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8.4.10 $A := \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$.

$$|A - \lambda I| = (1-\lambda)(-1-\lambda)$$
$$= -1 - \lambda + \lambda + \lambda^2$$
$$= -1 + \lambda^2$$
$$\Rightarrow \lambda \in \{\pm 1\}.$$

$\underline{\lambda_1 := 1}$

$$\left[\begin{array}{cc|c} 1 & 0 & x_1 \\ 2 & -1 & x_2 \end{array}\right]$$

$\Rightarrow x_1 = x_1 , \quad 2x_1 - x_2 = x_2$

$\Rightarrow x_1 \in \mathbb{R}, \quad x_2 = x_1$

Let $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ i.e. $Av_1 = \lambda_1 v_1$.

$\underline{\lambda_2 := -1}$

$$\left[\begin{array}{cc|c} 1 & 0 & -x_1 \\ 2 & -1 & -x_2 \end{array}\right]$$

$\Rightarrow \quad x_1 = -x_1, \qquad 2x_1 - x_2 = -x_2$

$\Rightarrow \quad x_1 = 0, \qquad x_2 \in \mathbb{R}$

Let $\quad V_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad$ i.e. $\quad A v_2 = \lambda_2 v_2$.

Eigenbasis $\longrightarrow \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$.

Diagonalization:

Let $\quad X = \begin{bmatrix} | & | \\ V_1 & V_2 \\ | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

$$X^{-1} = \frac{1}{|X|} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$D := X^{-1} A X = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

8.4.24 Let $\lambda_1, \dots, \lambda_n$ be the eigvals of $\underline{A}$.

[This is the symmetric coefficient of the quadratic form given.]

<u>$Q(\underline{x})$ is positive definite iff $\lambda_1, \dots, \lambda_n > 0$.</u>:

$Q(\underline{x}) = \underline{y}^T D \underline{y} = \lambda_1 y_1^2 + \dots + \lambda_n y_n^2$,

where $\underline{y} = X^{-1}\underline{x}$ (see pg. 343 of textbook).

$\underline{\Rightarrow}$:

Suppose $Q(\underline{x})$ is positive definite.

$\Rightarrow$ $Q(\underline{x}) = \lambda_1 y_1^2 + \dots + \lambda_n y_n^2 > 0$, for all $\underline{y} \neq \underline{0}$

In particular, the above line is true for $\underline{y} = [0, 0, \dots, 0, 1, 0, \dots, 0]^T$,

$\uparrow$ ith position

for all $i \in \{1, \dots, n\}$.

$\Rightarrow$ For all $i \in \{1, \dots, n\}$, $\lambda_i > 0$. ✓

$\Leftarrow$:

Suppose $\lambda_1, \dots, \lambda_n > 0$.

$\Rightarrow$ For all $\underline{y} \neq \underline{0}$,

$Q(\underline{x}) = \underbrace{\lambda_1}_{>0} \underbrace{y_1^2}_{\geq 0} + \dots + \underbrace{\lambda_n}_{>0} \underbrace{y_n^2}_{\geq 0}$, and

at least one of $y_1, \dots, y_n$, is nonzero, say y_i, so,

$$
\begin{aligned}
Q(x) &= \lambda_1 y_1^2 + \dots + \lambda_n y_n^2 \\
&\geq \lambda_i y_i^2 \\
&> 0
\end{aligned}
$$

✓

The cases (a) and (b) are proven similarly.

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8.5.5 Let $A := \begin{bmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix}$

$$\bar{A}^T = \begin{bmatrix} -i & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{bmatrix}$$

$\Rightarrow$ A is <u>not</u> Hermitian.
A <u>is</u> skew-Hermitian.

$$A\bar{A}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow$ A <u>is</u> unitary.

