Theorem: $\vec{F}$ is conservative $\overset{\text{iff}}{\Longleftrightarrow}$ $\vec{F} = -\nabla U^{(\vec{F})}_{P/O}$.

Proof: Makes use of Stokes theorem

$$\oint_S \vec{F}\cdot d\vec{r} = \int_\Sigma (\nabla\times\vec{F})\cdot n\,dA$$

S: enclosing curve

Σ: surface

The $(\Leftarrow)$ portion of the proof:

If $\vec{F} = -\nabla U$, then plug in

$$-\oint_S \nabla U\cdot dr = -\int_\Sigma (\nabla\times\nabla U)\cdot n\,dA$$

vector identity from calculus

$$\underbrace{\nabla\times}_{\text{curl}}\underbrace{\nabla U}_{\text{gradient of a scalar potential function.}} = 0$$

$$\Rightarrow \oint_S \vec{F}\cdot dr = 0$$

$\Rightarrow \vec{F}$ is conservative.

The $(\Rightarrow)$ portion of the proof:

If $\vec{F}$ is conservative,

$$U^{(\vec{F})}_{P/O} = -\int \vec{F}\cdot {}^{I}d\vec{r}_{P/O}$$

$$\nabla U_{P/O} = -\nabla\int \vec{F}_P\cdot {}^{I}d\vec{r}_{P/O}$$

$$\nabla U_{P/O} = -\vec{F}_P \Rightarrow \vec{F}_P = -\nabla U_{P/O} \checkmark$$

This theorem is providing a way to check if $\vec{F}$ is conservative. Take $\nabla\times\vec{F}$ to see if $\nabla\times\vec{F} = 0$

Curl operator

$$\vec{\nabla} \times$$

$$\left(\frac{\partial}{\partial x} \hat{e}_x + \frac{\partial}{\partial y} \hat{e}_y + \frac{\partial}{\partial z} \hat{e}_z \right) \times$$

Tutorial 5.1 Calculate the potential for a given conservative force.

$$\vec{F}_p = (-x+y)\hat{e}_x + (x-y+y^2)\hat{e}_y, \quad \text{Find } U_{P/O}^{(\vec{F}_p)}$$

$$U_{P/O}^{(\vec{F}_p)} = -\int \vec{F}_p \cdot d\vec{r}_{P/O} = -\int [(-x+y)\hat{e}_x + (x-y+y^2)\hat{e}_y] \cdot (dx\,\hat{e}_x + dy\,\hat{e}_y)$$

$$= -\int [(-x+y)dx + (x-y+y^2)dy]$$

(Note: It is hard to isolate x terms with dx only)

$$= \int x\,dx + (y-y^2)dy - \underbrace{(y\,dx + x\,dy)}_{d(xy)}$$

$$= \frac{1}{2}x^2 + \frac{1}{2}y^2 - \frac{1}{3}y^3 - xy + C$$

Potential energy Function.

Check that it works:

$$\vec{F}_p \stackrel{?}{=} -\nabla U_{P/O}^{(\vec{F}_p)}$$

$$= -\frac{\partial U}{\partial x}\hat{e}_x - \frac{\partial U}{\partial y}\hat{e}_y$$

$$= -(x-y)\hat{e}_x - (y-y^2-x)\hat{e}_y$$

$$= (-x+y)\hat{e}_x + (x-y+y^2)\hat{e}_y \quad \checkmark$$

Total Energy

$$E_{P/O} \triangleq T_{P/O} + \underbrace{U_{P/O}}_{\text{Total potential energy due to all conservative forces}}$$

How does this definition affect the work-energy formulas?

$$T_{P/O}(t_2) = T_{P/O}(t_1) + W_P^{(tot)} \qquad \text{Formula \#1}$$

$$+ \quad U_{P/O}(t_2) = U_{P/O}(t_1) - W_P^{(c)} \qquad \text{Formula \#2}$$

$$\boxed{E_{P/O}(t_2) = E_{P/O}(t_1) + W_P^{(nc)}} \qquad \text{Work-energy formula \#3}$$

Conservation?

