Laplace Transform

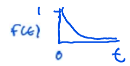
Let f(t) be a function of time t, such that f(t) = 0, for t < 0, then

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty f(t)e^{-st}dt$$

is the Laplace transform of f(t).

s is a complex variable

e^{-at} $a>0$



Example:

$$f(t) = e^{-at}$$

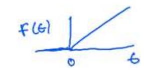
$$F(s) = \int_0^\infty e^{-at}e^{-st}dt = \int_0^\infty e^{-(a+s)t}dt$$

$$= -\left.\frac{e^{-(a+s)t}}{s+a}\right|_0^\infty$$

$$= -\left[\frac{e^{-(a+s)\infty}}{s+a} - \frac{1}{s+a}\right] = \frac{1}{s+a}$$

1

Laplace Transform



Example:

$$f(t) = at$$

$$F(s) = \int_0^\infty ate^{-st}dt$$

$$= -\left.\frac{ate^{-st}}{s}\right|_0^\infty + \int_0^\infty \frac{ae^{-st}}{s}dt$$

$$= -\left.\frac{ae^{-st}}{s^2}\right|_0^\infty = -\left[\frac{ae^{-\infty}}{s^2} - \frac{a}{s^2}\right] = \frac{a}{s^2}$$

Example (Step function):

Unit step: $f(t) = \begin{cases} a \text{ for } t>0 \\ 0 \text{ for } t<0 \end{cases}$

Derivative of ramp is step.

$\left(\frac{a}{s^2}\right)(s) = \frac{a}{s}$

ramp derivative

$$f(t) = \begin{cases} 0 & t<0 \\ A & t>0 \end{cases}$$

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty Ae^{-st}dt = \frac{A}{s}$$

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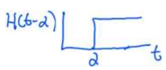
Laplace Transform (Impulse Function)

Example (Heaviside Step function):

$$\mathcal{H}(t-\alpha) = \begin{cases} 0 & t<\alpha \\ A & t>\alpha \end{cases} \qquad \mathcal{L}[\mathcal{H}(t-\alpha)] = \int_0^\infty \mathcal{H}(t-\alpha)e^{-st}dt$$



Define $\tau = t-\alpha \Rightarrow t = \tau+\alpha$

$dt = d\tau$

$$\mathcal{L}[\mathcal{H}(t-\alpha)] = \int_0^\infty \mathcal{H}(t-\alpha)e^{-st}dt \longrightarrow \mathcal{L}[\mathcal{H}(\tau)] = \int_{-\alpha}^\infty \mathcal{H}(\tau)e^{-s(\tau+\alpha)}d\tau$$

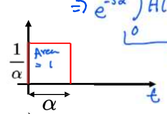
$\int_{-\alpha}^{\infty} He^{-s(\tau+\alpha)} = \int_{-\alpha}^{0} He^{-s(\tau+\alpha)}d\tau + \int_0^\infty He^{-s(\tau+\alpha)}d\tau$

$\Rightarrow e^{-s\alpha}\int_0^\infty H(\tau)e^{-s\tau}dt = \frac{e^{-s\alpha}}{s}$

pure delay

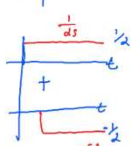
$$\mathcal{L}[\mathcal{H}(t-\alpha)] = \frac{e^{-s\alpha}}{s}$$

Define a pulse function as:



Addition of Heaviside and step function

$$f(t) = \begin{cases} 0 & t<0 \\ \frac{1}{\alpha} & t<\alpha \\ 0 & t>\alpha \end{cases} \qquad \mathcal{L}[f(t)] = \frac{1}{\alpha}\left(\frac{1}{s} - \frac{e^{-s\alpha}}{s}\right)$$



$= \frac{-e^{-s\alpha}}{\alpha s}$

In the limit as α goes to zero, the Laplace transform goes to unity.

Unit pulse: $\lim_{\alpha\to 0}\left(\frac{1-e^{-s\alpha}}{\alpha s}\right) = \infty \underset{\text{l'Hopital's}}{\Rightarrow} \lim_{\alpha\to 0}\left(\frac{se^{-s\alpha}}{s}\right) = 1$

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Derivative of step is unit impulse. $\left(\frac{1}{s}\right)(s) = 1$

(step)(derivative)

Laplace Transform of the Derivative of a Function

$$\mathcal{L}\left[\frac{df(t)}{dt}\right] = ?$$

Consider $\displaystyle\int_0^\infty f(t)e^{-st}dt = f(t)\left.\frac{e^{-st}}{-s}\right|_0^\infty - \int_0^\infty \left[\frac{df(t)}{dt}\right]\frac{e^{-st}}{-s}dt$

$$\Rightarrow F(s) = \frac{f(0)}{s} + \frac{1}{s}\mathcal{L}\left[\frac{df(t)}{dt}\right]$$

$$\Rightarrow \mathcal{L}\left[\frac{df(t)}{dt}\right] = sF(s) - f(0)$$

$$\frac{df(t)}{dt} = g(t)$$

$$\mathcal{L}\left[\frac{d^2f(t)}{dt^2}\right] = \mathcal{L}\left[\frac{dg(t)}{dt}\right] = s\mathcal{L}[g(t)] - g(0) = s\left(\mathcal{L}\left[\frac{df(t)}{dt}\right]\right) - \dot{f}(0)$$

$$= s^2F(s) - sf(0) - \dot{f}(0)$$

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Laplace Transform of the Derivative of a Function

Similarly

$$\mathcal{L}\left[\frac{d^n f(t)}{dt^n}\right] = s^n F(s) - s^{n-1} f(0) - s^{n-2}\dot{f}(0) - \ldots - \frac{d^{n-1} f}{dt^{n-1}}(0)$$

$s^3 F - s^2 f(0) - s\dot{f}(0) - \ddot{f}(0)$

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Laplace Transform of a Function multiplied by e^{-at}

Let f(t) be Laplace transformable, its Laplace transform being F(s), then the Laplace transform of $e^{-at} f(t)$

$= \int_0^\infty e^{-(s+a)t} f(t)\,dt$ $\quad \overset{s'=s+a}{\int_0^\infty f(t) e^{-s't} dt}$

$$\mathcal{L}\left(e^{-at} f(t)\right) = \int_0^\infty e^{-at} f(t) e^{-st} dt = F(s+a)$$

We see that multiplication of f(t) by e^{-at} *has the effect of replacing s by (s+a) in the Laplace Transform*

Example:

$$\mathcal{L}[\ sin(\omega t)] = \frac{\omega}{s^2+\omega^2}$$

$$\mathcal{L}[\ e^{-at} sin(\omega t)] = \frac{\omega}{(s+a)^2+\omega^2}$$

If $f(t) = e^{-3t} \sin(5t)$

$\Rightarrow \frac{5}{(s+3)^2+25}$

$e^{t} \sin(5t) \Rightarrow \frac{5}{(s-1)^2+25}$

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Complex-Differentiation Theorem

