ECE 68000: MODERN AUTOMATIC CONTROL

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Linearization

Linearization

- All physical systems are non-linear
- When we use a linear model of a physical system, we are employing some form of linearization
- A linear model accurately models a physical system in some range about an operating point about which the system is linearized

Taylor Linearization—Sec 2.3

- Non-linear time-invariant system

$$\left.\begin{array}{rcl} \frac{dx_1}{dt} & = & f_1(x_1, x_2, \ldots, x_n) \\ \frac{dx_2}{dt} & = & f_2(x_1, x_2, \ldots, x_n) \\ & \vdots & \\ \frac{dx_n}{dt} & = & f_n(x_1, x_2, \ldots, x_n) \end{array}\right\}$$

- Compact notation

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$$

Equilibrium state—continuous-time (CT) case

Definition

A point $\boldsymbol{x}_e$ is an equilibrium state, or an equilibrium point, of the system modeled by $\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t))$ if it has the property that when the system starts at $\boldsymbol{x}_e$, it will stay at $\boldsymbol{x}_e$ for all future time. The equilibrium points are obtained by solving the algebraic equation

$$\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{0}.$$

Equilibrium state—discrete-time (DT) case

Definition

A point $\boldsymbol{x}_e$ is an equilibrium state, or an equilibrium point, of the system modeled by $\boldsymbol{x}[k+1] = \boldsymbol{f}(\boldsymbol{x}[k])$ if it has the property that when the system starts at $\boldsymbol{x}_e$, it will stay at $\boldsymbol{x}_e$ for all future time, that is, $\boldsymbol{x}_e = \boldsymbol{f}(\boldsymbol{x}_e)$. The equilibrium points for DT systems are obtained by solving the algebraic equation

$$\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{x}).$$

r-neighborhood

Definition

The r-neighborhood is a set of points

$$\{\boldsymbol{x} : \|\boldsymbol{x} - \boldsymbol{x}_e\| < r\},$$

where $\|\cdot\|$ can be any p-norm on $\mathbb{R}^n$

Isolated Equilibrium State

Definition

An equilibrium state $\boldsymbol{x}_e$ is isolated if there is an $r > 0$ such that the r-neighborhood of $\boldsymbol{x}_e$ contains no equilibrium state other than $\boldsymbol{x}_e$.

Taylor linearization about equilibrium—continuous-time (CT) case

$$\frac{d}{dt}\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{x}_e) + \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}(\boldsymbol{x}_e)\,\delta\boldsymbol{x} + \text{higher-order terms}$$

where

$$\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}(\boldsymbol{x}_e)$$

is the Jacobian matrix of $\boldsymbol{f}$ with respect $\boldsymbol{x}$, evaluated at the equilibrium state $\boldsymbol{x}_e$, and $\delta\boldsymbol{x} = \boldsymbol{x} - \boldsymbol{x}_e$

Taylor linearization about equilibrium—manipulations

- Observe that

$$\frac{d}{dt}\boldsymbol{x} = \frac{d}{dt}\boldsymbol{x}_e + \frac{d}{dt}\delta\boldsymbol{x} = \frac{d}{dt}\delta\boldsymbol{x}$$

  because $\boldsymbol{x}_e$ is constant
- Use $\boldsymbol{f}(\boldsymbol{x}_e) = \boldsymbol{0}$ to obtain

$$\frac{d}{dt}\delta\boldsymbol{x} = \frac{\partial\boldsymbol{f}}{\partial\boldsymbol{x}}(\boldsymbol{x}_e)\,\delta\boldsymbol{x}$$

Taylor linearized model—continuous-time (CT) case

- Let
$$\boldsymbol{A} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}(\boldsymbol{x}_e)$$
- Linear model in $\delta\boldsymbol{x}$,
$$\frac{d}{dt}\delta\boldsymbol{x} = \boldsymbol{A}\,\delta\boldsymbol{x}$$

Taylor linearization—DT case

Expand the right-hand side of $\boldsymbol{x}[k+1] = \boldsymbol{f}(\boldsymbol{x}[k])$ into first-order Taylor series about $\boldsymbol{x}_e$,

$$\boldsymbol{x}[k+1] = \boldsymbol{f}(\boldsymbol{x}_e) + \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}(\boldsymbol{x}_e)\,\delta\boldsymbol{x}[k] + \text{higher-order terms}$$

where

$$\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}(\boldsymbol{x}_e)$$

is the Jacobian matrix of $\boldsymbol{f}$ with respect $\boldsymbol{x}$, evaluated at the equilibrium state $\boldsymbol{x}_e$, and $\delta\boldsymbol{x}[k] = \boldsymbol{x}[k] - \boldsymbol{x}_e$

Taylor linearization—DT case; manipulations

- Recall that

$$\boldsymbol{f}(\boldsymbol{x}_e) = \boldsymbol{x}_e$$

- Hence,

$$\begin{aligned}
\boldsymbol{x}[k+1] &= \boldsymbol{f}(\boldsymbol{x}_e) + \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}(\boldsymbol{x}_e)\,\delta\boldsymbol{x}[k] \\
&= \boldsymbol{x}_e + \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}(\boldsymbol{x}_e)\,\delta\boldsymbol{x}[k]
\end{aligned}$$

- Since $\delta\boldsymbol{x}[k+1] = \boldsymbol{x}[k+1] - \boldsymbol{x}_e$, therefore,

$$\delta\boldsymbol{x}[k+1] = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}(\boldsymbol{x}_e)\,\delta\boldsymbol{x}[k]$$

Taylor linearized model—DT case

- Let
$$\boldsymbol{A} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}(\boldsymbol{x}_e)$$
- Linear model in $\delta \boldsymbol{x}$,
$$\delta \boldsymbol{x}[k+1] = \boldsymbol{A}\,\delta \boldsymbol{x}[k]$$