$G(s) = \frac{\omega^2}{s^2+2\zeta\omega s+\omega^2}$ ω - natural frequency
ζ - Damping ratio $0 < \zeta < 1$

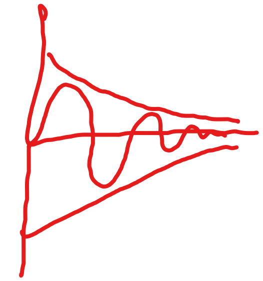
$\frac{C(s)}{R(s)} = G = \frac{\omega^2}{(s+\omega\zeta)^2+(\omega\sqrt{1-\zeta^2})^2} \Rightarrow$

Poles: $s_{1,2} = \frac{-2\zeta\omega \pm \sqrt{(2\zeta\omega)^2 - 4\omega^2}}{2}$

$s = -\zeta\omega \pm i\omega\sqrt{1-\zeta^2}$

$C(t) = \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega t} \sin(\omega\sqrt{1-\zeta^2}\, t)$

Transient Response



*If the output of control systems for an input (step, ramp, etc.) varies with respect to time, it is called the **time response** of the control system. The time response consists of two parts:*

- *Transient Response*
- *Steady State Response*

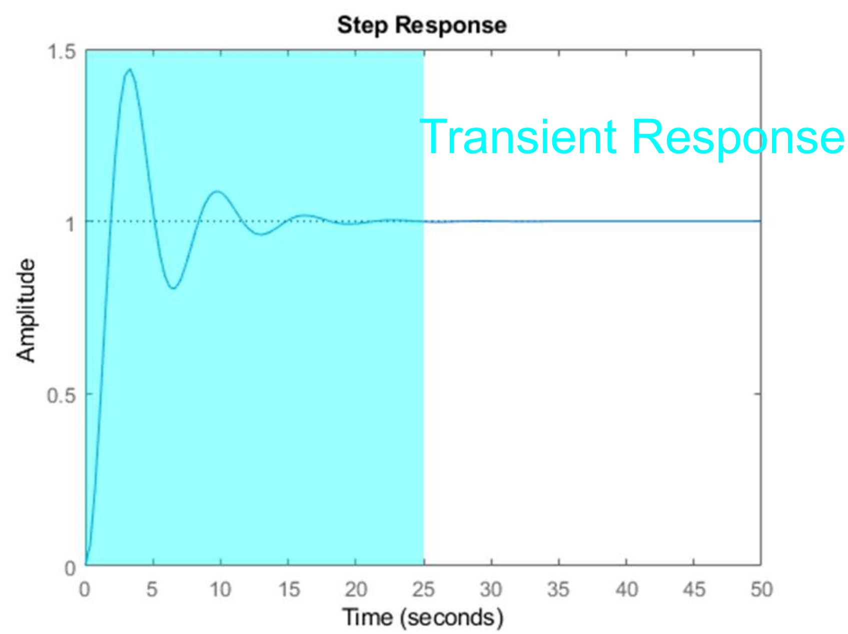
***Transient response** corresponds to the behavior of the system from the initial state to the final state.*

Ex) $m\ddot{x} + c\dot{x} + kx = \frac{1}{2}u$

$\frac{X}{U} = \frac{1/2}{ms^2+cs+k}$

***Steady State response** is the part of the time response that as time tends to infinity*

- Step Response is the time response when the input is a step
- Impulse response is the time response when the input is an impulse



First Order System

Transfer function:

$$\frac{Y(s)}{R(s)} = \frac{1}{Ts+1}$$

Response to a unit step input is: step : $R = \frac{1}{s}$

$$Y(s) = \frac{1}{Ts+1}\frac{1}{s}$$

Partial Fraction Expansion leads to: $Y = \frac{a_0}{s} + \frac{a_1}{Ts+1}$

$$Y(s) = \frac{1}{s} - \frac{T}{Ts+1} = \frac{1}{s} - \frac{1}{s+1/T}$$

Inverse Laplace transform leads to:

$$y(t) = 1 - e^{-\frac{t}{T}}$$

time constant

First Order System

At t = T, the output is:

$$y(t) = 1 - exp(-1) = 0.632$$

↑ 1 time constant

*T represents the time required for the system response to reach 63.2% of the final value. T is referred to as the **Time Constant** of the system*

The slope of the system response at time = 0 is:

$$\frac{dy(t)}{dt} = \frac{1}{T}e^{-\frac{0}{T}} = \frac{1}{T}$$

Response of the first order system to a unit ramp is: $R(s) = 1/s^2$

$$Y(s) = \frac{1}{Ts+1}\frac{1}{s^2} = \frac{1}{s^2} - \frac{T}{s} + \frac{T^2}{Ts+1}$$

$$y(t) = t - T + Te^{-\frac{t}{T}}$$

Tracking error as time tends to infinity is:

$$e(t \to \infty) = t - y(t) = T - \cancel{Te^{-\frac{t}{T}}} = T$$

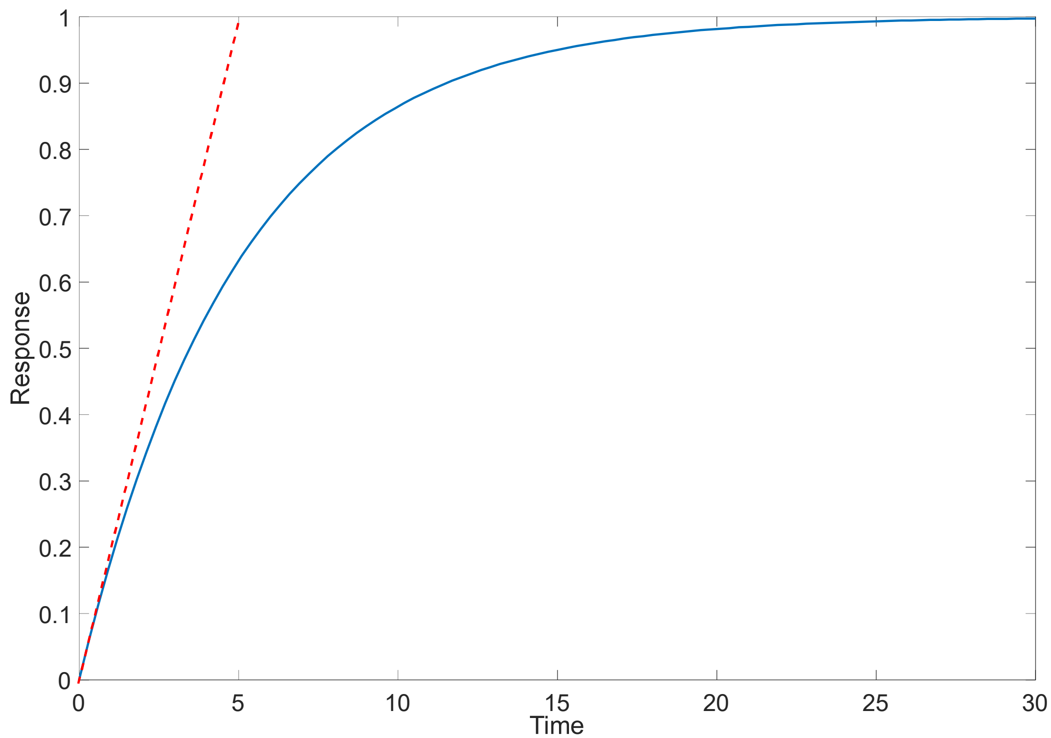
$\to 0$ as $t \to \infty$

First Order System

As has been stated, the slope of the system response at time = 0 is:

$$\frac{dy(t)}{dt} = \frac{1}{T}e^{-\frac{0}{T}} = \frac{1}{T}$$

For a system with T = 5, as seen from the figure, the time constant represents the elapsed time required for the system response to reach the final value, if the system had continued to decay at the initial rate.

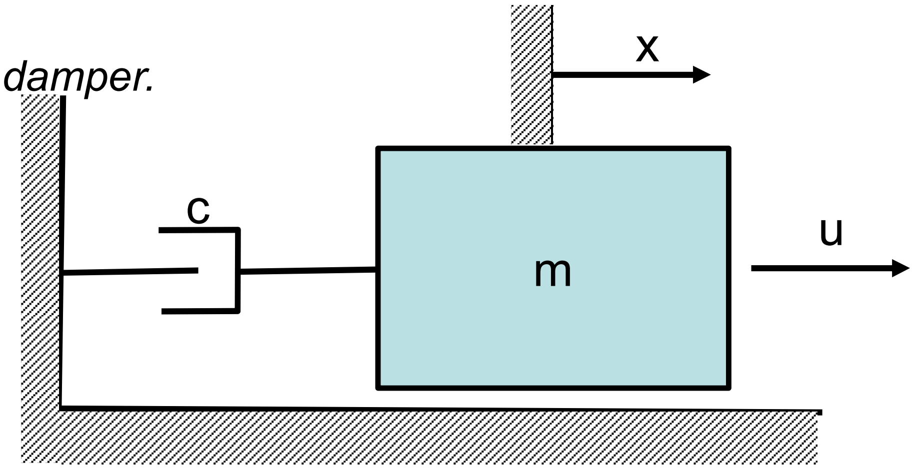


$m\ddot{x} + c\dot{x} = 0$

$\ddot{x} = \dot{v}$ $\dot{x} = v$

First Order System

Consider a mass attached to a wall via a damper.



$$m\dot{v} + cv = 0, \quad v(0) = v_0$$

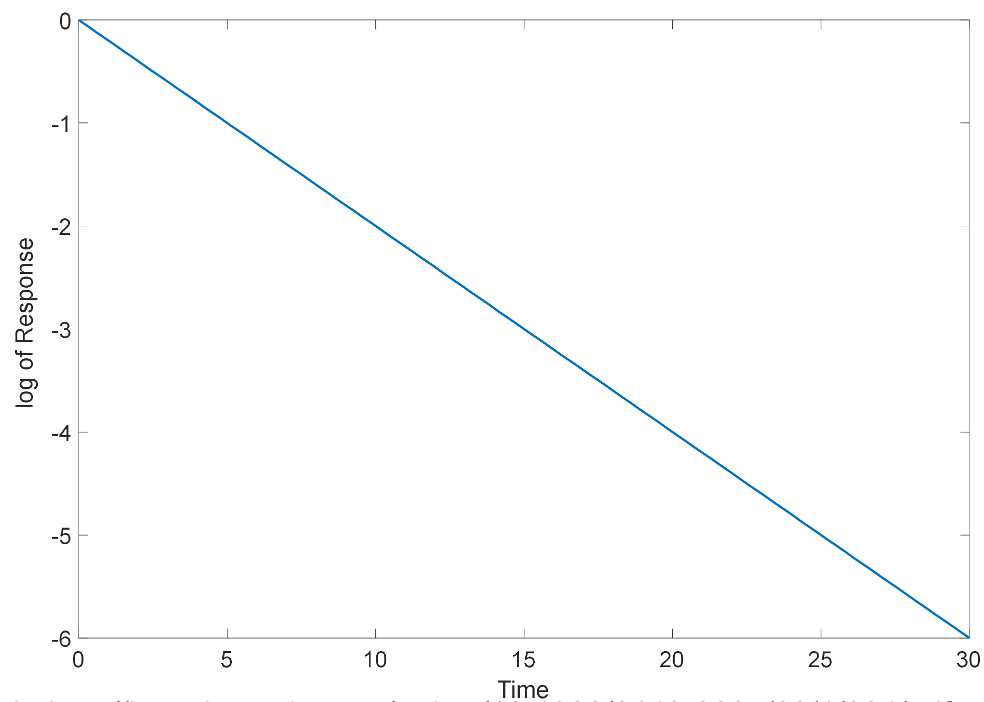
$$V(s) = \frac{mv_0}{ms + c}$$

$$v(t) = v_0 exp(-\frac{c}{m}t)$$

$$log(v(t)) = log(v_0) - \frac{c}{m}t$$

It is clear that the relationship between the natural log of v(t) versus time (t) is a straight line.

Consequently, one can plot the log of an evolving output versus time and if the curve is a straight line, that implies that the process is first order.



