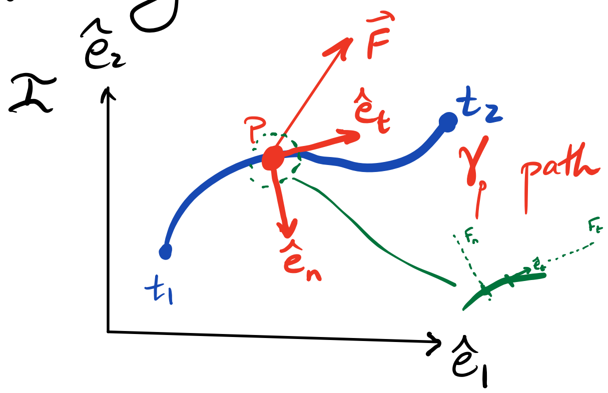
Ch 5: Energy of a Particle

"For certain classes of forces, we can solve for the position and/or velocity of a particle without finding the E.O.M. or trajectories"



$$\boxed{W_P^{(\vec{F})}(\vec{r}_{P/O};\gamma_P) \triangleq \int_{t_1}^{t_2} \vec{F}\cdot{}^{\mathcal{I}}d\vec{r}_{P/O}}$$

But recall ${}^{\mathcal{I}}\vec{v}_{P/O} = \frac{{}^{\mathcal{I}}d}{dt}\vec{r}_{P/O} \Rightarrow {}^{\mathcal{I}}d\vec{r}_{P/O} = {}^{\mathcal{I}}\vec{v}_{P/O}\,dt$

$$\boxed{W_P^{(\vec{F})}(\vec{r}_{P/O};\gamma_P) = \int_{t_1}^{t_2} \vec{F}\cdot{}^{\mathcal{I}}\vec{v}_{P/O}\,dt}$$

Alternate expression

Or we could consider a path frame,

$$\boxed{W_P^{(\vec{F})}(\vec{r}_{P/O}\,;\gamma_P) = \int_{\gamma_P} \vec{F}\cdot\hat{e}_t\,ds}$$

Alternate expression

Note that in a path frame

$$\vec{F} = F_t\,\hat{e}_t + F_n\,\hat{e}_n$$

Then we see $\vec{F}\cdot\hat{e}_t$ term ignores F_n contributions

key idea: Only the tangential component of the force does work on P. Normal component does no work.

Notes:

1. Work depends on the path
2. Even though work is a scalar quantity, it is frame dependent.
   - We will only be calculating work in an inertial frame.
3. Not all forces on a particle do work
4.

We can also examine the total work on P

$$W_P^{(tot)}(\vec{r}_{P/O};\gamma_P) \text{ total work } \longleftrightarrow \vec{F}_P = m_P\,{}^{\mathcal{I}}\vec{a}_{P/O}$$

$$W_P^{(tot)}(\vec{r}_{P/O};\gamma_P) = \int_{\gamma_P} m_P\,{}^{\mathcal{I}}\vec{a}_{P/O}\cdot{}^{\mathcal{I}}d\vec{r}_{P/O}$$

$$= \int_{\gamma_P} m_P \frac{{}^{\mathcal{I}}d}{\cancel{dt}}\left({}^{\mathcal{I}}\vec{v}_{P/O}\right)\cdot{}^{\mathcal{I}}d\vec{r}_{P/O}$$

$${}^{\mathcal{I}}d\vec{r}_{P/O} = {}^{\mathcal{I}}\vec{v}_{P/O}\,\cancel{dt}$$

$$= \int_{\gamma_P} m_P\,{}^{\mathcal{I}}d\left({}^{\mathcal{I}}\vec{v}_{P/O}\right)\cdot{}^{\mathcal{I}}\vec{v}_{P/O}$$

Tricky Product rule manipulation,

$${}^{\mathcal{I}}d\left({}^{\mathcal{I}}\vec{v}_{P/O}\cdot{}^{\mathcal{I}}\vec{v}_{P/O}\right) = {}^{\mathcal{I}}d\,{}^{\mathcal{I}}\vec{v}_{P/O}\cdot{}^{\mathcal{I}}\vec{v}_{P/O} + {}^{\mathcal{I}}\vec{v}_{P/O}\cdot{}^{\mathcal{I}}d\,{}^{\mathcal{I}}\vec{v}_{P/O} = 2\left({}^{\mathcal{I}}d\,{}^{\mathcal{I}}\vec{v}_{P/O}\cdot{}^{\mathcal{I}}\vec{v}_{P/O}\right)$$

