Optimal Estimation Methods

(Lecture 11 – Review of Dynamic Systems: Part II)

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Observability (i)

- Observability gives us an indication of the state quantities that can be monitored ("observed") from the measurements
  - Full observability means that one can reconstruct all of the initial time states from measurements in the future
  - For linear time-invariant systems there are many possible tests for observability
  - Let's derive one for linear time-invariant models
  - Let's begin with general single-input-single-output n^{th}-order linear ordinary differential equation

$$
\begin{aligned}
&\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0\, y \\
&= b_n \frac{d^n u}{dt^n} + b_{n-1}\frac{d^{n-1} u}{dt^{n-1}} + \cdots + b_1 \frac{du}{dt} + b_0\, u
\end{aligned}
$$

Observability (ii)

- An observable state-space form is given by the *observer canonical form*

$$\dot{\mathbf{x}}_o = F_o\,\mathbf{x}_o + B_o\,u$$
$$y_o = H_o\,\mathbf{x}_o + D_o\,u$$

where

$$F_o = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix}$$
$$B_o = \begin{bmatrix} (b_0 - b_n\,a_0) & (b_1 - b_n\,a_1) & \cdots & (b_{n-1} - b_n\,a_{n-1}) \end{bmatrix}^T$$
$$H_o = \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix}$$
$$D_o = b_n$$

Observability (iii)

- Clearly, since all states are "coupled" together in the F_o matrix, we only need to monitor one state (given as the last state by H_o) to observe *all* states
- The matrix F_o is called the *right companion matrix*
- A general single-output system (F, B, H, D) is "fully observable" if it can be converted into observer canonical form
- Let's use a transformation of state to see the conditions to make this happen

$$F_o = T^{-1} F\, T, \quad H\, T = H_o$$

where T is a nonsingular constant matrix

- Consider third-order case (general case is easy to see afterwards)
- Left multiplying both sides by T gives

$$T\, F_o = F\, T$$

- For the third-order case let T be partitioned into column vectors so that $T = [\mathbf{t}_1 \ \ \mathbf{t}_2 \ \ \mathbf{t}_3]$

Observability (iv)

- This leads directly to

$$\begin{bmatrix}\mathbf{t}_1 & \mathbf{t}_2 & \mathbf{t}_3\end{bmatrix}\begin{bmatrix}0 & 0 & -a_0\\ 1 & 0 & -a_1\\ 0 & 1 & -a_2\end{bmatrix} = F\begin{bmatrix}\mathbf{t}_1 & \mathbf{t}_2 & \mathbf{t}_3\end{bmatrix}$$

- Solving for t_2 and t_3 gives

$$\mathbf{t}_2 = F\,\mathbf{t}_1$$
$$\mathbf{t}_3 = F\,\mathbf{t}_2$$

- Using $H\,T = H_o$ leads to

$$H\,\mathbf{t}_1 = 0$$
$$H\,\mathbf{t}_2 = 0$$
$$H\,\mathbf{t}_3 = 1$$

- Substituting quantities gives

$$\begin{aligned} H\,\mathbf{t}_1 &= 0 \\ HF\,\mathbf{t}_1 &= 0 \\ HF\,\mathbf{t}_2 = HF^2\mathbf{t}_1 &= 1 \end{aligned} \quad \Longrightarrow \quad \mathbf{t}_1 = \begin{bmatrix} H \\ HF \\ HF^2 \end{bmatrix}^{-1}\begin{bmatrix}0\\0\\1\end{bmatrix}$$

Observability (v)

- Clearly, the original system can only be transformed into observer canonical form if the matrix inverse exists

- The extension to higher-order systems is given by the following $n \times n$ *observability matrix*

$$\boxed{\mathcal{O} = \begin{bmatrix} H \\ HF \\ HF^2 \\ \vdots \\ HF^{n-1} \end{bmatrix}}$$

- Must be rank n for the system to be fully observable
- Note that a similar condition also exists if we have multiple outputs

Example

$\dot{x} = Fx + Bu$
$y = Hx + Du$

- Consider the following system

Sensor placement

$$F = \begin{bmatrix} 0 & 1 \\ -2 & -f_{22} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

- Computing the observability matrix gives

$$\mathcal{O} = \begin{bmatrix} 1 & 1 \\ -2 & 1 - f_{22} \end{bmatrix}$$

- Clearly, the system is observable unless $f_{22} = 3$
- Let's compute transfer function with $f_{22} = 3$ $\frac{Y}{U} = H(sI-F)B + D$

$$\frac{Y(s)}{U(s)} = \frac{(b_{11} + b_{21})(s+1)}{(s+1)(s+3)}$$

- This clearly indicates that a "pole-zero cancellation" has occurred. Sensor can't see s+1 pole
- We cannot observe the state associated with $s + 1 = 0$

