbit Mechanics

Purdue University

PS6 Solutions

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Useful Constants

	Axial Rotaional Period (Rev/Day)	Mean Equatorial Radius (km)	Gravitational Parameter $\mu = Gm(km^3/sec^2)$	Semi-major Axis of orbit (km)	Orbital Period (sec)	Eccentricity of Orbit	Inclination of Orbit to Ecliptic (deg)
☉ Sun	0.0394011	695990	132712440017.99	-	-	-	-
☾ Moon	0.0366004	1738.2	4902.8005821478	384400 (around Earth)	2360592 27.32 Earth Days	0.0554	5.16
☾ Mercury	0.0170514	2439.7	22032.080486418	57909101	7600537 87.97 Earth Days	0.20563661	7.00497902
♀ Venus	0.0041149 (Retrograde)	6051.9	324858.59882646	108207284	19413722 224.70 Earth Days	0.00676399	3.39465605
⊕ Earth	1.0027378	6378.1363	398600.4415	149597898	31558205 365.26 Earth Days	0.01673163	0.00001531
♂ Mars	0.9747000	3397	42828.314258067	227944135	59356281 686.99 Earth Days	0.09336511	1.84969142
♃ Jupiter	2.4181573	71492	126712767.8578	778279959	374479305 11.87 Years	0.04853590	1.30439695
♄ Saturn	2.2522053	60268	37940626.061137	1427387908	930115906 29.47 Years	0.05550825	2.48599187
⛢ Uranus	1.3921114 (Retrograde)	25559	5794549.0070719	2870480873	2652503938 84.05 Years	0.04685740	0.77263783
Ψ Neptune	1.4897579	25269	6836534.0638793	4498337290	5203578080 164.89 Years	0.00895439	1.77004347
♇ Pluto	-0.1565620 (Retrograde)	1162	981.600887707	5907150229	7830528509 248.13 Years	0.24885238	17.14001206

- First three columns of the body data are consistent with GMAT 2020a default values, which are mainly from JPL's ephemerides file de405.spk
- The rest of the data are from JPL website(`https://ssd.jpl.nasa.gov/?planet_pos`, retrieved at 09/01/2020)
based on E.M. Standish's "Keplerian Elements for Approximate Positions of the Major Planets"

Problem 1

Problem Statement

Assume that a spacecraft is passing the Moon (radius $R_☾$ and $Gm_☾ = \mu_☾$) in a hyperbolic orbit (again!). The vehicle is sufficiently close that it is reasonable to model the orbit in terms of the relative two-body problem. At some instant, the spacecraft orbit is described such that,

$$r = -9R_☾\hat{p}$$

$$v^2 = \frac{41}{144}\frac{\mu_☾}{R_☾}$$

(a) The orbit is hyperbolic. How do you know that from this given information?

(b) Determine the following quantities for the hyperbola: $r_p, v_p, e, b, d_{aim}, p, h, \delta, v_\infty, \epsilon, \theta^*_\infty$ with all distances in terms of $R_☾$. (Observe that d_{aim} is the distance from the aim point to the center of the hyperbola; maybe a sketch of the corresponding right triangle will help!) Determine r, v, γ at $\theta^* = -120°$; determine r, v, γ at $\theta^* = +120°$.

(c) Plot the hyperbolic orbit in your MATLAB script between $\theta^* = \pm 120°$. Add the current r vector as well as the following to the plot: $\hat{e}, \hat{p}, \bar{v}, \theta^*, \gamma, b, |a|, \theta^*_\infty$, aim point, local horizon, d_{aim}. (Always include the local horizon!)

(d) Of course, the plane of the hyperbola is also oriented in space. Assume that $\omega = +90°, i = 45°, \Omega = 60°$ relative to lunar equatorial coordinates. Sketch the 3D orbit orientation with these parameters; include $\hat{r}, \hat{\theta}, \hat{h}$ and $\hat{x}, \hat{y}, \hat{z}$. Add $i, \omega, \Omega, AN, DN$ and line of nodes to the sketch. Add ω to the plot in (c). Bonus Option: You can plot GMAT to see the 3D hyperbola!

(e) Determine $\bar{r}, \bar{v}, \theta^* = -120°$ and $\theta^* = -120°$ in terms of lunar equatorial inertial coordinates $\hat{x}, \hat{y}, \hat{z}$.

Part (a)

We can determine what type of orbit the given spacecraft is in by looking at the relationship between its velocity and the circular velocity for an orbit at the same radius. The orbit is hyperbolic if:

$$v > \sqrt{2}v_c$$

$$v = \sqrt{\frac{41}{144}\frac{\mu_☾}{R_☾}} = 0.8962\ km/s$$

$$v_c = \sqrt{\frac{\mu_☾}{|\bar{r}|}} = 0.5598\ km/s \qquad \longrightarrow \qquad \sqrt{2}v_c = 0.7917\ km/s$$

$$\boxed{0.8962\ km/s > 0.7917\ km/s}$$

Therefore, we know that the spacecraft must be in a hyperbolic orbit. Alternatively, you could compare the velocity to escape velocity which would show that at a distance of $9R_☾$, $v > v_e$.

Part (b)

Calculating the orbital parameters:

$$\epsilon = \frac{v^2}{2} - \frac{\mu}{r} \boxed{= 0.0881\ km^2/s}$$

$$\epsilon = \frac{\mu}{2|a|} \qquad \longrightarrow \qquad a = 27811.2\ km = 16R_☾$$

$$p = |\bar{r}| = 15643.8\ km \boxed{= 9R_☾}$$

$$e = \sqrt{\frac{p}{|a|} + 1} \boxed{= 1.25}$$

$$r_p = |a|(e-1) = 6952.8\ km \boxed{= 4R_☾}$$

$$v_p = \sqrt{2\left(\epsilon + \frac{\mu}{r_p}\right)} \boxed{= 1.2596\ km/s}$$

$$b = |a|\sqrt{e^2 - 1} = 20858.4\ km \boxed{= 12R_☾}$$

$$d_{aim} = \sqrt{(r_p + |a|)^2 - b^2} = 27811.2\ km \boxed{= 16R_☾}$$

$$h = \sqrt{\mu p} \boxed{= 8757.76\ km^2/s}$$

$$\theta^*_\infty = \cos^{-1}\left(\frac{-1}{e}\right) = 2.4981\ rad \boxed{= 143.1301°}$$

$$\delta = 2(\theta^*_\infty - 90) \boxed{= 106.2602°}$$

$$v_\infty = \sqrt{2\epsilon} \boxed{= 0.4199\ km/s}$$

Since the hyperbola will be symmetric about $\theta^* = 0$, the magnitudes of r, v and γ will be the same at corresponding positive and negative true anomalys.

$$r = \frac{p}{1 + e\cos\theta^*} = 41717\ km \boxed{= 24R_☾}$$

$$v = \sqrt{2\left(\epsilon + \frac{\mu}{r}\right)} \boxed{= 0.6414\ km/s}$$

At $\theta^* = -120$ the spacecraft is descending:

$$\gamma = \cos^{-1}\left(\frac{h}{rv}\right) = 1.2373\ rad \boxed{= -70.8934°}$$

And at $\theta^* = +120$ the spacecraft is ascending:

$$\gamma = \cos^{-1}\left(\frac{h}{rv}\right) = 1.2373\ rad \boxed{= +70.8934°}$$

Part (c)

Using code from previous assignments we can easily plot the hyperbolic trajectory and annotate it with all of the requested parameters:

