#1) $A = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ $\quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad x(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad u = 1$

$$e^{At} = \mathcal{L}^{-1}(sI - A)^{-1}$$

$$sI - A = \begin{pmatrix} s & 1 \\ -1 & s+1 \end{pmatrix}$$

$$(sI - A)^{-1} = \begin{pmatrix} s+1 & -1 \\ 1 & s \end{pmatrix} \frac{1}{s^2+s+1}$$

$$\mathcal{L}^{-1}\left(\frac{s-a}{(s-a)^2+K^2}\right) = e^{at}\cos(Kt)$$

$$\mathcal{L}^{-1}\left(\frac{K}{(s-a)^2+K^2}\right) = e^{at}\sin(Kt)$$

$$(s-a)^2 = s^2 - 2as + a^2$$

$$(s-a)^2 + K^2 = s^2 - 2as + a^2 + K^2 = s^2 + s + 1$$

$$\therefore \ -2a = 1 \Rightarrow a = -1/2$$

$$a^2 + K^2 = 1 \Rightarrow K = \sqrt{1-(-1/2)^2} = \frac{\sqrt{3}}{2}$$

$$s^2 + s + 1 = (s + \tfrac{1}{2})^2 + 3/4$$

$$\frac{s+1}{s^2+s+1} = \frac{s+\frac{1}{2}}{(s+1/2)^2 + 3/4} + \frac{1}{\sqrt{3}} \frac{\sqrt{3}/2}{(s+1/2)^2 + 3/4}$$

$$\mathcal{L}^{-1}\left(\frac{s+1}{s^2+s+1}\right) = e^{-t/2}\left(\cos(\tfrac{\sqrt{3}}{2}t) + \frac{1}{\sqrt{3}}\sin(\tfrac{\sqrt{3}}{2}t)\right)$$

$$\frac{-1}{(s+\frac{1}{2})^2+\frac{3}{4}} = -\frac{2}{\sqrt{3}} \frac{\frac{\sqrt{3}}{2}}{(s^2+\frac{1}{2})^2+\frac{3}{4}}$$

$$\mathcal{L}^{-1}\left(\frac{-1}{(s^2+\frac{1}{2})^2+\frac{3}{4}}\right) = \frac{-2}{\sqrt{3}} e^{-t/2} \sin\left(\frac{\sqrt{3}}{2}t\right)$$

$$\mathcal{L}^{-1}\left(\frac{1}{(s^2+\frac{1}{2})^2+\frac{3}{4}}\right) = \frac{2}{3} e^{-t/2} \sin\left(\frac{\sqrt{3}}{2}t\right)$$

$$\frac{s}{(s^2+\frac{1}{2})^2+\frac{3}{4}} = \frac{s+\frac{1}{2}}{(s^2+\frac{1}{2})^2+\frac{3}{4}} - \frac{1}{\sqrt{3}} \frac{\frac{\sqrt{3}}{2}}{(s^2+\frac{1}{2})^2+\frac{3}{4}}$$

$$\mathcal{L}^{-1}\left(\frac{s}{(s^2+\frac{1}{2})^2+\frac{3}{4}}\right) = e^{-t/2}\left(\cos\left(\frac{\sqrt{3}}{2}t\right) - \frac{1}{\sqrt{3}} \sin\left(\frac{\sqrt{3}}{2}t\right)\right)$$

$$e^{At} = e^{-\frac{t}{2}} \begin{pmatrix} \cos\left(\frac{\sqrt{3}}{2}t\right) + \frac{1}{\sqrt{3}} \sin\left(\frac{\sqrt{3}}{2}t\right) & -\frac{2}{\sqrt{3}} \sin\left(\frac{\sqrt{3}}{2}t\right) \\ \frac{2}{\sqrt{3}} \sin\left(\frac{\sqrt{3}}{2}t\right) & \cos\left(\frac{\sqrt{3}}{2}t\right) - \frac{1}{\sqrt{3}} \sin\left(\frac{\sqrt{3}}{2}t\right) \end{pmatrix}$$

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)} Bu(\tau)\, d\tau$$

$$= e^{At}x(0) + \int_0^t e^{A\tau} Bu(t-\tau)\, d\tau$$

$$= e^{At}\begin{pmatrix}0\\0\end{pmatrix} + \int_0^t e^{A\tau}\begin{pmatrix}1\\0\end{pmatrix}(1)\, d\tau$$

$$x(t) = \int_0^t e^{-\frac{\tau}{2}} \begin{pmatrix} \cos\left(\frac{\sqrt{3}}{2}\tau\right) + \frac{1}{\sqrt{3}} \sin\left(\frac{\sqrt{3}}{2}\tau\right) \\ \frac{2}{\sqrt{3}} \sin\left(\frac{\sqrt{3}}{2}\tau\right) \end{pmatrix} d\tau$$

$$\int_0^\tau e^{-\tau/2}\cos\left(\tfrac{\sqrt3}{2}\tau\right) + \frac{e^{-\tau/2}}{\sqrt3}\sin\left(\tfrac{\sqrt3}{2}\tau\right) d\tau =$$

$$\int_0^\tau e^{-\tau/2}\cos\left(\tfrac{\sqrt3}{2}\tau\right)d\tau + \frac{1}{\sqrt3}\int_0^\tau e^{-\tau/2}\sin\left(\tfrac{\sqrt3}{2}\tau\right)d\tau$$

