AAE 532 - Orbit Mechanics

Purdue University

PS7 Solutions

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Useful Constants

	Axial Rotaional Period (Rev/Day)	Mean Equatorial Radius (km)	Gravitational Parameter $\mu = Gm(km^3/sec^2)$	Semi-major Axis of orbit (km)	Orbital Period (sec)	Eccentricity of Orbit	Inclination of Orbit to Ecliptic (deg)
☉ Sun	0.0394011	695990	132712440017.99	-	-	-	-
☾ Moon	0.0366004	1738.2	4902.8005821478	384400 (around Earth)	2360592 27.32 Earth Days	0.0554	5.16
☾ Mercury	0.0170514	2439.7	22032.080486418	57909101	7600537 87.97 Earth Days	0.20563661	7.00497902
♀ Venus	0.0041149 (Retrograde)	6051.9	324858.59882646	108207284	19413722 224.70 Earth Days	0.00676399	3.39465605
⊕ Earth	1.0027378	6378.1363	398600.4415	149597898	31558205 365.26 Earth Days	0.01673163	0.00001531
♂ Mars	0.9747000	3397	42828.314258067	227944135	59356281 686.99 Earth Days	0.09336511	1.84969142
♃ Jupiter	2.4181573	71492	126712767.8578	778279959	374479305 11.87 Years	0.04853590	1.30439695
♄ Saturn	2.2522053	60268	37940626.061137	1427387908	930115906 29.47 Years	0.05550825	2.48599187
⛢ Uranus	1.3921114 (Retrograde)	25559	5794549.0070719	2870480873	2652503938 84.05 Years	0.04685740	0.77263783
Ψ Neptune	1.4897579	25269	6836534.0638793	4498337290	5203578080 164.89 Years	0.00895439	1.77004347
♇ Pluto	-0.1565620 (Retrograde)	1162	981.600887707	5907150229	7830528509 248.13 Years	0.24885238	17.14001206

- First three columns of the body data are consistent with GMAT 2020a default values, which are mainly from JPL's ephemerides file de405.spk

- The rest of the data are from JPL website(`https://ssd.jpl.nasa.gov/?planet_pos`, retrieved at 09/01/2020)

based on E.M. Standish's "Keplerian Elements for Approximate Positions of the Major Planets"

Problem 1

Problem Statement

Assume a two-body model for a Earth orbiter. Currently, the vehicle is in an orbit such that $r_p = 1.5R_\oplus$ and $r_a = 8.0R_\oplus$. A single in-plane maneuver will be used to raise perigee and lower apogee. New values are specified as $r_p = 2.0R_\oplus$ and $r_a = 6.0R_\oplus$. It is also required that perigee advance by 90° ($\Delta\omega = +90°$).

(a) There are two options for the location to place the maneuver to satisfy the requirements. (Sketch the scenario.) At what θ^* in the original orbit should the maneuver take place? Be sure to select the location that results in the min $|\Delta\bar{v}|$ and justify the choice. Determine $\bar{r}_1$, $\bar{v}_1^-$, γ_1^- at the maneuver point.

(b) Determine the required maneuver ($\Delta\bar{v}$, that is, Δv, α). [Don't forget the vector diagram!] Compute the $\bar{v}_1^+$, γ_1^+.

(c) Plot the old and the new orbits. You can use either Matlab or GMAT. On the plot, mark $\bar{r}_0$, $\bar{r}_1$, $\bar{v}_1^-$, local horizon, γ_1^-, $\bar{v}_1^+$, γ_1^+, $\Delta\bar{v}$, α. Identify $\Delta\omega$ and the old and new true anomalies.

Part (a)

Let's begin by sketching the scenario described in the problem statement above: In order to determine the true

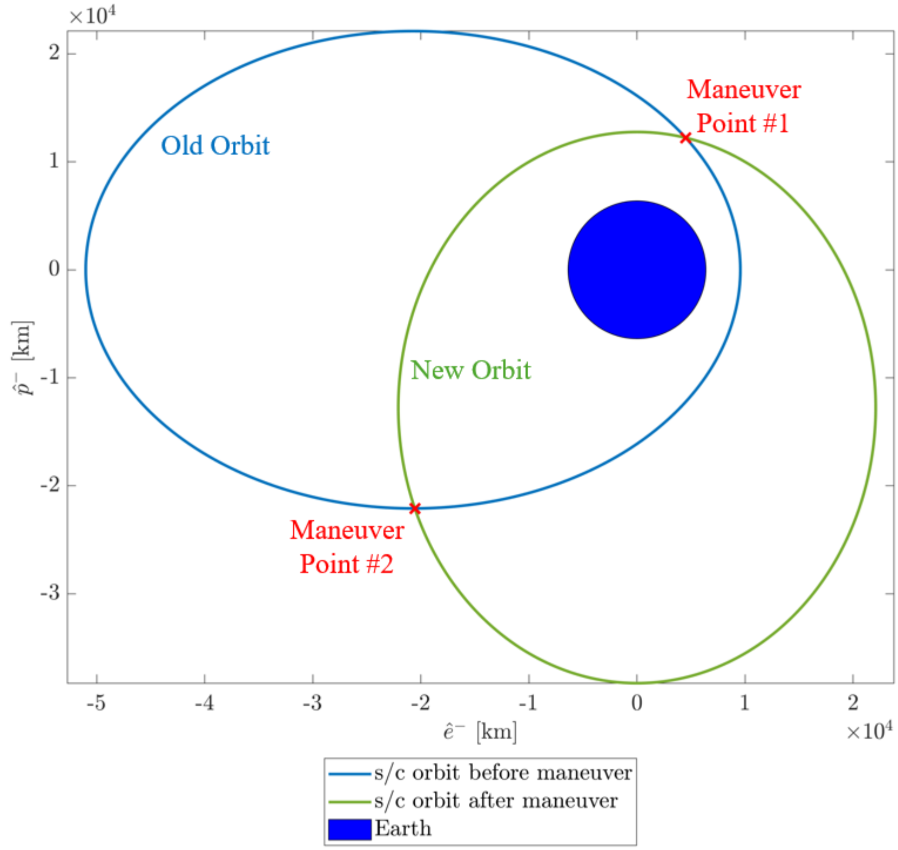

Figure 1: "Sketch" of the two orbits described in the problem statement with the maneuver points indicated by the red 'X'. Note that the new orbit's perigee is +90° with respect to the perigee of the old orbit.

anomalies for each maneuver point, one must first solve for some of the orbital elements and other parameters. Let's begin with solving for the semi-major axis in both the old and new orbits. Note that items with a minus superscript represent elements of the old orbit while items with a plus superscript represent elements of the new orbit:

$$a^- = \frac{r_p^- + r_a^-}{2} = 3.0296 \cdot 10^4 \text{ km} \tag{1}$$

$$a^+ = \frac{r_p^+ + r_a^+}{2} = 2.5513 \cdot 10^4 \text{ km} \tag{2}$$

Next, one can solve for the eccentricity of each orbit:

$$e^- = \frac{r_a^-}{a^-} - 1 = 0.6842 \tag{3}$$

$$e^+ = \frac{r_a^+}{a^+} - 1 = 0.5000 \tag{4}$$

One can now solve for the semi-latus rectum:

$$p^- = a^-(1 - (e^-)^2) = 1.6113 \cdot 10^4 \text{ km} \tag{5}$$

$$p^+ = a^+(1 - (e^+)^2) = 1.9134 \cdot 10^4 \text{ km} \tag{6}$$

One can also determine the specific energy of each orbit:

$$\mathcal{E}^- = -\frac{\mu}{2a^-} = -6.5784 \text{ km}^2/\text{s}^2 \tag{7}$$

$$\mathcal{E}^+ = -\frac{\mu}{2a^+} = -7.8119 \text{ km}^2/\text{s}^2 \tag{8}$$

Lastly, one can also determine the specific angular momentum:

$$h^- = \sqrt{\mu p^-} = 8.0142 \cdot 10^4 \text{ km}^2/\text{s} \tag{9}$$

$$h^+ = \sqrt{\mu p^+} = 8.7333 \cdot 10^4 \text{ km}^2/\text{s} \tag{10}$$

Now, let's determine the values of true anomaly θ^* at both maneuver locations for both orbits. To do so, recall that, at each maneuver point, the radius of each orbit is the same, which means one can set the conic equations for both orbits equal to one another:

$$\frac{p^-}{1 + e^- \cos\theta^{*-}} = \frac{p^+}{1 + e^+ \cos\theta^{*+}} \tag{11}$$

Which can be rewritten as:

$$\frac{p^-}{1 + e^- \cos\theta^{*-}} = \frac{p^+}{1 + e^+ \cos\left(\theta^{*-} - \Delta\omega\right)} \tag{12}$$

Rearranging:

$$0 = 1 + e^- \cos\theta^{*-} - \frac{p^-}{p^+}\Big(1 + e^+ \cos\left(\theta^{*-} - \Delta\omega\right)\Big) \tag{13}$$

Which can also be expressed as:

$$0 = 1 - \frac{p^-}{p^+} + \cos\theta^{*-}\left(e^- - \frac{p^-}{p^+}e^+ \cos\Delta\omega\right) - \frac{p^-}{p^+}e^+ \sin\theta^{*-} \sin\Delta\omega \tag{14}$$

The easiest way to solve equation (14) is through numerical means as the equation appears to be Transcendental in nature. Note that, from our sketch, it would appear that the maneuver points in the original orbit occur when the spacecraft is ascending and descending, which means, to obtain a true anomaly when the spacecraft is ascending, one can provide a guess with a positive value. However, to obtain a true anomaly when the spacecraft is descending, one can provide a guess with a negative value. Thus, the true anomalies for the maneuver point the original and new orbit for when the spacecraft is ascending in the original orbit are:

$$\boxed{\theta_1^{*-} = 69.7270°} \tag{15}$$

$$\boxed{\theta_1^{*+} = \theta_1^{*-} - \Delta\omega = -20.2730°} \tag{16}$$

And the true anomalies for the maneuver point the original and new orbit for when the spacecraft is descending in the original orbit are:

$$\boxed{\theta_2^{*-} = -132.9420°} \tag{17}$$

$$\boxed{\theta_2^{*+} = \theta_2^{*-} - \Delta\omega = 137.0580°} \tag{18}$$

Now, the question is which maneuver point do we select. Well, one can, first, examine the sketch and the velocity vector diagrams for each maneuver point: As one can see from figures 4 and 5, it would appear that the $\Delta\bar{v}$ for

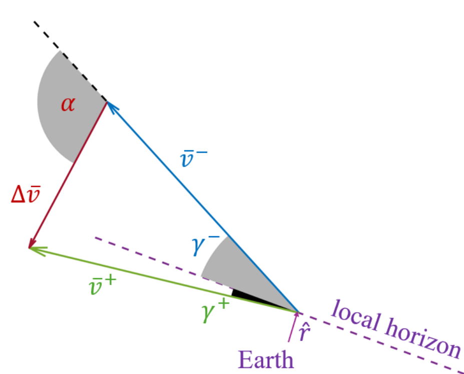

Figure 2: Sketch of the velocity vector diagrams at maneuver point #1.

