#1) $\dot{x} = \begin{pmatrix} \frac{4}{1+x_2} \\ x_1 x_2 \end{pmatrix} + \begin{pmatrix} x_2 \\ 1 \end{pmatrix} u \qquad u = -2$

a) $0 = \begin{pmatrix} \frac{4}{1+x_2} - 2x_2 \\ x_1 x_2 - 2 \end{pmatrix}$

$$2x_2 = \frac{4}{1+x_2}$$

$2x_2 + 2x_2^2 = 4 \implies x_2^2 + x_2 - 2 = 0$

$(x_2+2)(x_2-1) = 0$

$x_2 = -2$

$x_2 = 1$

$x_1 x_2 = 2$

$x_1 = \frac{2}{x_2} \quad \therefore \quad x_2 = -2,\ x_1 = -1 \ ; \quad x_2 = 1,\ x_1 = 2$

$$\boxed{x_{e_1} = \begin{pmatrix} -1 \\ -2 \end{pmatrix} \qquad x_{e_2} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}}$$

check: $\frac{4}{1-2} + (-2)(-2) = -4+4$

$(-2)(-1) - 2 = 0$ ✓

b) $f_1 = 4(1+x_2)^{-1} + x_2 u$

$f_2 = x_1 x_2 + u$

$\frac{\partial f_1}{\partial x_1} = 0 \qquad \frac{\partial f_1}{\partial x_2} = -4(1+x_2)^{-2} + u$

$\frac{\partial f_2}{\partial x_1} = x_2 \qquad \frac{\partial f_2}{\partial x_2} = x_1$

$\frac{\partial f_1}{\partial u} = x_2 \qquad \frac{\partial f_2}{\partial u} = 1$

#1) $\left.\frac{\partial f_1}{\partial x_2}\right|_{\left(\begin{bmatrix}-1\\-2\end{bmatrix},-2\right)} = -4(1-2)^{-2} - 2 = \frac{-4}{(-1)^2} - 2 = -6$

$\therefore A^{(1)} = \begin{pmatrix} 0 & -6 \\ -2 & -1 \end{pmatrix}$; $B^{(1)} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$\left.\frac{\partial f_1}{\partial x_2}\right|_{\left(\begin{bmatrix}3\\1\end{bmatrix},-2\right)} = -4(1+1)^{-2} - 2 = \frac{-4}{2^2} - 2 = -3$

$\therefore A^{(2)} = \begin{pmatrix} 0 & -3 \\ 1 & 2 \end{pmatrix}$, $B^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\boxed{\begin{aligned} \therefore\ \delta \dot{x}^{(1)} &= \begin{pmatrix} 0 & -6 \\ -2 & -1 \end{pmatrix} \delta x + \begin{pmatrix} -2 \\ 1 \end{pmatrix} \delta u \\ \delta \dot{x}^{(2)} &= \begin{pmatrix} 0 & -3 \\ 1 & 2 \end{pmatrix} \delta x + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \delta u \end{aligned}}$$

$\delta x = x - x_e$

$\delta u = u - u_e$

#2) $\dot{x} = \sqrt{2}\,x + 3u \qquad x(0) = 1$

$$J = \int_0^\infty \left(\tfrac{1}{4}x^2 + 9u^2\right) dt$$

a)

$\frac{1}{4}x^2 = x^T \frac{1}{4} x \quad \therefore \quad Q = \frac{1}{4}$

$9u^2 = u^T 9 u \quad \therefore \quad R = 9$

$A = \sqrt{2} \qquad B = 3$

ARE: $A^T P + PA + Q - PBR^{-1}B^T P = 0$

$$(\sqrt{2})P + P(\sqrt{2}) + \tfrac{1}{4} - P(3)(\tfrac{1}{9})(3)P = 0$$

$$2\sqrt{2}P + \tfrac{1}{4} - P^2 = 0$$

$$P^2 - 2\sqrt{2}P - \tfrac{1}{4} = 0$$

$$P = \frac{2\sqrt{2} \pm \sqrt{8-(4)(-\frac{1}{4})}}{2} = \frac{2\sqrt{2} \pm \sqrt{9}}{2} = \sqrt{2} \pm \frac{3}{2}$$

$$P = \sqrt{2} + \frac{3}{2}$$

$$u = -Kx = -R^{-1}B^T P x = -(\tfrac{1}{9})(3)(\sqrt{2} + \tfrac{3}{2})x$$

$$\boxed{u = -\tfrac{1}{3}(\sqrt{2} + \tfrac{3}{2})x}$$

b) $J_0 = x(0)^T P x(0) = (1)P(1)$

$$\boxed{J_0 = \sqrt{2} + \tfrac{3}{2}}$$

#3) $J = \frac{1}{2}\int_0^1 u^2 dt$ $\quad x(0) = \begin{pmatrix}0\\0\end{pmatrix}, \; x(1) = \begin{pmatrix}1\\3\end{pmatrix}$