$u = \cos\left(\tfrac{\sqrt3}{2}\tau\right)$ $\qquad dv = e^{-\tau/2}d\tau$

$du = -\frac{\sqrt3}{2}\sin\left(\tfrac{\sqrt3}{2}\tau\right)d\tau$ $\qquad v = -2e^{-\tau/2}$

$$uv - \int v\,du = -2e^{-\tau/2}\cos\left(\tfrac{\sqrt3}{2}\tau\right) - \int_0^t \sqrt3 e^{-\tau/2}\sin\left(\tfrac{\sqrt3}{2}\tau\right)d\tau$$

$u_2 = \sin\left(\tfrac{\sqrt3}{2}\tau\right)$ $\qquad dv_2 = \sqrt3\, e^{-\tau/2}d\tau$

$du_2 = \frac{\sqrt3}{2}\cos\left(\tfrac{\sqrt3}{2}\tau\right)d\tau$ $\qquad v_2 = -2\sqrt3\, e^{-\tau/2}$

$$\underbrace{-2e^{-\tau/2}\cos\left(\tfrac{\sqrt3}{2}\tau\right)}_{uv} - \left(-2\sqrt3 e^{-\tau/2}\sin\left(\tfrac{\sqrt3}{2}\tau\right) + \int 3\cos\left(\tfrac{\sqrt3}{2}\tau\right)e^{-\tau/2}d\tau\right)$$

$$\therefore \int e^{-\tau/2}\cos\left(\tfrac{\sqrt3}{2}\tau\right)d\tau = -\frac{e^{-\tau/2}\cos\left(\tfrac{\sqrt3}{2}\tau\right)}{2} + \frac{\sqrt3 e^{-\tau/2}\sin\left(\tfrac{\sqrt3}{2}\tau\right)}{2} \quad \underline{(1)}$$

$\int \frac{e^{-\tau/2}}{\sqrt3}\sin\left(\tfrac{\sqrt3}{2}\tau\right)d\tau \Rightarrow$ $\quad u = \frac{\sin\left(\tfrac{\sqrt3}{2}\tau\right)}{\sqrt3}$ $\qquad dv = e^{-\tau/2}d\tau$

$du = \frac{\cos\left(\tfrac{\sqrt3}{2}\tau\right)}{2}d\tau$ $\qquad v = -2e^{-\tau/2}$

$$\frac{-2}{\sqrt3}e^{-\tau/2}\sin\left(\tfrac{\sqrt3}{2}\tau\right) - \int -e^{-\tau/2}\cos\left(\tfrac{\sqrt3}{2}\tau\right)d\tau = uv - \int v\,du$$

$$u_2 = \cos\left(\tfrac{\sqrt{3}}{2}\tau\right) \qquad dv = -e^{-\tau/2}\,d\tau$$

$$du_2 = -\tfrac{\sqrt{3}}{2}\sin\left(\tfrac{\sqrt{3}}{2}\tau\right)d\tau \qquad v = 2e^{-\tau/2}$$

$$\frac{-2}{\sqrt{3}}e^{-\tau/2}\sin\left(\tfrac{\sqrt{3}}{2}\tau\right) - \left(2e^{-\tau/2}\cos\left(\tfrac{\sqrt{3}}{2}\tau\right) + \underbrace{\int \sqrt{3}\,e^{-\tau/2}\sin\left(\tfrac{\sqrt{3}}{2}\tau\right)d\tau}_{\sqrt{3}X}\right.$$

$$= \frac{1}{\sqrt{3}}X$$

$$\frac{-2}{\sqrt{3}}e^{-\tau/2}\sin\left(\tfrac{\sqrt{3}}{2}\tau\right) - 2e^{-\tau/2}\cos\left(\tfrac{\sqrt{3}}{2}\tau\right) = \frac{1}{\sqrt{3}}X + \sqrt{3}X$$

$$= \frac{4}{\sqrt{3}}X$$

$$\therefore \int \frac{1}{\sqrt{3}}e^{-\tau/2}\sin\left(\tfrac{\sqrt{3}}{2}\tau\right)d\tau = \frac{-e^{-\tau/2}\sin\left(\tfrac{\sqrt{3}}{2}\tau\right)}{2\sqrt{3}} - \frac{e^{-\tau/2}\cos\left(\tfrac{\sqrt{3}}{2}\tau\right)}{2} \quad (2)$$

$$\therefore \int_0^t e^{-\tau/2}\cos\left(\tfrac{\sqrt{3}}{2}\tau\right) + \frac{e^{-\tau/2}}{\sqrt{3}}\sin\left(\tfrac{\sqrt{3}}{2}\tau\right)d\tau = (1) + (2)\Big|_0^t =$$

$$\frac{-e^{-\tau/2}\cos\left(\tfrac{\sqrt{3}}{2}\tau\right)}{2} + \frac{\sqrt{3}}{2}e^{-\tau/2}\sin\left(\tfrac{\sqrt{3}}{2}\tau\right) - \frac{e^{-\tau/2}\sin\left(\tfrac{\sqrt{3}}{2}\tau\right)}{2\sqrt{3}} - \frac{e^{-\tau/2}\cos\left(\tfrac{\sqrt{3}}{2}\tau\right)}{2}\Bigg|_0^t$$

$$= e^{-\tau/2}\left(-\cos\left(\tfrac{\sqrt{3}}{2}\tau\right) + \frac{1}{\sqrt{3}}\sin\left(\tfrac{\sqrt{3}}{2}\tau\right)\right)\Big|_0^t$$

$$= e^{-t/2}\left(-\cos\left(\tfrac{\sqrt{3}}{2}t\right) + \tfrac{1}{\sqrt{3}}\sin\left(\tfrac{\sqrt{3}}{2}t\right)\right) + 1$$

