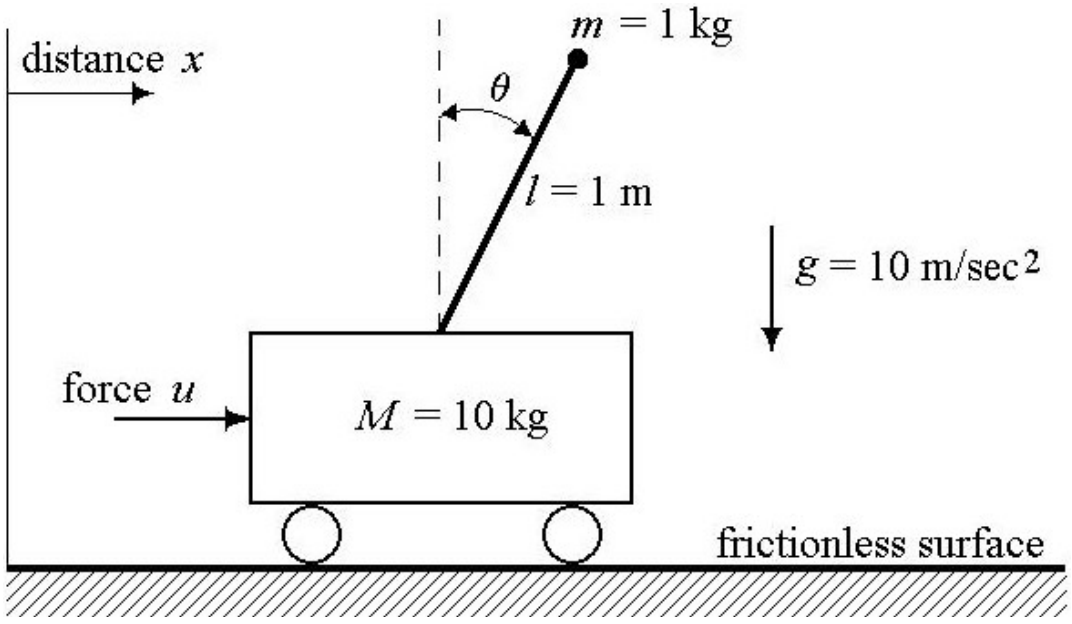
Case Study

We will use the Lagrange equations of motion to derive a state-space model of a dynamical system consisting of a cart with an inverted pendulum (point mass on a mass-less shaft) attached to it as depicted in the figure below.



The kinetic energy K is $K = K_1 + K_2$, where K_1 is the kinetic energy of the cart, while K_2 is the kinetic energy of pendulum. The kinetic energy of the cart, K_1, is due to its translational velocity $\dot{x}$. Hence,

$$K_1 = \frac{1}{2}M\dot{x}^2.$$

The kinetic energy of the pendulum is the sum of the translational and rotational energies,

$$K_2 = \frac{1}{2}m\|v\|^2 + \frac{1}{2}I_{cm}\dot{\theta}^2,$$

where I_{cm} is the rotational inertia with respect to the center of mass. In our example, $I_{cm} = 0$. Hence, the kinetic energy of this pendulum is

$$K_2 = \frac{1}{2}m\|v\|^2 = \frac{1}{2}m(\dot{x}_m^2 + \dot{y}_m^2),$$

where the horizontal position of the bob is $x_m = x + l\sin\theta$, and its vertical position is $y_m = l\cos\theta$. Therefore, the magnitude squared velocity of the bob is

$$\begin{aligned}\left(\frac{d}{dt}(x + l\sin\theta)\right)^2 + \left(\frac{d}{dt}(l\cos\theta)\right)^2 &= \left(\dot{x} + l\dot{\theta}\cos\theta\right)^2 + \left(-l\dot{\theta}\sin\theta\right)^2 \\ &= \dot{x}^2 + 2l\dot{x}\dot{\theta}\cos\theta + l^2\dot{\theta}^2\left(\cos^2\theta + \sin^2\theta\right) \\ &= \dot{x}^2 + 2l\dot{x}\dot{\theta}\cos\theta + l^2\dot{\theta}^2.\end{aligned}$$