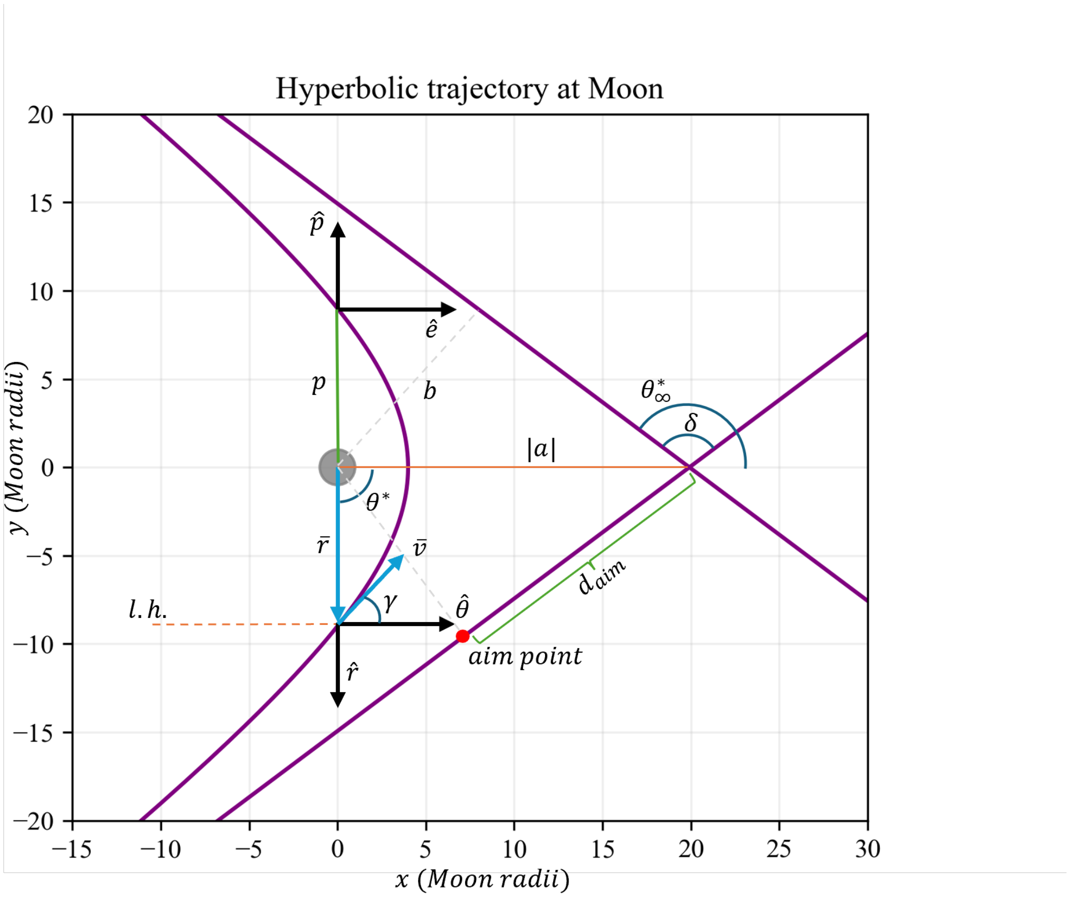

Figure 1: Hyperbolic trajectory.

Part (d)

We can sketch the hyperbola by noting that ω is the angle between the line of nodes and the inertial $\hat{z}$ axis, i is the angle between the trajectory and the $\hat{x} - \hat{y}$ plane, and Ω is the angle between the inertial $\hat{x}$ axis and the line of nodes. The sketch is defined relative to lunar equatorial coordinates, so the $\hat{x} - \hat{y}$ plane will be the same as the lunar equatorial plane. For clarity, I will use the GMAT trajectory as the sketch and add on the requested parameters:

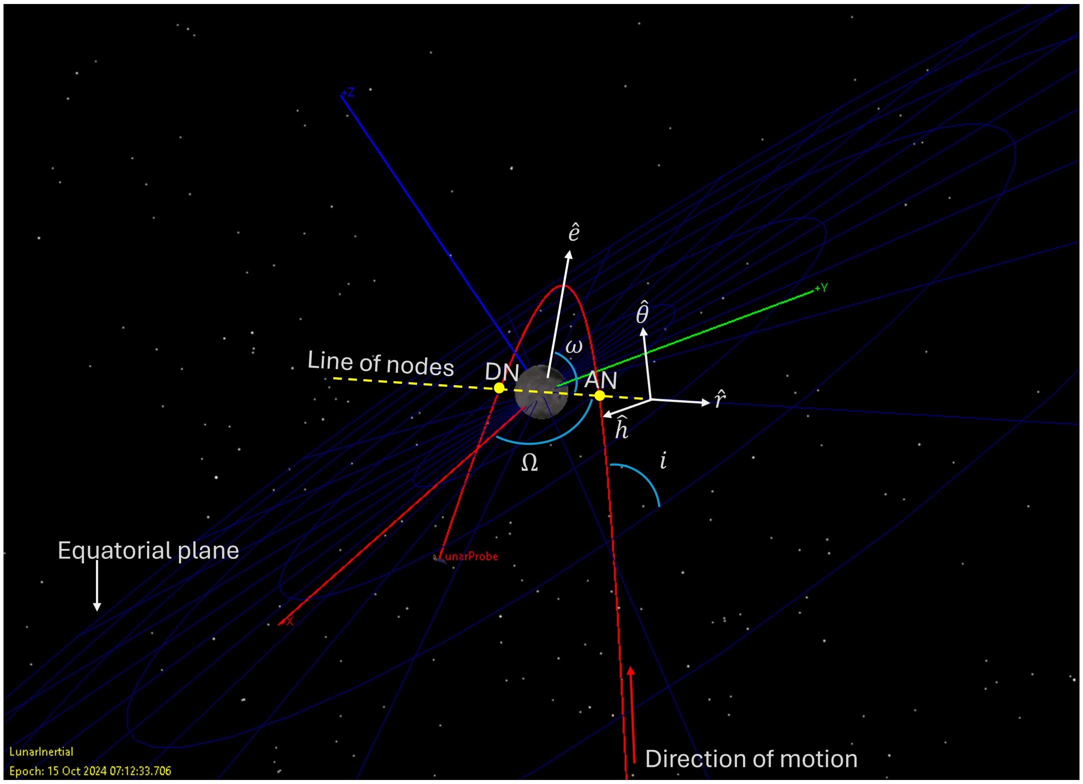

Figure 2: Hyperbolic trajectory oriented in space.

We can also look at some planar projections of the trajectory for further visualization. Note that no epoch was provided so October 14, 2024 was used.

