AAE 532 - Orbit Mechanics

Purdue University

PS4 Solutions

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Useful Constants

	Axial Rotaional Period (Rev/Day)	Mean Equatorial Radius (km)	Gravitational Parameter $\mu = Gm(km^3/sec^2)$	Semi-major Axis of orbit (km)	Orbital Period (sec)	Eccentricity of Orbit	Inclination of Orbit to Ecliptic (deg)
☉ Sun	0.0394011	695990	132712440017.99	-	-	-	-
☾ Moon	0.0366004	1738.2	4902.8005821478	384400 (around Earth)	2360592 27.32 Earth Days	0.0554	5.16
☾ Mercury	0.0170514	2439.7	22032.080486418	57909101	7600537 87.97 Earth Days	0.20563661	7.00497902
♀ Venus	0.0041149 (Retrograde)	6051.9	324858.59882646	108207284	19413722 224.70 Earth Days	0.00676399	3.39465605
⊕ Earth	1.0027378	6378.1363	398600.4415	149597898	31558205 365.26 Earth Days	0.01673163	0.00001531
♂ Mars	0.9747000	3397	42828.314258067	227944135	59356281 686.99 Earth Days	0.09336511	1.84969142
♃ Jupiter	2.4181573	71492	126712767.8578	778279959	374479305 11.87 Years	0.04853590	1.30439695
♄ Saturn	2.2522053	60268	37940626.061137	1427387908	930115906 29.47 Years	0.05550825	2.48599187
⛢ Uranus	1.3921114 (Retrograde)	25559	5794549.0070719	2870480873	2652503938 84.05 Years	0.04685740	0.77263783
Ψ Neptune	1.4897579	25269	6836534.0638793	4498337290	5203578080 164.89 Years	0.00895439	1.77004347
♇ Pluto	-0.1565620 (Retrograde)	1162	981.600887707	5907150229	7830528509 248.13 Years	0.24885238	17.14001206

- First three columns of the body data are consistent with GMAT 2020a default values, which are mainly from JPL's ephemerides file de405.spk

- The rest of the data are from JPL website(`https://ssd.jpl.nasa.gov/?planet_pos`, retrieved at 09/01/2020) based on E.M. Standish's "Keplerian Elements for Approximate Positions of the Major Planets"

Problem 1

Problem Statement

Missions to asteroids have been more frequently proposed in recent years; many asteroids of various sizes are, in fact, increasingly accessible. Consider the asteroid Anteros #1943. Its orbital characteristics include:

$$a = 1.43018128 \; AU; \qquad e = .2576460$$

On January 1, 1996 the asteroid was located in its orbit such that $\theta^* = 118.65°$.

(a) Determine the following: p, h, $\mathbb{P}$, r_p, r_a, ε; r_0, v_0, $\dot{r}_0$, $\dot{\theta}_0$, γ_0, E_0, and time till next perihelion. [Note that distances should be listed in AU! ALWAYS list angles in degrees.]

Is the asteroid currently ascending or descending? How do you know?

Write $\bar{r}_0$ and $\bar{v}_0$ in terms of components in the directions of both $\hat{r}$ and $\hat{\theta}$ as well as the unit vectors $\hat{e}$ and $\hat{p}$.

(b) PLOT the entire orbit in MATLAB. (Do not use polar plots; compute $\hat{e}$ and $\hat{p}$ components along the path and plot the orbit.)

On the plot, start by adding appropriate unit vectors. Mark the given state by adding to the plot: line of apsides, minor axis, $\bar{r}_0$, $\bar{v}_0$, θ_0^*, γ_0, E_0, local horizon.

(c) Time t_1 is exactly 6.4 months after the given state. Determine the following information at the new time: $\bar{r}_1$, $\bar{v}_1$, $\dot{r}_1$, $\dot{\theta}_1$, γ_1, E_1, θ_1^*. Add the new state to the previous plot.

(d) What quadrant includes the new state? How long until perihelion from the new t_1?

(e) Will the asteroid ever cross Earth orbit? How do you know?

Part (a)

Since the true anomaly is less than 180°, the asteroid is ascending. The desired values are

$$p = a(1-e^2) = \boxed{1.33524\ AU}$$

$$h = \sqrt{\mu p} = \boxed{2.30064 \times 10^{-7}\ AU^2/s = 5.14871 \times 10^{9}\ km^2/s}$$

$$\mathbb{P} = 2\pi\sqrt{\frac{a^3}{\mu}} = \boxed{624.72\ days = 1.71039\ years}$$

$$r_p = a(1-e) = \boxed{1.06170\ AU}$$

$$r_a = a(1+e) = \boxed{1.79866\ AU}$$

$$\varepsilon = \frac{-\mu}{2a} = \boxed{-1.38584 \times 10^{-14}\ AU^2/s^2 = -310.145\ km^2/s^2}$$

$$r_0 = \frac{p}{1+e\cos\theta^*} = \boxed{1.52343\ AU}$$

$$v_0 = \sqrt{\mu\left(\frac{2}{r} - \frac{1}{a}\right)} = \boxed{1.55960 \times 10^{-7}\ AU/s = 23.3313\ km/s}$$

$$\gamma_0 = +\cos^{-1}\left(\frac{h}{r_0 v_0}\right) = \boxed{14.4651°}$$

$$E_0 = 2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta^*}{2}\right)\right) = \boxed{104.6595°}$$

$$\dot{r}_0 = v_0 sin\left(\gamma_0\right) = \boxed{3.89574 \times 10^{-8}\ AU/s = 5.82794\ km/s}$$

$$\dot{\theta}_0 = \frac{h}{r_0^2} = \boxed{9.91290 \times 10^{-8}\ rad/s}$$

$$t_{p,next} = \mathbb{P} - \sqrt{\frac{a^3}{\mu}}\left(E_0 - e\sin E_0\right) = \boxed{467.8836\ days}$$

Finally, the vector values for $\bar{r}_0$ and $\bar{v}_0$ are expressed in rotating frame coordinates as

$$\bar{r}_0 = \boxed{1.52343\hat{r}\ AU}$$

$$\bar{v}_0 = v_0\sin\gamma_0\hat{r} + v_0\cos\gamma_0\hat{\theta} = \boxed{5.82795\hat{r} + 22.59174\hat{\theta}\ km/s}$$

The position and velocity are expressed in the inertial frame by applying a rotation matrix as follows

$$\begin{bmatrix}\hat{e}\\ \hat{p}\\ \hat{h}\end{bmatrix} = \begin{bmatrix}\cos\theta^* & -\sin\theta^* & 0\\ \sin\theta^* & \cos\theta^* & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\hat{r}\\ \hat{\theta}\\ \hat{h}\end{bmatrix} \tag{1}$$

resulting in

$$\boxed{\bar{r}_0 = -0.730422\hat{e} + 1.33691\hat{p}\ AU}$$

$$\boxed{\bar{v}_0 = -22.61997\hat{e} - 5.71739\hat{p}\ km/s}$$

Some important notes are:

- Convert μ into units of AU^3/s^2 when using distances in AU.
- Be careful of the units of E_0! In Kepler's time equation, eccentric anomaly must be in *radians*.
- As the asteroid is ascending, eccentric anomaly must be less than 180°, and the flight path angle is positive.