Demonstration of the exponential decay law using beer froth, http://iopscience.iop.org/article/10.1088/0143-0807/23/1/304/pdf

First Order System

Half-Life*: Time it takes for a quantity to reduce to half its initial value. For this example, velocity to decrease to half of its initial value.*

The log of the evolving output v(t) is given by the equation:

$$log(v(t)) = log(v_0) - \frac{c}{m}t$$

which can be rewritten as the equation:

$$log(v(t)) - log(v_0) = log(\frac{v(t)}{v_0}) = -\frac{c}{m}t$$

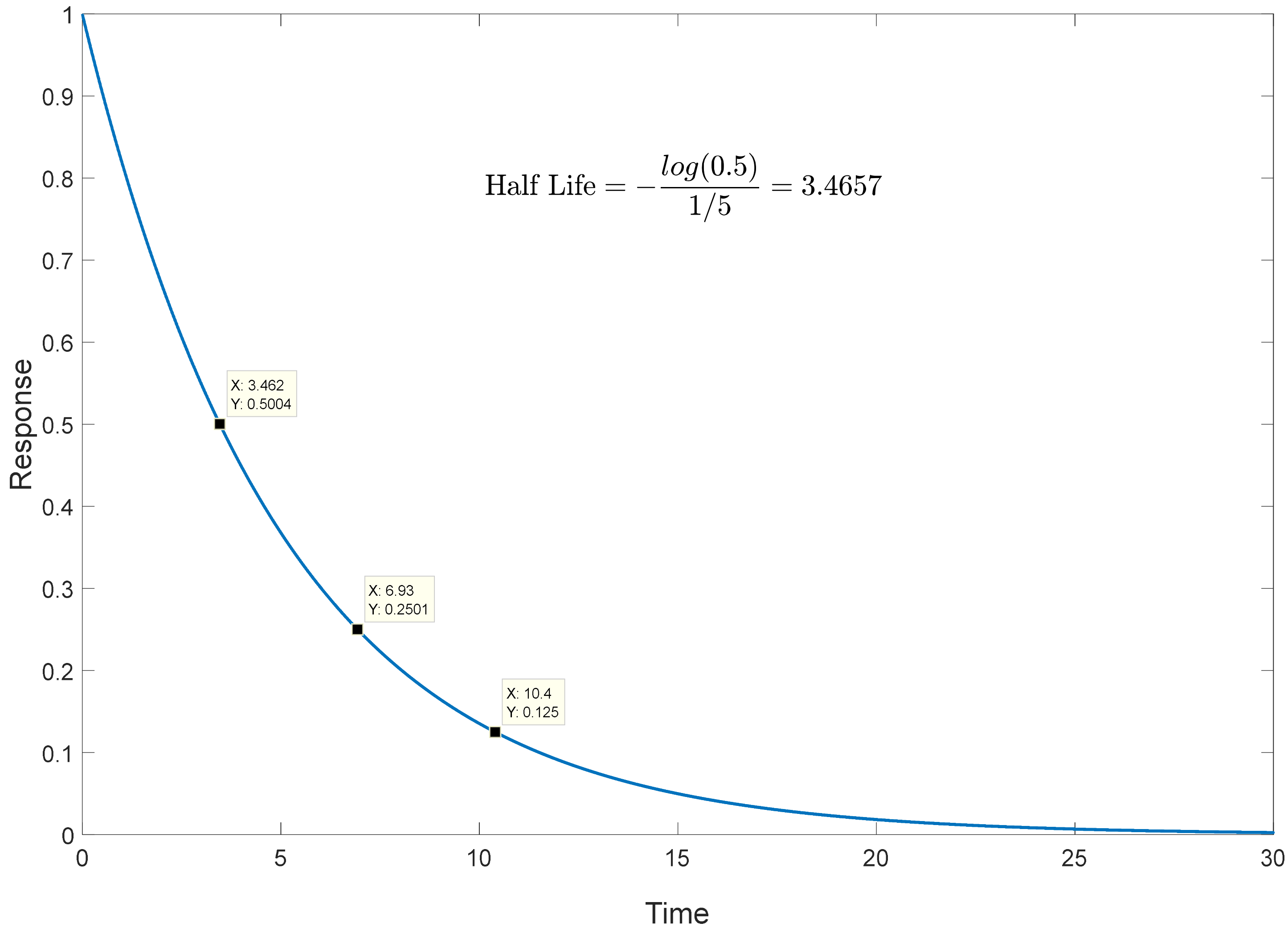
Have life is the time required for the output to reach half its initial value, therefore:

$$log(\frac{v(t)}{v_0}) = log(\frac{1}{2}) = -\frac{c}{m}t$$

First Order System

$$m\dot{v} + cv = 0, \quad v(0) = v_0$$

m = 5, c = 1, v_0 *= 1.*



First Order System

A biological half-life or elimination half-life is the time it takes for a substance (drug, radioactive nuclide, or other) to lose one-half of its pharmacologic, physiologic, or radiological activity. In a medical context, the half-life may also describe the time that it takes for the concentration of a substance in blood plasma to reach one-half of its steady-state value (the "plasma half-life").

The converse of half-life is doubling time. It is applied to population growth, inflation, resource extraction, consumption of goods, compound interest, the volume of malignant tumours, and many other things that tend to grow over time. When the relative growth rate (not the absolute growth rate) is constant, the quantity undergoes exponential growth and has a constant doubling time or period, which can be calculated directly from the growth rate.

https://en.wikipedia.org/wiki/Half-life,https://en.wikipedia.org/wiki/Doubling_time

Second Order System

Transfer function:

$$\frac{Y(s)}{R(s)} = \frac{\omega^2}{s^2 + 2\zeta\omega s + \omega^2}$$

ω	natural frequency
ζ	damping ratio

$$\frac{Y(s)}{R(s)} = \frac{\omega^2}{(s+\zeta\omega)^2 + (\omega\sqrt{1-\zeta^2})^2}$$

