ECE 602: LUMPED LINEAR SYSTEMS

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Stability of Continuous-Time LTI Systems

Stability of CT Autonomous Linear Systems

Continuous-time linear system has an equilibrium point at $x_e = 0$

$$\dot{x}(t) = A(t)x(t) \tag{1}$$

Definition (Asymptotic Stability)

System (1) is called asymptotically stable at $x_e = 0$ if its solution $x(t)$ starting from any initial condition $x(0) \in \mathbb{R}^n$ satisfies $x(t) \to 0$ as $t \to \infty$

Definition (Exponential Stability)

System (1) is called exponentially stable at $x_e = 0$ if its solution $x(t)$ starting from any initial condition $x(0) \in \mathbb{R}^n$ satisfies

$$\|x(t)\| \leq Ke^{-rt}\|x(0)\|, \quad \forall t \geq 0,$$

for some constants $K, r > 0$.

- Exponential stability $\Rightarrow$ asymptotic stability

Internal Stability vs Input/Output Stability

For CT LTI system $\dot{x} = Ax + Bu$, $y = Cx + Du$, its transfer function

$$H(s) = C(sI - A)^{-1}B + D$$

Input/Output Stability: poles of $H(s)$ all have negative real parts

- Assume $x(0) = 0$, and input $u(\cdot)$ is arbitrary
- Equivalently, bounded input $u(\cdot)$ results in bounded output $y(\cdot)$ (BIBO stability)

Internal Stability: stability notion to be studied in this lecture

- Assume $u(\cdot) \equiv 0$, and $x(0)$ is arbitrary
- Focus on state solution $x(\cdot)$ rather than output $y(\cdot)$
- A stronger notion of stability (more on this later)

Characterizing Stability of CT LTI Systems

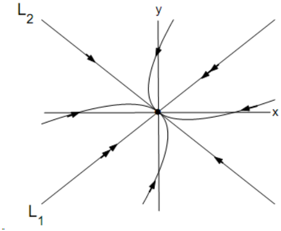
Theorem

For CT LTI system $\dot{x} = Ax$, the following statements are equivalent

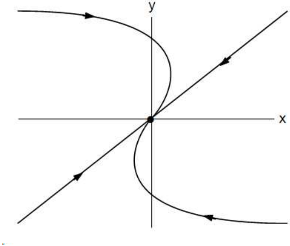
1. *System is asymptotically stable*
2. *System is exponentially stable*
3. *All eigenvalues of A are in the open left half of the complex plane $\mathbb{C}$*

Proof: Solution $x(t) = e^{At}x(0)$ is a linear combination of modes whose entries are $p(t)e^{\lambda_i t}$, where λ_i is an eigenvalue of A and $p(t)$ is a polynomial of t. ∎

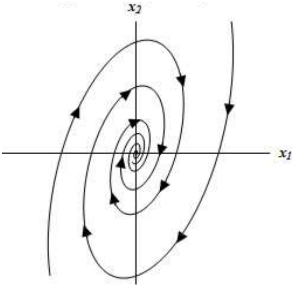
Phase Portraits of Stable 2D LTI Systems



Stable node



Stable degenerate node



Stable focus

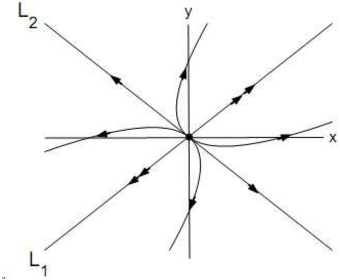
Unstable Systems

LTI system $\dot{x} = Ax$ is unstable if $\|x(t)\|$ is unbounded for **some** $x(0)$

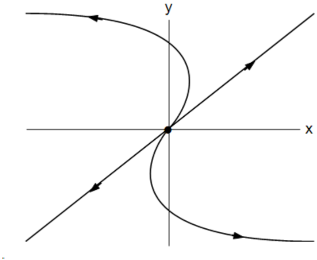
This is the case if **either** of the following holds:

1. A has eigenvalues on the open right half plane of $\mathbb{C}$
2. A has defective eigenvalues on the $j\omega$-axis
   - An eigenvalue is defective if it has a Jordan block of size > 1

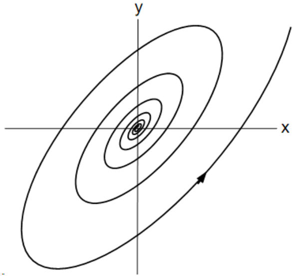
Phase Portraits of Untable 2D Systems



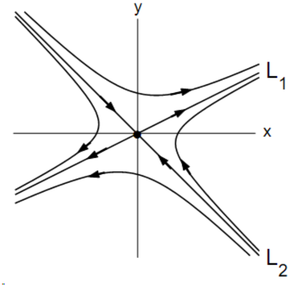
Unstable node



Unstable degenerate node



Unstable focus



Saddle

Marginally Stable Systems

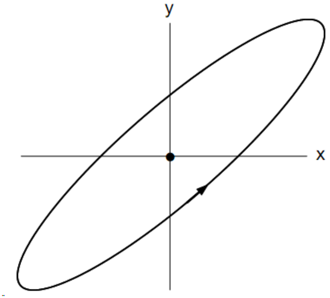
LTI system $\dot{x} = Ax$ is marginally stable if all solutions $x(t)$ are bounded and at least one solution $x(t)$ does not converge to zero

This is the case if **both** of the following hold:

1. A has no eigenvalues on the open right half of $\mathbb{C}$
2. A has eigenvalues on the $j\omega$-axis, all being non-defective

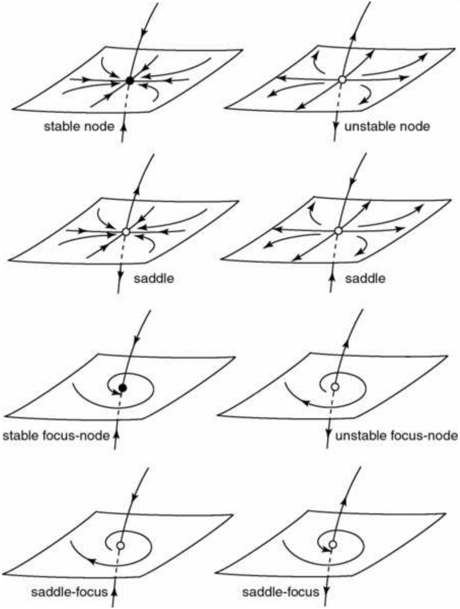
Note: some textbooks classify marginally stable systems as unstable

Phase Portraits of Marginally Stable 2D Systems



$$\dot{x} = Ax \text{ with } A = \underbrace{\begin{bmatrix} v_1 & v_2 \end{bmatrix}}_{T} \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} T^{-1}$$

Phase Portraits of 3D Systems



LTI system $\dot{x} = Ax$, where $A \in \mathbb{R}^{3\times 3}$

- System has three modes
- Each mode could be stable or unstable