ECE 68000: MODERN AUTOMATIC CONTROL

Professor Stan Żak

The Kronecker Product

Definition

- Let $\boldsymbol{A}$ be an $m \times n$ and $\boldsymbol{B}$ be a $p \times q$ matrices
- The *Kronecker product* of $\boldsymbol{A}$ and $\boldsymbol{B}$, denoted $\boldsymbol{A} \otimes \boldsymbol{B}$, is an $mp \times nq$ matrix defined as

$$\boldsymbol{A} \otimes \boldsymbol{B} = \begin{bmatrix} a_{11}\boldsymbol{B} & a_{12}\boldsymbol{B} & \cdots & a_{1n}\boldsymbol{B} \\ a_{21}\boldsymbol{B} & a_{22}\boldsymbol{B} & \cdots & a_{2n}\boldsymbol{B} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}\boldsymbol{B} & a_{m2}\boldsymbol{B} & \cdots & a_{mn}\boldsymbol{B} \end{bmatrix},$$

  where the symbol $\otimes$ reads "otimes"
- Thus, the matrix $\boldsymbol{A} \otimes \boldsymbol{B}$ consists of mn blocks
- In MATLAB, kron(A,B)

Example 1

- Let

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{B} = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$$

- Then

$$\begin{aligned} \boldsymbol{A} \otimes \boldsymbol{B} &= \begin{bmatrix} \boldsymbol{B} & 2\boldsymbol{B} & 3\boldsymbol{B} \\ 3\boldsymbol{B} & 2\boldsymbol{B} & \boldsymbol{B} \end{bmatrix} \\ &= \begin{bmatrix} 2 & 1 & 4 & 2 & 6 & 3 \\ 2 & 3 & 4 & 6 & 6 & 9 \\ 6 & 3 & 4 & 2 & 2 & 1 \\ 6 & 9 & 4 & 6 & 2 & 3 \end{bmatrix} \end{aligned}$$

Example 2

- Let $\boldsymbol{A}$ be an arbitrary 2×2 matrix
- Then

$$
\begin{aligned}
\boldsymbol{I}_2 \otimes \boldsymbol{A} &= \begin{bmatrix} \boldsymbol{A} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{A} \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \\
&= \begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & a_{11} & a_{12} \\ 0 & 0 & a_{21} & a_{22} \end{bmatrix}
\end{aligned}
$$

Example 2—contd.

- We have

$$\begin{aligned} \boldsymbol{A} \otimes \boldsymbol{I}_2 &= \begin{bmatrix} a_{11}\boldsymbol{I}_2 & a_{12}\boldsymbol{I}_2 \\ a_{21}\boldsymbol{I}_2 & a_{22}\boldsymbol{I}_2 \end{bmatrix} \\ &= \begin{bmatrix} a_{11} & 0 & a_{12} & 0 \\ 0 & a_{11} & 0 & a_{12} \\ a_{21} & 0 & a_{22} & 0 \\ 0 & a_{21} & 0 & a_{22} \end{bmatrix} \end{aligned}$$

- In general

$$\boldsymbol{A} \otimes \boldsymbol{B} \neq \boldsymbol{B} \otimes \boldsymbol{A}$$

Vectorization operator or stacking operator

- Let
$$\boldsymbol{X} = \left[\ \boldsymbol{x}_1,\quad \boldsymbol{x}_2,\quad \ldots,\quad \boldsymbol{x}_m\ \right]$$
be a $n \times m$ matrix, where $\boldsymbol{x}_i$, $i = 1, 2, \ldots, m$ are the columns of $\boldsymbol{X}$
- Each $\boldsymbol{x}_i$ consists of n elements
- Then, the *vectorization operator* or *stacking operator* is defined as
$$\begin{aligned} \mathrm{vec}(\boldsymbol{X}) &= \left[\ \boldsymbol{x}_1^\top,\quad \boldsymbol{x}_2^\top,\quad \ldots,\quad \boldsymbol{x}_m^\top\ \right]^\top \\ &= \left[\ x_{11}\quad x_{21}\quad \cdots\quad x_{12}\quad \cdots\quad x_{n2}\quad \cdots\quad x_{1m}\quad \cdots\quad x_{nm}\ \right]^\top \end{aligned}$$
is the column nm-vector formed from the columns of $\boldsymbol{X}$ taken in order
- In MATLAB, X(:)