Response to a unit step input is: $R = Y_s$

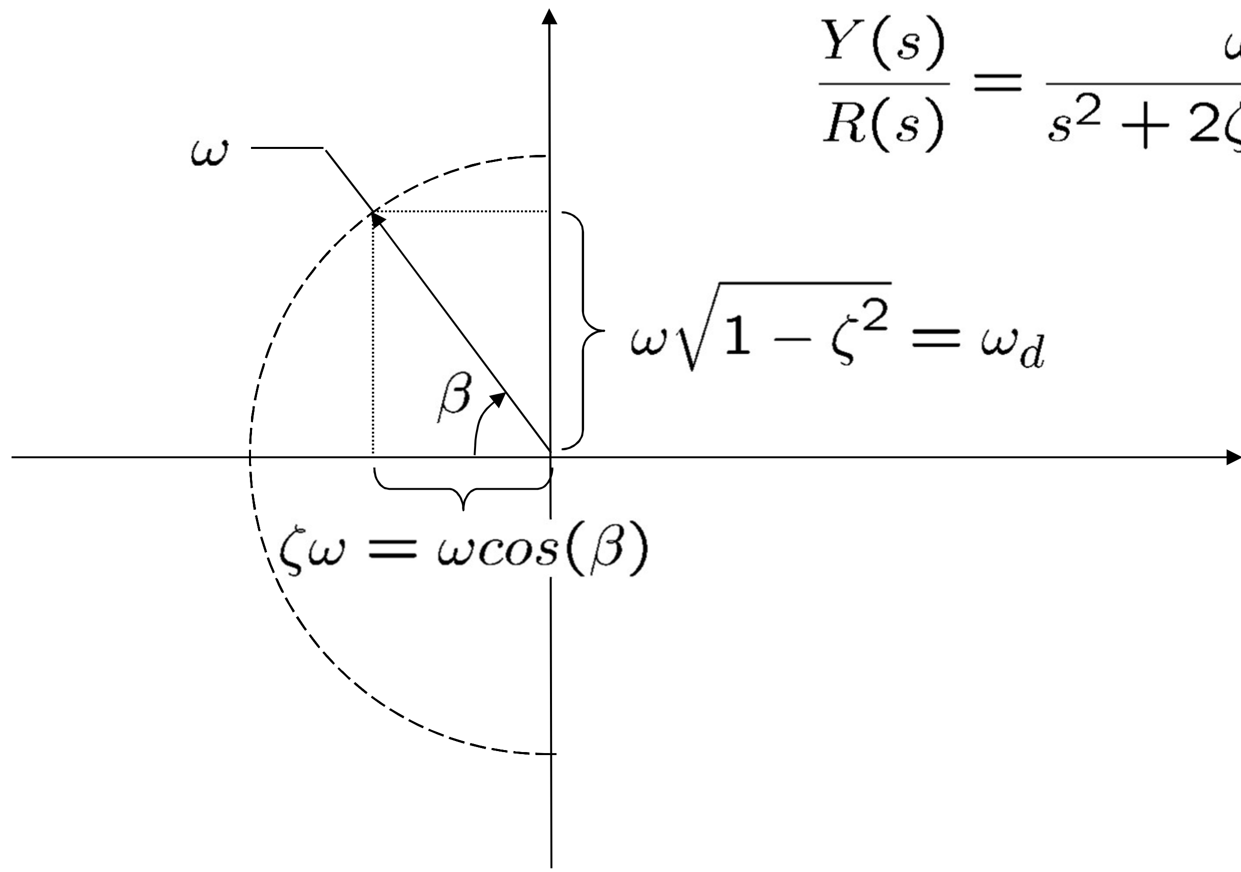
$$Y(s) = \frac{\omega^2}{(s+\zeta\omega)^2 + (\omega\sqrt{1-\zeta^2})^2}\frac{1}{s} = \frac{a_0}{s} + \frac{a_1 s + a_2}{s^2 + 2\zeta\omega s + \omega^2}$$

$$y(t) = 1 - e^{-\zeta\omega t}\left(\cos(\omega_d t) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_d t)\right)$$

S-plane representation

Laplace transform with zero initial conditions leads to:

$$\frac{Y(s)}{R(s)} = \frac{\omega^2}{s^2 + 2\zeta\omega s + \omega^2}$$



Poles Located at: $s = -\zeta\omega \pm j\omega\sqrt{1-\zeta^2}$

Transient Response

In analyzing and designing control systems, we must have a basis of comparison of performance of various control systems. This basis may be set up by specifying particular test input signals and by comparing the response of various systems to these input signals. Typical test signals: Step function, ramp function, impulse function, sinusoid function.

The time response of a control system consists of two parts: the transient and the steady-state response. Transient response corresponds to the behavior of the system from the initial state to the final state. By steady state, we mean the manner in which the system output behaves as time approaches infinity.

For a step input, the transient response can be characterized by:

Delay time t_d: time to reach half the final value for the first time.

Rise time t_r: time required for the response to rise from 10% to 90% for overdamped systems, and from 0% to 100% for underdamped systems

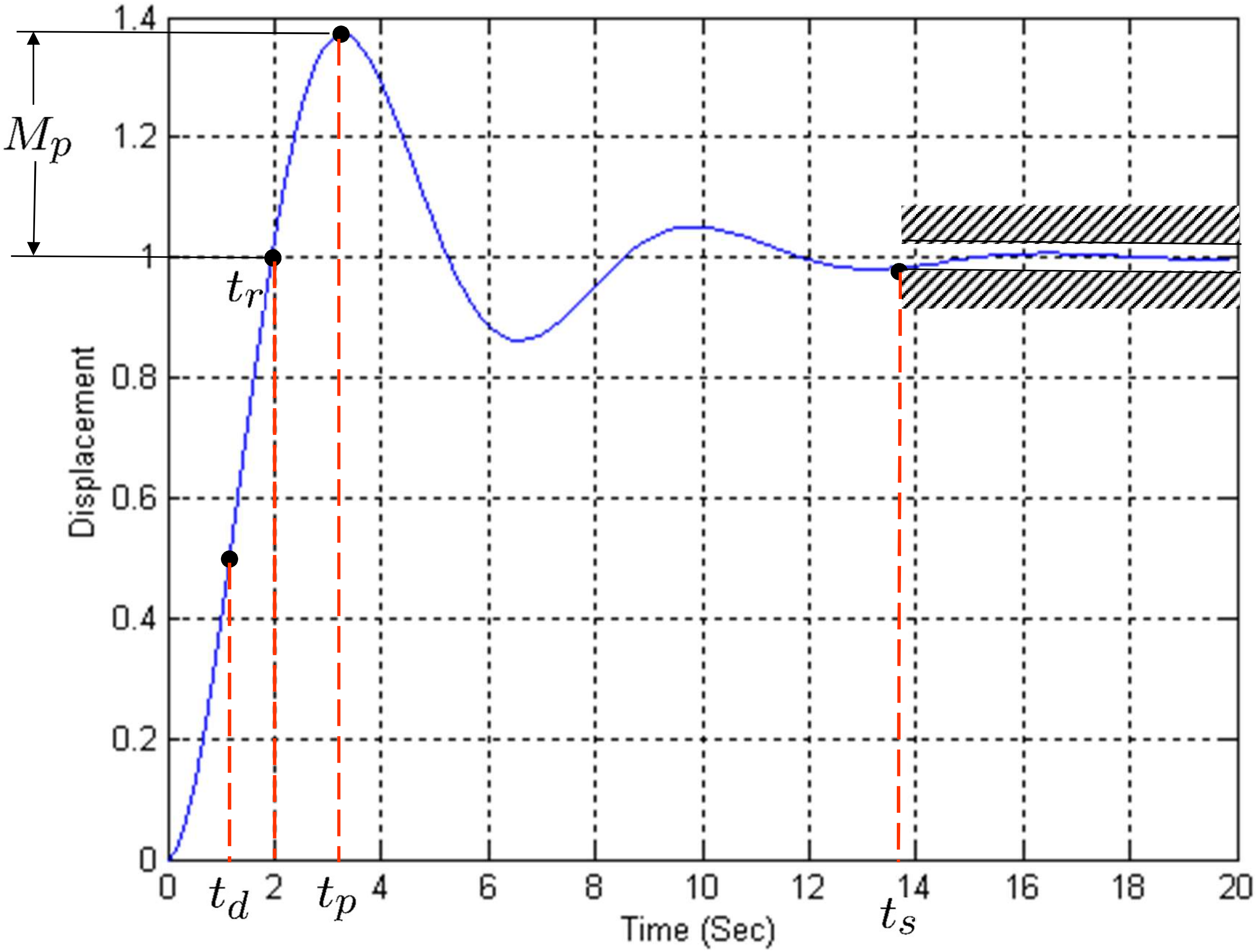
Peak time t_p: time required to reach the first peak of the overshoot

