4.6. Nonhomogeneous systems of DE.

$y' = Ay + g(t)$

(Ex) $y' = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} y + \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t}$?

(idea) $y'' - 4y = e^{3t}$

(1) Solve $y'' - 4y = 0$: $r^2 - 4 = 0$, $r = 2, -2$

$y_c(t) = C_1 e^{2t} + C_2 e^{-2t}$

(2) Find a particular solution $y_p(t)$

$y(t) = \underline{y_c(t) + y_p(t)}$

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1. Solve $y' = \underbrace{\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}}_{=A} y \rightarrow Y_c(t)$

2. Find $Y_p(t)$.

$Y(t) = Y_c(t) + Y_p(t)$: a general solution

(1)

$\det(A - \lambda I) = (1-\lambda)(-1-\lambda) - 3 = 0$

$\lambda^2 - 4 = 0 \quad \lambda = 2, -2$

$V_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad V_2 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

$Y_c(t) = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + C_2 \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-2t}$

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(2) $Y_p(t) = ?$

(idea) $\frac{d}{dt}(\text{polynomial}) = \text{a polynomial}$

$\underline{\frac{d}{dt} e^{at} = a e^{at}}$, $\frac{d}{dt} \sin(kt) = k\cos(kt)$

$\frac{d}{dt}\cos(kt) = -k\sin(kt)$

Set $\underline{Y_p(t) = Ve^{3t}}$

Ⓛ $Y_p'(t) = 3Ve^{3t}$

$=$

Ⓡ $= AY_p + \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} = AVe^{3t} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t}$

$$A V \cancel{e^{3t}} + \begin{bmatrix}1\\2\end{bmatrix} \cancel{e^{3t}} = 3V \cancel{e^{3t}}$$

$$(A-3I)V = \begin{bmatrix}-1\\-2\end{bmatrix} \quad : \quad A = \begin{bmatrix}1 & 1\\3 & -1\end{bmatrix}$$

$$\begin{bmatrix}-2 & 1\\3 & -4\end{bmatrix} V = \begin{bmatrix}-1\\-2\end{bmatrix} : V = \begin{bmatrix}-2 & 1\\3 & -4\end{bmatrix}^{-1} \begin{bmatrix}-1\\-2\end{bmatrix}$$

$$V = \frac{1}{5}\begin{bmatrix}-4 & -1\\-3 & -2\end{bmatrix}\begin{bmatrix}-1\\-2\end{bmatrix} = \frac{1}{5}\begin{bmatrix}6\\7\end{bmatrix} = \begin{bmatrix}6/5\\7/5\end{bmatrix}$$

$$\therefore \; Y_p(t) = \begin{bmatrix}6/5\\7/5\end{bmatrix} e^{3t}$$

$$\therefore \; Y(t) = C_1\begin{bmatrix}1\\1\end{bmatrix}e^{2t} + C_2\begin{bmatrix}1\\-3\end{bmatrix}e^{-2t} + \begin{bmatrix}6/5\\7/5\end{bmatrix}e^{3t}$$

(Ex) $Y' = \underbrace{\begin{bmatrix}1 & 1\\3 & -1\end{bmatrix}}_{A} Y + \begin{bmatrix}1\\2\end{bmatrix}e^{-2t}$

(1) $Y_c(t) = C_1\begin{bmatrix}1\\1\end{bmatrix}e^{2t} + C_2\begin{bmatrix}1\\-3\end{bmatrix}e^{-2t}$

(2) Set $Y_p(t) = Ve^{-2t}$ (Rule 1)

Ⓛ $Y_p' = -2Ve^{-2t}$ =

Ⓡ $= AY_p + \begin{bmatrix}1\\2\end{bmatrix}e^{-2t} = AVe^{-2t} + \begin{bmatrix}1\\2\end{bmatrix}e^{-2t}$

$-2V = AV + \begin{bmatrix}1\\2\end{bmatrix} \; : \; (A+2I)V = \begin{bmatrix}-1\\-2\end{bmatrix}$

$+2V \qquad +2V$

$\begin{bmatrix}3 & 1\\3 & 1\end{bmatrix}V = \begin{bmatrix}-1\\-2\end{bmatrix}$ has No solutions.