Recall

$$\|{}^{\mathcal{I}}\vec{v}_{P/O}\| = \sqrt{{}^{\mathcal{I}}\vec{v}_{P/O}\cdot{}^{\mathcal{I}}\vec{v}_{P/O}}$$

Plugging in,

$$= \frac{1}{2}\int_{\gamma_P} m_P\,{}^{\mathcal{I}}d\left({}^{\mathcal{I}}\vec{v}_{P/O}\cdot{}^{\mathcal{I}}\vec{v}_{P/O}\right) = \frac{1}{2}\int_{\gamma_P} m_P\,{}^{\mathcal{I}}d\left(\|{}^{\mathcal{I}}\vec{v}_{P/O}\|^2\right)$$

$$W_P^{(tot)}(\vec{r}_{P/O};\gamma_P) = \frac{1}{2}m_P\|{}^{\mathcal{I}}\vec{v}_{P/O}(t_2)\|^2 - \frac{1}{2}m_P\|{}^{\mathcal{I}}\vec{v}_{P/O}(t_1)\|^2$$

Dfn: The <u>kinetic energy</u> of particle P is given by

$$T_{P/O} \triangleq \frac{1}{2} m_P \| {}^{I}\vec{v}_{P/O} \|^2$$

Remember that you have to use an inertial velocity here.

Then, we get

$$W_P^{(tot)}(\vec{r}_{P/O}; \gamma_P) = T_{P/O}(t_2) - T_{P/O}(t_1)$$

Or,

$$\boxed{T_{P/O}(t_2) = T_{P/O}(t_1) + W_P^{(tot)}(\vec{r}_{P/O}; \gamma_P)}$$

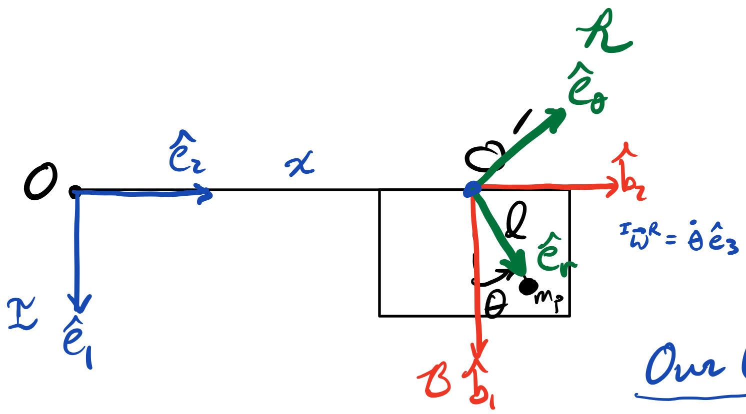
Work-Energy Formula #1 of 3

Conservation?

The kinetic energy is conserved if the total work done on P is zero.

Ex 5.2 (Similar to 4.6 in K+P) But the box travels with constant velocity here.

Find the kinetic energy $T_{P/O}$



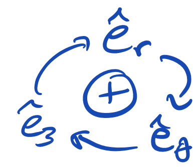
Our Guess:

$\frac{1}{2}m_p\dot{x}^2 + \frac{1}{2}m_p\ell\dot{\theta}^2$

Kinematics

$\vec{r}_{P/O} = \vec{r}_{P/O'} + \vec{r}_{O'/O} = \ell\hat{e}_r + x\hat{e}_2$

$^I\vec{V}_{P/O} = \ell\underbrace{\frac{^I d}{dt}\hat{e}_r} + \dot{x}\hat{e}_2$



$^I\vec{\omega}^R \times \hat{e}_r = (\underbrace{^I\vec{\omega}^B}_{=0} + \underbrace{^B\vec{\omega}^R}_{\dot{\theta}\hat{e}_3}) = \dot{\theta}\hat{e}_3 \times \hat{e}_r = \dot{\theta}\hat{e}_\theta$

$^I\vec{V}_{P/O} = \ell\dot{\theta}\hat{e}_\theta + \dot{x}\hat{e}_2$

We want $T_{P/O}$, but we need to be careful, because $\hat{e}_\theta$ and $\hat{e}_2$ are not perpendicular, so we can't just take the square root of the sum of squares.

