5.4.5

a) $e^{At} = S e^{\Lambda t} S^{-1}$

$\therefore \quad e^{A(t+T)} = S e^{\Lambda(t+T)} S^{-1}$

$e^{\Lambda(t+T)} = e^{\Lambda t} e^{\Lambda T}$

$\therefore \quad S e^{\Lambda(t+T)} S^{-1} = S e^{\Lambda t} e^{\Lambda T} S^{-1}$

$S^{-1} S = I \quad \therefore \quad \underbrace{S e^{\Lambda t} S^{-1}}_{e^{At}} \underbrace{S e^{\Lambda T} S^{-1}}_{e^{AT}} = S e^{\Lambda t} e^{\Lambda T}$

$= e^{At} e^{At}$

$\therefore \quad \boxed{e^{A(t+T)} = e^{At} e^{AT}}$

b) $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$

$e^{A} = I + A + \frac{A^2}{2!} + \dots$

$A^2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \therefore$

$e^{A} = I + A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

$e^{B} = I + B + \frac{B^2}{2!} + \dots$

$B^2 = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$$\therefore\ e^B = I + B = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$e^A e^B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

$$e^{A+B} = I + (A+B) + \frac{(A+B)^2}{2!} + \dots$$

$$(A+B) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

From example 3: $e^{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}} = \begin{pmatrix} \cos(1) & -\sin(1) \\ \sin(1) & \cos(1) \end{pmatrix}$

$$\therefore \boxed{e^{A+B} \neq e^A e^B}$$

5.4.6

$y'' + y = 0$

$u_1 = y$
$u_2 = y'$

$u_1' = u_2 = y'$

$u_2' = y'' = -y = -u_1$

$$\frac{du}{dt} = \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$\boxed{A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}$$

$\chi_A = \lambda^2 + 1 = 0$

$\lambda_1 = i, \ \lambda_2 = -i$

$(A - \lambda_1 I) v_1 = \vec{0}$

$\begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \vec{0}$ $\quad -ix_1 + x_2 = 0$

Let $x_1 = 1 \therefore x_2 = i$

$v_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$

5.4.6

$(A-\lambda_2 I)\, v_2 = \vec{0}$

$\begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

$i x_1 + x_2 = 0$ let $x_1 = 1$ $\therefore$ $x_2 = -i$

$v_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$

$$\boxed{\begin{matrix} \lambda_1 = i & v_1 = \begin{pmatrix} 1 \\ i \end{pmatrix} \\ \lambda_2 = -i & v_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix} \end{matrix}}$$

$u(t) = e^{At}\, u(0) = S e^{\Lambda t} S^{-1} y(0) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2$

$e^{\Lambda t} = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}$

$S = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$

$S^{-1} = \frac{1}{-2i}\begin{pmatrix} -i & -1 \\ -i & 1 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2i \\ 1/2 & -1/2i \end{pmatrix}$

$y(t) = c_1 e^{it}\begin{pmatrix} 1 \\ i \end{pmatrix} + c_2 e^{-it}\begin{pmatrix} 1 \\ -i \end{pmatrix}$

5.46

$$C = S^{-1} u(0) = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2i} \\ \frac{1}{2} & \frac{1}{2i} \end{pmatrix} \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$e^{it} = \cos(t) + i \sin(t)$$

$$e^{-it} = \cos(-t) + i \sin(-t)$$

$$e^{it} \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} \cos(t) + i\sin(t) \\ i\cos(t) - \sin(t) \end{pmatrix}$$

$$e^{-it} \begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} \cos(-t) + i\sin(-t) \\ -i\cos(-t) + \sin(-t) \end{pmatrix}$$

$$\cos(-t) = \cos(t) \quad , \quad \sin(-t) = -\sin(t)$$

$$e^{-it} \begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} \cos(t) - i\sin(t) \\ -i\cos(t) - \sin(t) \end{pmatrix}$$

$$u(t) = \overset{1}{\cancel{c_1}} e^{it} \begin{pmatrix} 1 \\ i \end{pmatrix} + \overset{1}{\cancel{c_2}} e^{-it} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$u(t) = \begin{pmatrix} \cos(t) + \cancel{i}\sin(t) \\ i\cancel{\cos(t)} - \sin(t) \end{pmatrix} + \begin{pmatrix} \cos(t) - i\cancel{\sin(t)} \\ -i\cancel{\cos(t)} - \sin(t) \end{pmatrix}$$

5.4.6

$$\boxed{U(t) = \begin{pmatrix} y \\ y' \end{pmatrix} = \begin{pmatrix} 2\cos(t) \\ -2\sin(t) \end{pmatrix}}$$

