5.1.1

$A = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}$

$\chi_A = |A - \lambda I| = \begin{vmatrix} 1-\lambda & -1 \\ 2 & 4-\lambda \end{vmatrix}$

$(1-\lambda)(4-\lambda) - (-1)(2) = 4 - \lambda - 4\lambda + \lambda^2 + 2$

$= \lambda^2 - 5\lambda + 6$

$\chi_A = 0 = \lambda^2 - 5\lambda + 6 \quad \Rightarrow (\lambda - 3)(\lambda - 2) = 0$

$\Rightarrow \boxed{\lambda_1 = 3, \quad \lambda_2 = 2}$

$(A - \lambda_1 I)V_1 = 0$

$\begin{pmatrix} -2 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$-2X_1 - X_2 = 0$ Let $X_1 = 1 \therefore X_2 = -2$

$\boxed{\text{For } \lambda_1 = 3, \ V_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}}$

$(A - \lambda_2 I)V_2 = 0$

$\begin{pmatrix} -1 & -1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

5.1.1

$-x_1 - x_2 = 0$ Let $x_1 = 1$ $\therefore$ $x_2 = -1$

$$\boxed{\text{For } \lambda_2 = 2, \quad v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

$\lambda_1 + \lambda_2 = 3 + 2 = 5$

$Tr(A) = 1 + 4 = 5$

$\therefore \boxed{\Sigma \lambda_i = Tr(A) = 5}$

$\det(A) = 4 - (-1)(2) = 6$

$(\lambda_1)(\lambda_2) = (3)(2) = 6$

$\therefore \boxed{\det(A) = (\lambda_1)(\lambda_2) = 6}$

5.1.4

$$\dot{u} = P u$$

$$P = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

Assume solution: $u(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$

$$|P - \lambda I| = 0 \quad \begin{vmatrix} 1/2 - \lambda & 1/2 \\ \frac{1}{2} & 1/2 - \lambda \end{vmatrix}$$

$$\chi_p = \lambda^2 - \lambda = 0 \Rightarrow \lambda(\lambda - 1) = 0$$

$$\lambda_1 = 0, \quad \lambda_2 = 1$$

$$(P - \lambda_1 I) v_1 = 0$$

$$\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \Rightarrow 1/2\, x_1 + 1/2\, x_2 = 0$$

Let $x_1 = 1 \therefore x_2 = -1$

$$v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$(P - \lambda_2 I) v_2 = 0$$

$$\begin{pmatrix} -1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \Rightarrow -\frac{x_1}{2} + \frac{x_2}{2} = 0$$

Let $x_1 = 1 \therefore x_2 = 1$

$$v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

5.1.4

$$u(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$u(0) = \begin{bmatrix} 5 \\ 3 \end{bmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}^{-1} = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 5 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{5}{2} - \frac{3}{2} \\ \frac{5}{2} + \frac{3}{2} \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$$

$$\boxed{u(t) = -1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + 4 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

5.4.10

$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda \Rightarrow \lambda_{A^1} = 0, \lambda_{A^2} = 2$

$B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$|B - \lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 0 \\ 0 & -\lambda \end{vmatrix} = \lambda^2 - \lambda \Rightarrow \lambda_{B1} = 0, \lambda_{2B} = 1$

$AB = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \quad |AB - \lambda I| = \begin{pmatrix} 1-\lambda & 0 \\ 1 & -\lambda \end{pmatrix}$

$= 0 = \lambda^2 - \lambda \Rightarrow \lambda_{AB}^1 = 0, \lambda_{AB}^2 = 1$

$(\lambda_{A_1})(\lambda_{B_1}) = (0)(0) = \lambda_{AB_1}$

$(\lambda_{A_2})(\lambda_{B_2}) = (2)(1) = \boxed{2 \neq \lambda_{AB_2} = 1}$

$A + B = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

$|(A+B) - \lambda I| = 0 = \lambda^2 - 3\lambda + 1 = 0$

$\lambda_{1,2} = \dfrac{3 \pm \sqrt{9-4}}{2} = \dfrac{3+\sqrt{5}}{2}, \dfrac{3-\sqrt{5}}{2}$

