#1a)

$$V(x) = x_1^2 - x_1^4 + x_2^2$$

Assume $x_e = \vec{0}$: For LPD - $V(x_e) = 0$ (Neccessary)

$x_e = \begin{pmatrix} x_{1e} \\ x_{2e} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\frac{\partial V}{\partial x}(x_e) = 0$ (Neccessary)

$\frac{\partial^2 V}{\partial x^2}(x_e) > 0$ (sufficient)

$V(0) = 0$ ✓

$$\frac{\partial V}{\partial x} = \begin{pmatrix} 2x_1 - 4x_1^3 & 2x_2 \end{pmatrix}$$

$$\frac{\partial V}{\partial x}(0) = \begin{pmatrix} 0 & 0 \end{pmatrix} \checkmark$$

$$\frac{\partial^2 V}{\partial x^2} = \begin{pmatrix} 2 - 12x_1^2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\frac{\partial^2 V}{\partial x^2}(0) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$\det(2) > 0$ & $\det\left(\frac{\partial^2 V}{\partial x^2}(0)\right) = 4 > 0$

$\therefore \frac{\partial^2 V}{\partial x^2}(0) > 0$ via Sylvester criteria

$\boxed{V(0) = 0,\ \frac{\partial V}{\partial x}(0) = 0,\ \&\ \frac{\partial^2 V}{\partial x^2}(0) > 0}$, therefore $V(x)$ is L.P.D about the origin.

#1b)

$$V(x) = x_1 + x_2^2, \quad x_e = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$V(0) = 0$ ✓

$$\frac{\partial V}{\partial x} = [1 \quad 2x_2]$$

$$\frac{\partial V}{\partial x}(0) = [1 \quad 0] \neq 0 \quad ✗$$

$\frac{\partial V}{\partial x}(0) \neq 0 \therefore$ Neccessary condition not met. $V(x)$ is <u>not</u> L.P.D. about origin

#1c) $V(x) = 2x_1^2 - x_1^3 + x_1 x_2 + x_2^2$ $\qquad x_e = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$V(0) = 0$ ✓

$$\frac{\partial V}{\partial x} = \begin{bmatrix} 4x_1 - 3x_1^2 + x_2 & x_1 + 2x_2 \end{bmatrix}$$

$$\frac{\partial V(0)}{\partial x} = \begin{bmatrix} 0 & 0 \end{bmatrix} \checkmark$$

$$\frac{\partial^2 V}{\partial x^2} = \begin{bmatrix} 4-6x_1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\frac{\partial^2 V}{\partial x^2}(0) = \begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix}$$

$\det(4) > 0$

$\det\left(\frac{\partial^2 V}{\partial x^2}(0)\right) = 7 > 0$

$\therefore \frac{\partial^2 V}{\partial x^2}(0) > 0$ Via sylvester criteria

$\boxed{V(0)=0, \ \frac{\partial V}{\partial x}(0) = 0, \ \& \ \frac{\partial^2 V}{\partial x^2}(0) > 0 \ \therefore V(x) \text{ is L.P.D about the origin.}}$

#2) $\dot{x}_1 = x_2$ $\quad\quad x_e = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\dot{x}_2 = -x_1^3$

Treat $\dot{x}_2$ as Conservative force so $U = -\int_{x_e}^{x_1} f(n)\, dn$

$$U = -\int_0^{x_1} -n^3\, dn = -\left[\frac{-n^4}{4} \Big|_0^{x_1} \right] = \frac{x_1^4}{4}$$

Treat x_2 as velocity with $m=1$: $T = \frac{1}{2} m \|V\|^2 = \frac{1}{2} x_2^2$

"Total Energy" Canidate Lyapunov function: $V = T + U$

$V = \frac{1}{2} x_2^2 + \frac{x_1^4}{4}$ $\quad\quad V(0) = 0$, $V(x) > 0$, & $\lim_{\|x\| \to \infty} V(x) \Rightarrow \infty$

For stability: V is L.P.D about x_e & $\frac{\partial V}{\partial x} \dot{x} \leq 0$ for $\|x - x_e\| < R$

$V(x_e) = V(0) = 0$ & $V(x) > 0$ when $x \neq x_e$. x_2^2 & x_1^4 are also both L.P.D. about x_e, Then $V(x)$ is L.P.D about the $x_e = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

$$\frac{\partial V}{\partial X} = [x_1^3 \quad x_2]$$

$$\frac{\partial V}{\partial X} \dot{x} = [x_1^3 \quad x_2] \begin{bmatrix} x_2 \\ -x_1^3 \end{bmatrix} = x_2 x_1^3 - x_2 x_1^3 = 0$$

The canidate Lyapunov function $V(x) = \frac{1}{2} x_2^2 + \frac{x_1^4}{4}$ is L.P.D about $x_e = \vec{0}$ & $\frac{\partial V}{\partial x} \dot{x} = 0 \leq 0$, therefore the origin is stable x_e.

