#1) $\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 3 + x_1 x_2 \\ -6 + 5x_1 x_2 \end{pmatrix} + \begin{pmatrix} -1 \\ x_2 \end{pmatrix} u$

a)

$$y = x_1^2 + x_2 u$$

$$\dot{x}_1 = 0 = 3 + x_1 x_2 - u_e$$
$$\dot{x}_2 = 0 = -6 + 5x_1 x_2 + x_2 u_e$$

$u_e = 3 \therefore \quad x_1 x_2 = 0$

$$-6 + 5x_1 x_2 + 3x_2 = 0 \Rightarrow -6 + 3x_2 = 0$$

$\therefore \quad x_2 = 2$ &

$x_1 = 0$

$$x_e = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \qquad \boxed{(x_e, u_e) = \left[\begin{pmatrix} 0 \\ 2 \end{pmatrix}, 3\right]}$$

b) $\frac{\partial f_1}{\partial x_1} = x_2 \Big|_{(x_e, u_e)} = 2 \qquad \frac{\partial f_1}{\partial u} = -1$

$\frac{\partial f_1}{\partial x_2} = x_1 \Big|_{(x_e, u_e)} = 0 \qquad \frac{\partial f_2}{\partial u} = x_2 \Big|_{(x_e, u_e)} = 2$

$\frac{\partial f_2}{\partial x_1} = 5x_2 \Big|_{(x_e, u_e)} = 10 \qquad \frac{\partial y}{\partial u} = x_2 \Big|_{(x_e, u_e)} = 2$

$\frac{\partial f_2}{\partial x_2} = 5x_1 + u \Big|_{(x_e, u_e)} = 3$

$\frac{\partial y}{\partial x_1} = 2x_1 \Big|_{(x_e, u_e)} = 0 \qquad \frac{\partial y}{\partial x_2} = u \Big|_{(x_e, u_e)} = 3$

#1)

$$A = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} \Bigg|_{(x_e, u_e)} \qquad B = \begin{pmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{pmatrix} \Bigg|_{(x_e, u_e)}$$

$$C = \left[\frac{\partial y}{\partial x_1} \quad \frac{\partial y}{\partial x_2} \right] \Bigg|_{(x_e, u_e)} \qquad D = \frac{\partial y}{\partial u} \Bigg|_{(x_e, u_e)}$$

$$A = \begin{pmatrix} 2 & 0 \\ 10 & 3 \end{pmatrix} \qquad B = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & 3 \end{pmatrix} \qquad D = 2$$

$$\Delta \dot{x} = A \Delta x + B \Delta u$$

$$\Delta y = C \Delta x + D \Delta u$$

#2) $A=\begin{pmatrix}-2 & -9\\ 0 & 1\end{pmatrix}$ $\quad B=\begin{pmatrix}0\\1\end{pmatrix}$

$C=\begin{pmatrix}1 & 3\end{pmatrix}$ $\quad X(0)=\begin{pmatrix}-3/2\\1\end{pmatrix}$ $\quad D=0$

$$y(t)=Ce^{At}x(0)+C\int_0^t e^{A\tau}BU(t-\tau)\,d\tau \quad + \cancel{DU(t-\tau)}^{\,0}$$

$$e^{At}=Te^{Jt}T^{-1}$$

$$\chi_A=\lambda^2+\lambda-2=0=(\lambda+2)(\lambda-1) \quad \lambda_1=-2,\ \lambda_2=1$$

A is diagonlizable, $\therefore\ e^{At}=Te^{\Lambda t}T^{-1}$

$$\Lambda=\begin{pmatrix}-2 & 0\\ 0 & 1\end{pmatrix}$$

$$(A-\lambda_1 I)V_1=0=\begin{pmatrix}0 & -9\\ 0 & -3\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=0 \Rightarrow X_1=1,\ X_2=0$$

$$V_1=\begin{pmatrix}1\\0\end{pmatrix}$$

$$(A-\lambda_2)V_2=0=\begin{pmatrix}-3 & -9\\ 0 & 0\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=0 \Rightarrow X_2=1,\ X_1=-3$$

$$T=\begin{pmatrix}1 & -3\\ 0 & 1\end{pmatrix} \qquad T^{-1}=\begin{pmatrix}1 & 3\\ 0 & 1\end{pmatrix}$$

$$e^{At}=\begin{pmatrix}1 & -3\\ 0 & 1\end{pmatrix}\begin{pmatrix}e^{-2t} & 0\\ 0 & e^{t}\end{pmatrix}\begin{pmatrix}1 & 3\\ 0 & 1\end{pmatrix}=\begin{pmatrix}e^{-2t} & -3e^{t}\\ 0 & e^{t}\end{pmatrix}\begin{pmatrix}1 & 3\\ 0 & 1\end{pmatrix}$$

