Simultaneous Estimation of the State, Unknown Input, and Output Disturbance in Discrete-Time Linear Systems

Badriah Alenezi, Mukai Zhang, Stefen Hui, and Stanislaw H. Żak

***Abstract*—A state observer and unknown input and output disturbance estimators are proposed for discrete-time linear systems corrupted by bounded unknown inputs and output disturbances. Necessary and sufficient conditions for the existence of the state observer and disturbance estimators are given. Relationships with the strong observer of Hautus are investigated. The state, unknown input, and output disturbance estimation errors are guaranteed to be ℓ_∞-stable with prescribed performance level. The design of the state observer and disturbance estimators are given in terms of linear matrix inequalities. The proposed estimators can be applied to detect adversarial attacks on the communication channels between the controller and actuators and between the plant sensors and the controller.**

***Index Terms*—Discrete-time (DT) linear systems, output disturbance estimators, unknown input observers.**

I. Introduction

A. Motivation and Literature Overview

An essential requirement for a modern control system is its safe operation when it is subjected to different types of attacks. The attacks may affect the communication channels between the plant sensors and the controller and between the controller and actuators.

In this article, we propose unknown input and output disturbance estimators for the detection of attacks on the plant input and output channels. Attacks on the plant actuators are modeled as unknown plant inputs.

The problem of designing observers for linear systems with unknown inputs has been studied since 1969 [1], see also [2] for an overview of early UIO developments, and [3] for a comparative study of various UIO architectures. Applications of unknown input observers in fault detection were reported in [4]–[8], stress estimation in humans in [9], and secure state estimation in cyber-physical system (CPS) in [10]–[13].

In [14], state and unknown input observers for a class of discrete-time (DT) nonlinear systems that satisfy prescribed performance levels were given. In [15], a state observer was proposed for DT systems in the presence of identical disturbances to the sensors and actuators of the system. For the same type of plant models, a delayed DT UIO with prescribed performance level was constructed in [16]. In [17], a DT UIO was proposed to estimate the system state in the presence of unknown inputs.

In [18], designs of estimators and controllers were proposed for linear systems with system actuators or sensors under attack. A secure state estimator was proposed in [19], when the communication channel between a sensor and a remote estimator is corrupted by jamming attacks. In [20], distributed attacks detectors and distributed state estimators were proposed for networked CPS under malicious attacks. A secure state estimator of distributed power systems under cyber-physical attacks and communication failure was presented in [12]. In [10] and [19], the vector recovery method was applied to detect malicious packet drop attacks in a CPS. The main idea of that method is to transform the CPS state and unknown input estimation problem into a zero-norm minimization problem with equality constraints. The resulting problem is then solved using the one-norm approximation of the zero-norm minimization problem. However, in order to obtain an accurate approximation of zero-norm minimization by the one-norm minimization [21], the unknown disturbance vectors must be sparse, which limits the applicability of the method.

Our objective is to simultaneously estimate the state, unknown input, and output disturbance when the unknown input and output estimator are bounded but not necessarily sparse. The unknown input can represent the attacks between the controller and actuator, while the output disturbances can represent the attacks between the sensor and the controller.

B. Article Contributions and Notation

Our article's contributions are the following.

1) We propose novel unknown input and output disturbance estimator architectures for DT systems corrupted by bounded unknown inputs and output disturbances.
2) We present a design method of the proposed UIO and unknown input and output disturbance estimators in the linear matrix inequality (LMI) format.
3) We specify guaranteed performance for the proposed estimators.

In our analysis, we use the following notation. For a vector $v \in \mathbb{R}^n$, we use standard notation for the Euclidean norm of a vector: $\|v\| = \sqrt{v^\top v}$. For a sequence of vectors $\{v[k]\}_{k=0}^{\infty}$, we denote $\|v[k]\|_\infty \triangleq \sup_{k\geq 0} \|v[k]\|$. We say that a sequence $\{v[k]\} \in l_\infty$ if $\|v[k]\|_\infty < \infty$.

II. Problem Statement

We consider a class of DT dynamical systems modeled by

$$\left.\begin{aligned} x[k+1] &= Ax[k] + B_1 u[k] + B_2 w[k] \\ y[k] &= Cx[k] + Dv[k], \end{aligned}\right\} \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B_1 \in \mathbb{R}^{n\times m_1}$, $B_2 \in \mathbb{R}^{n\times m_2}$, $C \in \mathbb{R}^{p\times n}$, and $D \in \mathbb{R}^{p\times r}$. The control input is $u[k] \in \mathbb{R}^{m_1}$. The unknown input and output disturbance to the system are modeled by $w[k] \in \mathbb{R}^{m_2}$ and $v[k] \in \mathbb{R}^r$, respectively. (See, for example, [22, Sec. 1.1.2 and Ch. 2] or [23] for a discussion on the modeling of DT systems.)

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Badriah Alenezi is with the Department of Electrical Engineering, Kuwait University, Safat 13060, Kuwait (e-mail: badriah.alenezi@ku.edu.kw).

Mukai Zhang and Stanislaw H. Żak are with the School of Electrical and Computer Engineering, Purdue University, West Lafayette, IN 47907 USA (e-mail: 1624837930@qq.com; zak@purdue.edu).

Stefen Hui is with the Department of Mathematics, San Diego State University, San Diego, CA 92182 USA (e-mail: shui@mail.sdsu.edu).

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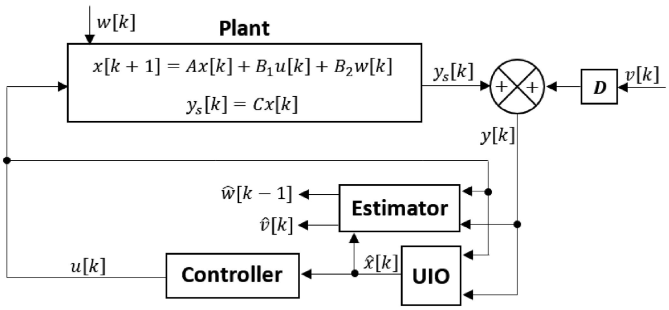

Fig. 1. Combined UIO-controller compensator and an estimator of unknown input and output disturbance for system modeled by (1).

