11.6 Fourier-Legendre series.

Topic: Legendre polynomials.

$$\begin{cases} \frac{d}{dx}\left((1-x^2)\frac{dy}{dx}\right) + \lambda y = 0, \quad -1 < x < 1 & \text{: SL eq.} \\ y(-1) = y(1) & \text{: periodic BC.} \end{cases}$$

$p(x) = 1-x^2, \quad q(x) = 0, \quad r(x) = 1$

(Rodrigue's formula)

$\lambda_n = 0, 1, 2, \cdots$

$$y_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left[(x^2-1)^n\right]$$

$n=0$: $y_0(x) = 1, \quad y_1(x) = \frac{1}{2}\frac{d}{dx}(x^2-1)$

$\underline{y_1(x) = x}, \quad \underline{y_2(x) = \frac{1}{2}(3x^2-1)}, \cdots$

(Bonnet's formula) recursive formula.

$y_0(x) = 1, \quad y_1(x) = x$

$$\underline{y_{n+1}(x) = \frac{2n+1}{n+1}x\,y_n(x) - \frac{n}{n+1}y_{n-1}(x)}$$

$n=2$

$$y_3(x) = \frac{5}{\cancel{3}}x\left(\frac{1}{\underline{2}}\right)(3x^2-1) - \frac{2}{3}x$$

$$= \frac{5}{6}(3x^3-x) - \frac{4}{6}x = \frac{15}{6}x^3 - \frac{5}{6}x - \frac{4}{6}x$$

$$= \frac{5}{2}x^3 - \frac{9}{6}x = \frac{5}{2}x^3 - \frac{3}{2}x = \frac{1}{2}(5x^3-3x)$$

(Infinite series)

Given a function $f(x)$,

$$F(x) = \sum_{n=0}^{\infty} a_n y_n(x) \quad : \quad a_n = ?$$

$$a_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)\,y_n(x)\,dx$$

Remark

$\{y_n\}$: orthogonal on $(-1, 1)$

HW#1 $f(x) = 63x^5 - 90x^3 + 35x$.

$\Rightarrow F(x) = ?$

(Ex) $f(x) = x^3$: $F(x) = ?$

$\{ y_0(x) = 1, \quad y_1(x) = x, \quad y_2(x) = \frac{1}{2}(3x^2 - 1)$

$y_3(x) = \frac{1}{2}(5x^3 - 3x)$

$2y_3(x) = 5x^3 - 3x = 5x^3 - 3y_1(x)$

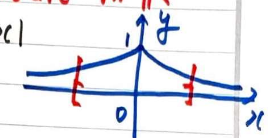
$5x^3 = 2y_3(x) + 3y_1(x)$

$x^3 = \frac{2}{5}y_3(x) + \frac{3}{5}y_1(x)$

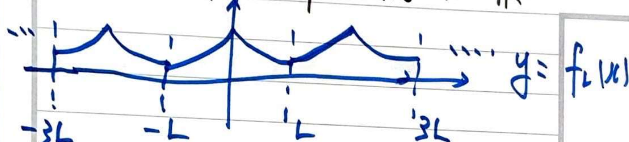
11.7. Fourier integral.

Q what if $f(x)$ is not periodic in $\mathbb{R}$

(Idea) (Ex) $f(x) = e^{-|x|}$



1. Let $f_L(x) = f(x)$, $-L < x < L$
2. Make $f_L(x)$ periodic in $\mathbb{R}$.



3. Compute the Fourier series $F_L(x)$ of $f_L(x)$
4. ~~Let $L \to \infty$~~ ?

Assume that $\int_{-\infty}^{\infty} |f(x)|\,dx$ is finite.

(ex) $f(x) = e^{-|x|}, e^{-x^2}, \cdots$ ✓

But $f(x) = x, x^2$: $\int_{-\infty}^{\infty} |f(x)|\,dx = \infty$

(3) $F_L(x) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L})]$

$= \frac{1}{2L}\int_{-L}^{L} \underbrace{f_L(x)}_{= f(x)}\,dx + \sum_{n=1}^{\infty} \frac{1}{L}\int_{-L}^{L} \underbrace{f_L(z)}_{= f(z)} \cos(\frac{n\pi z}{L})\,dz \cdot \cos(\frac{n\pi x}{L})$

$+ \sum_{n=1}^{\infty} \frac{1}{L}\int_{-L}^{L} \underbrace{f_L(z)}_{= f(z)} \sin(\frac{n\pi z}{L})\,dz \, \sin(\frac{n\pi x}{L})$.

Let $W_n = \frac{n\pi}{L}$: $\Delta W = W_{n+1} - W_n$

$\Delta W = \frac{(n+1)\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L}$: $\frac{1}{L} = \frac{\Delta W}{\pi}$

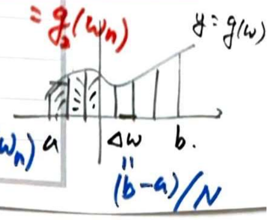
$$F_L(x) = \frac{\Delta W}{2\pi}\int_{-L}^{L} f(x)\,dx \quad -①$$

$$+ \sum_{n=1}^{\infty} \frac{\Delta W}{\pi} \underbrace{\int_{-L}^{L} f(z)\cos(W_n z)\,dz \cos(W_n x)}_{g_1(W_n)} \quad -②$$

