Optimal Estimation Methods

(Lecture 16 – Batch State Estimation: Part I)

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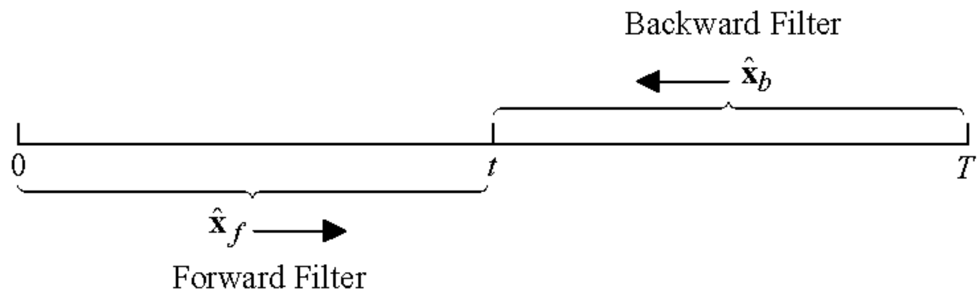
Introduction (i)

- Batch state estimators (smoothers) are used to provide even more filtered estimates from noisy measurements
  - Basically, smoothers are used to estimate the state quantities using measurements made before and after a certain time t
  - To accomplish this task, two filters are usually used: a forward-time filter and a backward-time filter



  - The disadvantage of batch estimation methods, such as smoothers, is they cannot be implemented in real time
    - However, they have the advantage of providing state estimates with a lower error-covariance than sequential methods
    - This may be extremely helpful when accuracy is an issue, but real time application is not required

Introduction (ii)

- Three types of smoothers
  - **Fixed-Interval Smoothing**. This smoother uses the entire batch of measurements over a fixed interval to estimate all the states in the interval. The times 0 and T are fixed and t varies from time 0 to T in this formulation. Since the entire batch of measurements is used to produce an estimate, this smoother provides the best possible estimate over the interval.
  - **Fixed-Point Smoothing**. This smoother estimates the state at a specific fixed point in time t, given a batch of measurements up to the current time T. This smoother is often used to estimate the state at only one time point in the interval.
  - **Fixed-Lag Smoothing**. This smoother estimates the state at a fixed time interval that lags the time of the current measurement at time T. This smoother is often used to refine the optimal forward filter estimate.

Introduction (iii)

- The fixed-point and fixed-lag smoothers are batch processes only in the sense that they require measurements up to the current time
- The derivation of all of these smoothers can be given from the Kalman filter
  - In fact, all smoothers use the Kalman filter for forward-time filtering
- The history of smoothing actually predates the Kalman filter
  - Wiener solved the original fixed-lag smoothing problem in the 1940's, but he only considered the stationary case where the smoother assumes that the entire past history of the input is available for weighting in its estimate
  - The first practical smoothing algorithms are attributed to Bryson and Frazier in 1962, as well as Rauch, Tung, and Striebel (RTS) in 1965
  - In particular, the RTS smoothing algorithm has maintained its popularity since the initial paper, and is likely the most widely used algorithm for smoothing to date
- Focus here is on fixed-interval smoothing

Fixed-Interval Smoothing

- Fixed-interval smoothing uses the entire batch of measurements over a fixed interval to estimate all the states in the interval
  - Fraser and Potter in 1969 have shown that this smoother can be derived from a combination of two Kalman filters
    - One of which works forward over the data and the other of which works backward over the fixed interval
    - Together these two filters use all the available information to provide optimal estimates
  - Earlier work (Bryson in 1962 and Rauch in 1965) gives the smoother estimate as a correction to the Kalman filter estimate for the same point, and others (Mayne in 1966 and Fraser in 1967) do not have the appearance of a correction to the Kalman filter estimate
  - All are mathematically equivalent, but the required computations are different for each approach

Discrete-Time (i)

- True system is modeled by

$$\mathbf{x}_{k+1} = \Phi_k \mathbf{x}_k + \Gamma_k \mathbf{u}_k + \Upsilon_k \mathbf{w}_k$$

$$\tilde{\mathbf{y}}_k = H_k \mathbf{x}_k + \mathbf{v}_k$$

where $\mathbf{w}_k \sim N(\mathbf{0}, Q_k)$ and $\mathbf{v}_k \sim N(\mathbf{0}, R_k)$

- The forward filter is just the Kalman filter
  - Propagation equations

forward

$$\hat{\mathbf{x}}^-_{f\,k+1} = \Phi_k \hat{\mathbf{x}}^+_{f\,k} + \Gamma_k \mathbf{u}_k$$

$$P^-_{f\,k+1} = \Phi_k P^+_{f\,k} \Phi_k^T + \Upsilon_k Q_k \Upsilon_k^T$$

  - Update equations

$$\hat{\mathbf{x}}^+_{f\,k} = \hat{\mathbf{x}}^-_{f\,k} + K_{f\,k}[\tilde{\mathbf{y}}_k - H_k \hat{\mathbf{x}}^-_{f\,k}]$$

$$P^+_{f\,k} = [I - K_{f\,k} H_k] P^-_{f\,k}$$

  - Gain

$$K_{f\,k} = P^-_{f\,k} H_k^T [H_k P^-_{f\,k} H_k^T + R_k]^{-1}$$

Discrete-Time (ii)

- Solve the truth model for $\mathbf{x}_k$

$$\mathbf{x}_k = \Phi_k^{-1}\mathbf{x}_{k+1} - \Phi_k^{-1}\Gamma_k\mathbf{u}_k - \Phi_k^{-1}\Upsilon_k\mathbf{w}_k \quad (1)$$

  - Clearly, the inverse of Φ must exist, meaning that the state matrix has no zero eigenvalues, but we shall see that the final form of the backward filter does not depend on this condition
- The backward estimate is provided by the backward-running filter just *before* the measurement at time t_k
  - Hence, the backward-time state propagation is given by

$$\hat{\mathbf{x}}_{b\,k}^- = \Phi_k^{-1}\hat{\mathbf{x}}_{b\,k+1}^+ - \Phi_k^{-1}\Gamma_k\mathbf{u}_k \quad (2)$$

- We seek a smoothed estimate that is a linear combination of the backward propagated estimate and forward updated estimate

$$\hat{\mathbf{x}}_k = M_k\hat{\mathbf{x}}_{f\,k}^+ + N_k\hat{\mathbf{x}}_{b\,k}^-$$

- Define the following error states for the smoother, forward filter and backward filter

$$\tilde{\mathbf{x}}_k = \hat{\mathbf{x}}_k - \mathbf{x}_k, \quad \tilde{\mathbf{x}}_{f\,k}^+ = \hat{\mathbf{x}}_{f\,k}^+ - \mathbf{x}_k \quad \tilde{\mathbf{x}}_{b\,k}^- = \hat{\mathbf{x}}_{b\,k}^- - \mathbf{x}_k$$

Discrete-Time (iii)

- Then we have

$$\tilde{\mathbf{x}}_k = [M_k + N_k - I]\mathbf{x}_k + M_k\tilde{\mathbf{x}}_{f\,k}^+ + N_k\tilde{\mathbf{x}}_{b\,k}^-$$

- An unbiased state estimate requires $N_k = I - M_k$
- Therefore the smoother estimate has the form

$$\hat{\mathbf{x}}_k = M_k\hat{\mathbf{x}}_{f\,k}^+ + [I - M_k]\hat{\mathbf{x}}_{b\,k}^-$$