$$\int_0^t e^{-\tau/2}\,\tfrac{2}{\sqrt{3}}\sin\left(\tfrac{\sqrt{3}}{2}\tau\right) = \frac{-e^{-\tau/2}\sin\left(\tfrac{\sqrt{3}}{2}\tau\right)}{\sqrt{3}} - e^{-\tau/2}\cos\left(\tfrac{\sqrt{3}}{2}\tau\right)\Big|_0^t$$

$$= -e^{-t/2}\left(\frac{\sin(\frac{\sqrt{3}}{2}t)}{\sqrt{3}} + \cos(\tfrac{\sqrt{3}}{2}t)\right) + 1$$

$$\boxed{x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} e^{-t/2}\left(-\cos(\frac{\sqrt{3}}{2}t) + \frac{1}{\sqrt{3}}\sin(\frac{\sqrt{3}}{2}t)\right) + 1 \\ 1 - e^{-t/2}\left(\frac{\sin(\frac{\sqrt{3}}{2}t)}{\sqrt{3}} + \cos(\frac{\sqrt{3}}{2}t)\right) \end{pmatrix}}$$

#2) $A = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ $B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $C = [0 \quad 1]$

$T = 0.2$

$A_d = e^{AT}$ $C_d = C$

$B_d = \int_0^T e^{A\tau} d\tau \, B$

$$A_d = e^{AT} = \underbrace{e^{-.2/2} \begin{pmatrix} \cos\left(\frac{.2\sqrt{3}}{2}\right) + \frac{1}{\sqrt{3}} \sin\left(\frac{.2\sqrt{3}}{2}\right) & -2/\sqrt{3} \sin\left(\frac{.2\sqrt{3}}{2}\right) \\ 2/\sqrt{3} \sin\left(\frac{.2\sqrt{3}}{2}\right) & \cos\left(\frac{.2\sqrt{3}}{2}\right) - 1/\sqrt{3} \sin\left(\frac{.2\sqrt{3}}{2}\right) \end{pmatrix}}_{\text{From Problem 1 } e^{At}, \text{ sub } t = T}$$

Let $e^{A\tau} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

$$\int_0^T e^{A\tau} d\tau = \begin{pmatrix} \int_0^T a_{11} d\tau & \int_0^T a_{12} d\tau \\ \int_0^T a_{21} d\tau & \int_0^T a_{22} d\tau \end{pmatrix}$$

$$\int_0^T e^{A\tau} d\tau \, B = \begin{pmatrix} \int_0^T a_{11} d\tau & \int_0^T a_{12} d\tau \\ \int_0^T a_{21} d\tau & \int_0^T a_{22} d\tau \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \int_0^T a_{11} d\tau \\ \int_0^T a_{21} d\tau \end{pmatrix} = B_d$$

$$\int_0^T a_{11}\, d\tau = \int_0^T e^{-\tau/2}\left(\cos\left(\tfrac{\sqrt{3}}{2}\tau\right) + \tfrac{1}{\sqrt{3}}\sin\left(\tfrac{\sqrt{3}}{2}\tau\right)\right) d\tau$$

From Problem 1: $\int_0^t e^{-\tau/2}\left(\cos\left(\tfrac{\sqrt{3}}{2}\tau\right) + \tfrac{1}{\sqrt{3}}\sin\left(\tfrac{\sqrt{3}}{2}\tau\right)\right) d\tau$

$$= e^{-t/2}\left(-\cos\left(\tfrac{\sqrt{3}}{2}t\right) + \tfrac{1}{\sqrt{3}}\sin\left(\tfrac{\sqrt{3}}{2}t\right)\right) + 1$$

sub T for t

$$\int_0^T a_{11}\, d\tau = e^{-.2/2}\left(-\cos\left(\tfrac{.2\sqrt{3}}{2}\right) + \tfrac{1}{\sqrt{3}}\sin\left(\tfrac{.2\sqrt{3}}{2}\right)\right) + 1$$

$$\int_0^T a_{21}\, d\tau = \int_0^T e^{-\tau/2}\,\tfrac{2}{\sqrt{3}}\sin\left(\tfrac{\sqrt{3}}{2}\tau\right) d\tau$$

From 1: $\int_0^t e^{-\tau/2}\,\tfrac{2}{\sqrt{3}}\sin\left(\tfrac{\sqrt{3}}{2}\tau\right) d\tau = -e^{-t/2}\left(\dfrac{\sin\left(\tfrac{\sqrt{3}}{2}t\right)}{\sqrt{3}} + \cos\left(\tfrac{\sqrt{3}}{2}t\right)\right) + 1$

sub T for t:

$$\int_0^T a_{21}\, d\tau = -e^{-.2/2}\left(\dfrac{\sin\left(\tfrac{.2\sqrt{3}}{2}\right)}{\sqrt{3}} + \cos\left(\tfrac{.2\sqrt{3}}{2}\right)\right) + 1$$

$$x[k+1] = A_d\, x[k] + B_d\, u[k]$$
$$y[k+1] = C_d\, x[k] + D_d\, u[k]$$

$$\boxed{A_d = \begin{pmatrix} 0.9813 & -0.1801 \\ 0.1801 & 0.8013 \end{pmatrix} \qquad B_d = \begin{pmatrix} 0.1987 \\ 0.0187 \end{pmatrix}}$$

$$\boxed{C_d = [0 \quad 1] \quad , \quad D_d = 0}$$

$$H(z) = C_d (zI - A_d)^{-1} B_d + D_d$$

$$(zI - A_d) = \begin{pmatrix} z - .9813 & .1801 \\ -.1801 & z - .8013 \end{pmatrix}$$