- Time-Varying Case
  - Definition: a system is observable if for any unknown $\mathbf{x}(t_0)$, knowledge of $\mathbf{y}(t)$ can uniquely determine $\mathbf{x}(t_0)$
  - Recall the solution for the state

$$\mathbf{x}(t) = \Phi(t, t_0)\,\mathbf{x}(t_0) + \int_{t_0}^{t} \Phi(t, \tau)\,B(\tau)\,\mathbf{u}(\tau)\;d\tau$$

  - Substituting this into $\mathbf{y}(t) = H(t)\mathbf{x}(t)$ gives

$$H(t)\,\Phi(t, t_0)\,\mathbf{x}(t_0) = \mathbf{p}(t) \qquad (1)$$

where

$$\mathbf{p}(t) \equiv \mathbf{y}(t) - H(t)\int_{t_0}^{t} \Phi(t, \zeta)\,B(\zeta)\,\mathbf{u}(\zeta)\;d\zeta$$

  - Note that τ was replaced with ζ; the reason for this will be seen soon (it's just a dummy integration variable)

Observability (ii)

- Left multiplying Eq. (1) by $\Phi^T(t, t_0) H^T(t)$ and integrating from t_0 to t_f gives

$$W_o(t_0, t_f)\,\mathbf{x}(t_0) = \int_{t_0}^{t_f} \Phi^T(\tau, t_0)\, H^T(\tau)\, \mathbf{p}(\tau)\; d\tau$$

where

$$\boxed{W_o(t_0, t_f) \equiv \int_{t_0}^{t_f} \Phi^T(\tau, t_0)\, H^T(\tau)\, H(\tau)\, \Phi(\tau, t_0)\; d\tau} \qquad (2)$$

is known as the continuous-time *observability Gramian*

- Clearly, this matrix must be nonsingular in order to determine $\mathbf{x}(t_0)$
  - Gives an observability condition for time-varying systems
- Computing the integral may be difficult because the state transition matrix is required
  - Fortunately, there is an easier approach to compute the continuous-time observability Gramian

Observability (iii)

- The goal is to determine the initial condition, so replace t_0 with t

$$W_o(t,\, t_f) \equiv \int_t^{t_f} \Phi^T(\tau, t)\, H^T(\tau)\, H(\tau)\, \Phi(\tau, t)\; d\tau$$

- We'll need the derivative of $\Phi(\tau, t) = \Phi^{-1}(t, \tau)$
- Take the derivative of $V\, V^{-1} = I$ for some matrix V

$$V\, \dot{V}^{-1} + \dot{V}\, V^{-1} = 0 \quad \rightarrow \quad \dot{V}^{-1} = -V^{-1} \dot{V}\, V^{-1}$$

- Letting $V \equiv \Phi(t, \tau)$ and noting $V^{-1} = \Phi(\tau\, , t)$ leads to

$$\begin{aligned} \dot{\Phi}(\tau, t) &= -\Phi(\tau, t) \dot{\Phi}(t, \tau)\, \Phi(\tau, t) \\ &= -\Phi(\tau, t)\, F(t)\, \cancelto{I}{\Phi(t, \tau)\, \Phi(\tau, t)} \\ &= -\Phi(\tau, t)\, F(t) \end{aligned}$$

where the following identities were used

$$\dot{\Phi}(t, \tau) = F(t)\, \Phi(t, \tau)$$

$$\Phi(t, \tau)\;\; \Phi(\tau, t) = \Phi(t, t) = I$$

Observability (iv)

- Then the derivative of the observability Gramian is given by

$$
\begin{aligned}
\dot{W}_o(t,\, t_f) = & -\cancelto{I}{\Phi^T(t,t)}\, H^T(t)\, H(t)\, \cancelto{I}{\Phi(t,t)} \\
& - F^T(t) \underbrace{\int_t^{t_f} \Phi^T(\tau,t)\, H^T(\tau)\, H(\tau)\, \Phi(\tau,t)\; d\tau}_{W_o(t,\,t_f)} \\
& - \underbrace{\int_t^{t_f} \Phi^T(\tau,t)\, H^T(\tau)\, H(\tau)\, \Phi(\tau,t)\; d\tau}_{W_o(t,\,t_f)}\; F(t)
\end{aligned}
$$

no state transition matrix.

or

$$
\boxed{\dot{W}_o(t,\, t_f) = -F^T(t)\, W_o(t,\, t_f) - W_o(t,\, t_f)\, F(t) - H^T(t)\, H(t)}
$$

- Note that the minus sign in front of $H^T(t)H(t)$ is due to the fact that the t appears at the bottom of the integral
- This is integrated backwards with final condition $W_o(t_f, t_f) = 0$

Observability (v)

- Time-Invariant Case
  - $W_o(t, t_f)$ reaches steady state very rapidly
  - Find the steady state condition by setting the derivative to zero

