ECE 68000: MODERN AUTOMATIC CONTROL

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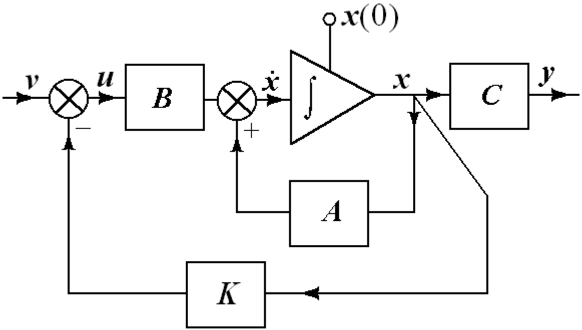
The Pole Placement Problem

Linear State-Feedback Control—Outline

- The pole allocation (shifting, placement) problem
- Pole placement problem for single-input plants
- Controller design for the system in the controller companion form
- The Ackermann's formula for pole placement
- Linear controller for nonlinear plants
- Example

Linear state-feedback

- The system: $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$
- The controller: $\boldsymbol{u} = -\boldsymbol{K}\boldsymbol{x} + \boldsymbol{v}$ where $\boldsymbol{K} \in \mathbb{R}^{m \times n}$ is a constant matrix and the vector $\boldsymbol{v}$ is an external input signal



The pole placement problem

- The closed-loop system

$$\dot{\boldsymbol{x}} = (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K})\,\boldsymbol{x} + \boldsymbol{B}\boldsymbol{v}$$

- The closed-loop poles are the roots of

$$\det\left(s\boldsymbol{I}_n - \boldsymbol{A} + \boldsymbol{B}\boldsymbol{K}\right) = 0$$

- The linear state-feedback control law design—select the gains

$$k_{ij}, \quad i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, n,$$

  so that the roots of the closed-loop characteristic equation $\det\left(s\boldsymbol{I}_n - \boldsymbol{A} + \boldsymbol{B}\boldsymbol{K}\right) = 0$ are in desirable locations in the complex plane

The pole placement problem—contd.

- The linear state-feedback control law design consists of selecting the gains

$$k_{ij}, \quad i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, n,$$

  so that the roots of the closed-loop characteristic equation $\det\left(s\boldsymbol{I}_n - \boldsymbol{A} + \boldsymbol{BK}\right) = 0$ are in desirable locations in the complex plane

- A designer selects the desired closed-loop poles:

$$s_1, s_2, \ldots, s_n$$

Closed-loop pole selection

- The desired closed-loop poles can be real or complex
- If they are complex, then they must come in complex conjugate pairs
- This is because we use only real gains k_{ij}
- Having selected the desired closed-loop poles, form the desired closed-loop characteristic polynomial (CLCP)

$$\begin{aligned} \alpha_c(s) &= (s - s_1)(s - s_2) \cdots (s - s_n) \\ &= s^n + \alpha_{n-1} s^{n-1} + \cdots + \alpha_1 s + \alpha_0 \end{aligned}$$

The closed-loop characteristic polynomial (CLCP)

- Our goal is to select a feedback matrix $\boldsymbol{K}$ such that

$$\begin{aligned} & \det\left(s\boldsymbol{I}_n - \boldsymbol{A} + \boldsymbol{B}\boldsymbol{K}\right) = \alpha_c(s) \\ = \quad & s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \alpha_0 \end{aligned}$$

- We first discuss the pole placement problem for the single-input plants
- In this case $\boldsymbol{K} = \boldsymbol{k} \in \mathbb{R}^{1\times n}$

Controller form

- The plant
$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}u(t),$$
where the pair $(\boldsymbol{A}, \boldsymbol{b})$ is assumed to be reachable, or equivalently in the CT case, controllable
- This means that
$$\operatorname{rank} \begin{bmatrix} \boldsymbol{b} & \boldsymbol{A}\boldsymbol{b} & \cdots & \boldsymbol{A}^{n-1}\boldsymbol{b} \end{bmatrix} = n.$$
- Select the last row of the inverse of the controllability matrix—call it $\boldsymbol{q}_1$