$$(zI - A_d)^{-1} = \frac{1}{(z - .9813)(z - .8013) + (.1801)^2} \begin{pmatrix} z - .8013 & -.1801 \\ .1801 & z - .9813 \end{pmatrix}$$

$$C_d (zI - A_d)^{-1} = [0 \quad 1] \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = (a_{21} \quad a_{22})$$

$$C_d (zI - A_d^{-1}) B_d = (a_{21} \quad a_{22}) \begin{pmatrix} .1987 \\ .0187 \end{pmatrix} = .1987 a_{21} + .0187 a_{22}$$

$$H(z) = \frac{(.1987)(.1801)}{(z - .9813)(z - .8013) + (.1801)^2} + \frac{(.0187)(z - .9813)}{(z - .9813)(z - .8013) + (.1801)^2}$$

$$\boxed{H(z) = \frac{.0187 z + .0175}{z^2 - 1.783 z + .8187}}$$

$$X[K] = \cancel{A_d^{K} X(0)}^{\,0} + \sum_{i=0}^{K-1} A_d^{(K-1-i)} B \cancel{u[i]}^{\,1} + \cancel{D u[K]}^{\,0}$$

$$\boxed{Y[K] = \sum_{i=0}^{K-1} \begin{pmatrix} 0.9813 & -0.1801 \\ 0.1801 & 0.8013 \end{pmatrix}^{(K-1-i)} \begin{pmatrix} .1987 \\ .0187 \end{pmatrix}}$$

Note: $H(z) = \dfrac{Y(z)}{U(z)}$ if $U(z)$ is step, $U(z) = \dfrac{z}{z-1}$

$$\therefore \; Y(z) = \frac{.0187 z^2 + .0175 z}{(z^2 - 1.783 z + .8187)(z-1)}$$

#3) $Q = \begin{pmatrix} \alpha^4 & \alpha^3 & \alpha^2 \\ \alpha^3 & \alpha^2 & \alpha \\ \alpha^2 & \alpha & 1 \end{pmatrix} \xRightarrow{E_{21}(-\frac{1}{\alpha}) E_{31}(-\alpha^{-2})} \begin{pmatrix} \alpha^4 & \alpha^3 & \alpha^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\boxed{\text{rank doesn't depend on alpha}}$ as Q is rank 1 due to all α terms in the non-pivot rows canceling out.

$(\alpha^4 \ \alpha^3 \ \alpha^2)\alpha^{-1} = \alpha^3 \ \alpha^2 \ \alpha$

$(\alpha^4 \ \alpha^3 \ \alpha^2)\alpha^{-2} = \alpha^2 \ \alpha \ 1$

$$|Q - \lambda I| = \chi_Q = \begin{vmatrix} \alpha^4-\lambda & \alpha^3 & \alpha^2 \\ \alpha^3 & \alpha^2-\lambda & \alpha \\ \alpha^2 & \alpha & 1-\lambda \end{vmatrix}$$

$$= (\alpha^4-\lambda)\begin{vmatrix} \alpha^2-\lambda & \alpha \\ \alpha & 1-\lambda \end{vmatrix} = (\alpha^4-\lambda)[(\alpha^2-\lambda)(1-\lambda) - \alpha^2]$$

$$= (\alpha^4-\lambda)[\cancel{\alpha^2} - \alpha^2\lambda - \lambda + \lambda^2 \cancel{-\alpha^2}] =$$

$$= -\cancel{\alpha^6\lambda} - \cancel{\alpha^4\lambda} + \alpha^4\lambda^2 + \alpha^2\lambda^2 + \lambda^2 - \lambda^3$$

$$-\alpha^3\begin{vmatrix} \alpha^3 & \alpha \\ \alpha^2 & 1-\lambda \end{vmatrix} = -\alpha^3[\cancel{\alpha^3} - \alpha^3\lambda - \cancel{\alpha^3}] = \cancel{+\alpha^6\lambda}$$

$$\alpha^2\begin{vmatrix} \alpha^3 & \alpha^2-\lambda \\ \alpha^2 & \alpha \end{vmatrix} = \alpha^2[\cancel{\alpha^4} - \cancel{\alpha^4} + \alpha^2\lambda] = \cancel{\alpha^4\lambda}$$

$$\chi_Q = \lambda^2[\alpha^4 + \alpha^2 + 1 - \lambda] = 0$$

$\lambda_1 = \lambda_2 = 0$

$\lambda_3 = 1 + \alpha^2 + \alpha^4$ (Positive)

$\boxed{\text{Q is positive semi-definite}}$ as all eigenvalues are greater than or equal to 0.

ii) $f = \gamma x_1^2 + x_2^2 + 4x_1x_2$

$$A = \begin{pmatrix} \gamma & 2 \\ 2 & 1 \end{pmatrix}$$

$$f = x^TAx = [x_1 \ x_2] \begin{pmatrix} \gamma & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= [\gamma x_1 + 2x_2 \quad 2x_1 + x_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \gamma x_1^2 + 2x_2x_1 + 2x_2x_1 + x_2^2 \checkmark$$

Positive definite if leading principal minors of A > 0 (Sylvester)

$$\underbrace{|\gamma|}_{LPM1} > 0, \qquad \underbrace{\begin{vmatrix} \gamma & 2 \\ 2 & 1 \end{vmatrix}}_{LPM2} = \gamma - 4 > 0$$

$\therefore \boxed{\gamma > 4}$

#4) $f = 2x_1x_3 - x_1^2 - x_2^2 - 5x_3^2 - 2\xi x_1x_2 - 4x_2x_3$

$$A = \begin{pmatrix} -1 & -\xi & 1 \\ -\xi & -1 & -2 \\ 1 & -2 & -5 \end{pmatrix}$$