$\dot{W} = 0$

$$\boxed{F^T W_o + W_o F = -H^T H} \quad *$$

  - This is known as the *matrix Lyapunov equation*
    - Extremely important equation
    - Many ways to solve this equation
    - MATLAB lyap command can be used
  - The matrix Lyapunov equation appears in many applications
    - Many of them are in control system designs
    - We'll see it again in the derivation of the Kalman filter too

Observability (vi)

- How does this relate to the observability matrix $\mathcal{O}$?
- From Eq. (2) $W_o(t_0, t_f)$ is singular iff there exists a nonzero $\mathbf{x}_a$ such that

$$H\, e^{F\,t}\, \mathbf{x}_a = \mathbf{0}, \quad \forall\, t \in [t_0,\, t_f] \qquad (3)$$

  - This implies that the integration over time does not buy you enough "movement" to obtain a nonsingular matrix
- Now use the series expansion

$$e^{Ft} = \sum_{k=0}^{\infty} \frac{1}{k!} t^k F^k$$

so that Eq. (3) becomes

$$H \sum_{k=0}^{\infty} \frac{1}{k!} t^k F^k\, \mathbf{x}_a = \mathbf{0}, \quad \forall\, t \in [0,\, t_f] \qquad (4)$$

Observability (vii)

- Equation (4) implies that

$$HF^k\mathbf{x}_a = \mathbf{0}, \quad \forall\, k \geq 0$$

- By the Cayley-Hamilton Theorem only the first $n-1$ powers are required because higher ones can be written in terms of these, so

$$HF^k\mathbf{x}_a = \mathbf{0}, \quad \forall\, k = 0, 1 \ldots, n-1$$

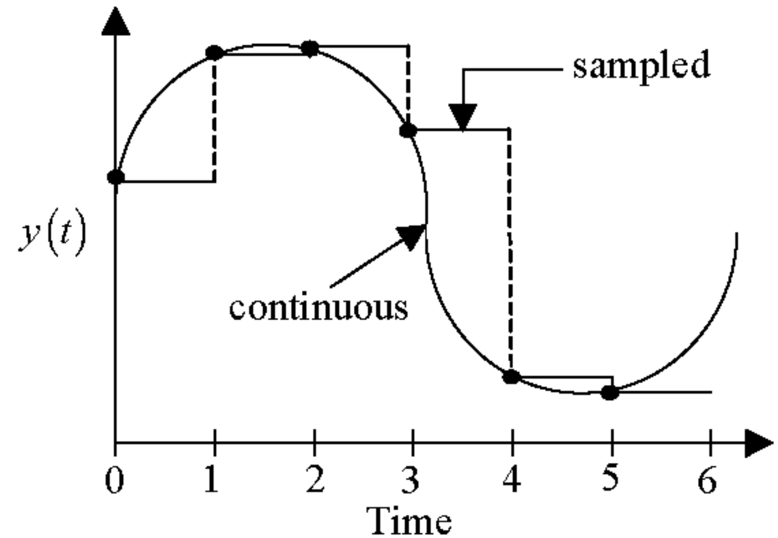
- This equation can be written in matrix form by

$$\begin{bmatrix} H \\ HF \\ \vdots \\ HF^{n-1} \end{bmatrix} \mathbf{x}_a \equiv \mathcal{O}\,\mathbf{x}_a = \mathbf{0}$$

- This can only be true if $\mathcal{O}$ is singular
  - So the Gramian condition is equivalent to the observability matrix condition in this case

Discrete-Time Systems (i)

- Discrete-time systems have now become standard in most dynamic applications with the advent of digital computers, which are used to process sampled-data systems
- The mechanism that acts on the sensor output and supplies numbers to the digital computer is the analog-to-digital (A/D) converter
- Then, the numbers are processed through numerical subroutines and sent to the dynamic system input through the digital-to-analog (D/A) converter
- This allows the use of software driven systems to accommodate the estimation/control aspect of a dynamic system, which can be modified simply by uploading new subroutines to the computer



- We shall only consider the most common sampled-type system given by a “zero-order hold” which holds the sampled point to a constant value throughout the interval
- Obviously, as the sample interval decreases the sampled signal more closely approximates the continuous signal

Discrete-Time Systems (ii)

- Consider the case where time is set to the first sample interval, denoted by Δt, and $F(t)$ and $B(t)$ are constants

$$\mathbf{x}(\Delta t) = e^{F\Delta t}\mathbf{x}(0) + \left[\int_0^{\Delta t} e^{F(\Delta t-\tau)}\, d\tau\right] B\,\mathbf{u}(0)$$

- The integral can be simplified by defining $\zeta \equiv \Delta t - \tau$, giving

$$\int_0^{\Delta t} e^{F(\Delta t-\tau)}\, d\tau = -\int_{\Delta t}^{0} e^{F\zeta}\, d\zeta = \int_0^{\Delta t} e^{F\zeta}\, d\zeta$$

