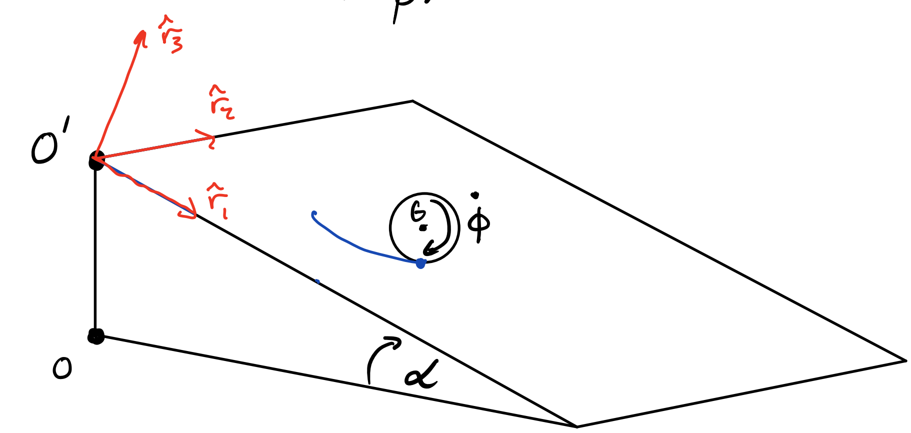
Ex. Penny rolling and spinning without slipping on a ramp.



Degrees of freedom and coordinates:

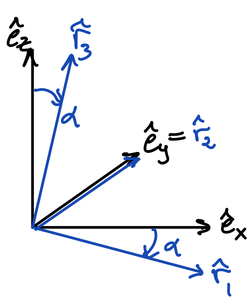
$$M = 6(1) - \overset{2}{\cancel{K_H}} - \overset{2}{\cancel{K_S}} \longrightarrow \begin{aligned} \dot{x} + a\dot{\phi}\sin\theta &= 0 \\ \dot{y} - a\dot{\phi}\cos\theta &= 0 \end{aligned}$$

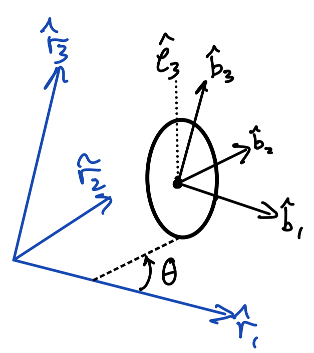
$$= 6-2-2=2$$

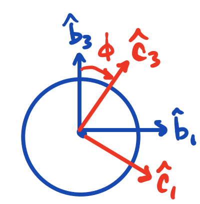
$$N_c = 6 - K_H = 6-2 = 4$$

N_c : x, y, ϕ, θ

Reference frames:







	$\hat{r}_1$	$\hat{r}_2$	$\hat{r}_3$
$\hat{e}_x$	$c\alpha$	0	$s\alpha$
$\hat{e}_y$	0	1	0
$\hat{e}_z$	$-s\alpha$	0	$c\alpha$

	$\hat{b}_1$	$\hat{b}_2$	$\hat{b}_3$
$\hat{r}_1$	$s\theta$	$c\theta$	0
$\hat{r}_2$	$-c\theta$	$s\theta$	0
$\hat{r}_3$	0	0	1

$${}^{I}\vec{\omega}^{B} = \dot{\theta}\hat{b}_3$$

	$\hat{c}_1$	$\hat{c}_2$	$\hat{c}_3$
$\hat{b}_1$	$c\phi$	0	$s\phi$
$\hat{b}_2$	0	1	0
$\hat{b}_3$	$-s\phi$	0	$c\phi$

$${}^{B}\vec{\omega}^{C} = \dot{\phi}\hat{b}_2 = \dot{\phi}\hat{c}_2$$

Kinematics:

$$\vec{r}_{G/O} = \vec{r}_{O'/O} + \vec{r}_{P/O'} + \vec{r}_{G/P}$$

$$= \vec{r}_{O'/O} + x\hat{r}_1 + y\hat{r}_2 + \vec{r}_{G/P}$$

$${}^{I}\vec{v}_{G/O} = \dot{x}\hat{r}_1 + \dot{y}\hat{r}_2$$

Kinetic energy

$$T_O = T_{G/O} + T_G$$

$$= \frac{m}{2}\|{}^{I}\vec{v}_{G/O}\|^2 + \frac{1}{2}{}^{I}\vec{\omega}^{C}\cdot\left(\mathbb{I}_G \cdot {}^{I}\vec{\omega}^{C}\right)$$

$$[\mathbb{I}_G]_C = \frac{m}{2}a^2\begin{bmatrix}\frac{1}{2} & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \frac{1}{2}\end{bmatrix}_C$$

