ECE 68000: MODERN AUTOMATIC CONTROL

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Linear Quadratic Regulator (LQR)

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- The plant

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}, \quad \boldsymbol{x}(0) = \boldsymbol{x}_0$$

  and the associated performance index

$$J = \int_0^\infty \left(\boldsymbol{x}^\top \boldsymbol{Q}\boldsymbol{x} + \boldsymbol{u}^\top \boldsymbol{R}\boldsymbol{u}\right) dt$$

  where $\boldsymbol{Q} = \boldsymbol{Q}^\top \succeq 0$ and $\boldsymbol{R} = \boldsymbol{R}^\top \succ 0$
- Objective: construct a stabilizing linear state-feedback controller,

$$\boldsymbol{u} = -\boldsymbol{K}\boldsymbol{x}$$

  that minimizes the performance index J
- Denote such a linear control law by $\boldsymbol{u}^*$

LQR development

- Assume that a linear state-feedback optimal controller exists such that the optimal closed-loop system

$$\dot{\boldsymbol{x}} = (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K})\boldsymbol{x}$$

  is asymptotically stable
- Hence there is a Lyapunov function $V = \boldsymbol{x}^\top \boldsymbol{P}\boldsymbol{x}$ for the closed-loop system, that is, for some $\boldsymbol{P} = \boldsymbol{P}^\top \succ 0$ the Lyapunov derivative dV/dt is negative definite

Theorem

If the state-feedback controller $\boldsymbol{u}^* = -\boldsymbol{K}\boldsymbol{x}$ is such that

$$\min_{\boldsymbol{u}} \left(\frac{dV}{dt} + \boldsymbol{x}^\top \boldsymbol{Q}\boldsymbol{x} + \boldsymbol{u}^\top \boldsymbol{R}\boldsymbol{u} \right) = 0,$$

for some $V = \boldsymbol{x}^\top \boldsymbol{P}\boldsymbol{x}$, then the controller is optimal

Proof of the LQR theorem

- Rewrite the condition of the theorem as

$$\left.\frac{dV}{dt}\right|_{\boldsymbol{u}=\boldsymbol{u}^*} + \boldsymbol{x}^\top \boldsymbol{Q}\boldsymbol{x} + \boldsymbol{u}^{*\top} \boldsymbol{R}\boldsymbol{u}^* = 0$$

- Hence,

$$\left.\frac{dV}{dt}\right|_{\boldsymbol{u}=\boldsymbol{u}^*} = -\boldsymbol{x}^\top \boldsymbol{Q}\boldsymbol{x} - \boldsymbol{u}^{*\top} \boldsymbol{R}\boldsymbol{u}^*$$

- Integrate both sides of the resulting equation with respect to time from 0 to ∞

$$V(\boldsymbol{x}(\infty)) - V(\boldsymbol{x}(0)) = -\int_0^\infty \left(\boldsymbol{x}^\top \boldsymbol{Q}\boldsymbol{x} + \boldsymbol{u}^{*\top} \boldsymbol{R}\boldsymbol{u}^*\right) dt$$

Proof of the LQR theorem—contd.

- Since the closed-loop system is asymptotically stable, $\boldsymbol{x}(\infty) = \mathbf{0}$, and

$$V(\boldsymbol{x}(0)) = \boldsymbol{x}_0^\top \boldsymbol{P} \boldsymbol{x}_0 = \int_0^\infty \left(\boldsymbol{x}^\top \boldsymbol{Q} \boldsymbol{x} + \boldsymbol{u}^{*\top} \boldsymbol{R} \boldsymbol{u}^* \right) dt$$

- Thus, we showed that if a linear state-feedback controller satisfies the assumption of the theorem, then the value of the performance index for such a controller is

$$J\left(\boldsymbol{u}^*\right) = \boldsymbol{x}_0^\top \boldsymbol{P} \boldsymbol{x}_0$$

- To show that such a controller is indeed optimal, use a proof by contradiction
- Assume that $\boldsymbol{u}^*$ is not optimal
- Suppose that $\tilde{\boldsymbol{u}}$ yields a smaller value of J, that is,

$$J\left(\tilde{\boldsymbol{u}}\right) < J\left(\boldsymbol{u}^*\right)$$

Proof of the LQR theorem—by contradiction

- Hence
$$\left.\frac{dV}{dt}\right|_{\boldsymbol{u}=\tilde{\boldsymbol{u}}} + \boldsymbol{x}^\top \boldsymbol{Q}\boldsymbol{x} + \tilde{\boldsymbol{u}}^\top \boldsymbol{R}\tilde{\boldsymbol{u}} \geq 0$$
that is,
$$\left.\frac{dV}{dt}\right|_{\boldsymbol{u}=\tilde{\boldsymbol{u}}} \geq -\boldsymbol{x}^\top \boldsymbol{Q}\boldsymbol{x} - \tilde{\boldsymbol{u}}^\top \boldsymbol{R}\tilde{\boldsymbol{u}}$$
- Integrating the above with respect to time from 0 to ∞ yields
$$V(\boldsymbol{x}(0)) \leq \int_0^\infty \left(\boldsymbol{x}^\top \boldsymbol{Q}\boldsymbol{x} + \tilde{\boldsymbol{u}}^\top \boldsymbol{R}\tilde{\boldsymbol{u}} \right) dt$$
implying that
$$J(\boldsymbol{u}^*) \leq J(\tilde{\boldsymbol{u}})$$
which is a contradiction, and the proof is complete.

