ECE 68000: MODERN AUTOMATIC CONTROL

Professor Stan Żak

Stability in the sense of Lyapunov (isL): A Tool for Analysis and Design of Dynamical Systems

A. M. Lyapunov's (1857--1918) Thesis

INT. J. CONTROL, 1992, VOL. 55, NO. 3, 529

ОБЩАЯ ЗАДАЧА

ОБЪ

УСТОЙЧИВОСТИ ДВИЖЕНІЯ.

РАЗСУЖДЕНІЕ

А. ЛЯПУНОВА.

Lyapunov's Thesis

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1892.

Lyapunov's Thesis Translated

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The general problem of the stability of motion

A. M. LYAPUNOV

Translated from Russian into French by Édouard Davaux, Marine Engineer at Toulon.†
Translated from French into English by A. T. Fuller.‡

Preface

In this work some methods are expounded for the resolution of questions concerning the properties of motion and, in particular, of equilibrium, which are known by the terms *stability* and *instability*.

The ordinary questions of this kind, those to which this work is devoted, lead to the study of differential equations of the form

$$\frac{dx_1}{dt} = X_1, \quad \frac{dx_2}{dt} = X_2, \quad \ldots, \quad \frac{dx_n}{dt} = X_n,$$

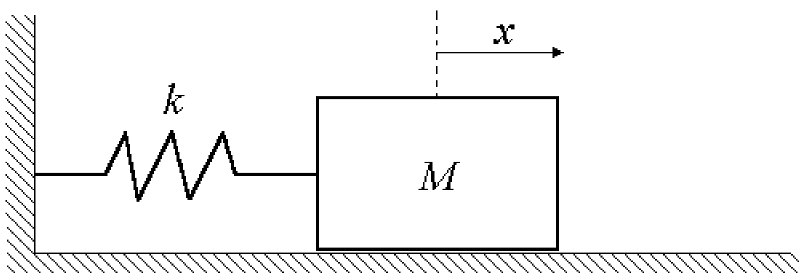
Some Details About Translation

† Mr Lyapunov has very graciously authorized the publication in French of his memoir *Obshchaya zadacha ob ustoichivosti dvizheniya* printed in 1892 by the Mathematical Society of Kharkov. The [French] translation has been reviewed and corrected by the author [Lyapunov], who has added a note based on an article which appeared in 1893 in *Communications de la Société mathématique de Kharkow*.

‡ [Comments in square brackets are by A.T.F.]

§ We have in mind the cases where there applies the known theorem of Lagrange on the maxima of the force-function [this is minus the potential energy function], relating to the stability of equilibrium; also, the cases where there applies a more general theorem of Routh on the maxima and minima of the integrals of the equations of motion, allowing the resolution of certain questions relative to the stability of motion (see *The advanced part of A Treatise on the Dynamics of a System of Rigid Bodies*, fourth edition, 1884, pp. 52, 53).

A Spring-Mass Mechanical System



x---displacement of the mass from the rest position

Modeling the Mass-Spring System

- Assume a linear mass, where *k* is the linear spring constant
- Apply Newton's law to obtain

$$M\ddot{x} + kx = 0$$

- Define state variables: x_1=x and x_2=dx/dt
- The model in state-space format:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{k}{M}x_1 \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix}$$

