ECE 68000: MODERN AUTOMATIC CONTROL

Professor Stan Żak

Observer and controller design Using linear matrix inequalities

Some useful matrix properties

- Matrix properties useful when converting LMIs into equivalent LMIs or converting some nonlinear matrix inequalities into linear matrix inequalities
- Start with a simple observation

Lemma

Let $\boldsymbol{P} = \boldsymbol{P}^\top$ be a nonsingular n-by-n matrix and let $\boldsymbol{x} = \boldsymbol{M}\boldsymbol{z}$, where $\boldsymbol{M} \in \mathbb{R}^{n\times n}$ such that $\det \boldsymbol{M} \neq 0$. Then,

$$\boldsymbol{x}^\top \boldsymbol{P}\boldsymbol{x} \geq 0 \;\; \textit{if and only if} \;\; \boldsymbol{z}^\top \boldsymbol{M}^\top \boldsymbol{P}\boldsymbol{M}\boldsymbol{z} \geq 0,$$

that is,

$$\boldsymbol{P} \succeq 0 \;\; \textit{if and only if} \;\; \boldsymbol{M}^\top \boldsymbol{P}\boldsymbol{M} \succeq 0.$$

- Similarly

$$\boldsymbol{P} \succ 0 \;\; \text{if and only if} \;\; \boldsymbol{M}^\top \boldsymbol{P}\boldsymbol{M} \succ 0$$

More useful matrix properties

- Suppose that we have a square block matrix

$$\begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{B}^\top & \boldsymbol{D} \end{bmatrix},$$

  where $\boldsymbol{A} = \boldsymbol{A}^\top$ and $\boldsymbol{D} = \boldsymbol{D}^\top$
- Then,

$$\begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{B}^\top & \boldsymbol{D} \end{bmatrix} \succeq 0 \iff \begin{bmatrix} \boldsymbol{O} & \boldsymbol{I} \\ \boldsymbol{I} & \boldsymbol{O} \end{bmatrix} \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{B}^\top & \boldsymbol{D} \end{bmatrix} \begin{bmatrix} \boldsymbol{O} & \boldsymbol{I} \\ \boldsymbol{I} & \boldsymbol{O} \end{bmatrix} \succeq 0,$$

  where $\boldsymbol{I}$ is an identity matrix of appropriate dimension
- In other words, $\begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{B}^\top & \boldsymbol{D} \end{bmatrix} \succeq 0$ if and only if $\begin{bmatrix} \boldsymbol{D} & \boldsymbol{B}^\top \\ \boldsymbol{B} & \boldsymbol{A} \end{bmatrix} \succeq 0$

The Schur complement

- Consider a square block matrix of the form

$$\begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} \end{bmatrix},$$

  where $\boldsymbol{A}_{11}$ and $\boldsymbol{A}_{22}$ are square and symmetric submatrices, and $\boldsymbol{A}_{12} = \boldsymbol{A}_{21}^\top$
- Suppose that the matrix $\boldsymbol{A}_{11}$ is invertible. Then,

$$\begin{bmatrix} \boldsymbol{I} & \boldsymbol{O} \\ -\boldsymbol{A}_{21}\boldsymbol{A}_{11}^{-1} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{21}^\top \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{I} & -\boldsymbol{A}_{11}^{-1}\boldsymbol{A}_{21}^\top \\ \boldsymbol{O} & \boldsymbol{I} \end{bmatrix}$$
$$= \begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{A}_{22} - \boldsymbol{A}_{21}\boldsymbol{A}_{11}^{-1}\boldsymbol{A}_{21}^\top \end{bmatrix}.$$

- The matrix

$$\boldsymbol{\Delta}_{11} = \boldsymbol{A}_{22} - \boldsymbol{A}_{21}\boldsymbol{A}_{11}^{-1}\boldsymbol{A}_{21}^\top$$

  is called the Schur complement of $\boldsymbol{A}_{11}$

The Schur complement—contd.

