#1) $V(x) = x_1^4 - x_1^2 x_2 + x_2^2$

For positive definite: $V(0) = 0$, $V(x) > 0$, & $\lim_{\|x\| \to \infty} V(x) = \infty$

If $V(x) = x^T P x$, where $P = P^T \succ 0$, then $V(x)$ is radially unbounded & $V(x) > 0$ as $x^T P x \geq \lambda_{min}(P) \|x\|^2$ where $\lambda_{min}(P) > 0$ if $P = P^T \succ 0$.

$V(0) = 0^4 - (0^2)(0) + 0^2 = 0$ ✓

$$X \triangleq \begin{pmatrix} x_1^2 \\ x_2 \end{pmatrix}$$

$$V(x) = X^T P X = X^T \begin{pmatrix} 1 & -1/2 \\ -1/2 & 1 \end{pmatrix} X$$ ← symmetric

$$P = \begin{pmatrix} 1 & -1/2 \\ -1/2 & 1 \end{pmatrix}$$

$1 > 0$, & $\det(P) = 1 - \frac{1}{4} = \frac{3}{4} > 0$, $\therefore$ $P = P^T \succ 0$, via Sylvester's criteria

$\boxed{\begin{array}{l} V(x) \text{ can be written as } \begin{bmatrix} x_1^2 \\ x_2 \end{bmatrix}^T \begin{pmatrix} 1 & -1/2 \\ -1/2 & 1 \end{pmatrix} \begin{bmatrix} x_1^2 \\ x_2 \end{bmatrix} \text{ where} \\ P \succ 0 \text{ \& } V(0) = 0 \therefore V(x) \text{ is a positive definite function.} \end{array}}$

#2) $\dot{x} = -(1+\sin(x))x$ $\qquad x_e = 0$

For Asymptotic stability: $V(x)$ is locally positive definite about $x_e$,

& $\dot{V} < 0$ for $x \neq x_e$ & $\|x - x_e\| < R$

Scalar system, choose $V(x) = x^2$ as canidate Lyapunov function

$V(0) = 0$ & $V(x) > 0$ for $x \neq 0$ $\therefore$ $V = x^2$ is L.P.D about $0$.

$\frac{\partial V}{\partial x} = 2x$

$\dot{V} = \frac{\partial V}{\partial x}\dot{x} = -2x^2(1+\sin(x))$

$-2x^2$ is always negative when $x \neq 0$, $\therefore$ for $\dot{V} < 0$, $(1+\sin(x))$ must be positive when $x \neq 0$. The minimum value of $\sin(x)$ is $-1$, which occurs when $x = -\frac{\pi}{2}$, $\therefore$ $(1+\sin(x)) = 0$ at $x = -\frac{\pi}{2}$, & when $x \neq -\frac{\pi}{2}$, $(1+\sin(x)) > 0$. Therefore $\dot{V} < 0$

when $x \neq 0$ & $\|x - x_e\| < \frac{\pi}{2}$.

$V(x)$ is L.P.D about the origin, & $\dot{V} < 0$ when $x \neq 0$ & $\|x - x_e\| < \frac{\pi}{2}$. Therefore $x_e = 0$ is Asymptotically stable.

#3) $\dot{x}_1 = x_2$ $x_e = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\dot{x}_2 = -x_1 + x_1^3 - x_2$

For Asymptotic stability : V is L.P.D & $\dot{V} < 0$ for $\|x - x_e\| < R$

Treat $\dot{x}_1$ as velocity $\therefore$ $T = \frac{1}{2} x_2^2$

Treat $\dot{x}_2$ as force $\therefore$ $U = -\int_0^{x_1} -x_1 + x_1^3 \, dx_1$

$$U = \frac{x_1^2}{2} - \frac{x_1^4}{4}$$

"Total Energy" $= \frac{1}{2} x_2^2 + \frac{1}{2} x_1^2 - \frac{x_1^4}{4}$

Add terms to account for damping, get $x^T P x$ form:

$\lambda x_2 x_1 + \frac{\lambda x_1^2}{2}$

$\therefore V(x) = \frac{1}{2} x_2^2 + \lambda x_2 x_1 + \frac{\lambda x_1^2}{2} + \frac{x_1^2}{2} - \frac{x_1^4}{4}$ $x \triangleq \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$V(x) = x^T \begin{pmatrix} \lambda/2 & \lambda/2 \\ \lambda/2 & \frac{1}{2} \end{pmatrix} x + \frac{x_1^2}{2} - \frac{x_1^4}{4}$$

$V(0) = 0$ ✓

$$\frac{\partial V}{\partial x} = [\lambda x_2 + \lambda x_1 + x_1 - x_1^3 \quad x_2 + \lambda x_1]$$

$$\frac{\partial V(0)}{\partial x} = [0 \quad 0]$$

$$\frac{\partial^2 V}{\partial x^2} = \begin{pmatrix} \lambda + 1 - 3x^2 & \lambda \\ \lambda & 1 \end{pmatrix}$$

$$\frac{\partial^2 V(0)}{\partial x^2} = \begin{pmatrix} \lambda+1 & \lambda \\ \lambda & 1 \end{pmatrix}$$

For $\frac{\partial^2 V(0)}{\partial x^2} \succ 0$ , $\lambda+1 > 0$ & $\det\left(\frac{\partial^2 V(0)}{\partial x^2}\right) > 0$

$0 < \lambda < 1$ $\therefore$ $\lambda+1 > 0$ ✓

$\det = \lambda + 1 - \lambda^2 > 0$ ✓

$\therefore \frac{\partial^2 V(0)}{\partial x^2} \succ 0.$

$V(0) = 0$, $\frac{\partial V(0)}{\partial x} = 0$, & $\frac{\partial^2 V}{\partial x^2}(0) \succ 0$ $\therefore$ $\underline{V \text{ is L.P.D.}}$

$$\dot{V} = \frac{\partial V}{\partial x}\dot{x} = [\lambda x_2 + \lambda x_1 + x_1 - x_1^3 \quad x_2 + \lambda x_1] \begin{bmatrix} x_2 \\ -x_1 + x_1^3 - x_2 \end{bmatrix}$$

$$\dot{V} = \lambda x_2^2 + \cancel{\lambda x_2 x_1} + \cancel{x_2 x_1} - \cancel{x_1^3 x_2} - \cancel{x_2 x_1} + \cancel{x_2 x_1^3} - x_2^2$$
$$- \lambda x_1^2 + \lambda x_1^4 - \cancel{\lambda x_1 x_2}$$

$$\dot{V} = x_2^2(\lambda - 1) + \lambda x_1^2(x_1^2 - 1)$$

$x_2^2(\lambda - 1) < 0$ for all $x_2$

$\lambda x_1^2(x_1^2 - 1) < 0$ for $\|x_1\| < 1$

$\therefore$ $\underline{\dot{V} < 0}$ when $x \neq x_e$ & $\underline{\|x - x_e\| < 1}$ $\left(x_e = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)$

$V(x) = \frac{1}{2}x_2^2 + \lambda x_2 x_1 + \frac{\lambda x_1^2}{2} + \frac{x_1^3}{2} - \frac{x_1^4}{4}$ is L.P.D &

$\dot{V} < 0$ for $\|x - x_e\| < 1$ $\therefore$ $x_e = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is an asymptotically stable equilibrium state.