5.4.10

$$\lambda_{A_1} + \lambda_{B_1} = 0 \neq \frac{3+\sqrt{5}}{2} = \lambda_{AB_1}$$

$$\lambda_{A_2} + \lambda_{B_2} = 3 \neq \frac{3+\sqrt{5}}{2} = \lambda_{AB_1}$$

b)

$$\lambda_{AB_1} + \lambda_{AB_2} = 3$$

$$\lambda_{A_1} + \lambda_{A_2} + \lambda_{B_1} + \lambda_{B_2} = 0 + 2 + 0 + 1 = 3$$

$$\therefore \boxed{\Sigma \lambda_{AB} = \Sigma \lambda_A + \Sigma \lambda_B = 3}$$

5.1.14

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \xrightarrow{E_{31}(-1),\ E_{21}(-1),\ E_{41}(-1)}$$

$$A \Rightarrow \begin{pmatrix} \boxed{1} & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad 1 \text{ Pivot}, \quad \boxed{\text{rank} = 1}$$

$$|A - \lambda I| = \begin{pmatrix} 1-\lambda & 1 & 1 & 1 \\ 1 & 1-\lambda & 1 & 1 \\ 1 & 1 & 1-\lambda & 1 \\ 1 & 1 & 1 & 1-\lambda \end{pmatrix}$$

$$= (1-\lambda)\begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} - \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} + \begin{vmatrix} 1 & 1-\lambda & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix}$$

$$- \begin{vmatrix} 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \\ 1 & 1 & 1 \end{vmatrix} = 0$$

$$(1-\lambda)\left[(1-\lambda)\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 1 & 1-\lambda \end{vmatrix} + \begin{vmatrix} 1 & 1-\lambda \\ 1 & 1 \end{vmatrix}\right]$$

$$- \left[\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 1 & 1-\lambda \end{vmatrix} + \begin{vmatrix} 1 & 1-\lambda \\ 1 & 1 \end{vmatrix}\right] + \left[\begin{vmatrix} 1 & 1 \\ 1 & 1-\lambda \end{vmatrix} - (1-\lambda)\begin{vmatrix} 1 & 1 \\ 1 & 1-\lambda \end{vmatrix} + \cancel{\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}}\right]$$

$$- \left[\begin{vmatrix} 1 & 1-\lambda \\ 1 & 1 \end{vmatrix} - (1-\lambda)\begin{vmatrix} 1 & 1-\lambda \\ 1 & 1 \end{vmatrix} + \cancel{\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}}\right] = 0$$

5.1.14

$(1-\lambda)[(1-\lambda)(\lambda^2-2\lambda) - (1-\lambda-1) + (1-(1-\lambda))]$

$= (1-\lambda)[(1-\lambda)(\lambda^2-2\lambda)+2\lambda)$

$= \lambda^4-2\lambda^3-2\lambda^3+4\lambda^2+\lambda^2-\cancel{2\lambda}+\cancel{2\lambda}-2\lambda^2 = \lambda^4-4\lambda^3+3\lambda^2$

$-[(\lambda^2-2\lambda) - (\cancel{1}-\lambda-\cancel{1}) + (\cancel{1}-\cancel{(1}-\lambda))] = \underline{-\lambda^2}$

$[(1-\lambda)-1 - (1-\lambda)((1-\lambda)-1)] = (-\lambda)-(1-\lambda)(-\lambda) = \cancel{-\lambda}\cancel{+\lambda}\underline{-\lambda^2}$

$-[\cancel{1}\cancel{(1}-\lambda) - (1-\lambda)(1-(1-\lambda))] = -\lambda+(1-\lambda)(\lambda) = -\lambda^2$

$\therefore\ \chi_A = \lambda^4-4\lambda^3+3\lambda^2-\lambda^2-\lambda^2-\lambda^2 = \lambda^4-4\lambda^3=0$

