MA 527

Lecture Notes (section 7.5 & 7.6)

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7.5 Existence, Uniqueness.

(Ex)

(1) $\begin{cases} x+2y=1 \\ 2x+4y=2 \end{cases}$: infinitely many sol.

(2) $\begin{cases} x+2y=1 \\ 2x+4y=5 \end{cases}$: No sol.

(1) $\left[\begin{array}{cc:c} 1 & 2 & 1 \\ 2 & 4 & 2 \end{array}\right] \xrightarrow{r_2-2r_1} \left[\begin{array}{cc:c} 1 & 2 & 1 \\ 0 & 0 & 0 \end{array}\right] \leftarrow 0=0.$

(2) $\left[\begin{array}{cc:c} 1 & 2 & 1 \\ 2 & 4 & 5 \end{array}\right] \longrightarrow \left[\begin{array}{cc:c} 1 & 2 & 1 \\ 0 & 0 & 3 \end{array}\right] \leftarrow 0=3.$

7.5. Existence & Uniqueness.

Def If $AX=b$ has a solution, then $AX=b$ is called consistent.

(1) is consistent

(2) is not consistent.

Thm 1 A : an $m\times n$ matrix.

(1) $AX=b$ is consistent iff $\operatorname{rank} A = \operatorname{rank}[A \vdots b]$

Thm 1

(2) $AX=b$ has only one solution iff $\operatorname{rank}A = \operatorname{rank}[A \vdots b] = n$

(3) If $\operatorname{rank}A = \operatorname{rank}[A \vdots b] \lneq n$, then $AX=b$ has infinitely many solutions.

Question : A is a 3×4 matrix.

How many solutions does $AX=0$ have?

$\underbrace{\operatorname{rank}A}_{\leq 3} + \dim \operatorname{Null}(A) = 4$: $\dim \operatorname{Null}(A) \geq 1$.

: Infinitely many solutions.

$$\left[\begin{array}{cc:c} 1 & 2 & 1 \\ 2 & 5 & 2 \end{array}\right] \xrightarrow{r_2 - 2r_1} \left[\begin{array}{cc:c} 1 & 2 & 1 \\ 0 & 1 & 0 \end{array}\right]$$

2×2

7.6 Determinants. : $\det A$

Q: Any other ways to find consist system?

A $\det A$.

A : an $n\times n$ matrix.

(1) $n=2$: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$|A| = \det A = ad - bc$: the determinant of A.

(Ex) 1. $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$: $\det A = 4 - 2^2 = 0$

: singular.

2. $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$: $\det B = 4 - 6 = -2$.

: nonsingular.

(Cramer's rule)

$$\begin{cases} ax + by = e_1 & -① \\ cx + dy = e_2 & -② \end{cases}$$

$d \cdot ① - b②$:

$$\begin{array}{rl} adx + bdy &= e_1 d \\ -)\ bcx + bdy &= e_2 b. \\ \hline (ad-bc)x &= e_1 d - e_2 b. \end{array}$$

$$x = \frac{e_1 d - e_2 b}{ad - bc} = \frac{\det \begin{bmatrix} e_1 & b \\ e_2 & d \end{bmatrix}}{\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}}$$

$$y = \frac{\det \begin{bmatrix} a & e_1 \\ c & e_2 \end{bmatrix}}{\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}}$$: Cramer's rule.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

General case: $AX = b$: $A = [a_1\ a_2\ \cdots\ a_n]$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \cdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix},\ X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},\ b = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

$$x_1 = \frac{\det\left[b\ a_2 \cdots a_n\right]}{\det A}, \quad \cdots$$

$$x_i = \frac{\det\left[a_1 \cdots, \overset{i\text{th}}{b}, a_{i+1} \cdots a_n\right]}{\det A}, \quad i = 1, 2, \cdots, n.$$