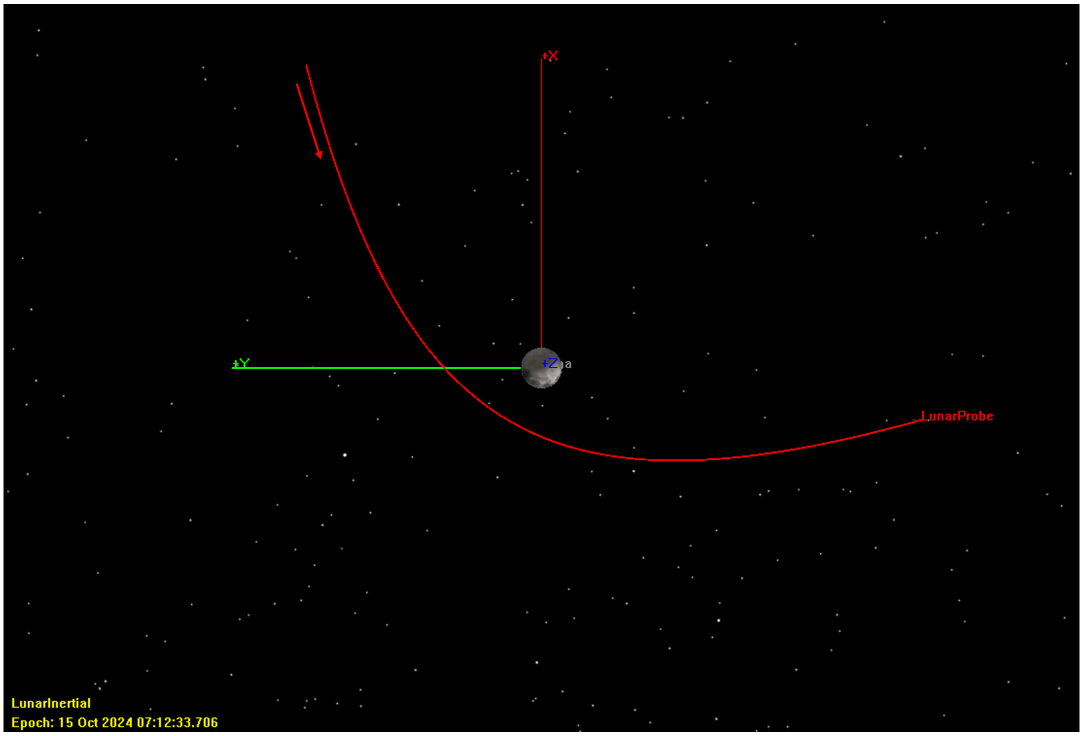

Figure 3: Hyperbolic trajectory $\hat{x} - \hat{y}$ plane projection.

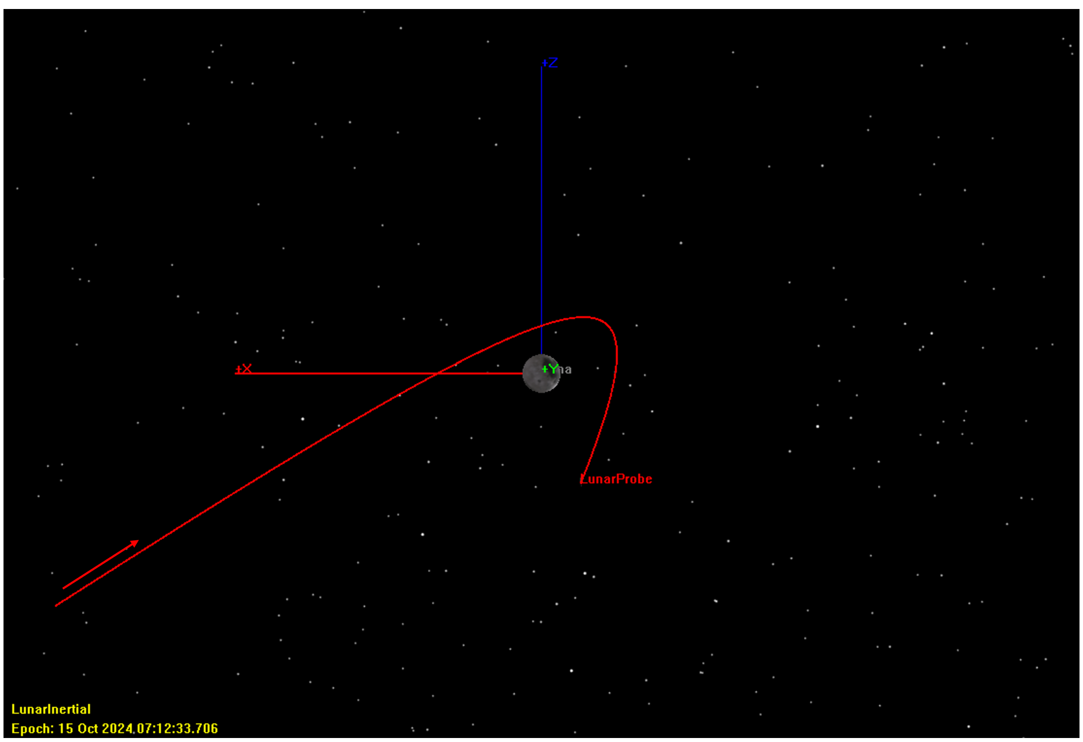

Figure 4: Hyperbolic trajectory $\hat{x} - \hat{z}$ plane projection.

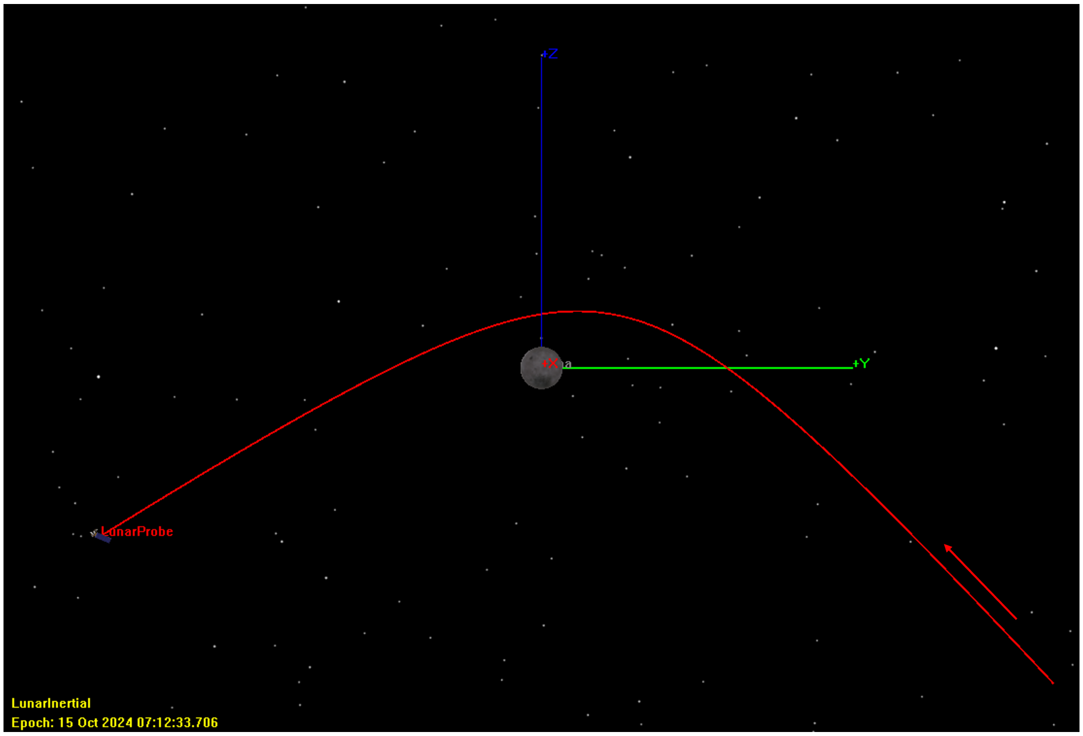

Figure 5: Hyperbolic trajectory $\hat{y} - \hat{z}$ plane projection.