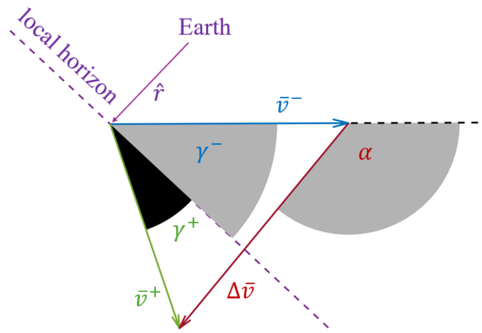

Figure 3: Sketch of the velocity vector diagrams at maneuver point #2.

maneuver point #1 would be smaller as the change in the flight path angle is larger for maneuver point #2 than the change in flight path angle for maneuver point #1 (as also can be seen by our sketch in figure 1). Therefore, without any calculations yet, we will select **maneuver point #1** as our maneuver point to minimize $\Delta\bar{v}$. Now, let's calculate $\bar{r}_1$, $\bar{v}_1^-$, and γ_1^- for the maneuver point. First, let's find the radius at the maneuver point using the conic equation:

$$r_1 = \frac{p^-}{1 + e^- \cos\theta_1^{*-}} = 1.3025 \cdot 10^4 \text{ km} \tag{19}$$

which means, in the rotating frame:

$$\boxed{\bar{r}_1 = r_1\hat{r}_1 = 1.3025 \cdot 10^4\, \hat{r}_1 \text{ km}} \tag{20}$$

One can next determine the speed v_1^- of the spacecraft before the maneuver:

$$v_1^- = \sqrt{2\left(\mathcal{E}^- + \frac{\mu}{r_1}\right)} = 6.9316 \text{ km/s} \tag{21}$$

Then, one can solve for the flight path angle γ_1^- before the maneuver:

$$\boxed{\gamma_1^- = \arccos\frac{h^-}{r_1 v_1^-} = 27.4214°} \tag{22}$$

So, one can then determine the velocity of the spacecraft $\bar{v}_1^-$ of the spacecraft before the maneuver:

$$\boxed{\begin{aligned}\bar{v}_1^- &= v_1^- \sin\gamma_1^- \hat{r}_1 + v_1^- \cos\gamma_1^- \hat{\theta}_1 \\ &= 3.1922\, \hat{r}_1 + 6.1528\, \hat{\theta}_1 \text{ km/s}\end{aligned}} \tag{23}$$

Alternatively, if we selected the other maneuver point (#2), then the radius at the maneuver point using the conic equation would be:

$$r_2 = \frac{p^-}{1 + e^- \cos\theta_2^{*-}} = 3.0181 \cdot 10^4 \text{ km} \tag{24}$$

which means, in the rotating frame:

$$\bar{r}_2 = r_2\hat{r}_2 = 3.0181 \cdot 10^4\, \hat{r}_2 \text{ km} \tag{25}$$

One can next determine the speed v_2^- of the spacecraft before the maneuver:

$$v_2^- = \sqrt{2\left(\mathcal{E}^- + \frac{\mu}{r_2}\right)} = 3.6410 \text{ km/s} \tag{26}$$

Then, one can solve for the flight path angle γ_2^- before the maneuver:

$$\gamma_2^- = \arccos\frac{h^-}{r_2 v_2^-} = -43.1731° \tag{27}$$

So, one can then determine the velocity of the spacecraft $\bar{v}_2^-$ of the spacecraft before the maneuver:

$$
\begin{aligned}
\bar{v}_2^- &= v_2^- \sin\gamma_2^- \hat{r}_2 + v_2^- \cos\gamma_2^- \hat{\theta}_2 \\
&= -2.4912\, \hat{r}_2 + 2.6553\, \hat{\theta}_2 \text{ km/s}
\end{aligned}
\tag{28}
$$

Part (b)

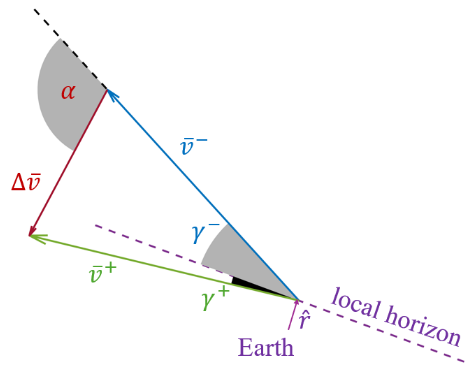

Figure 4: Sketch of the velocity vector diagrams at maneuver point #1.

Now, let's determine the required maneuver to transfer into the new orbit described. But first, one can first determine the speed v_1^+ of the spacecraft after the maneuver:

$$v_1^+ = \sqrt{2\left(\mathcal{E}^+ + \frac{\mu}{r_1}\right)} = 6.7513 \text{ km/s} \tag{29}$$

Before determining the velocity of the spacecraft though, one can calculate the flight path angle γ_1^+ of the spacecraft after the maneuver:

$$\boxed{\gamma_1^+ = \arccos \frac{h^+}{r_1 v_1^+} = -6.7260°} \tag{30}$$

So, one can now determine the velocity of the spacecraft $\bar{v}_1^+$ of the spacecraft after the maneuver. Note that, because the direction of the position vector remains the same and the maneuver remains within the same plane, the $\hat{r}$ and $\hat{\theta}$ remain the same. This, however, is not necessarily true for an out-of-plane maneuver:

$$\boxed{\begin{aligned} \bar{v}_1^+ &= v_1^+ \sin\gamma_1^+ \hat{r}_1 + v_1^+ \cos\gamma_1^+ \hat{\theta}_1 \\ &= -0.7907\,\hat{r}_1 + 6.7049\,\hat{\theta}_1 \text{ km/s} \end{aligned}} \tag{31}$$

Then, since the $\hat{r}$ and $\hat{\theta}$ remain the same, the $\Delta\bar{v}_1$ can be expressed in terms of the $\hat{r}$ and $\hat{\theta}$ directions (albeit, shifted to the end of $\bar{v}_1^-$):

$$\boxed{\Delta\bar{v}_1 = \bar{v}_1^+ - \bar{v}_1^- = -3.9830\,\hat{r}_1 + 0.5521\,\hat{\theta}_1 \text{ km/s}} \tag{32}$$

However, maneuver can also be expressed by its magnitude and angle with respect to the original velocity vector $\bar{v}_1^-$. So, the magnitude of the maneuver is:

$$\boxed{|\Delta\bar{v}_1| = \sqrt{(v_1^-)^2 + (v_1^+)^2 - 2v_1^- v_1^+ \cos\Delta\gamma_1} = 4.0210 \text{ km/s}} \tag{33}$$

where:

$$\Delta\gamma_1 = \gamma_1^+ - \gamma_1^- = -34.1474° \tag{34}$$

One can then determine the angle α with respect to the original velocity vector $\bar{v}_1^-$ using sine law:

$$\frac{\sin\beta}{v_1^+} = \frac{\sin|\Delta\gamma_1|}{|\Delta\bar{v}_1|} \tag{35}$$

where β is the supplementary angle to α. Thus:

$$\beta = \arcsin\left[\frac{v_1^+ \sin|\Delta\gamma_1|}{|\Delta\bar{v}_1|}\right] = 70.4698° \tag{36}$$

Since the angle α is pointing towards (counter-clockwise direction) the Earth, it will be a negative angle ($\alpha < 0$):

$$\boxed{\alpha = -(180° - \beta) = -109.5302°} \tag{37}$$

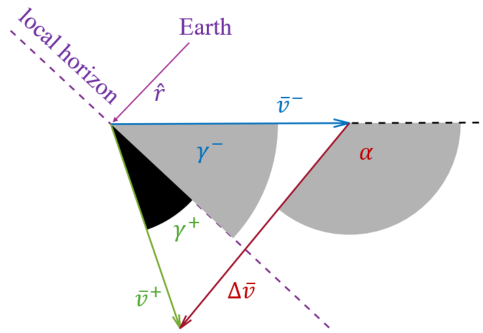

Figure 5: Sketch of the velocity vector diagrams at maneuver point #2.

