MA 527

Lecture Notes (section 7.4)

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7.4. Rank of matrix.

$AX = b.$ $\quad A_{n \times n}$ $\quad n = 10^6, 10^9.$

(Motivation)

PDE $\longrightarrow$ $AX = b.$

(Linear Independence)

(Ex) $a_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $a_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$:

$a_2 = 2a_1 \rightarrow \underline{2a_1 - a_2 = 0}$

$c_1 a_1 + c_2 a_2 = 0$ has nonzero solutions

: $c_1 = 2, \quad c_2 = -1.$

(Def) $a_1, a_2, \cdots, a_m$

(1) $c_1 a_1 + c_2 a_2 + \cdots + c_m a_m$ is called a linear combination of $a_1, \cdots, a_m$.

(2) If $c_1 a_1 + c_2 a_2 + \cdots + c_m a_m = 0$ has a nonzero solution $c_i \neq 0$, then $a_1, a_2, \cdots, a_m$ are called linearly dependent.

(3) If $c_1 \mathbf{a}_1 + c_2 \mathbf{a}_2 + \cdots + c_m \mathbf{a}_m = 0$ has only zero solution, $c_1 = 0, c_2 = 0, \cdots, c_m = 0$, then $\mathbf{a}_1, \cdots, \mathbf{a}_m$ are called linearly independent

(Ex) $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$: linearly independent ? Yes

① Set $c_1 \mathbf{a}_1 + c_2 \mathbf{a}_2 = 0$.

$c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$: $\begin{aligned} c_1 + 2c_2 &= 0 \\ 2c_1 \quad &= 0 \\ 3c_1 + c_2 &= 0 \end{aligned}$

$c_1 = 0, \quad c_2 = 0$

(2) $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$? : linearly dependent.

lin. independent ? (X)

$c_1 \mathbf{a}_1 + c_2 \mathbf{a}_2 = 0$: $c_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Pick

$c_1 = 0, \quad c_2 = 2, \cdots$

Q $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_m$: linearly independent. (row vectors) in $\mathbb{R}^n$

$\mathbf{v}_2, \mathbf{v}_1, \mathbf{v}_3, \cdots, \mathbf{v}_m$: lin. independent.

$A = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_m \end{bmatrix}_{m \times n}$: m row vectors of A are lin. independent.

7.4. Rank

Def rank A = the max number of lin. indep. rows of A.

Remark: Let $v_1, v_2, \cdots, v_m$ be lin. independent row vectors.

⇒

(1) $v_2, v_1, v_3, \cdots, v_m$: lin. indep.

(2) $v_1, c_2 v_2, v_3, \cdots, v_m$: lin. indep. ($c_2 \neq 0$)

(3) $v_1, v_2 - r v_1, v_3, \cdots, v_m$: lin. indep.

(proof) Set $\alpha_1 v_1 + \alpha_2 (v_2 - r v_1) + \alpha_3 v_3 + \cdots + \alpha_m v_m = 0$

$$\underbrace{(\alpha_1 - r\alpha_2)}_{=0} v_1 + \underbrace{\alpha_2}_{=0} v_2 + \cdots + \underbrace{\alpha_m}_{=0} v_m = 0$$

$\alpha_1 - r \underbrace{\alpha_2}_{=0} = 0$ ∴ $\alpha_1 = 0$.

∴ $v_1, v_2, \cdots, v_m$: lin. independent.

Thm (Theorem).

Any elementry row operations do not change Linear independence (dependence) of rows of a matrix A.

(Ex) $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \xrightarrow{r_2 - 3r_1} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ ←

$\operatorname{rank} A = 1.$

$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 5 \end{bmatrix} \xrightarrow{r_2 - 2r_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & -1 \end{bmatrix}$

$\longrightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & \frac{1}{4} \end{bmatrix}$ $\quad \operatorname{rank} A = 2.$

row-equivalent.

Thm 1 Row-equivalent matrices have the same rank.

Q1 Can we use columns? Yes

$\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \end{bmatrix}$. : lin. dependent.

Remark: $\operatorname{rank} A = \operatorname{rank} A^T$

Q2. $a_1, a_2, \cdots, a_k$: k vectors in $\mathbb{R}^n$.
(with n components)

If $k > n$, then $a_1, \cdots, a_n$ are linearly dependent.

Let $A = \begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix}_{k \times n} \longrightarrow \begin{bmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{bmatrix}$

$\operatorname{rank} A \leq n$

__Remark__ $A_{m \times n}$

(1) $\operatorname{rank} A \leq m$, $\operatorname{rank} A \leq n$

(2) $\operatorname{rank} A \leq \min\{m, n\}$.

(Vector space).

1. Motivation: $AX = b$. $A_{m \times n}$

$X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ $A = [a_1 \ a_2 \ \cdots \ a_n]$

columns.

$b = AX = [a_1 \ a_2 \ \cdots \ a_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$b = x_1 a_1 + x_2 a_2 + \cdots + x_n a_n$.

Q: the set of all the linear combinations of the columns of A.