Our objective is to construct an observer to estimate the system state in the presence of unknown input $w[k]$ and output disturbance $v[k]$. In addition, we want to obtain estimates of the unknown input and output disturbances.

We make the following assumptions.

Assumption 1: The pair (A, C) is detectable.

Assumption 2: The matrices B_2 and D have full column rank.

Assumption 3: The unknown sequences of inputs $\{w[k]\}_{k=0}^{\infty}$ and outputs $\{v[k]\}_{k=0}^{\infty}$ are in ℓ_∞, or in other words, $w[k]$ and $v[k]$ are uniformly bounded as functions of k.

III. Proposed UIO Architecture

In this section, we propose an observer architecture to estimate the state of the system given by (1). The estimated state is then used to synthesize a combined UIO-controller compensator as shown in Fig. 1.

We begin by representing $x[k]$ as

$$\begin{aligned} x[k] &= x[k] - MCx[k] + MCx[k] \\ &= (I - MC)x[k] + M(y[k] - Dv[k]) \end{aligned} \tag{2}$$

where $M \in \mathbb{R}^{n\times p}$ is to be determined. We select M such that

$$MD = O_{n\times r} \tag{3}$$

where $O_{n\times r}$ is an n-by-r matrix of zeros. We will discuss a method for finding M in the following section.

To proceed, let $z[k] = (I - MC)x[k]$. Taking into account (3), we represent (2) as

$$x[k] = z[k] + My[k]. \tag{4}$$

We will now show that an estimate of the state $x[k]$ can be obtained from

$$\hat{x}[k] = z[k] + My[k]. \tag{5}$$

The signal $z[k]$ is obtained from the equation, $z[k+1] = (I - MC)x[k+1]$. Substituting (1) into the above gives

$$z[k+1] = (I - MC)(Ax[k] + B_1u[k] + B_2w[k]). \tag{6}$$

We select M such that (3) is satisfied and

$$(I - MC)B_2 = O_{n\times m_2}. \tag{7}$$

Next, substituting (4) and (7) into (6), we obtain

$$z[k+1] = (I - MC)(Az[k] + AMy[k] + B_1u[k]).$$

To proceed, let $e[k] = x[k] - \hat{x}[k]$ be the state estimation error. Performing some manipulations gives

$$e[k+1] = (I - MC)Ae[k].$$

We can see from the above that we do not have any control over the estimation error convergence dynamics, which is determined by the matrix $(I - MC)A$. To improve the estimation error convergence dynamics, we add the innovation term $L(y[k] - \hat{y}[k])$, where $L \in \mathbb{R}^{n\times p}$ and $\hat{y}[k] = C\hat{x}[k] = C(z[k] + My[k])$. Then

$$z[k+1] = (I - MC)(A\hat{x}[k] + B_1u[k]) + L(y[k] - \hat{y}[k]). \tag{8}$$

Combining (5) and (8), we obtain the proposed UIO architecture

$$\left.\begin{aligned} z[k+1] =\ & (I - MC)(Az[k] + AMy[k] + B_1u[k]) \\ & + L(y[k] - Cz[k] - CMy[k]) \\ \hat{x}[k] =\ & z[k] + My[k]. \end{aligned}\right\} \tag{9}$$

The error dynamics for the UIO given by (9) are

$$\begin{aligned} e[k+1] &= x[k+1] - \hat{x}[k+1] \\ &= Ax[k] + B_1u[k] + B_2w[k] - z[k+1] - My[k+1]. \end{aligned}$$

Substituting (8) and $\hat{y}[k] = C(z[k] + My[k])$ into the above gives

$$\begin{aligned} e[k+1] = Ax[k] &+ B_1u[k] + B_2w[k] - (I - MC)(A\hat{x}[k] \\ &+ B_1u[k]) - L(y[k] - Cz[k] - CMy[k]) \\ &- M(Cx[k+1] + Dv[k+1]). \end{aligned}$$

By substituting the state dynamics of (1), (3), and (5) into the above, we obtain

$$\begin{aligned} e[k+1] = Ax[k] &+ B_1u[k] + B_2w[k] - (I - MC)(A\hat{x}[k] \\ &+ B_1u[k]) - L(y[k] - C\hat{x}[k]) \\ &- MC(Ax[k] + B_1u[k] + B_2w[k]). \end{aligned}$$

Performing some manipulations and taking into account (7) and the output dynamics of (1) gives $e[k+1] = ((I - MC)A - LC)e[k] - LDv[k]$. Let $\tilde{A} = (I - MC)A$. Then

$$e[k+1] = (\tilde{A} - LC)e[k] - LDv[k]. \tag{10}$$

Remark 1: Note that if an L exists such that $(\tilde{A} - LC)$ is Schur stable and $LD = O_{n\times r}$, then the error dynamics in (10) are asymptotically stable. However, for general systems, it may not be feasible to find a gain matrix L that simultaneously satisfies both conditions. For this reason, we do not impose the constraint $LD = O_{n\times r}$ in our development. We present conditions for the stability of the error dynamics (10) and conditions for the existence of the UIO gain matrix L in Section VII.

In the following section, we give an existence condition for a matrix M to satisfy (3) and (7).

IV. Solving for M

In this section, we present necessary and sufficient conditions for the existence of a matrix M with the required properties, that is, conditions for the existence of M that solves matrix equations (3) and (7) simultaneously.

Theorem 1: There exists a solution M to

$$(I - MC)B_2 = O_{n\times m_2} \tag{11a}$$

$$MD = O_{n\times r} \tag{11b}$$

if and only if

$$\operatorname{rank}\begin{bmatrix} CB_2 & D \\ B_2 & O_{n\times r} \end{bmatrix} = \operatorname{rank}\begin{bmatrix} CB_2 & D \end{bmatrix}. \tag{12}$$

Proof: We represent (11a) and (11b) as

$$M\begin{bmatrix} CB_2 & D \end{bmatrix} = \begin{bmatrix} B_2 & O_{n\times r} \end{bmatrix}. \tag{13}$$

A necessary and sufficient condition for M to solve (13) is that the space spanned by the rows of the matrix $[B_2 \quad O_{n\times r}]$ is in the range of the space spanned by the rows of the matrix $[CB_2 \quad D]$. This is equivalent to (12), which concludes the proof. ∎

We next express condition (12) in an equivalent form.