Now, alternatively, and to confirm our selection in part (a), one can calculate the required maneuver to transfer into the new orbit at the second maneuver point. One can start with the speed v_2^+ of the spacecraft after the maneuver:

$$v_2^+ = \sqrt{2\left(\mathcal{E}^+ + \frac{\mu}{r_2}\right)} = 3.2848 \text{ km/s} \tag{38}$$

Again, before determining the velocity of the spacecraft, one can calculate the flight path angle γ_2^+ of the spacecraft after the maneuver:

$$\gamma_2^+ = \arccos \frac{h^+}{r_2 v_2^+} = 28.2486° \tag{39}$$

One can now determine the velocity of the spacecraft $\bar{v}_2^+$ of the spacecraft after the maneuver. Again, note that, because the direction of the position vector remains the same and the maneuver remains within the same plane, the $\hat{r}$ and $\hat{\theta}$ remain the same. This, however, is not necessarily true for an out-of-plane maneuver:

$$\begin{aligned} \bar{v}_2^+ &= v_2^+ \sin\gamma_2^+ \hat{r}_2 + v_2^+ \cos\gamma_2^+ \hat{\theta}_2 \\ &= 1.5547\, \hat{r}_2 + 2.8936\, \hat{\theta}_2 \text{ km/s} \end{aligned} \tag{40}$$

Then, since the $\hat{r}$ and $\hat{\theta}$ remain the same, the $\Delta\bar{v}_2$ can be expressed in terms of the $\hat{r}$ and $\hat{\theta}$ directions (albeit, shifted to the end of $\bar{v}_1^-$):

$$\Delta\bar{v}_2 = \bar{v}_2^+ - \bar{v}_2^- = 4.0459\, \hat{r}_2 + 0.2382\, \hat{\theta}_2 \text{ km/s} \tag{41}$$

However, maneuver can also be expressed by its magnitude and angle with respect to the original velocity vector $\bar{v}_2^-$. So, the magnitude of the maneuver is:

$$|\Delta\bar{v}_2| = \sqrt{(v_2^-)^2 + (v_2^+)^2 - 2v_2^- v_2^+ \cos\Delta\gamma_2} = 4.0529 \text{ km/s} \tag{42}$$

where:

$$\Delta\gamma_2 = \gamma_2^+ - \gamma_2^- = 71.4217° \tag{43}$$

One can then determine the angle α with respect to the original velocity vector $\bar{v}_2^-$ using sine law:

$$\frac{\sin\beta}{v_2^+} = \frac{\sin|\Delta\gamma_2|}{|\Delta\bar{v}_2|} \tag{44}$$

where β is the supplementary angle to α. Thus:

$$\beta = \arcsin\left[\frac{v_2^+ \sin|\Delta\gamma_2|}{|\Delta\bar{v}_2|}\right] = 50.1970° \tag{45}$$

Since the angle α is pointing away (clockwise direction) from the Earth, it will be a positive angle ($\alpha > 0$):

$$\alpha = 180° - \beta = 129.8030° \tag{46}$$

As one can see from the result in equation (42), our selection in part (a) <u>was correct</u> as the $|\Delta\bar{v}|$ required for maneuver point #2 is <u>greater than</u> $|\Delta\bar{v}|$ required for maneuver point #1.

Part (c)

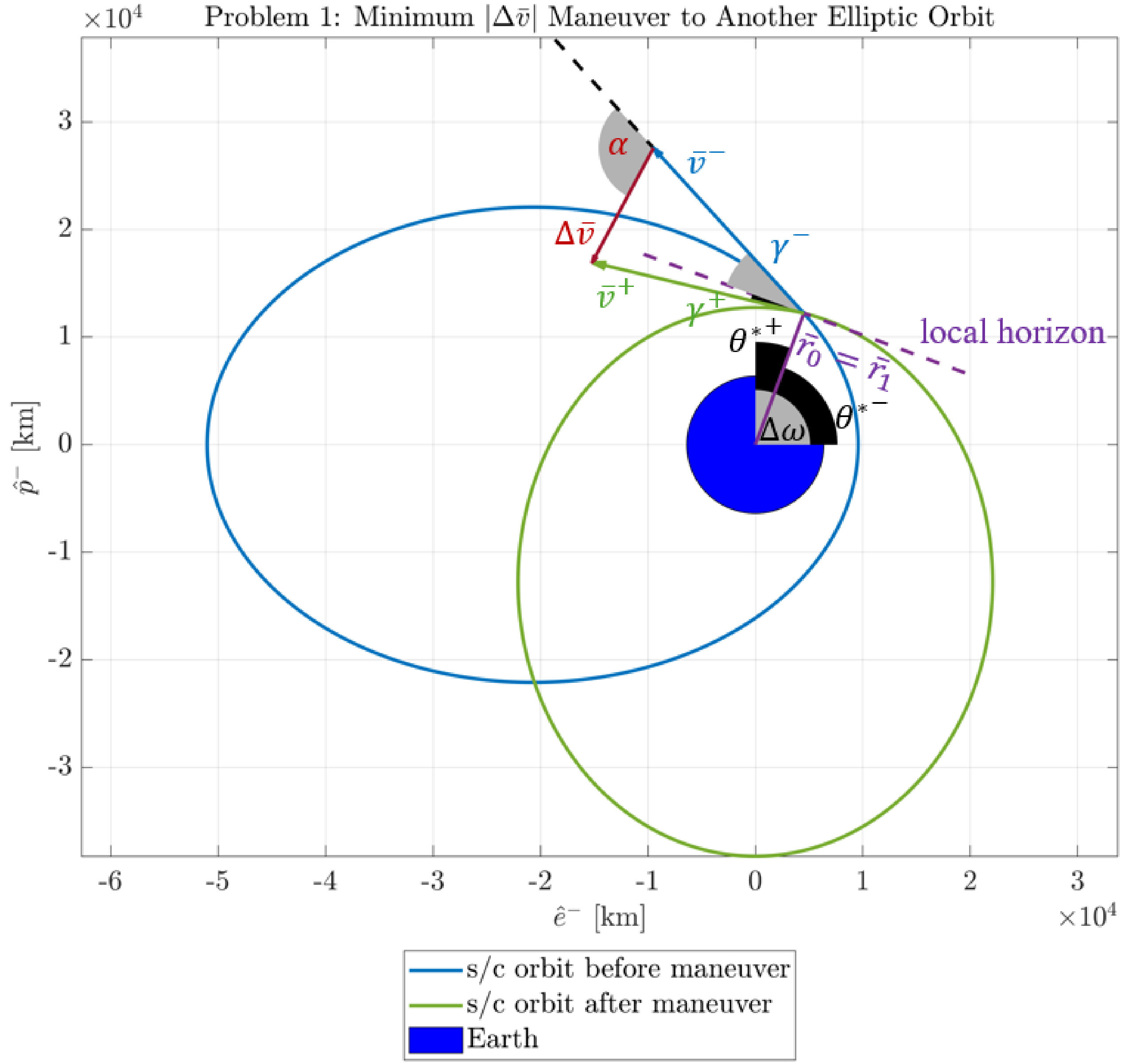

Figure 6: Plot of old and new orbits at maneuver point that minimizes $|\Delta\bar{v}|$.

The plot of the old and new orbits when performing a maneuver at the minimum $|\Delta\bar{v}|$ location are depicted in figure 6. Recognize that the maneuver point occurs closer to perigee in both orbits. Also recognize that the velocity vectors depicted in the figure are exaggerated, but are at the same scale with respect to each other.

Problem 2

Problem Statement

Assuming a relative 2B model, a spacecraft moves in Earth orbit characterized by the following:

$$
\begin{aligned}
a &= 8R_{\oplus} \\
e &= 0.7 \\
i &= 30^\circ \\
\Omega &= 60^\circ \\
\omega &= 90^\circ
\end{aligned}
$$

with respect to an Earth inertial equatorial coordinate system. To meet some specific objective, a maneuver is planned to adjust the orbit.

(a) Let a maneuver be implemented when $\theta^* = 150^\circ$. Apply a Δv such that the magnitude is 1 km/s and directed such that $\alpha = +45^\circ$ and $\beta = +60^\circ$. Express the Δv vector in terms of rotating orbit unit vectors $(\hat{r}, \hat{\theta})$; inertial orbit unit vectors $(\hat{e}, \hat{p})$; and VNC unit vectors. (Note that these are all associated with the ORIGINAL orbit!) Print the orbit as seen in GMAT; identify the maneuver location.

(b) Determine the state immediately following the maneuver: $\bar{r}, \bar{v}, r, v, M, E, \gamma, \theta^*, \Delta\omega$. Determine the characteristics of the new orbit: $a_N, e_N, i_N, \Omega_N, \omega_N, \Delta\omega$.

(c) Implement the maneuver and confirm the results with GMAT. Print the corresponding parts of the output file. Plot the initial/final orbit. Add the Equatorial Plane, the J2000 unit vectors, the Satellite Periapsis vector, and the Line of Nodes vector to a 3D GMAT plot; plot two different views. Indicate all of the appropriate information of the plot.

Part (a)

First we can sketch the vector diagram of the maneuver in both the $\hat{V}\hat{N}\hat{C}$ and $\hat{r}\hat{\theta}\hat{h}$ frames:

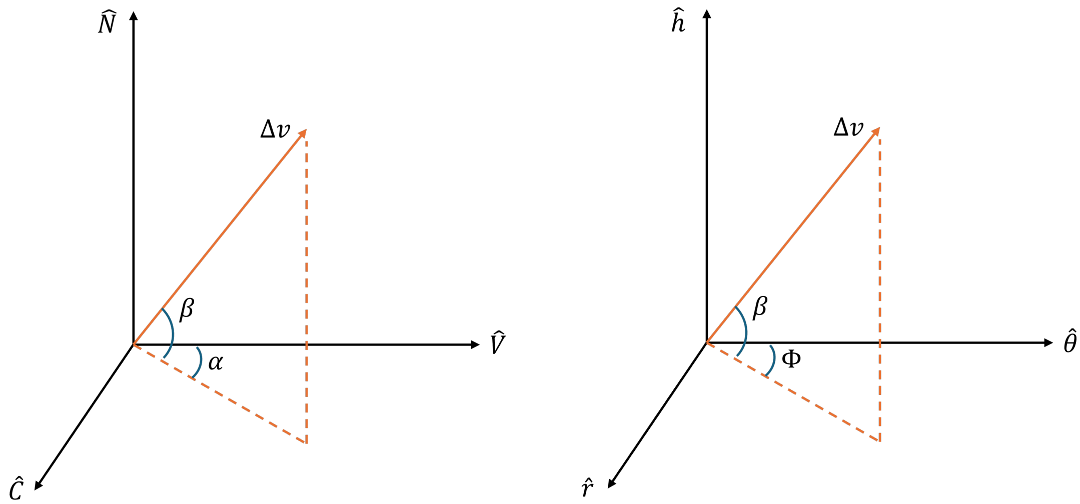

Figure 7: Δv diagrams.

