Gabriel Colangelo | Online | 1.2.3

4D Space so 4 Variables

3 non-Parallel planes so 3 linearly independent equations

Therefore 1 Free variable, so intersection is a line.

$u + v + w + z = 6$

$u + w + z = 4$

$u + w = 2$

$u = -1 \Rightarrow w = 2 - u = 3$

$z = 4 - u - w = 4 + 1 - 3 = 2$

$v = 6 - u - w - z = 6 + 1 - 3 - 2 = 2$

$$\boxed{\begin{pmatrix} u \\ v \\ w \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 3 \\ 2 \end{pmatrix}}$$

For no solution, let $\underline{u + w = 5}$. This causes a contradiction with $u + w = 2$.

1.2.8

$u + v + w = 2$ (1)

$u + 2v + 3w = 1$ (2)

$v + 2w = 0$ (3)

subtract (1) from (2)

$u + v + w = 2$ (1)

$v + 2w = -1$ (2)

$v + 2w = 0$ (3)

subtract (2) from (3)

$$\boxed{\begin{aligned} u + v + w &= 2 \\ v + 2w &= -1 \\ 0 &= 1 \end{aligned}}$$

To allow for solutions, $\boxed{\text{Change the zero to -1.}}$

$\therefore \; (v+2w) - (v+2w) = (-1) - (-1) = 0 = 0$

Choose free variable to be 0, $\therefore$ let $\underline{u=0} \Rightarrow$

$v + w = 2 \Rightarrow w = 2 - v$

$2v + 3w = 1 \Rightarrow 2v + 3(2-v) = 1 = 2v + 6 - 3v = 1$

$-v = -5 \Rightarrow v = 5$

$$\boxed{\underline{\text{Solution:}} \quad \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ -3 \end{pmatrix}}$$

1.3.11

$2x - 3y = 3$

$4x - 5y + z = 7$

$2x - y - 3z = 5$

subtract 2 times row 1 from row 2

$2x - 3y = 3$

$y + z = 1$

$2x - y - 3z = 5$

Subtract 1 times row 1 from row 3

$2x - 3y = 3$

$y + z = 1$

$2y - 3z = 2$

subtract 2 times row 2 from row 3

$2x - 3y = 3$

$y + z = 1$

$-5z = 0$

$z = 0 \Rightarrow y = 1 \Rightarrow x = \frac{3 + 3y}{2} = \frac{3+3}{2} = 3$

$$\boxed{\begin{aligned} x &= 3 \\ y &= 1 \\ z &= 0 \end{aligned}}$$

1.3.12

$2x + 5y + z = 0$ (1)
$4x + dy + z = 2$ (2)
$y - z = 3$ (3)

Subtract 2 times (1) from (2)

$2x + 5y + z = 0$
$(d-10)y - z = 2$
$y - z = 3$

If $d = 10$, then a row exchange must occur to obtain a triangular system. Therefore $\boxed{d = 10}$ forces a row exchange and the associated triangular system is:

$$\boxed{\begin{aligned} 2x + 5y + z &= 0 \\ y - z &= 3 \\ -z &= 2 \end{aligned}}$$

For a singular matrix (no third pivot) coefficient of z in last row must by 0.

$2x + 5y + z = 0$ (1)
$(d-10)y - z = 2$ (2)
$y - z = 3$ (3)

Subtract $\frac{1}{d-10}$ times (2) from (3)

1.3.12

$2x + 5y + z = 0$

$(d-10)y - z = 2$

$\left(-1 - \frac{1}{d-10}\right) z = 3 - \frac{2}{d-10}$

$-1 - \frac{1}{d-10} = 0 \Rightarrow \boxed{d = 11}$ - For no 3rd pivot

1.3.26

$x - y = 0$ (1)