#3) $\dot{x}_1 = x_2$ $\quad x_e = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\dot{x}_2 = -x_1 + x_1^3$

Treat x_2 as Velocity so $T = \frac{1}{2} m |v|^2$ $\therefore$ $m=1$ & $T = \frac{1}{2} x_2^2$

Treat $\dot{x}_2$ as conservative force so $U = -\int_{x_e}^{x} F(n)\,dn$

$$U = -\int_0^{x_1} -n + n^3\,dn = -\left[-\frac{n^2}{2} + \frac{n^4}{4}\right]_0^{x_1} = \frac{x_1^2}{2} - \frac{x_1^4}{4}$$

"Total Energy" Candidate Lyapunov function: $V = T + U = \frac{1}{2} x_2^2 + \frac{x_1^2}{2} - \frac{x_1^4}{4}$

$V(x) = \frac{1}{2} x_2^2 + \frac{x_1^2}{2} - \frac{x_1^4}{4}$

$V(x_e) = V(0) = 0$ ✓

$\frac{\partial V}{\partial x} = [x_1 - x_1^3 \quad x_2]$

$\frac{\partial V}{\partial x}(0) = [0 \quad 0]$ ✓

$$\frac{\partial^2 V}{\partial x^2} = \begin{bmatrix} 1 - 3x_1^2 & 0 \\ 0 & 1 \end{bmatrix} \qquad \frac{\partial^2 V}{\partial x^2}(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_{2\times 2} \succ 0$$

$\frac{\partial^2 V(0)}{\partial x^2} = I_{2\times 2}$; Identity matrix is positive definite.

$V(x_e) = 0$, $\frac{\partial V}{\partial x}(x_e) = 0$, & $\frac{\partial^2 V(0)}{\partial x^2} > 0$ $\therefore$ $\underline{V(x) \text{ is L.P.D}}$

For stability: V is L.P.D & $\dot{V} \leq 0$

$$\dot{V} = \frac{\partial V}{\partial x}\dot{x} = [x_1 - x_1^3 \quad x_2]\begin{bmatrix} x_2 \\ -x_1 + x_1^3 \end{bmatrix} = x_2 x_1 - x_2 x_1^3 + x_2 x_1^3 - x_2 x_1 = 0$$

$\dot{V} = 0$ ✓

$\boxed{\text{V is L.P.D about origin \& } \dot{V} = 0 \leq 0 \ \therefore \text{ origin is a stable equilibrium state.}}$

#4) $\dot{x}_1 = x_2^3$ $x_e = \begin{vmatrix} 0 \\ 0 \end{vmatrix}$

$\dot{x}_2 = -x_2^2 x_1$

For stability: $\dot{V} = \frac{\partial V}{\partial x} \dot{x} \leq 0$

$\therefore \left[\frac{\partial V}{\partial x} \quad \frac{\partial V}{\partial x_2}\right] \begin{bmatrix} x_2^3 \\ -x_2^2 x_1 \end{bmatrix} \leq 0$

$$\frac{\partial V}{\partial x_1} x_2^3 - \frac{\partial V}{\partial x_2} x_2^2 x_1 \leq 0$$

Set equal to zero: $\frac{\partial V}{\partial x_1} x_2^3 - \frac{\partial V}{\partial x_2} x_2^2 x_1 = 0$

Rearrange: $\left(\frac{\partial V}{\partial x_1}\right)\left(\frac{1}{x_1}\right) - \left(\frac{\partial V}{\partial x_2}\right)\left(\frac{1}{x_2}\right) = 0$

Solve P.D.E. using method of characteristics:

$x U_x + y U_y = 0 \Rightarrow \frac{dy}{dx} = \frac{y}{x}, \quad U = f\left(\frac{dy}{dx}\right)$

$\therefore \frac{dx_2}{dx_1} = \frac{-\left(\frac{1}{x_2}\right)}{\left(\frac{1}{x_1}\right)} = \frac{-x_1}{x_2} \Rightarrow x_2 dx_2 = -x_1 dx_1$

$\int x_2 dx_2 = -\int x_1 dx_1 \Rightarrow \frac{x_2^2}{2} = \frac{-x_1^2}{2}$

$\therefore V(x) = \frac{x_2^2}{2} + \frac{x_1^2}{2}$, Check $\left(\frac{\partial V}{\partial x_1}\right)\left(\frac{1}{x_1}\right) - \left(\frac{\partial V}{\partial x_2}\right)\left(\frac{1}{x_2}\right) = 0$

$(x_1)\left(\frac{1}{x_1}\right) - (x_2)\left(\frac{1}{x_2}\right) = 0$ ✓

Choose $V(x) = \frac{x_1^2}{2} + \frac{x_2^2}{2}$ as canidate Lyapunov function

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$V(x_e = \vec{0}) = 0^2 + 0^2 = 0$

