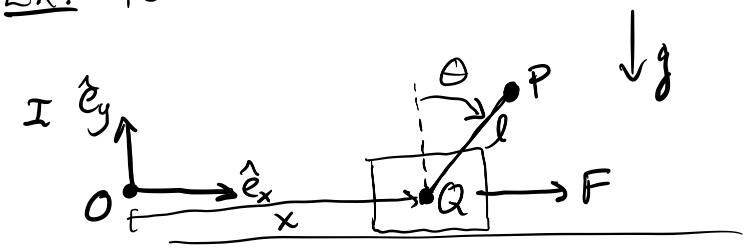
Ex. Pendulum on a cart



D.O.F. & Generalized Coordinates

Planar

$$M = 2N - K_H - \cancel{K_S}^{\,0}$$

$\hookrightarrow y_Q = 0$

$\hookrightarrow (x_P - x_Q)^2 + (y_P - y_Q)^2 = l^2$

$$M = 2(2) - 2 = 2$$

Kinematics

$$\vec{r}_{Q/O} = x\,\hat{e}_x$$

$${}^I\vec{V}_{Q/O} = \dot{x}\,\hat{e}_x$$

$$\vec{r}_{P/O} = x\,\hat{e}_x + l c\theta\,\hat{e}_y + l s\theta\,\hat{e}_x$$

$$= (x + l s\theta)\,\hat{e}_x + l c\theta\,\hat{e}_y$$

$${}^I\vec{V}_{P/O} = (\dot{x} + l c\theta\,\dot{\theta})\,\hat{e}_x - l s\theta\,\dot{\theta}\,\hat{e}_y$$

Note: There is no need to calculate accelerati for the Lagrangian method.

Kinetic Energy

$$T_O = T_{P/O} + T_{Q/O} = \frac{1}{2} m_P \|{}^I\vec{V}_{P/O}\|^2 + \frac{1}{2} m_Q \|{}^I\vec{V}_{Q/O}\|^2$$

$$= \frac{1}{2} m_P \left[(\dot{x} + l c\theta\,\dot{\theta})^2 + (l s\theta\,\dot{\theta})^2 \right] + \frac{1}{2} m_Q \dot{x}^2$$

Potential Energy

$$U_O = U_{P/O} + \cancel{U_{Q/O}}^{\,0}$$

$\hookrightarrow$ $$U_{P/O} = -\int \vec{F}_g \cdot {}^I d\vec{r}_{P/O}$$

$$= -\int (-m g\,\hat{e}_y) \cdot \left[\cancel{(dx + l c\theta\, d\theta)\,\hat{e}_x} - l s\theta\, d\theta\,\hat{e}_y \right]$$

$$= -m_P g l \int s\theta\, d\theta = m_P g l \cos\theta$$

Lagrangian

$L_0 = T_0 - U_0$

$L_0 = \frac{1}{2} m_p\left[(\dot{x} + \ell c\theta\dot{\theta})^2 + (\ell s\theta\dot{\theta})^2\right] + \frac{1}{2} m_Q \dot{x}^2 - m_p g \ell c\theta$

Calculate partial derivatives:

For $q_1 = x$,

$$\begin{cases} \dfrac{\partial L}{\partial \dot{q}_1} = \dfrac{\partial L}{\partial \dot{x}} = m_p(\dot{x} + \ell c\theta\dot{\theta}) + m_Q\dot{x} \\ \dfrac{\partial L}{\partial q_1} = \dfrac{\partial L}{\partial x} = 0 \end{cases}$$

(We will see that x here is what is known as a "cyclic" variable)

For $q_2 = \theta$,

$$\begin{cases} \dfrac{\partial L}{\partial \dot{\theta}} = m_p\left[(\dot{x} + \ell c\theta\dot{\theta})\ell c\theta + \ell^2 s^2\theta\dot{\theta}\right] = m_p\left[\dot{x}\ell c\theta + \ell^2\dot{\theta}\right] \\ \dfrac{\partial L}{\partial \theta} = m_p\left[-\dot{x}\ell s\theta\dot{\theta} + g\ell s\theta\right] \end{cases}$$

But we also need the $Q_j^{(nc,a)}$ forces.