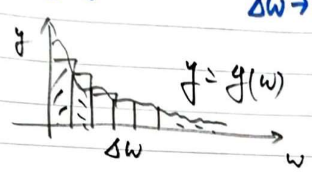
$$+ \sum_{n=1}^{\infty} \frac{\Delta W}{\pi} \underbrace{\int_{-L}^{L} f(z)\sin(W_n z)\,dz \sin(W_n x)}_{= g_2(W_n)} . \quad -③$$

<u>Remark</u> (Riemann integral)

1. $\int_a^b g(w)\,dw = \cancel{\Sigma} \lim_{\substack{N\to\infty \\ \Delta w \to 0}} \sum_{n=1}^{N} (\Delta w) g(w_n)$



2. $\int_0^\infty g(w)\,dw = \lim_{\Delta w \to 0} \sum_{n=1}^{\infty} \Delta w\, g(w_n)$



① $\underset{\Delta w \to 0}{\underline{L\to\infty}}$ $\left|\int_{-L}^{L} f(x)\,dx\right| \le \int_{-L}^{L} |f(x)|\,dx \le \underbrace{\int_{-\infty}^{\infty} |f(x)|}_{\text{finite}}$

$$\lim_{\substack{L\to\infty \\ \Delta w \to 0}} ① = 0$$

$② = \frac{1}{\pi}\sum_{n=1}^{\infty} \Delta w\, g_1(W_n)$: $\lim_{\substack{L\to\infty \\ \Delta w \to 0}} ②$

$$\lim_{\substack{L\to\infty \\ \Delta w \to 0}} ② = \frac{1}{\pi}\int_0^\infty \lim g_1(w)\,dw$$

$$\left(\lim_{\Delta w \to 0} g_1(W_n) = \int_{-\infty}^{\infty} f(z)\cos(wz)\,dz \cos(wx)\right)$$

$$\lim_{\Delta w \to 0} ② = \frac{1}{\pi}\int_0^\infty \underbrace{\boxed{\int_{-\infty}^{\infty} f(z)\cos(wz)\,dz}}_{= A(w)} \cos(wx)\,dw$$

$$\lim_{\Delta w \to 0} ③ = \frac{1}{\pi}\int_0^\infty \underbrace{\boxed{\int_{-\infty}^{\infty} f(z)\sin(wz)\,dz}}_{= B(w)} \sin(wx)\,dw$$

$$\lim_{L\to\infty} F_L(x) = \frac{1}{\pi}\int_0^\infty A(w)\cos(wx)\,dw + \frac{1}{\pi}\int_0^\infty B(w)\sin(wx)\,dw .$$

(Fourier integral) $\int_{-\infty}^{\infty} |f(x)|\,dx$ is finite

$$F(x) = \cancel{\tfrac{1}{\pi}} \int_0^\infty A(w)\cos(wx)\,dw + \cancel{\tfrac{1}{\pi}} \int_0^\infty B(w)\sin(wx)\,dw$$

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(z)\cos(wz)\,dz,$$

$$B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(z)\sin(wz)\,dz$$

Thm

H
- $f(x)$ is piecewise continuous in every finite interval
- $f(x)$ has a right-hand derivatives & a left-hand derivative at every point.
- $\int_{-\infty}^{\infty} |f(x)|\,dx$ is finite

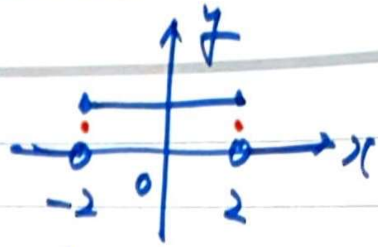
C (1) If $f(x)$ is continuous at $x_0 \in \mathbb{R}$,

$$f(x) = F(x)$$

(2) If $f(x)$ is discontinuous at $x_0 \in \mathbb{R}$,

then $$F(x_0) = \frac{1}{2}\left[\lim_{x \to x_0 -} f(x) + \lim_{x \to x_0 +} f(x)\right]$$

(Ex) $f(x) = \begin{cases} 1, & |x| \le 2 \\ 0, & |x| > 2 \end{cases}$



$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\,\cos(\omega x)\,dx$$

$$= \frac{1}{\pi}\int_{-2}^{2} \cos(\omega x)\,dx = \frac{2\sin(2\omega)}{\pi\omega}$$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} \underbrace{f(x)\,\sin(\omega x)}_{\text{odd}}\,dx = 0$$

$$F(x) = \int_0^{\infty}\left(\frac{2\sin(2\omega)}{\pi\omega}\right)\cos(\omega x)\,d\omega$$

Q1. $F(0) = \int_0^{\infty} \frac{2\sin(2\omega)}{\pi\omega}\,d\omega = ?$ $z = 2\omega$, $dz = 2d\omega$

$$F(0) = \int_0^{\infty}\frac{2\sin(z)}{\pi\, z/\cancel{2}}\,\frac{1}{\cancel{2}}\,dz = \frac{2}{\pi}\int_0^{\infty}\frac{\sin z}{z}\,dz$$

$Si(u) = \int_0^{u}\frac{\sin(z)}{z}\,dz$: the sine integral.

$f(0) = 1$: $\frac{2}{\pi}\int_0^{\infty}\frac{\sin z}{z}\,dz = 1$

$$\therefore \int_0^{\infty}\frac{\sin(z)}{z}\,dz = \frac{\pi}{2}.$$

Q2 $F(2) = \int_0^{\infty}\left(\frac{2\sin(2\omega)}{\pi\omega}\right)\cos(2\omega)\,d\omega$

$$F(2) = \frac{1}{2}(1+0) = \frac{1}{2}.$$