(Ex) $g(t) = \begin{cases} 0, & t<3 \\ -2, & 3 \le t < 6 \\ t-8, & 6 \le t < 12 \\ 2, & t \geq 12 \end{cases}$

$$g(t) = 0 + (-2-0)\,u(t-3) + \cancel{(t-8-(-2))}$$
$$+ (t-8+(+2))\,u(t-6)$$
$$+ (\underbrace{2-(t-8)}_{2-t+8})\,u(t-12).$$

$$g(t) = -2\,u(t-3) + (t-6)\,u(t-6)$$
$$- (\underbrace{t-10}_{-12+12})\,u(t-12).$$

$$L(g) = -2L(u(t-3)) + L((t-6)\,u(t-6))$$
$$- L((t-12)\,u(t-12) + 2\,u(t-12))$$

$$\begin{cases} L(u(t-a)) = \frac{1}{s}e^{-as}, & s>0 \quad \#25. \\ L(u(t-a)f(t-a)) = e^{-as}L(f(t)), & \#27 \end{cases}$$

$$L(g) = -2\cdot\frac{1}{s}e^{-3s} + e^{-6s}L(t)$$
$$- e^{-12s}L(t) - 2L(u(t-12))$$
$$= \frac{-2}{s}e^{-3s} + e^{-6s}\frac{1}{s^2} - e^{-12s}\frac{1}{s^2} - \frac{2}{s}e^{-12s}$$
$(s>0).$

(Inverse Laplace transform)

(Ex) (1) $L^{-1}\left(\frac{s}{s^2+6s+13}\right)$ 1. Partial fraction 2. Complete square.

$$\left(s^2+6s+\underset{9+4}{13} = (s+3)^2+4\right)$$

$$= L^{-1}\left(\frac{s+3-3}{(s+3)^2+2^2}\right)$$

$$= L^{-1}\left(\frac{s+3}{(s+3)^2+2^2}\right) - \frac{3}{2}L^{-1}\left(\frac{1\cdot 2}{(s+3)^2+2^2}\right)$$

#20 $(a=-3, b=2)$ #19.

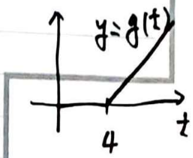
$$\underbrace{= e^{-3t}\cos(2t) - \frac{3}{2}e^{-3t}\sin(2t)}_{=f(t)}$$

(2) $L^{-1}\left(e^{-6s}\underbrace{\frac{s}{s^2+6s+13}}_{=F}\right)$ $\quad f(t) = L^{-1}(F(s))$.

$= u(t-6)\,f(t-6)$.

$= u(t-6)\left(e^{-3(t-6)}\cos(2(t-6)) - \frac{3}{2}e^{-3(t-6)}\sin(2(t-\right.$

(Ex) $\begin{cases} y'' + 9y = \underbrace{\begin{cases} 0, & t<4 \\ t-4, & t \geq 4. \end{cases}}_{= \text{ let } g(t)} \\ y(0) = 0 \\ y'(0) = 0 \end{cases}$



L:

$L(y'') + 9L(y) = L\left(0 + (t-4)\,u(t-4)\right)$

$s^2L(y) - s\underbrace{y(0)}_{0} - \underbrace{y'(0)}_{0} + 9L(y)$

$= L((t-4)\,u(t-4)) = e^{-4s}\frac{1}{s^2}$.

$(s^2+9)L(y) = e^{-4s}\frac{1}{s^2}$

$L(y) = e^{-4s}\frac{1}{s^2(s^2+9)}$ let $= F(s)$

(Partial fraction)

$\frac{1}{s^2(s^2+9)} = \frac{As+B}{s^2} + \frac{Cs+D}{s^2+9}$

$1 = (As+B)(s^2+9) + s^2(Cs+D)$

$1 = As^3 + Bs^2 + 9As + 9B + Cs^3 + Ds^2$.

$1 = \underbrace{(A+C)}_{0}s^3 + \underbrace{(B+D)}_{0}s^2 + \underbrace{9A}_{0}s + \underbrace{9B}_{1}$

$A=0,\ C=0,\quad B=\frac{1}{9},\quad B+D=0:\ D=-B=-\frac{1}{9}$

$\frac{1}{s^2(s^2+9)} = \frac{1}{9}\frac{1}{s^2} - \frac{1}{9}\frac{1}{s^2+9}$.

