ECE 602: LUMPED LINEAR SYSTEMS

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Solution of Continuous-Time Controlled LTI and LTV Systems

Continuous-Time Controlled LTI Systems

Continuous-time LTI system with control input u:

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du,$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input, $y \in \mathbb{R}^p$ is the output.

- $A \in \mathbb{R}^{n \times n}$ is the (state) dynamics matrix
- $B \in \mathbb{R}^{n \times m}$ is the input matrix
- $C \in \mathbb{R}^{p \times n}$ is the output matrix
- $D \in \mathbb{R}^{p \times m}$

Solution:

$$e^{-At}\left[\dot{x}(t) - Ax(t)\right] = e^{-At}Bu(t) \quad \Rightarrow \quad \frac{d}{dt}\left[e^{-At}x(t)\right] = e^{-At}Bu(t)$$

$$\Rightarrow \quad x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)\, d\tau$$

Solutions of CT Controlled LTI Systems

Given input $u(\cdot)$ and $x(0)$, the solution of the controlled LTI system is:

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau$$

$$y(t) = \underbrace{Ce^{At}x(0)}_{\text{zero-input response}} + \underbrace{C\int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau + Du(t)}_{\text{zero-state response}}, \quad t \geq 0$$

- Zero-input response: response due to the initial state $x(0)$ when $u = 0$
- Zero-state response: response due to the input $u(t)$ when $x(0) = 0$
- Equivalently, state solution is $x(t) = e^{At}x(0) + \int_0^t e^{A\tau}Bu(t-\tau)\,d\tau$
- Solution with the starting time t_0:

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^t e^{A(t-\tau)}Bu(\tau)\,d\tau$$

Example

$$\begin{cases} \dot{x} &= \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} x + \begin{bmatrix} -2 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 0 & 1 \end{bmatrix} x \end{cases} \qquad e^{At} = \begin{bmatrix} e^{-t} & 0 \\ \frac{e^{t}-e^{-t}}{2} & e^{t} \end{bmatrix}$$

Given the input $u(t) = 1$, $\forall t \geq 0$, and initial state $x(0) = 0$, the solution is

$$x(t) = \int_0^t \underbrace{\begin{bmatrix} e^{-\tau} & 0 \\ \frac{e^{\tau}-e^{-\tau}}{2} & e^{\tau} \end{bmatrix}}_{e^{A\tau}} \underbrace{\begin{bmatrix} -2 \\ 1 \end{bmatrix}}_{B} u(t-\tau)\, d\tau = \int_0^t \begin{bmatrix} -2e^{-\tau} \\ e^{-\tau} \end{bmatrix} d\tau = \begin{bmatrix} 2e^{-t} - 2 \\ 1 - e^{-t} \end{bmatrix}$$

$$y(t) = 1 - e^{-t}$$

- The unstable mode for eigenvalue 1 is not activated by u with $x(0) = 0$
- With $x(0) \neq 0$, the unstable mode may appear in $y(t)$
- These will be explained later using controllability and observability

LTI Solutions via Laplace Transform

Take the Laplace transform of the LTI system equations:

$$\begin{cases} \dot{x} & = Ax + Bu \\ y & = Cx + Du \end{cases} \quad \Rightarrow \quad \begin{cases} sX(s) - x(0) & = AX(s) + BU(s) \\ Y(s) & = CX(s) + DU(s) \end{cases}$$

Therefore,

$$X(s) = (sI_n - A)^{-1}x(0) + (sI_n - A)^{-1}BU(s)$$
$$Y(s) = \underbrace{C(sI_n - A)^{-1}x(0)}_{\text{zero-input response}} + \underbrace{C(sI_n - A)^{-1}BU(s) + DU(s)}_{\text{zero-state response}}$$

Taking the inverse Laplace transform to recover the previous results:

$$x(t) = e^{At}x(0) + \underbrace{e^{At} \star Bu(t)}_{\text{convolution}}$$
$$y(t) = Ce^{At}x(0) + Ce^{At} \star Bu(t) + Du(t)$$

Transfer Function Matrix of CT LTI Systems

The zero-state response (assuming $x(0) = 0$) is

$$Y(s) = \underbrace{\left[C(sI_n - A)^{-1}B + D\right]}_{H(s)} \cdot U(s)$$

- $H(s)$ is the transfer function matrix of the LTI system
- LTI system (A, B, C, D) is called a realization of $H(s)$

Example: $\dot{x} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} x + \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} u, \quad y = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x$

Algebraically Equivalent LTI Systems

LTI system $\begin{cases} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{cases}$ after a change of coordinates $x = T\tilde{x}$:

$$\begin{cases} \dot{\tilde{x}} &= \tilde{A}\tilde{x} + \tilde{B}u = (T^{-1}AT)\tilde{x} + (T^{-1}B)u \\ y &= \tilde{C}\tilde{x} + \tilde{D}u = (CT)\tilde{x} + Du \end{cases}$$

- LTI systems (A, B, C, D) and $(T^{-1}AT, T^{-1}B, CT, D)$ are called algebraically equivalent, and have the same transfer function matrix
- Matlab command `[Ap,Bp,Cp,Dp]=ss2ss(A,B,C,D,P)`

Solutions of CT Controlled LTV Systems

Continuous-time LTV system:

$$\begin{aligned}\dot{x}(t) &= A(t)x(t) + B(t)u(t)\\ y(t) &= C(t)x(t) + D(t)u(t),\end{aligned}$$

Its solution under input $u(\cdot)$ and initial state $x(0)$ is

$$x(t) = \Phi(t)x(0) + \int_0^t \Phi(t,\tau)B(\tau)u(\tau)\,d\tau$$

$$y(t) = \underbrace{C(t)\Phi(t)x(0)}_{\text{zero-input response}} + \underbrace{C(t)\int_0^t \Phi(t,\tau)B(\tau)u(\tau)\,d\tau + D(t)u(t)}_{\text{zero-state response}}$$

More generally, solution with the initial time t_0 is

$$x(t) = \Phi(t,t_0)x(t_0) + \int_{t_0}^t \Phi(t,\tau)B(\tau)u(\tau)\,d\tau$$

$$y(t) = C(t)\Phi(t,t_0)x(t_0) + C(t)\int_{t_0}^t \Phi(t,\tau)B(\tau)u(\tau)\,d\tau + D(t)u(t)$$