# ECE 602: LUMPED LINEAR SYSTEMS

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Fundamental Matrices and State Transition Matrices for CT LTV Systems

# Continuous-Time Autonomous LTV Systems

Consider an autonomous linear time-varying (LTV) system

$$\dot{x}(t) = A(t)x(t), \quad t \geq 0, \quad \text{with initial condition } x(0) \in \mathbb{R}^n$$

- $A(t) \in \mathbb{R}^{n \times n}$ varies with time $t \geq 0$
- $A(t)$ is assumed to have some nice properties (e.g. continuous, piecewise continuous) so that, for any $x(0)$, a unique solution $x(t)$ exists for all $t \geq 0$

**Example:** Nonlinear ODEs

1. $\dot{x}(t) = 1 + [x(t)]^2$, $x(0) = 0$, has solution $x(t) = \tan(t)$, $t \in [0, \frac{\pi}{2})$
2. $\dot{x}(t) = [x(t)]^{1/3}$, $x(0) = 0$, has two solutions: $x(t) \equiv 0$ and $x(t) = \left(\frac{2t}{3}\right)^{3/2}$

# Scalar Autonomous LTV Systems

Consider the scalar case with $x(t), a(t) \in \mathbb{R}$:

$$\dot{x}(t) = a(t)x(t), \quad x(0) \in \mathbb{R}$$

Solution is $x(t) = e^{\int_0^t a(\tau)\,d\tau} x(0), \quad t \geq 0$

**Conjecture**: the solution of LTV system $\dot{x}(t) = A(t)x(t)$ is

$$x(t) = e^{\int_0^t A(\tau)\,d\tau} x(0), \quad t \geq 0$$

Unfortunately, this is not true since generally $\frac{d}{dt} e^{\int_0^t A(\tau)\,d\tau} \neq A(t) e^{\int_0^t A(\tau)\,d\tau}$

# Solution Space

For LTV system $\dot{x}(t) = A(t)x(t)$, its solution space is

$$\mathbb{X} := \{x(t),\ t \geq 0 \,|\, \dot{x}(t) = A(t)x(t)\}$$

- $\mathbb{X}$ is an $n$-dimensional vector space: $x(0) \in \mathbb{R}^n \mapsto x(t) \in \mathbb{X}$ is a bijection
- A basis of $\mathbb{X}$ is given by $\{\phi_1(t), \ldots, \phi_n(t)\}$ where $\phi_i(t) \in \mathbb{R}^n$ is the solution with $x(0) = e_i = \begin{bmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{bmatrix}^T$, i.e.,

$$\dot{\phi}_i(t) = A(t)\phi_i(t), \quad \phi_i(0) = e_i$$

- Any solution $x(t) \in \mathbb{X}$ with initial condition $x(0)$ can be written as

$$x(t) = x_1(0)\phi_1(t) + \cdots + x_n(0)\phi_n(t) = \underbrace{\begin{bmatrix} \phi_1(t) & \cdots & \phi_n(t) \end{bmatrix}}_{\Phi(t)} x(0)$$

# Fundamental Matrix

The fundamental matrix $\Phi(t)$, $t \geq 0$, is

$$\Phi(t) := \begin{bmatrix} \phi_1(t) & \cdots & \phi_n(t) \end{bmatrix} \in \mathbb{R}^{n \times n}, \quad t \geq 0.$$

where $\phi_i(t)$ is the solution with initial condition $x(0) = e_i$, $i = 1, \ldots, n$

- $\Phi(t)$ describes how the state solution propagates an initial state at time $t = 0$ to the state at time $t$: $x(t) = \Phi(t)x(0)$
- $\Phi(t)$ is invertible for all $t$ (why?)
- $\Phi(t)$ is the solution of the matrix differential equation:

$$\dot{\Phi}(t) = A(t)\Phi(t), \quad \Phi(0) = I_n$$

- For LTI systems, $\Phi(t) = e^{At}$

# State Transition Matrix

State transition matrix of the LTV systems $\dot{x}(t) = A(t)x(t)$ is defined as

$$\Phi(t,\tau) = \Phi(t)\Phi(\tau)^{-1} \in \mathbb{R}^{n\times n}, \ \ \forall t,\tau \geq 0$$

- $\Phi(t,\tau)$ describes how the solution $x(t)$ propagates from time $\tau$ to time $t$:

$$x(t) = \Phi(t,\tau)x(\tau), \quad \forall t,\tau \geq 0.$$

- $\Phi(t_3,t_2)\Phi(t_2,t_1) = \Phi(t_3,t_1)$ for any $t_1,t_2,t_3 \geq 0$
- Given $t_0$, $\Phi(\cdot,t_0)$ is the solution of $\frac{d}{dt}\Phi(t,t_0) = A(t)\Phi(t,t_0)$, $\Phi(t_0,t_0) = I$
- For LTI system $\dot{x} = Ax$, $\Phi(t,\tau) = \Phi(t-\tau) = e^{A(t-\tau)}$