5.6.1

$B \sim A \qquad B \sim C$

$B = M^{-1}AM$

$C = N^{-1}BN \qquad \therefore \quad NCN^{-1} = B$

$NCN^{-1} = M^{-1}AM$

$C = N^{-1}M^{-1}AMN$

$(AB)^{-1} = B^{-1}A^{-1} \Rightarrow N^{-1}M^{-1} = (MN)^{-1}$

$\therefore \quad \boxed{C = (MN)^{-1}A(MN)}$ where $(MN) = T$

So $C = T^{-1}AT$ (Similiar) to A

Only I is similiar to I.

5.6.2

$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\chi_A = 0 = \lambda^2 - 1 = 0 \qquad \lambda_{1,2} = \pm 1$

Matrices similiar to A will have two λ's with $\lambda_1 = 1$ & $\lambda_2 = -1$, with dimension 2x2 & trace $= 0$ with det $= -1$.

$\boxed{A_1 = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}}$

$\chi_{A_1} = \lambda^2 - 1 = 0 \qquad \lambda_{1,2} = \pm 1$

$\boxed{A_2 = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}}$

$\chi_{A_2} = \lambda^2 - 1 = 0 \qquad \lambda_{1,2} = \pm 1$

5.6.14

$Tf = f'$ $\qquad\qquad f = f(x)$

$Tf(x) = \int_0^x f(t)\,dt$

Let $f = e^{\lambda x}$

$Tf = \lambda f$ $\qquad\qquad f' = \lambda e^{\lambda x}$

$\therefore \boxed{Tf = f' = \lambda f = \lambda e^{\lambda x}}$

$Tf = \lambda f$

$\frac{d}{dt}(Tf) = \frac{d}{dT}(\lambda f) \quad = f = \lambda f'$

$Tf(0) = \int_0^0 f\,dt = 0$

If $\lambda = 0$, $f = 0$.

If $\lambda \neq 0$, $f' = \frac{1}{\lambda} f$

$\therefore f = Ce^{1/\lambda t}$

$T(f(0)) = 0 = C \quad \therefore \quad C = 0$, $f = 0 \quad \therefore$

no eigenvalues

5.6.15

$A^T = \lambda A$

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{pmatrix}$$

$A^T - \lambda A = 0$

$$\begin{pmatrix} a(1-\lambda) & c-\lambda b \\ b-\lambda c & d(1-\lambda) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Assume $a_{ij} \neq 0$

$\therefore \boxed{\lambda_1 = 1, \lambda_2 = 1}$, $\lambda = c/b$, $\lambda = \frac{b}{c}$

$c = b\lambda \quad \therefore \quad \lambda = \frac{b}{b\lambda} = \frac{1}{\lambda} \quad \therefore \lambda^2 = 1$

$$\boxed{\lambda_{3,4} = \pm 1}$$

$$A = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\boxed{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}$$

$\lambda = 1, \frac{c}{b}$

$\lambda = -1, \frac{-c}{b}$

5.6.23

$$A(A-I)(A-2I) = A(A^2-2A-A+2I) = A^3-3A^2+2A$$

$$A = S^{-1}\Lambda S$$

$$= S^{-1}\begin{pmatrix}0&0&0\\0&1&0\\0&0&2\end{pmatrix} S$$

$$A^3 = S^{-1}\Lambda^3 S = S^{-1}\begin{pmatrix}0&0&0\\0&1&0\\0&0&8\end{pmatrix} S$$

$$A^2 = S^{-1}\Lambda^2 S = S^{-1}\begin{pmatrix}0&0&0\\0&1&0\\0&0&4\end{pmatrix} S$$

$$A^3-3A^2+2A = S^{-1}\begin{pmatrix}0&0&0\\0&1&0\\0&0&8\end{pmatrix} S - 3\left(S^{-1}\begin{pmatrix}0&0&0\\0&1&0\\0&0&4\end{pmatrix} S\right) + 2\left(S^{-1}\begin{pmatrix}0&0&0\\0&1&0\\0&0&2\end{pmatrix} S\right)$$

$$= S^{-1}\begin{pmatrix}0&0&0\\0&1&0\\0&0&8\end{pmatrix} + \begin{pmatrix}0&0&0\\0&-3&0\\0&0&-12\end{pmatrix} + \begin{pmatrix}0&0&0\\0&2&0\\0&0&4\end{pmatrix} S$$

$$= S^{-1}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix} S = \begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix} = A(A-I)(A-2I)$$

$$\therefore \boxed{\lambda_1 = \lambda_2 = \lambda_3 = 0}$$

B.1

$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

$\chi_A = \lambda^2 - 2\lambda = 0$

$\lambda_1 = 0 \;,\; \lambda_2 = 2$

$m^a(\lambda_1) = m^a(\lambda_2) = 1$

$B_1 = (A - \lambda_1 I) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} \boxed{1} & 1 \\ 0 & 0 \end{pmatrix}$ rank $= 1$
$\dim(\text{Ker}(B)) = n - r = 1$