Theorem 2: Equation (13) has a solution, if and only if

$$\operatorname{rank}\begin{bmatrix} CB_2 & D \end{bmatrix} = \operatorname{rank}(B_2) + \operatorname{rank}(D). \tag{14}$$

Proof: We have

$$\begin{aligned} \operatorname{rank}\begin{bmatrix} CB_2 & D \end{bmatrix} &= \operatorname{rank}\begin{bmatrix} CB_2 & D \\ B_2 & O \end{bmatrix} \\ &= \operatorname{rank}\left(\begin{bmatrix} I_p & -C \\ O & I_n \end{bmatrix}\begin{bmatrix} CB_2 & D \\ B_2 & O \end{bmatrix}\right) \\ &= \operatorname{rank}\begin{bmatrix} O & D \\ B_2 & O \end{bmatrix} = \operatorname{rank}(B_2) + \operatorname{rank}(D) \end{aligned}$$

which completes the proof. ∎

By Assumption 2 and condition (14), the matrix $[CB_2 \;\; D]$ has full column rank and is therefore left invertible. For example, $[CB_2 \;\; D]^\dagger$, where $[\cdot]^\dagger$ denotes the Moore–Penrose pseudoinverse, is a left inverse of $[CB_2 \;\; D]$. Therefore

$$M = [B_2 \;\; O_{n\times r}][CB_2 \;\; D]^\dagger \tag{15}$$

is a solution to (11a) and (11b).

Note that a class of solutions to (11a) and (11b) has the form

$$\begin{aligned} M = [B_2 \;\; O_{n\times r}]\big(&[CB_2 \;\; D]^\dagger \\ &+H_0(I - [CB_2 \;\; D][CB_2 \;\; D]^\dagger)\big) \end{aligned} \tag{16}$$

where $H_0 \in \mathbb{R}^{(m_2+r)\times p}$ is a design parameter matrix.

Remark 2: For the matrix $[CB_2 \;\; D]$ to have full column rank, it is necessary that $r \le p - m_2$, which means that there should be at least as many outputs as the combined number of unknown input and output disturbances.

V. UIO Synthesis Conditions

In our further discussion, we use the following lemma.

Lemma 1: If the pair $(\tilde{A}, MC)$ is detectable, then the pair $(\tilde{A}, C)$ is detectable. Furthermore, if M has full column rank, then the converse is also true. (Note that in our application, we need $MD = O$ and thus unless D is the zero matrix and $p = m_2$, M can never have full column rank.)

Proof: We prove the lemma by contraposition. Assume that the pair $(\tilde{A}, C)$ is nondetectable. Then, there exists an eigenvalue $|z_1| \ge 1$ such that $\operatorname{rank}\left[\begin{smallmatrix} z_1 I - \tilde{A} \\ C \end{smallmatrix}\right] < n$. Therefore, there exist a vector $v_1 \in \mathbb{C}^n$ such that $\left[\begin{smallmatrix} z_1 I - \tilde{A} \\ C \end{smallmatrix}\right] v_1 = 0$, and hence, $Cv_1 = 0$. Premultiplying the above equation by M gives $MCv_1 = 0$. We conclude from the above that $\left[\begin{smallmatrix} z_1 I - \tilde{A} \\ MC \end{smallmatrix}\right] v_1 = 0$. Thus, z_1 also an unobservable eigenvalue of the pair $(\tilde{A}, MC)$, that is, the pair $(\tilde{A}, MC)$ is nondetectable.

Note that

$$\begin{bmatrix} z_1 I - \tilde{A} \\ MC \end{bmatrix} = \begin{bmatrix} I & O \\ O & M \end{bmatrix}\begin{bmatrix} z_1 I - \tilde{A} \\ C \end{bmatrix}$$

and if M has full column rank, then $M^\dagger$ is its left pseudo-inverse and

$$\begin{bmatrix} z_1 I - \tilde{A} \\ C \end{bmatrix} = \begin{bmatrix} I & O \\ O & M^\dagger \end{bmatrix}\begin{bmatrix} z_1 I - \tilde{A} \\ MC \end{bmatrix}.$$

Hence, if M has full column rank, then the pair $(\tilde{A}, MC)$ is detectable $\iff$ the pair $(\tilde{A}, C)$ is detectable.

We now present a theorem that gives necessary and sufficient conditions for the existence of the proposed UIO.

Theorem 3: The proposed UIO given by (9) exists if and only if:

1) condition (14) is satisfied;
2) the pair $(\tilde{A}, C) = ((I - MC)A, C)$ is detectable.

Proof: By Theorem 1, condition (14) is necessary and sufficient for the existence of M that solves (11).

The detectability of the pair $(\tilde{A}, C)$ is necessary and sufficient for the existence of L such that $(\tilde{A} - LC)$ is Schur stable.

The following lemma gives conditions for the detectability of the pair $(\tilde{A}, MC)$ of the system given by:

$$\left.\begin{aligned} x[k+1] &= Ax[k] + B_1 u[k] + B_2 w[k] \\ My[k] &= MCx[k] + MDv[k]. \end{aligned}\right\} \tag{17}$$

We will use this lemma in our further analysis. To proceed, note that by construction, $MD = O$.

Lemma 2: If

1) $\operatorname{rank}(CB_2) = \operatorname{rank}(B_2) = m_2$;
2) $\operatorname{rank}(I_n - MC) = n - m_2$.

Then the following conditions are equivalent:

1) $(\tilde{A}, MC)$ is detectable;
2) $\operatorname{rank}\left[\begin{smallmatrix} z(I_n - MC) - \tilde{A} \\ MC \end{smallmatrix}\right] = n$ for all $|z| \ge 1$;
3) $\operatorname{rank}\left[\begin{smallmatrix} zI_n - \tilde{A} & -B_2 \\ MC & O_{n\times m_2} \end{smallmatrix}\right] = n + m_2$ for all $|z| \ge 1$.