see comments about inverse from previous sol'n.s

$$|A - \lambda I| = (i - \lambda)\left((0-\lambda)^2 + 1\right)$$

$$\Rightarrow \lambda \in \{i, -i\}$$ ← eigenvalues

$\underline{\lambda_1 = i}$:

$$\left[\begin{array}{ccc|c} i & 0 & 0 & i x_1 \\ 0 & 0 & i & i x_2 \\ 0 & i & 0 & i x_3 \end{array}\right]$$

$$\Rightarrow x_1 = x_1 , \quad x_3 = x_2$$

Let $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

<u>Aside:</u>

($v_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ would also work, but notice how we picked orthogonal eigenvalues)

$\underline{\lambda_2 = -i}$

$$\left[\begin{array}{ccc|c} i & 0 & 0 & -i x_1 \\ 0 & 0 & i & -i x_2 \\ 0 & i & 0 & -i x_3 \end{array}\right]$$

$\Rightarrow \quad x_1 = -x_1 \;, \quad x_3 = -x_2$

$\Rightarrow \quad x_1 = 0, \quad x_3 = -x_2$

Let $\quad V_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

Eigvals	Eigvects
i	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$
$-i$	$\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

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12. (a) $y_1 = y,\ y_2 = y',\ y_3 = y'',\ y_4 = y'''$

$$y_4 = -2y_3 + y_2 + 2y_1$$

$$\underset{\underline{y}'}{\begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix}} = \underset{A}{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & -2 \end{bmatrix}} \underset{\underline{y}}{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}}$$

$$|A - \lambda| = \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 2 & 1 & -2-\lambda \end{bmatrix}$$

$$= -\lambda(2\lambda + \lambda^2 - 1) - 1(-2)$$

$$= -\lambda^3 - 2\lambda^2 + \lambda + 2$$

$$= -\lambda^2(\lambda + 2) + \lambda + 2$$

$$= (\lambda + 2)(1 - \lambda^2)$$

$\Rightarrow \lambda \in \{-2, \pm 1\}.$

$\underline{\lambda_1 = -2}$:

$\left(\text{Supposing} \quad \underline{y} = \underline{v}\, e^{\lambda t} \right.$
$\left. \Rightarrow \ \dots \ \Rightarrow \ A\underline{v} = \lambda \underline{v} \right)$

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & -2y_1 \\ 0 & 0 & 1 & -2y_2 \\ 2 & 1 & -2 & -2y_3 \end{array}\right]$$

$$\Rightarrow \left.\begin{aligned} y_2 &= -2y_1 \\ y_3 &= -2y_2 \end{aligned}\right\} \quad \begin{aligned} y_2 &= -2y_1 \\ y_3 &= -2(-2y_1) = 4y_1 \end{aligned}$$

$$2y_1 + y_2 \cancel{-2y_3} = \cancel{-2y_3}$$

$$\Rightarrow 2y_1 + (-2y_1) = 0, \quad y_2 = -2y_1, \quad y_3 = 4y_1$$

Let $v_1 = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$

$\underline{\lambda_2 = -1}$:

$$\left.\begin{aligned} y_2 &= -y_1 \\ y_3 &= -y_2 \end{aligned}\right\} \quad \begin{aligned} y_2 &= -y_1 \\ y_3 &= -(-y_1) = y_1 \end{aligned}$$

$$2y_1 + y_2 - 2y_3 = -y_3$$

$$\Rightarrow \cancel{2y_1} + (-y_1) \cancel{-2y_1} = -y_1, \quad y_2 = -y_1, \quad y_3 = y_1$$

Let $V_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

$\underline{\lambda_3 = 1}$:

$$\left.\begin{aligned} y_2 &= y_1 \\ y_3 &= y_2 \end{aligned}\right\} \quad \begin{aligned} y_2 &= y_1 \\ y_3 &= y_1 \end{aligned}$$

$$2y_1 + y_2 - 2y_3 = y_3$$

$$\Rightarrow \cancel{2y_1} + y_1 - \cancel{2y_1} = y_1 \ , \quad y_2 = y_1 \ , \quad y_3 = y_1$$

Let $V_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\therefore$

General soln. :

$$\underline{y} = c_1 V_1 e^{\lambda_1 t} + C_2 V_2 e^{\lambda_2 t} + C_3 V_3 e^{\lambda_3 t}$$