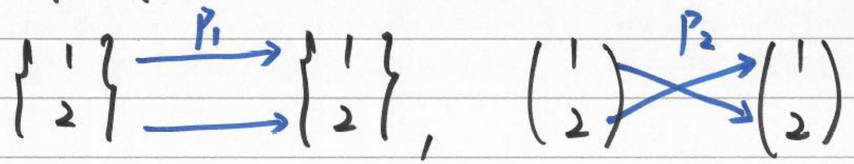
(2) $n = 3$: $A_{3\times 3} \to \det A = ?$

Def $A = [a_{ij}]_{n\times n}$

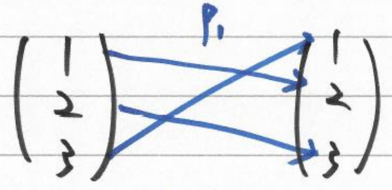
$$\det A = \sum_{p} \operatorname{sign}(p)\, a_{1p(1)} a_{2p(2)} \cdots a_{np(n)}$$

Remark: $p : \{1, 2, \cdots, n\} \to \{1, 2, \cdots, n\}$.

$n = 2$:



$n = 3$:



: even

$n = 3$: $A = \begin{bmatrix} 0 & 3 & 5 \\ 3 & 5 & 0 \\ 5 & 0 & 10 \end{bmatrix}$

$$\det A = \begin{vmatrix} 0 & 3 & 5 & 0 & 3 \\ 3 & 5 & 0 & 3 & 5 \\ 5 & 0 & 10 & 5 & 0 \end{vmatrix} = (0\cdot 5\cdot 10 + 3\cdot 0\cdot 5 + 0) - (\quad)$$

$$\det A = 0 - (\underset{125}{5^3} + 0 + \underset{90}{10 \cdot 9}) = \underline{-215}$$

Q: If we use 3 row operations for a matrix A, $\det A$ is same or different?

A, B : row-equivalent.

$$\det B = \det A \ ?$$

(Ex)

$$B = \begin{bmatrix} 3 & 5 & 0 \\ 0 & 3 & 5 \\ 5 & 0 & 10 \end{bmatrix} :$$

$$\det B = \begin{vmatrix} 3 & 5 & 0 & 3 & 5 \\ 0 & 3 & 5 & 0 & 3 \\ 5 & 0 & 10 & 5 & 0 \end{vmatrix}$$

$$= (90 + 125 + 0) - (0) = 215$$

(row op 1) $A_i \leftrightarrow A_j \longrightarrow B$

$$\underline{\det B = (-1) \det A.}$$

(row op 2)

$$C = \begin{bmatrix} 0 & 3 & 5 \\ 3 & 5 & 0 \\ 10 & 0 & 20 \end{bmatrix}$$

$$|C| = \begin{vmatrix} 0 & 3 & 5 & 0 & 3 \\ 3 & 5 & 0 & 3 & 5 \\ 10 & 0 & 20 & 10 & 0 \end{vmatrix} = 0 - (250 + 0 + 180)$$

$9 \cdot 20$

$$= -430. = 2 \det A.$$

(row op2)

$A = \begin{bmatrix} A_1 \\ \vdots \\ A_n \end{bmatrix}$: $C = \begin{bmatrix} kA_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix}$

$\det C = k \det A.$

$D = \begin{bmatrix} kA_1 \\ \vdots \\ kA_n \end{bmatrix}$: $\det D = k^n \det A.$

Remark: $\det A$.

$\#(+/\cdot) = n \cdot \#(\text{permutations}) = n \cdot (n!)$

ex $20! = 2.4 \times 10^{18}$

1. Cramer rule : bad for computation

GE : $\#(+/\cdot) \cong \frac{n^3}{3} \cdots$

2. #4 (p300)

$n = 10, 15, 20, 25$: flops

$\text{Time} = \frac{\#(+/\cdot)}{\text{flops}} \cong \frac{n \cdot n!}{\text{flops}}$