ECE 68000: MODERN AUTOMATIC CONTROL

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Implementing the bang-bang controller

Implementing the time optimal controller

- The plant

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}u \end{aligned}$$

- The performance index

$$J = \int_0^{t_f} dt$$

- The control is required to satisfy

$$|u(t)| \leq 1$$

  for all $t \in [0, t_f]$
- This constraint means that the control must have magnitude no greater than 1
- Objective: find admissible control minimizing J that transfers the system from a given initial $\boldsymbol{x}_0$ to the origin

Admissible control

- Find an admissible control minimizing the Hamiltonian

$$
\begin{aligned}
\arg_u \min H &= \arg_u \min(1 + p_1 x_2 + p_2 u) \\
&= \arg_u \min(p_2 u) \\
&= \begin{cases} u(t) = 1 & \text{if} \quad p_2 < 0 \\ u(t) = ? & \text{if} \quad p_2 = 0 \\ u(t) = -1 & \text{if} \quad p_2 > 0 \end{cases}
\end{aligned}
$$

- If $p_2 = 0$ is not sustained over an interval time, the control law is piecewise constant taking the values 1 or -1
- This control law has at most two intervals of constancy because the argument is a linear function, $-d_1 t + d_2$, that changes its sign at most once
- This type of control is called a *bang-bang control* because it switches back and forth between its extreme values

Bang-bang control

- Implement the bang-bang control law using a relay element, which is a signum function, as

$$u^*(t) = -\text{sgn}(p_2^*) = -\text{sgn}(-d_1 t + d_2),$$

where "sgn" is the label for the signum function defined as

$$\text{sgn}(z) = \begin{cases} \frac{z}{|z|} \quad \text{or} \quad \frac{|z|}{z} & \text{if} \quad z \neq 0 \\ 0 & \text{if} \quad z = 0 \end{cases} = \begin{cases} 1 & \text{if} \quad z > 0 \\ 0 & \text{if} \quad z = 0 \\ -1 & \text{if} \quad z < 0 \end{cases}$$

- System trajectories for $u = 1$ and $u = -1$ are families of parabolas
- Only one parabola from each family passes through the specified terminal point $\boldsymbol{x} = \begin{bmatrix} 0 & 0 \end{bmatrix}^\top$ in the state plane

Switching curve

- Segments of the two parabolas through the origin form the switching curve,

$$x_1 = -\frac{1}{2} x_2^2 \,\mathrm{sgn}(x_2)$$

- If an initial state is above the switching curve, then $u = -1$ is used until the switching curve is reached
- Then, $u = 1$ is used to reach the origin
- For an initial state below the switching curve, the control $u = 1$ is used first to reach the switching curve, and then the control is switched to $u = -1$
- Implement the above control action as $u = -\mathrm{sgn}(v)$, where

$$v = v(x_1, x_2) = x_1 + \frac{1}{2} x_2^2 \mathrm{sgn}(x_2)$$

is the equation describing the switching curve

The closed-loop system

- The switching curve

$$v = v(x_1, x_2) = x_1 + \frac{1}{2}x_2^2\mathrm{sgn}(x_2)$$

- The controller

$$\begin{aligned} u &= -\mathrm{sgn}(v) \\ &= -\mathrm{sgn}(x_1 + \frac{1}{2}x_2^2\mathrm{sgn}(x_2)) \end{aligned}$$

- Starting at an arbitrary initial state in the state plane, the trajectory will always be moving in an optimal fashion towards the origin
- Once the origin is reached, the trajectory will stay there

Implementing the controller

- Two sign functions in the control law
- The sign function is known as the relay in control's jargon
- Hard to handle for ode solvers
- Make it easy; smooth it out

$$\begin{aligned}\operatorname{sgn}(v) &= \frac{v}{|v|} \\ &\approx \frac{v}{|v|+\varepsilon}\end{aligned}$$

  where ε is a small positive constant
- Let's look at some smooth approximations of the signum function,
  ezplot('x/(abs(x)+0.2)',[-1, 1]); grid