6.1.3

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} = A$$

$$|A - \lambda I| = \lambda^2 - (a+c)\lambda + ac - b^2 = 0$$

$$\lambda_1 \lambda_2 = ac - b^2$$

$a > 0$

$ac > b^2$

$\therefore \quad (\lambda_1)(\lambda_2) > 0$

$$\boxed{\lambda_{1,2} = \frac{(a+c) \pm \sqrt{(-a-c)^2 - (4)(ac-b^2)}}{2}}$$

$$\lambda_1 + \lambda_2 = a + c$$

$$\frac{\lambda_1 + \lambda_2}{2} = a > 0 \quad \therefore \quad c > 0$$

$\therefore \quad \boxed{\lambda_1 \text{ \& } \lambda_2 > 0}$

6.1.4

a) $F = -1 + 4(e^x - x) - 5x\sin y + 6y^2$

$$\frac{\partial F}{\partial x} = 4e^x - 4 - 5\sin y \Big|_{x=0,y=0} = 4 - 4 - 5\sin(0) = 0$$

$$\frac{\partial F}{\partial y} = -5x\cos y + 12y \Big|_{x=0,y=0} = 0$$

$$\frac{\partial^2 F}{\partial x^2} = 4e^x \Big|_{x=0,y=0} = 4$$

$$\frac{\partial^2 F}{\partial x \partial y} = -5\cos y \Big|_{x=0,y=0} = -5$$

$$\frac{\partial^2 F}{\partial y^2} = 5x\sin y + 12 \Big|_{x=0,y=0} = 12$$

$$A = \begin{pmatrix} 4 & -5 \\ -5 & 12 \end{pmatrix}$$

$$|A - \lambda I| = \lambda^2 - 16\lambda + 23 = 0$$

$$\lambda_{1,2} = \frac{16 \pm \sqrt{16^2 - (4)(23)}}{2}$$

$$\lambda_1 = \frac{16 + \sqrt{164}}{2}, \quad \lambda_2 = \frac{16 - \sqrt{164}}{2}$$

6.1.4

$\lambda_1 = 14.4031$

$\lambda_2 = 1.5969$

λ_1 & $\lambda_2 > 0 \therefore$ $\boxed{\text{Hessian is Positive definite, miniumum at } x=y=0}$

b) $F = (x^2-2x)\cos y$

$$\frac{\partial F}{\partial x} = (2x-2)\cos y \Big|_{x=1, y=\pi} = 0$$

$$\frac{\partial F}{\partial y} = -(x^2-2x)\sin y \Big|_{x=1, y=\pi} = 0$$

$$\frac{\partial^2 F}{\partial x \partial y} = -(2x-2)\sin y \Big|_{x=1, y=\pi} = 0$$

$$\frac{\partial^2 F}{\partial x^2} = 2\cos y \Big|_{x=1, y=\pi} = -2$$

$$\frac{\partial^2 F}{\partial y^2} = -(x^2-2x)\cos y \Big|_{x=1, y=\pi} = -(1-2)(-1) = -1$$

$$A = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}$$

6.1.4

$|A-\lambda I| = \lambda^2+3\lambda+2=0 = (\lambda+2)(\lambda+1)=0$

$\lambda_1 = -2$, $\lambda_1 = -1$

λ_1 , $\lambda_2 < 0$

$\therefore$ Hessian is negative definite , maximum at $x=1$, $y=\pi$.

6.1.12

$$F = x^2y^2 - 2x - 2y$$

$$\frac{\partial F}{\partial x} = 2xy^2 - 2 \Big|_{x=1,y=1} = 2-2 = 0$$

$$\frac{\partial F}{\partial y} = 2x^2y - 2 \Big|_{x=1,y=1} \quad 2-2=0$$

$$\frac{\partial F}{\partial y}\Big|_{x=1,y=1} = \frac{\partial F}{\partial x}\Big|_{x=1,y=1} = 0$$

(1,1) is Stationary Point.

