Hamilton's Equations of Motion (Ch 8 Goldstein, Ch 5 Hand + Finch)

- No new physics, just additional, powerful methods for working with physical principles already established.
- Assume holonomic constraints and forces are derived from a potential (i.e. monogenic)
- Recall the Lagrangian formulation for a holonomic system with M D.O.F.

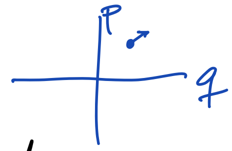
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \quad \text{for } i = 1, \ldots, M$$

- M 2nd-order differential equations requiring $2M$ initial conditions for $q_i(0)$ and $\dot{q}_i(0)$
- M dimensional configuration space: motion produces a trajectory $q_i(t)$ in this space.
- The Hamiltonian Formulation is based on $2M$ 1st-order equations of motion expressed in terms of $2M$ independent variables.

$$\{\underbrace{q_1, \ldots, q_M}_{\text{"canonical variables"}}, \underbrace{p_1, \ldots, p_M}_{\text{canonical or "conjugate momenta"}}\} \qquad p_i = \frac{\partial L}{\partial \dot{q}_i}$$

Note: We are considering p_i as a "dynamically independent partner to q_i"

• The solutions to the 2M E.O.M. live in a 2M-dimensional phase space.



• The transition from Lagrangian to Hamiltonian formulations corresponds to a change of cordinates

$$(q, \dot{q}, t) \longmapsto (q, p, t)$$

that uses the Legendre Transformation

• The Legendre Transformation is a general procedure for starting with a function of a variable and generating another function of another new variable.

E.g. $L(q, \dot{q}, t) \longrightarrow H(q, p, t)$

Consider a known function $A(x,y)$ of a "passive" variable x and an "active" variable y

Let $B(x,y,z) \triangleq yz - A(x,y)$

where z is a new variable.

$$dB = z\,dy + y\,dz \overbrace{- \frac{\partial A}{\partial x}\,dx - \frac{\partial A}{\partial y}\,dy}^{= -dA}$$

$$= \left(z - \frac{\partial A}{\partial y}\right)dy + y\,dz - \frac{\partial A}{\partial x}\,dx$$

→ Remember: we want to get rid of y

Now if we choose $z(x,y) = \dfrac{\partial A}{\partial y}$

$$\Rightarrow dB = y\,dz - \frac{\partial A}{\partial x}\,dx$$

But we know for B to be a perfect differential, we expect the form,

$$dB = \frac{\partial B}{\partial x} dx + \frac{\partial B}{\partial z} dz$$

Equating, if we require that

$$\frac{\partial B}{\partial z} = y \qquad \frac{\partial B}{\partial x} = -\frac{\partial A}{\partial x}$$

Then we would have B as a perfect differential and

$$B = B(x, y(x,z), z) = B(x,z)$$

using $z(x,y) = \frac{\partial A}{\partial y}$ to find $y = y(x,z)$.

Repeat the same calculations with (x,y,z) replaced by $(q, \dot{q}, p)$ and (A,B) replaced with (L,H), and

We need dL

$$L = L(q,\dot{q},t) \Rightarrow$$

$$dL = \sum_i \left(\frac{\partial L}{\partial q_i} dq_i + \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i \right) + \frac{\partial L}{\partial t} dt$$

Choose $p_i \triangleq \frac{\partial L}{\partial \dot{q}_i} \Rightarrow \dot{p}_i = \frac{\partial L}{\partial q_i}$ (Using Lagrange's Equation $\frac{d}{dt}(p_i) - \frac{\partial L}{\partial q_i} = 0$)

Plugging in,

$$dL = \sum_i \left(\dot{p}_i dq_i + p_i d\dot{q}_i \right) + \frac{\partial L}{\partial t} dt$$

Then take $H = H(q, p, t)$ by the Legendre transform

$$\boxed{H(q,p,t) = \sum_i \dot{q}_i p_i - L(q,\dot{q},t)}$$

$$dH = \sum_i (\dot{q}_i dp_i + p_i d\dot{q}_i) - dL$$

Use our dL from earlier and plug in

$$dH = \sum_i (\dot{q}_i dp_i - \dot{p}_i dq_i) - \frac{\partial L}{\partial t} dt$$