Then we can express the Δv in each frame as:

$$\Delta\bar{v}_{VNC} = \Delta v\left[\cos\beta\cos\alpha\hat{V} + \sin\beta\hat{N} + \cos\beta\sin\alpha\hat{C}\right]$$

$$\boxed{\Delta\bar{v}_{VNC} = 0.3536\hat{V} + 0.8660\hat{N} + 0.3536\hat{C}\ km/s}$$

$$\Delta\bar{v}_{r\theta h} = \Delta v\left[\cos\beta\sin\Phi\hat{r} + \cos\beta\cos\Phi\hat{\theta} + \sin\beta\hat{h}\right]$$

Where the angle Φ is:

$$\Phi = \alpha + \gamma$$

To find γ we will need to compute some additional parameters:

$$p = a(1-e^2) = 2.6023e4\ km$$

$$h = \sqrt{\mu p} = 1.0185e5\ km^2/s$$

$$r = \frac{p}{1 + e\cos\theta^*} = 6.6084e4\ km$$

$$v = \sqrt{\mu\left[\frac{2}{r} - \frac{1}{a}\right]} = 2.0619\ km/s$$

$$\gamma = \pm\cos^{-1}\left(\frac{h}{rv}\right) = 0.7266\ rad = 41.63°$$

Note that the flight path angle is positive because our true anomaly is between 0° and 180° meaning that the spacecraft is ascending. From this we determine $\Phi = 86.63°$ and:

$$\boxed{\Delta\bar{v}_{r\theta h} = 0.4991\hat{r} + 0.0294\hat{\theta} + 0.8660\hat{h}\ km/s}$$

To transform from the $\hat{r}\hat{\theta}\hat{h}$ to $\hat{e}\hat{p}\hat{h}$ frames:

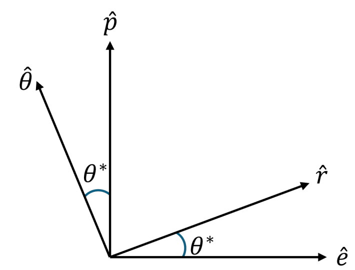

Figure 8: Frame rotation diagram.

$$^{eph}C^{r\theta h} = \begin{bmatrix} \cos\theta^* & -\sin\theta^* & 0 \\ \sin\theta^* & \cos\theta^* & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Delta\bar{v}_{eph} =^{eph} C^{r\theta h}\Delta\bar{v}_{r\theta h}$$

$$\boxed{\Delta\bar{v}_{eph} = -0.4470\hat{e} + 0.2241\hat{p} + 0.8660\hat{h}\ km/s}$$

To visualize the orbit in GMAT, we select an arbitrary epoch of Oct 25, 2024 and plot the orbit for one full period:

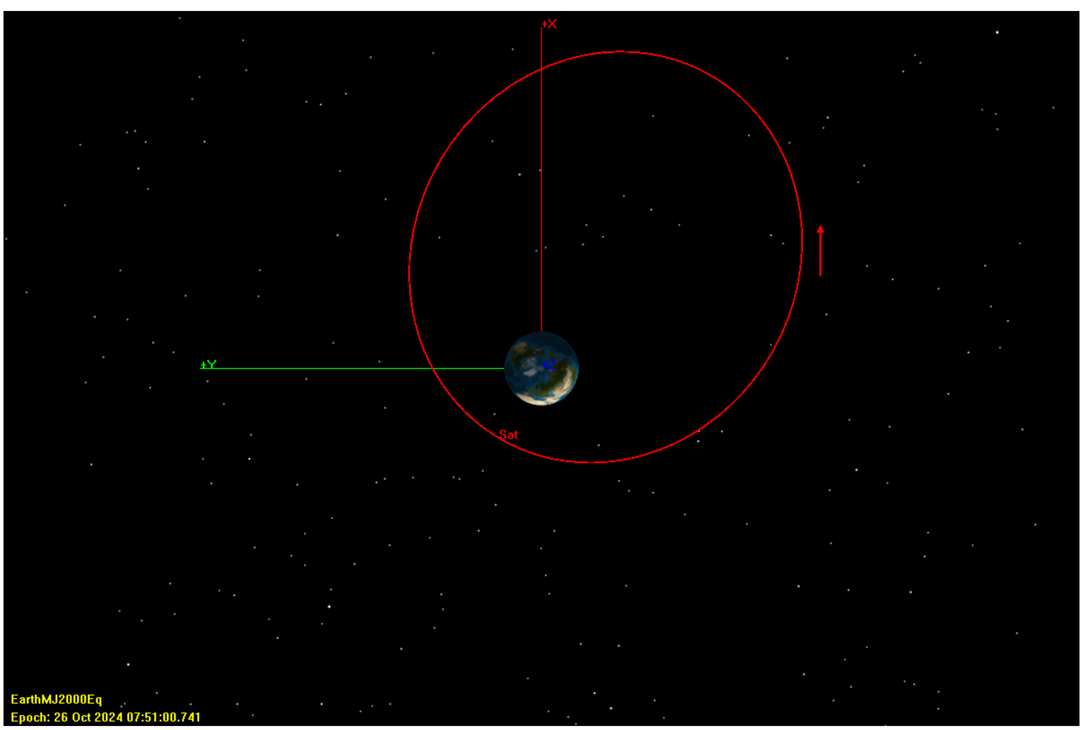

Figure 9: Orbit propagated in GMAT for one period.

Then, to observe where the maneuver will take place, we can propagate to a true anomaly of 150°. Note that as expected, the spacecraft is ascending and this location is before apoapsis.

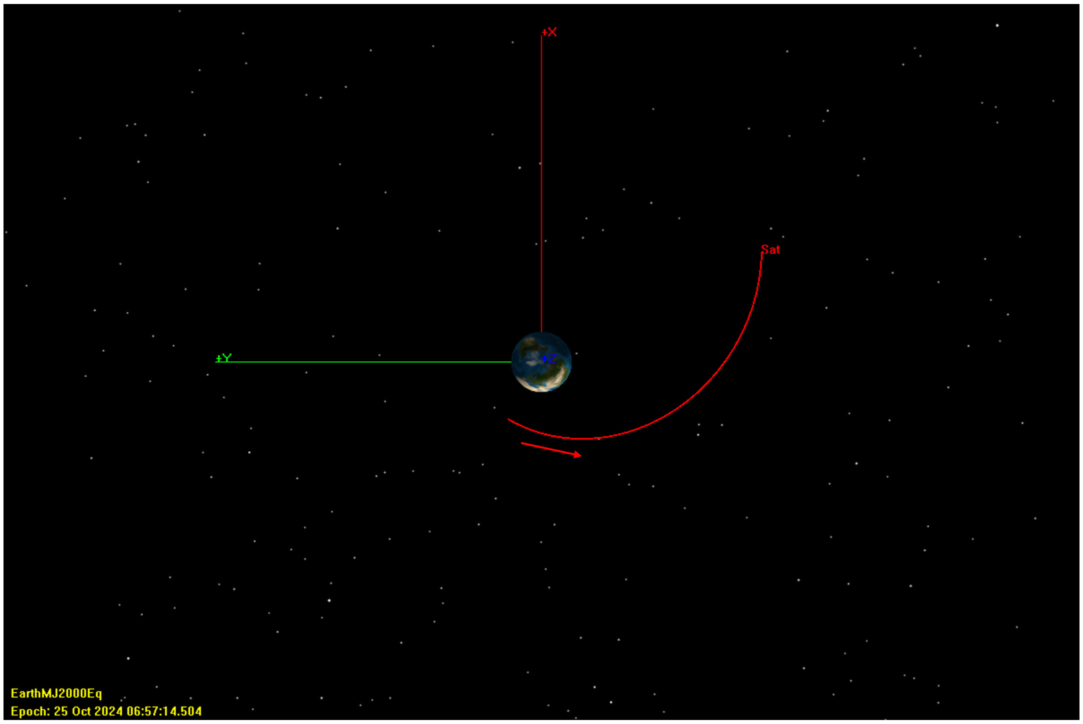

Figure 10: Orbit propagated in GMAT to the maneuver location.