A is negative semi-definite if $-A$ is positive semi definite

Sylvester: All principal minors are non negative for positive semi-definite

$$-A = \begin{pmatrix} 1 & \xi & -1 \\ \xi & 1 & 2 \\ -1 & 2 & 5 \end{pmatrix}$$

$\begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = 5-4 = 1 \geq 0$ ✓ $\quad |1| \geq 0$ ✓

$\begin{vmatrix} 1 & -1 \\ -1 & 5 \end{vmatrix} = 5-1 = 4 \geq 0$ ✓ $\quad |1| \geq 0$ ✓

$\begin{vmatrix} 1 & \xi \\ \xi & 1 \end{vmatrix} = 1 - \xi^2 \geq 0 \quad |5| \geq 0$ ✓

$\underline{-1 \leq \xi \leq 1}$

$\begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} - \xi \begin{vmatrix} \xi & 2 \\ -1 & 5 \end{vmatrix} - \begin{vmatrix} \xi & 1 \\ -1 & 2 \end{vmatrix} = \det(A) \geq 0$

$1 - \xi(5\xi + 2) - (2\xi + 1) = 1 - 5\xi^2 - 4\xi - 1 \geq 0$

$-\xi(5\xi + 4) \geq 0 \quad \therefore \quad \xi \leq 0 \quad \xi \geq -\frac{4}{5}$

f is negative semi-definite if

$$\boxed{0 \le q \le -4/5}$$

#5)

$$P = \begin{pmatrix} .475 & .45 & .175 \\ .45 & 1.25 & .25 \\ .175 & .25 & .375 \end{pmatrix}$$

$\lambda_1 = .2279$, $\lambda_2 = .3379$, $\lambda_3 = 1.5342$ > 0

$\therefore$ P is positie definite,

Continous time LTI system is asymptotically stable, the positive definitness of P implies that a lyapunov function exists for this system.

b) $V(x) = x^T x$ $\quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $P = I_{3x3}$

$V(x) = x_1^2 + x_2^2 + x_3^2$

$$A = \begin{pmatrix} -2 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & -2 & -2 \end{pmatrix} \qquad \begin{aligned} \dot{x}_1 &= -2x_1 \\ \dot{x}_2 &= x_1 + x_3 \\ \dot{x}_3 &= -2x_2 - 2x_3 \end{aligned} \qquad x_e = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$V(x_e) = 0^2 + 0^2 + 0^2 = 0$ ✓

$V(x) = x_1^2 + x_2^2 + x_3^2 > 0$, $x \neq x_e$ ✓

P is also positive definite, V(x) is Canidate Lyapunov

$\dot{V}(x) = 2x_1\dot{x}_1 + 2x_2\dot{x}_2 + 2x_3\dot{x}_3$

$\dot{V}(x) = (2x_1)(-2x_1) + (2x_2)(x_1+x_3) + (2x_3)(-2x_2-2x_3)$

$\dot{V}(x) = -4x_1^2 + 2x_2x_1 - 2x_2x_3 - 4x_3^2 = x^T - Q\, x$

$\therefore -Q = \begin{pmatrix} -4 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & -4 \end{pmatrix}$

Verify Q matches $\quad A^TP + PA = -Q$

$$\begin{pmatrix} -2 & 1 & 0 \\ 0 & 0 & -2 \\ 0 & 1 & -2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -2 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & -2 & -2 \end{pmatrix} = \begin{pmatrix} -4 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & -4 \end{pmatrix}$$

For Lyapunov Function, $\dot{V}(x) < 0$

$\therefore -Q$ must be negative definite which is the same as Q being positive definite.

$Q = \begin{pmatrix} 4 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 4 \end{pmatrix}$

Sylvester: $|4| > 0$ ✓ $\quad \begin{vmatrix} 4 & -1 \\ -1 & 0 \end{vmatrix} = 0 - 1 = -1 \not> 0$

The leading principal minors of Q are not positive, $\therefore$ Q is not positive definite & V(x) is $\therefore$ not a Lyapunov function for this system.

#6)

$$P = \begin{pmatrix} 4.0571 & -.1428 & .2721 \\ -.1428 & 2.3073 & .9904 \\ .2721 & .9904 & 2.0426 \end{pmatrix}$$

$\lambda_1 = 1.1451$, $\lambda_2 = 3.1686$, $\lambda_3 = 4.0933 > 0$

$\boxed{\text{P is positive definite, therefor the system is asymptotically stable, which implies the existence of a Lyapunov function for this system.}}$

b) $V(x) = x^T x \qquad P = I$

$V(x)$ is positive definite when $x \neq 0$ &

$V(0) = 0$ for $x = 0$. $V(x)$ is Lyapunov canidate function. (P is positive definite)

$$\Delta V(x[k]) = V(x[k+1]) - V(x[k]) < 0$$

$$x[k+1] = \begin{pmatrix} -.8 & 0 & 0 \\ .4 & 0 & .4 \\ 0 & -.8 & -.8 \end{pmatrix} x[k]$$

