ECE 68000: MODERN AUTOMATIC CONTROL

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Unknown Input Observer (UIO) Synthesis Examples

Synthesis of the UIO

- Proposed UIO

$$\begin{aligned} \boldsymbol{z}[k+1] &= (\boldsymbol{I}_n - \boldsymbol{M}\boldsymbol{C})(\boldsymbol{A}\boldsymbol{z}[k] + \boldsymbol{A}\boldsymbol{M}\boldsymbol{y}[k] + \boldsymbol{B}_1\boldsymbol{u}[k]) \\ &\quad + \boldsymbol{L}(\boldsymbol{y}[k] - \hat{\boldsymbol{y}}[k]) \\ \hat{\boldsymbol{x}}[k] &= \boldsymbol{z}[k] + \boldsymbol{M}\boldsymbol{y}[k] \end{aligned}$$

- Observation error dynamics:

$$\boldsymbol{e}[k+1] = (\boldsymbol{A}_1 - \boldsymbol{L}\boldsymbol{C})\boldsymbol{e}[k] - \boldsymbol{L}\boldsymbol{D}\boldsymbol{v}[k]$$

where $\boldsymbol{A}_1 = (\boldsymbol{I}_n - \boldsymbol{M}\boldsymbol{C})\boldsymbol{A}$

B. Alenezi, M. Zhang, S. Hui, and S. H. Żak, Simultaneous Estimation of the State, Unknown Input, and Output Disturbance in Discrete-Time Linear Systems, *IEEE Transactions on Automatic Control*, Vol. 66, No. 12, December 2021, pp. 6115–6122

More on the Synthesis of the UIO

- Observation error dynamics:

$$\boldsymbol{e}[k+1] = (\boldsymbol{A}_1 - \boldsymbol{L}\boldsymbol{C})\boldsymbol{e}[k] - \boldsymbol{L}\boldsymbol{D}\boldsymbol{v}[k]$$

  where $\boldsymbol{A}_1 = (\boldsymbol{I}_n - \boldsymbol{M}\boldsymbol{C})\boldsymbol{A}$

- Note that if an $\boldsymbol{L}$ exists such that $(\boldsymbol{A}_1 - \boldsymbol{L}\boldsymbol{C})$ is Schur stable and

$$\boldsymbol{L}\boldsymbol{D} = \boldsymbol{O},$$

  then the error dynamics become

$$\boldsymbol{e}[k+1] = (\boldsymbol{A}_1 - \boldsymbol{L}\boldsymbol{C})\boldsymbol{e}[k]$$

UIO Synthesis—Example 1

System model matrices

$$\boldsymbol{A} = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.5 \end{bmatrix} \ \boldsymbol{B}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},$$

$$\boldsymbol{C} = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \ \boldsymbol{D} = \begin{bmatrix} 0 \\ 0.2 \end{bmatrix}$$

Theorem

There exists a solution $\boldsymbol{M}$ *to*

$$\boldsymbol{M} \begin{bmatrix} \boldsymbol{C}\boldsymbol{B}_2 & \boldsymbol{D} \end{bmatrix} = \begin{bmatrix} \boldsymbol{B}_2 & \boldsymbol{O}_{n \times r} \end{bmatrix}$$

if and only if

$$rank \begin{bmatrix} \boldsymbol{C}\boldsymbol{B}_2 & \boldsymbol{D} \end{bmatrix} = rank(\boldsymbol{B}_2) + rank(\boldsymbol{D})$$

Example 1—Computing $\boldsymbol{M}$

- Check the rank condition

$$\operatorname{rank}(\boldsymbol{B}_2)+\operatorname{rank}(\boldsymbol{D}) = \operatorname{rank}\left[\begin{array}{cc} \boldsymbol{C}\boldsymbol{B}_2 & \boldsymbol{D} \end{array}\right] = \operatorname{rank}\begin{bmatrix} 2 & 1 \\ 1 & 0.2 \end{bmatrix} = 2$$

