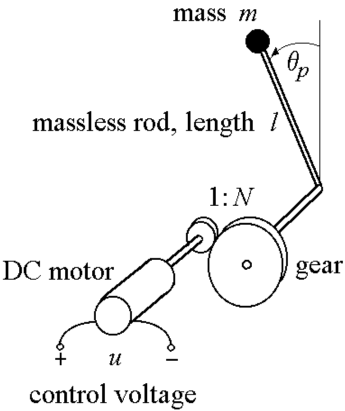
An Example of the Observer-Based Feedback Control Design

- **Objective**: Construct a combined observer-controller compensator to control one-link manipulator controlled by a DC motor via a gear
- Design steps:
  1. Construct simulation and design models
  2. Construct stabilizing state-feedback using the design model
  3. Construct an asymptotic state observer using the design model
  4. Combine the observer and controller into the combined observer-controller compensator
  5. Test the compensator on the non-linear simulation model

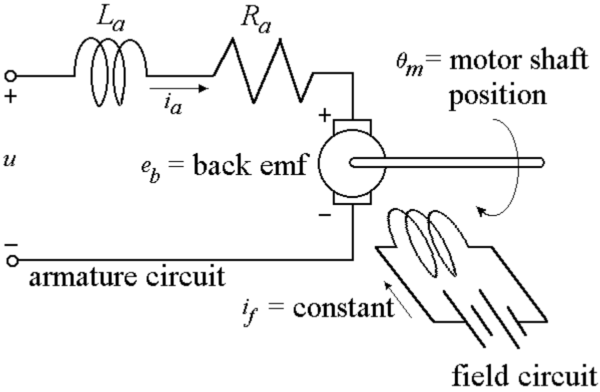
Modeling

- For the controller design purposes, first construct a truth/simulation model of the plant at hand
- Then construct its design model to be used to construct the controller
- The truth model is the simulation model that should include all the relevant characteristics of the physical system to be controlled
- The control designs are being simulated using the truth model

One-link manipulator controlled by a DC motor via a gear



Armature-controlled DC motor



Modeling assumptions

- The motor moment of inertia is negligible compared with that of the robot arm
- Model the arm as a point mass m attached to the end of a massless rod of length l; hence the arm inertia is $I_a = ml^2$
- The gear train has no backlash and all connecting shafts are rigid
- Counter-clockwise rotation of the arm is defined as positive and clockwise rotation of the arm as negative
- Counter-clockwise rotation of the motor shaft is defined to be negative, and clockwise rotation of the shaft as positive
- The torque delivered by the motor is

$$T_m = K_m i_a,$$

  where K_m is the motor-torque constant and i_a is the armature current

Gear and torque—mechanical part

- The gears are in contact and thus

$$\theta_p \times \text{radius of arm gear} = \theta_m \times \text{radius of motor gear},$$

  and the radii of the gears are proportional to their number of teeth
- Let N denote the gear ratio
- We have

$$\frac{\theta_p}{\theta_m} = \frac{\text{radius of motor gear}}{\text{radius of arm gear}} = \frac{\text{number of teeth of motor gear}}{\text{number of teeth of arm gear}} = \frac{1}{N}$$

- The work done by the gears must be equal

Robot arm dynamics

- Let T_p be the torque applied to the robot arm, then

$$T_p \theta_p = T_m \theta_m$$

- The torque applied to the pendulum is $T_p = NT_m = NK_m i_a$
- Use Newton's second law to write the equation modeling the arm dynamics,

$$I_a \frac{d^2\theta_p}{dt^2} = mgl \sin\theta_p + T_p$$

- Substituting and rearranging yields

$$ml^2 \frac{d^2\theta_p}{dt^2} = mgl \sin\theta_p + NK_m i_a$$

where $g = 9.8$ m/s^2 is the gravitational constant

DC motor—electrical part

- Recall that

$$\frac{\theta_p}{\theta_m} = \frac{\text{radius of motor gear}}{\text{radius of arm gear}} = \frac{\text{number of teeth of motor gear}}{\text{number of teeth of arm gear}} = \frac{1}{N}$$

