(Bonus problem). $HX=b$.

GE: big errors.

(Iterations)

cg (Conjugate Gradient), pcg, bicg.

SVD, pseudoinv

gmres, minres, lsqminres

TFQMR/ $n=10^5$ (GS) Gauss-Seidel, ...

regularization: $(H+\varepsilon I)X=b$, $\varepsilon=10^{-5}, 10^{-8}, 10^{-10}$

"Numerical Linear Algebra"

(Fourier integral) $\int_{-\infty}^{\infty} |f(x)|\,dx$ is finite

$$F(x) = \int_0^{\infty} A(\omega)\cos(\omega x)\,d\omega + \int_0^{\infty} B(\omega)\sin(\omega x)\,d\omega$$

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(z)\cos(\omega z)\,dz,$$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(z)\sin(\omega z)\,dz$$

Thm

$f(x)$ is piecewise continuous in every finite interval

(H) $f(x)$ has a right-hand derivatives & a left-hand derivative at every point.

$\int_{-\infty}^{\infty} |f(x)|\,dx$ is finite

(C) (1) If $f(x)$ is continuous at $x_0 \in \mathbb{R}$,

$f(x) = F(x)$

(2) If $f(x)$ is discontinuous at $x_0 \in \mathbb{R}$,

then $F(x_0) = \frac{1}{2}\left[\lim_{x\to x_0-} f(x) + \lim_{x\to x_0+} f(x)\right]$

11.7. Fourier integral.

(1) $f(x)$ is an even function, & $\int_{-\infty}^{\infty} |f(x)|\,dx < \infty$ finite.

$$\begin{cases} B(w) \equiv 0 \\ A(w) = \frac{2}{\pi}\int_0^{\infty} f(z)\cos(wz)\,dz \end{cases}$$

$$F(x) = \int_0^{\infty} A(w)\cos(wx)\,dw = F_c(x)$$

the Fourier Cosine integral of f

(2) $f(x)$ is odd in $\mathbb{R}$, $\int_{-\infty}^{\infty} |f(x)|\,dx < \infty$

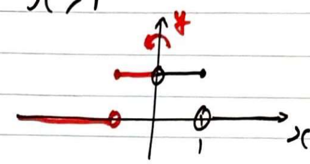
$$A(w) \equiv 0$$

$$B(w) = \frac{2}{\pi}\int_0^{\infty} f(z)\sin(wz)\,dz$$

$$F(x) = \int_0^{\infty} B(w)\sin(wx)\,dw = F_s(x)$$

(Ex) $f(x) = \begin{cases} 1, & 0 < x \le 1 \\ 0 & x > 1 \end{cases}$ (missing data)

(1) Even extension



$$f_E(x) = \begin{cases} f(x), & x > 0 \\ f(-x), & x < 0 \end{cases} \text{ : even.}$$

$$F(x) = \int_0^{\infty} A(w)\cos(wx)\,dw = F_c(x)$$

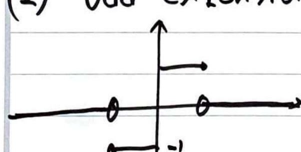
$$A(w) = \frac{2}{\pi}\int_0^{\infty} f(z)\cos(wz)\,dz$$

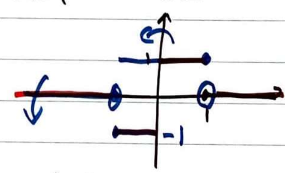
$$A(w) = \frac{2}{\pi}\int_0^{1}\cos(wz)\,dz = \frac{2}{\pi}\left[\frac{\sin(wz)}{w}\right]_0^1$$

$$= \frac{2}{w\pi}(\sin w - 0) = \frac{2\sin w}{\pi w}$$

$$F_c(x) = \int_0^{\infty}\left(\frac{2\sin w}{\pi w}\right)\cos(wx)\,dw$$

(2) Odd extension





$$f_O(x) = \begin{cases} f(x), & x > 0 \\ -f(-x), & x < 0 \end{cases} \text{ : odd}$$

$$A(w) = 0$$

$$B(w) = \frac{2}{\pi}\int_0^{\infty} f(x)\sin(wx)\,dx$$

$$= \frac{2}{\pi}\int_0^{1}\sin(wx)\,dx = \frac{2}{\pi}\left[-\frac{\cos(wx)}{w}\right]_0^1$$

$$= \frac{-2}{w\pi}(\cos(w) - 1) = \frac{2(1-\cos w)}{\pi w}$$

$$F(x) = \int_0^\infty \left(\frac{2(1-\cos w)}{\pi w}\right) \sin(wx)\, dx = F_s(x)$$