Percent Overshoot M_p:

$$M_p = \frac{y(t_p) - y(\infty)}{y(\infty)}$$

Settling time t_s: time required for the response curve to reach and stay within 2% or 5% of the final value. Is a function of the largest time constant of the control system.

Transient Response



Transient Response

For a step input, the transient response can be characterized by:

Rise time t_r: time required for the response to rise from 10% to 90% for overdamped systems, and from 0% to 100% for underdamped systems

$$y(t_r) = 1 = 1 - e^{-\zeta\omega t_r}\left(cos(\omega_d t_r) + \frac{\zeta}{\sqrt{1-\zeta^2}} sin(\omega_d t_r)\right)$$

Since $e^{-\zeta\omega t_r} \neq 0$

$$cos(\omega_d t_r) + \frac{\zeta}{\sqrt{1-\zeta^2}} sin(\omega_d t_r) = 0$$

$$\Rightarrow tan(\omega_d t_r) = -\frac{\sqrt{1-\zeta^2}}{\zeta} = -\frac{\omega_d}{\sigma}$$

Thus, the rise time is:

$$t_r = \frac{1}{\omega_d} tan^{-1}\left(\frac{\omega_d}{-\sigma}\right) = \frac{\pi - \beta}{\omega_d}$$

Transient Response

For a step input, the transient response can be characterized by:

Peak time t_p: time required to reach the first peak of the overshoot

$$\frac{dy}{dt} = 0 = \zeta\omega e^{-\zeta\omega t}\left(\cancel{cos(\omega_d t)} + \frac{\zeta}{\sqrt{1-\zeta^2}} sin(\omega_d t)\right)$$
$$- e^{-\zeta\omega t}\left(-\omega_d \sin(\omega_d t) + \frac{\omega_d \zeta}{\sqrt{1-\zeta^2}} \cancel{\cos(\omega_d t)}\right)$$

$$\frac{dy}{dt} = 0 = e^{-\zeta\omega t_p} sin(\omega_d t_p) \frac{\omega}{\sqrt{1-\zeta^2}}$$

Thus, the peak time is:

$$\omega_d t_p = 0, \pi, 2\pi, \ldots$$

$$t_p = \frac{\pi}{\omega_d}$$

Transient Response

For a step input, the transient response can be characterized by:

Percent Overshoot M_p:

$$M_p = y(t_p) - 1 = -e^{-\zeta\omega\frac{\pi}{\omega_d}}\left(cos(\pi) + \frac{\zeta}{\sqrt{1-\zeta^2}}sin(\pi)\right)$$
$$= e^{-\frac{\sigma}{\omega_d}\pi}$$
$$= e^{-\frac{\zeta}{\sqrt{1-\zeta^2}}\pi}$$

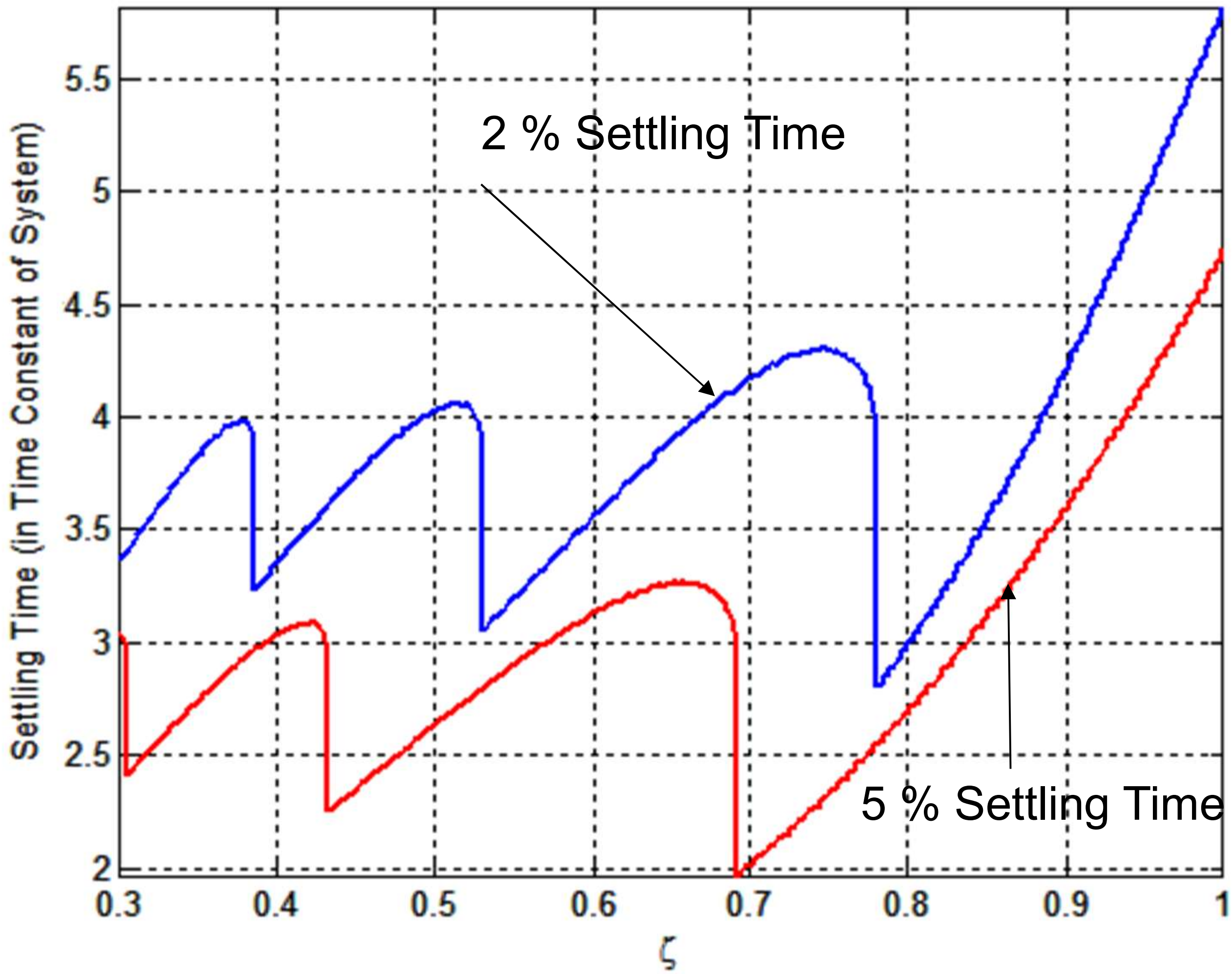
Maximum overshoot is function of damping ratio only.

