#1)

a) False

b) False

C) True

D) True

E) False

F) False

G) False

H) True

#2) $M = 2N - \cancel{K_S}^{0} - K_H = 2(2) - (2) = 2$ DOF

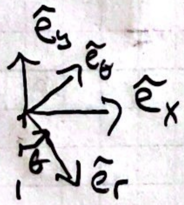
$N_C = 2N - K_H = 2$

$N_C : x, \theta$

<u>Kinematics</u>: $\vec{r}_{Q/O} = x\,\hat{e}_x$

$^{I}\vec{v}_{Q/O} = \dot{x}\,\hat{e}_x$

$\vec{r}_{P/O} = x\,\hat{e}_x + r\,\hat{e}_r$



$\hat{e}_r = \sin\theta\,\hat{e}_x - \cos\theta\,\hat{e}_y$

$\vec{r}_{P/O} = (x + r\sin\theta)\,\hat{e}_x - r\cos\theta\,\hat{e}_y$

$^{I}\vec{v}_{P/O} = (\dot{x} + r\dot{\theta}\cos\theta)\,\hat{e}_x + r\dot{\theta}\sin\theta\,\hat{e}_y$

$\vec{F}_s = -K(x - x_0)\,\hat{e}_x$

$\vec{F}_g = -m_p g\,\hat{e}_y$

$U^{(\vec{F}_s)} = -\int \vec{F}_s \cdot {}^{I}d\vec{r}_{P/O} = -\int -K(x-x_0)\,\hat{e}_x \cdot dx\,\hat{e}_x$

$U^{(\vec{F}_s)} = -\int -K(x-x_0)\,dx = \dfrac{K(x-x_0)^2}{2}$

$U^{(\vec{F}_g)} = -\int -m_p g\,\hat{e}_y \cdot (dx + r\,d\theta\cos\theta)\,\hat{e}_x + (r\,d\theta\sin\theta)\,\hat{e}_y$

$U^{(\vec{F}_g)} = -\int -m_p g r\,d\theta\sin\theta = m_p g r \int \sin\theta\,d\theta$

$U^{(\vec{F}_g)} = -m_p g r\cos\theta + C_1$ (let $C_1 = 0$)

$\therefore U^{(\vec{F}_s)} = -m_p g r \cos\theta$

$U_0 = U^{(\vec{F}_k)} + U^{(\vec{F}_s)} = \dfrac{K(x-x_0)^2}{2} - m_p g r \cos\theta$

$T_0 = \sum \frac{1}{2} m_i |{}^{\mathcal{I}}\vec{v}_{i/O}|^2$

$T_0 = \frac{1}{2} m_Q \dot{x}^2 + \frac{1}{2} m_p [(\dot{x} + r\dot{\theta}\cos\theta)^2 + (r\dot{\theta}\sin\theta)^2]$

$T_0 = \frac{1}{2} m_Q \dot{x}^2 + \frac{1}{2} m_p [\dot{x}^2 + r^2\dot{\theta}^2\cos^2\theta + 2r\dot{\theta}\dot{x}\cos\theta + r^2\dot{\theta}^2\sin^2\theta]$

$T_0 = \frac{1}{2} m_Q \dot{x}^2 + \frac{1}{2} m_p [\dot{x}^2 + r^2\dot{\theta}^2 + 2r\dot{\theta}\dot{x}\cos\theta]$

$\mathcal{L} = T_0 - U_0 = \dfrac{\dot{x}^2}{2}(m_Q + m_p) + \dfrac{m_p}{2}(r^2\dot{\theta}^2 + 2r\dot{\theta}\dot{x}\cos\theta) - \dfrac{K(x-x_0)^2}{2} + m_p g r\cos\theta$

Euler-Lagrange: $\dfrac{d}{dt}\left(\dfrac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) - \dfrac{\partial \mathcal{L}}{\partial q_j} = \cancelto{0}{Q_j^{(nc,a)}} + \cancelto{0}{Q_j^{(nc)}} + \cancelto{0}{Q_j^{(constraint)}}$

no non Conservative Forces

no non-holonomic constraints

$q_1 = x, \; q_2 = \theta$

$\dfrac{\partial \mathcal{L}}{\partial q_1} = -K(x-x_0), \quad \dfrac{\partial \mathcal{L}}{\partial \dot{q}_1} = \dot{x}(m_Q + m_p) + m_p r\dot{\theta}\cos\theta$

