Optimal Estimation Methods

(Lecture 10 – Review of Dynamic Systems: Part I)

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Second-Order Systems (i)

- Typical spring-mass-damper system

$$m\,\ddot{x} + c\,\dot{x} + k\,x = F$$

- Free response with $F = 0$

$$\text{let } x = Ae^{st},\ \dot{x} = sAe^{st} = s\,x,\ \ddot{x} = s^2 Ae^{st} = s^2 x$$

  - Leads to

$$(m\,s^2 + c\,s + k)Ae^{st} = 0$$

- Assuming that $Ae^{st} \neq 0$ gives the *characteristic equation*

$$m\,s^2 + c\,s + k = 0$$

  - Roots are given by

$$s_{1,2} = \frac{-c \pm \sqrt{c^2 - 4\,m\,k}}{2\,m}$$

Second-Order Systems (ii)

- Possibilities for $s_{1,2}$
  - Real and unequal with $c^2 - 4\,m\,k > 0$

$$x(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

  - Real and equal with $c^2 - 4\,m\,k = 0$

$$x(t) = A_1 e^{s_1 t} + t\,A_2 e^{s_2 t}$$

  - Complex conjugates with $c^2 - 4\,m\,k < 0$

$$x(t) = B\,e^{-a\,t}\sin(b\,t + \phi)$$

  where

$$a = \frac{c}{2\,m}, \quad b = \frac{\sqrt{4\,m\,k - c^2}}{2\,m}$$

- Constants found through initial conditions

Second-Order Systems (iii)

- Define the following

Natural Frequency

Damping Ratio

$$\omega_n = \sqrt{\frac{k}{m}} \quad \text{and} \quad \zeta = \frac{c}{2\sqrt{m\,k}}$$

  - Then

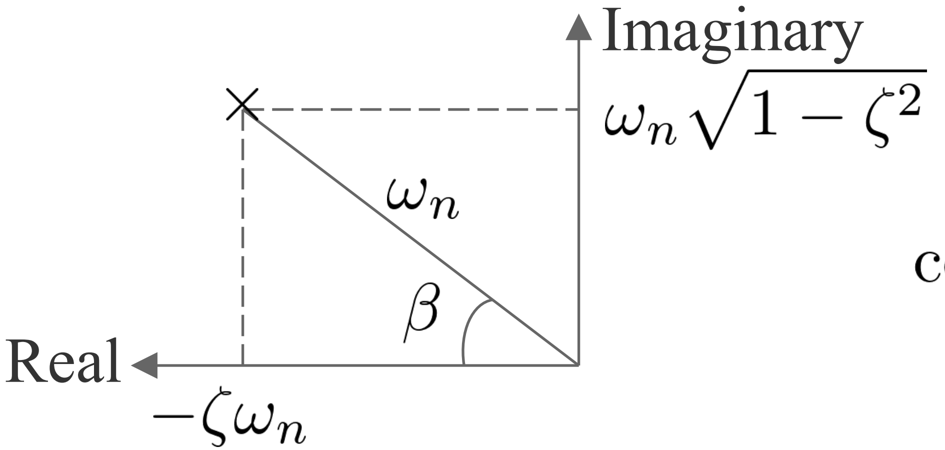
$$m\,s^2 + c\,s + k = 0 \Rightarrow s^2 + 2\,\zeta\omega_n s + \omega_n^2 = 0$$

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

- Three cases again
  - Real and unequal with $\zeta > 1$
  - Real and equal with $\zeta = 1$
  - Complex conjugates with $0 < \zeta < 1$
    - Note if $\zeta = 0$ then $s_{1,2} = \pm\omega_n\,j$
- Also $\omega_d \equiv \omega_n\sqrt{1-\zeta^2}$ is the *damped natural frequency*

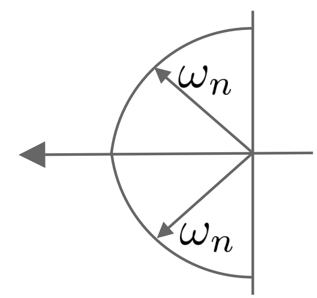
Second-Order Systems (iv)

