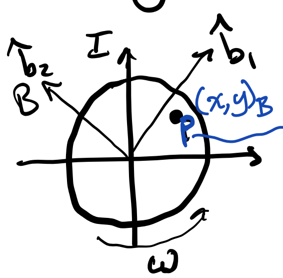
## Ex. Bug on a turn-table (Hand + Finch)




$$\vec{r}_{P/O} = x\hat{b}_1 + y\hat{b}_2$$

$$\frac{{}^B d}{dt}(\vec{r}_{P/O}) = {}^B\vec{v}_{P/O}$$

$${}^B\vec{v}_{P/O} = \dot{x}\hat{b}_1 + \dot{y}\hat{b}_2$$

$${}^I\vec{v}_{P/O} = \dot{x}\hat{b}_1 + \dot{y}\hat{b}_2 + \omega\hat{b}_3 \times (x\hat{b}_1 + y\hat{b}_2)$$

$$= (\dot{x} - \omega y)\hat{b}_1 + (\dot{y} + \omega x)\hat{b}_2$$

$$L = T_0 - \cancel{U_0}^{\,0}$$

$$= \frac{1}{2}m\left[(\dot{x} - \omega y)^2 + (\dot{y} + \omega x)^2\right]$$

Proceed to Hamiltonian formulation:

$$p_x = \frac{\partial L}{\partial \dot{x}} = m(\dot{x} - \omega y) \qquad p_y = \frac{\partial L}{\partial \dot{y}} = m(\dot{y} + \omega x)$$

Hamiltonian via Legendre:

$$H = p_x\dot{x} + p_y\dot{y} - L \quad \leftarrow H = \sum_i p_i\dot{q}_i - L$$

$$= p_x\left(\frac{p_x}{m} + \omega y\right) + p_y\left(\frac{p_y}{m} - \omega x\right) - \frac{1}{2m}\left(p_x^2 + p_y^2\right)$$

$$= \underbrace{\frac{1}{2m}\left(p_x^2 + p_y^2\right)}_{\text{Looks like Non-rotating frame's Hamiltonian}} + \underbrace{\omega\left(y p_x - x p_y\right)}_{\text{Term due to frame rotation}}$$

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{1}{m}p_x + \omega y \qquad \dot{y} = \frac{\partial H}{\partial p_y} = \frac{p_y}{m} - \omega x$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}$$

$$\dot{p}_x = -\frac{\partial H}{\partial x} = \omega p_y \qquad \dot{p}_y = -\frac{\partial H}{\partial y} = -\omega p_x$$

## Cyclic cordinates in the Hamiltonian Story

- Recall if $q_j$ is cyclic, it does not appear in the Lagrangian
- We have

By Hamilton's equations

$$\dot{p}_j = \underbrace{\frac{\partial L}{\partial q_j}}_{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0} \overset{0}{=} \frac{-\partial H}{\partial q_j} = 0$$

$p_j$

$\Rightarrow$ The cyclic cordinate $q_j$ will be absent from the Hamiltonian as well. And the corresponding conjugate momentum is conserved.

- The relationship between conservation laws and symmetry of the Lag./Ham. is preserved (Noether's theorem)

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## Routh's Procedure

- Often Hamiltonian formulations don't give much advantage in practical calculations, and end up defaulting to the same number of 2nd-order differential equations as the Lagrangian approach if you have to eliminate the conjugate momenta $p_j$ during the solution process.
  - ↳ However, the Hamiltonian approach is useful when cyclic cordinates are present.

Routh said we should "blend" the two approaches.

↳ Modify the Lagrangian so it is no longer a function of the generalized velocity of the cyclic variables, but it still involves the conjugate momenta, which are conserved. The $p_j$'s can be considered constants of integration, and the remaining integrations involve only non-cyclic variables.

Consider $q_m$ as cyclic,

$$p_m = \frac{\partial L}{\partial \dot{q}_m} = \alpha \text{ constant}$$

$L = L(q_1, \ldots, q_{m-1}, \dot{q}_1, \ldots, \dot{q}_m, t)$ Note: $q_m$ is missing

$H = H(q_1, \ldots, q_{m-1}, p_1, \ldots, p_{m-1}, \alpha, t)$ Note: $q_m$ is missing, $p_m$ is missing

$\alpha \leftarrow p_m$

This reduction of 1 DOF gives
2M−2 Eqn's from Ham. dynamics
along with $\dot{q}_m = \frac{\partial H}{\partial \alpha}$

Routh transforms from $(q, \dot{q})$ to $(q, p)$ only for the cyclic coordinates and uses Hamilton's Eqn's for those. For the remaining coordinates, he uses the Euler-Lagrange equations.

