ECE 602: LUMPED LINEAR SYSTEMS

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Solutions of Continuous-Time LTI Systems: Diagonalizable *A* Case

System Modes (Diagonalizable A)

Suppose $A \in \mathbb{R}^{n \times n}$ is diagonalizable: $A = T \Lambda T^{-1}$

- Diagonal entries of $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ are the eigenvalues of A
- Column of $T = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}$ are (right) eigenvectors of A
- Rows of $T^{-1} = \begin{bmatrix} w_1 & \cdots & w_n \end{bmatrix}^T$ are left eigenvectors of A

With $\{v_1, \ldots, v_n\}$ being a basis of $\mathbb{R}^n$, the corresponding basis of $\mathbb{X}$ is

$$\{x^{(i)}(t) = e^{At} v_i = e^{\lambda_i t} v_i,\ i = 1, \ldots, n\}$$

- Each $x^{(i)}(t)$, $i \in \{1, \ldots, n\}$, is called a mode of the system's solutions
- Modes are exactly the columns of $Te^{\Lambda t}$
- Any arbitrary solution $x(t)$ can be decomposed in terms of the modes

System Modes (Diagonalizable A)

Decompose solution $x(t)$ to $\dot{x} = Ax$ from an arbitrary $x(0)$ into modes:

- Express $x(0)$ as a linear combination of basis vectors $\{v_1, \ldots, v_n\}$

$$x(0) = \underbrace{\begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}}_{T} \underbrace{\begin{bmatrix} w_1^T \\ \vdots \\ w_n^T \end{bmatrix}}_{T^{-1}} x(0) = \left(w_1^T x(0)\right) v_1 + \cdots + \left(w_n^T x(0)\right) v_n$$

- Solution $x(t)$ as a linear combination of modes:

$$x(t) = \left(w_1^T x(0)\right) \underbrace{e^{\lambda_1 t} v_1}_{x^{(1)}(t)} + \cdots + \left(w_n^T x(0)\right) \underbrace{e^{\lambda_n t} v_n}_{x^{(n)}(t)}$$

Alternatively, $x(t) = e^{At} x(0) = T e^{\Lambda t} T^{-1} x(0)$ where modes are columns of $T e^{\Lambda t}$

Example

Example: $\dot{x} = Ax$ with $A = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{T} \underbrace{\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}}_{\Lambda} \underbrace{\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}}_{T^{-1}}$

Decoupled Form of Systems

LTI system $\dot{x} = Ax$ with diagonalizable $A = T\Lambda T^{-1}$

After a change of coordinates $\tilde{x} = T^{-1}x$, the system becomes decoupled:

$$\dot{\tilde{x}} = \Lambda \tilde{x} \quad \Leftrightarrow \quad \begin{cases} \dot{\tilde{x}}_1 = \lambda_1 \tilde{x}_1 \\ \vdots \\ \dot{\tilde{x}}_n = \lambda_n \tilde{x}_n \end{cases}$$

- A group of decoupled scalar linear ODEs with solutions $\tilde{x}_i(t) = e^{\tilde{\lambda}_i t}\tilde{x}_i(0)$

Real and Complex Modes

For a real eigenvalue λ_i, the mode $e^{\lambda_i t} v_i$ is called

- stable if $\lambda_i < 0$
- unstable if $\lambda_i > 0$
- marginally stable if $\lambda_i = 0$

For a complex eigenvalue $\lambda_i = \sigma_i + j\omega_i$ with $v_i = p_i + jq_i \in \mathbb{C}^n$

- Mode $e^{\lambda_i t} v_i$ is complex, and there is another mode $e^{\bar{\lambda}_i t} \bar{v}_i$
- Suppose $w_i^T x(0) = \alpha + j\beta$. Then a real solution in $\mathbb{X}$ is

$$2 \cdot \mathrm{Re}\left[\left(w_i^T x(0)\right) e^{\lambda_i t} v_i\right] = \begin{bmatrix} p_i & q_i \end{bmatrix} e^{\sigma_i t} \begin{bmatrix} \cos(\omega_i t) & \sin(\omega_i t) \\ -\sin(\omega_i t) & \cos(\omega_i t) \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$