If f(t) is Laplace transformable, then

$$\mathcal{L}[tf(t)] = -\frac{dF(s)}{ds}$$

$$\mathcal{L}\left[t^2 f(t)\right] = \frac{d^2 F(s)}{ds^2}$$

In general

$$\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n F(s)}{ds^n}$$

Example:

$$\mathcal{L}[\ sin(\omega t)] = \frac{\omega}{s^2+\omega^2}$$

$(\omega)(s^2+\omega^2)^{-1}$

$$\mathcal{L}[tsin(\omega t)] = -\frac{d}{ds}\left(\frac{\omega}{s^2+\omega^2}\right) = \frac{2s\omega}{(s^2+\omega^2)^2}$$

$\frac{(-1)(\omega)(2s)}{(s^2+\omega^2)^2}(-1)$

$= \frac{2s\omega}{(s^2+\omega^2)^2}$

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Final and Initial Value Theorem

<u>*Final Value Theorem:*</u>

If f(t) and $\frac{df(t)}{dt}$ *are Laplace transformable, if F(s) is the Laplace transform of f(t), and if* $\lim_{t\to\infty} f(t)$ *exists, then*

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

Consider

$$\lim_{s\to 0} \int_0^\infty \left[\frac{df(t)}{dt}\right] e^{-st} dt = \lim_{s\to 0} [sF(s) - f(0)]$$

Since

$$\lim_{s\to 0} e^{-st} = 1,$$

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Final and Initial Value Theorem

we have

$$\int_0^\infty \left[\frac{df(t)}{dt}\right] dt = f(t)|_0^\infty = f(\infty) - f(0) = \lim_{s\to 0} sF(s) - f(0)$$

Thus,

$$f(\infty) = \lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

Final Value theorem is applicable only if

1. *F(s) has no roots of the denominator (poles) in the complex right half plane*
2. *F(s) should have no poles on the imaginary axis, except at most one pole at s=0*

Initial Value Theorem:

If f(t) and $\frac{df(t)}{dt}$ *are Laplace transformable, if F(s) is the Laplace transform of*

f(t), and if $\lim_{s\to\infty} sF(s)$ *exists, then*

$$f(0^+) = \lim_{s\to\infty} sF(s)$$

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Inverse Laplace Transform

Example: (Distinct Poles)

$$F(s) = \frac{B(s)}{A(s)} = \frac{k(s+z_1)(s+z_2)\dots(s+z_m)}{(s+p_1)(s+p_2)\dots(s+p_n)} \text{ where } m < n$$

We can write

$$F(s) = \frac{B(s)}{A(s)} = \frac{a_1}{s+p_1} + \frac{a_2}{s+p_2} + \dots + \frac{a_n}{s+p_n}$$

where a_k – residue at the pole $s=p_k$.

$s+3 = a_1(s+2) + a_2(s+1)$

Example:

$$F(s) = \frac{(s+3)}{(s+1)(s+2)} = \frac{a_1}{(s+1)} + \frac{a_2}{(s+2)}$$

$\left(\frac{s+3}{s+2}\right) = a_1 + \frac{a_2(s+1)}{(s+2)}$

$$a_1 = \left[\cancel{(s+1)}\frac{(s+3)}{\cancel{(s+1)}(s+2)}\right]\Bigg|_{s=-1} = 2$$

Let $s=-1$

$\frac{2}{1} = a_1 + 0$

$$a_2 = \left[\cancel{(s+2)}\frac{(s+3)}{(s+1)\cancel{(s+2)}}\right]\Bigg|_{s=-2} = -1$$

$\frac{s+3}{s+1} = \frac{a_1(s+2)}{(s+1)} + a_2$

$s = -2$

$\frac{1}{-1} = a_2 + 0$

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Inverse Laplace Transform

therefore,

$$f(t) = \mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}[\frac{2}{(s+1)}] - \mathcal{L}^{-1}[\frac{-1}{(s+2)}]$$

$$f(t) = 2e^{-t} - e^{-2t} \text{ for } t > 0$$

Example:

$$F(s) = \frac{s^3 + 5s^2 + 9s + 7}{(s+1)(s+2)}$$

This has to be written as

$$s^2 + 3s + 2\,)\,s^3 + 5s^2 + 9s + 7\,(\,s+2$$
$$s^3 + 3s^2 + 2s$$
$$------$$
$$2s^2 + 7s + 7$$
$$2s^2 + 6s + 4$$
$$------$$
$$s + 3$$

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Inverse Laplace Transform

therefore,

$$F(s) = (s+2) + \frac{(s+3)}{(s+1)(s+2)}$$

$$f(t) = \frac{d}{dt}(\delta(t)) + 2\delta(t) + 2e^{-t} - e^{-2t} \text{ for } t > 0$$

Example: (Multiple Poles) - Repeated Roots

$$F(s) = \frac{s^2 + 2s + 3}{(s+1)^3}$$

$$F(s) = \frac{B(s)}{A(s)} = \frac{a_1}{(s+1)} + \frac{a_2}{(s+1)^2} + \frac{a_3}{(s+1)^3}$$

$$a_3 = \left[(s+1)^3 F(s)\right]\Big|_{s=-1}$$

$= s^2+2s+3 = a_1(s+1)^2 + \frac{a_2}{(s+1)} + a_3$

$$(s+1)^3\frac{B(s)}{A(s)} = a_1(s+1)^2 + a_2(s+1) + a_3$$

$$\therefore a_3 = \left[(s+1)^3\frac{B(s)}{A(s)}\right]\Bigg|_{s=-1}$$

$\Rightarrow a_3 = 2$

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$a_1(s^2+2s+1) + a_2(s+1)$

$a_1 s^2 + (2a_1 + a_2)s + (a_1 + a_2)$

$a_1 = 1$

$2(1) + a_2 = 2 \quad a_2 = 0$

Check

$3 = a_1 + a_2 + 2$

$3 = (1) + 0 + 2$ ✓

$s^2 + 2s + 3 = a_1(s+1)^2 + a_2(s+1) + 2$

$\frac{d}{ds} \Rightarrow 2s + 2 = 2a_1(s+1) + a_2$

$a_2 = 0$

$\frac{d}{ds} \Rightarrow 2 = 2a_1$

$a_1 = 1$

Inverse Laplace Transform

Consider,

$$\frac{d}{ds}\left[(s+1)^3\frac{B(s)}{A(s)}\right] = 2a_1(s+1) + a_2$$

$$\therefore a_2 = \frac{d}{ds}\left[(s+1)^3\frac{B(s)}{A(s)}\right]\Bigg|_{s=-1}$$

$$\therefore a_2 = \frac{d}{ds}\left[(s^2+2s+3)\right]\Big|_{s=-1} = 0$$

Similiarly

$$2a_1 = \frac{d^2}{ds^2}\left[(s+1)^3\frac{B(s)}{A(s)}\right]\Bigg|_{s=-1}$$

$$2a_1 = \frac{d^2}{ds^2}\left[(s^2+2s+3)\right]\Big|_{s=-1} \Rightarrow a_1 = 1$$

$F(s) = \frac{1}{s+1} + \frac{2}{(s+1)^3}$

OR compare coefficients $\quad \mathcal{L}^{-1}(F) = e^{-t} + t^2 e^{-t}$

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Solving Differential Equations

