ECE 68000: MODERN AUTOMATIC CONTROL

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Stability of Linear Systems

Asymptotic Stability of Linear Systems

- **Objective**: Introduce internal and external stability concepts of linear lumped continuous-time (CT) and discrete-time (DT) systems
- Consider a CT linear time-varying (LTV) system

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}(t)\boldsymbol{x}(t)$$

which has an equilibrium state at the origin of the state space, $\boldsymbol{x}_e = \boldsymbol{0}$

Definition (Asymptotic Stability)

The equilibrium state $\boldsymbol{x}_e = \boldsymbol{0}$ is asymptotically stable if any solution $\boldsymbol{x}(t)$ starting from any initial condition $\boldsymbol{x}(0) \in \mathbb{R}^n$ satisfies

$$\boldsymbol{x}(t) \to \boldsymbol{0} \quad as \quad t \to \infty$$

Exponential stability $\Longrightarrow$ Asymptotic stability

Definition (Exponential Stability)

The equilibrium state $\boldsymbol{x}_e = \mathbf{0}$ is exponentially stable of the system $\dot{\boldsymbol{x}} = \boldsymbol{A}(t)\boldsymbol{x}$ if any solution $\boldsymbol{x}(t)$ starting from any initial condition $\boldsymbol{x}(0) \in \mathbb{R}^n$ satisfies

$$\|\boldsymbol{x}(t)\| \leq Ke^{-rt}\|\boldsymbol{x}(0)\|$$

for some constants $K > 0$ and $r > 0$

Internal stability of CT linear time-invariant (LTI) systems

Theorem

For a CT LTI system $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$, $\boldsymbol{A} \in \mathbb{R}^{n \times n}$, *the following statements are equivalent*

1. *The equilibrium state* $\boldsymbol{x}_e = \boldsymbol{0}$ *is asymptotically stable*
2. *The equilibrium state* $\boldsymbol{x}_e = \boldsymbol{0}$ *is exponentially stable*
3. *All eigenvalues of* $\boldsymbol{A}$ *are in the open left-half complex plane, that is,* $\Re\lambda_i(\boldsymbol{A}) < 0$, $i = 1, 2, \ldots, n$

Internal stability of CT linear time-invariant (LTI) systems theorem restated

Theorem

For a CT LTI system $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$, $\boldsymbol{A} \in \mathbb{R}^{n\times n}$, the following statements are equivalent

1. *The system $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$ is asymptotically stable*
2. *The system $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$ is exponentially stable*
3. *All eigenvalues of $\boldsymbol{A}$ are in the open left-half complex plane, that is, $\Re\lambda_i(\boldsymbol{A}) < 0$, $i = 1, 2, \ldots, n$*

Instability of CT linear time-invariant (LTI) systems

- LTI CT system $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$, $\boldsymbol{A} \in \mathbb{R}^{n \times n}$, is unstable if $\|\boldsymbol{x}(t)\|$ is unbounded for some initial condition $\boldsymbol{x}(0)$
- This is the case if either of the following conditions holds:
  1. $\boldsymbol{A}$ has at least one eigenvalue in the open right-half complex plane, that is, there exists an eigenvalue such that $\Re\lambda_i(\boldsymbol{A}) > 0$
  2. There are eigenvalues of $\boldsymbol{A}$ with multiplicity greater than one on the imaginary axis

Marginally stable systems

- LTI CT system $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$, $\boldsymbol{A} \in \mathbb{R}^{n\times n}$, is marginally stable if all solution $\boldsymbol{x}(t)$ are bounded and at least one solution $\boldsymbol{x}(t)$ does not converge to zero
- This is the case if both of the following conditions hold:
  1. $\boldsymbol{A}$ has no eigenvalues in the open right-half complex plane, that is, $\boldsymbol{A}$ has no eigenvalue such that $\Re\lambda_i(\boldsymbol{A}) > 0$
  2. $\boldsymbol{A}$ has simple eigenvalues on the $j\omega$-axis
- Some texts classify marginally stable systems as unstable

