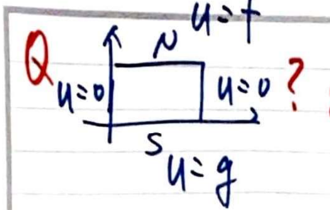
Q 

? : $u(x,y) = \underline{u_1(x,y)} + \underline{u_2(x,y)}$

12.7. Heat equation on very long bars.

No boundary effect/condition.

(IVP)
$$\begin{cases} u_t - k u_{xx} = 0, & -\infty < x < \infty \\ u(x,0) = f(x) \end{cases}$$

Fourier integral / Fourier transform.

(Fourier transform) $\mathcal{F}(f) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx$

Def If $U(x,t)$ is a solution of

(IVP)
$$\begin{cases} U_t - k U_{xx} = 0, & -\infty < x < \infty,\ t > 0 \\ U(x,0) = \underline{\delta(x)}, & -\infty < x < \infty, \end{cases}$$

Dirac delta

$U(x,t)$ is called the fundamental solution of the heat equation.

Thm $U(x,t) = \frac{1}{2\sqrt{\pi k t}} e^{-\frac{x^2}{4kt}}$

(Proof) $\mathcal{F}(U_t) - k\mathcal{F}(U_{xx}) = \mathcal{F}(0) = 0$

$$\frac{\partial}{\partial t} \underbrace{\mathcal{F}(U)}_{=\hat{U}} - k(-\omega^2) \underbrace{\mathcal{F}(U)}_{=\hat{U}} = 0$$

$$\frac{d}{dt}\hat{U} + k\omega^2 \hat{U} = 0 : \rho = e^{k\omega^2 t}$$

$$e^{k\omega^2 t} \frac{d}{dt}\hat{U} + \underline{e^{k\omega^2 t} k\omega^2} \hat{U} = 0$$

$$\frac{d}{dt}\left(e^{k\omega^2 t} \hat{U}\right) = 0 : e^{k\omega^2 t} \hat{U}(\omega,t) = C$$

$$\hat{U}(\omega,t) = C e^{-k\omega^2 t} : \underline{U(\omega,t) = C\mathcal{F}^{-1}(e^{-k\omega^2 t})}$$

$$\mathcal{F}^{-1}(\hat{f}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x}\, d\omega$$

Sec 11.4 $\mathcal{F}^{-1}(e^{-w^2/4}) = \sqrt{2}\, e^{-x^2}$

$$\left(w = \frac{z}{2\sqrt{kt}}\right)$$

$$U(x,t) = \frac{1}{2\sqrt{\pi k t}} e^{-\frac{x^2}{4kt}}$$

Thm (formula)

The solution $u(x,t)$ of the IVP

(IVP) $\begin{cases} u_t - k u_{xx} = 0, & -\infty < x < \infty,\ t > 0 \\ u(x,0) = f(x) \end{cases}$

is $u(x,t) = U(x,t) * f(x) = f(x) * U(x,t)$

$$u(x,t) = \int_{-\infty}^{\infty} f(z)\, U(x-z,t)\, dz$$

(Pf) $\frac{\partial u}{\partial t} = \int_{-\infty}^{\infty} f(z)\, U_t(x-z,t)\, dz$

$-k\frac{\partial^2 u}{\partial x^2} = \int_{-\infty}^{\infty} f(z) (-k) U_{xx}(x-z,t)\, dz$

$u_t - k u_{xx} = \int_{-\infty}^{\infty} f(z) \underbrace{(U_t - kU_{xx})(x-z,t)}_{=0} dz$

$= 0$

$u(x,0) = \int_{-\infty}^{\infty} f(z)\, \underbrace{U(x-z,0)}_{=\delta(x-z)}\, dz = f(x)$

∎

Remark (Fundamental solution of Δ) $\Delta = \nabla^2$

1. $\nabla^2 U = \delta(x,y) = \delta(x)\,\delta(y)$ (2-dim)

$$U(x,y) = \frac{1}{\sqrt{2\pi}} \ln\left(\sqrt{x^2+y^2}\right)$$

2. $\nabla^2 U = \delta(x,y,z) = \delta(x)\,\delta(y)\,\delta(z)$ (3-dim)

$$U(x,y,z) = \frac{-1}{4\pi\sqrt{x^2+y^2+z^2}}$$

(Practice test)

#14

$$\begin{cases} y'' + \lambda y = 0, & 0 < x < \pi \\ y'(0) = 0, & y(\pi) = 0 \end{cases}$$

$y'' + \lambda y = 0$: $\quad r^2 + \lambda = 0, \quad r = \pm\sqrt{-\lambda}$

(1) $-\lambda > 0\ (\lambda < 0)$:

$y(x) = C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x}$

$y'(x) = \sqrt{-\lambda}\, C_1 e^{\sqrt{-\lambda}x} - \sqrt{-\lambda}\, C_2 e^{-\sqrt{-\lambda}x}$

$y'(0) = \cancel{\sqrt{-\lambda}}\, C_1 - \cancel{\sqrt{-\lambda}}\, C_2 = 0 \swarrow$ BC

$C_1 = C_2$

$y(\pi) = C_1 e^{\sqrt{-\lambda}\pi} + C_2 e^{-\sqrt{-\lambda}\pi} = 0 \swarrow$ BC

$C_1 \left(\underbrace{e^{\sqrt{-\lambda}\pi} + e^{-\sqrt{-\lambda}\pi}}_{> 0}\right) = 0$: $C_1 = 0$

$C_2 = 0$

$y(x) \equiv 0$: No eigenfunction.