- The matrix rank condition satisfied
- Solve for

$$\begin{aligned} \boldsymbol{M} &= \left[\begin{array}{cc} \boldsymbol{B}_2 & \boldsymbol{O}_{n\times r} \end{array}\right]\left[\begin{array}{cc} \boldsymbol{C}\boldsymbol{B}_2 & \boldsymbol{D} \end{array}\right]^{\dagger} \\ &= \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & 0.2 \end{bmatrix}^{\dagger} = \begin{bmatrix} 0.5 & 0 \\ 0.5 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

- Construct $\boldsymbol{A}_1 = (\boldsymbol{I}_3 - \boldsymbol{M}\boldsymbol{C})\boldsymbol{A} = \begin{bmatrix} 0.5 & -0.5 & -0.25 \\ 0 & 0 & -0.25 \\ 0 & 0 & 0.5 \end{bmatrix}$

Example 1 Contd.

- Can we find an $\boldsymbol{L}$ such that $(\boldsymbol{A}_1 - \boldsymbol{LC})$ is Schur stable and

$$\boldsymbol{LD} = \boldsymbol{O}$$

so that the error dynamics would become

$$\boldsymbol{e}[k+1] = (\boldsymbol{A}_1 - \boldsymbol{LC})\boldsymbol{e}[k]?$$

- Used `cvx` to obtain $\boldsymbol{L} = \begin{bmatrix} -0.25 & 0 \\ -0.05 & 0 \\ 0.10 & 0 \end{bmatrix}$
- Eigenvalues of $(\boldsymbol{A}_1 - \boldsymbol{LC})$ at

$$0.5,\ 0.0,\ 0.5$$

UIO Synthesis—Example 2

System model matrices

$$\boldsymbol{A} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -0.3 \end{bmatrix} \boldsymbol{B}_2 = \begin{bmatrix} -2 \\ -3 \\ -4 \end{bmatrix},$$

$$\boldsymbol{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \boldsymbol{D} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

- The matrix rank condition satisfied
- Solve for $\boldsymbol{M} = \begin{bmatrix} -2 & 2 \\ -3 & 3 \\ -4 & 4 \end{bmatrix}$
- Construct

$$\boldsymbol{A}_1 = (\boldsymbol{I}_3 - \boldsymbol{M}\boldsymbol{C})\boldsymbol{A} = \begin{bmatrix} -3 & 4 & 0 \\ -3 & 4 & 0 \\ -4 & 8 & -0.3 \end{bmatrix}$$

Example 2 Contd.

- Can we find an $\boldsymbol{L}$ such that $(\boldsymbol{A}_1 - \boldsymbol{L}\boldsymbol{C})$ is Schur stable and

$$\boldsymbol{L}\boldsymbol{D} = \boldsymbol{O},$$

so that the error dynamics would become

$$\boldsymbol{e}[k+1] = (\boldsymbol{A}_1 - \boldsymbol{L}\boldsymbol{C})\boldsymbol{e}[k]?$$

- Used `cvx` to obtain $\boldsymbol{L} = \begin{bmatrix} -3.4974 & 3.4974 \\ -3.4936 & 3.4936 \\ -5.9137 & 5.9137 \end{bmatrix}$
- Eigenvalues of $(\boldsymbol{A}_1 - \boldsymbol{L}\boldsymbol{C})$ at

$$0.3000,\ 1.0000,\ 0.0038$$

- No such luck in this example

Conclusions from the examples

- In general it may not be possible find an $\boldsymbol{L}$ such that $(\boldsymbol{A}_1 - \boldsymbol{LC})$ is Schur stable and

$$\boldsymbol{LD} = \boldsymbol{O},$$

so that the error dynamics would become

$$\boldsymbol{e}[k+1] = (\boldsymbol{A}_1 - \boldsymbol{LC})\boldsymbol{e}[k]$$

- We thus need to analyze the error dynamics

$$\boldsymbol{e}[k+1] = (\boldsymbol{A}_1 - \boldsymbol{LC})\boldsymbol{e}[k] - \boldsymbol{LDv}[k]$$

Stability of the Observation Error Dynamics