$\lambda^3(\lambda-4)=0$

$\boxed{\lambda_1, \lambda_2, \lambda_3 = 0\ ,\ \lambda_4 = 4}$

$(A-4I)V_4=0$

$$\begin{pmatrix} -3 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \vec{0} \Rightarrow$$

$$\begin{pmatrix} -3 & 1 & 1 & 1 \\ 0 & -8/3 & 4/3 & 4/3 \\ 1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -3 \end{pmatrix} \Rightarrow \begin{pmatrix} -3 & 1 & 1 & 1 \\ 0 & -8/3 & 4/3 & 4/3 \\ 0 & 4/3 & -8/3 & 4/3 \\ 1 & 1 & 1 & -3 \end{pmatrix} \Rightarrow \begin{pmatrix} -3 & 1 & 1 & 1 \\ 0 & -8/3 & 4/3 & 4/3 \\ 0 & 4/3 & -8/3 & 4/3 \\ 0 & 4/3 & 4/3 & -8/3 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -3 & 1 & 1 & 1 \\ 0 & -8/3 & 4/3 & 4/3 \\ 0 & 0 & -2 & 2 \\ 0 & 4/3 & 4/3 & -8/3 \end{pmatrix} \Rightarrow \begin{pmatrix} -3 & 1 & 1 & 1 \\ 0 & -8/3 & 4/3 & 4/3 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 2 & -2 \end{pmatrix} \Rightarrow \begin{pmatrix} \boxed{-3} & 1 & 1 & 1 \\ 0 & \boxed{-8/3} & 4/3 & 4/3 \\ 0 & 0 & \boxed{-2} & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

x_1, x_2, x_3 are pivots, x_4 is free.

5.1.14

Preferred solution: $x_4 = 1$

$$-3x_1 + x_2 + x_3 + x_4 = 0$$
$$-\tfrac{8}{3}x_2 + \tfrac{4}{3}x_3 + \tfrac{4}{3}x_4 = 0$$
$$-2x_3 + 2x_4 = 0$$

$\therefore\ x_3 = x_4 = 1$

$$-\tfrac{8}{3}x_2 + \tfrac{4}{3} + \tfrac{4}{3} = 0$$
$$x_2 = 1$$

$$-3x_1 + 1 + 1 + 1 = 0$$
$\therefore\ x_1 = 1$

$$\boxed{v_4 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}}$$

↖ E-vec w/ $\lambda = 4$

5.1.14

$$C = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \xRightarrow{E_{31}(-1),\; E_{42}(-1)}$$

$$C = \begin{pmatrix} 0 & \boxed{1} & 0 & 1 \\ \boxed{1} & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$ 2 pivots, $\boxed{\text{rank} = 2}$

$$|C - \lambda I| = \begin{vmatrix} -\lambda & 1 & 0 & 1 \\ 1 & -\lambda & 1 & 0 \\ 0 & 1 & -\lambda & 1 \\ 1 & 0 & 1 & -\lambda \end{vmatrix}$$

$$= -\lambda \left[\begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} \right] - \begin{vmatrix} 1 & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} - \begin{vmatrix} 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \\ 1 & 0 & 1 \end{vmatrix}$$

$$= -\lambda \left[-\lambda \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 0 & -\lambda \end{vmatrix} \right] - \left[\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} - \begin{vmatrix} 0 & 1 \\ 1 & -\lambda \end{vmatrix} \right]$$

$$- \left[\begin{vmatrix} 1 & -\lambda \\ 0 & 1 \end{vmatrix} + \lambda \begin{vmatrix} 0 & -\lambda \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \right]$$

$$= (-\lambda)[-\lambda(\lambda^2 - 1) + \lambda] = -\lambda(-\lambda^3 + \lambda + \lambda) = \underline{\lambda^4 - 2\lambda^2}$$

$$-(\lambda^2 - 1 + 1) = \underline{-\lambda^2}$$

$$-(1 + \lambda(\lambda) - 1) = \underline{-\lambda^2}$$

$$|C - \lambda I| = \lambda^4 - 2\lambda^2 - \lambda^2 - \lambda^2 = \lambda^4 - 4\lambda^2 = 0$$