Definition of Vector Space

Let V be a set on which two operations (**vector addition** and **scalar multiplication**) are defined. If the listed axioms are satisfied for every $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and every scalar (real number) c and d, then V is called a **vector space.**

Addition:

1. $\mathbf{u} + \mathbf{v}$ is in V.	Closure under addition
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$	Commutative property
3. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$	Associative property
4. V has a **zero vector** $\mathbf{0}$ such that for every $\mathbf{u}$ in V, $\mathbf{u} + \mathbf{0} = \mathbf{u}$.	Additive identity
5. For every $\mathbf{u}$ in V, there is a vector in V denoted by $-\mathbf{u}$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.	Additive inverse

Scalar Multiplication:

6. $c\mathbf{u}$ is in V.	Closure under scalar multiplication
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$	Distributive property
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$	Distributive property
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$	Associative property
10. $1(\mathbf{u}) = \mathbf{u}$	Scalar identity

(Ex) $\mathbb{R}^n$: a vector space.

Def V: a set of objects (vectors, matrices, ⋯)

$\begin{cases} \mathbf{a}, \mathbf{b} \in V, & \mathbf{a} \oplus \mathbf{b} \in V \\ \mathbf{a} \in V, \beta \in \mathbb{R}: & \beta \odot \mathbf{a} \in V \end{cases}$

Assume that $\mathbf{a}, \mathbf{b}, \mathbf{c} \in V$ satisfy

(1) $\mathbf{a} \oplus \mathbf{b} = \mathbf{b} \oplus \mathbf{a}$: commutative.

(2) $(\mathbf{a} \oplus \mathbf{b}) \oplus \mathbf{c} = \mathbf{a} \oplus (\mathbf{b} \oplus \mathbf{c})$: associative.

(3) $\mathbf{a} \oplus 0 = \mathbf{a}$: 0 : the identity.

(4) $\mathbf{a} \oplus (-\mathbf{a}) = 0$

(5) $c \odot (\mathbf{a} \oplus \mathbf{b}) = (c \odot \mathbf{a}) \oplus (c \odot \mathbf{b})$

(6) $(c+k) \odot \mathbf{a} = c \odot \mathbf{a} \oplus k \odot \mathbf{a}$

distribution

(7) $c \odot (k \odot \mathbf{a}) = (ck) \odot \mathbf{a}$

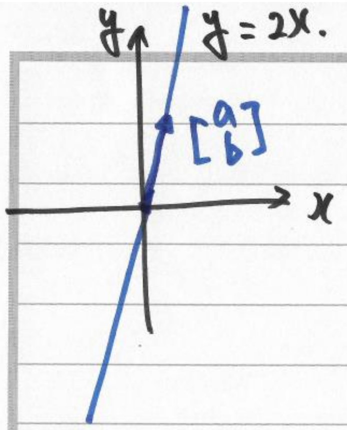
(8) $1 \odot \mathbf{a} = \mathbf{a}$

(Ex) $\mathbb{R}^n$, $M_{m\times n} = \{A : A \text{ is an } m\times n \text{ matrix}\}$.

: Vector spaces.

(Ex) $\mathbb{R}^2$: $W = \{ \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2 : b = 2a \}$.

$W \subset \mathbb{R}^2$: a subset.



W is a vector space.

W is called a subspace.

Thm $W \subset V$ for a vector space $(V, \oplus, \odot)$

W is a vector space (subspace of V)

iff ① for $\mathbb{a}, \mathbb{b} \in W$, $\mathbb{a} \oplus \mathbb{b} \in W$

② for $\beta \in \mathbb{R}$, $\mathbb{a} \in W$, $\beta \mathbb{a} \in W$

(Ex) $M_{2\times 2} = \{ A : A \text{ is a } 2\times 2 \text{ matrix} \}$.

: a vector.

(1) $W = \left\{ \begin{bmatrix} 0 & a \\ b & a+b \end{bmatrix} : a, b \in \mathbb{R} \right\}$.

: a subspace? Yes

(Proof) Take $\underbrace{\begin{bmatrix} 0 & a \\ b & a+b \end{bmatrix}}_{A}, \underbrace{\begin{bmatrix} 0 & c \\ d & c+d \end{bmatrix}}_{B} \in W$

① $A + B = \begin{bmatrix} 0 & a+c \\ b+d & a+b+c+d \end{bmatrix} \in W$

$a+b+c+d = (a+c) + (b+d)$.

② $\beta \in \mathbb{R}$, $\beta \begin{bmatrix} 0 & a \\ b & a+b \end{bmatrix} = \begin{bmatrix} 0 & \beta a \\ \beta b & \beta a + \beta b \end{bmatrix} \in W$

(n=2)

Q. $V = \mathbb{R}^n$, $A_{n\times n} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$W = \left\{ X \in \mathbb{R}^n \mid AX = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right\}$: not a subspace of $\mathbb{R}^2$.

$X_1, X_2 \in W$: $X_1 + X_2 \in W$?

$A(X_1+X_2) = AX_1 + AX_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

Q $V = \mathbb{R}^n$ $A_{n\times n}$

$W = \{ X \in \mathbb{R}^n \mid AX = 0 \}$: a subspace of $\mathbb{R}^n$! (Nullspace of A)

① Take $X_1, X_2 \in W$

$X_1 + X_2 \in W$

(Because $A(X_1+X_2) = \underbrace{AX_1}_{=0} + \underbrace{AX_2}_{=0} = 0$

② $\beta \in \mathbb{R}$: $\beta X_1 \in W$

$A(\beta X_1) = \beta \underbrace{AX_1}_{=0} = 0$