Finding $\boldsymbol{P}$

- It follows from the above theorem that the synthesis of the optimal control law involves finding an appropriate Lyapunov function, or equivalently, the matrix $\boldsymbol{P}$
- The appropriate $\boldsymbol{P}$ is found by minimizing

$$f(\boldsymbol{u}) = \frac{dV}{dt} + \boldsymbol{x}^\top \boldsymbol{Q}\boldsymbol{x} + \boldsymbol{u}^\top \boldsymbol{R}\boldsymbol{u}$$

- Apply the necessary condition for unconstrained minimization

$$\frac{\partial}{\partial \boldsymbol{u}} \left(\frac{dV}{dt} + \boldsymbol{x}^\top \boldsymbol{Q}\boldsymbol{x} + \boldsymbol{u}^\top \boldsymbol{R}\boldsymbol{u} \right) \bigg|_{\boldsymbol{u}=\boldsymbol{u}^*} = \boldsymbol{0}^\top$$

Finding $\boldsymbol{P}$—manipulations

- Differentiating yields

$$
\begin{aligned}
& \frac{\partial}{\partial \boldsymbol{u}}\left(\frac{dV}{dt} + \boldsymbol{x}^\top \boldsymbol{Q}\boldsymbol{x} + \boldsymbol{u}^\top \boldsymbol{R}\boldsymbol{u}\right) \\
= \; & \frac{\partial}{\partial \boldsymbol{u}}\left(2\boldsymbol{x}^\top \boldsymbol{P}\dot{\boldsymbol{x}} + \boldsymbol{x}^\top \boldsymbol{Q}\boldsymbol{x} + \boldsymbol{u}^\top \boldsymbol{R}\boldsymbol{u}\right) \\
= \; & \frac{\partial}{\partial \boldsymbol{u}}\left(2\boldsymbol{x}^\top \boldsymbol{P}\boldsymbol{A}\boldsymbol{x} + 2\boldsymbol{x}^\top \boldsymbol{P}\boldsymbol{B}\boldsymbol{u} + \boldsymbol{x}^\top \boldsymbol{Q}\boldsymbol{x} + \boldsymbol{u}^\top \boldsymbol{R}\boldsymbol{u}\right) \\
= \; & 2\boldsymbol{x}^\top \boldsymbol{P}\boldsymbol{B} + 2\boldsymbol{u}^\top \boldsymbol{R} \\
= \; & \boldsymbol{0}^\top
\end{aligned}
$$

Optimal control law

- Candidate for an optimal control law
$$\boldsymbol{u}^* = -\boldsymbol{R}^{-1}\boldsymbol{B}^\top \boldsymbol{P}\boldsymbol{x} = -\boldsymbol{K}\boldsymbol{x},$$
where $\boldsymbol{K} = \boldsymbol{R}^{-1}\boldsymbol{B}^\top \boldsymbol{P}$
- Note that
$$\begin{aligned} & \frac{\partial^2}{\partial \boldsymbol{u}^2}\left(\frac{dV}{dt} + \boldsymbol{x}^\top \boldsymbol{Q}\boldsymbol{x} + \boldsymbol{u}^\top \boldsymbol{R}\boldsymbol{u}\right) \\ = \; & \frac{\partial^2}{\partial \boldsymbol{u}^2}\left(2\boldsymbol{x}^\top \boldsymbol{P}\boldsymbol{A}\boldsymbol{x} + 2\boldsymbol{x}^\top \boldsymbol{P}\boldsymbol{B}\boldsymbol{u} + \boldsymbol{x}^\top \boldsymbol{Q}\boldsymbol{x} + \boldsymbol{u}^\top \boldsymbol{R}\boldsymbol{u}\right) \\ = \; & \frac{\partial}{\partial \boldsymbol{u}}\left(2\boldsymbol{x}^\top \boldsymbol{P}\boldsymbol{B} + 2\boldsymbol{u}^\top \boldsymbol{R}\right) \\ = \; & 2\boldsymbol{R} \\ \succ \; & 0. \end{aligned}$$
- The second order sufficiency condition for $\boldsymbol{u}^*$ to be optimal, that is, to minimize J satisfied

Closed-loop system driven by $\boldsymbol{u}^*$

- How to find appropriate $\boldsymbol{P}$?
- The optimal closed-loop system

$$\dot{\boldsymbol{x}} = \left(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^\top \boldsymbol{P}\right)\boldsymbol{x}, \quad \boldsymbol{x}(0) = \boldsymbol{x}_0$$