- Define the following covariance expressions Backward

$$P_k \equiv E\left\{\tilde{\mathbf{x}}_k\tilde{\mathbf{x}}_k^T\right\}, \quad P_{f\,k}^+ \equiv E\left\{\tilde{\mathbf{x}}_{f\,k}^+\tilde{\mathbf{x}}_{f\,k}^{+T}\right\}, \quad P_{b\,k}^- \equiv E\left\{\tilde{\mathbf{x}}_{b\,k}^-\tilde{\mathbf{x}}_{b\,k}^{-T}\right\}$$

- Then using $N_k = I - M_k$ in the error equation we have

$$P_k = M_k P_{f\,k}^+ M_k^T + [I - M_k]P_{b\,k}^-[I - M_k]^T$$

- An optimal expression for M_k is given by minimizing the trace of P_k
  - The necessary conditions, i.e., differentiating with respect to M_k, give

$$0 = 2M_k P_{f\,k}^+ - 2[I - M_k]P_{b\,k}^-$$

or

$$M_k = P_{b\,k}^-[P_{f\,k}^+ + P_{b\,k}^-]^{-1}$$

Discrete-Time (iv)

- Also $I - M_k$ is given by

$$\begin{aligned} I - M_k &= [P_{f\,k}^{+} + P_{b\,k}^{-}][P_{f\,k}^{+} + P_{b\,k}^{-}]^{-1} - P_{b\,k}^{-}[P_{f\,k}^{+} + P_{b\,k}^{-}]^{-1} \\ &= P_{f\,k}^{+}[P_{f\,k}^{+} + P_{b\,k}^{-}]^{-1} \end{aligned}$$

- Substituting these into P_k leads to

$$\boxed{P_k = \left[(P_{f\,k}^{+})^{-1} + (P_{b\,k}^{-})^{-1} \right]^{-1}} \qquad (3)$$

- Look at the scalar case

$$p_k = \frac{p_{f\,k}^{+}\, p_{b\,k}^{-}}{p_{f\,k}^{+} + p_{b\,k}^{-}}$$

- This indicates that the smoother error covariance is always less than or equal to either the forward or backward covariance
  - Therefore, the smoother estimate is always better than either filter alone

Discrete-Time (v)

- Can reduce the number of inverses by using the matrix inversion lemma and defining $\mathcal{P}_{b\,k}^{-} \equiv (P_{b\,k}^{-})^{-1}$

$$\boxed{P_k = P_{f\,k}^{+} - P_{f\,k}^{+}\mathcal{P}_{b\,k}^{-}[I + P_{f\,k}^{+}\mathcal{P}_{b\,k}^{-}]^{-1}P_{f\,k}^{+}}$$

- Substituting the M_k and $I - M_k$ expressions into the smoother state estimate leads to

$$\begin{aligned}\hat{\mathbf{x}}_k &= P_k\left[(P_{f\,k}^{+})^{-1}\hat{\mathbf{x}}_{f\,k}^{+} + (P_{b\,k}^{-})^{-1}\hat{\mathbf{x}}_{b\,k}^{-}\right] \\ &\equiv P_k\left[(P_{f\,k}^{+})^{-1}\hat{\mathbf{x}}_{f\,k}^{+} + \mathcal{P}_{b\,k}^{-}\hat{\mathbf{x}}_{b\,k}^{-}\right]\end{aligned}$$

  - This is also known as *Millman's theorem*, which is an exact analog to maximum likelihood of a scalar with independent measurements

- Substituting the P_k expression into $P_k(P_{f\,k}^{+})^{-1}$ gives

$$P_k(P_{f\,k}^{+})^{-1} = I - P_{f\,k}^{+}\mathcal{P}_{b\,k}^{-}[I + P_{f\,k}^{+}\mathcal{P}_{b\,k}^{-}]^{-1}$$

- Therefore the smoother state estimate becomes

$$\hat{\mathbf{x}}_k = [I - K_k]\hat{\mathbf{x}}_{f\,k}^{+} + P_k\mathcal{P}_{b\,k}^{-}\hat{\mathbf{x}}_{b\,k}^{-} \qquad (4)$$

where the smoother gain is given by

$$\boxed{K_k \equiv P_{f\,k}^{+}\mathcal{P}_{b\,k}^{-}[I + P_{f\,k}^{+}\mathcal{P}_{b\,k}^{-}]^{-1}}$$

- The backwards update covariance is given directly from the information form update

$$\boxed{\mathcal{P}_{b\,k}^{+} = \mathcal{P}_{b\,k}^{-} + H_k^T R_k^{-1} H_k}$$

Discrete-Time (vii)

- To derive a backward recursion for the backwards propagation covariance, first subtract Eq. (1) from Eq. (2), and use the error definitions $\tilde{\mathbf{x}}_{b\,k}^{-} = \hat{\mathbf{x}}_{b\,k}^{-} - \mathbf{x}_k, \;\; \tilde{\mathbf{x}}_{b\,k}^{+} = \hat{\mathbf{x}}_{b\,k}^{+} - \mathbf{x}_k$, giving

$$\tilde{\mathbf{x}}_{b\,k}^{-} = \Phi_k^{-1}\tilde{\mathbf{x}}_{b\,k+1}^{+} + \Phi_k^{-1}\Upsilon_k\mathbf{w}_k$$

- Since the error state is uncorrelated with $\mathbf{w}_k$, then

$$P_{b\,k}^{-} = \Phi_k^{-1}[P_{b\,k+1}^{+} + \Upsilon_k Q_k \Upsilon_k^T]\Phi_k^{-T}$$

- We want the inverse of this equation
- Using the matrix inversion lemma gives

$$\boxed{\mathcal{P}_{b\,k}^{-} = \Phi_k^T[I - K_{b\,k}\Upsilon_k^T]\mathcal{P}_{b\,k+1}^{+}\Phi_k} \qquad (5)$$

where the gain is given by

$$\boxed{K_{b\,k} = \mathcal{P}_{b\,k+1}^{+}\Upsilon_k[\Upsilon_k^T\mathcal{P}_{b\,k+1}^{+}\Upsilon_k + Q_k^{-1}]^{-1}}$$

Discrete-Time (viii)

- The inverse of Q_k must exist to implement this form
  - Fraser in 1967 showed that only those states that are controllable by the process noise driving the system are smoothable
  - Therefore, in practice Q_k must have an inverse, otherwise this controllability condition is violated
  - We'll show later it can be relaxed to just being positive semi-definite
- Boundary conditions
  - The forward filter is implemented using the same initial conditions as the standard Kalman filter (that's exactly what the forward filter is!)
  - Let t_N denote the terminal time
  - Since at time $t_k = t_N$ the smoother estimate must be the same as the forward filter, this clearly requires that $\hat{\mathbf{x}}_N = \hat{\mathbf{x}}_{f\,N}^+, \quad P_N = P_{f\,N}^+$
  - From Eq. (3) the covariance condition at the terminal time can only be satisfied when

$$(P_{b\,N}^-)^{-1} \equiv \mathcal{P}_{b\,N}^- = 0$$

  - Still need the terminal condition on the backward state update

$$\hat{\mathbf{x}}_{b\,k}^+ = \hat{\mathbf{x}}_{b\,k}^- + K_{b\,k}[\tilde{\mathbf{y}}_k - H_k\hat{\mathbf{x}}_{b\,k}^-]$$

Discrete-Time (ix)

- Use the information form instead to help the solution

$$\hat{\mathbf{x}}_{b\,k}^{+} = P_{b\,k}^{+}[\mathcal{P}_{b\,k}^{-}\hat{\mathbf{x}}_{b\,k}^{-} + H_k^T R_k^{-1}\tilde{\mathbf{y}}_k]$$

or

$$\mathcal{P}_{b\,k}^{+}\hat{\mathbf{x}}_{b\,k}^{+} = \mathcal{P}_{b\,k}^{-}\hat{\mathbf{x}}_{b\,k}^{-} + H_k^T R_k^{-1}\tilde{\mathbf{y}}_k$$

