#1) $PA_1 + A_1^T P + 2\alpha P \leq 0$

$PA_2 + A_2^T + 2\alpha P \leq 0$ (These Exist)

$\dot{x} = A(x)\, x$ $\quad A(x) = A_0 + \Psi(x)\,\Delta A$

$A_1 = A_0 + a\,\Delta A$

$A_2 = A_0 + b\,\Delta A$

$a \leq \Psi \leq b$

Choose Lyapunov Canidate : $V = x^T P x$ $\quad P = P^T$

$\therefore\ \dot{V} = \dot{x}^T P x + x^T P \dot{x} = 2x^T P \dot{x} = 2x^T P A(x) x$

$\dot{V} = 2x^T P (A_0 + \Psi(x)\,\Delta A)\, x$

$A(x) \leq A_1$

$A(x) \leq A_2$ (A(x) is bounded by a & b)

$\therefore\ \dot{V} = 2x^T P A(x) x \leq 2x^T P A_1 x$ &

$\dot{V} = 2x^T P A(x) x \leq 2x^T P A_2 x$

Manipulate $PA_i + A_i^T P + 2\alpha P \leq 0$

$PA_1 + A_1^T P \leq -2\alpha P$

$PA_2 + A_2^T P \leq -2\alpha P$

$2x^T P A_i x = x^T (PA_i + A_i^T P) x$ if $P = P^T$

$\therefore\ \dot{V} \leq 2x^T P A_1 x = x^T (PA_1 + A_1^T P) x \leq x^T(-2\alpha P) x$ &

$\dot{V} \leq 2x^T P A_2 x = x^T (PA_2 + A_2^T P) x \leq x^T(-2\alpha P) x$

$\therefore\ \dot{V} \leq -2\alpha x^T P x = -2\alpha V$ $\therefore$ system is GUES. with rate of Convergence α.

#2) $\dot{x}_1 = -2x_1 + x_2 + \gamma e^{-x_1^2} x_2$
$\dot{x}_2 = -x_1 - 3x_2 - \gamma e^{-x_1^2} x_1$

$A_0 = \begin{pmatrix} -2 & 1 \\ -1 & -3 \end{pmatrix}$ $\Delta A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$\Psi = \gamma e^{-x_1^2}$, $\Psi(0) = \gamma$, $\Psi(\infty) = 0$

$\therefore$ Ψ has max value of γ & min value of 0

$\therefore$ $0 \le \Psi \le \gamma$, $a = 0$ & $b = \gamma$
$a \le \Psi \le b$

$A(x) = A_0 + \Psi \Delta A$

$A_1 = A_0 + a \Delta A = A_0$
$A_2 = A_0 + b \Delta A = A_0 + \gamma \Delta A$

From MATLAB, no supremal γ exists. Any $\gamma > 0$ will make the system stable about the origin.

Ex) $\gamma = 11$, From MATLAB, $P = \begin{pmatrix} 120.2914 & 4.55 \\ 4.55 & 114.8985 \end{pmatrix}$

$\det(P) = 13800 > 0$ & $P(1,1) = 120.2914 > 0$.
$\therefore$ $P = P^T > 0$

Check LMI's satisfied,

#2)

$$A_1 = \begin{pmatrix} -2 & 1 \\ -1 & -3 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} -2 & 11.99 \\ -11.99 & -3 \end{pmatrix}$$

$$PA_1 + A_1^T P = \underbrace{\begin{pmatrix} -490.26 & -17.357 \\ -17.357 & -686.29 \end{pmatrix}}_{\text{has } \lambda_1 = -681.9 \ \& \ \lambda_2 = -488.7}$$

λ_1 & $\lambda_2 < 0$ $\therefore$ $PA_1 + A_1^T P \prec 0$, LMI satisfied for $\gamma = 11$

$$PA_2 + A_2^T P = \underbrace{\begin{pmatrix} -590.3 & 41.91 \\ 41.91 & -580.28 \end{pmatrix}}_{\lambda_1 = -627.5 \ \& \ \lambda_2 = -543.07}$$

λ_1 & $\lambda_2 < 0$ $\therefore$ $PA_2 + A_2^T P \prec 0$, LMI

satisfied for $\gamma = 11$.

Similiarly for $\gamma = 1$: $PA_1 + A_1^T P = \begin{pmatrix} -501 & .5 \\ .5 & -571 \end{pmatrix}$

$PA_1 + A_1^T P$ has $\lambda_1 = -571$ & $\lambda_2 = -501$ $\therefore$ $PA_1 + A_1^T P \prec 0$

$PA_2 + A_2^T P = \begin{pmatrix} -572 & 26.4 \\ 26.4 & -561 \end{pmatrix}$ with $\lambda_1 = -572$ & $\lambda_2 = -500$

$\therefore$ $PA_2 + A_2^T P \prec 0$, $\therefore$ LMI's $PA_i + A_i^T P < 0$ are also satisfied for $\gamma = 1$.