Part (b)

We know that prior to the maneuver:

$$\bar{r}^- = 6.6084e4\hat{r}\ km$$

$$\bar{v}^- = 2.0619\sin\gamma\hat{r} + 2.0169\cos\gamma\hat{\theta}\ km/s = 1.3698\hat{r} + 1.5412\hat{\theta}\ km/s$$

We can convert from the $\hat{r}\hat{\theta}\hat{h}$ frame to inertial coordinates using the direction cosine matrix:

$$^IC^R = \begin{bmatrix} \cos\Omega\cos\theta - \sin\Omega\cos i\sin\theta & -\cos\Omega\sin\theta - \sin\Omega\cos i\cos\theta & \sin\Omega\sin i \\ \sin\Omega\cos\theta + \cos\Omega\cos i\sin\theta & -\sin\Omega\sin\theta + \cos\Omega\cos i\cos\theta & -\cos\Omega\sin i \\ \sin i\sin\theta & \sin i\cos\theta & \cos i \end{bmatrix}$$

Where $\theta = \omega + \theta^* = 240°$. Therefore:

$$\Delta\bar{v}_{xyz} =^I C^R \Delta\bar{v}_{r\theta h} = 0.5982\hat{x} - 0.6041\hat{y} + 0.5265\hat{z}\ km/s$$

$$\bar{r}^-_{xyz} =^I C^R \bar{r}^-_{r\theta h} = 2.6402e4\hat{x} - 5.3397e4\hat{y} - 2.8615e4\hat{z}\ km$$

$$\bar{v}^-_{xyz} =^I C^R \bar{v}^-_{r\theta h} = 1.7925\hat{x} - 0.2846\hat{y} - 0.9784\hat{z}\ km/s$$

Since the maneuver is impulsive and applied instantaneously, the position before and after are the same. The velocity after the maneuver is found using vector addition.

$$\boxed{\bar{r}^+_{xyz} = \bar{r}^-_{xyz} = 2.6402e4\hat{x} - 5.3397e4\hat{y} - 2.8615e4\hat{z}\ km}$$

$$\boxed{\bar{v}^+_{xyz} = \bar{v}^-_{xyz} + \Delta\bar{v}_{xyz} = 2.3907\hat{x} - 0.8888\hat{y} - 0.4519\hat{z}\ km/s}$$

Then finding the magnitudes:

$$\boxed{r^+_{xyz} = ||\bar{r}^+_{xyz}|| = 6.6084e4\ km}$$

$$\boxed{v^+_{xyz} = ||\bar{v}^+_{xyz}|| = 2.5903\ km/s}$$

Now we can continue solving for the other parameters:

$$\bar{h}^+ = \bar{r}^+ \times \bar{v}^+ = -1.3015e3\hat{x} - 5.6479\hat{y} + 1.0419e5\hat{z}\ km^2/s$$

$$h^+ = 1.1852e5\ km^2/s$$

$$\gamma^+ = \pm\cos^{-1}\left(\frac{h^+}{r^+v^+}\right) = \pm 0.8060\ rad = \pm 46.18°$$

To determine the sign we do a quadrant check by taking the dot product of $\bar{r}$ and $\bar{v}$:

$$\bar{r}^+ \cdot \bar{v}^+ = 1.8689 > 0$$

Thus the spacecraft is still ascending in its new orbit and the flight path angle should be positive.

$$\boxed{\gamma^+ = +46.18°}$$

$$\frac{-\mu}{2a} = \frac{v^2}{2} - \frac{\mu}{r}$$

$$\boxed{a^+ = 7.4451e4\ km = 11.6729 R_\oplus}$$

$$e = \sqrt{1 - \frac{p}{a}} \qquad p = \frac{h^2}{\mu}$$

$$\boxed{e^+ = 0.7257}$$

$$\theta^* = \pm \cos^{-1}\left(\frac{1}{e}\left[\frac{p}{r} - 1\right]\right)$$

$$\boxed{\theta^*_+ = 2.2694\ rad = 130.03°}$$

$$E = \pm \cos^{-1}\left(\frac{1}{e}\left[1 - \frac{r}{a}\right]\right)$$

$$\boxed{E^+ = 1.4153\ rad = 81.09°}$$

The sign of the angles are positive since the spacecraft is ascending.

$$M = E - e \sin E$$

$$\boxed{M^+ = 0.6984\ rad = 40.01°}$$

Determining the directions of the $\hat{r}\hat{\theta}\hat{h}$ frame after the maneuver is performed:

$$\hat{r} = \frac{\bar{r}^+_{xyz}}{||r^+_{xyz}||} = 0.3995\hat{x} - 0.8080\hat{y} - 0.4330\hat{z}$$

$$\hat{h} = \frac{\bar{h}^+_{xyz}}{||h^+_{xyz}||} = -0.0110\hat{x} - 0.4765\hat{y} + 0.8791\hat{z}$$

$$\hat{\theta} = \hat{h} \times \hat{r} = 0.9167\hat{x} + 0.3465\hat{y} + 0.1993\hat{z}$$

We can relate these values to the corresponding elements of ${}^I C^R_+$ where $\hat{r}$ is the first column (elements C_{11}, C_{21}, C_{31}), $\hat{\theta}$ is the second column (elements C_{12}, C_{22}, C_{32}), and $\hat{h}$ is the third column (elements C_{13}, C_{23}, C_{33}). This gives us expressions between the values we have already solved for and the angles we still need to find.

$$\cos i =^I C^R_{+33}$$

$$\boxed{i^+ = 0.4968\ rad = 28.47°}$$

To find Ω^+ we need to evaluate both double valued expressions and pick the solution that appears in both:

$$\sin\Omega\sin i =^I C^R_{+13} \quad\longrightarrow\quad \Omega = 358.68°, 181.32°$$

$$-\cos\Omega\sin i =^I C^R_{+23} \quad\longrightarrow\quad \Omega = \pm 358.68°$$

Therefore,

$$\boxed{\Omega^+ = 358.68°}$$

Following the same process for θ^+:

$$\sin i\sin\theta =^I C^R_{+31} \quad\longrightarrow\quad \theta = -65.29°, -114.71°$$

$$\sin i\cos\theta =^I C^R_{+32} \quad\longrightarrow\quad \theta = \pm 65.29°$$

Therefore,

$$\boxed{\theta^+ = -65.29° = 294.71°}$$

Finally,

$$\boxed{\omega^+ = \theta^+ - \theta^*_+ = 164.69°}$$

$$\boxed{\Delta\omega = \omega^+ - \omega^- = 74.69°}$$

Part (c)

After implementing the maneuver we can visualize both orbits using GMAT:

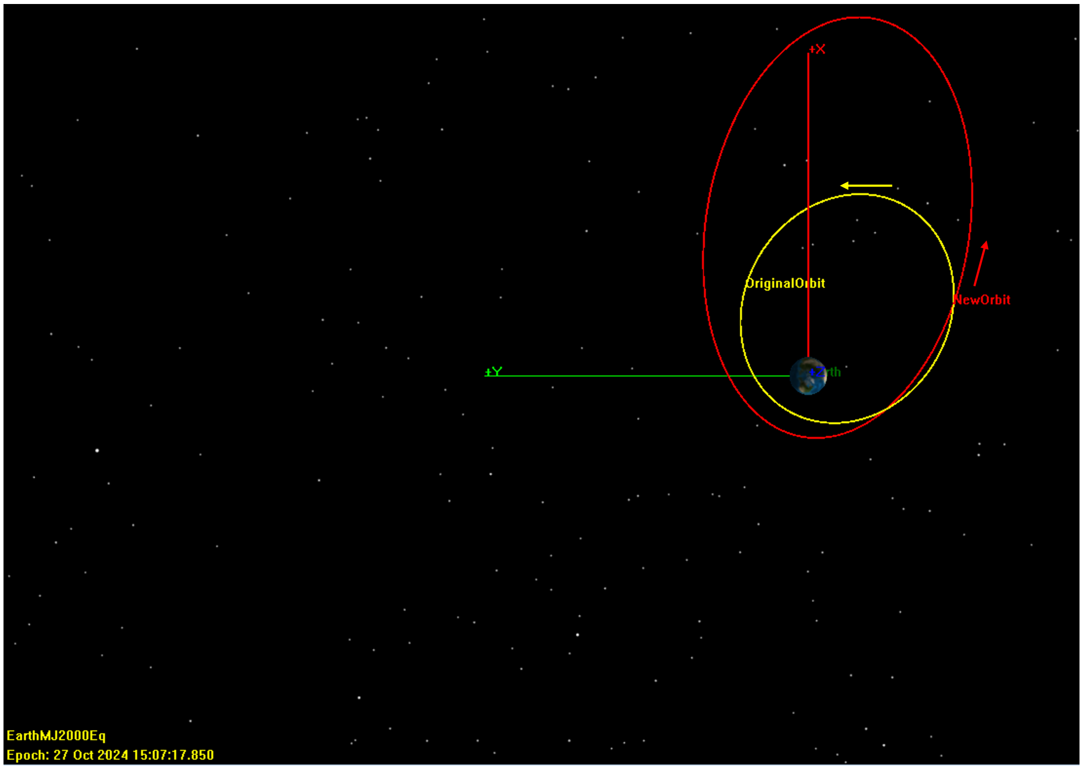

Figure 11: Both orbits projected onto xy plane.

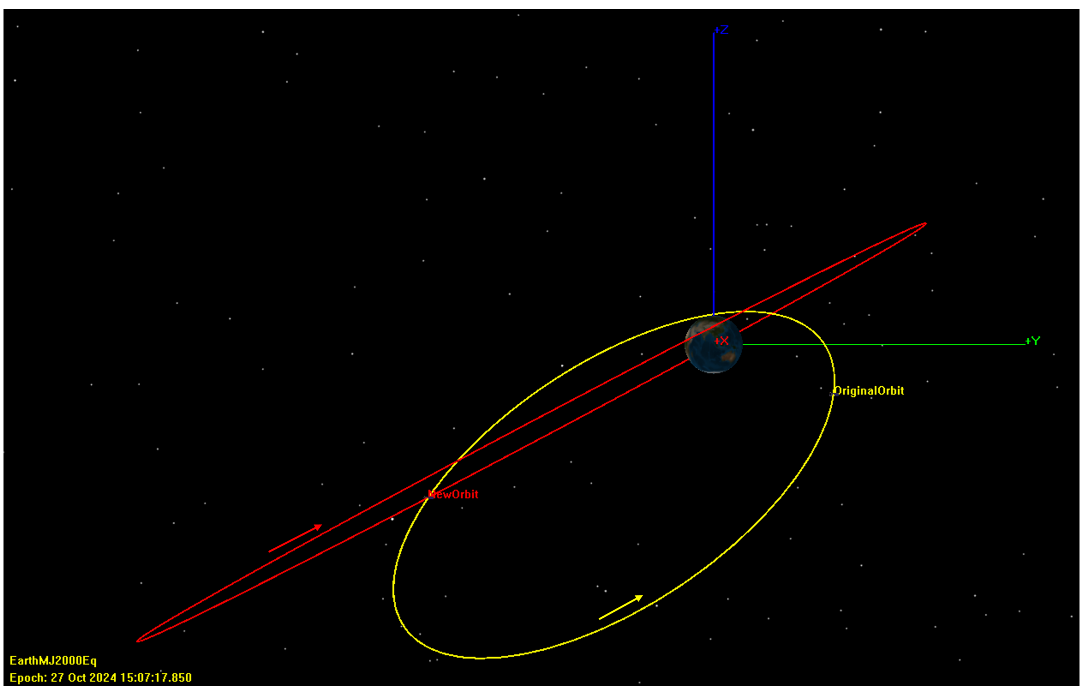

Figure 12: Both orbits projected onto yz plane.