- Define the following variables

$$\hat{\boldsymbol{\chi}}_{b\,k}^{+} \equiv \mathcal{P}_{b\,k}^{+}\hat{\mathbf{x}}_{b\,k}^{+}$$
$$\hat{\boldsymbol{\chi}}_{b\,k}^{-} \equiv \mathcal{P}_{b\,k}^{-}\hat{\mathbf{x}}_{b\,k}^{-}$$

- Then the update is written as

$$\boxed{\hat{\boldsymbol{\chi}}_{b\,k}^{+} = \hat{\boldsymbol{\chi}}_{b\,k}^{-} + H_k^T R_k^{-1}\tilde{\mathbf{y}}_k}$$

- Since $\mathcal{P}_{b\,N}^{-} = 0$ then the boundary condition on the backward update is given by

$$\hat{\boldsymbol{\chi}}_{b\,N}^{-} = \mathbf{0}$$

Discrete-Time (x)

- Backward propagation will complete the smoother equations
- Recall Eq. (2)

$$\hat{\mathbf{x}}_{bk}^{-} = \Phi_k^{-1}\hat{\mathbf{x}}_{b\,k+1}^{+} - \Phi_k^{-1}\Gamma_k \mathbf{u}_k$$

- Substitute this into $\hat{\boldsymbol{\chi}}_{b\,k}^{-} \equiv \mathcal{P}_{b\,k}^{-}\hat{\mathbf{x}}_{b\,k}^{-}$ and use $\hat{\mathbf{x}}_{b\,k+1}^{+} = (\mathcal{P}_{b\,k+1}^{+})^{-1}\hat{\boldsymbol{\chi}}_{b\,k+1}^{+}$

$$\hat{\boldsymbol{\chi}}_{b\,k}^{-} = \mathcal{P}_{b\,k}^{-}\Phi_k^{-1}\left[(\mathcal{P}_{b\,k+1}^{+})^{-1}\hat{\boldsymbol{\chi}}_{b\,k+1}^{+} - \Gamma_k \mathbf{u}_k\right]$$

- Substituting Eq. (5) gives the desired form

$$\boxed{\hat{\boldsymbol{\chi}}_{b\,k}^{-} = \Phi_k^{T}[I - K_{b\,k}\Upsilon_k^{T}][\hat{\boldsymbol{\chi}}_{b\,k+1}^{+} - \mathcal{P}_{b\,k+1}^{+}\Gamma_k \mathbf{u}_k]}$$

- The smoother state in Eq. (4) can now be rewritten as

$$\boxed{\hat{\mathbf{x}}_k = [I - K_k]\hat{\mathbf{x}}_{f\,k}^{+} + P_k\hat{\boldsymbol{\chi}}_{b\,k}^{-}}$$

- Note that equations for $\hat{\mathbf{x}}_{b\,k}^{+},\ \hat{\mathbf{x}}_{b\,k}^{-}$ are no longer needed

Discrete-Time Summary (i)

Model	$\mathbf{x}_{k+1} = \Phi_k \mathbf{x}_k + \Gamma_k \mathbf{u}_k + \Upsilon_k \mathbf{w}_k, \quad \mathbf{w}_k \sim N(\mathbf{0}, Q_k)$ $\tilde{\mathbf{y}}_k = H_k \mathbf{x}_k + \mathbf{v}_k, \quad \mathbf{v}_k \sim N(\mathbf{0}, R_k)$
Forward Initialize	$\hat{\mathbf{x}}_f(t_0) = \hat{\mathbf{x}}_{f0}$ $P_f(t_0) = E\{\tilde{\mathbf{x}}_f(t_0)\, \tilde{\mathbf{x}}_f^T(t_0)\}$
Gain	$K_{fk} = P_{fk}^- H_k^T [H_k P_{fk}^- H_k^T + R_k]^{-1}$
Forward Update	$\hat{\mathbf{x}}_{fk}^+ = \hat{\mathbf{x}}_{fk}^- + K_{fk}[\tilde{\mathbf{y}}_k - H_k \hat{\mathbf{x}}_{fk}^-]$ $P_{fk}^+ = [I - K_{fk} H_k] P_{fk}^-$
Forward Propagation	$\hat{\mathbf{x}}_{fk+1}^- = \Phi_k \hat{\mathbf{x}}_{fk}^+ + \Gamma_k \mathbf{u}_k$ $P_{fk+1}^- = \Phi_k P_{fk}^+ \Phi_k^T + \Upsilon_k Q_k \Upsilon_k^T$

Discrete-Time Summary (ii)

Backward Initialize	$\hat{\boldsymbol{\chi}}_{b\,N}^{-} = \mathbf{0}$ $\mathcal{P}_{b\,N}^{-} = 0$
Gain	$K_{b\,k} = \mathcal{P}_{b\,k+1}^{+}\Upsilon_k[\Upsilon_k^T\mathcal{P}_{b\,k+1}^{+}\Upsilon_k + Q_k^{-1}]^{-1}$
Backward Update	$\hat{\boldsymbol{\chi}}_{b\,k}^{+} = \hat{\boldsymbol{\chi}}_{b\,k}^{-} + H_k^T R_k^{-1}\tilde{\mathbf{y}}_k$ $\mathcal{P}_{b\,k}^{+} = \mathcal{P}_{b\,k}^{-} + H_k^T R_k^{-1} H_k$
Backward Propagation	$\hat{\boldsymbol{\chi}}_{b\,k}^{-} = \Phi_k^T[I - K_{b\,k}\Upsilon_k^T][\hat{\boldsymbol{\chi}}_{b\,k+1}^{+} - \mathcal{P}_{b\,k+1}^{+}\Gamma_k\mathbf{u}_k]$ $\mathcal{P}_{b\,k}^{-} = \Phi_k^T[I - K_{b\,k}\Upsilon_k^T]\mathcal{P}_{b\,k+1}^{+}\Phi_k$
Gain	$K_k = P_{f\,k}^{+}\mathcal{P}_{b\,k}^{-}[I + P_{f\,k}^{+}\mathcal{P}_{b\,k}^{-}]^{-1}$
Covariance	$P_k = [I - K_k]P_{f\,k}^{+}$
Estimate	$\hat{\mathbf{x}}_k = [I - K_k]\hat{\mathbf{x}}_{f\,k}^{+} + P_k\hat{\boldsymbol{\chi}}_{b\,k}^{-}$

Steady State Smoother (i)

- For autonomous systems, at steady state we have

$$\mathcal{P}_b^- = \Phi^T \mathcal{P}_b^+ \Phi - \Phi^T \mathcal{P}_b^+ \Upsilon \left[\Upsilon^T \mathcal{P}_b^+ \Upsilon + Q^{-1}\right]^{-1} \Upsilon^T \mathcal{P}_b^+ \Phi$$

  - Substituting $\mathcal{P}_b^- = \mathcal{P}_b^+ - H^T R^{-1} H$ gives

$$\mathcal{P}_b^+ = \Phi^T \mathcal{P}_b^+ \Phi - \Phi^T \mathcal{P}_b^+ \Upsilon \left[\Upsilon^T \mathcal{P}_b^+ \Upsilon + Q^{-1}\right]^{-1} \Upsilon^T \mathcal{P}_b^+ \Phi + H^T R^{-1} H$$

  - As before, form the following Hamiltonian matrix

$$\mathcal{H} \equiv \begin{bmatrix} \Phi^{-1} & \Phi^{-1} \Upsilon\, Q\, \Upsilon^T \\ H^T R^{-1} H\, \Phi^{-1} & \Phi^T + H^T R^{-1} H\, \Phi^{-1} \Upsilon\, Q\, \Upsilon^T \end{bmatrix}$$

