#1) $\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{pmatrix}3 + x_1x_2\\ -6 + 5x_1x_2\end{pmatrix} + \begin{pmatrix}-1\\ x_2\end{pmatrix}u$ $\qquad u_e = 3$

$\dot{x}_1 = 0 = 3 + x_1x_2 - u_e$

$\dot{x}_2 = 0 = -6 + 5x_1x_2 + x_2u_e$

$3 + x_1x_2 - 3 = 0 \quad \Rightarrow \quad x_1x_2 = 0$

$-6 + 5x_1x_2 + 3x_2 = 0 \Rightarrow -6 + 3x_2 = 0 \quad \Rightarrow x_{2e} = 2, \; x_{1e} = 0$

$\boxed{x_e = \begin{pmatrix}0\\ 2\end{pmatrix} \qquad u_e = 3}$ ← equilibrium pair (x_e, u_e)

$y_e = \cancel{x_{1e}^2}^{\,0} + x_{2e}u_e = 6$

$y_e = 6$

Check: $\dot{x}_1 = 3 + 0 - 3 = 0$

$\dot{x}_2 = -6 + 0 + (3)(2) = 0$ ✓.

#2)

$$\dot{x}_1 = 3 + x_1 x_2 - u \qquad x_{2e} = 2 \qquad x_{1e} = 0$$

$$\dot{x}_2 = -6 + 5x_1 x_2 + x_2 u \qquad u_e = 3$$

$$y = x_1^2 + x_2 u$$

$$\frac{\partial \dot{x}_1}{\partial x_1} = x_2 \Big|_{(x_e, u_e)} = 2 \qquad \frac{\partial \dot{x}_1}{\partial x_2} = x_1 \Big|_{(x_e, u_e)} = 0$$

$$\frac{\partial \dot{x}_1}{\partial u} = -1 \qquad \frac{\partial \dot{x}_2}{\partial x_1} = 5x_2 \Big|_{(x_e, u_e)} = 10$$

$$\frac{\partial \dot{x}_2}{\partial x_2} = 5x_1 + u \Big|_{(x_e, u_e)} = 3 \qquad \frac{\partial \dot{x}_2}{\partial u} = x_2 \Big|_{(x_e, u_e)} = 2$$

$$\frac{\partial y}{\partial x_1} = 2x_1 \Big|_{(x_e, u_e)} = 0 \qquad \frac{\partial y}{\partial x_2} = u \Big|_{(x_e, u_e)} = 3$$

$$\frac{\partial y}{\partial u} = x_2 \Big|_{(x_e, u_e)} = 2$$

$$\boxed{\begin{aligned} \delta\dot{x} &= \underbrace{\begin{pmatrix} 2 & 0 \\ 10 & 3 \end{pmatrix}}_{A} \delta x + \underbrace{\begin{pmatrix} -1 \\ 2 \end{pmatrix}}_{B} \delta u \\ \delta y &= \underbrace{\begin{pmatrix} 0 & 3 \end{pmatrix}}_{C} \delta x + \underbrace{2}_{D} \delta u \end{aligned}}$$

$$\delta x = x - x_e$$

$$\delta u = u - u_e$$

$$\delta y = y - y_e$$

#3) $A = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$ $B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$C = [B \ \ AB] = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ ← rank $= 2 = n$, Controllable

$C^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ $\therefore$ $q_1 = (0 \ \ 1)$

$\alpha_c = (s+1)(s+2) = s^2 + 3s + 2$

$\alpha_c(A) = A^2 + 3A + 2I$

$A^2 = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3 & 4 \end{pmatrix}$

$\alpha_c(A) = \begin{pmatrix} 1 & 0 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 3 & 0 \\ 3 & 6 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 6 & 12 \end{pmatrix}$

$K = q_1 \alpha_c(A) = (0 \ \ 1)\begin{pmatrix} 6 & 6 \\ 6 & 12 \end{pmatrix} = \underbrace{(6 \ \ 12)}_{\text{Gain}}$

$$\boxed{u = -(6 \ \ 12)x + r}$$

#4) $A = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$ $B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $CA = (0\ 1)\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} = (1\ 2)$

$A^T C^T = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$C = (0\ 1)$ $D = 3$

$$\boxed{\dot{\tilde{x}} = (A - LC)\tilde{x} + (B - LD)u + Ly}$$

observer dynamics symbolic

$O = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$ 2 Full rank, observable

