Student Name Solution

Orbit Mechanics
9/20/24

Exam 1

Please read the problems carefully.
Write clearly and use diagrams when necessary.
Use the following constant values when appropriate

Body	GM (km^3/s^2)	Radius (km)
Earth	4.0000×10^5	6400.0
Mars	4.2000×10^4	3400.0

Purdue Honor Pledge "As a Boilermaker pursuing academic excellence, I pledge to be honest and true in all that I do. Accountable together—We Are Purdue."

For this exam, I understand it is a take-home exam with the following requirements:

1. I can use my own class notes and my own previously completed assignments.
2. I am not allowed to search for any resources online.
3. I can use my own calculator. I cannot use Matlab or other commercial software.
4. I am expected to work the exam on my own. I am not allowed to work with another person. I am not allowed to contact another person for help while completing the exam.
5. If I have any questions during the exam period, I will email Prof Howell (howell@purdue.edu) AND/OR the TAs Mitch Dominguez (doming18@purdue.edu) or Samantha Ramsey (ramsey87@purdue.edu) or Tyler Hook (hook9@purdue.edu). Given this longer exam period, we will answer as soon as possible.

Signature ______________________________

(35 Points)

Problem 1: NASA's SETI Program (Search for ExtraTerrestial Intelligence) sought out signs of intelligent life in the universe by scanning the sky. Assume that we have information concerning a distant system of three bodies. At a certain instant, the three bodies are known to be described with the following characteristics:

Body*	GM (km^3/s^2)	Distance (km)
A	2.0000×10^8	$r_{AB} = 8.0000\times10^8$
B	5.0000×10^8	$r_{BC} = 4\sqrt{3}\times10^8$
C	1.0000×10^8	$r_{AC} = 4.0000\times10^8$

*Assume that the bodies are spherically symmetric

spherical body

(a) What is the significance of the assumption that bodies are spherically symmetric? assumes density is constant or a function only of radius; bodies can be then modelled as particles for purposes of the gravitational force.
Sketch (by hand) the three-body system.
Locate the system center of mass; add it to your sketch.

(b) The inverse square law of gravitation still applies! To consider the future locations of each body, you need the equations of motion.

(i) Since GM for body C is the smallest mass, write the differential equations for the motion of body C relative to body A;
Then, also write the EOMs for motion of Body C relative to body B.
Which one is correct? Why?

(ii) Write the governing vector differential equation for the relative motion that you selected.
use $\ddot{r}_{BC}$

(iii) Identify the independent variable and the set of scalar dependent variables.
independent variable: t
dependent variables: $x_{BC}, y_{BC}, z_{BC}, \dot{x}_{BC}, \dot{y}_{BC}, \dot{z}$ x_{BA}, y_{BA}, z_{BA} } 9!

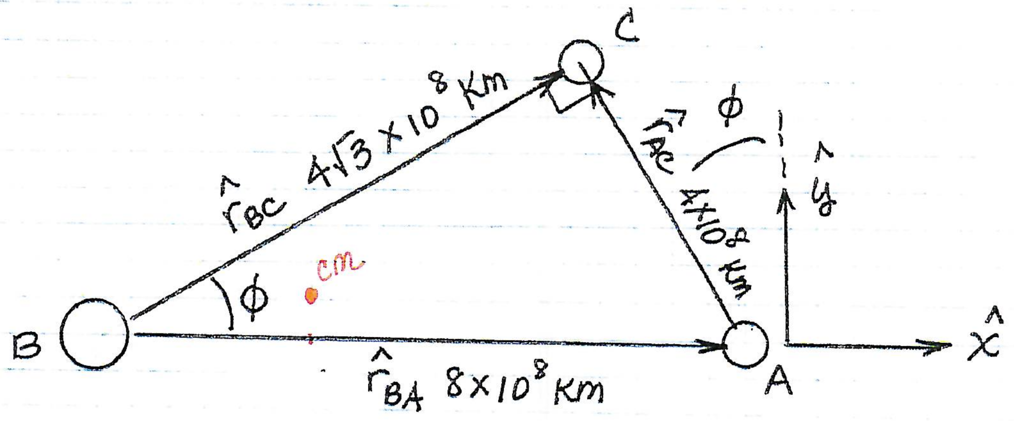
(c) For the vector differential equation in part (b), consider the acceleration on body C.