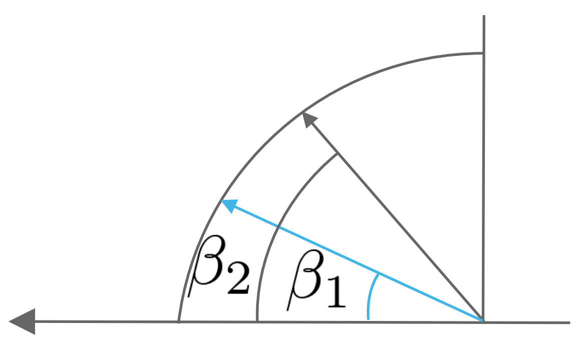
- Graphical Interpretation



$$\cos\beta = \frac{\zeta\omega_n}{\omega_n} = \zeta$$

- Roots lying on a circle centered at the origin have same natural frequency
- The closer a line is to the imaginary axis the smaller ζ becomes





$$\cos\beta_1 > \cos\beta_2$$

$$\text{so } \zeta_1 > \zeta_2$$

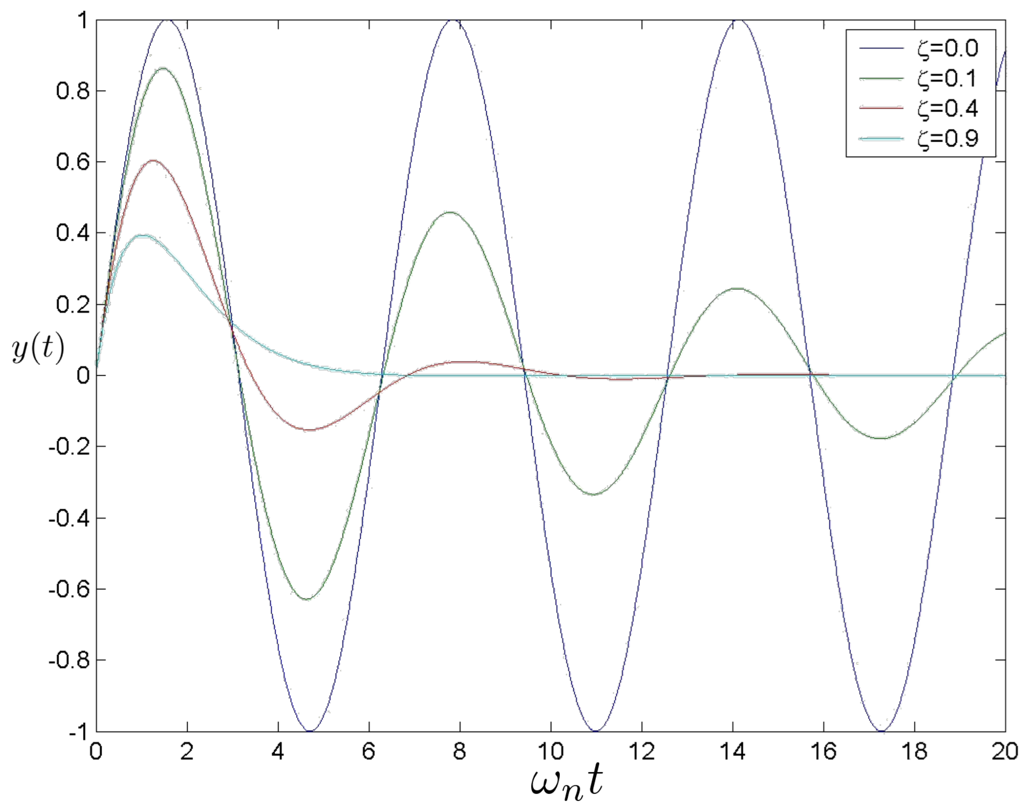
$$\text{note when } \beta = 90^o \text{ then } \zeta = 0$$

Optimal $\zeta = \frac{\sqrt{2}}{2}$

- Responses to various ζ values

$$Y(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$y(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_n \sqrt{1-\zeta^2}\, t), \text{ for } 0 < \zeta < 1$$



Spring Mass System (i)

- For spring-mass system we have (no forcing)

$$m\,\ddot{x} + k\,x = 0$$

- Characteristic equation becomes

$$m\,s^2 + k = 0$$

  - Roots are given by (purely imaginary)

$$s_{1,2} = \sqrt{\frac{k}{m}}\,j$$

- Assumed form of the solution

$$x(t) = B\sin(\sqrt{k/m}\,t + \phi)$$

  - Need to find constants B and ϕ
  - Derivative is given by

$$\dot{x}(t) = B\sqrt{k/m}\cos(\sqrt{k/m}\,t + \phi)$$

Spring Mass System (ii)

- Initial Conditions

$$x(0) \equiv x_0 = B\sin\phi \text{ and } \dot{x}(0) \equiv \dot{x}_0 = B\sqrt{k/m}\,\cos\phi$$

  - So

$$B = \frac{x_0}{\sin\phi}$$

  - Also we have

$$\frac{x_0}{\dot{x}_0} = \frac{1}{\sqrt{k/m}}\tan\phi \quad \text{so} \quad \phi = \tan^{-1}\left(\frac{x_0}{\dot{x}_0}\sqrt{\frac{k}{m}}\right)$$

Linear Systems (i)

- In-class assignment
  - Compute the transfer function of the following system
    - Take the Laplace Transform with zero initial conditions