Recall $Q_j^{(nc,a)} = \sum_{i=1}^{N} \vec{F}_i^{(nc,a)} \cdot \dfrac{\partial \vec{r}_{i/O}}{\partial q_j}$ for $j = 1, \ldots, N_c$

$\vec{F}_Q^{(nc,a)} = F\hat{e}_x \qquad \vec{F}_P^{(nc,a)} = 0$

Then, $Q_x = \vec{F}_Q^{(nc,a)} \cdot \frac{\partial}{\partial x}(\vec{r}_{Q/O}) + \vec{F}_P^{(nc,a)} \cdot \frac{\partial}{\partial x}(\vec{r}_{P/O})$

$$= F\hat{e}_x \cdot \frac{\partial}{\partial x}(x\hat{e}_x) + 0 \cdot \frac{\partial}{\partial x}(\vec{r}_{P/O})$$

$$= F$$

$$Q_\theta = \vec{F}_Q^{(nc,a)} \cdot \frac{\partial}{\partial \theta}(\vec{r}_{Q/O}) + \vec{F}_P^{(nc,a)} \cdot \frac{\partial}{\partial \theta}(\vec{r}_{P/O})$$

$$= \vec{F}_Q^{(nc,a)} \cdot 0 + 0 \cdot \frac{\partial(\vec{r}_{P/O})}{\partial \theta} = 0$$

Plug into the Euler-Lagrange Equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j^{(nc,a)}$$

For $q_1 = x$,

$$\Rightarrow \frac{d}{dt}\left(m_p(\dot{x} + lc\theta\dot{\theta}) + m_Q\dot{x}\right) = F$$

$$\boxed{m_p(\ddot{x} + lc\theta\ddot{\theta} - ls\theta\dot{\theta}^2) + m_Q\ddot{x} = F} \quad \text{Eqn ①}$$

For $q_2 = \theta$,

$$\Rightarrow \ddot{x}lc\theta - \cancel{\dot{x}ls\theta\dot{\theta}} + l^2\ddot{\theta} + \cancel{\dot{x}ls\theta\dot{\theta}} - gls\theta = 0$$

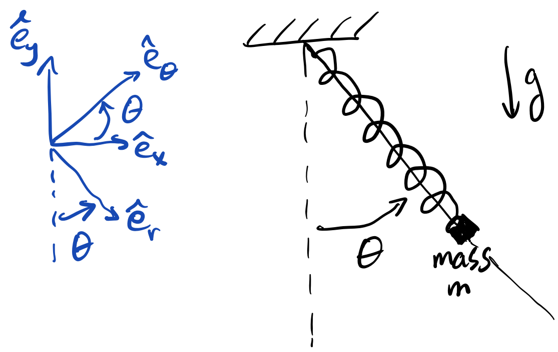
$$\Rightarrow \boxed{\ddot{x}lc\theta + l^2\ddot{\theta} - gls\theta = 0} \quad \text{Eqn ②}$$

2 Equations and 2 unknowns $\ddot{x}$ and $\ddot{\theta}$

Note: • F is not considered an unknown here.
• We didn't calculate accelerations or constraints! ☺

Spring-Pendulum Example

Bead of mass m on a massless rod with a spring.



Find the E.O.M.

$$M = 2N - \cancel{K_H}^{\,0} - \cancel{K_S}^{\,0}$$
$$= 2(1) = 2$$

Kinematics

$$\vec{r}_{P/O} = r\hat{e}_r$$
$$^{I}\vec{v}_{P/O} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$$

