ECE 68000: MODERN AUTOMATIC CONTROL

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Approximating Nonlinear Models With Linear Models: Taylor's Linearization

Overivew of Taylor's linearization

- Taylor's linearization yields models linear in $\delta\boldsymbol{x}$ and $\delta\boldsymbol{u}$
- These models, in general, are not linear in $\boldsymbol{x}$ and $\boldsymbol{u}$, but rather affine
- To illustrate the point, suppose that a nonlinear model of a plant has the form

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{G}(\boldsymbol{x})\boldsymbol{u}$$

- Let

$$\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{u}) = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{G}(\boldsymbol{x})\boldsymbol{u}.$$

- Then, we can represent $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{G}(\boldsymbol{x})\boldsymbol{u}$ as

$$\dot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x}, \boldsymbol{u})$$

Review of Taylor's linearization of systems linear in control

- Expanding $\boldsymbol{F}$ into a Taylor series around an operating pair $(\boldsymbol{x}_o, \boldsymbol{u}_o)$ yields

$$\begin{aligned}\dot{\boldsymbol{x}} &= \boldsymbol{F}(\boldsymbol{x}_o, \boldsymbol{u}_o) + \left.\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{x}}\right|_{\substack{\boldsymbol{x}=\boldsymbol{x}_o \\ \boldsymbol{u}=\boldsymbol{u}_o}} (\boldsymbol{x} - \boldsymbol{x}_o) + \left.\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{u}}\right|_{\substack{\boldsymbol{x}=\boldsymbol{x}_o \\ \boldsymbol{u}=\boldsymbol{u}_o}} (\boldsymbol{u} - \boldsymbol{u}_o) \\ &\quad + \text{higher order terms},\end{aligned}$$

  where

$$\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{u}) = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{G}(\boldsymbol{x})\boldsymbol{u} = \boldsymbol{f}(\boldsymbol{x}) + \sum_{k=1}^{m} u_k \boldsymbol{g}_k(\boldsymbol{x}),$$

  and $\boldsymbol{g}_k$ is the k-th column of $\boldsymbol{G}$
- Write expressions for the second and the third terms of the above as functions of $\boldsymbol{f}$ and $\boldsymbol{G}$

Taylor's linearization—contd.

- Let g_{ij} be the (i, j)-th element of the matrix $\boldsymbol{G}$
- Then,

$$\begin{aligned}\left.\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{x}}\right|_{\substack{\boldsymbol{x}=\boldsymbol{x}_o\\ \boldsymbol{u}=\boldsymbol{u}_o}} &= \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x}=\boldsymbol{x}_o} + \sum_{k=1}^{m} u_k \left.\frac{\partial \boldsymbol{g}_k}{\partial \boldsymbol{x}}\right|_{\substack{\boldsymbol{x}=\boldsymbol{x}_o\\ \boldsymbol{u}=\boldsymbol{u}_o}} \\ &= \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x}=\boldsymbol{x}_o} + \boldsymbol{H}(\boldsymbol{x}_o, \boldsymbol{u}_o),\end{aligned}$$

where the (i, j)-th element of the $n \times n$ matrix $\boldsymbol{H}$ is

$$\sum_{k=1}^{m} u_k \left.\frac{\partial g_{ik}(\boldsymbol{x})}{\partial x_j}\right|_{\substack{\boldsymbol{x}=\boldsymbol{x}_o\\ \boldsymbol{u}=\boldsymbol{u}_o}}$$

- We have

$$\left.\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{u}}\right|_{\substack{\boldsymbol{x}=\boldsymbol{x}_o\\ \boldsymbol{u}=\boldsymbol{u}_o}} = \boldsymbol{G}(\boldsymbol{x}_o)$$

Taylor's linearized model

- A pair $(\boldsymbol{x}_o^\top, \boldsymbol{u}_o^\top)^\top \in \mathbb{R}^{n+m}$ is an equilibrium pair if $\boldsymbol{F}(\boldsymbol{x}_o, \boldsymbol{u}_o) = \boldsymbol{0}$, that is, if at $(\boldsymbol{x}_o, \boldsymbol{u}_o)$ we have $\dot{\boldsymbol{x}} = \boldsymbol{0}$
- Let $\delta\boldsymbol{x} = \boldsymbol{x} - \boldsymbol{x}_o$ and $\delta\boldsymbol{u} = \boldsymbol{u} - \boldsymbol{u}_o$
- Note that

$$\frac{d\boldsymbol{x}_o}{dt} = \boldsymbol{0}$$

- Then, the linearized model about the equilibrium pair $(\boldsymbol{x}_o, \boldsymbol{u}_o) = (\boldsymbol{x}_e, \boldsymbol{u}_e)$ is obtained by neglecting higher order terms and observing that at the equilibrium, $\boldsymbol{F}(\boldsymbol{x}_e, \boldsymbol{u}_e) = \boldsymbol{0}$
- The linearized model has the form

$$\frac{d}{dt}\delta\boldsymbol{x} = \boldsymbol{A}\delta\boldsymbol{x} + \boldsymbol{B}\delta\boldsymbol{u},$$

where

$$\boldsymbol{A} = \left.\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{x}}\right|_{\substack{\boldsymbol{x}=\boldsymbol{x}_e \\ \boldsymbol{u}=\boldsymbol{u}_e}} \quad \text{and} \quad \boldsymbol{B} = \left.\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{u}}\right|_{\substack{\boldsymbol{x}=\boldsymbol{x}_e \\ \boldsymbol{u}=\boldsymbol{u}_e}}$$