$x_1 = y$

$x_2 = \dot{y}$

$$\ddot{y} + \dot{y} + y = u$$

$(s^2 + s + 1)Y(s) = U(s)$

$\frac{Y(s)}{U(s)} = \frac{1}{s^2+s+1}$

  - Then compute its state space model

$$\dot{x} = \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}}_{F} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{\underline{x}} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{B} \underbrace{\underline{u}}_{\underline{u}}$$

$$y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{H} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{\underline{x}} + \underbrace{0}_{D} \underline{u}$$

Linear Systems (ii)

- Take the Laplace Transform with zero initial conditions

$$(s^2 + s + 1)Y(s) = U(s)$$

- Transfer function is then given by

$$\frac{Y(s)}{U(s)} = \frac{1}{(s^2 + s + 1)}$$

- Define states up to the $n-1$ derivatives: $\ddot{y} + \dot{y} + y = u$

$$x_1 = y, \qquad \dot{x}_1 = x_2$$
$$x_2 = \dot{y}, \qquad \dot{x}_2 = -x_2 - x_1 + u$$

- State space model is then given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0]u$$

Linear Systems (iii)

- General form for single-input-single-output (SISO) systems

$$\dot{\mathbf{x}}(t) = F\,\mathbf{x}(t) + B\,u(t)$$
$$y(t) = H\,\mathbf{x}(t) + D\,u(t)$$

- Convert back to TF by taking the Laplace Transform

$$[sI]\mathbf{X}(s) = F\,\mathbf{X}(s) + B\,U(s)$$
$$Y(s) = H\,\mathbf{X}(s) + D\,U(s)$$

- From first equation we have

$$\mathbf{X}(s) = (sI - F)^{-1} B\,U(s)$$

- Substituting this into the second equation gives

$$\boxed{\frac{Y(s)}{U(s)} = H\,(sI - F)^{-1} B + D}$$

  - Note that $(sI - F)^{-1} = \mathrm{adj}(sI - F)/|sI - F|$
  - Then $|sI - F|$ gives the denominator of the transfer function
  - This shows that the eigenvalues of F are the roots of the characteristic equation! A

Linear Systems (iv)

- From last example

$$F = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 0$$

- First compute $(sI - F)^{-1}$

$$(sI - F)^{-1} = \begin{bmatrix} s & -1 \\ 1 & s+1 \end{bmatrix}^{-1} = \frac{1}{s^2 + s + 1} \begin{bmatrix} s+1 & 1 \\ -1 & s \end{bmatrix}$$

- Then

$$\begin{aligned} \frac{Y(s)}{U(s)} &= H\,(sI - F)^{-1} B + D \\ &= \frac{1}{s^2+s+1} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s+1 & 1 \\ -1 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 0 \\ &= \frac{1}{s^2+s+1} \begin{bmatrix} s+1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \frac{1}{s^2+s+1} \quad \checkmark \end{aligned}$$

- In-class assignment
  - Compute the transfer function of the following system

$$\mathrm{inv}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} D & -B \\ -C & A \end{pmatrix}$$

$$F = \begin{bmatrix} -0.5 & -1.5 \\ 0.5 & -0.5 \end{bmatrix}, \quad B = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad H = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & 1 \end{bmatrix}, \quad D = 0$$

$$sI - F = \begin{pmatrix} s+0.5 & 1.5 \\ -0.5 & s+0.5 \end{pmatrix} \Rightarrow \frac{1}{(s+.5)^2 + (1.5)(.5)} \begin{pmatrix} s+.5 & -1.5 \\ 0.5 & s+.5 \end{pmatrix}$$

$$s^2 + s + .25 + .75$$

$$\underbrace{\begin{pmatrix} \frac{s+.5}{s^2+s+1} & \frac{-1.5}{s^2+s+1} \\ \frac{0.5}{s^2+s+1} & \frac{s+.5}{s^2+s+1} \end{pmatrix}}_{(sI-F)^{-1}}$$

$$H(sI-F)^{-1}B + \cancel{0}$$

$$\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{pmatrix} \frac{s+.5}{s^2+s+1} & \frac{-1.5}{s^2+s+1} \\ \frac{0.5}{s^2+s+1} & \frac{s+.5}{s^2+s+1} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{pmatrix}$$

$$= \frac{1}{s^2+s+1}$$

Linear Systems (vi)

- First compute $(sI-F)^{-1}$

$$(sI-F)^{-1} = \begin{bmatrix} s+0.5 & 1.5 \\ -0.5 & s+0.5 \end{bmatrix}^{-1} = \frac{1}{s^2+s+1}\begin{bmatrix} s+0.5 & -1.5 \\ 0.5 & s+0.5 \end{bmatrix}$$

