11.1.12 Let $F(x)$ be the Fourier series of f.

By (6):

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{\pi^2}{2\pi}$$ (look @ area under graph)

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-x)\cos(nx)\,dx + \int_{0}^{\pi} x\cos(nx)\,dx\right]$$

$$= \frac{1}{\pi}\left[(-x)\frac{\sin(nx)}{n}\Big|_{-\pi}^{0} - \int_{-\pi}^{0}\frac{\sin(nx)}{n}(-1)\,dx\right.$$

$$\left. + (x)\frac{\sin(nx)}{n}\Big|_{0}^{\pi} - \int_{0}^{\pi}\frac{\sin(nx)}{n}(1)\,dx\right]$$

$$= \frac{1}{\pi}\left[0 - \left[\frac{\cos(nx)}{n^2}\right]_{-\pi}^{0} + 0 + \left[\frac{\cos(nx)}{n^2}\right]_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{-1}{n^2} + \frac{(-1)^n}{n^2} + \frac{(-1)^n}{n^2} - \frac{1}{n^2}\right]$$

$$= \frac{2}{\pi n^2}\left[-1 + (-1)^n\right]$$

$$= \begin{cases} 0, & \text{if } n \text{ even,} \\ \dfrac{-4}{\pi n^2}, & \text{if } n \text{ odd.} \end{cases}$$

Idea: Notice that if $g(x) = |x| \sin(nx)$

then $g(-x) = -g(x)$

$$\Rightarrow \int_{-\pi}^{0} g(x) = -\int_{0}^{\pi} g(x)\,dx$$ (picture graph of function)

$$\Rightarrow \int_{-\pi}^{\pi} g(x) = 0$$

$$\Rightarrow b_n = 0$$

Here, g is an "odd function".

More is discussed in section 11.2.

$$\therefore \boxed{F(x) = \frac{\pi}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1 + (-1)^n)}{n^2} \cos(nx).}$$

$$\int uv' = uv\Big| - \int vu'$$

11.2.20 In problem 11.2.11 we see that the function in question is even. Let's find it's Fourier series expansion, $F(x)$.

$$a_0 = \int_0^1 f(x)\,dx = \int_0^1 x^2\,dx = \frac{1}{3}$$

$$a_n = 2\int_0^1 x^2 \cos(n\pi x)\,dx$$

$$= 2\left[x^2 \frac{\sin(n\pi x)}{n\pi}\Bigg|_0^1 - \underbrace{\int_0^1 2x\frac{\sin(n\pi x)}{n\pi}\,dx}_{\frac{-2x\cos(n\pi x)}{n^2\pi^2}\Big|_0^1 + \int_0^1 \frac{2\cos(n\pi x)}{n^2\pi^2}\,dx} \right]$$

$$= 2\left[0 - \left(-\frac{2\cos(n\pi)}{n^2\pi^2} + \left[\frac{2\sin(n\pi x)}{n^3\pi^3}\right]_0^1 \right) \right]$$

$$= 2\left[\frac{2(-1)^n}{n^2\pi^2} - 0 \right]$$

$$= \frac{4(-1)^n}{n^2\pi^2}$$

$$\Rightarrow f(x) = F(x) = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2\pi^2} \cos(n\pi x)$$

$$\Rightarrow 1 = f(1) = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2\pi^2} (-1)^n$$

$$= \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\Rightarrow \frac{2}{3} = \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\Rightarrow \boxed{\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}.}$$

11.2.24 (a)

$$a_0 = \frac{1}{4}\int_0^4 f(x)\,dx = \frac{1}{4}(2)$$ (picture)

$$a_n = \frac{1}{2}\int_0^4 f(x)\cos\left(\frac{n\pi}{4}x\right)dx$$

$$= \frac{1}{2}\int_2^4 \cos\left(\frac{n\pi}{4}x\right)dx$$

$$= \frac{1}{2}\left[\frac{4}{n\pi}\sin\left(\frac{n\pi}{4}x\right)\right]_2^4$$

$$= \frac{1}{2}\left[0 - \frac{4}{n\pi}\sin\left(\frac{n\pi}{2}\right)\right]$$

$$= -\frac{2}{n\pi}\underbrace{\sin\left(\frac{n\pi}{2}\right)}$$

= 0, if n even

alternates between 1 and -1, if n odd

i.e.

