AAE 666 Homework 9 Solution

March 31, 2023

Exercise 1

We need to obtain the describing function for

$$\phi(y) = y^5$$

Here

$$\phi(a \sin\theta) = a^5 \sin^5\theta$$

and

$$\sin^5\theta = \sin^5\theta = \left(\frac{e^{j\theta} - e^{-j\theta}}{2j}\right)^5 = \frac{\left(e^{j\theta} - e^{-j\theta}\right)^5}{32j}$$

Using symbolic manipulation in Matlab:

$$(a-b)^5 = a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5$$

Hence

$$\left(e^{j\theta} - e^{-j\theta}\right)^5 = e^{j5\theta} - e^{-j5\theta} - 5(e^{j3\theta} - e^{-j3\theta}) + 10(e^{j\theta} - e^{-j\theta})$$

and

$$\sin^5\theta = \frac{5}{8}\sin\theta - \frac{5}{16}\sin 3\theta + \frac{1}{16}\sin 5\theta$$

Thus the describing function for ϕ is

$$\boxed{N(a) = \frac{5a^4}{8}}$$

Exercise 2

Obtain the transfer function for the system:

$$\ddot{y} - y = -y^3$$

where $u = -y^3 = -\phi(y)$. Thus, assuming zero initial condition we have,

$$s^2 Y - Y = U$$
$$\hat{G}(s) = \frac{Y}{U} = \frac{1}{s^2 - 1}$$

Using describing function to estimate:

$$\begin{aligned}
\phi(a \sin \omega t) &= a^3 \sin^3 \omega t \\
&= \frac{a^3}{4} \sin \omega t - \frac{a^3}{4} \sin 3\omega t \\
b_1(a) &= \frac{3a^3}{4} \\
N(a) &= \frac{b_1(a)}{a} = \frac{3a^2}{4}
\end{aligned}$$

Using the describing function condition

$$1 + \hat{G}(i\omega) N(a) = 0$$

results in

$$\begin{aligned}
1 - \frac{1}{(\omega^2 + 1)} \frac{3a^2}{4} &= 0 \\
\omega^2 + 1 &= \frac{3a^2}{4} \\
\omega &= \frac{\sqrt{3a^2 - 4}}{2}
\end{aligned}$$

where ω is real if $a > \frac{2}{\sqrt{3}}$ and we end up with periodic solution with approximate periods of $T = \frac{2\pi}{\omega} = \frac{4\pi}{\sqrt{3a^2 - 4}}$.

Exercise 3

The system:

$$\ddot{y} + \mu\left(\frac{\dot{y}^3}{3} - \dot{y}\right) + y = 0$$

can be described by

$$\ddot{y} - \mu \dot{y} + y = \mu u \tag{1}$$
$$u = -\frac{\dot{y}^3}{3} \tag{2}$$

The linear system (1) has transfer function

$$\hat{G}(s) = \frac{\mu}{s^2 - \mu s + 1}$$

We now obtain the describingh describing function of the nonlinear term with

$$y = a \sin \omega t$$

we have

$$\begin{aligned}
\dot{y}^3 &= a^3\omega^3 \cos^3 \omega t \\
&= a^3\omega^3 (\frac{3}{4}\cos \omega t + \frac{1}{4}\cos 3\omega t) \\
&\approx \frac{3}{4} a^3 \omega^3 \cos \omega t
\end{aligned}$$

Hence the describing function is given by

$$N(a, \omega) = \jmath \frac{a^2\omega^3}{4}$$

Using the describing function condition

$$1 + \hat{G}(i\omega) N(a, \omega) = 0$$

results in

$$1 + \frac{\mu}{1 - \omega^2 - \jmath\mu\omega} \frac{\jmath a^2 \omega^3}{4} = 0$$

that is

$$1 - \omega^2 + \jmath\mu\omega(-1 + \frac{a^2\omega^2}{4}) = 0$$

or (with $\omega \neq 0$)

$$1 - \omega^2 = 1, \qquad -1 + \frac{a^2\omega^2}{4} = 0$$

which results in

$$\omega = 1, a = 2$$

and we end up with periodic solution with approximate periods of

$$T = \frac{2\pi}{\omega} = 2\pi$$

Exercise 4

Obtain the transfer function for the system:

$$\dot{x} = -x + 2u$$

where we let $u = -sgm(x(t-h))$ and $y = x(t-h)$, which gives $\phi = sgm(x(t-h)) = smg(y)$. Thus, assuming zero initial condition we have,

$$G(s) = \frac{X}{U} = \frac{2}{s+1}e^{-sh}$$

We also have the following describing function for the signum function.