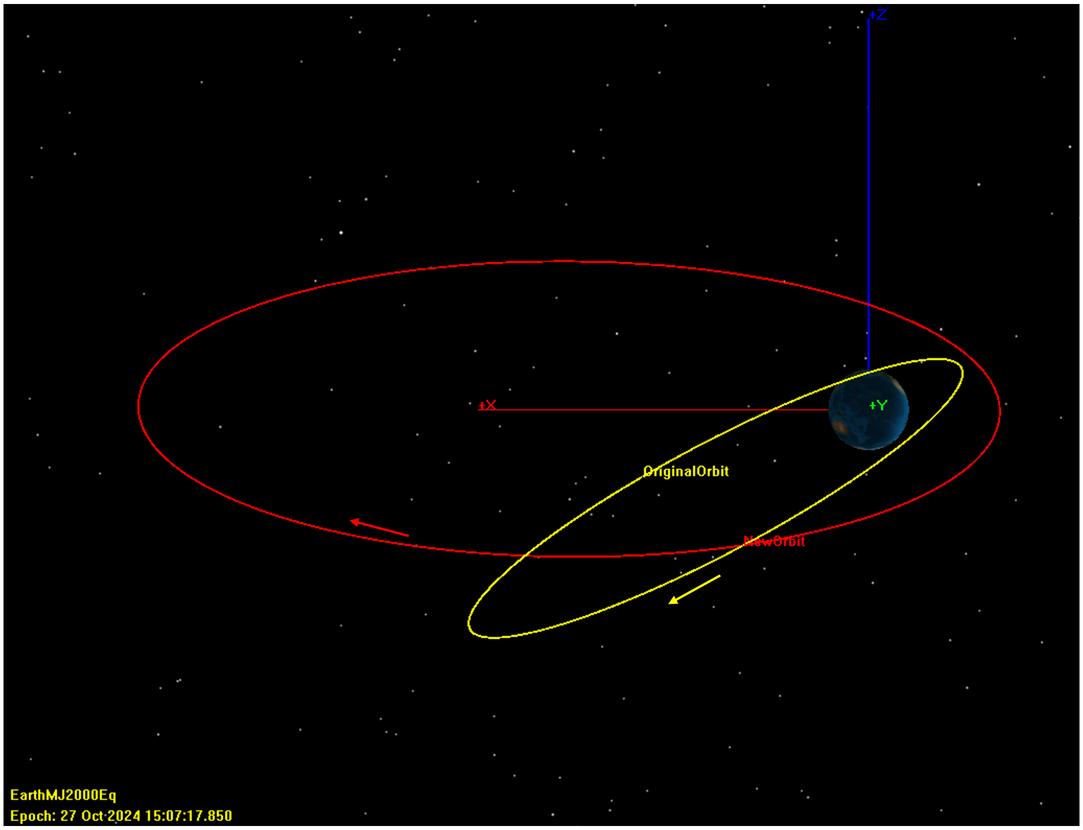

Figure 13: Both orbits projected onto xz plane.

Additionally, we can annotate a couple views to show the equatorial plane, line of nodes of the new orbit, and the periapsis vectors of both orbits to show how it has changed:

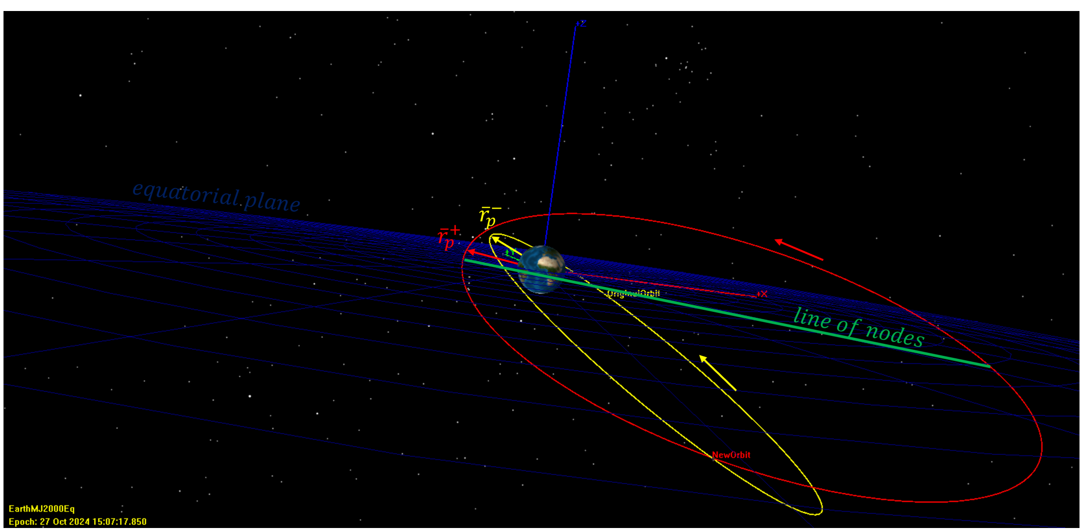

Figure 14: Annotated orbits.

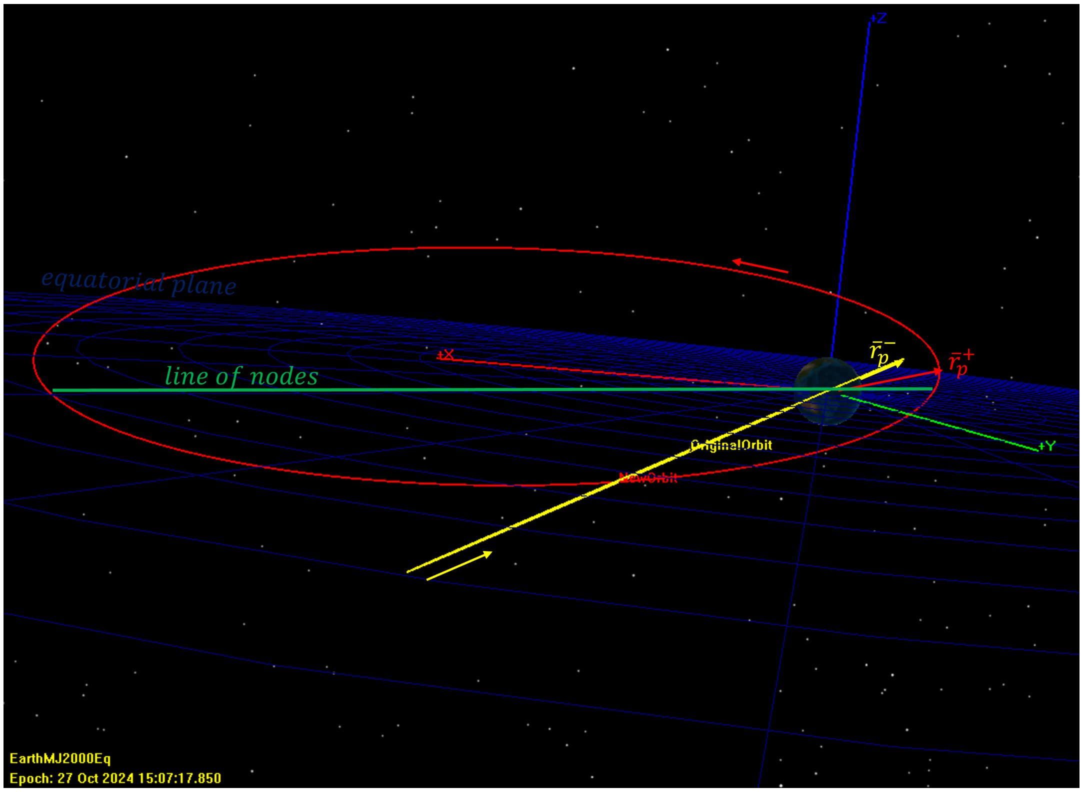

Figure 15: Annotated orbits.

We can also generate an output file to analyze the tabulated results. Note that the center highlighted lines show the location where the maneuver was performed:

A1ModJulian	EarthMJ2000Eq.FPA	EarthMJ2000Eq.RAAN	Earth.TA
30608.50042863868	90.00000000000001	60.00000000000001	0
30608.50112308312	89.38486668654267	59.99957027234648	1.493912509814561
30608.50297964156	87.74330359548172	59.99842833255446	5.481410662523039
30608.50619200655	84.93082101233907	59.99651831127554	12.32070675172917
...			
30608.77740653507	47.78894141471872	59.98612095564702	148.4901678380517
30608.79017984193	48.37651570896068	59.98597076782217	150.0000000270583
30608.79087428637	43.81702857003345	358.6823879387325	130.1196403147524
30608.79290654285	43.79051700183143	358.6823781634332	130.3904160087095
...			
30611.11497076707	44.08465665531124	358.6676969624193	127.8288758110353
30611.13049634576	43.83557646164443	358.6676107849726	129.9999999937481

Problem 3

Problem Statement

The New Horizons spacecraft is the only spacecraft from Earth to visit the Pluto System and it is now moving through the Kuiper Belt. After launch January 19, 2006, the spacecraft encountered Jupiter with a closest approach on February 28, 2007. New Horizons then crossed the orbit of Uranus on March 18, 2011, and Pluto closest approach occurred July 14, 2015. New Horizons then continued on to reach Kuiper belt objects. On January 1, 2019, New Horizons accomplished a flyby of Ultima Thule (officially named 2014 MU69), a Kuiper Belt Object that is approximately 4 billion miles (6.4 billion kilometers) from Earth. On April 17, 2021, New Horizons passed a distance of 50 AU from the Sun!