Part (e)

Starting in the $\hat{r}, \hat{\theta}$ frame with:

$$\bar{r} = r\hat{r} + 0\hat{\theta} + 0\hat{h}$$

$$\bar{v} = v \sin\gamma \hat{r} + v\cos\gamma\hat{\theta} + 0\hat{h}$$

We can get the position and velocity vectors in the $\hat{x}, \hat{y}, \hat{z}$ frame using the direction cosine matrix:

$$^{I}C^{R} = \begin{bmatrix} \cos\Omega\cos\theta - \sin\Omega\cos i\sin\theta & -\cos\Omega\sin\theta - \sin\Omega\cos i\cos\theta & \sin\Omega\sin i \\ \sin\Omega\cos\theta + \cos\Omega\cos i\sin\theta & -\sin\Omega\sin\theta + \cos\Omega\cos i\cos\theta & -\cos\Omega\sin i \\ \sin i\sin\theta & \sin i\cos\theta & \cos i \end{bmatrix}$$

Which yields:

$\theta^* = -120°$:

$$\boxed{\bar{r} = 17.7408\hat{x} + 13.7574\hat{y} - 8.4853\hat{z}\ R_☾}$$

$$\boxed{\bar{v} = -0.5069\hat{x} - 0.1923\hat{y} + 0.3428\hat{z}\ km/s}$$

$\theta^* = +120°$:

$$\boxed{\bar{r} = -3.0438\hat{x} - 22.2426\hat{y} - 8.4853\hat{z}\ R_☾}$$

$$\boxed{\bar{v} = 0.0869\hat{x} - 0.5351\hat{y} - 0.3428\hat{z}\ km/s}$$

Problem 2

Problem Statement

Assume that a relative two-body model can be employed for some preliminary assessment. A vehicle is successfully launched into an equatorial Earth orbit such that $r_p = 7000$ km and $r_a = 45000$ km. Currently, the vehicle is located at perigee. A single in-plane maneuver is used to circularize the orbit. Consider two options:

(a) Circularize at apogee.

- (i) What is the "wait time" till the maneuver?
- (ii) Determine $\bar{r}_1^-$, $\bar{v}_1^-$, γ_1^- at the maneuver point. What is $\bar{v}_1^+$, γ_1^+? [Include VECTOR diagrams!] Construct the required maneuver (Δv, α). Recall that $\Delta v = |\Delta \bar{v}|$.
- (iii) Plot the old and new orbits on the same figure using your MATLAB script. On the plot, mark $\bar{r}_0$, $\bar{r}_1$, $\bar{v}_1^-$, local horizon, γ_1^-, $\bar{v}_1^+$, γ_1^+, $\Delta \bar{v}$, α. Identify the maneuver point.
- (iv) What percent of the maneuver magnitude is used to shift energy? What percent to shift direction?

(b) The option in (a) circularizes the orbit but the radius of the new orbit is too large. Assume that the radius of the new circular orbit is specified as 25000 km.

- (i) Determine the true anomaly in the original orbit, i.e., θ^*, at which the maneuver will occur. Where do the orbits intersect?
- (ii) Now determine the "wait time" till the maneuver from perigee in the original orbit to the appropriate θ^* location?
- (iii) Again, compute the required maneuver (Δv, α). [Include VECTOR diagrams!] Plot the old and new orbits in MATLAB. On the plot, mark $\bar{r}_0$, $\bar{r}_1$, $\bar{v}_1^-$, local horizon, γ_1^-, $\bar{v}_1^+$, γ_1^+, $\Delta \bar{v}$, α.
- (iv) What percent of the maneuver magnitude is used to shift energy? What percent to shift direction?

(c) How do the final orbits compare in the two cases? How do the costs (Δv's) compare? It may not actually be fair to compare the Δv's. Why not?

Part (a.i)

Recall that the spacecraft is initially at perigee. If the maneuver occurs at apogee, then, because of the symmetry of a conic orbit, one can note that the "wait time" till the maneuver is half of the orbit period, where the orbit period is defined as:

$$\mathcal{P} = \frac{2\pi}{n} \tag{1}$$

where:

$$n = \sqrt{\frac{\mu}{a^3}} = 1.5059 \cdot 10^{-4} \tag{2}$$

So, the wait time is:

$$\boxed{\begin{aligned}(t - t_p) = \frac{\mathcal{P}}{2} &= 2.086128 \cdot 10^4 \text{ s} \\ &= 5.7948 \text{ hrs}\end{aligned}} \tag{3}$$

Alternatively, one can use Kepler's equation to solve for the wait time since one would know that the eccentric anomaly at apogee is 180° (and where the reference location is perigee).

Part (a.ii)

Now, let's determine the $\bar{r}_1^-$, $\bar{v}_1^-$, γ_1^- at the maneuver point. Note that we are located at apogee, meaning that the position magnitude of the spacecraft can be expressed as:

$$r^- = r_a = a(1-e) = 45000 \text{ km} \tag{4}$$

where the position vector is then:

$$\boxed{\bar{r}^- = \bar{r}^+ = 45000 \text{ km } \hat{r}} \tag{5}$$

Realize that the position vector before and after the maneuver will remain the same! Next, let's now solve for the semi-major axis and eccentricity of the orbit before the maneuver. Recognize first that:

$$a = \frac{r_p + r_a}{2} = 26000 \text{ km} \tag{6}$$

So, with semi-major axis known, one can use the expression for r_a or r_p to solve for eccentricity. Using r_a:

$$e = \frac{r_a}{a} - 1 = 0.7308 \tag{7}$$

Note that, because the spacecraft is located at apogee, the flight path angle before the maneuver, by definition, is:

$$\boxed{\gamma^- = 0°} \tag{8}$$

Then, to calculate $\bar{v}_1^-$, one can first calculate the velocity magnitude of the spacecraft. Recall that the specific energy of the spacecraft is defined as:

$$\mathcal{E} = -\frac{\mu}{2a} = -7.6654 \text{ km}^2/\text{s}^2 \tag{9}$$

So, the speed of the spacecraft is:

$$v^- = \sqrt{2\left(\mathcal{E} + \frac{\mu}{r}\right)} = 1.5443 \text{ km/s} \tag{10}$$

Recognize that one can express the velocity vector as:

$$\bar{v} = v \sin\gamma\, \hat{r} + v\cos\gamma\, \hat{\theta} \tag{11}$$

where, since the flight path angle is zero at apogee, then:

$$\boxed{\bar{v}^- = 1.5443 \text{ km/s } \hat{\theta}} \tag{12}$$

Now, let's determine $\bar{v}_1^+$ and γ_1^+ as well as Δv and α. First, before solving for any quantities, one should create a vector diagram. Recall that we are circularizing the spacecraft's orbit trajectory. As a result, for this specific maneuver case, the direction of $\bar{v}_1^+$ will be along the same direction as $\bar{v}_1^-$: Again, since we are circularizing the orbit,

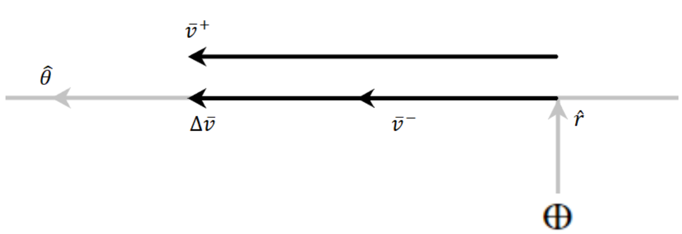

Figure 6: Vector diagram of the velocities before and after the first maneuver to circularize the orbit at apogee.

the flight path angle after the maneuver, by definition of a circular orbit, will be:

$$\boxed{\gamma^+ = 0°} \tag{13}$$

And the speed of the spacecraft in a circular orbit after the maneuver should be:

$$v^+ = \sqrt{\frac{\mu}{r}} = 2.9762 \text{ km/s} \tag{14}$$

Since the flight path angle is zero, one can recall equation (11) to then note that the velocity vector of the spacecraft after the maneuver is:

$$\boxed{\bar{v}^+ = 2.9762 \text{ km/s } \hat{\theta}} \tag{15}$$

Recognize that, since this is an in-plane maneuver, $\hat{\theta}$ will remain in the same direction before and after the maneuver. Thus, $\Delta\bar{v}$ can be found in terms of $\hat{r}$-$\hat{\theta}$-$\hat{h}$ as:

$$\boxed{\Delta\bar{v} = \bar{v}^+ - \bar{v}^- = 1.4319 \text{ km/s } \hat{\theta}} \tag{16}$$

