ECE 68000: MODERN AUTOMATIC CONTROL

Professor Stan Żak

Taylor’s linearization of controlled time-varying nonlinear systems

Taylor linearization of controlled nonlinear systems

- Nonlinear controlled system model

$$\left.\begin{array}{rcl} \frac{dx_1}{dt} & = & f_1(t, x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_m) \\ \frac{dx_2}{dt} & = & f_2(t, x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_m) \\ & \vdots & \\ \frac{dx_n}{dt} & = & f_n(t, x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_m) \end{array}\right\}$$

- Represent the above system in vector form

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x}, \boldsymbol{u})$$

Equilibrium pair

Let $\boldsymbol{u}_e = \begin{bmatrix} u_{1e} & u_{2e} & \cdots & u_{me} \end{bmatrix}^\top$ be a constant input that forces the system $\dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x}, \boldsymbol{u})$ to settle into a constant, equilibrium, state $\boldsymbol{x}_e = \begin{bmatrix} x_{1e} & x_{2e} & \cdots & x_{ne} \end{bmatrix}^\top$ at time t_0, that is, $\boldsymbol{u}_e$ and $\boldsymbol{x}_e$ satisfy

$$\boldsymbol{f}(t, \boldsymbol{x}_e, \boldsymbol{u}_e) = \boldsymbol{0} \quad \text{for all} \quad t \geq t_0$$

- The pair $(\boldsymbol{x}_e, \boldsymbol{u}_e)$ is the equilibrium pair at t_0
- If the pair $(\boldsymbol{x}_e, \boldsymbol{u}_e)$ is an equilibrium pair at t_0, then it is also an equilibrium pair for all $\tilde{t}_0 \geq t_0$

Perturbing the equilibrium

- Perturb the equilibrium state as

$$\boldsymbol{x} = \boldsymbol{x}_e + \delta\boldsymbol{x}, \quad \boldsymbol{u} = \boldsymbol{u}_e + \delta\boldsymbol{u}.$$

- Perform Taylor's expansion

$$\begin{aligned}
\frac{d}{dt}\boldsymbol{x} &= \boldsymbol{f}(t, \boldsymbol{x}_e + \delta\boldsymbol{x}, \boldsymbol{u}_e + \delta\boldsymbol{u}) \\
&= \boldsymbol{f}(t, \boldsymbol{x}_e, \boldsymbol{u}_e) + \frac{\partial\boldsymbol{f}}{\partial\boldsymbol{x}}(t, \boldsymbol{x}_e, \boldsymbol{u}_e)\,\delta\boldsymbol{x} \\
&+ \frac{\partial\boldsymbol{f}}{\partial\boldsymbol{u}}(t, \boldsymbol{x}_e, \boldsymbol{u}_e)\,\delta\boldsymbol{u} + \text{higher order terms}
\end{aligned}$$

The Jacobian matrix of $\boldsymbol{f}$ with respect $\boldsymbol{x}$

$$\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}(t, \boldsymbol{x}_e, \boldsymbol{u}_e) = \left. \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \right|_{\substack{\boldsymbol{x}=\boldsymbol{x}_e \\ \boldsymbol{u}=\boldsymbol{u}_e}}$$

The Jacobian matrices are evaluated at the equilibrium pair, $\begin{bmatrix} \boldsymbol{x}_e^\top & \boldsymbol{u}_e^\top \end{bmatrix}^\top$

The Jacobian matrix of $\boldsymbol{f}$ with respect $\boldsymbol{u}$

$$\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}(t, \boldsymbol{x}_e, \boldsymbol{u}_e) = \left. \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_m} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_m} \end{bmatrix} \right|_{\substack{\boldsymbol{x}=\boldsymbol{x}_e \\ \boldsymbol{u}=\boldsymbol{u}_e}}$$

The Jacobian matrices are evaluated at the equilibrium pair, $\begin{bmatrix} \boldsymbol{x}_e^\top & \boldsymbol{u}_e^\top \end{bmatrix}^\top$

Some manipulations

- Note that

$$\frac{d}{dt}\boldsymbol{x} = \frac{d}{dt}\boldsymbol{x}_e + \frac{d}{dt}\delta\boldsymbol{x} = \frac{d}{dt}\delta\boldsymbol{x}$$

  because $\boldsymbol{x}_e$ is constant
- Furthermore

$$\boldsymbol{f}(t, \boldsymbol{x}_e, \boldsymbol{u}_e) = \boldsymbol{0} \text{ for all } t \geq t_0$$

- Let

$$\boldsymbol{A}(t) = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}(t, \boldsymbol{x}_e, \boldsymbol{u}_e) \quad \text{and} \quad \boldsymbol{B}(t) = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}(t, \boldsymbol{x}_e, \boldsymbol{u}_e)$$

Some manipulations—contd.