#4) $\dot{x}_1 = x_2$ $\qquad\qquad$ $x_e = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\dot{x}_2 = x_1 - x^3 + u$

$u = -K_1 x_1 - K_2 x_2$

Closed loop system: $\dot{x}_1 = x_2$

$$\dot{x}_2 = x_1 - x_1^3 \underbrace{- K_1 x_1}_{\text{stiffness}} \underbrace{- K_2 x_2}_{\text{Damping}}$$

Kinetic Energy term: $T = \frac{1}{2} \dot{x}_1^T \dot{x}_1 = \frac{1}{2} x_2^2$

Potential Energy term: $U = -\int_0^{x_1} x_1 - K_1 x_1 - x_1^3 \, dx_1$

$$U = -\frac{x_1^2}{2} + \frac{K_1 x_1^2}{2} + \frac{x_1^4}{4}$$

Total Energy terms: $\frac{1}{2} x_2^2 + \frac{K_1 x_1^2}{2} + \frac{x_1^4}{4} - \frac{x_1^2}{2}$

<u>For G.A.S.</u>: $V(x)$ is positive definite & $\dot{V} < 0$ for all $x \neq 0$

Add terms to set $x^T P x$: $\lambda K_2 x_2 x_1 + \frac{\lambda K_2^2 x_1^2}{2}$ $\quad (0 < \lambda < 1)$

$$\therefore V(x) = \frac{1}{2} x_2^2 + \lambda K_2 x_2 x_1 + \frac{\lambda K_2^2 x_1^2}{2} + \frac{K_1 x_1^2}{2} + \frac{x_1^4}{4} - \frac{x_1^2}{2}$$

$$V(x) = x^T P x + \frac{K_1 x_1^2}{2} + \frac{x_1^4}{4} - \frac{x_1^2}{2} \quad \text{where } x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\& \; P = P^T = \begin{pmatrix} \lambda K_2^2 & \lambda K_2 \\ \lambda K_2 & 1 \end{pmatrix} \frac{1}{2}$$

For $P > 0$: $\lambda K_2^2 > 0$ & $\det(P) > 0$

$$\det(P) = \lambda K_2^2 - \lambda^2 K_2^2 = \lambda K_2^2 (1 - \lambda) > 0 \quad \text{If } K_2 > 0$$

$\therefore P = P^T > 0$ If $K_2 > 0$.

$$\therefore V(x) = x^T P x + \frac{K_1 x_1^2}{2} + \frac{x_1^4}{4} - \frac{x_1^2}{2} \geq x^T P x - \frac{(K_1 - 1)^2}{4}$$

$$V(x) \geq x^T P x - \frac{(K_1 - 1)^2}{4} \quad \therefore \underline{\lim_{\|x\| \to \infty} V(x) = \infty}$$

$\therefore$ <u>$V(x)$ is a radially unbounded function</u>

$V(0) = 0^2 + 0^2 + 0^2 + 0^2 + 0^4 - 0^2 = 0$

$V(0) = 0$ ✓

$\frac{\partial V}{\partial x} = [\lambda K_2 x_2 + \lambda K_2^2 x_1 + K_1 x_1 + x_1^3 - x_1 \quad x_2 + \lambda K_2 x_1]$

$\frac{\partial V(0)}{\partial x} = [0 \quad 0]$

$\frac{\partial V(0)}{\partial x} = 0$ ✓

$\frac{\partial^2 V}{\partial x^2} = \begin{pmatrix} \lambda K_2^2 + K_1 + 3x^2 - 1 & \lambda K_2 \\ \lambda K_2 & 1 \end{pmatrix}$

$K_2 > 0$ from radially unbounded test

For $\frac{\partial^2 V}{\partial x^2} > 0$, $\lambda K_2^2 + K_1 + 3x^2 - 1 > 0$ &

$\lambda K_2^2 + K_1 + 3x^2 - 1 - \lambda^2 K_2^2 = \det\left(\frac{\partial^2 V}{\partial x^2}\right) > 0$ $\therefore$ $K_1$ must be greater than, $K_1 > 1 + \lambda^2 K_2^2 - \lambda K_2^2$. $\therefore$ $\frac{\partial^2 V}{\partial x^2} > 0$ for all $x$ & $V(x) > 0$.

$V(0) = 0$, $\frac{\partial V(0)}{\partial x} = 0$, & $\frac{\partial^2 V}{\partial x^2} > 0$ for all $x$ $\therefore$ $V(x) > 0$ &

$\lim_{\|x\| \to \infty} V(x) = \infty$ (V is radially unbounded) $\therefore$ V(x) is a positive definite function if $K_2 > 0$ & $K_1 > 1 + \lambda^2 K_2^2 - \lambda K_2^2$.

$$\dot{V} = \frac{\partial V}{\partial x}\dot{x}$$

$$\frac{\partial V}{\partial x} = \left[\lambda K_2 x_2 + \lambda K_2^2 x_1 + K_1 x_1 + x_1^3 - x_1 \quad x_2 + \lambda K_2 x_1\right]$$

$$\frac{\partial V}{\partial x}\dot{x} = \left[\lambda K_2 x_2 + \lambda K_2^2 x_1 + K_1 x_1 + x_1^3 - x_1 \quad x_2 + \lambda K_2 x_1\right]\begin{bmatrix} x_2 \\ x_1 - x_1^3 - K_1 x_1 - K_2 x_2 \end{bmatrix}$$

$$\dot{V} = \lambda K_2 x_2^2 + \cancel{\lambda K_2^2 x_1 x_2} + \cancel{K_1 x_1 x_2} + \cancel{x_2 x_1^3} - \cancel{x_1 x_2}$$
$$+ \cancel{x_2 x_1} - \cancel{x_2 x_1^3} - \cancel{x_2 K_1 x_1} - K_2 x_2^2 + \lambda K_2 x_1^2 + \lambda K_2 x_1^4 - \lambda K_2 K_1 x_1^2$$
$$- \cancel{\lambda K_2^2 x_2 x_1}$$

$$\dot{V} = K_2(\lambda x_2^2 - x_2^2 + \lambda x_1^2 - \lambda x_1^4 - \lambda K_1 x_1^2)$$
$$\dot{V} = K_2 x_2^2(\lambda - 1) + K_2 \lambda x_1^2(1 - K_1) - K_2 \lambda x_1^4$$

$K_2 x_2^2(\lambda - 1) < 0$ if $K_2 > 0$ as $0 < \lambda < 1$

$-K_2 \lambda x_1^4 < 0$ as $x_1^4 > 0$ & $K_2 > 0$ & $0 < \lambda < 1$

$K_2 \lambda x_1^2 (1 - K_1)$ is $< 0$ when $K_1 > 1$ as $K_2 \lambda x_1^2 > 0$

$\therefore$ For $\dot{V} < 0$, $K_1 > 1$ & $K_2 > 0$

If control gains $K_1 > 1$ & $K_2 > 0$, it can be shown that the control law $u = -K_1 x_1 - K_2 x_2$ results in a closed loop system that is G.A.S. about the origin as $V(x) = \frac{1}{2}x_2^2 + \lambda K_2 x_2 x_1 + \frac{\lambda K_2^2 x_1^2}{2} + \frac{K_1 x_1^2}{2} + \frac{x_1^4}{4} - \frac{x_1^2}{2}$ is positive definite & $\dot{V} < 0$ for all $x \neq 0$.