#3) $\gamma = 1$ $\quad A_0 = \begin{pmatrix} 0 & 1 \\ -2 & -1 \end{pmatrix}$ $\quad \Delta A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$a = -\gamma \, , \; b = \gamma$

$A_1 = A_0 - \gamma \Delta A$

$A_2 = A_0 + \gamma \Delta A$

LMI's: $PA_1 + A_1^T P \leq -2\alpha P$

$PA_2 + A_2^T P \leq -2\alpha P$

From MATLAB, $\alpha_{max} = 0.122$

<u>Check:</u>

$PA_1 + A_1^T P + 2\alpha P \leq 0$

$PA_2 + A_2^T P + 2\alpha P \leq 0$

Eisenvalues of $PA_1 + A_1^T P + 2\alpha P$ are $\lambda_1 = -4.13$ & $\lambda_2 = 0$.

Eisenvalues of $PA_2 + A_2^T P + 2\alpha P$ are $\lambda_1 = -1.60$ & $\lambda_2 = 0$.

$P = P^T \succ 0$ & both $PA_1 + A_1^T P + 2\alpha P$ & $PA_2 + A_2^T + 2\alpha P$ are negative semi-definite. Therefore the largest rate of exponential convergence is $\alpha = 0.122$

#4) $\ddot{\theta}_1 + 2\dot{\theta}_1 - \dot{\theta}_2 + 2K\theta_1 - K\theta_2 - \sin\theta_1 = 0$

$\ddot{\theta}_2 - \dot{\theta}_1 + \dot{\theta}_2 - K\theta_1 + K\theta_2 - \sin\theta_2 = 0$

$\ddot{\theta}_1 = -2\dot{\theta}_1 + \dot{\theta}_2 - 2K\theta_1 + K\theta_2 + \sin\theta_1$

$\ddot{\theta}_2 = \dot{\theta}_1 - \dot{\theta}_2 + K\theta_1 - K\theta_2 + \sin\theta_2$

$X_1 = \theta_1 \quad X_2 = \theta_2 \quad X_3 = \dot{\theta}_1 \quad X_4 = \dot{\theta}_2$

$\dot{x}_1 = X_3$

$\dot{X}_2 = X_4$

$\dot{X}_3 = -2X_3 + X_4 - 2KX_1 + KX_2 + \sin(X_1)$

$\dot{X}_4 = X_3 - X_4 + KX_1 - KX_2 + \sin(X_2)$

$\Psi_1(x) = \begin{cases} \frac{\sin(X_1)}{X_1} & \text{if } X_1 \neq 0 \\ 1 & \text{if } X_1 = 0 \end{cases}$

$-1 \leq \sin(X_1) \leq 1$ $\qquad A(x) = A_0 + \Psi_1 \Delta A_1 + \Psi_2 \Delta A_2$

$\therefore\ a_1 = -1$ & $b_1 = 1$ $\qquad \dot{x} = A(x)x$

$\Psi_2(x) = \begin{cases} \frac{\sin(X_2)}{X_2} & \text{if } X_2 \neq 0 \\ 1 & \text{if } X_2 = 0 \end{cases}$

$\therefore\ a_2 = -1$ & $b_2 = 1$

$$A_0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2K & K & -2 & 1 \\ K & -K & 1 & -1 \end{pmatrix}$$

$$\Delta A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \Delta A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$A_1 = A_0 + a_1 \Delta A_1 + a_2 \Delta A_2$$

$$A_2 = A_0 + a_1 \Delta A_1 + b_2 \Delta A_2$$

$$A_3 = A_0 + b_1 \Delta A_1 + a_2 \Delta A_2$$

$$A_4 = A_0 + b_1 \Delta A_1 + b_2 \Delta A_2$$

$$PA_i + A_i^T P < 0$$

<u>From MATLAB</u>: Any $K > 18$ will produce a system that is G.E.S. about the zero solution.