Analysis of Taylor's linearized model

- The result of Taylor's linearization of a nonlinear model about an operating non-equilibrium pair is, in general, affine, rather than a linear model in $\boldsymbol{x}$ and $\boldsymbol{u}$
- Even if the operating pair is an equilibrium pair, Taylor linearization will not yield, in general, a model linear in $\boldsymbol{x}$ and $\boldsymbol{u}$
- Indeed, suppose that the operating point $(\boldsymbol{x}_e, \boldsymbol{u}_e)$ is an equilibrium pair, that is,

$$\boldsymbol{f}(\boldsymbol{x}_e) + \boldsymbol{G}(\boldsymbol{x}_e)\boldsymbol{u}_e = \boldsymbol{0}$$

Taylor's linearized model, in general, not linear in $\boldsymbol{x}$ and $\boldsymbol{u}$

- The resulting linearized model has the form

$$\begin{aligned}\frac{d}{dt}(\boldsymbol{x}-\boldsymbol{x}_e) &= \boldsymbol{f}(\boldsymbol{x}_e)+\boldsymbol{G}(\boldsymbol{x}_e)\boldsymbol{u}_e+\boldsymbol{A}(\boldsymbol{x}-\boldsymbol{x}_e)+\boldsymbol{B}(\boldsymbol{u}-\boldsymbol{u}_e)\\ &= \boldsymbol{A}(\boldsymbol{x}-\boldsymbol{x}_e)+\boldsymbol{B}(\boldsymbol{u}-\boldsymbol{u}_e)\end{aligned}$$

- Represent the above as

$$\dot{\boldsymbol{x}}=\boldsymbol{A}\boldsymbol{x}+\boldsymbol{B}\boldsymbol{u}-(\boldsymbol{A}\boldsymbol{x}_e+\boldsymbol{B}\boldsymbol{u}_e)$$

- The term $(\boldsymbol{A}\boldsymbol{x}_e+\boldsymbol{B}\boldsymbol{u}_e)$ does not have to be equal zero, and hence the above model is not linear, but rather affine in $\boldsymbol{x}$ and $\boldsymbol{u}$
- Note that Taylor's linearization yields a linear system in $\boldsymbol{x}$ and $\boldsymbol{u}$ if the equilibrium pair is $(\boldsymbol{x}_e^\top,\boldsymbol{u}_e^\top)=(\boldsymbol{0}^\top,\boldsymbol{0}^\top)$

Example

- Consider a subsystem of the inverted pendulum on a cart model

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{g\sin(x_1) - mlax_2^2\sin(2x_1)/2}{4l/3 - mla\cos^2(x_1)} \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{a\cos(x_1)}{4l/3 - mla\cos^2(x_1)} \end{bmatrix} u,$$

where $g = 9.8 \text{ m/sec}^2$, $m = 2$ kg, $M = 8$ kg, $a = 1/(m+M)$, $l = 0.5$ m

- Linearize the above about $x_1 = \pi/4$
- For $x_1 = \pi/4$ to be a component of an equilibrium state, we have to have $x_2 = 0$
- Hence, the equilibrium state is

$$\boldsymbol{x}_e = \begin{bmatrix} \pi/4 & 0 \end{bmatrix}^\top$$

Example—contd.

- Compute corresponding $u = u_e$
- Performing simple calculations, we get $u_e = 98$
- We obtain

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ 22.4745 & 0 \end{bmatrix}.$$

- Next,

$$\boldsymbol{B} = \begin{bmatrix} 0 \\ -0.1147 \end{bmatrix}$$

- Note that

$$\boldsymbol{A}\boldsymbol{x}_e + \boldsymbol{B}u_e = \begin{bmatrix} 0 \\ 6.4108 \end{bmatrix} \neq \mathbf{0}$$

  in

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} - (\boldsymbol{A}\boldsymbol{x}_e + \boldsymbol{B}\boldsymbol{u}_e)$$

Use MATLAB to compute A and B

```
clear all
clc
syms x1 x2 u
F=[x2
  (9.8*sin(x1)-0.1*x2^2*sin(2*x1)/2-0.1*cos(x1)*u)...
  /(2/3-0.1*cos(x1)^2)];
A=jacobian([x2;F],[x1 x2]);
A=eval(subs(A,[x1 x2 u],[pi/4 0 98]))
B=jacobian([x2;F],u);
B=eval(subs(B,[x1 x2 u],[pi/4 0 98]))
```