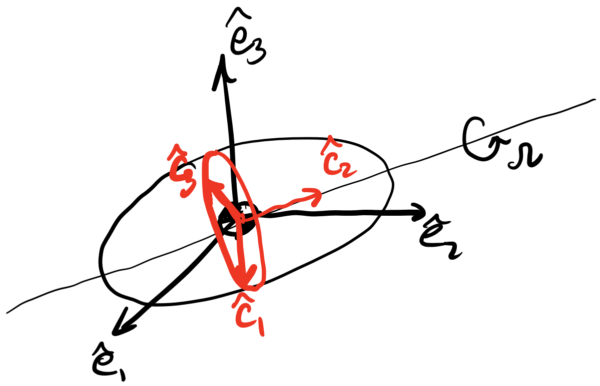
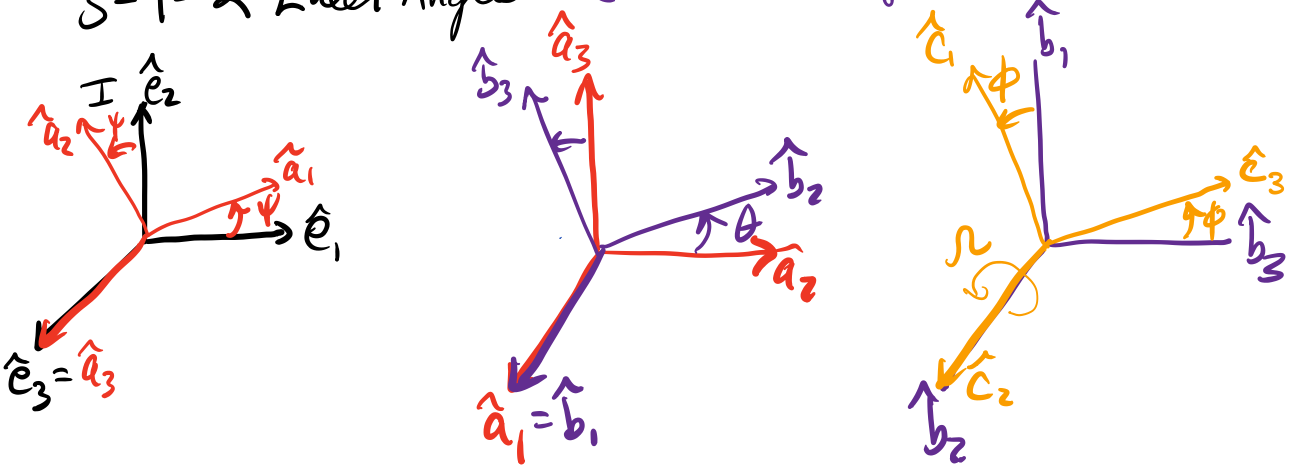
Ex 11.9 Dynamics of a spinning symmetric body



Find the E.O.M.

3-1-2 Euler Angles (Elementary rotation view)



(Note: For spinning body problems, it is sometimes easier to work in the "next to last" frame, which is the B frame here.)

To write down the dynamics:

$$\frac{{}^{B}d}{dt}\left({}^{I}\vec{h}_G\right) + {}^{I}\vec{\omega}^{B} \times {}^{I}\vec{h}_G = \vec{M}_G$$

$${}^{I}\vec{\omega}^{B} = {}^{I}\vec{\omega}^{A} + {}^{A}\vec{\omega}^{B} = \dot{\psi}\hat{a}_3 + \dot{\theta}\hat{b}_1$$

$$= \underbrace{\dot{\theta}}_{\omega_1}\hat{b}_1 + \underbrace{\dot{\psi}s\theta}_{\omega_2}\hat{b}_2 + \underbrace{\dot{\psi}c\theta}_{\omega_3}\hat{b}_3$$

Define

$$\left[{}^{I}\vec{\omega}^{B}\right]_B = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}_B$$

$$
{}^I\vec{h}_G = \mathbb{I}_G \circ {}^I\vec{\omega}^C = \mathbb{I}_G \circ \left({}^I\vec{\omega}^B + \underbrace{{}^B\vec{\omega}^C}_{\Omega \hat{b}_2}\right)
$$

We need the moment of inertia matrix in the B-frame.
What is it in the **C** frame?

$$
[\mathbb{I}_G]_C = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}_C
$$

Note: C Frame is a principle axis frame

Further symmetry gives $I_1 = I_3$

If you do the calculation,

$$
[\mathbb{I}_G]_B = ({}^BC^C)[\mathbb{I}_G]_C({}^BC^C)^T = [\mathbb{I}_G]_C
$$

This M.O.I. matrix is unaffected by ${}^BC^C$

$$
[{}^I\vec{h}_G]_B = [\mathbb{I}_G]_B\,[{}^I\vec{\omega}^C]_B = \begin{bmatrix} I_1\,\dot{\theta} \\ I_2(\dot{\psi}s\theta + \Omega) \\ I_3\,\dot{\psi}c\theta \end{bmatrix}_B
$$