For $K = 18.1$, $P = P^T = \begin{pmatrix} 2039 & -554 & 15 & 10 \\ -554 & 1485 & 10 & 25 \\ 15 & 10 & 82 & 51 \\ 10 & 25 & 51 & 134 \end{pmatrix}$

$P > 0$ with $\lambda_1 = 50$, $\lambda_2 = 165$, $\lambda_3 = 114$, & $\lambda_4 = 2382$

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Homework 6 Gabriel Colangelo

```
clear
close all
clc
```

Problem 2

Polytopic nonlinear system, A(x) = A0 + psi*delta_A

```
A0          = [-2 1; -1 -3];
delta_A     = [0 1; -1 0];

% Initialize gamma
gamma       = 1;

% Initialize counter
count       = 0;

% Initialize while loop logic
tfeas       = -1;

% Options for feasp - silent
opts        = [0;0;0;0;1];

% Create iterative loop
while tfeas < 0

    % Increase counter
    count   = count + 1;

    % Create counter break
    if count > 1000
        disp('Supremal value of gamma not found')
        fprintf('\n')
        break
    end

    % LMI toolbox setup
    setlmis([]);

    % Matrices for extreme values of psi, a = 0, b = gamma
    A1      = A0;
    A2      = A0 + gamma*delta_A;

    % Positive definite matrix
    P       = lmivar(1, [2,1]);

    % Create LMI's: P*Ai + Ai'*P < 0
    lmi1    = newlmi;
    lmiterm([lmi1,1,1,P],1,A1,'s');

    lmi2    = newlmi;
    lmiterm([lmi2,1,1,P],1,A2,'s');

    Plmi    = newlmi;
```

```
    lmiterm([-Plmi,1,1,P],1,1);
    lmiterm([Plmi,1,1,0],1);
    lmis    = getlmis;

    % Solve LMIS
    [tfeas, xfeas] = feasp(lmis,opts);

    % Create P matrix
    P       = dec2mat(lmis,xfeas,P);

    % If feasible increase gamma, save latest gamma
    if tfeas < 0
        gamma_max   = gamma;
        gamma       = gamma + .01;
    end

end

% Check P and lyapunov equation
disp('Maximum P is')
disp(P)

disp('Determinant of P is')
disp(det(P))

% Output LMI's
disp('P*A1 + A1''P =')
disp(P*A1 + A1'*P)

disp('P*A2 + A2''P =')
disp(P*A2 + A2'*P)

disp('Eigenvalues of P*A1 + A1''P')
disp(eig(P*A1 + A1'*P))

disp('Eigenvalues of P*A2 + A2''P')
disp(eig(P*A2 + A2'*P))
```

```
Supremal value of gamma not found

Maximum P is
  120.2914    4.5500
    4.5500  114.8985

Determinant of P is
   1.3801e+04

P*A1 + A1'P =
 -490.2658  -17.3571
  -17.3571 -680.2908

P*A2 + A2'P =
 -590.2751   41.9115
   41.9115 -580.2814

Eigenvalues of P*A1 + A1'P
 -681.8632
 -488.6933

Eigenvalues of P*A2 + A2'P
 -627.4866
 -543.0700
```

Problem 3

Polytopic nonlinear system, A(x) = A0 + psi*delta_A

```
A0          = [0 1; -2 -1];
delta_A     = [0 0; 1 0];

% Set Gamma
gamma       = 1;

% Matrices for extreme values of psi, a = -gamma, b = gamma
A1          = A0 - gamma*delta_A;
A2          = A0 + gamma*delta_A;

% Initialize alpha
alpha       = 0;

% Initialize counter
count       = 0;

% Initialize while loop logic
tfeas       = -1;

% Create iterative loop
while tfeas < 0

    % Increase counter
    count   = count + 1;

    % Create counter break
    if count > 1000
        disp('Supremal value of alpha not found')
        fprintf('\n')
        break
    end

    % LMI toolbox setup
    setlmis([]);

    % Positive definite matrix
    P       = lmivar(1, [2,1]);

    % Create LMI's: P*Ai + Ai'*P <= -2*alpha*P
    lmi1    = newlmi;
    lmiterm([lmi1,1,1,P],1,A1,'s');
    lmiterm([-lmi1 1 1 P], -2*alpha,1); % -2*alpha*P term, RHS

    lmi2    = newlmi;
    lmiterm([lmi2,1,1,P],1,A2,'s');
    lmiterm([-lmi2 1 1 P], -2*alpha,1); % -2*alpha*P term, RHS

    Plmi    = newlmi;
    lmiterm([-Plmi,1,1,P],1,1);
    lmiterm([Plmi,1,1,0],1);

    lmis    = getlmis;

    % Solve LMIS
    [tfeas, xfeas] = feasp(lmis,opts);

    % Create P matrix
    P       = dec2mat(lmis,xfeas,P);

    % If feasible increase alpha, save latest alpha
    if tfeas < 0
        alpha_max   = alpha;
        alpha       = alpha + .001;
    end
end
```

```
% Check P and lyapunov equation
disp('Maximum P is')
disp(P)

disp('Eigenvalues of P are')
disp(eig(P))

% Check LMI solver results
disp('Eigenvalues of P*A1 + A1''*P + 2*alpha*P are')
disp(eig(P*A1 + A1'*P + 2*alpha_max*P))

disp('Eigenvalues of P*A2 + A2''*P + 2*alpha*P are')
disp(eig(P*A2 + A2'*P + 2*alpha_max*P))

fprintf('The largest rate of exponential convergence is %.3f \n',alpha_max)
```

```
Maximum P is
    2.5243    0.6339
    0.6339    1.2615

Eigenvalues of P are
    0.9983
    2.7876

Eigenvalues of P*A1 + A1'*P + 2*alpha*P are
   -4.1362
    0.0014

Eigenvalues of P*A2 + A2'*P + 2*alpha*P are
   -1.5971
   -0.0023

The largest rate of exponential convergence is 0.122
```