Now, compare this to the generic exact differential of H:

$$dH = \sum_i \left(\frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i \right) + \frac{\partial H}{\partial t} dt$$

Equating coefficients,

$$\dot{q}_i = \frac{\partial H}{\partial p_i}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}$$

$$\frac{\partial H}{\partial t} = \frac{-\partial L}{\partial t}$$

Hamilton's Equations

$2M+1$ equations

Also, remember that

$p_i \triangleq \frac{\partial L}{\partial \dot{q}_i}$ conjugate momentum

$H \triangleq \sum_i \dot{q}_i p_i - L(q,\dot{q},t)$ Legendre Transform

$= \dot{q}^T p - L$

We could have also derived Hamilton's equations from Hamilton's principle.

$$\delta I \equiv \delta \int_{t_1}^{t_2} L \, dt = 0$$

But, we have that

$$L(q,\dot{q},t) = \dot{q}^T p - H(q,p,t) \quad \begin{pmatrix}\text{Re-arranged} \\ \text{Legendre}\end{pmatrix}$$

Plugging in,

$$\delta \int_{t_1}^{t_2} (\dot{q}^T p - H(q,p,t))\, dt = 0$$

Recall that the original Hamilton's principle looked at variations in the q_i's in the configuration space. Since we have done the Legendre transform, we have introduced p_i's and we want to treat them independently from the q_i's. This means we need to take variations in both q_i's and p_i's.

It also means these variations are occuring in the (q,p) phase space.

The

$$\delta \int_{t_1}^{t_2} \underbrace{\left(p_i \dot{q}_i - H(q,p,t)\right)}_{\text{some function}} dt = 0$$ expression

can be seen as

$$\delta \int_{t_1}^{t_2} \underbrace{f(q, \dot{q}, p, \cancel{\dot{p}}, t)}_{\text{some function}} dt = 0$$

Variations in δq and δp lead to

$$\frac{\delta f}{\delta q_i} = \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{q}_i}\right) - \frac{\partial f}{\partial q_i} = 0 \quad \text{for } i = 1, \dots, m$$

and

$$\frac{\delta f}{\delta p_i} = \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{p}_i}\right) - \frac{\partial f}{\partial p_i} = 0 \quad \text{for } i = 1, \dots,$$

The first set leads to

$$\frac{d}{dt}(p_i) + \frac{\partial H}{\partial q_i} = 0 \Rightarrow \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

The 2nd set leads to

$$-\dot{q}_i + \frac{\partial H}{\partial p_i} = 0 \Rightarrow \dot{q}_i = \frac{\partial H}{\partial p_i}$$

Matrix (Symplectic) Notation for Hamiltonian Dynamics

Let $\underline{\eta} \triangleq (q_1, \ldots, q_M, p_1, \ldots, p_M)^T \in \mathbb{R}^{2M}$

Also, define the "Hamiltonian matrix"

$$J \triangleq \begin{bmatrix} O_{M\times M} & I_{M\times M} \\ -I_{M\times M} & O_{M\times M} \end{bmatrix} \in \mathbb{R}^{2M\times 2M}$$

Then we can re-write the dynamics as:

$$\dot{\underline{\eta}} = \begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} O & I \\ -I & O \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{bmatrix}$$

Recall

$$\dot{q} = \frac{\partial H}{\partial p}$$

$$\dot{p} = \frac{-\partial H}{\partial q}$$

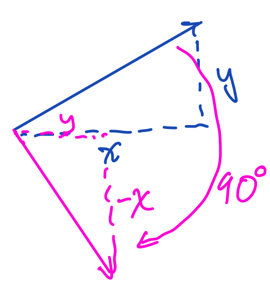
$$\boxed{\dot{\underline{\eta}} = J \nabla_\eta H}$$

