AAE 532 - Orbit Mechanics

Purdue University

PS3 Solutions

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Useful Constants

	Axial Rotaional Period (Rev/Day)	Mean Equatorial Radius (km)	Gravitational Parameter $\mu = Gm(km^3/sec^2)$	Semi-major Axis of orbit (km)	Orbital Period (sec)	Eccentricity of Orbit	Inclination of Orbit to Ecliptic (deg)
☉ Sun	0.0394011	695990	132712440017.99	-	-	-	-
☾ Moon	0.0366004	1738.2	4902.8005821478	384400 (around Earth)	2360592 27.32 Earth Days	0.0554	5.16
☾ Mercury	0.0170514	2439.7	22032.080486418	57909101	7600537 87.97 Earth Days	0.20563661	7.00497902
♀ Venus	0.0041149 (Retrograde)	6051.9	324858.59882646	108207284	19413722 224.70 Earth Days	0.00676399	3.39465605
⊕ Earth	1.0027378	6378.1363	398600.4415	149597898	31558205 365.26 Earth Days	0.01673163	0.00001531
♂ Mars	0.9747000	3397	42828.314258067	227944135	59356281 686.99 Earth Days	0.09336511	1.84969142
♃ Jupiter	2.4181573	71492	126712767.8578	778279959	374479305 11.87 Years	0.04853590	1.30439695
♄ Saturn	2.2522053	60268	37940626.061137	1427387908	930115906 29.47 Years	0.05550825	2.48599187
⛢ Uranus	1.3921114 (Retrograde)	25559	5794549.0070719	2870480873	2652503938 84.05 Years	0.04685740	0.77263783
Ψ Neptune	1.4897579	25269	6836534.0638793	4498337290	5203578080 164.89 Years	0.00895439	1.77004347
♇ Pluto	-0.1565620 (Retrograde)	1162	981.600887707	5907150229	7830528509 248.13 Years	0.24885238	17.14001206

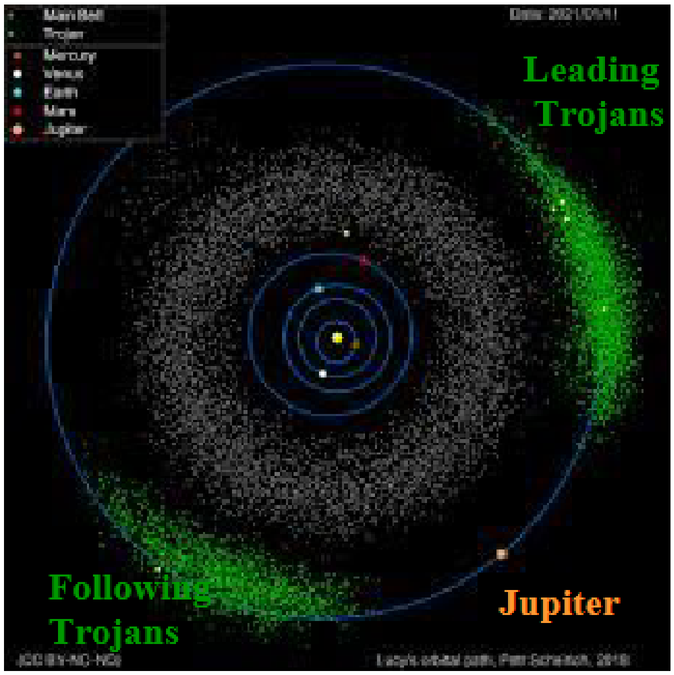
- First three columns of the body data are consistent with GMAT 2020a default values, which are mainly from JPL's ephemerides file de405.spk

- The rest of the data are from JPL website(`https://ssd.jpl.nasa.gov/?planet_pos`, retrieved at 09/01/2020)
based on E.M. Standish's "Keplerian Elements for Approximate Positions of the Major Planets"

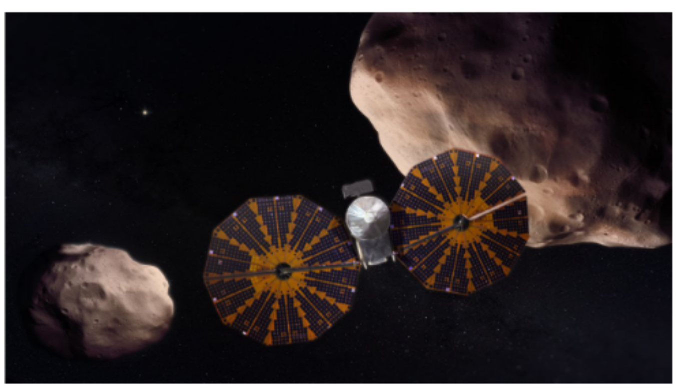
Problem 1

Problem Statement

Lucy is the first space mission to study the Trojan asteroids. The mission takes its name from the fossilized human ancestor (called "Lucy" by her discoverers) whose skeleton offered unique insight into humanity's evolution. The Trojan asteroids may, in fact, be left over material from planetary formation. They are comprised of two groups of asteroids that generally move relative to the Sun in Jupiter's orbit...just ahead or just behind Jupiter. Lucy is the first space mission that is focused on examining some of the Trojan asteroids and can offer some evolutionary insight!



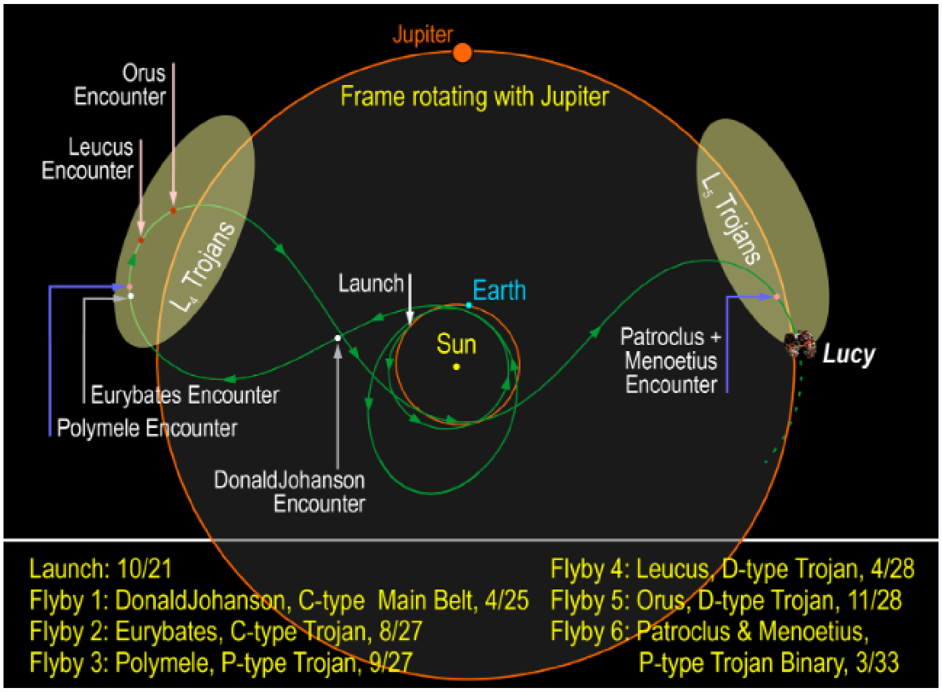
Lucy was launched from Florida on October 16, 2021 with a launch mass of 1550 kg. With multiple Earth gravity assists as part of the baseline mission, Lucy will complete a 12-year journey to a variety of different asteroids — including a Main Belt asteroid and six Trojans. It will visit both Trojan 'camps' and deliver the first close-up view of all three major types of bodies in the swarms.



