Case Study

For the nonlinear dynamical system,

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}2 - x_1x_2\\ -4 + x_1^2\end{bmatrix} + \begin{bmatrix}-1\\ x_2\end{bmatrix}u$$
$$y = x_1 + x_2,$$

1. find the equilibrium pair (x_e, u_e) corresponding to $u_e = 2$ for which $x_{1e} = 0$, where x_{1e} denotes the first component of x_e;
2. find the corresponding linearized model about the equilibrium pair found above;
3. design the state-feedback controller, $\delta u = -k\delta x$, such that the closed-loop poles of the linearized system are located at $\{-2, -3\}$;
4. apply the resulting controller to the nonlinear plant model and write down the equations of the closed-loop system where the obtained controller drives the nonlinear plant;
5. design an asymptotic state observer with the observer poles located at -3 and -4. Denote the observer state vector by z. Write down the equations of the observer dynamics;
6. let $\delta u = -kz + v$. Find the transfer function of the closed-loop system, $\delta Y(s)/V(s)$.

Please take a moment to work through this problem on your own. When you are ready, scroll down the page to view the solution.

Explanation:

1. We set $\dot{x}_2 = 0$ and substitute the given values into the resulting algebraic equation to obtain
$$-4 + 2x_2 = 0.$$
$$x_{2e} = 2$$
$$(x_e, u_e)$$
$$x_e = \begin{bmatrix}0\\ 2\end{bmatrix}, \quad u_e = 2.$$

2. Let $\dot{x}_1 = f_1$ and $\dot{x}_2 = f_2$. The linearized model has the form,
$$\frac{d}{dt}\delta x = A\delta x + b\delta u,$$
where
$$A = \begin{bmatrix}\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2}\\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2}\end{bmatrix} = \begin{bmatrix}-x_2 & -x_1\\ 2x_1 & u\end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix}\frac{\partial f_1}{\partial u}\\ \frac{\partial f_2}{\partial u}\end{bmatrix} = \begin{bmatrix}-1\\ x_2\end{bmatrix}$$
evaluated at the equilibrium pair about which we linearize the nonlinear system. We have,

$$\frac{d}{dt}\delta x = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}\delta x + \begin{bmatrix} -1 \\ 2 \end{bmatrix}\delta u.$$

The linearized output map has the form,

$$\delta y = \begin{bmatrix} 1 & 1 \end{bmatrix}\delta x,$$

where $\delta y = y - y_e = y - 2$.

3. We already have one pole in the desired location. We just shift the unstable pole to -3. By inspection,

$$k = \begin{bmatrix} 0 & 2.5 \end{bmatrix}.$$

4. The resulting controller applied to the nonlinear plant has the form,

$$u = -k\delta x + u_e.$$

The closed-loop system has the form,

$$\begin{aligned}
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 2 - x_1 x_2 \\ -4 + x_1^2 \end{bmatrix} + \begin{bmatrix} -1 \\ x_2 \end{bmatrix}\left(-\begin{bmatrix} 0 & 2.5 \end{bmatrix}\delta x + u_e\right) \\
&= \begin{bmatrix} 2 - x_1 x_2 \\ -4 + x_1^2 \end{bmatrix} + \begin{bmatrix} -1 \\ x_2 \end{bmatrix}\left(-2.5(x_2 - 2) + 2\right) \\
&= \begin{bmatrix} 2.5x_2 - x_1 x_2 - 5 \\ x_1^2 + 7x_2 - 2.5x_2^2 - 4 \end{bmatrix}
\end{aligned}$$

5. We can apply Ackermann's formula to to the pair $(A^\top, c^\top)$ to obtain the estimator gain vector l. We form the
controllability matrix of the pair $(A^\top, c^\top)$, then find the last row of its inverse and call it q_1. We have

$$\begin{bmatrix} c^\top & A^\top c^\top \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -2 \\ 1 & 2 \end{bmatrix}^{-1} = \frac{1}{4}\begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}.$$

Hence, $q_1 = \frac{1}{4}\begin{bmatrix} -1 & 1 \end{bmatrix}$. The desired characteristic polynomial of $A - lc$ is

$$\det(sI_2 - A + lc) = (s+3)(s+4) = s^2 + 7s + 12.$$

Therefore, the observer gain l is

$$
\begin{aligned}
l^\top &= q_1\left(\left(A^\top\right)^2 + 7A^\top + 12I_2\right)\\
&= \frac{1}{4}\begin{bmatrix}-1 & 1\end{bmatrix}\begin{bmatrix}2 & 0\\ 0 & 30\end{bmatrix}\\
&= \begin{bmatrix}-\frac{1}{2} & \frac{15}{2}\end{bmatrix}.
\end{aligned}
$$

Hence,

$$
l = \begin{bmatrix}-\frac{1}{2}\\ \frac{15}{2}\end{bmatrix},
$$

and the observer dynamics are described by

$$
\begin{aligned}
\dot{z} &= (A - lc)z + b\delta u + l\delta y\\
&= \begin{bmatrix}-1.5 & 0.5\\ -7.5 & -5.5\end{bmatrix} z + \begin{bmatrix}-1\\ 2\end{bmatrix}\delta u + \begin{bmatrix}-\frac{1}{2}\\ \frac{15}{2}\end{bmatrix}\delta y.
\end{aligned}
$$

6. We have

$$
A - bk = \begin{bmatrix}-2 & 2.5\\ 0 & -3\end{bmatrix}.
$$

Therefore

$$
\det\left(sI_2 - A + bk\right) = s^2 + 5s + 6.
$$

Hence, the transfer function $\delta Y(s)/V(s)$ is

$$
\frac{\delta Y(s)}{V(s)} = c(sI_2 - A + bk)^{-1}b = \frac{s+6}{s^2+5s+6}.
$$