(Ex) $L: \begin{cases} y'' + 3y' + 2y = 10\,\delta(t-1) \\ y(0) = 0, \quad y'(0) = 1. \end{cases}$

$$L(y'') + 3L(y') + 2L(y) = 10\,L(\delta(t-1))$$

$$s^2 L(y) - s\,y(0) - y'(0) + 3(sL(y) - y(0)) + 2L(y) = 10\,e^{-s},$$

$$(s^2+3s+2)L(y) = 1 + 10e^{-s}$$

$$L(y) = \frac{1}{s^2+3s+2} + 10e^{-s}\frac{1}{s^2+3s+2}.$$

$$= \frac{1}{(s+1)(s+2)} + 10e^{-s}\frac{1}{(s+1)(s+2)}$$

$$y(t) = L^{-1}\left(\frac{1}{(s+1)(s+2)}\right) + 10L^{-1}\left(e^{-s}\frac{1}{(s+1)(s+2)}\right)$$

$$\left(\begin{array}{l} \dfrac{1}{(s+1)(s+2)} = \dfrac{A}{s+1} + \dfrac{B}{s+2} : \begin{pmatrix} A = 1 \\ B = -1. \end{pmatrix} \\ 1 = A(s+2) + B(s+1) \end{array}\right)$$

let $F(s)$

$$y(t) = L^{-1}\left(\frac{1}{s+1} - \frac{1}{s+2}\right) + 10L^{-1}\left(e^{-s}\boxed{\left(\frac{1}{s+1} - \frac{1}{s+2}\right)}\right)$$

$$= \underline{e^{-t} - e^{-2t}} + 10\,u(t-1)\left(e^{-t} - e^{-2t}\right)\Big|_{t-1}$$

#27

$$= e^{-t} - e^{-2t} + 10\,u(t-1)\left(e^{-(t-1)} - e^{-2(t-1)}\right).$$

6.5. Convolution.

(motivation) #19

$$L(e^{2t}\sin(4t)) = \frac{4}{(s-2)^2+4^2}, \quad s-2>0 \ (s>2)$$

$$L(e^{2t}\sin(4t)) \overset{?}{=} L(e^{2t})\cdot L(\sin(4t)) = \frac{1}{s-2}\cdot\frac{4}{s^2+4^2}. \quad \neq$$

Question: $L^{-1}(F(s)\cdot G(s)) = L^{-1}(F(s)) \,?\, L^{-1}(G(s))$

A: $L^{-1}(F(s)\cdot G(s)) = \underbrace{L^{-1}(F) * L^{-1}(G)}_{\text{the convolution of } L^{-1}(F) \,\&\, L^{-1}(G)}$

Def $(f*g)(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$

: the convolution of f & g.

(Properties)

1. $f*g(t) = g*f(t)$

2. $L(f*g) = L(f)\cdot L(g)$

$L^{-1}(F(s)\,G(s)) = L^{-1}(F) * L^{-1}(G)$

---

1. $f*g(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$ $\left(\begin{array}{l} z = t-\tau \\ dz = -d\tau \\ \tau = t-z \end{array}\right.$

$= \int_t^0 f(t-z)\, g(z)\, (-1)\, dz.$

$f*g(t) = \int_0^t g(z) f(t-z)\, dz = g*f(t)$

(Ex) (1) $t * t^{10} = \int_0^t \tau (t-\tau)^{10}\, d\tau$

$(t-\tau)^{10} = \sum_{k=0}^{10} \binom{10}{k} t^k (-\tau)^{10-k}$

$t*t^{10} = t^{10} * t = \int_0^t \tau^{10}(t-\tau)\, d\tau$

$= \int_0^t (t\tau^{10} - \tau^{11})\, d\tau = \left[t\frac{\tau^{11}}{11} - \frac{\tau^{12}}{12} \right]_0^t$

$= \frac{t\cdot t^{11}}{11} - \frac{t^{12}}{12} - 0 = t^{12}\left(\frac{1}{11} - \frac{1}{12}\right)$

$= \frac{t^{12}}{132}$ ✓

(2) $y + \int_0^t (t-\tau)\, y(\tau)\, d\tau = 1$ ✓

: an integral equation

① $y' + \underline{\frac{d}{dt}\int_0^t (t-\tau)\, y(\tau)\, d\tau} = 0$

$\| $

$(t-t)\,y(t) + \int_0^t \frac{d}{dt}(t-\tau)\, y(\tau)\, d\tau$

$= \int_0^t y(\tau)\, d\tau$

$y' + \int_0^t y(\tau)\, d\tau = 0$

$y'' + y(t) = 0,\ y(0) = 1,\ y'(0) = 0$

$$L^{-1}\left(\frac{1}{(s^2+1)(s^2+4)}\right) = L^{-1}\left(\frac{1}{s^2+1}\right) * L^{-1}\left(\frac{1 \cdot 2}{s^2+2^2}\right)\frac{1}{2}$$

$$= \frac{1}{2}\sin(t) * \sin(2t) = \frac{1}{2}\int_0^t \sin(\tau)\sin(2(t-\tau))\,d\tau$$

6.6. DE with variable coefficients.

Q $L^{-1}\left(\frac{12s}{(s^2+36)^2}\right) = ?$ Quick way?