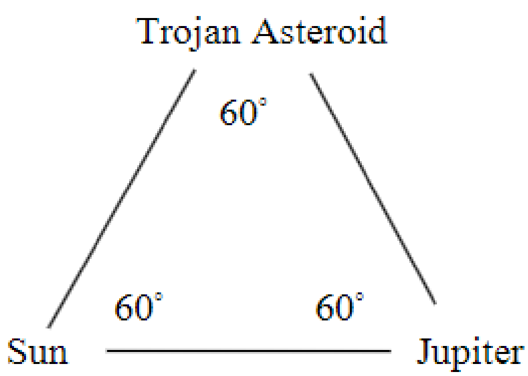
The Lucy spacecraft is over 13 meters (over 45 feet) from tip to tip, but most of that is the solar panels (each about 6 meters (20 feet) in diameter) to power the spacecraft to reach the orbit of Jupiter. All of the instruments, as well as 2-meter (6.5 ft)-high gain antenna, is located on the smaller spacecraft body.

The spacecraft path is viewed (next page) in a Sun-Jupiter frame. After launch in 2021, Lucy required a little over 2 years, or about 746 days, before it reached its first asteroid as it passed the tiny main belt asteroid Dinkinesh on

November 1, 2023. "Dinkinesh" (discovered in 1999) is the Ethiopian name for the human-ancestor fossil, i.e., Lucy, which was found in that country. (The Lucy spacecraft discovered that Dinkinesh has a moon of its own!) The next target, the main belt asteroid DonaldJohanson, named for one of the co-discoverers of the Lucy fossil, will occur April 20, 2025. Then, it requires over 3000 days before Lucy reaches its first Trojan asteroid, Eurybates. Lucy will continue its tour of the Trojan Asteroids, encountering 3 more Trojans, before it flies by the final pair, Patroclus and Menoetius in 2033.



Consider the time when Lucy reaches the asteroid 21900 Orus in November 2028. Orus is about 60 km in diameter; for the mass of the asteroid, assume $\mu = 10 km^3/sec^2$. For a preliminary assessment, the positions of a sample Trojan asteroid (e.g., Orus), the Sun, and Jupiter can be modeled as located at the vertices of an equilateral triangle as envisioned below:



Assume that the distance between the Sun and Jupiter is equal to the semi-major axis of Jupiter's orbit. Consider the total acceleration on the asteroid. (The spacecraft will actually be influenced by the same forces at arrival near Orus as the asteroid but, for now, just focus on the asteroid.)

(a) Write the expression for the acceleration of the asteroid relative to the Sun where Jupiter is a perturbing body. [Note that the definition of a set of unit vectors is always required.] Write this expression in the form $\ddot{\bar{r}}_{\odot \to asteroid}$ =(sum of terms). Label each of the following terms in this expression: dominant term, direct perturbing terms, indirect perturbing terms. Determine the magnitude and direction of each of the terms in the expression, as well as the net perturbing accelerations and the total net acceleration.

(b) Re-formulate the problem and write the expression for the acceleration of the asteroid relative to Jupiter. Again, determine the magnitude and direction of each of the terms in the expression, as well as the net perturbing accelerations for each body and the total net acceleration.

(c) Which individual term is the largest in each formulation? How does the magnitude of the dominant term in each formulation compare?
Determine the net perturbing acceleration in each case. Which has the largest impact in each formulation, Sun or Jupiter?
Is the net total acceleration on the asteroid the same in each formulation? Should it be the same? Why or why not?
Which formulation is correct? Why?
From the results here, is it reasonable to model the motion of the asteroid as a two-body problem, i.e., Sun-asteroid or Jupiter-asteroid? Why or why not?

Part (a)

First, let's clearly state our assumptions for this problem:

(1) It is a three-body system. We have Sun, Jupiter and the asteroid. We ignore all other planetary bodies within the solar system.

(2) All bodies are approximated as point masses.

(3) Three bodies form an equilateral triangle at the moment, length being the semi-major axis of Jupiter.

Next, we can define the unit vectors. In Figure 1, note that S, J, O represent Sun, Jupiter and Orus respectively. Although we can define the inertial directions arbitrarily, the most straight-forward way to define from the given configuration is assigning the first unit vector, $\hat{\imath}$, parallel to the vector from the Sun to Jupiter with the same direction. And the third unit vector $\hat{k}$ is assumed to be coming out of the page. Also note that these unit vectors do not require a specific origin as they represent the inertial directions only. For example, we could attach these unit vectors to the center of mass, but also to one of the bodies.

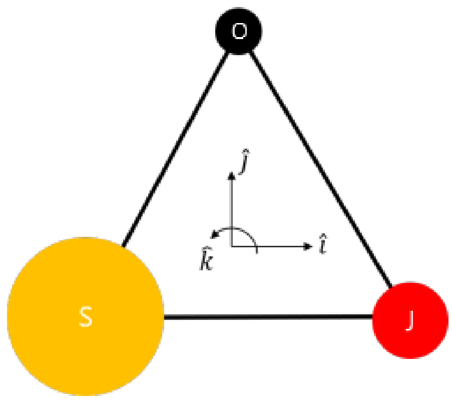


Figure 1: Inertial unit vectors within the Sun-Jupiter-Orus system.