Let $q_{s+1}, \dots, q_m$ be cyclic

where $s$ is the # of non-cyclic cordinates

$$R = R(q_1, \dots, q_s, \dot{q}_1, \dots, \dot{q}_m, \underbrace{p_{s+1}, \dots, p_m}_{\text{from cyclic variables}}, t)$$

$$R = \sum_{i=s+1}^{m} p_i \dot{q}_i - L$$

* Legendre transformation over cyclic cordinates only.

This leads to,

$$\boxed{\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{q}_i}\right) - \frac{\partial R}{\partial q_i} = 0}$$

for $i = 1, \dots, s$

$s$ $2^{nd}$-order ODE's

Number of $2^{nd}$-order ODEs

And,

$p_i = \text{constant}$ with $\boxed{\dot{q}_i = \frac{\partial R}{\partial p_i}}$

for $i = s+1, \dots, m$

$(m-s)$ $1^{st}$ order ODE's.

Routh's procedure automatically reduces the problem for cyclic cordinates. This makes the analysis and integration easier for engineering problems with many DOF's.

Ex. Kepler's Problem (Particle in a plane under the influence of a central force)

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \underbrace{\frac{k}{r^n}}_{\text{potential function } V(r)}$$

Here $\theta$ is cyclic $\Rightarrow p_\theta$ is conserved $\rightarrow p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$

$$R = p_\theta \dot{\theta} - L$$

$$= (mr^2\dot{\theta})\dot{\theta} - \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{k}{r^n}$$

$$= \frac{mr^2}{2}\dot{\theta}^2 - \frac{m}{2}\dot{r}^2 - \frac{k}{r^n}$$

Put in terms of $p_\theta$, $\rightarrow$ conserved, so treat as constant

$$R(r, \dot{r}, p_\theta) = \frac{p_\theta^2}{2mr^2} - \frac{m}{2}\dot{r}^2 - \frac{k}{r^n}$$

$$\Rightarrow \boxed{\ddot{r} - \frac{p_\theta}{mr^3} + \frac{nk}{r^{n+1}} = 0}$$

From Hamilton's equations on $\theta$,

$\dot{p}_\theta = 0$ and $\boxed{\dot{\theta} = \frac{p_\theta}{mr^2}}$ where $p_\theta$ is constant

# Canonical Transformations ~~and the Poisson Bracket~~

(Goldstein Ch 9, Hand + Finch Ch 6)

Motivation: Imagine you have all cyclic variables in your Hamiltonian and the Hamiltonian is conserved. Then,

Action-Angle variables

"angles"

$Q_i \to \theta_i$

$P_i \to I_i$

"actions"

$$P_i = \alpha_i \quad \text{constants}$$

$$H = H(\alpha_1, \dots, \alpha_m)$$

This leads to

$$\dot{q}_i = \frac{\partial H}{\partial \alpha_i} \triangleq \omega_i \quad \text{constant}$$

Integration:

constant of integration due to I.C.

$$q_i(t) = \omega_i t + \beta_i \quad \text{Simple!}$$

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- Note: The number of cyclic coordinates in a set of generalized coordinates varies depending on the choice of generalized coordinates.

  Ex. Kepler problem
  
  ↳ Particle in a plane, central force.

  Cartesian: $x, y \Rightarrow$ no cyclic coordinates

  Polar: $(r, \theta) \Rightarrow \theta$ is cyclic

Goal: Find a transformation of coordinates to a description of the motion in which the Hamiltonian is conserved and all coordinates are cyclic

**Dfn:** A transformation is said to be <u>canonical</u> if, after a transformation, Hamilton's equations are still the correct dynamical description of the motion.

$$(q,p) \xrightarrow[\text{transformation}]{\text{canonical}} (Q,P)$$

$$Q_i = Q_i(q,p,t)$$
$$P_i = P_i(q,p,t)$$

So, we want a new Hamiltonian,

$$K = K(Q,P,t)$$

and

$$\dot{Q}_i = \frac{\partial K}{\partial P_i} \quad \text{and} \quad \dot{P}_i = -\frac{\partial K}{\partial Q_i}$$

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Let's determine the constraints we need to satisfy for a transformation to be "canonical"

Recall that the Lagrangian

$$L = L(q,\dot{q},t)$$
$$F = F(q,t) \quad \text{Note } F \text{ doesn't depend on } \dot{q}$$
$$L' = L \pm \frac{dF}{dt}$$

} Still satisfies the E.L. equations, giving the same EOM's.