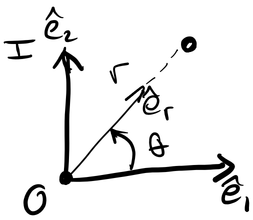
$E_{P/O}$ is conserved when $W_P^{(nc)} = 0$.

Summary of work-energy formulas

1. Total work + kinetic energy
2. Conservative work + Potential energy
3. Nonconservative work + total energy

$\boxed{\text{Tutorial 5.2}}$

Energy of orbital motion



$$\vec r_{P/O} = r\hat e_r$$

$${}^I\vec v_{P/O} = \dot r \hat e_r + r\dot\theta \hat e_\theta$$

What is the total energy of the particle?

$$E_{P/O} = T_{P/O} + U_{P/O}$$

$$= \frac{1}{2} m_p \|{}^I\vec v_{P/O}\|^2 + U_{P/O}$$

$$U_{P/O} \rightarrow -\frac{\overbrace{G m_O}^{\mu} m_p}{r}$$

$$= \frac{1}{2} m_p (\dot r^2 + r^2 \dot\theta^2) - \frac{\mu m_p}{r}$$

However, gravity is a conservative force

$\Rightarrow$ No non-conservative forces here $\Rightarrow$ $E_{P/O}$ is conserved.

Recall that angular momentum is also conserved:

$${}^I\vec h_{P/O} = \vec r_{P/O} \times m_p {}^I\vec v_{P/O} = m_p \underbrace{r^2\dot\theta}_{h_0} \hat e_3 \quad \text{conserved}$$

h_0 specific angular momentum.

We can define a specific energy (that is, "per unit mass")

$$\mathcal{E}_p \triangleq \frac{E_{pro}}{m_p} = \frac{1}{2}\dot{r}^2 + \underbrace{\frac{h_0^2}{2r^2} - \frac{\mu}{r}}$$

(we used h_0 to eliminate $\dot{\theta}$)

$$\mathcal{E}_p = \frac{1}{2}\dot{r}^2 + U_{eff}(r)$$

Effective potentials can sometimes make complicated systems look like a single D.O.F. They can also be used to identify "turning points" in the motion.

At $\dot{r}=0$, at periapsis,

$$\mathcal{E}_p = \frac{h_0^2}{2r_p^2} - \frac{\mu}{r_p}$$

This can be manipulated (see p 175) to give:

$$e = \sqrt{1 + \frac{2\mathcal{E}_p h_0^2}{\mu^2}}$$

eccentricity

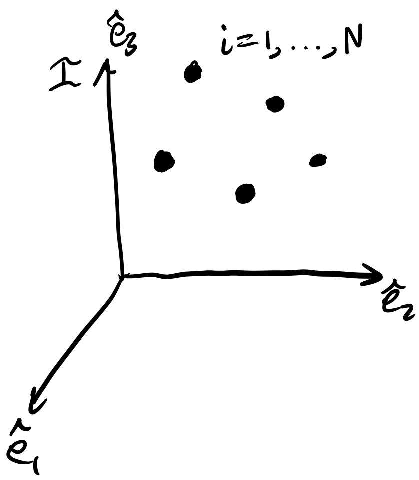
We can do other work to find:

$$a = \frac{-\mu}{2\mathcal{E}_p}$$

semi-major axis

$\mathcal{E}_p$ and h_0 entirely determine to size and shape of the elliptical orbit.

$\boxed{\text{Ch 6:}}$ Linear Momentum of Multiparticle Systems



Degrees of Freedom?

$$M = 3N - K$$

As N grows M grows quickly

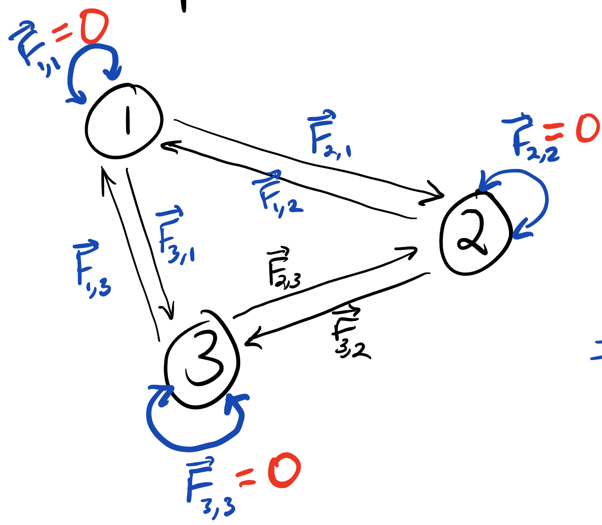
We want to work towards versions of N2L for multiparticle systems.