(See page 660 of K+P, and re-label axes)

$${}^{I}\vec{\omega}^{C} = {}^{I}\vec{\omega}^{B} + {}^{B}\vec{\omega}^{C} = \dot{\theta}\hat{b}_3 + \dot{\phi}\hat{c}_2$$

$$= \dot{\theta}\left(-s\phi\hat{c}_1 + c\phi\hat{c}_3\right) + \dot{\phi}\hat{c}_2$$

$$= -\dot{\theta}s\phi\hat{c}_1 + \dot{\phi}\hat{c}_2 + \dot{\theta}c\phi\hat{c}_3$$

$$T_G = \frac{1}{2}\left[{}^{I}\vec{\omega}^{C}\right]_C^T[\mathbb{I}_G]_C\left[{}^{I}\vec{\omega}^{C}\right]_C$$

$$= \frac{m}{4}a^2\begin{bmatrix}-\dot{\theta}s\phi & \dot{\phi} & \dot{\theta}c\phi\end{bmatrix}\begin{bmatrix}\frac{1}{2} & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \frac{1}{2}\end{bmatrix}_C\begin{bmatrix}-\dot{\theta}s\phi\\ \dot{\phi}\\ \dot{\theta}c\phi\end{bmatrix}_C$$

$$= \frac{m}{4}a^2\left(\frac{\dot{\theta}^2}{2}s^2\phi + \dot{\phi}^2 + \frac{\dot{\theta}^2}{2}c^2\phi\right)$$

$$= \frac{m}{8}a^2\dot{\theta}^2 + \frac{m}{4}a^2\dot{\phi}^2$$

Overall,

$$T_0 = \frac{m}{2}(\dot{x}^2+\dot{y}^2) + \left(\frac{m}{4}a^2\dot{\phi}^2 + \frac{m}{8}a^2\dot{\theta}^2\right)$$

Potential energy:

$$U_{G/O}^{(\vec{F}_g)} = -\int \vec{F}_g \cdot {}^{I}d\vec{r}_{G/O}$$

$$= -\int -mg(-s\alpha\,\hat{r}_1 + c\alpha\,\hat{r}_2)\cdot(dx\,\hat{r}_1 + dy\,\hat{r}_2)$$

$$= -mgx\sin\alpha$$

Lagrangian $\quad L = T_0 - U_0$

$$L = \frac{m}{2}(\dot{x}^2+\dot{y}^2) + \frac{ma^2}{4}\left(\dot{\phi}^2 + \frac{1}{2}\dot{\theta}^2\right) + mgx\sin\alpha$$

Generalized forces due to constraints:

$f_1: \dot{x} + a\dot{\phi}s\theta = 0 \qquad a_{1x} = 1 \qquad a_{1y} = 0 \qquad a_{1\phi} = +as\theta \quad a_{1\theta} = 0$

$f_2: \dot{y} - a\dot{\phi}c\theta = 0 \qquad a_{2x} = 0 \qquad a_{2y} = 1 \quad a_{2\phi} = -ac\theta \quad a_{2\theta} = 0$

$$Q_j^{(\text{nonhol})} = \sum_{i=1}^{K_s} \mu_i a_{ij}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j^{(\text{nonhol})}$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} \qquad \frac{\partial L}{\partial x} = mg\sin\alpha$$

$$\frac{\partial L}{\partial \dot{y}} = m\dot{y} \qquad \frac{\partial L}{\partial y} = 0$$

$$\frac{\partial L}{\partial \dot{\phi}} = \frac{ma^2}{2}\dot{\phi} \qquad \frac{\partial L}{\partial \phi} = 0$$

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{ma^2}{4}\dot{\theta} \qquad \frac{\partial L}{\partial \theta} = 0$$

x: $\quad m\ddot{x} - mg\sin\alpha = \mu_1$

y: $\quad m\ddot{y} = \mu_2$

ϕ: $\quad \frac{m a^{\cancel{2}}}{2}\ddot{\phi} = \cancel{a} s\theta \mu_1 - \cancel{a}\, c\theta \mu_2$

θ: $\quad \frac{ma^2}{4}\ddot{\theta} = 0$

unknown's?