(3) Set $Y_p(t) = V_1 t e^{-2t} + V_2 e^{-2t}$ (Rule 2)

Ⓛ $Y_p' = V_1(e^{-2t} + (-2)te^{-2t}) - 2V_2e^{-2t}$

Ⓡ $= AY_p + \begin{bmatrix}1\\2\end{bmatrix}e^{-2t}$

$= AV_1te^{-2t} + AV_2e^{-2t} + \begin{bmatrix}1\\2\end{bmatrix}e^{-2t}$ =

$V_1\cancel{e^{-2t}} - 2V_1 t\cancel{e^{-2t}} - 2V_2\cancel{e^{-2t}}$

$= AV_1 t\cancel{e^{-2t}} + AV_2\cancel{e^{-2t}} + \begin{bmatrix}1\\2\end{bmatrix}\cancel{e^{-2t}}$

① $AV_1 = -2V_1$ ② $V_1 - 2V_2 = AV_2 + \begin{bmatrix}1\\2\end{bmatrix}$

$$V_1 = \left(-\frac{1}{4}\right)\begin{bmatrix}1\\-3\end{bmatrix} = \begin{bmatrix}-1/4\\3/4\end{bmatrix}$$

$$3v_1 + v_2 = -\frac{1}{4} - 1 = -\frac{5}{4}$$

$$v_2 = -\frac{5}{4} - 3v_1 \ : \quad v_1 = 0 : v_2 = -\frac{5}{4}$$
$$v_1 = 1 : v_2 = -\frac{17}{4}, \cdots$$

$$V_2 = \begin{bmatrix}v_1\\v_2\end{bmatrix} = \begin{bmatrix}0\\-\frac{5}{4}\end{bmatrix}$$

$$\therefore Y_p(t) = \begin{bmatrix}-\frac{1}{4}\\3/4\end{bmatrix} t e^{-2t} + \begin{bmatrix}0\\-\frac{5}{4}\end{bmatrix} e^{-2t}$$

$$Y(t) = C_1\begin{bmatrix}1\\1\end{bmatrix}e^{2t} + C_2\begin{bmatrix}1\\-3\end{bmatrix}e^{-2t} + Y_p(t)$$

(Ex) $y' = \begin{bmatrix}1 & 1\\3 & -1\end{bmatrix} y + \underbrace{\begin{bmatrix}2t\\5t\end{bmatrix}}_{g(t)}$

(1) $Y_c(t) = C_1\begin{bmatrix}1\\1\end{bmatrix}e^{2t} + C_2\begin{bmatrix}1\\-3\end{bmatrix}e^{-2t}$

(2) $Y_p(t) \overset{let}{=} V_1 t + V_2$

$g(t) = \begin{bmatrix}2\\5\end{bmatrix} t$

Ⓛ $Y_p' = V_1$, Ⓡ $= AY_p + \begin{bmatrix}2\\5\end{bmatrix}t$

$= AV_1 t + AV_2 + \begin{bmatrix}2\\5\end{bmatrix}t$

$V_1 = \left(AV_1 t + \begin{bmatrix}2\\5\end{bmatrix}t\right) + AV_2$

① $AV_1 + \begin{bmatrix}2\\5\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$ ② $AV_2 = V_1$

$$\begin{bmatrix}1 & 1\\3 & -1\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix} = \begin{bmatrix}-2\\-5\end{bmatrix} : \quad \begin{aligned} v_1 + v_2 &= -2 \\ +\ 3v_1 - v_2 &= -5 \\ \hline 4v_1 &= -7. \end{aligned}$$