Now, one can move on to the relative acceleration of the asteroid with respect to the Sun:

$$\ddot{\bar{r}}_{\odot\to asteroid} = -\frac{\mu_{\odot} + \mu_{asteroid}}{r^3_{\odot\to asteroid}}\bar{r}_{\odot\to asteroid} + \left(\frac{\mu_{♃}}{r^3_{asteroid\to ♃}}\bar{r}_{asteroid\to ♃} - \frac{\mu_{♃}}{r^3_{\odot\to ♃}}\bar{r}_{\odot\to ♃}\right) \tag{1}$$

where the orange term represents the dominant acceleration term due to the central body (the Sun), the red term is the direct perturbing acceleration of Jupiter acting on the asteroid, and the blue term is the indirect perturbing acceleration of Jupiter acting on the Sun. Note that the indirect term itself does not have a negative sign, but when we compute the net perturbing force from Jupiter, we subtract the indirect term from the direct term, which is why we have the negative sign. Since we have the equation, we can compute values and vectors needed. First from the table of constants:

$$\mu_{\odot} = 132712440017.99 \text{ km}^3/\text{sec}^2 = 1.3271 \cdot 10^{11} \text{ km}^3/\text{sec}^2$$
$$\mu_{mathJupiter} = 126712767.8578 \text{ km}^3/\text{sec}^2 = 1.2671 \cdot 10^{8} \text{ km}^3/\text{sec}^2$$
$$\mu_{asteroid} = 10 \text{ km}^3/\text{sec}^2$$
$$a_{♃} = 778279959 \text{ km} = 7.7828 \cdot 10^{8} \text{ km}$$

Then, recalling the orientation of the system in figure 1, the relative position vectors are:

$$\bar{r}_{\odot\to asteroid} = a_{♃}\cos(60°)\hat{\imath} + a_{♃}\sin(60°)\hat{\jmath} = 3.8914 \cdot 10^{8} \text{ km } \hat{\imath} + 6.7401 \cdot 10^{8} \text{ km } \hat{\jmath}$$
$$\bar{r}_{asteroid\to ♃} = a_{♃}\cos(60°)\hat{\imath} - a_{♃}\sin(60°)\hat{\jmath} = 3.8914 \cdot 10^{8} \text{ km } \hat{\imath} - 6.7401 \cdot 10^{8} \text{ km } \hat{\jmath}$$
$$\bar{r}_{\odot\to ♃} = 7.7828 \cdot 10^{8} \text{ km } \hat{\imath}$$
$$r_{\odot\to asteroid} = r_{asteroid\to ♃} = r_{\odot\to ♃} = a_{♃} = 7.7828 \cdot 10^{8} \text{ km}$$

So, when plugging in these vectors into equation 1 above, the acceleration terms become:

$$\boxed{\begin{aligned}
-\frac{\mu_{\odot}+\mu_{asteroid}}{r^3_{\odot\rightarrow asteroid}}\bar{r}_{\odot\rightarrow asteroid} &= -1.0955\cdot10^{-7}\hat{i}-1.8975\cdot10^{-7}\hat{j}\ [km/s^2]\ \text{ dominant acceleration}\\
\frac{\mu_{\jupiter}}{r^3_{asteroid\rightarrow\jupiter}}\bar{r}_{asteroid\rightarrow\jupiter} &= 1.0460\cdot10^{-10}\hat{i}-1.8117\cdot10^{-10}\hat{j}\ [km/s^2]\ \text{ Jupiter direct perturbing}\\
\frac{\mu_{\jupiter}}{r^3_{\odot\rightarrow\jupiter}}\bar{r}_{\odot\rightarrow\jupiter} &= 2.0919\cdot10^{-10}\hat{i}\ [km/s^2]\ \text{ Jupiter indirect perturbing}\\
\frac{\mu_{\jupiter}}{r^3_{asteroid\rightarrow\jupiter}}\bar{r}_{asteroid\rightarrow\jupiter}-\frac{\mu_{\jupiter}}{r^3_{\odot\rightarrow\jupiter}}\bar{r}_{\odot\rightarrow\jupiter} &= -1.0460\cdot10^{-10}\hat{i}-1.8117\cdot10^{-10}\hat{j}\ [km/s^2]\ \text{ Jupiter net perturbing}\\
\ddot{\bar{r}}_{\odot\rightarrow asteroid} &= -1.0965\cdot10^{-7}\hat{i}-1.8993\cdot10^{-7}\hat{j}\ [km/s^2]\ \text{ total acceleration}
\end{aligned}}$$

A good way to check that our answer is correct, is to see if the directions of the accelerations are consistent with our intuition. For example, the dominant term is the Sun pulling the asteroid towards the Sun. So it should have negative $\hat{i}$ direction and also negative $\hat{j}$ direction. Indeed it does, which aligns with our intuition. We can perform a similar analysis for the perturbing accelerations as well.

Part (b)

Now, let's re-formulate the problem with respect to Jupiter and consider the Sun as the perturbing body. One can rewrite the relative acceleration equation of the asteroid with respect to Jupiter as:

$$\ddot{\bar{r}}_{♃\to asteroid} = -\frac{\mu_{♃} + \mu_{asteroid}}{r^3_{♃\to asteroid}}\bar{r}_{♃\to asteroid} + \left(\frac{\mu_{\odot}}{r^3_{asteroid\to\odot}}\bar{r}_{asteroid\to\odot} - \frac{\mu_{\odot}}{r^3_{♃\to\odot}}\bar{r}_{♃\to\odot}\right) \tag{2}$$

Then, recalling the orientation of the system in figure 1, the relative position vectors are:

$$\begin{aligned}
\bar{r}_{♃\to asteroid} &= -a_{♃}\cos(60°)\hat{i} + a_{♃}\sin(60°)\hat{j} = -3.8914 \cdot 10^8 \text{ km } \hat{i} + 6.7401 \cdot 10^8 \text{ km } \hat{j} \\
\bar{r}_{asteroid\to\odot} &= -a_{♃}\cos(60°)\hat{i} - a_{♃}\sin(60°)\hat{j} = -3.8914 \cdot 10^8 \text{ km } \hat{i} - 6.7401 \cdot 10^8 \text{ km } \hat{j} \\
\bar{r}_{♃\to\odot} &= -7.7828 \cdot 10^8 \text{ km } \hat{i} \\
r_{♃\to asteroid} &= r_{asteroid\to\odot} = r_{♃\to\odot} = a_{♃} = 7.7828 \cdot 10^8 \text{ km}
\end{aligned}$$