  - Take an eigenvalue/eigenvector decomposition

$$\mathcal{H} = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} \begin{bmatrix} \Lambda & 0 \\ 0 & \Lambda^{-1} \end{bmatrix} \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}^{-1}$$

    where Λ is a diagonal matrix of the n eigenvalues outside of the unit circle, and W_{11}, W_{21}, W_{12}, and W_{22} are block elements of the eigenvector matrix

Steady State Smoother (ii)

- Using the same approach as before the steady-state covariance for the update is given by

$$\boxed{\mathcal{P}_b^+ = W_{21} W_{11}^{-1}}$$

- The following equation for the propagated value is also required

$$\mathcal{P}_b^- = \mathcal{P}_b^+ - H^T R^{-1} H$$

- Compute the steady-state forward covariance and gain from before
- Then the steady-state gain for the backwards filter can be computed, as well as the steady-state smoother gain and covariance

$$K_b = \mathcal{P}_b^+ \Upsilon [\Upsilon^T \mathcal{P}_b^+ \Upsilon + Q^{-1}]^{-1}$$

$$K = P_f^+ \mathcal{P}_b^- [I + P_f^+ \mathcal{P}_b^-]^{-1}$$

$$P = [I - K] P_f^+$$

RTS Smoother (i)

- Rauch, Tung, and Striebel (RTS) in 1965 combined the backward filter and smoother into one single backward recursion
  - Their derivation is based on maximum likelihood
  - We'll derive their equations from the previous derivation
  - First rewrite the previously derived smoother covariance

$$P_k = P_{f\,k}^{+} - P_{f\,k}^{+}\mathcal{P}_{b\,k}^{-}[I + P_{f\,k}^{+}\mathcal{P}_{b\,k}^{-}]^{-1}P_{f\,k}^{+}$$

as

$$P_k = P_{f\,k}^{+} - P_{f\,k}^{+}[P_{f\,k}^{+} + P_{b\,k}^{-}]^{-1}P_{f\,k}^{+} \quad (1)$$

  - Now substitute the following previously derived equation into the inverse

$$P_{b\,k}^{-} = \Phi_k^{-1}[P_{b\,k+1}^{+} + \Upsilon_k Q_k \Upsilon_k^T]\Phi_k^{-T}$$

and factor out Φ_k to obtain

$$[P_{f\,k}^{+} + P_{b\,k}^{-}]^{-1} = \Phi_k^T[\Phi_k P_{f\,k}^{+}\Phi_k^T + P_{b\,k+1}^{+} + \Upsilon_k Q_k \Upsilon_k^T]^{-1}\Phi_k \quad (2)$$

RTS Smoother (ii)

- Solving the following forward time propagation for $P_{f\,k}^{+}$

$$P_{f\,k+1}^{-} = \Phi_k P_{f\,k}^{+} \Phi_k^T + \Upsilon_k Q_k \Upsilon_k^T$$

gives

$$P_{f\,k}^{+} = \Phi_k^{-1}(P_{f\,k+1}^{-} - \Upsilon_k Q_k \Upsilon_k^T)\Phi_k^{-T}$$

- Substituting into the inverse in Eq. (2) gives

$$[P_{f\,k}^{+} + P_{b\,k}^{-}]^{-1} = \Phi_k^T [P_{f\,k+1}^{-} + P_{b\,k+1}^{+}]^{-1} \Phi_k \qquad (3)$$

- Recall the information updates for both the forward and backward filters

$$\mathcal{P}_{f\,k}^{+} = \mathcal{P}_{f\,k}^{-} + H_k^T R_k^{-1} H_k$$
$$\mathcal{P}_{b\,k}^{+} = \mathcal{P}_{b\,k}^{-} + H_k^T R_k^{-1} H_k$$

- Solve the first for $H_k^T R_k^{-1} H_k$ and substitute into the second, and then take the inverse

$$P_{b\,k}^{+} = [\mathcal{P}_{b\,k}^{-} + \mathcal{P}_{f\,k}^{+} - \mathcal{P}_{f\,k}^{-}]^{-1} \qquad (4)$$

RTS Smoother (iii)

- Now use the following expression in Eq. (4)

$$P_k^{-1} = (P_{f\,k}^{+})^{-1} + (P_{b\,k}^{-})^{-1} \rightarrow \mathcal{P}_{b\,k}^{-} = P_k^{-1} - \mathcal{P}_{f\,k}^{+}$$

to give

$$P_{b\,k}^{+} = [P_k^{-1} - \mathcal{P}_{f\,k}^{-}]^{-1}$$

- Taking one time-step ahead and substituting into Eq. (3) gives

$$[P_{f\,k}^{+} + P_{b\,k}^{-}]^{-1} = \Phi_k^T \left\{ P_{f\,k+1}^{-} + [P_{k+1}^{-1} - \mathcal{P}_{f\,k+1}^{-}]^{-1} \right\}^{-1} \Phi_k$$

- Factoring $\mathcal{P}_{f\,k+1}^{-}$ yields

$$[P_{f\,k}^{+} + P_{b\,k}^{-}]^{-1} = \Phi_k^T \mathcal{P}_{f\,k+1}^{-} \left\{ \mathcal{P}_{f\,k+1}^{-} + \mathcal{P}_{f\,k+1}^{-} [P_{k+1}^{-1} - \mathcal{P}_{f\,k+1}^{-}]^{-1} \mathcal{P}_{f\,k+1}^{-} \right\}^{-1} \mathcal{P}_{f\,k+1}^{-} \Phi_k$$

- Using the matrix inversion lemma gives

$$[P_{f\,k}^{+} + P_{b\,k}^{-}]^{-1} = \Phi_k^T \mathcal{P}_{f\,k+1}^{-} [P_{f\,k+1}^{-} - P_{k+1}] \mathcal{P}_{f\,k+1}^{-} \Phi_k$$

RTS Smoother (iv)

- Substituting this equation into Eq. (1) gives

$$\boxed{P_k = P^+_{f\,k} - \mathcal{K}_k [P^-_{f\,k+1} - P_{k+1}]\,\mathcal{K}_k^T} \qquad (5)$$

where the gain matrix is defined as

$$\boxed{\mathcal{K}_k \equiv P^+_{f\,k}\Phi_k^T (P^-_{f\,k+1})^{-1}} \qquad (6)$$

- Note that the covariance smoother is no longer a function of the backward covariance update or propagation
- Therefore, the smoother covariance can be solved directly from knowledge of the forward covariance alone, which provides a very computationally efficient algorithm
- We shall now see that the smoother covariance is not needed for the smoothed estimate equation
  - Only needed for analysis purposes to get 3σ bounds on the smoothed estimates

- Let's prove that the smoother state estimate is given by

$$\boxed{\hat{\mathbf{x}}_k = \hat{\mathbf{x}}_{f\,k}^{+} + \mathcal{K}_k[\hat{\mathbf{x}}_{k+1} - \hat{\mathbf{x}}_{f\,k+1}^{-}]}$$

- Compare this equation to the previously derived equation

$$\begin{aligned}\hat{\mathbf{x}}_k &= [I - K_k]\hat{\mathbf{x}}_{f\,k}^{+} + P_k \mathcal{P}_{b\,k}^{-} \hat{\mathbf{x}}_{b\,k}^{-} \\ &= [I - K_k]\hat{\mathbf{x}}_{f\,k}^{+} + P_k \hat{\boldsymbol{\chi}}_{b\,k}^{-}\end{aligned}$$

