ECE 602: LUMPED LINEAR SYSTEMS

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Fundamental Matrices for Two Special Cases of CT LTV Systems

Special Case I: Upper (or Lower) Triangular $A(t)$

If $A(t)$ is upper or lower triangular, we can solve a sequence of scalar ODEs

Example: LTV system: $\dot{x}(t) = \begin{bmatrix} -1 & 1 \\ 0 & -2t \end{bmatrix} x(t)$, i.e., $\begin{cases} \dot{x}_1 = -x_1 + x_2 \\ \dot{x}_2 = -2t\, x_2 \end{cases}$

- We first solve for $x_2(t)$ from the second ODE:
$$x_2(t) = e^{\int_0^t (-2\tau)\, d\tau} x_2(0) = e^{-t^2} x_2(0)$$
- Then the first ODE becomes $\dot{x}_1 = -x_1 + e^{-t^2} x_2(0)$, whose solution is
$$x_1(t) = e^{-t} x_1(0) + \int_0^t e^{-t+\tau-\tau^2}\, d\tau x_2(0)$$
- Therefore, the fundamental matrix is $\Phi(t) = \begin{bmatrix} e^{-t} & \int_0^t e^{-t+\tau-\tau^2}\, d\tau \\ 0 & e^{-t^2} \end{bmatrix}$
- State transition matrix is $\Phi(t,\tau) = \Phi(t)\Phi(\tau)^{-1}$

Special Case II: Commutative $A(t)$

Proposition

If $A(\tau)$ and $A(t)$ commute for all $\tau, t \geq 0$, then

$$\Phi(t) = e^{\int_0^t A(s)\, ds}, \quad t \geq 0.$$

- Thus, the solution to LTV system $\dot{x}(t) = A(t)x(t)$ is

$$x(t) = e^{\int_0^t A(\tau)\, d\tau} x(0), \quad t \geq 0$$

- State transition matrix is $\Phi(t, \tau) = e^{\int_\tau^t A(s)\, ds}$

Example

Consider the LTV system: $\dot{x}(t) = A(t)x(t) = \begin{bmatrix} -e^{-t} & \frac{1}{t+1} \\ 0 & -e^{-t} \end{bmatrix} x(t)$

- Write $A(t) = -e^{-t} \cdot I + N(t)$ where $N(t) = \frac{1}{t+1} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \frac{1}{t+1} J$
- $A(t)$ and $A(\tau)$ commute since $N(t)N(\tau) = N(\tau)N(t) = \mathbf{0}$
- The state transition matrix $\Phi(t, \tau)$ is then given by

$$\Phi(t,\tau) = e^{\int_\tau^t A(s)\, ds} = e^{(e^{-t} - e^{-\tau})I + \ln \frac{t+1}{\tau+1} J} = e^{(e^{-t} - e^{-\tau})} I \cdot e^{\ln \frac{t+1}{\tau+1} J}$$
$$= e^{(e^{-t} - e^{-\tau})} \begin{bmatrix} 1 & \ln \frac{t+1}{\tau+1} \\ 0 & 1 \end{bmatrix}$$

- Fundamental matrix is $\Phi(t) = \Phi(t, 0) = e^{(e^{-t} - 1)} \begin{bmatrix} 1 & \ln(t+1) \\ 0 & 1 \end{bmatrix}$