The transformation matrix

- Let $\boldsymbol{q}_1$ be the last row of the inverse of the controllability matrix
- Form the matrix

$$\boldsymbol{T} = \begin{bmatrix} \boldsymbol{q}_1 \\ \boldsymbol{q}_1 \boldsymbol{A} \\ \vdots \\ \boldsymbol{q}_1 \boldsymbol{A}^{n-1} \end{bmatrix}$$

$\boldsymbol{T}$ is non-singular

- Indeed, because

$$
\begin{aligned}
\boldsymbol{T}\begin{bmatrix} \boldsymbol{b} & \cdots & \boldsymbol{A}^{n-1}\boldsymbol{b} \end{bmatrix} &= \begin{bmatrix} \boldsymbol{q}_1 \\ \boldsymbol{q}_1\boldsymbol{A} \\ \vdots \\ \boldsymbol{q}_1\boldsymbol{A}^{n-1} \end{bmatrix}\begin{bmatrix} \boldsymbol{b} & \cdots & \boldsymbol{A}^{n-1}\boldsymbol{b} \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & x \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & x & \cdots & x & x \end{bmatrix}
\end{aligned}
$$

- The symbol x denotes "don't care", unspecified, scalars in our present discussion

State-space transformation

- The state variable transformation, $\tilde{\boldsymbol{x}} = \boldsymbol{T}\boldsymbol{x}$
- Hence, $\dot{\tilde{\boldsymbol{x}}} = \boldsymbol{T}\dot{\boldsymbol{x}}$ because $\boldsymbol{T}$ is a constant matrix
- Pre-multiply $\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}u(t)$ by $\boldsymbol{T}$ to obtain

$$\boldsymbol{T}\dot{\boldsymbol{x}}(t) = \boldsymbol{T}\boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{T}\boldsymbol{b}u(t)$$

- Take into account that $\dot{\tilde{\boldsymbol{x}}} = \boldsymbol{T}\dot{\boldsymbol{x}}$ and that $\boldsymbol{x} = \boldsymbol{T}^{-1}\tilde{\boldsymbol{x}}$
- In the new coordinates, the system model is

$$\begin{aligned}\dot{\tilde{\boldsymbol{x}}}(t) &= \boldsymbol{T}\boldsymbol{A}\boldsymbol{T}^{-1}\tilde{\boldsymbol{x}}(t) + \boldsymbol{T}\boldsymbol{b}u(t) \\ &= \tilde{\boldsymbol{A}}\tilde{\boldsymbol{x}}(t) + \tilde{\boldsymbol{b}}u(t)\end{aligned}$$

The system in new coordinates

- The matrices $\tilde{\boldsymbol{A}}$ and $\tilde{\boldsymbol{b}}$ have particular structures
- Because $\boldsymbol{q}_1$ is the last row of the inverse of the controllability matrix, we have

$$\boldsymbol{q}_1\boldsymbol{b} = \boldsymbol{q}_1\boldsymbol{A}\boldsymbol{b} = \cdots = \boldsymbol{q}_1\boldsymbol{A}^{n-2}\boldsymbol{b} = 0$$

  and

$$\boldsymbol{q}_1\boldsymbol{A}^{n-1}\boldsymbol{b} = 1$$

- Hence,

$$\tilde{\boldsymbol{b}} = \boldsymbol{T}\boldsymbol{b} = \begin{bmatrix} \boldsymbol{q}_1\boldsymbol{b} \\ \boldsymbol{q}_1\boldsymbol{A}\boldsymbol{b} \\ \vdots \\ \boldsymbol{q}_1\boldsymbol{A}^{n-2}\boldsymbol{b} \\ \boldsymbol{q}_1\boldsymbol{A}^{n-1}\boldsymbol{b} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