① $L^{-1}\left(\frac{12}{s^2+36}\cdot\frac{s}{s^2+36}\right) = L^{-1}\left(\frac{12}{s^2+36}\right) * L^{-1}\left(\frac{s}{s^2+36}\right)$

② Hint: $L^{-1}\left(\frac{6}{s^2+36}\right) = \sin(6t)$.

<u>Remark</u> $L\left(\int_0^t f(t)\,dt\right) = \frac{1}{s}L(f(t))$.

$$\underbrace{\int_0^t f(\tau)\,d\tau}_{\int_0^t f(\tau)\cdot 1\,d\tau = f*1} = \frac{1}{s}L(f). \quad \#32.$$

(Ex) $L^{-1}\left(\frac{1}{s(s^2+1)}\right) = L^{-1}\left(\frac{1}{s}\frac{1}{s^2+1}\right)$

$= L^{-1}\left(\frac{1}{s}\right) * L^{-1}\left(\frac{1}{s^2+1}\right) = 1 * \sin(t) = \sin(t) * 1$

$= \int_0^t \sin(\tau)\cdot 1\,d\tau = \left[-\cos(\tau)\right]_0^t$

$= -\cos(t) - (-1) = 1 - \cos(t)$.

② Use "$*$":

$y + t * y = 1$

L: $L(y) + L(t*y) = L(1) = \frac{1}{s}$

$L(y) + \underbrace{L(t)}_{=\frac{1}{s^2}}\cdot L(y) = \frac{1}{s}$

$L(y) + \frac{1}{s^2}L(y) = \frac{1}{s}$

$s^2L(y) + L(y) = s^2\cdot\frac{1}{s} = s$

$(s^2+1)L(y) = s$: $L(y) = \frac{s}{s^2+1}$

$y(t) = \cos(t)$.

$$\frac{d}{ds}\left(\frac{6}{s^2+36}\right) = \frac{d}{ds}\left(6\cdot(s^2+36)^{-1}\right)$$

$$= 6\cdot(-1)\,(s^2+36)^{-2}\,\frac{d}{ds}(s^2+36)$$

$$= -6\,(s^2+36)^{-2}\cdot 2s = \frac{-12s}{(s^2+36)^2}$$

$$\frac{12s}{(s^2+36)^2} = \frac{d}{ds}\left(\frac{-6}{s^2+36}\right)$$

(formula) $\underline{L(t\,f(t)) = (-1)\frac{d}{ds}L(f(t))}$

(Proof) $(-1)\frac{d}{ds}L(f(t)) = -\frac{d}{ds}\int_0^{\infty} e^{-st} f(t)\,dt$

$$(-1)\frac{d}{ds}L(f(t)) = (-1)\frac{d}{ds}\int_0^{\infty} e^{-st} f(t)\,dt$$

$$= -\int_0^{\infty}\frac{d}{ds}e^{-st} f(t)\,dt = -\int_0^{\infty} e^{-st}(-t)\,f(t)\,dt$$

$$= \int_0^{\infty} e^{-st}\,t\,f(t)\,dt = \underline{L(t\,f(t))}$$

Remark: $F(s) = L(f(t))$

$$\underline{L^{-1}(F'(s)) = -t\,f(t) = -t\,L^{-1}(F)}$$ ✓

(Ex) $L^{-1}\left(\frac{12s}{(s^2+36)^2}\right) = L^{-1}\left(\frac{d}{ds}\left(\frac{-6}{s^2+36}\right)\right)$

$$= -t\,L^{-1}\left(\frac{-6}{s^2+36}\right) = t\,L^{-1}\left(\frac{6}{s^2+36}\right) = t\sin(6t)$$

1. Bessel DE : $t^2 y'' + t y' + (t^2 - \alpha^2) y = 0$

Remark $L(t^2 f(t)) = L(t \cdot t\, f(t))$

$= (-1) \frac{d}{ds} L(t f(t)) = (-1)^2 \frac{d^2}{ds^2} L(f(t))$

$L(t^n f(t)) = (-1)^n \frac{d^n}{ds^n} L(f(t))$ #30

(Ex) $y'' + t y = 0, \quad y(0) = 0, \quad y'(0) = 1$

$L(y'') + L(ty) = 0$

$s^2 L(y) - s y(0) - y'(0) + (-1) \frac{d}{ds} L(y) = 0$

$-\frac{d}{ds} L(y) + s^2 L(y) = 1 \quad : \quad Y = L(y)$

$Y' - s^2 Y = -1, \quad \rho = e^{\int -s^2 ds} = e^{-s^3/3}$

$\frac{d}{ds}\left(e^{-s^3/3} Y\right) = -e^{-s^3/3}$

$Y(s) = -e^{s^3/3}\left(\int e^{-s^3/3} ds + C\right)$

$y(t) = L^{-1}(Y)$