$$e^{At} = \begin{pmatrix} e^{-2t} & 3e^{-2t} - 3e^{t} \\ 0 & e^{t} \end{pmatrix}$$

$$Ce^{At}x(0) = [1 \quad 3] \begin{pmatrix} e^{-2t} & 3e^{-2t} - 3e^{t} \\ 0 & e^{t} \end{pmatrix} \begin{pmatrix} -3/2 \\ 1 \end{pmatrix}$$

$$= [e^{-2t} \quad 3e^{-2t}] \begin{pmatrix} -3/2 \\ 1 \end{pmatrix} = \frac{-3}{2} e^{-2t} + 3e^{-2t}$$

$$= \frac{3}{2} e^{-2t} \quad (1)$$

$$\int_0^t e^{A\tau} Bu(t-\tau)\, d\tau = \int_0^t \begin{pmatrix} e^{-2\tau} & 3e^{-2\tau} - 3e^{\tau} \\ 0 & e^{\tau} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} (1)\, d\tau$$

$$= \int_0^t \begin{pmatrix} 3e^{-2\tau} - 3e^{\tau} \\ e^{\tau} \end{pmatrix} d\tau = \begin{pmatrix} \frac{-3}{2} e^{-2t} - 3e^{t} + \frac{3}{2} + 3 \\ e^{t} - 1 \end{pmatrix}$$

$$C\int_0^t e^{A\tau} Bu(t-\tau)\, d\tau = [1 \quad 3] \begin{pmatrix} -\frac{3}{2} e^{-2t} - 3e^{t} + \frac{9}{2} \\ e^{t} - 1 \end{pmatrix}$$

$$= \frac{-3}{2} e^{-2t} - 3e^{t} + \frac{9}{2} + 3e^{t} - 3 = -\frac{3}{2} e^{-2t} + \frac{3}{2} \quad (2)$$

$$y(t) = \frac{3}{2} e^{-2t} - \frac{3}{2} e^{-2t} + \frac{3}{2} = (1) + (2)$$

$$\boxed{y(t) = \frac{3}{2}}$$

#3)

a) $G(z) = \begin{pmatrix} \frac{2z-1}{3z+6} \\ \frac{1}{z+2} \end{pmatrix}$ $\qquad \frac{2z-1}{3z+6} = \frac{2z-1}{3(z+2)}$

$$\begin{array}{r|l} & \frac{2}{3} \\ 3z+6 & 2z-1 \\ & -(2z+4) \\ \hline & -5 \end{array}$$

$$G(z) = \begin{pmatrix} \frac{-5}{3z+6} \\ \frac{1}{z+2} \end{pmatrix} + \begin{pmatrix} \frac{2}{3} \\ 0 \end{pmatrix} = G_{sp} + G(\infty)$$

$$G_{sp} = \frac{1}{z+2} \begin{pmatrix} \frac{-5}{3} \\ 1 \end{pmatrix}$$

$$\dot{x}_1 = A\, x_1 + B\, u$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = C\, x_1 + D\, u$$

$$\boxed{A = (-2) \qquad B = (1) \qquad C = \begin{pmatrix} \frac{-5}{3} \\ 1 \end{pmatrix} \qquad D = \begin{pmatrix} \frac{2}{3} \\ 0 \end{pmatrix}}$$

#3)
$$G = \begin{pmatrix} \frac{2s-1}{3s+6} & \frac{1}{s+2} \end{pmatrix}$$

3)

$$G_1 = \frac{2s-1}{3s+6} \Rightarrow G_{sp} + G(\infty) = \frac{-5}{3s+6} + \frac{2}{3} = \frac{-5/3}{(s+2)} + \frac{2}{3}$$

$$G_2 = \frac{1}{s+2}$$

$A_1 = -2$, $B_1 = 1$

$C_1 = -5/3$, $D_1 = 2/3$

$A_2 = -2$, $B_2 = 1$

$C_2 = 1$, $D_2 = 0$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + B \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \qquad A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}$$

$$y = C \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + D \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \qquad C = [C_1 \ C_2], \quad D = (D_1 \ D_2)$$

$$\boxed{A = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad C = [-5/3 \ \ 1] \qquad D = [2/3 \ \ 0]}$$

#4)

$$\dot{x} = \begin{pmatrix} 0 & 0 \\ 1 & \frac{1}{t} \end{pmatrix}$$

a)

$$A(t)A(\tau) = \begin{pmatrix} 0 & 0 \\ 1 & \frac{1}{t} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & \frac{1}{\tau} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \frac{1}{t} & \frac{1}{t\tau} \end{pmatrix}$$

$$A(\tau)A(t) = \begin{pmatrix} 0 & 0 \\ 1 & \frac{1}{\tau} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & \frac{1}{t} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \frac{1}{\tau} & \frac{1}{t\tau} \end{pmatrix}$$

