11. 4. Trigonometric polynomials

$f(x)$: periodic with $p = 2L$

$$F(x) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L})]$$

Q How do we approximate $F(x)$ (or $f(x)$) ?

$$F_N(x) = a_0 + \sum_{n=1}^{N} [a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L})]$$

: the best approximation ?

<u>Def</u> 1. $G_N(x) = A_0 + \sum_{n=1}^{N} (A_n \cos(\frac{n\pi x}{L}) + B_n \sin(\frac{n\pi x}{L}))$

: a trigonometric polynomial of degree N.

$A_0, A_n, B_n \in \mathbb{R}$: any real numbers.

$$\left(\begin{array}{l} \|f - F\|_{L^2} = \left(\int_{-L}^{L} (f(x) - F(x))^2 dx\right)^{\frac{1}{2}} = 0 \\ \underset{G_N}{\text{minimize}} \|f - G_N\|_{L^2} \end{array} \right)$$

<u>Def</u> 2. $E_G = \int_{-\pi}^{\pi} (f(x) - G_N(x))^2 dx$

: the square error of $G_N(x)$

<u>Thm 1</u> E_G is the minimum

iff $G_N(x) = F_N(x)$

Let $L = \pi$ ($p = 2\pi$)

(Parseval's Theorem / identity)

Ⓗ $f(x)$ is periodic in $\mathbb{R}$ and square integrable : $\int_{-\pi}^{\pi} f(x)^2 dx$ is finite

Ⓒ $\int_{-\pi}^{\pi} f(x)^2 dx = \int_{-\pi}^{\pi} F(x)^2 dx$

$$= \pi (2a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2))$$

<u>Remark</u>

$$2a_0^2 + \sum_{n=1}^{\infty} [a_n^2 + b_n^2] = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)^2 dx.$$

(Ex) $f(x)$ is periodic in $\mathbb{R}$ with $p = 2\pi$
& one full period $f(x) = x, \quad -\pi \le x < \pi$

(1) $F(x) = \sum_{n=1}^{\infty} \underbrace{\left[\frac{2(-1)^{n+1}}{n}\right]}_{= b_n} \sin(nx)$: $a_0 = 0$, $a_n = 0$.

HW: Compute $\boxed{\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.}$

(2) Parseval's identity.

$a_0 = 0, \quad a_n = 0$

① $\sum_{n=1}^{\infty} b_n^2 = \sum_{n=1}^{\infty} \left[\frac{2^2(-1)^{2(n+1)}}{n^2}\right]$

② $= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{1}{\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{\pi^3 + (+\pi)^3}{2\pi}$ $\quad \frac{2\pi^3}{3\pi}$

$\sum_{n=1}^{\infty} \frac{4}{n^2} = \frac{2\pi^2}{3}$ iff $\underline{\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{4}\frac{2\pi^2}{3} = \frac{\pi^2}{6}.}$

11.5 Sturm-Liouville problems

Topic: Boundary value problem.

(Ex) $\begin{cases} -y'' = ky, & 0 < x < 1 \\ y(0) = 0, & y(1) = 0 \end{cases}$

Def 1. $(p(x)y')' + (q(x) + \lambda r(x))y = 0, \quad a < x < b$

: a Sturm-Liouville equation.

2. BC: $\begin{cases} k_1 y(a) + k_2 y'(a) = 0, & x = a \\ l_1 y(b) + l_2 y'(b) = 0, & x = b \end{cases}$

Remark

1. $k_1 \ne 0, \; k_2 = 0, \; l_1 \ne 0, \; l_2 = 0$

   BC: $\underline{y(a) = 0, \quad y(b) = 0}$: Dirichlet BC.

2. $k_1 = 0, \; k_2 \ne 0, \; l_1 = 0, \; l_2 \ne 0$

   BC: $\underline{y'(a) = 0, \quad y'(b) = 0}$: Neumann BC.

3. $k_1 \ne 0, \; k_2 = 0, \; l_1 = 0, \; l_2 \ne 0$

   BC: $y(a) = 0, \quad y'(b) = 0.$: mixed BC.

