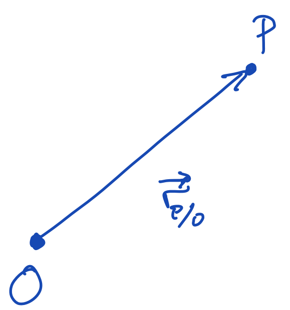
Lecture 1

Dynamics is the discipline that develops predictive models of the position and velocity of an object's motion.

Mathematical review (KP ch 1, Appendix B)

Vector (and more broadly, tensor) calculus gives us the tools to develop a coordinate-free approach to Newtonian Mechanics.

Vector - A quantity that has both magnitude and direction.



$\vec{r}_{P/O}$ is the position of point P w.r.t. point O.

The length of a vector $\vec{a} \in \mathbb{R}^3$ is the magnitude or Euclidean norm $\|\vec{a}\|$.

"in" or "member of"

set of tuples of 3 real numbers

$$\vec{a} = (\underbrace{a_1, a_2, a_3}_{\text{components}})^T \Longrightarrow \|\vec{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

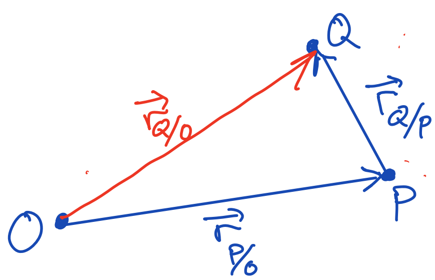
Therefore, the zero vector has length zero, and a unit vector has length equal to one.

To form a unit vector:

hat means unit vector

$$\hat{a} = \frac{\vec{a}}{\|\vec{a}\|}$$

Vector addition occurs by linking vectors from head to tail. The resultant goes from the start point to the finish point.



$$\vec{r}_{P/O} + \vec{r}_{Q/P} = \vec{r}_{Q/O}$$

$$\left(\frac{\cancel{P}}{O}\right)\left(\frac{Q}{\cancel{P}}\right) = \frac{Q}{O}$$

Properties of vector addition:

1. Commutativity $\quad \vec{a} + \vec{b} = \vec{b} + \vec{a}$
2. Associativity $\quad (\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$
3. zero vector $\quad \vec{a} + \vec{0} = \vec{a}$
4. Negativity $\quad \vec{a} + (-\vec{a}) = \vec{0}$

Vector notation

Sometimes $\vec{a}$, $\underset{\sim}{a}$, $a \in \mathbb{R}^n$ — Just watch out for notational laziness

Function nomenclature

$$\vec{f} : \mathbb{R} \longrightarrow \mathbb{R}^3$$

OR

$$x \in \mathbb{R} \longmapsto \vec{f}(x) \in \mathbb{R}^3$$

OR

$$x \in \mathbb{R} \text{ and } \vec{f}(x) \in \mathbb{R}^3$$

is called a vector-valued, scalar function.

$$\vec{f} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

is called a vector-valued, vector function.

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}$$

is called a scalar-valued, vector function.

"maps from" example

$$f : \mathbb{R}^3 \longrightarrow \mathbb{R}$$

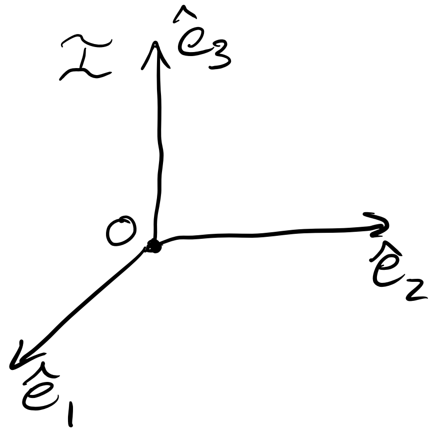
$$f(\vec{x}) = \begin{bmatrix} 3 & 4 & 7 \end{bmatrix} \vec{x}$$

If we let $\vec{x} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$,

$$f\left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 3 & 4 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = (3)(1) + (4)(0) + (7)(-1)$$