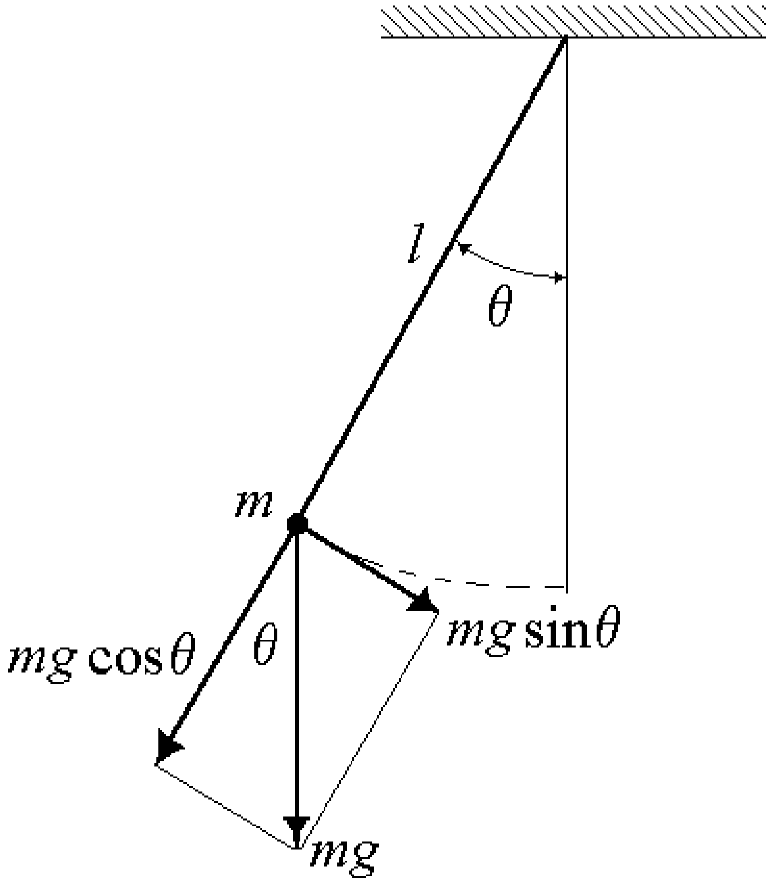
Analysis of the Spring-Mass System Model

- The spring-mass system model is linear time-invariant (LTI)
- Representing the LTI system in standard state-space format

$$\begin{aligned}
\dot{\boldsymbol{x}} &= \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \\
&= \begin{bmatrix} x_2 \\ -\frac{k}{M} x_1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 \\ -\frac{k}{M} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
&= \boldsymbol{A}\boldsymbol{x}
\end{aligned}$$

Modeling of the Simple Pendulum

The simple pendulum



The Simple Pendulum Model

- Apply Newton's second law

$$J\ddot{\theta} = -mgl\sin\theta$$

where J is the moment of inertia,

$$J = ml^2$$

- Combining gives

$$\ddot{\theta} = -\frac{g}{l}\sin\theta$$

State-Space Model of the Simple Pendulum

- Represent the second-order differential equation as an equivalent system of two first-order differential equations
- First define state variables,

  $x_1=\theta$ and $x_2=d\theta/dt$
- Use the above to obtain state–space model (nonlinear, time invariant)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{g}{l}\sin x_1 \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix}$$

Analysis of Dynamical Systems

- Sometimes not interested in detailed solutions
- Rather one seeks to characterize the system behavior---equilibrium states and their stability properties

☞ A device needed for nonlinear system analysis summarizing the system behavior, suppressing detail

Dynamical System Models

- Linear time-invariant (LTI) system model

$$\dot{x} = Ax, \quad A \in \Re^{n\times n}$$

- Nonlinear system model

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} f_1(t, x_1, \ldots, x_n) \\ f_2(t, x_1, \ldots, x_n) \\ \vdots \\ f_n(t, x_1, \ldots, x_n) \end{bmatrix}$$

- Shorthand notation of the above model

$$\dot{x} = f(t, x), \quad x \in \Re^n$$

More Notation

- System model

$$\dot{x}(t) = f\left(t, x(t)\right), \quad x(t_0) = x_0$$

- Solution

$$x(t) = x(t; t_0, x_0)$$

- Example: LTI model,

$$\dot{x} = Ax, \quad x(0) = x_0$$

- Solution of the LTI modeling equation

$$x(t) = e^{At} x_0$$

Equilibrium State

A vector x_e is an equilibrium state for a dynamical system model

$$\dot{x}(t) = f(t, x(t))$$

if once the state vector equals to x_e it remains equal to x_e for all future time.

