AAE 666

# Homework One: Solutions

## Exercise 1

$$2\ddot{q}_1 + \ddot{q}_2 + sinq_1 = 0$$
$$\ddot{q}_1 + 2\ddot{q}_2 + sinq_2 = 0$$
$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} = \begin{bmatrix} -sinq_1 \\ -sinq_2 \end{bmatrix}$$
$$\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} -sinq_1 \\ -sinq_2 \end{bmatrix}$$
$$\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} -sinq_1 \\ -sinq_2 \end{bmatrix}$$
$$\ddot{q}_1 = -\frac{2}{3} sinq_1 + \frac{1}{3} sinq_2$$
$$\ddot{q}_2 = \frac{1}{3} sinq_1 - \frac{1}{3} sinq_2$$

Define state variables:

$$x_1 = q_1$$
$$x_2 = \dot{q}_1$$
$$x_3 = q_2$$
$$x_4 = \dot{q}_2$$

Fianlly:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{2}{3} sinx_1 + \frac{1}{3} sinx_3$$
$$\dot{x}_3 = x_4$$
$$\dot{x}_4 = \frac{1}{3} sinx_1 - \frac{1}{3} sinx_3$$

## Exercise 2

$$\ddot{q}_1 + \dot{q}_2 + q_1^3 = 0$$
$$\dot{q}_1 + \dot{q}_2 + q_2^3 = 0$$
$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} -q_1^3 \\ -\dot{q}_1 - q_2^3 \end{bmatrix}$$
$$\begin{bmatrix} \ddot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -q_1^3 \\ -\dot{q}_1 - q_2^3 \end{bmatrix}$$
$$\begin{bmatrix} \ddot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -q_1^3 \\ -\dot{q}_1 - q_2^3 \end{bmatrix}$$
$$\ddot{q}_1 = -q_1^3 + \dot{q}_1 + q_2^3$$
$$\dot{q}_2 = -\dot{q}_1 - q_2^3$$

Define state variables:

$$x_1 = q_1$$
$$x_2 = \dot{q}_1$$
$$x_3 = q_2$$

Finally:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1^3 + x_2 + x_3^3$$
$$\dot{x}_3 = -x_2 - x_3^3$$

## Exercise 3

$$\ddot{q}_1 + q_1 + 2\dot{q}_2 = 0$$
$$\ddot{q}_1 + \dot{q}_2 + q_2 = 0$$
$$\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} -q_1 \\ -q_2 \end{bmatrix}$$
$$\begin{bmatrix} \ddot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -q_1 \\ -q_2 \end{bmatrix}$$
$$\begin{bmatrix} \ddot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -q_1 \\ -q_2 \end{bmatrix}$$
$$\ddot{q}_1 = q_1 - 2q_2$$
$$\dot{q}_2 = -q_1 + q_2$$

Define state variables:

$$x_1 = q_1$$
$$x_2 = \dot{q}_1$$
$$x_3 = q_2$$

Fianlly:

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_1 - 2x_3 \\
\dot{x}_3 &= -x_1 + x_3
\end{aligned}$$

# Exercise 4

$$q_1(k+2) + q_1(k) + 2q_2(k+1) = 0$$
$$q_1(k+2) + q_1(k+1) + q_2(k) = 0$$
$$\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} q_1(k+2) \\ q_2(k+1) \end{bmatrix} = \begin{bmatrix} -q_1(k) \\ -q_1(k+1) - q_2(k) \end{bmatrix}$$
$$\begin{bmatrix} q_1(k+2) \\ q_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} -q_1(k) \\ -q_1(k+1) - q_2(k) \end{bmatrix}$$
$$\begin{bmatrix} q_1(k+2) \\ q_2(k+1) \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} 0 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -q_1(k) \\ -q_1(k+1) - q_2(k) \end{bmatrix}$$
$$q_1(k+2) = -q_1(k+1) - q_2(k)$$
$$q_2(k+1) = -\frac{1}{2}q_1(k) + \frac{1}{2}q_1(k+1) + \frac{1}{2}q_2(k)$$

Define state variables:

$$\begin{aligned}
x_1(k) &= q_1(k) \\
x_2(k) &= q_1(k+1) \\
x_3(k) &= q_2(k)
\end{aligned}$$

Fianlly:

$$\begin{aligned}
x_1(k+1) &= x_2(k) \\
x_2(k+1) &= -x_2(k) - x_3(k) \\
x_3(k+1) &= -\frac{1}{2}x_1(k) + \frac{1}{2}x_2(k) + \frac{1}{2}x_3(k)
\end{aligned}$$

# Exercise 5

When the system is at its equilibrium, it means for all k

$$x(k+1) = x(k)$$

This means when $x = x^e$

$$\begin{aligned}
x(k+1) - x(k) &= -\frac{g(x(k))}{g'(x(k))} \\
0 &= -\frac{g(x^e)}{g'(x^e)}
\end{aligned}$$

And this is only true if $g(x^e) = 0$.

## Exercise 6

The first nonlinear system is given by:

$$\dot{x} = -\alpha \mathrm{sgm}(x)$$

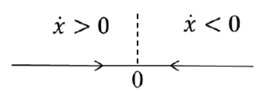


Figure 1: Exercise 6 Plot

## Exercise 7

$$\dot{x} = x^2(x^2 - 1) = x^2(x - 1)(x + 1)$$

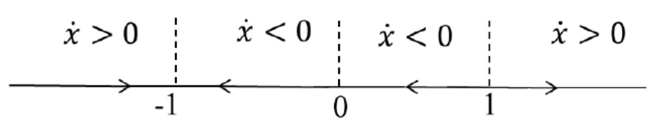


Figure 2: Exercise 7 Plot

## Exercise 8

$$\dot{x} = -x^3$$

When $x(t) \neq 0$,

$$-\frac{\dot{x}}{x^3} = 1$$

that is,

$$\frac{d}{dt}\left(\frac{1}{2x^2}\right) = 1$$

Integrate form 0 to $t$ and obtain:

$$\frac{1}{2x(t)^2} - \frac{1}{2x(0)^2} = t$$

Hence

$$x(t)^2 = \frac{x_0^2}{1 + 2x_0^2 t}$$

where $x_0 = x(0)$ and

$$x(t) = \frac{x_0}{\sqrt{1 + 2x_0^2 t}}$$

for $t \geq 0$.