Consider the differential equation

$$\ddot{x} + 3\dot{x} + 2x = 0, \quad x(0) = a, \dot{x}(0) = b$$

Define

$$X(s) = \mathcal{L}[x(t)]$$

$$\mathcal{L}[\dot{x}(t)] = sX(s) - x(0)$$

$$\mathcal{L}[\ddot{x}(t)] = s^2X(s) - sx(0) - \dot{x}(0)$$

Therefore

$$\left[s^2X(s) - sx(0) - \dot{x}(0)\right] + 3\left[sX(s) - x(0)\right] + 2X(s) = 0$$

$$(s^2+3s+2)X(s) = sx(0) + \dot{x}(0) + 3x(0)$$

$a_1 = 2a+b$

$a_2 = -(a+b)$

$$X(s) = \frac{(s+3)a+b}{s^2+3s+2} = \frac{2a+b}{s+1} - \frac{a+b}{s+2} = \frac{a_1}{s+1} + \frac{a_2}{s+2}$$

$$x(t) = (2a+b)e^{-t} - (a+b)e^{-2t} \quad \forall t > 0$$

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Solving Differential Equations

Consider the differential equation

$$\ddot{x} + 2\dot{x} + 5x = 3, \quad x(0) = 0, \dot{x}(0) = 0$$

$X(s) = \frac{a_1}{s} + \frac{a_2 s + a_3}{s^2+2s+5}$

Laplace transformation leads to:

$$(s^2+2s+5)X(s) = \frac{3}{s}$$

$\frac{3}{s^2+2s+5} = a_1 + \frac{a_2 s^2 + a_3 s}{s^2+2s+5}\Big|_{s=0}$

$a_1 = \frac{3}{5}$

Therefore

$$X(s) = \frac{3}{s(s^2+2s+5)} = \frac{3}{5s} - \frac{3}{10((s+1)^2+2^2)} - \frac{3(s+1)}{5((s+1)^2+2^2)}$$

$$x(t) = \frac{3}{5} - \frac{3}{10}e^{-t}sin(2t) - \frac{3}{5}e^{-t}cos(2t) \quad \forall t > 0$$

$3 = a_1(s^2+2s+5) + (a_2 s + a_3)s$

$3 = (\frac{3}{5} + a_2)s^2 + (\frac{6}{5} + a_3)s + 3$

$X = \frac{3}{5s} - \frac{\frac{3}{5}s + \frac{6}{5}}{(s+1)^2+4} = \frac{3}{5s} - \frac{\frac{3}{5}(s+1) + \frac{3}{5}}{(s+1)^2+4}$

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Transfer Function

The transfer function of a linear time-invariant system is defined to be the ratio of the Laplace transform of the output to the Laplace transform of the input under the assumption of zero initial conditions.

Consider the differential equation:

$$a_0y^n + a_1y^{n-1} + \ldots + a_ny = b_0u^m + b_1u^{m-1} + \ldots + b_mu,$$

Where y is the output of the system and u is the input. The transfer function is given by taking the Laplace transform of the above equation:

$$\left(a_0s^n + a_1s^{n-1} + \ldots + a_n\right)Y(s) = \left(b_0s^m + b_1s^{m-1} + \ldots + b_m\right)U(s),$$

$$G(s) = \frac{Y(s)}{U(s)} = \frac{\left(b_0s^m + b_1s^{m-1} + \ldots + b_m\right)}{\left(a_0s^n + a_1s^{n-1} + \ldots + a_n\right)}$$

•The Transfer function is a property of the system itself and not a function of the input function

•Transfer function of many physically different systems can be identical. eg. Spring-mass-dashpot and RLC circuit.

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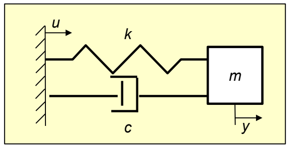
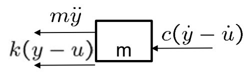
Example

For the spring-mass dashpot system



Force balance leads to the free body diagram



$$m\ddot{y} + c\dot{y} + ky = ku + c\dot{u}$$

Assuming the initial conditions to be zero, and taking the Laplace transform of the above equation leads to:

$$\frac{Y(s)}{U(s)} = G(s) = \frac{cs + k}{ms^2 + cs + k}$$

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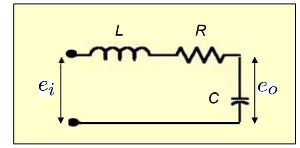
Example

For the R-L-C system



Kirchoff's law gives us

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = e_i$$

$$\frac{1}{C}\int i\,dt = e_o$$

Assuming the initial conditions to be zero, and taking the Laplace transform of the above equation leads to:

$$\left(Ls + R + \frac{1}{Cs}\right) I(s) = E_i(s)$$

$$\frac{1}{Cs} I(s) = E_o(s)$$

If e_i and e_o are the inputs and outputs respectively, we have

$$\frac{I(s)}{E_i(s)} = \frac{1}{Ls + R + \frac{1}{Cs}} = \frac{Cs}{LCs^2 + RCs + 1} \qquad \frac{E_o(s)}{I(s)} = \frac{1}{Cs}$$

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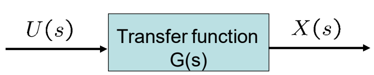
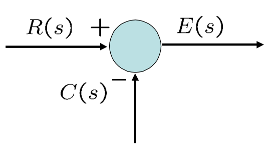
Block Diagrams

A block diagram of a system is a pictorial representative of the function performance by each component and of the flow of signals. The block is a symbol for the mathematical operation on the input signal to the block.



Error Detector: The error detector produces a signal which is the difference between the reference input and the feedback signal of the control system.



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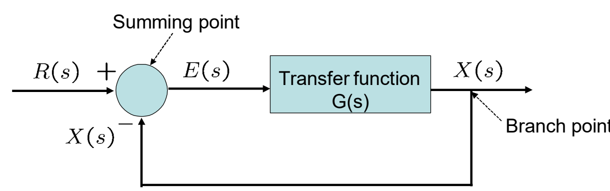
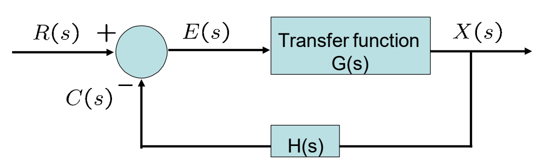
Block Diagram (Closed Loop)



When the output signal is fed back to the summing point for comparison with the input, it is necessary to convert the form of the output signal to that of the input signal. This is accomplished by the feedback element whose transfer function is H(S)



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Block Diagram (Closed Loop)

The ratio of the feedback signal to the actuating error signal E(s) is called the open-loop transfer function. That is

$$\frac{C(s)}{E(s)} = G(s)H(s) \quad \text{Open Loop TF}$$

The ratio of the output X(s) to the actuating signal E(s) is called the feedforward transfer function:

$$\frac{X(s)}{E(s)} = G(s) \quad \text{Feedforward TF}$$

We can see from the closed loop block diagram:

$$X(s) = G(s)E(s)$$

$$E(s) = R(s) - C(s) = R(s) - H(s)X(s)$$

Eliminating E(s) from the above equations, we have

$$X(s) = G(s)\left(R(s) - H(s)X(s)\right)$$

$$\Rightarrow \frac{X(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} \quad \text{Closed Loop TF}$$