$\dfrac{\partial \mathcal{L}}{\partial q_2} = -m_p r\dot{\theta}\dot{x}\sin\theta - m_p g r\sin\theta$

$\dfrac{\partial \mathcal{L}}{\partial \dot{q}_2} = m_p r^2\dot{\theta} + m_p r\dot{x}\cos\theta$

$\dfrac{d}{dt}\left(\dfrac{\partial \mathcal{L}}{\partial \dot{q}_1}\right) = \ddot{x}(m_Q + m_p) + m_p r\ddot{\theta}\cos\theta - m_p r\dot{\theta}^2\sin\theta$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_2}\right) = m_p r^2 \ddot{\theta} + m_p r \ddot{x} \cos\theta - m_p r \dot{x} \dot{\theta} \sin\theta$$

$$\boxed{q_1 : \ \ddot{x}(M_Q + m_p) + m_p r \ddot{\theta} \cos\theta - m_p r \dot{\theta}^2 \sin\theta + K(x - x_0) = 0}$$

$$q_2 : \ m_p r^2 \ddot{\theta} + m_p r \ddot{x} \cos\theta - m_p r \dot{x} \dot{\theta} \sin\theta + m_p r \dot{\theta} \dot{x} \sin\theta + m_p g r \sin\theta = 0$$

$$\boxed{q_2 : \ m_p r^2 \ddot{\theta} + m_p r \ddot{x} \cos\theta + m_p g r \sin\theta = 0}$$

#3)

a) $M = 3N - K_s - K_H = 3(1) - 0 - 2$

$$\boxed{M = 1 \text{ DOF}}$$

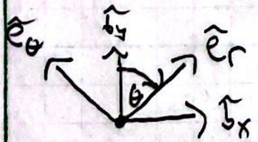
b) $\boxed{\text{No}}$

c) ${}^I\vec{\omega}^B = \omega \hat{b}_y = \omega \hat{e}_y$ ${}^B\vec{\omega}^R = -\dot{\theta}\,\hat{b}_z$

$\vec{r}_{P/O} = r\,\hat{e}_r$

${}^I\vec{\omega}^R = {}^I\vec{\omega}^B + {}^B\vec{\omega}^R = \omega\hat{b}_y - \dot{\theta}\,\hat{b}_z$

${}^I\vec{v}_{P/O} = {}^B\frac{d}{dt}(\vec{r}_{P/O}) + {}^I\vec{\omega}^B \times \vec{r}_{P/O} =$



$\hat{e}_r = \cos\theta\,\hat{b}_y + \sin\theta\,\hat{b}_x$

$\vec{r}_{P/O} = r\cos\theta\,\hat{b}_y + r\sin\theta\,\hat{b}_x$

${}^I\vec{v}_{P/O} = -r\dot{\theta}\sin\theta\,\hat{b}_y + r\dot{\theta}\cos\theta\,\hat{b}_x - \omega r\sin\theta\,\hat{b}_z$

$\vec{F}_g = -mg\,\hat{e}_y = -mg\,\hat{b}_y$

$U_0 = -\int -mg\hat{b}_y \cdot (-r\,d\theta\sin\theta\,\hat{b}_y + r\,d\theta\cos\theta\,\hat{b}_x - d\omega\, r\sin\theta\,\hat{b}_z)$

$U_0 = \int -mgr\sin\theta\,d\theta = mgr\cos\theta + C_1$

Let $C_1 = 0$

$U_0 = mgr\cos\theta$

$T_0 = \frac{1}{2}m[r^2\dot{\theta}^2\sin^2\theta + r^2\dot{\theta}^2\cos^2\theta + \omega^2r^2\sin^2\theta] = \frac{1}{2}m_i \|{}^{I}\vec{v}_{P/O}\|^2$