So the kinetic energy of the pendulum is

$$K_2 = \frac{1}{2}m\left(\dot{x}^2 + 2l\dot{x}\dot{\theta}\cos\theta + l^2\dot{\theta}^2\right).$$

The total kinetic energy of the system is

$$K = K_1 + K_2 = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\dot{x}^2 + 2l\dot{x}\dot{\theta}\cos\theta + l^2\dot{\theta}^2\right).$$

The potential energy of the system U is due only to the point mass m and is equal to

$$U = mgl\cos\theta.$$

Hence, the Lagrangian L is

$$L = K - U = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\dot{x}^2 + 2l\dot{x}\dot{\theta}\cos\theta + l^2\dot{\theta}^2\right) - mgl\cos\theta.$$

We can now determine the equations that model the system using Lagrange's equations. The first Lagrange equation is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = u.$$

We note that

$$\frac{\partial L}{\partial \dot{x}} = M\dot{x} + m(\dot{x} + l\dot{\theta}\cos\theta),$$

and hence

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = M\ddot{x} + m\ddot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta.$$

Because $\frac{\partial L}{\partial x} = 0$, we obtain

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} &= M\ddot{x} + m\ddot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta \\ &= u.\end{aligned}$$

We now write the second Lagrange equation,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0.$$

Note that

$$\frac{\partial L}{\partial \dot{\theta}} = m(l\dot{x}\cos\theta + l^2\dot{\theta}),$$

and thus

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = ml\left(\ddot{x}\cos\theta - \dot{x}\dot{\theta}\sin\theta\right) + ml^2\ddot{\theta}.$$

We also have,

$$\frac{\partial L}{\partial \theta} = -ml\dot{x}\dot{\theta}\sin\theta + mgl\sin\theta.$$

The second Lagrange equation is

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} &= ml\left(\ddot{x}\cos\theta - \dot{x}\dot{\theta}\sin\theta\right) + ml^2\ddot{\theta} + ml\dot{x}\dot{\theta}\sin\theta - mgl\sin\theta \\ &= ml\ddot{x}\cos\theta + ml^2\ddot{\theta} - mgl\sin\theta \\ &= \ddot{x}\cos\theta + l\ddot{\theta} - g\sin\theta \\ &= 0.\end{aligned}$$

We represent the obtained Lagrange equations in matrix form

$$\begin{bmatrix} M+m & ml\cos\theta \\ \cos\theta & l \end{bmatrix}\begin{bmatrix}\ddot{x} \\ \ddot{\theta}\end{bmatrix} = \begin{bmatrix} ml\dot{\theta}^2\sin\theta + u \\ g\sin\theta \end{bmatrix}.$$

The above representation is convenient from the calculation point of view. Indeed, we can use, the following MATLAB commands to calculate the vector $\begin{bmatrix}\ddot{x} & \ddot{\theta}\end{bmatrix}^\top$:

```
syms M m theta l u thetadot g
D=[M+m m*l*cos(theta);cos(theta) l];
v=[u+m*l*thetadot^2*sin(theta);g*sin(theta)];
D_inv=inv(D);
g=D_inv*v;
simplify(g);
pretty(ans)
```

to obtain the desired result. Let

$$\Delta = M + m - m\cos^2\theta.$$

Then, we have

$$\begin{bmatrix}\ddot{x} \\ \ddot{\theta}\end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} u + ml\dot{\theta}^2\sin\theta - mg\cos\theta\sin\theta \\ \frac{1}{l}\left(-u\cos\theta - ml\dot{\theta}^2\cos\theta\sin\theta + gM\sin\theta + gm\sin\theta\right)\end{bmatrix}.$$

We define the following state variables:

$$x_1 = x, \quad x_2 = \dot{x}, \quad x_3 = \theta, \quad x_4 = \dot{\theta}.$$

Then, we represent the above two second-order differential modeling equations in state-space format,

$$\begin{cases} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & \frac{mlx_4^2 \sin x_3 - mg \cos x_3 \sin x_3 + u}{M + m - m \cos^2 x_3} \\ \dot{x}_3 & = & x_4 \\ \dot{x}_4 & = & \frac{-mlx_4^2 \cos x_3 \sin x_3 + gM \sin x_3 + gm \sin x_3 - \cos x_3 u}{l(M + m - m \cos^2 x_3)} \end{cases}$$

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