So, when plugging in these vectors into equation 2 above, the acceleration terms become:

$$\boxed{\begin{aligned}
-\frac{\mu_{♃} + \mu_{asteroid}}{r^3_{♃\to asteroid}}\bar{r}_{♃\to asteroid} &= 1.0460 \cdot 10^{-10}\hat{i} - 1.8117 \cdot 10^{-10}\hat{j}\ [km/s^2] \text{ dominant acceleration} \\
\frac{\mu_{\odot}}{r^3_{asteroid\to\odot}}\bar{r}_{asteroid\to\odot} &= -1.0955 \cdot 10^{-7}\hat{i} - 1.8975 \cdot 10^{-7}\hat{j}\ [km/s^2] \text{ Sun direct perturbing} \\
\frac{\mu_{\odot}}{r^3_{♃\to\odot}}\bar{r}_{♃\to\odot} &= -2.1910 \cdot 10^{-7}\hat{i}\ [km/s^2] \text{ Sun indirect perturbing} \\
\frac{\mu_{\odot}}{r^3_{asteroid\to\odot}}\bar{r}_{asteroid\to\odot} - \frac{\mu_{\odot}}{r^3_{♃\to\odot}}\bar{r}_{♃\to\odot} &= 1.0955 \cdot 10^{-7}\hat{i} - 1.8975 \cdot 10^{-7}\hat{j}\ [km/s^2] \text{ Sun net perturbing} \\
\ddot{\bar{r}}_{♃\to asteroid} &= 1.0965 \cdot 10^{-7}\hat{i} - 1.8993 \cdot 10^{-7}\hat{j}\ [km/s^2] \text{ total acceleration}
\end{aligned}}$$

Part (c)

Formulation		$\ddot{\vec{r}}_{\odot \to asteroid}$	$\ddot{\vec{r}}_{\text{♃} \to asteroid}$
Dominant	Vector	$-1.0955 \cdot 10^{-7}\hat{i} - 1.8975 \cdot 10^{-7}\hat{j}$	$1.0460 \cdot 10^{-10}\hat{i} - 1.8117 \cdot 10^{-10}\hat{j}$
	Magnitude	$2.1910 \cdot 10^{-7}$	$2.0919 \cdot 10^{-10}$
Net Perturbing	Vector	$-1.0460 \cdot 10^{-10}\hat{i} - 1.8117 \cdot 10^{-10}\hat{j}$	$1.0955 \cdot 10^{-7}\hat{i} - 1.8975 \cdot 10^{-7}\hat{j}$
	Magnitude	$2.0919 \cdot 10^{-10}$	$2.1910 \cdot 10^{-7}$
Net Total	Vector	$-1.0965 \cdot 10^{-7}\hat{i} - 1.8993 \cdot 10^{-7}\hat{j}$	$1.0965 \cdot 10^{-7}\hat{i} - 1.8993 \cdot 10^{-7}\hat{j}$
	Magnitude	$2.1931 \cdot 10^{-7}$	$2.1931 \cdot 10^{-7}$

Table 1: : Summary of accelerations from two formulations (Unit: km/s^2)

Discussions regarding Table 1:

- Dominant term is the largest magnitude in the first formulation (w.r.t. the Sun), and the perturbing forces are the largest magnitudes in the second formulation. Note that the direct/indirect perturbing forces have the same magnitude since they are in the equilateral configuration as depicted in figure 1.
- The dominant term in the first formulation is on the order of 1e-7, whereas in the second formulation it is on the order of 1e-10. We have about 1000 times larger dominant term in the first formulation.
- In both formulations, the Sun has larger impact on the asteroid, whether is a dominant acceleration (first formulation) or is a perturbing acceleration (second formulation).
- The net acceleration vectors are different in each formulation. But we notice that the magnitude is the same, which is a direct result from the equilateral configuration. They should not be the same in general. Depending on the reference body that we are looking at, the acceleration of the asteroid with respect to the body is going to be different.
- Both formulations are mathematically correct! We gain different information from each formulation since each model represents two different relative accelerations with two different central bodies.
- Representing the motion of the asteroid as a two-body problem **is not** reasonable. Using the first formulation, one will notice that the net perturbing acceleration of Jupiter is only 3 orders of magnitude smaller than that of the dominant term of the Sun. This may not seem a lot, but it can have a drastic impact, particularly over larger time frames observed within a planetary system environment. Thus, a more accurate system to use to model the motion of the asteroid would be the three-body problem.

Problem 2

Problem Statement

An Introductory Manual for the General Mission Analysis Tool (GMAT) software is posted under GMAT on Brightspace. GMAT is open source and is easily downloaded. To obtain some practice using GMAT, step through the manual carefully. Complete all the steps and view the final orbits.

(a) Now use an Epoch of 13 Sept 2024. Produce a satellite orbit with a 'semi-major axis' of 60,000 km, 'eccentricity' of 0.7, and an 'inclination' of 60 deg. Note that you are now using an Earth point mass model (In the Resource tree, for the 'LowEarthProp' propagator, replace Gravity Model with 'JGM-2' but set degree and order to zero to render a point mass model. Under 'DefaultOtbitView' it will be a cleaner image if you do NOT enable the constellations).

Plot images from GMAT. Also print the summary of the orbit details from the "Report" option. Under the Resources tree, right click on 'Output'. You are offered the opportunity to add a report file with numerical data from the simulation. The report will appear in the Output tree. From the output data, determine:

- radius at closest approach or Rad. Peri.
- radius at farthest excursion or Rad. Apo.
- energy
- semi-major axis
- semi-latus rectum
- angular momentum
- the Cartesian components of position and velocity at the initial time

With what reference frame are these associated? What are the units associated with each quantity?

(b) Given the orbit in the scenario in part (a), set the inclination to zero. Also define Ascending Node (RAAN) and Argument Of Periapsis (AOP) to be zero. Add a second spacecraft with an orbit of a different color. With the same eccentricity and zero inclination, try a different semi-major axis, i.e., 40,000 km. Add a third satellite with a = 75,000 km. [Thus, you will have three satellites with e = 0.7 and a = 40,000 km, 60,000 km and 75,000 km.]

Repeat the exercise for $a = 60,000$ km and three eccentricities, i.e., $e = 0.2$, 0.65, 0.88. In each case, hold the inclination fixed at 0 deg.

Repeat the exercise for $e = 0.65$ and three semi-major axes, i.e., $a = 20,000$ km, $35,000$ km, $75,000$ km.

You should have three plots from GMAT; use a view that is looking down on the orbit plane for variations in semi-major axis and eccentricity

Part (a)

First, let's summarize the values that we are using for the initial state. Note that we are using the values from the tutorial unless stated otherwise.

