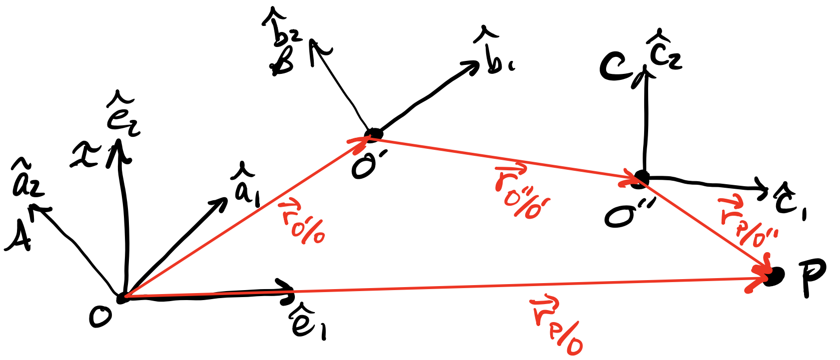
$\boxed{8.2.2}$ Addition of angular velocity vectors



Compare to Ex 2.5, which is a three-link robotic manipulator.
Note: the picture drawn here is setup more generally though.

$$\vec{r}_{P/O} = \vec{r}_{O''/O} + \vec{r}_{P/O''}$$

$$\vec{r}_{P/O} = \vec{r}_{O'/O} + \vec{r}_{O''/O'} + \vec{r}_{P/O''}$$

Apply T.E. to put in terms of frames A, B, and C.

$$\underbrace{\frac{{}^{I}d}{dt}(\vec{r}_{P/O})}_{{}^{I}\vec{v}_{P/O}} = \underbrace{\frac{{}^{I}d}{dt}(\vec{r}_{O'/O})}_{\substack{\text{Apply T.E. again} \\ {}^{A}\vec{v}_{O'/O} + {}^{I}\vec{\omega}^{A} \times \vec{r}_{O'/O}}} + {}^{B}\vec{v}_{O''/O'} + {}^{I}\vec{\omega}^{B} \times \vec{r}_{O''/O'} + {}^{C}\vec{v}_{P/O''} + {}^{I}\vec{\omega}^{C} \times \vec{r}_{P/O''}$$

Note that the angular velocities at all directly referenced to I.

(*)

$$ {}^{I}\vec{v}_{P/O} = {}^{A}\vec{v}_{O'/O} + {}^{I}\vec{\omega}^{A} \times \vec{r}_{O'/O} + {}^{B}\vec{v}_{O''/O'} + {}^{I}\vec{\omega}^{B} \times \vec{r}_{O''/O'} + {}^{C}\vec{v}_{P/O''} + {}^{I}\vec{\omega}^{C} \times \vec{r}_{P/O''}$$

Now rewrite these kinematics to express the intermediate angular velocities (e.g. ${}^{A}\vec{\omega}^{B}$) (See 3rd–5th eqns on pg 308)

$$ {}^{I}\vec{v}_{P/O} = \underbrace{\frac{{}^{I}d}{dt}(\vec{r}_{O'/O}) + \frac{{}^{I}d}{dt}(\vec{r}_{O''/O'}) + \frac{{}^{I}d}{dt}(\vec{r}_{P/O''})}$$

We previously applied T.E. referenced to I for each term.

Instead, use intermediate frames for reference.

$$\frac{{}^I d}{dt}(\quad) = \frac{{}^A d}{dt}(\quad) + {}^I\vec{\omega}^A \times (\quad)$$

Then

$$\frac{{}^A d}{dt}(\quad) = \frac{{}^B d}{dt}(\quad) + {}^A\vec{\omega}^B \times (\quad)$$

Following that process leads to,

$$ {}^I\vec{V}_{P/O} = {}^A\vec{V}_{O'/O} + {}^I\vec{\omega}^A \times \vec{r}_{O'/O} + {}^B\vec{V}_{O''/O'} + {}^A\vec{\omega}^B \times \vec{r}_{O''/O'} + {}^I\vec{\omega}^A \times \vec{r}_{O''/O'} + {}^C\vec{V}_{P/O''} + {}^B\vec{\omega}^C \times \vec{r}_{P/O''} + {}^A\vec{\omega}^B \times \vec{r}_{P/O''} + {}^I\vec{\omega}^A \times \vec{r}_{P/O''}$$

