ECE 68000: MODERN AUTOMATIC CONTROL

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The Hamilton-Jacobi-Bellman (HJB) Equation

Problem statement

- Plant

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x}, \boldsymbol{u}), \quad \boldsymbol{x}(t_0) = \boldsymbol{x}_0,$$

- Associated performance index to be minimized

$$J(t_0, \boldsymbol{x}(t_0), \boldsymbol{u}) = \Phi(t_f, \boldsymbol{x}(t_f)) + \int_{t_0}^{t_f} F(\tau, \boldsymbol{x}(\tau), \boldsymbol{u}(\tau)) d\tau.$$

- Define

$$J(t, \boldsymbol{x}(t), \boldsymbol{u}(\tau)) = \Phi(t_f, \boldsymbol{x}(t_f)) + \int_{t}^{t_f} F(\tau, \boldsymbol{x}(\tau), \boldsymbol{u}(\tau)) d\tau$$

  with $t \leq \tau \leq t_f$

- Let

$$J^*(t, \boldsymbol{x}(t)) = \min_{\boldsymbol{u}} J(t, \boldsymbol{x}(t), \boldsymbol{u}(\tau))$$

- Subdivide the interval $[t, t_f]$ as

$$[t, t_f] = [t, t + \Delta t] \cup [t + \Delta t, t_f]$$

Use PO

- Write

$$J^*(t, \boldsymbol{x}(t), \boldsymbol{u}(\tau)) = \min_{\boldsymbol{u}} \left\{ \int_t^{t+\Delta t} F d\tau + \int_{t+\Delta t}^{t_f} F d\tau + \Phi(t_f, \boldsymbol{x}(t_f)) \right\}$$

- By the PO the trajectory on the interval $[t + \Delta t, t_f]$ must be optimal
- Hence,

$$J^*(t, \boldsymbol{x}(t)) = \min_{\boldsymbol{u}} \left\{ \int_t^{t+\Delta t} F d\tau + J^*(t + \Delta t, \boldsymbol{x}(t + \Delta t)) \right\}$$

Taylor's expansion

- Expand $J^*(t + \Delta t, \boldsymbol{x}(t + \Delta t))$ into a Taylor series about $(t, \boldsymbol{x}(t))$

$$
\begin{aligned}
J^*(t, \boldsymbol{x}(t)) &= \min_{\boldsymbol{u}} \left\{ \int_t^{t+\Delta t} F d\tau + J^*(t, \boldsymbol{x}(t)) \right. \\
&\quad \left. + \frac{\partial J^*}{\partial t} \Delta t + \frac{\partial J^*}{\partial \boldsymbol{x}} \left(\boldsymbol{x}(t + \Delta t) - \boldsymbol{x}(t) \right) + \text{H.O.T.} \right\}
\end{aligned}
$$

  where H.O.T. stands for higher order terms

- Cancel the terms $J^* = J^*(t, \boldsymbol{x}(t))$ out and use the fact that $\boldsymbol{x}(t + \Delta t) - \boldsymbol{x}(t) \approx \dot{\boldsymbol{x}} \Delta t$ to obtain

$$
0 = \min_{\boldsymbol{u}} \left\{ \int_t^{t+\Delta t} F d\tau + \frac{\partial J^*}{\partial t} \Delta t + \frac{\partial J^*}{\partial \boldsymbol{x}} \dot{\boldsymbol{x}} \Delta t + \text{H.O.T.} \right\}
$$

The Hamilton-Jacobi-Bellman equation

- By assumption Δt is small, hence

$$0 = \min_{\boldsymbol{u}} \left\{ F\Delta t + \frac{\partial J^*}{\partial t}\Delta t + \frac{\partial J^*}{\partial \boldsymbol{x}}\boldsymbol{f}\Delta t + \text{H.O.T.} \right\}$$

- Divide by Δt and letting $\Delta t \to 0$ yields

$$0 = \frac{\partial J^*}{\partial t} + \min_{\boldsymbol{u}} \left\{ F + \frac{\partial J^*}{\partial \boldsymbol{x}}\boldsymbol{f} \right\}$$

- Let $H = F + \frac{\partial J^*}{\partial \boldsymbol{x}}\boldsymbol{f}$ denote the Hamiltonian
- Then, we obtain

$$\boxed{0 = \frac{\partial J^*}{\partial t} + \min_{\boldsymbol{u}} H}$$

subject to the boundary condition

$$J^*(t_f, \boldsymbol{x}(t_f)) = \Phi(t_f, \boldsymbol{x}(t_f))$$

The HJB equation is a PDE

- The partial differential equation for the optimal cost $J^*(t, \boldsymbol{x}(t))$ is called the Hamilton-Jacobi-Bellman (HJB) equation

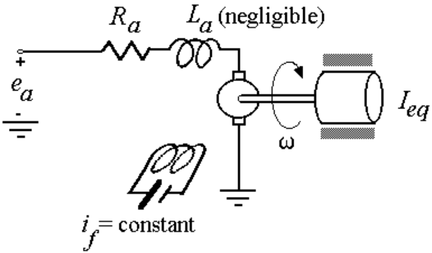
$$\boxed{0 = \frac{\partial J^*}{\partial t} + \min_{\boldsymbol{u}} H}$$

subject to the boundary condition

$$J^*(t_f, \boldsymbol{x}(t_f)) = \Phi(t_f, \boldsymbol{x}(t_f))$$

- It provides the solution to the optimal control problem for general nonlinear dynamical systems
- Analytical solution to the HJB equation is difficult to obtain in most cases