$T_0 = \frac{1}{2}m[r^2\dot{\theta}^2 + \omega^2r^2\sin^2\theta]$

$\mathcal{L} = T_0 - U_0 =$

$$\boxed{\mathcal{L} = \frac{mr^2\dot{\theta}^2}{2} + \frac{m\omega^2r^2\sin^2\theta}{2} - mgr\cos\theta}$$

$$\frac{\partial\mathcal{L}}{\partial q_1} = \frac{m\omega^2r^2\sin(2\theta)}{2} + mgr\sin\theta$$

$$\frac{\partial\mathcal{L}}{\partial \dot{q}_1} = mr^2\dot{\theta} \qquad \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial \dot{q}_1}\right) = mr^2\ddot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial \dot{q}_1}\right) - \frac{\partial\mathcal{L}}{\partial q_1} = 0$$

$$mr^2\ddot{\theta} - \frac{m\omega^2}{2}r^2\sin(2\theta) - mgr\sin\theta = 0$$

$$\boxed{\ddot{\theta} = \frac{\omega^2}{2}\sin(2\theta) + \frac{g\sin\theta}{r}}$$

Equation of Motion

d) $H = \dot{q}^T p - L$

$$p = \frac{\partial\mathcal{L}}{\partial \dot{q}_1} = mr^2\dot{\theta} \Rightarrow \dot{\theta} = \frac{p}{mr^2} = \dot{q}_1$$

$$H = \left(\frac{p}{mr^2}\right)(p) - L$$

$$L = \left(\frac{mr^2}{2}\right)\left(\frac{p}{mr^2}\right)^2 + \frac{m\omega^2}{2} r^2 \sin^2(q_1) - mgr\cos(q_1)$$

$$L = \frac{p^2}{2mr^2} + \frac{m\omega^2}{2} r^2 \sin^2(q_1) - mgr\cos(q_1)$$

$$H = \frac{p^2}{mr^2} - \frac{p^2}{2mr^2} - \frac{m\omega^2}{2} r^2 \sin^2(q_1) + mgr\cos(q_1)$$

$$\boxed{H = \frac{p^2}{2mr^2} - \frac{m\omega^2}{2} r^2 \sin^2(q_1) + mgr\cos(q_1)}$$

$$\boxed{\dot{q}_1 = \dot{\theta} = \frac{p}{mr^2}} = \frac{\partial H}{\partial p}$$

$$\boxed{\dot{p}_1 = -\frac{\partial H}{\partial q_1} = \frac{m\omega^2}{2} r^2 \sin(2q_1) + mgr\sin(q_1)}$$

E) $\boxed{\mathcal{L} = \frac{mr^2\dot{\theta}^2}{2} + \frac{m\dot{\beta}^2 r^2 \sin^2\theta}{2} - mgr\cos\theta}$

F) $\frac{\partial \mathcal{L}}{\partial \beta} = 0 \therefore \beta$ is cyclic $\quad \frac{\partial \mathcal{L}}{\partial \dot{\beta}} = m\dot{\beta} r^2 \sin^2\theta = P_\beta$

$P_\beta = m\dot{\beta} r^2 \sin^2\theta$ is conserved because β is a cyclic variable,

therefore it's conjugate momentum is conserved.

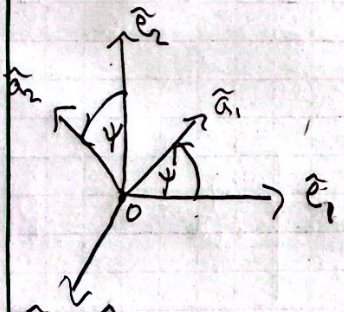
Scanned with CamScanner

#4) $\;{}^{I}\vec{v}_{O/O} = \begin{bmatrix} 100 \\ 10 \\ -10 \end{bmatrix}_B \qquad {}^{I}\vec{\omega}^{B} = \begin{bmatrix} 0.1 \\ 0 \\ 0.1 \end{bmatrix}_B$

$\vec{r}_{B/O} = \begin{bmatrix} 30 \\ 20 \\ -120 \end{bmatrix}_I \qquad$ 3-2-1 Euler $(\psi, \theta, \phi) = (5^\circ, 10^\circ, 5^\circ)$

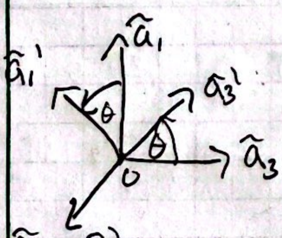
$$[{}^{I}\vec{v}_{O/O}]_I = {}^{I}C^{B}\,[{}^{I}\vec{v}_{P/O}]_B$$

$$[{}^{I}\vec{\omega}^{B}]_I = {}^{I}C^{B}\,[{}^{I}\vec{\omega}^{B}]_B$$



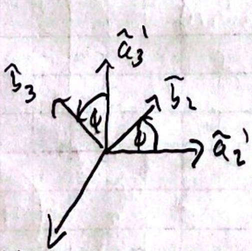
$\hat{e}_3 = \hat{a}_3$

$$\begin{array}{c|ccc} & \hat{a}_1 & \hat{a}_2 & \hat{a}_3 \\ \hline \hat{e}_1 & C\psi & -S\psi & 0 \\ \hat{e}_2 & S\psi & C\psi & 0 \\ \hat{e}_3 & 0 & 0 & 1 \end{array} = {}^{I}C^{A}$$