We want to apply Euler's Equations:

$$
\frac{{}^Bd}{dt}[{}^I\vec{h}_G]_B + [{}^I\vec{\omega}^B \times {}^I\vec{h}_G]_B = [\vec{M}_G]_B
$$

$$\frac{{}^{B}d}{dt}\left[{}^{I}\vec{h}_G\right]_B = \begin{bmatrix} I_1\ddot{\theta} \\ \frac{d}{dt}\left(I_2(\dot{\psi}s\theta+\Omega)\right) \\ I_1(\ddot{\psi}c\theta - \dot{\psi}s\theta\dot{\theta}) \end{bmatrix}_B$$

Cross product equivalent matrix

$$\underbrace{\left[{}^{I}\vec{\omega}^{B}\times\right]_B}_{\begin{bmatrix}\omega_1\\\omega_2\\\omega_3\end{bmatrix}_B}\left[{}^{I}\vec{h}_G\right]_B = \underbrace{\begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}}_{\text{Skew-symmetric matrix } S=-S^T}\begin{bmatrix} I_1\dot{\theta} \\ I_2(\dot{\psi}s\theta+\Omega) \\ I_1\dot{\psi}c\theta \end{bmatrix}_B$$

What happens if we apply a moment $\left[\vec{M}_G\right]_B = \begin{bmatrix} M_1 \\ 0 \\ 0 \end{bmatrix}_B$

$$\begin{cases} I_1\ddot{\theta} + (I_1 - I_2)\dot{\psi}^2 s\theta c\theta - I_2\Omega\dot{\psi}c\theta = M_1 \\ \frac{d}{dt}\left(I_2(\dot{\psi}s\theta+\Omega)\right) = 0 \Rightarrow I_2(\dot{\psi}s\theta+\Omega) = \text{constant based on I.C.'s} \\ I_1\ddot{\psi}c\theta + (I_2 - 2I_1)\dot{\psi}\dot{\theta}s\theta + I_2\Omega\dot{\theta} = 0 \end{cases}$$

Equations of motion provide
- Precession → slow rotation of the symmetry axis (change in ψ variable)
- Nutation → wobble of the symmetry axis (change in θ variable)

These equations provide
- Precession
- Nutation

These are dynamical effects exhibited by a spinning body subject to a constant torque.

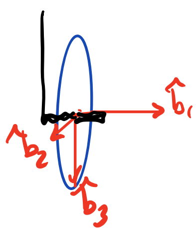
For Earth: Precession of the equinoxes → Period 26,000 years
Chandler Wobble

"Qualitative analysis" of a rapidly spinning rigid body

$$\frac{{}^{B}d}{dt}\left({}^{I}\vec{h}_G\right) \simeq 0 \quad \text{For a rapidly spinning body}$$

$$\Rightarrow \quad {}^{I}\vec{\omega}^{B} \times {}^{I}\vec{h}_G = \vec{M}_G$$

This is an approximate constraint on the dynamic system that holds and can be used to analyze gyroscopic effects.

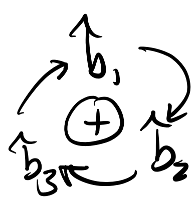


$$\underline{{}^{I}\vec{\omega}^{B}} \times {}^{I}\vec{h}_G = \vec{M}_G$$

What cross $\hat{b}_1$ is $\hat{b}_2$?

What happens when I apply a moment $\vec{M} = M\hat{b}_2$

This causes ${}^{I}\vec{\omega}^{B}$ to have a component in the $\hat{b}_3$ direction



See Walter Lewin video on youtube at 35:00

Euler's Equations

↳ Application of Euler's 2nd law to a spinning rigid body

$$\frac{{}^{B}d}{dt}\left({}^{I}\vec{h}_G\right) + {}^{I}\vec{\omega}^{B} \times {}^{I}\vec{h}_G = \vec{M}_G$$

$$[\mathbb{I}_G]_B = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}_B$$

(Frame B is a principle axes frame)

with $I_3 \geq I_2 \geq I_1$

$${}^{I}\vec{h}_G = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}_B \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}_B = \begin{bmatrix} I_1\omega_1 \\ I_2\omega_2 \\ I_3\omega_3 \end{bmatrix}_B \Rightarrow \frac{{}^{B}d}{dt}\left({}^{I}\vec{h}_G\right) = \begin{bmatrix} I_1\dot{\omega}_1 \\ I_2\dot{\omega}_2 \\ I_3\dot{\omega}_3 \end{bmatrix}_B$$

$$\left[{}^{I}\vec{\omega}^{B} \times {}^{I}\vec{h}_G\right]_B = \left[{}^{I}\vec{\omega}^{B}\times\right]_B \left[{}^{I}\vec{h}_G\right]_B = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \begin{bmatrix} I_1\omega_1 \\ I_2\omega_2 \\ I_3\omega_3 \end{bmatrix}$$

Collect all the terms:

$$\boxed{\begin{aligned} I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 &= M_1 \\ I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 &= M_2 \\ I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 &= M_3 \end{aligned}}$$

Euler's Equations for a spinning rigid body.