- Then

$$\begin{aligned} \frac{Y(s)}{U(s)} &= H\,(sI-F)^{-1}B + D \\ &= \frac{1}{s^2+s+1}\cdot\frac{1}{2}\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} s+0.5 & -1.5 \\ 0.5 & s+0.5 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + 0 \\ &= \frac{1}{s^2+s+1}\cdot\frac{1}{2}\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} s+2 \\ -s \end{bmatrix} \\ &= \frac{1}{s^2+s+1} \end{aligned}$$

- Same transfer function as before
  - There are actually an infinite amount of state space *realizations* that give the same transfer function

Linear Systems (vii)

- Numerator dynamics; consider the following TF

$$\frac{Y(s)}{U(s)} = \frac{s+1}{(s^2+s+1)} = \frac{Y(s)}{Z(s)} \cdot \frac{Z(s)}{U(s)}$$

- Let

$$\frac{Y(s)}{Z(s)} = s+1 \quad (1) \qquad\qquad \frac{Z(s)}{U(s)} = \frac{1}{(s^2+s+1)} \quad (2)$$

- From Eq. (2) we have

$$\ddot{z} + \dot{z} + z = u, \quad \text{or} \quad \ddot{z} = -\dot{z} - z + u$$

- State space

$$x_1 = z, \qquad \dot{x}_1 = x_2$$

$$x_2 = \dot{z}, \qquad \dot{x}_2 = -x_2 - x_1 + u$$

  - Same as the first example
  - This is because the dominator is identical

Linear Systems (viii)

- From Eq. (1) we have

$$y = \dot{z} + z, \quad \text{or} \quad y = x_2 + x_1$$

- State space model is then given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0]u$$

  - Same as the first example, except that the H matrix is different
  - Note that the eigenvalues of the F matrix are the same as before, because it's the same matrix (again due to the fact that the dominator is identical)

- Another example; consider the following TF

$$\frac{Y(s)}{U(s)} = \frac{s^2+3s+2}{(s^2+s+1)} = \frac{Y(s)}{Z(s)} \cdot \frac{Z(s)}{U(s)}$$

- Let

$$\frac{Y(s)}{Z(s)} = s^2+3s+2 \quad (1) \qquad\qquad \frac{Z(s)}{U(s)} = \frac{1}{(s^2+s+1)} \quad (2)$$

- From Eq. (2) we have

$$\ddot{z} + \dot{z} + z = u, \quad \text{or} \quad \ddot{z} = -\dot{z} - z + u$$

- State space

$$x_1 = z, \qquad \dot{x}_1 = x_2$$

$$x_2 = \dot{z}, \qquad \dot{x}_2 = -x_2 - x_1 + u$$

  - Same as the first example
  - This is again because the dominator is identical
  - So the F and B matrices are the same

- From Eq. (1) we have $y = \ddot{z} + 3\dot{z} + 2z$
- Now use the following

$$\ddot{z} = \dot{x}_2 = -x_2 - x_1 + u$$

$$\dot{z} = x_2$$

$$z = x_1$$

- Then y becomes

$$y = -x_2 - x_1 + u + 3x_2 + 2x_1 = x_1 + 2x_2 + u$$

- State space model is then

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [1]u$$

Order num can't be > order of den. not realizable

- When the numerator has the same order as the denominator then the D matrix is nonzero (always happens)
  - This is called "Direct Transmission" because on output is detected directly when an input is applied

Linear Systems (xi)

- Past example, slight return; consider the following TF

$$\frac{Y(s)}{U(s)} = \frac{s^2 + 3s + 2}{(s^2 + s + 1)} = \frac{Y(s)}{Z(s)} \cdot \frac{Z(s)}{U(s)}$$

- Let

$$\frac{Y(s)}{Z(s)} = s^2 + 3s + 2 \quad (1) \qquad\qquad \frac{Z(s)}{U(s)} = \frac{1}{(s^2 + s + 1)} \quad (2)$$

- From Eq. (2) we have

$$\ddot{z} + \dot{z} + z = u, \quad \text{or} \quad \ddot{z} = -\dot{z} - z + u$$

- Use different states now

$$\begin{aligned} x_1 &= \dot{z}, & \dot{x}_1 &= -x_1 - x_2 + u \\ x_2 &= z, & \dot{x}_2 &= x_1 \end{aligned}$$

  - A little different than before
  - So the F and B matrices are different, but the eigenvalues of F are the same

- From Eq. (1) we have $y = \ddot{z} + 3\dot{z} + 2z$
- Now use the following

$$\ddot{z} = \dot{x}_1 = -x_1 - x_2 + u$$
$$\dot{z} = x_1$$
$$z = x_2$$

- Then y becomes

$$y = -x_1 - x_2 + u + 3x_1 + 2x_2 = 2x_1 + x_2 + u$$

- State space model is then

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [1]u$$

- The D matrix is the same as before, but others have changed
- The transfer function is identical though
  - Just a different realization of the same system