Part (b)

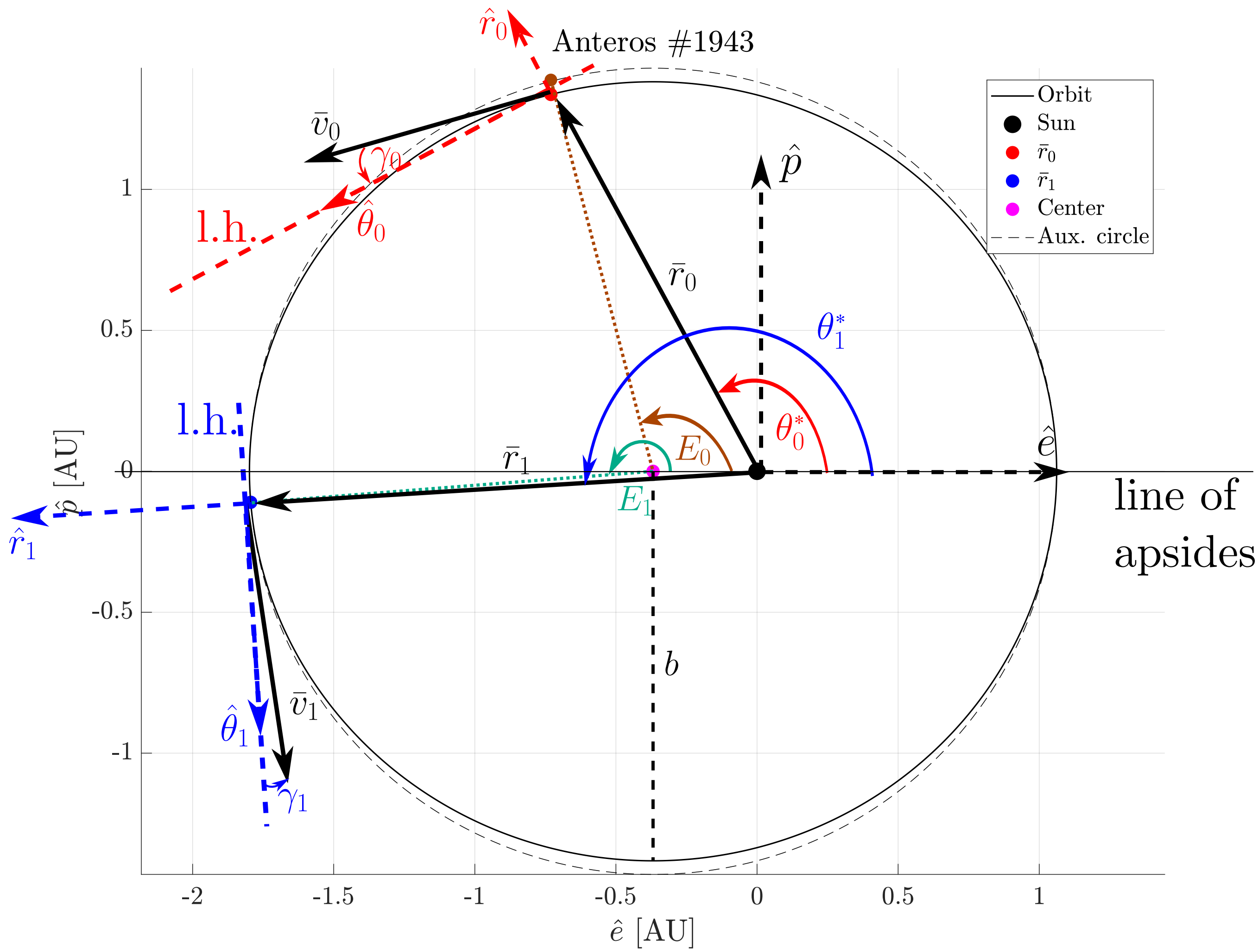

Figure 1: Asteroid Anteros #1943 locations and relevant quantities at times t_0 and t_1

Part (c)

Many definitions of one month exist. For the purposes of this problem, assume that 1 month is equal to exactly 30 days. The time past periapsis for the asteroid in part (a), at time t_0, is

$$t_0 = \sqrt{\frac{a^3}{\mu}}\,(E_0 - e\sin E_0) = 156.8359\ days$$

Thus, the time t_1 is 348.8359 days past periapsis. The mean anomaly M_1 is then computed using

$$M_1 = \sqrt{\frac{\mu}{a^3}}\,(t_1 - t_p) = 201.0197°$$

Kepler's time equation relates mean anomaly to eccentric anomaly

$$M_1 = E_1 - e\sin E_1$$

Newton's method is leveraged to solve for the eccentric anomaly iteratively, as in

$$E_{n+1} = E_n - \frac{E_n - e\sin E_n - M}{1 - e\cos E_n}$$

where the notation E_i refers to the value of E at each iteration i. By iterating until the update to E_{n+1} is smaller than some tolerance (in this case, 10^{-12}), the value of eccentric anomaly is found to within numerical precision. To initiate the Newton solver, an initial guess for the eccentric anomaly is provided. While there are many methods for generating this initial guess, the simplest method is to set $E_0 = M$. Information from the iterative procedure initialized using $E_0 = M$ is shown in Table 1. Converting the final result to degrees, the eccentric anomaly at time t_1 is

$$\boxed{E_1 = 196.7623°}$$

From the eccentric anomaly, the true anomaly θ_1^* is calculated, as in

$$\theta_1^* = 2\tan^{-1}\left(\sqrt{\frac{1+e}{1-e}}\tan\left(\frac{E_1}{2}\right)\right) = \boxed{192.9160°}$$

Using the same equations as in Part (a), the following values are also computed

$$r_1 = 1.78300\ AU$$
$$v_1 = 19.35982\ km/s$$
$$\gamma_1 = \boxed{-4.39749°}$$
$$\dot{r}_1 = \boxed{-1.484422\ km/s}$$
$$\dot{\theta}_1 = \boxed{7.23674 \times 10^{-8}\ rad/s}$$
$$\bar{r}_1 = \boxed{1.78300\hat{r}\ AU = -1.73789\hat{e} - 0.39854\hat{p}\ AU}$$
$$\bar{v}_1 = \boxed{-1.48442\hat{r} + 19.302820\hat{\theta}\ km/s = 5.76148\hat{e} - 18.48264\hat{p}\ km/s}$$

These final states are marked on the plot in Part (b).

Iteration	E_i [rad]	$\lvert\Delta E\rvert$ [rad]
0	3.508455308734311	–
1	3.433957464860005	0.074497843874306
2	3.434149779095297	0.000192314235293
3	3.434149780197008	0.000000001101711
4	3.434149780197008	0

Table 1: Iterating for eccentric anomaly at time t_1

Part (d)

Since θ_1^* is greater than 180° and less than 270°, the asteroid is in the third quadrant. Applying the same equation as in Part (a), the time until perihelion from t_1 is 275.8836 days, which is exactly $30 \times 6.4 = 192$ days less than the result obtained from Part (a), as expected.

