11.1. Fourier Series

$f(x) = \sin(x)$

$p = 4\pi$: $\sin(x + 4\pi) = \sin(x)$

2π,

$\vdots$

Period: $p > 0$ is the smallest period.

Need inner product & orthogonal functions.

(inner product)

$f(x), g(x)$: periodic & piecewise continuous functions with $p = 2\pi$

$$(f, g) = \int_{-\pi}^{\pi} f(x)\, g(x)\, dx$$

Thm1 (n, m: integers)

(a) $\int_{-\pi}^{\pi} \cos(nx) \cos(mx)\, dx = \begin{cases} 0 & \text{if } n \neq m \\ \pi & \text{if } n = m \end{cases}$

(b) $\int_{-\pi}^{\pi} \sin(nx) \sin(mx)\, dx = \begin{cases} 0 & \text{if } n \neq m \\ \pi & \text{if } n = m \end{cases}$

(c) $\int_{-\pi}^{\pi} \sin(nx) \cos(mx)\, dx = 0$ for any $n, m = 0, 1, 2, \ldots$

Remark: $\{1, \cos(nx), \sin(nx)\}_{n=1}^{\infty}$: orthogonal.

(Proof)

(Id) 1. $\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$

2. $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

3. $\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$

(pf) (a) $n \neq m$

$\int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,dx$

$= \frac{1}{2}\int_{-\pi}^{\pi} \left(\cos(n+m)x + \cos(n-m)x\right) dx$

$z = (n+m)x$, $dz = (n+m)\,dx$ $\qquad$ $z = (n-m)x$, $dz = (n-m)\,dx.$

$= \frac{1}{2}\left(\int_{-(n+m)\pi}^{(n+m)\pi} \cos(z)\frac{1}{n+m}\,dz + \int_{-(n-m)\pi}^{(n-m)\pi} \cos(z)\frac{1}{n-m}\,dz\right.$

$= \frac{1}{2}\left[\frac{\sin(z)}{n+m}\right]_{-(n+m)\pi}^{(n+m)\pi} + \frac{1}{n-m}\left[\sin(z)\right]_{-(n-m)\pi}^{(n-m)\pi}$

$= 0$

$n = m$: $\int_{-\pi}^{\pi} \cos^2(nx)\,dx = \int_{-\pi}^{\pi} \frac{1}{2}\left(1+\cos(2nx)\right) dx$

$= \frac{1}{2}\left[x + \frac{\sin(2nx)}{2n}\right]_{-\pi}^{\pi}$

$= \frac{1}{2}\left(\pi + \underbrace{\frac{\sin(2n\pi)}{2n}}_{=0} - \left(-\pi + \underbrace{\frac{\sin(-2n\pi)}{2n}}_{=0}\right)\right)$

$= \frac{2\pi}{2} = \pi.$

Remark: $\{1, \cos(nx), \sin(nx)\}$

: orthogonal wrt $(\cdot,\cdot)$ on $(-\pi, \pi)$

Fourier series: $F(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n\cos(nx) + b_n\sin(nx)\right]$

$f(t)$ is continuous & periodic with $p = 2\pi$

Q $a_n = ?$ $b_n = ?$

Want $f(x) = F(x)$.

$(f(x), 1) = (F(x), 1)$

$= (a_0, 1) + \sum_{n=1}^{\infty} a_n \underbrace{(\cos(nx), 1)}_{=0}$

$+ \sum_{n=1}^{\infty} b_n \underbrace{(\sin(nx), 1)}_{=0}$

$\int_{-\pi}^{\pi} f(x)\cdot 1\,dx = \int_{-\pi}^{\pi} a_0\,dx = a_0\, 2\pi$

$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx.$

$(f(x), \cos(mx)) = (F(x), \cos(mx))$

$\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx = \underbrace{(a_0, \cos(mx))}_{=0}$

$+\sum_{n=1}^{\infty}[a_n \underbrace{(\cos(nx), \cos(mx))}_{\begin{cases}\pi & \text{if } n=m \\ 0 & \text{if } n\neq m\end{cases}} + b_n \underbrace{(\sin(nx), \cos(mx))}_{0}$

$\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx = \pi a_m.$

$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx$ $\quad m=1,2,\cdots$

$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)\,dx,$

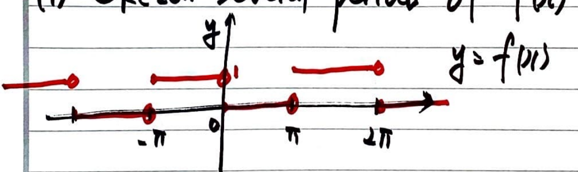
(Ex) 1. $f(x)=1.$ $x\in\mathbb{R}$: periodic, $p>0$.