$$\dot{x} = \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}x + \begin{pmatrix}0\\1\end{pmatrix}u$$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = u$$

$$H = \frac{1}{2}u^2 + p^T\underbrace{\left(\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}x + \begin{pmatrix}0\\1\end{pmatrix}u\right)}_{f} = F + p^T f$$

$$\frac{\partial H}{\partial u} = 0 = u + p^T\begin{pmatrix}0\\1\end{pmatrix} = u + (p_1 \;\; p_2)\begin{pmatrix}0\\1\end{pmatrix} = u + p_2$$

$$\therefore \; u = -p_2$$

$$\dot{p} = -\left(\frac{\partial H}{\partial x}\right)^T = -\left(p^T\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\right)^T = -\left([p_1 \;\; p_2]\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\right)^T$$

$$\dot{p} = -(0 \;\; p_1)^T = \begin{pmatrix}0\\-p_1\end{pmatrix} = \begin{pmatrix}\dot{p}_1\\\dot{p}_2\end{pmatrix}$$

$$\frac{dp_1}{dt} = 0 \quad \therefore \quad p_1 = p_1(0)$$

$$\frac{dp_2}{dt} = -p_1 = -p_1(0) \quad \therefore \quad \int_{p_{20}}^{p_2} dp_2 = \int_0^t -p_1(0)\,dt$$

$$p_2 = -p_1(0)t + p_2(0)$$

$$\therefore \; u = p_1(0)t - p_2(0)$$

#3)

$$\frac{dX_2}{dt} = U = P_1(0)t - P_2(0)$$

$$\int_0^{X_2} dX_2 = \int_0^t P_1(0)t - P_2(0)\,dt$$

$$X_2 = \frac{P_1(0)t^2}{2} - P_2(0)t$$

$$\frac{dX_1}{dt} = X_2 \quad \therefore \quad \int_0^{X_1} dX_1 = \int_0^t \frac{P_1(0)t^2}{2} - P_2(0)t\,dt$$

$$X_1 = \frac{P_1(0)t^3}{6} - \frac{P_2(0)t^2}{2}$$

$$X_1(1) = 1 = \frac{P_1(0)}{6} - \frac{P_2(0)}{2}$$

$$X_2(1) = 3 = \frac{P_1(0)}{2} - P_2(0)$$

$$P_2(0) = -3 + \frac{P_1(0)}{2}$$

$$1 = \frac{P_1(0)}{6} - \frac{1}{2}\left[-3 + \frac{P_1(0)}{2}\right] = \frac{P_1(0)}{6} + \frac{3}{2} - \frac{P_1(0)}{4}$$

$$-\frac{1}{2} = \frac{2P_1(0)}{12} - \frac{3P_1(0)}{12} = -\frac{P_1(0)}{12}$$

$\therefore$ $P_1(0) = 6$ & $P_2(0) = 0$, $U = P_1(0)t + P_2(0)$

$$\therefore \boxed{U = 6t}$$

#4) $J = (x[2]-1)^2 + 2\sum_0^1 u[k]^2$ $\quad x[0]=10$

$x[k+1] = b\,u[k]$

$J^*[2] = (x[2]-1)^2$

$x[2] = b\,u[1]$

$J^*[2] = (b\,u[1]-1)^2$

$J^*[1] = J^*[2] + 2u[1]^2 = (b\,u[1]-1)^2 + 2u[1]^2$

$$\frac{\partial J^*[1]}{\partial u[1]} = 2b(b\,u[1]-1) + 4u[1] = 0$$

$$= 2b^2u[1] - 2b + 4u[1] = 0$$

$$u[1](2b^2+4) = 2b$$

$$\boxed{u[1] = \frac{2b}{2b^2+4}}$$

$$\therefore J^*[1] = \left(\frac{2b^2}{2b^2+4} - 1\right)^2 + \frac{8b^2}{(2b^2+4)^2}$$