#5) $V(x) = x_1 - x_1^3 + x_1^4 - x_2^2 + x_2^4$

For radially unbounded: $\lim_{\|x\| \to \infty} V(x) = \infty$

$\|x\| = \sqrt{x_1^2 + x_2^2}$

$\|x\| = \infty = \sqrt{0^2 + \infty^2}$

$\|x\| = \infty = \sqrt{\infty^2 + 0^2}$

$\|x\| = \infty = \sqrt{\infty^2 + \infty^2}$

$\therefore \|x\| \to \infty$ if $(x_1, x_2) = (\infty, 0)$ or $(0, \infty)$ or $(\infty, \infty)$

$\lim_{(x_1,x_2) \to (0,\infty)} V(x) = 0 - 0^3 + 0^4 - \infty^2 + \infty^4 = \infty$ ✓

$\lim_{(x_1,x_2) \to (\infty,0)} V(x) = \infty - \infty^3 + \infty^4 - 0 - 0 = \infty$ ✓

$\lim_{(x_1,x_2) \to (\infty,\infty)} V(x) = \infty - \infty^3 + \infty^4 - \infty^2 + \infty^4 = \infty$ ✓

$\therefore \boxed{\lim_{\|x\| \to \infty} V(x) = \infty \text{ \& } V(x) \text{ is radially unbounded}}$

#6) $\dot{x}_1 = x_2$

$\dot{x}_2 = x_1 - x_1^3 - cx_2 + 1$

$V(x) = \frac{1}{2}\lambda c^2 x_1^2 + \lambda c x_1 x_2 + \frac{1}{2}x_2^2 - \frac{1}{2}x_1^2 + \frac{1}{4}x_1^4$

<u>For bounded:</u> $\dot{V} \leq 0$ for $\|x\| \geq R$ & $V$ is radially unbounded

$\|x\| \to \infty$ if $(x_1, x_2) = (0, \infty), (\infty, 0)$, or $(\infty, \infty)$

$\lim_{(x_1,x_2)\to(0,\infty)} V(x) = 0 + 0 + \infty^2 - 0 - 0 = \infty$ ✓

$\lim_{(x_1,x_2)\to(\infty,0)} V(x) = \infty^2 + 0 + 0 - \infty^2 + \infty^4 = \infty$ ✓

$\lim_{(x_1,x_2)\to(\infty,\infty)} V(x) = \infty^2 + \infty^2 + \infty^2 - \infty^2 + \infty^4 = \infty$ ✓

$\therefore \lim_{|x|\to\infty} V(x) = \infty$ & <u>$V(x)$ is radially unbounded</u>

$$\frac{\partial V}{\partial x} = [\lambda c^2 x_1 + \lambda c x_2 - x_1 + x_1^3,\ \lambda c x_1 + x_2]$$

$$\frac{\partial V}{\partial x}\dot{x} = \cancel{x_2 x_1 \lambda c^2} + \lambda c x_2^2 - \cancel{x_2 x_1} + \cancel{x_1^3 x_2} + \cancel{x_2 x_1} - \cancel{x_2 x_1^3}$$
$$- c x_2^2 + x_2 + \lambda c x_1^2 - \lambda c x_1^4 - \cancel{\lambda c^2 x_1 x_2} + \lambda c x_1$$

$$= \lambda c x_2^2 - c x_2^2 + x_2 + \lambda c x_1^2 - \lambda c x_1^4 + \lambda c x_1$$

$$= x_2^2 c(\lambda - 1) + x_2 + \lambda c x_1^2 - \lambda c x_1^4 + \lambda c x_1$$

If $\lim_{\|x\|\to\infty} \dot{V}(x) = -\infty$, then $\dot{V} \leq 0$ for $\|x\| \geq R$

Note: $c(\lambda - 1) < 0$ as $0 < \lambda < 1$ & $c > 0$

$$\lim_{(x_1,x_2)\to(\infty,0)} \dot{V}(x) = 0 + 0 + \infty^2 - \infty^4 + \infty = -\infty \checkmark$$

$$\lim_{(x_1,x_2)\to(0,\infty)} \dot{V}(x) = -\infty^2 + \infty + 0 + 0 + \infty = -\infty \checkmark \quad (c(\lambda-1) < 0)$$

$$\lim_{(x_1,x_2)\to(0,\infty)} \dot{V}(x) = -\infty^2 + \infty + \infty^2 - \infty^4 + \infty = -\infty \checkmark$$

$\therefore \lim_{|x|\to\infty} \dot{V}(x) = -\infty \quad \therefore \underline{\dot{V} \leq 0 \text{ for } |x| \geq R}$

$V(x)$ is radially unbounded & $\dot{V} \leq 0$ for $|x| \geq R$ $\therefore$ all solutions are bounded.

#7) $\dot{x} = \cos(x) - x^3 + 100$

For bounded solutions: $V$ is radially unbounded &

$\dot{V} \leq 0$ for $||x|| \geq R$

Scalar system, choose $V = x^2$ as candidate Lyapunov function

$\dot{V} = 2x\dot{x} = 2x(\cos(x) - x^3 + 100)$

$-1 \leq \cos(x) \leq 1$

$\therefore -x^3 + 99 \leq \cos(x) - x^3 + 100 \leq -x^3 + 101$

$-x^3 + 101 \leq 0$ for $x \geq \sqrt[3]{101}$

$-x^3 + 101 \geq 0$ for $x \leq \sqrt[3]{101}$

$2x \geq 0$ for $x \geq 0$

$2x \leq 0$ for $x \leq 0$

$\therefore \dot{V} \leq 0$ if $||x|| \geq \sqrt[3]{101}$ as $x \geq \sqrt[3]{101}$

makes $2x > 0$ & $\cos(x) - x^3 + 100 < 0$ & $x \leq -\sqrt[3]{101}$

makes $\cos(x) - x^3 + 100 > 0$ & $2x < 0$.

$\lim_{x \to \infty} V(x) = \lim_{x \to \infty} x^2 = \infty$ ✓ $\therefore$ $V(x)$ is radially unbounded

$V(x)$ is radially unbounded & $\dot{V} \leq 0$ for $||x|| \geq \sqrt[3]{101}$,

$\therefore$ all solutions are bounded.

#8) $\dot{x}_1 = x_2$

$\dot{x}_2 = \cos(x_1) - x_1^3 + 100$

Treat $\dot{x}_1$ as velocity, $\therefore$ $T = \frac{x_2^2}{2}$

Treat $\dot{x}_2$ as conservative force, $\therefore$ $U = -\int_0^{x_1} \cos(x_1) - x_1^3 + 100\, dx_1$

For bounded solutions: $\dot{V} \leq 0$ for $\|x\| > R$ & $V$ is radially unbounded.