Euler-Lagrange Equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j^{(nc,a)} \quad \text{where} \quad Q_j^{(nc,a)} = \sum_{i=1}^{N} \vec{F}_i^{(nc,a)} \cdot \frac{\partial \vec{r}_{i/O}}{\partial q_j}$$

for $j = 1, \dots, N_c$

$$N_c = 2 \Rightarrow q = (q_1, q_2)^T = (r, \theta)^T$$

Find Lagrangian $L_O = T_O - U_O$

$$T_O = \frac{1}{2} m_P \|{}^{I}\vec{v}_{P/O}\|^2 = \frac{1}{2} m_P \left(\dot{r}^2 + (r\dot{\theta})^2\right)$$

$$U_O = \frac{1}{2}kr^2 - mgr\cos\theta$$

(Is this right? Built based on K+P Tutorial 5.3)

$$L_O = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{1}{2}kr^2 + mgr\cos\theta$$

For $q_1 = r$,

$$\frac{d}{dt}\left(\frac{\partial L_O}{\partial \dot{r}}\right) - \frac{\partial L_O}{\partial r} = 0$$

and

for $q_2 = \theta$,

$$\frac{d}{dt}\left(\frac{\partial L_O}{\partial \dot{\theta}}\right) - \frac{\partial L_O}{\partial \theta} = 0$$

$$\frac{\partial L_0}{\partial \dot r} = m\dot r \qquad \frac{\partial L_0}{\partial \dot\theta} = mr^2\dot\theta$$

$$\frac{\partial L_0}{\partial r} = mr\dot\theta^2 - kr + mg\cos\theta \qquad \frac{\partial L_0}{\partial \theta} = -mgr\sin\theta$$

$$\frac{d}{dt}(mr^2\dot\theta) + mgr\sin\theta = 0$$

$$\boxed{m\ddot r - mr\dot\theta^2 + kr - mg\cos\theta = 0} \qquad \boxed{mr^2\ddot\theta + 2mr\dot r\dot\theta + mgr\sin\theta = 0}$$

What if we don't include the spring potential in the potential energy? (Instead, let's treat it as an applied force)

$$T_0 = \frac{m}{2}(\dot r^2 + r^2\dot\theta^2)$$

$$U_0 = -mgr\cos\theta$$

$$Q_r = -kr\hat e_r \cdot \frac{\partial(r\hat e_r)}{\partial r} = -kr$$

$$Q_\theta = 0$$

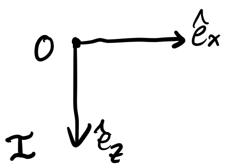
Spring force appeared in $\frac{d}{dt}\left(\frac{\partial L_0}{\partial \dot r}\right) - \frac{\partial L_0}{\partial r} = Q_r$

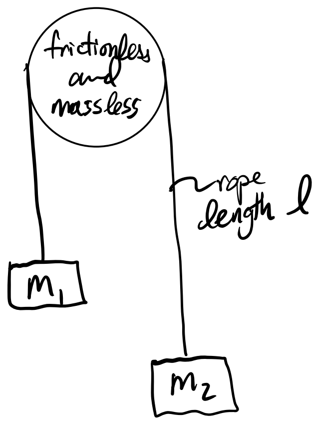
$$\Rightarrow \boxed{m\ddot r - mr\dot\theta^2 - mg\cos\theta = -kr}$$

Similarly, we could have done this for gravity as well. In which case, we would have been implementing the first form of the E.L. equations.

$$\frac{d}{dt}\left(\frac{\partial T_0}{\partial \dot q_j}\right) - \frac{\partial T_0}{\partial q_j} = Q_j^{(a)}$$

Ex. Atwood's Machine





DOF:

$$M = 2N - K_H - \cancel{K_S}^{\,0}$$
$$= 2(2) - 3 - 0 = 1$$

$\hookrightarrow$ $x_1 = \text{const.}$
$x_2 = \text{const.}$
$\|\vec{r}_{1/O} + \vec{r}_{2/O}\| = \ell$