- So the following needs to be true

$$-K_k \hat{\mathbf{x}}_{f\,k}^{+} + P_k \hat{\boldsymbol{\chi}}_{b\,k}^{-} = \mathcal{K}_k[\hat{\mathbf{x}}_{k+1} - \hat{\mathbf{x}}_{f\,k+1}^{-}]$$

- Substituting the following equation

$$K_k \equiv P_{f\,k}^{+} \mathcal{P}_{b\,k}^{-} [I + P_{f\,k}^{+} \mathcal{P}_{b\,k}^{-}]^{-1}$$

with Eqs. (5) and (6), and simplifying gives

$$\begin{aligned}&- \mathcal{P}_{b\,k}^{-}[I + P_{f\,k}^{+} \mathcal{P}_{b\,k}^{-}]^{-1} \hat{\mathbf{x}}_{f\,k}^{+} + \hat{\boldsymbol{\chi}}_{b\,k}^{-} - \Phi_k^T \mathcal{P}_{f\,k+1}^{-}[P_{f\,k+1}^{-} - P_{k+1}] \mathcal{P}_{f\,k+1}^{-} \Phi_k P_{f\,k}^{+} \hat{\boldsymbol{\chi}}_{b\,k}^{-} \\ &= \Phi_k^T \mathcal{P}_{f\,k+1}^{-}[\hat{\mathbf{x}}_{k+1} - \hat{\mathbf{x}}_{f\,k+1}^{-}] \qquad (7)\end{aligned}$$

RTS Smoother (vi)

- We will return to Eq. (7), but for the time being let's concentrate on determining a more useful expression for $\hat{\mathbf{x}}_{k+1}$, which will be used to help the proof
- Take one time-step ahead of the following previously derived equation

$$\hat{\mathbf{x}}_k = P_k \left[\mathcal{P}_{f\,k}^{+} \hat{\mathbf{x}}_{f\,k}^{+} + \hat{\boldsymbol{\chi}}_{b\,k}^{-} \right]$$

to give

$$\hat{\mathbf{x}}_{k+1} = P_{k+1} \mathcal{P}_{f\,k+1}^{+} \hat{\mathbf{x}}_{f\,k+1}^{+} + P_{k+1} \hat{\boldsymbol{\chi}}_{b\,k+1}^{-} \qquad (8)$$

- Take one time-step ahead of the following previously derived equation and solve for $\hat{\boldsymbol{\chi}}_{b\,k+1}^{-}$

$$\hat{\boldsymbol{\chi}}_{b\,k}^{+} = \hat{\boldsymbol{\chi}}_{b\,k}^{-} + H_k^T R_k^{-1} \tilde{\mathbf{y}}_k$$

to give

$$\hat{\boldsymbol{\chi}}_{b\,k+1}^{-} = \hat{\boldsymbol{\chi}}_{b\,k+1}^{+} - H_{k+1}^T R_{k+1}^{-1} \tilde{\mathbf{y}}_{k+1} \qquad (9)$$

RTS Smoother (vii)

- Take one time-step ahead of

$$\hat{\mathbf{x}}_{f\,k}^{+} = \hat{\mathbf{x}}_{f\,k}^{-} + K_{f\,k}[\tilde{\mathbf{y}}_k - H_k \hat{\mathbf{x}}_{f\,k}^{-}]$$

$$K_{f\,k} = \mathcal{P}_{f\,k}^{+} H_k^T R_k^{-1}$$

to give

$$\hat{\mathbf{x}}_{f\,k+1}^{+} = \hat{\mathbf{x}}_{f\,k+1}^{-} + \mathcal{P}_{f\,k+1}^{+} H_{k+1}^T R_{k+1}^{-1}[\tilde{\mathbf{y}}_{k+1} - H_{k+1} \hat{\mathbf{x}}_{f\,k+1}^{-}] \quad (10)$$

- Substituting Eqs. (9) and (10) into Eq. (8) gives

$$\hat{\mathbf{x}}_{k+1} = P_{k+1}\left[\mathcal{P}_{f\,k+1}^{+} - H_{k+1}^T R_{k+1}^{-1} H_{k+1}\right]\hat{\mathbf{x}}_{f\,k+1}^{-} + P_{k+1}\hat{\boldsymbol{\chi}}_{b\,k+1}^{+}$$

- Using a one time-step ahead of the following equation

$$\mathcal{P}_{f\,k}^{+} = \mathcal{P}_{f\,k}^{-} + H_k^T R_k^{-1} H_k$$

simplifies the above expression to

$$\hat{\mathbf{x}}_{k+1} = P_{k+1}\mathcal{P}_{f\,k+1}^{-}\hat{\mathbf{x}}_{f\,k+1}^{-} + P_{k+1}\hat{\boldsymbol{\chi}}_{b\,k+1}^{+}$$

- Subtracting $\hat{\mathbf{x}}_{f\,k+1}^{-}$ from both sides and factoring out $\mathcal{P}_{f\,k+1}^{-}$ yields

$$\hat{\mathbf{x}}_{k+1} - \hat{\mathbf{x}}_{f\,k+1}^{-} = [P_{k+1} - P_{f\,k+1}^{-}]\mathcal{P}_{f\,k+1}^{-}\hat{\mathbf{x}}_{f\,k+1}^{-} + P_{k+1}\hat{\boldsymbol{\chi}}_{b\,k+1}^{+} \quad (11)$$

RTS Smoother (viii)

- Next, rewrite the forward-time prediction as

$$\hat{\mathbf{x}}_{f\,k}^{+} = \Phi_k^{-1}\hat{\mathbf{x}}_{f\,k+1}^{-} - \Phi_k^{-1}\Gamma_k \mathbf{u}_k \qquad (12)$$

- Substituting Eqs. (11) and (12) into Eq. (7), and multiplying by $\mathcal{P}_{b\,k}^{-}$ yields

$$\begin{aligned}
&-[P_{b\,k}^{-} + P_{f\,k}^{+}]^{-1}\Phi_k^{-1}\hat{\mathbf{x}}_{f\,k+1}^{-} + [P_{b\,k}^{-} + P_{f\,k}^{+}]^{-1}\Phi_k^{-1}\Gamma_k \mathbf{u}_k \\
&+ \hat{\boldsymbol{\chi}}_{b\,k}^{-} - \Phi_k^T \mathcal{P}_{f\,k+1}^{-}[P_{f\,k+1}^{-} - P_{k+1}]\mathcal{P}_{f\,k+1}^{-}\Phi_k P_{f\,k}^{+}\hat{\boldsymbol{\chi}}_{b\,k}^{-} \\
&= \Phi_k^T \mathcal{P}_{f\,k+1}^{-}[P_{k+1} - P_{f\,k+1}^{-}]\mathcal{P}_{f\,k+1}^{-}\hat{\mathbf{x}}_{f\,k+1}^{-} + \Phi_k^T \mathcal{P}_{f\,k+1}^{-} P_{k+1}\hat{\boldsymbol{\chi}}_{b\,k+1}^{+}
\end{aligned}$$

- Using the following previously derived equation

$$[P_{f\,k}^{+} + P_{b\,k}^{-}]^{-1} = \Phi_k^T \mathcal{P}_{f\,k+1}^{-}[P_{f\,k+1}^{-} - P_{k+1}]\mathcal{P}_{f\,k+1}^{-}\Phi_k$$

simplifies the above expression to

$$[P_{b\,k}^{-} + P_{f\,k}^{+}]^{-1}\Phi_k^{-1}\Gamma_k \mathbf{u}_k + \hat{\boldsymbol{\chi}}_{b\,k}^{-} - [P_{b\,k}^{-} + P_{f\,k}^{+}]^{-1} P_{f\,k}^{+}\hat{\boldsymbol{\chi}}_{b\,k}^{-} = \Phi_k^T \mathcal{P}_{f\,k+1}^{-} P_{k+1}\hat{\boldsymbol{\chi}}_{b\,k+1}^{+}$$