Consider,

$$L' = L(q,\dot{q},t) - \frac{d\underbrace{F(q,Q,t)}_{}}{dt}$$

This form is a choice

The Function $F(q, Q, t)$ can be used to generate a new equivalent description of the system; It is called a generating function.

In the original description of the dynamics

$$H = p_i \dot{q}_i - L$$

We want,

$$K = P_i \dot{Q}_i - L'$$

$$K = P_i \dot{Q}_i - L + \frac{dF(q,Q,t)}{dt}$$

Notation comment

on this page and a few others, we are using index notation. So, $p_i q_i = \sum_i p_i q_i$

Expand $\frac{dF(q,Q,t)}{dt}$

$$\frac{dF(q,Q,t)}{dt} = \frac{\partial F}{\partial q_i}\dot{q}_i + \frac{\partial F}{\partial Q_i}\dot{Q}_i + \frac{\partial F}{\partial t}$$

Manipulate and equate the two forms above for $L$,

$$p_i \dot{q}_i - H = P_i \dot{Q}_i - K + \frac{\partial F}{\partial q_i}\dot{q}_i + \frac{\partial F}{\partial Q_i}\dot{Q}_i + \frac{\partial F}{\partial t}$$

Collecting terms,

$$\underline{\left(p_i - \frac{\partial F}{\partial q_i}\right)}\dot{q}_i - H = \underline{\left(P_i + \frac{\partial F}{\partial Q_i}\right)}\dot{Q}_i - K + \frac{\partial F}{\partial t}$$

$q_i$ and $Q_i$ variables are independent, so the coefficients must vanish independently,

$$\boxed{p_i = \frac{\partial F}{\partial q_i} \quad \text{and} \quad P_i = -\frac{\partial F}{\partial Q_i}}$$

The generating function determines the transformation equations needed to preserve the Hamiltonian struct.

constraints that our generating function must satisfy

These equations of the transformation are 2 Eqn, 2 unknowns for $Q_i(q,p)$

$P_i(q,p)$

Now find the new Hamiltonian:

$$K = P_i\dot{Q}_i - L'$$

$$= \frac{-\partial F}{\partial Q_i}\dot{Q}_i - L + \frac{dF}{dt}$$

$$= \cancel{\frac{-\partial F}{\partial Q_i}\dot{Q}_i} - L + \underbrace{\frac{\partial F}{\partial q_i}}_{p_i}\dot{q}_i + \cancel{\frac{\partial F}{\partial Q_i}\dot{Q}_i} + \frac{\partial F}{\partial t}$$

$$K = p_i\dot{q}_i - L + \frac{\partial F}{\partial t}$$

$$\boxed{K = H + \frac{\partial F}{\partial t}}$$

$$K(Q,P,t) = H(q,p,t) + \frac{\partial F}{\partial t} \quad \text{where } F = F(q,Q,t)$$

$q = q(Q,P)$ $\quad$ $p = p(Q,P)$

Then, we get $\dot{Q}_i = \frac{\partial K}{\partial P_i}$ and $\dot{P}_i = \frac{-\partial K}{\partial Q_i}$

as required for a canonical transformation

The above derivation depended on the <u>choice</u> of the form of the generating function F.
That was the derivation for a "Type 1" generating function

$$F_1 = F_1(q, Q, t)$$

**Ex.**

$$F_1(q,Q,t) = \underbrace{qQ}_{\text{You make this up}}$$

constraint on $F_1$:

$$p = \frac{\partial F_1}{\partial q} = Q$$

$$P = \frac{-\partial F}{\partial Q} = -q$$

Since F is time-invariant here, $K = H(-P, Q)$

⇒ This transformation swaps the generalized coordinates and their corresponding momenta (with a sign change)

i.e. $(q, p)$ becomes $(-P, Q)$

In general, there are 4 types of generating functions:

<u>invertibility condition</u>

① $F_1 = F_1(q,Q,t) \Rightarrow p = \frac{\partial F_1}{\partial q}$ and $P = \frac{-\partial F_1}{\partial Q}$ $\qquad \frac{\partial^2 F_1}{\partial q \partial Q} \neq 0$

② $F_2 = F_2(q,P,t) \Rightarrow p = \frac{\partial F_2}{\partial q}$ and $Q = \frac{\partial F_2}{\partial P}$ $\qquad \frac{\partial^2 F_2}{\partial q \partial P} \neq 0$