5.1.14

$\lambda^2(\lambda^2-4)=0 = \lambda^2(\lambda+2)(\lambda-2)=0$

$$\boxed{\lambda_{1,2}=0,\quad \lambda_3=-2,\quad \lambda_4=2}$$

$$\begin{pmatrix} 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \end{pmatrix} = (C-\lambda_3 I)\,V_3 = 0$$

$$\Rightarrow \begin{pmatrix} 2 & 1 & 0 & 1 \\ 0 & 3/2 & 1 & -1/2 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \end{pmatrix} \Rightarrow \begin{pmatrix} 2 & 1 & 0 & 1 \\ 0 & 3/2 & 1 & -1/2 \\ 0 & 1 & 2 & 1 \\ 0 & -1/2 & 1 & 3/2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 2 & 1 & 0 & 1 \\ 0 & 3/2 & 1 & -1/2 \\ 0 & 0 & 4/3 & 4/3 \\ 0 & -1/2 & 1 & 3/2 \end{pmatrix} \Rightarrow \begin{pmatrix} 2 & 1 & 0 & 1 \\ 0 & 3/2 & 1 & -1/2 \\ 0 & 0 & 4/3 & 4/3 \\ 0 & 0 & 4/3 & 4/3 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 2 & 1 & 0 & 1 \\ 0 & 3/2 & 1 & -1/2 \\ 0 & 0 & 4/3 & 4/3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$x_1, x_2\ x_3$ are pivots, x_4 is free, let $x_4=1$

$4/3\,x_3 + 4/3 = 0 \qquad x_3=-1$

$3/2\,x_2 - 1 - 1/2 = 0 \qquad x_2 = 1$

$2x_1 + 1 + 1 = 0 \qquad x_1 = -1$

$$\boxed{V_3 = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}}$$ ← Evec of $\lambda_3=-2$

5.1.14

$(C - \lambda_4 I) V_4 = 0$

$$\begin{pmatrix} -2 & 1 & 0 & 1 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 1 & 0 & 1 & -2 \end{pmatrix} \Rightarrow \begin{pmatrix} -2 & 1 & 0 & 1 \\ 0 & -3/2 & 1 & 1/2 \\ 0 & 0 & -1/3 & 1/3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \vec{0}$$

x_1, x_2, x_3 are pivots, let $x_4 = 1$

$-\frac{1}{3}x_3 + \frac{1}{3} = 0 \Rightarrow x_3 = 1$

$-\frac{3}{2}x_2 + 1 + \frac{1}{2} = 0 \Rightarrow x_2 = 1$

$-2x_1 + 1 + 1 = 0 \Rightarrow x_1 = 1$

$\therefore \boxed{V_4 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}}$ ← E-vec for $\lambda = 2$

5.2.2

$\lambda_1 = 1$, $\lambda_2 = 4$

$X_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, $X_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$S = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$ $\quad \Lambda = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$

$$S^{-1} = \frac{1}{3-2} \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}$$

$$A = S \Lambda S^{-1} = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 8 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}$$

$$\boxed{A = \begin{pmatrix} -5 & 18 \\ -3 & 10 \end{pmatrix}}$$

5.2.5

$A_1 = \begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix}$

$\chi_{A_1} = 0 = \lambda^2 \Rightarrow \lambda_1 = \lambda_2 = 0$

$(A_1 - \lambda_{1,2} I) v_{1,2} = 0$

$\begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \quad \Rightarrow \quad x_1 = x_2 = 1$

$v_2 = v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\boxed{v_1, v_2 \text{ aren't linearly independent}, \therefore A_1 \text{ isn't diagonalizable}}$

$A_2 = \begin{pmatrix} 2 & 0 \\ 2 & -2 \end{pmatrix} \qquad \chi_{A_2} = 0 = \lambda^2 - 4 = (\lambda+2)(\lambda-2)$

$\lambda_1 = -2, \lambda_2 = 2$, distinct λ_1 & λ_2 $\therefore$ diagonalizable

$A_3 = \begin{pmatrix} 2 & 0 \\ 2 & 2 \end{pmatrix} \qquad \chi_{A_3} = \lambda^2 - 4\lambda + 4 = 0 \Rightarrow (\lambda-2)(\lambda-2) = 0$