- The optimal controller satisfies

$$\left.\frac{dV}{dt}\right|_{\boldsymbol{u}=\boldsymbol{u}^*} + \boldsymbol{x}^\top \boldsymbol{Q}\boldsymbol{x} + \boldsymbol{u}^{*\top}\boldsymbol{R}\boldsymbol{u}^* = 0$$

that is,

$$2\boldsymbol{x}^\top \boldsymbol{P}\boldsymbol{A}\boldsymbol{x} + 2\boldsymbol{x}^\top \boldsymbol{P}\boldsymbol{B}\boldsymbol{u}^* + \boldsymbol{x}^\top \boldsymbol{Q}\boldsymbol{x} + \boldsymbol{u}^{*\top}\boldsymbol{R}\boldsymbol{u}^* = 0$$

Algebraic Riccati equation (ARE)

- Substitute the expression for $\boldsymbol{u}^*$ into the above equation and represent it as

$$
\begin{aligned}
\boldsymbol{x}^\top \left(\boldsymbol{A}^\top \boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}\right)\boldsymbol{x} - 2\boldsymbol{x}^\top \boldsymbol{P}\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^\top \boldsymbol{P}\boldsymbol{x} \\
+\boldsymbol{x}^\top \boldsymbol{Q}\boldsymbol{x} + \boldsymbol{x}^\top \boldsymbol{P}\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^\top \boldsymbol{P}\boldsymbol{x} \\
= 0
\end{aligned}
$$

- Factoring out $\boldsymbol{x}$ yields

$$
\boldsymbol{x}^\top \left(\boldsymbol{A}^\top \boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} + \boldsymbol{Q} - \boldsymbol{P}\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^\top \boldsymbol{P}\right)\boldsymbol{x} = 0
$$

- The above equation should hold for any $\boldsymbol{x}$
- For this to be true we have to have

$$
\boxed{\boldsymbol{A}^\top \boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} + \boldsymbol{Q} - \boldsymbol{P}\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^\top \boldsymbol{P} = \boldsymbol{O}}
$$

CARE

- The equation

$$\boxed{\boldsymbol{A}^\top \boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} + \boldsymbol{Q} - \boldsymbol{P}\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^\top \boldsymbol{P} = \boldsymbol{O}}$$

  is the *algebraic Riccati equation* (ARE) or *continuous-time algebraic Riccati equation* (CARE)
- In sum, the synthesis of the optimal linear state feedback controller minimizing the performance index

$$J = \int_0^\infty \left(\boldsymbol{x}^\top \boldsymbol{Q}\boldsymbol{x} + \boldsymbol{u}^\top \boldsymbol{R}\boldsymbol{u}\right) dt$$

  subject to

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}, \quad \boldsymbol{x}(0) = \boldsymbol{x}_0$$

  requires solving the ARE

Example

- The plant:
$$\dot{x} = 2u_1 + 2u_2, \qquad x(0) = 3,$$
and the associated performance index
$$J = \int_0^\infty \left(x^2 + ru_1^2 + ru_2^2\right) dt,$$
where $r > 0$ is a parameter
- Find the solution to the ARE, where
$$\boldsymbol{A} = 0, \quad \boldsymbol{B} = \begin{bmatrix} 2 & 2 \end{bmatrix}, \quad \boldsymbol{Q} = 1, \quad \boldsymbol{R} = r\boldsymbol{I}_2$$
- The ARE for this problem is
$$\boldsymbol{O} = \boldsymbol{A}^\top \boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} + \boldsymbol{Q} - \boldsymbol{P}\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^\top \boldsymbol{P} = 1 - \frac{8}{r}p^2,$$
whose solution is
$$p = \sqrt{\frac{r}{8}}$$

Example—contd.

- Write the equation of the closed-loop system driven by the optimal controller,

$$\boldsymbol{u} = -\boldsymbol{R}^{-1}\boldsymbol{B}^\top \boldsymbol{P}\boldsymbol{x} = -\frac{1}{\sqrt{2r}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} x$$

- The closed-loop optimal system is described by

$$\dot{x} = \begin{bmatrix} 2 & 2 \end{bmatrix} \boldsymbol{u} = -\frac{4}{\sqrt{2r}} x$$

- Find the value of J for the optimal closed-loop system
- We have

$$J = \boldsymbol{x}(0)^\top \boldsymbol{P}\boldsymbol{x}(0) = \frac{9}{2}\sqrt{\frac{r}{2}}$$

Example—verification

- Verify that $J = \frac{9}{2}\sqrt{\frac{r}{2}}$ as follows:

$$
\begin{aligned}
\min J &= \min \int_0^\infty \left(x^2 + ru_1^2 + ru_2^2\right) dt \\
&= \int_0^\infty 9 \exp\left(-\frac{8t}{\sqrt{2r}}\right)(1+1)\, dt = 9\frac{\sqrt{2r}}{4} = \frac{9}{2}\sqrt{\frac{r}{2}}
\end{aligned}
$$