$x_1[k+1] = -.8\, x_1[k]$, $x_2[k+1] = .4x_1[k] + .4\, x_3[k]$

$x_3[k+1] = -.8x_2[k] - .8x_3[k]$

$$V(x[k+1]) = [-.8x_1 \quad .4(x_1+x_3) \quad -.8(x_2+x_3)] \begin{bmatrix} -.8x_1 \\ .4(x_1+x_3) \\ -.8(x_2+x_3) \end{bmatrix}$$

$$V(x[k+1]) = .64\,x_1[k]^2 + .16\,(x_1[k]+x_3[k])^2 + .64\,(x_2[k]+x_3[k])^2$$

$$V(x[k+1]) = .8\,x_1[k]^2 + .32\,x_1[k]x_3[k] + .8\,x_3[k]^2 + .64\,x_2[k]^2$$
$$+ 1.28\,x_2[k]x_3[k]$$

$$V(x[k]) = x_1[k]^2 + x_2[k]^2 + x_3[k]^2$$

$$\Delta V(x[k]) = -.2\,x_1[k]^2 - .36\,x_2[k]^2 - .2\,x_3[k]^2 + .32\,x_1[k]x_3[k]$$
$$+ 1.28\,x_2[k]\,x_3[k]$$

$$\Delta V(x[k]) = x[k]^T - Q\,x[k]$$

$$-Q = \begin{pmatrix} -.2 & 0 & .16 \\ 0 & -.36 & .64 \\ .16 & .64 & -.2 \end{pmatrix}$$

Check $A^T P A - P = -Q$ gives same $-Q$

$$\begin{pmatrix} -.8 & .4 & 0 \\ 0 & 0 & -.8 \\ 0 & .4 & -.8 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -.8 & 0 & 0 \\ .4 & 0 & .4 \\ 0 & -.8 & -.8 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} .8 & 0 & .16 \\ 0 & .64 & .64 \\ .16 & .64 & .8 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -.2 & 0 & .16 \\ 0 & -.36 & .64 \\ .16 & .64 & -.2 \end{pmatrix} \checkmark$$

If $V(x)$ is Lyapunov function, Q is positive definite

$$Q = \begin{pmatrix} .2 & 0 & -.16 \\ 0 & .36 & -.64 \\ -.16 & -.64 & .2 \end{pmatrix}$$

$|.2| > 0$ ✓

$$\begin{vmatrix} .2 & 0 \\ 0 & .36 \end{vmatrix} = .072 > 0 \checkmark$$

$$\begin{vmatrix} .2 & 0 & .16 \\ 0 & .36 & -.64 \\ -.16 & -.64 & .2 \end{vmatrix} = .2\begin{vmatrix} .36 & -.64 \\ -.64 & .2 \end{vmatrix} - .16\begin{vmatrix} 0 & .36 \\ -.16 & -.64 \end{vmatrix}$$

$$= (.2)(.072 - .4096) - .16(0 + .0576)$$

$$= -0.0767 \not> 0$$

$$\boxed{\text{Leading principal minors of } Q \text{ aren't positive, } \therefore Q \text{ isn't positive definite \& } V(x) = x^T x \text{ is NOT a Lyapunov function for this system.}}$$

Contents


```
clear
close all
clc
```

Problem 1

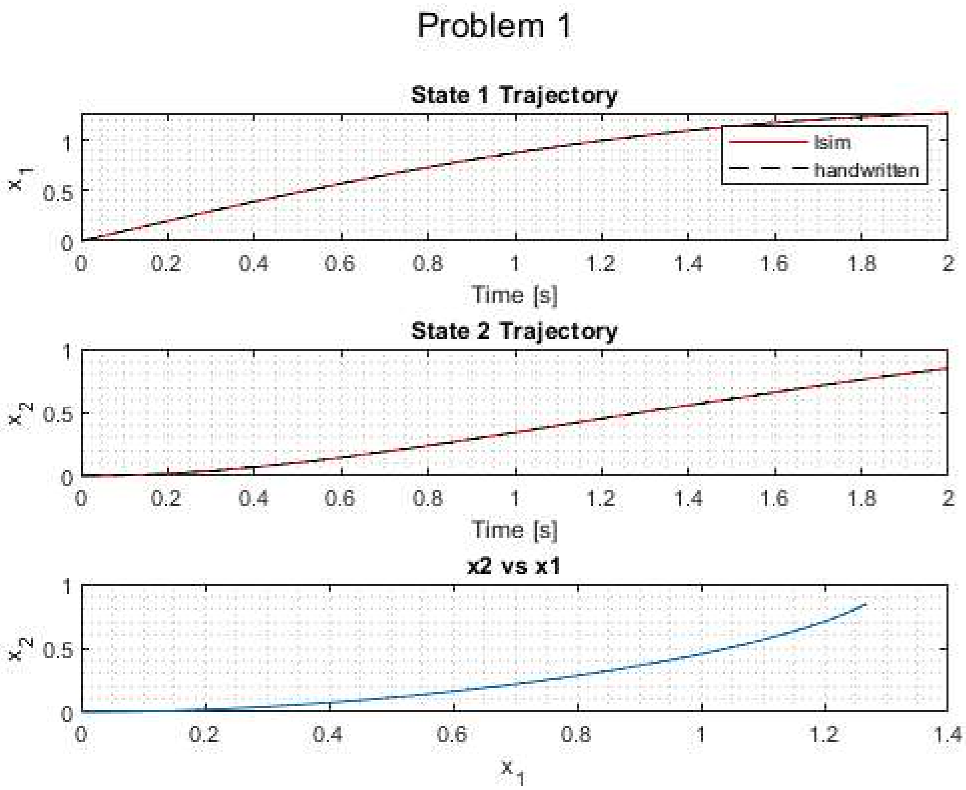
State Space Matrices

```
A       = [0 -1; 1 -1];
B       = [1; 0];
C       = eye(2);               % Output both states
D       = zeros(2,1);

t       = (0:.005:2)';          % time vector
x0      = zeros(2,1);           % Initial contiosn x(0) = 0
u       = ones(length(t),1);    % Step Input

sys     = ss(A,B,C,D);          % System state space object

[~,T,X] = lsim(sys,u,t,x0);     % lsim output
```

