ECE 68000: MODERN AUTOMATIC CONTROL

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Stability of dynamical systems: The Lyapunov perspective

Lyapunov Function Definition

A function that allows one to deduce stability is termed a Lyapunov function

Lyapunov Function Properties for Continuous Time Systems

- Continuous-time system

$$\dot{x}(t) = f(x(t))$$

- Equilibrium state of interest

$$x_e$$

Three Properties of a Lyapunov Function

We seek an aggregate summarizing function V

- V is continuous
- V has a unique minimum with respect to all other points in some neighborhood of the equilibrium of interest
- Along any trajectory of the system, the value of V never increases

Lyapunov Theorem for Continuous Systems

- Continuous-time system

$$\dot{x}(t) = f(x(t))$$

- Equilibrium state of interest

$$x_e = 0$$

Lyapunov Theorem---Negative Rate of Increase of V

- If $\boldsymbol{x}(t)$ is a trajectory, then $V(\boldsymbol{x}(t))$ represents the corresponding values of V along the trajectory
- In order for $V(\boldsymbol{x}(t))$ not to increase, we require

$$\dot{V}(x(t)) \leq 0$$

The Lyapunov Derivative

- Use the chain rule to compute the derivative of $V(\boldsymbol{x}(t))$

$$\dot{V}\left(x\left(t\right)\right) = \nabla V\left(x\right)^{\mathrm{T}} \dot{x}$$

- Use the plant model to obtain

$$\dot{V}\left(x\left(t\right)\right) = \nabla V\left(x\right)^{\mathrm{T}} f\left(x\right)$$

- Recall

$$\nabla V\left(x\right) = \begin{bmatrix} \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} & \cdots & \frac{\partial V}{\partial x_n} \end{bmatrix}^{\mathrm{T}}$$

Lyapunov Theorem for LTI Systems

The system d$\boldsymbol{x}$/dt=$\boldsymbol{Ax}$ is asymptotically stable, that is, the equilibrium state $\boldsymbol{x}_e=\mathbf{0}$ is asymptotically stable (a.s), if and only if any solution converges to $\boldsymbol{x}_e=\mathbf{0}$ as t tends to infinity for any initial $\boldsymbol{x}_0$

Lyapunov Theorem Interpretation

- View the vector $\boldsymbol{x}(t)$ as defining the coordinates of a point in an n-dimensional state space
- In an a.s. system the point $\boldsymbol{x}(t)$ converges to $\boldsymbol{x}_e = \mathbf{0}$

Lyapunov Theorem for n=2

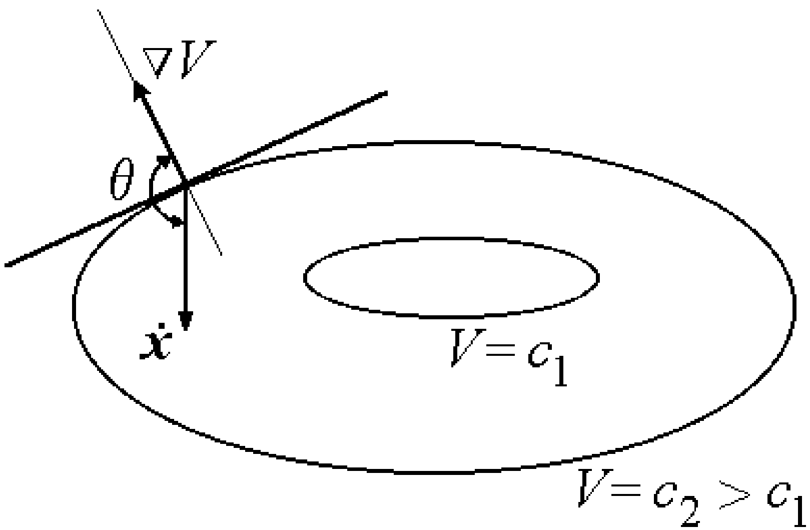
If a trajectory is converging to $\boldsymbol{x}_e = \mathbf{0}$, it should be possible to find a nested set of closed curves $V(x_1, x_2) = c$, $c \geq 0$, such that decreasing values of c yield level curves shrinking in on the equilibrium state $\boldsymbol{x}_e = \mathbf{0}$

Lyapunov Theorem and Level Curves

- The limiting level curve $V(x_1,x_2)=V(\mathbf{0})=0$ is 0 at the equilibrium state $\boldsymbol{x}_e=\mathbf{0}$
- The trajectory moves through the level curves by cutting them in the inward direction ultimately ending at $\boldsymbol{x}_e=\mathbf{0}$

The trajectory is moving in the direction of decreasing V

Note that $\dot{V} = \|\nabla V\| \|\dot{\boldsymbol{x}}\| \cos\theta < 0$



Level Sets

- The level curves can be thought of as contours of a cup-shaped surface
- For an a.s. system, that is, for an a.s. equilibrium state $\boldsymbol{x}_e=\mathbf{0}$, each trajectory falls to the bottom of the cup

Positive Definite Function---General Definition

The function V is positive definite in S, with respect to $\boldsymbol{x}_e$, if V has continuous partials, $V(\boldsymbol{x}_e)=0$, and $V(\boldsymbol{x})>0$ for all $\boldsymbol{x}$ in S, where $\boldsymbol{x} \neq \boldsymbol{x}_e$

Positive Definite Function With Respect to the Origin

Assume, for simplicity, $\boldsymbol{x}_e=\mathbf{0}$, then the function V is positive definite in S if V has continuous partials, $V(\mathbf{0})=0$, and $V(\boldsymbol{x})>0$ for all $\boldsymbol{x}$ in S, where $\boldsymbol{x}\neq\mathbf{0}$

Example: Positive Definite Function

Positive definite function of two variables

$$
\begin{aligned}
V(x_1, x_2) &= 2x_1^2 + 3x_2^2 \\
&= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
&= \boldsymbol{x}^T \boldsymbol{P} \boldsymbol{x} \\
&> 0 \quad \text{for all} \quad \boldsymbol{x} \neq \boldsymbol{0}
\end{aligned}
$$

Positive Semi-Definite Function--- General Definition

The function V is positive semi-definite in S, with respect to $\boldsymbol{x}_e$, if V has continuous partials, $V(\boldsymbol{x}_e)=0$, and $V(\boldsymbol{x})\geq 0$ for all $\boldsymbol{x}$ in S

Positive Semi-Definite Function With Respect to the Origin

Assume, for simplicity, $\boldsymbol{x}_e=\mathbf{0}$, then the function V is positive semi-definite in S if V has continuous partials, $V(\mathbf{0})=0$, and $V(\boldsymbol{x})\geq 0$ for all $\boldsymbol{x}$ in S

Example: Positive Semi-Definite Function

An example of positive semi-definite function of two variables

$$\begin{aligned} V(x_1, x_2) &= x_1^2 \\ &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \boldsymbol{x}^T \boldsymbol{P} \boldsymbol{x} \\ &\geq 0 \quad \text{in} \quad R^2 \end{aligned}$$