11.9 Fourier transform.

Fourier integral: Given a function $f(x)$

$$F(x) = \int_0^\infty \left(\underline{A(\omega)}\cos(\omega x) + \underline{B(\omega)}\sin(\omega x)\right)d\omega$$

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(z)\cos(\omega z)\,dz, \quad B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(z)\sin(\omega z)\,dz$$

$$= \frac{1}{\pi}\int_0^\infty\int_{-\infty}^{\infty}\Big(f(z)\cos(\omega z)\cos(\omega x) + f(z)\sin(\omega z)\sin(\omega x)\Big)dz\,d\omega$$

$$= \frac{1}{\pi}\int_0^\infty\int_{-\infty}^{\infty} f(z)\big(\underline{\cos(\omega z)\cos(\omega x) + \sin(\omega z)\sin(\omega x)}\big)\,dz\,d\omega$$

$(\cos\theta = \cos(-\theta))$ $\quad \cos(\omega z - \omega x) = \cos(\omega(x-z))$

$$F(x) = \frac{1}{\pi}\int_0^\infty\underbrace{\int_{-\infty}^{\infty} f(z)\cos(\omega(x-z))\,dz}_{\text{an even function of } \omega}\,d\omega$$

$\int_{-\infty}^{\infty} g(\omega)\,d\omega = 2\int_0^\infty g(\omega)\,d\omega$ if g is even.

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(z)\cos(\omega(x-z))\,dz\,d\omega \,:$$

$$i\frac{1}{2\pi}\int_{-\infty}^{\infty}\underbrace{\int_{-\infty}^{\infty} f(z)\sin(\omega(x-z))\,dz}_{\text{odd}}\,d\omega = 0$$

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(z)\cos(\omega(x-z))\,dz\,d\omega + i\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(z)\sin(\omega(x-z))\,dz\,d\omega$$

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(z)\big(\cos(\omega(x-z)) + \cancel{i\sin}\; + i\sin(\omega(x-z))\big)\,dz\,d\omega$$

$e^{i\omega(x-z)} = \cos(\omega(x-z)) + i\sin(\omega(x-z))$ Euler formula : $\frac{1}{2\pi} = \left(\frac{1}{\sqrt{2\pi}}\right)^2$

$$F(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(z)e^{i\omega x}\underline{e^{-i\omega z}}\,dz\,d\omega$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(z)e^{-i\omega z}\,dz\right)e^{i\omega x}\,d\omega$$

$$= \hat{f}(\omega) = \mathcal{F}(f)$$

Def 1. $\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(z)e^{-i\omega z}\,dz$

the Fourier transform of f.

2. $\mathcal{F}^{-1}(\hat{f})_{(x)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w)\, e^{iwx}\, dw$

: the inverse Fourier transform of $\hat{f}(w)$

(Ex) $f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| \geq a \end{cases} \quad (a > 0)$

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(z)\, e^{-iwz}\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{-iwz}\, dz = \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-iwz}}{-iw} \right]_{-a}^{a}$$

$$= \frac{-1}{iw\sqrt{2\pi}} \left(e^{-iwa} - e^{iwa} \right)$$

$$\hat{f}(w) = \frac{-1}{iw\sqrt{2\pi}} \Big(\cancel{\cos(wa)} - i\sin(wa) - (\cancel{\cos(wa)} + i\sin(wa)) \Big)$$

$$= \frac{+1}{\cancel{i}w\sqrt{2\pi}} (+2\cancel{i}) \sin(wa) = \frac{2\sin(wa)}{w\sqrt{2\pi}}$$

Q $f(x) \equiv 1$.

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iwx}\, dx = \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-iwx}}{-iw} \right]_{-\infty}^{\infty}$$

$$= \frac{1}{-iw\sqrt{2\pi}} \left(\lim_{x\to\infty} e^{-iwx} - \lim_{x\to-\infty} e^{-iwx} \right)$$

$e^{-iwx} = \cos(wx) - i\sin(wx)$: No limit

$\hat{f}(w) = \mathcal{F}(1)$ doesn't exist.

: not defined

Q Which functions have Fourier transform.

Thm 1

Ⓗ $\left(\begin{array}{l} f(x) \text{ is piecewise continuous on every finite interval.} \\ \int_{-\infty}^{\infty} |f(x)|\, dx \text{ is finite (: } f \text{ is integrable)} \end{array} \right.$

Ⓒ $\mathcal{F}(f)$ exists.