No non-holonomic constraints
$\Rightarrow N_c = M$

Kinematics

$$\vec{r}_{1/O} = z\hat{e}_z \qquad \vec{r}_{2/O} = (\ell - z)\hat{e}_z$$

$$^{I}\vec{v}_{1/O} = \dot{z}\hat{e}_z \qquad ^{I}\vec{v}_{2/O} = -\dot{z}\hat{e}_z$$

$$T_O = \frac{m_1}{2}\|^{I}\vec{v}_{1/O}\|^2 + \frac{m_2}{2}\|^{I}\vec{v}_{2/O}\|^2 = \frac{m_1 + m_2}{2}\dot{z}^2$$

$$U_O = U_{1/O} + U_{2/O} = -\int \vec{F}_g \cdot {}^{I}d\vec{r}_{1/O} + -\int \vec{F}_g \cdot {}^{I}d\vec{r}_{2/O}$$
$$= -\int m_1 g\hat{e}_z \cdot dz\hat{e}_z + \ldots$$
$$= -m_1 g z - m_2 g(\ell - z)$$

$$L_O = T_O - U_O = \frac{m_1 + m_2}{2}\dot{z}^2 + m_1 g z + m_2 g(\ell - z)$$

Euler-Lagrange: $\displaystyle \frac{d}{dt}\left(\frac{\partial L_O}{\partial \dot{q}_j}\right) - \frac{\partial L_O}{\partial q_j} = Q_j^{(nc,a)}$ for $j = 1, \ldots, N_c$

$$\frac{\partial L_O}{\partial \dot{z}} = (m_1 + m_2)\dot{z} \qquad \frac{\partial L_O}{\partial z} = m_1 g - m_2 g = (m_1 - m_2)g$$

E.O.M. $\boxed{(m_1 + m_2)\ddot{z} - (m_1 - m_2)g = 0}$

Non-uniqueness of a Lagrangian

The Lagrangian of a system is <u>not</u> unique

Consider $L = L(q, \dot{q}, t)$ where $q = (q_1, \ldots, q_{N_c})^T$, $\dot{q} = (\dot{q}_1, \ldots, \dot{q}_{N_c})^T$

Let $F(q)$ be a scalar-valued function of just the generalized coordinates.

Then define
$$L' = L + \frac{dF}{dt} = L + \frac{\partial F}{\partial q}\dot{q}$$

Does L' satisfy the EL equations?

$$\frac{\partial L'}{\partial \dot{q}} = \underbrace{\frac{\partial L}{\partial \dot{q}} + \frac{\partial F}{\partial q}}$$

$$\frac{\partial L'}{\partial q} = \frac{\partial L}{\partial q} + \frac{\partial}{\partial q}\left(\frac{dF}{dt}\right) = \underbrace{\frac{\partial L}{\partial q} + \frac{d}{dt}\left(\frac{\partial F}{\partial q}\right)}$$

E.L. Eqn:
$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}}\right) - \frac{\partial L'}{\partial q} \overset{?}{=} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} + \cancel{\frac{\partial F}{\partial q}}\right) - \left(\frac{\partial L}{\partial q} + \cancel{\frac{d}{dt}\left(\frac{\partial F}{\partial q}\right)}\right) \overset{?}{=} \text{E.L. of } L$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = \text{E.L. of } L$$

$\Rightarrow$ Two different Lagrangians can lead to the same equations of motion.

Alternative way to derive the E.L. equations

Calculus of Variations & Hamilton's principle } See Ch2 of Goldstein

Hamilton's principle – The "action integral" $I = \int_{t_1}^{t_2} L\, dt$ has a stationary value for the actual path of motion of a system $\Rightarrow \boxed{\delta I = 0}$

$L = L(q_1, \ldots, q_{N_c})$

Recall $\delta L = \sum_k \frac{\delta L}{\delta q_k} \delta q_k$

$$\delta I = \delta \int L\, dt = \int \delta L\, dt = \int \sum_k \frac{\delta L}{\delta q_k} \delta q_k\, dt$$

Hand & Finch provide a trick to side-step many of the calculus of variations that is need for these computations.

Def'n of a notation for the variational derivative

$$\frac{\delta L}{\delta q_k} \triangleq \frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)$$

See discussion in Hand & Finch on the $\frac{\delta L}{\delta q_k}$ notation.