Or:

$$\boxed{\Delta v = 1.4319 \text{ km/s}} \tag{17}$$

Given that all of the velocity vectors align along $\hat{\theta}$, the angle α can be expressed as:

$$\boxed{\alpha = 0°} \tag{18}$$

Part (a.iii)

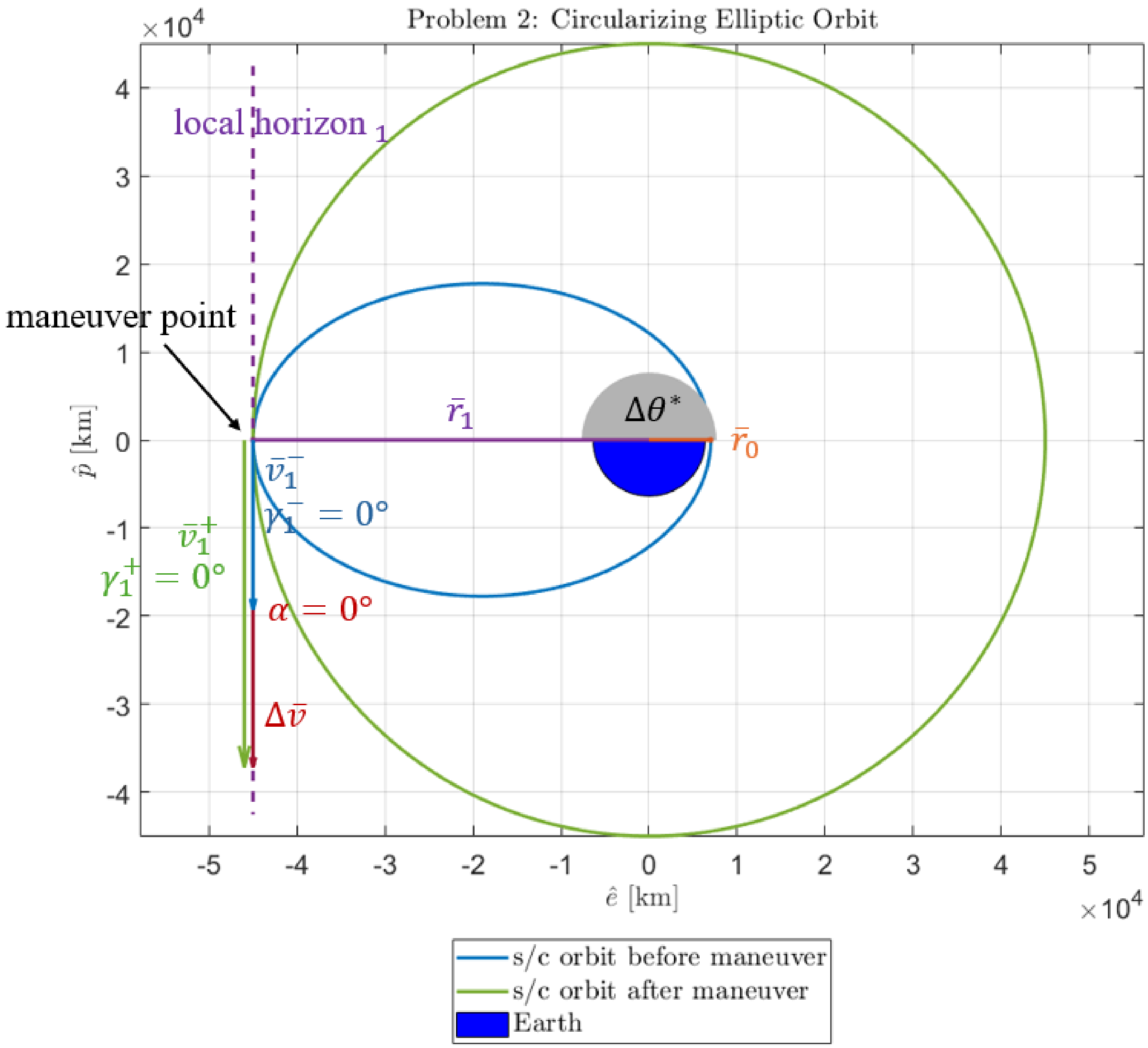

Figure 7: Plot of old and new orbits when circularizing at apogee.

The plot of the old and new orbits when circularizing at apogee are depicted in figure 7. Recognize that the maneuver point occurs at apogee. Also recognize that the velocity vectors depicted in the figure are exaggerated, but are at the same scale with respect to each other. Note that the change in true anomaly $\Delta\theta^*$ between the start point (perigee) and the maneuver point (apogee) is not necessary for the problem, but was depicted for reference.

Part (a.iv)

To determine the percent of the maneuver magnitude is used to shift energy and what percent is used to shift direction, one can look to how much the change in velocity between $\bar{v}_1^-$ and $\bar{v}_1^+$ is of $\Delta\bar{v}$. For this case, because the maneuver occurs at apogee and one is circularizing the orbit:

$$\boxed{\%_{energy} = \frac{|v_1^+ - v_1^-|}{\Delta v} \cdot 100 = 100\%} \tag{19}$$

$$\boxed{\%_{direction} == 100 - \%_{energy} = 0\%} \tag{20}$$

Part (b.i)

Now, the maneuver occurs at a radius of 25000 km. Let's determine the true anomaly θ^* at which the maneuver will occur. Recall the conic equation is:

$$r = \frac{p}{1 + e\cos\theta^*} \tag{21}$$

One can rearrange the conic equation and solve for true anomaly θ^*:

$$\boxed{\theta^* = \pm \arccos\left[\frac{1}{e}\left(\frac{p}{r} - 1\right)\right] = \pm 134.8508°} \tag{22}$$

For convenience, we will take the true anomaly at which the maneuver will occur to be when the spacecraft is ascending in its orbit, meaning that $\boxed{\theta^* = 134.8508°}$. Note that, given the quadrant ambiguity, the orbits intersect at **two** locations, the first location being where the spacecraft is ascending in its initial orbit and the second location being where the spacecraft is descending in its initial orbit.

Part (b.ii)

Recall that the spacecraft is initially at perigee. One can use Kepler's equation to solve for the wait time. First, let's determine the eccentric anomaly of the spacecraft at the maneuver point, using the following relationship:

$$E = \arccos\left[\frac{a-r}{ae}\right] = \pm 86.9830° \tag{23}$$

Since we have chosen that the maneuver location is when the spacecraft is ascending, $E = 86.9830°$, meaning that, using Kepler's equation, the mean anomaly is:

$$M = E - e\sin E = 45.1711° \tag{24}$$

This means that the wait time for the spacecraft till the maneuver is:

$$\boxed{\begin{aligned}(t-t_p) = \frac{M}{n} &= 5.2351 \cdot 10^3 \text{ s} \\ &= 1.4542 \text{ hrs}\end{aligned}} \tag{25}$$

Alternatively, if we chose the maneuver point to be where the spacecraft is descending, then the wait time is:

$$\boxed{\begin{aligned}(t-t_p) = (\frac{\mathcal{P}}{2} + \frac{M}{n}) + \frac{\mathcal{P}}{2} &= 3.6487 \cdot 10^4 \text{ s} \\ &= 10.1354 \text{ hrs}\end{aligned}} \tag{26}$$

Where $M = -45.1711°$.