(since we assumed $\underline{y} = \underline{v} e^{\lambda t}$
$\Rightarrow \underline{y}' = \underline{v} \lambda e^{\lambda t}$)

$$\Rightarrow \boxed{y_1 = c_1 e^{-2t} + C_2 e^{-t} + C_3 e^{t}.}$$

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4.3.18

$$y_1' = \overbrace{\frac{12}{200}}^{\text{percentage of gallons of water flowing in per min.}} \cdot \overbrace{0}^{\text{pounds of salt flowing in per gallon of water}} + \frac{4}{200} y_2 - \frac{16}{200} y_1$$

$$y_2' = -\frac{4}{200} y_2 + \frac{16}{200} y_1 - \frac{12}{200} \cdot y_2$$

$$\underbrace{\begin{bmatrix} y_1' \\ y_2' \end{bmatrix}}_{\underline{y}'} = \underbrace{\begin{bmatrix} -16/200 & 4/200 \\ 16/200 & -16/200 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{\underline{y}}$$

Similarly, to previous problems, for A,

Eigvals	Eigvects
$\lambda_1 = -3/25$	$v_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$
$\lambda_2 = -1/25$	$v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\therefore$ General soln. is

$$\underline{y} = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t},$$

where $c_1, c_2 \in \mathbb{R}$.

Applying initial conditions:

@ $t=0$: $\begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 100 \\ 200 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 100 \\ 200 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\Rightarrow \dots \Rightarrow c_1 = 0, \quad c_2 = 100$$

$\Rightarrow$ Solution is

$$\boxed{\begin{aligned} y_1 &= 100e^{-\frac{1}{25}t}, \\ y_2 &= 200e^{-\frac{1}{25}t}. \end{aligned}}$$

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4.4.5 Let $A = \begin{bmatrix} -2 & 2 \\ -2 & -2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$p = a_{11} + a_{22} = -4$

$q = |A| = 8$

$\Delta = p^2 - 4q = -16$

$p < 0, \quad q > 0, \quad \Delta < 0$

$\Rightarrow$ attractive spiral point @ origin.

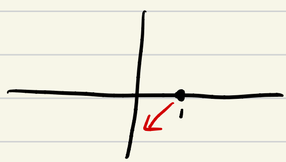
To check clockwise v.s. anti-clockwise:

test a non-origin pt.

@ $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} -2 + 0 \\ -2 + 0 \end{bmatrix}$

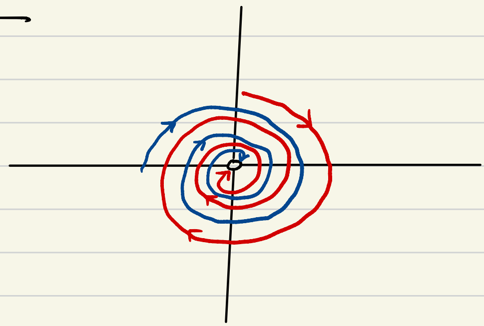
$\Rightarrow$ trajectory at this point is $\swarrow$

Picture:



$\therefore$ Spiral is clockwise.

Sketch:



Omitting details from a process we're seen before:

Eig vals	Eigvects
$-2+2i$	$\begin{bmatrix} -i \\ 1 \end{bmatrix}$
$-2-2i$	$\begin{bmatrix} i \\ 1 \end{bmatrix}$

So, a general soln. is

$$\boxed{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c_1 \begin{bmatrix} -i \\ 1 \end{bmatrix} e^{(-2+2i)t} + c_2 \begin{bmatrix} i \\ 1 \end{bmatrix} e^{(-2-2i)t}}$$

<u>Aside</u>:

$$\underbrace{\begin{bmatrix} -i \\ 1 \end{bmatrix} e^{(-2+2i)t}} \quad \text{is a soln}$$

$$= e^{-2t} \begin{bmatrix} -i \\ 1 \end{bmatrix} (\cos(2t) + i\sin(2t))$$

$$= e^{-2t} \begin{bmatrix} -i\cos(2t) + \sin(2t) \\ \cos(2t) + i\sin(2t) \end{bmatrix}$$

$$= e^{-2t} \left(\begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix} + i \begin{bmatrix} -\cos(2t) \\ \sin(2t) \end{bmatrix} \right)$$

So, considering linearity of differential operator (see superposition principle)

$$e^{-2t}\begin{bmatrix}\sin(2t)\\ \cos(2t)\end{bmatrix} \quad \text{and} \quad e^{-2t}\begin{bmatrix}-\cos(2t)\\ \sin(2t)\end{bmatrix}$$

are lin. indep. solns.

