ECE 602: LUMPED LINEAR SYSTEMS

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Determining Local Stability from Linearized Dynamics

Hartman-Grobman Theorem

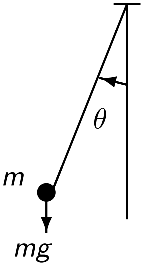
Theorem

Suppose $\dot{x} = f(x)$ with a smooth $f(\cdot)$ has an equilibrium point $x_{\rm e}$, and $Df(x_{\rm e})$ has no eigenvalue with real part equal to zero. Then, there exists a homeomorphism ϕ between a neighborhood $\mathcal{N}_e$ of x_e and a neighborhood $\mathcal{N}_0$ of 0 in $\mathbb{R}^n$ that maps solutions of $\dot{x} = f(x)$ inside $\mathcal{N}_e$ to solutions of $\dot{x} = Df(x_{\rm e})x$ inside $\mathcal{N}_0$.

- Thus, $\dot{x} = f(x)$ is locally asymptotically stable at x_e if and only if the linearized system $\dot{z} = Df(x_{\rm e})z$ is stable.
- Examples: Eigenvalues of $Df(x_{\rm e})$ are:
(i) $\{-1, -2 \pm j\}$; (ii) $\{1, -2 \pm j\}$; $\{\pm j, -2 \pm j\}$

- D. M. Grobman, "*Homeomorphisms of systems of differential equations*," Doklady Akademii Nauk SSSR. 128: 880–881, 1959.
- P. Hartman, "*On local homeomorphisms of Euclidean spaces*," Bol. Soc. Math. Mexicana. 5: 220–241, 1960

Example: Simple Pendulum



Dynamics: $\ddot{\theta} = -mg\ell \sin\theta - \eta\dot{\theta}$

- $\eta > 0$ is damping coefficient

State $x = \begin{bmatrix} \theta & \dot{\theta} \end{bmatrix}^T$ has dynamics

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = f(x) = \begin{bmatrix} x_2 \\ -mg\ell \sin x_1 - \eta x_2 \end{bmatrix}$$

Two equilibrium points $x_{\mathrm{e}1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $x_{\mathrm{e}2} = \begin{bmatrix} \pi \\ 0 \end{bmatrix}$, with linearized dynamics:

$$\frac{d}{dt}z(t) = \underbrace{\begin{bmatrix} 0 & 1 \\ -mg\ell & -\eta \end{bmatrix}}_{Df(x_{\mathrm{e}1})} z(t), \qquad \frac{d}{dt}z(t) = \underbrace{\begin{bmatrix} 0 & 1 \\ mg\ell & -\eta \end{bmatrix}}_{Df(x_{\mathrm{e}2})} z(t)$$

Example

Nonlinear system $\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 - x_1^3 - \alpha x_2 \end{cases}$ where $\alpha \neq 0$.

Inconclusive Cases

What if $Df(x_{\mathrm{e}})$ has eigenvalues on the $j\omega$-axis?

1. $\dot{x} = -x^3$

2. $\dot{x} = x^3$

3. Simple pendulum at $x_{\mathrm{e}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ without damping ($\eta = 0$):

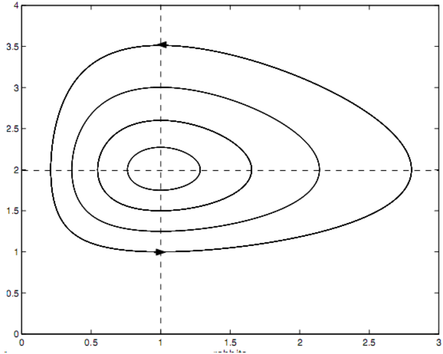
$$Df(x_{\mathrm{e}}) = \begin{bmatrix} 0 & 1 \\ -mg\ell & 0 \end{bmatrix}$$

Example: Lotka-Volterra Model

Population model of two species:

- x_1, x_2: populations of prey and predator
- Prey has unlimited food and predator total dependence on prey

$$\begin{cases} \frac{dx_1}{dt} & = 4x_1 - 2x_1x_2 \\ \frac{dx_2}{dt} & = -x_2 + x_1x_2 \end{cases} \text{ with equilibrium points } x_{\mathrm{e},1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \; x_{\mathrm{e},2} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$



Linearization of Controlled Nonlinear Systems

A controlled nonlinear time-invariant system

$$\dot{x}(t) = f\left(x(t), u(t)\right)$$

has an equilibrium point x_{e} if $f(x_{\mathrm{e}}, 0) = 0$.

- $x(t) \equiv x_{\mathrm{e}}$ is a solution under $u(t) \equiv 0$.
- If $x - x_{\mathrm{e}}$ and u are small, then $x - x_{\mathrm{e}} \approx z$ where z is the solution of

$$\dot{z} = \underbrace{\frac{\partial f}{\partial x} f(x_{\mathrm{e}}, 0)}_{A}\, z + \underbrace{\frac{\partial f}{\partial u} f(x_{\mathrm{e}}, 0)}_{B}\, u$$

Linearization around a Trajectory

Suppose the nonlinear time-varying system

$$\frac{d}{dt}x(t) = f(x, u, t), \quad x(0) = x_0, \qquad y(t) = g(x, u, t)$$

has (nominal) solutions $x^*(t)$ and $y^*(t)$ under nominal input $u^*(t)$

Suppose input is perturbed slightly: $u(t) = u^*(t) + \delta u(t)$. The resulting $x(t) = x^*(t) + \delta x(t)$ and $y(t) = y^*(t) + \delta y(t)$ satisfy approximately

$$\begin{cases} \frac{d}{dt}\delta x(t) = A(t)\delta x(t) + B(t)\delta u(t) \\ \delta y(t) = C(t)\delta x(t) + D(t)\delta u(t) \end{cases}$$

with

$$\begin{aligned} A(t) &= \tfrac{\partial}{\partial x} f\left(x^*(t), u^*(t), t\right), & B(t) &= \tfrac{\partial}{\partial u} f\left(x^*(t), u^*(t), t\right) \\ C(t) &= \tfrac{\partial}{\partial x} g\left(x^*(t), u^*(t), t\right) & D(t) &= \tfrac{\partial}{\partial u} g\left(x^*(t), u^*(t), t\right) \end{aligned}$$