$J^*[0] = J^*[1] + 2u[0]^2$

$$\frac{\partial J^*[0]}{\partial u[0]} = 0 = 4u[0] = 0$$

$$\boxed{u[0] = 0}$$

$$J^* = \left(\frac{2b^2}{2b^2+4} - \frac{2b^2+4}{2b^2+4}\right)^2 = \left(\frac{-4}{2b^2+4}\right)^2 = \frac{16}{(2b^2+4)^2}$$

$$\boxed{J^* = \frac{16}{(2b^2+4)^2}}$$

#5) $J_0 = 3\sum_0^{\infty} \|x\|^2$ $\quad A = \begin{pmatrix} -.5 & 0 \\ 0 & .5 \end{pmatrix}$ $\quad x(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$3\|x\|^2 = 3x^Tx = 3(Ix)^T(Ix) = 3x^TI^TIx = 3x^Tx$$

$$3I = Q$$

$$A^TPA - P = -Q$$

$$\begin{pmatrix} -.5 & 0 \\ 0 & .5 \end{pmatrix}\begin{pmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{pmatrix}\begin{pmatrix} -.5 & 0 \\ 0 & .5 \end{pmatrix} - \begin{pmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{pmatrix} = \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix}$$

$$\begin{pmatrix} -.5P_{11} & -.5P_{12} \\ .5P_{12} & .5P_{22} \end{pmatrix}\begin{pmatrix} -.5 & 0 \\ 0 & .5 \end{pmatrix} = \begin{pmatrix} .25P_{11} & -.25P_{12} \\ -.25P_{12} & .25P_{22} \end{pmatrix}$$

$$\begin{pmatrix} -.75P_{11} & 1.25P_{12} \\ 1.25P_{12} & -.75P_{22} \end{pmatrix} = \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix}$$

$$P_{12} = 0$$

$$\frac{3}{4}P_{11} = 3 \qquad P_{11} = 4$$

$$P_{22} = 4$$

$$\boxed{P = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}}$$

$$J_0 = \begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 4 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\boxed{J_0 = 4}$$

#6) $\dot{x} = 2u_1 + 2u_2$ $\quad x(0)=3$

$\dot{x} = [2 \quad 2] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ $\quad J = \int_0^\infty (x^2 + ru_1^2 + ru_2^2)\,dt$

a)

$Q = 1$ $\quad \underset{1\times2\,(2\times2)\,(2\times1)}{U^T R U} = 1$ $\quad [u_1 \quad u_2] \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} = [ru_1 \quad ru_2] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = ru_1^2 + ru$

$R = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}$

ARE: $\cancel{A^TP}^{0} + \cancel{PA}^{0} + Q - PBR^{-1}B^TP = 0$

$1 - P[2 \quad 2] \begin{pmatrix} \frac{1}{r} & 0 \\ 0 & \frac{1}{r} \end{pmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} P = 0$

$1 - P\left[\frac{2}{r} \quad \frac{2}{r}\right] \begin{bmatrix} 2 \\ 2 \end{bmatrix} P = 0$

$1 - P^2\left[\frac{4}{r} + \frac{4}{r}\right] = 0$

$1 - P^2 \frac{8}{r} = 0$

$$\boxed{P = \sqrt{\frac{r}{8}}}$$

b) $U = -Kx = -R^{-1}B^TPx = \underset{2\times2}{\begin{pmatrix} -\frac{1}{r} & 0 \\ 0 & -\frac{1}{r} \end{pmatrix}} \underset{2\times1}{\begin{bmatrix} 2 \\ 2 \end{bmatrix}} \sqrt{\frac{r}{8}}\, x$

$U = \begin{pmatrix} -\frac{2}{r} \\ -\frac{2}{r} \end{pmatrix} \sqrt{\frac{r}{8}}\, x$

$\dot{x} = Ax + Bu$

$\dot{x} = [2 \quad 2] \begin{bmatrix} -\frac{2}{r} \\ -\frac{2}{r} \end{bmatrix} \sqrt{\frac{r}{8}}\, x$

$\dot{x} = \left(-\frac{4}{r} - \frac{4}{r}\right) \sqrt{\frac{r}{8}}\, x$

$$\boxed{\dot{x} = -\frac{8}{r} \sqrt{\frac{r}{8}}\, x}$$

c) $J = x(0)^T P\, x(0)$

$= (3)\left(\sqrt{\frac{r}{8}}\right)(3)$

$$\boxed{J = 9\sqrt{\frac{r}{8}}}$$

#7) $\dot{x}_1 = x_2 - 2$ $\quad J = \int_0^{t_f} dt$

$\dot{x}_2 = u$

$|u| \leq 1$

a) $H = F + p^T f = 1 + [p_1 \;\; p_2] \begin{pmatrix} x_2 - 2 \\ u \end{pmatrix}$

$H = 1 + p_1(x_2 - 2) + p_2 u$

$\dot{p} = -\left(\frac{\partial H}{\partial x}\right)^T = -(0 \;\; p_1)^T = \begin{pmatrix} 0 \\ -p_1 \end{pmatrix}$