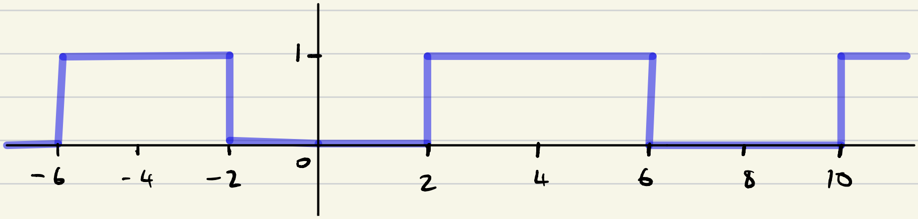
n	-3	-1	1	3	5
$\sin(\frac{n\pi}{2})$	1	-1	1	-1	1

, etc.

Fourier cosine series:

$$\frac{1}{2} - \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin\left(\frac{n\pi}{2}\right) \cos\left(\frac{n\pi}{4} x\right)$$

Sketch:



(b)

$$b_n = \frac{1}{2} \int_0^4 f(x) \sin\left(\frac{n\pi x}{4}\right) dx$$

$$= \frac{1}{2} \int_2^4 \sin\left(\frac{n\pi x}{4}\right) dx$$

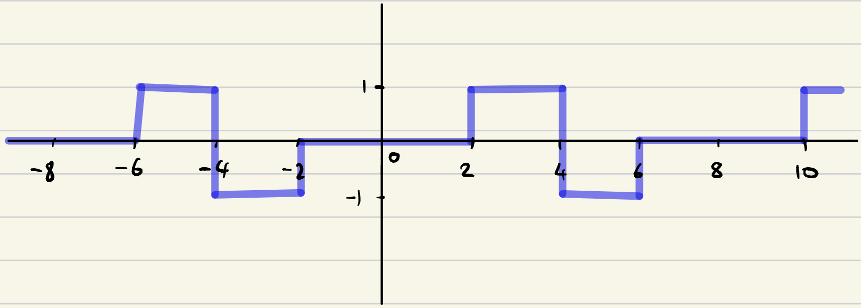
$$= \frac{1}{2} \left[-\frac{4}{n\pi} \cos\left(\frac{n\pi x}{4}\right) \right]_2^4$$

$$= \frac{-2}{n\pi} \left[\cos(n\pi) - \cos\left(\frac{n\pi}{2}\right) \right]$$

Fourier sine series:

$$\sum_{n=1}^{\infty} \frac{-2}{n\pi}\left[\cos(n\pi) - \cos\left(\frac{n\pi}{2}\right)\right]\sin\left(\frac{n\pi x}{4}\right)$$

Sketch:



11.2.29 (a)

$$a_0 = \frac{1}{\pi}\int_0^{\pi} \sin(x)\,dx$$

$$= \frac{1}{\pi}\Big[-\cos(x)\Big]_0^{\pi}$$

$$= \frac{2}{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} \sin(x)\cos(nx)\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi}\big(\sin((1+n)x)+\sin((1-n)x)\big)\,dx$$

$$= \frac{1}{\pi}\left[-\frac{\cos((1+n)x)}{1+n} - \frac{\cos((1-n)x)}{1-n}\right]_0^{\pi}, \quad n \neq 1$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{1+n} + \frac{1}{1+n} + \frac{(-1)^n}{1-n} + \frac{1}{1-n}\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n(1-n+1+n) + (1-n) + (1+n)}{1-n^2}\right]$$

$$= \frac{2}{\pi}\left[\frac{(-1)^n + 1}{1-n^2}\right], \quad \text{if } n \neq 1$$

If $n=1$:

$$a_n = \frac{1}{\pi}\int_0^{\pi}\left(\sin((1+n)x)+\sin((1-n)x)\right)dx$$

$$= \frac{1}{\pi}\int_0^{\pi}(\sin(2x)+0)\,dx$$

$$= \frac{1}{\pi}\left[-\frac{\cos(2x)}{2}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{1}{2}+\frac{1}{2}\right]$$