Part (e)

Taking values from the Useful Constants table on page 2, Earth's aphelion is computed to be

$$r_{a,\oplus} = 1.0167\ AU$$

which is smaller than the perihelion for the asteroid. Thus, the asteroid will *never* intersect Earth's orbit, assuming each body remains in the same orbit for all time.

Problem 2

Problem Statement

A satellite is in orbit about the Earth characterized by $e = 0.75$ and $a = 8R_{\oplus}$.

(a) Create one plot with three curves for r as a function of θ^*, E, M. Which angular rate is faster/slower? Which slopes are shallow/steep? In class, we noted that solving Kepler's equation was more challenging closer to the apse points. From this plot, why is that true?

(b) The vehicle is currently located at the point in the orbit such that $M_0 = 90°$. Use the plot from part (a) to tell you approximately where this point is located in the orbit. Then, determine the location in terms of E and θ^*. Determine the additional orbit parameters and satellite state information at this location: p,h,r_p,r_a, period, $\mathcal{E}$, r, v, γ, $(t - t_p)$.

(c) Write $\bar{\mathbf{r}}_0$, $\bar{\mathbf{v}}_0$ in terms of components in the directions of $\hat{\mathbf{r}}$ and $\hat{\theta}$; $\hat{\mathbf{e}}$ and $\hat{\mathbf{p}}$.

(d) Determine θ^* in precisely 2 hours. Use f and g relationships to write $\bar{\mathbf{r}}$, $\bar{\mathbf{v}}$ in terms of $\bar{\mathbf{r}}_0$, $\bar{\mathbf{v}}_0$. Prove that $f(\theta^* - \theta_0^*)$, $g(\theta^* - \theta_0^*)$ produce the same results as $f(E - E_0)$, $g(E - E_0)$. Which vector basis do you use to write $\bar{\mathbf{r}}$, $\bar{\mathbf{v}}$?

(e) PLOT the entire orbit in MATLAB. (Do <u>not</u> use polar plots; compute $\hat{\mathbf{e}}$ and $\hat{\mathbf{p}}$ components along the path and plot the orbit.)
On the plot, start by adding appropriate unit vectors. Mark the given state by adding to the plot: line of apsides, minor axis, $\bar{\mathbf{r}}_0$, $\bar{\mathbf{v}}_0$, θ_0^*, γ_0, E_0, local horizon.

Part (a)

To plot the magnitude of the radius vector as a function of true anomaly θ^*, one can recall the conic equation where:

$$r = \frac{p}{1 + e\cos\theta^*} \tag{2}$$

One can also plot the magnitude of the radius vector as a function of eccentric anomaly E by using the following relation:

$$r = a(1 - e\cos E) \tag{3}$$

And lastly, since we know radius as a function of eccentric anomaly, one can relate the same radius values to mean anomaly through eccentric anomaly by using Kepler's equation:

$$M = E - e\sin E \tag{4}$$

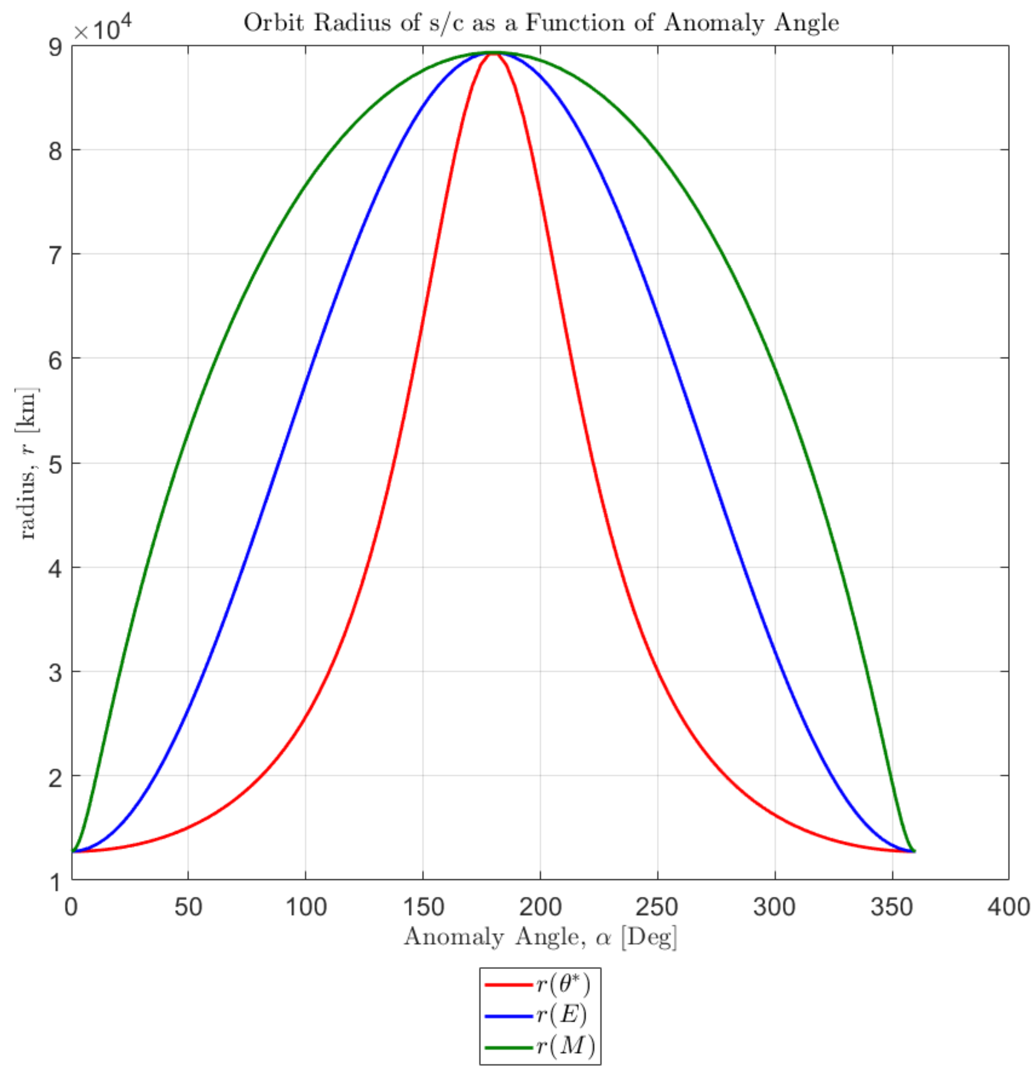

Figure 2: Depiction of the orbit radius of the spacecraft as a function of the anomaly angles θ^*, E, M.

True anomaly has the fastest angular rate. Conversely, mean anomaly has the slowest angular rate, which is equivalent to the mean motion of the spacecraft.

