(Final exam).

1. Open from 8:00pm (7/31 Wed) to 8:00pm (8/3 Sat)

2. 2 hours 30 min

- Auto-grading system in the Brightspace

Be careful when you choose your answer.
(No partial credit)

- Go to "Quizzes":

3. Begin the test (at least) before 5:30pm 8/3 (Sat)

4. Your grades will be posted next Monday.
Do not talk about any problems in the final exam.

#17. $f(x) = \begin{cases} 0, & 0 < x < 1 \\ 1, & 1 < x < 3 \\ 0, & x > 3. \end{cases}$

$F(\omega)$: the Fourier sine transform.

$$F(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin(\omega x)\, dx$$

The inverse Fourier sine transform

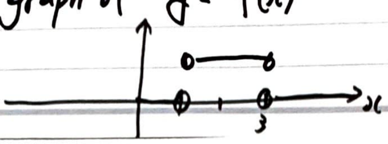
$$\mathcal{F}_s^{-1}(F(w))_{(x)} = \sqrt{\frac{2}{\pi}} \int_0^\infty F(w) \sin(wx)\, dw$$

$g(z)$ = the Fourier sine transform of $F(w)$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty F(w) \sin(wz)\, dw$$

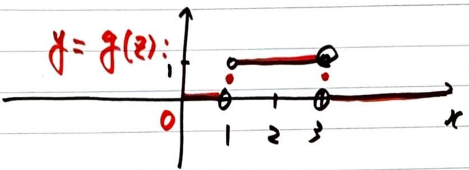
: the inverse Fourier sine transform of $F(w)$

The graph of $y = f(x)$



$$g(z) = \mathcal{F}_s(F) = \frac{2}{\pi} \int_0^\infty \int_0^\infty f(x) \sin(wx)\, dx \sin(wz)\, dw$$

: the Fourier sine integral of $f(x)$



$g(2) = 1, \quad g(3) = \frac{1}{2}(1+0) = \frac{1}{2}.$

#18. let $f(x) = e^{-x}$: continuous on $(0, \infty)$

$$\hat{f}_c(w) = \mathcal{F}_c(f) = \sqrt{\frac{2}{\pi}}\left(\frac{1}{1+w^2}\right)$$

$$f(x) = \mathcal{F}_c^{-1}(\hat{f}_c)_{(x)} = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_c(w) \cos(wx)\, dw$$

$\parallel$

the Fourier cosine integral.

$$e^{-x} = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}}\left(\frac{1}{1+w^2}\right) \cos(wx)\, dw$$

$$= \frac{2}{\pi} \int_0^\infty \frac{\cos(wx)}{1+w^2}\, dw$$

$$x = 2: \quad \frac{2}{\pi} \int_0^\infty \frac{\cos(2w)}{1+w^2}\, dw = e^{-2}$$

#19. $\mathcal{F}(e^{-x^2/2}) = e^{-w^2/2}$ — ①

$\mathcal{F}(x e^{-x^2/2}) = ?$

①: $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} e^{-iwx} \, dx = e^{-w^2/2}$

$\frac{d}{dw} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \quad " \quad dx = \frac{d}{dw} e^{-w^2/2}$

$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} e^{-iwx} (-ix) \, dx = e^{-w^2/2} \left(-\frac{2w}{2}\right)$

$(-i) \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-x^2/2} e^{-iwx} \, dx}_{= \mathcal{F}(x e^{-x^2/2})} = -w e^{-w^2/2}$

$\mathcal{F}(x e^{-x^2/2}) = \frac{+w}{+i} e^{-w^2/2} = -iw e^{-w^2/2}$

$\left(\frac{1}{i} = -i\right)$

#20. $f(x) = \begin{cases} \sin(2x), & 0 \le x \le \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < x < \pi \end{cases}$

(IBVP) $\begin{cases} u_{tt} = u_{xx}, & 0 < x < \pi, \ t > 0 \\ u(0,t) = 0, \ u(\pi, t) = 0, & t \ge 0 \\ u(x,0) = f(x), \ u_t(x,0) = 0 \end{cases}$

$c = 1, \quad L = \pi$

d'Alembert formula (Example 1)