"Total Energy" Lyapunov Function: $V = T + U$

$U = -\left[\sin(x_1) - \frac{x_1^4}{4} + 100x_1\right]\Big|_0^{x_1} = -\sin(x_1) + \frac{x_1^4}{4} - 100x_1$

$V(x) = \frac{x_2^2}{2} + \frac{x_1^4}{4} - \sin(x_1) - 100x_1$

$\dot{V} = \frac{\partial V}{\partial x}\dot{x}$

$\frac{\partial V}{\partial x} = \left[x_1^3 - \cos(x_1) - 100 \quad x_2\right]$

$\dot{V} = \cancel{x_2 x_1^3} - \cancel{x_2\cos(x_1)} - \cancel{100x_2} + \cancel{x_2\cos(x_1)} - \cancel{x_2 x_1^3} + \cancel{100x_2}$

$\dot{V} = 0 \leq 0$ for all $x$ $\therefore$ $\dot{V} \leq 0$ for $\|x\| >$ any $R$

For radially unbounded: $\lim_{\|x\| \to \infty} V(x) = \infty$

$\|x\| = \infty$ if $(x_1, x_2) = (0, \infty)$, $(\infty, 0)$, or $(\infty, \infty)$

$\lim_{(x_1, x_2) \to (0, \infty)} V(x) = \infty^2 + 0 + 0 + 0 = \infty$ ✓

$$\lim_{(x_1,x_2)\to(\infty,0)} V(x) = 0 + \infty^4 - \infty = \infty \checkmark$$

$$\lim_{(x_1,x_2)\to(\infty,\infty)} V(x) = \infty^2 + \infty^4 - \infty = \infty \checkmark$$

$\therefore \lim_{\|x\|\to\infty} V(x) = \infty$ & $V(x)$ is radially unbounded

$$\boxed{\begin{array}{l} V(x) = \frac{x_2^2}{2} + \frac{x_1^4}{4} - \sin(x_1) - 100x_1 \text{ is radially unbounded \&} \\ \dot{V} = 0 \leq 0 \text{ for all } x, \therefore \text{ all solutions are bounded.} \end{array}}$$

#9) $\dot{x} = -(2 + \sin(x))x \qquad x_e = 0$

For G.E.S. $\beta_1 \|x - x_e\|^2 \leq V(x) \leq \beta_2 \|x - x_e\|^2$

& $\dot{V} \leq -2\alpha V \qquad \beta_1, \beta_2, \alpha > 0$

Scalar system, choose $V(x) = x^2$ as candidate Lyapunov function.

$\beta_1 \|x - 0\|^2 \leq x^2 \leq \beta_2 \|x - 0\|^2$

$\beta_1 x^2 \leq x^2 \leq \beta_2 x^2 \quad \therefore \beta_1 = \beta_2 = 1.$

$\dot{V} = 2x\dot{x} = -2x^2(2 + \sin(x))$

$-1 \leq \sin(x) \leq 1$

$\therefore \dot{V} \leq -2x^2(2-1) \quad \therefore \dot{V} \leq -2x^2 = -2\alpha V$

$\therefore \alpha = 1.$

$\boxed{\text{$x_e = 0$ is a globally exponentially stable equilibrium state with rate of convergence $\alpha = 1$.}}$

#10) $\dot{x} = -(2+\sin(x))(x-1) \quad x_e = 1$

For G.E.S.: $\beta_1 \|x - x_e\|^2 \leq V(x) \leq \beta_2 \|x - x_e\|^2$ &

$\dot{V} \leq -2\alpha V \qquad (\beta_1, \beta_2, \alpha) > 0$

Scalar system, choose $V(x) = (x-1)^2$ as candidate Lyapunov function.

$\beta_1 \|x-1\|^2 \leq (x-1)^2 \leq \beta_2 \|x-1\|^2$

$\therefore \ \underline{\beta_1 = \beta_2 = 1}$

$\dot{V} = \frac{\partial V}{\partial x} \dot{x} = -2(x-1)^2(2+\sin(x))$

$-1 \leq \sin(x) \leq 1$

$\therefore \ \dot{V} = -2(x-1)^2(2+\sin(x)) \leq -2(x-1)^2(2-1)$

$\dot{V} = -2(x-1)^2(2+\sin(x)) \leq -2(x-1)^2$

$-2(x-1)^2 = (-2)(\alpha)(V)$

$\therefore \ \underline{\alpha = 1}$

$x_e = 1$ is a G.E.S equilibrium state with a rate of convergence $\alpha = 1$.

#11)

$$\dot{x}_1 = -x_1 + (I_2 - I_3)x_2x_3 \qquad x_e = \begin{pmatrix}0\\0\\0\end{pmatrix}$$
$$\dot{x}_2 = -2x_2 + (I_3 - I_1)x_1x_3$$
$$\dot{x}_3 = -3x_3 + (I_1 - I_2)x_1x_2$$

For G.E.S.: $\beta_1 \|x - x_e\|^2 \leq V(x) \leq \beta_2 \|x - x_e\|^2$

System appears similar to stabilized attitude dynamics problem, without $I_i$ term in denominator, $\therefore$ choose $V(x) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2)$ as candidate Lyapunov function. (Total Kinetic Energy - like function), $V = x^T P x$

$$\|x - x_e\|^2 = \|x\|^2 = x^T x = [x_1 \;\; x_2 \;\; x_3]\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

$$\|x - x_e\|^2 = x_1^2 + x_2^2 + x_3^2$$

$$\beta_1(x_1^2 + x_2^2 + x_3^2) \leq \frac{1}{2}(x_1^2 + x_2^2 + x_3^2) \leq \beta_2(x_1^2 + x_2^2 + x_3^2)$$

$\therefore$ $\underline{\beta_1 = \beta_2 = \frac{1}{2}}$. Note $V = x^T P x$ $\therefore$ $\lambda_{min}(P) = \beta_1$ & $\lambda_{max}(P) = \beta_2$ where $P = \frac{1}{2} I_{3\times3}$

$$\dot{V} = \frac{\partial V}{\partial x}\dot{x} = [x_1 \;\; x_2 \;\; x_3]\begin{bmatrix}-x_1 + (I_2 - I_3)x_2x_3\\ -2x_2 + (I_3 - I_1)x_1x_3\\ -3x_3 + (I_1 - I_2)x_1x_2\end{bmatrix}$$

$$\dot{V} = -x_1^2 + (I_2 - I_3)x_1x_2x_3 - 2x_2^2 + (I_3 - I_1)x_1x_2x_3 - 3x_3^2 + (I_1 - I_2)x_1x_2x_3 = -x_1^2 - 2x_2^2 - 3x_3^2$$

$$\dot{V} = \dot{x}^T P x + x^T P \dot{x} = 2x^T P \dot{x} = 2[x_1 \;\; x_2 \;\; x_3]\begin{pmatrix}1/2 & 0 & 0\\ 0 & 1/2 & 0\\ 0 & 0 & 1/2\end{pmatrix}\dot{x}$$

$$\dot{V} = x^T I_{3\times3}\dot{x} = -x_1^2 - 2x_2^2 - 3x_3^2$$

$$\dot{V} = 2x^T P \dot{x} \leq -2x^T Q x$$

$$\dot{V} = -x_1^2 - 2x_2^2 - 3x_3^2 = -2x^T \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3/2 \end{pmatrix} x$$

$$\therefore \quad Q = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3/2 \end{pmatrix}$$

$Q$ & $P$ are both positive definite & symmetric $\therefore$

$$\alpha = \lambda_{min}(P^{-1}Q)$$

$$P^{-1} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$P^{-1}Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$ ← eigenvalues are on diagonal

