Recall

$$\widehat{[{}^{I}V^{B}]}_I = \frac{{}^{I}d}{dt}({}^{I}g^{B})({}^{I}g^{B})^{-1}$$

$$\widehat{[{}^{I}V^{B}]}_B = ({}^{I}g^{B})^{-1}\frac{{}^{I}d}{dt}({}^{I}g^{B})$$

What if we want to convert between these two?
From the two formulas,

$$\frac{{}^{I}d}{dt}({}^{I}g^{B}) = \widehat{[{}^{I}V^{B}]}_I({}^{I}g^{B}) = ({}^{I}g^{B})\widehat{[{}^{I}V^{B}]}_B$$

$$\Rightarrow \boxed{\begin{array}{l} \widehat{[{}^{I}V^{B}]}_I = {}^{I}g^{B}\,\widehat{[{}^{I}V^{B}]}_B({}^{I}g^{B})^{-1} \\ \text{and} \\ \widehat{[{}^{I}V^{B}]}_B = ({}^{I}g^{B})^{-1}\,\widehat{[{}^{I}V^{B}]}_I({}^{I}g^{B}) \end{array}}$$

If you would rather work with twist coordinates instead of twist matrices, another way to transform twists is using an adjoint transformation

From the traditional Kinematics

$$[{}^{I}\vec{\omega}^{B}]_I = {}^{I}C^{B}[{}^{I}\vec{\omega}^{B}]_B$$

and

$$[{}^{B}\vec{V}_{o'/o}]_I = [-{}^{I}\vec{\omega}^{B}\times\vec{r}_{o'/o} + {}^{I}\vec{V}_{o'/o}]_I$$

$$= [\vec{r}_{o'/o}]_I \times {}^{I}C^{B}[{}^{I}\vec{\omega}^{B}]_B + {}^{I}C^{B}[{}^{I}\vec{V}_{o'/o}]_B$$

Stacking these together and factoring,

$$\underbrace{\begin{bmatrix} {}^{B}\vec{v}_{o'/o} \\ {}^{I}\vec{\omega}^{B} \end{bmatrix}_I}_{[{}^{I}V^{B}]_I} = \underbrace{\begin{bmatrix} {}^{I}C^{B} & \vec{r}_{o'/o} \times {}^{I}C^{B} \\ 0_{3\times3} & {}^{I}C^{B} \end{bmatrix}}_{\substack{6\times6 \text{ matrix} \\ \triangleq Ad_{({}^{I}g^{B})}}} \underbrace{\begin{bmatrix} {}^{I}\vec{v}_{o'/o} \\ {}^{I}\vec{\omega}^{B} \end{bmatrix}_B}_{[{}^{I}V^{B}]_B}$$

Twist coordinates

Converts twist coordinates between frames.

- Ad_g is the 6×6 matrix that transforms twist coordinates from one reference frame to another. It is called the adjoint transformation associated with $g \in SE(3)$

For $g = (R, \vec{r}) \in SE(3)$,

then
$$Ad_g = \begin{bmatrix} R & \vec{r} \times R \\ 0_{3\times3} & R \end{bmatrix}$$

What is $(Ad_g)^{-1}$?

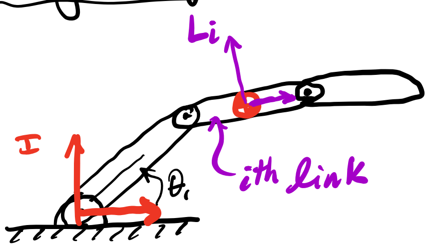
$$(Ad_g)^{-1} = \begin{bmatrix} R^T & -(R^T\vec{r}) \times R^T \\ 0_{3\times3} & R^T \end{bmatrix} = Ad_{g^{-1}}$$

If $\hat{\xi} \in se(3)$ is a twist with twist coordinates $\xi \in \mathbb{R}^6$ then for $g \in SE(3)$, the transformed quantity,

$$({}^{I}g^{B})\,\hat{\xi}\,({}^{I}g^{B})^{-1} \in se(3)$$

is a twist with twist coordinates $Ad_g \xi \in \mathbb{R}^6$

Ex Lagrangian for an open-chain robot (MLS Section 4.3.1 p186)



Configuration of link i:

$${}^{I}g^{L_i}(\underline{\theta}) = e^{\hat{\xi}_1\theta_1} \cdots e^{\hat{\xi}_i\theta_i}\, {}^{I}g^{L_i}(0)$$

What is the body velocity of Link L_i?

$$\left[\widehat{{}^{I}V^{L_i}}\right]_{L_i} = ({}^{I}g^{L_i})^{-1}\, \frac{{}^{I}d}{dt}({}^{I}g^{L_i})$$

Note that ${}^{I}g^{L_i}$ is a function of the θ_j joint angles for $j = 1, \ldots, i$

$$\frac{{}^{I}d}{dt}({}^{I}g^{L_i}) = \sum_{j=1}^{i} \frac{\partial\, {}^{I}g^{L_i}}{\partial \theta_j}\dot{\theta}_j \qquad \text{(chain rule)}$$

$$\widehat{\left[{}^{I}V^{L_i}\right]}_{L_i} = \sum_{j=1}^{i} \left(\left({}^{I}g^{L_i}(\underline{\theta})\right)^{-1} \frac{\partial\, {}^{I}g^{L_i}}{\partial \theta_j} \right) \dot{\theta}_j$$

Define

$$\left[{}^{I}J^{L_i}\right]_{L_i} \triangleq \left[\left(({}^{I}g^{L_i})^{-1} \frac{\partial\, {}^{I}g^{L_i}}{\partial \theta_1} \right)^{\vee} \quad \cdots \quad \left(({}^{I}g^{L_i})^{-1} \frac{\partial\, {}^{I}g^{L_i}}{\partial \theta_n} \right)^{\vee} \right]$$

Body Jacobian

Then we can write

$$\widehat{\left[{}^{I}V^{L_i}\right]}_{L_i} = \left[{}^{I}J^{L_i}\right]_{L_i} \dot{\underline{\theta}}$$

Kinetic energy:

$$T_i = \frac{1}{2} \left[{}^{I}V^{L_i}\right]_{L_i}^{T} M_i \left[{}^{I}V^{L_i}\right]_{L_i}$$

$$= \frac{1}{2} \dot{\underline{\theta}}^{T} \left({}^{I}J^{L_i}\right)^{T} M_i \left({}^{I}J^{L_i}\right) \dot{\underline{\theta}}$$

Potential: $V_i = m_i g h_i(\underline{\theta})$

Lagrangian, $L = \sum_{i=1}^{n} (T_i - V_i)$