ECE 68000: MODERN AUTOMATIC CONTROL

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Application of the Lyapunov continuous matrix equation to evaluate performance indices

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Lyapunov's thm:

A constant square matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ has its eigenvalues in the open left half-complex plane if and only if for any real, symmetric, positive definite $\boldsymbol{Q} \in \mathbb{R}^{n\times n}$, the solution $\boldsymbol{P} = \boldsymbol{P}^\top$ to the Lyapunov matrix equation

$$\boldsymbol{A}^\top \boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} = -\boldsymbol{Q}$$

is positive definite

A variant of Lyapunov's theorem

Theorem

A matrix $\boldsymbol{A}$ *is asymptotically stable if and only if for any* $\boldsymbol{Q} = \boldsymbol{C}^\top \boldsymbol{C}$, $\boldsymbol{C} \in \mathbb{R}^{p \times n}$, *such that the pair* $(\boldsymbol{A}, \boldsymbol{C})$ *is observable, the solution* $\boldsymbol{P}$ *to the Lyapunov matrix equation* $\boldsymbol{A}^\top \boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} = -\boldsymbol{C}^\top \boldsymbol{C}$ *is positive definite*

Proof of the variant of Lyapunov's theorem

- $(\Rightarrow)$ The necessity condition for asymptotic stability
- If $\boldsymbol{A}$ is asymptotically stable, that is, $\Re\lambda_i(\boldsymbol{A}) < 0$, then we will show that

$$\boldsymbol{P} = \int_0^\infty e^{\boldsymbol{A}^\top t}\boldsymbol{C}^\top \boldsymbol{C} e^{\boldsymbol{A}t} dt$$

  is the symmetric positive definite solution to the Lyapunov equation
- Indeed, the integrand is a sum of terms of the form

$$t^k e^{\lambda_i t},$$

  where λ_i is the i-th eigenvalue of $\boldsymbol{A}$
- Since $\Re\lambda_i(\boldsymbol{A}) < 0$, the integral exists because the terms in $\boldsymbol{P}$ do not diverge when $t \to \infty$
- Indeed, let $-\alpha_i = \Re\lambda_i$, where $\alpha_i > 0$, then

$$\lim_{t\to\infty} t^k e^{-\alpha_i t} = \lim_{t\to\infty} \frac{t^k}{e^{\alpha_i t}}$$

Proof of the necessity condition

- Differentiating the numerator and denominator k times and using L'Hospital's rule gives

$$\begin{aligned}\lim_{t\to\infty}\frac{t^k}{e^{\alpha_i t}} &= \lim_{t\to\infty}\frac{kt^{k-1}}{\alpha_i e^{\alpha_i t}}\\ &= \lim_{t\to\infty}\frac{k(k-1)t^{k-2}}{\alpha_i^2 e^{\alpha_i t}}\\ &\vdots\\ &= \lim_{t\to\infty}\frac{k!}{\alpha_i^k e^{\alpha_i t}}\\ &= 0\end{aligned}$$

- Thus, the formula for $\boldsymbol{P}$ is well defined
- Next, note that $\boldsymbol{P} = \boldsymbol{P}^\top$
- The matrix $\boldsymbol{P}$ is also positive definite because the pair $(\boldsymbol{A}, \boldsymbol{C})$ is observable

Proof of the necessity condition—contd.

- Substituting the expression for $\boldsymbol{P}$ into the Lyapunov equation and performing manipulations yields

$$
\begin{aligned}
\boldsymbol{A}^\top \boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} &= \int_0^\infty \boldsymbol{A}^\top e^{\boldsymbol{A}^\top t} \boldsymbol{C}^\top \boldsymbol{C} e^{\boldsymbol{A} t} dt \\
&\quad + \int_0^\infty e^{\boldsymbol{A}^\top t} \boldsymbol{C}^\top \boldsymbol{C} e^{\boldsymbol{A} t} \boldsymbol{A} dt \\
&= \int_0^\infty \frac{d}{dt} \left(e^{\boldsymbol{A}^\top t} \boldsymbol{C}^\top \boldsymbol{C} e^{\boldsymbol{A} t} \right) dt \\
&= e^{\boldsymbol{A}^\top t} \boldsymbol{C}^\top \boldsymbol{C} e^{\boldsymbol{A} t} \Big|_0^\infty \\
&= -\boldsymbol{C}^\top \boldsymbol{C}
\end{aligned}
$$

