

### **ECE 68000: MODERN AUTOMATIC CONTROL**

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The Kronecker Product

#### **Definition**

- Let **A** be an  $m \times n$  and **B** be a  $p \times q$  matrices
- The *Kronecker product* of **A** and **B**, denoted  $A \otimes B$ , is an  $mp \times nq$  matrix defined as

$$m{A} \otimes m{B} = \left[ egin{array}{cccc} a_{11} m{B} & a_{12} m{B} & \cdots & a_{1n} m{B} \\ a_{21} m{B} & a_{22} m{B} & \cdots & a_{2n} m{B} \\ dots & dots & \ddots & dots \\ a_{m1} m{B} & a_{m2} m{B} & \cdots & a_{mn} m{B} \end{array} 
ight],$$

where the symbol ⊗ reads "otimes"

- Thus, the matrix  $A \otimes B$  consists of mn blocks
- In MATLAB, kron(A,B)

### Example 1

Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$$
 and  $B = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$ 

Then

$$\begin{array}{rcl}
\mathbf{A} \otimes \mathbf{B} & = & \begin{bmatrix} \mathbf{B} & 2\mathbf{B} & 3\mathbf{B} \\ 3\mathbf{B} & 2\mathbf{B} & \mathbf{B} \end{bmatrix} \\
& = & \begin{bmatrix} 2 & 1 & 4 & 2 & 6 & 3 \\ 2 & 3 & 4 & 6 & 6 & 9 \\ 6 & 3 & 4 & 2 & 2 & 1 \\ 6 & 9 & 4 & 6 & 2 & 3 \end{bmatrix}
\end{array}$$

### Example 2

- Let A be an arbitrary  $2 \times 2$  matrix
- Then

$$egin{array}{lll} m{I}_2 \otimes m{A} & = & \left[ egin{array}{ccc} m{A} & m{O} \\ m{O} & m{A} \end{array} 
ight] \ & = & \left[ egin{array}{cccc} 1 & 0 & 0 \\ 0 & 1 \end{array} 
ight] \otimes \left[ egin{array}{cccc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} 
ight] \ & = & \left[ egin{array}{cccc} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & a_{11} & a_{12} \\ 0 & 0 & a_{21} & a_{22} \end{array} 
ight] \end{array}$$

### Example 2—contd.

We have

$$egin{array}{lll} m{A} \otimes m{I}_2 & = & \left[ egin{array}{ccc} a_{11} m{I}_2 & a_{12} m{I}_2 \ a_{21} m{I}_2 & a_{22} m{I}_2 \end{array} 
ight] \ & = & \left[ egin{array}{cccc} a_{11} & 0 & a_{12} & 0 \ 0 & a_{11} & 0 & a_{12} \ a_{21} & 0 & a_{22} & 0 \ 0 & a_{21} & 0 & a_{22} \end{array} 
ight] \end{array}$$

In general

$$A \otimes B \neq B \otimes A$$

### Vectorization operator or stacking operator

Let

$$X = [x_1, x_2, \ldots, x_m]$$

be a  $n \times m$  matrix, where  $x_i$ , i = 1, 2, ..., m are the columns of X

- Each  $x_i$  consists of n elements
- Then, the *vectorization operator* or *stacking operator* is defined as

$$\operatorname{vec}(\boldsymbol{X}) = \begin{bmatrix} \boldsymbol{x}_1^{\top}, & \boldsymbol{x}_2^{\top}, & \dots, & \boldsymbol{x}_m^{\top} \end{bmatrix}^{\top}$$

$$= \begin{bmatrix} x_{11} & x_{21} & \cdots & x_{12} & \cdots & x_{n2} & \cdots & x_{1m} & \cdots & x_{nm} \end{bmatrix}^{\top}$$

is the column *nm*-vector formed from the columns of *X* taken in order

• In MATLAB, X(:)

# Converting a matrix-matrix equation into a matrix-vector equation

- Let now *A* be an  $n \times n$ , and *C* and *X* be  $n \times m$  matrices
- Then, the matrix equation

$$AX = C$$

can be written as

$$(\boldsymbol{I}_m \otimes \boldsymbol{A}) \operatorname{vec}(\boldsymbol{X}) = \operatorname{vec}(\boldsymbol{C})$$

• Indeed, write AX = C as

$$\begin{array}{rcl}
\mathbf{AX} &=& \mathbf{A} \left[ \begin{array}{cccc} \mathbf{x}_1, & \mathbf{x}_2, & \dots, & \mathbf{x}_m \end{array} \right] \\
&=& \left[ \begin{array}{ccccc} \mathbf{Ax}_1, & \mathbf{Ax}_2, & \dots, & \mathbf{Ax}_m \end{array} \right] \\
&=& \left[ \begin{array}{ccccc} \mathbf{c}_1, & \mathbf{c}_2, & \dots, & \mathbf{c}_m \end{array} \right]
\end{array}$$