$$N(a) = \frac{4}{\pi a}$$

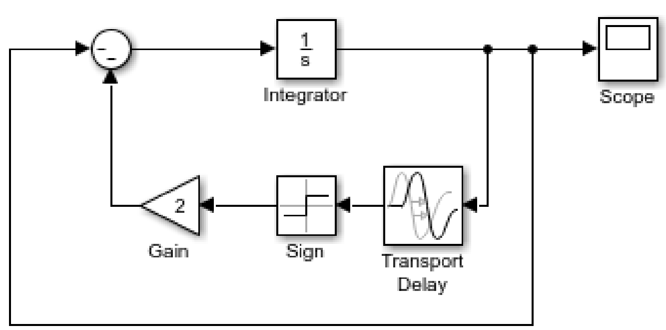
Together with the transfer function after substituting s with $i\omega$, we come to:

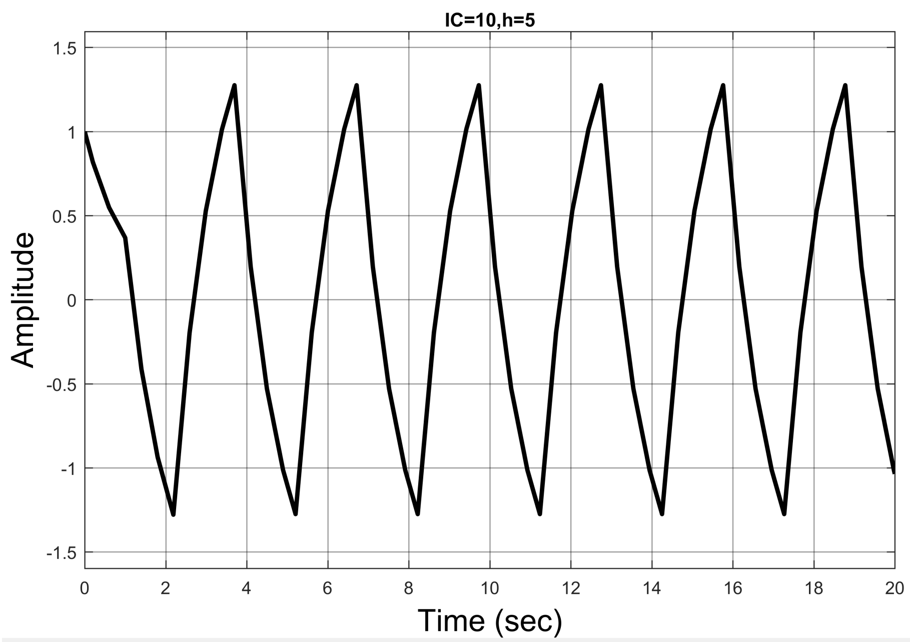
$$\begin{aligned}
1 + G(i\omega)N(a) &= 0 \\
1 + \frac{8}{i\pi a\omega + \pi a}e^{-i\omega h} &= 0 \\
i\pi a\omega + \pi a &= -8e^{-i\omega h} \\
i\pi a\omega + \pi a &= -8(cos(-i\omega h) + isin(-\omega h)) \\
i\pi a\omega + \pi a &= -8(cos(i\omega h) - isin(\omega h)) \\
i(\pi a\omega - 8sin(\omega h)) + \pi a + 8cos(\omega h) &= 0
\end{aligned}$$

