(Test) Examity:

1. During the ~~t~~ test, you can open pdf files of textbook, notes, HW

2. Digital textbook (the website) is allowed.

#6 $Y=\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$: $Y'=\underbrace{\begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}}_{=A} Y$

λ: $|A-\lambda I| = (-1-\lambda)^2 - (-1) = (\lambda+1)^2+1 = 0$

$\lambda = -1 \pm i$. $(\alpha=-1, \beta=1)$

V: $\lambda = -1+i$: $A-(-1+i)I = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}$

$\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$: $-iv_1 + v_2 = 0$

$v_2 = iv_1$

$V_1 = \begin{bmatrix} 1 \\ i \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{=a} + i\underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{=b}$

$Y(t) = c_1 e^{-t}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\cos(t) - \begin{bmatrix} 0 \\ 1 \end{bmatrix}\sin(t)\right)$

$+ c_2 e^{-t}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\sin(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\cos(t)\right)$

(2) $(0,0)$: an asymptotically stable spiral point.

#7. $\begin{cases} y_1' = \sin(y_1) + y_2 \overset{\text{let}}{=} f_1(y_1,y_2) \\ y_2' = 4y_1 + \sin(y_2) = f_2(y_1,y_2) \end{cases}$

(1) $J = \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{bmatrix} = \begin{bmatrix} \cos(y_1) & 1 \\ 4 & \cos(y_2) \end{bmatrix}$

$J(0,0) = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$: $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

(2) $|J-\lambda I| = (1-\lambda)^2 - 4 = \lambda^2 - 2\lambda - 3 = 0$

$(\lambda+1)(\lambda-3) = 0$: $\lambda = -1, 3$

$(0,0)$: a unstable saddle point.

#8 the plot: (0,0): a unstable improper node ($\lambda_1>0, \lambda_2>0$)

A: $\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$ $(2-\lambda)^2+1=0$: $\lambda = 2 \pm i$

B: $\begin{bmatrix} 2 & 2 \\ -2 & 6 \end{bmatrix}$ $(2-\lambda)(6-\lambda)+4=0$

$\lambda^2-8\lambda+16=0$ $\lambda = 4, 4.$

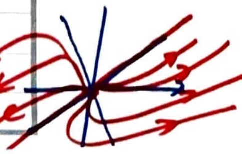
C $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$: $(2-\lambda)^2-1=0$

$\lambda^2-4\lambda+3=0$ $(\lambda-1)(\lambda-3)=0$

$\lambda=1, 3$

•. Case1 : C

Case 3



#9. $Y' = \underbrace{\begin{bmatrix} 8 & -1 \\ 1 & 10 \end{bmatrix}}_{A} Y$

λ:

$|A-\lambda I| = (8-\lambda)(10-\lambda)+1=0$

$\lambda^2-18\lambda+81=0$ $(\lambda-9)^2=0$: $\lambda = 9, 9$

V_1: $(A-9I)V = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$v_1+v_2=0$: $v_2=-v_1$

Let $V_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$: $Y_1(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}e^{9t}$

$Y_2(t) \overset{let}{=} \begin{bmatrix} 1 \\ -1 \end{bmatrix}te^{9t} + V_2e^{9t}$

① $Y_2' = \begin{bmatrix} 1 \\ -1 \end{bmatrix}e^{9t} + \cancel{9\begin{bmatrix} 1 \\ -1 \end{bmatrix}te^{9t}} + 9V_2e^{9t}$

② $= AY_2 = A\begin{bmatrix} 1 \\ -1 \end{bmatrix}te^{9t} + AV_2e^{9t}$ =

$= \cancel{9\begin{bmatrix} 1 \\ -1 \end{bmatrix}te^{9t}} + AV_2e^{9t}$

$\begin{bmatrix} 1 \\ -1 \end{bmatrix}\cancel{e^{9t}} + 9V_2\cancel{e^{9t}} = AV_2\cancel{e^{9t}}$

$(A-9I)V_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$: $\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$(-1)v_1 - v_2 = 1$: $-v_2 = 1+v_1$

$v_2 = -1-v_1$: $V_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \dots$

$$y_2(t) = \begin{bmatrix}1\\-1\end{bmatrix} t e^{9t} + \begin{bmatrix}0\\-1\end{bmatrix} e^{9t}$$

$$y(t) = c_1 y_1 + c_2 y_2 \quad \text{(A)}$$

(d) $(0,0)$: a degenerate node (proper)