(i) Evaluate the dominant acceleration, the direct acceleration, the indirect acceleration and the net perturbing acceleration, each as a magnitude and direction.

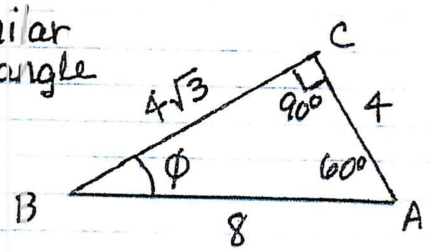
(ii) Compare the magnitudes of each of the quantities in part (c)(i).

(iii) Discuss the dominant term in comparison to the perturbing term. For the two bodies involved in the dominant term, is the perturbing term – at this instant – tending to increase or decrease the distance between them? Why?

1.



Similar triangle



$$r_{AC}^2 = r_{BC}^2 + r_{BA}^2 - 2r_{BC}r_{BA}\cos\phi$$

$$\phi = 30^\circ$$

$$(4\sqrt{3})^2 + 4^2 = 8^2 \quad \text{right triangle}$$

$$G\mathcal{M} = 8\times10^8\ \text{Km}^3/\text{s}^2$$

$$\bar{r}_{B\to cm} = \frac{1}{G\mathcal{M}}\left[\mu_A(8\times10^8)\,\hat{x} + \mu_C(4\sqrt{3}\times10^8)(C_\phi\,\hat{x} + S_\phi\,\hat{y})\right]$$

$$\boxed{\bar{r}_{B\to cm} = 2.7500\times10^8\ \text{Km}\ \hat{x} + .43301\times10^8\ \text{Km}\ \hat{y}}$$

(b) Both representations are correct! There are no assumptions made to produce either equation.

for motion of C relative to B:

$$\ddot{\bar{r}}_{BC} = -\frac{(\mu_B+\mu_C)}{r_{BC}^2}\hat{r}_{BC} + \mu_A\left(\frac{\hat{r}_{CA}}{r_{CA}^2} - \frac{\hat{r}_{BA}}{r_{BA}^2}\right)$$

for motion of C relative to A:

$$\ddot{\bar{r}}_{AC} = -\frac{(\mu_A+\mu_C)}{r_{AC}^2}\hat{r}_{AC} + \mu_B\left(\frac{\hat{r}_{CB}}{r_{CB}^2} - \frac{\hat{r}_{AB}}{r_{AB}^2}\right)$$

(b) $\ddot{r}_{BC} = -\dfrac{(\mu_B + \mu_C)}{r_{BC}^2}\hat{r}_{BC} + \mu_A\left(\dfrac{\hat{r}_{CA}}{r_{CA}^2} - \dfrac{\hat{r}_{BA}}{r_{BA}^2}\right)$

dominant $= -\dfrac{6\times10^8\ Km^3/s^2}{(4\sqrt{3}\times10^8)^2}(C\phi\hat{x} + S\phi\hat{y}) \Rightarrow 1.2500\times10^{-9}\ \dfrac{Km}{s^2}$ mag

direct $= \dfrac{2\times10^8\ Km^3/s^2}{(4\times10^8)^2}(S\phi\hat{x} - C\phi\hat{y}) \rightarrow 1.2500\times10^{-9}\ \dfrac{Km}{s^2}$ mag

indirect $= \dfrac{2\times10^8\ Km^3/s^2}{(8\times10^8)^2}\hat{x} \rightarrow 3.1250\times10^{-10}\ \dfrac{Km}{s^2}$ mag