Hand Written Solutions for State Trajectories

```
x1      = exp(-t/2).*(-cos((sqrt(3)*t)/2) + (1/sqrt(3))*sin((sqrt(3)*t)/2)) + 1;
x2      = 1 - exp(-t/2).*(1/sqrt(3) * sin((sqrt(3)*t)/2) + cos((sqrt(3)*t)/2));

figure
subplot(3,1,1)
plot(T,X(:,1),'-r',t,x1,'--k')
xlabel('Time [s]')
title('State 1 Trajectory')
legend('lsim','handwritten')
grid minor
ylabel('x_1')
subplot(3,1,2)
plot(T,X(:,2),'-r',t,x2,'--k')
title('State 2 Trajectory')
xlabel('Time [s]')
grid minor
ylabel('x_2')
subplot(3,1,3)
plot(x1,x2)
title('x2 vs x1')
grid minor
xlabel('x_1')
```

```
ylabel('x_2')
sgtitle('Problem 1')
```



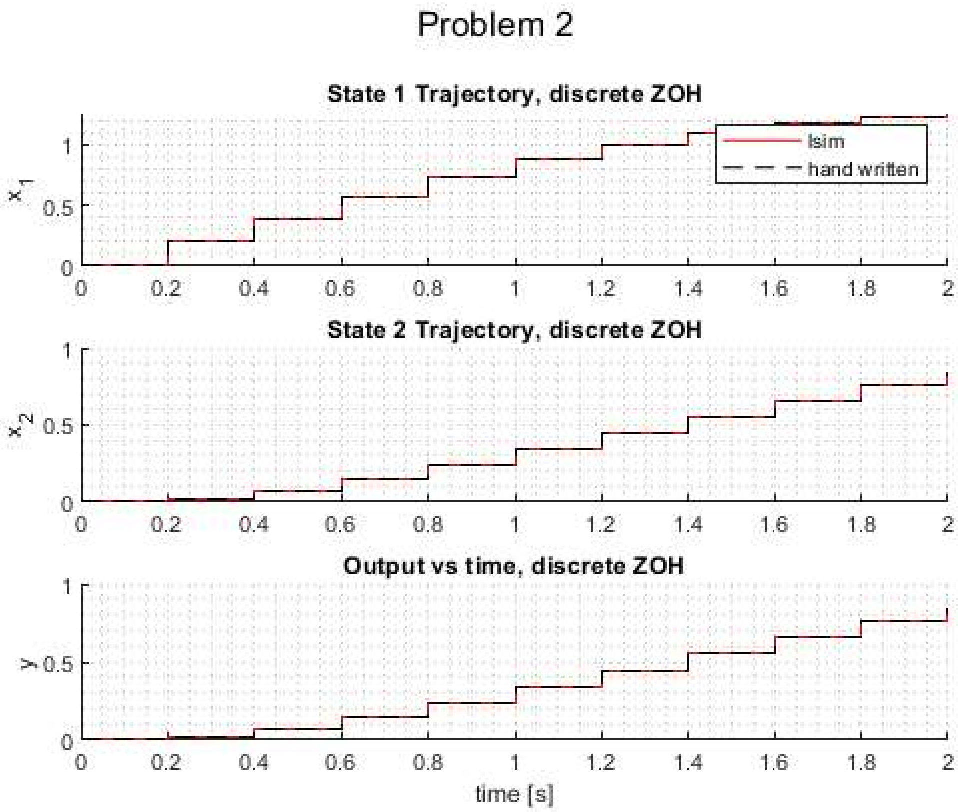
Problem 2

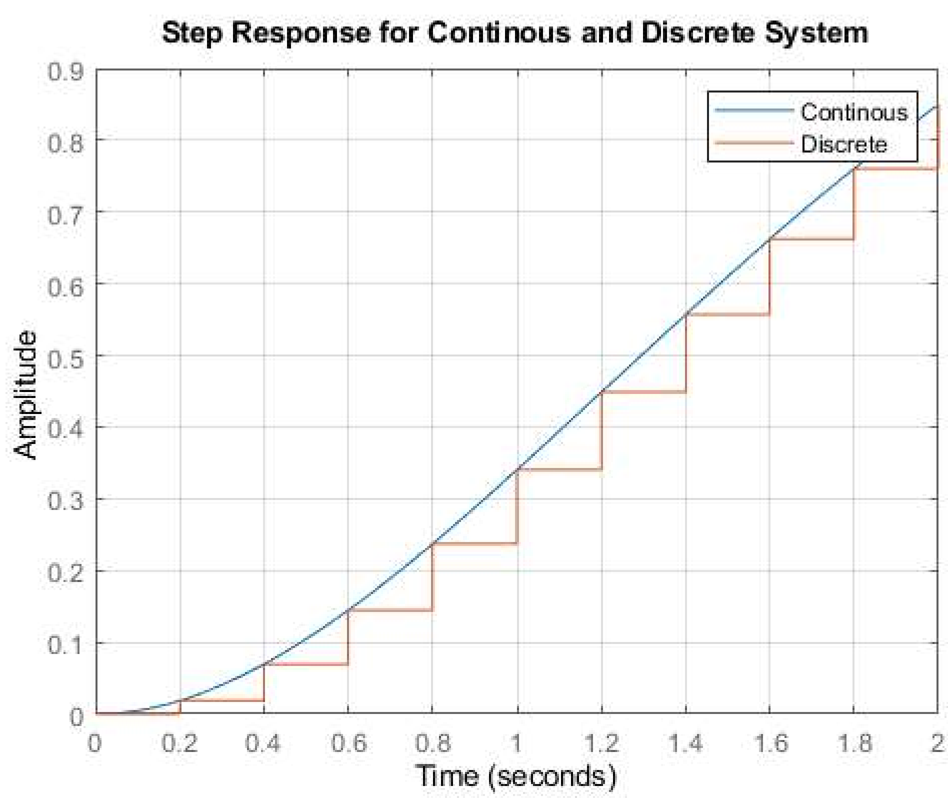
```
Ts      = 0.2;  % step size [s]

disp('Hand Written Discrete State Space Model')

% Hand Written discrete A matrix
Ad      = exp(-.2/2)*[cos(.2*sqrt(3)/2)+ (1/sqrt(3)*sin(.2*sqrt(3)/2)),...
          -2*sin(.2*sqrt(3)/2)/sqrt(3);2*sin(.2*sqrt(3)/2)/sqrt(3),...
          cos(.2*sqrt(3)/2)- (1/sqrt(3)*sin(.2*sqrt(3)/2))]

% Hand Written discrete B matrix
Bd      = [exp(-.2/2)*(-cos(.2*sqrt(3)/2) + sin(.2*sqrt(3)/2)/sqrt(3)) + 1;...
          -exp(-.2/2)*(sin(.2*sqrt(3)/2)/sqrt(3) + cos(.2*sqrt(3)/2)) + 1]

% Hand Written discrete C matrix
Cd      = [0 1]

% Hand Written discrete D matrix
Dd      = 0

% Continous time ss object
sysc    = ss(A,B,[0 1],0);

disp('c2d Discrete State Space Model')
sysd            = c2d(sysc, Ts, 'zoh')          % c2d discrete ss object

time            = (0:Ts:2)';                    % Time Vector
K               = round(time/Ts);               % Integer Steps
```

```
x               = zeros(2,length(K));           % Initialize State Vector
x(:,1)          = x0;
u               = ones(1,length(K));            % Step Input
zeroStateSum    = 0;
y               = zeros(1,length(K));

% x[k] = A^k*x0 + sum(Ad^(k-1-i))*B*u[i] + D*u[k]
for count = 2:length(K)
    k               = K(count);
    zeroStateSum    = Ad^(k - 1)*Bd*u(k) + zeroStateSum;
    x(:,count)      = (Ad^k)*x0 + zeroStateSum;
    y(:,count)      = Cd*x(:,count);
end

% Discrete lsim output
[Y,T,X]     = lsim(sysd,u,time,x0);

figure
subplot(3,1,1)
hold on
stairs(T,X(:,1),'-r')
stairs(time,x(1,:),'--k')
hold off
grid minor
title('State 1 Trajectory, discrete ZOH')
ylabel('x_1')
legend('lsim','hand written')
subplot(3,1,2)
hold on
stairs(T,X(:,2),'-r')
stairs(time,x(2,:),'--k')