- First compute $(sI-F)^{-1}$

$$(sI-F)^{-1}=\begin{bmatrix} s+1 & 1 \\ -1 & s \end{bmatrix}^{-1}=\frac{1}{s^2+s+1}\begin{bmatrix} s & -1 \\ 1 & s+1 \end{bmatrix}$$

- Then

$$\begin{aligned}\frac{Y(s)}{U(s)} &= H\,(sI-F)^{-1}B+D \\ &= \frac{1}{s^2+s+1}\begin{bmatrix} 2 & 1 \end{bmatrix}\begin{bmatrix} s & -1 \\ 1 & s+1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}+1 \\ &= \frac{1}{s^2+s+1}\begin{bmatrix} 2 & 1 \end{bmatrix}\begin{bmatrix} s \\ 1 \end{bmatrix}+1 \\ &= \frac{2s+1}{s^2+s+1}+1 \\ &= \frac{2s+1+s^2+s+1}{s^2+s+1} \\ &= \frac{s^2+3s+2}{s^2+s+1} \quad \checkmark\end{aligned}$$

General State Space Model

- General ODE

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0\, y$$
$$= b_n \frac{d^n u}{dt^n} + b_{n-1}\frac{d^{n-1}u}{dt^{n-1}} + \cdots + b_1 \frac{du}{dt} + b_0\, u$$

- State space matrices

$$x_1 = y$$
$$x_2 = \frac{dy}{dt}$$
$$\vdots$$
$$x_n = \frac{d^{n-1}y}{dt^{n-1}}$$

$$F = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix}^T$$

$$H = \begin{bmatrix} (b_0 - b_n\, a_0) & (b_1 - b_n\, a_1) & \cdots & (b_{n-1} - b_n\, a_{n-1}) \end{bmatrix}$$

$$D = b_n$$

Similarity Transformation (i)

- Consider a multi-input-multi-output (MIMO) model

$$\dot{\mathbf{x}} = F\,\mathbf{x} + B\,\mathbf{u} \quad (1)$$

$$\mathbf{y} = H\,\mathbf{x} + D\,\mathbf{u}$$

  - Now define $\mathbf{x} = T\mathbf{z}$, where T is constant and nonsingular
  - Then $\dot{\mathbf{x}} = T\dot{\mathbf{z}}$
  - Multiplying both sides by T^{-1}, and substituting the state space model from Eq. (1) gives

$$\dot{\mathbf{z}} = A\,\mathbf{z} + G\,\mathbf{u}$$

$$\mathbf{y} = C\,\mathbf{z} + D\,\mathbf{u}$$

where

$$A \equiv T^{-1} F\,T$$

$$G \equiv T^{-1} B$$

$$C \equiv H\,T$$

Similarity Transformation (ii)

- Compute the transfer function matrix from $\mathbf{U}(s)$ to $\mathbf{Y}(s)$

$$
\begin{aligned}
\mathbf{Y}(s) &= \left[C\left(sI - A\right)^{-1}G + D\right]\mathbf{U}(s) \\
&= \left[H\,T\left(sI - T^{-1}F\,T\right)^{-1}T^{-1}B + D\right]\mathbf{U}(s) \\
&= \left\{H\,T\left[T^{-1}\left(sI - F\right)T\right]^{-1}T^{-1}B + D\right\}\mathbf{U}(s) \\
&= \left[H\,T\,T^{-1}\left(sI - F\right)^{-1}T\,T^{-1}B + D\right]\mathbf{U}(s) \\
&= \left[H\left(sI - F\right)^{-1}B + D\right]\mathbf{U}(s)
\end{aligned}
$$

- Same transfer function as before
- Similarity transformation does not change transfer function!
- Many possible forms
  - Diagonal form for A (assuming distinct eigenvalues) can be found by using the eigenvector matrix of F for T

Cayley-Hamilton Theorem (i)

- Cayley-Hamilton (CH) Theorem states that a matrix satisfies its own characteristic equation
  - Say characteristic equation of a matrix A is given by

$$\lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \cdots + a_1\lambda + a_0 = 0$$

  - Then the CH Theorem states

$$A^n + a_{n-1}A^{n-1} + a_{n-2}A^{n-2} + \cdots + a_1 A + a_0 I = 0$$

  - Proof for diagonalizable case (general case can be proven too)
    - Compute the following

$$M = A^n + a_{n-1}A^{n-1} + a_{n-2}A^{n-2} + \cdots + a_1 A + a_0 I \qquad (1)$$

