AAE 666

Homework Six: Solutions

Exercise 1

As a candidate Lyapunov function for GUES, consider $V(x) = x^T P x$. Then

$$\begin{aligned}\dot{V} &= 2x'P\dot{x}\\ &= 2x'PA_0x + 2\psi(t,x)x'P\Delta A x\,.\end{aligned}$$

For each fixed x, the above expression for $\dot{V}$ is a linear affine function of the scalar $\psi(t,x)$. Hence an upper bound for $\dot{V}$ occurs when $\psi(t,x) = a$ or $\psi(t,x) = b$ which results in

$$\dot{V} \leq 2x'PA_1x = x'(PA_1 + A_1'P)x \leq -2\alpha x'Px$$

or

$$\dot{V} \leq 2x'PA_2x = x'(PA_2 + A_2'P)x \leq -2\alpha x'Px$$

respectively. We now obtain that, for all t, x,

$$\dot{V} \leq -2\alpha x^T P x = -2\alpha V\,.$$

This guarantees the system is GUES with rate α.

Exercise 2

The system given is:

$$\begin{aligned}\dot{x_1} &= -2x_1 + x_2 + \gamma e^{-x_1^2}x_2\\ \dot{x_2} &= -x_1 - 3x_2 - \gamma e^{-x_1^2}x_2\end{aligned}$$

Writing these system of equations in the form $\dot{x} = A(x)x$, where $A = A_0 + \psi(x)\Delta A$, we have,

$$A_0 = \begin{bmatrix} -2 & 1 \\ -1 & -3 \end{bmatrix} \quad \& \quad \Delta A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \& \quad \psi(x) = \gamma e^{-x_1^2}$$

The limits of our scalar non-linear function are: $0 \leq \psi(x) \leq \gamma$ as $e^{-x_1^2} \to 0$ as $|x_1| \to \infty$ and $e^{-x_1^2} \to 1$ as $|x_1| \to 0$. Therefore, comparing with $a \leq \psi(x) \leq b$, we have $a = 0$ and $b = \gamma$. Therefore, since $A_1 = A_0 + a\Delta A$ and $A_2 = A_0 + b\Delta A$,

$$A_1 = \begin{bmatrix} -2 & 1 \\ -1 & -3 \end{bmatrix} \quad \& \quad A_2 = \begin{bmatrix} -2 & 2 \\ -2 & -3 \end{bmatrix}$$

We then solve the following feasibility determination problem (or convex optimization problem) which are formulated as the following LMIs:

$$PA_1 + A_1^T P < 0; \quad PA_2 + A_2^T P < 0; \quad P > I; \quad P = P^T$$

We solve these LMIs as an optimization problem in cvxpy (which can do many of the same things that the LMI toolbox on matlab can). The question asks us to find the supremal value of γ such that the system is to be stable about the origin, so we iterate the optimization over a range of values from 0 to see if it becomes unfeasible about any value. We observe that the LMIs are feasible upto an arbritrarily chosen large value of gamma, and hence, stability of the system does on depend on the magnitude of $|\gamma|$ as long as $\gamma > 0$. Here is the code for reference:

```
#AAE666 assignment 6, exercise 2
#creating the input matrices
A0 = np.array(([-2,1],[-1,-3]))
DelA = np.array(([0,1],[-1,0]))
a = 0
A1 = A0 + a*DelA
```
✓ 0.0s

```
#iterating over a range of gamma values to see if the LMIs become unfeasible.
count = 0
P = cp.Variable((2,2))
for i in range(1000):
    gamma = i + 1
    A2 = A0 + gamma*DelA
    objective = cp.Maximize(0) #dummy objective
    constraints = [P@A1+A1.T@P << 0,P@A2+A2.T@P << 0,P >> np.eye(2),P == P.T]
    prob = cp.Problem(objective,constraints)
    result = prob.solve()
    if prob.status == "optimal":
        count = count + 1
if count == 1000:
    print("All the gammas you input are feasible!")
```
✓ 14.8s

```
All the gammas you input are feasible!
```

As a side note, we can check and confirm stability using the candidate Lyapunov function:

$$V(x) = \frac{1}{2}(x_1^2 + x_2^2) = \frac{1}{2}|x|^2$$

Then, $\beta_1, \beta_2 > 0$ exist such that $\beta_1|x|^2, \leq V(x) \leq \beta_2|x|^2$

$$\dot{V}(x) = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} -2x_1 + x_2 + \gamma e^{-x_1^2} x_2 \\ -x_1 + 3x_2 - \gamma e^{-x_1^2} x_1 \end{bmatrix} = -2(x_1^2 + x_2^2) - x_2^2 \leq -4V(x)$$

Therefore, the system is **GES** about the origin with a rate of convergence $\alpha = 2$.