Check: $y = 2\cos(t)$

$y' = -2\sin(t)$

$y'' = -2\cos(t)$

$$y'' + y = -2\cos(t) + 2\cos(t) = 0 \checkmark$$

5.4.8

$\frac{dr}{dt} = 4r - 2w$ $\quad U = \begin{pmatrix} r \\ w \end{pmatrix}$

$\frac{dw}{dt} = r + w$

$$\frac{du}{dt} = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix} u$$

a) $\chi_A = \lambda^2 - 5\lambda + 6 = 0$

$(\lambda - 3)(\lambda - 2)$

$\lambda_1 = 3$

$\lambda_2 = 2$

$\therefore$ $\boxed{\text{System is unstable}}$

b) $(A - \lambda_1 I)V_1 = 0$

$$\begin{pmatrix} 1 & -2 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$x_1 - 2x_2 = 0$

Let $x_2 = 1$ $\therefore$ $x_1 = 2$

$V_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

5.4.8

$(A-\lambda_2 I)\,v_2 = 0$

$$\begin{pmatrix} 2 & -2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$2x_1 - 2x_2 = 0$

Let $x_2 = 1 \quad \therefore \quad x_1 = 1 \qquad v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$u(t) = S e^{\Lambda t} S^{-1} u(0) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$$

$$u(0) = \begin{pmatrix} 300 \\ 200 \end{pmatrix}$$

$$C = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = S^{-1} u(0)$$

$$S = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$S^{-1} = \frac{1}{2-1}\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 300 \\ 200 \end{pmatrix} = \begin{pmatrix} 100 \\ 100 \end{pmatrix}$$

$$u(t) = 100 e^{3t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + 100 e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

5.4.8

$$U(t) = \begin{bmatrix} 200e^{3t} + 100e^{2t} \\ 100e^{3t} + 100e^{2t} \end{bmatrix}$$

$$\boxed{\begin{aligned} r(t) &= 200e^{3t} + 100e^{2t} \\ w(t) &= 100e^{3t} + 100e^{2t} \end{aligned}}$$

c)

$$\lim_{t \to \infty} \left(\frac{r}{w}\right) = \frac{200\cancel{e^{3t}}}{100\cancel{e^{3t}}}$$

$$\boxed{\frac{r}{w} = 2}$$

5.4.9

a) $A = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$

$\chi_A = \lambda^2 - 7\lambda - 2 = 0$

$\lambda_{1,2} = \dfrac{7 \pm \sqrt{49+8}}{2}$

$\lambda_1 = 7.27$, $\lambda_2 = -0.27$

$\boxed{\text{Unstable}}$

b) $A = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}$

$\chi_A = \lambda^2 - 7 = 0$

$\lambda_1 = -\sqrt{7}$, $\lambda_2 = \sqrt{7}$

$\boxed{\text{Unstable}}$

c) $A = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$ $\chi_A = \lambda^2 + \lambda - 3 = 0$

$\lambda_{1,2} = \dfrac{-1 \pm \sqrt{1+12}}{2} \Rightarrow \lambda_1 = 1.3$

$\lambda_2 = -2.3$

$\boxed{\text{Unstable}}$

5.4.4

d) $A = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}$

$\chi_A = \lambda^2 + 2\lambda = 0$

$\chi_A = \lambda(\lambda+2) = 0$ $\qquad \lambda_1 = 0$

$\lambda_2 = -2$

$\boxed{\text{neutrally stable}}$

5.4.14

$$\frac{d^2 u}{dt^2} = \begin{pmatrix} -5 & 4 \\ 4 & -5 \end{pmatrix} u$$

$$\chi_A = \lambda^2 + 10\lambda + 9$$
$$(\lambda + 9)(\lambda + 1) = 0$$

$$\boxed{\begin{aligned} \lambda_1 &= -9 \\ \lambda_2 &= -1 \end{aligned}}$$

$\omega_1 = \sqrt{-\lambda_1} = \sqrt{9}$ $\quad \omega_2 = \sqrt{-\lambda_2} = \sqrt{1}$

$$\boxed{\begin{aligned} \omega_1 &= 3 \\ \omega_2 &= 1 \end{aligned}}$$

$$(A - \lambda_1 I) v_1 = 0$$

$$\begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$4x_1 + 4x_2 = 0$ $\quad$ let $x_1 = 1 \therefore x_2 = -1$