This is called the closed loop transfer function

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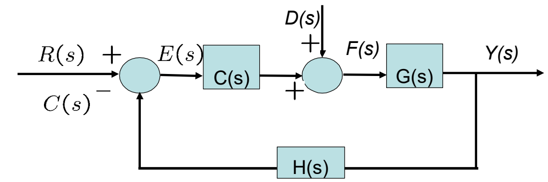
Block Diagram Reduction



$$E = R - HY \qquad U = CE$$

$$F = D + U \qquad Y = GF$$

$$E = R - HGF = R - HG(D + U)$$

$$E = R - HGD - HGCE$$

$$E = \frac{1}{1 + HGC}R - \frac{HG}{1 + HGC}D$$

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Block Diagram Reduction

$$Y = GD + CGE$$

$$Y = GD + CG\left(\frac{1}{1 + HGC}R - \frac{HG}{1 + HGC}D\right)$$

$$Y = G\frac{1 + HGC}{1 + HGC}D + CG\left(\frac{1}{1 + HGC}R - \frac{HG}{1 + HGC}D\right)$$

$$Y = \frac{G}{1 + HGC}D + \frac{CG}{1 + HGC}R$$

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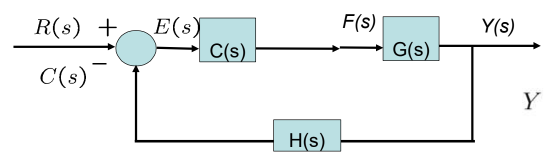
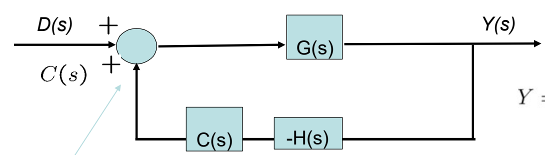
Block Diagram Reduction (Superposition)

Assume D = 0,



$$Y = \frac{CG}{1 + HGC}R$$

Assume R = 0,



$$Y = \frac{G}{1 - (-H)GC}D$$

Note: Positive Feedback)

$$Y = \frac{G}{1 + HGC}D + \frac{CG}{1 + HGC}R$$

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First Order System

Transfer function:

$$\frac{Y(s)}{R(s)} = \frac{1}{Ts+1}$$

Response to a unit step input is: step : $R = \frac{1}{s}$

$$Y(s) = \frac{1}{Ts+1}\frac{1}{s}$$

Partial Fraction Expansion leads to: $Y = \frac{a_0}{s} + \frac{a_1}{Ts+1}$

$$Y(s) = \frac{1}{s} - \frac{T}{Ts+1} = \frac{1}{s} - \frac{1}{s+1/T}$$

Inverse Laplace transform leads to:

$$y(t) = 1 - e^{-\frac{t}{T}}$$

time constant

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First Order System

At t = T, the output is:

$$y(t) = 1 - exp(-1) = 0.632$$

1 time constant

T represents the time required for the system response to reach 63.2% of the final value. T is referred to as the ***Time Constant*** *of the system*

The slope of the system response at time = 0 is:

$$\frac{dy(t)}{dt} = \frac{1}{T}e^{-\frac{0}{T}} = \frac{1}{T}$$

Response of the first order system to a unit ramp is: $R(s) = \frac{1}{s^2}$

$$Y(s) = \frac{1}{Ts+1}\frac{1}{s^2} = \frac{1}{s^2} - \frac{T}{s} + \frac{T^2}{Ts+1}$$

$$y(t) = t - T + Te^{-\frac{t}{T}}$$

Tracking error as time tends to infinity is:

$$e(t \to \infty) = t - y(t) = T - T\cancel{e^{-\frac{t}{T}}} = T$$

→ 0 as $t \to \infty$

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First Order System

Half-Life*: Time it takes for a quantity to reduce to half its initial value. For this example, velocity to decrease to half of its initial value.*

The log of the evolving output v(t) is given by the equation:

$$log(v(t)) = log(v_0) - \frac{c}{m}t$$

which can be rewritten as the equation:

$$log(v(t)) - log(v_0) = log(\frac{v(t)}{v_0}) = -\frac{c}{m}t$$

Have life is the time required for the output to reach half its initial value, therefore:

$$log(\frac{v(t)}{v_0}) = log(\frac{1}{2}) = -\frac{c}{m}t$$

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Second Order System

Transfer function:

$$\frac{Y(s)}{R(s)} = \frac{\omega^2}{s^2 + 2\zeta\omega s + \omega^2}$$

ω	natural frequency
ζ	damping ratio

$$\frac{Y(s)}{R(s)} = \frac{\omega^2}{(s+\zeta\omega)^2 + (\omega\sqrt{1-\zeta^2})^2}$$

Response to a unit step input is: $R = 1/s$

$$Y(s) = \frac{\omega^2}{(s+\zeta\omega)^2 + (\omega\sqrt{1-\zeta^2})^2}\frac{1}{s} = \frac{a_0}{s} + \frac{a_1 s + a_2}{s^2 + 2\zeta\omega s + \omega^2}$$

$$y(t) = 1 - e^{-\zeta\omega t}\left(cos(\omega_d t) + \frac{\zeta}{\sqrt{1-\zeta^2}}sin(\omega_d t)\right)$$

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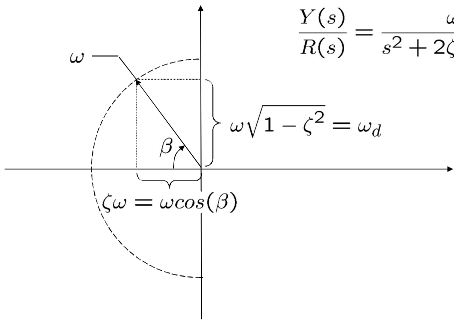
S-plane representation

Laplace transform with zero initial conditions leads to:

$$\frac{Y(s)}{R(s)} = \frac{\omega^2}{s^2 + 2\zeta\omega s + \omega^2}$$



Poles Located at: $s = -\zeta\omega \pm j\omega\sqrt{1-\zeta^2}$

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Transient Response

In analyzing and designing control systems, we must have a basis of comparison of performance of various control systems. This basis may be set up by specifying particular test input signals and by comparing the response of various systems to these input signals. Typical test signals: Step function, ramp function, impulse function, sinusoid function.

The time response of a control system consists of two parts: the transient and the steady-state response. Transient response corresponds to the behavior of the system from the initial state to the final state. By steady state, we mean the manner in which the system output behaves as time approaches infinity.

For a step input, the transient response can be characterized by:

Delay time t_d: time to reach half the final value for the first time.

Rise time t_r: time required for the response to rise from 10% to 90% for overdamped systems, and from 0% to 100% for underdamped systems

Peak time t_p: time required to reach the first peak of the overshoot

Percent Overshoot M_p: $M_p = \frac{y(t_p) - y(\infty)}{y(\infty)}$

Settling time t_s: time required for the response curve to reach and stay within 2% or 5% of the final value. Is a function of the largest time constant of the control system.