net perturbing = direct − indirect

$= 1.25\times10^{-9}(-C\phi\hat{y} + S\phi\hat{x}) - 3.125\times10^{-10}\hat{x}$

$\boxed{\text{net pert} = -1.0826\times10^{-9}\hat{y} + 3.1250\times10^{-10}\hat{x}\ Km/s^2}$

net perturbing mag $= 1.1268\times10^{-9}\ Km/s^2$

(c)(iii) dominant and direct possess same mag but NOT same direction so they cannot offset

net perturbing accel is same order of magnitude as dominant (with slightly larger mag)

could not reasonably reduce to a 2BP

modeling as a relative 2BP will not yield useful results

The large mass B is instantaneously attracting mass C to draw closer; note that it is only for this instant!

$$\ddot{\bar{r}}_{AC} = -\frac{(\mu_A+\mu_C)}{r_{AC}^2}\hat{r}_{AC} + \mu_B\left(\frac{\hat{r}_{CB}}{r_{CB}^2} - \frac{\hat{r}_{AB}}{r_{AB}^2}\right)$$

$$\text{dominant} = -\frac{3\times10^8\ km^3/s^2}{(4\times10^8)^2}(-s_\phi\hat{x} + c_\phi\hat{y}) \Rightarrow 1.875\times10^{-9}\ \frac{km}{s^2}\ \text{mag}$$

$$\text{direct} = \frac{5\times10^8\ km^3/s^2}{(4\sqrt{3}\times10^8)^2}(-c_\phi\hat{x} - s_\phi\hat{y}) \Rightarrow 1.0417\times10^{-9}\ \frac{km}{s^2}\ \text{mag}$$

$$\text{indirect} = \frac{5\times10^8\ km^3/s^2}{(8\times10^8)^2}(-\hat{x}) \Rightarrow 7.8125\times10^{-10}\ \frac{km}{s^2}\ \text{mag}$$

$$\text{net perturbing} = \text{direct} - \text{indirect}$$

$$= 1.0417\times10^{-9}(-c_\phi\hat{x} - s_\phi\hat{y}) - 7.8125\times10^{-10}(-\hat{x})$$

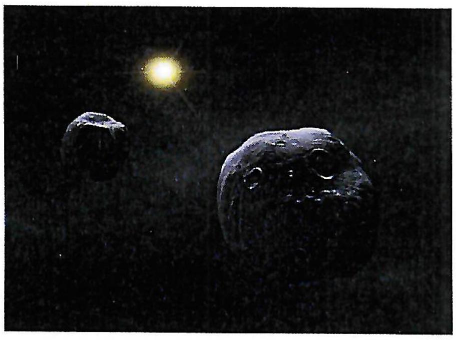
$$\boxed{\text{net pert} = -1.2089\times10^{-10}\hat{x} - 5.2085\times10^{-10}\hat{y}\ km/s^2}$$

$$\text{net pert mag} = 5.3470\times10^{-10}\ km/s^2$$

In this representation, dominant > pert accel
Since the $\hat{y}$ component of perturbing acceleration is large, at this instant, the perturbation from B seems to be increase the distance from A.
Even so, the dominant term is greater but only by one order of magnitude. Modeling as a relative 2BP still not going to produce accurate results.

(30 Points)