The equilibrium state satisfies

$$f(t, x(t)) = 0$$

Formal Definition of Equilibrium

- A state $\boldsymbol{x}_e$ is called an equilibrium state of $d\boldsymbol{x}/dt=\boldsymbol{f}(t,\boldsymbol{x})$, or simply an equilibrium, at time t_0 if for all $t \geq t_{0,}$

$$\boldsymbol{f}(t, \boldsymbol{x}_e)=0$$

- Note that if $\boldsymbol{x}_e$ is an equilibrium of our system at t_0, then it is also an equilibrium for all $\tau \geq t_0$

Equilibrium States for LTI Systems

- For the time-invariant system

$$d\boldsymbol{x}/dt=\boldsymbol{f}(\boldsymbol{x})$$

a point is an equilibrium at some time τ if and only if it is an equilibrium at all times

- Equilibrium states for time-invariant systems are obtained by solving

$$\boldsymbol{f}(\boldsymbol{x})=0$$

Equilibrium State for LTI Systems

- LTI model

$$\dot{x} = f(t, x) = Ax$$

- Any equilibrium state x_e must satisfy

$$Ax_e = 0$$

- If A^{-1} exist, then we have unique equilibrium state

$$x_e = 0$$

Equilibrium States of Nonlinear Systems

- A nonlinear system may have a number of equilibrium states
- The origin, $\boldsymbol{x}=\mathbf{0}$, may or may not be an equilibrium state of a nonlinear system

Translating the Equilibrium of Interest to the Origin

- If the origin is not the equilibrium state, it is always possible to translate the origin of the coordinate system to that state
- So, no loss of generality is lost in assuming that the origin is the equilibrium state of interest

Example of a Nonlinear System with Multiple Equilibrium Points

- Nonlinear system model

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_1 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 - x_2 - x_1^2 \end{bmatrix}$$

- Two isolated equilibrium states

$$x_e^{(1)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad x_e^{(2)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Isolated Equilibrium

An equilibrium state $\boldsymbol{x}_e$ in R^n is an isolated equilibrium state if there is an r>0 such that the r-neighborhood of $\boldsymbol{x}_e$ contains no equilibrium states other than $\boldsymbol{x}_e$

Neighborhood of $\boldsymbol{x}_e$

The *r*-neighborhood of $\boldsymbol{x}_e$ can be a set of points of the form

$$B_r(\boldsymbol{x}_e) = \{\boldsymbol{x} : \|\boldsymbol{x} - \boldsymbol{x}_e\| < r\}$$

where ||.|| can be any *p*-norm on R^n

Remarks on Stability

- Stability properties characterize the system behavior if its initial state is close but not at the equilibrium point of interest
- When an initial state is close to the equilibrium, the state may remain close, or it may move away from the equilibrium

An Informal Definition of Stability

An equilibrium state is stable if whenever the initial state is near that point, the state remains near it, perhaps even tending toward the equilibrium point as time increases

Stability Intuitive Interpretation

(1)

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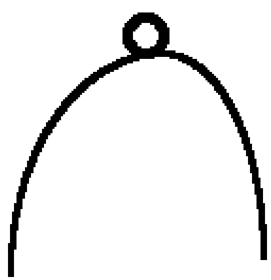

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Formal Definition of Stability

An equilibrium state x_e is stable, in the sense of Lyapunov, if for any positive scalar ε there exist a positive scalar

