Holonomic constraints

- We might want to do this approach if not all of our q_j's are independent

Constraints: $f_i^{(hol)}(q_1, \ldots, q_{N_c}) = 0$ for $i = 1, \ldots, K_H$

Let's form an effective Lagrangian,

$$L' = L + \sum_{i=1}^{K_H} \lambda_i f_i$$

Then

$$I' = \int L' dt$$ Action integral

Hamiltons principle says $\delta I' = 0$

$$\delta I' = \int \delta L' dt = \int \left[\frac{\delta L}{\delta q_j} + \sum_{i=1}^{K_H} \lambda_i \frac{\delta f_i}{\delta q_j} \right] \delta q_j \, dt = 0$$

Although the variations would not be independent and vanish individually, λ_i can be chosen such that all the coefficients of the δq_j's vanish (which is what would occur if they were independent).

Then,

$$\frac{\delta L}{\delta q_j} + \sum_{k=1}^{K_H} \lambda_i \frac{\partial f_i}{\partial q_j} = 0$$

and $f_i(q_1, \ldots, q_{N_c}) = 0$ for $i = 1, \ldots, K_H$

Notice that

$$\underbrace{\frac{\delta L}{\delta q_j}}_{\substack{\text{L.H.S. of the} \\ \text{EL for } q_j \\ \frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)}} = \underbrace{-\sum_{i=1}^{K_H} \lambda_i \frac{\partial f_i}{\partial q_j}}_{\triangleq\, Q_j^{(hol)} \text{ generalized force}}$$

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j^{(hol)} \quad \text{with} \quad Q_j^{(hol)} = \sum_{i=1}^{K_H} \lambda_i \frac{\partial f_i}{\partial q_j}}$$

Nonholonomic Constraints

We have already seen that for holonomic constraints, we have the same number of coordinates as degrees of freedom so $N_c = M$.

Our coordinates are chosen to be independent so that when we apply Hamilton's principle, we get an E.L. equation for each coordinate through the reasoning,

$$\delta I = \int \sum_{j}^{N_c} \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} - Q_j^{(nc,a)} \right) \delta q_j \, dt = 0$$

If we can't find independent q_j's or don't want to, we saw that we could introduce the additional freedom we needed to vary the coordinates independently by bringing in Lagrange multipliers

$$\delta I' = \int \sum_{j}^{N_c} \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} - Q_j^{(nc,a)} - \sum_{i=1}^{K_H} \lambda_i \frac{\partial f_i}{\partial q_j} \right) \delta q_j \, dt = 0$$

What happens if we have a nonholonomic constraint?

$$f(q_1, \dots, q_{N_c}, \dot{q}_1, \dots, \dot{q}_{N_c}, t) = 0$$

In this case, you can't simply use the notation of the variational derivative from Hand and Finch

$$\frac{\delta f}{\delta q} = \frac{\partial f}{\partial q} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{q}}\right)$$

You would need additional tools from the Calculus of Variations. Instead, Hand + Finch present "A method that works."

The method involves "freezing time" and manipulating the constraint equations to get constraints that only contain variations in the generalized coordinates, not their rates of change.

This method can be applied to the useful (but restricted) class of nonholonomic constraints that are linear in $\dot{q}_j$

That is,

$$f_i^{(\text{nonhol})} = \sum_{j=1}^{N_c} a_{ij}\,\dot{q}_j + \underbrace{a_{i,0}}_{\substack{\text{Anything left that}\\ \text{doesn't multiply}\\ \text{a } \dot{q}_j}} = 0$$

"Method that works"

Multiply both sides by dt,

$$\left(\sum_{j=1}^{N_c} a_{ij}\frac{dq_j}{dt} + a_{i,0}\right)dt = 0$$

$$\sum_{j=1}^{N_c} a_{ij}\,dq_j + a_{i,0}\,dt = 0$$

"Freeze time" by setting $dt = 0$,

$$\sum_{j=1}^{N_c} a_{ij}\,dq_j = 0$$

Replace the d with δ,
(Recall that $d \neq \delta$ in general since δ involves freezing time)

$$\sum_{j=1}^{N_c} a_{ij}\,\delta q_j = 0$$

This now looks like a constraint with virtual displacements and no rates of change.

If we had made an effective Lagrangian,

$$L' = L + \sum_{i=1}^{K_s} \mu_i f_i^{(\text{nonhol})}$$

Lagrange multipliers

and proceeded with Hamilton's principle,

$$\delta I' = \int \sum_{j=1}^{N_c} \frac{\delta L'}{\delta q_j} \delta q_j \, dt = 0$$

$$= \int \sum_{j=1}^{N_c} \left(\frac{\delta L}{\delta q_j} + \sum_{i=1}^{K_s} \mu_i \frac{\delta f_i}{\delta q_j} \right) \delta q_j \, dt = 0$$

Get this from the "Method that works" constraint equation by treating it like partial differentiation.

$$\frac{\delta f_i}{\delta q_j} = a_{ij}$$

Then, we have,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = \sum_{i=1}^{K_s} \mu_i a_{ij}$$

Note that in Goldstein, this equation appears in the form (adjusting indices),

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = -\sum_{i=1}^{K_s} \mu_i \frac{\partial f_i}{\partial \dot{q}_j}$$

Sign of Lagrange multiplier is flipped (doesn't matter)

equivalent to the a_{ij} terms

Now, we are ready for an overall summary of the Euler-Lagrange equations

$$\boxed{\begin{gathered}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j^{(nc,a)} + Q_j^{(hol)} + Q_j^{(nonhol)} \\[2em]
Q_j^{(nc,a)} = \sum_{i=1}^{N} \vec{F}_i^{(nc,a)} \cdot \frac{\partial \vec{r}_{i/O}^{\,I}}{\partial q_j} \qquad Q_j^{(hol)} = \sum_{i=1}^{K_h} \lambda_i \frac{\partial f_i^{(hol)}}{\partial q_j} \\[1em]
Q_j^{(nonhol)} = \sum_{i=1}^{K_s} \mu_i a_{ij} \qquad \text{where } f_i^{(nonhol)} = \sum_{j=1}^{N_c} a_{ij}\dot{q}_j + a_{i,0}
\end{gathered}}$$