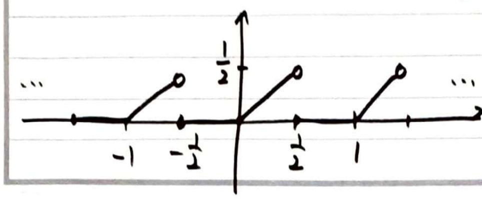
$\{1, \cos(\frac{n\pi x}{L}), \sin(\frac{n\pi x}{L})\}$: orthogonal on $(-L, L)$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

(Ex) $f(x) = \begin{cases} 0, & -\frac{1}{2} < x < 0 \\ x, & 0 \le x < \frac{1}{2} \end{cases}$: periodic in $\mathbb{R}$ with $p = 1$, $L = \frac{1}{2}$



$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = 1\cdot\int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)\,dx$$

$$= \int_0^{1/2} x\,dx = \left[\frac{x^2}{2}\right]_0^{1/2} = \frac{1}{2}\cdot\left(\frac{1}{2}\right)^2 = \frac{1}{8}.$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{1/2}\right)dx.$$

$$= \frac{1}{1/2}\int_0^{1/2} \underset{u}{x}\,\underbrace{\cos(2n\pi x)}_{v'}\,dx$$

$$\int_a^b u(x)v'(x)\,dx = [u(x)v(x)]_a^b - \int_a^b u'(x)v(x)\,dx$$

$$= 2\left(\left[x\frac{\sin(2n\pi x)}{2n\pi}\right]_0^{1/2} - \int_0^{1/2}\frac{\sin(2n\pi x)}{2n\pi}\,dx\right)$$

$$= 2\left(\left(\frac{1}{2}\cancel{\frac{\sin(2n\pi\frac{1}{2})}{2n\pi}}^{\,0} - 0\right) + \frac{1}{2n\pi}\left[\frac{+\cos(2n\pi x)}{2n\pi}\right]_0^{1/2}\right)$$

$$a_n = 2\cdot\frac{1}{\underbrace{(2n\pi)^2}_{4n^2\pi^2}}\left[\cos(n\pi) - 1\right] = \frac{\cos(n\pi) - 1}{2n^2\pi^2}$$

$$b_n = \frac{1}{1/2}\int_0^{1/2} x\sin\left(\frac{n\pi x}{1/2}\right)dx$$

$$= 2\int_0^{1/2} \underset{u}{x}\,\underbrace{\sin(2n\pi x)}_{v'}\,dx$$

$$= 2\left(\left[x\left(-\frac{\cos(2n\pi x)}{2n\pi}\right)\right]_0^{1/2} + \int_0^{1/2}\frac{+\cos(2n\pi x)}{2n\pi}\,dx\right)$$

$$= \cancel{2}\left(\frac{1}{\cancel{2}}\frac{(-1)\cos(n\pi)}{2n\pi} - 0 + \frac{1}{2n\pi}\left[\cancel{\frac{\sin(2n\pi x)}{2n\pi}}^{\,0}\right]_0^{1/2}\right)$$

$$= \frac{-\cos(n\pi)}{2n\pi}.$$

$$F(x) = \frac{1}{8} + \sum_{n=1}^{\infty}\left[\frac{\cos(n\pi)-1}{2n^2\pi^2}\right]\cos(2n\pi x)$$
$$+ \sum_{n=1}^{\infty}\left[\frac{-\cos(n\pi)}{2n\pi}\right]\sin(2n\pi x)$$

Remark

1. Simplify $F(x)$: $\cos(n\pi) = (-1)^n$

$a_n = \dfrac{(-1)^n - 1}{2n^2\pi^2}$: n is even $(-1)^n = 1$

$a_n = a_{2k} = 0$

n is odd : $(-1)^n = -1$: $a_n = a_{2k-1} = \dfrac{-2}{2(2k-1)^2\pi^2}$

$a_{2k-1} = \dfrac{-1}{(2k-1)^2\pi^2}$, $k = 1, 2, \cdots$

$$F(x) = \frac{1}{8} + \sum_{k=1}^{\infty}\left[\frac{-1}{(2k-1)^2\pi^2}\right]\cos(2(2k-1)\pi x)$$
$$- \sum_{n=1}^{\infty}\frac{(-1)^n}{2n\pi}\sin(2n\pi x)$$

Q1. $F(\frac{1}{2}) = ?$ Ⓐ $f(\frac{1}{2}) = 0$

$L = \frac{1}{2}$, $R = 0$

$F(\frac{1}{2}) = \frac{1}{2}(\frac{1}{2} + 0) = \frac{1}{4}.$

Q2 $F(\frac{1}{4}) = f(\frac{1}{4}) = \frac{1}{4}.$

Q3 $\displaystyle\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = ?$

Ⓐ $\displaystyle F(0) = \frac{1}{8} + \left(\frac{-1}{\pi^2}\right)\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}$

