Ch 9 (cont'd)

Work and energy of a rigid body

Recall that kinetic energy separated for a collection of particles:

$$T_O = T_{G/O} + T_G$$

Using the definition of a rigid body (i.e. no relative motion of the internal particles), the transport theorem, and the definition of moment of inertia, T_G can be rewritten as,

$$\boxed{T_G = \frac{1}{2} I_G \left({}^{I}\vec{\omega}^{B} \cdot {}^{I}\vec{\omega}^{B} \right) = \frac{1}{2} I_G \, \| {}^{I}\vec{\omega}^{B} \|^2}$$

We can also substitute the angular momentum expression to obtain:

$$\boxed{T_G = \frac{1}{2} \, {}^{I}\vec{h}_G \cdot {}^{I}\vec{\omega}^{B}}$$

For the work-energy formulas, we begin with the multiparticle formulas and impose "no internal relative motion" constraints.

↳ There is no internal work within a rigid body

The total-work-kinetic energy formula still holds for a rigid body:

$$\boxed{T_O(t_2) = T_O(t_1) + W^{(tot)}} \quad (\#1 \text{ of } 3)$$

Since there is no internal work we can just write $W = W^{(tot)}$

The work W separates, just as kinetic energy does

$$W = \underbrace{W_{G/O}}_{\int_{t_1}^{t_2} \vec{F}_G \cdot {}^{I}\vec{V}_{G/O}\,dt} + \underbrace{W_G}_{\int_{t_1}^{t_2} \vec{M}_G \cdot {}^{I}\vec{\omega}^{B}\,dt}$$

rotational work about COM.

In can be shown that

$$W_{G/O} = \Delta T_{G/O} = T_{G/O}(t_2) - T_{G/O}(t_1)$$

and

$$W_G = \Delta T_G = T_G(t_2) - T_G(t_1)$$

Notes:

- If the total external force is zero, the translational kinetic energy of G, is conserved
- If the total external moment about G is zero, the rotational kinetic energy is conserved

The potential energy for a conservative force acting on the rigid body is,

$$U_O(t) = \int_{\mathcal{B}} U_{dm/O}(\vec{r}_{dm/O})\, dm$$

Then

$$\boxed{U_O(t_2) = U_O(t_1) - W^{(c,ext)}_{(t_1,t_2)}}$$ (#2 of 3)

Note that there is no internal potential energy change and no internal conservative work

Lastly, define total energy,

$$E_O(t) \triangleq T_{G/O}(t) + T_G(t) + U_O(t)$$

Then,

$$\boxed{E_O(t_2) = E_O(t_1) + W^{(nc)}_{G/O} + W^{(nc)}_G}$$

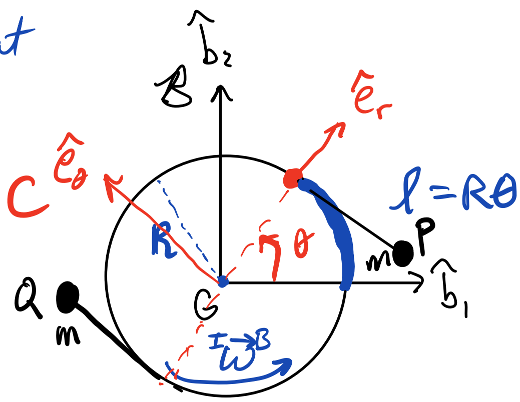
(#3 of 3)

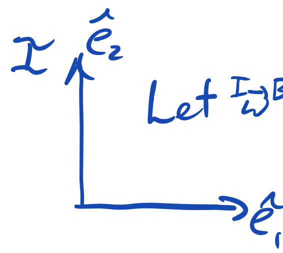
Ex. Yo-Yo De-spin

$^{I}\vec{h}_G^{(sys)} = ?$

B: body frame that rotates

C: polar frame that tracks to contact point





Let $^{I}\vec{\omega}^{B} = \omega \hat{e}_3$

Objective is to slow down the spin of a satellite.