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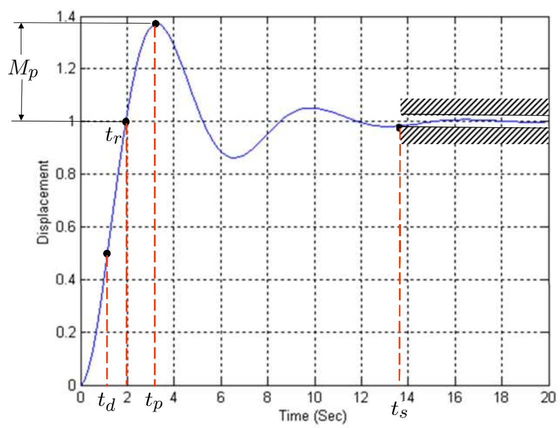
Transient Response



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Transient Response

For a step input, the transient response can be characterized by:

Rise time t_r: time required for the response to rise from 10% to 90% for overdamped systems, and from 0% to 100% for underdamped systems

$$y(t_r) = 1 = 1 - e^{-\zeta\omega t_r}\left(cos(\omega_d t_r) + \frac{\zeta}{\sqrt{1-\zeta^2}} sin(\omega_d t_r)\right)$$

Since $e^{-\zeta\omega t_r} \neq 0$

$$cos(\omega_d t_r) + \frac{\zeta}{\sqrt{1-\zeta^2}} sin(\omega_d t_r) = 0$$

$$\Rightarrow tan(\omega_d t_r) = -\frac{\sqrt{1-\zeta^2}}{\zeta} = -\frac{\omega_d}{\sigma}$$

Thus, the rise time is:

$$t_r = \frac{1}{\omega_d} tan^{-1}\left(\frac{\omega_d}{-\sigma}\right) = \frac{\pi - \beta}{\omega_d}$$

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Transient Response

For a step input, the transient response can be characterized by:

Peak time t_p: time required to reach the first peak of the overshoot

$$\frac{dy}{dt} = 0 = \zeta\omega e^{-\zeta\omega t}\left(\cancel{cos(\omega_d t)} + \frac{\zeta}{\sqrt{1-\zeta^2}}sin(\omega_d t)\right)$$
$$-e^{-\zeta\omega t}\left(-\omega_d \sin(\omega_d t) + \frac{\omega_d\zeta}{\sqrt{1-\zeta^2}}\cancel{cos(\omega_d t)}\right)$$

$$\frac{dy}{dt} = 0 = e^{-\zeta\omega t_p} sin(\omega_d t_p)\frac{\omega}{\sqrt{1-\zeta^2}}$$

Thus, the peak time is:

$$\omega_d t_p = 0, \pi, 2\pi, \ldots$$

$$t_p = \frac{\pi}{\omega_d}$$

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Transient Response

For a step input, the transient response can be characterized by:

Percent Overshoot M_p:

$$M_p = y(t_p) - 1 = -e^{-\zeta\omega\frac{\pi}{\omega_d}}\left(cos(\pi) + \frac{\zeta}{\sqrt{1-\zeta^2}}sin(\pi)\right)$$
$$= e^{-\frac{\sigma}{\omega_d}\pi}$$
$$= e^{-\frac{\zeta}{\sqrt{1-\zeta^2}}\pi}$$

Maximum overshoot is function of damping ratio only.

Maximum percent overshoot is: $M_p = e^{-\frac{\zeta}{\sqrt{1-\zeta^2}}\pi} \times 100$

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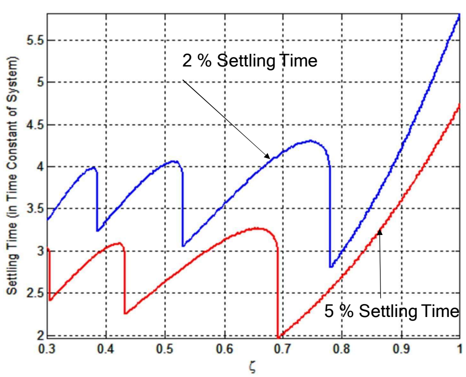
Transient Response

For a step input, the transient response can be characterized by:

Settling time t_s:



2% Settling time:

$$t_s = 4T = \frac{4}{\sigma} = \frac{4}{\zeta\omega}$$

5% Settling time:

$$t_s = 3T = \frac{3}{\sigma} = \frac{3}{\zeta\omega}$$

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Routh Stability Criterion

Consider the closed loop transfer function of the form:

$$G(s) = \frac{b_0 s^m + b_1 s^{m-1} + \ldots + b_{m-1}s + b_m}{a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n}$$

where a's and b's are constant and where m< n.

If any of the coefficients are zero or negative in the presence of at least one positive coefficient, there is a root or roots which are imaginary or which have positive real parts.

If all the coefficients are positive, arrange the coefficients of the polynomial in rows and columns according to the pattern:

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Routh Stability Criterion

s^n	a_0	a_2	a_4	a_6	…
s^{n-1}	a_1	a_3	a_5	a_7	…
s^{n-2}	b_1	b_2	b_3	b_4	…
s^{n-3}	c_1	c_2	c_3	c_4	…
s^{n-4}	d_1	d_2	d_3	d_4	…
…	…	…	…	…	…
s^2	e_1	e_2			
s^1	f_1				
s^0	g_1				

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1}$$
$$b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1}$$
$$b_3 = \frac{a_1 a_6 - a_0 a_7}{a_1}$$
…
$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1}$$
$$c_2 = \frac{b_1 a_5 - a_1 b_3}{b_1}$$
…
$$d_1 = \frac{c_1 b_2 - b_1 c_2}{c_1}$$

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Routh Stability Criterion

Routh's stability criterion states that the number of roots of the system G(s) with positive real parts is equal to the number of changes in the sign of the coefficients of the first column of the array.

The necessary and sufficient condition that all poles of G(s) lie in the left half plane is that all the coefficient of the denominator of G(s) be positive and all terms in the first column of the array have positive signs.

Special Case:

If a first-column term in any row is zero, but the remaining terms are not zero or there is no remaining term, then the zero term is replaced by a very small positive number ϵ *and the rest of the array is evaluated.*

$$s^3 + 2s^2 + s + 2 = 0$$

s^3	1	1
s^2	2	2
s	$0 \approx \epsilon$	
s^0	2	

If the sign of the coefficient above the zero is the same as that below it, it indicates that there are a pair of poles on the imaginary axis.

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Routh Stability Criterion

Special Case:

If all the coefficients in any derived row are zero, it indicates that there are roots of equal magnitude lying radially opposite in the s-plane, eg. two real roots with equal magnitudes and opposite signs and/or two conjugate imaginary roots

$$s^5 + 2s^4 + 24s^3 + 48s^2 - 25s - 50 = 0$$

s^5	1	24	-25
s^4	2	48	-50
s^3	0	0	

In such a case, the evaluation of the rest of the array can be continued by forming an auxiliary polynomial with the coefficient of the last row and the coefficients of the derivative of this polynomial in the next row

$$P(s) = 2s^4 + 48s^2 - 50$$
$$\frac{dP(s)}{ds} = 8s^3 + 96s$$

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Routh Stability Criterion

The closed loop system can be represented as:

$$\frac{X_1(s)}{U(s)} = \frac{(K_1 + K_2 s)(s^2+1)}{s^4 + 2s^2 + (K_2 s + K_1)(s^2+1)}$$

The Routh table for the closed loop system is:

s^4	1	$2 + K_1$	K_1
s^3	K_2	K_2	
s^2	$\dfrac{K_2 + K_1 K_2}{K_2}$	K_1	
s^1	$\dfrac{K2}{1 + K1}$		
s^0	K_1		