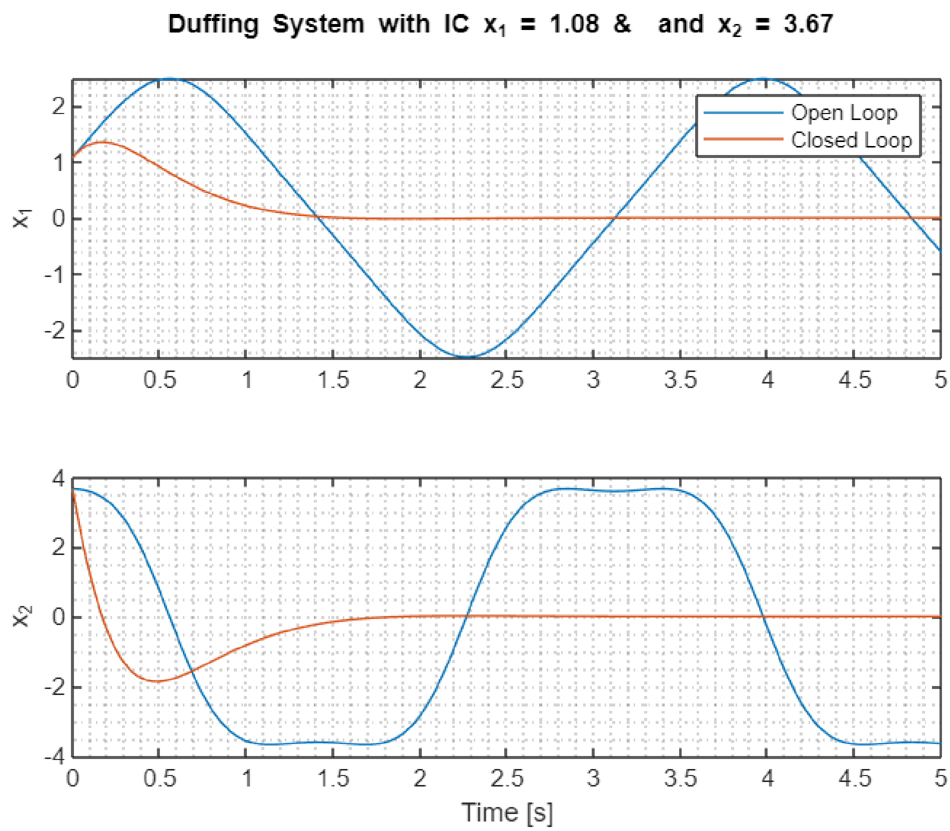
$$\lambda_1(P^{-1}Q) = 1, \quad \lambda_2(P^{-1}Q) = 2, \quad \lambda_3 = (P^{-1}Q) = 3$$

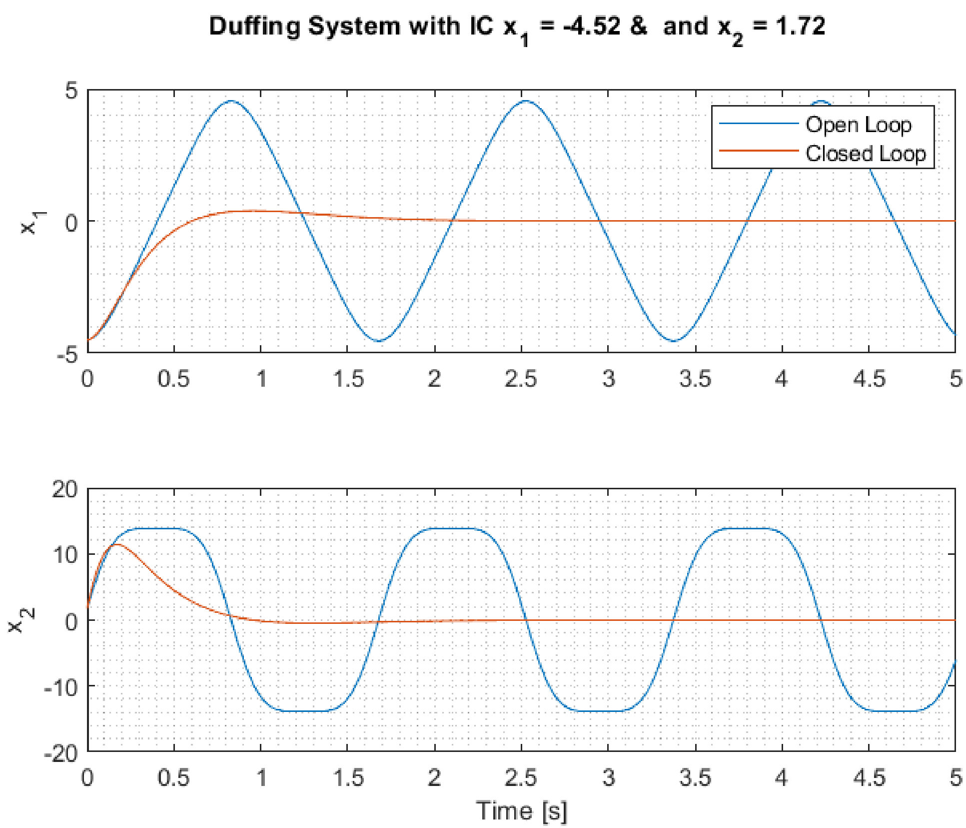
$$\therefore \quad \underline{\alpha = 1}$$

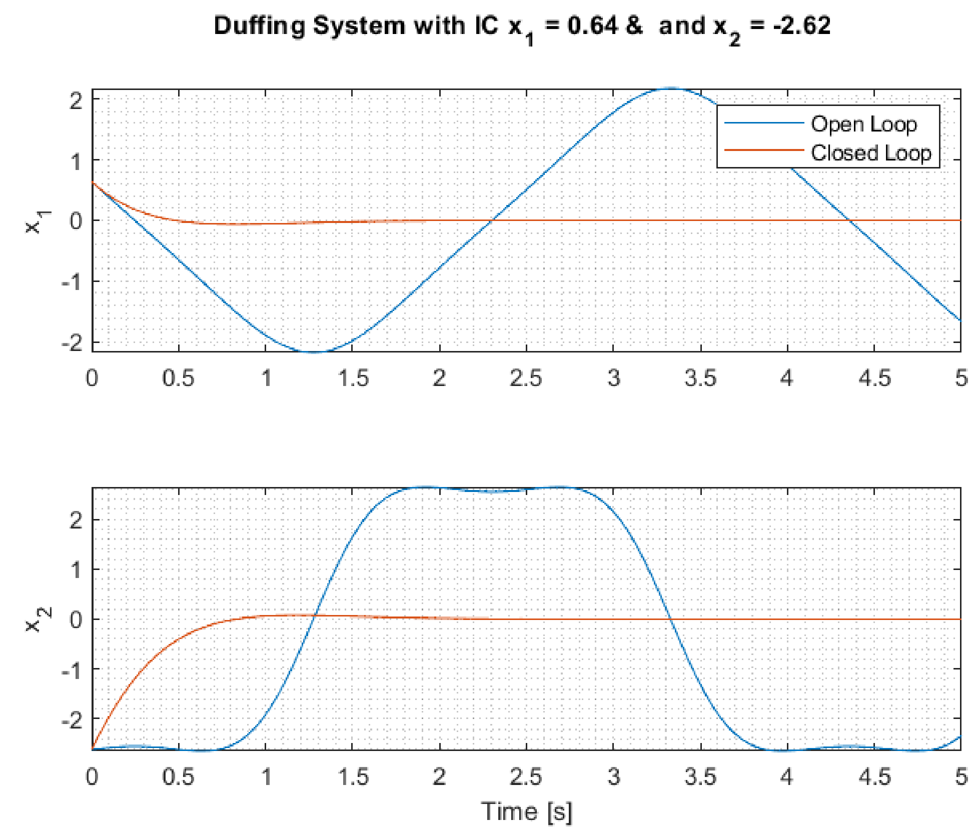
The system is G.E.S. about the zero state, with a rate of convergence of $\alpha = 1$.