No external forces or moments $\Rightarrow$ Total angular momentum is conserved.

$$\vec{r}_{P/G} = R\hat{e}_r - R\theta\hat{e}_\theta$$

$$^{I}\vec{v}_{P/G} = R\left(^{I}\vec{\omega}^{C} \times \hat{e}_r\right) - R\dot{\theta}\hat{e}_\theta - R\theta\left(^{I}\vec{\omega}^{C} \times \hat{e}_\theta\right)$$

$$^{I}\vec{\omega}^{C} = {}^{I}\vec{\omega}^{B} + {}^{B}\vec{\omega}^{C} = (\omega + \dot{\theta})\hat{c}_3 \quad \text{(addition of angular velocity)}$$

$$^{I}\vec{v}_{P/G} = R\theta(\omega + \dot{\theta})\hat{e}_r + R\,{}^{I}\vec{\omega}^{B}\hat{e}_\theta$$

Similarly for Q,

$$^{I}\vec{v}_{Q/G} = -R\theta(\omega + \dot{\theta})\hat{e}_r - R\omega\hat{e}_\theta$$

Total angular momentum is, $^{I}\vec{h}_G^{(sys)} = {}^{I}\vec{h}_G^{(DISK)} + {}^{I}\vec{h}_G^{(particles)}$

$\hookrightarrow \sum_i \vec{r}_{i/G} \times m_i \, {}^{I}\vec{v}_{i/G}$

$$^{I}\vec{h}_G^{(disk)} = I_G \omega \hat{c}_3$$

$$^{I}\vec{h}_G^{(sys)} = I_G \omega \hat{c}_3 + \left(\vec{r}_{P/G} \times m\, {}^{I}\vec{v}_{P/G}\right) + \left(\vec{r}_{Q/G} \times m\, {}^{I}\vec{v}_{Q/G}\right)$$

$$= \left(I_G \omega + 2mR^2\left(\omega + \theta^2(\omega + \dot{\theta})\right)\right)\hat{c}_3$$

ω and $\dot{\theta}$ are both unknowns. We can use conservation of total energy to get another equation. This leads to a set of equations (differential in $\dot{\theta}$) that can be solved for $\theta(t)$ and $\omega(t)$.

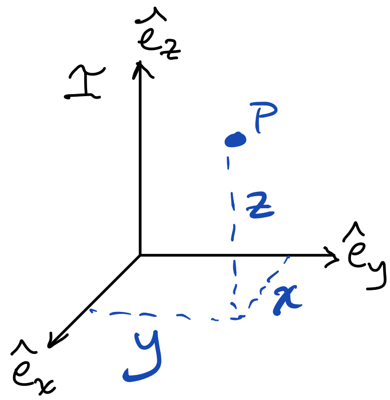
Chapter 10 - Particle Kinematics & Kinetics in 3D