$\lambda_1 = \lambda_2 = 2$

$(A_3 - \lambda_{1,2} I) v_{3,4} = 0$

$\boxed{v_3, v_4 \text{ aren't linearly Independent}, \therefore A_3 \text{ isn't diagonalizable}}$

5.2.12

a) False, the eigenvalues could be non-zero.

b) True, repeated eigenvalues will produce non-linearly independent eigenvectors.

c) True, eigenvectors aren't linearly Independent

5.3.4

$G_{k+2} = \frac{1}{2} G_{k+1} + \frac{1}{2} G_k$

$G_{k+1} = G_{k+1}$

$$\begin{bmatrix} G_{k+2} \\ G_{k+1} \end{bmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} G_{k+1} \\ G_k \end{pmatrix}$$

a) $\chi_A = \lambda^2 - \frac{1}{2}\lambda - 1/2$

$\chi_A = (\lambda - 1)(\lambda + 1/2) = 0$

$$\boxed{\begin{aligned} \lambda_1 &= 1 \\ \lambda_2 &= -1/2 \end{aligned}}$$

$(A - \lambda_1 I) V_1 = 0$

$\begin{pmatrix} -1/2 & 1/2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$ $\quad -\frac{x_1}{2} + \frac{x_2}{2} = 0$

Let $x_1 = 1 \therefore x_2 = 1$

$$\boxed{V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

$(A - \lambda_2 I) V_2 = 0$

$\begin{pmatrix} 1 & 1/2 \\ 1 & -1/2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$ $\quad x_1 + \frac{x_2}{2} = 0$

$x_1 = 1 \therefore x_2 = -2$

$$\boxed{V_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}}$$

5.3.4

b) $S = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$ $\quad \Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$

$$S^{-1} = \frac{1}{-3}\begin{pmatrix} -2 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2/3 & 1/3 \\ 1/3 & -1/3 \end{pmatrix}$$

$$\lim_{n \to \infty} \left(S \Lambda^n S^{-1}\right) = \lim_{n \to \infty} \left(S \begin{bmatrix} 1^n & 0 \\ 0 & -\frac{1}{2}^n \end{bmatrix} S^{-1}\right)$$

$$= S \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} S^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 2/3 & 1/3 \\ 1/3 & -1/3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2/3 & 1/3 \\ 1/3 & -1/3 \end{pmatrix} = \begin{pmatrix} 2/3 & 1/3 \\ 2/3 & 1/3 \end{pmatrix}$$

$$\boxed{\lim_{n \to \infty} (A^n) = \begin{pmatrix} 2/3 & 1/3 \\ 2/3 & 1/3 \end{pmatrix}}$$

c) $\begin{bmatrix} G_{k+2} \\ G_{k+1} \end{bmatrix} = A^n \begin{bmatrix} G_1 \\ G_0 \end{bmatrix}$

$$\begin{pmatrix} 2/3 \\ 2/3 \end{pmatrix} = \begin{pmatrix} 2/3 & 1/3 \\ 2/3 & 1/3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\boxed{\begin{bmatrix} G_{k+2} \\ G_{k+1} \end{bmatrix} = \begin{pmatrix} 2/3 \\ 2/3 \end{pmatrix}}$$

5.3.8

$$\begin{pmatrix} d_{n+1} \\ s_{n+1} \\ w_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 1/4 & 0 \\ 0 & 3/4 & 1/2 \\ 0 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} d_n \\ s_n \\ w_n \end{pmatrix}$$

$$|(A-\lambda I)| = \left| \begin{pmatrix} 1-\lambda & 1/4 & 0 \\ 0 & 3/4-\lambda & 1/2 \\ 0 & 0 & 1/2-\lambda \end{pmatrix} \right| = 0$$

$$(1-\lambda) \begin{vmatrix} 3/4-\lambda & 1/2 \\ 0 & 1/2-\lambda \end{vmatrix} - \frac{1}{4} \begin{vmatrix} 0 & 1/2 \\ 0 & 1/2-\lambda \end{vmatrix} + 0$$