$L^{-1}(F) = L^{-1}\left(\frac{1}{9}\frac{1}{s^2} - \frac{1}{9}\frac{1}{s^2+9}\right)$

$= \frac{1}{9}L^{-1}\left(\frac{1}{s^2}\right) - \frac{1}{9}L^{-1}\left(\frac{1\cdot 3}{s^2+3^2}\right)\frac{1}{3}$.

$= \frac{1}{9}t - \frac{1}{27}\sin(3t) = f(t)$

$\therefore\ y(t) = L^{-1}(e^{-4s}F(s)) =$

$$y(t) = u(t-4)\left(\frac{1}{9}(t-4) - \frac{1}{27}\sin(3(t-4))\right)$$

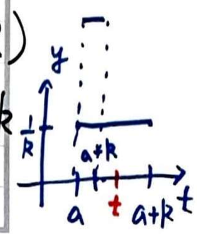
6.4. Dirac delta. $\delta(t-a) = \delta_a(t)$.

(motivation) $\begin{cases}\text{point source, point mass} \\ \text{Impulse}\end{cases}$

Q How do we model these problems?

<u>Def</u> $a>0$, $k>0$ $(a, k \in \mathbb{R})$

Let $f_k(t-a) = \begin{cases} \frac{1}{k}, & a \le t \le a+k \\ 0, & \text{otherwise} \end{cases}$



(Properties)

1. $\int_0^\infty f_k(t-a)\,dt = \int_a^{a+k} \frac{1}{k}\,dt = \frac{1}{k}\cdot k = 1$

for any $k>0$ & $a>0$.

2. For $t \neq a$, $\lim_{k\to 0} f_k(t-a) = 0$

For $t = a$, $\lim_{k\to 0} f_k(t-a) = \infty$

3. For a continuous function $g(t)$

$$\lim_{k\to 0}\int_0^\infty f_k(t-a)\,g(t)\,dt = g(a)$$

(Proof)

$$\int_0^\infty f_k(t-a)\,g(t)\,dt = \int_a^{a+k}\frac{1}{k}\,g(t)\,dt$$

$= \frac{1}{k}\int_a^{a+k} g(t)\,dt$: Let $G(t) = \int g(t)\,dt$

$\underline{G'(t) = g(t)}$

$$\lim_{k\to 0}\int_0^\infty f_k(t-a)\,g(t)\,dt = \lim_{k\to 0}\frac{1}{k}\Big[G(t)\Big]_a^{a+k}$$

$$= \lim_{k\to 0}\frac{G(a+k) - G(a)}{k} = G'(a) = g(a).$$

Def $\delta(t-a) = \lim_{k\to 0} f_k(t-a)$

(Properties)

1. $\int_0^\infty \delta(t-a)\,dt = 1$

2. $\delta(t-a) = \begin{cases} 0, & t \neq a \\ \infty, & t = a \end{cases}$

3. $g(t)$: a continuous function

   $\int_0^\infty \delta(t-a)\, g(t)\,dt = g(a).$

4. $L(\delta(t-a)) = \int_0^\infty e^{-st}\delta(t-a)\,dt = e^{-as}.$

(Ex) L: $\begin{cases} y'' + y = \delta(t-2) \\ y(0) = 0, \ y'(0) = 0 \end{cases}$

$L(y'') + L(y) = L(\delta(t-2))$

$s^2 L(y) - s\underbrace{y(0)}_{0} - \underbrace{y'(0)}_{0} + L(y) = e^{-2s}$

$(s^2+1)\, L(y) = e^{-2s}$; $L(y) = e^{-2s}\dfrac{1}{s^2+1}$

$y(t) = L^{-1}\left(e^{-2s}\underbrace{\dfrac{1}{s^2+1}}_{F}\right) = u(t-2)\, f(t-2)$ #27.

$\left(L^{-1}(F) = \sin(t) = f(t)\right)$ $= u(t-2)\sin(t-2)$

#14 (p231)

$f(t)$ is piecewise continuous and periodic with period $p > 0$:

$f(t+p) = f(t)$, for any $t \in \mathbb{R}$

$L(f) = ? = \dfrac{1}{1-e^{-ps}}\int_0^p e^{-st} f(t)\,dt$

$L(f) = \int_0^\infty e^{-st} f(t)\,dt = \int_0^p \text{''}\,dt + \int_p^{2p} \text{''}\,dt + \cdots$

$= \sum_{n=1}^{\infty} \int_{(n-1)p}^{np} e^{-st} f(t)\,dt$ $\quad r = t-(n-1)p,\ dr = dt$

$= \sum_{n=1}^{\infty} \int_0^p e^{-s(r+(n-1)p)} f(r+(n-1)p)\,dr.$