Proof: First, we show that conditions 1 and 2 are equivalent. The pair $(\tilde{A}, MC)$ being detectable is equivalent to

$$\operatorname{rank}\begin{bmatrix} zI_n - \tilde{A} \\ MC \end{bmatrix} = n \text{ for all } |z| \ge 1$$

which is equivalent to

$$\begin{aligned} n &= \operatorname{rank}\left(\begin{bmatrix} I_n & -zI_n \\ O & I_n \end{bmatrix}\begin{bmatrix} zI_n - \tilde{A} \\ MC \end{bmatrix}\right) \\ &= \operatorname{rank}\begin{bmatrix} z(I_n - MC) - \tilde{A} \\ MC \end{bmatrix}, \text{ for all } |z| \ge 1 \end{aligned}$$

which proves that conditions 1 and 2 are equivalent.

Next, we will show that conditions 2 and 3 are equivalent. Since B_2 has full column rank, then ${B_2}^\dagger$ is such that ${B_2}^\dagger B_2 = I_{m_2}$. Then $\ker({B_2}^\dagger) \cap \ker(I_n - MC) = \{0\}$ and hence $\operatorname{rank}\left[\begin{smallmatrix} I_n - MC \\ {B_2}^\dagger \end{smallmatrix}\right] = n$. Let

$$S = \begin{bmatrix} I_n - MC & O_{n\times n} \\ {B_2}^\dagger & O_{m_2\times n} \\ O_{n\times n} & I_n \end{bmatrix}, \; T = \begin{bmatrix} I_n & O_{n\times m_2} \\ {B_2}^\dagger(zI_n - A) & I_{m_2} \end{bmatrix}.$$

Then $\operatorname{rank}(S) = 2n$ and $\operatorname{rank}(T) = n + m_2$. Since $S \in \mathbb{R}^{(2n+m_2)\times(2n)}$, and $T \in \mathbb{R}^{(n+m_2)\times(n+m_2)}$, we conclude that S has full column rank and T has full rank. We have, using the above and $(I - MC)B_2 = O$

$$\begin{aligned}\operatorname{rank}\begin{bmatrix} zI_n - A & -B_2 \\ MC & O\end{bmatrix} &= \operatorname{rank}\left(S\begin{bmatrix} zI_n - A & -B_2 \\ MC & O\end{bmatrix}T\right)\\ &= \operatorname{rank}\begin{bmatrix} z(I_n - MC) - \tilde{A} & O \\ O & -I_{m_2} \\ MC & O\end{bmatrix}\\ &= \operatorname{rank}\begin{bmatrix} z(I_n - MC) - \tilde{A} \\ MC\end{bmatrix} + m_2.\end{aligned}$$

This shows that conditions 2 and 3 are equivalent.

The following theorem gives us some insight into the role of system zeros on the existence of the proposed UIO.

Theorem 4: If

1) the matrix rank condition (14) is satisfied;
2) the matrix $\left[\begin{smallmatrix} -B_2 \\ D\end{smallmatrix}\right]$ is defined and has full column rank;
3) $\operatorname{rank}\left[\begin{smallmatrix} I-MC & O \\ O & M\end{smallmatrix}\right] = n$;
4) $\operatorname{rank}\left[\begin{smallmatrix} z(I_n-MC)-\tilde{A} \\ MC\end{smallmatrix}\right] = n$ for all $|z| \geq 1$.

Then

$$\operatorname{rank}\begin{bmatrix} zI_n - A & -B_2 \\ C & D\end{bmatrix} = n + m_2 \text{ for all } |z| \geq 1. \tag{18}$$

Proof: By Theorem 1, if matrix rank condition (14) is satisfied then there exists a solution M that satisfies

$$\begin{bmatrix} I - MC & O \\ O & M\end{bmatrix}\begin{bmatrix} -B_2 \\ D\end{bmatrix} = O.$$

Let $\tilde{M} = \left[\begin{smallmatrix} I-MC & O \\ O & M\end{smallmatrix}\right]$. There exists $M_1 \in \mathbb{R}^{(p-m_2)\times(n+p)}$ such that $M_1\begin{bmatrix} -B_2 \\ D\end{bmatrix} = O$ and $\operatorname{rank}\begin{bmatrix}\tilde{M} \\ M_1\end{bmatrix} = n+p-m_2$. Then since $\begin{bmatrix} -B_2 \\ D\end{bmatrix}$ has full column rank, it has a left inverse $\begin{bmatrix} -B_2 \\ D\end{bmatrix}^{\dagger}$. It is then elementary that $\ker\begin{bmatrix}\tilde{M} \\ M_1\end{bmatrix} \cap \ker\begin{bmatrix} -B_2 \\ D\end{bmatrix}^{\dagger} = \{0\}$ and $\operatorname{rank}\begin{bmatrix}\tilde{M} \\ M_1 \\ \begin{bmatrix} -B_2 \\ D\end{bmatrix}^{\dagger}\end{bmatrix} = n+p$. Let

$$S = \begin{bmatrix}\tilde{M} \\ M_1 \\ \begin{bmatrix} -B_2 \\ D\end{bmatrix}^{\dagger}\end{bmatrix},\quad T = \begin{bmatrix} I_n & O \\ -\begin{bmatrix} -B_2 \\ D\end{bmatrix}^{\dagger}\begin{bmatrix} zI_n - A \\ C\end{bmatrix} & I_{m_2}\end{bmatrix}.$$

Then

$$\begin{aligned}\operatorname{rank} S\begin{bmatrix} zI_n - A & -B_2 \\ C & D\end{bmatrix}T &= \operatorname{rank}\begin{bmatrix}\tilde{M}\begin{bmatrix} zI_n - A \\ C\end{bmatrix} & O \\ M_1\begin{bmatrix} zI_n - A \\ C\end{bmatrix} & O \\ O & I_{m_2}\end{bmatrix}\\ &= \operatorname{rank}\begin{bmatrix}\tilde{M}\begin{bmatrix} zI_n - A \\ C\end{bmatrix} \\ M_1\begin{bmatrix} zI_n - A \\ C\end{bmatrix}\end{bmatrix} + m_2.\end{aligned} \tag{19}$$

Note that $\tilde{M}\begin{bmatrix} zI_n - A \\ C\end{bmatrix} = \begin{bmatrix} z(I-MC) - \tilde{A} \\ MC\end{bmatrix}$. Hence (18) holds if

$$\operatorname{rank}\begin{bmatrix} z(I-MC) - \tilde{A} \\ MC\end{bmatrix} = n, \text{ for all } |z| \geq 1.$$

VI. Relations With the Strong Observer of Hautus

In this section, we discuss relationships between our UIO existence conditions and the strong observer existence conditions of Hautus [24]. Our conditions are applicable to a general class of linear systems when the unknown input and output disturbance are different as given by (1). Hautus, on the other hand, gives the strong observer existence conditions for a class of linear systems with the same unknown input and output disturbance. The Hautus' necessary and sufficient conditions for the existence of his strong observer are

$$\operatorname{rank}\begin{bmatrix} CB_2 & D \\ D & O\end{bmatrix} = \operatorname{rank}(D) + \operatorname{rank}\begin{bmatrix} B_2 \\ D\end{bmatrix} \tag{20}$$

and system zeros of the system defined by quadruple (A, B_2, C, D) are in the open unit disk.