$$\frac{\partial^2 F}{\partial x^2} = 2y^2 \Big|_{(1,1)} = 2$$

$$\frac{\partial^2 F}{\partial x \partial y} = 4xy \Big|_{(1,1)} = 4$$

$$\frac{\partial^2 F}{\partial y^2} = 2x^2 \Big|_{(1,1)} = 2$$

$$A = \begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix}$$

$$|A - \lambda I| = \lambda^2 - 4\lambda - 12 = 0$$

$$(\lambda-6)(\lambda+2) = 0$$

6.1.12

$\lambda_1 = 6, \lambda_2 = -2 \quad \therefore$

$\boxed{\text{Hessian is indefinite, (1,1) is a saddle point \& not a minimum.}}$

6.1.17

$(AB)^T = B^T A^T$

$(x^T A^T) = (Ax)^T$

$\therefore\ x^T A^T Ax = (Ax)^T (Ax)$

$||x||^2 = x^T x$

$\therefore\ ||Ax||^2 = (Ax)^T (Ax)$

If $A^T A$ is positive definite, then

$||Ax||^2 > 0$ unless $x = 0$

then $||A\vec{0}||^2 = 0$

6.2.2

$$A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

$\det(A_1) = 2 > 0$

$\det(A_2) = 4 - 1 = 3 > 0$

$$\det(A_3) = 2\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} - (-1)\begin{vmatrix} -1 & -1 \\ -1 & 2 \end{vmatrix} - 1\begin{vmatrix} -1 & 2 \\ -1 & -1 \end{vmatrix}$$

$$= (2)(3) + (-2-1) - (1+2) = 0$$

$\therefore$ $\boxed{\text{A isn't positive definite}}$

$$B = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix}$$

$\det(B_1) = 2 > 0$

$\det(B_2) = 4-1 = 3 > 0$

$$\det(B_3) = 2\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} - (-1)\begin{vmatrix} -1 & 1 \\ -1 & 2 \end{vmatrix} - \begin{vmatrix} -1 & 2 \\ -1 & 1 \end{vmatrix}$$

$$= (2)(3) + (-2+1) - (-1+2)$$
$$= 4 > 0$$

$\boxed{\text{B is positive definite.}}$

6.2.2

$$C^2 = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 5 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 2 & 5 \end{pmatrix}$$

$$\det(C_1^2) = 5$$

$$\det(C_2^2) = 10 - 4 = 6$$

$$\det(C_3^2) = (5)(6) - 2\begin{vmatrix} 2 & 2 \\ 1 & 5 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix}$$

$$= 30 - (2)(8) + 2 = 16$$

$$\boxed{C \text{ is positive definite.}}$$

6.2.7

Because entries of Λ are positive & R is symmetric.

$$A = \begin{pmatrix} 10 & 6 \\ 6 & 10 \end{pmatrix}$$

$$|A - \lambda I| = \lambda^2 - 20\lambda + 64 = 0$$
$$(\lambda - 16)(\lambda - 4)$$
$$\lambda_1 = 16, \quad \lambda_2 = 4$$

$$\Lambda = \begin{pmatrix} 16 & 0 \\ 0 & 4 \end{pmatrix}$$

$$(A - \lambda_1 I)x_1 = 0$$

$$\begin{pmatrix} -6 & 6 \\ 6 & -6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \Rightarrow x_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$(A - \lambda_2 I)x_2 = 0$$
$$\begin{pmatrix} 6 & 6 \\ 6 & 6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \Rightarrow x_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$q_1 = \frac{x_1}{\|x_1\|} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$b_2 = x_2 - (q_1^T x_2)q_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} - \left[\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right]\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$b_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$q_2 = \frac{b_2}{\|b_2\|} = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

↖ Makes sense as e-vec of symmetric real matrix are orthogonal

6.2.7

$$Q = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$Q^T = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$\sqrt{\Lambda} = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$$

$$R = Q\sqrt{\Lambda}Q^T = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} Q^T$$

$$R = \begin{pmatrix} 4/\sqrt{2} & -2/\sqrt{2} \\ 4/\sqrt{2} & 2/\sqrt{2} \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$\boxed{R = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}}$$

$$R^2 = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 10 & 6 \\ 6 & 10 \end{pmatrix} \checkmark$$

$$A = \begin{pmatrix} 10 & -6 \\ -6 & 10 \end{pmatrix}$$

$$|A - \lambda I| = \lambda^2 - 20\lambda + 64 = 0$$

$$\lambda_1 = 16, \quad \lambda_2 = 4$$

6.2.7

$(A-\lambda_1 I)x_1 = 0$

$$\begin{pmatrix} -6 & -6 \\ -6 & -6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \vec{0} \Rightarrow \begin{pmatrix} -1 \\ 1 \end{pmatrix} = x_1$$