    - Need to show that $M = 0$
    - Take an eigenvalue/eigenvector decomposition of A

$$A = V \Lambda V^{-1} \quad \rightarrow \quad V^{-1} A V = \Lambda \qquad (2)$$

where Λ is a diagonal matrix of the eigenvalues and V is the eigenvector matrix

Cayley-Hamilton Theorem (ii)

- Left multiply Eq. (1) by V^{-1} and right multiply by V

$$V^{-1}M\,V = V^{-1}A^nV + a_{n-1}V^{-1}A^{n-1}V + \cdots + a_0 I$$

- But

$$V^{-1}A^jV = \underbrace{(V^{-1}A\,V)(V^{-1}A\,V)\cdots(V^{-1}A\,V)}_{j \text{ times}} = \Lambda^j$$

where Eq. (2) was used

- So

$$V^{-1}M\,V = \begin{bmatrix} \lambda_1^n & & 0 \\ & \ddots & \\ 0 & & \lambda_n^n \end{bmatrix} + a_{n-1}\begin{bmatrix} \lambda_1^{n-1} & & 0 \\ & \ddots & \\ 0 & & \lambda_n^{n-1} \end{bmatrix} + \cdots + a_0 \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

- Because every eigenvalue satisfies its characteristic equation then the right-hand-side of this equation is zero
- This can only be true if $M = 0$ since $V \neq 0$

Cayley-Hamilton Theorem (iii)

- CH Theorem shows that every $f(A)$ function of a square matrix can be computed as a linear combination of the first $n-1$ powers of A

$$f(A) = \gamma_0 I + \gamma_1 A + \cdots + \gamma_{n-1} A^{n-1}$$

- Why is this true?
- From the CH Theorem we have

$$A^n = \underbrace{-a_{n-1}A^{n-1} - a_{n-2}A^{n-2} - \cdots - a_0 I}_{\text{RHS}}$$

- So then we have

$$\begin{aligned} A^{n+1} &= -a_{n-1}A^n - a_{n-2}A^{n-1} - \cdots - a_0 A \\ &= -a_{n-1}[\text{RHS}] - a_{n-2}A^{n-1} - \cdots - a_0 A \\ &= \gamma_{n-1}A^{n-1} - \gamma_{n-2}A^{n-2} + \cdots + \gamma_0 I \end{aligned}$$

where γ_i's are a function of the a_i's

Cayley-Hamilton Theorem (iv)

- One way to compute the coefficients for distinct eigenvalues is to note that $f(A)$ is the same linear combination of the powers of A as that $f(\Lambda)$ is of the powers of Λ
- For non-distinct eigenvalues first define

$$h(\lambda) \equiv \gamma_0 + \gamma_1 \lambda + \cdots + \gamma_{n-1} \lambda^{n-1}$$

- The n unknowns γ_i's can be solved using the following n equations

$$f^{(\ell)}(\lambda) = h^{(\ell)}(\lambda) \text{ for } \ell = 0,\, 1,\, \ldots,\, n_i - 1 \text{ and } i = 1,\, 2,\, \ldots,\, m$$

where

$$f^{(\ell)}(\lambda) \equiv \left. \frac{d^\ell f(\lambda)}{d\lambda^\ell} \right|_{\lambda = \lambda_i}$$

$$n = \sum_{i-1}^{m} n_i$$

Example I (i)

$$A = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad \text{compute } A^{100}$$

- Let $f(A) = A^{100}$, then

$$A^{100} = \gamma_0 I + \gamma_1 A + \gamma_2 A^2$$

- Eigenvalue matrix is given by $\Lambda = \text{diag}([1 \quad +j \quad -j])$
- They are distinct so we can now use

$$F(\Lambda) = \begin{bmatrix} 1^{100} & 0 & 0 \\ 0 & j^{100} & 0 \\ 0 & 0 & (-j)^{100} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \gamma_0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \gamma_1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & -j \end{bmatrix} + \gamma_2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example I (ii)

- Gives the following three equations

$$\begin{aligned} \gamma_0 + \gamma_1 + \gamma_2 &= 1 \\ \gamma_0 + \gamma_1 j - \gamma_2 &= 1 \\ \gamma_0 - \gamma_1 j + \gamma_2 &= 1 \end{aligned}$$

- By inspection the solutions are

$$\gamma_0 = 1, \quad \gamma_1 = 0, \quad \gamma_2 = 0$$

- Then

$$\begin{aligned} A = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}^{100} &= \gamma_0 I + \gamma_1 A + \gamma_2 A^2 \\ = 1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 0 \begin{bmatrix} 1 & 0 & 4 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} &+ 0 \begin{bmatrix} 1 & 0 & 4 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}^2 \\ = I \end{aligned}$$

Example II (i)

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix}, \quad \text{compute } e^{At}$$