Exercise 3

See example in notes, with addtional code:
lmiterm([lmi1,1,p],2α,1)
lmiterm([lmi2,1,p],2α,1)
Solving LMI in MATLAB iteratively yields $\alpha \leq 0.122$

Code

```
clear all
close all
clc
% AAE 666 Homework 5 Exercise 3
% Solve for alpha iteratively
% Coder: Siwei Fan
gamma=1;
alpha = 0.00;
flag = 0;
while flag == 0;
% for i = 1:30
    gamma = gamma;
    alpha = alpha + 0.0001

A0 = [0 1; -2 -1];
DelA = [0 0; 1 0];

A1 = A0 - gamma*DelA;
A2 = A0 + gamma*DelA;

setlmis([])
p=lmivar(1, [2,1]);
%
lmi1=newlmi;
lmiterm([lmi1,1,1,p],1,A1,'s')
lmiterm([lmi1,1,1,p],2*alpha,1)
%
lmi2=newlmi;
lmiterm([lmi2,1,1,p],1,A2,'s')
lmiterm([lmi2,1,1,p],2*alpha,1)
%
Plmi= newlmi;
lmiterm([-Plmi,1,1,p],1,1)
lmiterm([Plmi,1,1,0],1)
%
lmis = getlmis;
%
```

```
[tfeas, xfeas] = feasp(lmis);
%
P = dec2mat(lmis,xfeas,p);
if tfeas >=0
    flag = 1;
end
end
```

Exercise 4

Let $x_1 = \theta_1, x_2 = \theta_2, x_3 = \dot{\theta}_1, x_4 = \dot{\theta}_2$, then we have the system as:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2k & k & -2 & 1 \\ k & -k & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ sinx_1 \\ sinx_2 \end{bmatrix}$$

Then, we can write:

$$\dot{x} = A(x)x$$
$$A(x) = A_0 + \phi_1(x)\Delta A_1 + \phi_2(x)\Delta A_2$$
$$\Delta A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\Delta A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

where $sin(x)/x$ has minmum value of -0.217. then, $-0.217 \leq \phi_1(x) = \frac{sinx_1}{x_1} \leq 1$ and $-0.217 \leq \phi_2(x) = \frac{sinx_2}{x_2} \leq 1$. Solving LMI in MATLAB iteratively gives minimum k = 7.8. Any k value larger than 7.8 is ok.

Code

```
close all
clear all
clc

% AAE 666 Homework 5 Exercise 4
% Solve for minimum k value that results in GES
% Coder: Siwei Fan

k = 7.8;
```

```
A0 = [0 0 1 0 ;
      0 0 0 1;
      -2*k k -2 1;
      k -k 1 -1];
DelA1 = [0 0 0 0; 0 0 0 0 ; 1 0 0 0 ; 0 0 0 0];
DelA2 = [0 0 0 0; 0 0 0 0 ; 0 0 0 0 ; 0 1 0 0];
a1 = -0.2172;
a2 = -0.2172;
b1 = 1;
b2 = 1;
A1 = A0+a1*DelA1+a2*DelA2;
A2 = A0+a1*DelA1+b2*DelA2;
A3 = A0+b1*DelA1+a2*DelA2;
A4 = A0+b1*DelA1+b2*DelA2;

setlmis([]);
p=lmivar(1,[4,1]);
lmi1=newlmi;lmiterm([lmi1,1,1,p],1,A1,'s');
lmi2=newlmi;lmiterm([lmi2,1,1,p],1,A2,'s');
lmi3=newlmi;lmiterm([lmi3,1,1,p],1,A3,'s');
lmi4=newlmi;lmiterm([lmi4,1,1,p],1,A4,'s');
Plmi= newlmi;
lmiterm([-Plmi,1,1,p],1,1)
lmiterm([Plmi,1,1,0],1)
lmis = getlmis;
[tfeas,xfeas]= feasp(lmis);
P=dec2mat(lmis,xfeas,p)
eig(P);
```