- Recall from Taylor's expansion

$$
\begin{aligned}
\frac{d}{dt}\boldsymbol{x} &= \boldsymbol{f}(t, \boldsymbol{x}_e, \boldsymbol{u}_e) + \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}(t, \boldsymbol{x}_e, \boldsymbol{u}_e)\, \delta \boldsymbol{x} \\
&+ \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}(t, \boldsymbol{x}_e, \boldsymbol{u}_e)\, \delta \boldsymbol{u} + \text{higher order terms}
\end{aligned}
$$

- Neglecting higher order terms, we arrive at the linear approximation

$$
\frac{d}{dt}\delta \boldsymbol{x} = \boldsymbol{A}(t)\delta \boldsymbol{x} + \boldsymbol{B}(t)\delta \boldsymbol{u}
$$

Linearizing the output map

- Output map

$$\left.\begin{array}{rcl} y_1 & = & h_1(t, x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_m) \\ y_2 & = & h_2(t, x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_m) \\ & \vdots & \\ y_p & = & h_p(t, x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_m) \end{array}\right\}$$

- In vector notation

$$\boldsymbol{y} = \boldsymbol{h}(t, \boldsymbol{x}, \boldsymbol{u})$$

Taylor's expansion of the output map

- Taylor's series expansion used to yield the linear approximation of the nonlinear output equations
- Perturb from the equilibrium

$$\boldsymbol{y} = \boldsymbol{y}_e + \delta \boldsymbol{y}$$

  where $\boldsymbol{y}_e = \boldsymbol{h}(t, \boldsymbol{x}_e, \boldsymbol{u}_e)$
- We obtain

$$\delta \boldsymbol{y} = \boldsymbol{C}(t)\delta \boldsymbol{x} + \boldsymbol{D}(t)\delta \boldsymbol{u}$$

Output matrices

$$\boldsymbol{C}(t) = \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}}(t, \boldsymbol{x}_e, \boldsymbol{u}_e) = \left[\begin{array}{ccc} \frac{\partial h_1}{\partial x_1} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial h_p}{\partial x_1} & \cdots & \frac{\partial h_p}{\partial x_n} \end{array} \right] \Bigg|_{\substack{\boldsymbol{x}=\boldsymbol{x}_e \\ \boldsymbol{u}=\boldsymbol{u}_e}}$$

and

$$\boldsymbol{D}(t) = \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{u}}(t, \boldsymbol{x}_e, \boldsymbol{u}_e) = \left[\begin{array}{ccc} \frac{\partial h_1}{\partial u_1} & \cdots & \frac{\partial h_1}{\partial u_m} \\ \vdots & & \vdots \\ \frac{\partial h_p}{\partial u_1} & \cdots & \frac{\partial h_p}{\partial u_m} \end{array} \right] \Bigg|_{\substack{\boldsymbol{x}=\boldsymbol{x}_e \\ \boldsymbol{u}=\boldsymbol{u}_e}}$$

are the Jacobian matrices of $\boldsymbol{h}$ with respect $\boldsymbol{x}$ and $\boldsymbol{u}$ evaluated at the equilibrium pair $\left[\begin{array}{cc} \boldsymbol{x}_e^\top & \boldsymbol{u}_e^\top \end{array} \right]^\top$

Example

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 2u \times 1(t-2) - x_1 x_2 \\ -4 + x_1^2 \end{bmatrix}$$

where $1(t)$ is the unit step function

- Find the equilibrium states at $t_0 = 3$ corresponding to $u_e = 1$
- Construct the corresponding linear models about the equilibrium states found above

Example solution

- The system model for $t \geq t_0$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 2u - x_1 x_2 \\ -4 + x_1^2 \end{bmatrix}.$$

- To find equilibrium states at $t_0 = 3$ corresponding to $u_e = 1$, solve the algebraic equations

$$\dot{x}_1 = \dot{x}_2 = 0$$

that is,

$$2u - x_1 x_2 = 0 \quad \text{and} \quad -4 + x_1^2 = 0$$

Computing equilibrium states

- The equilibrium states, corresponding to $u_e = 1$

$$\boldsymbol{x}_e^{(1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{x}_e^{(2)} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

- Linearized models

$$\frac{d}{dt}\delta\boldsymbol{x} = \boldsymbol{A}\,\delta\boldsymbol{x} + \boldsymbol{b}\delta u$$

Computing the linearized model matrices

- $$\boldsymbol{A} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} -x_2 & -x_1 \\ 2x_1 & 0 \end{bmatrix}$$
- $$\boldsymbol{b} = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$
- Matrices are evaluated at the equilibrium pairs

Linearized systems

Two linear models:

1. about $\boldsymbol{x}_e^{(1)}$

$$\frac{d}{dt}\delta\boldsymbol{x} = \begin{bmatrix} -1 & -2 \\ 4 & 0 \end{bmatrix} \delta\boldsymbol{x} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} \delta u$$

2. about $\boldsymbol{x}_e^{(2)}$

$$\frac{d}{dt}\delta\boldsymbol{x} = \begin{bmatrix} 1 & 2 \\ -4 & 0 \end{bmatrix} \delta\boldsymbol{x} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} \delta u$$

Linearized systems analysis

- For any $t_0 < 2$, we have two states corresponding to $u = 1$ such that $\dot{x}_1 = \dot{x}_2 = 0$:

$$\boldsymbol{x}^{(1)} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{x}^{(2)} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

- However, the above states do not satisfy the algebraic equations $\dot{x}_1 = \dot{x}_2 = 0$ for all $t \geq t_0$ when $t_0 < 2$