$\therefore$ The general soln. is

$$\boxed{\begin{aligned} y_1 &= \alpha_1 e^{-2t}\sin(2t) - \alpha_2 e^{-2t}\cos(2t), \\ y_2 &= \alpha_1 e^{-2t}\cos(2t) + \alpha_2 e^{-2t}\sin(2t). \end{aligned}}$$

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4.5.7 Find critical pts. :

$$-y_1 + y_2 - y_2^2 = 0 = -y_1 - y_2$$

$$\Rightarrow \quad 2y_2 - y_2^2 = 0, \qquad y_1 = -y_2$$

$$\Rightarrow \quad y_2 = 0 \quad \text{and} \quad y_1 = 0$$

OR

$$y_2 = 2 \quad \text{and} \quad y_1 = -2$$

Critical points: $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix} \right\}$

Jacobian, $J(y_1, y_2) = \begin{bmatrix} \frac{\partial F_1}{\partial y_1} & \frac{\partial F_1}{\partial y_2} \\ \frac{\partial F_2}{\partial y_1} & \frac{\partial F_2}{\partial y_2} \end{bmatrix}$

$$= \begin{bmatrix} -1 & 1 - 2y_2 \\ -1 & -1 \end{bmatrix}$$

@ $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$:

$$J(0,0) = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$$

$\Rightarrow p = -2 < 0$

$q = 2 > 0$

$\Delta = -4 < 0$

$\therefore$ Attractive spiral pt.

@ $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$:

$$J(-2,2) = \begin{bmatrix} -1 & -3 \\ -1 & -1 \end{bmatrix}$$

$\Rightarrow p = 2 > 0$

$q = -2 < 0$

$\therefore$ Saddle pt.

4.5.11 Let $y_1 = y$, $y_2 = y'$, $y_3 = y''$.

$$\left.\begin{aligned} y_1' &= y_2 \\ y_2' &= -\cos(y_1) \end{aligned}\right\} \text{system}$$

<u>Critical pts:</u>

$y_2 = 0 = -\cos(y_1)$

$\Rightarrow$ Critical pts are $\begin{bmatrix} \pi(z+\frac{1}{2}) \\ 0 \end{bmatrix}$, $z \in \mathbb{Z}$.

<u>Jacobian:</u>

$$J(y_1, y_2) = \begin{bmatrix} 0 & 1 \\ \sin(y_1) & 0 \end{bmatrix}$$

$p = 0$

$q = -\sin(y_1)$

and $\sin(\pi(z+\frac{1}{2})) = \begin{cases} 1, & z \text{ even}, \\ -1, & z \text{ odd}. \end{cases}$

$\Rightarrow p = 0$

$q \quad \begin{matrix} > 0, & \text{when } z \text{ is odd} \\ < 0, & \text{―― '' ―― even} \end{matrix}$

$\Rightarrow$ saddle pt., when z is even

centre, when z is odd

$\therefore$ Saddle pt.s @ $\begin{bmatrix} \pi(z + 1/2) \\ 0 \end{bmatrix}$,

where $z \in \{\ldots, -4, -2, 0, 2, 4, \ldots\}$.

Centres @ $\begin{bmatrix} \pi(z + 1/2) \\ 0 \end{bmatrix}$,

where $z \in \{\ldots, -3, -1, 1, 3, \ldots\}$

Remark:

Geometrically, if we have centres along the y_1-axis, in order for the trajectories to "smoothly" morph into each other, we need a saddle pt. in between them.