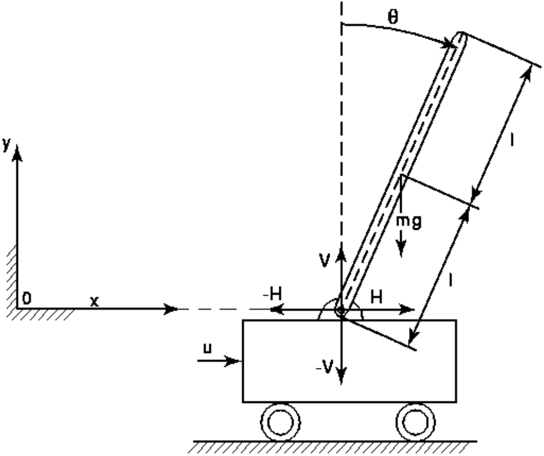
Stabilizing Feedback Control Design

- **Objective**: Construct a stabilizing controller to stabilize a stick balancer system, that is, stabilize the inverted pendulum on a cart system
- Design steps:
  1. Construct simulation and design models
  2. Construct stabilizing state-feedback
  3. Test the controller on the non-linear simulation model
  4. Re-design if needed

Modeling

- For the controller design purposes, first construct a truth/simulation model of the plant at hand
- The truth model is the simulation model that should include all the relevant characteristics of the physical system to be controlled
- The truth model may be too complicated for use in the controller design
- Need to develop a simplified model that can be used to design a controller. Such a simplified model is called the design model
- The design model should capture the essential feature of the process
- The control designs are being simulated using the truth model

Stick balancer



The cart with an inverted pendulum modeling using Newton's laws

- $H = H(t)$ and $V = V(t)$ are horizontal and vertical reaction forces, respectively
- x and y are the coordinates of the fixed, non-rotating coordinate frame $x - y$
- The angular displacement of the stick from the vertical position is $\theta = \theta(t)$
- The mass of the cart is M, while the mass of the stick is m
- The length of the stick is $2l$, and its center of gravity is at its geometric center
- The control force applied to the cart is u
- We assume that the wheels of the cart do not slip

Modeling equations: Apply Newton's second law along the x axis

- Let (x_G, y_G) be the coordinates of the center of gravity of the stick
- Then,

$$\left\{ \begin{array}{lcl} x_G & = & x + l\sin(\theta) \\ y_G & = & l\cos(\theta) \end{array} \right.$$

- Apply Newton's second law along the x axis

$$m\frac{d^2}{dt^2}\left(x + l\sin(\theta)\right) = H$$

Modeling equations: Apply Newton's second law to the cart

- Differentiate

$$m\left(\ddot{x} + l\left(-\dot{\theta}^2\sin(\theta) + \ddot{\theta}\cos(\theta)\right)\right) = H$$

- Newton's second law applied to the cart

$$M\frac{d^2x}{dt^2} = u - H$$

- Combine

$$m\ddot{x} + ml\ddot{\theta}\cos(\theta) - ml\dot{\theta}^2\sin(\theta) = u - M\ddot{x}$$

- Re-arrange

$$\boxed{(M+m)\,\ddot{x} + ml\cos(\theta)\ddot{\theta} = ml\dot{\theta}^2\sin(\theta) + u}$$

Summing the torques about the center of gravity of the stick

- The vertical motion of the center of gravity of the stick—apply Newton's second law along the y axis

$$m\frac{d^2}{dt^2}\left(l\cos(\theta)\right) = V - mg$$

Differentiate

$$ml\left(-\dot{\theta}^2\cos(\theta) - \ddot{\theta}\sin(\theta)\right) = V - mg$$

- Summing the torques about the center of gravity of the stick

$$I_{cm}\frac{d^2\theta}{dt^2} = Vl\sin(\theta) - Hl\cos(\theta)$$

where I_{cm} is the moment of inertia of the stick with respect to its center of mass