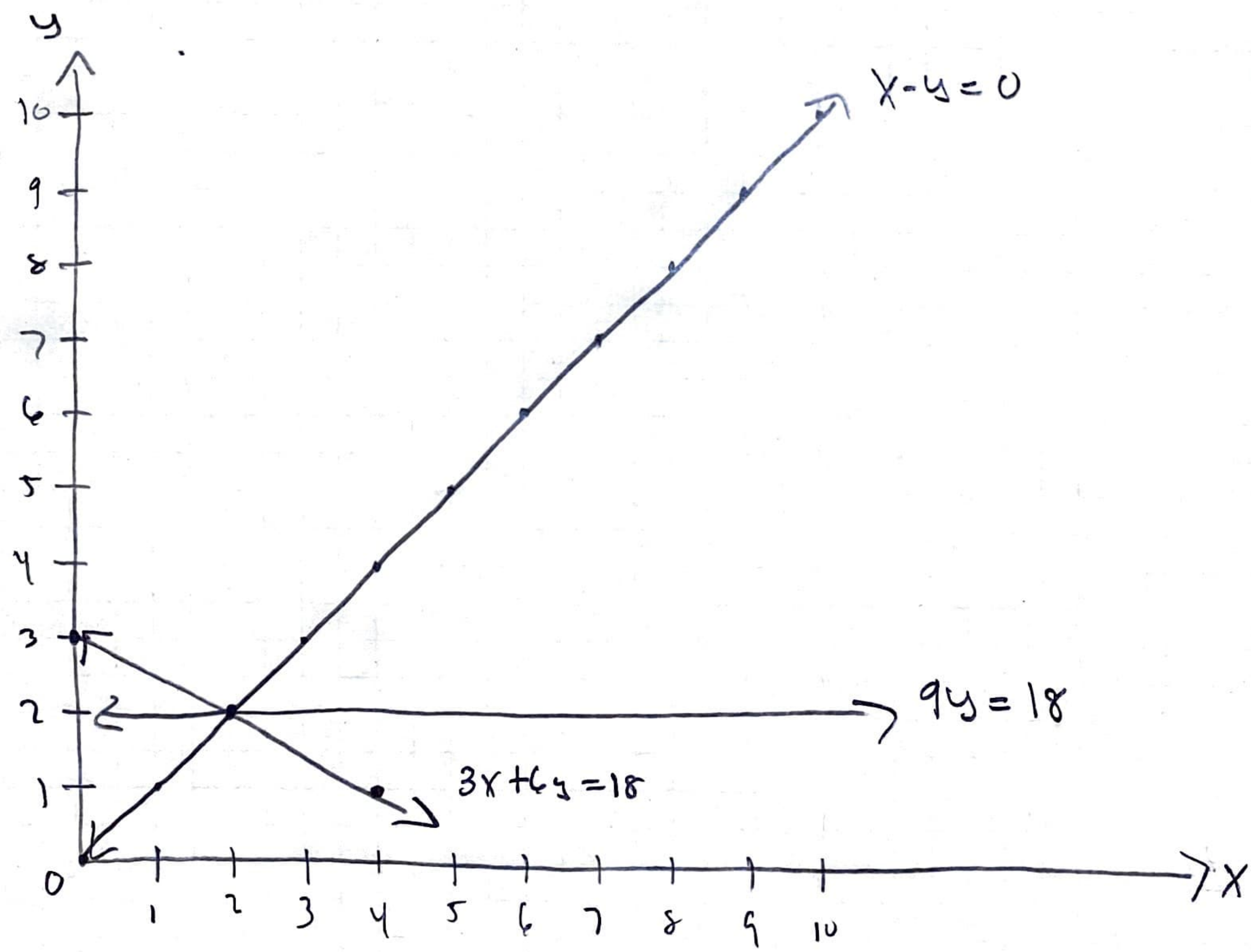
$3x + 6y = 18$ (2)

Subtract 3 times (1) from (2)

$x - y = 0$

$9y = 18 \Rightarrow y = 2$

$$\boxed{\begin{array}{l} x = 2 \\ y = 2 \end{array}}$$



1.4.2

$$\begin{pmatrix} 4 & 1 \\ 5 & 1 \\ 6 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

row 1: $(4)(1) + (1)(3) = 7$

row 2: $5(1) + (1)(3) = 8$

row 3: $6(1) + (1)(3) = 9$

$$\begin{pmatrix} 4 & 1 \\ 5 & 1 \\ 6 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \boxed{\begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

row 1: $(1)(0) + (2)(1) + (3)(0) = 2$

row 2: $(4)(0) + (5)(1) + (6)(0) = 5$

row 3: $(7)(0) + (8)(1) + (9)(0) = 8$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \boxed{\begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix}}$$

$$\begin{pmatrix} 4 & 3 \\ 6 & 6 \\ 8 & 9 \end{pmatrix}\begin{pmatrix} 1/2 \\ 1/3 \end{pmatrix}$$

row 1: $(4)(1/2) + (3)(1/3) = 3$

row 2: $(6)(1/2) + (6)(1/3) = 5$

row 3: $(8)(1/2) + (9)(1/3) = 7$

$$\begin{pmatrix} 4 & 3 \\ 6 & 6 \\ 8 & 9 \end{pmatrix}\begin{pmatrix} \frac{1}{2} \\ \frac{1}{3} \end{pmatrix} = \boxed{\begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix}}$$

