#1) $\dot{x} = \begin{pmatrix}0&0\\1&0\end{pmatrix}x + \begin{pmatrix}1\\1\end{pmatrix}u \qquad T_s = 2$

1) $\dot{x} \approx \dfrac{x[k+1]-x[k]}{T_s} \approx Ax + Bu$

$$x[k+1] = x[k] + AT_s x[k] + BT_s u[k]$$
$$= (I + AT_s)x[k] + BT_s u[k]$$
$$= \left[\begin{pmatrix}1&0\\0&1\end{pmatrix} + \begin{pmatrix}0&0\\2&0\end{pmatrix}\right]x[k] + \begin{bmatrix}2\\2\end{bmatrix}u[k]$$

$$\boxed{x[k+1] = \begin{pmatrix}1&0\\2&1\end{pmatrix}x[k] + \begin{bmatrix}2\\2\end{bmatrix}u[k]}$$

2) $\Phi = e^{AT_s} = I + AT_s + \dfrac{A^2T_s^2}{2!} + \dots$

$A^2 = \begin{pmatrix}0&0\\0&0\end{pmatrix} \quad \therefore \quad \Phi = I + AT_s$

$\Phi = \begin{pmatrix}1&0\\0&1\end{pmatrix} + \begin{pmatrix}0&0\\2&0\end{pmatrix} = \begin{pmatrix}1&0\\2&1\end{pmatrix}$

$\Gamma = \displaystyle\int_0^{T_s} e^{A\eta} B\, d\eta \qquad e^{A\eta} = \begin{pmatrix}1&0\\ \eta&1\end{pmatrix} \qquad e^{A\eta}B = \begin{pmatrix}1&0\\ \eta&1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\ n+1\end{pmatrix}$

$\Gamma = \displaystyle\int_0^{T_s} \begin{pmatrix}1\\ n+1\end{pmatrix} d\eta = \begin{pmatrix}n\\ \frac{n^2}{2}+n\end{pmatrix}\Bigg|_0^{T_s} = \begin{pmatrix}2\\4\end{pmatrix} \qquad \begin{matrix}n & 0\\ \frac{n^2}{2} & n\end{matrix} = \begin{pmatrix}2&0\\2&2\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = ($

$$\boxed{x[k+1] = \underbrace{\begin{pmatrix}1&0\\2&1\end{pmatrix}}_{\Phi}x[k] + \underbrace{\begin{pmatrix}2\\4\end{pmatrix}}_{\Gamma}u[k]}$$

H2) $Q = \begin{pmatrix} 2 & 2 & 1 & 1 \\ 2 & 3 & 0 & 1 \\ 1 & 0 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$ $\quad C^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \frac{1}{2-1}$

$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

1) $\Delta 22 = A - BC^{-1}B^T = \begin{pmatrix} 2 & 2 \\ 2 & 3 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

$= \begin{pmatrix} 2 & 2 \\ 2 & 3 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 3 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$

$$\boxed{\Delta 22 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}}$$

2) $\Delta 22 = \Delta 22^T$, use sylvester criteria, $C = C^T$

$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ $\quad 1 > 0$

$\det(\Delta 22) = 1 - 1 = 0 \not> 0$

$\Delta 22$ is positive semi-definite

$C = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ $\quad 2 > 0$

$\det(C) = 2 - 1 = 1 > 0$, C is positive definite

$\therefore$ $\boxed{\text{Q is positive semi-definite as } \Delta 22 \text{ is positive semi-definite \& C is positive definite}}$

#3) $\Phi = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad \Gamma = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\quad 1 \leq y[k] \leq 2$

$C = [1 \quad 0]$ $\quad N_p = 2$

$$\Phi_a = \begin{pmatrix} \Phi & 0 \\ C\Phi & I \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \qquad \Gamma_a = \begin{bmatrix} \Gamma \\ C\Gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$C\Phi = [1 \quad 0] \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = [0 \quad 1]$$

$$C\Gamma = [1 \quad 0] \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0 \qquad C_a = [0 \quad 0 \quad 1]$$

$$\begin{bmatrix} -Z \\ Z \end{bmatrix} \Delta U \leq \begin{bmatrix} -y^{min} + W x_a[k] \\ y^{max} - W x_a[k] \end{bmatrix}$$

$$Z = \begin{pmatrix} C_a\Gamma_a & 0 \\ C_a\Phi_a\Gamma_a & C_a\Gamma_a \end{pmatrix} = \qquad C_a\Gamma_a = [0 \quad 0 \quad 1] \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 0$$

$$C_a\Phi_a\Gamma_a = [0 \quad 0 \quad 1] \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = (0 \quad 1 \quad 1) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 1$$

$$Z = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\Phi_a^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$C_a\Phi_a^2 = [0 \quad 0 \quad 1] \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} = [0 \quad 1 \quad 1]$$

$$W = \begin{bmatrix} C_a\Phi_a \\ C_a\Phi_a^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\boxed{\begin{gathered} \begin{pmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} \Delta U \leq \begin{bmatrix} -1 + \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} x_a[k] \\ 2 - \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} x_a[k] \end{bmatrix} \qquad x_a = \begin{pmatrix} x_{a1} \\ x_{a2} \\ x_{a3} \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} \Delta U \leq \begin{bmatrix} -1 + x_{a2} + x_{a3} \\ -1 + x_a + x_{a3} \\ 2 - x_{a2} - x_{a3} \\ 2 - x_{a2} - x_{a3} \end{bmatrix} \end{gathered}}$$