δ= δ(ε)

such that if

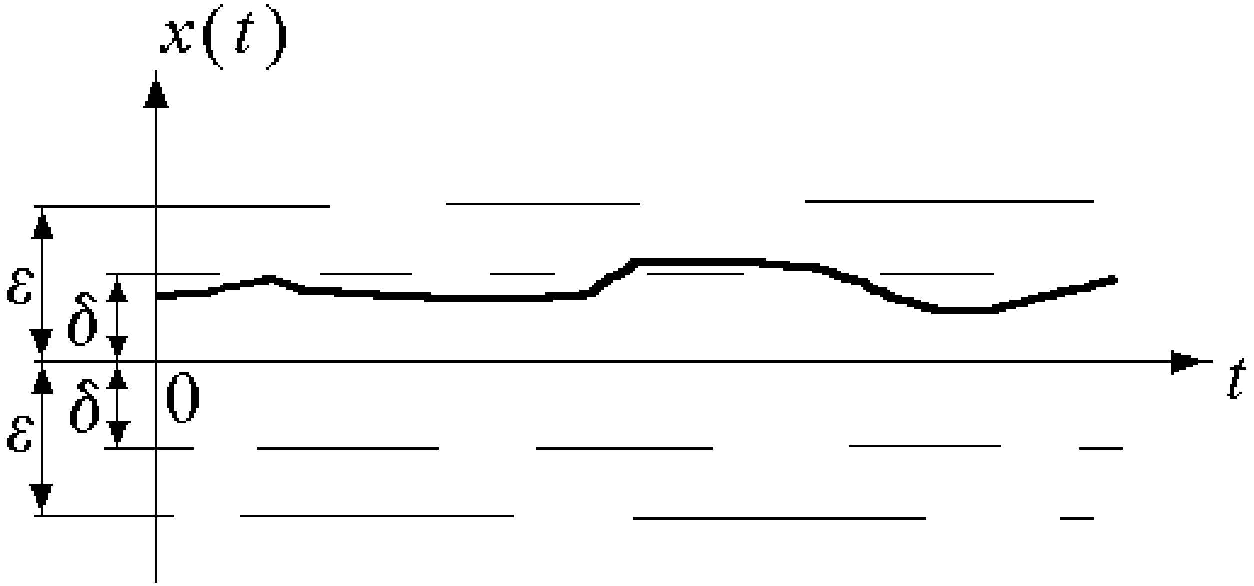
$$\left\| x(t_0) - x_e \right\| < \delta$$

then

$$\left\| x(t; t_0, x_0) - x_e \right\| < \varepsilon$$

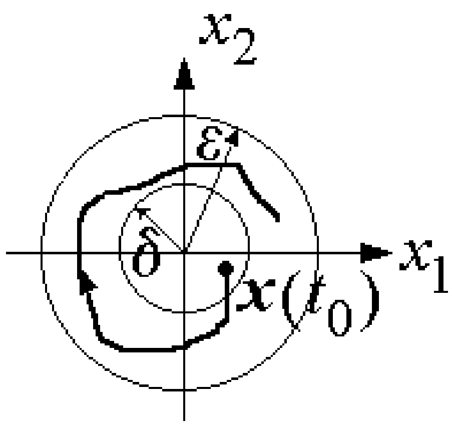
for all $t \geq t_0$

Stability Concept in 1D



(a)

Stability Concepts in 2D



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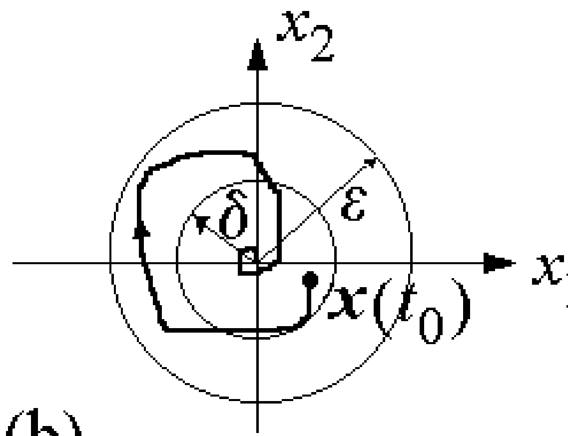