$A(\tau)\,A(t)$ don't commute $\therefore$ $\phi(t,\tau) \neq e^{\int_\tau^t A(s)\,ds}$

A is lower triangular, solve scalar ODE's:

$\dot{X}_1 = 0$

$$\frac{dX_1}{dt} = 0 \Rightarrow \int_{X_1(1)}^{X} dX_1 = \cancel{\int 0\, dt}$$

$X_1 = X_1(1)$ $\quad \underline{(1)}$

$$\dot{X}_2 = X_1 + \frac{X_2}{t} = X_1(1) + \frac{X_2}{t}$$

$$\dot{X}_2 - \frac{1}{t}X_2 = X_1(1)$$

Method of Integrating factors: $P(t) = \frac{-1}{t}$, $Q(t) = X_1(1)$

$$\rho = e^{\int P(t)dt} = e^{\int -\frac{1}{t}dt}$$

$$\int \frac{-1}{t}dt = -\ln(t)$$

$$e^{-\ln(t)} = \frac{1}{t} = \rho$$

$$\underbrace{\frac{\dot{x}_2}{t} + \frac{x_2}{t^2}} = \frac{x_1(1)}{t}$$

$$\frac{d}{dt}\left(\frac{x_2}{t}\right) = \frac{d}{dt}(\rho x_2)$$

$$\frac{d}{dt}\left(\frac{x_2}{t}\right) = \frac{x_1(1)}{t}$$

$$\int d\left(\frac{x_2}{t}\right) = x_1(1)\int_1^t \frac{dt}{t}$$

$$\frac{x_2}{t} = x_1(1)\ln(t) + x_2(1)$$

$$x_2 = t\ln(t)\, x_1(1) + t\, x_2(1) \quad (2)$$

$$\phi(t) = \begin{pmatrix} 1 & 0 \\ t\ln(t) & t \end{pmatrix}$$

$$\phi(\tau) = \begin{pmatrix} 1 & 0 \\ \tau\ln(\tau) & \tau \end{pmatrix}$$

$$\phi(\tau)^{-1} = \frac{1}{\tau}\begin{pmatrix} \tau & 0 \\ -\tau\ln(\tau) & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\ln(\tau) & \frac{1}{\tau} \end{pmatrix}$$

$$\phi(t,\tau) = \phi(t)\ \phi(\tau)$$

$$\phi(t,\tau) = \begin{pmatrix} 1 & 0 \\ t\ln(t) - t\ln(\tau) & \frac{t}{\tau} \end{pmatrix}$$

$$\boxed{\phi(t,\tau) = \begin{pmatrix} 1 & 0 \\ t\ln\left(\frac{t}{\tau}\right) & \frac{t}{\tau} \end{pmatrix}}$$

b) $x(t) = \phi(t,\tau)\, x(\tau)$

$x(2) = \phi(2,1)\, x(1)$

$x(1) = \phi(2,1)^{-1}\, x(2)$

$$x(1) = \begin{pmatrix} 1 & 0 \\ 2\ln(2) & 2 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 4 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 2 & 0 \\ -2\ln(2) & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\ln(2) & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 0 \\ 4 \end{pmatrix}$$

$$\boxed{x(1) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}}$$

#5)

a) $\Phi(t,\tau) = \begin{pmatrix} e^{\cos t - \cos\tau} & 0 \\ 0 & e^{-\sin t + \sin\tau} \end{pmatrix}$

$$\frac{d}{dt}(\Phi(t,0)) = A(t)\,\Phi(t,0)$$

$$\Phi(t,0) = \begin{pmatrix} e^{\cos t - 1} & 0 \\ 0 & e^{-\sin t} \end{pmatrix}$$

$$\frac{d}{dt}(\Phi(t,0)) = \begin{pmatrix} -\sin t\, e^{\cos t - 1} & 0 \\ 0 & -\cos t\, e^{-\sin t} \end{pmatrix}$$

$$\begin{pmatrix} -\sin t\, e^{(\cos t - 1)} & 0 \\ 0 & -\cos t\, e^{-\sin t} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} e^{\cos t - 1} & 0 \\ 0 & e^{-\sin t} \end{pmatrix}$$

$-\sin t\, e^{(\cos t - 1)} = a_{11}\, e^{\cos t - 1} \Rightarrow a_{11} = -\sin t$