Mean anomaly, towards the periapse location(s), has the steepest curve while near the apoapse location, it has one of the most shallow curves. For true anomaly, it is almost the exact opposite as, towards the periapse location(s), it has the most shallow curve while near the apoapse location, it has the most steep curve. Eccentric anomaly, as one can see in figure 2, has the middle ground in terms of slope. It begins somewhat shallow near the periapse location, increases steepness in transit towards the apoapse location only to become more shallow near apoapsis.

It is more challenging to solve Kepler's equation near the apse points, particularly periapsis as, since the spacecraft moves faster at periapsis, the rapid change in eccentric anomaly makes it difficult to relate to mean anomaly. In addition, when solving for mean anomaly iteratively, such as using a Newton-Raphson method, one must be careful about critical points. As one can see from figure 2, periapsis and apoapsis are critical points where the slope of the plot becomes undefined or is zero. When the slope of a function is zero or is undefined, Newton-Raphson tends to bounces around that critical point, failing to converge to a root.

Part (b)

As one can see in figure 3 below, one can draw a line at roughly where mean anomaly is 90° and then draw a horizontal line and two other vertical lines to help estimate the corresponding values for true and eccentric anomaly. Using this estimation method, one would approximate eccentric and true anomaly to be $E \approx 125°$ and $\theta^* \approx 155°$. Now, using a Newton-Raphson method, one can solve iteratively to find the true value of eccentric anomaly when

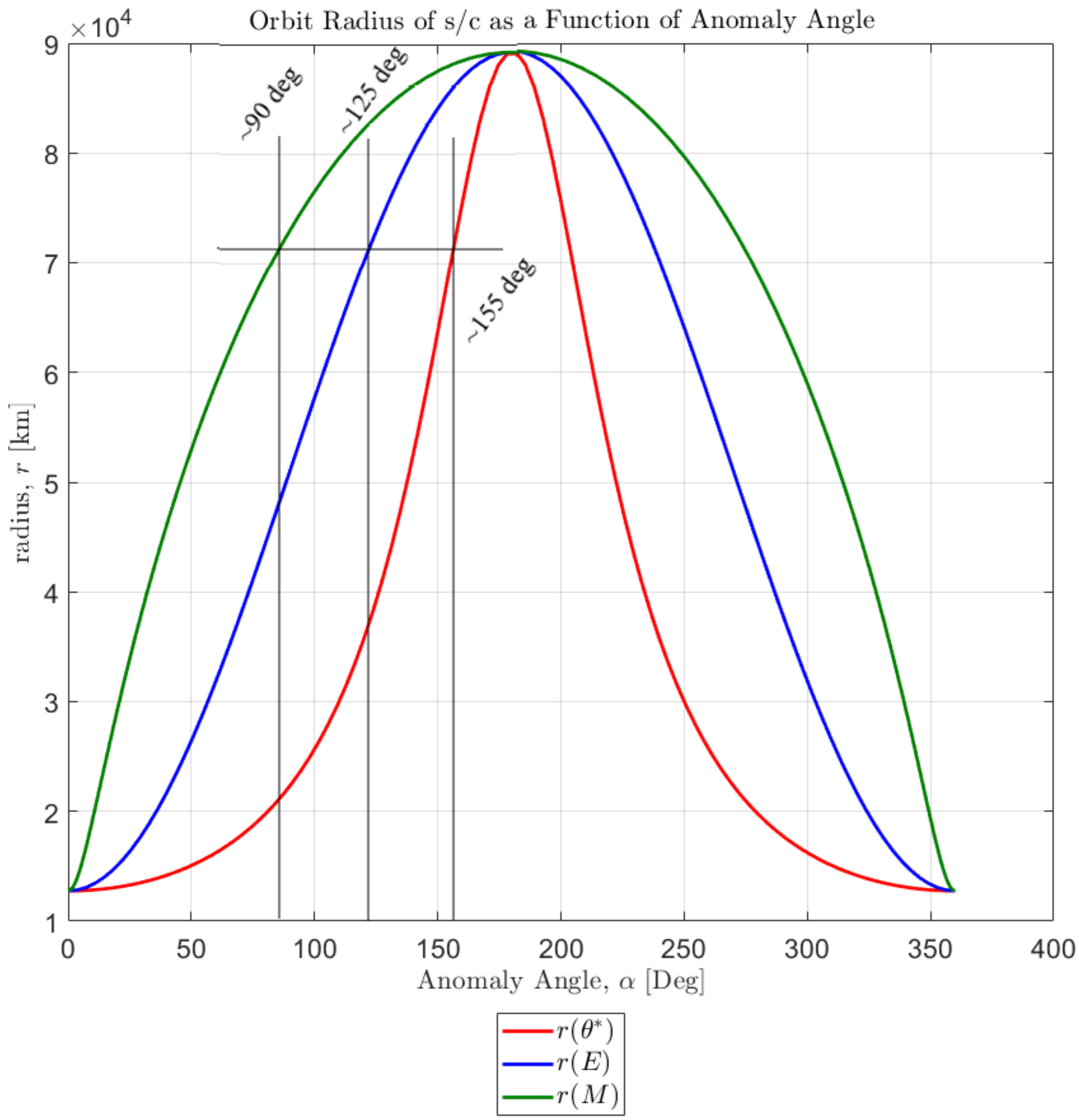

Figure 3: Estimating the corresponding θ^* and E given $M_0 = 90°$.

mean anomaly is 90° and find:

$$\boxed{E = 125.1401°} \tag{5}$$

which is very close to our guesstimate from above. Next, one can solve for true anomaly using the following relationship with eccentric anomaly:

$$\boxed{\theta^* = 2 \cdot \arctan\left[\sqrt{\frac{1+e}{1-e}} \cdot \tan\frac{E}{2}\right] = 157.8026°} \tag{6}$$

which is also pretty close to our estimated value for true anomaly from above. One can also find the orbit radius of the spacecraft at this location, by using the conic equation or the following relationship:

$$\boxed{r = a(1 - e\cos E) = 7.3051 \cdot 10^4 \text{ km}} \tag{7}$$

One can easily solve for the semi-latus rectum since we are given semi-major axis and eccentricity at the beginning of the problem:

$$\boxed{p = a(1 - e^2) = 2.2323 \cdot 10^4 \text{ km}} \tag{8}$$

Since we just solved for semi-latus rectum, one can obtain the specific angular momentum magnitude through the following equation:

$$\boxed{h = \sqrt{\mu p} = 9.4330 \text{ km}^2/\text{s}} \tag{9}$$

And, again, since we are given semi-major axis and eccentricity, one can solve for the periapsis and apoapsis radii using the following relationships, derived from the conic equation:

$$\boxed{r_p = a(1 - e) = 1.2756 \cdot 10^4 \text{ km}} \tag{10}$$

$$\boxed{r_a = a(1 + e) = 8.9294 \cdot 10^4 \text{ km}} \tag{11}$$

One can then solve for mean motion, which then can be used to solve for the orbit period of the spacecraft:

$$n = \sqrt{\frac{\mu}{a^3}} = 5.4776 \cdot 10^{-5} \text{ rad/s} \tag{12}$$

$$\boxed{\begin{aligned} \mathcal{P} = \frac{2\pi}{n} &= 1.1471 \cdot 10^5 \text{ s} \\ &= 31.8628 \text{ hrs} \\ &= 1.3276 \text{ days} \end{aligned}} \tag{13}$$