(Ex) $f(x) = \begin{cases} 0, & x < 0 \\ e^{-ax}, & x \geq 0 \end{cases} \quad (a > 0)$

(1) $\int_{-\infty}^{\infty} |f(x)|\, dx = \int_0^{\infty} e^{-ax}\, dx = \left[\frac{e^{-ax}}{-a} \right]_0^{\infty}$

$= \frac{1}{-a} \Big(\underbrace{\lim_{x\to\infty} e^{-ax}}_{=0} - 1 \Big) = \frac{1}{a}$ is finite

(2) $\mathcal{F}(f) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx$

$= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-ax} e^{-i\omega x}\, dx = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-(a+i\omega)x}\, dx$

$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-(a+i\omega)x}}{-(a+i\omega)} \right]_0^{\infty}$

$\mathcal{F}(f) = \frac{1}{-(a+i\omega)\sqrt{2\pi}} \Big(\underbrace{\lim_{x\to\infty} e^{-ax} e^{-i\omega x}}_{=0} - 1 \Big)$

Remark: $|e^{-i\omega x}| = |\cos(\omega x) - i \sin(\omega x)| = 1.$

$\mathcal{F}(f) = \frac{1}{(a+i\omega)\sqrt{2\pi}}.$

(Ex) $\underline{\mathcal{F}(e^{-x^2}) = \frac{1}{\sqrt{2}} e^{-\frac{\omega^2}{4}}}.$ ✓ (MA525)

(Properties).

1. $\mathcal{F}(af + bg) = a\,\mathcal{F}(f) + b\,\mathcal{F}(g)$

2. $\mathcal{F}(f') = i\omega\, \mathcal{F}(f).$

   $\mathcal{F}(f'') = (i\omega)^2 \mathcal{F}(f) = -\omega^2 \mathcal{F}(f).$

3. $f * g\,(x) = \int_{-\infty}^{\infty} f(z)\, g(x-z)\, dz$

   $\mathcal{F}(f * g) = \sqrt{2\pi}\, \mathcal{F}(f) \cdot \mathcal{F}(g)$

12.1. Partial Differential equations (PDE).

Def A PDE = an equation involving one or more partial derivatives

The order of a PDE = the order of the highest order derivatives of the PDE.

(Ex) 1. 1st order PDE.

(1) $\frac{\partial u}{\partial t} + a(x,t)\frac{\partial u}{\partial x} = 0$: transport equation

(2) $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$: Burger's equation. (nonlinear)

2. 2nd order PDE.

(1) Wave equation: (Sec 12.2)

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0$$

Q $u(x,t) = ?$

A $u(x,t) = \cos(ct)\sin(x)$

$\frac{\partial u}{\partial t} = -c\sin(ct)\sin(x)$, $\frac{\partial u}{\partial x} = \cos(ct)\cos(x)$

$\frac{\partial^2 u}{\partial t^2} = -c^2\cos(ct)\sin(x)$,

$\frac{\partial^2 u}{\partial x^2} = -\cos(ct)\sin(x)$

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = -c^2 \cos(ct)\sin(x) + c^2(+1)\cos(ct)\sin(x) = 0$$

(2) $\frac{\partial u}{\partial t} - k\,u_{xx} = 0$: (1-dim) heat equation

$\frac{\partial u}{\partial t} - \nabla\cdot(k\nabla u) = 0$: d-dim "

$\boxed{u_{xx} = \frac{\partial^2 u}{\partial x^2}}$

Q $u(x,t) = ?$

Ⓐ $\underline{u(x,t) = e^{-kt}\sin(x)}$

$$\frac{\partial u}{\partial t} = -k e^{-kt}\sin(x)$$

$$\frac{\partial^2 u}{\partial x^2} = e^{-kt}(-1)\sin(x)$$

$$\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = -k e^{-kt}\sin(x) + k e^{-kt}(+1)\sin(x)$$

$= 0$

(3) $\Delta u = \nabla^2 u = \underline{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}$

: Laplace equation.

Q $u(x,y) = 0$

Ex $u(x,y) = x^2 - y^2$

$\frac{\partial^2 u}{\partial x^2} = 2$, $\frac{\partial^2 u}{\partial y^2} = -2$: $\nabla^2 u = 2-2 = 0$