Then, plugging in,

$$\delta I = \int_{t_1}^{t_2} \sum_{k=1}^{N_c} \left(\frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)\right) \delta q_k\, dt = 0$$

See the reasoning from D'Alembert's principle to conclude that

$$\underbrace{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0}_{\substack{\text{E.L. eqns} \\ \text{when there are no} \\ \text{nonconservative, applied forces.}}} \quad \text{for } k = 1, \ldots, N_c$$

What happens if you can't easily find independent generalized coordinates or you want to work in a set that is not independent?

↳ We can use the method of Lagrange multipliers and Hamilton's principle.

Method of Lagrange Multipliers (for holonomic constraints)

- Allows for the treatment of the q_j's independently, provided we introduce a new freedom in form of a Lagrange multiplier.

If q_j's are not independent, then there exists the constraint

$$f(q_1, \ldots, q_{N_c}) = 0$$

Define the effective Lagrangian,

$$L' \triangleq L + \underset{\substack{\uparrow \\ \text{Lagrange multiplier}}}{\lambda} f$$

Aside on optimization

→ Concepts from Calc III
→ Course on optimization

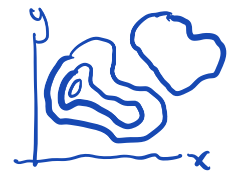
$$\min_{\underline{x}} \; J(\underline{x})$$ } Unconstrained optimization problem

First-order necessary conditions for optimality,

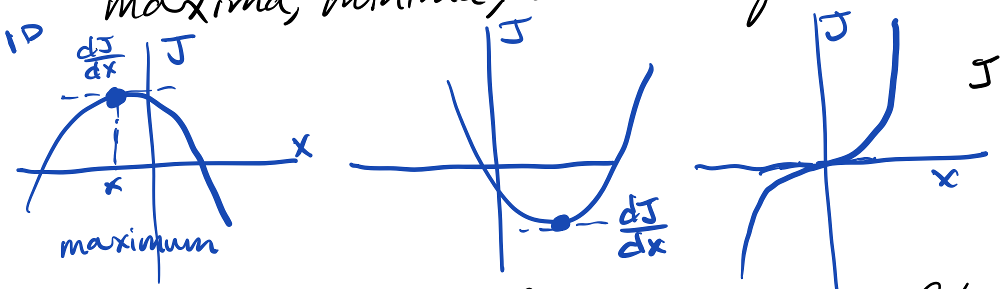
$$\nabla J = \underline{0} \qquad \begin{bmatrix} \frac{\partial J}{\partial x_1} \\ \vdots \\ \frac{\partial J}{\partial x_n} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

In terms of 2D, $J(x,y)$,

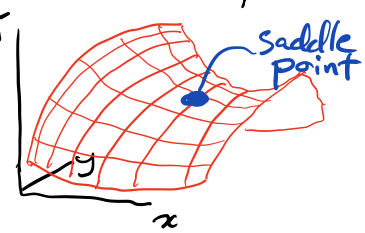
$$\frac{\partial J}{\partial x} = 0 \quad \text{and} \quad \frac{\partial J}{\partial y} = 0$$



These 1st-order conditions provide **stationary points** or critical points that are candidate solutions. They are only candidates, because they could be maxima, minima, or saddle points (inflection point in 1D)



For 2D for example,



We must look at the 2nd order conditions to determine if a stationary point is actually a minimum.