$$\|^I\vec{V}_{P/O}\| \neq \sqrt{(\ell\dot{\theta})^2 + (\dot{x})^2}$$

$$\|{}^I\vec{V}_{P/O}\| = \sqrt{{}^I\vec{V}_{P/O} \cdot {}^I\vec{V}_{P/O}}$$

Convert to all one basis,

$${}^I\vec{V}_{P/O} = \ell\dot{\theta}\left(-\sin\theta\,\hat{b}_1 + \cos\theta\,\hat{b}_2\right) + \dot{x}\hat{b}_2$$

$$= -\ell\dot{\theta}\sin\theta\,\hat{b}_1 + (\ell\dot{\theta}\cos\theta + \dot{x})\hat{b}_2$$

$$T_{P/O} = \frac{m_P}{2}\|{}^I\vec{V}_{P/O}\|^2 = \frac{1}{2}m_P\,{}^I\vec{V}_{P/O} \cdot {}^I\vec{V}_{P/O}$$

$$= \frac{1}{2}m_P\left[(-\ell\dot{\theta}\sin\theta)^2 + (\ell\dot{\theta}\cos\theta + \dot{x})^2\right]$$

$$= \frac{1}{2}m_P\left[\ell^2\dot{\theta}^2 + \dot{x}^2 + \underbrace{2c\theta\,\ell\dot{\theta}\dot{x}}\right]$$

Cross terms that involve the motion of frames B and R

What if $\dot{x}$ is constant?

We could also calculate $T_{P/O'}$

$$T_{P/O'} = \frac{m_P}{2}\|{}^I\vec{V}_{P/O'}\|^2 = m_P\ell^2\dot{\theta}^2$$

Compare to other expression

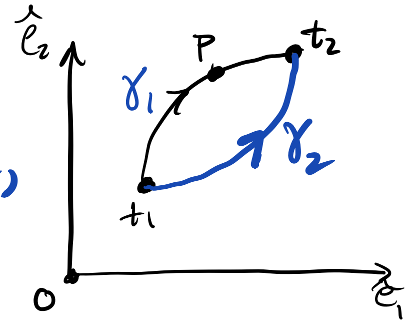
$\Rightarrow$ The answers are different in differing inertial frames.

Also note: $T_{P/O} \neq T_{P/O'} + T_{O'/O}$ Kinetic energies don't add this way

(Recall kinetic energy is a quadratic function of velocity)

An important class of forces

<u>Dfn</u>: A <u>conservative force</u> is one in which the work performed only depends on the end points of the path.



Let $\vec{F}$ be a conservative force,

$$W_p^{(\vec{F})}(\vec{r}_{P/O};\gamma_p) = W_p^{(\vec{F})}(t_1,t_2)$$

Simplifying notation since work only depends on the end points here.

$$W_p^{(\vec{F})}(t_1,t_2) = \int_{t_1}^{t_2} \vec{F}\cdot {}^{I}d\vec{r}_{P/O} = -\int_{t_2}^{t_1} \vec{F}\cdot {}^{I}d\vec{r}_{P/O} = -W_p^{(\vec{F})}(t_2,t_1)$$

Further, we could consider a closed circuit

$$\oint \vec{F}\cdot {}^{I}d\vec{r}_{P/O} = \underbrace{\int_{t_1}^{t_2} \vec{F}\cdot {}^{I}d\vec{r}_{P/O}}_{W_p^{(\vec{F})}(t_1,t_2)} + \underbrace{\int_{t_2}^{t_1} \vec{F}\cdot {}^{I}d\vec{r}_{P/O}}_{W_p^{(\vec{F})}(t_2,t_1) = -W_p^{(\vec{F})}(t_1,t_2)}$$

$$\Rightarrow \boxed{\oint \vec{F}\cdot {}^{I}d\vec{r}_{P/O} = 0 \quad \text{if } \vec{F} \text{ is conservative}}$$

We will make use of the nice properties of conservative forces, and to begin doing this, we will divide forces into two groups:

$\vec{F}^{(c)}$ conservative forces

$\vec{F}^{(nc)}$ nonconservative forces

Then work can be divided up as well:

$$W_P^{(tot)}(\vec{r}_{P/O}, \gamma_P) = W_P^{(c)}(t_1, t_2) + W_P^{(nc)}(\vec{r}_{P/O}; \gamma_P)$$

Dfn: Let the <u>potential energy</u> associated with a conservative force $\vec{F}$ be

$$U_{P/O}^{(\vec{F})}(\vec{r}_{P/O}) \triangleq - \int \vec{F} \cdot {}^{I}d\vec{r}_{P/O}$$

recall conservative $\vec{F}$

Indefinite integral results in constant of integration. Potential energy is therefore arbitrary up to an additive constant.