Part (b.iii)

Now, let's determine the $\bar{r}_1^-$, $\bar{v}_1^-$, γ_1^- at the maneuver point. Again, for the calculations below, we assume that the maneuver point occurs when the spacecraft is ascending. However, one can also equally compute the values for the maneuver point if the spacecraft was descending as well. Recall that the position vector is:

$$\boxed{\bar{r}^- = \bar{r}^+ = 25000 \text{ km } \hat{r}} \tag{27}$$

Realize that the position vector before and after the maneuver will remain the same! Next, let's now solve for the semi-latus rectum and specific angular momentum of the spacecraft in the original orbit:

$$p = a(1 - e^2) = 1.2115 \cdot 10^4 \text{ km} \tag{28}$$

$$h = \sqrt{\mu p} = 6.9492 \cdot 10^4 \text{ km}^2/\text{s} \tag{29}$$

Next, to calculate $\bar{v}_1^-$, one can first calculate the velocity magnitude of the spacecraft. Recall that the specific energy of the spacecraft is defined as:

$$\mathcal{E} = -\frac{\mu}{2a} = -7.6654 \text{ km}^2/\text{s}^2 \tag{30}$$

So, the speed of the spacecraft is:

$$v^- = \sqrt{2\left(\mathcal{E} + \frac{\mu}{r}\right)} = 4.0691 \text{ km/s} \tag{31}$$

Next, before calculating the velocity vector, one can calculate the the flight path angle before the maneuver, using the following equation:

$$\boxed{\gamma^- = \arccos \frac{h}{r^- v^-} = \pm 46.9113°} \tag{32}$$

Since the spacecraft is **ascending**, $\boxed{\gamma^- = 46.9113°}$. Now, one can recall that one can use equation (11) from above to express the velocity vector:

$$\boxed{\bar{v}^- = 2.9716\,\hat{r} + 2.7797\,\hat{\theta} \text{ km/s}} \tag{33}$$

Note that, if the spacecraft was **descending**, then:

$$\boxed{\bar{v}^- = -2.9716\,\hat{r} + 2.7797\,\hat{\theta} \text{ km/s}} \tag{34}$$

Now, let's determine $\bar{v}_1^+$ and γ_1^+ as well as Δv and α. First, before solving for any quantities, one should create a vector diagram. Recall that we are circularizing the spacecraft's orbit trajectory So, the velocity vector diagram would look like:

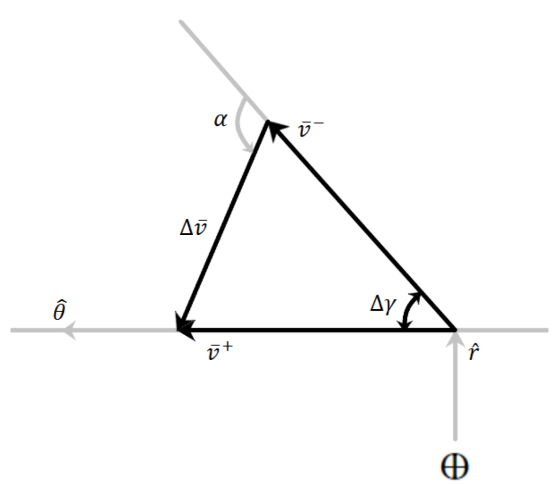

Figure 8: Vector diagram of the velocities before and after the first maneuver to circularize the orbit at $r = 25000$ **km.**

Again, since we are circularizing the orbit, the flight path angle after the maneuver, by definition of a circular orbit, will be:

$$\boxed{\gamma^+ = 0°} \tag{35}$$

And the speed of the spacecraft in a circular orbit after the maneuver should be:

$$v^+ = \sqrt{\frac{\mu}{r}} = 3.9930 \text{ km/s} \tag{36}$$

Since the flight path angle is zero, one can recall equation (11) to then note that the velocity vector of the spacecraft after the maneuver is:

$$\boxed{\bar{v}^+ = 3.9930 \text{ km/s } \hat{\theta}} \tag{37}$$

Recognize that, since this is an in-plane maneuver, $\hat{\theta}$ will remain in the same direction before and after the maneuver. Thus, $\Delta\bar{v}$ can be found in terms of $\hat{r}$-$\hat{\theta}$-$\hat{h}$ as:

$$\boxed{\Delta\bar{v} = \bar{v}^+ - \bar{v}^- = -2.9716\,\hat{r} + 1.2133\,\hat{\theta} \text{ km/s}} \tag{38}$$

Or:

$$\boxed{\Delta v = 3.2098 \text{ km/s}} \tag{39}$$

Note that, if the spacecraft was <u>descending</u> in the original orbit, then:

$$\boxed{\Delta\bar{v} = \bar{v}^+ - \bar{v}^- = 2.9716\,\hat{r} + 1.2133\,\hat{\theta} \text{ km/s}} \tag{40}$$

Lastly, one can determine the angle α. One can do this through using sine law:

$$\frac{\sin\beta}{v^+} = \frac{\sin\Delta\gamma}{\Delta v} \tag{41}$$

where β is the supplementary angle to α and $\Delta\gamma = |\gamma^+ - \gamma^-|$. Thus:

$$\beta = \arcsin\left[\frac{v^+ \sin\Delta\gamma}{\Delta v}\right] = 65.2987° \tag{42}$$

$$\boxed{\alpha = 180° - \beta = -114.7013°} \tag{43}$$

Note that the negative angle means that α should point in the direction towards Earth, and as one can see in figure 9, it does. If the spacecraft was descending in its original orbit, then α would be positive as, given the maneuver, α would be pointing away from the Earth.

The plot of the old and new orbits when circularizing at apogee are depicted in figure 9. Recognize that the maneuver point occurs at $r = 25000$ km. Also recognize that the velocity vectors depicted in the figure are exaggerated, but are at the same scale with respect to each other. Note that the change in true anomaly $\Delta\theta^*$ between the start point (perigee) and the maneuver point is not necessary for the problem, but was depicted for reference.