- Using the following previously derived equation (i.e. Eq. (2))

$$[P_{f\,k}^{+} + P_{b\,k}^{-}]^{-1} = \Phi_k^T[\Phi_k P_{f\,k}^{+}\Phi_k^T + P_{b\,k+1}^{+} + \Upsilon_k Q_k \Upsilon_k^T]^{-1}\Phi_k$$

and left multiplying both sides of the resulting equation by $[P_{f\,k+1}^{-} + P_{b\,k+1}^{+}]\Phi_k^{-T}$ yields

$$\Gamma_k \mathbf{u}_k + \left([P_{f\,k+1}^{-} + P_{b\,k+1}^{+}]\Phi_k^{-T} - \Phi_k P_{f\,k}^{+}\right)\hat{\boldsymbol{\chi}}_{b\,k}^{-} = [P_{f\,k+1}^{-} + P_{b\,k+1}^{+}]\mathcal{P}_{f\,k+1}^{-} P_{k+1}\hat{\boldsymbol{\chi}}_{b\,k+1}^{+}$$

- Next, rewrite the forward-time covariance prediction as

$$P_{f\,k}^{+} = \Phi_k^{-1} P_{f\,k+1}^{-}\Phi_k^{-T} - \Phi_k^{-1}\Upsilon_k Q_k \Upsilon_k^T \Phi_k^{-T}$$

- Substituting this into the above equation gives

$$\Gamma_k \mathbf{u}_k + \left([P_{f\,k+1}^{-} + P_{b\,k+1}^{+}]\Phi_k^{-T} - P_{f\,k+1}^{-}\Phi_k^{-T} + \Upsilon_k Q_k \Upsilon_k^T \Phi_k^{-T}\right)\hat{\boldsymbol{\chi}}_{b\,k}^{-}$$
$$= [P_{f\,k+1}^{-} + P_{b\,k+1}^{+}]\mathcal{P}_{f\,k+1}^{-} P_{k+1}\hat{\boldsymbol{\chi}}_{b\,k+1}^{+}$$

RTS Smoother (x)

- Using a one time-step ahead of the previously derived equation

$$P_{b\,k}^{+} = [P_k^{-1} - \mathcal{P}_{f\,k}^{-}]^{-1}$$

and solving for $\hat{\boldsymbol{\chi}}_{b\,k}^{-}$ yields

$$\hat{\boldsymbol{\chi}}_{b\,k}^{-} = \Phi_k^T [I + \mathcal{P}_{b\,k+1}^{+} \Upsilon_k Q_k \Upsilon_k^T]^{-1} [\hat{\boldsymbol{\chi}}_{b\,k+1}^{+} - \mathcal{P}_{b\,k+1}^{+} \Gamma_k \mathbf{u}_k]$$

- Using the matrix inversion lemma gives

$$\hat{\boldsymbol{\chi}}_{b\,k}^{-} = \Phi_k^T [I - K_{b\,k} \Upsilon_k^T][\hat{\boldsymbol{\chi}}_{b\,k+1}^{+} - \mathcal{P}_{b\,k+1}^{+} \Gamma_k \mathbf{u}_k]$$

- This is the same equation derived previously, which completes the proof
- Again, the RTS smoother does not need the covariance expression
  - Only needed for analysis purposes

RTS Smoother Summary (i)

Model	$\mathbf{x}_{k+1} = \Phi_k \mathbf{x}_k + \Gamma_k \mathbf{u}_k + \Upsilon_k \mathbf{w}_k, \quad \mathbf{w}_k \sim N(\mathbf{0}, Q_k)$ $\tilde{\mathbf{y}}_k = H_k \mathbf{x}_k + \mathbf{v}_k, \quad \mathbf{v}_k \sim N(\mathbf{0}, R_k)$
Forward Initialize	$\hat{\mathbf{x}}_f(t_0) = \hat{\mathbf{x}}_{f0}$ $P_f(t_0) = E\{\tilde{\mathbf{x}}_f(t_0)\, \tilde{\mathbf{x}}_f^T(t_0)\}$
Gain	$K_{fk} = P_{fk}^- H_k^T [H_k P_{fk}^- H_k^T + R_k]^{-1}$
Forward Update	$\hat{\mathbf{x}}_{fk}^+ = \hat{\mathbf{x}}_{fk}^- + K_{fk}[\tilde{\mathbf{y}}_k - H_k \hat{\mathbf{x}}_{fk}^-]$ $P_{fk}^+ = [I - K_{fk} H_k] P_{fk}^-$
Forward Propagation	$\hat{\mathbf{x}}_{fk+1}^- = \Phi_k \hat{\mathbf{x}}_{fk}^+ + \Gamma_k \mathbf{u}_k$ $P_{fk+1}^- = \Phi_k P_{fk}^+ \Phi_k^T + \Upsilon_k Q_k \Upsilon_k^T$

RTS Smoother Summary (ii)

Smoother Initialize	$\hat{\mathbf{x}}_N = \hat{\mathbf{x}}_{fN}^{+}$ $P_N = P_{fN}^{+}$
Gain	$\mathcal{K}_k \equiv P_{fk}^{+} \Phi_k^T (P_{fk+1}^{-})^{-1}$
Covariance	$P_k = P_{fk}^{+} - \mathcal{K}_k [P_{fk+1}^{-} - P_{k+1}] \mathcal{K}_k^T$
Estimate	$\hat{\mathbf{x}}_k = \hat{\mathbf{x}}_{fk}^{+} + \mathcal{K}_k [\hat{\mathbf{x}}_{k+1} - \hat{\mathbf{x}}_{fk+1}^{-}]$

- Note how much simpler the backwards equations are compared to the other smoother
- The covariance expression is not needed either to obtain the state estimate
- In this form Q_k can be positive semi-definite, not just non-singular!

RTS Smoother Stability (i)

- As before, we consider only the homogenous part to prove stability

$$\hat{\mathbf{x}}_k = P_{fk}^{+}\Phi_k^T \mathcal{P}_{fk+1}^{-}\hat{\mathbf{x}}_{k+1}$$

  - Note that this is a backwards recursion
  - Consider the following candidate Lyapunov function

$$V(\hat{\mathbf{x}}) = \hat{\mathbf{x}}_{k+1}^T \mathcal{P}_{fk+1}^{+}\hat{\mathbf{x}}_{k+1}$$

  - The increment, now going backwards in time, is given by

$$\Delta V(\hat{\mathbf{x}}) = \hat{\mathbf{x}}_k^T \mathcal{P}_{fk}^{+}\hat{\mathbf{x}}_k - \hat{\mathbf{x}}_{k+1}^T \mathcal{P}_{fk+1}^{+}\hat{\mathbf{x}}_{k+1}$$

  - Substitute the above smoother state equation

$$\Delta V(\hat{\mathbf{x}}) = \hat{\mathbf{x}}_{k+1}^T \left[\mathcal{P}_{fk+1}^{-}\Phi_k P_{fk}^{+}\Phi_k^T \mathcal{P}_{fk+1}^{-} - \mathcal{P}_{fk+1}^{+}\right]\hat{\mathbf{x}}_{k+1}$$

  - Next, rewrite the forward-time covariance prediction as

$$P_{fk}^{+} = \Phi_k^{-1}P_{fk+1}^{-}\Phi_k^{-T} - \Phi_k^{-1}\Upsilon_k Q_k \Upsilon_k^T \Phi_k^{-T}$$

RTS Smoother Stability (ii)