$$= (1-\lambda)(3/4-\lambda)(1/2-\lambda) = 0$$

$$\lambda_1 = 1, \quad \lambda_2 = 3/4, \quad \lambda_3 = 1/2$$

$$(A-\lambda_1 I)V_1 = 0$$

$$\begin{pmatrix} 0 & 1/4 & 0 \\ 0 & -1/4 & 1/2 \\ 0 & 0 & -1/2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \quad \Rightarrow$$

$$\begin{pmatrix} 0 & 1/4 & 0 \\ 0 & 0 & 1/2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

x_1 is free, let $x_1 = 1$

$\therefore \; x_2 = x_3 = 0$

$$V_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$(A-\lambda_2 I)V_2 = 0$$

$$\begin{pmatrix} 1/4 & 1/4 & 0 \\ 0 & 0 & 1/2 \\ 0 & 0 & -1/4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

5.3.8

$$\begin{pmatrix} 1/4 & 1/4 & 0 \\ 0 & 0 & 1/2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

x_2 is free, let $x_2 = 1$.

$\frac{x_1}{4} + \frac{x_2}{4} = 0 \quad \therefore x_1 = -1, \; x_3 = 0$

$$V_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$(A - \lambda_3 I) V_3 = 0$

$$\begin{pmatrix} 1/2 & 1/4 & 0 \\ 0 & 1/4 & 1/2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

x_3 is free, let $x_3 = 1$

$\frac{x_1}{2} + \frac{x_2}{4} = 0$

$\frac{x_2}{4} + \frac{1}{2} = 0 \therefore x_2 = -2, \; x_1 = 1$

$$V_3 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$\therefore \; S = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/4 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}$$

$$S^{-1} = \frac{C^T}{\det(S)}$$

5.3.8

$C_{11} = \begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} = 1$

$C_{12} = -\begin{vmatrix} 0 & -2 \\ 0 & 1 \end{vmatrix} = 0$

$C_{13} = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0$

$C_{21} = -\begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} = 1$

$C_{22} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$

$C_{23} = -\begin{vmatrix} 1 & -1 \\ 0 & 0 \end{vmatrix} = 0$

$C_{31} = \begin{vmatrix} -1 & 1 \\ 1 & -2 \end{vmatrix} = 1$

$C_{32} = -\begin{vmatrix} 1 & 1 \\ 0 & -2 \end{vmatrix} = 2$

$C_{33} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$

$\det(S) = (1)C_{11} + (-1)(C_{12}) + (1)(C_{13}) = 1$

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}$$

5.3.8

$$S^{-1} = C^T = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A^k = S \Lambda^k S^{-1} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1^k & 0 & 0 \\ 0 & 3/4^k & 0 \\ 0 & 0 & 1/2^k \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A^k = \begin{pmatrix} 1^k & (-1)(3/4)^k & (1/2)^k \\ 0 & (3/4)^k & (-2)(1/2)^k \\ 0 & 0 & (1/2)^k \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A^k = \begin{pmatrix} 1^k & (1)^k + (-1)(3/4)^k & 1^k + (2)(-1)(3/4)^k + (1/2)^k \\ 0 & (3/4)^k & (2)(3/4)^k + (-2)(1/2)^k \\ 0 & 0 & (1/2)^k \end{pmatrix}$$

$$u_k = A^k u_0$$

$$\lim_{(k \to \infty)} (u_k = A^k u_0) = \begin{pmatrix} d_\infty \\ s_\infty \\ w_\infty \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} d_0 \\ s_0 \\ w_0 \end{pmatrix}$$

Steady state is eisenvector associated with $\lambda = 1$,

$\therefore$ $\boxed{\begin{pmatrix} d_\infty \\ s_\infty \\ w_\infty \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}$ or $d_\infty = d_0 + s_0 + w_0$ with everyone dead.