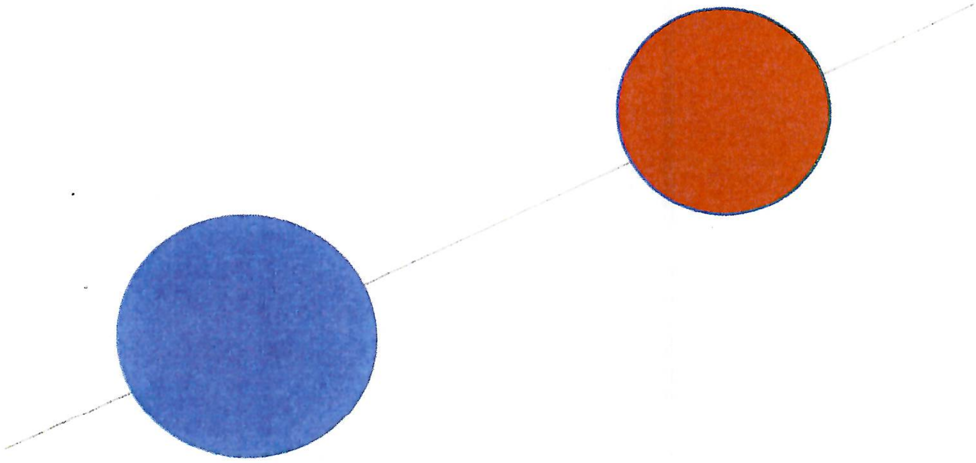
Problem 2: Both NASA and ESA have been considering missions to binary asteroids. One candidate is asteroid 90 Antiope. It is comprised of a larger primary B (blue) and a secondary R (red). Assume a similar binary and consider the motion of the two primaries relative to each other in their periodic orbit. Thus, the binary is modeled as a relative two-body problem for the motion of R relative to B due to the mutual gravity. Both bodies as assumed to be spherically symmetric. Ignore all other forces.



	Primary B	Secondary R
Gm	.040 km^3/s^2	.030 km^3/s^2
Diameter	90 km	80 km
Semi-major axis of orbit for R relative to B = 200 km		
Eccentricity of relative orbit = 0.25		

$\mu_B = .04\ km^3/s^2$

$\mu_R = .03\ Km^3/s^2$



(a) For the relative motion, determine the values of the following quantities:

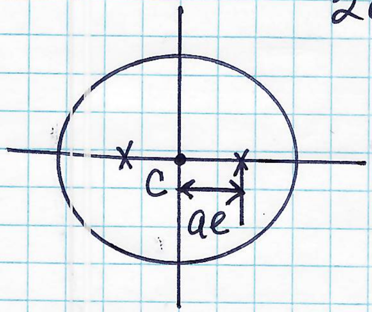
Semi-latus rectum
Specific energy
Distance between the foci
Apoapsis distance
Specific angular momentum
Period (in hours)
Average angular velocity
Angular velocity and speed at periaspsis

(a) $a = 200 \text{ Km}$ $\quad \mu_B = .04 \text{ Km}^3/s^2$

$e = 1/4$ $\quad \mu_R = .03 \text{ Km}^3/s^2$

$p = a(1-e^2) = \boxed{187.5 \text{ Km}}$

$\mathcal{E} = -\dfrac{(\mu_B + \mu_R)}{2a} \Longrightarrow \boxed{\mathcal{E} = -1.75 \times 10^{-4} \text{ Km}^2/s^2}$



$\boxed{\underbrace{\text{distance betw foci}}_{2ae} = 100 \text{ Km}}$

$r_a = a(1+e) \longrightarrow \boxed{r_a = 250 \text{ Km}}$

$h = \sqrt{\mu p} = \sqrt{(\mu_B + \mu_R) p} \Longrightarrow \boxed{h = 3.6228 \text{ Km}^2/s}$

Avg ang. velocity = mean motion

$n = \sqrt{\dfrac{(\mu_B + \mu_R)}{a^3}} \Longrightarrow \boxed{n = 9.3541 \times 10^{-5} \text{ rad/s}}$

$\mathbb{P} = \dfrac{2\pi}{n} = 67170.4 \text{ s} \Longrightarrow \boxed{\mathbb{P} = 18.658 \text{ hr}}$

$h = r^2\dot{\theta}$ where $r, \dot{\theta}$ are instantaneous quantities

at r_p, $\dot{\theta} = \dfrac{h}{r^2} \Longrightarrow \boxed{\dot{\theta}_{r_p} = 1.6101 \times 10^{-4} \text{ rad/s}}$