Next, one can solve for energy, using the expression:

$$\boxed{\mathcal{E} = -\frac{\mu}{2a} = -3.9059 \text{ km}^2/\text{s}^2} \tag{14}$$

Afterthat, one can solve for the orbit velocity of the spacecraft, using the vis-viva equation:

$$\boxed{v = \sqrt{2(\mathcal{E} + \frac{\mu}{r}} = 1.7610 \text{ km/s}} \tag{15}$$

One can also solve for the flight path angle γ. Realize that, since the spacecraft is ascending in its orbit, the flight path angle will be positive (+). Thus, using the following relation, the flight path angle is:

$$\boxed{\gamma = +\arccos\left(\frac{h}{rv}\right) = 42.8379°} \tag{16}$$

Lastly, one can solve for the time since periapsis $(t - t_p)$. Realize that it is "time since" periapsis and not "time until" periapsis because the spacecraft is ascending. This also means that our time value can be taken to be positive to indicate time that has elapsed since our last periapsis passage. So, recalling Kepler's equation:

$$\boxed{\begin{aligned} (t - t_p) = \frac{M_0}{n} &= 2.8677 \cdot 10^4 \text{ s} \\ &= 7.9657 \text{ hrs} \end{aligned}} \tag{17}$$

Part (c)

Now, let's write $\bar{\mathbf{r}}_0$, $\bar{\mathbf{v}}_0$ in terms of components in the directions of $\hat{\mathbf{r}}$ and $\hat{\theta}$ as well as $\hat{\mathbf{e}}$ and $\hat{\mathbf{p}}$. Note that, in the Radial-Transverse-Normal (also known as RTN or $\hat{\mathbf{r}} - \hat{\theta} - \hat{\mathbf{h}}$) reference frame, writing the radius vector is easy as it is the orbit radius magnitude in the radial direction:

$$\boxed{\bar{\mathbf{r}}_0 = 7.3052 \cdot 10^4 \, \hat{\mathbf{r}} \text{ [km]}} \tag{18}$$

Recall that the velocity vector, in the RTN frame, can be expressed as:

$$\bar{\mathbf{v}} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\theta} \tag{19}$$

Which means we need to define $\dot{r}$ and $\dot{\theta}$. Recall that, for $\dot{\theta}$:

$$\dot{\theta} = \frac{h}{r^2} = 1.7676 \cdot 10^{-5} \text{ rad/s} \tag{20}$$

Then, remembering that the spacecraft is still ascending in its orbit, then:

$$\dot{r} = +\sqrt{v^2 - r^2\dot{\theta}^2} = +1.1973 \text{ km/s} \tag{21}$$

Which means that the velocity vector is then:

$$\boxed{\bar{\mathbf{v}}_0 = 1.1973 \, \hat{\mathbf{r}} + 1.2913 \, \hat{\theta} \text{ [km/s]}} \tag{22}$$

Now, to write $\bar{\mathbf{r}}_0$, $\bar{\mathbf{v}}_0$ in terms of components in the directions of $\hat{\mathbf{e}}$ and $\hat{\mathbf{p}}$, one must note the transformation from the RTN frame to the Perifocal Coordinate System (also known as PQW or $\hat{\mathbf{e}} - \hat{\mathbf{p}} - \hat{\mathbf{h}}$). This transformation simply involves the angle true anomaly and can be expressed as:

$$\begin{bmatrix} \hat{\mathbf{e}} \\ \hat{\mathbf{p}} \\ \hat{\mathbf{h}} \end{bmatrix} = \begin{bmatrix} \cos\theta^* & -\sin\theta^* & 0 \\ \sin\theta^* & \cos\theta^* & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{r}} \\ \hat{\theta} \\ \hat{\mathbf{h}} \end{bmatrix} \tag{23}$$

Thus, plugging in $\theta^* = 157.8026°$ from above, the position and velocity vectors in the PQW frame are:

$$\boxed{\bar{\mathbf{r}}_0 = -6.7638 \cdot 10^4 \, \hat{\mathbf{e}} + 2.7599 \cdot 10^4 \, \hat{\mathbf{p}} \text{ [km]}} \tag{24}$$

$$\boxed{\bar{\mathbf{v}}_0 = -1.5964 \, \hat{\mathbf{e}} - 0.7432 \, \hat{\mathbf{p}} \text{ [km/s]}} \tag{25}$$

Part (d)

Next, let's determine the true anomaly θ^* in precisely 2 hours. To do so, we will again make use of Kepler's equation, where:

$$\begin{aligned}(t - t_p) = (t - t_{p,0}) + 2 \text{ hrs } &= 3.5877 \cdot 10^4 \text{ s} \\ &= 9.9657 \text{ hrs}\end{aligned} \tag{26}$$

$$M = n(t - t_p) = 112.5969° \tag{27}$$

So, using the same Newton-Raphson method used previously in part (b), the corresponding eccentric anomaly given the mean anomaly above is:

$$E = 140.1388° \tag{28}$$

So, one can solve for true anomaly using the following relationship with eccentric anomaly:

$$\boxed{\theta^* = 2 \cdot \arctan\left[\sqrt{\frac{1+e}{1-e}} \cdot \tan\frac{E}{2}\right] = 164.3925°} \tag{29}$$

Now, we would like to use the f and g relationships to write $\bar{\mathbf{r}}$ and $\bar{\mathbf{v}}$ in terms of $\bar{\mathbf{r}}_0$ and $\bar{\mathbf{v}}_0$. Recall that the f, g, $\dot{f}$, and $\dot{g}$ are defined, in terms of <u>true</u> anomaly, as:

$$f = 1 - \frac{r}{p}\Big[1 - \cos(\theta^* - \theta_0^*)\Big] \tag{30}$$

$$g = \frac{r r_0}{\sqrt{\mu p}} \sin(\theta^* - \theta_0^*) \tag{31}$$

$$\dot{f} = \left[\frac{\bar{\mathbf{r}}_0 \cdot \bar{\mathbf{v}}_0}{p r_0}\Big[1 - \cos(\theta^* - \theta_0^*)\Big] - \frac{1}{r_0}\sqrt{\frac{\mu}{p}} \sin(\theta^* - \theta_0^*)\right] \tag{32}$$

$$\dot{g} = 1 - \frac{r_0}{p}\Big[1 - \cos(\theta^* - \theta_0^*)\Big] \tag{33}$$

Meaning that we can write $\bar{\mathbf{r}}$ and $\bar{\mathbf{v}}$ as:

$$\bar{\mathbf{r}} = f\bar{\mathbf{r}}_0 + g\bar{\mathbf{v}}_0 \tag{34}$$

$$\bar{\mathbf{v}} = \dot{f}\bar{\mathbf{r}}_0 + \dot{g}\bar{\mathbf{v}}_0 \tag{35}$$