Problem 4

```
% State dependent A(x) = A0 + psi1*delta_A1 + psi2*delta_A2
delta_A1        = zeros(4);
delta_A2        = zeros(4);
delta_A1(3,1)   = 1;
delta_A2(4,2)   = 1;

% Bounds on psi1 and psi2
a1              = -1;
b1              = 1;
a2              = -1;
b2              = 1;

% Initialize spring constant
K               = 0.1;

% Initialize counter
count           = 0;

% Initialize while loop logic
tfeas           = 1;

% Create iterative loop
while tfeas > 0

    % Increase counter
    count   = count + 1;

    % Create counter break
```

```
    if count > 1000
        disp('Stable spring constant not found')
        fprintf('\n')
        break
    end

    % LMI toolbox setup
    setlmis([]);

    % Constant matrix
    A0      = [0 0 1 0; 0 0 0 1; -2*K K -2 1; K -K 1 -1];

    % Extreme matrices
    A1      = A0 + a1*delta_A1 + a2*delta_A2;
    A2      = A0 + a1*delta_A1 + b2*delta_A2;
    A3      = A0 + b1*delta_A1 + a2*delta_A2;
    A4      = A0 + b1*delta_A1 + b2*delta_A2;

    % Positive definite matrix
    P       = lmivar(1, [4,1]);

    % Create LMI's: P*Ai + Ai'*P < 0
    lmi1    = newlmi;
    lmiterm([lmi1,1,1,P],1,A1,'s');

    lmi2    = newlmi;
    lmiterm([lmi2,1,1,P],1,A2,'s');

    lmi3    = newlmi;
    lmiterm([lmi3,1,1,P],1,A3,'s');

    lmi4    = newlmi;
    lmiterm([lmi4,1,1,P],1,A4,'s');

    Plmi    = newlmi;
    lmiterm([-Plmi,1,1,P],1,1);
    lmiterm([Plmi,1,1,0],1);
    lmis    = getlmis;

    % Solve LMIS
    [tfeas, xfeas] = feasp(lmis,opts);

    % Create P matrix
    P       = dec2mat(lmis,xfeas,P);

    % If not feasible, increase K
    if tfeas > 0
       K       = K + .05;
    end

end

% Check P and lyapunov equation
fprintf('\n')
disp('Final P is')
disp(P)

disp('Eigenvalues of P are')
disp(eig(P))

% Output LMI's
disp('Eigenvalues of P*A1 + A1''P')
disp(eig(P*A1 + A1'*P))

disp('Eigenvalues of P*A2 + A2''P')
disp(eig(P*A2 + A2'*P))

disp('Eigenvalues of P*A3 + A3''P')
```

```
disp(eig(P*A3 + A3'*P))

disp('Eigenvalues of P*A2 + A2''P')
disp(eig(P*A3 + A3'*P))

fprintf(['A spring constant value that guarantees' ...
        ' the system is globally exponentially stable about' ...
        ' the zero solution is K = %.1f \n'],K)
```

```
Final P is
   1.0e+03 *

    2.0397   -0.5542    0.0152    0.0102
   -0.5542    1.4858    0.0101    0.0254
    0.0152    0.0101    0.0824    0.0516
    0.0102    0.0254    0.0516    0.1339

Eigenvalues of P are
   1.0e+03 *

    0.0505
    0.1648
    1.1442
    2.3823

Eigenvalues of P*A1 + A1'P
 -873.3868
 -562.9016
 -236.8597
   -0.0297

Eigenvalues of P*A2 + A2'P
 -886.2393
 -428.6464
 -232.2910
  -24.5844

Eigenvalues of P*A3 + A3'P
 -872.0881
 -479.8083
 -238.6097
  -21.7355

Eigenvalues of P*A2 + A2'P
 -872.0881
 -479.8083
 -238.6097
  -21.7355

A spring constant value that guarantees the system is globally exponentially stable about the zero solution is K = 18.1
```

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