#10. $\mathcal{L}$: $\begin{cases} y'' - 2y' - 3y = \delta(t-2) \\ y(0) = 5, \quad y'(0) = 0 \end{cases}$

$$\mathcal{L}(y'') - 2\mathcal{L}(y') - 3\mathcal{L}(y) = \mathcal{L}(\delta(t-2))$$

$$s^2\mathcal{L}(y) - s\underbrace{y(0)}_{5} - \underbrace{y'(0)}_{0} - 2(s\mathcal{L}(y) - \underbrace{y(0)}_{5}) - 3\mathcal{L}(y) = e^{-2s}.$$

$$(s^2 - 2s - 3)\mathcal{L}(y) - \underline{5s + 10} = e^{-2s}$$

$$\mathcal{L}(y) = \frac{5s-10}{s^2-2s-3} + e^{-2s}\frac{1}{(s-3)(s+1)}$$

$$\frac{5s-10}{(s-3)(s+1)} = \frac{A}{s-3} + \frac{B}{s+1} \qquad A = \tfrac{5}{4}, \quad B = \tfrac{15}{4}$$

$$5s - 10 = A(s+1) + B(s-3)$$

$$= (A+B)s + A - 3B$$

$$\begin{array}{rl} & A + B = 5 \\ - & A - 3B = -10. \\ \hline & 4B = 15 \end{array} \qquad \begin{array}{l} B = \tfrac{15}{4}, \quad A = 5 - B \\ A = 5 - \tfrac{15}{4} = \tfrac{5}{4} \end{array}$$

$$y(t) = \mathcal{L}^{-1}\left(\frac{5/4}{s-3} + \frac{15/4}{s-(-1)}\right) + \mathcal{L}^{-1}\left(e^{-2s}\left(\frac{1/4}{s-3} - \frac{1/4}{s+1}\right)\right)$$

$$= \frac{5}{4}e^{3t} + \frac{15}{4}e^{-t} + u(t-2)\left(\frac{1}{4}e^{3t} - \frac{1}{4}e^{-t}\right)_{t-2}$$

$$= \frac{5}{4}e^{3t} + \frac{15}{4}e^{-t} + \frac{u(t-2)}{4}\left(e^{3(t-2)} - e^{-(t-2)}\right)$$

#11. $f(t) = \begin{cases} 3, & 0 \le t < 2 \\ t+2, & 2 \le t < 5 \\ 4 & t \ge 5 \end{cases}$

$$f(t) = 3 + (t+2-3)u(t-2) + (4-(t+2))u(t-5)$$

$$f(t) = 3 + (t\underset{-2+2}{-1})\,u(t-2) + (\underbrace{-t+2}_{-\left(\substack{t-2 \\ -5+5}\right)})\,u(t-5)$$

$$f(t) = 3 + (t-2)\,u(t-2) + u(t-2)$$
$$- (t-5)\,u(t-5) - 3\,u(t-5)$$

$$L(f) = \frac{3}{s} + e^{-2s}\frac{1}{s^2} + \frac{1}{s}e^{-2s}$$
$$- e^{-5s}\frac{1}{s^2} - \frac{3}{s}e^{-5s}, \quad s>0$$

D

#12 (1) $F(s) = \dfrac{2s+4}{(s+1)^2+4} = \dfrac{2(s+1)+2}{(s+1)^2+2^2}$

$$L^{-1}(F) = 2L^{-1}\left(\frac{s+1}{(s+1)^2+2^2}\right) + L^{-1}\left(\frac{2}{(s+1)^2+2^2}\right)$$

$a=-1,\ b=2$

$$= 2e^{-t}\cos(2t) + e^{-t}\sin(2t).$$

(2) $F(s) = e^{-3s}\dfrac{1}{(s-5)^2} = G(s)$ #27

$L^{-1}(G) = L^{-1}\left(\dfrac{1}{(s-5)^2}\right)$ #23 $n=1$.

$$= e^{5t}\,t \overset{\text{let}}{=} g(t)$$

$$L^{-1}(F) = u(t-3)\,g(t-3) = \underline{u(t-3)\,e^{5(t-3)}(t-3)}$$

#13 $\mathbb{P}_3$

A. $\{1+t^2\}$: the set of one polynomial, $1+t^2$: not a vector space

B.. $at+b,\ at^2+bt+c \in \mathbb{P}_3$

$p(t) = a_1 t + b_1 t^2 + (a_1+b_1)t^3 \in W_2$

$W_2 \overset{\text{let}}{=} \{at+bt^2+(a+b)t^3 \mid a, b \in \mathbb{R}\}$

$q(t) = a_2 t + b_2 t^2 + (a_2+b_2)t^3$

① $p(t)+q(t) = (a_1+a_2)t + (b_1+b_2)t^2 + (a_1+b_1+a_2+b_2)t^3$
$\in W_2.$

$\beta \in \mathbb{R}$:

② $\beta \cdot p(t) = \beta a_1 t + \beta b_1 t^2 + (\beta a_1 + \beta b_1)t^3 \in W_2$

$\therefore W_2$: a subspace of $\mathbb{P}_3$

(C) $2p(t) = 2a + 2bt + 2abt^2$ (X)

$p(t) = a+bt+abt^2$: $2a \cdot 2b = 4ab$

(D) $p(t), q(t) \in$ " : $p(2)=0,\ q(2)=0$

$p(2)+q(2) = 0+0=0$

$\beta p(2) = \beta \cdot 0 = 0.$