So, given that $\theta_0^* = 157.8026°$ and $\theta^* = 164.3925°$, $\bar{\mathbf{r}}$ and $\bar{\mathbf{v}}$ can be expressed as:

$$\boxed{\bar{\mathbf{r}} = 0.9762\,\bar{\mathbf{r}}_0 + 7.1456 \cdot 10^3\,\bar{\mathbf{v}}_0 \text{ [km]}} \tag{36}$$

$$\boxed{\bar{\mathbf{v}} = -6.2839 \cdot 10^{-6}\,\bar{\mathbf{r}}_0 + 0.9784\,\bar{\mathbf{v}}_0 \text{ [km/s]}} \tag{37}$$

Or, written in the PQW frame:

$$\boxed{\bar{\mathbf{r}} = -7.7436 \cdot 10^4\,\hat{\mathbf{e}} + 2.1631 \cdot 10^4\,\hat{\mathbf{p}} \text{ [km]}} \tag{38}$$

$$\boxed{\bar{\mathbf{v}} = -1.1369\,\hat{\mathbf{e}} - 0.9006\,\hat{\mathbf{p}} \text{ [km/s]}} \tag{39}$$

Note that one could write $\bar{\mathbf{r}}$ and $\bar{\mathbf{v}}$ in terms of components in the RTN frame, but one must realize that the unit vectors $\hat{\mathbf{r}} - \hat{\theta} - \hat{\mathbf{h}}$ are from the original location $\bar{\mathbf{r}}_0$ and $\bar{\mathbf{v}}_0$, meaning that the position vector $\bar{\mathbf{r}}$ does not completely point in the radial direction. Thus, we normally write $\bar{\mathbf{r}}$ and $\bar{\mathbf{v}}$ in terms of the PQW frame for convenience, even though you can express the vectors in any set of unit vectors you desire.

Now, let's prove that $f(\theta^* - \theta_0^*)$, $g(\theta^* - \theta_0^*)$ produce the same results as $f(E - E_0)$, $g(E - E_0)$. Recall that the f, g, $\dot{f}$, and $\dot{g}$ are defined, in terms of <u>eccentric</u> anomaly, as:

$$f = 1 - \frac{a}{r_0}\Big[1 - \cos(E - E_0)\Big] \tag{40}$$

$$g = (t - t_0) - \sqrt{\frac{a^3}{\mu}}\Big[(E - E_0) - \sin(E - E_0)\Big] \tag{41}$$

$$\dot{f} = \frac{\sqrt{\mu a}}{r r_0} \sin\left(E - E_0\right) \tag{42}$$

$$\dot{g} = 1 - \frac{a}{r}\Big[1 - \cos\left(E - E_0\right)\Big] \tag{43}$$

So, given that $E = 125.1401°$ and $E_0 = 140.1388°$, the values for f and g in terms of eccentric anomaly are:

$$\boxed{\begin{aligned} f(E) &= 0.9762 \text{ [unitless]} \\ g(E) &= 7.1456 \cdot 10^3 \text{ [s]} \\ \dot{f}(E) &= -6.2839 \cdot 10^{-6} \text{ [1/s]} \\ \dot{g}(E) &= 0.9784 \text{ [unitless]} \end{aligned}} \tag{44}$$

Note that the values in equation (44) above match the f and g coefficient values found in terms of true anomaly expressed in equations (36) and (37), proving that $f(\theta^* - \theta_0^*)$, $g(\theta^* - \theta_0^*)$ produce the same results as $f(E - E_0)$, $g(E - E_0)$.

Part (e)

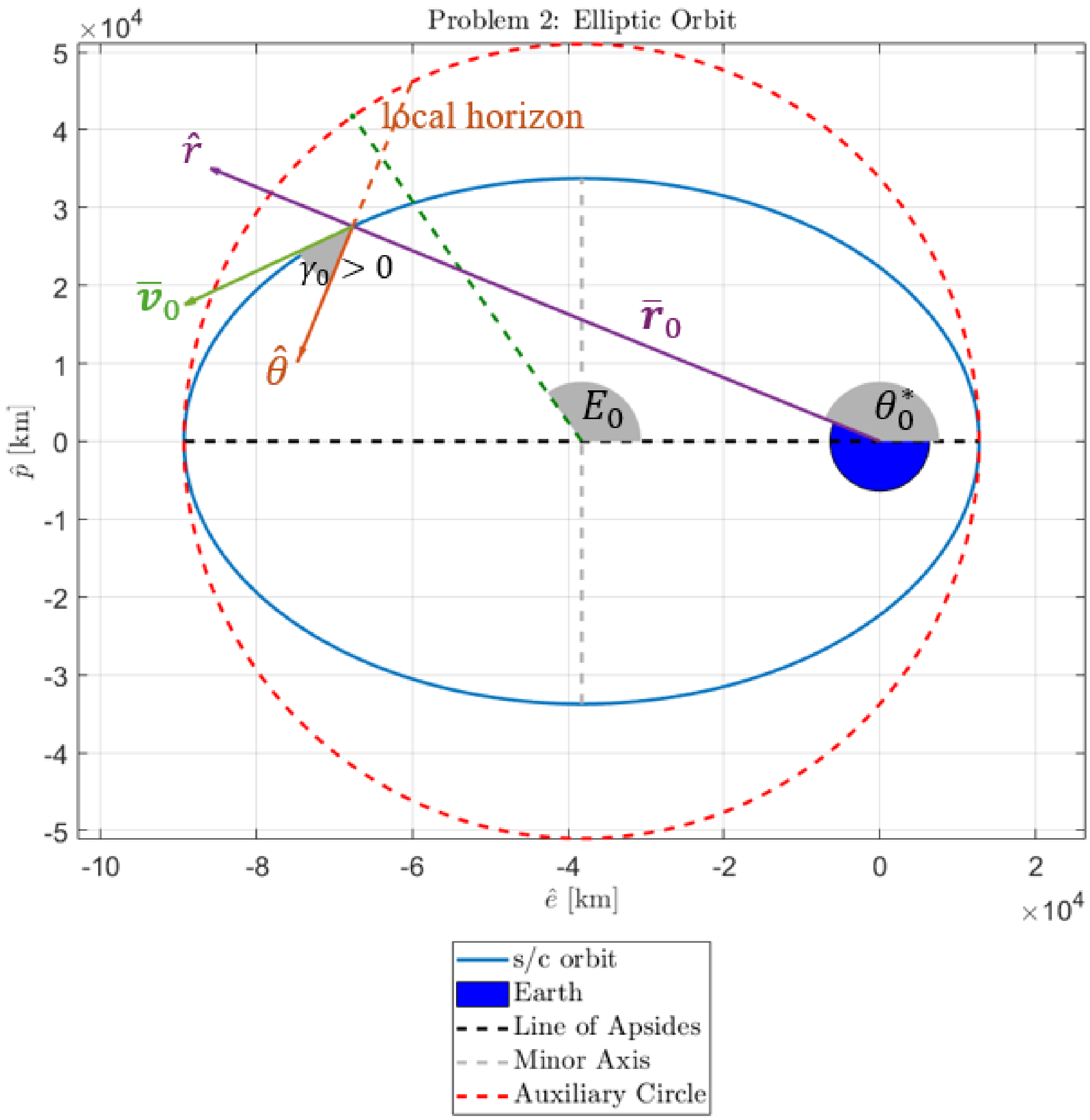

Figure 4: Plot of the s/c at its initial location.