Comment on J: You can think of J as resembling a 90° rotation

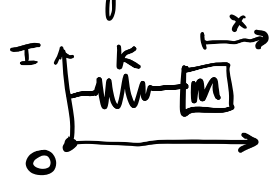
If we were in $\mathbb{R}^2$, a rotation matrix looks like

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \quad \text{Let } \theta = 90° \Rightarrow \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ -x \end{bmatrix}$$



Ex. Simple Harmonic motion



$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}Kx^2$$
$$= \frac{1}{2}m\dot{q}^2 - \frac{1}{2}Kq^2$$
$$q = x$$
$$p \triangleq \frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

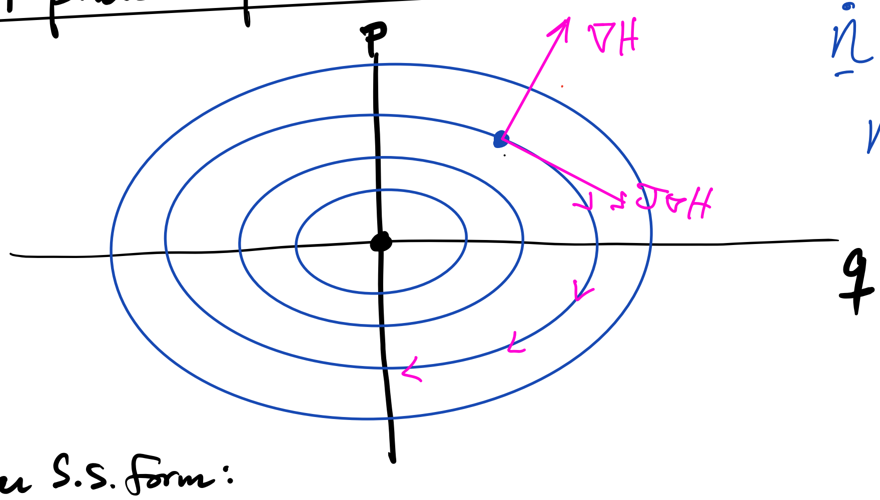
Legendre Transform:

$$H = p\dot{q} - L$$
$$= (m\dot{x})(\dot{x}) - \frac{m\dot{x}^2}{2} + \frac{K}{2}x^2$$
$$= \frac{m}{2}\dot{x}^2 + \frac{K}{2}x^2 \quad (\text{In this case } E_0 = H)$$

However, we want to write H in terms of q and p,
Recall, $p = m\dot{x}$

$$H(q,p) = \frac{p^2}{2m} + \frac{K}{2}q^2$$

Plot phase-space level curves of H



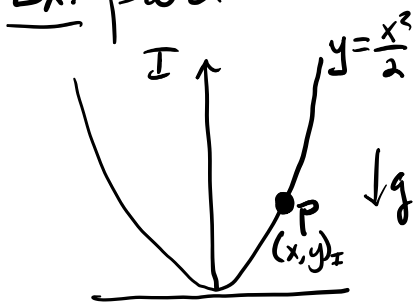
$$\dot{\eta} = J\nabla H$$
$$\eta = \begin{pmatrix} q \\ p \end{pmatrix}$$

In linear S.S. form:

$$\frac{d}{dt}\begin{bmatrix} q \\ p \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{m} \\ -k & 0 \end{bmatrix}\begin{bmatrix} q \\ p \end{bmatrix}$$

Eq. pt at $(0,0)$ is a "center"

Ex. particle on a parabolic wire



$$\vec{r}_{P/O} = x\,\hat{e}_x + y\,\hat{e}_y$$

$$\vec{v}_{P/O} = \dot{x}\,\hat{e}_x + \dot{y}\,\hat{e}_y$$

$$L = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) - mgy$$

We know that this is 1DOF. We can plug in the constraint to eliminate y

Note: $y = \frac{x^2}{2} \Rightarrow \dot{y} = x\dot{x}$

Plugging in,

$$L = \tfrac{1}{2}m\left(\dot{x}^2 + x^2\dot{x}^2\right) - mg\frac{x^2}{2}$$

$$= \tfrac{1}{2}m\dot{x}^2\left(1 + x^2\right) - mg\frac{x^2}{2}$$

$$= \tfrac{1}{2}m\dot{q}^2\left(1 + q^2\right) - mg\frac{q^2}{2} \qquad \text{(since } q = x\text{)}$$