- This leads directly to

$$\mathbf{x}(\Delta t) = \Phi\,\mathbf{x}(0) + \Gamma\,\mathbf{u}(0)$$

where

$$\Phi \equiv e^{F\Delta t}$$

$$\Gamma \equiv \left[\int_0^{\Delta t} e^{Ft}\, dt\right] B \qquad (1)$$

Discrete-Time Systems (iii)

- Expanding for $k+1$ samples gives

$$\mathbf{x}[(k+1)\Delta t] = \Phi\,\mathbf{x}(k\,\Delta t) + \Gamma\,\mathbf{u}(k\,\Delta t)$$

- It is common convention to drop Δt notation so that the entire discrete state-space representation is given by

$$\boxed{\begin{aligned}\mathbf{x}_{k+1} &= \Phi\,\mathbf{x}_k + \Gamma\,\mathbf{u}_k \\ \mathbf{y}_k &= H\,\mathbf{x}_k + D\,\mathbf{u}_k\end{aligned}} \qquad (2)$$

- Notice that the output system matrices H and D are unaffected by the conversion to a discrete-time system
- The system can be shown to be stable if all eigenvalues of Φ lie within the unit circle $|\lambda| < 1$

Example (i)

- Choose the following matrices

$$F = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- Use the Laplace Transform to compute Φ

$$\Phi = e^{F\Delta t} = \{\mathcal{L}^{-1}[sI - F]^{-1}\}\big|_{\Delta t} = \left\{\mathcal{L}^{-1}\begin{bmatrix} \frac{1}{s+1} & 0 \\ \frac{1}{s(s+1)} & \frac{1}{s} \end{bmatrix}\right\}\Bigg|_{\Delta t} = \begin{bmatrix} e^{-\Delta t} & 0 \\ 1 - e^{-\Delta t} & 1 \end{bmatrix}$$

- The matrix Γ is computed using

$$\Gamma = \int_0^{\Delta t} \begin{bmatrix} e^{-t} \\ 1 - e^{-t} \end{bmatrix} dt = \begin{bmatrix} 1 - e^{-\Delta t} \\ \Delta t + e^{-\Delta t} - 1 \end{bmatrix}$$

- If we choose $\Delta t = 0.1$ seconds then

$$\Phi = \begin{bmatrix} 0.9048 & 0 \\ 0.0952 & 1 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 0.0952 \\ 0.0048 \end{bmatrix}$$

Example (ii)

- Determining analytical expressions for Φ and Γ can be tedious and difficult for large-order systems
- Fortunately, several numerical approaches exist for computing these matrices
  - Moler, C., and Van Loan, C., "Nineteen Dubious Ways to Compute the Exponential of a Matrix," *SIAM Review*, Vol. 20, No. 4, 1978, pp. 801-836.
- Series approach is often used

Best approach, especially for small Δt

$$\Phi = I + F\Delta t + \frac{1}{2!}F^2\Delta t^2 + \frac{1}{3!}F^3\Delta t^3 + \cdots$$

- Use Eq. (1) to determine Γ

$$\Gamma = \left[I\Delta t + \frac{1}{2!}F\Delta t^2 + \frac{1}{3!}F^2\Delta t^3 + \cdots \right] B$$

- From last example, three terms are sufficient giving

$$\Phi = \begin{bmatrix} 0.9048 & 0 \\ 0.0952 & 1 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 0.0952 \\ 0.0048 \end{bmatrix}$$

Observability (i)

- The discrete system is observable if there exists a finite k such that knowledge of the outputs to $k-1$ is sufficient to determine the initial state of the system
  - Expand Eq. (2) for a single output and no input and the solve for the initial condition

$$
\begin{aligned}
y_0 &= H\mathbf{x}_0 \\
y_1 &= H\mathbf{x}_1 = H\Phi\,\mathbf{x}_0 \\
y_2 &= H\mathbf{x}_2 = H\Phi^2\,\mathbf{x}_0 \\
&\vdots \\
y_{n-1} &= H\mathbf{x}_{n-1} = H\Phi^{n-1}\,\mathbf{x}_0
\end{aligned}
\quad \Longrightarrow \quad
\mathbf{x}_0 = \begin{bmatrix} H \\ H\Phi \\ H\Phi^2 \\ \vdots \\ H\Phi^{n-1} \end{bmatrix}^{-1} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{bmatrix}
$$

Observability (ii)

- Clearly, the initial state can be obtained only if the following observability matrix is nonsingular

$$\mathcal{O}_d = \begin{bmatrix} H \\ H\Phi \\ H\Phi^2 \\ \vdots \\ H\Phi^{n-1} \end{bmatrix}$$