Parameter	Value
Epoch	13 Sep 2024 00:00:00
Semi-major axis	60000 km
Eccentricity	0.7
Inclination	45°
RAAN	292.8362°
AOP	218.9805°
TA	180°

Since the propagation starts with a true anomaly of 180 deg, we know that the simulation begins at apoapsis. Propagating to the stopping condition of periapsis therefore yields one half of an orbit:

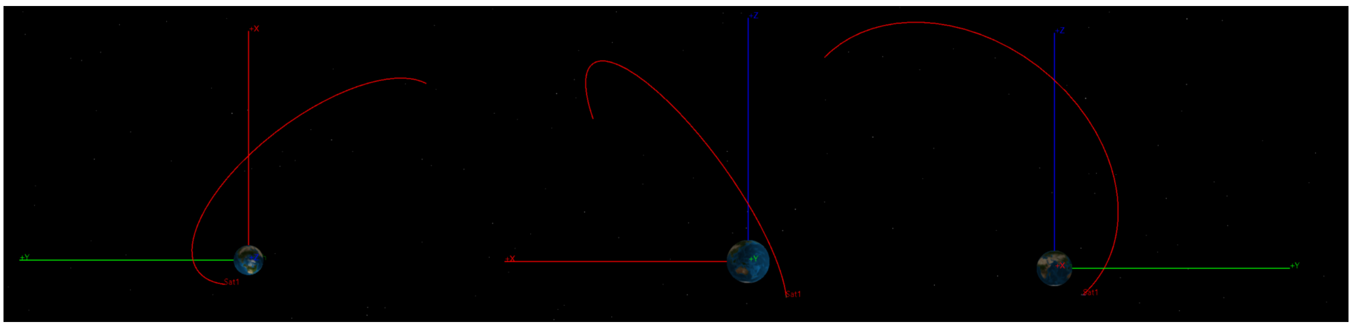


Figure 2: Orbit in EarthJ2000Eq inertial view projected into the xy, xz, and yz planes.

Note that a minimum of half an orbit propagation is required for us to be able to accurately assess the geometry of the orbit. By beginning at either apoapsis or periapsis the remainder of the orbit is simple a reflection about the plane of symmetry. We can generate a report file to include all of the requested parameters like the one pictured on the following page. As the simulation progresses, the values for many of the parameters start to change slightly. This is due to a combination of numerical error and perturbation effects. Since we set up our propagator to include solar radiation pressure and the gravitational effects of the Sun and moon (Luna), the orbital elements of the orbit start to shift over time. We could adjust the propagator such that we model Earth as a point mass with no external perturbations but we would still see a slight variation in these values due to numerical errors, though they would remain significantly closer to the values shown at the initial time. At the start of the simulation:

Parameter	Value
Rad.Peri. (km)	18000
Rad.Apo. (km)	102000
Energy (km^2/s^2)	-3.3217
SMA (km)	60000
Semi-latus Rectum (km)	30600
Angular Momentum (km^2/s)	110440.8145
Initial Position Components (km)	60339.78 $\hat{x}$ - 60624.87 $\hat{y}$ + 55567.39 $\hat{z}$
Initial Velocity Components (km/s)	0.123520 $\hat{x}$ + 0.791054 $\hat{y}$ + 0.728924 $\hat{z}$

Where all values are expressed in the J200 reference frame which is inertially fixed with the xy plane passing through Earth's equator and the z axis pointing out the north pole.

Sat.UTCGregorian	Sat.Earth.RadPer	Sat.Earth.RadApo	Sat.Earth.Energy	Sat.Earth.SMA	Sat.Earth.SemilatusRectum	Sat.Earth.HMAG
13 Sep 2024 00:00:00.000	18000	102000	-3.32167034583334	60000	30599.99999999999	110440.8145112123
13 Sep 2024 00:01:00.000	17999.98273535538	102000.00001332	-3.321670823360179	59999.99137433771	30599.97505318435	110440.7694924672
13 Sep 2024 00:04:04.880	17999.92972580426	102000.000221992	-3.3216722849205 3	59999.96497389814	30599.89846371726	110440.6312798549
13 Sep 2024 00:13:54.200	17999.76266217812	102000.0025803724	-3.321676844069866	59999.88262127527	30599.65716228323	110440.1958284878
13 Sep 2024 00:41:44.166	17999.30520545363	102000.0233567433	-3.321688931729323	59999.66428109847	30598.99706705175	110439.0046151451
13 Sep 2024 01:26:44.166	17998.61641303143	102000.101564003	-3.321705833359694	59999.35898851721	30598.0052635577	110437.2147741576
13 Sep 2024 02:11:44.166	17997.99167404083	102000.2357820139	-3.321719411613592	59999.11372802735	30597.108528577	110435.5964719447
13 Sep 2024 02:56:44.166	17997.43188019766	102000.4269427151	-3.321729615927863	59998.92941145639	30596.3081951604	110434.1521218006
13 Sep 2024 03:41:44.166	17996.93736451843	102000.6759380148	-3.321736412332337	59998.80665126663	30595.60478702026	110432.8826757039
13 Sep 2024 04:26:44.166	17996.50793815264	102000.9836555234	-3.321739781442286	59998.74579683801	30594.99807332217	110431.7877230912
13 Sep 2024 05:11:44.166	17996.14292122277	102001.3510042057	-3.321739716895316	59998.74696271421	30594.4871143765	110430.8655732469
13 Sep 2024 05:56:44.166	17995.84116726443	102001.7789299839	-3.32173622424317	59998.81004862416	30594.0702976409	110430.1133202429
13 Sep 2024 06:41:44.166	17995.60108083329	102002.2684205465	-3.321729320331113	59998.93475068991	30593.74536337295	110429.5268892294
13 Sep 2024 07:26:44.166	17995.42062776856	102002.8204975823	-3.321719033228326	59999.12056267543	30593.50941911396	110429.1010625969
13 Sep 2024 08:11:44.166	17995.29733747552	102003.4361931992	-3.32170540281667	59999.36676533734	30593.35894193603	110428.8294840785
13 Sep 2024 08:56:44.166	17995.22829640613	102004.1165051328	-3.321688482209846	59999.67240076949	30593.28976702101	110428.7046382054
13 Sep 2024 09:41:44.166	17995.21013167149	102004.8623220537	-3.321668340272959	60000.03622686258	30593.29706064011	110428.7178016289
13 Sep 2024 10:26:44.166	17995.23898340892	102005.6743050677	-3.321645065665316	60000.45664423833	30593.37527491531	110428.8589615795
13 Sep 2024 11:11:44.166	17995.3104641458	102006.5527030207	-3.321618773074687	60000.93158358324	30593.51808081521	110429.11669506
13 Sep 2024 11:56:44.166	17995.41960297825	102007.4970649735	-3.321589612716898	60001.45833397587	30593.71827458266	110429.4780001032
13 Sep 2024 12:41:44.166	17995.56077199146	102008.5057885814	-3.321557784867262	60002.03328028645	30593.96765111918	110429.9280674076
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60577.89872244752	-58573.30477320629	57326.52791216758	0.0665947092173009 4	0.8471729063507432	0.6757455013634981
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60602.74824829986	-53683.94497987354	60652.48074015641	-0.0578629374185734 5	0.962456640225238	0.55449979175238 3
60361.02849656062	-51010.73664127383	62063.24825237971	-0.121358824620457	1.017413832929123	0.4900510369830365
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32326.42195115387	3639.578265983409	54036.38394331514	-1.431870454944648	1.547214341374986	-1.245906103310564
29014.0718127916 4	7057.691961339679	51046.03072157368	-1.550942513461898	1.526207638720419	-1.450055957313114
25444.96842335197	10389.82139897905	47588.14330869776	-1.680177180499196	1.4844225735168	-1.684380205086093
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12993.86485711966	19166.56550612493	33612.18924570858	-2.135085772581321	1.101000632350472	-2.667944767675204