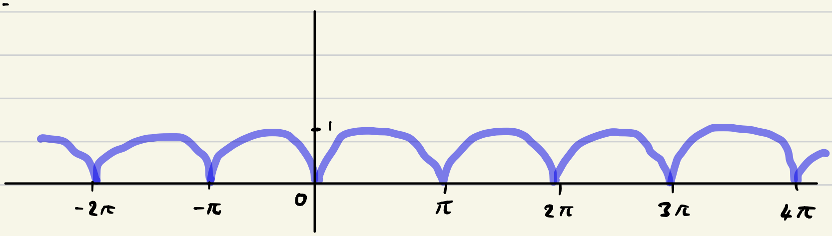
$$= 0$$

Fourier cosine series:

$$\frac{2}{\pi} + \sum_{n=2}^{\infty}\frac{2}{\pi}\left[\frac{(-1)^n+1}{1-n^2}\right]\cos(nx)$$

note

Sketch:



(b) This is just $\sin(x)$, $x \in \mathbb{R}$.

I don't know if uniqueness of Fourier coefficients was discussed, but clearly, $\sin(x)$ satisfies the defn. of a Fourier series.

Aside: [The uniqueness of coefficients can be seen using Thm 1 (orthog. of trig. system), uniform convergence of the series and a that the sum and integral sign may be interchanged under this uniformly convergent condition].

11.3.6 Using method of undetermined coefficients, a particular solution is

$$y_p = A\cos(\alpha t) + B\sin(\alpha t) + C\cos(\beta t) + D\sin(\beta t)$$

$$y_p'' = -\alpha^2(A\cos\alpha t + B\sin\alpha t) - \beta^2(C\cos\beta t + D\sin\beta t)$$

$\Rightarrow$ (substituting into O.D.E.) :

$$(*) \begin{cases} -\alpha^2(A\cos\alpha t + B\sin\alpha t) - \beta^2(C\cos\beta t + D\sin\beta t) \\ +\omega^2\left[A\cos(\alpha t) + B\sin(\alpha t) + C\cos(\beta t) \right. \\ \left. + D\sin(\beta t)\right] = \sin\alpha t + \sin\beta t \end{cases}$$

$\Rightarrow$ Letting

$$B(-\alpha^2 + \omega^2) = 1$$

$$D(-\beta^2 + \omega^2) = 1$$

$$A = 0$$

$$C = 0$$

is allowable for y_p.

$$\Rightarrow y_p = \frac{1}{\omega^2 - \alpha^2} \sin(\alpha t) + \frac{1}{\omega^2 - \beta^2} \sin(\beta t)$$

$$y_h = E\cos(\omega t) + F\sin(\omega t)$$

(well-known result)

$\therefore$ General solution is

$$y = y_h + y_p$$

$$= E\cos(\omega t) + F\sin(\omega t) + \frac{1}{\omega^2 - \alpha^2} \sin(\alpha t) + \frac{1}{\omega^2 - \beta^2} \sin(\beta t).$$

11.4.12 For 11.1.14, the Fourier series is

$$a_0 = \frac{\pi^2}{3},$$

$$a_n = \frac{4(-1)^n}{n^2},$$

$$b_n = 0.$$

Parseval's identity says that

$$2\frac{\pi^4}{9} + \sum_{n=1}^{\infty} \frac{16}{n^4} = \frac{1}{\pi}\int_{-\pi}^{\pi} x^4\,dx$$

$$= \frac{2\pi^4}{5}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

N	1	2	3	4	5
$\sum_{n=1}^{N} 1/n^4$	1	1.0625	1.0748...	1.078...	1.0803...

11.5.13 Using 11.5.6,

$$p = e^{\int 8 dx} = e^{8x}$$

$$q = 16e^{8x}$$

$$r = e^{8x}$$

$$\Rightarrow \left[e^{8x}y'\right]' + \left[16e^{8x} + \lambda e^{8x}\right]y = 0$$

We use ideas from section 2.4:

Here $m = 1$, $c = 8$, $k = \lambda + 16$,

$\alpha = 4$, $\beta = \frac{1}{2}\sqrt{8^2 - 4(\lambda+16)} = \sqrt{-\lambda}$

Case 1:

$8^2 > 4(\lambda + 16) \Rightarrow 0 > \lambda$

General soln.:

$$y = c_1 e^{-(4-\sqrt{-\lambda})t} + c_2 e^{-(4+\sqrt{-\lambda})t}$$

$$y(0) = 0 = c_1 + c_2$$

$$y(\pi) = 0 = c_1 e^{-(4-\sqrt{-\lambda})\pi} + c_2 e^{-(4+\sqrt{-\lambda})\pi}$$

$$\Rightarrow 0 = c_1\left(e^{-(4-\sqrt{-\lambda})\pi} - e^{-(4+\sqrt{-\lambda})\pi}\right)$$