- Substitute this into the increment to give

$$\Delta V(\hat{\mathbf{x}}) = \hat{\mathbf{x}}_{k+1}^T \left[\mathcal{P}_{f\,k+1}^- - \mathcal{P}_{f\,k+1}^- \Upsilon_k Q_k \Upsilon_k^T \mathcal{P}_{f\,k+1}^- - \mathcal{P}_{f\,k+1}^+ \right] \hat{\mathbf{x}}_{k+1}$$

- Using a one time-step ahead of the following equation

$$\mathcal{P}_{f\,k}^+ = \mathcal{P}_{f\,k}^- + H_k^T R_k^{-1} H_k$$

gives

$$\mathcal{P}_{f\,k+1}^+ = \mathcal{P}_{f\,k+1}^- + H_{k+1}^T R_{k+1}^{-1} H_{k+1}$$

- Substitute this into the increment to give

$$\Delta V(\hat{\mathbf{x}}) = -\hat{\mathbf{x}}_{k+1}^T \left[H_{k+1}^T R_{k+1}^{-1} H_{k+1} + \mathcal{P}_{f\,k+1}^- \Upsilon_k Q_k \Upsilon_k^T \mathcal{P}_{f\,k+1}^- \right] \hat{\mathbf{x}}_{k+1}$$

- Clearly, if R_{k+1} is positive definite and Q_k is at least positive semi-definite, then the Lyapunov condition is satisfied
  - So the discrete-time RTS smoother is stable
  - Similar conditions as in the Kalman filter

Example (i)

- Single-axis attitude estimation problem from before

$$\mathbf{x}_{k+1} = \Phi\,\mathbf{x}_k + \Gamma\,\tilde{\omega}_k + \mathbf{w}_k$$

$$\tilde{y}_k = H\,\mathbf{x}_k + v_k$$

$$\tilde{\omega} = \dot{\theta} + \beta + \eta_v$$

$$\dot{\beta} = \eta_u$$

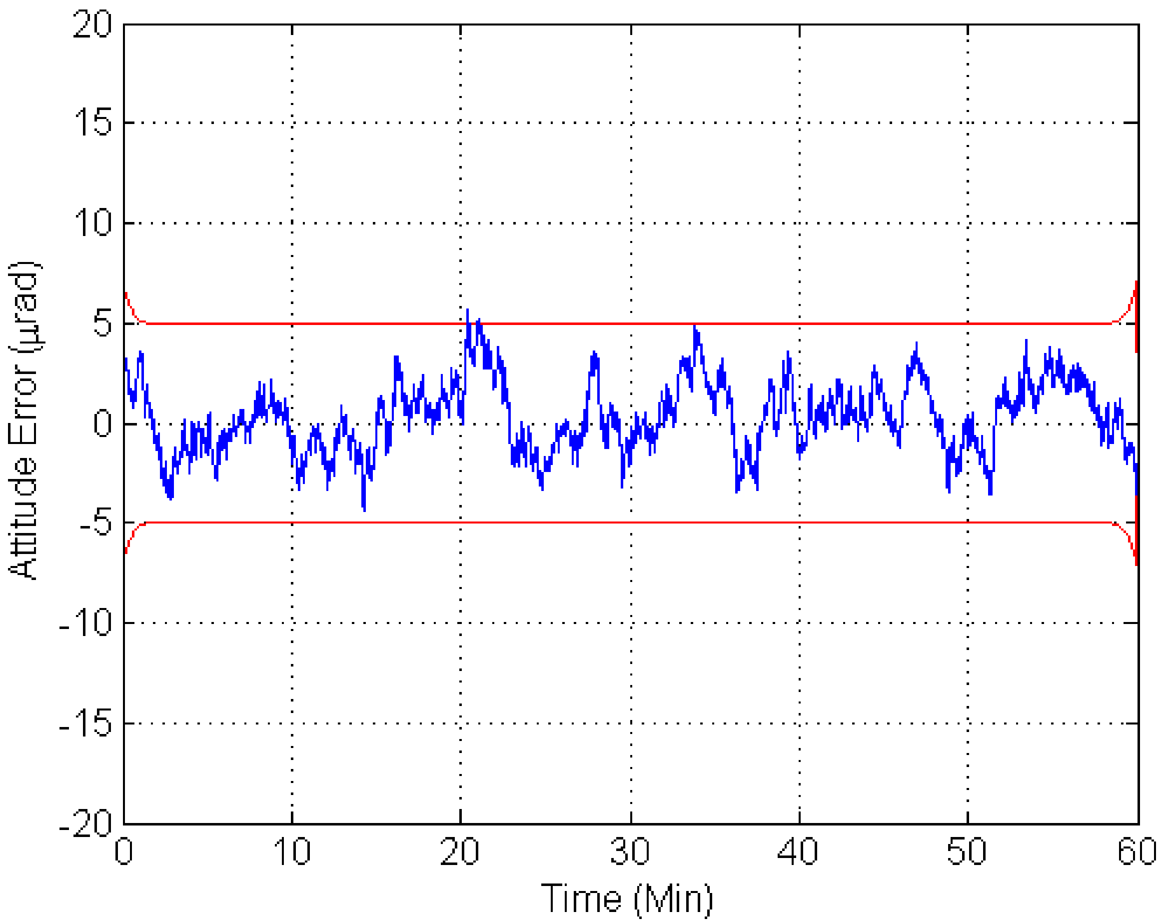
where $\mathbf{x} = \begin{bmatrix}\theta & \beta\end{bmatrix}^T$, $\Gamma = \begin{bmatrix}\Delta t & 0\end{bmatrix}^T$, $H = \begin{bmatrix}1 & 0\end{bmatrix}$, $E\left\{\mathbf{w}_k \mathbf{w}_k^T\right\} = Q$,
and $E\left\{v_k^2\right\} = \sigma_n^2$

$$Q = \begin{bmatrix} \sigma_v^2 \Delta t + \frac{1}{3}\sigma_u^2 \Delta t^3 & -\frac{1}{2}\sigma_u^2 \Delta t^2 \\ -\frac{1}{2}\sigma_u^2 \Delta t^2 & \sigma_u^2 \Delta t \end{bmatrix}$$

- Ran a time-varying case but also compared the computed 3σ bounds to the steady state solution from the smoother
  - Solve the following Lyapunov equation