# Gabriel Colangelo Homework 4

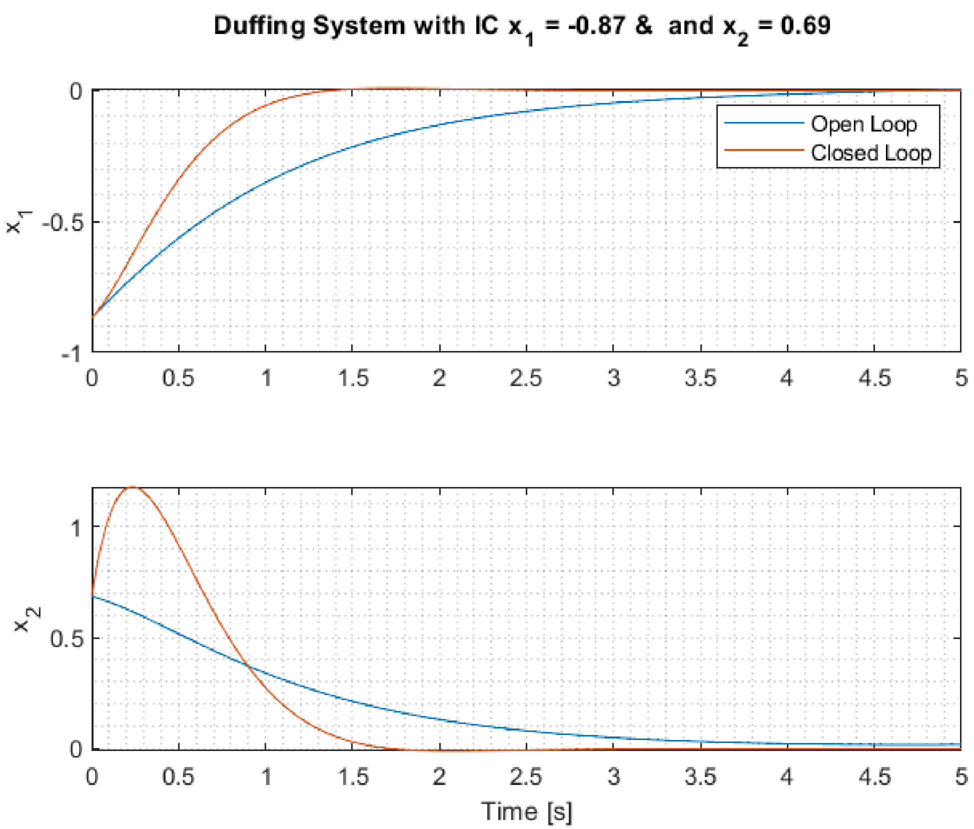
```
clear
close all
clc

% Control gains
k1      = 10;   % From Lyapunov analysis k1 > 1
k2      = 5;    % From Lyapunov analysis k2 > 0

% Initial conditions
IC      = zeros(2,1) + 2*randn(2,10);

% sim time
time    = (0:.005:5)';

% ODE45 solver options
options = odeset('AbsTol',1e-8,'RelTol',1e-8);

% Loop through all IC's
for i = 1:length(IC)
    % Open loop system
    [~, X_open]   = ode45(@(t,x) DuffingSystem(t,x), time, IC(:,i), options);

    % Closed loop system
    [~, X_cl]     = ode45(@(t,x) ControlledDuffingSystem(t,x,[k1 k2]),...
                    time, IC(:,i), options);

    % Generate Plots
    title_str     = sprintf(['Duffing System with IC x_1 = %.2f & ',...
                            ' and x_2 = %.2f \n'],IC(1,i),IC(2,i));

    figure(i)
    subplot(211)
    plot(time,X_open(:,1), time, X_cl(:,1))
    ylabel('x_1')
    grid minor
    legend('Open Loop','Closed Loop')
    title(title_str)
    subplot(212)
    plot(time,X_open(:,2), time, X_cl(:,2))
    ylabel('x_2')
    grid minor
    xlabel('Time [s]')

end
```


Duffing System with IC x1 = 1.08 & and x2 = 3.67
Open Loop
Closed Loop
x1
x2
Time [s]

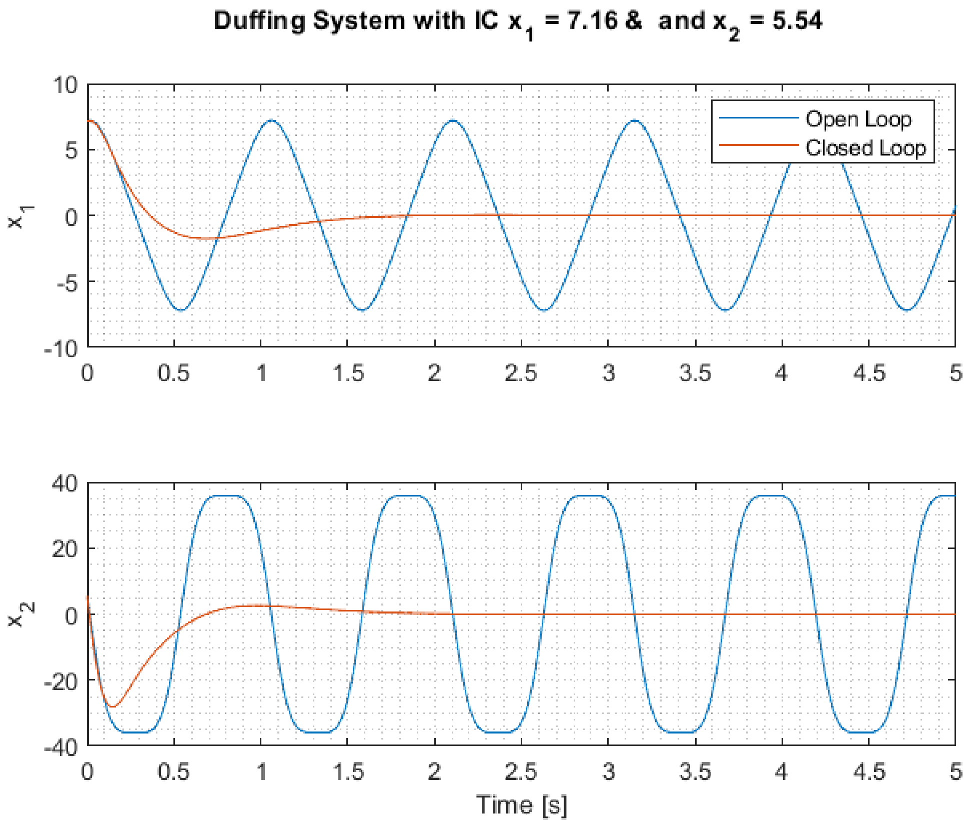

Duffing System with IC x1 = -4.52 & and x2 = 1.72
Open Loop
Closed Loop
x1
x2
Time [s]