$(A-\lambda_2 I)x_2 = 0$

$$\begin{pmatrix} 6 & -6 \\ -6 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix} = x_2$$

$$q_1 = \frac{x_1}{\|x_1\|} = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$q_2 = \frac{x_2}{\|x_2\|} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$Q = \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$\sqrt{\Lambda} = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} \sqrt{16} & 0 \\ 0 & \sqrt{4} \end{pmatrix}$$

$$R = Q\sqrt{\Lambda}Q^T = \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} Q^T$$

$$= \begin{pmatrix} -4/\sqrt{2} & 2/\sqrt{2} \\ 4/\sqrt{2} & 2/\sqrt{2} \end{pmatrix} \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

6.2.7

$$\boxed{R = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}}$$

$$R^2 = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 10 & -6 \\ -6 & 10 \end{pmatrix} \checkmark$$

6.2.10

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$$

$$|A - \lambda I| = 0 = \lambda^2 - 5\lambda + 4$$

$$\lambda_1 = 4, \ \lambda_2 = 1$$

$$(A - \lambda_1 I) X_1 = 0$$

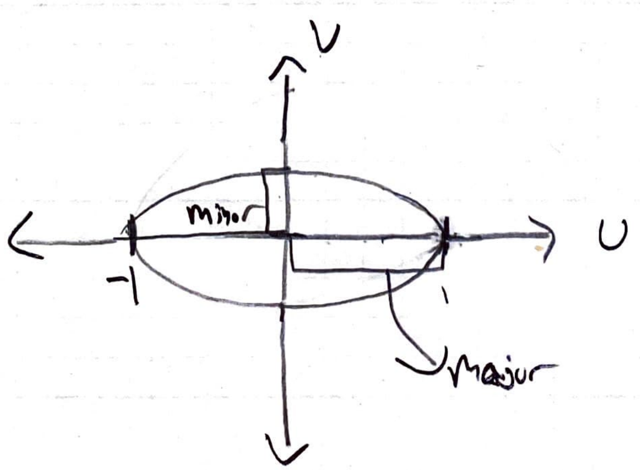
$$\begin{pmatrix} -3 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \implies X_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$(A - \lambda_2 I) X_2 = 0$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \implies x_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\boxed{\begin{array}{l} \lambda_1 = 4, \ X_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \lambda_2 = 1, \ X_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{array}}$$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \leftarrow$ major axis as λ_2 is smallest



6.2.14

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 3 & 4 & 9 \end{pmatrix}$$

$$\xrightarrow{E_{21}(-2)} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 3 & 4 & 9 \end{pmatrix} \xrightarrow{E_{31}(-3)} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & -2 & 0 \end{pmatrix}$$

$$\xrightarrow{E_{32}(2)} \begin{pmatrix} \boxed{1} & 2 & 3 \\ 0 & \boxed{1} & -2 \\ 0 & 0 & \boxed{-4} \end{pmatrix}$$

$A^T = A$, 1 Negative Pivot, 2 Positive Pivots, $\therefore$

1 Negative λ & 2 Positive λ. Therefore

$\boxed{\text{A is indefinite}}$

$$B = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 2 & 6 & -2 & 0 \\ 0 & -2 & 5 & -2 \\ 0 & 0 & -2 & 3 \end{pmatrix}$$

$$\xrightarrow{E_{21}(-2)} \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & -2 & 5 & -2 \\ 0 & 0 & -2 & 3 \end{pmatrix} \xrightarrow{E_{32}(1)} \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & -2 & 3 \end{pmatrix}$$

$$\xrightarrow{E_{43}(\frac{2}{3})} \begin{pmatrix} \boxed{1} & 2 & 0 & 0 \\ 0 & \boxed{2} & -2 & 0 \\ 0 & 0 & \boxed{3} & -2 \\ 0 & 0 & 0 & \boxed{\frac{5}{3}} \end{pmatrix}$$