- Discrete-time observability Gramian and recursion are given by

recursion

$$W_{d_0} \equiv \sum_{i=0}^{N} \Phi^T(i,0) H_i^T H_i \Phi(i,0)$$

$$W_{d_k} = \Phi_k^T W_{d_{k+1}} \Phi_k + H_k^T H_k$$

- For time-invariant systems at steady-state we have

$$\boxed{W_d = \Phi^T W_d \Phi + H^T H}$$

→ Check if non-singular   det(W) ≠ 0

  - This equation is known as a discrete-time matrix Lyapunov equation
  - MATLAB dlyap command can be used

Nonlinear Systems (i)

- Consider the following general nonlinear system

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}, \mathbf{u})$$
$$\mathbf{y} = \mathbf{h}(t, \mathbf{x}, \mathbf{u})$$

  - Some of the nonlinear systems of differential equations encountered in applications can be solved for an exact analytical solution
  - Unfortunately, only a minority of these systems have known analytical solutions
    - No standardized methods exist for finding exact analytical solutions though
  - In many cases a reference motion may be known, which is “close” to the actual state history
    - In these cases the departure of the actual state history from a known reference motion may be adequate to describe the nonlinear equation solution

Nonlinear Systems (ii)

- The nominal reference $\mathbf{x}_N$ trajectory is found analytically by

$$\mathbf{x}_N(t) = \mathbf{x}_N(t_0) + \int_0^t \mathbf{f}(\tau, \mathbf{x}_N, \mathbf{u}_N)\, d\tau$$

$$\mathbf{y}(t) = \mathbf{h}(t, \mathbf{x}_N, \mathbf{u}_N)$$

- Now, we assume that the actual quantities are given by the nominal quantities plus a perturbation

$$\mathbf{x}(t) = \mathbf{x}_N(t) + \boldsymbol{\delta}\mathbf{x}(t), \quad \mathbf{u}(t) = \mathbf{u}_N(t) + \boldsymbol{\delta}\mathbf{u}(t), \quad \mathbf{y}(t) = \mathbf{y}_N(t) + \boldsymbol{\delta}\mathbf{y}(t)$$

- A Taylor series expansion of $\mathbf{f}(t, \mathbf{x}, \mathbf{u})$ and $\mathbf{h}(t, \mathbf{x}, \mathbf{u})$ leads to

$$\boldsymbol{\delta}\dot{\mathbf{x}}(t) = F(t)\, \boldsymbol{\delta}\mathbf{x}(t) + B(t)\, \boldsymbol{\delta}\mathbf{u}(t)$$

$$\boldsymbol{\delta}\mathbf{y}(t) = H(t)\, \boldsymbol{\delta}\mathbf{x}(t) + D(t)\, \boldsymbol{\delta}\mathbf{u}(t)$$

where

$$F(t) = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\mathbf{x}_N, \mathbf{u}_N}, \quad B(t) = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\right|_{\mathbf{x}_N, \mathbf{u}_N}$$

$$H(t) = \left.\frac{\partial \mathbf{h}}{\partial \mathbf{x}}\right|_{\mathbf{x}_N, \mathbf{u}_N}, \quad D(t) = \left.\frac{\partial \mathbf{h}}{\partial \mathbf{u}}\right|_{\mathbf{x}_N, \mathbf{u}_N}$$

Nonlinear Systems (iii)

- Some remarks
  - Oftentimes an approximate analytical solution is available for the linearized system
  - However, the perturbation approach suffers from one fundamental drawback
    - For each specification of the functions, lengthy algebraic developments must be carried through to obtain only an **approximate** solution
    - In many cases the practical constraints imposed by “having but one life to give” and the desirability of constructing general-purpose algorithms make the analytical perturbation approach unattractive
  - In any given application to nonlinear problems, one must realistically face the problems of choosing suitable nominal trajectories to linearize about, and analyze the effects of errors introduced through the linearization
    - Fortunately, we will see that this can work well for many estimation problems involving nonlinear models

Example (i)

- Consider the following nonlinear differential equations

$$\dot{\alpha} = \dot{\theta} - \alpha^2\dot{\theta} - 0.09\alpha\dot{\theta} - 0.88\alpha + 0.47\alpha^2 + 3.85\alpha^3 - 0.02\theta^2$$
$$\ddot{\theta} = -0.396\dot{\theta} - 4.208\alpha - 0.47\alpha^2 - 3.564\alpha^3$$

where α is the angle of attack and θ is the pitch angle

- These equation describe the behavior when an aircraft operates at high angles of attack, in which the lift coefficient cannot be accurately represented as a linear function of angle of attack
- The state vector is given by $\mathbf{x} = \begin{bmatrix}\alpha & \theta & \dot{\theta}\end{bmatrix}^T \equiv \begin{bmatrix}x_1 & x_2 & x_3\end{bmatrix}^T$
- The linearized state matrix is given by

$$F = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ 0 & 0 & 1 \\ f_{31} & 0 & f_{33} \end{bmatrix}$$