Regrouping and using the distributive property of cross product,

$$ {}^I\vec{V}_{P/O} = {}^A\vec{V}_{O'/O} + {}^I\vec{\omega}^A \times \vec{r}_{O'/O} + {}^B\vec{V}_{O''/O'} + \left({}^I\vec{\omega}^A + {}^A\vec{\omega}^B\right) \times \vec{r}_{O''/O'} + {}^C\vec{V}_{P/O''} + \left({}^I\vec{\omega}^A + {}^A\vec{\omega}^B + {}^B\vec{\omega}^C\right) \times \vec{r}_{P/O''} \qquad (**)$$

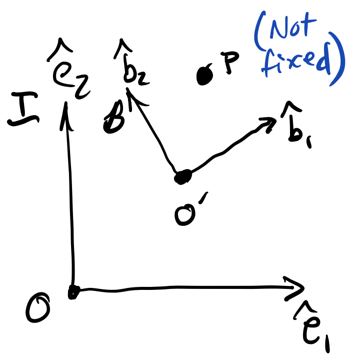
Now compare (*) and (**) equations and equate terms to see

$$ {}^I\vec{\omega}^B = {}^I\vec{\omega}^A + {}^A\vec{\omega}^B$$

$$ {}^I\vec{\omega}^C = {}^I\vec{\omega}^A + {}^A\vec{\omega}^B + {}^B\vec{\omega}^C$$

This shows that the angular velocity vectors add for frames in relative motion.

8.3 Planar Kinetics in a rotating frame



$$ {}^I\vec{v}_{P/O} = {}^I\vec{v}_{O'/O} + {}^B\vec{v}_{P/O'} + {}^I\vec{\omega}^B \times \vec{r}_{P/O'} $$

Differentiating,

$$ {}^I\vec{a}_{P/O} = \frac{{}^I d}{dt}\left({}^I\vec{v}_{O'/O}\right) + \frac{{}^I d}{dt}\left({}^B\vec{v}_{P/O'} + {}^I\vec{\omega}^B \times \vec{r}_{P/O'}\right) $$

Using T.E.

$$ = {}^I\vec{a}_{O'/O} + \frac{{}^B d}{dt}\left({}^B\vec{v}_{P/O'} + {}^I\vec{\omega}^B \times \vec{r}_{P/O'}\right) + {}^I\vec{\omega}^B \times \left({}^B\vec{v}_{P/O'} + {}^I\vec{\omega}^B \times \vec{r}_{P/O'}\right) $$

Distribute the cross product and define

angular acceleration $\quad {}^I\vec{\alpha}^B \triangleq \frac{{}^I d}{dt}\left({}^I\vec{\omega}^B\right) = \frac{{}^B d}{dt}\left({}^I\vec{\omega}^B\right) + \underbrace{{}^I\vec{\omega}^B \times {}^I\vec{\omega}^B}_{=0}$

Then we get the general vector expression,

$$ \boxed{{}^I\vec{a}_{P/O} = {}^I\vec{a}_{O'/O} + {}^B\vec{a}_{P/O'} + {}^I\vec{\alpha}^B \times \vec{r}_{P/O'} + \underbrace{2\,{}^I\vec{\omega}^B \times {}^I\vec{v}_{P/O'}}_{\text{Coriolis Acceleration}} + \underbrace{{}^I\vec{\omega}^B \times \left({}^I\vec{\omega}^B \times \vec{r}_{P/O'}\right)}_{\text{centripetal acceleration}}} $$

Note: You can use this formula or just follow the process of deriving the inertial acceleration in coordinates (like you have been).

Apparant Forces

The above frame acceleration terms give rise to the apparant force terms if you simply move them to the other side of the equation (N2L).

↳ Note: We don't include Apparant Forces in FBD's.

