Observer-Based Feedback Control Design

- **Objective**: Construct Combined Observer-Controller Compensator to control linear lumped continuous-time (CT) or discrete-time (DT) system using only input-output signals
- We consider linear time-varying (LTI) system

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}u(t) \\ y(t) &= \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t) \end{aligned}$$

or

$$\begin{aligned} \boldsymbol{x}[k+1] &= \boldsymbol{A}\boldsymbol{x}[k] + \boldsymbol{b}u[k] \\ \boldsymbol{y}[k] &= \boldsymbol{C}\boldsymbol{x}[k] + \boldsymbol{D}\boldsymbol{u}[k] \end{aligned}$$

- We assume that the system at hand is both reachable and observable

Combined Observer-Controller Compensator

- The equation of the observer,

$$\dot{\tilde{\boldsymbol{x}}}(t) = \boldsymbol{A}\tilde{\boldsymbol{x}}(t) + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{L}\left(\boldsymbol{y}(t) - \tilde{\boldsymbol{y}}(t)\right)$$

- In the stability analysis of the closed-loop system driven by the combined observer-controller compensator, take into account that $\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t)$ and $\tilde{\boldsymbol{y}}(t) = \boldsymbol{C}\tilde{\boldsymbol{x}}(t)$
- Substituting the above expressions for $\boldsymbol{y}(t)$ and $\tilde{\boldsymbol{y}}(t)$ gives

$$\begin{aligned} \dot{\tilde{\boldsymbol{x}}}(t) &= \boldsymbol{A}\tilde{\boldsymbol{x}}(t) + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{L}\left(\boldsymbol{C}\boldsymbol{x}(t) - \boldsymbol{C}\tilde{\boldsymbol{x}}(t)\right) \\ &= (\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C})\tilde{\boldsymbol{x}}(t) + \boldsymbol{L}\boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) \end{aligned}$$

- Note that we do not implement the observer using the above representation; it is only for the stability analysis of the closed-loop system

Combined Observer-Controller Compensator Analysis

- Equivalent observer implementation format

$$\dot{\tilde{x}}(t) = (\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C})\tilde{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{L}\boldsymbol{y}(t)$$

- Equations of the closed-loop system

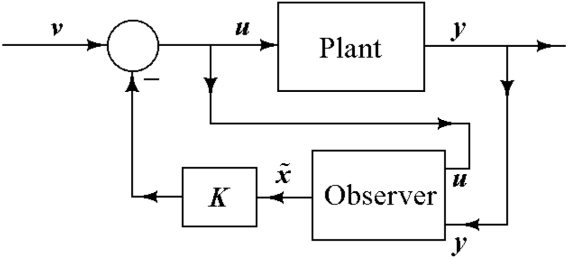
$$\left.\begin{array}{rcl} \begin{bmatrix} \dot{x}(t) \\ \dot{\tilde{x}}(t) \end{bmatrix} & = & \begin{bmatrix} \boldsymbol{A} & \boldsymbol{O} \\ \boldsymbol{L}\boldsymbol{C} & \boldsymbol{A} - \boldsymbol{L}\boldsymbol{C} \end{bmatrix} \begin{bmatrix} x(t) \\ \tilde{x}(t) \end{bmatrix} + \begin{bmatrix} \boldsymbol{B} \\ \boldsymbol{B} \end{bmatrix} \boldsymbol{u}(t) \\ \boldsymbol{y}(t) & = & \begin{bmatrix} \boldsymbol{C} & \boldsymbol{O} \end{bmatrix} \begin{bmatrix} x(t) \\ \tilde{x}(t) \end{bmatrix} \end{array}\right\}$$

Controller implementation

- The control law

$$\boldsymbol{u}(t) = -\boldsymbol{K}\tilde{\boldsymbol{x}}(t) + \boldsymbol{v}(t)$$

instead of the actual state-feedback control law



Closed-loop system

- Closing the loop

$$\left.\begin{array}{rcl} \begin{bmatrix} \dot{\boldsymbol{x}}(t) \\ \dot{\tilde{\boldsymbol{x}}}(t) \end{bmatrix} & = & \begin{bmatrix} \boldsymbol{A} & \boldsymbol{O} \\ \boldsymbol{LC} & \boldsymbol{A}-\boldsymbol{LC} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(t) \\ \tilde{\boldsymbol{x}}(t) \end{bmatrix} + \begin{bmatrix} \boldsymbol{B} \\ \boldsymbol{B} \end{bmatrix} (-\boldsymbol{K}\tilde{\boldsymbol{x}}(t) + \boldsymbol{v}(t)) \\ \boldsymbol{y}(t) & = & \begin{bmatrix} \boldsymbol{C} & \boldsymbol{O} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(t) \\ \tilde{\boldsymbol{x}}(t) \end{bmatrix} \end{array}\right\}$$