- Eigenvalue matrix is given by $\Lambda = \mathrm{diag}([1 \;\; 1 \;\; 2])$
  - Note that there are repeated eigenvalues
- Let $f(\lambda) = e^{\lambda t}$, then $f'(\lambda) = t\,e^{\lambda t}$ (note: derivative w.r.t. λ to not t)

$$h(\lambda) \equiv \gamma_0 + \gamma_1 \lambda + \gamma_2 \lambda^2, \quad h'(\lambda) = \gamma_1 + 2\gamma_2 \lambda$$

- Need to solve the following equations

$$\begin{aligned} f(1) = h(1): &\quad e^t = \gamma_0 + \gamma_1 + \gamma_2 \\ f'(1) = h'(1): &\quad t\,e^t = \gamma_1 + 2\gamma_2 \\ f(2) = h(2): &\quad e^{2\,t} = \gamma_0 + 2\gamma_1 + 4\gamma_2 \end{aligned}$$

Example II (ii)

- Solving these equations gives

$$\gamma_0 = -2\,t\,e^t + e^{2\,t}$$
$$\gamma_1 = 3\,t\,e^t + 2\,e^t - 2\,e^{2\,t}$$
$$\gamma_2 = e^{2\,t} - e^t - t\,e^t$$

- Then

$$\begin{aligned} e^{At} &= \gamma_0 I + \gamma_1 A + \gamma_2 A^2 \\ &= \gamma_0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \gamma_1 \begin{bmatrix} 0 & 0 & -2 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix} + \gamma_2 \begin{bmatrix} -2 & 0 & -6 \\ 0 & 1 & 0 \\ 3 & 0 & 7 \end{bmatrix} \\ &= \begin{bmatrix} 2\,e^t - e^{2\,t} & 0 & 2\,e^t - 2\,e^{2\,t} \\ 0 & e^t & 0 \\ e^{2\,t} - e^t & 0 & 2\,e^{2\,t} - e^t \end{bmatrix} \end{aligned}$$

Properties of e^{At} (i)

- Look at an expansion of $e^{\lambda t}$

$$e^{\lambda t} = 1 + \lambda t + \frac{\lambda^2 t^2}{2!} + \cdots$$

  - Then

$$e^{At} = I + At + A^2 \frac{t^2}{2!} + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!} t^k A^k \qquad (1)$$

  - Some useful properties

$$e^0 = I$$

$$e^{A(t_1 + t_2)} = e^{At_1} e^{At_2}$$

$$\left[e^{At}\right]^{-1} = e^{-At}$$

    - The last one is especially useful because you don't need to take the inverse of an entire matrix!

Properties of e^{At} (ii)

- Take time derivative of Eq. (1)

$$\frac{d}{dt}e^{At} = \sum_{k=1}^{\infty} \frac{1}{(k-1)!} t^{k-1} A^k \qquad (2)$$

- This can be rewritten as

$$\frac{d}{dt}e^{At} = A\left(\sum_{k=0}^{\infty} \frac{1}{k!} t^k A^k\right) = \left(\sum_{k=0}^{\infty} \frac{1}{k!} t^k A^k\right) A$$

  - Note that because of the A^k term in Eq. (2), when the sum was rewritten an extra A matrix is needed, which can be pulled out
  - Also, note that the A can be pulled out on either the right or left side
- Then from Eq. (1)

$$\frac{d}{dt}e^{At} = A\,e^{At} = e^{At}A \quad (3)$$

- Note that $e^{(A+B)t} \neq e^{At}e^{Bt}$ in general unless $AB = BA$, i.e. the matrices A and B *commute*

Properties of e^{At} (iii)

- Laplace Transform of the time derivative of a function

$$\mathcal{L}[f'(t)] = s\mathcal{L}[f(t)] - f(0)$$

  - Using this concept on the time derivative of e^{At} gives

$$A\,\mathcal{L}[e^{At}] = s\mathcal{L}[e^{At}] - e^{0}$$

  where Eq. (3) was used
    - Note that the first form of Eq. (3) was specifically used so that A appears of the left (can be pulled on the right too since s is a scalar)
  - Noting that $e^{0} \ = I$ and collecting terms yields

$$(sI - A)\,\mathcal{L}[e^{At}] = I$$

  - Then

$$\mathcal{L}[e^{At}] = (sI - A)^{-1}$$

  - Taking the inverse Laplace Transform of both sides gives

$$\boxed{e^{At} = \mathcal{L}^{-1}\left[(sI - A)^{-1}\right]}$$

Example

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix}, \quad \text{compute } e^{At}, \quad (sI - A) = \begin{bmatrix} s & 0 & 2 \\ 0 & s-1 & 0 \\ -1 & 0 & s-3 \end{bmatrix}$$