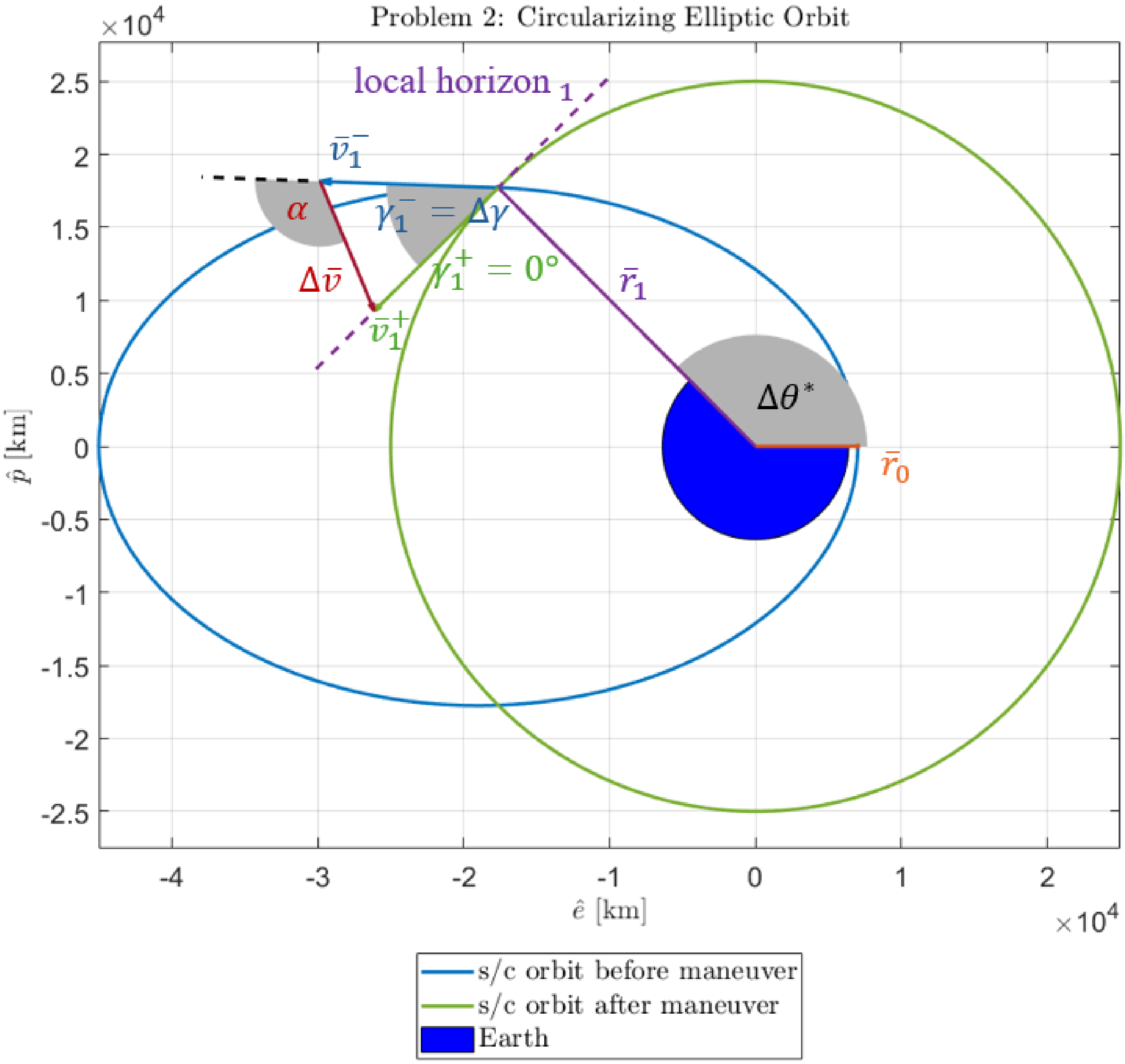

Figure 9: Plot of old and new orbits when circularizing at $r = 25000$ km.

Part (b.iv)

To determine the percent of the maneuver magnitude is used to shift energy and what percent is used to shift direction, one can look to how much the change in velocity between $\bar{v}_1^-$ and $\bar{v}_1^+$ is of $\Delta\bar{v}$:

$$\boxed{\%_{energy} = \frac{|v_1^+ - v_1^-|}{\Delta v} \cdot 100 = 2.3698\%} \tag{44}$$

$$\boxed{\%_{direction} == 100 - \%_{energy} = 97.6302\%} \tag{45}$$

Part (c)

Looking at the final orbits in the two cases, there are some differences. For the first case, the new orbit is 20000 km higher than the radius of the second case. Also, for the first case, the Δv required for the maneuver was $\Delta v = 1.4319$ km/s as opposed to $\Delta v = 3.2098$ km/s required for the second case. It is obvious that the first case has a better Δv requirement. However, it is important to note that optimizing Δv may not always take priority. One could favor a particular altitude or a shorter wait time to set up for a follow-up maneuver. In addition, as pointed out by our calculations for percent energy and direction, the first case purely changes energy while the second case changes mostly the direction of the velocity vector of the spacecraft, meaning that each maneuver is designed to do two different tasks, which isn't a fair comparison.

Problem 3

Problem Statement

A vehicle is launched successfully into an orbit with $e = 0.4$ and $a = 4R_{\oplus}$. At this distance from the Earth, it is reasonable to model the orbit as a relative 2BP. The vehicle is currently located at $\theta^* = 225°$. A single in-plane maneuver will be implemented when $\theta^* = 120°$. Let the maneuver be defined as $|\Delta\bar{v}| = 1.0\ km/s$, $\alpha = -90°$

(a) Establish the current orbit characteristics: p, b, $\mathbb{P}$, $(t - t_p)$, r_p, r_a. (Is r_p acceptable?)

(b) Determine the "wait time" till the maneuver.

(c) What are the conditions in the states (i.e., r^-, v^-, γ^-) at the maneuver point prior to maneuver implementation?

(d) Determine the r^+, v^+, γ^+ in the new orbit immediately after the maneuver. Also, produce the following characteristics for the new orbit: a, e, i, θ^*, $\mathbb{P}$, r_p, p, $\Delta\omega$. What is the time in the new orbit since perigee, i.e., $(t - t_p)$?

[VECTOR diagrams are required!!!!]

(e) Plot the old and new orbits. You can use Matlab or GMAT. Observe the velocity vectors at the maneuver point.

Determine the difference in the components of the velocity vectors at the intersection. What vector basis can you use to produce the 'components?

Does the difference in the velocity vectors equal the maneuver that you intended? Should it? Why or why not?

Part (a)

The current orbit characteristics are:

$$
\begin{aligned}
p &= a(1-e^2) && = \boxed{21430.538\ km} \\
b &= a\sqrt{1-e^2} && = \boxed{23382.634\ km} \\
\mathbb{P} &= 2\pi\sqrt{\frac{a^3}{\mu}} && = \boxed{11.26521\ hr} \\
r_p &= a(1-e) && = \boxed{15307.527\ km} \\
r_a &= a(1+e) && = \boxed{35717.563\ km}
\end{aligned}
$$

The perigee radius r_p is acceptable because it is greater than the radius of the Earth. Finally, the time since perigee at the current location is found by first computing eccentric anomaly, as in

$$E = 2\tan^{-1}\left[\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta^*}{2}\right)\right] = 244.6448°$$

Time since perigee is then

$$t - t_p = \sqrt{\frac{a^3}{\mu}}\,(E - e\sin E) = \boxed{8.30357\ hr}$$

Part (b)

The time since perigee for the maneuver location $(t-t_p)_m$ is computed in the same way as in Part (a). The wait time is then

$$t_{wait} = (\mathbb{P} - (t-t_p)_0) - (t-t_p)_m = \boxed{5.2911\ hr}$$

Part (c)

$$
\begin{aligned}
r^- &= \frac{p}{1+e\cos\theta^{*-}} && = \boxed{26788.172\ km} \\
v^- &= \sqrt{\frac{2\mu}{r} - \frac{\mu}{a}} && = \boxed{3.75975\ km/s} \\
\gamma^- &= \cos^{-1}\left(\frac{\sqrt{\mu p}}{r^- v^-}\right) && = \boxed{23.41322°}
\end{aligned}
$$

Part (d)

From the vector diagram in Figure 10, the following values are found

$$
\begin{aligned}
r^{+} &= r^{-} && = \boxed{26788.172\ km} \\
\beta &= 180° - |\alpha| && = 90° \\
v^{+} &= \sqrt{(v^{-})^2 + (\Delta v)^2 - 2(v^{-})(\Delta v)\cos\beta} && = \boxed{3.89047\ km/s} \\
\Delta\gamma^{+} &= \sin^{-1}\frac{\Delta v}{v^{+}}\sin\beta && = -14.894413° \\
\gamma^{+} &= \gamma^{-} + \Delta\gamma && = \boxed{8.518811°}
\end{aligned}
$$