The second and fifth row indicate that :

$$K_1 > 0 \text{ and } K_2 > 0$$

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Routh Stability Criterion

The third and fourth row requires:

$$(1+K_1) > 0 \text{ or } K_1 > -1$$

The requirement:

$$\begin{aligned} K_1 > 0 &\text{ and } K_1 > -1 \\ \Rightarrow \qquad & K_1 > 0 \end{aligned}$$

Results in a stable controller for all gains which satisfy the inequality constraints:

$$K_1 > 0 \text{ and } K_2 > 0$$

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Routh Stability Criterion (Relative Stability)

Relative Stability: The Routh criteria is a tools which provides a binary answer to the question of *absolute stability*, i.e., whether the system is stable or not. Relative stability permits comparing two system and gauging which system is relatively more stable. A simple characterization of relative stability is the distance of the pole with the largest real value from the imaginary axis. Closer the pole is to the imaginary axis, smaller is its relative stability. Once can use the Routh Criteria to determine the number of poles that lie to the right of a shifted imaginary axis. Substitute s = z − σ (σ = constant) into the characteristic equation of the system and rewrite the characteristic equation in terms of z. Applying the Routh Criteria to new polynomial in z permits one to determine the number of poles to the right of the vertical line s = −σ. Thus, this test reveals the number of roots which lie to the right of the vertical line s = −σ

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Static Error Coefficients

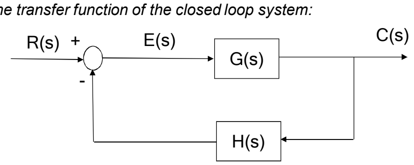
Consider the open loop transfer function:

$$G(s)H(s) = \frac{K(T_a s+1)(T_b s+1)...(T_{m-1}s+1)(T_m s+1)}{s^N(T_1 s+1)(T_2 s+1)...(T_n s+1)}$$

which includes N poles at the origin of the s-plane.

A system is called type 0, type 1, type 2,…., if N = 0,1,2,…, respectively.

The transfer function of the closed loop system:



$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

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Static Error Coefficients

and the error transfer function can be calculated as:

$$\begin{aligned} E(s) &= R(s) - H(s)C(s) \\ \frac{E(s)}{R(s)} &= 1 - H(s)\frac{C(s)}{R(s)} \\ \frac{E(s)}{R(s)} &= 1 - \frac{G(s)H(s)}{1+G(s)H(s)} \\ \frac{E(s)}{R(s)} &= \frac{1}{1+G(s)H(s)} \end{aligned}$$

The steady state error can be calculated as:

$$e_{ss} = \lim_{t\to\infty} e_{ss}(t) = \lim_{s\to 0}\frac{sR(s)}{1+G(s)H(s)}$$

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Static Position Error Coefficients K_p

The steady-state error of a system subject to an unit step input is:

$$e_{ss} = \lim_{s\to 0} \frac{s}{1+G(s)H(s)}\frac{1}{s}$$
$$e_{ss} = \frac{1}{1+G(0)H(0)}$$

The static position error coefficient K_p is defined as:

$$K_p = \lim_{s\to 0} G(s)H(s) = G(0)H(0)$$

The steady state error is given by the equation:

$$e_{ss} = \frac{1}{1+K_p}$$

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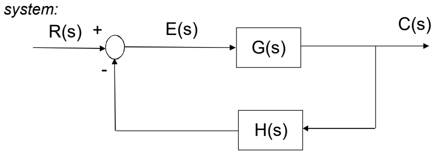
Root Locus

The locus of the roots of the closed loop system as a function of a system parameter, as it is varied from 0 to infinity results in the moniker, Root-Locus.

The root-locus permits determination of the closed-loop poles given the open-loop poles and zeros of the system.

For the system:



the closed loop transfer function is:

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

The characteristic equation:

$$1+G(s)H(s)=0$$

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Root Locus

when solved results in the poles of the closed-loop system.

The characteristic equation can be written as:

$$G(s)H(s) = -1$$
$$G(s)H(s) = e^{\pi j}$$

Since G(s)H(s) is a complex quantity, the equation can be rewritten as:

Angle criterion: $\angle G(s)H(s) = \pm\pi(2k+1),\ k=0,1,2,3,\ldots$

Magnitude criterion: $|G(s)H(s)| = 1$

The values of s which satisfy the magnitude and angle criterion lie on the root locus.

Solving the angle criterion alone results in the root-locus. The magnitude criterion locates the closed loop poles on the locus.

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Root Locus

Often the open-loop transfer function G(s)H(s) involved a gain parameter K, resulting in the characteristic equation:

$$1 + \frac{K(s+z_1)(s+z_2)\ldots(s+z_m)}{(s+p_1)(s+p_2)\ldots(s+p_n)} = 0$$

Then the root-loci are the loci of the closed loop poles as K is varied from 0 to infinity.

To sketch the root-loci, we require the poles and zeros of the open-loop system. Now, the angle and magnitude criterion can be schematically represented as, for the system:

$$G(s)H(s) = \frac{K(s+z_1)}{(s+p_1)(s+p_2)(s+p_3)(s+p_4)}$$

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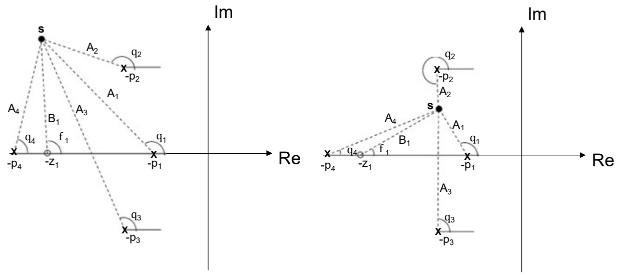
Root Locus



$$\angle G(s)H(s) = \phi_1 - \theta_1 - \theta_2 - \theta_3 - \theta_4$$

$$|G(s)H(s)| = \frac{KB_1}{A_1A_2A_3A_4}$$

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Root Locus (Example)

Asymptotes of the root-loci as s tends to infinity can be determined as follows: Select a test point far from the origin:

$$\lim_{s\to\infty} G(s) = \lim_{s\to\infty} \frac{K}{s(s+1)(s+2)} = \lim_{s\to\infty} \frac{K}{s^3}$$

and the angle criterion is:

$$-3\angle s = \pm\pi(2k+1) \quad \text{k=0,1,2,3,...}$$

or, the angle of the asymptotes are:

$$\text{angle of asymptotoes } = \pm\frac{\pi(2k+1)}{3} \quad \text{k=0,1,2,3,...}$$

For this example, they are 60º, -60º, -180º.

