ECE 602: LUMPED LINEAR SYSTEMS

Professor Stan Żak

Canonical Forms of Single-Input Single-Output (SISO) Linear Systems

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- **Objective**: Introduce canonical forms of linear lumped both continuous-time (CT) and discrete-time (DT) systems. We will use these special forms to easily design controllers and observers for SISO systems
- We consider linear time-varying (LTV) systems

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}u(t) \\ y(t) &= \boldsymbol{c}\boldsymbol{x}(t) \end{aligned}$$

or

$$\begin{aligned} \boldsymbol{x}[k+1] &= \boldsymbol{A}\boldsymbol{x}[k] + \boldsymbol{b}u[k] \\ y[k] &= \boldsymbol{c}\boldsymbol{x}[k] \end{aligned}$$

- We assume that the system is both reachable and observable

Controller form

- Consider a linear time-invariant (LTI) CT system

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}u(t),$$

or LTI DT system

$$\boldsymbol{x}[k+1] = \boldsymbol{A}\boldsymbol{x}[k] + \boldsymbol{b}u[k]$$

where the pair $(\boldsymbol{A}, \boldsymbol{b})$ is reachable

- This means that

$$\operatorname{rank} \begin{bmatrix} \boldsymbol{b} & \boldsymbol{A}\boldsymbol{b} & \cdots & \boldsymbol{A}^{n-1}\boldsymbol{b} \end{bmatrix} = n.$$

- Select the last row of the inverse of the controllability matrix, let $\boldsymbol{q}_1$ be that row

Constructing the similarity transformation that brings the system into controller form

- Form the matrix

$$\boldsymbol{T} = \begin{bmatrix} \boldsymbol{q}_1 \\ \boldsymbol{q}_1\boldsymbol{A} \\ \vdots \\ \boldsymbol{q}_1\boldsymbol{A}^{n-1} \end{bmatrix}$$

- $\boldsymbol{T}$ is invertible. Indeed, since

$$\boldsymbol{T}\begin{bmatrix} \boldsymbol{b} & \boldsymbol{A}\boldsymbol{b} & \cdots & \boldsymbol{A}^{n-1}\boldsymbol{b} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & x \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & x & \cdots & x & x \end{bmatrix},$$

this implies that $\boldsymbol{T}$ must be nonsingular, where the symbol x means "don't care"scalars not important in the analysis

Transforming the input vector b

- The state variable transformation $\tilde{\boldsymbol{x}} = \boldsymbol{T}\boldsymbol{x}$
- The system in the new coordinates

$$\dot{\tilde{\boldsymbol{x}}}(t) = \boldsymbol{T}\boldsymbol{A}\boldsymbol{T}^{-1}\tilde{\boldsymbol{x}}(t) + \boldsymbol{T}\boldsymbol{b}u(t) = \tilde{\boldsymbol{A}}\tilde{\boldsymbol{x}}(t) + \tilde{\boldsymbol{b}}u(t)$$

- The matrices $\tilde{\boldsymbol{A}}$ and $\tilde{\boldsymbol{b}}$ have particular structures
- We first analyze the structure of $\tilde{\boldsymbol{b}}$
- Note that $\boldsymbol{q}_1$ is the last row of the inverse of the controllability matrix
- Therefore, $\boldsymbol{q}_1\boldsymbol{b} = \boldsymbol{q}_1\boldsymbol{A}\boldsymbol{b} = \cdots = \boldsymbol{q}_1\boldsymbol{A}^{n-2}\boldsymbol{b} = 0 \; \boldsymbol{q}_1\boldsymbol{A}^{n-1}\boldsymbol{b} = 1$
- Hence,

$$\tilde{\boldsymbol{b}} = \boldsymbol{T}\boldsymbol{b} = \begin{bmatrix} \boldsymbol{q}_1\boldsymbol{b} \\ \vdots \\ \boldsymbol{q}_1\boldsymbol{A}^{n-2}\boldsymbol{b} \\ \boldsymbol{q}_1\boldsymbol{A}^{n-1}\boldsymbol{b} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