Since our plant model is more general than that of Hautus, we cannot apply his theorem directly. However, our plant model can be put into the Hautus form as

$$\left.\begin{aligned} x[k+1] &= Ax[k] + B_1u[k] + \begin{bmatrix} B_2 & O\end{bmatrix}\begin{bmatrix} w[k] \\ v[k]\end{bmatrix} \\ y[k] &= Cx[k] + \begin{bmatrix} O & D\end{bmatrix}\begin{bmatrix} w[k] \\ v[k]\end{bmatrix}\end{aligned}\right\} \tag{21}$$

where O denote zero matrices of zeros. We will show that if a Hautus observer can be constructed for this model, then so can our observer but we will give an example where our observer can be constructed but not a Hautus observer.

Suppose the Hautus conditions are satisfied. Then applying (20) to (21) gives

$$\operatorname{rank}\begin{bmatrix} CB_2 & O & O & D \\ O & D & O & O\end{bmatrix} = \operatorname{rank}\begin{bmatrix} O & D\end{bmatrix} + \operatorname{rank}\begin{bmatrix} B_2 & O \\ O & D\end{bmatrix} \tag{22}$$

which clearly is the same as the matrix rank condition (14).

Next, the system zeros condition for the system of (21) is

$$\operatorname{rank}\begin{bmatrix} zI - A & -B_2 & O \\ C & O & D\end{bmatrix} = n + m_2 + r \text{ for all } |z| \geq 1. \tag{23}$$

If the rank condition (18) is not satisfied, and B_2 and D have the same number of columns, then there are β_1, β_2, not both zero, such that $\begin{bmatrix} zI - A & -B_2 \\ C & D\end{bmatrix}\begin{bmatrix}\beta_1 \\ \beta_2\end{bmatrix} = 0$. Then clearly $\begin{bmatrix} zI - A & -B_2 & O \\ C & O & D\end{bmatrix}\begin{bmatrix}\beta_1 \\ \beta_2 \\ \beta_2\end{bmatrix} = 0$. It follows that condition (23) implies the system zeros condition (18). Therefore the Hautus existence conditions imply our existence conditions.

The following example shows that there are general systems given by (1) such that our proposed UIO can be constructed but the strong observer of Hautus cannot be constructed for the equivalent augmented system given by (21).

Example 1: We consider a plant model (1) with

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -0.3\end{bmatrix},\quad B_2 = \begin{bmatrix} -2 \\ -3 \\ -4\end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, D = \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

Recall that in the plant model given by (1), the unknown input $w[k]$ is different from the output disturbance $v[k]$.

To proceed, we first verify the UIO existence conditions given in Theorem 3. The matrix rank condition (14) is satisfied with

$$\text{rank}\begin{bmatrix} CB_2 & D \end{bmatrix} = \text{rank}\begin{bmatrix} -2 & 2 \\ -3 & 2 \end{bmatrix} = \text{rank}(B_2) + \text{rank}(D) = 2.$$

We solve (15) to obtain $M = \begin{bmatrix} -2 & 2 \\ -3 & 3 \\ -4 & 4 \end{bmatrix}$. Next, we construct the matrix $\tilde{A} = (I_3 - MC)A$ to obtain $\tilde{A} = \begin{bmatrix} -3 & 4 & 0 \\ -3 & 4 & 0 \\ -4 & 8 & -0.3 \end{bmatrix}$.

It is easy to check that the pair $(\tilde{A}, C)$ is detectable. Hence, there exists a matrix L such that $(\tilde{A} - LC)$ is Schur stable. Therefore, both existence conditions of our proposed observer are satisfied.

We next represent the system of this example in the Hautus form given by (21) and check the Hautus existence conditions. It is easy to check that the matrix rank condition of Hautus is satisfied. However, the system zeros condition is not satisfied for $z_1 = 1$. Indeed

$$\det\begin{bmatrix} I_3 - A & -B & O \\ C & O & D \end{bmatrix} = 0.$$

In conclusion, one can construct our proposed observer for this example but not the strong observer of Hautus using (21).

The following theorem shows that the matrix rank condition for the existence of our proposed UIO is also sufficient for the matrix rank condition of the strong observer of Hautus.

Theorem 5: The matrix rank condition (14) implies the matrix rank condition (20) of Hautus.

Proof: See the Appendix. ■

In the next section, we analyze the stability of the error dynamics given by (10).

VII. Stability of the Error Dynamics

In this section, we analyze the error dynamics stability and give the conditions for finding the UIO gain matrix L in terms of LMIs. To proceed, we define ℓ_∞-stability with performance level γ following [14].

Definition 1: The error system, $e[k+1] = f(k, e[k], v[k])$, is globally uniformly ℓ_∞-stable with performance level γ if the following conditions are satisfied.

1) The undisturbed system, (that is, $v[k] = 0$ for all $k \geq 0$) is globally uniformly exponentially stable with respect to the origin.
2) For zero initial condition, $e[k_0] = 0$, and every bounded unknown input $v[k]$, we have $\|e[k]\| \leq \gamma \|v[k]\|_\infty$ for all $k \geq k_0$.
3) For every initial condition, $e[k_0] = e_0$, and every bounded unknown input $v[k]$, we have $\limsup_{k\to\infty} \|e[k]\| \leq \gamma \|v[k]\|_\infty$.

We now present a lemma from [14] that we use in our proof of the theorem that gives the design condition of the proposed UIO in LMI format.

