Case Study

Use the Kronecker product to convert the "matrix-matrix" equation,

$$AXB = C$$

into the "matrix-vector" equation, where X is the unknown matrix variable.

- Assume that the matrices A, B, C, and X are square and of the same size. Formulate the necessary and sufficient condition for the existence of the unique solution to
$$AXB = C$$
 in terms of the eigenvalues of A and B.

- Does the equation,

$$\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \begin{bmatrix} -2 & 0 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ -4 & 2 \end{bmatrix},$$

  have a solution? If yes, find a solution, if not then explain why not.

Please take a moment to work through this problem on your own. When you are ready, scroll down the page to view the solution.

Explanation:

- We use the Kronecker product to represent the equation $AXB = C$ in the form

$$\left(B^\top \otimes A\right) \operatorname{vec}(X) = \operatorname{vec}(C),$$

  where $\operatorname{vec}(\cdot)$ is the stacking operator, that is, the vectorizing operator. There exists a unique solution to the above equation if and only if the square matrix $B^\top \otimes A$ is invertible. This matrix is invertible if and only if all its eigenvalues are non-zero. The eigenvalues of the matrix $B^\top \otimes A$ are the products of the eigenvalues of A and B. In summary, a necessary and sufficient condition for the matrix equation $AXB = C$ to have a unique solution is that none of the eigenvalues of the matrices A and B are equal zero.

- We represent $AXB = C$ in the form,

$$\left(\begin{bmatrix} -2 & -3 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\right) \operatorname{vec}\left(\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}\right) = \operatorname{vec}\left(\begin{bmatrix} -4 & 2 \\ -4 & 2 \end{bmatrix}\right).$$

  Performing the required operations, we obtain

$$\begin{bmatrix} 0 & -2 & 0 & -3 \\ 0 & -2 & 0 & -3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \\ x_{12} \\ x_{22} \end{bmatrix} = \begin{bmatrix} -4 \\ -4 \\ 2 \\ 2 \end{bmatrix}.$$

There are multiple solutions to the above system of equations. One of the solutions is

$$\boxed{X = \begin{bmatrix} 0 & 0 \\ -1 & 2 \end{bmatrix}}$$