Figure 3: Tabulated data output from GMAT

Part (b)

For the first case:

	SMA (km)	Eccentricity
Orbit 1	60000	0.7
Orbit 2	40000	0.7
Orbit 3	75000	0.7

Note that in addition to RAAN and AOP I have also set the true anomaly to 0 deg so that with a stopping condition of periapsis we are able to propagate a full orbit. Since Orbit 3 has the largest semi-major axis, it will also have the largest period. So we select this orbits perisapsis as the stopping condition to ensure that all 3 orbits are propagated for a minimum of one full period.

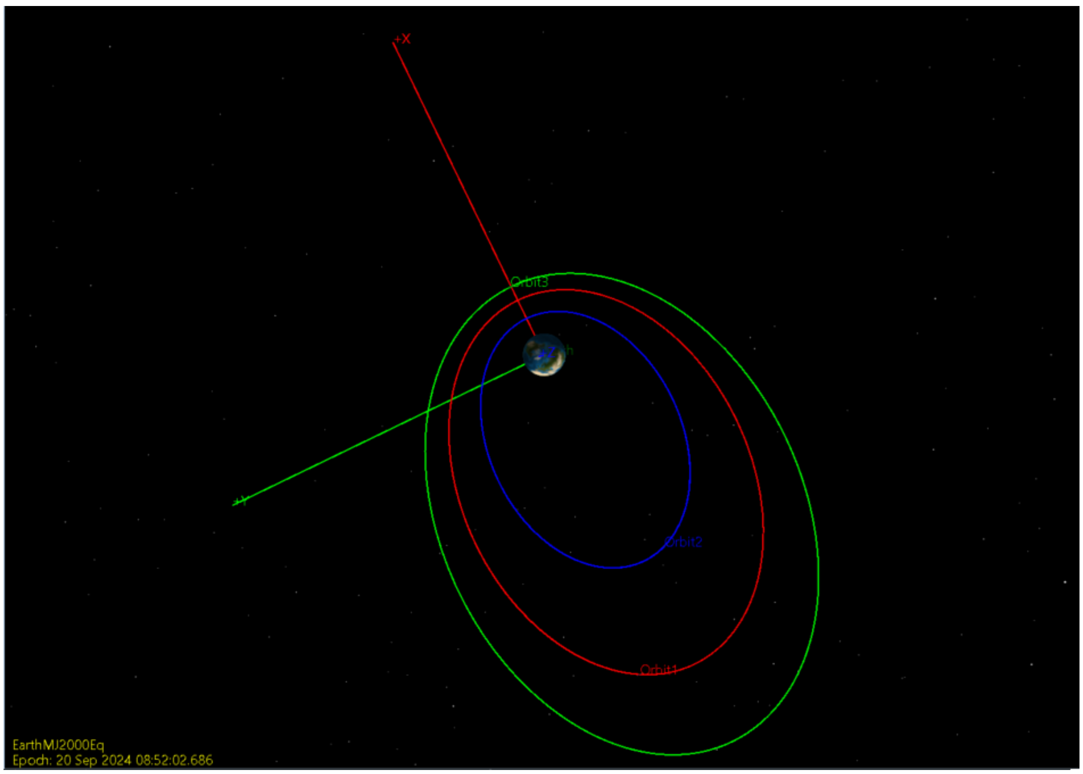

Figure 4: First case: three orbits in EarthJ2000Eq inertial view

For the second case:

	SMA (km)	Eccentricity
Orbit 1	60000	0.2
Orbit 2	60000	0.65
Orbit 3	60000	0.88

This time, all 3 orbits have the same semi-major axis; therefore, regardless of the different eccentricities they will all complete one orbit in the same amount of time. In the orbit viewer this is confirmed as all three spacecraft reach periapsis at the same time. You could also step through the propagation to see how each spacecraft progresses through its individual orbit to see how their orbital rates vary along the way.

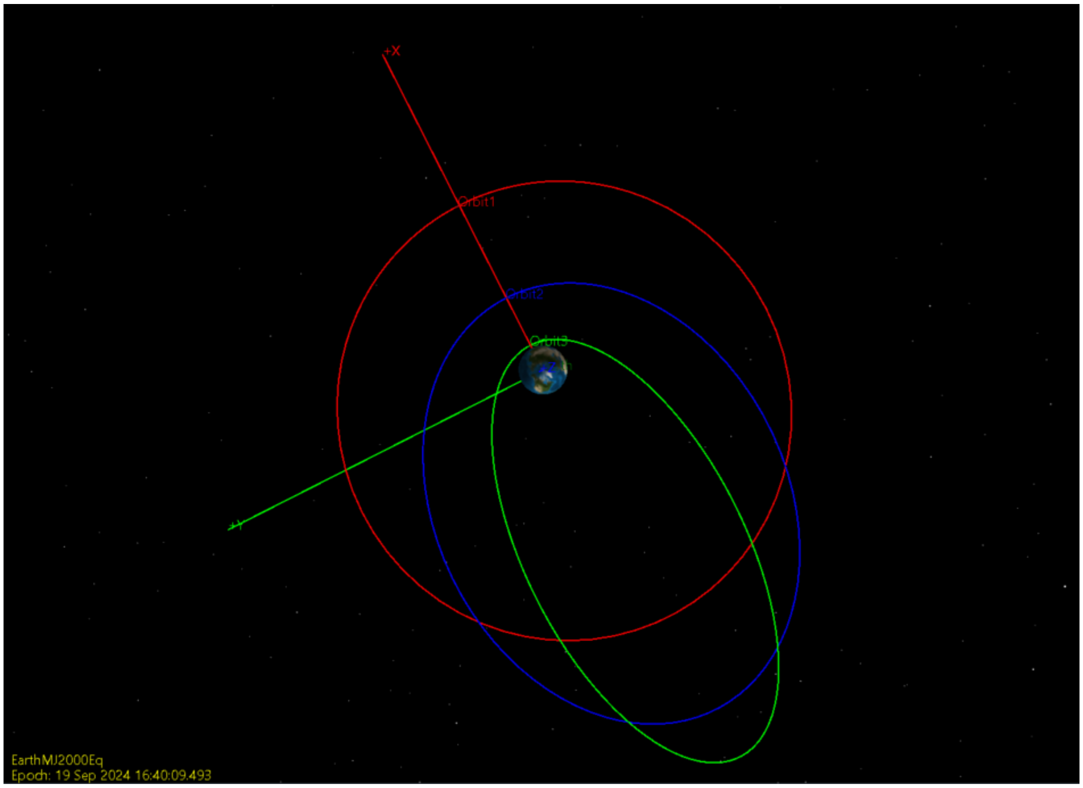

Figure 5: Second case: three orbits in EarthJ2000Eq inertial view

For the third case:

	SMA (km)	Eccentricity
Orbit 1	20000	0.65
Orbit 2	35000	0.65
Orbit 3	75000	0.65

This is similar to the first case since all three orbits have the same eccentricity but their semi-major axes vary significantly. Again, we choose the periapsis of Orbit 3 as the stopping condition. Notice that the spacecraft in the two smaller orbits complete multiple revolutions about the Earth in the time it takes the third spacecraft to complete one.