(a) How far is New Horizons from the Sun on October 15, 2024? (Cite your source.) Do you have an image to share?

(b) Consider all the dates given above. Create a timeline by converting all the dates to Julian Days (JD). Assume that all events occur at noon. Determine the number of Julian days and Julian years that occur between each event as well as the mission length to the 10-15-24 date. [Note that a Julian Day is exactly 86400 seconds and a Julian year is exactly 365.25 Julian days.]

(c) Begin considering a transfer by examining a Hohmann transfer from $\oplus$ to ♇. Assume that planetary orbits are coplanar and circular. Compare total $|\Delta\bar{v}_{total}|$ for both a departure and an arrival maneuver and the TOF (time- of-flight in Julian years). Ignore the local gravity fields. [Do not forget vector diagrams! Each planar $\Delta\bar{v}$ still requires $|\Delta\bar{v}|, \alpha, \beta$.] Of course, New Horizons did not employ an arrival maneuver as it passed by the system. Focus on just the departure, i.e., $|\Delta\bar{v}_1|$ which is the maneuver necessary at Earth departure. Even though a Hohmann transfer is the minimum cost two-impulse maneuver, is it likely that we could generally use such a transfer path to get to Pluto? Why not? Why did New Horizons not use a Hohmann transfer? Compare the Hohmann TOF to the actual Earth-to-Pluto TOF used by New Horizons. In the US, interplanetary missions are planned to be no longer than 14 years. Why might that be the plan in the US?

(d) For the Hohmann transfer, what is the phase angle at Earth departure and the synodic period?

(e) Reconsider and improve the model for Pluto's orbit. Assume it is eccentric with the eccentricity as given in the Table of Constants. Now, consider a Hohmann transfer to Pluto's perihelion. Re-compute the total $|\Delta\bar{v}|$ and TOF for a Hohmann transfer from Earth to Pluto perihelion. Does it save $|\Delta\bar{v}|$? TOF? Upon arrival in the Pluto system, observe that the spacecraft hyperbolic passage by Pluto also allowed close encounters with Pluto's five known moons, including Charon, Hydra, Kerberos, Styx, and Nix!

Part (a)

Using the Skylive website which was introduced earlier in the semester, we can set the ephemeris date to October 15, 2024 at 00:00 UTC and see that the New horizons spacecraft is approximately 9001.27Mkm or 60.1698AU from the Earth, and 8990.41Mkm or 60.09718AU from the Sun. The spacecraft is located closer to the Sun than Earth due to their relative locations in their orbits about the Sun. Currently, they are on opposite sides of the Sun from one another but a few months from now when Earth has progressed more in its orbit, the spacecraft will be closer to the Earth than it is the Sun. This is why if we were to visualize the spacecrafts position relative to Earth as a function of time, we would see oscillating behavior occurring simultaneously as the distance grows. We can also visualize the spacecrafts current location in orbit:

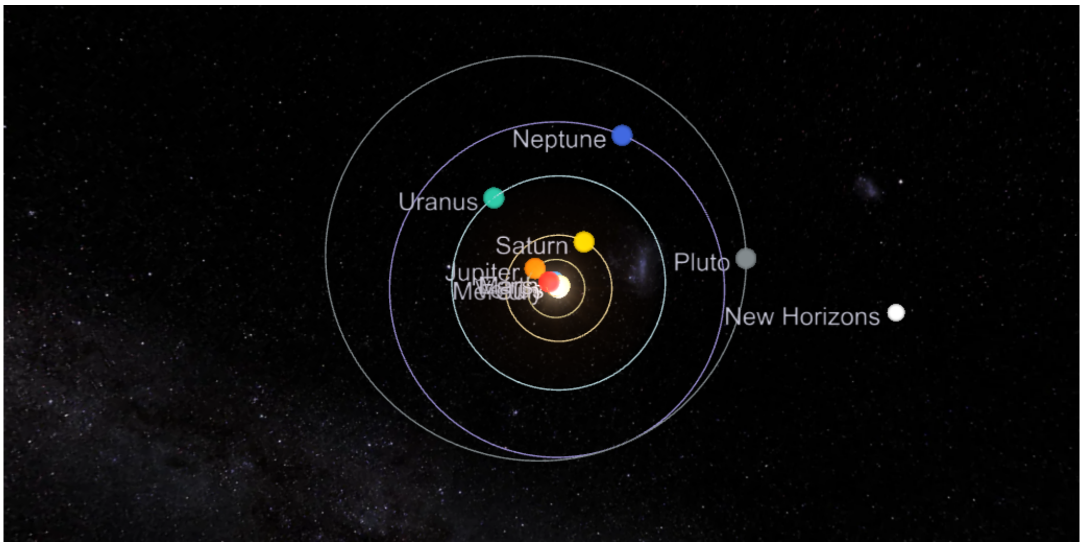

Figure 16: New Horizons location in the solar system on 10/15/2024.

Source: https://theskylive.com/newhorizons-info

Part (b)

Using the JPL date/time converter we can create a table with all the given dates. Note that the last two columns give the amount of time that has passed since the previous event.

Event	Calender Date	Julian Date	Julian days passed	Julian years passed
Launch	1/19/2006	2453755	-	-
Jupiter close approach	2/28/2007	2454160	405	1.1088
Uranus crossing	3/18/2011	2455639	1479	4.0493
Pluto close approach	7/14/2015	2457218	1579	4.3231
Ultima Thule flyby	1/1/2019	2458485	1267	3.4689
50 AU reached	4/17/2021	2459322	837	2.2916
Reference date	10/15/2024	2460599	1277	3.4962
Total mission	-	-	6844	18.7379

Source: https://ssd.jpl.nasa.gov/tools/jdc/#/cd

Part (c)

We start with the assumption that the orbits of the Earth and Pluto are coplanar and circular.

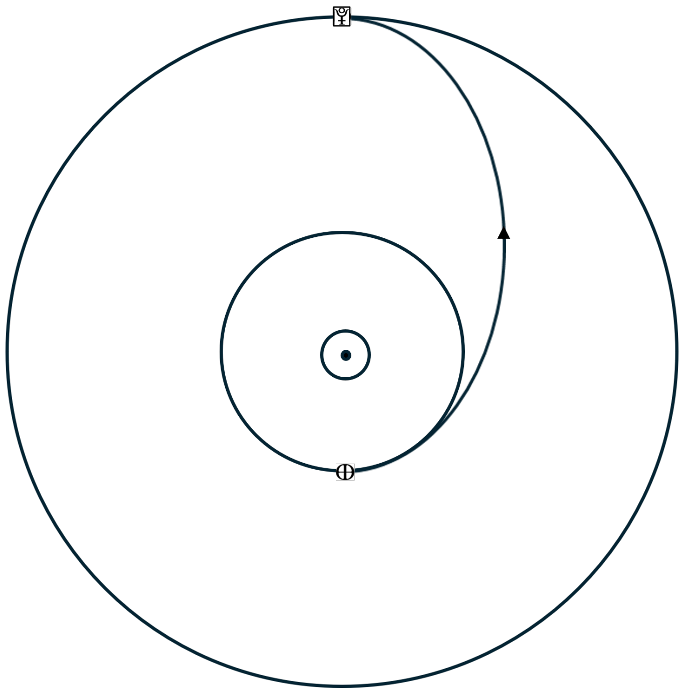

Figure 17: Hohmann transfer diagram from Earth to Pluto.

$$a_h = \frac{a_\oplus + a_{♇}}{2} = 20.2434\ AU$$

$$|v_p^{\odot s}| = \sqrt{2\left(\frac{\mu_\odot}{a_\oplus} - \frac{\mu_\odot}{2a_h}\right)} = 41.5985\ km/s$$

$$|v_a^{\odot s}| = \sqrt{2\left(\frac{\mu_\odot}{a_{♇}} - \frac{\mu_\odot}{2a_h}\right)} = 1.0535\ km/s$$

Now we can look at the vector diagrams for the velocity at departure and arrival:

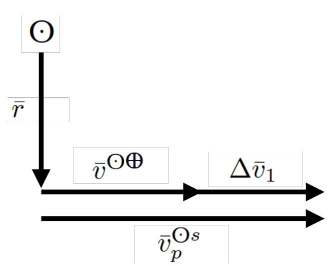

Figure 18: Vector diagram for Earth departure.

$$v^{\odot\oplus} = \sqrt{\frac{\mu_\odot}{a_\oplus}} = 29.7847\ km/s$$

$$\Delta\bar{v}_1 = \bar{v}_p^{\odot s} - \bar{v}^{\odot\oplus} \qquad \longrightarrow \qquad \Delta v_1 = v_p^{\odot s} - v^{\odot\oplus} = 11.8138\ km/s$$

$$\alpha_1 = \beta_1 = 0°$$

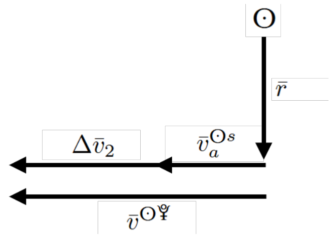

Figure 19: Vector diagram for Pluto arrival.

$$v^{\odot ♇} = \sqrt{\frac{\mu_\odot}{a_{♇}}} = 4.7399\ km/s$$

$$\Delta v_2 = v^{\odot ♇} - v_a^{\odot s} = 3.6864\ km/s$$

$$\alpha_2 = \beta_2 = 0°$$

Combining the two maneuvers gives us a total Δv of:

$$\boxed{\Delta v_{total} = 15.5002\ km/s}$$

And a total time of flight of:

$$TOF = \frac{P_h}{2} = \pi\sqrt{\frac{a_h^3}{\mu_\odot}}\ \boxed{= 45.5412\ yrs}$$

Although a Hohmann transfer is the minimum cost two-impulse maneuver, there are many reasons why it is not likely to be used in an actual mission design to Pluto. First, we can consider the cost associated with the transfer. Since New Horizons did not perform a maneuver on its arrival at Pluto, we focus on the Earth departure Δv of 11.8138 km/s. This is an extremely large maneuver that is not likely practical for a deep space mission but could be reduced through the use of planetary flybys of additional bodies on the way to Pluto. There are many reasons why this type of transfer was not used for the New Horizons mission, including the large Δv and long time of flight, but also because the assumptions we made when assessing this problem are not applicable to real life applications. While it is often reasonable for preliminary analysis to assume circular coplanar orbits, this is not the case for Pluto which we saw in Problem Set 1 has an orbit that is significantly more eccentric and inclined than the primary planetary bodies in our solar system.