Maximum percent overshoot is: $M_p = e^{-\frac{\zeta}{\sqrt{1-\zeta^2}}\pi} \times 100$

Transient Response

For a step input, the transient response can be characterized by:

Settling time t_s:



2% Settling time:

$$t_s = 4T = \frac{4}{\sigma} = \frac{4}{\zeta\omega}$$

5% Settling time:

$$t_s = 3T = \frac{3}{\sigma} = \frac{3}{\zeta\omega}$$

Transient Response

```
zetavec = linspace(0.3,1,501);
tvec = linspace(0,30,1001);
ind = 1;
for zeta = zetavec,
    sys = tf([1],[1 2*zeta 1]);
    [y,t] = step(sys,tvec);
    err = 1-y;
    ii = find(abs(err) > 0.02);
    tind(ind) = t(max(ii))*zeta;
    ind = ind+1;
end;

plot(zetavec,tind);
```

Routh Stability Criterion

Consider the closed loop transfer function of the form:

$$G(s) = \frac{b_0 s^m + b_1 s^{m-1} + \ldots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n}$$

where a's and b's are constant and where m< n.

If any of the coefficients are zero or negative in the presence of at least one positive coefficient, there is a root or roots which are imaginary or which have positive real parts.

If all the coefficients are positive, arrange the coefficients of the polynomial in rows and columns according to the pattern:

Routh Stability Criterion

s^n	a_0	a_2	a_4	a_6	…
s^{n-1}	a_1	a_3	a_5	a_7	…
s^{n-2}	b_1	b_2	b_3	b_4	…
s^{n-3}	c_1	c_2	c_3	c_4	…
s^{n-4}	d_1	d_2	d_3	d_4	…
…	…	…	…	…	…
s^2	e_1	e_2			
s^1	f_1				
s^0	g_1				

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1}$$

$$b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1}$$

$$b_3 = \frac{a_1 a_6 - a_0 a_7}{a_1}$$

…

$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1}$$

$$c_2 = \frac{b_1 a_5 - a_1 b_3}{b_1}$$

…

$$d_1 = \frac{c_1 b_2 - b_1 c_2}{c_1}$$

Routh Stability Criterion

Routh's stability criterion states that the number of roots of the system G(s) with positive real parts is equal to the number of changes in the sign of the coefficients of the first column of the array.

The necessary and sufficient condition that all poles of G(s) lie in the left half plane is that all the coefficient of the denominator of G(s) be positive and all terms in the first column of the array have positive signs.

Special Case:

If a first-column term in any row is zero, but the remaining terms are not zero or there is no remaining term, then the zero term is replaced by a very small positive number ε *and the rest of the array is evaluated.*

$$s^3 + 2s^2 + s + 2 = 0$$

$$\begin{array}{lcc} s^3 & 1 & 1 \\ s^2 & 2 & 2 \\ s & 0 \approx \epsilon & \\ s^0 & 2 & \end{array}$$

If the sign of the coefficient above the zero is the same as that below it, it indicates that there are a pair of poles on the imaginary axis.

Routh Stability Criterion

Special Case:

If all the coefficients in any derived row are zero, it indicates that there are roots of equal magnitude lying radially opposite in the s-plane, eg. two real roots with equal magnitudes and opposite signs and/or two conjugate imaginary roots

$$s^5 + 2s^4 + 24s^3 + 48s^2 - 25s - 50 = 0$$

$$\begin{array}{llll} s^5 & 1 & 24 & -25 \\ s^4 & 2 & 48 & -50 \\ s^3 & 0 & 0 & \end{array}$$

In such a case, the evaluation of the rest of the array can be continued by forming an auxiliary polynomial with the coefficient of the last row and the coefficients of the derivative of this polynomial in the next row

$$P(s) = 2s^4 + 48s^2 - 50$$

$$\frac{dP(s)}{ds} = 8s^3 + 96s$$

Routh Stability Criterion

$$
\begin{array}{lrrr}
s^5 & 1 & 24 & -25 \\
s^4 & 2 & 48 & -50 \\
s^3 & 8 & 96 & \\
s^2 & 24 & -50 & \\
s & 112.7 & 0 & \\
s^0 & -50 & &
\end{array}
$$

One change in sign, implying one pole with a positive real part. Solving the characteristic equation, we have:

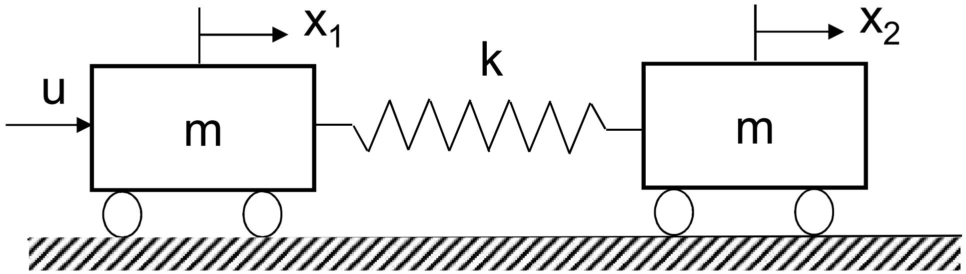
$$s = \pm 1, \;\; s = \pm j5$$

Thus, it is clear that there is one unstable poles.