Converting a matrix-matrix equation into a matrix-vector equation

- Let now $\boldsymbol{A}$ be an $n \times n$, and $\boldsymbol{C}$ and $\boldsymbol{X}$ be $n \times m$ matrices
- Then, the matrix equation

$$\boldsymbol{AX} = \boldsymbol{C}$$

can be written as

$$(\boldsymbol{I}_m \otimes \boldsymbol{A}) \operatorname{vec}(\boldsymbol{X}) = \operatorname{vec}(\boldsymbol{C})$$

- Indeed, write $\boldsymbol{AX} = \boldsymbol{C}$ as

$$\begin{aligned} \boldsymbol{AX} &= \boldsymbol{A} \left[\begin{array}{cccc} \boldsymbol{x}_1, & \boldsymbol{x}_2, & \ldots, & \boldsymbol{x}_m \end{array} \right] \\ &= \left[\begin{array}{cccc} \boldsymbol{Ax}_1, & \boldsymbol{Ax}_2, & \ldots, & \boldsymbol{Ax}_m \end{array} \right] \\ &= \left[\begin{array}{cccc} \boldsymbol{c}_1, & \boldsymbol{c}_2, & \ldots, & \boldsymbol{c}_m \end{array} \right] \end{aligned}$$

Manipulations

- Represent
$\begin{bmatrix} \boldsymbol{A}\boldsymbol{x}_1, & \boldsymbol{A}\boldsymbol{x}_2, & \ldots, & \boldsymbol{A}\boldsymbol{x}_m \end{bmatrix} = \begin{bmatrix} \boldsymbol{c}_1, & \boldsymbol{c}_2, & \ldots, & \boldsymbol{c}_m \end{bmatrix}$ as
$$\begin{bmatrix} \boldsymbol{A}\boldsymbol{x}_1 \\ \boldsymbol{A}\boldsymbol{x}_2 \\ \vdots \\ \boldsymbol{A}\boldsymbol{x}_m \end{bmatrix} = \begin{bmatrix} \boldsymbol{c}_1 \\ \boldsymbol{c}_2 \\ \vdots \\ \boldsymbol{c}_m \end{bmatrix}$$
- Equivalently
$$\begin{bmatrix} \boldsymbol{A} & \boldsymbol{O} & \cdots & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{A} & \cdots & \boldsymbol{O} \\ & & \ddots & \\ \boldsymbol{O} & \boldsymbol{O} & \cdots & \boldsymbol{A} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \vdots \\ \boldsymbol{x}_m \end{bmatrix} = \begin{bmatrix} \boldsymbol{c}_1 \\ \boldsymbol{c}_2 \\ \vdots \\ \boldsymbol{c}_m \end{bmatrix}$$
- That is,
$$(\boldsymbol{I}_m \otimes \boldsymbol{A}) \operatorname{vec}(\boldsymbol{X}) = \operatorname{vec}(\boldsymbol{C})$$

Converting another matrix-matrix equation into a matrix-vector equation

- Let now $\boldsymbol{B}$ be a $m \times m$, and $\boldsymbol{C}$ and $\boldsymbol{X}$ be $n \times m$ matrices
- Then, the matrix equation

$$\boldsymbol{X}\boldsymbol{B} = \boldsymbol{C}$$

can be written as

$$\left(\boldsymbol{B}^\top \otimes \boldsymbol{I}_n\right) \mathrm{vec}(\boldsymbol{X}) = \mathrm{vec}(\boldsymbol{C})$$

