ECE 68000: MODERN AUTOMATIC CONTROL

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Two-Point Boundary-Value Problem (TPBVP)

Problem Statement

- Minimize the quadratic performance index

$$J = \frac{1}{2}\boldsymbol{x}(t_f)^\top \boldsymbol{F}\boldsymbol{x}(t_f) + \frac{1}{2}\int_{t_0}^{t_f} \left(\boldsymbol{x}(t)^\top \boldsymbol{Q}\boldsymbol{x}(t) + \boldsymbol{u}(t)^\top \boldsymbol{R}\boldsymbol{u}(t)\right) dt$$

  subject to

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t), \quad \boldsymbol{x}(t_0) = \boldsymbol{x}_0, \quad \boldsymbol{x}(t_f) \quad \text{is free}$$

- The control input $\boldsymbol{u}$ unconstrained
- $\boldsymbol{F} = \boldsymbol{F}^\top \succeq 0$ and $\boldsymbol{Q} = \boldsymbol{Q}^\top \succeq 0$ and $\boldsymbol{R} = \boldsymbol{R}^\top \succ 0$
- The Hamiltonian function

$$H = \frac{1}{2}\boldsymbol{x}^\top \boldsymbol{Q}\boldsymbol{x} + \frac{1}{2}\boldsymbol{u}^\top \boldsymbol{R}\boldsymbol{u} + \boldsymbol{p}^\top \left(\boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}\right),$$

  where $\boldsymbol{p} \in \mathbb{R}^n$ is the costate vector

Optimal controller

- The optimal controller must minimize the Hamiltonian function
- Since the control vector is unconstrained, the necessary condition for optimality of the control $\boldsymbol{u}$ is

$$\frac{\partial H}{\partial \boldsymbol{u}} = \mathbf{0}^\top$$

- Evaluating yields

$$\frac{\partial H}{\partial \boldsymbol{u}} = \boldsymbol{u}^\top \boldsymbol{R} + \boldsymbol{p}^\top \boldsymbol{B} = \mathbf{0}^\top$$

- The optimal controller

$$\boldsymbol{u} = -\boldsymbol{R}^{-1}\boldsymbol{B}^\top \boldsymbol{p}$$

- Substituting

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} - \boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^\top \boldsymbol{p}, \quad \boldsymbol{x}(t_0) = \boldsymbol{x}_0$$

Combined plant dynamics and costate equation

- Costate equation,

$$\begin{aligned}\dot{\boldsymbol{p}} &= -\left(\frac{\partial H}{\partial \boldsymbol{x}}\right)^{\top} \\ &= -\boldsymbol{Q}\boldsymbol{x} - \boldsymbol{A}^{\top}\boldsymbol{p}\end{aligned}$$

- Combine with the plant dynamics

$$\begin{bmatrix} \dot{\boldsymbol{x}} \\ \dot{\boldsymbol{p}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} & -\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^{\top} \\ -\boldsymbol{Q} & -\boldsymbol{A}^{\top} \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{p} \end{bmatrix}$$

- The state vector at time t_f must satisfy

$$\boldsymbol{p}(t_f) = \frac{1}{2}\left(\nabla \boldsymbol{x}^{\top}\boldsymbol{F}\boldsymbol{x}\right)\Big|_{t=t_f} = \boldsymbol{F}\boldsymbol{x}(t_f)$$

Two-point boundary value problem formulation

- In sum, we reduced the linear quadratic control problem to solving $2n$ linear differential equations with mixed boundary conditions, which is an example of a *two-point boundary value problem*, or TPBVP for short
- Solve the above TPBVP
- If we had the initial conditions $\boldsymbol{x}(t_0)$ and $\boldsymbol{p}(t_0)$, then

$$\begin{bmatrix} \boldsymbol{x}(t_f) \\ \boldsymbol{p}(t_f) \end{bmatrix} = e^{\boldsymbol{H}(t_f - t_0)} \begin{bmatrix} \boldsymbol{x}(t_0) \\ \boldsymbol{p}(t_0) \end{bmatrix},$$

  where

$$\boldsymbol{H} = \begin{bmatrix} \boldsymbol{A} & -\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^\top \\ -\boldsymbol{Q} & -\boldsymbol{A}^\top \end{bmatrix}$$

- In this particular TPBVP, $\boldsymbol{p}(t_0)$ is unknown and instead we are given $\boldsymbol{p}(t_f)$

Preparation to solving TPBVP

- To proceed, let

$$\begin{bmatrix} \boldsymbol{x}(t_f) \\ \boldsymbol{p}(t_f) \end{bmatrix} = e^{\boldsymbol{H}(t_f - t)} \begin{bmatrix} \boldsymbol{x}(t) \\ \boldsymbol{p}(t) \end{bmatrix},$$

  where

$$e^{\boldsymbol{H}(t_f - t)} = \begin{bmatrix} \Phi_{11}(t_f, t) & \Phi_{12}(t_f, t) \\ \Phi_{21}(t_f, t) & \Phi_{22}(t_f, t) \end{bmatrix}$$