Lemma 3: Suppose that for the error dynamics, $e[k+1] = f(k, e[k], v[k])$, there exists a function $V : \mathbb{R}^n \to \mathbb{R}$ and scalars $\alpha \in (0,1)$, $\beta_1, \beta_2 > 0$ and $\mu_1, \mu_2 \geq 0$ such that

$$\beta_1 \|e[k]\|^2 \leq V(e[k]) \leq \beta_2 \|e[k]\|^2 \tag{24}$$

and

$$V(e[k+1]) - V(e[k]) \leq -\alpha(V(e[k]) - \mu_1 \|v[k]\|^2) \tag{25a}$$

$$\|e[k]\|^2 \leq \mu_2 V(e[k]) \tag{25b}$$

for all $k \geq 0$. Then the error system is globally uniformly ℓ_∞-stable with performance level $\gamma = \sqrt{\mu_1 \mu_2}$ with respect to the output disturbance $v[k]$.

To proceed, we consider the error dynamics equation given by (10). Let $E = \tilde{A} - LC$, and $N = -LD$. Then, we have the following theorem.

Theorem 6: Suppose Assumption 3 is satisfied. If there exist matrices $P = P^\top \succ 0$, L, and $\alpha \in (0,1)$ such that

$$\begin{bmatrix} E^\top PE - (1-\alpha)P & * \\ N^\top PE & N^\top PN - \alpha I \end{bmatrix} \preceq 0 \tag{26}$$

then the state observation error dynamics are ℓ_∞-stable with performance level $\gamma = 1/\sqrt{\lambda_{\min}(P)}$.

Proof: We verify the conditions of Lemma 3. Since $P = P^\top \succ 0$, conditions (24) and (25b) are satisfied with $\beta_1 = \lambda_{\min}(P)$, $\beta_2 = \lambda_{\max}(P)$, and $\mu_2 = 1/\lambda_{\min}(P)$.

Let $V(e[k]) = e[k]^\top Pe[k]$ be a Lyapunov function candidate for the estimation error dynamics given by (10) and let $\Delta[k] = V(e[k+1]) - V(e[k])$. Then

$$\Delta[k] = e[k]^\top (E^\top PE - P)e[k] + 2e[k]^\top E^\top PNv[k] + v[k]^\top N^\top PNv[k].$$

Let $\zeta = [e[k]^\top \ v[k]^\top]^\top$. Premultiplying and postmultiplying the matrix inequality (26) by $\zeta^\top$ and ζ, respectively, and taking into account the above equality, we obtain $\Delta[k] + \alpha(V(e[k]) - \|v[k]\|^2) \preceq 0$. Therefore, condition (25a) in Lemma 3 holds with $\mu_1 = 1$ and the observer error given by (10) satisfies

$$\limsup_{k\to\infty} \|e[k]\| \leq \gamma \limsup_{k\to\infty} \|v[k]\|_\infty$$

where $\gamma = 1/\sqrt{\lambda_{\min}(P)}$. In summary, the state error dynamics (10) are ℓ_∞-stable with performance level γ.

We present a method to solve matrix inequality (26) in Theorem 6 using an LMI. Let $Z = PL$. Then solving the matrix inequality (26) is equivalent to solving the following LMI for P and Z:

$$\begin{bmatrix} -P & * \\ \Omega_{21} & \Omega_{22} \end{bmatrix} \preceq 0 \tag{27}$$

where

$$\Omega_{21} = \begin{bmatrix} \tilde{A}^\top P - C^\top Z^\top \\ -D^\top Z^\top \end{bmatrix}, \ \Omega_{22} = \begin{bmatrix} -(1-\alpha)P & O_{n\times r} \\ O_{r\times n} & -\alpha I \end{bmatrix}.$$

Since $P = P^\top \succ 0$, taking the Schur complement of (27) gives

$$\Omega_{22} + \Omega_{21} P^{-1} \Omega_{21}^\top \preceq 0$$

which, in turn, yields matrix inequality (26).

VIII. Unknown Input and Output Disturbance Reconstruction

In this section, we propose estimators for the unknown input $w[k]$ and the output disturbance $v[k]$ of system model (1).

A. Unknown Input Reconstruction

By Assumption 2, the matrix B_2 has full column rank, therefore, $B_2^\dagger = (B_2^\top B_2)^{-1}(B_2)^\top$ exists. Premultiplying both sides of the state dynamics given by (1) by the matrix $B_2^\dagger$, we obtain $B_2^\dagger x[k+1] = B_2^\dagger Ax[k] + B_2^\dagger B_1 u[k] + B_2^\dagger B_2 w[k]$. Since $B_2^\dagger B_2 = I_{m_2}$, we rewrite the above equation as $w[k] = B_2^\dagger x[k+1] - B_2^\dagger Ax[k] - B_2^\dagger B_1 u[k]$.

Using this equation, we obtain the following unknown input estimator, $\hat{w}[k] = B_2^{\dagger}\hat{x}[k+1] - B_2^{\dagger}A\hat{x}[k] - B_2^{\dagger}B_1 u[k]$. The above unknown input estimator depends on $\hat{x}[k+1]$. Therefore, we delay the argument by one sampling period time-delay to obtain our proposed estimator of $w[k]$ of the form

$$\hat{w}[k-1] = B_2^{\dagger}\hat{x}[k] - B_2^{\dagger}A\hat{x}[k-1] - B_2^{\dagger}B_1 u[k-1]. \tag{28}$$

To prove ℓ_∞-stability of the unknown input estimates, we let $e_w[k] = w[k] - \hat{w}[k]$ be the unknown input estimation error. Then, we have $e_w[k] = B_2^{\dagger}e[k+1] - B_2^{\dagger}Ae[k]$. By Theorem 6, we have $\limsup_{k\to\infty}\|e[k]\| \le \gamma\|v[k]\|_\infty$. Hence the bound on the unknown input estimation steady-state error satisfies

$$\begin{aligned}\limsup_{k\to\infty}\|e_w[k]\| &\le \|B_2^{\dagger}\|(\gamma\|v[k+1]\|_\infty + \|A\|\gamma\|v[k]\|_\infty)\\ &\le \|B_2^{\dagger}\|(1+\|A\|)\sqrt{\mu_2}\|v[k]\|_\infty.\end{aligned}$$

Thus, the unknown input estimator estimates the unknown input with performance level $\gamma_w = \|B_2^{\dagger}\|(1+\|A\|)\sqrt{\mu_2}$.