$V(0) = 0$ ✓

$\frac{\partial V}{\partial x} = \left[\frac{\partial V}{\partial x_1} \quad \frac{\partial V}{\partial x_2}\right] = \left[x_1 \quad x_2\right]$

$$\frac{\partial V(0)}{\partial x} = [0 \quad 0]$$

$$\frac{\partial^2 V}{\partial x^2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \geq 0 \quad \text{(Identity matrix is positive definite)}$$

$$\therefore \frac{\partial^2 V(0)}{\partial x^2} > 0$$

$V(0) = 0$, $\frac{\partial V(0)}{\partial x} = 0$, & $\frac{\partial^2 V(0)}{\partial x^2} > 0$ $\therefore$ $\underline{V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2}$ $\underline{\text{is L.P.D about origin.}}$

$$\dot{V} = \frac{\partial V}{\partial x}\dot{x} = [x_1 \quad x_2]\begin{bmatrix} x_2^3 \\ -x_2^3 x_1 \end{bmatrix} = x_2^3 x_1 - x_2^3 x_1 = 0$$

$$\dot{V} = 0 \leq 0 \quad \checkmark$$

$\boxed{\text{Using } V(x) = \frac{x_1^2}{2} + \frac{x_2^2}{2}, \text{ it can be shown that } V(x) \text{ is L.P.D about the origin } \& \ \dot{V}(x) = 0 \leq 0 \ \therefore V(x) \text{ is a Lyapunov function that proves the system is stable about the zero state.}}$

#5) $\dot{x} = -(2 + \cos(x))\,x \qquad x_e = 0$

For Global Asymptotic Stability: V must be positive definite & $\dot{V} < 0$ when $x \neq 0$.

Scalar system, use $V(x) = x^2$ as candidate Lyapunov function.

For Positive Definite $V(x)$: $V(0) = 0$, $V(x) > 0$, & $\lim_{\|x\| \to \infty} V(x) = \infty$

$V(0) = 0^2 = 0$ ✓

$V(x) = x^2 > 0$ ✓ (Always positive when $x \neq 0$)

$\lim_{\|x\| \to \infty} x^2 = \|\infty\|^2 = \infty$ ✓

$V(0) = 0$, $V(x) > 0$ for $x \neq 0$, & $\lim_{\|x\| \to \infty} V(x) = \infty$ $\therefore$ $V(x) = x^2$ is a positive definite function.

$$\frac{\partial V}{\partial x} = 2x$$

$$\dot{V} = \frac{\partial V}{\partial x} f(x) = \frac{\partial V}{\partial x}\dot{x} = (2x)[-x(2 + \cos(x))]$$

$$\dot{V} = -2x^2(2 + \cos(x))$$

$-2x^2$ is always negative, for $x \neq 0$. For $\dot{V} < 0$, $(2 + \cos(x))$ must always be positive. The smallest value of $\cos(x)$ is -1 $\therefore$ the minimum value of $(2 + \cos(x))$ is 1. Then $\dot{V} < 0$ for all x, $x \neq 0$.

$$\dot{V} = -2x^2(2 + \cos x) < 0 \ ✓$$

$\boxed{V(x) = x^2 \text{ is Positive Definite } \& \ \dot{V} < 0, \therefore \text{ the system is globally asymptotically stable about the origin.}}$

#6) $\dot{x} = -(2+\cos(x))(x-1)$ $\qquad x_e = 1$

Cannot use $V(x) = x^2$ as Canidate Lyapunov function due to $V(x_e) = 1 \neq 0$

Try: $V(x) = (x-1)^2$ as Canidate Lyapunov function

$V(x_e) = V(1) = (1-1)^2 = 0$ ✓

$V(x) > 0$ for $x \neq x_e = 1$ ✓

$\lim_{\|x\| \to \infty} V(x) = (\infty - 1)^2 = \infty$ ✓

$V(x) = (x-1)^2$ is 0 at $x = x_e$, $V(x) > 0$ for $x \neq 1$, &

$\lim_{\|x\| \to \infty} V(x) = \infty$, There for $V(x) = (x-1)^2$ is Positive Definite.

$$\frac{\partial V}{\partial x} = 2(x-1)$$

For G.A.S., $\frac{\partial V}{\partial x} f(x) < 0$ for $x \neq x_e$

$$\frac{\partial V}{\partial x} f(x) = -2(x-1)(2+\cos(x))(x-1)$$

$$\frac{\partial V}{\partial x} f(x) = -2(x-1)^2(2+\cos(x))$$

From Problem 5, it was shown $(2+\cos(x))$ is always > 0 as it has a minimum value of 1. For $\frac{\partial V}{\partial x} f(x)$ to be G.A.S., then $-2(x-1)^2$ must always be negative when $x \neq 1$. $(x-1)^2$ is > 0 for $x \neq 1$ $\therefore$ $-2(x-1)^2$ is < 0 for $x \neq 1$.

$\therefore$ $\dot{V} = \frac{\partial V}{\partial x} f(x) < 0$ for $x \neq 1$.

Using $V(x) = (x-1)^2$ as a canidate Lyapunov function, it was shown that $V(x)$ is positive definite & $\dot{V} < 0$ for $x \neq x_e = 1$. Therefore this system is G.A.S. about 1.