MA 527

Lecture Notes (section 8.4)

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8.4. Eigenbases and diagonalization

(Ex) $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ $\lambda = -1, 5$

$\lambda = -1$: $X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda = 5$: $X_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. : linearly independent.

$\left\{ \underline{\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}} \right\}$: a basis of $\mathbb{R}^2$.

: an eigenbasis of $\mathbb{R}^2$.

Set $P = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$. $P^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$

$$P^{-1}AP = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}}_{= \begin{bmatrix} -1 & 5 \\ 1 & 10 \end{bmatrix}}$$

$$= \frac{1}{3}\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 5 \\ 1 & 10 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -3 & 0 \\ 0 & 15 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix}$$

$\therefore$ $\underline{P^{-1}AP = \mathrm{diag}(-1, 5)}$: diagonalization.

Q. Is every matrix diagonalizable? No.

Def A: an $n\times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ and corresponding eigenvectors $X_1, X_2, \cdots, X_n$: $AX_i = \lambda_i X_i$ $(i=1,\cdots,n)$

(1) If $\{X_1, X_2, \cdots, X_n\}$ is a basis of $\mathbb{R}^n$, then it is called an eigenbasis of $\mathbb{R}^n$.

(2) A has an eigenbasis of $\mathbb{R}^n$
$\Rightarrow$ A is called diagonalizable.

Q: Which matrix is diagonalizable?

(Ex) (1) $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}_{2\times 2}$: $\lambda = 1, 1$

$$|A-\lambda I| = \begin{vmatrix} 1-\lambda & 0 \\ 2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 = 0$$ ← multiplicity.

$\lambda = 1$: Solve $(A - 1\cdot I)X = 0$

$$\begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{aligned} 2x_1 &= 0 \\ x_1 &= 0 \end{aligned}$$

$$X = \begin{bmatrix} 0 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} 0 \\ 1 \end{bmatrix} : X_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

A is not diagonalizable.

Thm I

Ⓗ $A_{n\times n}$ has n distinct eigenvalues

Ⓒ A has an eigenbasis $\{X_1, \cdots, X_n\}$

i.e. A is diagonalizable.

(Ex) $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ $\quad A - \lambda I = \begin{bmatrix} (1-\lambda) & 0 & 2 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix}$

$|A - \lambda I| = (1-\lambda)^2(3-\lambda) = 0 : \quad \lambda = \underline{1}, 3.$

$\lambda = 1 :$ $\begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} : x_3 = 0$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2 \\ 0 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$\lambda = 3 :$ $A - 3I = \begin{bmatrix} -2 & 0 & 2 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} -2 & 0 & 2 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\quad \begin{matrix} -2x_1 + 2x_3 = 0 \\ -2x_2 = 0. \end{matrix}$

$x_2 = 0.$ $\quad x_3 = x_1$

$$X = \begin{bmatrix} x_1 \\ 0 \\ x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} : \quad X_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$: an eigenbasis of $\mathbb{R}^3$.

<u>Thm 4</u> (Diagonalization).

Ⓗ $A_{n\times n}$: a matrix with eigenvalue $\lambda_1, \cdots, \lambda_n$ and an eigenbasis $\{X_1, \cdots, X_n\}$

$P = [X_1 X_2 \cdots X_n]$, $\quad D = P^{-1}AP$

Ⓒ $D = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$.

(Proof) $AX_i = \lambda_i X_i$, $\quad i = 1, 2, \cdots, n.$

$$AP = A[X_1 X_2 \cdots X_n] = [\lambda_1 X_1 \cdots \lambda_n X_n]$$

$$AP = P \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} = PD$$

$$P^{-1}AP = D.$$

(Similar matrices).

<u>Def</u> $B_{n\times n}$ is called <u>similar</u> to a matrix $A_{n\times n}$ if $\underline{B = Q^{-1}AQ}$ for some nonsingular matrix Q.

Q: $B = Q^{-1}AQ$:

A, B have the same eigenvalues?

$\det(B-\lambda I) = \det(Q^{-1}AQ - \lambda Q^{-1}IQ)$.

$= \det(Q^{-1}(A-\lambda I)Q) = \cancel{\det} Q^{-1} \det(A-\lambda I) \cancel{\det} Q$.

$= \det(A-\lambda I) = 0$

<u>Thm 3</u>

(1) If B is similar to A ($: B = Q^{-1}AQ$), B has the same eigenvalues as A.

(2) Ⓗ $\begin{cases} B = Q^{-1}AQ \\ \lambda, X : \text{an eigenvalue and an eigenvector of } A \text{ s.t. } AX = \lambda X \end{cases}$

Ⓒ $\underline{Y = Q^{-1}X}$ is an eigenvector of B $\sim \lambda$.

i.e. $\underline{BY = \lambda Y}$

(Proof (2))

$BY = (Q^{-1}A\underline{Q)(Q^{-1}}X)$

$= Q^{-1}AX = Q^{-1}\lambda X$

$= \lambda Y$.

(Ex) $A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$ $Q = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

(1) $Q^{-1} = \frac{1}{1}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

$B = Q^{-1}AQ = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

$= \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$

(2) $\lambda = ?$ $\lambda = 1, -1.$: eigenvalues of A
" of B

(3) $\lambda = 1$: $(A - I) = \begin{bmatrix} 0 & 0 \\ 2 & -2 \end{bmatrix}$

$2x_1 - 2x_2 = 0$: $x_1 = x_2$

$X = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$: $X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = -1$: $\begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix} X = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $2x_1 = 0$, $x_1 = 0$

$X = \begin{bmatrix} 0 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} 0 \\ 1 \end{bmatrix}$: $X_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

B: $Y_1 = Q^{-1}X_1 = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$Y_2 = Q^{-1}X_2 = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

* The spectrum of A : $\{\lambda_1, \cdots, \lambda_n\}$
the set of eigenvalues.

Q: $A^T = A$: symmetric

A has real eigenvalues. $\lambda_1, \cdots, \lambda_n$

$x_1, \cdots, x_n$: eigenvectors of A ?

Thm 2 (Spectral Theorem).

Ⓗ A : an $n \times n$ symmetric real matrix

Ⓒ A has an orthogonal eigen basis of $\mathbb{R}^n$.

Q: If A is symmetric, then is A diagonalizable? "Yes"