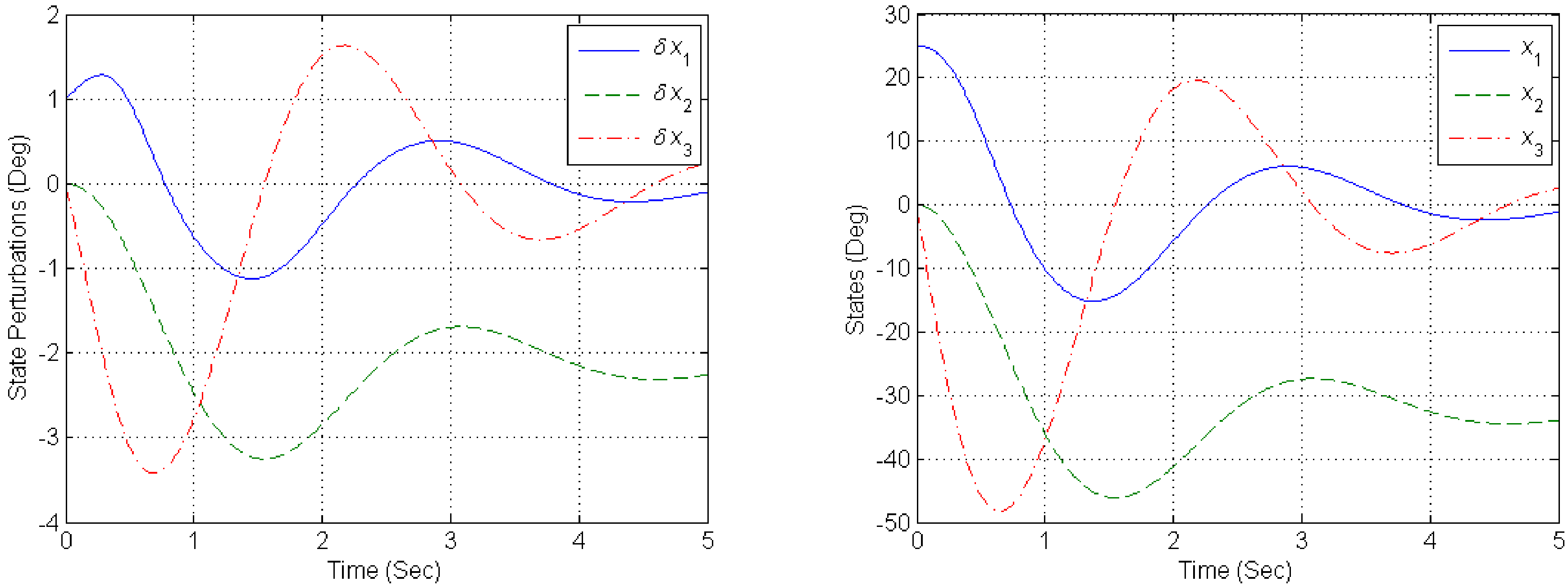
Example (ii)

where

$$f_{11} = -2x_1x_3 - 0.09x_3 - 0.88 + 0.94x_1 + 11.55x_1^2$$
$$f_{12} = -0.04x_2$$
$$f_{13} = 1 - x_1^2 - 0.09x_1$$
$$f_{31} = -4.208 - 0.94x_1 - 10.692x_1^2$$
$$f_{33} = -0.396$$

- For the actual system the initial angle of attack is 25 degrees and the pitch and pitch rate are both zero
- The nominal state quantities are found by integrating the nonlinear equations with initial conditions given by 24 degrees for the angle of attack and zero for both the pitch and pitch rate
- Then, the linearized system is integrated with initial conditions given by $\delta\mathbf{x}(t_0) = [1\times\pi/180 \quad 0 \quad 0]^T$
  - Note, the 1 degree error in angle of attack is reflected in the perturbed initial condition

Example (iii)



- These trajectories closely match the actual state trajectories
- Although the nominal trajectory typically involves the integration of the full nonlinear equations, the exercise of performing the linearization still remains useful, as will be demonstrated in the extended Kalman filter

Example (iv)

```
% Initialize Variables
m=501;dt=.01;
t=[0:dt:m*dt-dt]';
x0=[25*pi/180;0;0];
x=zeros(m,3);x(1,:)=x0';
xn=zeros(m,3);xn(1,:)=[x0(1)-1*pi/180;0;0]';
dx=zeros(m,3);dx(1,:)=[1*pi/180;0;0]';

% True and Nominal Values
c=[1;1;0.09;0.88;0.47;3.85;0.01;.396;4.208;0.47;3.564];
cn=c*1;

% Main Loop for Integration
for i=1:m-1,

 f1=dt*f8_fun(x(i,:),c);
 f2=dt*f8_fun(x(i,:)+0.5*f1',c);
 f3=dt*f8_fun(x(i,:)+0.5*f2',c);
 f4=dt*f8_fun(x(i,:)+f3',c);
 x(i+1,:)=x(i,:)+1/6*(f1'+2*f2'+2*f3'+f4');
```