$a(1-e) = 150 \text{ Km}$

at apse points $|\bar{v}| = r\dot{\theta} \longrightarrow \boxed{|\bar{v}_p| = .02415 \text{ Km/s}}$

$$\underbrace{\text{(b) Determine the location in the orbit where the speed equal to the circular velocity.}}^{\text{occurs at end of minor axis for ALL ellipses!}}$$

(b) Determine the location in the orbit where the speed equal to the circular velocity. What is the speed at this location? True anomaly?

(c) On the next page, the bodies are located at periapsis. Sketch (as accurately as possible) the orbit of R relative to B on the figure. Add the following to the sketch:

(i) Locate the center of mass at periapsis; mark it in the figure.
(ii) $r_p, r_a, a, p, \hat{e}, \hat{p}$ *, attracting focus, center, vacant focus*
(iii) $\hat{r}, \hat{\theta}$ when the secondary R is at the end of the minor axis and descending, $\hat{r}, \hat{\theta}$ local horizon, flight path angle

(b) @ θ^* minor axis, $E = \pm 90° \Rightarrow r = a(1 - e c_E)$

$r = a!$

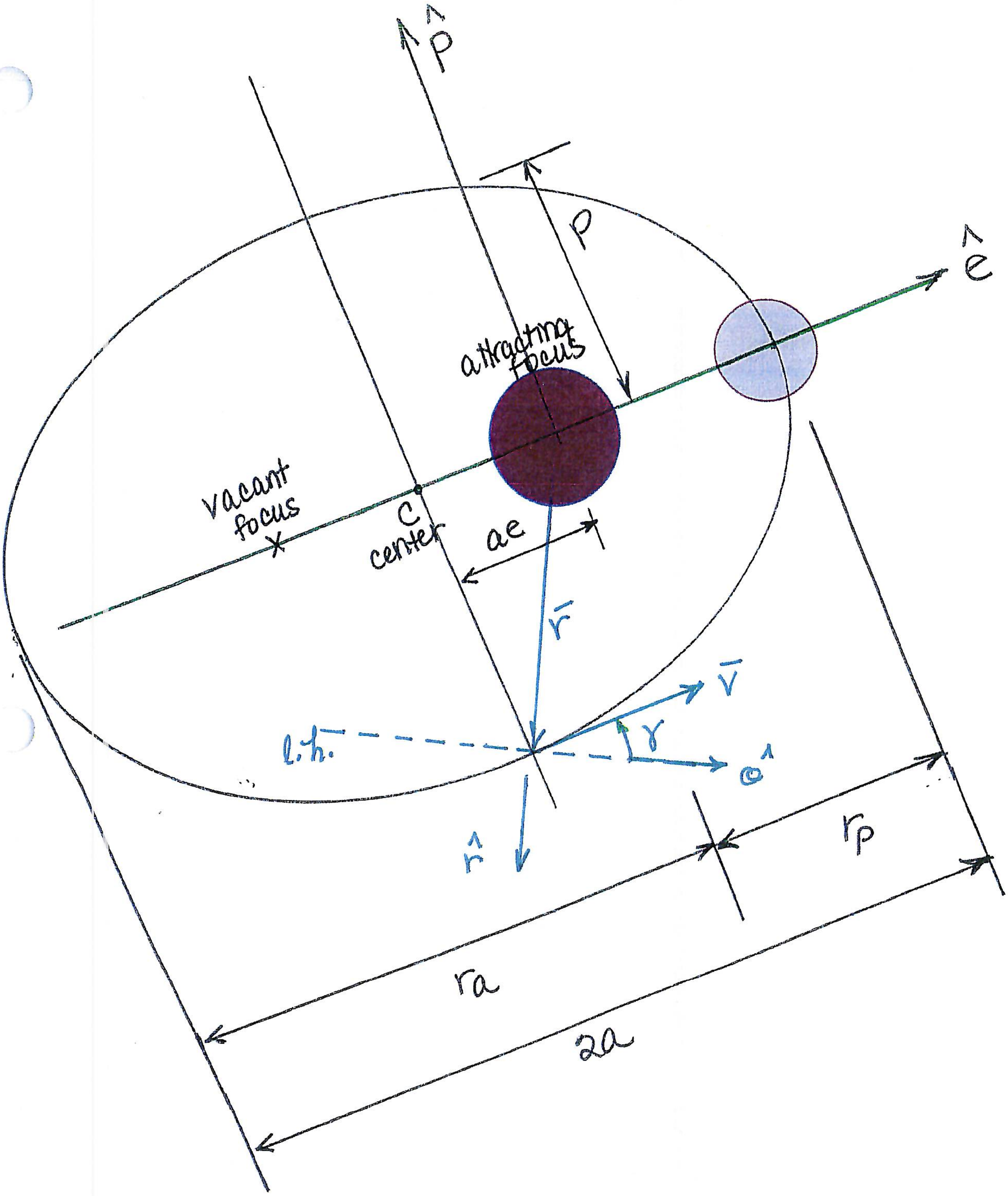
$$\frac{V^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a} \Rightarrow V = V_c = \sqrt{\frac{\mu}{r}} \qquad \boxed{V_c = .01871 \text{ Km/s}}$$