3D coordinate systems

Cartesian $(x, y, z)_{\mathcal{I}}$

$$\vec{r}_{P/O} = x\hat{e}_x + y\hat{e}_y + z\hat{e}_z$$

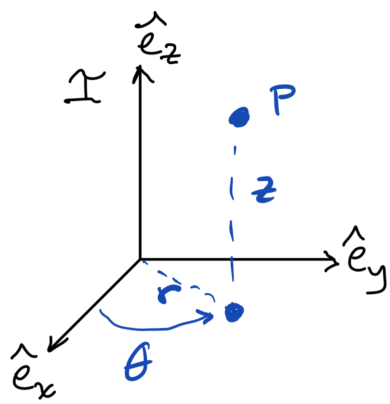
∴ simple time derivatives



Cylindrical $(r, \theta, z)_{\mathcal{I}}$

$$\vec{r}_{P/O} = r\cos\theta\,\hat{e}_x + r\sin\theta\,\hat{e}_y + z\hat{e}_z$$

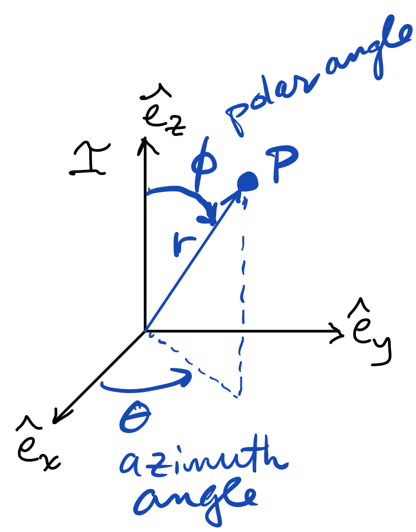
∴ messy time derivatives



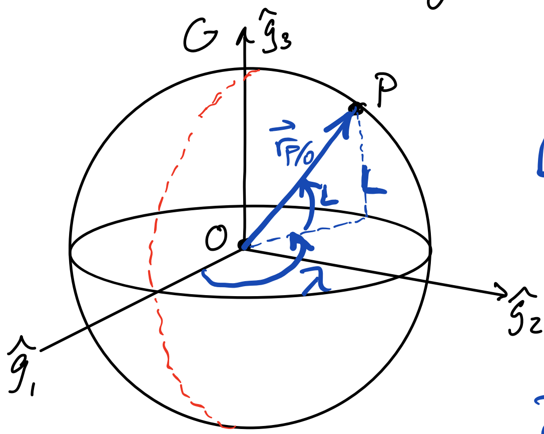
Spherical $(r, \theta, \phi)_{\mathcal{I}}$

$$\vec{r}_{P/O} = r\cos\theta\sin\phi\,\hat{e}_x + r\sin\theta\sin\phi\,\hat{e}_y + r\cos\phi\,\hat{e}_z$$

∴ Extremely messy time derivatives!



Example: Latitude and Longitude



Latitude is a complement of ϕ

$L = \frac{\pi}{2} - \phi$

λ: longitude

Frame $G = (O, \hat{g}_1, \hat{g}_2, \hat{g}_3)$ → Geographic ref. frame with $\hat{g}_1$ pointing to the great circle which goes through Greenwich, England.

Note: not inertial

$^{I}\vec{\omega}^{G} = \Omega_E \hat{g}_3$

Cylindrical and Spherical Reference Frames

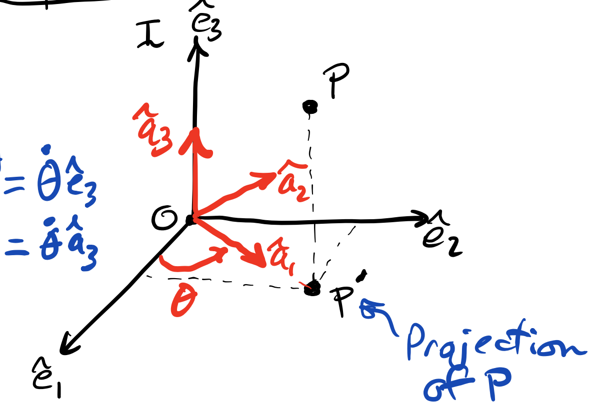
- These are generalizations of the 2D polar frame
- Make kinematic and dynamic calculations easier.
- angular velocity vector is crucial to kinematic derivations since it is used to differentiate unit vectors of a non-inertial frame.

Cylindrical frame – Just a polar frame that tracks the projection of $\vec{r}_{P/O}$ in $(\hat{e}_x, \hat{e}_y)$ plane, in combination with an $\hat{e}_z$ direction vector to complete the 3D reference frame

Spherical frame — A frame that tracks particle P that is constructed using **intermediate frames** that are related by **elementary rotations**.

Spherical Frame

$${}^{I\to A}\vec{\omega} = \dot{\theta}\,\hat{e}_3 = \dot{\theta}\,\hat{a}_3$$

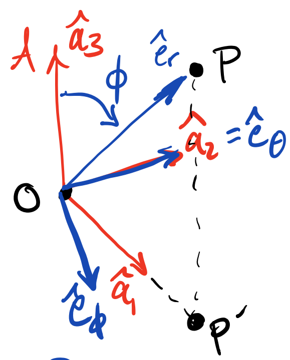


$A = (O, \hat{a}_1, \hat{a}_2, \hat{a}_3)$ "Intermediate frame"

$$\begin{array}{c|ccc} & \hat{a}_1 & \hat{a}_2 & \hat{a}_3 \\ \hline \hat{e}_1 & c\theta & -s\theta & 0 \\ \hat{e}_2 & s\theta & c\theta & 0 \\ \hat{e}_3 & 0 & 0 & 1 \end{array} \Bigg\} \; {}^{I}C^{A}$$