- Each of the blocks $\Phi_{ij}(t_f, t)$, $i, j = 1, 2$, is $n \times n$
- Thus we have

$$\begin{bmatrix} \boldsymbol{x}(t_f) \\ \boldsymbol{p}(t_f) \end{bmatrix} = \begin{bmatrix} \Phi_{11}(t_f, t) & \Phi_{12}(t_f, t) \\ \Phi_{21}(t_f, t) & \Phi_{22}(t_f, t) \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(t) \\ \boldsymbol{p}(t) \end{bmatrix}$$

Solving TPBVP—use boundary condition

- Represent

$$
\begin{aligned}
\boldsymbol{x}(t_f) &= \boldsymbol{\Phi}_{11}(t_f, t)\boldsymbol{x}(t) + \boldsymbol{\Phi}_{12}(t_f, t)\boldsymbol{p}(t) \\
\boldsymbol{p}(t_f) &= \boldsymbol{\Phi}_{21}(t_f, t)\boldsymbol{x}(t) + \boldsymbol{\Phi}_{22}(t_f, t)\boldsymbol{p}(t)
\end{aligned}
$$

- Use the boundary condition

$$
\boldsymbol{p}(t_f) = \boldsymbol{F}\boldsymbol{x}(t_f) = \boldsymbol{F}\boldsymbol{\Phi}_{11}(t_f, t)\boldsymbol{x}(t) + \boldsymbol{F}\boldsymbol{\Phi}_{12}(t_f, t)\boldsymbol{p}(t)
$$

- Subtracting

$$
\begin{aligned}
\boldsymbol{0} &= \left(\boldsymbol{F}\boldsymbol{\Phi}_{11}(t_f, t) - \boldsymbol{\Phi}_{21}(t_f, t)\right)\boldsymbol{x}(t) \\
&\quad + \left(\boldsymbol{F}\boldsymbol{\Phi}_{12}(t_f, t) - \boldsymbol{\Phi}_{22}(t_f, t)\right)\boldsymbol{p}(t)
\end{aligned}
$$

Optimal controller

- We obtain

$$\begin{aligned}
\boldsymbol{p}(t) &= \left(\boldsymbol{\Phi}_{22}(t_f, t) - \boldsymbol{F}\boldsymbol{\Phi}_{12}(t_f, t)\right)^{-1} \left(\boldsymbol{F}\boldsymbol{\Phi}_{11}(t_f, t) \right. \\
&\qquad \left. - \boldsymbol{\Phi}_{21}(t_f, t)\right) \boldsymbol{x}(t) \\
&= \boldsymbol{P}(t)\boldsymbol{x}(t)
\end{aligned}$$

  where

$$\boldsymbol{P}(t) = \left(\boldsymbol{\Phi}_{22}(t_f, t) - \boldsymbol{F}\boldsymbol{\Phi}_{12}(t_f, t)\right)^{-1} \left(\boldsymbol{F}\boldsymbol{\Phi}_{11}(t_f, t) - \boldsymbol{\Phi}_{21}(t_f, t)\right)$$

- Substituting yields the optimal state-feedback controller

$$\boldsymbol{u}(t) = -\boldsymbol{R}^{-1}\boldsymbol{B}^{\top}\boldsymbol{P}(t)\boldsymbol{x}(t)$$

Solving the costate equation

- The matrix $\boldsymbol{P}(t)$ satisfies a matrix differential equation
- Indeed, differentiating

$$\dot{\boldsymbol{P}}\boldsymbol{x} + \boldsymbol{P}\dot{\boldsymbol{x}} - \dot{\boldsymbol{p}} = \boldsymbol{0}$$

- Substituting yields

$$\left(\dot{\boldsymbol{P}} + \boldsymbol{P}\boldsymbol{A} - \boldsymbol{P}\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^{\top}\boldsymbol{P} + \boldsymbol{Q} + \boldsymbol{A}^{\top}\boldsymbol{P}\right)\boldsymbol{x} = \boldsymbol{0}$$

- This must hold throughout $t_0 \leq t \leq t_f$
- Hence, $\boldsymbol{P}$ must satisfy

$$\boxed{\dot{\boldsymbol{P}} = \boldsymbol{P}\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^{\top}\boldsymbol{P} - \boldsymbol{A}^{\top}\boldsymbol{P} - \boldsymbol{P}\boldsymbol{A} - \boldsymbol{Q}}$$

  subject to the boundary condition

$$\boldsymbol{P}(t_f) = \boldsymbol{F}$$

- We have the matrix *Riccati differential equation*
- Note that because $\boldsymbol{F}$ is symmetric, so is $\boldsymbol{P} = \boldsymbol{P}(t)$