Going back to look at conservative forces, let $\vec{F}$ be conservative, then,

$$W_P^{(c)}(t_1, t_2) = \int_{t_1}^{t_2} \vec{F} \cdot {}^{I}d\vec{r}_{P/O} = -U_{P/O}^{(\vec{F})}(t_2) + U_{P/O}^{(\vec{F})}(t_1)$$

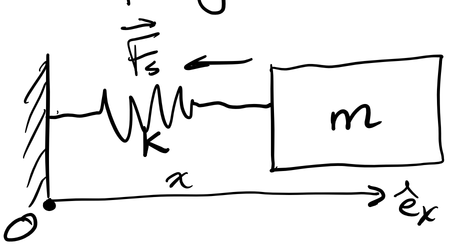
$$\boxed{U_{P/O}^{(\vec{F})}(t_2) = U_{P/O}^{(\vec{F})}(t_1) - W_P^{(c)}(t_1, t_2)}$$

Work-Energy Formula #2 of 3

Conservation?

$U_{P/O}$ is conserved if $W_P^{(c)}(t_1, t_2) = 0$.

<u>Ex.</u> mass-spring Find $U_{P/O}^{(\vec{F}_s)}$



$\vec{r}_{P/O} = x\hat{e}_x$

${}^{I}\vec{v}_{P/O} = \frac{{}^{I}d\vec{r}_{P/O}}{dt} = \dot{x}\hat{e}_x$

$\Rightarrow {}^{I}d\vec{r}_{P/O} = \frac{dx}{\cancel{dt}}\cancel{dt}\,\hat{e}_x$

$$U_{P/O}^{(\vec{F}_s)} = -\int \vec{F}_s \cdot {}^{I}d\vec{r}_{P/O} = -\int(-kx\hat{e}_x)\cdot dx\,\hat{e}_x = \int kx\,dx$$

$$= \frac{1}{2}kx^2 + C$$

where $C = U_{P/O}(0)$

Ex. Gravitational force. Find $U_{P/O}^{(\vec{F}_G)}$

$$\vec{F}_G = -\frac{G m_O m_P}{r^2}\hat{e}_r$$

kinematics:

$$\vec{r}_{P/O} = r\hat{e}_r$$

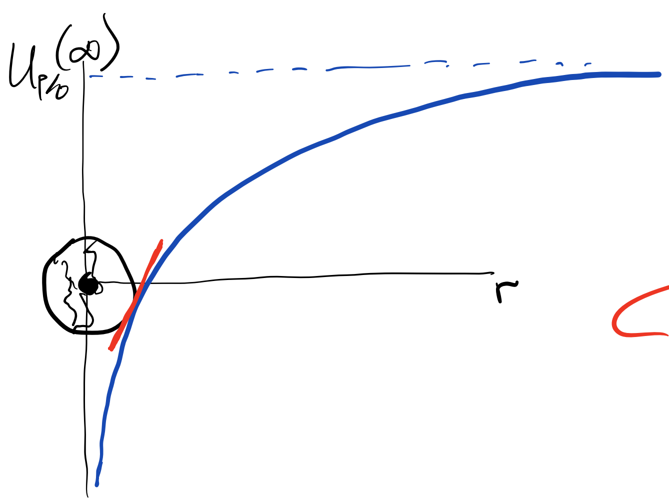
$$^{I}\vec{V}_{P/O} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta \Rightarrow {}^{I}d\vec{r}_{P/O} = dr\,\hat{e}_r + r\,d\theta\,\hat{e}_\theta$$

$$U_{P/O}^{(\vec{F}_G)} = -\int \vec{F}_G \cdot {}^{I}d\vec{r}_{P/O}$$

$$U_{P/O}^{(\vec{F}_G)} = -\int -\frac{G m_O m_P}{r^2}\hat{e}_r \cdot (dr\,\hat{e}_r + r\,d\theta\,\hat{e}_\theta)$$

$$= \int \frac{G m_O m_P}{r^2}\,dr = \frac{-G m_O m_P}{r} + \underbrace{C}_{U_{P/O}(\infty)}$$

Note: the integration contributes a minus sign here



$$\vec{F}_G = -\nabla U_{P/O}^{(\vec{F}_G)}$$

$$= -\frac{G m_O m_P}{r^2}\hat{e}_r$$

Near earth's surface, $r = (R_e + h) \approx R_e$ since $R_e \gg h$.

$$\frac{G m_O}{R_e^2} \approx 9.8\,\frac{m}{s^2}$$