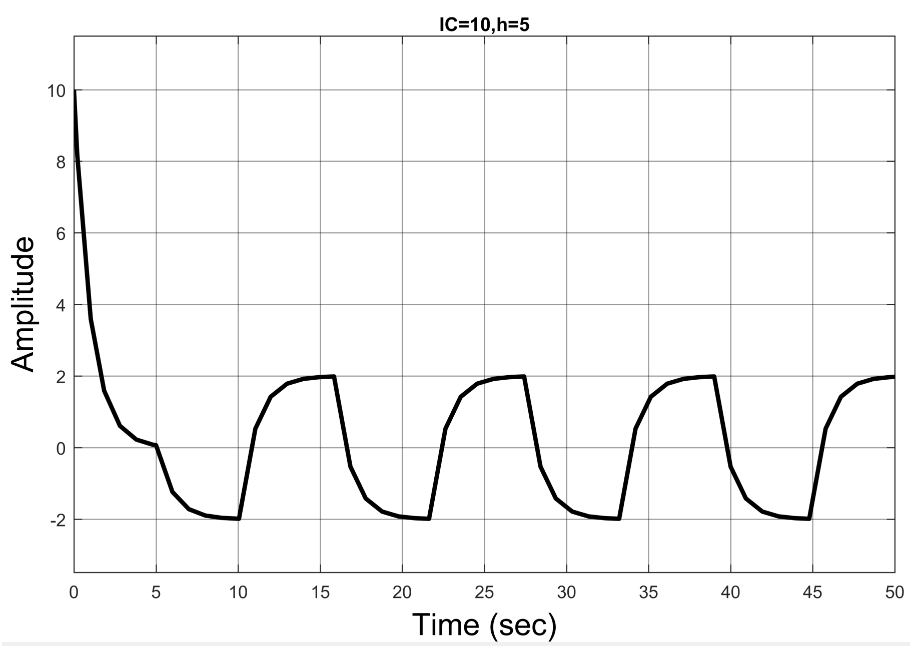
where manipulation is required to obtain explicit expression for ω and a

$$\begin{aligned}
a = \frac{8sin(\omega h)}{\pi\omega} &= -\frac{-8}{\pi}cos(\omega h) \\
\omega &= -tan(\omega h) \\
h &= \frac{-tan^{-1}(\omega)}{\omega}
\end{aligned}$$

Finally, we end up with periodic solution with approximate periods of $T = \frac{2\pi}{\omega} = \frac{2\pi}{-tan(\omega h)}$. Some simulation results are provided below with different delay h which is a given parameter.




IC=10,h=5
Amplitude
Time (sec)
1.5
1
0.5
0
-0.5
-1
-1.5
0
2
4
6
8
10
12
14
16
18
20


IC=10,h=5
Amplitude
Time (sec)
10
8
6
4
2
0
-2
0
5
10
15
20
25
30
35
40
45
50

Exercise 5

Obtain the transfer function for the system:

$$\ddot{q} = u$$

where $u = -k_P q - k_D \dot{q} - sat(k_I \int q)$.

$$G(s) = \frac{k_I}{s^3 + k_D s^2 + k_P s}$$

a) When $k_P = 1, k_D = 2$, use *rlocus* in MATLAB to plot the root locus of the following transfer function.

$$G(s) = \frac{k_I}{s^3 + 2s^2 + s}$$

It is found that the maximum value of k_I is 2 for which the closed loop system is asymptotically stable about $q(t) = 0$. a) When $k_P = 1, k_D = 2$, use we first find the describing function for the saturation function.

$$N(a) = \begin{cases} 1, & \text{if } 0 \le a \le 1 \\ \frac{2}{\pi}[sin^{-1}(\frac{1}{a}) + \frac{\sqrt{a^2-1}}{a^2}], & \text{if } a > 1 \end{cases}$$

(i) for $0 \le a \le 1$

$$\begin{aligned} 1 + G(i\omega)N(a) &= 0 \\ 1 + \frac{k_I}{-i\omega^3 - 2\omega^2 + i\omega} &= 0 \\ -i\omega^3 - 2\omega^2 + i\omega &= -k_I \\ k_I - i\omega^3 - 2\omega^2 + i\omega &= 0 \\ k_I - i\omega(1 - \omega^2) - 2\omega^2 &= 0 \end{aligned}$$

which gives $\omega = 1$, and $k_I = 2$.

(ii) for $a > 1$

$$\begin{aligned} 1 + G(i\omega)N(a) &= 0 \\ 1 + \frac{k_I}{-i\omega^3 - 2\omega^2 + i\omega}\frac{2}{\pi}[sin^{-1}(\frac{1}{a}) + \frac{\sqrt{a^2-1}}{a^2}] &= 0 \\ \frac{2}{\pi}[sin^{-1}(\frac{1}{a}) + \frac{\sqrt{a^2-1}}{a^2}]k_I - i\omega(\omega^2 - 1) - 2\omega^2 &= 0 \end{aligned}$$

which gives $\omega = 1$, and k_I is computed as the following:

$$k_I = \frac{\pi}{sin^{-1}\frac{1}{a} + \frac{\sqrt{a^2-1}}{a^2}}$$

since $a > 1$, thus $k_I > 2$.

As a result, the smallest k_I value for periodic solution is 2.