ECE 602: LUMPED LINEAR SYSTEMS

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Fundamental Matrices of Continuous-Time LTI Systems

Continuous-Time Autonomous LTI Systems

The scalar ODE $\dot{x} = ax$ has the solution $x(t) = e^{at}x(0),\ t \geq 0$.

The autonomous linear time-invariant (LTI) system

$$\dot{x} = Ax$$

with $A \in \mathbb{R}^{n \times n}$ and the initial condition $x(0)$ has the solution:

$$x(t) = e^{At}x(0), \quad t \geq 0$$

Proof: Use the property $\frac{d}{dt}e^{At} = Ae^{At}$. ■

Fundamental Matrix

The fundamental matrix $\Phi(t)$ of the LTI system $\dot{x} = Ax$ is defined as

$$\Phi(t) := e^{At}$$

- Solution to the LTI system is given by $x(t) = \Phi(t)x(0)$
- Solution $x(t)$ satisfies $x(t + t_0) = \Phi(t)x(t_0)$ for any t_0
- $\Phi(t)$ propagates the solution at any time along the solution t time forward
- $\Phi(t_1 + t_2) = \Phi(t_1)\Phi(t_2) = \Phi(t_2)\Phi(t_1)$ for all t_1 and t_2

Solution Space

Solution space of the LTI system $\dot{x} = Ax$ is the set of all its solutions:

$$\mathbb{X} := \{x(t),\ t \geq 0 \,|\, \dot{x} = Ax\}$$

$\mathbb{X}$ is an n-dimensional vector space

- One-to-one correspondence between $x(\cdot) \in \mathbb{X}$ and initial states $x(0) \in \mathbb{R}^n$
- A basis $\{v_1, v_2, \ldots, v_n\}$ of $\mathbb{R}^n$ yields a basis $\{e^{At}v_1, e^{At}v_2, \ldots, e^{At}v_n\}$ of $\mathbb{X}$
- Any solution $x(t) \in \mathbb{X}$ from $x(0) = \alpha_1 v_1 + \cdots + \alpha_n v_n$ can be written as:

$$x(t) = \alpha_1(e^{At}v_1) + \cdots + \alpha_n(e^{At}v_n)$$

- What would be a good choices of basis?

LTI System after a Change of Coordinates

Change of coordinates by a nonsingular $T = \begin{bmatrix} t_1 & \cdots & t_n \end{bmatrix} \in \mathbb{R}^{n \times n}$:

$$\tilde{x} = T^{-1} x$$

- $\tilde{x}$ is the coordinates of x in the new basis $\{t_1, \ldots, t_n\}$ of $\mathbb{R}^n$

LTI system $\dot{x} = Ax$ in the new $\tilde{x}$-coordinates:

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} := \left(T^{-1}AT\right)\tilde{x}, \quad \tilde{x}(0) = T^{-1}x(0)$$

Fundamental matrix in the new coordinates:

$$\widetilde{\Phi}(t) = e^{\tilde{A}t} = T^{-1}\Phi(t)T$$