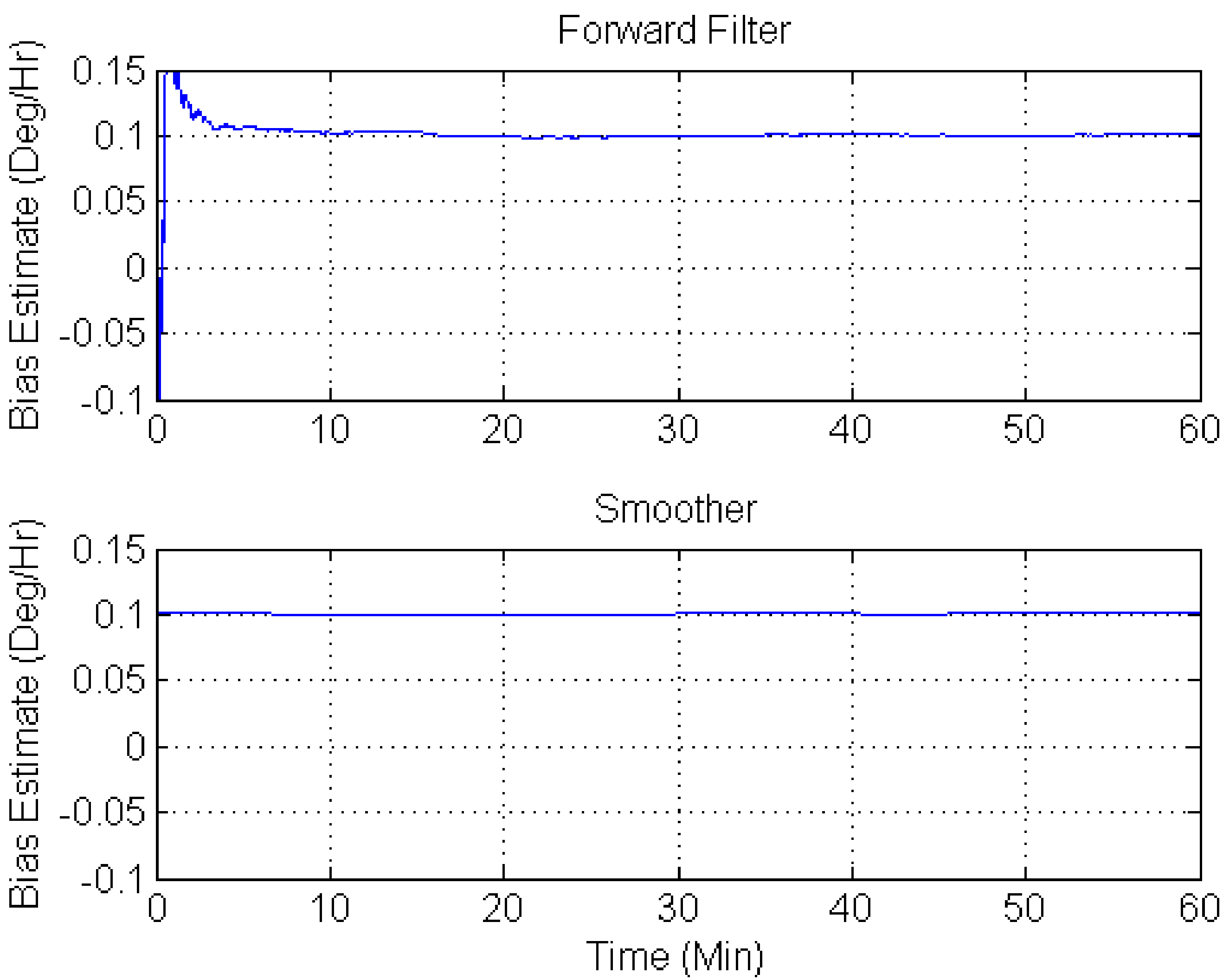
$$P = K\,P\,K^T + \left(P_f^+ - K\,P_f^- K^T\right), \quad K = P_f^+ \Phi^T \mathcal{P}_f^-$$

Example (ii)



From the steady-state covariance a 3σ attitude bound of 4.9216 μrad is found, which is verified by figure

Example (iii)


Forward Filter
Bias Estimate (Deg/Hr)
0.15
0.1
0.05
0
-0.05
-0.1
0
10
20
30
40
50
60
Smoother
Bias Estimate (Deg/Hr)
0.15
0.1
0.05
0
-0.05
-0.1
0
10
20
30
40
50
60
Time (Min)

Example (iv)

```
% True Rate
dt=1;tf=3600;t=[0:dt:tf]';m=length(t);
wtrue=0.0011;

% Gyro and Attitude Parameters
sigu=sqrt(10)*1e-10;
sigv=sqrt(10)*1e-7;
sign=17*1e-6;

% Measurements with Bias
ym=t*wtrue+sign*randn(m,1);
bias=lsim(0,1,1,0,sigu*randn(m,1),t,0.1*pi/180/3600);
wm=wtrue+sigv*randn(m,1)+bias;

% Discrete-Time Process Noise Covariance
q=[sigv^2*dt+1/3*sigu^2*dt^3 -1/2*sigu^2*dt^2;-
1/2*sigu^2*dt^2 sigu^2*dt];
phi=[1 -dt;0 1];gam=[dt;0];
```

Example (v)

```
% Initial Covariance
poa=1e-4;
pog=1e-12;
p=[poa 0;0 pog];
pcov=zeros(m,2);pcov(1,:)=[poa pog];

% Initial Condition and H Matrix (constant)
x0=[wm(1);0];xe=zeros(m,2);xe(1,:)=x0';x=x0;
h=[1 0];

% Preallocate Variables
xup=zeros(m,2);xprop=zeros(m,2);
pup=zeros(m,3);pprop=zeros(m,3);
psmooth=zeros(m,2);
xs=zeros(m,2);
```

Example (vi)

```
% Main Forward Loop
for i = 1:m-1

% Kalman Gain
gain=p*h'*inv(h*p*h'+sign^2);

% Update
x=x+gain*(ym(i)-h*x);
p=[eye(2)-gain*h]*p;
xup(i+1,:)=x';pup(i+1,:)=[p(1,1) p(1,2) p(2,2)];

% Propagate
x=phi*x+gam*wm(i);
p=phi*p*phi'+q;
xprop(i+1,:)=x';pprop(i+1,:)=[p(1,1) p(1,2) p(2,2)];

% Store Variables
xe(i+1,:)=x';pcov(i+1,:)=diag(p)';

end
```

Example (vii)

```
% Initialize RTS Smoother
xs(m,:)=xe(m,:);
ps=p;

% Main Backward Loop
for i=m:-1:2,

% RTS Gain
pf_plus=[pup(i,1) pup(i,2);pup(i,2) pup(i,3)];
pf_minus=[pprop(i,1) pprop(i,2);pprop(i,2) pprop(i,3)];
gain_s=pf_plus*phi'*inv(pf_minus);

% Smoother State and Covariance
xs(i-1,:)=xup(i,:)+(gain_s*(xs(i,:)-xprop(i,:))')';
ps=pf_plus-gain_s*(pf_minus-ps)*gain_s';

% Store RTS Covariance
psmooth(i-1,:)=diag(ps)';

end
```

Example (viii)

```
% 3-Sigma Outliers
sig3=pcov.^(0.5)*3;
sig3s=psmooth.^(0.5)*3;

plot(t/60,[sig3s(:,1) xs(:,1)-t*wtrue -sig3s(:,1)]*1e6)
axis([0 60 -20 20]);grid
set(gca,'Fontsize',12);set(gca,'Fontname','Helvetica');
xlabel('Time (Min)');
hh=get(gca,'Ylabel');
set(hh,'String','\fontsize{12} {Attitude Error ({\mu}rad)}');

disp(' Press any key to continue')
pause
```

Example (ix)

```
subplot(211)
plot(t/60,xe(:,2)*180*3600/pi);grid
axis([0 60 -0.1 0.15])
set(gca,'Ytick',[-0.1 -0.05 0 0.05 0.1 0.15])
set(gca,'Fontsize',12);set(gca,'Fontname','Helvetica');
xlabel('Time (Min)');
ylabel('Bias Estimate (Deg/Hr)')

subplot(212)
plot(t/60,xs(:,2)*180*3600/pi);grid
axis([0 60 -0.1 0.15])
set(gca,'Ytick',[-0.1 -0.05 0 0.05 0.1 0.15])
set(gca,'Fontsize',12);set(gca,'Fontname','Helvetica');
xlabel('Time (Min)');
ylabel('Bias Estimate (Deg/Hr)')
```

Example (x)

```
% Compute Steady-State Covariances
pf_minus_steady=dare(phi',h',q,sign^2);
gainnn=pf_minus_steady*h'*inv(h*pf_minus_steady*h'+sign^2);
pf_plus_steady=[eye(2)-gainnn*h]*pf_minus_steady;

gainss=pf_plus_steady*phi'*inv(pf_minus_steady);
ps_steady=dlyap(gainss,pf_plus_steady-gainss*pf_minus_steady*gainss');
```