MA 527

Lecture Notes (section 8.3)

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$\bar{A}\bar{X}_1 = \overline{1+i}\,\bar{X}_1$ iff $A\bar{X}_1 = (1-i)\bar{X}_1$

$\therefore X_2 = \bar{X}_1$.

8.3. Symmetric matrices.

Def $A = [a_{ij}]_{n\times n}$

(1) If $A^T = A$, then A is called symmetric

(2) If $A^T = -A$, then A is called skew-symmetric.

(Ex) $A = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 2 & 5 \\ 2 & -5 & 0 \end{bmatrix}$: skew-symmetric?

No!

$A^T = \begin{bmatrix} 0 & -1 & 2 \\ 1 & \underline{2} & -5 \\ -2 & 5 & 0 \end{bmatrix}$: $-A = \begin{bmatrix} 0 & -1 & 2 \\ 1 & \underline{-2} & -5 \\ -2 & 5 & 0 \end{bmatrix}$

Remark: If $A_{n\times n}$ is skew-symmetric, then each diagonal element of A is zero.

Remark A : an $n \times n$ matrix.

$$A = S + K \quad \left(\begin{array}{l} S: \text{symmetric} \\ K: \text{skew-symmetric} \end{array} \right)$$

1. $S = \frac{1}{2}(A + A^T)$:

$$S^T = \frac{1}{2}(A + A^T)^T = \frac{1}{2}(A^T + A) = S.$$

2. $K = \frac{1}{2}(A - A^T)$:

$$K^T = \frac{1}{2}(A - A^T)^T = \frac{1}{2}(A^T - A) = -K.$$

3. $A = S + K.$

(Orthogonal matrix).

Def (任) $\mathbb{a}, \mathbb{b} \in \mathbb{R}^n$: $\mathbb{a} = [a_1, \cdots, a_n]$, $\mathbb{b} = [b_1, \cdots, b_n]$

(1) $\mathbb{a} \cdot \mathbb{b} = 0 \Rightarrow$ $\mathbb{a}$ & $\mathbb{b}$ are called orthogonal: $\mathbb{a} \perp \mathbb{b}$

(2) $\left(\begin{array}{l} \mathbb{a} \cdot \mathbb{b} = 0 \; : \; \text{orthogonal.} \\ |\mathbb{a}| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2} = 1 \\ |\mathbb{b}| = 1 \end{array} \right.$

$\mathbb{a}$ & $\mathbb{b}$ are called orthonormal.

8.3. Orthogonal matrix.

1. symmetric matrix : $A^T = A$
2. skew-symmetric " : $A^T = -A$ $\quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
3. Orthogonal matrices.

Def $\mathbb{a}, \mathbb{b} \in \mathbb{R}^n$

(1) If $\mathbb{a} \cdot \mathbb{b} = 0$, then $\mathbb{a}$ & $\mathbb{b}$ are called <u>orthogonal</u>.

(2) If $\mathbb{a} \cdot \mathbb{b} = 0$, $|\mathbb{a}| = 1$ & $|\mathbb{b}| = 1$, then $\mathbb{a}$ & $\mathbb{b}$ are called <u>orthonormal</u>.

(3) $\mathbb{a}_1, \mathbb{a}_2, \cdots, \mathbb{a}_k$: orthonormal

iff $\mathbb{a}_i \cdot \mathbb{a}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

(Ex) $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$: $A^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

$\det A = \frac{1}{2} - (-\frac{1}{2}) = 1$

$A^{-1} = \frac{1}{1} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$: orthogonal.

$\therefore \underline{A^{-1} = A^T}$: $AA^T = I$, $A^TA = I$.

Remark: (1) QR - decomposition.

(2) $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$: orthonormal.

$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$: "

Def A : an $n \times n$ matrix

If $A^{-1} = A^T$, A is called orthogonal.

Remark $A = \begin{bmatrix} \mathbb{r}_1 \\ \vdots \\ \mathbb{r}_n \end{bmatrix}_{n \times n}$: orthogonal

$$\mathbb{r}_i \cdot \mathbb{r}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

(Ex) $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$: orthogonal

$$\cos^2\theta + \sin^2\theta = 1.$$

$$A^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \qquad \det A = 1$$

$$A^{-1} = \frac{1}{1}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$=$

Question: $A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ $(a, b \in \mathbb{R})$

Find conditions of a & b to make A
(1) symmetric (2) skew-symmetric
(3) orthogonal.

(1) $A^T = A$ iff $-b = b$ iff $2b = 0$

$\therefore \underline{b = 0}$

$A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$: eigenvalues : $\underline{\lambda = a}$

(2) $A^T = -A$: $a = -a$: $\underline{a = 0}$

$A = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$ $\lambda = ?$

$|A - \lambda I| = \begin{vmatrix} -\lambda & b \\ -b & -\lambda \end{vmatrix} = \lambda^2 + b^2 = 0$

$\underline{\lambda = \pm bi}$: pure-imaginary.

(3) $A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$: orthogonal.

$\underline{A^T = A^{-1}}$: $\det A = a^2 + b^2$

$A^T = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, $A^{-1} = \frac{1}{a^2+b^2}\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

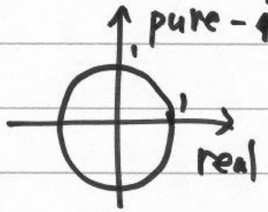
$\therefore a^2 + b^2 = 1.$

$$A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \quad : \quad a^2+b^2=1$$

$$|A-\lambda I| = \begin{vmatrix} a-\lambda & b \\ -b & a-\lambda \end{vmatrix} = (a-\lambda)^2+b^2=0$$

$$\lambda = a \pm bi \quad : \quad \underline{|\lambda| = \sqrt{a^2+b^2} = 1}$$

$\mathbb{C}$



in complex plane.

<u>*Thm*</u>

(1) Every eigenvalue of a symmetric matrix $A_{n\times n}$ is <u>real.</u>

(2) Every eigenvalue of a skew-symmetric matrix $A_{2\times 2}$ is <u>pure-imaginary.</u>

(3) If λ is an eigenvalue of an orthogonal matrix $A_{n\times n}$, $|\lambda| = 1$

Remark : $A_{n\times n}$ is orthogonal

$\underline{\det A = 1 \text{ or } -1.}$

(Proof) $A^T = A^{-1}$ iff $AA^T = I$

$\det(AA^T) = \det I = 1.$ iff $\underbrace{\det A \cdot \det A^T}_{=(\det A)^2} = 1.$

(Proof) (1) Assume that $A_{n\times n}$ is symmetric.

Let λ and X be an eigenvalue and eigenvector of A such that $AX = \lambda X$. —①

Since A is a symmetric real matrix, $\bar{A}^T = A$.

$$① : \ \bar{X}^T A X = \lambda \bar{X}^T X = \lambda |X|^2 \text{ : scalar} \quad -②$$

$$\text{Then} \ \ \underline{\overline{\bar{X}^T A X}^{\,T}} = \overline{\lambda |X|^2}^{\,T} = \bar{\lambda} |X|^2 \quad -③$$

$$\bar{X}^T \bar{A}^T X = \bar{X}^T A X$$

$$\text{Thererefore,} \ \ \lambda |X|^2 = \bar{X}^T A X = \bar{\lambda} |X|^2 \ (\text{by } ②, ③)$$

$\therefore \lambda = \bar{\lambda}$ $\quad$ $\therefore \lambda$ is real.