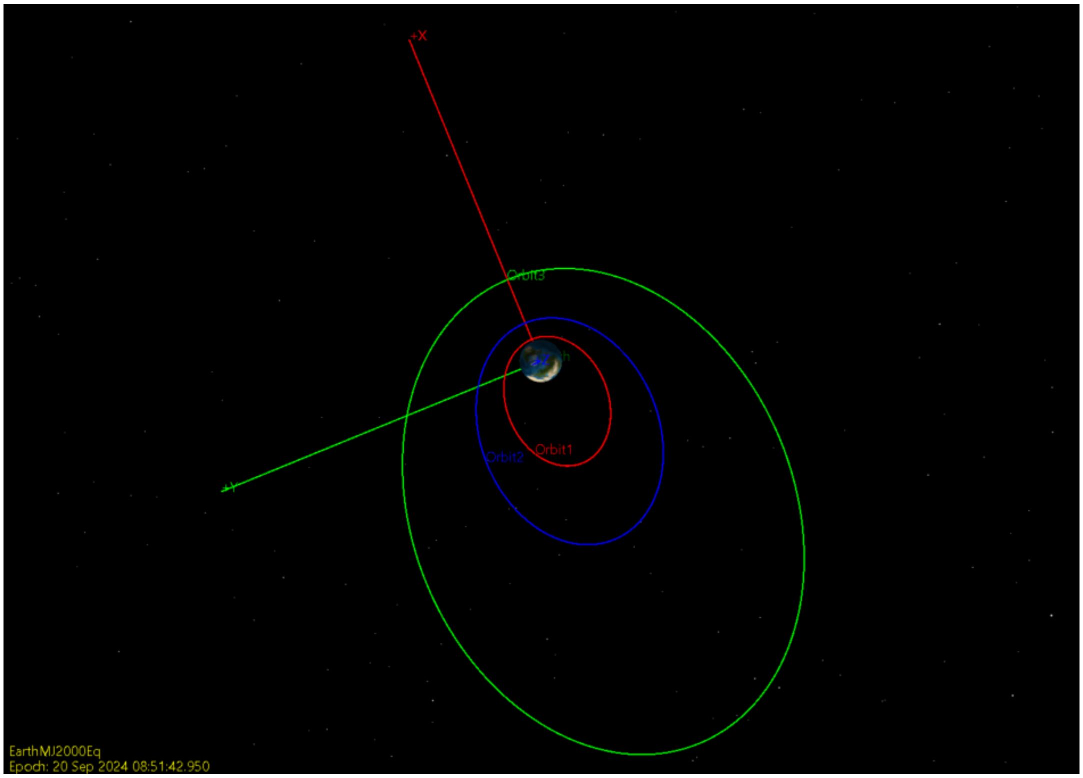

Figure 6: Third case: three orbits in EarthJ2000Eq inertial view

Problem 3

Problem Statement

Consider only the relative two-body problem (Earth and spacecraft). An Earth- orbiting vehicle is tracked from ground stations; the spacecraft mass is 300 kg. At a certain instant (t_0), the following position and velocity information is obtained relative to an inertial observer:

- Altitude = 8560 km
- Radial component of velocity = -2.11 km/s
- Transverse component of velocity = 4.89 km/s

Our tracking system is perfect so these states are completely without error!!

(a) ALWAYS sketch the scenario with unit vectors and vector definitions! Roughly identify the relative position and velocity vectors in the orbit at this instant.

(b) Compute the total system angular momentum $\bar{C}_3$, specific angular momentum, total kinetic energy for the system, total energy C_4, specific energy, areal velocity.

(c) What is the value and the units for the coefficient by which to multiply C_4 to produce specific energy?

(d) Within the context of the relative two-body problem, determine the following orbital characteristics: p, e, a, $\mathbb{P}$, γ, θ^*. Write the position and velocity vectors in terms of the inertial unit vectors $\hat{e}$ and $\hat{p}$.

(e) Compare this relative velocity to the circular relative velocity at this altitude.

Part (a)

The position of the spacecraft in rotating frame coordinates is

$$\bar{r} = (R_{\oplus} + 8560)\hat{r}\ km = 14938.1363\hat{r}\ km$$

and the velocity is

$$\bar{v} = -2.11\hat{r} + 4.89\hat{\theta}\ km/s$$

Since the radial velocity is negative, the spacecraft is descending, i.e., approaching periapsis. Additionally, the spacecraft is in an elliptical orbit, since the radial and transverse velocities have different magnitudes.