(2) $\lambda = 0$: $\quad y'' = 0$: $y'(x) = C_1$

$y(x) = C_1 x + C_2$: $y'(x) = C_1$

$y'(0) = C_1 = 0 \swarrow$ BC. : $y(x) = C_2$

$y(\pi) = C_2 = 0 \swarrow$ BC : $y(x) \equiv 0$

No eigenfunction.

(3) $-\lambda < 0\ (\lambda > 0)$: $\quad r = \pm\sqrt{-\lambda} = \pm\sqrt{\lambda}\, i$

$y(x) = C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x)$

$y'(x) = -\sqrt{\lambda}\, C_1 \sin(\sqrt{\lambda}x) + \sqrt{\lambda}\, C_2 \cos(\sqrt{\lambda}x)$

$y'(0) = \sqrt{\lambda}\, C_2 = 0 \swarrow$ BC : $C_2 = 0$

$y(x) = C_1 \cos(\sqrt{\lambda}x)$

$y(\pi) = C_1 \cos(\sqrt{\lambda}\pi) = 0$

If $C_1 = 0$, No eigenfunctions. : $C_1 \neq 0$

$\cos(\sqrt{\lambda}\pi) = 0$: $\sqrt{\lambda}\pi = n\pi - \frac{\pi}{2}$

$(n = 1, 2, 3, \cdots)$

$\sqrt{\lambda} = n - \frac{1}{2}$: $\lambda_n \overset{\text{let}}{=} \left(n - \frac{1}{2}\right)^2$, $n = 1, 2, \cdots$

$y_n(x) = \cos\left(\left(n - \frac{1}{2}\right)x\right)$: eigenfunctions.

#15. $f(x) = \begin{cases} 0, & 0 < x < \pi \\ 1, & \pi \leq x < 2\pi \end{cases}$: missing data

$F_s(x)$: the Fourier sine series.

$L = 2\pi$: Use $f_o(x)$: the odd extension.

$$F_s(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{nx}{2}\right)$$

$$b_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$b_n = \frac{2}{2\pi}\left(\int_0^{\pi} 0\, dx + \int_{\pi}^{2\pi} 1 \cdot \sin\left(\frac{nx}{2}\right) dx\right)$$

$$= \frac{1}{\pi}\left[\frac{-\cos\left(\frac{nx}{2}\right)}{n/2}\right]_{\pi}^{2\pi} = \frac{1}{\pi}\left(\frac{-2}{n}\right)\left[\cos\left(\frac{n2\pi}{2}\right) - \cos\left(\frac{n\pi}{2}\right)\right]$$

$$b_n = -\frac{2}{n\pi}\left(\cos(n\pi) - \cos\left(\frac{n\pi}{2}\right)\right]$$

$$F_s(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\left(\cos\left(\frac{n\pi}{2}\right) - \cos(n\pi)\right) \sin\left(\frac{nx}{2}\right)$$

#16. $\{1, x^7, \cos(x)\}$: orthogonal on $(-\pi, \pi)$

$(f, g) = \int_{-\pi}^{\pi} f(x)\, g(x)\, dx$: an inner product.

$$(1, x^7) = \int_{-\pi}^{\pi} x^7\, dx = 0$$

$$(x^7, \cos(x)) = \int_{-\pi}^{\pi} x^7 \cos(x)\, dx = 0$$

$$(1, \cos(x)) = \int_{-\pi}^{\pi} 1 \cdot \cos(x)\, dx = \left[\sin(x)\right]_{-\pi}^{\pi}$$

$$= \sin\pi - \sin(-\pi) = 0$$

$f(x) = C_1 \cdot 1 + C_2 x^7 + C_3 \cos(x)$: $C_2 = ?$

$$(f(x), x^7) = C_1 \underbrace{(1, x^7)}_{=0} + C_2 (x^7, x^7) + C_3 \underbrace{(\cos(x), x^7)}_{=0}$$

$$= C_2 \int_{-\pi}^{\pi} x^7 \cdot x^7\, dx = C_2 \int_{-\pi}^{\pi} x^{14}\, dx$$

$$(f, x^7) = C_2\left[\frac{x^{15}}{15}\right]_{-\pi}^{\pi} = \frac{C_2}{15}\left(\pi^{15} + (+\pi)^{15}\right)$$

$$= \frac{2\pi^{15}}{15} C_2$$

$$C_2 = \frac{15}{2\pi^{15}}(f, x^7) = \frac{15}{2\pi^{15}}\int_{-\pi}^{\pi} f(x)\, x^7\, dx.$$

#17. $f(x) = \begin{cases} 0, & 0 < x < 1 \\ 1, & 1 < x < 3 \\ 0, & x > 3. \end{cases}$

$F(\omega)$: the Fourier sine transform.

$$F(\omega) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\,\mathrm{Sin}(\omega x)\, dx$$

The inverse Fourier sine transform

$$\mathcal{F}_s^{-1}(F(\omega))_{(x)} = \sqrt{\frac{2}{\pi}}\int_0^\infty F(\omega)\,\mathrm{Sin}(\omega x)\, d\omega$$

$g(z)$ = the Fourier sine transform of $F(\omega)$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty F(\omega)\,\mathrm{Sin}(\omega z)\, d\omega$$

: the inverse Fourier sine transform