Necessary $\nabla^2 J \geq 0$

Sufficient $\nabla^2 J > 0$

Hessian

$$\nabla^2 J = \begin{bmatrix} \frac{\partial^2 J}{\partial x_1^2} & \cdots & \frac{\partial^2 J}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 J}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 J}{\partial x_n^2} \end{bmatrix}$$

Aside on Lagrange Multipliers

Optimize a cost function that is subject to a constraint

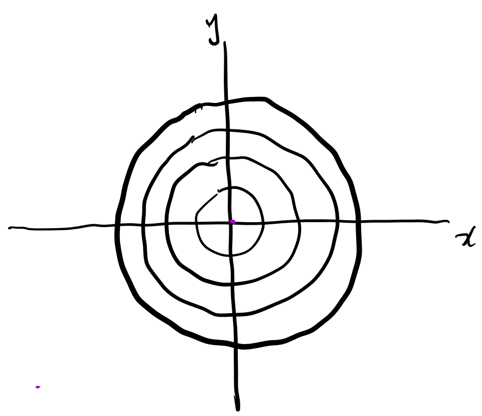
$$\min_{\underline{x}} \; J(\underline{x})$$
$$\text{s.t.} \; \underline{f}(\underline{x}) = 0$$

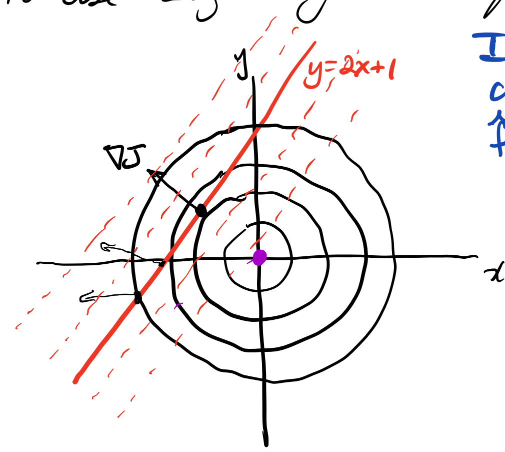
Constrained optimization problem.

Ex. (Hand + Finch)

$$\min_{(x,y)} \; J(x,y) = x^2 + y^2$$
$$\text{s.t.} \quad y = 2x + 1$$

Note: We could solve this problem by plugging in the constraint into the cost function to eliminate a variable. This is not always possible or you may not want to do it. So, we are going to use Lagrange multipliers





Imagine curves $\hat{f} = f - c$

Intuition: If I slide the bead along, I want to find a place where J doesn't change. That occurs when $\nabla J = \lambda \nabla \hat{f} = \lambda \nabla f$

The intuition tells us that we need,

$$\left\{ \begin{array}{c} \nabla J = \lambda \nabla f \\ \text{and} \\ f(\underline{x}) = 0 \end{array} \right\}$$

$$\nabla J - \lambda \nabla f = 0$$
$$\nabla(\underbrace{J - \lambda f}_{\triangleq J'}) = 0$$

Let's define an effective cost function,

$$J' = J + \lambda f$$

We want to solve

$$\min_{(x,y,\lambda)} J'(x,y,\lambda)$$

Recall $f(x)=0$
$y=2x+1$

Return to example:

$$J' = (x^2+y^2) + \lambda(y-2x-1)$$

x and y are not independent, but we are going to treat them as if they are. For a minimum, we would need,

$$\frac{\partial J'}{\partial x} = 0 \text{ and } \frac{\partial J'}{\partial y} = 0$$

The trick to ensure we can treat them independently is to choose λ to ensure that these equations hold.

$$\begin{cases} \nabla_{\underline{x}} J' = 0 \Rightarrow \nabla J + \sum_{k=1}^{M} \lambda_k \nabla f_k = 0 \\ \nabla_{\lambda} J' = 0 \Rightarrow f_1(\underline{x}) = \dots = f_M(\underline{x}) = 0 \end{cases}$$

Return to example:

① $\frac{\partial J'}{\partial x} = 2x - 2\lambda = 0$

② $\frac{\partial J'}{\partial y} = 2y + \lambda = 0$

③ $\frac{\partial J'}{\partial \lambda} = y - 2x - 1 = 0$

These three equations in three unknowns can be solved for $x = \frac{-2}{5}$ $y = \frac{1}{5}$ $\lambda = \frac{-2}{5}$ $J_{min} = \frac{1}{5}$

Note :

1) We could alternately define $J' = J - \lambda f$
The sign of the Lagrange multiplier would change.