$v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\quad \lambda_1 = -9$

5.4.14

$(A - \lambda_2 I)v_2 = 0$

$\begin{pmatrix} -4 & 4 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

$-4x_1 + 4x_2 = 0$ let $x_1 = 1$ $\therefore$ $x_2 = 1$

$v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\lambda_2 = -1$

$$\boxed{u(t) = [a_1 \cos(3t) + b_1 \sin(3t)]\begin{pmatrix} 1 \\ -1 \end{pmatrix} + [a_2 \cos(t) + b_2 \sin(t)]\begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

5.5.6

$$x = \begin{pmatrix} 2-4i \\ 4i \end{pmatrix} \qquad y = \begin{pmatrix} 2+4i \\ 4i \end{pmatrix}$$

$$\bar{x}^T y = [2+4i \quad -4i] \begin{bmatrix} 2+4i \\ 4i \end{bmatrix}$$

$$\bar{x}^T y = (2+4i)(2+4i) + (-4i)(4i)$$

$$= (4 + 16i + \cancel{16}i^2) + (-\cancel{16}i^2) =$$

$$\boxed{\bar{x}^T y = 4 + 16i}$$

$$\|x\|^2 = \bar{x}^T x = [2+4i \quad -4i] \begin{bmatrix} 2-4i \\ 4i \end{bmatrix}$$

$$\|x\|^2 = (2+4i)(2-4i) + (-4i)(4i)$$

$$= (4 - 16i^2) + (-16i^2) = 4 - 32i^2 = 4 + 32 = 36$$

$$\boxed{\|x\| = 6}$$

$$\|y\|^2 = \bar{y}^T y = [2-4i \quad -4i] \begin{bmatrix} 2+4i \\ 4i \end{bmatrix} = 36$$

$$\boxed{\|y\| = 6}$$

5.5.8

a) $A = \begin{pmatrix} 1 & i & 0 \\ i & 0 & 1 \end{pmatrix}$

$A \xrightarrow{E_{21}(-i)} = \begin{pmatrix} 1 & i & 0 \\ 0 & 1 & 1 \end{pmatrix}$

$Ax = 0$

$\begin{pmatrix} 1 & i & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \vec{0}$

$x_1 + i x_2 = 0$

$x_2 + x_3 = 0$ $\quad$ x_1 & x_2 are pivots, x_3 is free, let $x_3 = 1$

$\therefore$ $x_2 = -1$ & $x_1 = i$

$$\boxed{X = \text{span}\left\{ \begin{pmatrix} i \\ -1 \\ 1 \end{pmatrix} \right\} = N(A)}$$

b) $\bar{X}^T = [-i \;\; -1 \;\; 1]$

$(\bar{X}^T)\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} = [-i \;\; -1 \;\; 1]\begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} = -i - i = -2i \neq 0$

$$\therefore \quad \boxed{N(A) \not\perp C(A^T)}$$

5.5.8

$$A^H = \begin{pmatrix} 1 & -i \\ -i & 0 \\ 0 & 1 \end{pmatrix}$$

$$x^H \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} = [i \ -1 \ 1] \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix} = -i + i = 0$$

$$x^H \begin{pmatrix} -i \\ 0 \\ 1 \end{pmatrix} = -1 + 1 = 0$$

$$\therefore \boxed{N(A) \perp C(A^H)}$$

5.5.19

$$U = \begin{pmatrix} 1/\sqrt{3} & i/\sqrt{2} & c_1 \\ 1/\sqrt{3} & 0 & c_2 \\ i/\sqrt{3} & 1/\sqrt{2} & c_3 \end{pmatrix}$$

Colums are orthoromel

$\therefore\ c_i^H c_j = 0$

$$\begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & -i/\sqrt{3} \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = 0$$

$$c_1/\sqrt{3} + \frac{c_2}{\sqrt{3}} - \frac{i c_3}{\sqrt{3}} = 0$$

$$\begin{bmatrix} -i/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0$$

$$-i/\sqrt{2}\, c_1 + c_3/\sqrt{2} = 0 \quad \Rightarrow \quad c_3 = i c_1$$

$$c_1 + c_2 - i c_3 = 0 \Rightarrow c_1 + c_2 + c_1 = 0 \Rightarrow c_2 = -2c_1$$

$$\begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix}^H \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 1$$

$$\begin{bmatrix} c_1 & -2c_1 & -ic_1 \end{bmatrix} \begin{bmatrix} c_1 \\ -2c_1 \\ ic_1 \end{bmatrix} = 1$$

$$c_1^2 + 4c_1^2 + c_1^2 = 1 \quad \therefore \quad c_1 = \pm\sqrt{1/6}$$

5.5.14

$\therefore \; C_2 = -2C_1 = -2\left(\frac{1}{\sqrt{6}}\right) = \frac{-2}{\sqrt{6}}$