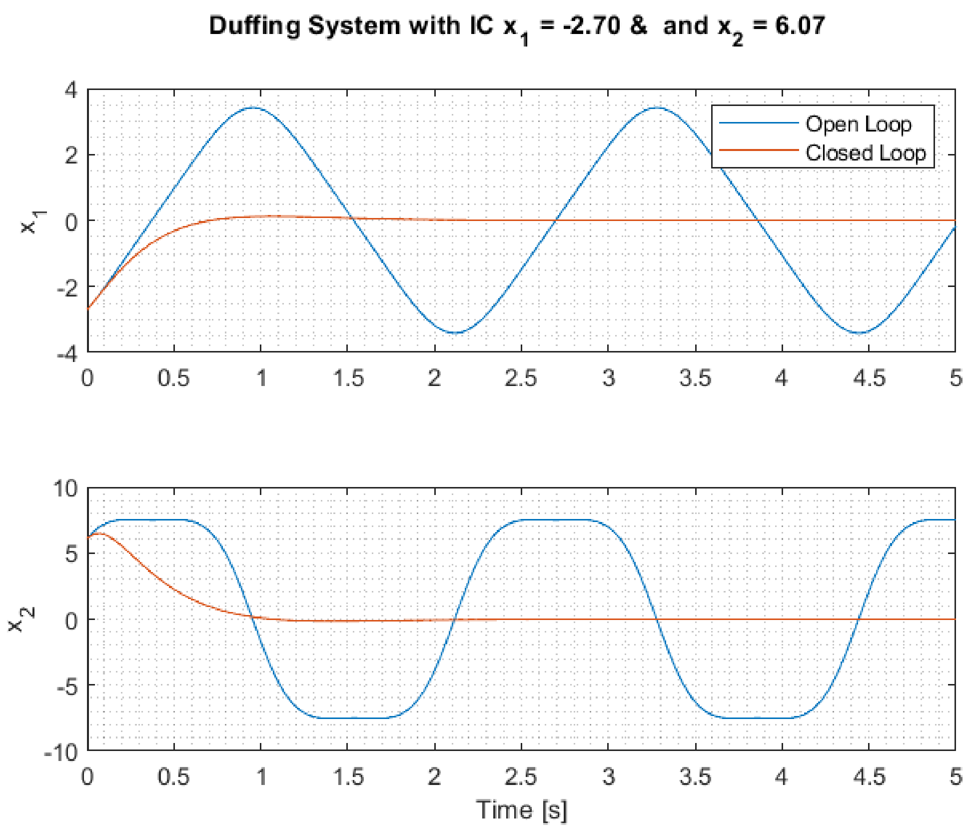


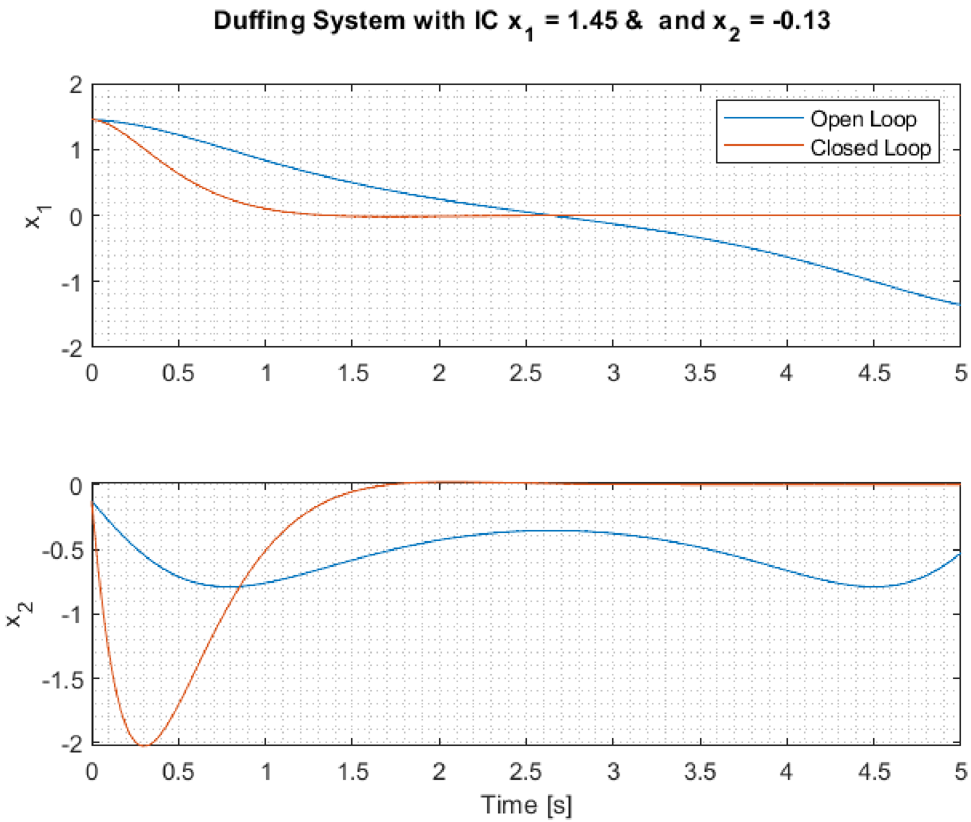
Duffing System with IC $x_1$ = 0.64 & and $x_2$ = -2.62
Open Loop
Closed Loop
$x_1$
$x_2$
Time [s]