Ex. Carousel

↳ The ball is thrown subject to no horizontal forces (ignoring drag)

$$\vec{F}_P = 0 \Rightarrow {}^{I}\vec{a}_{P/O} = 0$$

$\Rightarrow$ The ball goes straight (Although it doesn't look like it does in the rotating frame of the carousel)

What does the carousel rider see?

Set ${}^{I}\vec{a}_{P/O} = 0$, Solve ${}^{B}\vec{a}_{P/O'}$

re-arranging,

$${}^{B}\vec{a}_{P/O'} = -2\,{}^{I}\vec{\omega}^{B} \times {}^{B}\vec{v}_{P/O'} - {}^{I}\vec{\omega}^{B} \times ({}^{I}\vec{\omega}^{B} \times \vec{r}_{P/O'}) \underbrace{- {}^{I}\vec{\alpha}^{B} \times \vec{r}_{P/O'}}_{\text{Zero in this example}} - {}^{I}\vec{a}_{O'/O}$$

Ex. Merry-go-round

What apparent force is req'd to keep a point P stationary in the frame of a rotating observer?

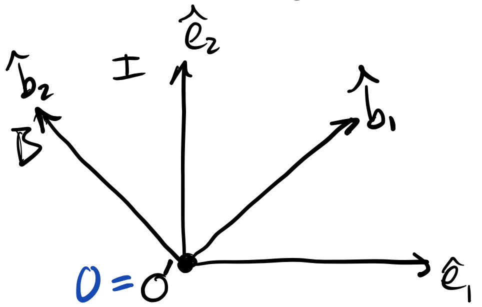


Fig 8.12

We want ${}^{B}\vec{a}_{P/O'} = 0$

Since B is not accelerating in translation or rotation,

${}^{I}\vec{a}_{O'/O} = 0 \qquad {}^{I}\vec{\alpha}^{B} = 0$

Since P is fixed in B,

${}^{B}\vec{V}_{P/O'} = {}^{B}\vec{a}_{P/O'} = 0$

Then, we have,

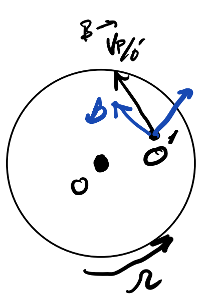
$$m_p \cancelto{0}{{}^{B}\vec{a}_{P/O'}} = \vec{F}_p - \cancelto{0}{m_p {}^{I}\vec{a}_{O'/O}} - \cancelto{0}{m_p {}^{I}\vec{\alpha}^{B}} \times \vec{r}_{P/O'} - 2 m_p {}^{I}\vec{\omega}^{B} \times \cancelto{0}{{}^{B}\vec{V}_{P/O'}}$$
$$- m_p {}^{I}\vec{\omega}^{B} \times ({}^{I}\vec{\omega}^{B} \times \vec{r}_{P/O'})$$

Then,

$$\vec{F}_p = m_p {}^{I}\vec{\omega}^{B} \times ({}^{I}\vec{\omega}^{B} \times \vec{r}_{P/O'})$$

Qualitatively examining acceleration terms

Ex. Carousel – In what direction does the ball appear to accelerate (based on Coriolis contribution)

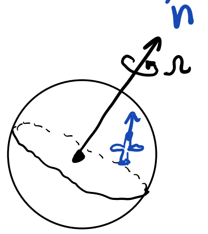


From ${}^{B}\vec{a}_{P/O'}$ the coriolis term is,

$$-2\,\underbrace{{}^{I}\vec{\omega}^{B}}_{\Omega \hat{e}_3} \times {}^{B}\vec{V}_{P/O'} = \text{appears to accelerate to the right in the frame of the rider.}$$

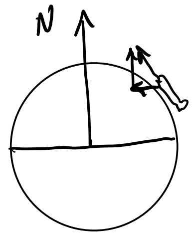
Ex. Coriolis for an airplane for an observer on earth.

$$-2\,{}^{I}\vec{\omega}^{B} \times {}^{B}\vec{v}_{P/O'}$$



Flying north and east

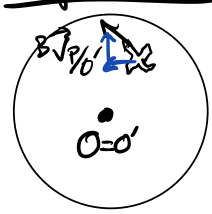
Side View



(Breaking into components)

Notice that the component of velocity parallel to N direction does not contribute to the cross product.