Manipulating the torque balance equation

- Manipulate

$$
\begin{aligned}
I_{cm}\ddot{\theta} &= \left(mg - ml\dot{\theta}^2\cos(\theta) - ml\ddot{\theta}\sin(\theta)\right) l\sin(\theta) \\
&\quad - (u - M\ddot{x})\, l\cos(\theta)
\end{aligned}
$$

- More manipulations

$$I_{cm}\ddot{\theta} = mgl\sin(\theta) - ml^2\ddot{\theta} - m\ddot{x}l\cos(\theta)$$

Stick balancer modeling equations

- Re-arrange to obtain

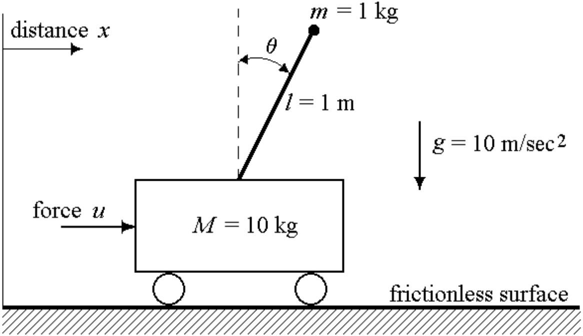
$$\boxed{ml\cos(\theta)\ddot{x} + \left(I_{cm} + ml^2\right)\ddot{\theta} = mgl\sin(\theta)}$$

- Combine the equations in boxes

$$\begin{bmatrix} M+m & ml\cos\theta \\ ml\cos\theta & I_{cm}+ml^2 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} ml\dot{\theta}^2\sin\theta + u \\ mgl\sin\theta \end{bmatrix}$$

Simple stick balancer

- Note that for a point mass on a mass-less shaft $I_{cm} = 0$



Point mass on a mass-less shaft moving on a cart

- Modeling equations simplify

$$\begin{bmatrix} M+m & ml\cos\theta \\ \cos\theta & l \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} ml\dot{\theta}^2\sin\theta + u \\ g\sin\theta \end{bmatrix}$$

- Solve for $\begin{bmatrix} \ddot{x} & \ddot{\theta} \end{bmatrix}^\top$ using MATLAB

```
D=sym('[M+m m*l*cos(theta);cos(theta) l]');
v=sym('[u+m*l*thetadot^2*sin(theta);g*sin(theta)]');
D_inv=inv(D);
g=symmul(D_inv,v);
simplify(g);
pretty(ans)
```

Non-linear state-space model

- Let $\Delta = M + m - m\cos^2\theta$
- Then

$$\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} u + ml\dot{\theta}^2\sin\theta - mg\cos\theta\sin\theta \\ \frac{1}{l}(-u\cos\theta - ml\dot{\theta}^2\cos\theta\sin\theta + gM\sin\theta + gm\sin\theta) \end{bmatrix}$$

- Non-linear state-space model, which is our simulation model,

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{mlx_4^2\sin x_3 - mg\cos x_3\sin x_3 + u}{M + m - m\cos^2 x_3} \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= \frac{-mlx_4^2\cos x_3\sin x_3 + gM\sin x_3 + gm\sin x_3 - \cos x_3 u}{l(M + m - m\cos^2 x_3)} \end{cases}$$

The linearized model about $x = \mathbf{0}$, $u = 0$

- The linearized model, which is our design model,

$$
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 11 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} u
$$

$$
y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}
$$

- The state-feedback control law $u = -\boldsymbol{k}\boldsymbol{x}$ such that the closed-loop poles are located at $\{-1, -2, -1 \pm j\}$ is

$$
u = -\boldsymbol{k}\boldsymbol{x} = -\begin{bmatrix} -0.4 & -1 & -21.4 & -6 \end{bmatrix} \boldsymbol{x}
$$

- Can use MATLAB's functions `acker` or `place` to compute the feedback gain