The structure of $\tilde{\boldsymbol{A}}$

- Represent $\boldsymbol{T}\boldsymbol{A}\boldsymbol{T}^{-1} = \tilde{\boldsymbol{A}}$ as

$$\boldsymbol{T}\boldsymbol{A} = \tilde{\boldsymbol{A}}\boldsymbol{T}$$

- The left-hand side

$$\boldsymbol{T}\boldsymbol{A} = \begin{bmatrix} \boldsymbol{q}_1\boldsymbol{A} \\ \boldsymbol{q}_1\boldsymbol{A}^2 \\ \vdots \\ \boldsymbol{q}_1\boldsymbol{A}^{n-1} \\ \boldsymbol{q}_1\boldsymbol{A}^n \end{bmatrix}$$

- By the Cayley-Hamilton theorem

$$\boldsymbol{A}^n = -a_0\boldsymbol{I}_n - a_1\boldsymbol{A} - \cdots - a_{n-1}\boldsymbol{A}^{n-1},$$

and hence

$$\boldsymbol{q}_1\boldsymbol{A}^n = -a_0\boldsymbol{q}_1 - a_1\boldsymbol{q}_1\boldsymbol{A} - \cdots - a_{n-1}\boldsymbol{q}_1\boldsymbol{A}^{n-1}$$

The structure of $\tilde{\boldsymbol{A}}$—contd.

- Compare both sides of $\boldsymbol{TA} = \tilde{\boldsymbol{A}}\boldsymbol{T}$, and take into account the Cayley-Hamilton theorem to obtain

$$\tilde{\boldsymbol{A}} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix}$$

Controller form

- We say that the pair

$$(\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{b}}) = \left(\begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \right)$$

  is in the controller form
- This form is also labeled in the literature as the controller canonical form. We use the shorter label
- The coefficients of the characteristic polynomial of $\boldsymbol{A}$ are immediately apparent by inspecting the last row of $\tilde{\boldsymbol{A}}$

Example

- System model:

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} u$$

- The controllability matrix of the pair $(\boldsymbol{A}, \boldsymbol{b})$

$$\begin{bmatrix} \boldsymbol{b} & \boldsymbol{A}\boldsymbol{b} & \boldsymbol{A}^2\boldsymbol{b} & \boldsymbol{A}^3\boldsymbol{b} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

- The pair $(\boldsymbol{A}, \boldsymbol{b})$ is controllable because the controllability matrix is non-singular
- The last row of the inverse of the controllability matrix to be $\boldsymbol{q}_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$

Transforming $\boldsymbol{A}$ into the new coordinates

- The transformation matrix

$$\boldsymbol{T} = \begin{bmatrix} \boldsymbol{q}_1 \\ \boldsymbol{q}_1\boldsymbol{A} \\ \boldsymbol{q}_1\boldsymbol{A}^2 \\ \boldsymbol{q}_1\boldsymbol{A}^3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

- $\boldsymbol{A}$ in the new coordinates

$$\begin{aligned} \boldsymbol{TAT}^{-1} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -3 & 0 \end{bmatrix} \end{aligned}$$

The input matrix $\boldsymbol{b}$ in the new coordinates

$$\boldsymbol{T}\boldsymbol{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Computing the controller gain $\boldsymbol{k}$ for single-input systems

- Our goal: Construct feedback matrix $\boldsymbol{k}$ such that

$$\det\left(s\boldsymbol{I}_n - \boldsymbol{A} + \boldsymbol{B}\boldsymbol{k}\right) = s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \alpha_0.$$

- This problem is also referred to as the pole placement problem
- For the single-input plants $\boldsymbol{K} = \boldsymbol{k} \in \mathbb{R}^{1\times n}$
- The solution to the problem is easily obtained if the pair $(\boldsymbol{A}, \boldsymbol{b})$ is already in the controller companion form