Legendre

$$H = p\dot{q} - L$$

We need p,

$$p = \frac{\partial L}{\partial \dot{q}} = m\dot{q}\left(1 + q^2\right)$$

$$H = m\dot{q}\left(1 + q^2\right)\dot{q} - \tfrac{1}{2}m\dot{q}^2\left(1 + q^2\right) + mg\frac{q^2}{2}$$

$$= \frac{m}{2}\left(1 + q^2\right)\dot{q}^2 + mg\frac{q^2}{2} \qquad \text{(Note } E_0 = H \text{ here)}$$

However, we want H in terms of p and q:

$$H = \frac{p^2}{2m(1+q^2)} + \frac{mg}{2}q^2$$

level-sets are not elliptic if you plot in Matlab

E.O.M.'s?

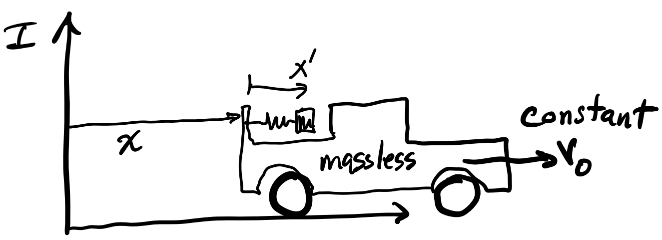
$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m(1+q^2)}$$

$$\dot{p} = \frac{-\partial H}{\partial q} = -q\left(\frac{-p^2}{m(1+q^2)^2} + mg\right)$$

$$\dot{\eta} = J\nabla H$$

$$\begin{bmatrix}\dot{q}\\ \dot{p}\end{bmatrix} = \begin{bmatrix}0 & I\\ -I & 0\end{bmatrix}\begin{bmatrix}\frac{\partial H}{\partial q}\\ \frac{\partial H}{\partial p}\end{bmatrix}$$

Ex. Mass-spring on a car



Find the Hamiltonians using x and x'

$$x = v_0 t + x' \implies \dot{x} = v_0 + \dot{x}'$$

In terms of the x variable,

$$L(x, \dot{x}, t) = T_0 - U_0 = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}k(x - v_0 t)^2$$

$$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

$$H(x,p,t) = p\dot{x} - L(x,\dot{x},t)$$

← Legendre transformation

$$= p\left(\frac{p}{m}\right) - \frac{1}{2}m\dot{x}^2 + \frac{1}{2}k(x - v_0 t)^2$$

$q = x$

$$= \cancel{\frac{1}{2}m\dot{x}^2} + \frac{1}{2}k(x - v_0 t)^2$$

$$\frac{p^2}{2m}$$

$$H(q,p,t) = \frac{p^2}{2m} + \frac{k}{2}(q - v_0 t)^2$$

Note: H is not conserved.

$H = E_0$ $\quad \frac{\partial H}{\partial t} \neq 0$

Let's consider H' in terms of x'

$$L'(x', \dot{x}', t) = \frac{m}{2}(\dot{x}' + v_0)^2 - \frac{1}{2}k(x')^2$$

$$p' = \frac{\partial L}{\partial \dot{x}'} = (\dot{x}' + v_0)m$$

Legendre,

$$H' = p'(\dot{x}') - L'$$

$$H' = p'\left(\frac{p'}{m} - v_0\right) - \frac{m}{2}\underbrace{(\dot{x}' + v_0)}_{\frac{p'}{m}}{}^2 + \frac{k}{2}(x')^2$$

$H' \neq E_0$ (But $\frac{\partial H'}{\partial t} = 0$, so H' is conserved!)

Notes:

- Hamiltonian dynamics applies to systems even if the total energy is not conserved.
- Both Hamiltonians give the correct EOM's
- H is not unique
- H is not always the total energy
- H may be conserved for one choice of cordinates, but vary for another.