$\therefore\; m^g(\lambda_1) = 1$

$B_2 = (A - 2I) = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \Rightarrow \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$ rank $= 1$
$\dim(\text{Ker}(B_2)) = n - r = 1$

$\therefore\; m^g(\lambda_2) = 1$

$m^g(\lambda_1) = m^a(\lambda_1) \;\therefore\; J_{\lambda_1, k} = J_{0,1} = 0$

$m^a(\lambda_2) = m^a(\lambda_2) \;\therefore\; J_{\lambda_2, k} = J_{2,1} = 2$

$\therefore\; \boxed{J = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}}$ $\leftarrow$ diagonlizable due to $m^a(\lambda_i) = m^g(\lambda_i)$

$B = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

B-1

$$(B-\lambda I) = \begin{pmatrix} -\lambda & 1 & 2 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{pmatrix}$$

$$-\lambda \begin{vmatrix} -\lambda & 0 \\ 0 & -\lambda \end{vmatrix} - \cancel{\begin{vmatrix} 0 & 0 \\ 0 & -\lambda \end{vmatrix}}^{0} + 2 \cancel{\begin{vmatrix} 0 & -\lambda \\ 0 & 0 \end{vmatrix}}^{0}$$

$$-\lambda(\lambda^2) = 0 \qquad \lambda_1 = \lambda_2 = \lambda_3 = 0$$

$$m^a(\lambda_1) = 3$$

$$B_1 = (B-\lambda I) = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \leftarrow \text{rank} = 1$$

$$\therefore \dim(\ker(B_1)) = n - r = 3 - 1 = 2$$

$$\therefore m^g(\lambda) = 2$$

$$J_{\lambda,k} = J_{0,1} = \left(\begin{array}{cc|c} 0 & 1 & \\ & 0 & \\ \hline & & 0 \end{array}\right) \quad \therefore$$

$$\boxed{J = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}$$

$$B_1 v_1 = 0$$

$$\begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \vec{0}$$

x_1 & x_3 are free, x_2 is pivot

B.1

Preferred solution: let $X_1 = 1$, $X_3 = 0$

$$X_2 + 2X_3 = 0$$
$$X_2 = 0$$

$$w_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Preferred solution $X_1 = 0$, $X_3 = 1$

$$X_2 + 2 = 0$$
$$X_2 = -2$$
$$w_3 = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$$

$$N(B_1) = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \right\}$$

$$Aw_2 = w_1$$

$$\begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$w_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

B.2

$$U_2 = e^{8t}(tx_1 + x_2)$$

$$\frac{dU_2}{dt} = (e^{8t})(tx_1) + x_2 e^{8t}$$

$$= (8e^{8t})(tx_1) + x_1 e^{8t} + 8x_2 e^{8t}$$

$$= e^{8t}(\underbrace{8tx_1}_{\text{term 3}} + \underbrace{x_1}_{\text{term 1}} + \underbrace{8x_2}_{\text{term 2}})$$

$$AU_2 = Ae^{8t}tx_1 + Ae^{8t}x_2$$

$$Ax_1 = 8x_1$$
$$Ax_2 = 8x_2 + x_1$$

$$AU_2 = (e^{8t})(t)\,8x_1 + (e^{8t})(8x_2 + x_1)$$

$$= \underbrace{e^{8t}x_1}_{\text{term 1}} + \underbrace{8e^{8t}x_2}_{\text{term 2}} + \underbrace{8tx_1e^{8t}}_{\text{term 3}}$$

$$\therefore \frac{dU_2}{dt} = AU_2$$

B.3

$$e^{Bt} = I + Bt + \frac{B^2t^2}{2!} + \dots$$

$$B^2 = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\therefore \quad e^{Bt} = I + Bt$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & t & 2t \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} 1 & t & 2t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

$$J = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$e^{Jt} = \begin{pmatrix} e^{0t} & te^{0t} & 0 \\ 0 & e^{0t} & 0 \\ 0 & 0 & e^{0t} \end{pmatrix} = \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \quad M^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

B.3

$$Me^{Jt}M^{-1} = \begin{pmatrix} 1 & t & 2t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = e^{Bt}$$

B.7

$J = M^{-1}AM$

$J^2 = (M^{-1}A\cancel{M})(\cancel{M^{-1}}AM)$

$J^2 = M^{-1}A^2M$

If $A^2 = A$, then $\boxed{J^2 = M^{-1}AM = J}$

If $J = 0$ then $\underline{J^2 = 0^2 = 0 = J}$

If $J = 1$ then $\underline{J^2 = 1^2 = 1 = J}$

5.20

$K^H = -K$

a) For invertible $\det \neq 0$, $\therefore$

$\det(K - I) \neq 0$

For $\det(K - \lambda I)$, $\lambda \in \mathbb{C}$ $\therefore$ the

$\det(K - I) \neq 0.$

b) $U^H = U^{-1}$, $U^H U = I$, $UU^H = I$

$K = -K^H = U \Lambda U^H$

c) $e^{\Lambda 6} = e^{(U^H K U) 6}$

d) $e^{Kt} = e^{U \Lambda U^H}$