The structure of $\tilde{A}$

- Represent $\boldsymbol{T}\boldsymbol{A}\boldsymbol{T}^{-1} = \tilde{\boldsymbol{A}}$ as

$$\boldsymbol{T}\boldsymbol{A} = \tilde{\boldsymbol{A}}\boldsymbol{T}$$

- The left-hand side

$$\boldsymbol{T}\boldsymbol{A} = \begin{bmatrix} \boldsymbol{q}_1\boldsymbol{A} \\ \boldsymbol{q}_1\boldsymbol{A}^2 \\ \vdots \\ \boldsymbol{q}_1\boldsymbol{A}^{n-1} \\ \boldsymbol{q}_1\boldsymbol{A}^n \end{bmatrix}$$

- By the Cayley-Hamilton theorem

$$\boldsymbol{A}^n = -a_0\boldsymbol{I}_n - a_1\boldsymbol{A} - \cdots - a_{n-1}\boldsymbol{A}^{n-1},$$

and hence

$$\boldsymbol{q}_1\boldsymbol{A}^n = -a_0\boldsymbol{q}_1 - a_1\boldsymbol{q}_1\boldsymbol{A} - \cdots - a_{n-1}\boldsymbol{q}_1\boldsymbol{A}^{n-1}$$

The structure of $\tilde{A}$—Contd

- Compare both sides of $\boldsymbol{TA} = \tilde{\boldsymbol{A}}\boldsymbol{T}$, and take into account the Cayley-Hamilton theorem to obtain

$$\tilde{\boldsymbol{A}} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix}$$

Controller form

- We say that the pair

$$(\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{b}}) = \left(\begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \right)$$

  is in the controller form
- This form is also labeled in the literature as the controllable canonical form. We use the shorter label
- Note that the coefficients of the characteristic polynomial of $\boldsymbol{A}$ are immediately apparent by inspecting the last row of $\tilde{\boldsymbol{A}}$

Example

- System model:

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} u$$

- The controllability matrix of the pair $(\boldsymbol{A}, \boldsymbol{b})$

$$\begin{bmatrix} \boldsymbol{b} & \boldsymbol{A}\boldsymbol{b} & \boldsymbol{A}^2\boldsymbol{b} & \boldsymbol{A}^3\boldsymbol{b} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

- The pair $(\boldsymbol{A}, \boldsymbol{b})$ is controllable because the controllability matrix is nonsingular
- The last row of the inverse of the controllability matrix to be $\boldsymbol{q}_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$

Transforming the matrix A into the new coordinates

- The transformation matrix is

$$T = \begin{bmatrix} q_1 \\ q_1 A \\ q_1 A^2 \\ q_1 A^3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

- The matrix A in the new coordinates

$$\begin{aligned} TAT^{-1} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -3 & 0 \end{bmatrix} \end{aligned}$$

The input matrix b in the new coordinates

$$\boldsymbol{T}\boldsymbol{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Transforming the system into the observer form

- Consider an observable dynamical system model

$$\left.\begin{array}{rcl} \dot{\boldsymbol{x}}(t) & = & \boldsymbol{A}\boldsymbol{x}(t) \\ \boldsymbol{y}(t) & = & \boldsymbol{c}\boldsymbol{x}(t) \end{array}\right\}$$

- The pair $(\boldsymbol{A}, \boldsymbol{c})$ is observable
- We omitted the input variable $\boldsymbol{u}$ for convenience because it plays no role in the subsequent discussion
- Consider the dual system

$$\dot{\boldsymbol{z}}(t) = \boldsymbol{A}^\top \boldsymbol{z}(t) + \boldsymbol{C}^\top \boldsymbol{v}(t)$$

- Because the pair $(\boldsymbol{A}, \boldsymbol{c})$ is observable, the dual pair $\left(\boldsymbol{A}^\top, \boldsymbol{c}^\top\right)$ is reachable
- Therefore, the pair $\left(\boldsymbol{A}^\top, \boldsymbol{c}^\top\right)$ can be transformed into the controller form

Observer form

Take the dual of the result to get

$$
\begin{aligned}
\dot{\hat{\boldsymbol{x}}}(t) &= \hat{\boldsymbol{A}}\hat{\boldsymbol{x}} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & 0 & \cdots & 0 & -a_1 \\ \vdots & \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -a_{n-2} \\ 0 & 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix} \\
\boldsymbol{y}(t) &= \hat{\boldsymbol{C}}\hat{\boldsymbol{x}} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \hat{\boldsymbol{x}}
\end{aligned}
$$