PURDUE UNIVERSITY

ECE 68000: MODERN AUTOMATIC CONTROL

Professor Stan Żak

Animation of the simple pendulum motion with MATLAB

Animation of the simple pendulum motion using MATLAB—Outline

- Solving numerically differential equations using the forward Euler method
- Intro to handle graphics
- Rotation matrix
- Animating the pendulum motion using the handle graphics

Solving numerically differential equations—preparation

- Suppose that we wish to obtain the solution of the modeling equation

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(t, \boldsymbol{x}(t), \boldsymbol{u}(t))$$

  in the time interval $\left[\ t_0,\quad t_f\ \right]$, subject to the initial condition $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$, and the input vector $\boldsymbol{u}(t)$
- Divide the interval $\left[\ t_0,\quad t_f\ \right]$ into N equal subintervals of width

$$h = \frac{t_f - t_0}{N}.$$

Derivative approximation

- We call $h = \frac{t_f - t_0}{N}$ the *step length*
- Set
$$t_k = t_0 + kh.$$
- Approximate the derivative $\frac{d\boldsymbol{x}(t)}{dt}$ at time t_k by
$$\frac{d\boldsymbol{x}(t_k)}{dt} \approx \frac{\boldsymbol{x}(t_{k+1}) - \boldsymbol{x}(t_k)}{h}$$
- We have
$$\frac{\boldsymbol{x}(t_{k+1}) - \boldsymbol{x}(t_k)}{h} = \boldsymbol{f}(t_k, \boldsymbol{x}(t_k), \boldsymbol{u}(t_k)).$$

The forward Euler method

- For $k = 1, 2, \ldots, N$, we have

$$\boldsymbol{x}(t_{k+1}) = \boldsymbol{x}(t_k) + h\boldsymbol{f}(t_k, \boldsymbol{x}(t_k), \boldsymbol{u}(t_k))$$

- The above is known as the *forward Euler* algorithm
- If h is sufficiently small, we can approximately determine the state at time $t_1 = t_0 + h$ from the initial condition $\boldsymbol{x}_0$ to get

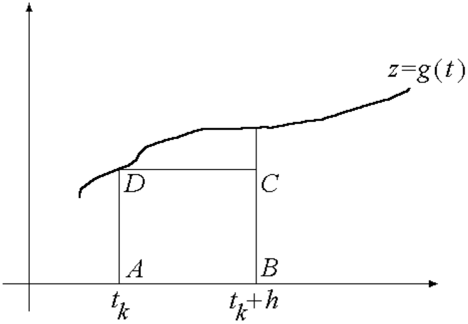
$$\boldsymbol{x}(t_1) = \boldsymbol{x}_0 + h\boldsymbol{f}(t_0, \boldsymbol{x}_0, \boldsymbol{u}(t_0))$$

Use of the forward Euler method

- Once we have determined the approximate solution at time t_1, we determine the approximate solution at time $t_2 = t_1 + h$, and so on
- The forward Euler method can also be arrived at when instead of considering the differential equation $\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(t, \boldsymbol{x}(t), \boldsymbol{u}(t))$, we rather start with its equivalent integral representation

$$\boldsymbol{x}(t) = \boldsymbol{x}_0 + \int_{t_0}^{t} \boldsymbol{f}(\tau, \boldsymbol{x}(\tau), \boldsymbol{u}(\tau)) d\tau.$$

Rectangular rule for numerical integration



Rectangular rule for numerical integration discussion

- The area under the curve $z = g(t)$ between $t = t_k$ and $t = t_k + h$ is approximately equal to the area of the rectangle $ABCD$
- That is,

$$\int_{t_k}^{t_k+h} g(t)dt \approx hg(t_k)$$

Basics of of MATLAB's handle graphics

- handle—a floating-point number that MATLAB assigns to every object in the figure window such as a line, text, label, axes, figure, etc.
- Thus a handle can be viewed as an object's identifier, that is, an object's ID in MATLAB
- Two ways to get hold of handles

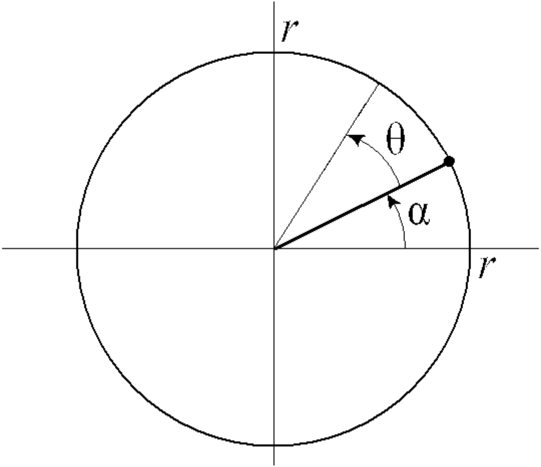
Creating handles

- Can create a handle explicitly
- For example, we can make a plot and get a handle at the same time with a command, **hand=plot(time,force)**
- Can also obtain a handle by using handle-returning functions, such as **gca(graphics current axis)** which returns the handle to the current axis in the current figure.

Using handles

- Each object has different properties
- If we wish to change a property, we need to get hold of the object in question
- Use object's handle to get hold of it and then change the object's property using the commands,
  get(handle,'PropertyName')
  set(handle,'PropertyName','PropertyValue')

Point mass on a circle



Rotating a point mass on a circle

- Suppose we have a point mass on a circle of radius r
- The point mass is connected to the circle center; call it hinge
- The line segment, bar, connecting the hinge and the mass forms the angle α
- Therefore, the point mass has the coordinates

$$\boldsymbol{x}_{present} = \left[\begin{array}{c} r\cos(\alpha) \\ r\sin(\alpha) \end{array} \right]$$

Rotating a point mass on a circle—manipulations

- Suppose the bar is rotated by an angle θ from its initial displacement angle α
- The new angle of the bar, with the positive x-axis, is $\alpha + \theta$
- Use the well-known trigonometric identities:

$$\begin{aligned}
\cos(\alpha + \theta) &= \cos(\alpha)\cos(\theta) - \sin(\alpha)\sin(\theta) \\
\sin(\alpha + \theta) &= \sin(\alpha)\cos(\theta) + \cos(\alpha)\sin(\theta)
\end{aligned}$$

Rotating a point mass on a circle—more manipulations

- The mass in new coordinates,

$$
\begin{aligned}
\boldsymbol{x}_{next} &= \left[\begin{array}{cc} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{array}\right] \boldsymbol{x}_{present} \\
&= \left[\begin{array}{cc} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{array}\right] \left[\begin{array}{c} r\cos(\alpha) \\ r\sin(\alpha) \end{array}\right]
\end{aligned}
$$

The rotation matrix

The matrix

$$\boldsymbol{R}(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

is called the rotation matrix

Ready to animate!

- Use the state-space model of the pendulum;

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{l}\sin(x_1) \end{aligned}$$

- Approximate the above system of differential equations using Euler's forward method,

$$\begin{aligned} \frac{x_{1next} - x_{1present}}{h} &= x_{2present} \\ \frac{x_{2next} - x_{2present}}{h} &= -\frac{g}{l}\sin(x_{1present}), \end{aligned}$$

  where h is the step size
- Combine the above with the handle graphics