What if the body is free spinning, $\vec{M}_G = 0$?

$$\frac{{}^{I}d}{dt}\left({}^{I}\vec{h}_G\right) = 0 \Rightarrow {}^{I}\vec{h}_G \text{ is conserved}$$

We also know that T_G is conserved

Rewrite when $\vec{M}_G = 0$,

$$I_1 \dot{\omega}_1 = (I_2 - I_3)\omega_2\omega_3$$
$$I_2 \dot{\omega}_2 = (I_3 - I_1)\omega_1\omega_3$$
$$I_3 \dot{\omega}_3 = (I_1 - I_2)\omega_1\omega_2$$

What are the equilibria?

$$0 = (I_2 - I_3)\omega_2\omega_3$$
$$0 = (I_3 - I_1)\omega_1\omega_3$$
$$0 = (I_1 - I_2)\omega_1\omega_2$$

$I_3 > I_2 > I_1 > 0$

Trivial case: $\omega_1 = \omega_2 = \omega_3 = 0$

Other Cases:

1) $\omega_1 \neq 0,\ \omega_2 = \omega_3 = 0$
2) $\omega_2 \neq 0,\ \omega_1 = \omega_3 = 0$
3) $\omega_3 \neq 0,\ \omega_1 = \omega_2 = 0$

If $\|\vec{h}_G\|$ is conserved,

$$\|\vec{h}_G\|^2 = (I_1\omega_1)^2 + (I_2\omega_2)^2 + (I_3\omega_3)^2 \triangleq h_G^2$$

Divide by h_G^2

$$1 = \frac{\omega_1^2}{(h_G/I_1)^2} + \frac{\omega_2^2}{(h_G/I_2)^2} + \frac{\omega_3^2}{(h_G/I_3)^2}$$

3D momentum Ellipsoid

We can similarly analyze kinetic energy

$$T_G = \frac{1}{2}\begin{bmatrix}\omega_1 & \omega_2 & \omega_3\end{bmatrix}\begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3\end{bmatrix}\begin{bmatrix}\omega_1 \\ \omega_2 \\ \omega_3\end{bmatrix} = \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2)$$

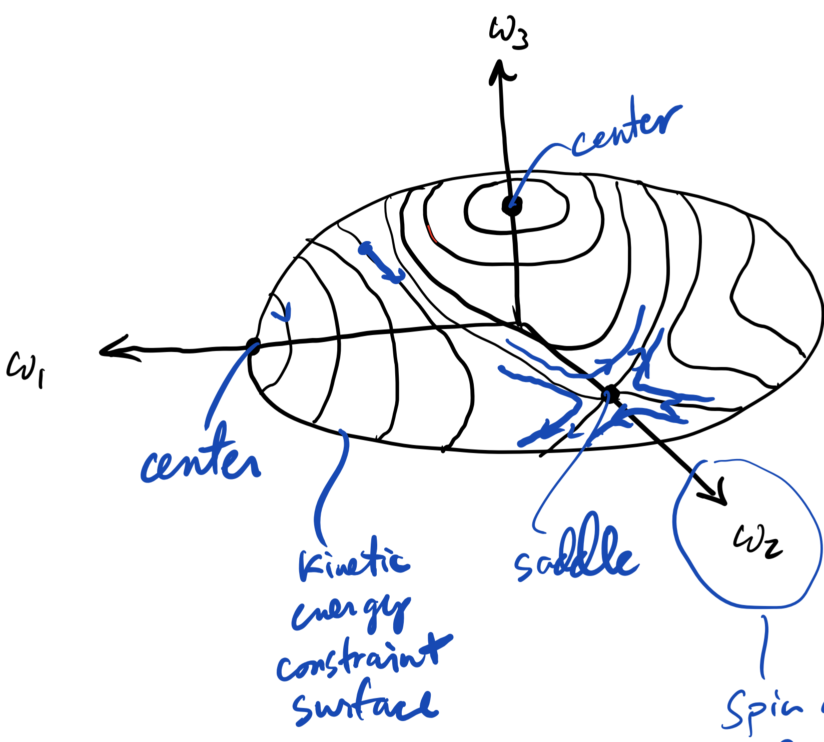
Dividing by T_G

$$1 = \frac{\omega_1^2}{2(T_G/I_1)} + \frac{\omega_2^2}{2(T_G/I_2)} + \frac{\omega_3^2}{2(T_G/I_3)}$$

3D Kinetic energy ellipsoid.

These ellipsoids are due to the conserved quantities and these equations are constraint equations that define constraint surfaces on which the solution must live.

$I_3 > I_2 > I_1$



Dissipation disturbs trajectories

Spin about 2 axis is unstable

Dzhanibekov effect

- Falling wing nut video (Viral video lab)
- Tennis Racket effect (Intermediate axis theorem) Veritasium youtube video 1:40