N2L still applies, so let's start there,

$$\vec{F}_i = \overbrace{\vec{F}_i^{(ext)}}^{\text{external}} + \overbrace{\sum_{j=1}^{N} \underbrace{\vec{F}_{ij}}_{\text{Force on particle } i \text{, due to particle } j}}^{\text{internal}} = m_i \, {}^{\mathcal{I}}\vec{a}_{i/O}$$

What can we say about the F_{ij} terms?

Internal Force Graph for $N=3$ $\rightarrow$ $\sum_{j=1}^{N} \vec{F}_{i,j}$



And look at particle 3: $i=3$, Then

$$\sum_{j=1}^{3} \vec{F}_{3,j} = \vec{F}_{3,1} + \vec{F}_{3,2} + \vec{F}_{3,3}$$

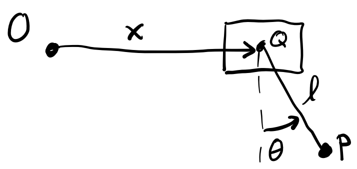
In general, for N particles, these forces show:

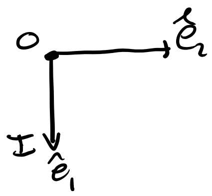
$\vec{F}_{i,i} = 0$ (Assume particles don't exert force on themselves)

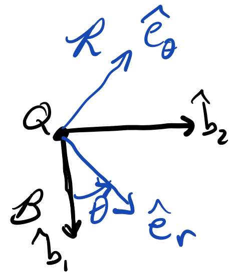
$\vec{F}_{i,j} = -\vec{F}_{j,i}$ (N3L)

Overall, to solve multiparticle system problems, just look at each mass separately and apply N2L.

Example: The overhead crane





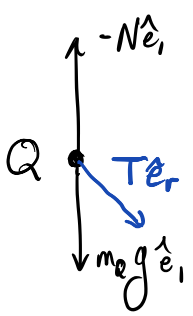


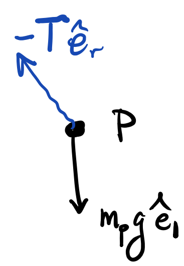
Kinematics:

$$\vec r_{Q/O} = x\hat e_2$$
$${}^I\vec v_{Q/O} = \dot x \hat e_2$$
$${}^I\vec a_{Q/O} = \ddot x \hat e_2$$

$$\vec r_{P/O} = \vec r_{Q/O} + \vec r_{P/Q}$$
$$= x\hat e_2 + \ell\hat e_r$$
$${}^I\vec v_{P/O} = \dot x\hat e_2 + \ell\dot\theta \hat e_\theta$$
$${}^I\vec a_{P/O} = \ddot x \hat e_2 + \ell\ddot\theta\hat e_\theta - \ell\dot\theta^2\hat e_r$$

FBD's:





N2L on each particle:

Eqn 1: $(m_Q g - N)\hat e_1 + T\hat e_r = m_Q \ddot x \hat e_2$ (N2L on Q)

Eqn 2: $m_P g\hat e_1 - T\hat e_r = m_P(\ddot x\hat e_2 + \ell\ddot\theta\hat e_\theta - \ell\dot\theta^2 \hat e_r)$ (N2L on P)

Two Vector Differential Equations that represent 4 scalar, ODE's.

Unknown's? $T, N, \ddot x, \ddot\theta$

Note: The second-order time derivative terms should be treated as unknowns because they will need to be solved for or isolated to put the system in 1st order form.

Breaking the connection between particles allows us to handle them individually, but introduces constraint forces that must be solved for.

Are there ways to study the motion of the system without having to solve for internal constraint forces?