$\left.\ddot{x}, \ddot{y}, \ddot{\phi}, \ddot{\theta}, \mu_1, \mu_2\right\}$ 6 unknowns

4 EL equations

2 nonholonomic constraints

An algebraically easier approach is to plug in the nonholonomic constraints into the Lagrangian before solving for the EL equations.

Then

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + \frac{ma^2}{4}\left(\dot{\phi}^2 + \frac{1}{2}\dot{\theta}^2\right) + mgx\sin\alpha$$

$\dot{x} = -a\dot{\phi}s\theta$ $\qquad$ $\dot{y} = a\dot{\phi}c\theta$

$$L = \frac{ma^2}{2}\dot{\phi}^2 s^2\theta + \frac{ma^2}{2}\dot{\phi}^2 c^2\theta + \frac{ma^2}{4}\left(\dot{\phi}^2 + \frac{1}{2}\dot{\theta}^2\right) + mgx\sin\alpha$$

$$L = \frac{3ma^2}{4}\dot{\phi}^2 + \frac{ma^2}{8}\dot{\theta}^2 + mgx\sin\alpha$$

$$\frac{\partial L}{\partial \dot{\phi}} = \frac{6}{4}ma^2\dot{\phi} \qquad \frac{\partial L}{\partial \phi} = 0$$

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{ma^2}{4}\dot{\theta} \qquad \frac{\partial L}{\partial \theta} = 0$$

$$\frac{\partial L}{\partial \dot{x}} = 0 \qquad \frac{\partial L}{\partial x} = mg\sin\alpha$$

$$\frac{\partial L}{\partial \dot{y}} = 0 \qquad \frac{\partial L}{\partial y} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_j}\right) - \frac{\partial L}{\partial q_j} = Q_j$$

E.O.M.'s:

ϕ: $\quad \frac{3}{2} m a^{\cancel{2}} \ddot{\phi} = +\cancel{a}\, s\theta\, \mu_1 - \cancel{a}\, c\theta\, \mu_2$

θ: $\quad \frac{m a^2}{4} \ddot{\theta} = 0$

x: $\quad -mg\sin\alpha = \mu_1$

y: $\quad 0 = \mu_2$

$$\boxed{\begin{aligned} \mu_1 &= -mg\sin\alpha \\ \ddot{\phi} &= \frac{2}{3ma}\sin\theta\,\mu_1 \\ \ddot{\theta} &= 0 \end{aligned}}$$

"Jumping Around" Roadmap

Topics from Ch 8 Goldstein — Cyclic Variables & Conservation

Ch 4 & 5 Goldstein — Rigid body motion

Murray, Li, Sastry "Mathematical Intro to Robotics" Ch 1 & 2 — Geometric approach to rigid body motion

Ch 8 Goldstein / H+F 5 — Hamilton's Equations

Ch 9 Goldstein / H+F 6 — Canonical Transformation

Conservation Laws and Symmetry (Ch2, mostly Ch8 Goldstein)

Consider the Lagrangian for a point mass in a potential field

$$L = \frac{1}{2} m \dot{x}^2 - U(x)$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} = \text{linear momentum in the } x \text{ direction} \triangleq \vec{p}_{P/O} \cdot \hat{e}_x$$

Def'n: A generalized momentum associated with q_j is given by

$$\boxed{p_j \triangleq \frac{\partial L}{\partial \dot{q}_j}}$$

This is also called a canonical or conjugate momentum.

Note: p_j does not always correspond to your intuitive expectation of momentum.

Def'n: A "cyclic coordinate" (or "ignorable" coordinate) q_j does not appear in the Lagrangian. (i.e. $\frac{\partial L}{\partial q_j} = 0$) and it should be independent of the other coordinates.

For the special case of no generalized forces present,

$$\frac{d}{dt}\Big(\underbrace{\boxed{\frac{\partial L}{\partial \dot{q}_j}}}_{\text{conjugate momentum}}\Big) - \underbrace{\frac{\partial L}{\partial q_j}}_{0} = 0 \Rightarrow \frac{d}{dt}\left(p_j\right) = 0$$

$\Rightarrow p_j$ is a conserved quantity!

<u>Theorem</u>: When no generalized forces are present, the conjugate momentum of a cyclic coordinate is conserved.

(Note: the cordinate needs to be independent of the other coordinates)

The conservation theorem is closely related to symmetries in the system, which is formalized by Noether's theorem:

<u>Noether's Theorem</u>: If a system is invariant under displacement or rotation of a particular cordinate, then the corresponding conjugate momentum is conserved.