The following calculations assume values for the new orbit, unless otherwise denoted with a (−) or (+) superscript

$$
\begin{aligned}
a &= \frac{-\mu}{2\left(\frac{v^2}{2} - \frac{\mu}{r}\right)} && = 27257.147\ km \\
e &= \sqrt{\left(\frac{rv^2}{\mu} - 1\right)^2 \cos^2\gamma + \sin^2\gamma} && = 0.149108 \\
i &= i^{-} && = 0 \\
\mathbb{P} &= 2\pi\sqrt{\frac{a^3}{\mu}} && = 12.440249\ hr \\
r_p &= a(1-e) && = 23192.8836\ km \\
p &= a(1-e^2) && = 26651.132\ km \\
\theta^{*} &= \tan^{-1}\left[\frac{\frac{rv^2}{\mu}\cos\gamma\sin\gamma}{\frac{rv^2}{\mu}\cos^2\gamma - 1}\right] && = 91.96612° \\
\Delta\omega &= \theta^{*-} - \theta^{*+} && = 28.03388° \\
t - t_p &= \sqrt{\frac{a^3}{\mu}}\,(E - e\sin E) && = 2.58784\ hr
\end{aligned}
$$

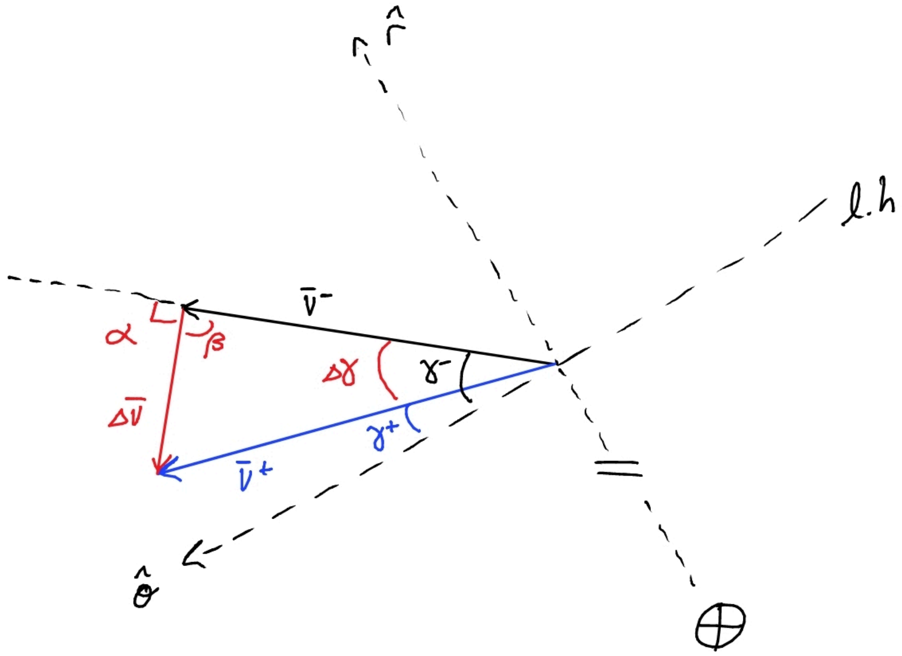

Figure 10: Vector diagram for the orbital maneuver

Part (e)

The $\hat{r}$ and $\hat{\theta}$ vectors are the same before and after the maneuver. Using the equations

$$v_r = \frac{\mu e}{\sqrt{\mu p}} \sin\theta^*$$

$$v_\theta = \frac{\sqrt{\mu p}}{r}$$

the radial and transverse components of the velocity vector is found before and after the maneuver. The difference in the vectors equals 1 km/s and the Δv vector is oriented 90° from the original velocity vector, indicating that the maneuver is implemented as intended. The original and final orbits are plotted in Figure 11.

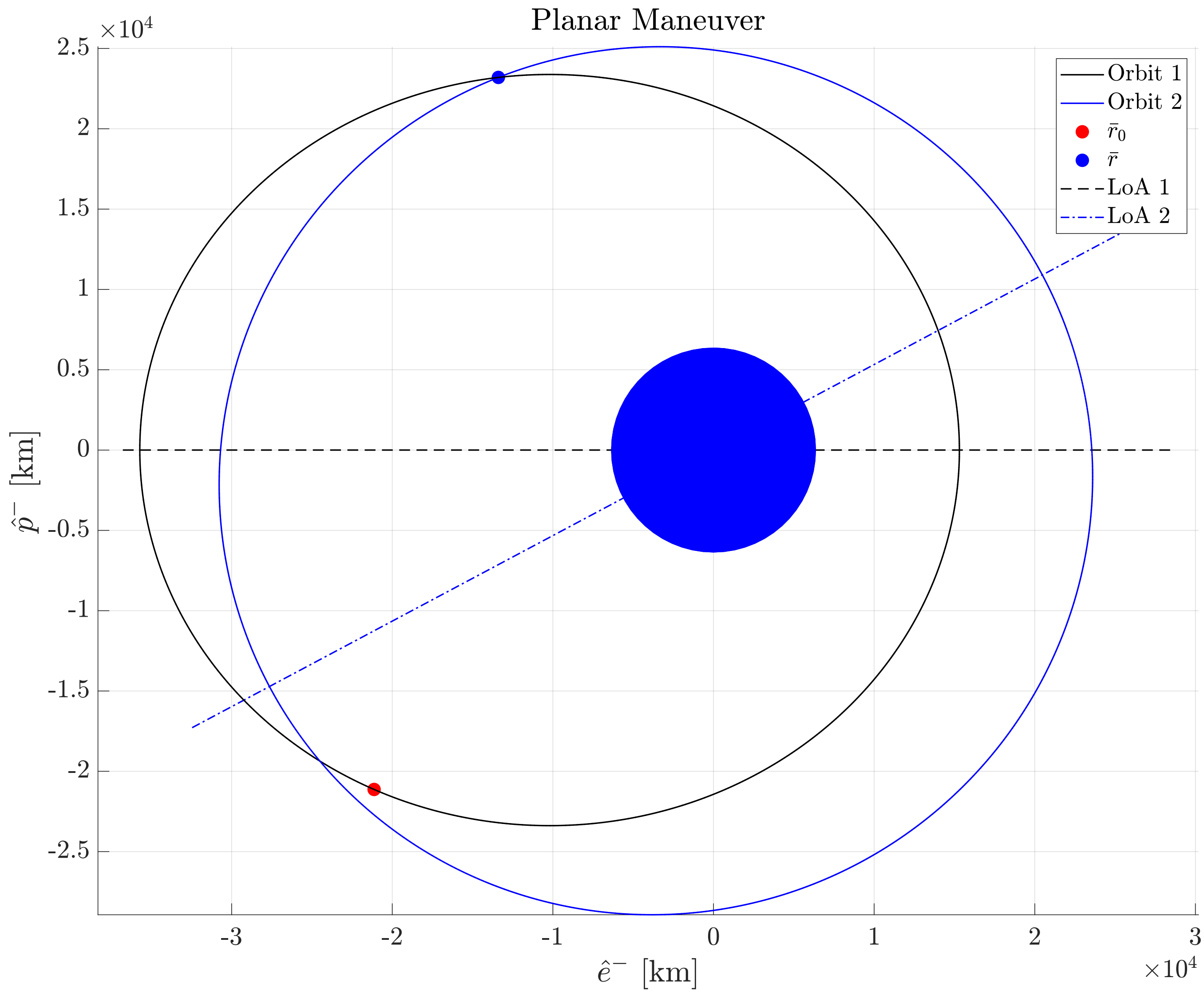

Figure 11: Original and final orbits