Top View



$$-2\Omega\hat{n} \times {}^{B}\vec{v}_{P/O'}$$

= to the right of the plane's velocity vector ${}^{B}\vec{v}_{P/O'}$

Ch 9 Dynamics of a planar, rigid body

Many concepts we have already covered apply or can be extended to rigid bodies in 2D or 3D.

Rigid Body Dynamics

A rigid body is a collection of particles with the requirement that they are all fixed relative to one another.

for all

That is, $\frac{{}^{I}d}{dt}\|\vec{r}_{i/j}\| = 0 \quad \forall (i,j)$ pairs

This provides enough constraints to reduce the total degrees of freedom of the multiparticle system so that it behaves as a single object.

↳ Planar R.B. = 3 DOF

↳ 3D R.B. = 6 DOF

Euler's Laws

① $\vec{F}_G = m_G\, {}^{I}\vec{a}_{G/O}$

② $\frac{{}^{I}d}{dt}\left({}^{I}\vec{h}_O\right) = \vec{M}_O$

Note: The internal forces between particles cancel due to N3L, so we don't need to use an (ext) superscript!

We can use the separation of angular momentum,

$${}^{I}\vec{h}_O = {}^{I}\vec{h}_{G/O} + {}^{I}\vec{h}_G$$

This leads to two additional expressions for Euler's 2nd law:

$$\frac{{}^{I}d}{dt}\left({}^{I}\vec{h}_{G/O}\right) = \vec{M}_{G/O}$$

$$\frac{{}^{I}d}{dt}\left({}^{I}\vec{h}_{G}\right) = \vec{M}_{G}$$

The last of these three is often the most useful, because working in the body frame is often easiest for analyzing angular momentum of a body and the total moment acting on the body.

To do this, apply the T.E.

$$\boxed{\frac{{}^{B}d}{dt}\left({}^{I}\vec{h}_{G}\right) + {}^{I}\vec{\omega}^{B} \times {}^{I}\vec{h}_{G} = \vec{M}_{G}}$$

Euler's Equation

In would be convenient to write $\frac{{}^{I}d}{dt}\left({}^{I}\vec{h}_{G}\right)$ in terms of angular velocity since we often measure angular rates in the Body frame.

Think of this:

translational N2L

$$\frac{{}^{I}d}{dt}\left({}^{I}\vec{p}_{G/O}\right) = m_G \frac{{}^{I}d}{dt}\left({}^{I}\vec{v}_{G/O}\right)$$

inertial term

Rotational case:

$$\frac{{}^{I}d}{dt}\left({}^{I}\vec{h}_{G}\right) = (?)\frac{{}^{I}d}{dt}\left({}^{I}\vec{\omega}^{B}\right)$$

Can we do this?

Moment of Inertia

↳ Goal: We would like to write ${}^{I}\vec{h}_G$ in terms of ${}^{I}\vec{\omega}^{B}$

Angular momentum of the c.o.m. of a multiparticle system,

$$ {}^{I}\vec{h}_G = \sum_{i=1}^{N} \vec{r}_{i/G} \times m_i \, {}^{I}\vec{V}_{i/G} $$

Apply Transport equation to reveal ${}^{I}\vec{\omega}^{B}$

$$ {}^{I}\vec{h}_G = \sum_{i=1}^{N} \vec{r}_{i/G} \times m_i \left(\frac{{}^{B}d}{dt} \vec{r}_{i/G} + {}^{I}\vec{\omega}^{B} \times \vec{r}_{i/G} \right) $$

Apply a vector triple product identity,

$$ {}^{I}\vec{h}_G = \sum_{i=1}^{N} m_i \Big[\underbrace{(\vec{r}_{i/G} \cdot \vec{r}_{i/G})\, {}^{I}\vec{\omega}^{B}} - \underbrace{\vec{r}_{i/G} (\vec{r}_{i/G} \cdot {}^{I}\vec{\omega}^{B})} \Big] $$