1.4.3

$$\begin{bmatrix}1 & -2 & 7\end{bmatrix}\begin{bmatrix}1\\-2\\7\end{bmatrix} = (1)(1) + (-2)(-2) + (7)(7)$$

$$\begin{bmatrix}1 & -2 & 7\end{bmatrix}\begin{bmatrix}1\\-2\\7\end{bmatrix} = \boxed{54}$$

$$\begin{bmatrix}1 & -2 & 7\end{bmatrix}\begin{bmatrix}3\\5\\1\end{bmatrix} = (1)(3) + (-2)(5) + (7)(1)$$

$$\begin{bmatrix}1 & -2 & 7\end{bmatrix}\begin{bmatrix}3\\5\\1\end{bmatrix} = \boxed{0}$$

$$\begin{bmatrix}1\\-2\\7\end{bmatrix}\begin{bmatrix}3 & 5 & 1\end{bmatrix} = \begin{pmatrix}(1)(3) & (1)(5) & (1)(1)\\(-2)(3) & (-2)(5) & (-2)(1)\\(7)(3) & (7)(5) & (7)(1)\end{pmatrix}$$

$$\begin{bmatrix}1\\-2\\7\end{bmatrix}\begin{bmatrix}3 & 5 & 1\end{bmatrix} = \boxed{\begin{pmatrix}3 & 5 & 1\\-6 & -10 & -2\\21 & 35 & 7\end{pmatrix}}$$

1.4.13

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$a^2 + bc = -1$

$ab + bd = 0$

$ca + dc = 0$

$cb + d^2 = -1$

$bc = -1 - a^2 = -1 - d^2 \quad \therefore \quad a = d$

$ab + bd = 2bd = 0$

$ca + dc = 2cd = 0$

Let $a = d = 0$.

$bc = -1$, let $c = 1$, $b = -1 \quad \therefore$

$$\boxed{A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$a^2 + bc = 0$

$ab + bd = 0$

$ca + dc = 0$

$cb + d^2 = 0$

$cb = -a^2 = -d^2 \quad \therefore \quad a = d = cb$

$ab + bd = 2bd = 0$

$ca + dc = 2cd = 0$

1.4.13

Let $a = d = 0$

$cb = 0$, let $c = 0$, $b = 1$

$\therefore$ $\boxed{B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = -\begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$C \quad D \quad = - \quad D \quad C$

$$\begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix} = -\begin{pmatrix} ea + fc & eb + fd \\ ga + hc & gb + hd \end{pmatrix}$$

$ae + bg = -ea - fc \Rightarrow 2ae = -bg - fc$

$af + bh = -eb - fd$

$ce + dg = -ga - hc$

$cf + dh = -gb - hd \Rightarrow 2dh = -bg - fc$

$\therefore$ $dh = ae$

Let $d = h = a = e = 0$

$\therefore$ $-bg = fc$

Let $f = c = g = 1$ $\therefore$ $b = -1$

1.4.13

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$-\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\boxed{C = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad D = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

$ae + bg = 0$

$af + bh = 0$

$ce + dg = 0$

$cf + dh = 0$

Let $a = 1 = c$:

$e = -bg \,,\; f = -bh \qquad -b = \frac{f}{h} = \frac{e}{g}$

$e = -dg \,,\; f = -dh \qquad -d = \frac{f}{h} = \frac{e}{g}$

$\therefore\; b = d$, let $h = g = 1 \;\therefore\; f = e = -b = -d$

Let $f = e = 1 \;\therefore\; b = d = -1$

1.4.13

$$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\boxed{E = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \qquad F = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}}$$

1.4.17

$(A+B)^2 = (A+B)(A+B)$

$(A+B)(A+B) = A^2 + AB + BA + B^2$ ✓

$A^2 + AB = A(A+B)$

$BA + B^2 = B(A+B)$

$A(A+B) + B(A+B)$ ✓

$(A+B)(B+A) = AB + A^2 + B^2 + BA$

$= A^2 + AB + BA + B^2$ ✓

$\therefore$

$$\boxed{\begin{aligned} (A+B)^2 &= \\ & A^2 + AB + BA + B^2 \\ & A(A+B) + B(A+B) \\ & (A+B)(B+A) \end{aligned}}$$

$AB \neq BA \quad \therefore (A+B)^2 \neq A^2 + 2AB + B^2$

1.4.29

$$\boxed{E_{13} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

E_{31} - Adds row 1 to row 3: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

Matrix E adds row 1 to row 3 and row 3 to row 1:

$$\boxed{E = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}}$$

$E_{13}\ E_{31}$ adds row 1 to row 3, then adds row 3 to row 1.

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \boxed{\begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}}$$

E_{31} Acts First