$\hat{a}_2 = \hat{a}_2'$

$$\begin{array}{c|ccc} & \hat{a}_1' & \hat{a}_2' & \hat{a}_3' \\ \hline \hat{a}_1 & C\theta & 0 & S\theta \\ \hat{a}_2 & 0 & 1 & 0 \\ \hat{a}_3 & -S\theta & 0 & C\theta \end{array} = {}^{A}C^{A'}$$



$\hat{a}_1' = \hat{b}_1$

$$\begin{array}{c|ccc} & \hat{b}_1 & \hat{b}_2 & \hat{b}_3 \\ \hline \hat{a}_1' & 1 & 0 & 0 \\ \hat{a}_2' & 0 & C\phi & -S\phi \\ \hat{a}_3' & 0 & S\phi & C\phi \end{array} = {}^{A'}C^{B}$$

$${}^{I}C^{B} = {}^{I}C^{A}\;{}^{A}C^{A'}\;{}^{A'}C^{B}$$

$$\left[\widehat{{}^{I}V^{B}} \right]_I = \begin{bmatrix} \left[{}^{I}\hat{\omega}^{B} \right]_I & -{}^{I}\vec{\omega}^{B} \times \vec{r}_{o'/o} + \left[{}^{I}\vec{V}_{o'/o} \right]_I \\ 0_{1\times 3} & 0 \end{bmatrix}$$

$$\boxed{\left[\widehat{{}^{I}V^{B}} \right]_I = \begin{pmatrix} 0 & -0.0807 & 0.0014 & 97.3731 \\ 0.0807 & 0 & -0.1161 & 2.8838 \\ -0.0014 & 0.1161 & 0 & -28.5968 \\ 0 & 0 & 0 & 0 \end{pmatrix}}$$

Contents

MAE562 Final Exam Problem 4

```
clear
clc

B_V         = [100 10 -10]';                                    % Intertial velocity expressed in body frame
B_w         = [0.1 0 0.1]';                                     % Angular velocity of B wrt I expressed in body frame
r           = [30 20 -120]';                                    % Position vector expressed in inertial frame
```

3-2-1 Euler Angle

```
psi         = 5;                                                % Rotation about 3 axis [deg]
theta       = 10;                                               % Rotation about 2 axis [deg]
phi         = 5;                                                % Rotation about 1 axis [deg]

IcA         = [cosd(psi) -sind(psi) 0; sind(psi) cosd(psi) 0; 0 0 1];   % Rotation matrix from Inertial to first intermediate frame
AcA2        = [cosd(theta) 0 sind(theta); 0 1 0; -sind(theta) 0 cosd(theta)];  % Rotation matrix from first intermediate frame to second intermediate frame
A2cB        = [1 0 0; 0 cosd(phi) -sind(phi); 0 sind(phi) cosd(phi)];   % Rotation matrix second intermediate frame to body frame

IcB         = IcA*AcA2*A2cB;                                    % 3-2-1 Euler Rotation Matrix

I_V         = IcB*B_V;                                          % Intertial velocity expressed in inertial frame
I_w         = IcB*B_w;                                          % Angular velocity of B wrt I expressed in inertial frame
```

Spatial Velocity

```
skew_w_I    = [ 0 -I_w(3) I_w(2); I_w(3) 0 -I_w(1); -I_w(2) I_w(1) 0];     % Skew symmetric matrix of angular velocity of B wrt I expressed in inertial frame

I_V_B       = [skew_w_I, -cross(I_w,r)+I_V; zeros(1,4)];        % Spatial Velocity

disp('The spatial velocity matrix is: ')
disp(I_V_B)
```

```
The spatial velocity matrix is:
         0   -0.0807    0.0014   97.3731
    0.0807         0   -0.1161    2.8838
   -0.0014    0.1161         0  -28.5968
         0         0         0         0
```

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