To draw the asymptotes, we must find the point where they intersect the real-axis. Since, the transfer function for a test point s far from the origin permits us to approximate the transfer function as:

$$G(s) = \frac{K}{s(s+1)(s+2)} = \frac{K}{s^3+3s^2+\ldots} \approx \frac{K}{(s+1)^3}$$

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Root Locus (Example)

The angle criterion for the approximated system is:

$$\angle\frac{K}{(s+1)^3} = -3\angle(s+1) = \pm\pi(2k+1)$$

$$\Rightarrow \angle(s+1) = \pm\frac{\pi}{3}(2k+1) \quad \text{k=0,1,2,3,...}$$

Substituting $s = \sigma + j\omega$ *, we have*

$$\angle\sigma + j\omega + 1 = \pm\frac{\pi}{3}(2k+1)$$

$$tan^{-1}\left(\frac{\omega}{\sigma+1}\right) = \frac{\pi}{3}, -\frac{\pi}{3}, -\pi$$

Taking the tangent of both sides, we have:

$$\left(\frac{\omega}{\sigma+1}\right) = \sqrt{3}, -\sqrt{3}, 0$$

which can be written as:

$$\sigma + 1 - \frac{\omega}{\sqrt{3}} = 0, \sigma + 1 + \frac{\omega}{\sqrt{3}} = 0, \omega = 0$$

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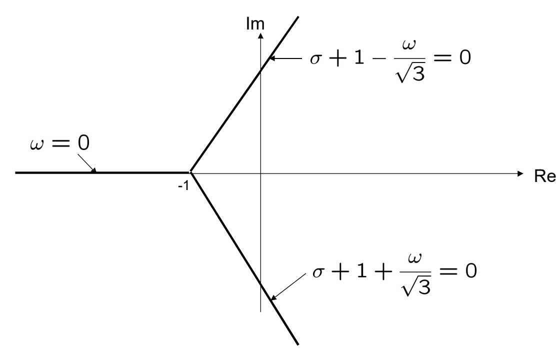
Root Locus (Example)

The three equations result is three straight lines shown below, which are the asymptotes, which meet at the point s=-1. Thus, the abscissa of the intersection of the asymptotes and the real-axis is determined by setting the denominator of the approximated transfer function to zero and solving for s.



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Root Locus (Example)

The root loci starting from 0 and -1, break away from the real axis as K is increased. The breakaway point corresponds to a point in the s-plane where multiple roots of the characteristic equation occur.

Representing the characteristic equation as:

$$f(s) = B(s) + KA(s) = 0 \qquad (1)$$

We know that f(s) has multiple roots at $s = s_1$, if

$$\left.\frac{df(s)}{ds}\right|_{s=s_1} = \frac{dB(s)}{ds} + K\frac{dA(s)}{ds} = 0$$

$$\Rightarrow K = -\frac{B'(s)}{A'(s)}, \text{ where } A'(s) = \frac{dA(s)}{ds}, B'(s) = \frac{dB(s)}{ds}$$

Which when substituted into Equation 1, gives us:

$$f(s) = B(s) - \frac{B'(s)}{A'(s)}A(s) = 0$$

$$B(s)A'(s) - B'(s)A(s) = 0$$

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Root Locus (Example)

Consider the equation:

$$f(s) = B(s) + KA(s) = 0$$

which can be rewritten as:

$$K = -\frac{B(s)}{A(s)}$$

The derivative of K with respect to s gives:

$$\frac{dK}{ds} = \frac{B(s)A'(s) - B'(s)A(s)}{A^2(s)}$$

Which when set to zero gives the same constraint as before:

$$B(s)A'(s) - B'(s)A(s) = 0$$

Therefore, the breakaway point can be determined by setting:

$$\frac{dK}{ds} = \frac{B(s)A'(s) - B'(s)A(s)}{A^2(s)} = 0$$

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Root Locus (Example)

For the current example, we have:

$$\frac{K}{s(s+1)(s+2)} + 1 = 0$$

$$K = -(s^3 + 3s^2 + 2) = 0$$

The derivative of K with respect to s gives:

$$\frac{dK}{ds} = -(3s^2 + 6s + 2) = 0$$

or

$$s = -0.4226, s = -1.5774$$

Since the breakaway point must lie between 0 and -1, s = -0.4226 corresponds to the actual breakaway point and the corresponding gain is:

$$K = 0.3849$$

The gain corresponding to the point s = -1.5774 is:

$$K = -0.3849$$

Which belongs to the complementary root-locus.

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Root Locus (Example)

Locations where the Root-loci crosses the imaginary axis can be determined by substituting s = jwin the characteristic equation as solving for K and s.

For the current example, we have

$$(j\omega)^3 + 3(j\omega)^2 + 2(j\omega) + K = 0$$

or

$$K - 3\omega^2 + j(2\omega - \omega^3) = 0$$

Equating the real and imaginary parts to zero, we have

$$K - 3\omega^2 = 0, \text{ and } (2\omega - \omega^3) = 0$$

which results in the solutions:

$$\omega = \pm\sqrt{2}, K = 6, \text{ or } \omega = 0, K = 0$$

Thus, the root locus cross the imaginary axis at $s = \pm j\sqrt{2}$

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Root Locus (Example)

Consider the system with an open-loop transfer function:

$$G(s) = \frac{K(s+2)}{s^2+2s+3}$$

which has open loop poles at:

$$s = -1 + j\sqrt{2} \quad s = -1 - j\sqrt{2}$$

and a zero at $s = -2$

From the rule about existence of root-loci on the real axis, it is clear that to the left of the point s=-2, the entire real axis is part of the root-loci.

Since root-loci start at open-loop poles and end at open-loop zeros, we know that the two root-loci which start at the complex open loop poles should break in on the real axis and one of the root-loci should move towards s = -2, and the other towards $-\infty$

How does the root locus depart from $s = -1 \pm j\sqrt{2}$ *?*

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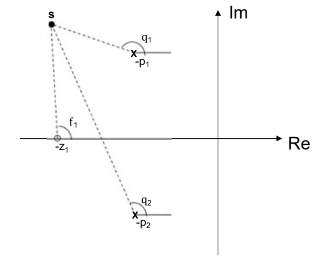
Root Locus (Example)

Consider a test point s in the vicinity of the complex open-loop pole p_1*:*



The angular contribution of the pole p_2 *and the zero* z_1 *to the test point s can be considered to be the same as the angle made by the lines connecting the pole* p_2 *and zero* z_1 *to the pole* p_1 *(if the test point is close to* p_1*).*

Then, the angle of departure t1 is given by the angle criterion:

$$\phi_1 - (\theta_1 + \theta_2) = \pm\pi(2k+1)$$

$$\theta_1 = \pi - \theta_2 + \phi_1 = \pi - \frac{\pi}{2} + atan(\frac{\sqrt{2}}{1})$$

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Root Locus (sketching rules)

Rule 3: A point on the real axis lies on a locus if number of open-loop poles plus zeros on the real axis to the right of this point are odd.

Rule 4: The (n-m) branches of the root loci which tend to infinity do so along straight line asymptotes whose angles are:

$$\phi_A = \frac{(2k+1)\pi}{(n-m)} \quad k = 0,1,2,3,\ldots$$

Rule 5: The asymptotes cross the real axis at a point given by:

$$-\sigma_A = \frac{\sum_{j=1}^{n}(-p_j) - \sum_{i=1}^{m}(-z_i)}{(n-m)}$$

$$= \frac{\sum \text{real part of poles} - \sum \text{real part of zeros}}{\text{number of poles - number of zeros}}$$

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Root Locus (sketching rules)

Rule 6: Break points (points at which multiple roots of the characteristic equation occur) of root locus are solutions of dk/ds = 0.

$$\phi_p = \pm\pi(2k+1) + \phi \quad k = 0,1,2,3,\ldots$$

Rule 7: The angle of departure from an open-loop pole is given by:

$$\phi_z = \pm\pi(2k+1) - \phi \quad k = 0,1,2,3,\ldots$$

where f is the net angle contribution at the pole of all other open-loop poles and zeros. Similarly, the angle of arrival at an open-loop zero is given by:

where f is the net angle contribution at the pole of all other open-loop poles and zeros.