$C_3 = i C_1 = i/\sqrt{6}$

$$\boxed{\text{Column } 3 = \begin{pmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ i/\sqrt{6} \end{pmatrix}}$$

2 Degrees of freedom in column 3 arising from the positive & negative square roots.

5.5.20

$K = \begin{bmatrix} i & i \\ i & i \end{bmatrix}$

$(K - \lambda I) = 0$

$\left| \begin{pmatrix} i-\lambda & i \\ i & i-\lambda \end{pmatrix} \right| = 0 = (i-\lambda)(i-\lambda) + i^2 = 0$

$i^2 - 2i\lambda + \lambda^2 - i^2 = 0$

$\lambda^2 - 2i\lambda = 0$

$\lambda(\lambda - 2i) = 0$ $\quad\quad \lambda_1 = 0$

$\lambda_2 = 2i$

$(K - \lambda_1 I) v_1 = 0$

$\begin{pmatrix} i & i \\ i & i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$ $\quad\quad x_1 + x_2 = 0$ $\quad\quad$ let $x_1 = 1 \therefore x_2 = -1$

$v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$(K - \lambda_2 I) v_2 = 0$

$\begin{pmatrix} -i & i \\ i & -i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$ $\quad\quad$ let $x_1 = 1 \therefore x_2 = 1$

$v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

5.5.20

$$S = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$\Lambda = \begin{pmatrix} 0 & 0 \\ 0 & 2i \end{pmatrix}$$

$$S^{-1} = \frac{1}{1-{}^{-}1} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

$$\boxed{K = S\Lambda S^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 2i \end{pmatrix} \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix}}$$

$$e^{Kt} = S e^{\Lambda t} S^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} e^{0} & 0 \\ 0 & e^{2i} \end{pmatrix} \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

$$e^{Kt} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{2it} \end{pmatrix} \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & e^{2it} \\ -1 & e^{2it} \end{pmatrix} \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(1+e^{2it}) & \frac{1}{2}(-1+e^{2it}) \\ \frac{1}{2}(-1+e^{2it}) & \frac{1}{2}(1+e^{2it}) \end{pmatrix}$$

5.5.20

$$\boxed{e^{Kt} = \frac{1}{2}\begin{pmatrix} 1+e^{2it} & -1+e^{2it} \\ -1+e^{2it} & 1+e^{2it} \end{pmatrix}}$$

$$\left(e^{Kt}\right)^H = \frac{1}{2}\begin{pmatrix} 1+e^{-2it} & -1+e^{-2it} \\ -1+e^{-2it} & 1+e^{-2it} \end{pmatrix}$$

$$\left(e^{Kt}\right)^H e^{Kt} = \frac{1}{4}\begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$$

$$U_{11} = (1+e^{-2it})(1+e^{2it}) + (-1+e^{-2it})(-1+e^{2it})$$

$$= 1+\cancel{e^{2it}}+\cancel{e^{-2it}}+e^0+1-\cancel{e^{2it}}-\cancel{e^{-2it}}+e^0 = 4$$

$$U_{12} = (1+e^{-2it})(-1+e^{2it}) + (-1+e^{-2it})(1+e^{2it})$$

$$= (\cancel{-1}+\cancel{e^{2it}}-\cancel{e^{-2it}}+\cancel{e^0}) + (\cancel{-1}-\cancel{e^{2it}}+\cancel{e^{-2it}}+\cancel{e^0}) = 0$$

By inspection $U_{11} = U_{22}$ & $U_{12} = U_{21}$

For unitary: $U^H U = I$

5.5.20

$$(e^{KT})^H (e^{KT}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \therefore \boxed{e^{Kt} \text{ is unitary}}$$

$$\frac{d}{dt}(e^{Kt}) = K e^{Kt}$$

$$\frac{d}{dt}(e^{Kt})\Big|_{t=0} = K e^{K0}$$

$$e^{K0} = \frac{1}{2}\begin{pmatrix} 1+1 & -1+1 \\ -1+1 & 1+1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\frac{d}{dt}(e^{Kt})\Big|_{t=0} = (K)(I) = K$$

$$\boxed{\frac{d}{dt}(e^{Kt})\Big|_{t=0} = \begin{pmatrix} i & i \\ i & i \end{pmatrix}}$$

