ECE 602: LUMPED LINEAR SYSTEMS

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From a Transfer Function Matrix to a State-Space Description: Multi-Input Multi-Output Case

From a Transfer Function Matrix to a State-Space Description

- **Objective**: Find a state-space realization a transfer function matrix of a linear lumped system either discrete or continuous time-invariant multi-input multi-output (MIMO) system
- A transfer function $\boldsymbol{G}(s)$ is realizable if there exists a quadruple of constant matrices $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D})$ such that $\boldsymbol{G}(s) = \boldsymbol{C}\left(s\boldsymbol{I}_n - \boldsymbol{A}\right)^{-1}\boldsymbol{B} + \boldsymbol{D}$. We call such a quadruple $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D})$ a realization of $\boldsymbol{G}(s)$

Definition

The dimension of a realization is the size of the matrix $\boldsymbol{A}$, that is, if $\boldsymbol{A}$ is an n-by-n matrix then we say that the dimension of the corresponding realization is n

Transfer function matrix of a multi-input multi-output (MIMO) system

Theorem

A transfer function matrix $\boldsymbol{G}(s)$ is realizable if and only if $\boldsymbol{G}(s)$ is a proper rational matrix.

- A method of constructing a state-space realization of a given transfer function matrix used by the MATLAB's function tf2ss
- Let $\boldsymbol{g}_i(s)$ be the i-th column of a proper tf matrix $\boldsymbol{G}(s)$
- Note that the tf of an MIMO system can be expressed as

$$
\begin{aligned}
\boldsymbol{Y}(s) &= \boldsymbol{G}(s)\boldsymbol{U}(s) \\
&= \boldsymbol{g}_1(s)U_1(s) + \boldsymbol{g}_2(s)U_2(s) + \cdots + \boldsymbol{g}_m(s)U_m(s),
\end{aligned}
$$

where $U_i(s)$ is the i-th component of the input vector $\boldsymbol{U}(s)$

Column-by-column realization

- Construct a state-space realization of each column $\boldsymbol{g}_i(s)$
- Combine obtained realizations in a state-space realization of $\boldsymbol{G}(s)$
- If the quadruple $(\boldsymbol{A}_i, \boldsymbol{b}_i, \boldsymbol{C}_i, \boldsymbol{d}_i)$ is the realization of $\boldsymbol{g}_i(s)$, then a realization $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D})$ of $\boldsymbol{G}(s)$ has the form,

$$\begin{aligned}
\boldsymbol{A} &= \begin{bmatrix} \boldsymbol{A}_1 & \boldsymbol{O} & \cdots & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{A}_2 & \cdots & \boldsymbol{O} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{O} & \boldsymbol{O} & \cdots & \boldsymbol{A}_m \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} \boldsymbol{b}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \boldsymbol{b}_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \boldsymbol{b}_m \end{bmatrix}, \\
\boldsymbol{C} &= \begin{bmatrix} \boldsymbol{C}_1 & \boldsymbol{C}_2 & \cdots & \boldsymbol{C}_m \end{bmatrix}, \text{ and } \boldsymbol{D} = \begin{bmatrix} \boldsymbol{d}_1 & \boldsymbol{d}_2 & \cdots & \boldsymbol{d}_m \end{bmatrix}
\end{aligned}$$

Example

- Use the column-by-column method to construct a realization of the tf

$$\boldsymbol{G}(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{2}{s+1} \\ \frac{-1}{s^2+3s+2} & \frac{1}{s+2} \end{bmatrix} = \begin{bmatrix} \boldsymbol{g}_1(s) & \boldsymbol{g}_2(s) \end{bmatrix}$$

- This tf matrix is strictly proper, therefore

$$\boldsymbol{D} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

A very nice property of the realization with a pair (A, b) in the controller form

- Let $D(s) = \det[s\boldsymbol{I}_n - \boldsymbol{A}]$
- Then

$$
\begin{aligned}
& [s\boldsymbol{I}_n - \boldsymbol{A}]^{-1}\boldsymbol{b} \\
= & \begin{bmatrix} s & -1 & \cdots & 0 \\ 0 & s & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 \\ a_0 & a_1 & \cdots & s + a_{n-1} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = \frac{1}{D(s)} \begin{bmatrix} 1 \\ s \\ s^2 \\ \vdots \\ s^{n-2} \\ s^{n-1} \end{bmatrix}
\end{aligned}
$$

Realization of the first column $g_1(s)$

- Represent $\boldsymbol{g}_1(s)$ as

$$\begin{aligned}
\boldsymbol{g}_1(s) &= \begin{bmatrix} \frac{1}{s+1} \\ \frac{-1}{s^2+3s+2} \end{bmatrix} \\
&= \frac{1}{s^2+3s+2} \begin{bmatrix} s+2 \\ -1 \end{bmatrix} \\
&= \frac{1}{s^2+3s+2} \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right) \\
&= \frac{1}{s^2+3s+2} \left(\boldsymbol{N}_1 s + \boldsymbol{N}_0 \right).
\end{aligned}$$

- Hence

$$\boldsymbol{A}_1 = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \boldsymbol{b}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \boldsymbol{C}_1 = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}, \boldsymbol{d}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Realization of the first column $g_2(s)$

- Represent the second column of $\boldsymbol{G}(s)$ as

$$
\begin{aligned}
\boldsymbol{g}_2(s) &= \begin{bmatrix} \frac{2}{s+1} \\ \frac{1}{s+2} \end{bmatrix} \\
&= \frac{1}{s^2+3s+2} \begin{bmatrix} 2(s+2) \\ s+1 \end{bmatrix} \\
&= \frac{1}{s^2+3s+2} \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} s + \begin{bmatrix} 4 \\ 1 \end{bmatrix} \right) \\
&= \frac{1}{s^2+3s+2} \left(\boldsymbol{N}_1 s + \boldsymbol{N}_0 \right).
\end{aligned}
$$

- Hence

$$
\boldsymbol{A}_2 = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \ \boldsymbol{b}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ \boldsymbol{C}_2 = \begin{bmatrix} 4 & 2 \\ 1 & 1 \end{bmatrix}, \ \boldsymbol{d}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

Matrices A and B of realization of $G(s)$

$$
\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_1 & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{A}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & \vdots & 0 & 0 \\ -2 & -3 & \vdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \vdots & 0 & 1 \\ 0 & 0 & \vdots & -2 & -3 \end{bmatrix}
$$

$$
\boldsymbol{B} = \begin{bmatrix} \boldsymbol{b}_1 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{b}_2 \end{bmatrix} = \begin{bmatrix} 0 & \vdots & 0 \\ 1 & \vdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \vdots & 0 \\ 0 & \vdots & 1 \end{bmatrix}
$$

Overall realization of $G(s)$—Matrices C and D

$$
\begin{aligned}
\boldsymbol{C} &= \begin{bmatrix} \boldsymbol{C}_1 & \boldsymbol{C}_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & \vdots & 4 & 2 \\ -1 & 0 & \vdots & 1 & 1 \end{bmatrix} \\
\boldsymbol{D} &= \begin{bmatrix} \boldsymbol{d}_1 & \boldsymbol{d}_2 \end{bmatrix} = \begin{bmatrix} 0 & \vdots & 0 \\ 0 & \vdots & 0 \end{bmatrix}
\end{aligned}
$$

When a state-space realization is minimal?

- Recall that the dimension of a realization is the size n of the matrix $\boldsymbol{A}$
- A natural question arises of how to find a minimal realization for a given transfer function matrix
- A realization is minimal if and only if it is both reachable and observable.