hold off
grid minor
title('State 2 Trajectory, discrete ZOH')
ylabel('x_2')
subplot(3,1,3)
hold on
stairs(T,Y,'-r')
stairs(time,y,'--k')
hold off
grid minor
title('Output vs time, discrete ZOH')
ylabel('y')
xlabel('time [s]')
sgtitle('Problem 2')

figure
hold on
step(sysc,2)
step(sysd,2)
grid minor
title('Step Response for Continous and Discrete System')
legend('Continous','Discrete')
```

```
Hand Written Discrete State Space Model

Ad =

    0.9813   -0.1801
```

```
    0.1801    0.8013


Bd =

    0.1987
    0.0187


Cd =

     0     1


Dd =

     0

c2d Discrete State Space Model

sysd =

  A =
            x1       x2
   x1   0.9813  -0.1801
   x2   0.1801   0.8013

  B =
            u1
   x1   0.1987
   x2  0.01867

  C =
       x1  x2
   y1   0   1

  D =
       u1
   y1   0

Sample time: 0.2 seconds
Discrete-time state-space model.
```





Problem 5

```
% Continous Time LTI A Matrix
A_CT        = [-2 0 0; 1 0 1; 0 -2 -2];
```

```
% Symmetric Positive Definite Matrix
Q           = eye(size(A_CT));

% Solve Lyapunov Equation for P
P_CT         = lyap(A_CT',Q)

% Check Eigenvalues of P for Positive Definite
lambda_CT    = eig(P_CT)

if (lambda_CT > 0)
    disp('P is positive definite, Continous LTI System is asymptotically stable')
else
    disp('P is not positive definite, Continous LTI System is not asymptotically stable')
end
```

```
P_CT =

    0.4750    0.4500    0.1750
    0.4500    1.2500    0.2500
    0.1750    0.2500    0.3750


lambda_CT =

    0.2279
    0.3379
    1.5342

P is positive definite, Continous LTI System is asymptotically stable
```

Problem 6

```
% Discrte Time LTI A matrix
A_DT        = [-.8 0 0; .4 0 .4; 0 -.8 -.8];

% Symmetric Positive Definite Matrix
Q           = eye(size(A_DT));

% Solve Discrete Lyapunov Equation for P
P_DT        = dlyap(A_DT',Q)

% Check Eigenvalues of P for Positive Definite
lambda_DT    = eig(P_DT)

if (lambda_DT > 0)
    disp('P is positive definite, Discrete LTI System is asymptotically stable')
else
    disp('P is not positive definite, Discrete LTI System is not asymptotically stable')
end
```

```
P_DT =

    4.0571   -0.1428    0.2721
   -0.1428    2.3073    0.9904
    0.2721    0.9904    2.0426
```

```
lambda_DT =

    1.1451
    3.1686
    4.0933

P is positive definite, Discrete LTI System is asymptotically stable
```

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