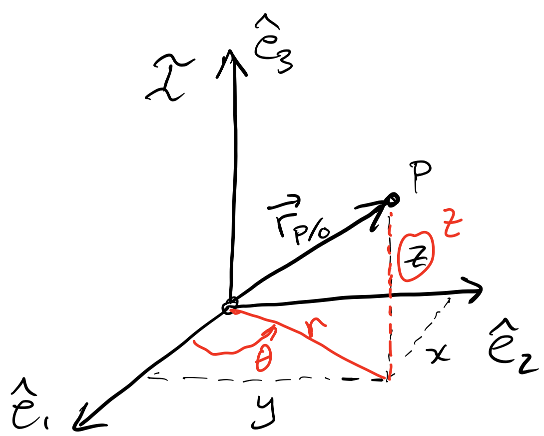
$$= \underline{\underline{-4}}$$

Reference Frame – is a point of view from which observations are made.



$$\mathcal{I} = (O, \hat{e}_1, \hat{e}_2, \hat{e}_3)$$

Coordinate System – is a set of scalars that locate the position of a point in a reference frame



cartesian coordinates

$(x, y, z)_{\mathcal{I}}$

another example:
spherical coordinates,
cylindrical coordinates,
and others

$(r, \theta, z)_{\mathcal{I}}$

Note that Reference frames and coordinate systems are separate concepts (although the definition of a coordinate system is built on a choice of reference frame)

↳ Don't say "coordinate Frame" or "reference system"

Components of a vector:

$$\vec{r}_{P/O} = x\hat{e}_1 + y\hat{e}_2 + z\hat{e}_3$$

Addition of components:

$$\vec{a} = a_1\hat{e}_1 + a_2\hat{e}_2 + a_3\hat{e}_3$$
$$+ \quad \vec{b} = b_1\hat{e}_1 + b_2\hat{e}_2 + b_3\hat{e}_3$$
$$(\vec{a}+\vec{b}) = (a_1+b_1)\hat{e}_1 + (a_2+b_2)\hat{e}_2 + (a_3+b_3)\hat{e}_3$$

Scalar product (or Dot product or inner product)

$$\vec{a}\cdot\vec{b} = \|\vec{a}\|\,\|\vec{b}\|\cos\theta$$ where θ is the angle between $\vec{a}$ and $\vec{b}$

↳ Properties:

Alternate notation:
$\langle\vec{a},\vec{b}\rangle$
$\vec{a}^T\vec{b}$

1. $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$
2. $\vec{a}\cdot(k\vec{b}) = (k\vec{a})\cdot\vec{b} = k(\vec{a}\cdot\vec{b})$ when $k \in \mathbb{R}$
3. $(\vec{a}+\vec{b})\cdot\vec{c} = \vec{a}\cdot\vec{c} + \vec{b}\cdot\vec{c}$

$\vec{a}$ and $\vec{b}$ are <u>orthogonal</u> if $\vec{a}\cdot\vec{b}=0$

Note: for a basis set of unit vectors, $\hat{e}_i\cdot\hat{e}_j=\begin{cases}1 & \text{if } i=j\\ 0 & \text{otherwise}\end{cases}$

In coordinates:

$$\vec{a} = a_1\hat{e}_1 + a_2\hat{e}_2 + a_3\hat{e}_3$$
$$\vec{b} = b_1\hat{e}_1 + b_2\hat{e}_2 + b_3\hat{e}_3$$