To $\to I$, From $\to A$

$${}^{A\to B}\vec{\omega} = \dot{\phi}\,\hat{a}_2 = \dot{\phi}\,\hat{e}_\theta$$



See Fig 10.4

$B = (O, \hat{e}_\phi, \hat{e}_\theta, \hat{e}_r)$

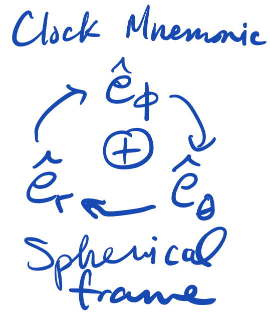
$$\begin{array}{c|ccc} & \hat{e}_\phi & \hat{e}_\theta & \hat{e}_r \\ \hline \hat{a}_1 & c\phi & 0 & s\phi \\ \hat{a}_2 & 0 & 1 & 0 \\ \hat{a}_3 & -s\phi & 0 & c\phi \end{array} \Bigg\} \; {}^{A}C^{B}$$

intermediate frame $\to A$; Spherical frame here $\to B$

Overall, our transformation can be obtained by using both elementary transformations:

$${}^{I}C^{B} = \left({}^{I}C^{A}\right)\left({}^{A}C^{B}\right) \qquad \left[\vec{r}_{P/O}\right]_I = {}^{I}C^{B}\left[\vec{r}_{P/O}\right]_B$$

$$
{}^{I}C^{B} = \begin{bmatrix} c\theta c\phi & -s\theta & c\theta s\phi \\ s\theta c\phi & c\theta & s\theta s\phi \\ -s\phi & 0 & c\phi \end{bmatrix}
$$



What are our kinematics in a spherical frame?

$$\vec{r}_{P/O} = r\hat{e}_r$$

$$ {}^{I}\vec{v}_{P/O} = \dot{r}\hat{e}_r + r\,\underbrace{\frac{{}^{I}d}{dt}(\hat{e}_r)}$$

→ We know how differentiate unit vectors using the angular velocity,

$$\frac{{}^{I}d}{dt}(\hat{e}_r) = \underbrace{{}^{I}\vec{\omega}^{B}} \times \hat{e}_r$$

→ Addition property

$$
\begin{aligned}
{}^{I}\vec{\omega}^{B} &= {}^{I}\vec{\omega}^{A} + {}^{A}\vec{\omega}^{B} \\
&= \dot{\theta}\,\hat{a}_3 + \dot{\phi}\,\hat{e}_\theta \\
&= \dot{\theta}\left(-s\phi\,\hat{e}_\phi + c\phi\,\hat{e}_r\right) + \dot{\phi}\,\hat{e}_\theta
\end{aligned}
$$

$$\Rightarrow \frac{{}^{I}d}{dt}(\hat{e}_r) = \dot{\theta}\,s\phi\,\hat{e}_\theta + \dot{\phi}\,\hat{e}_\phi$$

Similarly, we could find,

$$\frac{{}^I d}{dt}(\hat{e}_\theta) = {}^I\vec{\omega}^B \times \hat{e}_\theta = -\dot{\theta}\, s\phi\, \hat{e}_r - \dot{\theta}\, c\phi\, \hat{e}_\phi$$

$$\frac{{}^I d}{dt}(\hat{e}_\phi) = {}^I\vec{\omega}^B \times \hat{e}_\phi = -\dot{\phi}\, \hat{e}_r + \dot{\theta}\, c\phi\, \hat{e}_\theta$$

Returning to complete kinematics:

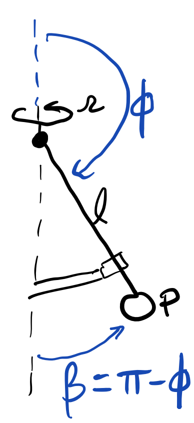
$${}^I\vec{v}_{P/O} = \dot{r}\,\hat{e}_r + r\dot{\theta}\, s\phi\, \hat{e}_\theta + r\dot{\phi}\,\hat{e}_\phi$$