Hamiltonian Function and Conservation

Consider $L = L(q_1, \dots, q_{N_c}, \dot{q}_1, \dots, \dot{q}_{N_c}, t)$

$$\frac{d}{dt}L = \sum_{j=1}^{N_c}\left(\frac{\partial L}{\partial q_j}\dot{q}_j + \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j\right) + \frac{\partial L}{\partial t}$$

- If there are no generalized forces present,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0 \Rightarrow \frac{\partial L}{\partial q_j} = \boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)}$$

Substituting,

$$\frac{dL}{dt} = \sum_{j=1}^{N_c}\underbrace{\left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\dot{q}_j + \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j\right)}_{\text{Looks like a product rule}} + \frac{\partial L}{\partial t}$$

$$\frac{dL}{dt} = \sum_{j=1}^{N_c}\frac{d}{dt}\left(\dot{q}_j\frac{\partial L}{\partial \dot{q}_j}\right) + \frac{\partial L}{\partial t}$$

Bring this to the RHS

$$0 = \frac{d}{dt}\Bigg(\underbrace{\sum_{j=1}^{N_c} \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L}_{\triangleq H}\Bigg) + \frac{\partial L}{\partial t}$$

called the Hamiltonian

Then,

$$\boxed{\frac{dH}{dt} = -\frac{\partial L}{\partial t}}$$

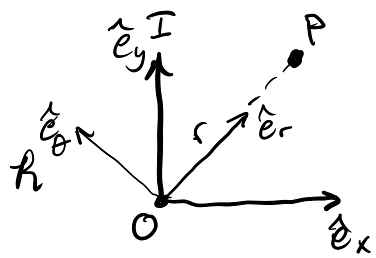
Note: H is conserved if $\frac{\partial L}{\partial t} = 0$

The Hamiltonian is an energy-like quantity that is often (but not always) equal to the total energy of the system

(Recall $E = T_0 + U_0$)

Ex. Simple Satellite

Can we find some conserved quantities?



$$\vec{r}_{P/O} = r\hat{e}_r$$

$$^I\vec{v}_{P/O} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$$

$$T_{P/O} = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2)$$

$$U_{P/O} = -\frac{G m_O m_P}{r}$$

$$L = T_O - U_O = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{G m_O m_P}{r}$$

Note: θ is cyclic and $\frac{\partial L}{\partial t} = 0$

$$P_\theta = \frac{\partial L}{\partial \dot{\theta}} = m r^2 \dot{\theta}$$ is conserved "Integral of motion"

This conservation also establishes Kepler's 2nd law of planetary motion.

↳ The radius vector sweeps out equal areas in equal times

$$dA = \frac{1}{2} r (r\,d\theta) \qquad \frac{dA}{dt} = \frac{r^2}{2}\frac{d\theta}{dt} = \frac{r^2\dot{\theta}}{2} = \underbrace{\frac{P_\theta}{2m}}_{\text{constant}}$$

Another integral of motion is $\left(\text{Since } \frac{\partial L}{\partial t} = 0\right)$

(Legendre Transformation)

$$H = \sum_{j=1}^{2} \dot{q}_j \underbrace{\frac{\partial L}{\partial \dot{q}_j}}_{p_j} - L \qquad H = \sum_{j=1}^{N_c} \dot{q}_j p_j - L, \quad H = \dot{q}^T p - L$$

$$= \dot{r}(m\dot{r}) + \dot{\theta}(mr^2\dot{\theta}) - \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{G m_0 m_p}{r}$$

is conserved

We can compare to E_0:

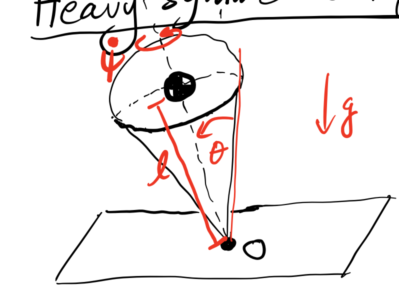
$E_0 = T_0 + U_0$ (The expression would match in this case)

$\Longrightarrow E_0$ is conserved.