- Recall the Schur complement of $\boldsymbol{A}_{11}$,

$$\boldsymbol{\Delta}_{11} = \boldsymbol{A}_{22} - \boldsymbol{A}_{21}\boldsymbol{A}_{11}^{-1}\boldsymbol{A}_{21}^{\top}$$

- Hence,

$$\begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{21}^{\top} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} \end{bmatrix} \succ 0 \ \text{ if and only if } \begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{\Delta}_{11} \end{bmatrix} \succ 0,$$

that is,

$$\begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{21}^{\top} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} \end{bmatrix} \succ 0 \ \text{ if and only if } \ \boldsymbol{A}_{11} \succ 0 \ \text{ and } \ \boldsymbol{\Delta}_{11} \succ 0$$

The Schur complement—Example

- Consider the following symmetric matrix,

$$\boldsymbol{Q} = \left[\begin{array}{cc|cc} 2 & 2 & 1 & 1 \\ 2 & 3 & 0 & 1 \\ \hline 1 & 0 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{array}\right] = \left[\begin{array}{c|c} \boldsymbol{Q}_{11} & \boldsymbol{Q}_{12} \\ \hline \boldsymbol{Q}_{21} & \boldsymbol{Q}_{22} \end{array}\right].$$

1. Compute the Schur complement, Δ_{22}, of $\boldsymbol{Q}_{22}$;
2. Determine if $\boldsymbol{Q}$ is positive definite, positive semi-definite, negative definite, negative semi-definite, or indefinite?

The Schur complement—Example solution

- The Schur complement of $\boldsymbol{Q}_{22}$ is

$$\begin{aligned}
\Delta_{22} &= \boldsymbol{Q}_{11} - \boldsymbol{Q}_{12}\boldsymbol{Q}_{22}^{-1}\boldsymbol{Q}_{21} \\
&= \begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}
\end{aligned}$$

- The matrix $\boldsymbol{Q}$ is positive semi-definite, and not positive definite, because $\boldsymbol{Q}_{11}$ is positive definite but Δ_{22} is only positive semi-definite

Observer Design

Plant model:

$$\begin{aligned}\dot{\boldsymbol{x}} &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} \\ \boldsymbol{y} &= \boldsymbol{C}\boldsymbol{x}\end{aligned}$$

Linear observer:

$$\dot{\tilde{\boldsymbol{x}}} = \boldsymbol{A}\tilde{\boldsymbol{x}} + \boldsymbol{B}\boldsymbol{u} + \boldsymbol{L}(\boldsymbol{y} - \boldsymbol{C}\tilde{\boldsymbol{x}})$$

Goal: Design $\boldsymbol{L}$ to ensure asymptotic stability of the error dynamics

- Matrix inequality for observer design:

$$(\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C})^\top \boldsymbol{P} + \boldsymbol{P}(\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C}) \prec 0,\ \boldsymbol{P} = \boldsymbol{P}^\top \succ 0$$

Observer Design—Contd.

$$A^\top P + PA - C^\top L^\top P - PLC \prec 0, \;\; P \succ 0$$

- To-do: Find L, P
- Problem: Bi-linear matrix inequality in L and P
- **Technique #1**: Choose $Y = PL$
- LMIs:

$$\underbrace{A^\top P + PA}_{\text{linear in } P} - \underbrace{C^\top Y^\top - YC}_{\text{linear in } Y} \prec 0, \;\; P \succ 0$$

- For robustness of solution, rewrite as

$$A^\top P + PA - C^\top Y^\top - YC + 2\alpha P \preceq 0, \;\; P \succ 0$$

  with fixed $\alpha > 0$
- Get back $L = P^{-1} Y$ ($P \succ 0$, hence invertible)

State/Output Feedback Control

LTI System with output feedback control:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \\ u &= -Ky \end{aligned}$$

Goal: Design K to ensure asymptotic stability of $(A - BKC)$

- Matrix inequality for output-feedback controller design:

$$(A - BKC)^\top P + P(A - BKC) \prec 0, \;\; P \succ 0$$

- Simpler case: state-feedback $(C = I)$

$$(A - BK)^\top P + P(A - BK) \prec 0, \;\; P \succ 0$$

Simpler Case: State-Feedback Control

$$(A - BK)^\top P + P(A - BK) \prec 0, \;\; P \succ 0$$

- To-do: Find K, P
- Problem: Bi-linear matrix inequality in K and P
- **Technique #2**: Congruence transformation with $S \triangleq P^{-1}$ and $Z \triangleq KS$
- New inequalities

$$SA^\top + AS - SK^\top B^\top - BKS \prec 0$$

- LMIs:

$$\underbrace{SA^\top + AS}_{\text{linear in } S} - \underbrace{Z^\top B^\top - BZ}_{\text{linear in } Z} \prec 0, \;\; P \succ 0$$

- Get back $P = S^{-1}$, $K = ZP$