Example (v)

```
f1=dt*f8_fun(xn(i,:),cn);
f2=dt*f8_fun(xn(i,:)+0.5*f1',cn);
f3=dt*f8_fun(xn(i,:)+0.5*f2',cn);
f4=dt*f8_fun(xn(i,:)+f3',cn);
xn(i+1,:)=xn(i,:)+1/6*(f1'+2*f2'+2*f3'+f4');

x1=xn(i,1);x2=xn(i,2);x3=xn(i,3);
a11=-2*c(2)*x1*x3-c(3)*x3-c(4)+2*c(5)*x1+3*c(6)*x1*x1;
a12=-2*c(7)*x2;
a13=c(1)-c(2)*x1^2-c(3)*x1;
a21=0;a22=0;a23=1;
a31=-c(9)-2*c(10)*x1-3*c(11)*x1^2;
a32=0;
a33=-c(8);
a=[a11 a12 a13;a21 a22 a23;a31 a32 a33];

phi=c2d(a,zeros(3,1),dt);
dx(i+1,:)=(phi*dx(i,:)')';

end
```

Example (vi)

```
% Plot Results
plot(t,dx(:,1)*180/pi,t,dx(:,2)*180/pi,'--',t,dx(:,3)*180/pi,'-.');
set(gca,'Fontsize',12);
ylabel('State Perturbations (Deg)')
xlabel('Time (Sec)');
ax=legend('{{\it \delta} {\it x}_1}','{{\it \delta} {\it x}_2}','{{\it \delta} {\it x}_3}');
leg=findobj(ax,'type','text');
set(leg,'FontUnits','points','fontsize',12);grid

disp(' Press any key to continue')
pause

plot(t,x(:,1)*180/pi,t,x(:,2)*180/pi,'--',t,x(:,3)*180/pi,'-.');
set(gca,'Fontsize',12);
ylabel('States (Deg)')
xlabel('Time (Sec)');
ax=legend('{{\it x}_1}','{{\it x}_2}','{{\it x}_3}');
leg=findobj(ax,'type','text');
set(leg,'FontUnits','points','fontsize',12);grid
```

Example (vii)

```
function f=f8_fun(x,c)

% Main Function Routine for f8 Aircraft

% Coefficients
c1=c(1);c2=c(2);c3=c(3);c4=c(4);c5=c(5);c6=c(6);c7=c(7);
c8=c(8);c9=c(9);c10=c(10);c11=c(11);

% Function
f=zeros(3,1);
f(1)=c1*x(3)-c2*x(1)^2*x(3)-c3*x(1)*x(3)-c4*x(1)+c5*x(1)^2+c6*x(1)^3-c7*x(2)^2;
f(2)=x(3);
f(3)=-c8*x(3)-c9*x(1)-c10*x(1)^2-c11*x(1)^3;
```

Stability (i)

- For linear systems stability found through eigenvalues of F (continuous-time) and Φ (discrete-time)
- Nonlinear systems are much more difficult to assess
  - Many theories exist, but we will focus on Lyapunov's direct method, which can provide global stability
  - This concept is closely related to the energy of a system, which is a scalar function
  - The scalar function must in general be continuous and have continuous derivatives with respect to all components of the state vector
  - Lyapunov showed that if the total energy of a system is dissipated, then the state is confined to a volume bounded by a surface of constant energy, so that the system must eventually settle to an equilibrium point
  - This concept is valid for both linear and nonlinear systems

Stability (ii)

- A system is asymptotically stable system if a function V satisfies

Can be used to show global stability but not instability.

  - $V(\mathbf{x}_e) = 0$
  - $V(\mathbf{x}) > 0$ for $\mathbf{x} \neq \mathbf{x}_e$
  - $\dot{V}(\mathbf{x}) < 0$

  where $\mathbf{x}_e$ is an equilibrium point
- For example, consider a linear system with $V(\mathbf{x}) = \mathbf{x}^T P \mathbf{x}$, where is constant and positive definite, i.e. $P > 0$
- Take the derivative and substitute $\dot{\mathbf{x}} = F\mathbf{x}$

$$\begin{aligned} \dot{V}(\mathbf{x}) &= \dot{\mathbf{x}}^T P \mathbf{x} + \mathbf{x}^T P \dot{\mathbf{x}} \\ &= \mathbf{x}^T (F^T P + PF)\mathbf{x} \end{aligned}$$