Proof of the sufficiency condition

- ($\Leftarrow$) The sufficiency part can be proven noting that the observability of the pair $(\boldsymbol{A}, \boldsymbol{C})$ implies that $\boldsymbol{x}(t)^\top \boldsymbol{C}^\top \boldsymbol{C}\boldsymbol{x}(t)$ is not identically zero along any nonzero solution of $\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t)$

Evaluating performance indices

- Apply the Lyapunov theory to evaluate the indices

$$J_r = \int_0^{\infty} t^r \boldsymbol{x}(t)^\top \boldsymbol{Q}\boldsymbol{x}(t)dt, \quad r = 0, 1, \ldots$$

  subject to

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0,$$

  where $\boldsymbol{Q} = \boldsymbol{C}^\top \boldsymbol{C}$ such that $(\boldsymbol{A}, \boldsymbol{C})$ is observable, and $\boldsymbol{A}$ asymptotically stable
- Reformulate the problem: Evaluate the indices

$$J_r = \int_0^{\infty} t^r \boldsymbol{y}(t)^\top \boldsymbol{y}(t)dt, \quad r = 0, 1, \ldots$$

  subject to

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0, \\ \boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{x}(t) \end{aligned}$$

Evaluating J_0

- Evaluate

$$\begin{aligned} J_0 &= \int_0^\infty \boldsymbol{y}(t)^\top \boldsymbol{y}(t) dt = \int_0^\infty \boldsymbol{x}(t)^\top \boldsymbol{C}^\top \boldsymbol{C}\boldsymbol{x}(t) dt \\ &= \int_0^\infty \boldsymbol{x}(t)^\top \boldsymbol{Q}\boldsymbol{x}(t) dt, \end{aligned}$$

- Recall that

$$\frac{d}{dt}\left(\boldsymbol{x}(t)^\top \boldsymbol{P}\boldsymbol{x}(t)\right) = -\boldsymbol{x}(t)^\top \boldsymbol{Q}\boldsymbol{x}(t),$$

where $\boldsymbol{P}$ and $\boldsymbol{Q} = \boldsymbol{C}^\top \boldsymbol{C}$ satisfy $\boldsymbol{A}^\top \boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} = -\boldsymbol{Q}$

- Hence,

$$J_0 = -\int_0^\infty \frac{d}{dt}\left(\boldsymbol{x}(t)^\top \boldsymbol{P}\boldsymbol{x}(t)\right) dt$$

Evaluating J_0—manipulations

- Integrate both sides of $J_0 = -\int_0^\infty \frac{d}{dt}\left(\boldsymbol{x}(t)^\top \boldsymbol{P}\boldsymbol{x}(t)\right) dt$ with respect to t to obtain

$$\begin{aligned} J_0 &= -\int_0^\infty \frac{d}{dt}\left(\boldsymbol{x}(t)^\top \boldsymbol{P}\boldsymbol{x}(t)\right) dt \\ &= -\left(\boldsymbol{x}(t)^\top \boldsymbol{P}\boldsymbol{x}(t)\right)\Big|_0^\infty \\ &= \lim_{t\to\infty}\left(-\boldsymbol{x}(t)^\top \boldsymbol{P}\boldsymbol{x}(t)\right) + \boldsymbol{x}(0)^\top \boldsymbol{P}\boldsymbol{x}(0) \\ &= \boldsymbol{x}(0)^\top \boldsymbol{P}\boldsymbol{x}(0) \end{aligned}$$

  since by assumption $\boldsymbol{A}$ is asymptotically stable and therefore, $\lim_{t\to\infty}\boldsymbol{x}(t) = \mathbf{0}$ for all $\boldsymbol{x}(0)$
- Furthermore, $\boldsymbol{P} = \boldsymbol{P}^\top \succ 0$
- Thus $J_0 > 0$ for all $\boldsymbol{x}(0) \neq \mathbf{0}$