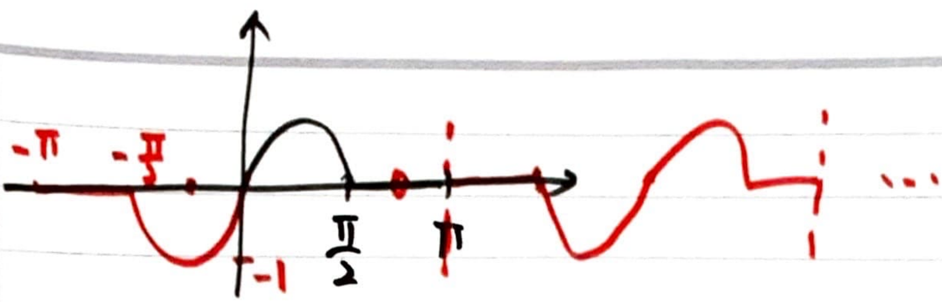
$u(x,t) = \frac{1}{2}[f_0(x+ct) + f_0(x-ct)]$

$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{cn\pi t}{L}\right)$

$u(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) = f(x)$

: the Fourier sine series of $f(x)$

↑ $f_0(x)$: the odd extension



$$u\left(\frac{\pi}{4}, \frac{\pi}{2}\right) = \frac{1}{2}\left[f_0(x+t) + f(x-t)\right]\Big|_{\left(\frac{\pi}{4}, \frac{\pi}{2}\right)}$$

$$= \frac{1}{2}\left[f_0\left(\frac{\pi}{4}+\frac{\pi}{2}\right) + f_0\left(\frac{\pi}{4}-\frac{\pi}{2}\right)\right]$$

$$= \frac{1}{2}\left[\underbrace{f_0\left(\frac{3\pi}{4}\right)}_{=0} + \underbrace{f_0\left(-\frac{\pi}{4}\right)}_{=-1}\right] = -\frac{1}{2}.$$

#21.
$$\begin{cases} u_t - 4u_{xx} = 0, & 0 < x < \pi, \quad t > 0 \\ u(0,t) = 0, & u(\pi, t) = 0 \\ u(x,0) = f(x) = x(\pi - x), & 0 \le x \le \pi \end{cases}$$

$k = 4, \quad L = \pi$

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-k\left(\frac{n\pi}{\cancel{L}}\right)^2 t} \sin\left(\frac{n\pi x}{\cancel{L}}\right)$$

$$= \sum_{n=1}^{\infty} A_n e^{-4n^2 t} \sin(nx)$$

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin(nx) = f(x).$$

$$A_n = \frac{4}{\pi}\left[\frac{1-(-1)^n}{n^3}\right].$$

$\therefore\ u(x,t) = \sum_{n=1}^{\infty} \frac{4}{\pi}\left[\frac{1-(-1)^n}{n^3}\right] e^{-4n^2 t} \sin(nx)$

#22. $f(x)$ is periodic in $\mathbb{R}$ with

$p = 2\pi$ & one full period: $f(x) = x$, $-\pi < x < \pi$

Use Parseval's identity:

$$\int_{-\pi}^{\pi} f(x)^2 \, dx = \int_{-\pi}^{\pi} F(x)^2 \, dx$$

$$F(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

$$\int_{-\infty}^{\infty} F(x)^2 \, dx = \left(2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)\right)\pi$$

$$F(x) = \sum_{n=1}^{\infty} \underbrace{\frac{2(-1)^{n+1}}{n}}_{\text{"}b_n.} \sin(nx) : \quad \underline{a_0 = 0,\ a_n = 0}$$

$$\int_{-\pi}^{\pi} f(x)^2 \, dx = \pi \sum_{n=1}^{\infty} b_n^2 = \pi \sum_{n=1}^{\infty} \frac{4 \cdot 1}{n^2}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{4\pi}\int_{-\pi}^{\pi} x^2 \, dx = \frac{1}{4\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{12\pi}\left(\pi^3 + (+\pi)^3\right) = \frac{\cancel{2}\,\pi^{\cancel{3}\,2}}{6\cancel{12}\,\cancel{\pi}}$$

$$= \frac{\pi^2}{6}$$

$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$: Find z s.t. $\zeta(z) = 0$.