5.3.11

$$A = \begin{pmatrix} .2 & .4 & .3 \\ .4 & .2 & .3 \\ .4 & .4 & .4 \end{pmatrix}$$

a) columns are linearly dependent $\therefore$

$$\boxed{\lambda_1 = 0}$$

$(A - \lambda_1 I) V_1 = 0$

$$\begin{pmatrix} .2 & .4 & .3 \\ .4 & .2 & .3 \\ .4 & .4 & .4 \end{pmatrix} \Rightarrow \begin{pmatrix} .2 & .4 & .3 \\ 0 & -0.6 & -0.3 \\ .4 & .4 & .4 \end{pmatrix} \Rightarrow \begin{pmatrix} .2 & .4 & .3 \\ 0 & -0.6 & -0.3 \\ 0 & -0.4 & -0.2 \end{pmatrix}$$

$$= \begin{pmatrix} .2 & .4 & .3 \\ 0 & -0.6 & -0.3 \\ 0 & 0 & 0 \end{pmatrix}$$

X_3 is free, let $X_3 = 1$. X_1 & X_2 are pivots.

$.2X_1 + .4X_2 + .3 = 0$

$-0.6X_2 = .3 \;\therefore\; X_2 = -1/2 ,\; X_1 = -1/2$

$$\boxed{V_1 = \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix}} \quad \text{or} \quad V_1 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

b) $|A - \lambda I| = 0$

$$\begin{pmatrix} .2-\lambda & .4 & .3 \\ .4 & .2-\lambda & .3 \\ .4 & .4 & .4-\lambda \end{pmatrix}$$

5.3.11

$$(.2-\lambda)\begin{vmatrix} .2-\lambda & .3 \\ .4 & .4-\lambda \end{vmatrix} - .4\begin{vmatrix} .4 & .3 \\ .4 & .4-\lambda \end{vmatrix} + .3\begin{vmatrix} .4 & .2-\lambda \\ .4 & .4 \end{vmatrix}$$

$$(.2-\lambda)(\lambda^2-.6\lambda-.04) - .4(.16-.4\lambda-.12) + .3(.16-(.08-.4\lambda))$$

$$(.2-\lambda)(\lambda^2-.6\lambda-.04) - .064 + .16\lambda + .048 + .048 - .024 + .12\lambda$$

$$(.2-\lambda)(\lambda^2-.6\lambda-.04) + .008 + .28\lambda = 0$$

$$= .2\lambda^2 - .12\lambda - \cancel{.008} - \lambda^3 + .6\lambda^2 - .04\lambda + \cancel{.008} + .28\lambda$$

$$= -\lambda^3 + .8\lambda^2 + .2\lambda = 0$$

$$-\lambda(\lambda^2 - .8\lambda - .2) = 0$$

$$-\lambda(\lambda-1)(\lambda+.2) = 0$$

$$\boxed{\begin{aligned} \lambda_1 &= 0 \\ \lambda_2 &= 1 \\ \lambda_3 &= -0.2 \end{aligned}}$$

c) $(A-\lambda_2 I)V_2 = 0$

$$\begin{pmatrix} -0.8 & .4 & .3 \\ .4 & -0.8 & .3 \\ .4 & .4 & -0.6 \end{pmatrix} \Rightarrow \begin{pmatrix} -0.8 & .4 & .3 \\ 0 & -0.6 & .45 \\ .4 & .4 & -0.6 \end{pmatrix} \Rightarrow$$

$$\begin{pmatrix} -.8 & .4 & .3 \\ 0 & -.6 & .45 \\ 0 & .6 & .45 \end{pmatrix} = \begin{pmatrix} -.8 & .4 & .3 \\ 0 & -.6 & .45 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$$

X_1 & X_2 are pivots, let $X_3 = 1$.