(Ex) $y'' + ky = 0$: $p(x) = 1$, $q(x) = 0$
(Spring) : SL. $r(x) = 1$, $\lambda = k$

$my'' + cy' + ky = 0.$

$c = 0$: No damping. : $my'' + ky = 0$

$y'' + \frac{k}{m} y = 0$ iff $-y'' = \frac{k}{m} y$

(BVP) $-y'' = \frac{k}{m} y$, $y(0) = 0$, $y(1) = 0$
$(0 < x < 1)$

$-\frac{d^2}{dx^2} y = \frac{k}{m} y$: $AX = \lambda X$

k/m : an eigenvalue of "$-\frac{d^2}{dx^2}$"

$\neq 0$

$y(t)$: an eigenfunction of "$-\frac{d^2}{dx^2}$"

(Ex) $\begin{cases} -y'' = ky, & 0 < x < 1 \\ y(0) = 0, & y(1) = 0 \end{cases}$

$y'' + ky = 0$: Assume $y(x) = e^{\lambda x}$

$\lambda^2 + k = 0$: $\lambda = \pm\sqrt{-k}$

(1) $-k > 0$ ($k < 0$) :

$y(x) = C_1 e^{\sqrt{-k}x} + C_2 e^{-\sqrt{-k}x}$

$y(0) = C_1 + C_2 = 0$ ← BC. $C_2 = -C_1$

$y(1) = C_1 e^{\sqrt{-k}} + C_2 e^{-\sqrt{-k}} = 0$

$C_1 (\underbrace{e^{\sqrt{-k}} - e^{-\sqrt{-k}}}_{> 0}) = 0$: $C_1 = 0$, $C_2 = 0$

$y(x) \equiv 0$: identically zero

No eigenfunctions.

(2) $k = 0$: $y'' = 0$ $y'(x) = C_1$, $y(x) = C_1 x + C_2$

$y(0) = C_2 = 0$ ← BC, $y(x) = C_1 x$

$y(1) = C_1 = 0$ ← $y(x) = 0$

No eigenfunctions

(3) $-k < 0$ ($k > 0$) $\lambda = \pm\sqrt{-k} = \pm\sqrt{k}\, i$

$y(x) = C_1 \cos(\sqrt{k}x) + C_2 \sin(\sqrt{k}x)$

BC: $y(0) = C_1 = 0$ ← BC

$y(x) = C_2 \sin(\sqrt{k}\, x)$

$y(1) = C_2 \sin\sqrt{k} = 0$ ✓ BC : $C_2 \neq 0$

$\sin(\sqrt{k}) = 0$ iff $\sqrt{k} = n\pi > 0$ (n: an integer)

$k = (n\pi)^2, \quad n = 1, 2, 3, \cdots$

Let $\underline{k_n = (n\pi)^2}$, $\underline{y_n(x) = \sin(n\pi x)}$

$n = 1, 2, 3, \cdots$

eigenvalues of "$-\frac{d^2}{dx^2}$" eigenfunctions of "$-\frac{d^2}{dx^2}$"

(Ex) $\begin{cases} -y'' = ky, & 0 < x < 1 \\ y(0) = 0, & y'(1) = 0 \end{cases}$

$\lambda = \pm\sqrt{-k}$

(1) $-k > 0$: No eigenfunctions

(2) $k = 0$: 〃

(3) $-k < 0$ ($k > 0$) $\lambda = \pm\sqrt{k}\, i$

$y(x) = C_1 \cos(\sqrt{k}\, x) + C_2 \sin(\sqrt{k}\, x)$

BC: $y(0) = \underline{C_1 = 0}$, $y(x) = C_2 \sin(\sqrt{k}\, x)$

$y'(x) = \sqrt{k}\, C_2 \cos(\sqrt{k}\, x)$

$y'(1) = \sqrt{k}\, C_2 \cos(\sqrt{k}) = 0$: $C_2 \neq 0$, $\sqrt{k} > 0$

$\cos\sqrt{k} = 0$: $\sqrt{k} = n\pi - \frac{\pi}{2}$, $n = 1, 2, \cdots$

let $k_n = \left(n\pi - \frac{\pi}{2}\right)^2$, $y_n(x) = \sin\left(n\pi - \frac{\pi}{2}\right)x$

$(n = 1, 2, 3, \cdots)$

Remark $\{\sin(n\pi x)\}$, $\{\sin(n - \frac{1}{2})\pi x\}$

: orthogonal on $(0, 1)$.

Q SL problems generate $\{y_n\}$: orthogonal?

Ⓐ Yes

Every SL problem generates orthogonal eigenfunctions. $\{y_n\}$.

$(y_n, y_m)_r = \int_a^b y_n(x)\, y_m(x)\, \underbrace{r(x)}_{\text{weight}}\, dx = 0$