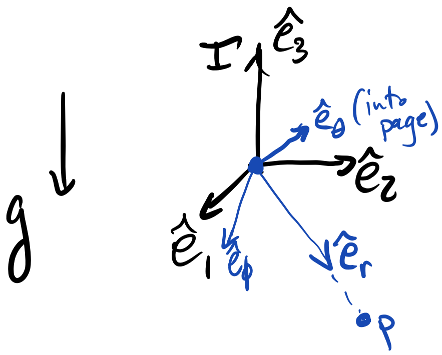
$$\begin{aligned} {}^I\vec{a}_{P/O} = &\left(2\dot{r}\dot{\phi} + r\ddot{\phi} - r\dot{\theta}^2 c\phi\, s\phi\right)\hat{e}_\phi \\ &+ \left(2\dot{r}\dot{\theta}\, s\phi + 2r\dot{\theta}\dot{\phi}\, c\phi + r\ddot{\theta}\, s\phi\right)\hat{e}_\theta \\ &+ \left(\ddot{r} - r\dot{\phi}^2 - r\dot{\theta}^2 \sin^2\phi\right)\hat{e}_r \end{aligned}$$

Much more manageable than the expressions for ${}^I\vec{a}_{P/O}$ expressed in the <u>inertial frame</u> using spherical coordinates.

<u>Ex.</u> <u>"Simplified" Flyball Governor</u>
- No linkages
- Ω is constant

See full Flyball Governer example in <u>Tutorial 10.2</u> to see a slightly different approach/model.
- Uses angular momentum





Kinematics

$$^I\vec\omega^B = \Omega \hat a_3 + \dot\phi \hat e_\theta$$

$$= \Omega(-s\phi\, \hat e_\phi + c\phi\, \hat e_r) + \dot\phi \hat e_\theta$$

$$\vec r_{P/O} = \ell \hat e_r$$

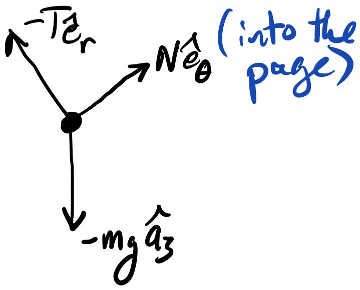
$$^I\vec v_{P/O} = \ell\, {}^I\vec\omega^B \times \hat e_r = \ell(\Omega s\phi\, \hat e_\theta + \dot\phi\, \hat e_\phi)$$

$$^I\vec a_{P/O} = \ell\Omega\dot\phi c\phi\, \hat e_\theta + \ell\Omega s\phi \frac{^Id}{dt}(\hat e_\theta) + \ell\ddot\phi\, \hat e_\phi + \ell\dot\phi \frac{^Id}{dt}(\hat e_\phi)$$

$$\vdots$$

$$= (-\ell\Omega^2 s\phi c\phi + \ell\ddot\phi)\hat e_\phi + (2\ell\Omega c\phi\dot\phi)\hat e_\theta + (-\ell\Omega^2 s^2\phi - \ell\dot\phi^2)\hat e_r$$

FBD



$$\vec F_P = m_P\, {}^I\vec a_{P/O} \Rightarrow -T\hat e_r + N\hat e_\theta - mg(-s\phi\, \hat e_\phi + c\phi\, \hat e_r) = m_P(\dots)$$

$$\hat e_r:\ -T - m_P g c\phi = m_P(-\ell\Omega^2 s^2\phi - \ell\dot\phi^2)$$

$$\hat e_\theta:\ N = 2 m_P \ell\Omega c\phi\dot\phi$$

$$\hat e_\phi:\ mg s\phi = m_P(-\ell\Omega^2 s\phi c\phi + \ell\ddot\phi)$$

Unknowns: $\ddot\phi$, T, N

Note: θ doesn't appear here
$\theta(t) = \Omega t + \theta(0)$

EOM: $\ddot\phi = \frac{g}{\ell} s\phi + \Omega^2 s\phi c\phi$

But, in terms of β for convenience, $\beta = \pi - \phi$

$$\boxed{\ddot{\beta} = -\frac{g}{\ell} s\beta + \Omega^2 s\beta c\beta}$$

Equil points

$$0 = \underbrace{\left(-\frac{g}{\ell} + \Omega^2 c\beta^*\right)}\underbrace{s\beta^*}$$

$\sin\beta^* = 0$

$\beta^* = 0, \pi$

$-\frac{g}{\ell} + \Omega^2 c\beta^* = 0$

$\beta^* = \pm\cos^{-1}\left(\frac{g}{\ell\Omega^2}\right)$

$$\beta^* = 0, \pi, \pm \underbrace{\cos^{-1}\left(\frac{g}{\ell\Omega^2}\right)}$$

only occurs for $\Omega \geq \sqrt{\frac{g}{\ell}}$

Bifurcation Diagram