5.15

$$A = \begin{pmatrix} 0 & -i & 0 \\ i & 1 & i \\ 0 & -i & 0 \end{pmatrix}$$

$$(A-\lambda I) = \begin{pmatrix} -\lambda & -i & 0 \\ i & 1-\lambda & i \\ 0 & -i & -\lambda \end{pmatrix}$$

$$|A-\lambda I| = -\lambda \begin{vmatrix} 1-\lambda & i \\ -i & -\lambda \end{vmatrix} + i \begin{vmatrix} i & i \\ 0 & -\lambda \end{vmatrix} = 0$$

$$-\lambda[(1-\lambda)(-\lambda) - (i)(-i)] + i(-\lambda i) = 0$$

$$-\lambda[-\lambda+\lambda^2-1] + \lambda = 0$$

$$-\lambda^3 + \lambda^2 + \lambda + \lambda = 0$$

$$-\lambda^3 + \lambda^2 + 2\lambda = 0$$

$$-\lambda(\lambda^2 - \lambda - 2) = 0 = -\lambda(\lambda-2)(\lambda+1)$$

$$\boxed{\begin{matrix} \lambda_1 = 0 \\ \lambda_2 = 2 \\ \lambda_3 = -1 \end{matrix}}$$

$$(A-\lambda_1 I)V_1 = 0$$

$$\begin{pmatrix} 0 & -i & 0 \\ i & 1 & i \\ 0 & -i & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \vec{0}$$

x_1 & x_2 are pivots, x_3 is free

let $\underline{x_3 = 1}$

$-ix_2 = 0 \quad \therefore \underline{x_2 = 0}$

$ix_1 + ix_3 = 0 \quad \therefore \quad \underline{x_1 = -x_3 = -1}$

5.15

$$\boxed{\text{For } \lambda_1 = 0, \quad V_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}$$

$(A - \lambda_2 I) V_2 = 0$

$$\begin{pmatrix} -2 & -i & 0 \\ i & -1 & i \\ 0 & -i & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \vec{0}$$

$(-2) \cdot \left(\frac{1}{2} i\right) \quad \left(\frac{1}{2} i\right)(-i)$

$(-\frac{1}{2})(2)(i) = i$

$(i)(-2i) = 2$

$$\Rightarrow \begin{pmatrix} -2 & -i & 0 \\ 0 & -\frac{1}{2} & i \\ 0 & -i & -2 \end{pmatrix} \Rightarrow \begin{pmatrix} -2 & -i & 0 \\ 0 & -\frac{1}{2} & i \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow$ x_3 is free let $x_3 = 1$ $\therefore$

$-2x_1 - ix_2 = 0$

$-\frac{1}{2}x_2 + ix_3 = 0 \qquad x_2 = (-i)(-2) = 2i$

$-2x_1 = ix_2 = 2i^2 = -2 \quad \therefore x_1 = 1$

$$\boxed{\text{For } \lambda_2 = 2, \quad V_2 = \begin{pmatrix} 1 \\ 2i \\ 1 \end{pmatrix}}$$

$(A - \lambda_3 I) V_3 = 0$

$$\begin{pmatrix} 1 & -i & 0 \\ i & 2 & i \\ 0 & -i & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

5.15

$$\begin{pmatrix} 1 & -i & 0 \\ i & 2 & i \\ 0 & -i & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & -i & 0 \\ 0 & 1 & i \\ 0 & -i & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & -i & 0 \\ 0 & 1 & i \\ 0 & 0 & 0 \end{pmatrix}$$

x_3 is free let $x_3 = 1$

$x_1 - i x_2 = 0$

$x_2 + i x_3 = 0 \qquad x_2 = -i$

$x_1 - i(-i) = 0 \qquad x_1 = 1$

$$\therefore \boxed{\text{For } \lambda_3 = -1, \quad V_3 = \begin{pmatrix} 1 \\ -i \\ 1 \end{pmatrix}}$$

$$A^H = \begin{pmatrix} 0 & -i & 0 \\ i & 1 & i \\ 0 & -i & 0 \end{pmatrix} = A \qquad \therefore A \text{ is Hermetian}$$

So the $\boxed{\text{eigenvectors should be orthogonal}}$, true see below

$$V_1^H V_2 = [-1 \ 0 \ 1] \begin{bmatrix} 1 \\ 2i \\ 1 \end{bmatrix} = -1 + 1 = 0 \quad \therefore V_1 \perp V_2$$

$$V_1^H V_3 = [-1 \ 0 \ 1] \begin{bmatrix} 1 \\ -i \\ 1 \end{bmatrix} = -1 + 1 = 0 \quad \therefore V_1 \perp V_3$$

$$V_2^H V_3 = [1 \ -2i \ 1] \begin{bmatrix} 1 \\ -i \\ 1 \end{bmatrix} = 1 + (2i^2) + 1 = 2 - 2 = 0 \quad \therefore V_2 \perp V_3$$