$p=2\pi$: $F(x) = a_0 + \sum_{n=1}^{\infty}[a_n\cos(nx) + b_n\sin(nx)]$

$$\begin{cases} a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} 1\,dx = \frac{2\pi}{2\pi} = 1. \\ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} 1\cdot\cos(nx)\,dx = 0 \\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} 1\cdot\sin(nx)\,dx = 0 \end{cases}$$ $\therefore$ $\underline{F(x)=1}$

(Ex) $f(x)$ is periodic with $p=2\pi$ & one full period: $f(x) = \begin{cases}1, & -\pi\le x<0 \\ 0, & 0\le x<\pi\end{cases}$

(1) Sketch several periods of $f(x)$



$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{0} 1\,dx = \frac{\pi}{2\pi} = \frac{1}{2}$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{0}\cos(nx)\,dx$ $\quad z=nx,\ dz=n\,dx$

$= \frac{1}{\pi}\int_{-n\pi}^{0}\cos(z)\cdot\frac{1}{n}\,dz = \frac{1}{n\pi}\left[\sin(z)\right]_{-n\pi}^{0}$

$= \frac{1}{n\pi}(0 - \sin(-n\pi)) = 0$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{0}\sin(nx)\,dx$$

$$= \frac{1}{\pi}\left[-\frac{\cos(nx)}{n}\right]_{-\pi}^{0} = \frac{-1}{n\pi}\left[1-\cos(n\pi)\right]$$

$$= \frac{\cos(n\pi)-1}{n\pi}$$

$$F(x) = \frac{1}{2} + \sum_{n=1}^{\infty}\left[\frac{\cos(n\pi)-1}{n\pi}\right]\sin(nx)$$

Remark:

1. $\cos(n\pi) = (-1)^n$.

$n=0$: $\cos 0 = 1$, $(-1)^0 = 1$.

$n=1$: $\cos(\pi) = -1$, $(-1)^1 = -1$

$n=2$: $\cos(2\pi) = 1$, $(-1)^2 = 1$, ...

$$F(x) = \frac{1}{2} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n-1}{n\pi}\right]\sin(nx).$$

n: an even integer: $(-1)^n = 1$, $\underline{b_{2k} = 0.}$
$(n=2k,\ k\in\mathbb{Z}_+)$

n: an odd integer: $(-1)^n = -1$,
$(n=2k-1,\ k\in\mathbb{Z}_+)$ $\quad b_{2k-1} = \frac{-2}{(2k-1)\pi}$

$$F(x) = \frac{1}{2} + \sum_{k=1}^{\infty}\left(\frac{-2}{(2k-1)\pi}\right)\sin(2k-1)x.$$

Remark 2: $f(0) = 0$, $f(\pi) = 1$
$F(0) = \frac{1}{2}$, $F(\pi) = \frac{1}{2}$ different!

$$\Rightarrow F(0) = \frac{1}{2} = \frac{1}{2}(1+0),\ F(\pi) = \frac{0+1}{2}$$

Thm

(H) $f(x)$ is periodic in $\mathbb{R}$ with $p = 2\pi$

(C) 1. If $f(x)$ is continuous at $x_0 \in [-\pi, \pi]$,
then $F(x_0) = f(x_0)$

2. If $f(x)$ is not continuous at $x_0 \in [-\pi, \pi]$
with $L = \lim_{x\to x_0^-} f(x)$ & $R = \lim_{x\to x_0^+} f(x)$,

$\cos(nMx)$ have period $\underline{p=2L}$:

$n=1,2,\cdots$

$$\cos(nM(x+2L)) = \cos(nMx + nM2L)$$
$$= \cos(nMx).$$

$n=1$: $\cos(Mx + \underbrace{M2L}_{=2\pi}) = \cos(Mx)$

$$M = \frac{2\pi}{2L} = \frac{\pi}{L}. : \cos\left(\frac{n\pi x}{L}\right), \sin\left(\frac{n\pi x}{L}\right).$$

$$F(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

$a_n, b_n = ?$

$\left\{1, \cos\left(\frac{n\pi x}{L}\right), \sin\left(\frac{n\pi x}{L}\right)\right\}$: orthogonal on $\underline{(-L, L)}$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

(Ex) $f(x) = \begin{cases} 0, & -\frac{1}{2} \le x < 0 \\ x, & 0 \le x < \frac{1}{2} \end{cases}$: periodic in $\mathbb{R}$ with $p=1$

$L = \frac{1}{2}$