# Exercise 9

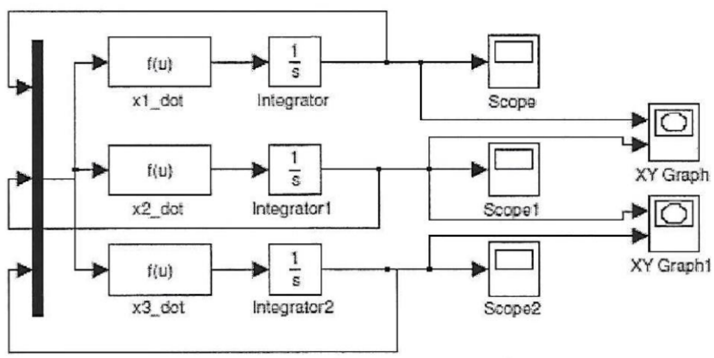


Figure 3: Exercise 9 Simulink Model

The difference in time history started after 40 seconds for all states. For the 60 seconds simulated, the history differs mostly in the frequency. The amplitude of the response remain roughly the same.

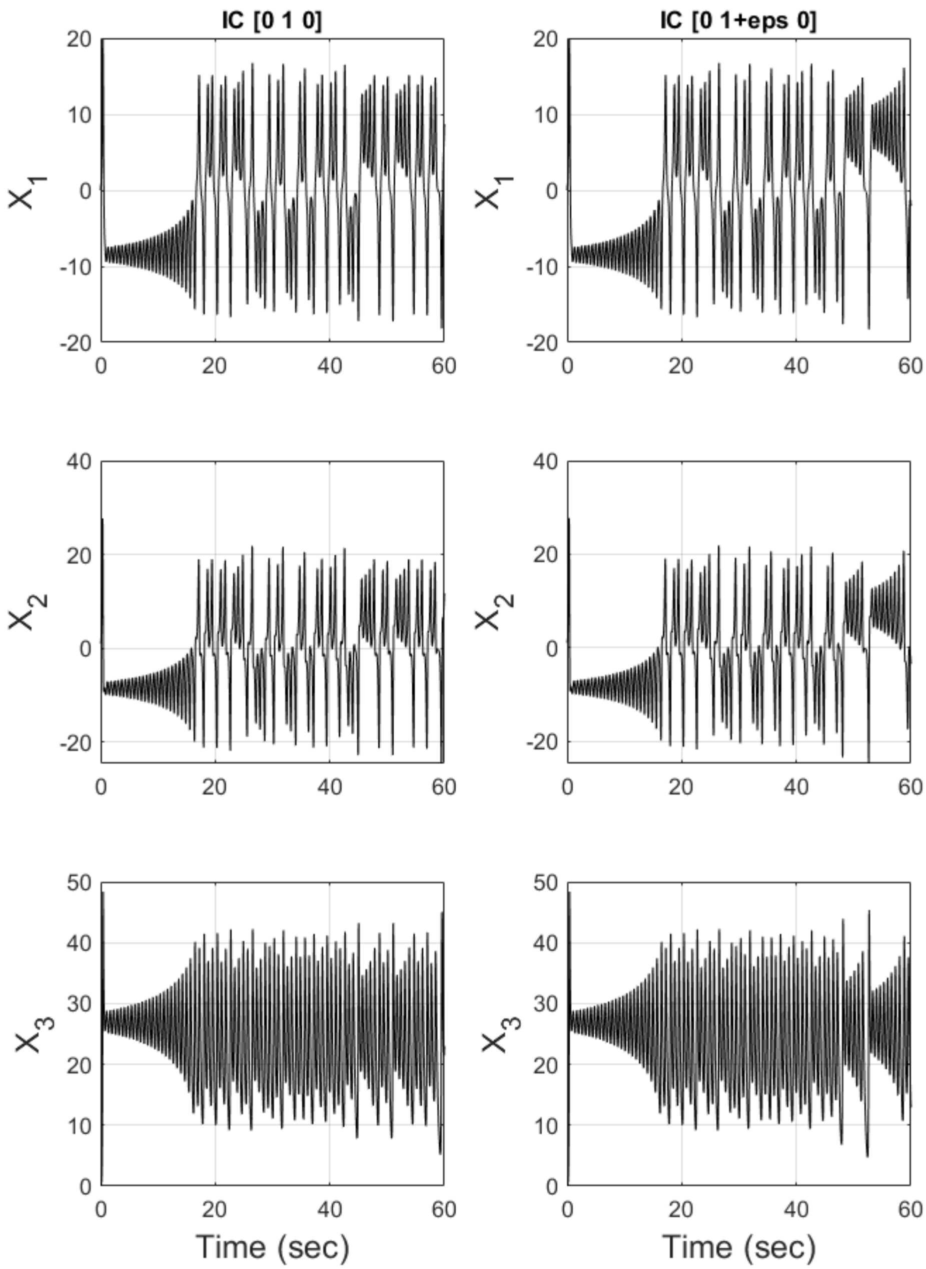


Figure 4: Exercise 9 Simulation Results