- Therefore

$$\frac{d\theta_m}{dt} = N\frac{d\theta_p}{dt}$$

- Apply Kirchhoff's voltage law (KVL) to the armature circuit

$$L_a\frac{di_a}{dt} + Ri_a + e_b = L_a\frac{di_a}{dt} + Ri_a + K_bN\frac{d\theta_p}{dt} = u$$

where K_b is the back emf constant

One-link robot state-space model

- Preparing to construct a third-order state-space model of the one-link robot
- Select the following state variables:

$$x_1 = \theta_p, \quad x_2 = \frac{d\theta_p}{dt} = \omega_p, \quad x_3 = i_a$$

- The model in state-space format

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{g}{l} \sin x_1 + \frac{NK_m}{ml^2} x_3 \\ -\frac{K_b N}{L_a} x_2 - \frac{R_a}{L_a} x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{L_a} \end{bmatrix} u$$

One-link robot state-space model simulation model

- Reasonable parameters of the robot are: $l = 1\,\text{m}$, $m = 1\,\text{kg}$, $N = 10$, $K_m = 0.1\,\text{Nm/A}$, $K_b = 0.1\,\text{Vs/rad}$, $R_a = 1\,\Omega$, $L_a = 100\,\text{mH}$
- With the above parameters the robot model takes the form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ 9.8 \sin x_1 + x_3 \\ -10\,x_2 - 10\,x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} u$$

- Assume the output, $y = x_1$

Design model

- Target equilibrium state of the robot

$$\boldsymbol{x}_e = \begin{bmatrix} 30\pi/180 \\ 0 \\ 0 \end{bmatrix}$$

- Compute the resulting $u_e = -4.9$
- The linearized model about the operating pair $\begin{bmatrix} \boldsymbol{x}_e^\top & u_e \end{bmatrix}^\top$ is

$$\frac{d}{dt}\delta\boldsymbol{x} = \boldsymbol{A}\delta\boldsymbol{x} + \boldsymbol{b}\delta u$$

where

$$\delta\boldsymbol{x} = \boldsymbol{x} - \boldsymbol{x}_e, \quad \delta u = u - u_e$$

and

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 \\ 8.487 & 0 & 1 \\ 0 & -10 & -10 \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix}$$

State-feedback control law

- Design a state feedback control law $\delta u = -\boldsymbol{k}\,\delta\boldsymbol{x}$ such that the closed-loop poles are located at $-2, -2 \pm j$
- Can use MATLAB's functions `acker` or `place`
- The resulting controller's gain

$$\boldsymbol{k} = \left[\begin{array}{ccc} 6.0922 & 1.1487 & -0.4000 \end{array}\right]$$

- Represent the control law $\delta u = -\boldsymbol{k}\,\delta\boldsymbol{x}$ as

$$u = -\boldsymbol{k}\boldsymbol{x} + \boldsymbol{k}\boldsymbol{x}_e + u_e = -\boldsymbol{k}\boldsymbol{x} + 0.24987$$

- Simulation—the initial state $\boldsymbol{x}(0) = \left[\begin{array}{ccc} -\pi & 0 & 0 \end{array}\right]^\top$

Combined observer-controller compensator

- Design the asymptotic state observer using the linearized model
- The observer's poles are to be located at $-5, -5 \pm j$
- The observer's gain vector is

$$\boldsymbol{l} = \begin{bmatrix} 5 \\ 24.487 \\ -80 \end{bmatrix}$$

- The asymptotic state observer

$$\dot{\boldsymbol{z}} = (\boldsymbol{A} - \boldsymbol{l}\boldsymbol{c})\boldsymbol{z} + \boldsymbol{b}\delta u + \boldsymbol{l}\delta y$$

where $y = x_1$ and $\delta y = y - x_{1e}$

- The combined observer-controller compensator

$$\begin{cases} u &= u_e - \boldsymbol{k}\boldsymbol{z} \\ \dot{\boldsymbol{z}} &= (\boldsymbol{A} - \boldsymbol{l}\boldsymbol{c})\boldsymbol{z} + \boldsymbol{b}\left(u - u_{1e}\right) + \boldsymbol{l}\left(y - x_{1e}\right) \end{cases}$$