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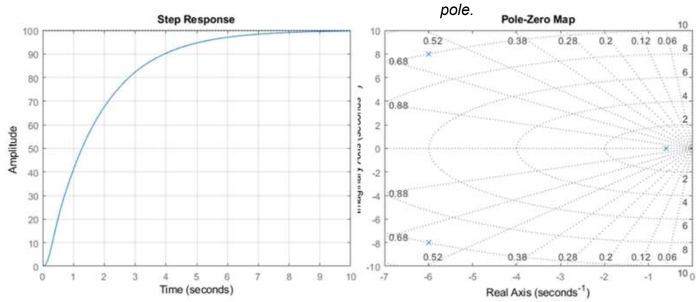
Dominant Poles

- *The "slowest poles" of a system (those closest to the imaginary axis in the s-plane) give rise to the longest lasting terms in the transient response of the system.*
- *If a pole or set of poles are very slow compared to others in the transfer function, then they generally dominate the transient response and are referred to as the "Dominant Poles".*

Example: $G(s) = \dfrac{6000}{(s^2 + 12s + 100)(s + 0.6)}$

The step response appears like a first order system response since the pole at s=-0.6 is the dominant pole.



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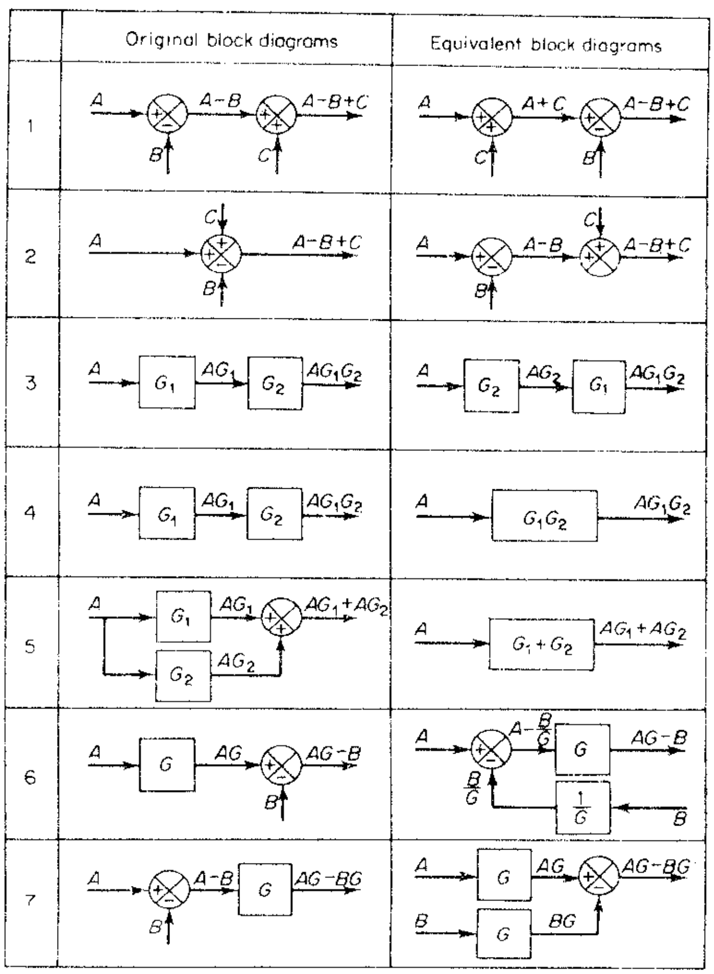
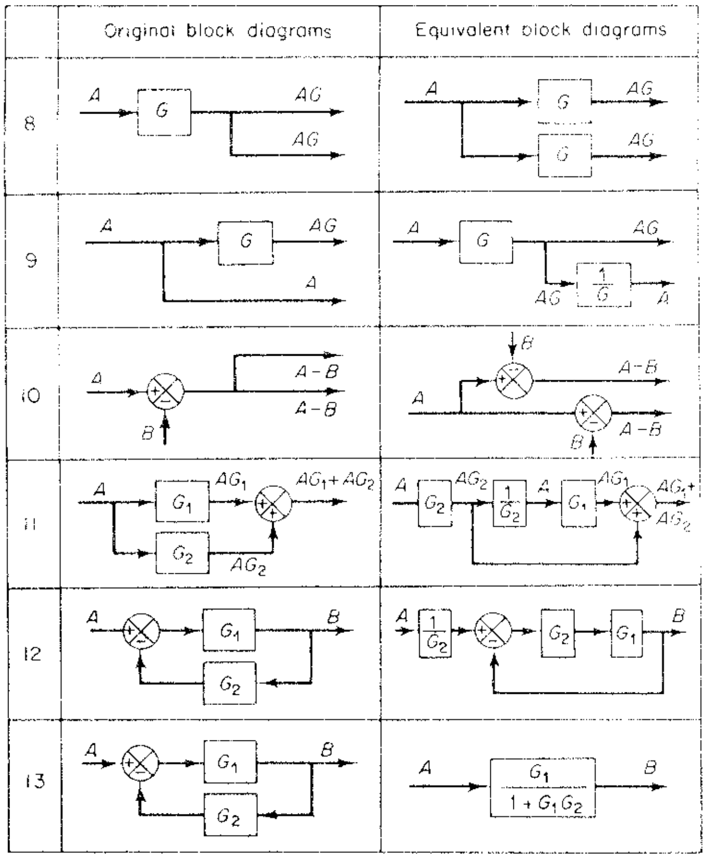
A general rule for simplifying a block diagram is to move branch points and summing points, interchange summing points, and then reduce internal feedback loops.

As an example of the use of the rules in Table 4-3, consider the system shown in Fig. 4-20 (a). By moving the summing point of the negative feedback loop containing H_2 outside the positive feedback loop containing H_1, we obtain Fig. 4-20 (b). Eliminating the positive feedback loop, we obtain Fig. 4-20 (c). Then, eliminating the loop containing H_2/G_1, we obtain Fig. 4-20 (d). Finally, by eliminating the feedback loop, we obtain Fig. 4-20 (e).

Notice that the numerator of the closed-loop transfer function $C(s)/R(s)$ is the product of the transfer functions of the feedforward path. The denominator of $C(s)/R(s)$ is equal to

Table 4-3. RULES OF BLOCK DIAGRAM ALGEBRA





$$1 - \sum (\text{product of the transfer functions around each loop})$$
$$= 1 - (G_1G_2H_1 - G_2G_3H_2 - G_1G_2G_3)$$
$$= 1 - G_1G_2H_1 + G_2G_3H_2 + G_1G_2G_3$$

(The positive feedback loop yields a negative term in the denominator.)

4-5 DERIVING TRANSFER FUNCTIONS OF PHYSICAL SYSTEMS

Control systems may consist of components of different types, such as electrical, mechanical, hydraulic, pneumatic, or thermal. A control engineer must be familiar with the fundamental laws underlying these components.

In Sections 4-2 and 4-3, we derived transfer functions for several systems. In this section we shall present additional examples showing the derivation of transfer functions for various types of physical systems.

In deriving transfer functions, note the following:

1. In approximating physical systems by linear lumped-parameter models,