B. Output Disturbance Reconstruction

By Assumption 2, the matrix D has full column rank, therefore it has left inverse. Premultiplying both sides of the output equation of model (1) by $D^{\dagger}$ gives $D^{\dagger}y[k] = D^{\dagger}Cx[k] + D^{\dagger}Dv[k]$. Rearranging the above equation, we obtain $v[k] = D^{\dagger}y[k] - D^{\dagger}Cx[k]$. From the above, we obtain the output disturbance estimator

$$\hat{v}[k] = D^{\dagger}y[k] - D^{\dagger}C\hat{x}[k]. \tag{29}$$

To prove the ℓ_∞-stability of the output disturbance estimate, we let $e_v[k] = v[k] - \hat{v}[k]$ be the output disturbance estimation error. Then, we have $e_v[k] = -D^{\dagger}Ce[k]$. By Theorem 6

$$\limsup_{k\to\infty}\|e[k]\| \le \gamma\|v[k]\|_\infty.$$

We use this to obtain a bound on the output disturbance steady-state estimation error

$$\begin{aligned}\limsup_{k\to\infty}\|e_v[k]\| &\le \|D^{\dagger}\|\|C\|\gamma\|v[k]\|_\infty\\ &\le \|D^{\dagger}\|\|C\|\sqrt{\mu_2}\|v[k]\|_\infty.\end{aligned}$$

Thus, the performance level of the output disturbance estimator is $\gamma_v = \|D^{\dagger}\|\|C\|\sqrt{\mu_2}$.

IX. Example

We consider a DT dynamical system model with

$$A = \begin{bmatrix}0.5 & 0 & 0\\ 0 & 0.5 & 0\\ 0 & 0 & 0.5\end{bmatrix},\quad B_2 = \begin{bmatrix}1\\1\\0\end{bmatrix}$$

$$C = \begin{bmatrix}0 & 2 & 1\\ 1 & 0 & 0\end{bmatrix},\ D = \begin{bmatrix}0\\0.2\end{bmatrix}.$$

In this example, the control input is set to zero. The matrix rank condition (14) is satisfied, thus we can proceed with our proposed UIO design. Solving (15) for M yields $M = \begin{bmatrix}0.5 & 0\\ 0.5 & 0\\ 0 & 0\end{bmatrix}$. Next, we construct the matrix $\tilde{A} = (I_3 - MC)A$ to obtain $\tilde{A} = \begin{bmatrix}0.5 & -0.5 & -0.25\\ 0 & 0 & -0.25\\ 0 & 0 & 0.5\end{bmatrix}$. It is easy to check that the pair $(\tilde{A}, C)$ is detectable.

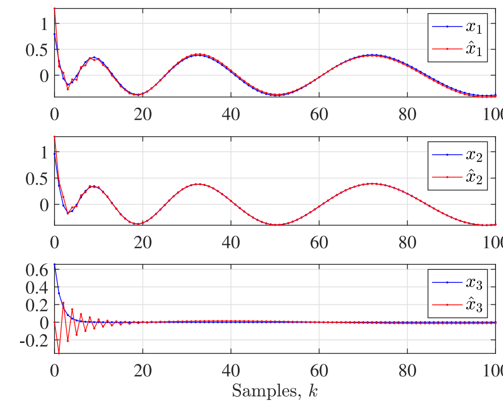

Fig. 2. State and its estimate.

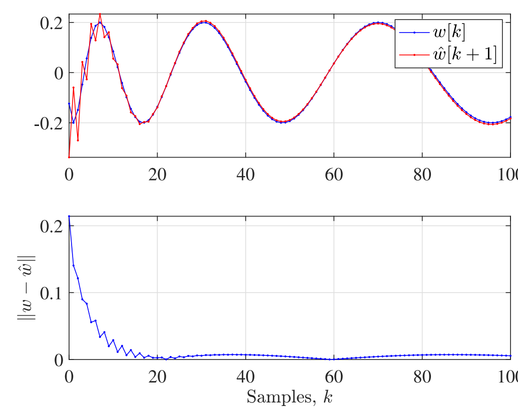

Fig. 3. Top plot shows the unknown input and its estimate. Bottom plot shows the unknown input reconstruction error norm.

We select $\alpha = 0.95$ and solve LMI (27) to obtain $P = P^\top \succ 0$, where the eigenvalues of P are

$$10^{10} \times \{\,0.713\,221\ 0.910\,700\ 6.899\,859\,\}$$

and the observer gain matrix

$$L = \begin{bmatrix}-0.066\,086 & 0.783\,868\\ -0.151\,477 & -0.027\,697\\ 0.739\,215 & 1.017\,137\end{bmatrix}.$$

The performance level is $\gamma = 1/\sqrt{\lambda_{\min}(P)} = 1.184\,099 \times 10^{-5}$.

In our simulation, we randomly generated the state initial condition $x[0] = \begin{bmatrix}0.967\,710 & 0.086\,768 & 0.173\,477\end{bmatrix}^\top$ and set the observer state initial condition and $\hat{x}[-1]$ to be zero. We used as unknown input and output disturbances $w[k] = 0.2\cos\sqrt{5k}$ and $v[k] = 0.25\cos\sqrt{k}$. Fig. 2 shows that the states are being estimated correctly with negligible estimation error. Figs. 3 and 4 show plots of the unknown input and the output disturbances and their estimates.

Remark 3: In this example, it is possible to find gain matrix L such that $(\tilde{A} - LC)$ is Schur stable and $LD = O$.

Appendix

We will use the following notation for the various matrix rank conditions:

1) $\mathcal{S} \iff \operatorname{rank}\begin{bmatrix}CB_2\end{bmatrix} = \operatorname{rank}B_2$;

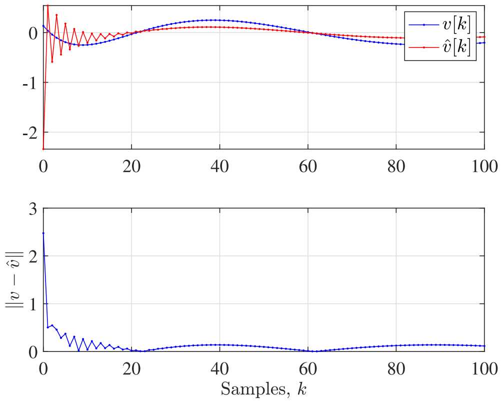

Fig. 4. Top plot shows the sensor disturbance and its estimate. Bottom plot shows the output disturbance reconstruction error norm.