- $(sI - A)^{-1}$ can be shown to be given by

$$(sI - A)^{-1} = \begin{bmatrix} \frac{2}{s-1} - \frac{1}{s-2} & 0 & \frac{2}{s-1} - \frac{2}{s-2} \\ 0 & \frac{1}{s-1} & 0 \\ \frac{1}{s-2} - \frac{1}{s-1} & 0 & \frac{2}{s-2} - \frac{1}{s-1} \end{bmatrix}$$

  - Check $(sI - A)^{-1}(sI - A) = I$ to make sure it's correct
- Taking the inverse Laplace Transform gives

$$e^{At} = \begin{bmatrix} 2\,e^{t} - e^{2\,t} & 0 & 2\,e^{t} - 2\,e^{2\,t} \\ 0 & e^{t} & 0 \\ e^{2\,t} - e^{t} & 0 & 2\,e^{2\,t} - e^{t} \end{bmatrix}$$

  - Same result as before

State Space Solution (i)

- Consider the following time-varying system

$$\dot{\mathbf{x}}(t) = F(t)\,\mathbf{x}(t) + B(t)\,\mathbf{u}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0$$

  - Claim the solution for $\mathbf{x}(t)$ is given by

$$\mathbf{x}(t) = \Phi(t, t_0)\,\mathbf{x}(t_0) + \int_{t_0}^{t} \Phi(t,\tau)\,B(\tau)\,\mathbf{u}(\tau)\;d\tau$$

  where $\Phi(t,\tau)$ is the *state transition matrix*

  - Some useful properties

$$\begin{aligned} \frac{d}{dt}\Phi(t,\tau) &= F(t)\,\Phi(t,\tau) \\ \Phi(\tau,\tau) &= I \\ \Phi(\tau,t) &= \Phi^{-1}(t,\tau) \\ \Phi(t_2,t_0) &= \Phi(t_2,t_1)\Phi(t_1,t_0) \end{aligned}$$

State Space Solution (ii)

- Take derivative to prove it is indeed the solution

$$
\begin{aligned}
\dot{\mathbf{x}}(t) &= \frac{d}{dt}\Phi(t,t_0)\,\mathbf{x}(t_0) + \frac{d}{dt}\int_{t_0}^{t}\Phi(t,\tau)\,B(\tau)\,\mathbf{u}(\tau)\;d\tau \\
&= F(t)\,\Phi(t,t_0)\,\mathbf{x}(t_0) + \cancelto{I}{\Phi(t,t)}\,B(t)\,\mathbf{u}(t) + \int_{t_0}^{t}F(t)\,\Phi(t,\tau)\,B(\tau)\,\mathbf{u}(\tau)\;d\tau \\
&= F(t)\underbrace{\left[\Phi(t,t_0)\,\mathbf{x}(t_0) + \int_{t_0}^{t}\Phi(t,\tau)\,B(\tau)\,\mathbf{u}(\tau)\;d\tau\right]}_{\mathbf{x}(t)} + B(t)\,\mathbf{u}(t) \\
&= F(t)\,\mathbf{x}(t) + B(t)\,\mathbf{u}(t) \;\;\checkmark
\end{aligned}
$$

- Check initial conditions

$$
\begin{aligned}
\mathbf{x}(t_0) &= \cancelto{I}{\Phi(t_0,t_0)}\,\mathbf{x}(t_0) + \cancelto{0}{\int_{t_0}^{t_0}\Phi(t_0,\tau)\,B(\tau)\,\mathbf{u}(\tau)\;d\tau} \\
&= \mathbf{x}_0 \;\;\checkmark
\end{aligned}
$$

Time Invariant Case (i)

- State transition matrix reduces down to $\Phi(t,\tau) = e^{F(t-\tau)}$
  - This is *matrix exponential* of F
  - Can be found using the inverse Laplace Transform

$$e^{Ft} = \mathcal{L}^{-1}\left[(sI - F)^{-1}\right]$$

  - Example

$$F = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

  - Then

$$(sI - F)^{-1} = \frac{1}{s^2}\begin{bmatrix} s & 1 \\ 0 & s \end{bmatrix} = \begin{bmatrix} \frac{1}{s} & \frac{1}{s^2} \\ 0 & \frac{1}{s} \end{bmatrix}$$

  - Inverse Laplace Transform gives

$$e^{Ft} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

Time Invariant Case (ii)

- Sometimes using the series expansion is easier though

$$e^{Ft} = I + Ft + F^2\frac{t^2}{2!} + \cdots$$

- Look at this matrix again

$$F = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

- Since $F^2 = 0$ then

$$\begin{aligned} e^{Ft} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix} + 0 + 0 + \cdots \\ &= \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \end{aligned}$$

- Same result as before