Assuming $c_1 \neq 0$ (otherwise we have trivial soln.),

$$\Rightarrow e^{-(4-\sqrt{-\lambda})\pi} = e^{-(4+\sqrt{-\lambda})\pi}$$

$$\Rightarrow e^{-\sqrt{-\lambda}\,\pi} = e^{\sqrt{-\lambda}\,\pi}$$

bijectivity of $z \mapsto e^z$, $z \in \mathbb{R}$.

$$\Rightarrow -\sqrt{-\lambda} = \sqrt{-\lambda} \quad \lightning$$

(contradiction)

$$\Rightarrow y = 0 \; ; \text{ trivial soln.}$$

Case 2:

$8^2 = 4(\lambda+16) \Rightarrow 0 = \lambda$

Gen. soln.:

$y = (c_1 + c_2 t)e^{-4t}$

$y(0) = 0 = c_1$

$y(\pi) = 0 = c_2 \pi e^{-4\pi}$

$\Rightarrow c_2 = 0$

$\Rightarrow y = 0$; trivial soln.

Case 3:

$8^2 < 4(\lambda+16) \Rightarrow 0 < \lambda$

Gen. soln.:

$y = e^{-4t}\left(c_1 \cos(\sqrt{\lambda}\, t) + c_2 \sin(\sqrt{\lambda}\, t)\right)$

$y(0) = 0 = c_1$

$y(\pi) = 0 = e^{-4\pi} c_2 \sin(\sqrt{\lambda}\,\pi)$

$\Rightarrow$ (Assuming $c_2 \neq 0$),

$$\sin(\sqrt{\lambda}\,\pi) = 0$$

$$\Rightarrow \sqrt{\lambda}\,\pi = \pi n, \quad n \in \mathbb{Z}$$

$$\Rightarrow \sqrt{\lambda} = n, \quad n \in \mathbb{Z}$$

$$\Rightarrow \lambda = n^2, \quad n \in \mathbb{Z}.$$

Letting $c_2 = 1$ (we have a choice up to scalar multiple),

$\Rightarrow$

Eigenvalues	Eigenfunctions
$\lambda \in \mathbb{N}$	$e^{-4t}\sin(\lambda t)$

Verifying orthogonality:

P.T.O.

If $n \neq m$:

$$\int_0^{\pi} \underbrace{e^{8t}}_{r(t)} e^{-4t} \sin(nt) e^{-4t} \sin(mt)\, dt$$

$$= \int_0^{\pi} \sin(nt)\sin(mt)\, dt$$

$$= \frac{1}{2} \int_0^{\pi} \left[\cos((n-m)t) - \cos((n+m)t)\right] dt$$

$$= \frac{1}{2}\left[\frac{\sin((n-m)\pi)}{n-m} - \frac{\sin((n+m)\pi)}{n+m} \right]$$

$$= 0$$

If $n = m$:

$$\int_0^{\pi} (\sin(nt))^2\, dt = \ldots = \frac{\pi}{2} \neq 0$$

11.6.3 We know that

$$0 = a_5 = a_6 = \cdots$$

since $\deg(1-x^4) = 4$.

Also,

$a_1 = a_3 = 0$, since if m is odd,

$$\underbrace{\underbrace{(1-x^4)}_{\text{even}}\underbrace{P_m}_{\text{even}}}_{\text{even}}.$$

$$a_0 = \frac{1}{2}\int_{-1}^{1}(1-x^4)1\,dx$$

$$= \frac{1}{2}\left[x - \frac{x^5}{5}\right]_{-1}^{1}$$

$$= \frac{4}{5}$$

$$a_2 = \frac{5}{2}\int_{-1}^{1}(1-x^4)\frac{1}{2}(3x^2-1)\,dx$$

$$= \cdots$$

$$= -\frac{4}{7}$$

$$a_4 = \frac{9}{2}\int_{-1}^{1}(1-x^4)\,\frac{1}{8}(35x^4-30x^2+3)\,dx$$

$$= \cdot -$$

$$= -\frac{8}{35}$$

$$\therefore \quad 1-x^4 = \frac{4}{5}P_0 - \frac{4}{7}P_2 - \frac{8}{35}P_4$$