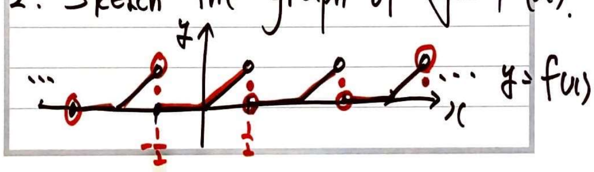
$\;\;\;\;\; \parallel$

$\;\;\; f(0) = 0$

$\displaystyle \frac{+1}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} = +\frac{1}{8}$: $\displaystyle\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} = \frac{\pi^2}{8}.$

HW: $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2} = ?$

2. Sketch the graph of $y = F(x)$.



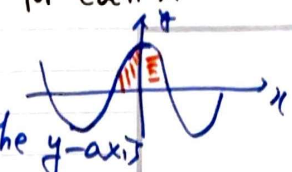
(Even/odd functions)

Q what if $f(x)$ is periodic in $\mathbb{R}$ and an even/odd function on $[-L, L]$?

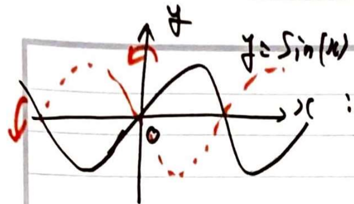
<u>Def</u> (1) $f(x)$ is called an even function if $f(-x) = f(x)$ for each x

ex $f(x) = \cos x$.

: symmetric wrt the y-axis



(2) $f(x)$ is called an odd function if $f(-x) = -f(x)$: ex $f(x) = \sin(x)$

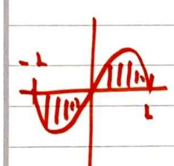


: symmetric wrt the origin

(Fourier series of an even function)

$f(x)$ is periodic in $\mathbb{R}$ & even on $[-L, L]$

$$b_n = \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)}_{e} \cdot \underbrace{\sin\left(\frac{n\pi x}{L}\right)}_{o} dx = 0$$



$$F(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) = F_c(x)$$

: the Fourier cosine series of $f(x)$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{L}\int_{0}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)\cos\left(\frac{n\pi x}{L}\right)}_{e.} dx$$

$$a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right) dx$$

(Fourier series of an odd function)

Let $f(x)$ be periodic in $\mathbb{R}$ and odd in $[-L, L]$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = 0$$

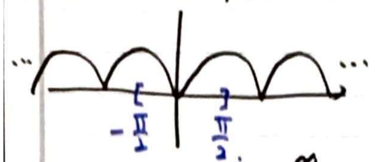
$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right) dx = 0$$

$$F(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) = F_s(x)$$: the Fourier sine series of $f(x)$

$$b_n = \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)\sin\left(\frac{n\pi x}{L}\right)}_{e.} dx$$

$$b_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right) dx$$

(Ex) Let $f(x) = |\sin(x)|$. $p = \pi$, $L = \frac{\pi}{2}$.



: even on $[-\frac{\pi}{2}, \frac{\pi}{2}]$

$$F(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{\pi/2}\right)$$

$$F(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(2nx)$$

$$a_0 = \frac{1}{\pi/2}\int_0^{\pi/2} \sin(x)\, dx = \frac{2}{\pi}\left[-\cos(x)\right]_0^{\pi/2}$$

$$= -\frac{2}{\pi}[0-1] = \frac{2}{\pi}$$

$$a_n = \frac{2}{\pi/2}\int_0^{\pi/2} \sin(x)\cos(2nx)\, dx$$

$$\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$$

$$= \frac{\cancel{4}\,2}{\pi}\int_0^{\frac{\pi}{2}} \frac{1}{\cancel{2}}\Big(\sin(\overbrace{x+2nx}^{=(2n+1)x}) + \sin(\overbrace{x-2nx}^{(1-2n)x})\Big)\, dx$$

$$= \frac{2}{\pi}\left[-\frac{\cos(2n+1)x}{2n+1} + \frac{\cos(1-2n)x}{-1+2n}\right]_0^{\pi/2}$$

$$a_n = \frac{2}{\pi}\Bigg[\frac{-1}{2n+1}\underbrace{\cos(2n+1)\frac{\pi}{2}}_{=0} + \frac{1}{2n-1}\underbrace{\cos(1-2n)\frac{\pi}{2}}_{=0} - \left(\frac{-1}{2n+1} + \frac{1}{2n-1}\right)\Bigg]$$

$$= \frac{2}{\pi}\left(\frac{1}{2n+1} - \frac{1}{2n-1}\right) = \frac{2}{\pi}\,\frac{\overbrace{\cancel{2n}-1-(\cancel{2n}+1)}^{=-2}}{(2n+1)(2n-1)}$$

$$a_n = \frac{-4}{\pi}\,\frac{1}{4n^2-1}$$

$$F(x) = \frac{2}{\pi} + \sum_{n=1}^{\infty}\left[\frac{-4}{\pi(4n^2-1)}\right]\cos(2nx).$$