Internal Stability versus Input/Output Stability

- Consider a CT LTI single-input single-output (SISO) system modeled by a proper transfer function $G(s)$, where

$$Y(s) = G(s)U(s)$$

Definition (BIBO Stability)

The system is BIBO stable $\iff$ for any bounded input $u(\cdot)$, the output $y(\cdot)$ is bounded

BIBO Stability Test

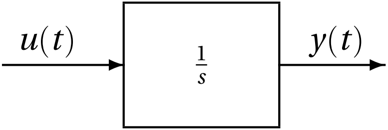
Theorem

A continuous MIMO LTI system with a proper rational transfer function matrix, $\boldsymbol{G}(s)$, is BIBO stable if and only if every pole of $G_{ij}(s)$ has a negative real part, where

$$\boldsymbol{G}(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) & \cdots & G_{1m}(s) \\ \vdots & \vdots & \ddots & \vdots \\ G_{p1}(s) & G_{p2}(s) & \cdots & G_{pm}(s) \end{bmatrix}$$

Example

- Is the following system BIBO stable?



- Ans: NO
- Indeed, let $u(t) = 1(t)$ (bounded input), then $y(t) = t^2 1(t)$ (unbounded output)

Internal stability of DT Linear Systems

Discrete-time (DT) LTV system $\boldsymbol{x}[k+1] = \boldsymbol{A}[k]\boldsymbol{x}[k],\ k = 0, 1, \ldots$

Definition (Asymptotic Stability)

LTV system is asymptotically stable at time k_0 if its solution $\boldsymbol{x}[k]$ starting from any initial condition $\boldsymbol{x}[k_0] \in \mathbb{R}^n$ at time k_0 satisfies

$$\boldsymbol{x}[k] \to 0 \quad \text{as} \quad k \to \infty$$

Definition (Exponential Stability)

LTV system is exponentially stable at time k_0 if its solution $\boldsymbol{x}[k]$ starting from any initial condition $\boldsymbol{x}[k_0] \in \mathbb{R}^n$ at time k_0 satisfies

$$\|\boldsymbol{x}[k]\| \leq K r^{k-k_0} \|\boldsymbol{x}[k_0]\| \ \text{for all} \ k = k_0, k_0 + 1, \ldots$$

for some constants $K > 0$ and $0 \leq r < 1$

Stability of autonomous DT linear time-invariant (LTI) systems

Theorem

For a DT LTI system $\boldsymbol{x}[k+1] = \boldsymbol{A}\boldsymbol{x}[k]$, $\boldsymbol{A} \in \mathbb{R}^{n \times n}$, *the following statements are equivalent*

1. *The system is asymptotically stable*
2. *The system is exponentially stable*
3. *All eigenvalues of* $\boldsymbol{A}$ *are inside of the open unit disk in the complex plane, that is,* $|\lambda_i(\boldsymbol{A})| < 1$, $i = 1, 2, \ldots, n$

- Note: starting time k_0 does not matter

Marginal Stability of DT LTI Systems

System $\boldsymbol{x}[k+1] = \boldsymbol{A}\boldsymbol{x}[k]$, $\boldsymbol{A} \in \mathbb{R}^{n \times n}$, is marginally stable if both of the following hold:

1. $\boldsymbol{A}$ has no eigenvalues outside the closed unit disk, that is, there are no eigenvalues such that $|\lambda_i(\boldsymbol{A})| > 1$
2. $\boldsymbol{A}$ has simple eigenvalues on the unit circle

Instability of DT LTI Systems

System $\boldsymbol{x}[k+1] = \boldsymbol{A}\boldsymbol{x}[k]$, $\boldsymbol{A} \in \mathbb{R}^{n \times n}$, is unstable if either of the following is true:

1. $\boldsymbol{A}$ has eigenvalues outside the closed unit disk, that is, for some eigenvalues $|\lambda_i(\boldsymbol{A})| > 1$
2. $\boldsymbol{A}$ has eigenvalues on the unit circle with multiplicity greater than one