Preparing to compute $\boldsymbol{k}$ for the plant model in the controller form

- We will be computing the matrix $\boldsymbol{A} - \boldsymbol{bk}$
- Note that

$$
\begin{aligned}
\boldsymbol{bk} &= \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 & \cdots & k_{n-1} & k_n \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ k_1 & k_2 & \cdots & k_{n-1} & k_n \end{bmatrix}
\end{aligned}
$$

Computing $\boldsymbol{k}$ for the plant model in controller form

- If the plant model in the controller form, then

$$\boldsymbol{A} - \boldsymbol{b}\boldsymbol{k} = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -a_0 - k_1 & -a_1 - k_2 & \cdots & -a_{n-2} - k_{n-1} & -a_{n-1} - k_n \end{bmatrix}$$

- Hence, the desired gains are

$$\begin{aligned} k_1 &= \alpha_0 - a_0, \\ k_2 &= \alpha_1 - a_1, \\ &\vdots \\ k_n &= \alpha_{n-1} - a_{n-1} \end{aligned}$$

Computing the controller gain $\boldsymbol{k}$ for the system not in the controller form

- If the pair $(\boldsymbol{A}, \boldsymbol{b})$ is not in the controller form, transform it into the controller form, then compute the gain vector $\tilde{\boldsymbol{k}}$ such that

$$\det\left(s\boldsymbol{I}_n - \tilde{\boldsymbol{A}} + \tilde{\boldsymbol{b}}\tilde{\boldsymbol{k}}\right) = s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \alpha_0$$

- Thus,

$$\tilde{\boldsymbol{k}} = \left[\begin{array}{cccc} \alpha_0 - a_0 & \alpha_1 - a_1 & \cdots & \alpha_{n-1} - a_{n-1} \end{array}\right].$$

- Then,

$$\boldsymbol{k} = \tilde{\boldsymbol{k}}\boldsymbol{T},$$

  where $\boldsymbol{T}$ is the transformation that brings the pair $(\boldsymbol{A}, \boldsymbol{b})$ into the controller form

Computing $\boldsymbol{k}$ in one shot

- Represent the formula for the gain matrix in an alternative way
- Note that

$$
\begin{aligned}
\tilde{\boldsymbol{k}}\boldsymbol{T} &= \begin{bmatrix} \alpha_0 - a_0 & \alpha_1 - a_1 & \cdots & \alpha_{n-1} - a_{n-1} \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_1 \\ \boldsymbol{q}_1\boldsymbol{A} \\ \vdots \\ \boldsymbol{q}_1\boldsymbol{A}^{n-1} \end{bmatrix} \\
&= \boldsymbol{q}_1\left(\alpha_0\boldsymbol{I}_n + \alpha_1\boldsymbol{A} + \cdots + \alpha_{n-1}\boldsymbol{A}^{n-1}\right) \\
&\quad -\boldsymbol{q}_1\left(a_0\boldsymbol{I}_n + a_1\boldsymbol{A} + \cdots + a_{n-1}\boldsymbol{A}^{n-1}\right)
\end{aligned}
$$

Computing $\boldsymbol{k}$ in one shot—contd.

- By the Cayley-Hamilton theorem,

$$\boldsymbol{A}^n = -\left(a_0\boldsymbol{I}_n + a_1\boldsymbol{A} + \cdots + a_{n-1}\boldsymbol{A}^{n-1}\right)$$

- Hence,

$$\boldsymbol{k} = \boldsymbol{q}_1\alpha_c\left(\boldsymbol{A}\right)$$

- The above expression for the gain row vector was proposed by Ackermann in 1972, and is now referred to as the Ackermann's formula for pole placement

Example

- Dynamical system

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u$$

- Use the Ackermann's formula to design a state-feedback controller, $u = -\boldsymbol{k}\boldsymbol{x}$, such that the closed-loop poles are located at $\{-1, -2\}$
- Form the controllability matrix of the pair $(\boldsymbol{A}, \boldsymbol{b})$ and then find the last row of its inverse denoted $\boldsymbol{q}_1$
- The controllability matrix is

$$\begin{bmatrix} \boldsymbol{b} & \boldsymbol{A}\boldsymbol{b} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$$