As one can see in figure 4, the spacecraft is located at its initial position, specified in part (b) above. The spacecraft is still ascending in its orbit, as evident by its positive flight path angle γ_0. The auxiliary circle is depicted as well for reference, although it's not necessary per the problem statement.

Problem 3

Problem Statement

As part of some interplanetary mission, assume that a spacecraft is arriving along a hyperbola relative to Venus. The hyperbola is defined such that:

$$r_p = 10R_{♀} \qquad\qquad |a| = 50R_{♀}$$

At the current time, the spacecraft is located at $\theta^* = -110°$. (Of course, if the vehicle does not maneuver, it will pass by Venus and then move away from this planetary system!)

(a) Determine the characteristics of the hyperbolic arrival path: $p, e, h, \varepsilon, v_\infty, \theta^*_\infty, \delta$. At the current time, determine the values for $r, v, \gamma, \theta^*, H$. How much time until the spacecraft passes through periapsis?

(b) Along the approach path, what is the distance between Venus and the aim point, i.e., b, the aiming radius? At this location, determine r, v, γ, θ^*. Plot the hyperbolic pass between, at least, $-120° \leq 120°$. Add the angles δ, θ^*_∞ to the plot as well as the distances p, b, and r_p. On the plot, mark $\theta^*, \bar{r}, \bar{v}, \gamma$ at the aim point, unit vectors, major/minor axis; also sketch the local horizon.

(c) Confirm your results in GMAT. Use Sept 27 2024 as the start date and assume a two-body relative model with point masses. For simplicity, set inclination, argument of periapsis, and longitude of the ascending node (RAAN) all equal to zero. Produce a script that will plot the hyperbola using Venus as the central body. Plot the path between (at least) $-120° \leq 120°$. Is it consistent with the data you computed in (a) and (b)? Compare the quantities.

(d) Determine the velocity at periapsis. What is the velocity that is required to be in a circular orbit about Venus at this altitude? How much Δv in km/s must be delivered by the spacecraft to drop into this circular Venusian orbit? Is the amount of Δv ‘a lot’ or ‘not so much’? Why do you think so?

Part (a)

The spacecraft is on a hyperbolic trajectory relative to Venus with:

$$\theta^* = -110^\circ \qquad r_p = 10R_{♀} = 60519\ km \qquad |a| = 50R_{♀} = 302595\ km$$

Note that many of the equations are the same as for other conic sections:

$$e = 1 + \frac{r_p}{|a|} \quad \boxed{= 1.2}$$

$$p = a(e^2 - 1) = 133141.8\ km \quad \boxed{= 22R_{♀}}$$

$$h = \sqrt{\mu p} \quad \boxed{= 207971.7735\ km^2/s}$$

$$\varepsilon = \frac{\mu}{2|a|} \quad \boxed{= 0.53678778\ km^2/s^2}$$

$$v_\infty = \sqrt{2\varepsilon} \quad \boxed{= 1.0361349\ km/s}$$

$$\theta^*_\infty = \cos^{-1}\left(\frac{-1}{e}\right) = 2.5559071\ rad \quad \boxed{= 146.44269^\circ}$$

$$\delta = 2\sin^{-1}\left(\frac{1}{e}\right) = 1.9702215\ rad \quad \boxed{= 112.88538^\circ}$$

At the current time:

$$r = \frac{|a|(e^2-1)}{1 + e\cos\theta^*} = 225826.422\ km \quad \boxed{= 37.31496R_{♀}}$$

$$v = \sqrt{2\mu\left(\frac{1}{r} + \frac{1}{2|a|}\right)} \quad \boxed{= 1.9876217\ km/s}$$

$$\gamma = -\cos^{-1}\left(\frac{\sqrt{\mu|a|(e^2-1)}}{rv}\right) = -1.08904\ rad \quad \boxed{= -62.397426^\circ}$$

$$\theta^* \quad \boxed{= -110^\circ\ \text{(given)}}$$

$$H = \cosh^{-1}\left(\frac{a-r}{ea}\right) \quad \boxed{= -0.921273}$$

Note that the hyperbolic anomaly is a non-dimensional number. Then, the time until periapsis crossing:

$$t_p - t = \sqrt{\frac{|a|^3}{\mu}}(H - e\sinh H) = 101458.86735\ s \quad \boxed{= 28.18302\ hr}$$

Part (b)

We can compute the aiming radius, b, as:

$$b = |a|\sqrt{e^2 - 1} = 200718.81569\ km \quad \boxed{= 33.16625 R_♀}$$

Then, geometrically we can see that

$$\theta^* = -\frac{\delta}{2} \quad \boxed{= -56.44269°}$$

$$r = \frac{p}{1 + e\cos\theta^*} = 80045.57339\ km \quad \boxed{= 13.226519 R_♀}$$

$$v = \sqrt{2\mu\left(\frac{1}{r} + \frac{1}{2|a|}\right)} \quad \boxed{= 3.031570\ km/s}$$

$$\gamma = -\cos^{-1}\left(\frac{\sqrt{\mu|a|(e^2-1)}}{rv}\right) = -0.541305\ rad \quad \boxed{= -31.01451°}$$

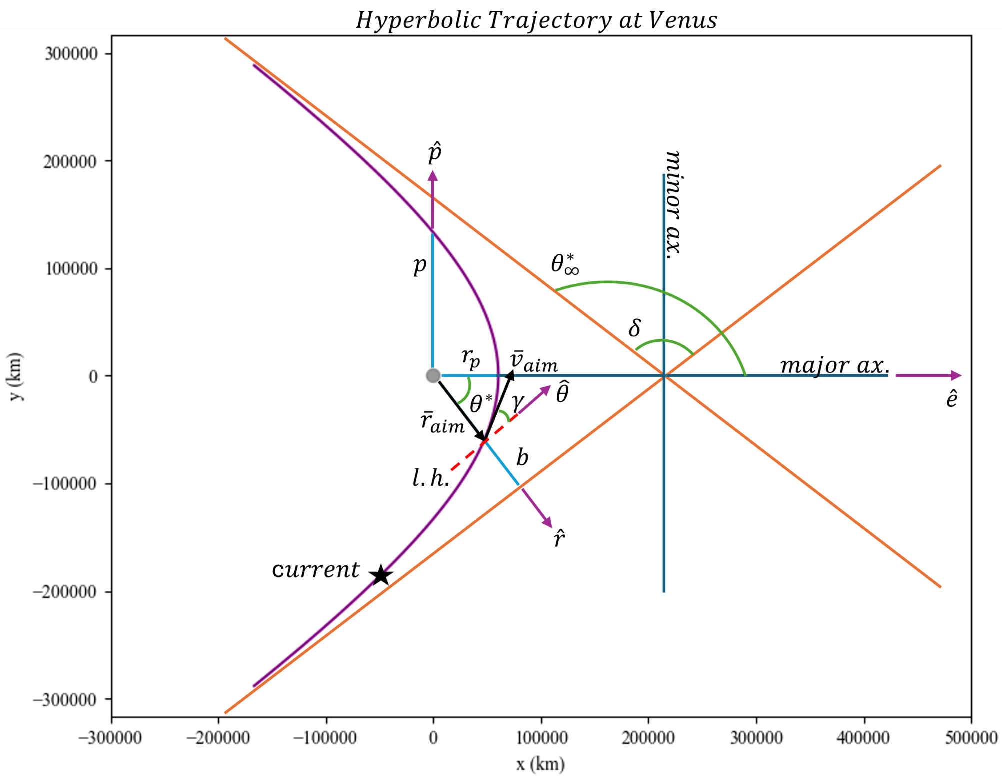

Figure 5: Annotated Venus trajectory plot.