$$\vec{a}\cdot\vec{b} = (a_1\hat{e}_1 + a_2\hat{e}_2 + a_3\hat{e}_3)\cdot(b_1\hat{e}_1 + b_2\hat{e}_2 + b_3\hat{e}_3)$$
$$= (a_1\hat{e}_1\cdot b_1\hat{e}_1) + \cancel{(a_1\hat{e}_1\cdot b_2\hat{e}_2)} + \cancel{(a_1\hat{e}_1\cdot b_3\hat{e}_3)}$$ ← This line in red
$$+\cancel{(a_2\hat{e}_2\cdot b_1\hat{e}_1)} + (a_2\hat{e}_2\cdot b_2\hat{e}_2) + \cancel{(a_2\hat{e}_2\cdot b_3\hat{e}_3)}$$
$$+\cancel{(a_3\hat{e}_3\cdot b_1\hat{e}_1)} + \cancel{(a_3\hat{e}_3\cdot b_2\hat{e}_2)} + (a_3\hat{e}_3\cdot b_3\hat{e}_3)$$
$$= (a_1b_1) + (a_2b_2) + (a_3b_3)$$

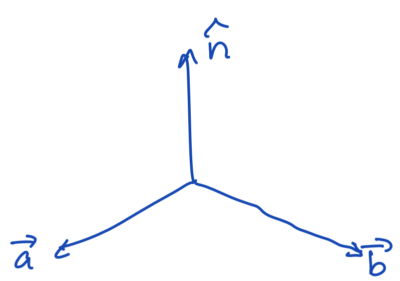
The Euclidean norm can be represented by the dot product

$$\|\vec{a}\| = \sqrt{\vec{a}\cdot\vec{a}} = \sqrt{a_1^2+a_2^2+a_3^2}$$
$$= \sqrt{\langle\vec{a},\vec{a}\rangle}$$
$$= \sqrt{\vec{a}^T\vec{a}}$$

Vector Product

$$\vec{c} = \vec{a}\times\vec{b} = \|\vec{a}\|\,\|\vec{b}\|\sin\theta\,\hat{n}$$

where θ is the angle between $\vec{a}$ and $\vec{b}$ and $\hat{n}$ is the unit vector orthogonal to $\vec{a}$ and $\vec{b}$ in a "right-handed" sense.

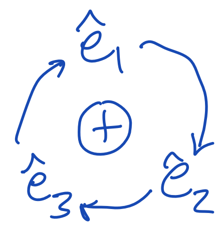


Properties:

1. $\vec{a} \times \vec{a} = 0$
2. $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
3. $(k\vec{a}) \times \vec{b} = k(\vec{a} \times \vec{b}) = \vec{a} \times (k\vec{b})$ for $k \in \mathbb{R}$
4. $(\vec{a} + \vec{b}) \times \vec{c} = (\vec{a} \times \vec{c}) + (\vec{b} \times \vec{c})$

Cross product of basis vectors:

$$\hat{e}_1 \times \hat{e}_2 = \hat{e}_3$$
$$\hat{e}_2 \times \hat{e}_3 = \hat{e}_1$$
$$\hat{e}_3 \times \hat{e}_1 = \hat{e}_2$$



Cross product in coordinates:

$$(\vec{a} \times \vec{b}) = (a_1 \hat{e}_1 + a_2 \hat{e}_2 + a_3 \hat{e}_3) \times (b_1 \hat{e}_1 + b_2 \hat{e}_2 + b_3 \hat{e}_3)$$
$$= (a_2 b_3 - a_3 b_2)\hat{e}_1 + (a_3 b_1 - a_1 b_3)\hat{e}_2 + (a_1 b_2 - a_2 b_1)\hat{e}_3$$

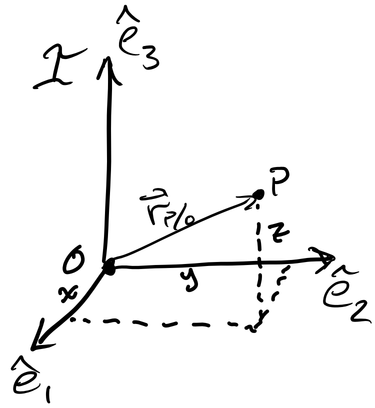
Scalar-triple product:

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c} = \vec{b} \cdot (\vec{c} \times \vec{a}) = \vec{c} \cdot (\vec{a} \times \vec{b})$$

Vector-triple product:

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$$

"Basis-vector representation" for expressing vectors in coordinates



$\mathcal{I} = (O, \hat{e}_1, \hat{e}_2, \hat{e}_3)$

$(x, y, z)_{\mathcal{I}}$

$$\vec{r}_{P/O} = x\hat{e}_1 + y\hat{e}_2 + z\hat{e}_3$$

"Array representation" for expressing vectors in coordinates

$$\left[\vec{r}_{P/O}\right]_{\mathcal{I}} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{\mathcal{I}}$$

expressed in frame $\mathcal{I}$

Mathematical Review (cont'd)

Mostly important for moment inertia tensors

A **tensor** $\underset{\sim}{T}$ is a linear operator that associates a vector $\vec{a}$ to another vector $\vec{b}$.

$$\vec{b} = \underset{\sim}{T} \circ \vec{a}$$

Properties:

1. $\underset{\sim}{T} \circ (\vec{a} + \vec{b}) = \underset{\sim}{T} \circ \vec{a} + \underset{\sim}{T} \circ \vec{b}$
2. $\underset{\sim}{T} \circ (k\vec{a}) = k\underset{\sim}{T} \circ \vec{a}$ for $k \in \mathbb{R}$
3. Zero tensor: $\underset{\sim}{0} \circ \vec{a} = \vec{0}$
4. Identity tensor: $\underset{\sim}{U} \circ \vec{a} = \vec{a}$

Notes:

- If $\underset{\sim}{T} \circ \underset{\sim}{T}^{-1} = \underset{\sim}{U}$ then $\underset{\sim}{T}$ is invertible
- If $\underset{\sim}{T} = \underset{\sim}{T}^{T}$ then $\underset{\sim}{T}$ is symmetric
- If $\underset{\sim}{T} = -\underset{\sim}{T}^{T}$ then $\underset{\sim}{T}$ is skew-symmetric

Tensor product $\otimes$

If $\underset{\sim}{T} = \vec{a} \otimes \vec{b}$,

$$\underset{\sim}{T} \circ \vec{c} = (\vec{a} \otimes \vec{b}) \circ \vec{c} = (\vec{b} \cdot \vec{c})\vec{a} \neq \vec{c} \cdot (\vec{a} \otimes \vec{b})$$

"Basis-vector representation" of a tensor

If $\vec{a} = a_1\hat{e}_1 + a_2\hat{e}_2 + a_3\hat{e}_3$

$\vec{b} = b_1\hat{e}_1 + b_2\hat{e}_2 + b_3\hat{e}_3$

tensor product

$$\underset{\sim}{T} = \vec{a} \otimes \vec{b} = \sum_{i=1}^{3}\sum_{j=1}^{3} a_i b_j (\hat{e}_i \otimes \hat{e}_j)$$

Ex. In coordinates, $\hat{e}_1 \otimes \hat{e}_2$ can be calculated as

$$[\hat{e}_1]_I [\hat{e}_2]_I^T = \begin{bmatrix}1\\0\\0\end{bmatrix}\begin{bmatrix}0 & 1 & 0\end{bmatrix} = \begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}$$

Note: $(\hat{e}_i \otimes \hat{e}_j) \cdot \hat{e}_k = \begin{cases} \hat{e}_i & \text{if } j=k \\ 0 & \text{if } j\neq k \end{cases}$

Example:

$$\underset{\sim}{U} = \sum_{i=1}^{3}\sum_{j=1}^{3} U_{ij}\, \hat{e}_i \otimes \hat{e}_j \quad \text{where } U_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$$

"Array Representation" for expressing tensors

$$\underset{\sim}{T} \cdot \vec{c} = \vec{b}$$

Use brackets to denote the associated basis:

$$[\underset{\sim}{T} \cdot \vec{c}]_I = [\vec{b}]_I$$

$$[\underset{\sim}{T}]_I [\vec{c}]_I = [\vec{b}]_I$$

$$\begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}_I \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}_I = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}_I$$

Key idea for mathematical review:

Vectors and tensors are coordinate-free quantities while their array representations depend on a coordinate system.