- Indeed, write $\boldsymbol{X}\boldsymbol{B} = \boldsymbol{C}$ as

$$\begin{bmatrix} \boldsymbol{X}\boldsymbol{b}_1 & \boldsymbol{X}\boldsymbol{b}_2 & \cdots & \boldsymbol{X}\boldsymbol{b}_m \end{bmatrix} = \begin{bmatrix} \boldsymbol{c}_1 & \boldsymbol{c}_2 & \cdots & \boldsymbol{c}_m \end{bmatrix}$$

- Represent the above as

$$\begin{bmatrix} \boldsymbol{X}\boldsymbol{b}_1 \\ \boldsymbol{X}\boldsymbol{b}_2 \\ \vdots \\ \boldsymbol{X}\boldsymbol{b}_m \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}_1 b_{11} + \boldsymbol{x}_2 b_{21} + \cdots + \boldsymbol{x}_m b_{m1} \\ \boldsymbol{x}_1 b_{12} + \boldsymbol{x}_2 b_{22} + \cdots + \boldsymbol{x}_m b_{m2} \\ \vdots \\ \boldsymbol{x}_1 b_{1m} + \boldsymbol{x}_2 b_{2m} + \cdots + \boldsymbol{x}_m b_{mm} \end{bmatrix} = \begin{bmatrix} \boldsymbol{c}_1 \\ \boldsymbol{c}_2 \\ \vdots \\ \boldsymbol{c}_m \end{bmatrix}$$

More manipulations

- Represent $\begin{bmatrix} \boldsymbol{x}_1 b_{11} + \boldsymbol{x}_2 b_{21} + \cdots + \boldsymbol{x}_m b_{m1} \\ \boldsymbol{x}_1 b_{12} + \boldsymbol{x}_2 b_{22} + \cdots + \boldsymbol{x}_m b_{m2} \\ \vdots \\ \boldsymbol{x}_1 b_{1m} + \boldsymbol{x}_2 b_{2m} + \cdots + \boldsymbol{x}_m b_{mm} \end{bmatrix} = \begin{bmatrix} \boldsymbol{c}_1 \\ \boldsymbol{c}_2 \\ \vdots \\ \boldsymbol{c}_m \end{bmatrix}$ as

$$\begin{bmatrix} b_{11}\boldsymbol{I}_n & b_{21}\boldsymbol{I}_n & \cdots & b_{m1}\boldsymbol{I}_n \\ b_{12}\boldsymbol{I}_n & b_{22}\boldsymbol{I}_n & \cdots & b_{m2}\boldsymbol{I}_n \\ \vdots & \vdots & \ddots & \vdots \\ b_{1m}\boldsymbol{I}_n & b_{2m}\boldsymbol{I}_n & \cdots & b_{mm}\boldsymbol{I}_n \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \vdots \\ \boldsymbol{x}_m \end{bmatrix} = \begin{bmatrix} \boldsymbol{c}_1 \\ \boldsymbol{c}_2 \\ \vdots \\ \boldsymbol{c}_m \end{bmatrix}$$

- The above can be written in compact from as

$$\left(\boldsymbol{B}^\top \otimes \boldsymbol{I}_n\right) \mathrm{vec}(\boldsymbol{X}) = \mathrm{vec}(\boldsymbol{C})$$

The Sylvester matrix equation

- Using the above two facts, represent the Sylvester matrix equation
$$\boldsymbol{A}\boldsymbol{X} + \boldsymbol{X}\boldsymbol{B} = \boldsymbol{C},$$
as
$$\left(\boldsymbol{I}_m \otimes \boldsymbol{A} + \boldsymbol{B}^\top \otimes \boldsymbol{I}_n\right) \mathrm{vec}(\boldsymbol{X}) = \mathrm{vec}(\boldsymbol{C})$$
- The continuous Lyapunov equation, $\boldsymbol{A}^\top \boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} = -\boldsymbol{Q}$, can be represented as
$$\left(\boldsymbol{I}_n \otimes \boldsymbol{A}^\top + \boldsymbol{A}^\top \otimes \boldsymbol{I}_n\right) \mathrm{vec}(\boldsymbol{P}) = -\mathrm{vec}(\boldsymbol{Q})$$