### **Manipulations**

Represent

$$\begin{bmatrix} \mathbf{A}\mathbf{x}_1, & \mathbf{A}\mathbf{x}_2, & \dots, & \mathbf{A}\mathbf{x}_m \end{bmatrix} = \begin{bmatrix} \mathbf{c}_1, & \mathbf{c}_2, & \dots, & \mathbf{c}_m \end{bmatrix}$$
 as

$$\left[egin{array}{c} oldsymbol{Ax}_1 \ oldsymbol{Ax}_2 \ dots \ oldsymbol{Ax}_m \end{array}
ight] = \left[egin{array}{c} oldsymbol{c}_1 \ oldsymbol{c}_2 \ dots \ oldsymbol{c}_m \end{array}
ight]$$

Equivalently

$$\left[egin{array}{cccc} m{A} & m{O} & \cdots & m{O} \ m{O} & m{A} & \cdots & m{O} \ & & \ddots & & \\ m{O} & m{O} & \cdots & m{A} \end{array}
ight] \left[egin{array}{c} m{x}_1 \ m{x}_2 \ dots \ m{x}_m \end{array}
ight] = \left[egin{array}{c} m{c}_1 \ m{c}_2 \ dots \ m{c}_m \end{array}
ight]$$

• That is,

$$(I_m \otimes A) \operatorname{vec}(X) = \operatorname{vec}(C)$$

## Converting another matrix-matrix equation into a matrix-vector equation

- Let now **B** be a  $m \times m$ , and **C** and **X** be  $n \times m$  matrices
- Then, the matrix equation

$$XB = C$$

can be written as

$$(\boldsymbol{B}^{\top} \otimes \boldsymbol{I}_n) \operatorname{vec}(\boldsymbol{X}) = \operatorname{vec}(\boldsymbol{C})$$

• Indeed, write XB = C as

$$\begin{bmatrix} \boldsymbol{X}\boldsymbol{b}_1 & \boldsymbol{X}\boldsymbol{b}_2 & \cdots & \boldsymbol{X}\boldsymbol{b}_m \end{bmatrix} = \begin{bmatrix} \boldsymbol{c}_1 & \boldsymbol{c}_2 & \cdots & \boldsymbol{c}_m \end{bmatrix}$$

• Represent the above as

$$egin{bmatrix} oldsymbol{X}oldsymbol{b}_1 \ oldsymbol{X}oldsymbol{b}_2 \ oldsymbol{:} \ oldsymbol{X}oldsymbol{b}_m \end{bmatrix} = egin{bmatrix} oldsymbol{x}_1b_{11} + oldsymbol{x}_2b_{21} + \cdots + oldsymbol{x}_mb_{m1} \ oldsymbol{x}_1b_{12} + oldsymbol{x}_2b_{22} + \cdots + oldsymbol{x}_mb_{m2} \ oldsymbol{:} \ oldsymbol{:} \ oldsymbol{c} \ oldsymbol{c} \ oldsymbol{:} \ oldsymbol{x}_1b_{1m} + oldsymbol{x}_2b_{2m} + \cdots + oldsymbol{x}_mb_{mm} \end{bmatrix} = egin{bmatrix} oldsymbol{c} oldsymbol{c} \ oldsymbol{c} \ oldsymbol{c} \ oldsymbol{:} \ oldsymbol{c} \ oldsymbol{c} \ oldsymbol{x} \ oldsymbol{:} \ oldsymbol{c} \ oldsymbol{c} \ oldsymbol{:} \ oldsymbol{c} \ oldsymbol{c} \ oldsymbol{c} \ oldsymbol{x} \ oldsymbol{c} \ oldsymbol{c$$

### More manipulations

• Represent 
$$\begin{bmatrix} x_1b_{11} + x_2b_{21} + \cdots + x_mb_{m1} \\ x_1b_{12} + x_2b_{22} + \cdots + x_mb_{m2} \\ \vdots \\ x_1b_{1m} + x_2b_{2m} + \cdots + x_mb_{mm} \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix}$$
 as

$$\begin{bmatrix} b_{11}\mathbf{I}_n & b_{21}\mathbf{I}_n & \cdots & b_{m1}\mathbf{I}_n \\ b_{12}\mathbf{I}_n & b_{22}\mathbf{I}_n & \cdots & b_{m2}\mathbf{I}_n \\ \vdots & \vdots & \ddots & \vdots \\ b_{1m}\mathbf{I}_n & b_{2m}\mathbf{I}_n & \cdots & b_{mm}\mathbf{I}_n \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_m \end{bmatrix} = \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \vdots \\ \mathbf{c}_m \end{bmatrix}$$

• The above can be written in compact from as

$$(\boldsymbol{B}^{\top} \otimes \boldsymbol{I}_n) \operatorname{vec}(\boldsymbol{X}) = \operatorname{vec}(\boldsymbol{C})$$

### The Sylvester matrix equation

 Using the above two facts, represent the Sylvester matrix equation

$$AX + XB = C$$

as

$$(I_m \otimes A + B^{\top} \otimes I_n) \operatorname{vec}(X) = \operatorname{vec}(C)$$

• The continuous Lyapunov equation,  $A^{T}P + PA = -Q$ , can be represented as

$$(\boldsymbol{I}_n \otimes \boldsymbol{A}^\top + \boldsymbol{A}^\top \otimes \boldsymbol{I}_n) \operatorname{vec}(\boldsymbol{P}) = -\operatorname{vec}(\boldsymbol{Q})$$