#4) $x[k+1] = x[k] + 2u[k]$ $\quad x[0] = 3$ $\quad x[3] = 9$

$J = \frac{1}{2} \sum_{k=0}^{2} u[k]^2$

$x[3] = 9 = x[2] + 2u[2]$

$x[2] = x[1] + 2u[1]$

$x[1] = x[0] + 2u[0]$ $\quad \therefore \; x[2] = x[0] + 2u[0] + 2u[1]$

$x[3] = x[0] + 2u[0] + 2u[1] + 2u[2] = 9$

$\therefore \; 9 = 3 + 2u[0] + 2u[1] + 2u[2]$

$$\therefore \; \underbrace{\begin{bmatrix} 2 & 2 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} u[0] \\ u[1] \\ u[2] \end{bmatrix}}_{u} = \underbrace{6}_{b}$$

$\lambda = 1\times1$
$A = 1\times3$
$u = 3\times1$

Lagrangian: $L = \frac{1}{2} u^T u + \lambda^T (Au - b)$

$\frac{\partial L}{\partial u} = u^T + \lambda^T A = 0$ $\qquad \underset{1\times1 \;\; 1\times3}{u^T = (-\lambda^T A)} = \underset{3\times1 \;\; 1\times1}{-A^T \lambda}$

$\frac{\partial L}{\partial \lambda} = \underset{1\times3 \, 3\times1}{Au - b = 0}$ $\qquad u = A^{-1} b$

$A^{-T} b = -A\lambda \;\Rightarrow\; \lambda = -(AA^T)^{-1}$

$\therefore \; u = A(AA^T)^{-1} b$

$u = A(AA^T)^{-1} b = [2 \; 2 \; 2] \left([2 \; 2 \; 2] \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \right)^{-1} 6$

$u = [2 \; 2 \; 2] \left(\frac{1}{12} \right) 6 = [1 \; 1 \; 1]$

$$\boxed{u = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}$$

#5) $J = \sum_{k=0}^{1} 2x[k]^2 + u[k]^2$ $\quad x[0] = 5$

$x[k+1] = 4x[k] + 3u[k]$

$J^*(2) = 0$

$J^*(1) = 2x[1]^2 + u[1]^2$

$\frac{\partial J^*(1)}{\partial u[1]} = 2u[1] = 0$

$(20 + 3u)(20 + 3u) = 400 + 9u^2 +$

$J^*(1) = 2x[1]^2$

$x[1] = 4x[0] + 3u[0] = 20 + 3u[0]$

$J^*(1) = 2(20 + 3u[0])^2 = 2(400 + 9u[0]^2 + 120u[0])$

$J^*(1) = 800 + 18u[0]^2 + 240u[0]$

$J^*(0) = 2x[0]^2 + u[0]^2 + J^*(1)$

$= 50 + u[0]^2 + 800 + 18u[0]^2 + 240u[0]$

$\frac{\partial J^*(0)}{\partial u[0]} = 0 = 2u[0] + 36u[0] + 240$

$$\boxed{\begin{aligned} u[0] &= \frac{-240}{56} \\ u[1] &= 0 \end{aligned}}$$

#6) $(X_1 - 2)^2 + (X_2 - 1)^2$

$X_2 - X_1^2 \geq 0 \quad \Rightarrow \quad 0 \geq -X_2 + X_1^2$

$2 - X_1 - X_2 \geq 0 \quad\quad 0 \geq X_2 + X_1 - 2$

$X_1 \geq 0 \quad\quad 0 \geq -X_1$

KKT: $\mu \geq 0$

$\nabla f + [\nabla g_1 \quad \nabla g_p] \mu = 0$

$\mu^T g(x) = 0$

Lagrangian: $(X_1 - 2)^2 + (X_2 - 1)^2 + \mu^T \begin{pmatrix} X_2 - X_1^2 \\ 2 - X - X_2 \\ X_1 \geq 0 \end{pmatrix}$

$\nabla f = [2(X_1 - 2) \quad 2(X_2 - 1)]$

$\nabla^2 f = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ ← Hessian positive definite, x^* is Minimum

$\mu \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix} = 0$

$\nabla g = \begin{bmatrix} -2X_1 & -1 \\ 1 & 1 \\ -1 & 0 \end{bmatrix} \quad\quad \nabla f \; [2(X_1 - 2) \quad 2(X_2 - 1)]$

$2(X_1 - 2) + \mu_1 2X_1 - \mu_2 X_2$

RAN OUT of TIME