2) $\mathcal{G} \iff \operatorname{rank}\begin{bmatrix} CB_2 & D \end{bmatrix} = \operatorname{rank} B_2 + \operatorname{rank} D$;
3) $\mathcal{M} \iff \operatorname{rank}(CB_2 + D) = \operatorname{rank}\begin{bmatrix} B_2 \\ D \end{bmatrix}$, when $CB_2 + D$ is defined;
4) $\mathcal{H} \iff$ Hautus' matrix rank condition

$$\operatorname{rank}\begin{bmatrix} CB_2 & D \\ D & O \end{bmatrix} = \operatorname{rank}\begin{bmatrix} B_2 \\ D \end{bmatrix} + \operatorname{rank} D.$$

For any matrix M, we let

$$\mathfrak{c}(M) = \text{Number of columns of } M$$
$$\ker M = \{v : Mv = 0\}.$$

We will make use of the well known equality

$$\mathfrak{c}(M) = \operatorname{rank} M + \dim\ker M. \tag{30}$$

First note that $\mathcal{G}$ implies that $\operatorname{rank}(CB_2) = \operatorname{rank} B_2$ and thus $CB_2 v = 0$ if and only if $B_2 v = 0$, or equivalently, $\ker(CB_2) = \ker B_2$.

If u, v are column vectors, we let $u \oplus v = \begin{bmatrix} u \\ v \end{bmatrix}$ and if U, V are vector spaces of column vectors, we let

$$U \oplus V = \{u \oplus v : u \in U, v \in V\}.$$

It is easy to see that $\dim(U \oplus V) = \dim U + \dim V$. Note that if $\{u_1, \ldots, u_p\}$ and $\{v_1, \ldots, v_q\}$ are bases of U, V, respectively, then $\{u_1 \oplus 0, \ldots, u_p \oplus 0, 0 \oplus v_1, \ldots, 0 \oplus v_q\}$ is a basis for $U \oplus V$.

Lemma 4: If $\mathcal{G}$, then $\ker(\begin{bmatrix} CB_2 & D \end{bmatrix}) = \ker B_2 \oplus \ker D$.

Proof: We have from $\mathcal{G}$ and equality (30) that

$$\begin{aligned}
&\dim\ker(\begin{bmatrix} CB_2 & D \end{bmatrix}) \\
&= \mathfrak{c}(\begin{bmatrix} CB_2 & D \end{bmatrix}) - \operatorname{rank}(\begin{bmatrix} CB_2 & D \end{bmatrix}) \\
&= \mathfrak{c}(CB_2) + \mathfrak{c}(D) - \operatorname{rank}(B_2) - \operatorname{rank} D \\
&= \dim\ker B_2 + \dim\ker D \\
&= \dim(\ker B_2 \oplus \ker D).
\end{aligned}$$

Since it is immediate that $\ker B_2 \oplus \ker D \subset \ker(\begin{bmatrix} CB_2 & D \end{bmatrix})$, we must have $\ker(\begin{bmatrix} CB_2 & D \end{bmatrix}) = \ker B_2 \oplus \ker D$.

Lemma 5: If $\mathcal{G}$, then $\ker(CB_2 + D) = \ker B_2 \cap \ker D$.

Proof: Clearly we have

$$\ker B_2 \cap \ker D \subset \ker(CB_2 + D).$$

Suppose $(CB_2 + D)v = 0$. Then $[CB_2 \quad D]\begin{bmatrix} v \\ v \end{bmatrix} = 0$. It follows from Lemma 4 that $\begin{bmatrix} v \\ v \end{bmatrix} \in \ker B_2 \oplus \ker D$. Therefore $B_2 v = 0$ and $Dv = 0$ and the lemma follows.

Lemma 6: If $\mathcal{G}$, then $\mathcal{M}$.

Proof: Observe that $\ker\begin{bmatrix} B_2 \\ D \end{bmatrix} = \ker B_2 \cap \ker D$. Thus by Lemma 5, we have $\ker(CB_2 + D) = \ker\begin{bmatrix} B_2 \\ D \end{bmatrix}$, which is equivalent to the claim of the lemma since the two matrices have the same number of columns.

Proof of Theorem 5: Assume $\mathcal{G}$. Then the above lemmas hold (and all will be used). The Hautus matrix rank condition is

$$\operatorname{rank}\begin{bmatrix} CB_2 & D \\ D & O \end{bmatrix} = \operatorname{rank}\begin{bmatrix} B_2 \\ D \end{bmatrix} + \operatorname{rank} D.$$

By Lemma 6, this is equivalent to

$$\operatorname{rank}\begin{bmatrix} CB_2 & D \\ D & O \end{bmatrix} = \operatorname{rank}(CB_2 + D) + \operatorname{rank} D$$

which is equivalent to

$$\operatorname{rank}\begin{bmatrix} CB_2 & D \\ D & O \end{bmatrix} = \operatorname{rank}\begin{bmatrix} CB_2 + D & O \\ O & D \end{bmatrix}.$$

We prove the above by showing that the two matrices have the same kernel. Suppose

$$\begin{bmatrix} CB_2 & D \\ D & O \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = 0.$$

Then $\begin{bmatrix} CB_2 & D \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = 0$, and $Du = 0$. By Lemma 4, $CB_2 u = 0$, $Dv = 0$, $Du = 0$, and it follows that

$$\begin{bmatrix} CB_2 + D & O \\ O & D \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = 0.$$

Conversely, suppose

$$\begin{bmatrix} CB_2 + D & O \\ O & D \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = 0.$$

Then $(CB_2 + D)u = 0$, and $Dv = 0$. By Lemma 5, $CB_2 u = 0$, $Du = 0$, and $Dv = 0$. Thus

$$\begin{bmatrix} CB_2 & D \\ D & O \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = 0.$$

Therefore the two matrices in question have the same kernel and therefore the same rank since they clearly have the same number of columns.

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