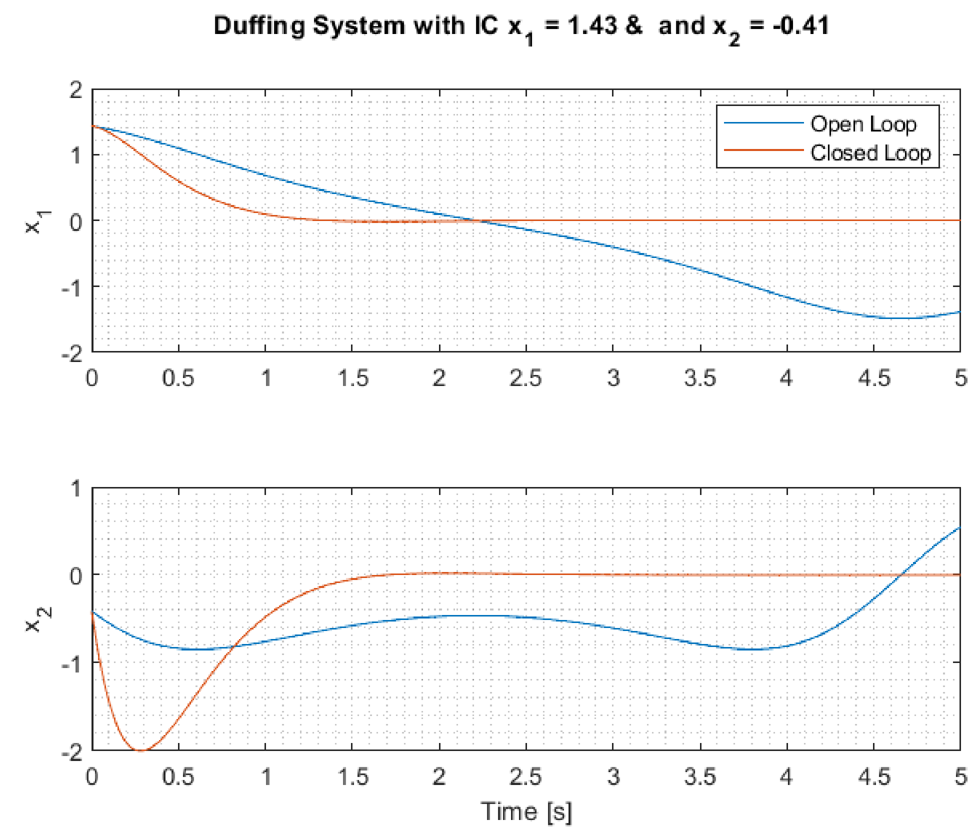



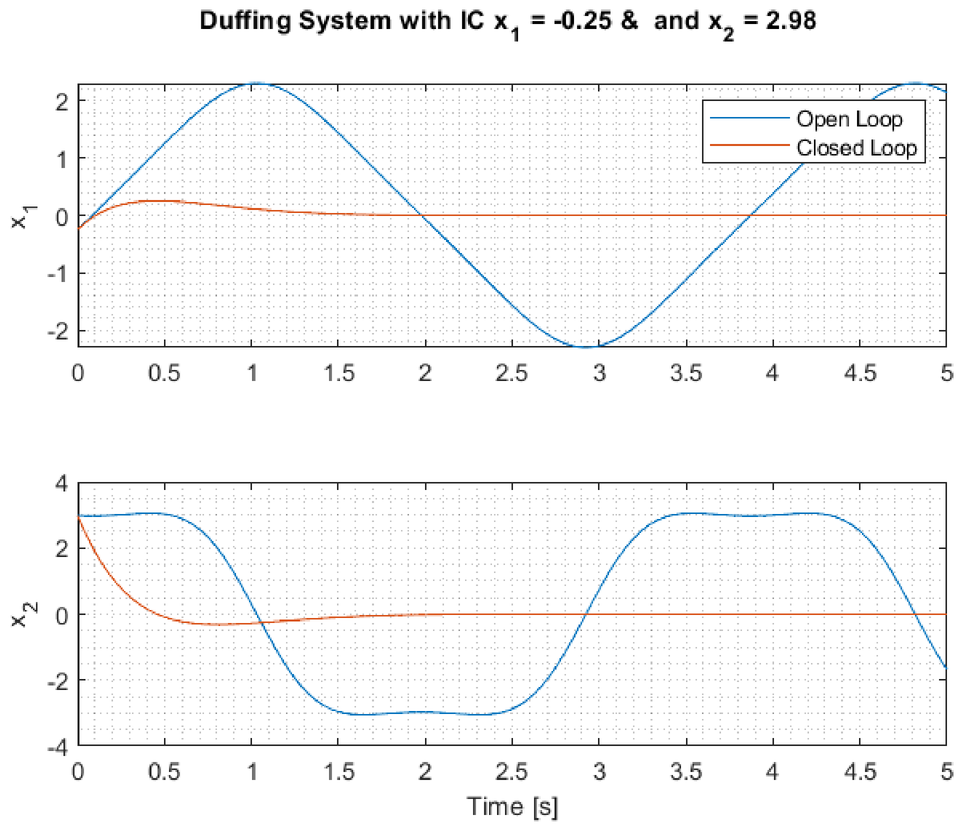
Duffing System with IC $x_1$ = -0.87 & and $x_2$ = 0.69
Open Loop
Closed Loop
$x_1$
$x_2$
Time [s]

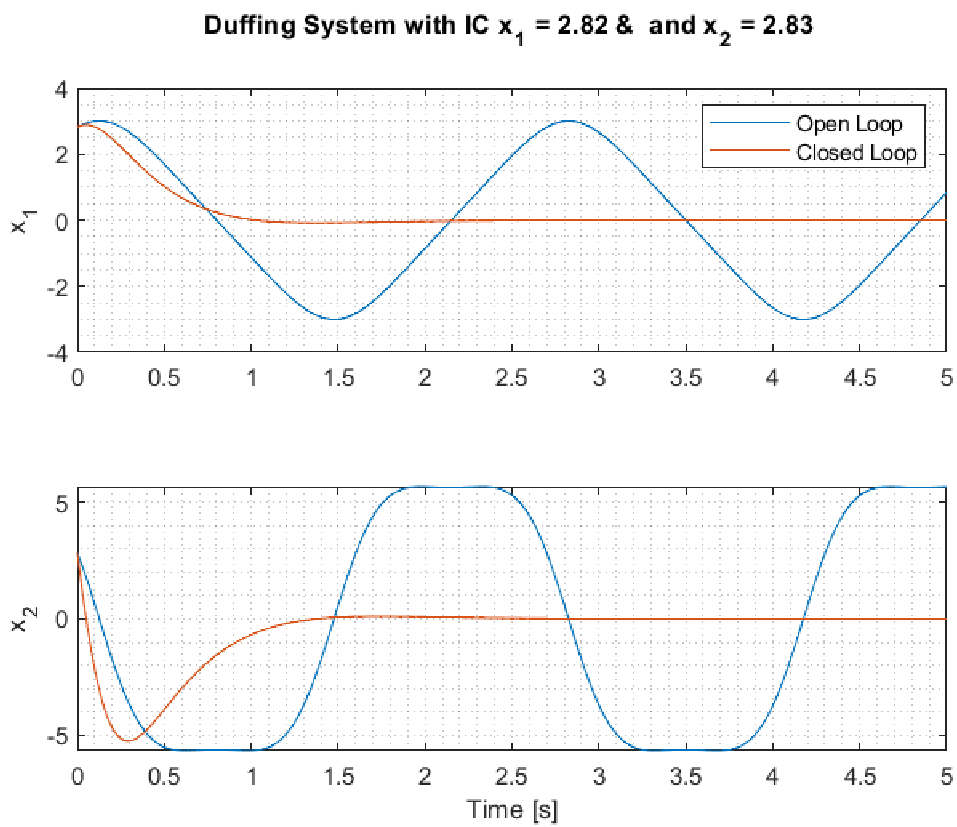

Duffing System with IC $x_1$ = 7.16 &  and $x_2$ = 5.54
Open Loop
Closed Loop
$x_1$
$x_2$
Time [s]


Duffing System with IC $x_1$ = -2.70 &  and $x_2$ = 6.07
Open Loop
Closed Loop
$x_1$
$x_2$
Time [s]


Duffing System with IC x1 = 1.45 & and x2 = -0.13
Open Loop
Closed Loop
x1
x2
Time [s]


Duffing System with IC x1 = 1.43 & and x2 = -0.41
Open Loop
Closed Loop
x1
x2
Time [s]




```
function xdot = DuffingSystem(t,x)
    % State space model
```

```
    xdot(1,1)   = x(2,1);
    xdot(2,1)   = x(1,1) - x(1,1)^3;
end

function xdot = ControlledDuffingSystem(t,x,K)
    % Control law, u = -k1*x1 - k2*x2
    u           = -K*x;

    % State space model
    xdot(1,1)   = x(2,1);
    xdot(2,1)   = x(1,1) - x(1,1)^3 + u;
end
```