11.8 Fourier cosine/sine transform.

Topic/motive Laplace transform

$$f(t) \underset{L^{-1}}{\overset{L}{\rightleftarrows}} L(f) = F(s)$$

$$g(x) \underset{\mathcal{F}_c^{-1}\ /\ \mathcal{F}_s^{-1}}{\overset{\mathcal{F}_c / \mathcal{F}_s}{\rightleftarrows}} \hat{g}_c(w) \text{ or } \hat{g}_s(w)$$

(Missing data) $f(x)$ is defined on $(0, \infty)$

Let $f(x)$ be continuous on $(0, \infty)$

$$F_c(x) = \int_0^\infty A(w) \cos(wx)\, dw$$

$$A(w) = \frac{2}{\pi} \int_0^\infty f(x) \cos(wx)\, dx \ : \ \frac{2}{\pi} = \sqrt{\frac{2}{\pi}}^{\,2}$$

Let $\hat{f}_c(w) = \sqrt{\frac{\pi}{2}} A(w)$

<u>Def</u>

1. $\hat{f}_c(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos(wx)\, dx$

   : the Fourier cosine transform of f

2. $\mathcal{F}_c^{-1}(\hat{f}_c)(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_c(w) \cos(wx)\, dw$

   : the inverse Fourier cosine transform of $\hat{f}_c$

3. $\hat{f}_s(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin(wx)\, dx$

   : the Fourier sine transform of f

4. $\mathcal{F}_s^{-1}(\hat{f}_s)(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_s(w) \sin(wx)\, dw$

   : the inverse Fourier sine transform of $\hat{f}_s$

<u>Remark</u> $\mathcal{F}_c(f) = \hat{f}_c, \quad \mathcal{F}_s(f) = \hat{f}_s$

(Ex) $f(x) = \begin{cases} 1, & 0 \le x < 1 \\ 0, & x \ge 1. \end{cases}$

(1) $\hat{f}_c(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos(wx)\, dx$

$$= \sqrt{\frac{2}{\pi}} \int_0^1 \cos(wx)\, dx = \sqrt{\frac{2}{\pi}} \left[\frac{\sin(wx)}{w}\right]_0^1$$

$$= \sqrt{\frac{2}{\pi}} \frac{\sin w}{w}$$

(2) $\hat{f}_s(w) = \sqrt{\frac{2}{\pi}}\,\frac{1-\cos(w)}{w}$

<u>Remark</u> $\mathcal{F}_c^{-1}(\hat{f}_c)_{(x)} = \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_c(w)\cos(wx)\,dw$

$$\mathcal{F}_c^{-1}(\hat{f}_c) = \sqrt{\frac{2}{\pi}}\int_0^\infty \left(\sqrt{\frac{2}{\pi}}\,\frac{\sin(w)}{w}\right)\cos(wx)\,dw = ?$$

$$= F_c(x)$$

$$\mathcal{F}_c^{-1}(\hat{f}_c) = F_c(x) = \begin{cases} 1, & 0 \le x < 1 \\ \frac{1}{2}, & x = 1 \\ 0 & x > 1 \end{cases}$$

$$\frac{2}{\pi}\int_0^\infty \frac{\sin(w)}{w}\cos(wx)\,dw = \begin{cases} 1, & 0 \le x < 1 \\ \frac{1}{2}, & x = 1 \\ 0, & x > 1 \end{cases}$$

$$\int_0^\infty \frac{\sin(w)\cos(wx)}{w}\,dw = \begin{cases} \frac{\pi}{2}, & 0 \le x < 1 \\ \frac{\pi}{4}, & x = 1 \\ 0, & x > 1 \end{cases}$$

(Ex2) $f(x) = \begin{cases} x, & 0 < x < 1 \\ 0, & x \geq 1 \end{cases}$

$$\hat{f}_s(w) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin(wx)\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^1 \underbrace{x}_{u}\,\underbrace{\sin(wx)}_{v'}\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left(\left[x\,\frac{(-1)\cos(wx)}{w}\right]_0^1 + \int_0^1 \frac{(+1)\cos(wx)}{w}\,dx\right)$$

$$= \sqrt{\frac{2}{\pi}}\left(\frac{(-1)\cos w}{w} - 0 + \frac{1}{w}\left[\frac{\sin(wx)}{w}\right]_0^1\right)$$

$$= \sqrt{\frac{2}{\pi}}\left(-\frac{\cos w}{w} + \frac{1}{w^2}(\sin w - 0)\right)$$

$$\hat{f}_s(w) = \sqrt{\frac{2}{\pi}}\left(\frac{\sin w}{w^2} - \frac{\cos w}{w}\right)$$

$$\mathcal{F}_s^{-1}(\hat{f}_s)_{(x)} = \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_s(w)\sin(wx)\,dw$$

$$= \begin{cases} x, & 0 < x < 1, \\ x/2 = \frac{1}{2}, & x = 1 \end{cases} \quad \mathcal{F}^{-1}(\hat{f}_s)(x) = 0,\ x > 1.$$