$v_1 = -\frac{7}{4}$, $v_2 = -2 - v_1$

$v_2 = -2 + \frac{7}{4} = -\frac{1}{4}$

$$V_1 = \begin{bmatrix}-\frac{7}{4}\\-1/4\end{bmatrix}$$

② $AV_2 = V_1$: $\begin{bmatrix}1 & 1\\3 & -1\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix} = \begin{bmatrix}-7/4\\-1/4\end{bmatrix}$

$$v_1 + v_2 = -\frac{7}{4} \quad -①$$
$$3v_1 - v_2 = -\frac{1}{4} \quad -②$$

① + ② :

$$4v_1 = -\frac{8}{4} = -2$$
$$v_1 = -\frac{1}{2}.$$

$$v_2 = -\frac{7}{4} - v_1 = -\frac{7}{4} + \frac{1}{2} = -\frac{5}{4}$$

$$V_2 = \begin{bmatrix} -\frac{1}{2} \\ -\frac{5}{4} \end{bmatrix} : Y_p(t) = \begin{bmatrix} -\frac{7}{4} \\ -\frac{1}{4} \end{bmatrix} t + \begin{bmatrix} -\frac{1}{2} \\ -\frac{5}{4} \end{bmatrix}$$

$$Y(t) = Y_c(t) + Y_p(t)$$

6.1. Laplace transform.

Application: ODE, PDE, Signal processing.

Def ▪ $L(f) = \int_0^\infty e^{-st} f(t)\,dt$

$$= \lim_{T\to\infty} \int_0^T e^{-st} f(t)\,dt$$

: the Laplace transform of f.

(Ex) 1. $f(x) = 1$:

$(u = -st, \; du = -s\,dt)$

$$L(1) = \int_0^\infty e^{-st}\,dt$$
$$= \lim_{T\to\infty} \int_0^T e^{-st}\,dt = \lim_{T\to\infty} \int_0^{-sT} e^u \cdot \frac{1}{-s}\,du$$

$$L(1) = -\frac{1}{s} \lim_{T\to\infty} \left[e^u\right]_0^{-sT} = -\frac{1}{s} \lim_{T\to\infty} (\cancel{e^{-sT}} - 1)$$

$\to 0$ $(s > 0)$

$$L(1) = \frac{1}{s}, \quad s > 0$$

(2) $f(t) = e^{at}$

$$L(f) = L(e^{at}) = \int_0^\infty e^{-st} e^{at}\,dt$$
$$= \int_0^\infty e^{(a-s)t}\,dt \quad : \; u = (a-s)t, \; du = (a-s)\,dt$$
$$= \lim_{T\to\infty} \int_0^T e^{(a-s)t}\,dt = \lim \int_0^{(a-s)T} e^u \frac{1}{a-s}\,du$$
$$= \lim_{T\to\infty} \left[\frac{1}{a-s} e^u\right]_0^{(a-s)T}$$

$$L(e^{at}) = \frac{1}{a-s}\lim_{T\to\infty}\left[\cancel{e^{(a-s)T}}^{\to 0} - 1\right]$$

$(a-s<0)$
$(s>a)$

$$= \frac{1}{a-s}(-1) = \frac{1}{s-a}$$

$$L(t^n) = \frac{n!}{s^{n+1}}, \quad s>0 \quad \text{(n: an integer)}$$

(Pf) $L(t^n) = \int_0^\infty \underbrace{e^{-st}}_{\text{"}g'\text{"}} \underbrace{t^n}_{\text{"}f\text{"}} dt$

$$\int_a^b f(t)\,g'(t)\,dt = \left[f(t)\,g(t)\right]_a^b - \int_a^b f'(t)\,g(t)\,dt$$

$$= \left[t^n \frac{e^{-st}}{-s}\right]_0^\infty - \int_0^\infty n t^{n-1} \frac{e^{-st}}{-s}\,dt \quad (s>0)$$

Remark $\lim_{t\to\infty} \frac{t^n}{e^{st}} = 0 \quad (s>0).$

$$L(t^n) = \frac{n}{s}\int_0^\infty e^{-st}\,t^{n-1}\,dt = \frac{n}{s}L(t^{n-1})$$

$$= \frac{n}{s}\frac{n-1}{s}L(t^{n-2}) = \cdots$$

$$= \frac{n\cdot(n-1)(n-2)\cdots 1}{s^n} L(1) = \frac{1}{s}$$

$$L(t^n) = \frac{n!}{s^{n+1}}, \quad s>0$$