6.2.14

$B = B^T$, all pivots are > 0 $\therefore$ all $\lambda > 0$

$\boxed{B \text{ is positive definite}}$

$C = -B$, therefore all pivots < 0 & all $\lambda < 0$

$\boxed{C \text{ is negative definite}}$

$D = A^{-1}$ $\qquad A^{-1} = \frac{C^T}{\det(A)}$

$$\det(A) = 1\begin{vmatrix} 5 & 4 \\ 4 & 9 \end{vmatrix} - 2\begin{vmatrix} 2 & 4 \\ 3 & 9 \end{vmatrix} + 3\begin{vmatrix} 2 & 5 \\ 3 & 4 \end{vmatrix}$$

$$= 45 - 16 - 2(18-12) + 3(8-15)$$

$$= 45 - 16 - 12 - 21 = -4$$

$C_{11} = \begin{vmatrix} 5 & 4 \\ 4 & 9 \end{vmatrix}$ $\quad C_{12} = -\begin{vmatrix} 2 & 4 \\ 3 & 9 \end{vmatrix}$ $\quad C_{13} = \begin{vmatrix} 2 & 5 \\ 3 & 4 \end{vmatrix}$

$C_{21} = -\begin{vmatrix} 2 & 3 \\ 4 & 9 \end{vmatrix}$ $\quad C_{22} = \begin{vmatrix} 1 & 3 \\ 3 & 9 \end{vmatrix}$ $\quad C_{23} = -\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$

$C_{31} = \begin{vmatrix} 2 & 3 \\ 5 & 4 \end{vmatrix}$ $\quad C_{32} = -\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix}$ $\quad C_{33} = \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix}$

$$C = \begin{pmatrix} 29 & -6 & -7 \\ -6 & 0 & 2 \\ -7 & 2 & 1 \end{pmatrix} \qquad C^T = C$$

6.2.14

$$D = A^{-1} = \begin{pmatrix} -29/4 & 6/4 & 7/4 \\ 6/4 & 0 & -1/2 \\ 7/4 & -1/2 & -1/4 \end{pmatrix}$$

$$E_{21}\left(\frac{6}{29}\right) \longrightarrow \begin{pmatrix} \boxed{-29/4} & 6/4 & 7/4 \\ 0 & \boxed{\frac{36}{116}} & -16/116 \\ 7/4 & -1/2 & -1/4 \end{pmatrix}$$

$42/116 \cdot 58/16$

$D = D^T$, First and second pivots have different signs, $\therefore$ at least one $\lambda < 0$ & one $\lambda > 0$. $\therefore$

$\boxed{D \text{ is indefinite}}$

6.2.23

a) All eigenvalues are greater than 0, therefore an inverse must exist.

b) Projection matrices all have det = 0 except for I.

c) The eigenvalues of a diagonal matrix are the diagonal entries.

d) The even powers of negative definite matrices can give a positive determinate.

6.2.24

$$A=\begin{pmatrix} s & -4 & -4 \\ -4 & s & -4 \\ -4 & -4 & s \end{pmatrix}$$

$\det(A_1) = s > 0$

$\det(A_2) = s^2-16 > 0 \quad s^2 > 16$

$$\det(A_3) = s(s^2-16) + 4\begin{vmatrix} -4 & -4 \\ -4 & s \end{vmatrix} - 4\begin{vmatrix} -4 & s \\ -4 & -4 \end{vmatrix} > 0$$

$= s^3 - 16s + 4(-4s-16) - 4(16+4s)$

$= s^3 - 16s - 16s - 64 - 64 - 16s$

$= s^3 - 48s - 128 > 0$

$$= \underbrace{(s+4)^2}_{\text{Always Positive}}\underbrace{(s-8)}_{s>8} > 0$$

$$\boxed{s > 8}$$

$$B=\begin{pmatrix} t & 3 & 0 \\ 3 & t & 4 \\ 0 & 4 & t \end{pmatrix}$$

$\det(B_1) = t > 0$

6.2.24

$\det(B_2) = t^2 - 16 > 0 \quad t^2 > 16$

$\det(B_3) = t(t^2-16) - 3\begin{vmatrix} 3 & 4 \\ 0 & t \end{vmatrix} > 0$

$= t^3 - 16t - 9t \quad > 0$

$= t(\underbrace{t^2-25}) \quad > 0$

$\therefore \quad t^2 > 25 \quad \Rightarrow t = \pm 5$

$t = -5 \not> 0 \quad t = -5$ isn't a solution

$\boxed{t > 5}$