Part (c)

First, we set the spacecraft up by defining the semimajor axis, eccentricity, and true anomaly. To verify part (a), we begin with $\theta^* = -110°$ and generate an output report to provide all of the requested parameters. To get values at ∞, we propagate for a longer time period to see what values each parameter approaches as the propagation goes towards infinity.

Parameter	Calculated	GMAT
e	1.2	1.2
$p\ (km)$	133141.8	133141.8
$h\ (km^2/s)$	207971.7735	207971.7735
$\varepsilon\ (km^2/s^2)$	0.53678778	0.53678778
$r\ (km)$	225826.422	225826.422
$v\ (km/s)$	1.9876217	1.9876217
$\gamma\ (deg)$	-62.397426	152.397426
$H\ (nd)$	-0.921273	-0.921273
$v_\infty\ (km/s)$	1.0361349	1.046607
$\theta^*_\infty\ (deg)$	146.44269	146.0585

At the initial time, all values appear to be exact matches for those that we calculated in the previous parts. The exception is flight path angle, which is offset by 270°. GMAT's documentation provides information about how the program defines flight path angle from which we can determine that this offset is acceptable. As we approach infinity, we can see not only from the final values provided in the table above but also by the progression of those values throughout the course of the simulation that they are indeed approaching the theoretical values at infinity.

To verify the values in part (b), we start the simulation at the true anomaly we calculated for the aiming radius location.

Parameter	Calculated	GMAT
$r\ (km)$	80045.57339	80045.57339
$v\ (km/s)$	3.031570	3.031570
$\gamma\ (deg)$	-31.01451	121.01451
$\theta^*\ (deg)$	-56.44269	303.55730

Again we see that the values match perfectly aside from the 270° offset in flight path angle. Thus we can conclude that all of the values are consistent with those we computed.

Now we can set the initial true anomaly to $-120°$ and the simulation stopping condition to be a true anomaly of $+120°$ to generate the following plot of the orbit:

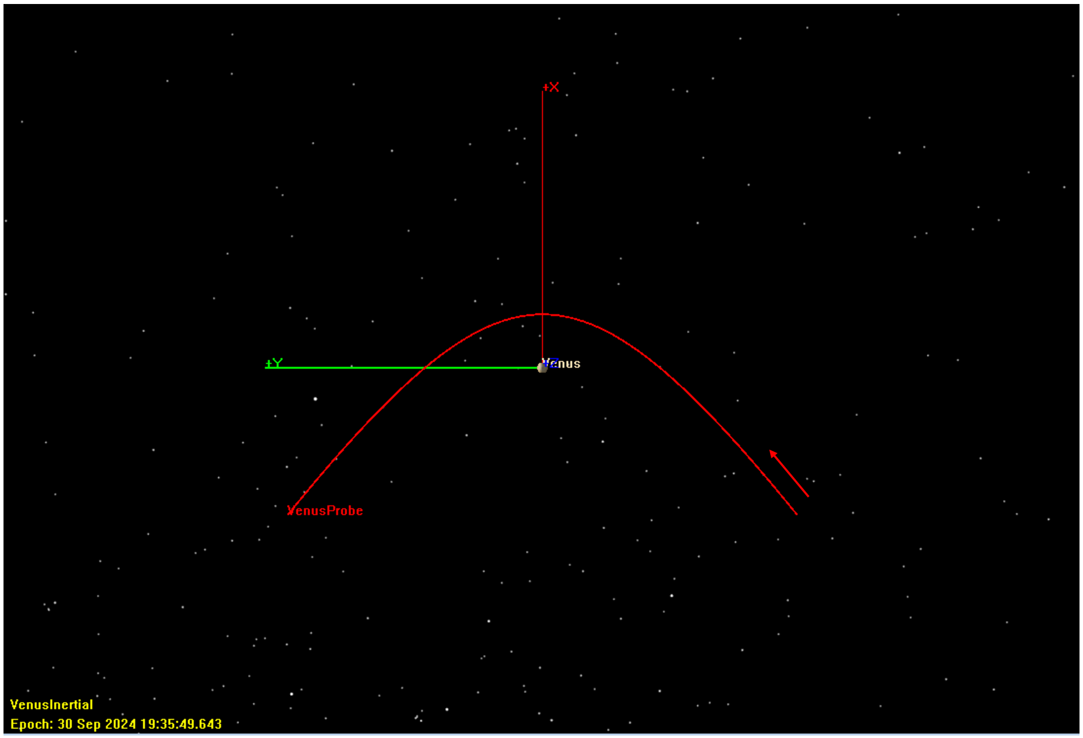

Figure 6: Hyperbolic trajectory at Venus with arrow showing direction of motion.

Part (d)

We can calculate the velocity at periapsis as follows:

$$v_p = \sqrt{2\varepsilon + \frac{2\mu}{r_p}} \quad \boxed{= 3.4364708\ km/s}$$

The velocity required for a circular orbit at this altitude is:

$$v_c = \sqrt{\frac{\mu}{r_p}} \quad \boxed{= 2.316868\ km/s}$$

And the Δv required to insert into a circular orbit at periapsis is the difference between these two:

$$\Delta v = v_p - v_c \quad \boxed{= 1.119603\ km/s}$$

While not infeasible, this is a relatively large maneuver for most interplanetary probes. When we design exploration missions, every kg of fuel on board the spacecraft means less scientific instruments that can be carried so it is important to minimize the required Δv. For example, the Saturn orbit insertion maneuver for Cassini was around 611 m/s [1], though it did not insert into a circular orbit it is still only about half the magnitude of our calculated Δv. We can also compare to the Messenger mission which went to Mercury. The Mercury insertion maneuver was around 862 m/s [2], again significantly less than that of our computed Venus insertion maneuver. The Lucy mission performed flybys of Jupiter's Trojan asteroids while in two consecutive elliptical orbits about the sun. The trajectory consists of numerous small maneuvers, totaling 1664.7 m/s for the entire mission [3]. In this context, we can conclude that a single maneuver of 1119.603 m/s is quite large.

Sources:

[1] https://dataverse.jpl.nasa.gov/file.xhtml?fileId=58832&version=2.0

[2] https://messenger.jhuapl.edu/About/Mission-Design.html

[3] https://ntrs.nasa.gov/api/citations/20190032357/downloads/20190032357.pdf