Computing $\boldsymbol{k}$ using Ackermann's formula

- The controllability matrix inverse is

$$\begin{bmatrix} \boldsymbol{b} & \boldsymbol{A}\boldsymbol{b} \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}.$$

- Hence, $\boldsymbol{q}_1 = \begin{bmatrix} 1 & -2 \end{bmatrix}$
- The desired closed-loop characteristic polynomial is

$$\alpha_c(s) = (s+1)(s+2) = s^2 + 3s + 2$$

- Therefore,

$$\begin{aligned}
\boldsymbol{k} &= \boldsymbol{q}_1 \alpha_c(\boldsymbol{A}) \\
&= \boldsymbol{q}_1 \left(\boldsymbol{A}^2 + 3\boldsymbol{A} + 2\boldsymbol{I}_2 \right) \\
&= \boldsymbol{q}_1 \left(\begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix} + 3 \begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix} + 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \\
&= \boldsymbol{q}_1 \left(\begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix} + \begin{bmatrix} 3 & -3 \\ 3 & -6 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \right)
\end{aligned}$$

Applying Ackermann's formula to find the controller gain $\boldsymbol{k}$

- Continuing

$$\begin{aligned} \boldsymbol{k} &= \boldsymbol{q}_1 \alpha_c(\boldsymbol{A}) \\ &= \begin{bmatrix} 1 & -2 \end{bmatrix} \begin{bmatrix} 5 & -2 \\ 2 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \end{aligned}$$

The Pole Placement Theorem

Theorem

The pole placement problem is solvable for all choices of n desired closed-loop poles, symmetric with respect to the real axis, if and only if the given pair $(\boldsymbol{A}, \boldsymbol{B})$ is reachable

- Use MATLAB's function place to generate the gain matrix $\boldsymbol{K}$ for single-input or multi-input system
- A general proof of the pole placement theorem first published by W. M. Wonham in December 1967 in the IEEE Transactions on Automatic Control

Implementing state-feedback for nonlinear systems

- Non-linear system $\dot{\boldsymbol{x}} = f(\boldsymbol{x}, \boldsymbol{u})$ linearized about $(\boldsymbol{x}_e, \boldsymbol{u}_e)$
- Linearized model

$$\frac{d}{dt}\delta x = \boldsymbol{A}\delta\boldsymbol{x} + \boldsymbol{B}\delta\boldsymbol{u},$$

  where

$$\delta\boldsymbol{x} = \boldsymbol{x} - \boldsymbol{x}_e \quad \text{and} \quad \delta\boldsymbol{u} = \boldsymbol{u} - \boldsymbol{u}_e$$

- The state-feedback control law designed for the linearized system

$$\delta\boldsymbol{u} = -\boldsymbol{K}\delta\boldsymbol{x} + \boldsymbol{v}$$

State-feedback for nonlinear systems

- We have

$$\begin{aligned}\delta\boldsymbol{u} &= \boldsymbol{u} - \boldsymbol{u}_e \\ &= -\boldsymbol{K}\delta\boldsymbol{x} + \boldsymbol{v} \\ &= -\boldsymbol{K}(\boldsymbol{x} - \boldsymbol{x}_e) + \boldsymbol{v} \\ &= -\boldsymbol{K}\boldsymbol{x} + \boldsymbol{K}\boldsymbol{x}_e + \boldsymbol{v}\end{aligned}$$

- The controller applied to $\dot{\boldsymbol{x}} = f(\boldsymbol{x}, \boldsymbol{u})$

$$\boldsymbol{u} = -\boldsymbol{K}\boldsymbol{x} + (\boldsymbol{K}\boldsymbol{x}_e + \boldsymbol{u}_e) + \boldsymbol{v}$$