Additionally, we can compare the time of flight for the Hohmann transfer (45.5412 Julian years) to the actual time New Horizons took to reach Pluto (9.4812 Julian years) which was significantly shorter. Interplanetary missions in the US are generally planned for a maximum duration of 14 years. Some reasons for this include the degradation of hardware and software over time. Technology, especially in the space industry, advances quickly but once a spacecraft is designed and launched, hardware and software on the ground has to be maintained to be compatible with what is onboard the spacecraft. There are other factors to consider as well like what kind of power supply system is on board the spacecraft and whether it can deliver adequate power to the spacecraft for the entire mission duration. The mission objectives could also determine what an appropriate duration is since the science objectives might need to be accomplished within a certain amount of time. Aside from the many technical implications of longer missions, there are also issues relating to the maintenance of on the ground personnel, funding, political factors, and more.

Part (d)

First we can annotate the original Hohmann transfer diagram to include the phase angle:

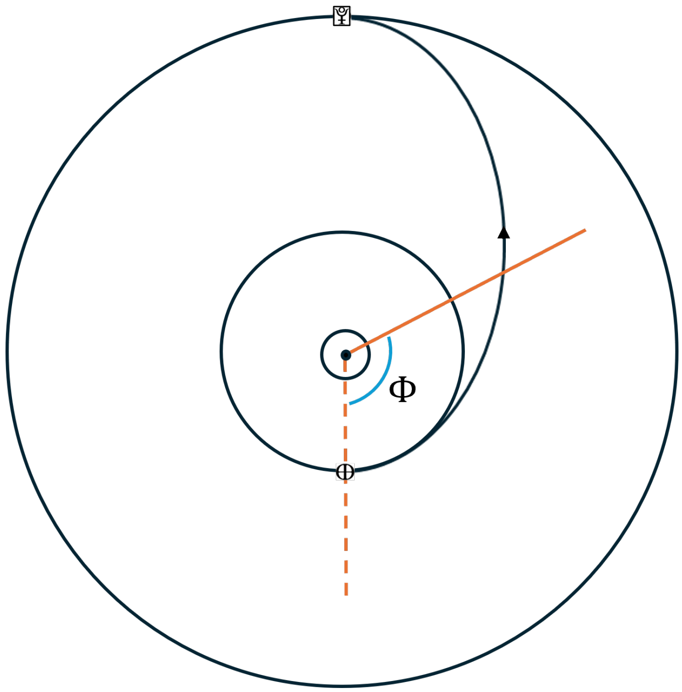

Figure 20: Hohmann transfer diagram from Earth to Pluto.

$$\Phi = 180^\circ - n_{♇}(TOF)$$

$$n_{♇} = \sqrt{\frac{\mu_\odot}{a_{♇}^3}}$$

$$\boxed{\Phi = 113.9276^\circ}$$

The synodic period is the amount of time it takes for a given alignment to repeat and can be calculated as:

$$t_{synodic} = \frac{2\pi}{n_\oplus - n_{♇}} \boxed{= 1.0041\ years}$$

Part (e)

If we improve the model for Pluto's orbit to include eccentricity we can modify the Hohmann transfer diagram:

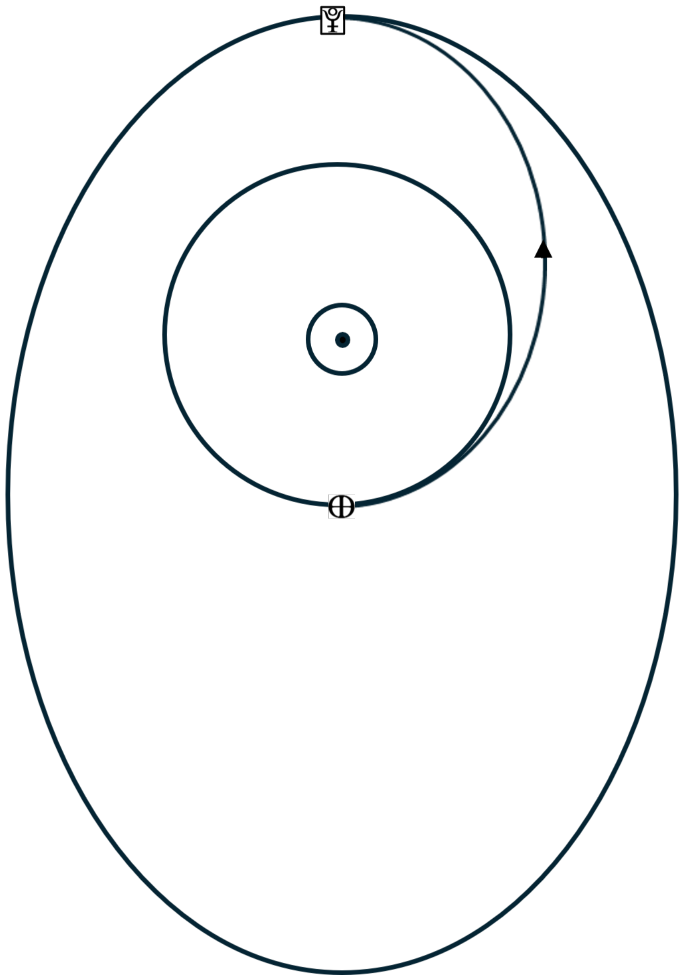

Figure 21: Modified Hohmann transfer diagram from Earth to Pluto.

We can recompute the cost of a transfer to Pluto at perihelion:

$$a_h = \frac{a_{\oplus} + r_{p♇}}{2} = 15.3304\ AU$$

$$|v_p^{\odot s}| = \sqrt{2\left(\frac{\mu_{mathSun}}{a_{\oplus}} - \frac{\mu_{\odot}}{2a_h}\right)} = 41.4293\ km/s$$

$$|v_a^{\odot s}| = \sqrt{2\left(\frac{\mu_{mathSun}}{r_{p♇}} - \frac{\mu_{\odot}}{2a_h}\right)} = 1.3968\ km/s$$

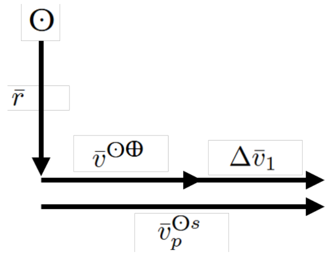

Figure 22: Vector diagram for Earth departure.

$$v^{\odot\oplus} = \sqrt{\frac{\mu_\odot}{a_\oplus}} = 29.7847\ km/s$$

$$\Delta v_1 = v^{\odot s} - v^{\odot\oplus} = 11.6446\ km/s$$

$$\alpha_1 = \beta_1 = 0°$$

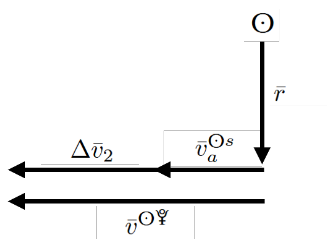

Figure 23: Vector diagram for Pluto arrival.

$$v^{\odot\text{♇}} = \sqrt{\frac{\mu_\odot}{a_{\text{♇}}}} = 4.7399\ km/s$$

$$\Delta v_2 = v^{\odot\text{♇}} - v_a^{\odot s} = 4.7149\ km/s$$

$$\alpha_2 = \beta_2 = 0°$$

Combining the two maneuvers gives us a total Δv of:

$$\boxed{\Delta v_{total} = 16.3595\ km/s}$$

And a total time of flight of:

$$TOF = \frac{P_h}{2} = \pi\sqrt{\frac{a_h^3}{\mu_\odot}}\ \boxed{= 30.0124\ yrs}$$

We can see that the Δv change is small, costing about an additional 1 km/s. Both Δv values are large and pretty similar, so computing the Hohmann transfer such that it takes into account the eccentricity of Pluto's orbit doesn't make much of a difference in terms of cost. There is a much larger improvement in the time of flight though, with the required time of the new transfer taking about 15 years less to reach Pluto. While this is still too long of a time of flight to be reasonable for an actual mission, it is a significant improvement over the previous calculation. In general, this result makes sense. Mission design usually involves some kind of trade off between cost and time of flight, so since we have improved the time of flight of the mission the maneuver cost has increased slightly. Note however, that this change is dependent on Pluto being located at perihelion at the time we arrive. If Pluto were to be at aphelion, the time of flight would increase significantly. So we must also consider the phasing of the planets when planning missions.