Case Study

The objective of this case study is to investigate the problem of finding performance level γ of the observation error of a projection operator unknown input observer (UIO) for a discrete time (DT) system.

We first design a UIO for the following model of a DT dynamical system:

$$
\begin{aligned}
x[k+1] &= Ax[k] + B_1 u_1[k] + B_2 u_2[k] \\
&= \begin{bmatrix} 0.75 & 1 & 0 \\ 1 & 0.5 & 1 \\ 0 & 0 & 0.25 \end{bmatrix} x[k] + \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} u_2[k] \\
y[k] &= Cx[k] + Dv[k] \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0.1 \\ 0.05 \end{bmatrix} v[k].
\end{aligned}
$$

Note that in the above model, we have $B_1 = O$.

Explanation: We prepared the following script to design the UIO:

```
clear all; clc;
% System data
A=[0.75 1 0;1 0.5 1;0 0 0.25];
B2=[1 -1 0]';
C=[1 0 0;0 1 1];
D=[.1; .05];
disp('rank(obsv(A,C))')
rank(obsv(A,C))
% Dimensions
n = size(A,1);
m2=size(B2,2);
p = size(C,1);
r=size(D,2);
disp('rank(B2)')
rank(B2)
disp('rank(C*B2)')
rank(C*B2)
% Design process
%disp('M')
M=[B2 zeros(n,r)]*pinv([C*B2 D])
disp('rank(M)')
rank(M)
disp('(eye(n)-M*C)*B2')
(eye(n)-M*C)*B2
%disp('A1')
A1 = (eye(n)-M*C)*A
disp('rank(obsv(A1,C))')
rank(obsv(A1,C))
cvx_begin sdp quiet
variable P(n,n) symmetric
variable Y(n,p)
 [-P, A1'*P-C'*Y'; P*A1-Y*C, -P]<= 0
%Y*D==0
P >= 0.01*eye(n)
cvx_end
%disp('L')
L = P\Y
% Checking if design objectives satisfied
disp('eig(A1-L*C)')
eig(A1-L*C)
abs(eig(A1-L*C))
disp('M*D')
M*D
disp('L*D')
L*D
E=A1-L*C;
N=-L*D;
alpha=.5;
DeM=[E'*P*E-(1-alpha)*P  E'*P*N;N'*P*E  N'*P*N-alpha*eye(r)]
disp('eig(DeM)')
eig(DeM)
%disp('P')
P
disp('eig(P)')
eig(P)
mineig=min(eig(P))
gamma=1/sqrt(mineig)
```

Note that in the script we have `%Y*D==0`. We tried to satisfy this condition without success. Therefore we commented it and as a result we got

$$LD \neq O.$$

The UIO has the form

$$\begin{aligned} z[k+1] &= (I_n - MC)(Az[k] + AMy[k] + B_1 u[k]) \\ &\quad + L(y[k] - \hat{y}[k]) \\ \hat{x}[k] &= z[k] + My[k], \end{aligned}$$

where

$$M = \begin{bmatrix} 0.3333 & -0.6667 \\ -0.3333 & 0.6667 \\ 0 & 0 \end{bmatrix} \text{ and } L = \begin{bmatrix} 1.1690 & 0.9172 \\ 0.5879 & 0.3345 \\ -0.0034 & 0.1242 \end{bmatrix}.$$

The poles of the UIO are located in the open uint disk in the complex plane at $\{0.2891,\ -0.0001,\ 0.0000\}$.

We also satisfy the conditions:

$$(I_3 - MC)B_2 = O \text{ and } MD = O.$$

However, as mentioned above, we were unable to select L such that $LD = O$. Instead, we obtained

$$LD = \begin{bmatrix} 0.1628 \\ 0.0755 \\ 0.0059 \end{bmatrix}$$

Therefore, the observation error dynamics are governed by the difference equation,

$$e[k+1] = (A_1 - LC)e[k] - LDv[k].$$

We were able to find α such that

$$\begin{bmatrix} E^\top PE - (1-\alpha)P & E^\top PN \\ N^\top PE & N^\top PN - \alpha I \end{bmatrix} \preceq 0,$$

where $E = A_1 - LC$ and $-LD$. The parameter value is $\alpha = 0.5$.

For this value of α, we have

$$\operatorname{eig} \begin{bmatrix} E^\top PE - (1-\alpha)P & E^\top PN \\ N^\top PE & N^\top PN - \alpha I \end{bmatrix} < 0,$$

that is, all the eigenvalues of the above symmetric matrix are negative, meaning that the above matrix is negative definite. Indeed, the eigenvalues are located at $\{-0.1265,\ -4.2856,\ -5.3451,\ -5.3153\}$.

By the theorem, we proved in this module, this implies that the observer error satisfies

$$\limsup_{k\to\infty} \|e[k]\| \leq \gamma \limsup_{k\to\infty} \|v[k]\|_\infty$$

where $\gamma = 1/\sqrt{\lambda_{\min}(P)}$. That is, the state error dynamics are ℓ_∞-stable with performance level γ.

We obtained

$$P = \begin{bmatrix} 10.6325 & -0.1179 & 0.0890 \\ -0.1179 & 10.6036 & 0.0277 \\ 0.0890 & 0.0277 & 10.7070 \end{bmatrix}.$$

The eigenvalues of P are located at $\{10.4713,\ 10.6902,\ 10.7816\}$. Therefore,

$$\gamma = 1/\sqrt{\lambda_{\min}(P)} = 0.3090.$$

This is not the only value of α for which

$$\begin{bmatrix} E^\top PE - (1-\alpha)P & E^\top PN \\ N^\top PE & N^\top PN - \alpha I \end{bmatrix} \preceq 0,$$

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for the same P. Satisfaction of the above LMI for some α and P is sufficient for the state error dynamics to be ℓ_∞-stable with performance level $\gamma = 1/\sqrt{\lambda_{\min}(P)}$.

Now this course is almost over. We learned a lot of different things. It is now appropriate to pose some open research problems. Here we state the following hypothesis:

If there exist P. L, and M such that in the observation error dynamics

$$e[k+1] = (A_1 - LC)e[k] - LDv[k],$$

the matrix $(A_1 - LC)$ is Schur matrix, then there exists $\alpha \in (0,\, 1)$ such that

$$\begin{bmatrix} E^\top PE - (1-\alpha)P & E^\top PN \\ N^\top PE & N^\top PN - \alpha I \end{bmatrix} \preceq 0.$$

That is, the existence of P. L, and M such that in the observation error dynamics

$$e[k+1] = (A_1 - LC)e[k] - LDv[k],$$

the matrix $(A_1 - LC)$ is Schur matrix is necessary and sufficient for the observation error dynamics to be ℓ_∞-stable with performance level $\gamma = 1/\sqrt{\lambda_{\min}(P)}$.