ECE 602: LUMPED LINEAR SYSTEMS

Professor Jianghai Hu

Solution of Discrete-Time LTI Systems

Discrete-Time Autonomous LTI Systems

Discrete-time LTI system

$$x[k+1] = Ax[k],\ k = 0, 1, \dots$$

with initial condition $x[0]$ has the solution

$$x[k] = A^k x[0] := \Phi[k]x[0], \quad k = 0, 1, \dots$$

$\Phi[k] = A^k$ is called the fundamental matrix of the DT LTI system

- For any k_0, $x[k_0 + k] = \Phi[k]x[k_0]$
- $\Phi[k]$ propagates the solution from any initial time to k steps later
- $\Phi[k+\ell] = \Phi[k] \cdot \Phi[\ell] = \Phi[\ell] \cdot \Phi[k], \quad k, \ell = 0, 1, \dots$
- $\Phi[k]$ may be singular (different from $\Phi(t)$ for CT LTI systems)

System Modes: Diagonalizable A Case

Suppose $A \in \mathbb{R}^{n \times n}$ is diagonalizable: $A = T \Lambda T^{-1}$

- Diagonal entries of $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ are the eigenvalues of A
- Column of $T = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}$ are (right) eigenvectors of A
- Rows of $T^{-1} = \begin{bmatrix} w_1 & \cdots & w_n \end{bmatrix}^T$ are left eigenvectors of A

The solution to $x[k+1] = Ax[k]$ with initial state $x[0]$ is

$$x[k] = T \Lambda^k T^{-1} x[0] = \left(w_1^T x[0] \right) \lambda_1^k v_1 + \cdots + \left(w_n^T x[0] \right) \lambda_n^k v_n$$

- $\lambda_1^k v_1, \ldots, \lambda_n^k v_n$ are the modes of the system
- Any solution $x[\cdot]$ is a linear combination of these n modes

System Modes: General A Case

Using the JCF: $A = TJT^{-1} = \begin{bmatrix} T_1 & \cdots & T_r \end{bmatrix} \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_r \end{bmatrix} \begin{bmatrix} S_1^T \\ \vdots \\ S_r^T \end{bmatrix}$

Solution to $x[k+1] = Ax[k]$ with the initial state $x[0]$:

$$x(t) = TJ^k T^{-1} x[0] = \sum_{i=1}^{r} T_i J_i^k \left(S_i^T x[0] \right)$$

- Columns of $T_i J_i^k \in \mathbb{R}^{n \times n_i}$ are modes corresponding to eigenvalue λ_i, whose weights in $x[k]$ are given by entries of vector $S_i^T x[0] \in \mathbb{R}^{n_i}$

$$J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix} \quad \Rightarrow \quad J_i^k = \begin{bmatrix} \lambda_i^k & k\lambda_i^{k-1} & \cdots & \frac{k(k-1)\cdots(k-n_i+2)}{(n_i-1)!}\lambda_i^{k-n_i+1} \\ & \lambda_i^k & \ddots & \vdots \\ & & \ddots & k\lambda_i^{k-1} \\ & & & \lambda_i^k \end{bmatrix}$$

Example

$$x[k+1] = Ax[k] \text{ with } A = \underbrace{\begin{bmatrix} -1 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{T} \underbrace{\begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}}_{J} \underbrace{\begin{bmatrix} 0 & -1 & 0 \\ 1 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix}}_{T^{-1}}$$