Example: an armature controlled DC motor



Use the HJB equation to construct optimal controller for a DC motor

- An armature controlled DC motor, where the system parameters are: $R_a = 2\,\Omega$, $L_a \approx 0\,\mathrm{H}$, $K_b = 2\,\mathrm{V/rad/sec}$, $K_i = 2\,\mathrm{Nm/A}$, and the equivalent moment of inertia referred to the motor shaft is $I_{eq} = 1\,\mathrm{kg{\cdot}m^2}$
- The friction is assumed to be negligible
- Construct a mathematical of the DC motor

Control problem statement

- Use the Hamilton-Jacobi-Bellman equation to construct the optimal state feedback controller, $e_a = k(t)\omega$, that minimizes the performance index

$$J = \frac{1}{2}\omega(10)^2 + \int_0^{10} R_a i_a(t)^2 dt,$$

  where i_a is the armature current
- The final state is free
- There are no constraints on e_a
- Assume $J^* = \frac{1}{2}p(t)\omega^2$

DC motor modeling

- Apply Kirchhoff's voltage law to the armature circuit

$$R_a i_a + K_b \omega = e_a$$

- The torque developed by the motor, T_m

$$I_{eq}\dot{\omega} = T_m = K_i i_a$$

- Substitute i_a and divide both sides by I_{eq}

$$\dot{\omega} = -\frac{K_i K_b}{R_a I_{eq}}\omega + \frac{K_i}{I_{eq} R_a} e_a$$

- Substitute the parameter values

$$\dot{\omega} = -2\omega + e_a$$

- Represent the performance index, J, in terms of ω and e_a
- Apply Ohm's law to the armature circuit

$$e_a - K_b \omega = R_a i_a$$

Performance index

- Penalty functional

$$J = \frac{1}{2}\omega(10)^2 + \int_0^{10} \frac{(e_a - K_b\omega)^2}{R_a} dt$$

- The Hamiltonian function

$$\begin{aligned} H &= \frac{(e_a - K_b\omega)^2}{R_a} + \frac{\partial J^*}{\partial \omega}\left(-\frac{K_i K_b}{R_a I_{eq}}\omega + \frac{K_i}{I_{eq} R_a} e_a\right) \\ &= \frac{(e_a - 2\omega)^2}{2} + \frac{\partial J^*}{\partial \omega}(-2\omega + e_a) \end{aligned}$$

Applying the HJB equation

- Since there are no constraints on e_a, can find the optimal control by solving

$$0 = \frac{\partial H}{\partial e_a} = \frac{2\left(e_a - K_b\omega\right)}{R_a} + \frac{K_i}{I_{eq}R_a}\frac{\partial J^*}{\partial \omega}$$

- Hence,

$$e_a^* = 2\omega - \frac{\partial J^*}{\partial \omega}$$

- Substituting gives

$$0 = \frac{\partial J^*}{\partial t} + \frac{1}{2}\left(-\frac{\partial J^*}{\partial \omega}\right)^2 + \frac{\partial J^*}{\partial \omega}\left(-\frac{\partial J^*}{\partial \omega}\right) = \frac{\partial J^*}{\partial t} - \frac{1}{2}\left(\frac{\partial J^*}{\partial \omega}\right)^2$$

Preparations to solve the HJB equation

- By assumption $J^* = \frac{1}{2}p(t)\omega^2$
- Hence,
$$\frac{\partial J^*}{\partial t} = \frac{1}{2}\dot{p}\omega^2 \quad \text{and} \quad \frac{\partial J^*}{\partial \omega} = p\omega$$
- Substituting the above into the HJB equation, we get
$$\frac{1}{2}\left(\dot{p} - p^2\right)\omega^2 = 0$$
- Solve the nonlinear differential equation
$$\dot{p} - p^2 = 0$$
- Assume $p = \rho\frac{\dot{w}}{w}$

Solving the HJB equation

- Therefore,
$$\dot{p} = \rho \frac{\ddot{w}w - \dot{w}^2}{w^2}$$
- Substitute the expressions for p and $\dot{p}$
$$\frac{\rho \ddot{w} w - \rho \dot{w}^2 - \rho^2 \dot{w}^2}{w^2} = 0$$
- Select $\rho = -1$ to eliminate the nonlinear terms in the numerator
$$\ddot{w} = 0,$$
whose solution is
$$w = w(t) = C_1 t + C_2,$$
where C_1 and C_2 are integration constants
- Hence,
$$p = -\frac{\dot{w}}{w} = -\frac{C_1}{C_1 t + C_2}$$

Using the boundary condition

- Since
$$J^*(10) = \frac{1}{2}\omega(10)^2 = \frac{1}{2}p(10)\omega(10)^2,$$
conclude that $p(10) = 1$
- Use this information to eliminate one of the integration constants
- We obtain
$$p(10) = 1 = -\frac{C_1}{10C_1 + C_2}$$
- Hence,
$$C_2 = -11C_1,$$
and
$$p = p(t) = -\frac{C_1}{C_1 t + C_2} = -\frac{C_1}{C_1 t - 11C_1} = -\frac{1}{t-11}$$

The solution

- Therefore,

$$\boxed{e_a^* = 2\omega - \frac{\partial J^*}{\partial \omega} = 2\omega - p\omega = \left(2 + \frac{1}{t - 11}\right)\omega}$$

- In our example, the final time t_f fixed and $t_f < \infty$
- Can solve the linear quadratic regulator design for $t_f = \infty$ using the HJB equation