ECE 68000: MODERN AUTOMATIC CONTROL

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Computing the exponential matrix

Solution of Uncontrolled System

- Consider a dynamical system's linear time-invariant (LTI) model

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t)$$

  subject to an initial condition

$$\boldsymbol{x}(0) = \boldsymbol{x}_0,$$

  where $\boldsymbol{A} \in \mathbb{R}^{n \times n}$

- The solution

$$\boxed{\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}_0}$$

The exponential matrix

- The exponential matrix

$$e^{\boldsymbol{A}t} = \boldsymbol{I}_n + t\boldsymbol{A} + \frac{t^2}{2!}\boldsymbol{A}^2 + \frac{t^3}{3!}\boldsymbol{A}^3 + \cdots$$

- Note that

$$\frac{d}{dt}e^{\boldsymbol{A}t} = \boldsymbol{A}e^{\boldsymbol{A}t} = e^{\boldsymbol{A}t}\boldsymbol{A}$$

- For more general $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$, the solution to $\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t)$ subject to $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$ is

$$\boxed{\boldsymbol{x}(t) = e^{\boldsymbol{A}(t-t_0)}\boldsymbol{x}_0}$$

The state transition matrix

- The matrix $e^{\boldsymbol{A}(t-t_0)}$ is often written as

$$e^{\boldsymbol{A}(t-t_0)} = \boldsymbol{\Phi}(t, t_0)$$

- It is called the *state transition matrix* because it relates the state at any instant of time t_0 to the state at any other time t as

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}(t-t_0)}\boldsymbol{x}(t_0)$$

Computing the exponential matrix

- In the case when the matrix $\boldsymbol{A}$ is of low dimensions, we can use the formula that results from applying the Laplace transform to the time-invariant system $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}, \quad \boldsymbol{x}(0) = \boldsymbol{x}_0$
- The Laplace transform of the above matrix differential equation is

$$s\boldsymbol{X}(s) - \boldsymbol{x}_0 = \boldsymbol{A}\boldsymbol{X}(s)$$

- Hence

$$\boldsymbol{X}(s) = (s\boldsymbol{I}_n - \boldsymbol{A})^{-1}\boldsymbol{x}_0,$$

The inverse Laplace transform method

- Recall, $\mathcal{L}\left(e^{at}\right) = \frac{1}{s-a}$
- Apply the inverse Laplace transform to

$$\left(s\boldsymbol{I}_n - \boldsymbol{A}\right)^{-1}$$

- We obtain

$$\boxed{e^{\boldsymbol{A}t} = \mathcal{L}^{-1}\left\{\left(s\boldsymbol{I}_n - \boldsymbol{A}\right)^{-1}\right\}}$$

where $\mathcal{L}^{-1}$ denotes the inverse Laplace transform operator

Example

- Use the inverse Laplace method to compute $e^{\boldsymbol{A}t}$ for

$$\boldsymbol{A} = \begin{bmatrix} 0 & 3 \\ 2 & 1 \end{bmatrix}$$

- We have

$$\begin{aligned}
e^{\boldsymbol{A}t} &= \mathcal{L}^{-1}\left([s\boldsymbol{I}_2 - \boldsymbol{A}]^{-1}\right) \\
&= \mathcal{L}^{-1}\left(\begin{bmatrix} s & -3 \\ -2 & s-1 \end{bmatrix}^{-1}\right) \\
&= \mathcal{L}^{-1}\left(\frac{1}{(s-3)(s+2)}\begin{bmatrix} s-1 & 3 \\ 2 & s \end{bmatrix}\right)
\end{aligned}$$

Example—contd.

- Hence

$$
\begin{aligned}
e^{\boldsymbol{A}t} &= \mathcal{L}^{-1}\left(\frac{1}{(s-3)(s+2)}\begin{bmatrix} s-1 & 3 \\ 2 & s \end{bmatrix}\right) \\
&= \begin{bmatrix} 0.4e^{3t}+0.6e^{-2t} & 0.6\left(e^{3t}-e^{-2t}\right) \\ 0.4\left(e^{3t}-e^{-2t}\right) & 0.6e^{3t}+0.4e^{-2t} \end{bmatrix}
\end{aligned}
$$