ECE 602: LUMPED LINEAR SYSTEMS

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Solution of Discrete-Time Controlled LTI and LTV Systems

Solutions of Discrete-Time Controlled LTI Systems

Discrete-time LTI system with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$:

$$\begin{aligned} x[k+1] &= Ax[k] + Bu[k] \\ y[k] &= Cx[k] + Du[k], \end{aligned}$$

Under input $u[k]$, $k = 0, 1, \ldots$, and initial state $x[0]$, the solution is

$$x[k] = A^k x[0] + \sum_{i=0}^{k-1} A^{k-1-i} Bu[i]$$

$$y[k] = \underbrace{CA^k x[0]}_{\text{zero-input response}} + \underbrace{C \sum_{i=0}^{k-1} A^{k-1-i} Bu[i] + Du[k]}_{\text{zero-state response}}, \quad k = 0, 1, \ldots$$

Transfer Matrix of DT LTI Systems

Taking z-transform of state and output equations:

$$X(z) = (zI_n - A)^{-1} z\, x[0] + (zI_n - A)^{-1} BU(z)$$

$$Y(z) = \underbrace{C(zI_n - A)^{-1} z\, x[0]}_{\text{zero-input response}} + \underbrace{C(zI_n - A)^{-1} BU(z) + DU(z)}_{\text{zero-state response}}$$

Transfer function matrix is $H(z) = C(zI_n - A)^{-1} B + D$

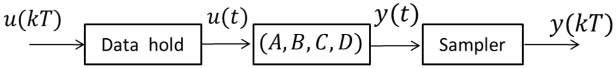
Algebraically Equivalent DT LTI Systems

$\begin{cases} x[k+1] &= Ax[k] + Bu[k] \\ y[k] &= Cx[k] + Du[k] \end{cases}$ after a change of coordinates $x = T\tilde{x}$:

$$\begin{cases} \tilde{x}[k+1] &= (T^{-1}AT)\tilde{x}[k] + (T^{-1}B)u[k] \\ y[k] &= (CT)\tilde{x}[k] + Du[k] \end{cases}$$

- LTI systems (A, B, C, D) and $(T^{-1}AT, T^{-1}B, CT, D)$ are algebraically equivalent and have the same transfer function matrix

Discretization of CT LTI Systems

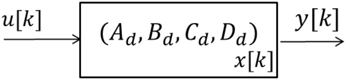


Given a continuous-time LTI system $\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$, and a sampling period $T > 0$

- input at sampling times is specified by a sequence $u(kT) = u[k]$, $k = 0, 1, \ldots$, and kept constant between sample times: $u(t) \equiv u(kT)$, $\forall\, t \in [kT, (k+1)T)$ (This is called 0^{th} order data hold)
- State and output are sampled every T time:

$$x[k] = x(kT), \quad y[k] = y(kT), \quad k = 0, 1, \ldots$$

Discretization of CT LTI Systems (cont.)



Sampled system dynamics is equivalent to a discrete-time LTI system:

$$\begin{cases} x[k+1] &= \underbrace{e^{AT}}_{A_d} x[k] + \underbrace{\left(\int_0^T e^{A\tau}\, d\tau\right) B}_{B_d} u[k] \\ y[k] &= \underbrace{C}_{C_d} x[k] + \underbrace{D}_{D_d} u[k] \end{cases}$$

- Matlab command `[Ad,Bd]=c2d(A,B,T,'zoh')`
- Other data hold options available

Solutions of DT Controlled LTV Systems

Discrete-time LTV system

$$\begin{aligned} x[k+1] &= A[k]x[k] + B[k]u[k] \\ y[k] &= C[k]x[k] + D[k]u[k], \end{aligned}$$

Its state solution and output under input $u[\cdot]$ are

$$x[k] = \Phi[k]x[0] + \sum_{i=0}^{k-1} \Phi[k, i+1]B[i]u[i]$$

$$y[k] = C[k]\Phi[k]x[0] + C[k]\sum_{i=0}^{k-1} \Phi[k, i+1]B[i]u[i] + D[k]u[k]$$