- Closed-loop system,

$$\begin{array}{rcl} \begin{bmatrix} \dot{\boldsymbol{x}}(t) \\ \dot{\tilde{\boldsymbol{x}}}(t) \end{bmatrix} & = & \begin{bmatrix} \boldsymbol{A} & -\boldsymbol{BK} \\ \boldsymbol{LC} & \boldsymbol{A}-\boldsymbol{LC}-\boldsymbol{BK} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(t) \\ \tilde{\boldsymbol{x}}(t) \end{bmatrix} + \begin{bmatrix} \boldsymbol{B} \\ \boldsymbol{B} \end{bmatrix} \boldsymbol{v}(t) \\ \boldsymbol{y}(t) & = & \begin{bmatrix} \boldsymbol{C} & \boldsymbol{O} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(t) \\ \tilde{\boldsymbol{x}}(t) \end{bmatrix} \end{array}$$

- To analyze the above closed-loop system, it is convenient to perform a change of coordinates

Closed-loop system analysis

- Use the transformation

$$\begin{bmatrix} x \\ x - \tilde{x} \end{bmatrix} = \begin{bmatrix} I_n & O \\ I_n & -I_n \end{bmatrix} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix}$$

- Note that

$$\begin{bmatrix} I_n & O \\ I_n & -I_n \end{bmatrix}^{-1} = \begin{bmatrix} I_n & O \\ I_n & -I_n \end{bmatrix}$$

Closed-loop system in the new coordinates

- The closed-loop system in the new coordinates

$$\begin{aligned}
\begin{bmatrix} \dot{\boldsymbol{x}}(t) \\ \dot{\boldsymbol{x}}(t) - \dot{\tilde{\boldsymbol{x}}}(t) \end{bmatrix} &= \begin{bmatrix} \boldsymbol{A} - \boldsymbol{BK} & \boldsymbol{BK} \\ \boldsymbol{O} & \boldsymbol{A} - \boldsymbol{LC} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(t) \\ \boldsymbol{x}(t) - \tilde{\boldsymbol{x}}(t) \end{bmatrix} \\
&\quad + \begin{bmatrix} \boldsymbol{B} \\ \boldsymbol{O} \end{bmatrix} \boldsymbol{v}(t) \\
\boldsymbol{y}(t) &= \begin{bmatrix} \boldsymbol{C} & \boldsymbol{O} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(t) \\ \boldsymbol{x}(t) - \tilde{\boldsymbol{x}}(t) \end{bmatrix}
\end{aligned}$$

- Note that $\boldsymbol{e}(t) = \boldsymbol{x}(t) - \tilde{\boldsymbol{x}}(t)$ is the estimation error

Transfer function of the closed-loop system

- The subsystem corresponding to the error component $\boldsymbol{e}(t) = \boldsymbol{x}(t) - \tilde{\boldsymbol{x}}(t)$ is unreachable
- The $2n$ poles of the closed-loop system are equal to the individual eigenvalues of both $\boldsymbol{A} - \boldsymbol{LC}$ and $\boldsymbol{A} - \boldsymbol{BK}$
- Thus the design of the observer is separated from the construction of the controller—the separation principle
- The closed-loop transfer function relating $\boldsymbol{Y}(s)$ and $\boldsymbol{V}(s)$ is

$$
\begin{aligned}
\boldsymbol{Y}(s) &= \begin{bmatrix} \boldsymbol{C} & \boldsymbol{O} \end{bmatrix} \begin{bmatrix} s\boldsymbol{I}_n - \boldsymbol{A} + \boldsymbol{BK} & -\boldsymbol{BK} \\ \boldsymbol{O} & s\boldsymbol{I}_n - \boldsymbol{A} + \boldsymbol{LC} \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{B} \\ \boldsymbol{O} \end{bmatrix} \boldsymbol{V}(s) \\
&= \boldsymbol{C} \left(s\boldsymbol{I}_n - \boldsymbol{A} + \boldsymbol{BK} \right)^{-1} \boldsymbol{B}\boldsymbol{V}(s)
\end{aligned}
$$

- The closed-loop system driven by the combined observer-controller compensator has is the same transfer function as the system driven by the state-feedback control law $\boldsymbol{u}(t) = -\boldsymbol{K}\boldsymbol{x}(t) + \boldsymbol{v}(t)$

Observer pole selection

- The combined observer-controller compensator yields the same closed-loop transfer function as the actual state-feedback control law
- It is recommended that the real parts of the observer poles, that is, the real parts of the eigenvalues of the matrix $\boldsymbol{A} - \boldsymbol{LC}$, be a factor of 2 to 6 times deeper in the open left-half plane than the real parts of the controller poles which are the eigenvalues of the matrix $\boldsymbol{A} - \boldsymbol{BK}$
- Such a choice ensures a faster decay of the observer error $\boldsymbol{e}(t) = \boldsymbol{x}(t) - \tilde{\boldsymbol{x}}(t)$ compared with the desired controller dynamics
- This in turn causes the controller poles to dominate the closed-loop system response

Combined observer-controller compensator design final comments

- The observer poles represent a measure of the speed with which the estimation error $\boldsymbol{e}(t) = \boldsymbol{x}(t) - \tilde{\boldsymbol{x}}(t)$ decays to zero, one would tend to assign observer poles deep in the left-hand plane
- However, fast decay requires large gains which may lead to saturation of some signals and unpredictable nonlinear effects
- If the observer poles were slower than the controller poles, the closed-loop system response would be dominated by the observer, which is undesirable
- As it is usual in engineering practice, the term compromise could be used to describe the process of constructing the final compensator structure