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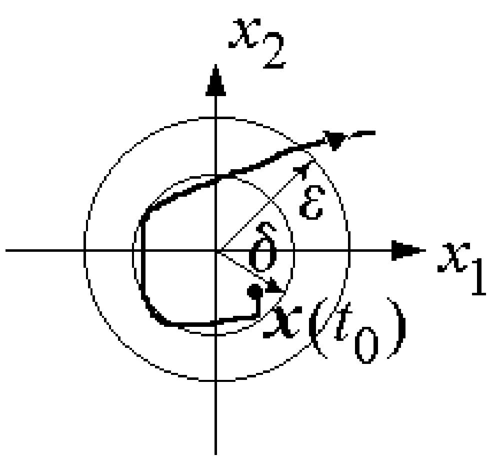

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Further Discussion of Lyapunov Stability

Think of a contest between you, the control system designer, and an adversary (nature?)---B. Friedland (ACSD, p. 43, Prentice-Hall, 1996)

Lyapunov Stability Game

- The adversary picks a region in the state space of radius ε
- You are challenged to find a region of radius δ such that if the initial state starts out inside your region, it remains in his region---if you can do this, your system is stable, in the sense of Lyapunov

Lyapunov Stability---Is It Any Good?

- Lyapunov stability is weak---it does not even imply that $\boldsymbol{x}(t)$ converges to $\boldsymbol{x}_e$ as t approaches infinity
- The states are only required to hover around the equilibrium state
- The stability condition bounds the amount of wiggling room for $\boldsymbol{x}(t)$

Asymptotic Stability i.s.L

The property of an equilibrium state of a differential equation that satisfies two conditions:

- (stability) small perturbations in the initial condition produce small perturbations in the solution;

Second Condition for Asymptotic Stability of an Equilibrium

- (attractivity of the equilibrium point) there is a domain of attraction such that whenever the initial condition belongs to this domain the solution approaches the equilibrium state at large times

Asymptotic Stability in the sense of Lyapunov (i.s.L.)

- The equilibrium state is asymptotically stable if
  - it is stable, and
  - convergent, that is,

$$x(t;t_0,x_0) \to x_e \text{ as } t \to \infty$$

Convergence Alone Does Not Guarantee Asymptotic Stability

Note: it is not sufficient that just

$$x(t;t_0,x_0) \rightarrow x_e \; as \; t \rightarrow \infty$$

for asymptotic stability. We need stability too! Why?

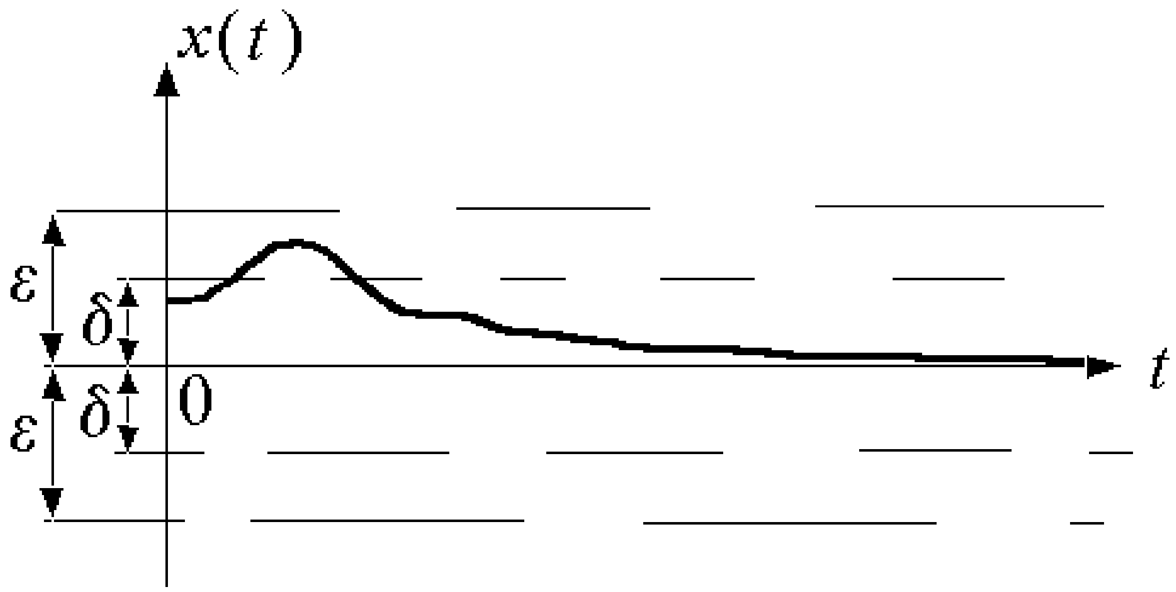
How Long to the Equilibrium?