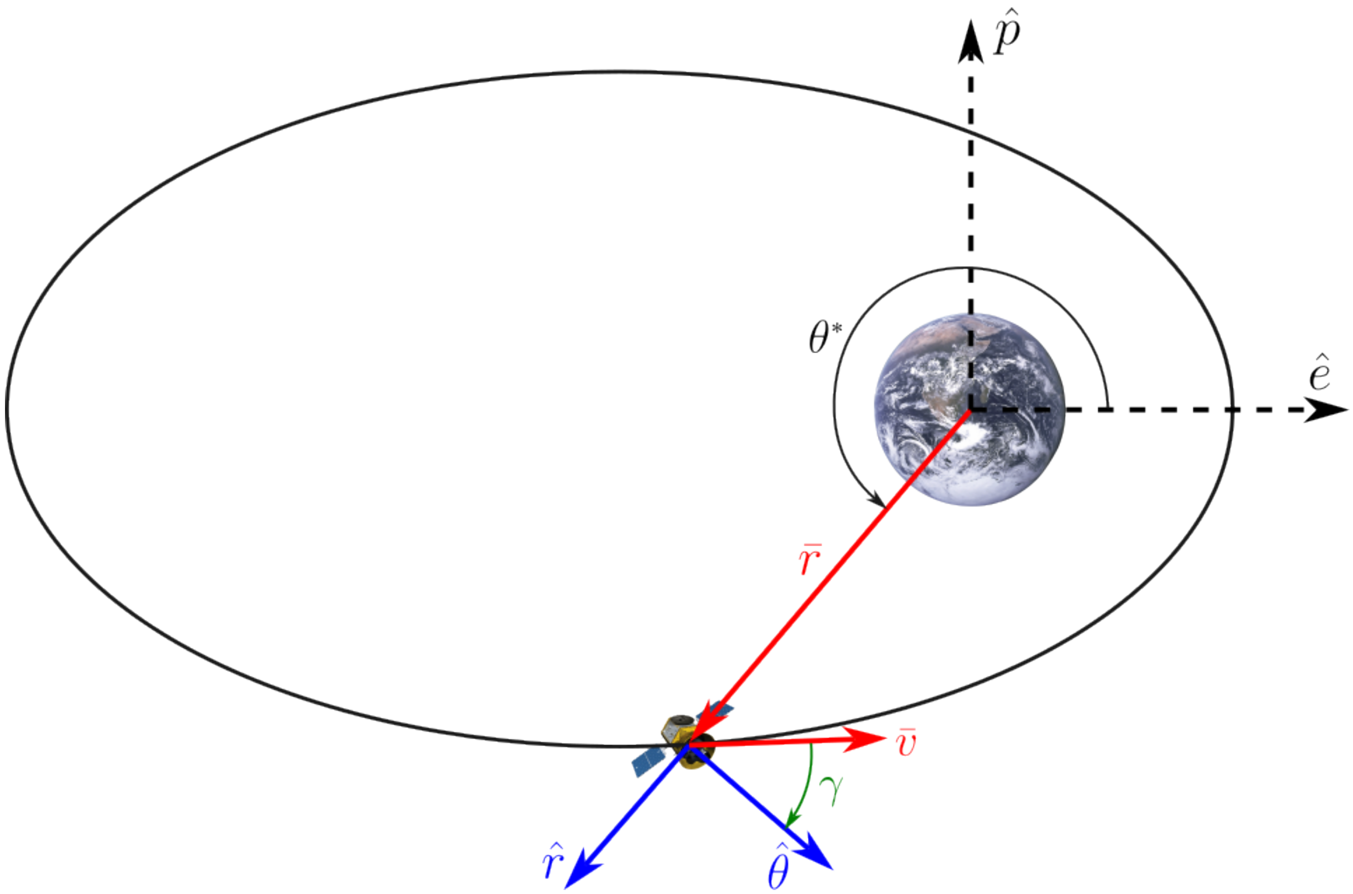

Figure 7: Diagram of spacecraft orbit

Part (b)

The system angular momentum C_3 is computed using

$$\bar{C}_3 = \frac{m_1 m_2}{m_1 + m_2} \bar{r} \times \bar{v} = 2.1914 \times 10^7 \ kg \cdot km^2/s \tag{3}$$

The specific angular momentum h is then obtained by multiplying $\frac{m_1+m_2}{m_1 m_2}$ to C_3, as in

$$\bar{h} = \frac{m_1 + m_2}{m_1 m_2} \bar{C}_3 = \bar{r} \times \bar{v} = 73047.5 \ km^2/s \tag{4}$$

From the specific angular momentum, the areal velocity is then computed as

$$\dot{A} = \frac{h}{2} = 36523.7 \ km^2/s \tag{5}$$

The total kinematic energy of the system is

$$T = \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} \bar{v} \cdot \bar{v} = 4254.63 \ kg \cdot km^2/s^2 \tag{6}$$

the total energy C_4 is

$$C_4 = T - U = T - \frac{G m_1 m_2}{r} = -3750.39 \ kg \cdot km^2/s^2 \tag{7}$$

and the specific energy is

$$\varepsilon = C_4 \frac{m_1 + m_2}{m_1 m_2} = \frac{v^2}{2} - \frac{\mu}{r} = -12.5013 \ km^2/s^2 \tag{8}$$

Part (c)

As shown in Equation (8), C_4 is multiplied by $\frac{m_1+m_2}{m_1 m_2}$, to obtain the specific energy ε. In this problem, the coefficient has the value

$$\frac{m_1 + m_2}{m_1 m_2} = 0.003333 \ kg^{-1} \tag{9}$$

Part (d)

The semimajor axis of the orbit is obtained from the specific energy ε

$$a = \frac{-\mu}{2\varepsilon} = 15942.3\ km \tag{10}$$

From the magnitude of the specific angular momentum $\bar{h}$, the semilatus rectum p is obtained, as in

$$p = \frac{h^2}{\mu} = 13386.7\ km \tag{11}$$

The eccentricity of the orbit is then

$$e = \sqrt{1 - \frac{p}{a}} = 0.4004 \tag{12}$$

and the period is

$$\mathbb{P} = 2\pi\sqrt{\frac{a^3}{\mu}} = 20032.7\ s = 5.5646\ hr \tag{13}$$

The true anomaly is obtained from

$$\theta^* = \cos^{-1}\left(\frac{1}{e}\left(\frac{p}{r} - 1\right)\right) = 254.97° \tag{14}$$

The sign ambiguity from the inverse cosine is resolved by selecting the true anomaly on the descending leg of the orbit. The flight path angle γ, marked in Figure 7, is computed as

$$\gamma = \tan^{-1}\left(\frac{\bar{v}\cdot\hat{r}}{\bar{v}\cdot\hat{\theta}}\right) = -23.340° \tag{15}$$

Since the spacecraft is descending, the flight path angle is negative.

With true anomaly now defined, the direction cosine matrix ${}^{\mathcal{I}}C^{\mathcal{R}}$ between the perifocal ($\mathcal{I}$) and rotating ($\mathcal{R}$) frames is defined as

$${}^{\mathcal{I}}C^{\mathcal{R}} = \begin{bmatrix} \cos\theta^* & \sin\theta^* & 0 \\ -\sin\theta^* & \cos\theta^* & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{16}$$

Thus, the position and velocity vectors are expressed in terms of the inertial frame unit vectors $\hat{e}$ and $\hat{p}$ as

$$\bar{r} = -3874.93\hat{e} - 14426.8\hat{p}\ km \tag{17}$$

$$\bar{v} = 5.270\hat{e} + 0.769\hat{p}\ km/s \tag{18}$$

Part (e)

The relative velocity is

$$v = \|\bar{v}\| = 5.326\ km/s \tag{19}$$

and the circular relative velocity at the same altitude as the spacecraft is

$$v_c = \sqrt{\frac{\mu}{r}} = 5.166\ km/s \tag{20}$$

The spacecraft's orbital velocity is greater than the circular velocity at this altitude.