③ $F_3 = F_3(p,Q,t) \Rightarrow q = \frac{-\partial F_3}{\partial p}$ and $P = \frac{-\partial F_3}{\partial Q}$ $\qquad \frac{\partial^2 F_3}{\partial p \partial Q} \neq 0$

④ $F_4 = F_4(p,P,t) \Rightarrow q = \frac{-\partial F_4}{\partial p}$ and $Q = \frac{\partial F_4}{\partial P}$ $\qquad \frac{\partial^2 F_4}{\partial p \partial P} \neq 0$

Trivial example,

$$F_2 = qP$$

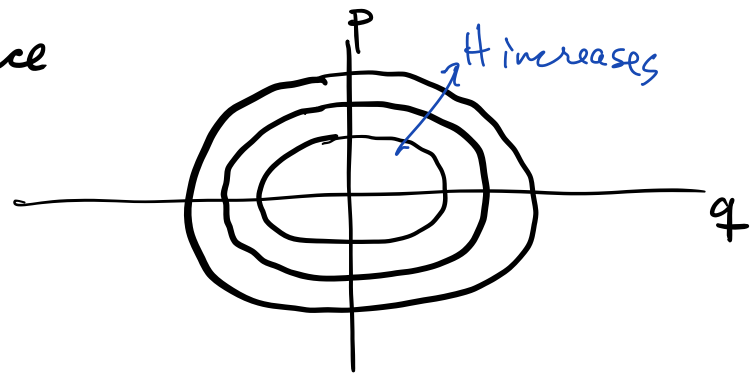
$$\Rightarrow \quad Q = \frac{\partial F_2}{\partial P} = q, \qquad p = \frac{\partial F_2}{\partial q} = P$$

identity transformation $(q,p) \to (Q,P)$

<u>Useful Example</u> Harmonic oscillator

$$H = \frac{1}{2m}\left(p^2 + Kmq^2\right)$$ where

Phase-Space




Choose, $F_1 = F_1(q, Q, t) = \frac{1}{2}\omega q^2 \cot(2\pi Q)$

$$\Rightarrow \quad p = \frac{\partial F_1}{\partial q} = \omega q \cot(2\pi Q), \qquad P = \frac{-\partial F_1}{\partial Q} = \frac{\pi \omega q^2}{\sin^2(2\pi Q)}$$

Invert these for $q(Q,P)$ and $p(Q,P)$ then substitute into $K = H(q,p) + \frac{\partial F_1}{\partial t}$

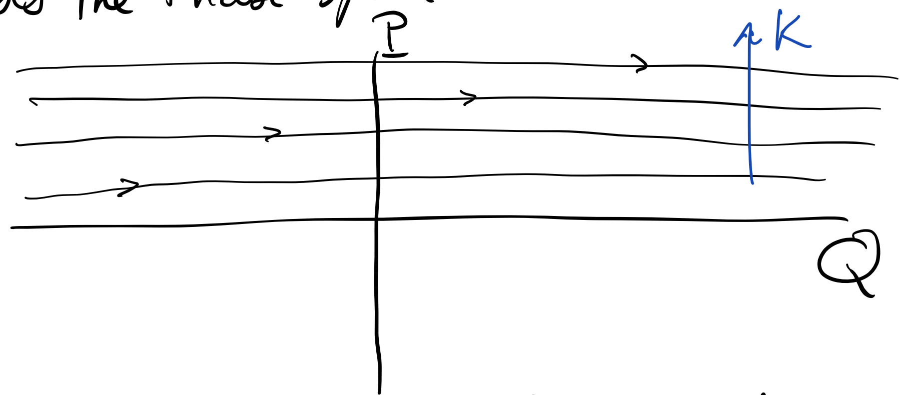
This yields $K(Q,P) = \frac{\omega}{2\pi} P$, $\omega = \sqrt{\frac{K}{m}}$, $Q$ is cyclic

$\Rightarrow P$ is conserved

Equation of motion:

$$\dot{Q} = \frac{\partial K}{\partial \mathcal{P}} = \frac{\omega}{2\pi} \Rightarrow Q(t) = \frac{\omega}{2\pi} t$$

What does the Phase space look like?




We can transform back to $(q, p)$ now that we know $\mathcal{P}$ is conserved.

$$q = \sqrt{\frac{\mathcal{P}}{2\pi\omega}} \sin \omega t \qquad p = \sqrt{\frac{\omega \mathcal{P}}{\pi}} \cos \omega t$$

$(q, p)$ in terms of $\mathcal{P}$ constant, after integrating in the $(Q, \mathcal{P})$ coordinates.