$$E = \pm 90° \quad \tan\frac{\theta^*}{2} = \left(\frac{1+e}{1-e}\right)^{1/2} \tan\frac{E}{2} \qquad \boxed{\theta^* = \pm 104.48°}$$

$$E = 90°, 270°$$

(c) $$\bar{r}_{Bcm} = \frac{\mu_R r_p}{(\mu_B + \mu_R)}$$

$$\boxed{\underbrace{r_{Bcm} = 64.29 \text{ Km}}_{\text{from attracting focus}}}$$



$\hat{p}$
p
$\hat{e}$
attracting focus
vacant focus
c
center
ae
$\bar{r}$
$\bar{v}$
γ
l.h.
$\hat{r}$
$\hat{\theta}$
r_p
r_a
2a

$\mu_♂ = 4.2 \times 10^4 \, km^3/s^2$ $R_♂ = 3460 \, Km$

(35 points)

Problem 3: The year is 20?? and a spacecraft is in orbit about Mars. Assume that the problem is modeled in terms of the relative two-body model. The orbit is characterized with the following quantities:

Spacecraft Mass = 8000.0 kg (<<< mass Mars!)
System Linear Momentum $\bar{C}_1 = \bar{0}$

Specific Angular Momentum $|\bar{h}| = 2.5000 \times 10^4 \, km^2/s$

Specific Energy $|\mathcal{E}| = 1.2351 \, km^2/s^2$

(a) System linear momentum is zero. What does that information tell you? Why?

Lin Mom $= \bar{0} \Rightarrow \sum \bar{r}_i m_i = m \bar{V}_{cm} = 0 \longrightarrow \bar{V}_{cm} = \bar{0}$.
So cm moves with velocity constant in mag & direction (not accelerating).
An appropriate inertial frame moves with the cm.

(b) Determine the following information about the orbit if $\mathcal{E} < 0$:

a, e, p, r_p, r_a, period (express distance in terms of $R_♂$)

(c) Make one change to the orbital characteristics: $\mathcal{E} = 0$. Repeat (b). How does the orbit change?

At the location where $\theta^* = -90°$, determine r, v.