↳ $\vec{r}_{i/G} \left(\vec{r}_{i/G}^{\,T} \, {}^{I}\vec{\omega}^{B} \right) = \underbrace{\left(\vec{r}_{i/G} \vec{r}_{i/G}^{\,T} \right)}_{\text{outer product or Tensor product}} {}^{I}\vec{\omega}^{B}$

In tensor notation

$$ (\vec{r}_{i/G} \cdot \vec{r}_{i/G}) \underset{\text{Unity Tensor}}{U} \overset{\text{Tensor operation}}{\circ} {}^{I}\vec{\omega}^{B} \qquad (\vec{r}_{i/G} \underset{\text{tensor product}}{\otimes} \vec{r}_{i/G}) \circ {}^{I}\vec{\omega}^{B} $$

Factor out ${}^{I}\vec{\omega}^{B}$,

$$ {}^{I}\vec{h}_G = \underbrace{\sum_{i=1}^{N} m_i \Big[(\vec{r}_{i/G} \cdot \vec{r}_{i/G}) U - (\vec{r}_{i/G} \otimes \vec{r}_{i/G}) \Big]}_{\triangleq \text{ Moment of Inertia Tensor } \mathbb{I}_G} \circ \, {}^{I}\vec{\omega}^{B} $$

Finally,

$$\boxed{{}^{I}\vec{h}_G = \mathbb{I}_G \circ {}^{I}\vec{\omega}^{B}}$$

New compact way to write the angular momentum about G. This is a vector-tensor equation, so it is coordinate independent

In body-frame cordinates,

$$[{}^{I}\vec{h}_G]_B = \underbrace{[\mathbb{I}_G]_B}_{\text{Moment of inertia matrix}} [{}^{I}\vec{\omega}^{B}]_B$$

For a continuous rigid ~~body~~ $\mathcal{B}$,

$$\mathbb{I}_G = \int_{\mathcal{B}} [(\vec{r}_{dm/G} \cdot \vec{r}_{dm/G})\mathbb{U} - (\vec{r}_{dm/G} \otimes \vec{r}_{dm/G})]\, dm$$

where we have passed from particle $\bullet \rightsquigarrow \square$ dm differential mass element

In body frame,

$$\mathbb{I}_G = \sum_{i=1}^{3} \sum_{j=1}^{3} I_{ij}\, \hat{b}_i \otimes \hat{b}_j$$

E.g. $\hat{b}_1 \otimes \hat{b}_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

To express in coordinates in the B frame,

$$[\mathbb{I}_G]_B = \int_{\mathcal{B}} \left(\|\vec{r}_{dm/G}\|^2 I - [\vec{r}_{dm/G}]_B [\vec{r}_{dm/G}]_B^T \right) \underbrace{\rho\, dV}_{dm}$$

In cartesian coordinates,

$$I_{11} = \int_{\mathcal{B}} (y^2 + z^2)\, dm \qquad I_{12} = I_{21} = -\int_{\mathcal{B}} xy\, dm$$

$$I_{22} = \int_{\mathcal{B}} (x^2 + z^2)\, dm$$

$$[\mathbb{I}_G]_B = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix}$$

with $[\mathbb{I}_G]_B = [\mathbb{I}_G]_B^T$

real symmetric matrix

"product of inertia"

Properties of Moment of Inertia Matrices

1. All the e-values are real
2. N linearly independent e-vectors → Full rank (invertible)
3. (Normalized) e-vectors form an orthonormal **basis.**

For N=3, the normalized e-vectors represent the unit vectors of a special body frame known as the <u>Principal Axes Frame</u>. The principal axes are often easy to find if there is symmetry.

Suppose B is an arbitrary body frame.
Let C be the principal axes frame.

$$[\mathbb{I}_G]_C = ({}^C R^B)[\mathbb{I}_G]_B({}^C R^B)^T$$

Change of Basis

$$\begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}_C$$

This matrix has been diagonalized!

- The columns of ${}^B R^C = ({}^C R^B)^T$ are the of $[\mathbb{I}_G]_B$ (principal axes)
- The diagonal entries of $[\mathbb{I}_G]_C$ are e-values of $[\mathbb{I}_G]_B$ (the principal moments of inertia)