Re-visit Rigid Body Motion in Ch 4+5 Goldstein

"Heavy symmetric Top"

→ Assume point O is fixed in I
→ Assume $I_1 = I_2$ symmetry

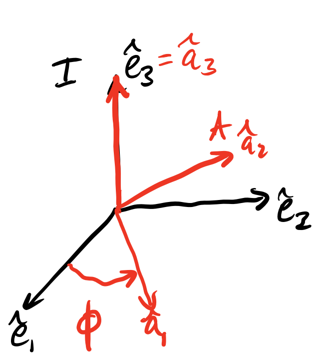


DOF

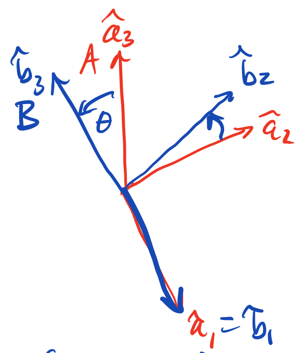
$$M = 6(1) - \cancel{K_H}^{\,3} - \cancel{K_S}^{\,0} = N_c = 3$$

Setup Frames

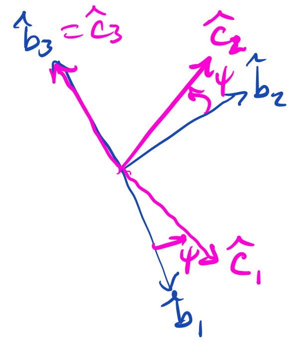
3-1-3 Euler angles for the coordinates (ϕ, θ, ψ)



(locates top)



(tilts top)



(spins top)

C is fixed body frame

$${}^{I}\vec{\omega}^{C} = \dot\phi\,\hat a_3 + \dot\theta\,\hat b_1 + \dot\psi\,\hat c_3$$

$$= \dot\phi\,(c\theta\,\hat b_3 + s\theta\,\hat b_2) + \dot\theta\,\hat b_1 + \dot\psi\,\hat b_3$$

$$= \dot\theta\,\hat b_1 + \dot\phi\, s\theta\,\hat b_2 + (\dot\phi\, c\theta + \dot\psi)\,\hat b_3 \triangleq \omega_1 \hat b_1 + \omega_2 \hat b_2 + \omega_3 \hat b_3$$

$$T_O = T_{G/O} + T_G$$

$$= \frac{1}{2} m \,\|{}^{I}\vec{v}_{G/O}\|^2 + \frac{1}{2}\,{}^{I}\vec{\omega}^{C} \cdot \mathbb{I}_G \cdot {}^{I}\vec{\omega}^{C}$$

For simplicity, we can combine the translational and rotational contributions using the parallel axis theorem (See K+P)

$$T_O = \frac{1}{2}\,{}^{I}\vec{\omega}^{C} \cdot \mathbb{I}_O \circ {}^{I}\vec{\omega}^{C}$$

$${}^{B}C^{C}\,[\mathbb{I}_O]_C\,({}^{B}C^{C})^T = [\mathbb{I}_O]_B$$

Let

$$[\mathbb{I}_O]_C = [\mathbb{I}_O]_B = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_1 & 0 \\ 0 & 0 & I_3 \end{bmatrix}_B$$

(Just a trick to make the algebra easier)

$$T_O = \frac{I_1}{2}(\dot\theta^2 + \dot\phi^2 s^2\theta) + \frac{I_3}{2}(\dot\psi + \dot\phi c\theta)^2$$

$$U_O = mglc\theta$$

$$L = T_O - U_O = \frac{I_1}{2}(\dot\theta^2 + \dot\phi^2 s^2\theta) + \frac{I_3}{2}(\dot\psi + \dot\phi c\theta)^2 - mglc\theta$$

Conservation? (ϕ, θ, ψ) and $\frac{\partial L}{\partial t} = 0$

cyclic variables

$\Rightarrow$ P_ϕ and P_ψ are conserved and H is conserved.

$$\left.\begin{aligned} P_\psi &\triangleq \frac{\partial L}{\partial \dot\psi} = I_3(\dot\psi + \dot\phi c\theta) = I_3\omega_3 \\ P_\phi &\triangleq \frac{\partial L}{\partial \dot\phi} = (I_1 s^2\theta + I_3 c^2\theta)\dot\phi + I_3\dot\psi c\theta \end{aligned}\right\} \text{constants}$$

Lastly,

$$H = \frac{I_1}{2}(\dot\theta^2 + \dot\phi^2 s^2\theta) + \frac{I_3}{2}(\dot\psi + \dot\phi c\theta)^2 + mglc\theta$$

$H = E_O$, so E_O is conserved.