- Asymptotic stability does not imply anything about how long it takes to converge to a prescribed neighborhood of $\boldsymbol{x}_e$
- Exponential stability provides a way to express the rate of convergence

Asymptotic Stability of Linear Systems

- An LTI system is asymptotically stable, meaning, the equilibrium state at the origin is asymptotically stable, if and only if the eigenvalues of ***A*** have negative real parts
- For LTI systems asymptotic stability is equivalent with convergence (stability condition automatically satisfied)

Asymptotic Stability in 1D



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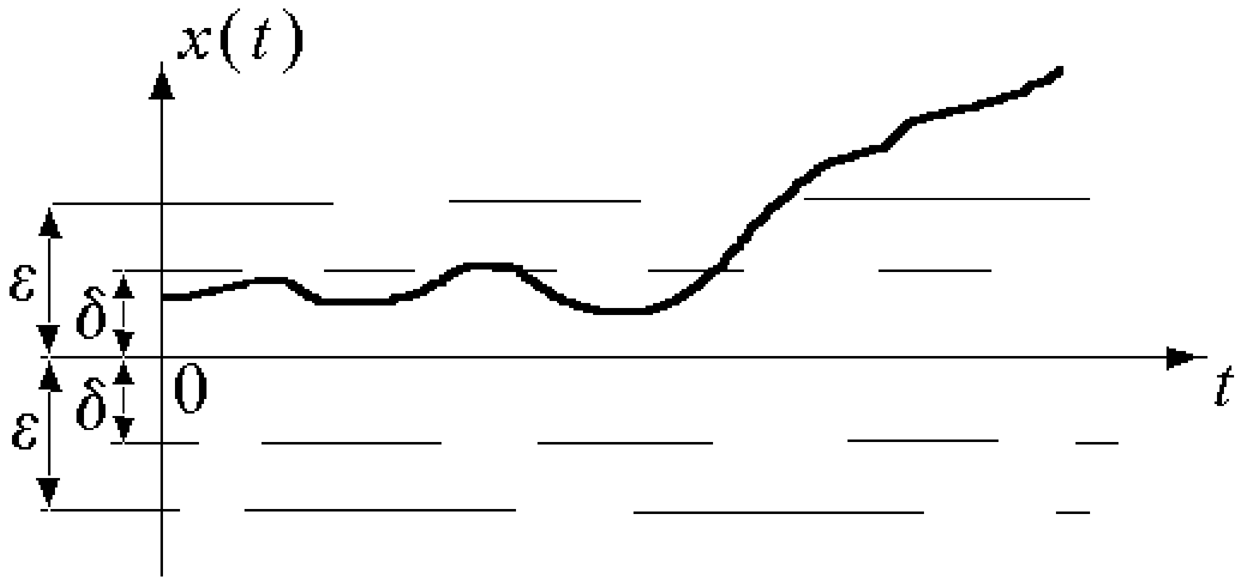
Instability in 1D



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Benefits of the Lyapunov Theory

- Solution to differential equation are not needed to infer about stability properties of equilibrium state of interest

☞ **Lyapunov's theory** useful in designing robust and adaptive controllers