$\dot{p}_1 = 0 \quad \therefore \quad p_1 = p_1(0)$

$\dot{p}_2 = -p_1 \quad \therefore \quad \int dp_2 = \int -p_1(0)\,dt \qquad p_2 = -p_1(0)t + p_2(0)$

$\arg_u \min(H) = \arg_u \min(p_2) = \arg_u \min(-p_1(0)t + p_2(0))$

$\underline{u=1}$: $\dot{x}_2 = 1 \quad \therefore \quad x_2 = t + x_2(0)$

$\dot{x}_1 = x_2 - 2 \quad \therefore \quad \dot{x}_1 = t + x_2(0) - 2$

$u = t + x_{20} - 2 \qquad du = dt$

$\therefore \quad x_1 = \frac{(t + x_{20} - 2)^2}{2} = \frac{(t + x_{20})^2 + 4 + x_1(0)}{2}$

$$\boxed{\begin{aligned} x_2 &= t + x_2(0) \\ x_1 &= \frac{x_2^2 + 4 + x_1(0)}{2} \end{aligned}}$$

b) $\boxed{u = x_1 - \frac{x_2^2}{2}\operatorname{sign}(x_2 - 2) = 0}$

c)

$$\boxed{Y = \begin{cases} 1 & \text{sign}(X_2 - 2) < 0 \\ \text{sign}(X_2 - 2) & 0 \\ -1 & \text{sign}(X_2 - 2) > 0 \end{cases}}$$

#8)

$$\det(A)\det(B) = \det(AB)$$

$$\det(A)\det(B) - \det(AB) = 0$$

$$(\det(A) \otimes I)(I \otimes \det(B)) - \mathrm{Vec}(AB) = \mathrm{Vec}(0)$$

I Have No Clue

Forgot determinate rules

$$AX = C \Rightarrow (I \otimes A) = \mathrm{vec}(C)$$

$$XB = C \Rightarrow (B^T \otimes I) = \mathrm{vec}(C)$$

#9) $\dot{x}_1 = \frac{4}{1+x_2} + x_2 u$ $\qquad x_{e1} = -1, \; U_e = -2$

$\dot{x}_2 = x_1 x_2 + u$

$0 = (-1) x_2 - 2$

$x_2 = -2 \qquad x_e = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$

$$A = \left[Df(x_0) + \frac{(f(x_0) - Df(x_0) x_0) x_0^T}{\|x_0\|^2} \right] x$$

$\nabla f_1 = [0 \quad -4(1+x_2)^{-2} + u] \Big|_{(x_e, u_e)} = [0 \quad -6]$

$\nabla f_2 = [x_2 \quad x_1] \Big|_{(x_e, u_e)} = [-2 \quad -1]$

$f_1(x_0) = 0$

$f_2(x_0) = 0$

$\|x_0\|^2 = x_0^T x_0 = (-1 \; -2) \begin{pmatrix} -1 \\ -2 \end{pmatrix} = 1 + 4 = 5$

$$A = \begin{pmatrix} 0 & -6 \\ -2 & -1 \end{pmatrix} - \frac{\begin{pmatrix} 0 & -6 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ -2 \end{pmatrix} (-1 \; -2)}{5} = \frac{\begin{pmatrix} 12 \\ 4 \end{pmatrix} (-1 \; -2)}{5} = \begin{pmatrix} \frac{-12}{5} & \frac{-24}{5} \\ \frac{-4}{5} & \frac{-8}{5} \end{pmatrix}$$

$$A = \begin{pmatrix} \frac{12}{5} & -6 + \frac{24}{5} \\ -2 + \frac{4}{5} & -1 + \frac{8}{5} \end{pmatrix} = \begin{pmatrix} \frac{12}{5} & -6/5 \\ -6/5 & 3/5 \end{pmatrix}$$

$$B = G(x_e) = \begin{pmatrix} x_2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$\boxed{\dot{x} \approx \begin{pmatrix} 12/5 & -6/5 \\ -6/5 & 3/5 \end{pmatrix} x + \begin{pmatrix} -2 \\ 1 \end{pmatrix} u}$$

#10) $A_1 = \begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix}$ $A_2 \begin{pmatrix} -2 & 4 \\ 0 & -1 \end{pmatrix}$

$A_1 + A_2 = \begin{pmatrix} -3 & 4 \\ 2 & -2 \end{pmatrix} \Rightarrow \underline{\lambda^2 + 5\lambda - 2}$

There is a sign change in the characteristic polynomial, therefore the system is not hurwitz & a quadratic lyapunov function doesn't exist

1	-2
5	
-2	