- Define the following Lyapunov equation

$$F^T P + PF = -Q$$

  - If $Q > 0$ then the system is stable (positive definite)

Example (i)

- Given the following system

$$F = \begin{bmatrix} -a & b \\ -b & -a \end{bmatrix}$$

  - Find conditions on a and b to ensure the system is stable
  - Setting $Q = I$, Lyapunov's equation leads to

$$-a\,p_{11} - b\,p_{12} - a\,p_{11} - b\,p_{12} = -1$$
$$-a\,p_{12} - b\,p_{22} - a\,p_{12} + b\,p_{11} = 0$$
$$-a\,p_{22} + b\,p_{12} - a\,p_{22} + b\,p_{12} = -1$$

  where p_{11}, p_{22} and p_{12} are elements of the P matrix

  - Solving these equations gives

$$P = \begin{bmatrix} \frac{1}{2a} & 0 \\ 0 & \frac{1}{2a} \end{bmatrix}$$

The matrix P is positive definite when $a > 0$, which gives the range for stability of the overall system matrix

  - Easier to solve then looking at the characteristic equation

Don't have to compute eigenvalues!

Example (ii)

- Given the following system

$$\dot{x}_1 = -x_1 + g(x_2), \quad \dot{x}_2 = -x_2 + h(x_1)$$

  where $|g(u)| \leq |u|/2$ and $|h(u)| \leq |u|/2$

- Try the following candidate Lyapunov function $V = (x_1^2 + x_2^2)/2$

$$\begin{aligned} \dot{V} &= x_1\dot{x}_1 + x_2\dot{x}_2 \\ &= -x_1^2 - x_2^2 + x_1 g(x_2) + x_2 h(x_1) \\ &\leq -x_1^2 - x_2^2 + |x_1 x_2| \\ &\leq -(x_1^2 + x_2^2)/2 \\ &< 0, \quad \text{for all } x_1 \text{ and } x_2 \neq 0 \end{aligned}$$

  where the following was used

$$|x_1 x_2| \leq (x_1^2 + x_2^2)/2 \quad \text{derived from } (|x_1| - |x_2|)^2 \geq 0$$

- Thus, the system is asymptotically stable

Discrete-Time Stability

- A system is asymptotically stable system if a function V satisfies

$$\bullet\ V(\mathbf{x}_e) = 0$$

$$\bullet\ V(\mathbf{x}_k) > 0 \text{ for } \mathbf{x}_k \neq \mathbf{x}_e$$

$$\bullet\ \Delta V(\mathbf{x}_k) = V[\mathbf{f}(\mathbf{x}_k)] - V(\mathbf{x}_k) \leq 0$$

  where $\mathbf{x}_e$ is an equilibrium point
- For example, consider a linear system with $V(\mathbf{x}) = \mathbf{x}^T P \mathbf{x}$, where is constant and positive definite, i.e. $P > 0$
- Take the increment with substitute $\mathbf{x}_{k+1} = \Phi \mathbf{x}_k$

$$\begin{aligned}\Delta V(\mathbf{x}_k) &= V(\Phi \mathbf{x}_k) - V(\mathbf{x}_k) \\ &= \mathbf{x}_k^T (\Phi^T P \Phi - P) \mathbf{x}_k\end{aligned}$$

- Define the following discrete-time Lyapunov equation

$$\Phi^T P \Phi - P = -Q$$

  - If $Q > 0$ then the system is stable Positive definite

Recursive Least Squares

- Consider the following recursive least squares equations

$$\hat{\mathbf{x}}_{k+1} = [I - K_{k+1} H_{k+1}] \hat{\mathbf{x}}_k$$

$$K_{k+1} = P_k H_{k+1}^T \left[H_{k+1} P_k H_{k+1}^T + W_{k+1}^{-1} \right]^{-1}$$

$$P_{k+1} = [I - K_{k+1} H_{k+1}] P_k$$

Analytically stable

  - Try the following candidate Lyapunov function

$$V(\hat{\mathbf{x}}_k) = \hat{\mathbf{x}}_k^T P^{-1} \hat{\mathbf{x}}_k$$

  - The increment is given by

$$\Delta V(\hat{\mathbf{x}}_k) = \hat{\mathbf{x}}_{k+1}^T P_{k+1}^{-1} \hat{\mathbf{x}}_{k+1} - \hat{\mathbf{x}}_k^T P_k^{-1} \hat{\mathbf{x}}_k$$

  - Substituting the least squares equations gives

$$\Delta V(\hat{\mathbf{x}}_k) = -\hat{\mathbf{x}}_k^T H_{k+1}^T [H_{k+1} P_k H_{k+1}^T + W_{k+1}^{-1}]^{-1} H_{k+1} \hat{\mathbf{x}}_k$$

  - This is stable if $W_{k+1} > 0$ and $m \leq n$
  - These conditions are true in general, so the recursive least square process will always converge