Evaluating J_1

- Evaluate
$$J_1 = \int_0^\infty t\boldsymbol{x}(t)^\top \boldsymbol{Q}\boldsymbol{x}(t)dt$$
subject to
$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0,$$
where $\boldsymbol{Q} = \boldsymbol{C}^\top \boldsymbol{C}$ such that $(\boldsymbol{A}, \boldsymbol{C})$ is observable, and $\boldsymbol{A}$ asymptotically stable
- Reformulate the above problem as: evaluate
$$J_1 = \int_0^\infty t\boldsymbol{y}(t)^\top \boldsymbol{y}(t)dt$$
subject to
$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0, \\ \boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{x}(t), \end{aligned}$$
where $(\boldsymbol{A}, \boldsymbol{C})$ is observable, and $\boldsymbol{A}$ asymptotically stable

Evaluating J_1—manipulations

- Note that

$$\begin{aligned}\frac{d}{dt}\left(t\boldsymbol{x}(t)^\top \boldsymbol{P}\boldsymbol{x}(t)\right) &= \boldsymbol{x}(t)^\top \boldsymbol{P}\boldsymbol{x}(t) + t\frac{d}{dt}\left(\boldsymbol{x}(t)^\top \boldsymbol{P}\boldsymbol{x}(t)\right)\\ &= \boldsymbol{x}(t)^\top \boldsymbol{P}\boldsymbol{x}(t) - t\boldsymbol{x}(t)^\top \boldsymbol{Q}\boldsymbol{x}(t),\end{aligned}$$

  where $\boldsymbol{P}$ satisfies the Lyapunov matrix equation, $\boldsymbol{A}^\top\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} = -\boldsymbol{Q}$
- Hence,

$$\begin{aligned}J_1 &= \int_0^\infty t\boldsymbol{x}(t)^\top \boldsymbol{Q}\boldsymbol{x}(t)dt\\ &= \int_0^\infty \boldsymbol{x}(t)^\top \boldsymbol{P}\boldsymbol{x}(t)dt - \int_0^\infty \frac{d}{dt}\left(t\boldsymbol{x}(t)^\top \boldsymbol{P}\boldsymbol{x}(t)\right)dt\end{aligned}$$

$J_1 = \boldsymbol{x}(0)^\top \boldsymbol{P}_1 \boldsymbol{x}(0)$

- Because $\boldsymbol{P} = \boldsymbol{P}^\top \succ 0$, we can solve

$$\boldsymbol{A}^\top \boldsymbol{P}_1 + \boldsymbol{P}_1 \boldsymbol{A} = -\boldsymbol{P}$$

  for $\boldsymbol{P}_1 = \boldsymbol{P}_1^\top \succ 0$
- Represent the above as

$$\frac{d}{dt}\left(\boldsymbol{x}(t)^\top \boldsymbol{P}_1 \boldsymbol{x}(t)\right) = -\boldsymbol{x}(t)^\top \boldsymbol{P} \boldsymbol{x}(t)$$

- Substituting yields

$$\begin{aligned}
J_1 &= -\int_0^\infty \frac{d}{dt}\left(\boldsymbol{x}(t)^\top \boldsymbol{P}_1 \boldsymbol{x}(t)\right) dt - \left(t\boldsymbol{x}(t)^\top \boldsymbol{P} \boldsymbol{x}(t)\right)\Big|_0^\infty \\
&= \lim_{t\to\infty}\left(-\boldsymbol{x}(t)^\top \boldsymbol{P}_1 \boldsymbol{x}(t)\right) + \boldsymbol{x}(0)^\top \boldsymbol{P}_1 \boldsymbol{x}(0) - 0 + 0 \\
&= \boldsymbol{x}(0)^\top \boldsymbol{P}_1 \boldsymbol{x}(0)
\end{aligned}$$