Sketch the orbit; mark $\bar{r}, \bar{v}, \hat{r}, \hat{\theta}, \gamma, a, p$, local horizon

(b) $h = 25{,}000 \, Km^2/s = \sqrt{\mu p} \Rightarrow \boxed{p = 4.377 R_♂}$

$\mathcal{E} = -1.2351 \, Km/s^2 = -\frac{\mu}{2a} \Rightarrow \boxed{a = 17002.67 \, Km = 5 R_♂}$

$p = a(1-e^2) \Rightarrow \boxed{e = .3530}$

$r_p = a(1-e)$, $r_a = a(1+e)$ $\Rightarrow \boxed{r_p = 3.235 R_♂}$ $\boxed{r_a = 6.765 R_♂}$

$n = \sqrt{\frac{\mu}{a^3}} \Rightarrow n = 9.246 \times 10^{-5} \, rad/s$

$\mathbb{P} = \frac{2\pi}{n} \Rightarrow \mathbb{P} = 67{,}956.07 \, sec \longrightarrow \boxed{\mathbb{P} = 18.88 \, hR}$

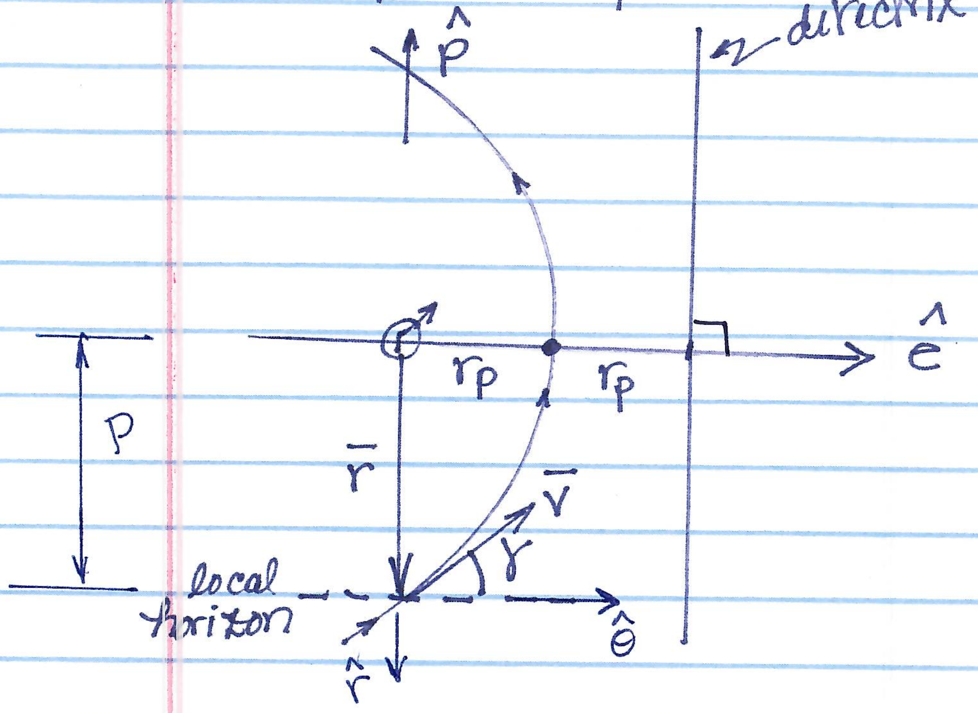
(c) $\mathcal{E} = 0 \Rightarrow$ parabola!

$\mathbb{P}$ undefined
r_a undefined

$a = \infty$ $e = 1$ $p = 4.377 R_♂$ (same)

$r = \frac{p}{1 + e\cos\theta^*}$ @ $\theta^* = 0°$ $r_p = \frac{p}{2}$ $\boxed{r_p = 2.1885 R_♂}$

$\theta^* = -90^\circ \qquad r = p$



$\bar{r} = -4.377 R_♂ \hat{p}$

$v = \sqrt{2}\, v_c \Longrightarrow \sqrt{2}\sqrt{\dfrac{\mu_♂}{p}} \Longrightarrow v = 2.3758\ \text{Km/s}$

$h = rv\cos\gamma \Longrightarrow \gamma = -45^\circ$

despending !