Routh Stability Criterion

The Routh criterion can be used for control system analysis, eg. for the determination of the range of gains of the feedback controller which guarantee stability.

Consider the two-mass-spring system



Whose equations of motion are:

$$m\ddot{x}_1 + k(x_1 - x_2) = u$$
$$m\ddot{x}_2 - k(x_1 - x_2) = 0$$

Assuming m = k =1, the transfer function relating the input u to the output x_1 is:

$$\frac{X_1(s)}{U(s)} = \frac{s^2+1}{s^2(s^2+2)}$$

Assuming a PD controller: $u = -K_1 x_1 - K_2 \dot{x}_1$

Routh Stability Criterion

The closed loop system can be represented as:

$$\frac{X_1(s)}{U(s)} = \frac{(K_1 + K_2 s)(s^2 + 1)}{s^4 + 2s^2 + (K_2 s + K_1)(s^2 + 1)}$$

The Routh table for the closed loop system is:

$$\begin{array}{llll}
s^4 & 1 & 2 + K_1 & K_1 \\
s^3 & K_2 & K_2 & \\
s^2 & \dfrac{K_2 + K_1 K_2}{K_2} & K_1 & \\
s^1 & \dfrac{K2}{1 + K1} & & \\
s^0 & K_1 & &
\end{array}$$

The second and fifth row indicate that :

$$K_1 > 0 \text{ and } K_2 > 0$$

Routh Stability Criterion

The third and fourth row requires:

$$(1 + K_1) > 0 \text{ or } K_1 > -1$$

The requirement:

$$\begin{aligned} & K_1 > 0 \text{ and } K_1 > -1 \\ \Rightarrow \quad & K_1 > 0 \end{aligned}$$

Results in a stable controller for all gains which satisfy the inequality constraints:

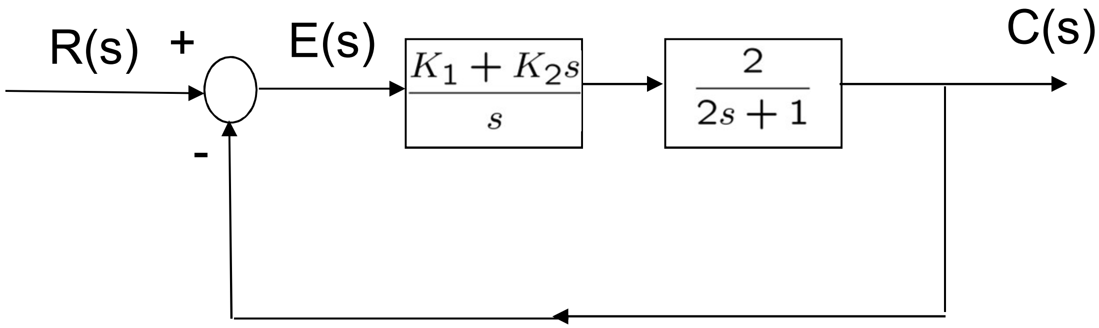
$$K_1 > 0 \text{ and } K_2 > 0$$

Routh Stability Criterion

Consider the flow control problem, with a transfer function:

$$\frac{H(s)}{Q_i(s)} = \frac{R}{RCs + 1}$$

With R = 2, C=1, and a PI controller, the closed loop system is:



The closed loop system is:

$$\frac{H(s)}{Q_i(s)} = \frac{2(K_1 + K_2 s)}{2s^2 + (2K_2 + 1)s + 2K_1}$$

Routh Stability Criterion

The Routh table for the closed loop system is:

$$\begin{array}{lcc} s^2 & 2 & 2K_1 \\ s^1 & (2K_2+1) & \\ s^0 & 2K_1 & \end{array}$$

The second and third row indicate that :

$$K_1 > 0 \text{ and } K_2 > -\frac{1}{2}$$

The poles of the closed loop system is:

$$s = -\frac{(2K_2+1)}{4} \pm \frac{\sqrt{(2K_2+1)^2 - 16K_1}}{4}$$

which corroborates the constraint:

$$K_1 > 0 \text{ and } K_2 > -\frac{1}{2}$$

Routh Stability Criterion (Relative Stability)

Relative Stability: The Routh criteria is a tools which provides a binary answer to the question of *absolute stability*, i.e., whether the system is stable or not. Relative stability permits comparing two system and gauging which system is relatively more stable. A simple characterization of relative stability is the distance of the pole with the largest real value from the imaginary axis. Closer the pole is to the imaginary axis, smaller is its relative stability. Once can use the Routh Criteria to determine the number of poles that lie to the right of a shifted imaginary axis. Substitute $s = z - \sigma$ (σ = constant) into the characteristic equation of the system and rewrite the characteristic equation in terms of z. Applying the Routh Criteria to new polynomial in z permits one to determine the number of poles to the right of the vertical line $s = -\sigma$. Thus, this test reveals the number of roots which lie to the right of the vertical line $s = -\sigma$

Static Error Coefficients

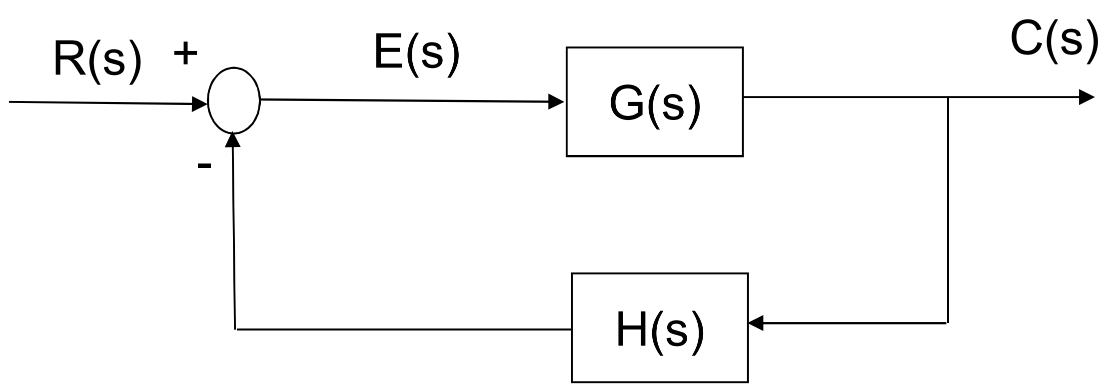
Consider the open loop transfer function:

$$G(s)H(s) = \frac{K(T_a s + 1)(T_b s + 1)...(T_{m-1} s + 1)(T_m s + 1)}{s^N (T_1 s + 1)(T_2 s + 1)...(T_n s + 1)}$$

which includes N poles at the origin of the s-plane.