$0 = a_{12}\, e^{-\sin t} \Rightarrow a_{12} = 0$

$0 = a_{21}\, e^{\cos t - 1} \Rightarrow a_{21} = 0$

$-\cos t\, e^{-\sin t} = a_{22}\, e^{-\sin t} \Rightarrow a_{22} = -\cos t$

$$\boxed{A(t) = \begin{pmatrix} -\sin(t) & 0 \\ 0 & -\cos(t) \end{pmatrix}}$$

b) $T = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}$, $J = \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix}$

$$\phi(t,\tau) = \phi(t)\ \phi(\tau)^{-1}$$

$$\phi(t) = e^{At} = T e^{Jt} T^{-1} \qquad (LTI)$$

$$\phi(t) = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} e^{-2t} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^{-2t} & 0 \\ -3e^{-2t} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} e^{-2t} & 0 \\ -3e^{-2t}+3 & 1 \end{pmatrix}$$

$$\phi(\tau) = \begin{pmatrix} e^{-2\tau} & 0 \\ -3e^{-2\tau}+3 & 1 \end{pmatrix}$$

$$\phi^{-1}(\tau) = \frac{1}{e^{-2\tau}} \begin{pmatrix} 1 & 0 \\ 3e^{-2\tau}-3 & e^{-2\tau} \end{pmatrix} = \begin{pmatrix} \frac{1}{e^{-2\tau}} & 0 \\ \frac{3e^{-2\tau}-3}{e^{-2\tau}} & 1 \end{pmatrix}$$

$$\phi(t,\tau) = \begin{pmatrix} e^{-2t} & 0 \\ -3e^{-2t}+3 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{e^{-2\tau}} & 0 \\ \frac{3e^{-2\tau}-3}{e^{-2\tau}} & 1 \end{pmatrix}$$

$$\boxed{\phi(t,\tau) = \begin{pmatrix} \frac{e^{-2t}}{e^{-2\tau}} & 0 \\ \frac{-3e^{-2t}+3e^{-2\tau}}{e^{-2\tau}} & 1 \end{pmatrix}}$$

c) $\phi(1,3) = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ $\quad x(1) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$x(t) = \phi(t,\tau)\, x(\tau)$

$x(1) = \phi(1,3)\, x(3)$

$x(3) = \phi(1,3)^{-1} x(1)$

$= \frac{1}{2-1} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$$\boxed{x(3) = \begin{pmatrix} -1 \\ 2 \end{pmatrix}}$$

d) $\phi(1,3) = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ $\quad \phi(2,3) = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$

$x(1) = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$

$x(3) = \phi(1,3)^{-1} x(1) = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 7 \\ -5 \end{pmatrix}$

$x(2) = \phi(2,3)\, x(3) = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} 7 \\ -5 \end{pmatrix}$

$$\boxed{x(2) = \begin{pmatrix} -1 \\ 3 \end{pmatrix}}$$

e) $X(t) = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t^2-t & t \end{pmatrix}$

$$X(t) = \begin{pmatrix} 2-t^2+t & -t \\ -1+t^2-t & t \end{pmatrix}$$

$$\phi(t,\tau) = X(t)\,X(\tau)^{-1}$$

$$X(\tau) = \begin{pmatrix} 2-\tau^2+\tau & -\tau \\ -1+\tau^2-\tau & \tau \end{pmatrix}$$

$$X(\tau)^{-1} = \frac{1}{2\tau-\cancel{\tau^3}+\cancel{\tau^2}-(\tau-\cancel{\tau^3}+\cancel{\tau^2})} \begin{pmatrix} \tau & \tau \\ 1-\tau^2+\tau & 2-\tau^2+\tau \end{pmatrix}$$

$$X(\tau)^{-1} = \begin{pmatrix} 1 & 1 \\ \frac{1-\tau^2+\tau}{\tau} & \frac{2-\tau^2+\tau}{\tau} \end{pmatrix}$$

$$\phi(t,\tau) = \begin{pmatrix} 2-t^2+t & -t \\ -1+t^2-t & t \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \frac{1-\tau^2+\tau}{\tau} & \frac{2-\tau^2+\tau}{\tau} \end{pmatrix}$$

$$= \begin{pmatrix} (2-t^2+t)\;\frac{-t+t\tau^2-t\tau}{\tau} & 2-t^2+t\;\frac{-2t+t\tau^2-t\tau}{\tau} \\ (-1+t^2-t)+\frac{t-t\tau^2+t\tau}{\tau} & -1+t^2-t+\frac{2t}{\tau}-\frac{t\tau^2}{\tau}+\frac{t\tau}{\tau} \end{pmatrix}$$

$$\boxed{\phi(t,\tau) = \begin{pmatrix} 2-t^2-\frac{t}{\tau}+t\tau & 2-t^2-\frac{2t}{\tau}+t\tau \\ -1+t^2+\frac{t}{\tau}-t\tau & -1+t^2+\frac{2t}{\tau}-t\tau \end{pmatrix}}$$