5.3.11

$$X_2 = \frac{-.45}{-.6} = 3/4$$

$$X_1 = \frac{(.4)(3/4) + .3}{.8} = 3/4$$

$$V_2 = \begin{pmatrix} 3/4 \\ 3/4 \\ 1 \end{pmatrix}$$

$$(A - \lambda_3 I) V_3 = 0$$

$$\begin{pmatrix} .4 & .4 & .3 \\ .4 & .4 & .3 \\ .4 & .4 & .8 \end{pmatrix} \Rightarrow \begin{pmatrix} .4 & .4 & .3 \\ 0 & 0 & 0 \\ 0 & 0 & .5 \end{pmatrix}$$

X_2 is free, let $X_2 = 1$ $\therefore$ $X_3 = 0$ & $X_1 = -1$

$$V_3 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$S = \begin{pmatrix} -1/2 & 3/4 & -1 \\ -1/2 & 3/4 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$S^{-1} = \frac{C^T}{\det(S)}$$

5.3.11

$$C_{11} = \begin{vmatrix} 3/4 & 1 \\ 1 & 0 \end{vmatrix} = -1$$

$$C_{12} = -\begin{vmatrix} -1/2 & 1 \\ 1 & 0 \end{vmatrix} = 1$$

$$C_{13} = \begin{vmatrix} -1/2 & 3/4 \\ 1 & 1 \end{vmatrix} = -5/4$$

$$C_{21} = -\begin{vmatrix} 3/4 & -1 \\ 1 & 0 \end{vmatrix} = -1$$

$$C_{22} = \begin{vmatrix} -1/2 & -1 \\ 1 & 0 \end{vmatrix} = 1$$

$$C_{23} = -\begin{vmatrix} -1/2 & 3/4 \\ 1 & 1 \end{vmatrix} = 5/4$$

$$C_{31} = \begin{vmatrix} 3/4 & -1 \\ 3/4 & 1 \end{vmatrix} = 6/4$$

$$C_{32} = -\begin{vmatrix} -1/2 & -1 \\ -1/2 & 1 \end{vmatrix} = 1$$

$$C_{33} = \begin{vmatrix} -1/2 & 3/4 \\ -1/2 & 3/4 \end{vmatrix} = 0$$

$$C = \begin{pmatrix} -1 & 1 & -5/4 \\ -1 & 1 & 5/4 \\ 6/4 & 1 & 0 \end{pmatrix}$$

$$\det(S) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$\det(S) = (-1/2)(-1) + (3/4)(1) + (-1)(-5/4) = 5/2$$

5.3.11

$$S^{-1} = \frac{C^T}{\det(S)} = \cdots$$

$$C^T = \begin{pmatrix} -1 & -1 & 6/4 \\ -1 & 1 & 1 \\ -5/4 & 5/4 & 0 \end{pmatrix}$$

$$S^{-1} = \begin{pmatrix} -2/5 & -2/5 & 3/5 \\ 2/5 & 2/5 & 2/5 \\ -1/2 & 1/2 & 0 \end{pmatrix}$$

$$\Lambda = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -0.2 \end{pmatrix}$$

$$\Lambda^k = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1^k & 0 \\ 0 & 0 & -0.2 \end{pmatrix}$$

$$\lim_{k \to \infty} (A^k U_0) = \lim_{k \to \infty} \left(S \Lambda^k S^{-1} \right) U_0$$

$$= \begin{pmatrix} -1/2 & 3/4 & -1 \\ -1/2 & 3/4 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 6 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -2/5 & -2/5 & 3/5 \\ 2/5 & 2/5 & 2/5 \\ -1/2 & 1/2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -1/2 & 3/4 & -1 \\ -1/2 & 3/4 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 2/5 & 2/5 & 2/5 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 6/20 & 6/20 & 6/20 \\ 6/20 & 6/20 & 6/20 \\ 2/5 & 2/5 & 2/5 \end{pmatrix}$$

5.3.11

$$U_\infty = \lim_{k \to \infty} (A^k) U_0$$

$$= \begin{pmatrix} 3/10 & 3/10 & 3/10 \\ 3/10 & 3/10 & 3/10 \\ 2/5 & 2/5 & 2/5 \end{pmatrix} \begin{pmatrix} 0 \\ 10 \\ 0 \end{pmatrix}$$

$$\boxed{U_\infty = \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix}}$$

$\leftarrow$ $U_\infty = (4)(V_2)$

$\therefore$ Steady state is eigenvector of $\lambda = 1$.