A system is called type 0, type 1, type 2,…., if N = 0,1,2,…, respectively.

The transfer function of the closed loop system:



$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

Static Error Coefficients

and the error transfer function can be calculated as:

$$\begin{aligned}
E(s) &= R(s) - H(s)C(s) \\
\frac{E(s)}{R(s)} &= 1 - H(s)\frac{C(s)}{R(s)} \\
\frac{E(s)}{R(s)} &= 1 - \frac{G(s)H(s)}{1+G(s)H(s)} \\
\frac{E(s)}{R(s)} &= \frac{1}{1+G(s)H(s)}
\end{aligned}$$

The steady state error can be calculated as:

$$e_{ss} = \lim_{t\to\infty} e_{ss}(t) = \lim_{s\to 0} \frac{sR(s)}{1 + G(s)H(s)}$$

Static Position Error Coefficients K_p

The steady-state error of a system subject to an unit step input is:

$$\begin{aligned} e_{ss} &= \lim_{s \to 0} \frac{s}{1 + G(s)H(s)} \frac{1}{s} \\ e_{ss} &= \frac{1}{1 + G(0)H(0)} \end{aligned}$$

The static position error coefficient K_p is defined as:

$$K_p = \lim_{s \to 0} G(s)H(s) = G(0)H(0)$$

The steady state error is given by the equation:

$$e_{ss} = \frac{1}{1 + K_p}$$

Static Position Error Coefficients K_p

For a type 0 system, the static position error coefficient K_p is:

$$K_p = \lim_{s \to 0} \frac{K(T_a s + 1)(T_b s + 1)...(T_{m-1} s + 1)(T_m s + 1)}{(T_1 s + 1)(T_2 s + 1)...(T_n s + 1)} = K$$

For a type 1 or higher system, the static position error coefficient K_p is:

$$K_p = \lim_{s \to 0} \frac{K(T_a s + 1)(T_b s + 1)...(T_{m-1} s + 1)(T_m s + 1)}{s(T_1 s + 1)(T_2 s + 1)...(T_n s + 1)} = \infty$$

The steady state error is finite for a type 0 system and is zero for system of type 1 or higher.

Static Velocity Error Coefficients K_v

The steady-state error of a system subject to an unit ramp input is:

$$e_{ss} = \lim_{s \to 0} \frac{s}{1 + G(s)H(s)} \frac{1}{s^2}$$

$$e_{ss} = \lim_{s \to 0} \frac{1}{sG(s)H(s)}$$

The static velocity error coefficient K_v is defined as:

$$K_v = \lim_{s \to 0} sG(s)H(s)$$

The steady state error is given by the equation:

$$e_{ss} = \frac{1}{K_v}$$

Static Velocity Error Coefficients K_v

For a type 0 system, the static velocity error coefficient K_v is:

$$K_v = \lim_{s \to 0} \frac{sK(T_a s + 1)(T_b s + 1)...(T_{m-1} s + 1)(T_m s + 1)}{(T_1 s + 1)(T_2 s + 1)...(T_n s + 1)} = 0$$

For a type 1 system, the static velocity error coefficient K_v is:

$$K_v = \lim_{s \to 0} \frac{sK(T_a s + 1)(T_b s + 1)...(T_{m-1} s + 1)(T_m s + 1)}{s(T_1 s + 1)(T_2 s + 1)...(T_n s + 1)} = K$$

For a type 2 or higher system, the static velocity error coefficient K_v is:

$$K_v = \lim_{s \to 0} \frac{sK(T_a s + 1)(T_b s + 1)...(T_{m-1} s + 1)(T_m s + 1)}{s^N(T_1 s + 1)(T_2 s + 1)...(T_n s + 1)} = \infty$$

The steady state error is infinite for a type 0 system, finite for a type 1 system and is zero for system of type 2 or higher.

Static Acceleration Error Coefficients K_a

The steady-state error of a system subject to an unit parabolic input is:

$$e_{ss} = \lim_{s \to 0} \frac{s}{1 + G(s)H(s)} \frac{1}{s^3}$$
$$e_{ss} = \lim_{s \to 0} \frac{1}{s^2 G(s)H(s)}$$

The static acceleration error coefficient K_a is defined as:

$$K_a = \lim_{s \to 0} s^2 G(s)H(s)$$

The steady state error is given by the equation:

$$e_{ss} = \frac{1}{K_a}$$

Static Acceleration Error Coefficients K_a

For a type 0 system, the static velocity error coefficient K_a is:

$$K_a = \lim_{s \to 0} \frac{s^2 K(T_a s + 1)(T_b s + 1)...(T_{m-1} s + 1)(T_m s + 1)}{(T_1 s + 1)(T_2 s + 1)...(T_n s + 1)} = 0$$

For a type 1 system, the static velocity error coefficient K_a is:

$$K_a = \lim_{s \to 0} \frac{s^2 K(T_a s + 1)(T_b s + 1)...(T_{m-1} s + 1)(T_m s + 1)}{s(T_1 s + 1)(T_2 s + 1)...(T_n s + 1)} = 0$$

For a type 2 system, the static velocity error coefficient K_a is:

$$K_a = \lim_{s \to 0} \frac{s^2 K(T_a s + 1)(T_b s + 1)...(T_{m-1} s + 1)(T_m s + 1)}{s^2(T_1 s + 1)(T_2 s + 1)...(T_n s + 1)} = K$$

The steady state error is infinite for a type 0 & 1 system, finite for a type 2 system and is zero for system of type 3 or higher.

Static Error Coefficients

	Step Input	Ramp Input	Parabolic Input
Type 0 System	$\frac{1}{1+K_p}$	∞	∞
Type 1 System	0	$\frac{1}{K_v}$	∞
Type 2 System	0	0	$\frac{1}{K_a}$