(A^T, C^T) is controllable if (A, C) is observable

$\alpha_c(A^T) = (s+3)(s+4) = s^2 + 7s + 12 = (A^T)^2 + 7A^T + 12I$

$(A^T)^2 = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 0 & 4 \end{pmatrix}$

$\alpha_c(A^T) = \begin{pmatrix} 1 & 3 \\ 0 & 4 \end{pmatrix} + \begin{pmatrix} 7 & 7 \\ 0 & 14 \end{pmatrix} + \begin{pmatrix} 12 & 0 \\ 0 & 12 \end{pmatrix} = \begin{pmatrix} 20 & 10 \\ 0 & 30 \end{pmatrix}$

$\mathcal{C} = [C^T\ A^T C^T] = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$

$\mathcal{C}^{-1} = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}$ $q_1 = [1\ 0]$

$L^T = q_1\,\alpha_c(A^T) = [1\ 0]\begin{pmatrix} 20 & 10 \\ 0 & 30 \end{pmatrix} = (20\ 10)$

$$\boxed{L = \begin{pmatrix} 20 \\ 10 \end{pmatrix}}$$

#4) $\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} - \begin{pmatrix} 20 \\ 10 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} - \begin{pmatrix} 0 & 20 \\ 0 & 10 \end{pmatrix}$

$= (A - LC) = \begin{pmatrix} 1 & -20 \\ 1 & -8 \end{pmatrix} = \lambda^2 + 7\lambda + 12$ ✓

$B - LD = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 20 \\ 10 \end{pmatrix} 3 = \begin{pmatrix} -59 \\ -30 \end{pmatrix}$

$$\boxed{\dot{\tilde{x}} = \begin{pmatrix} 1 & -20 \\ 1 & -8 \end{pmatrix} \tilde{x} + \begin{pmatrix} -59 \\ -30 \end{pmatrix} u + \begin{pmatrix} 20 \\ 10 \end{pmatrix} y}$$

#5) $Q = \left(\begin{array}{cc:cc} 1 & 2 & 6 & 0 \\ 0 & 2 & 0 & 6 \\ \hdashline 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{array}\right)$

$$P = \frac{Q + Q^T}{2} = \frac{\left(\begin{array}{cc:cc} 1 & 2 & 6 & 0 \\ 0 & 2 & 0 & 0 \\ \hdashline 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{array}\right) + \left(\begin{array}{cc:cc} 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ \hdashline 6 & 0 & 3 & 0 \\ 0 & 6 & 0 & 4 \end{array}\right)}{2}$$

$$P = \frac{\left(\begin{array}{cc:cc} 2 & 2 & 6 & 0 \\ 2 & 4 & 0 & 6 \\ \hdashline 6 & 0 & 6 & 0 \\ 0 & 6 & 0 & 4 \end{array}\right)}{2} = \left(\begin{array}{cc:cc} 1 & 1 & 3 & 0 \\ 1 & 2 & 0 & 3 \\ \hdashline 3 & 0 & 3 & 0 \\ 0 & 3 & 0 & 2 \end{array}\right) = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

$P \succeq 0$ if $A_{11} \succeq 0$ & $\Delta 11 \succeq 0$

$A_{11} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ $\quad$ $\det(A_{11}) = 2 - 1 = 1 > 0$ $\quad$ $1 > 0$ $\quad \therefore$ $\underline{A_{11} \succeq 0 \text{ as all leading}}$ $\underline{\text{principal minors} > 0}$

$$\Delta 11 = A_{22} - A_{21} A_{11}^{-1} A_{12}$$

$$= \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} - \begin{pmatrix} 6 & -3 \\ -3 & 3 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} - \begin{pmatrix} 18 & -9 \\ -9 & 9 \end{pmatrix}$$

$$= \begin{pmatrix} -15 & 9 \\ 9 & -7 \end{pmatrix}$$

$-\Delta 11 = \begin{pmatrix} 15 & -9 \\ -9 & 7 \end{pmatrix}$ $\quad$ $15 > 0$

$\det(-\Delta 11) = (15)(7) - 81 > 0$

#5) $-A_{11}$ is positive definite via sylvester criteria, $\therefore$ A_{11} is negative definite. A_{11} is negative definite & A_{11} is positive definite, $\therefore$ P is indefinite. P is symmetric form of Q

$\therefore$

$$\boxed{Q \text{ is indefinite}}$$

#6) $J_0 = \int_0^\infty y^2\, dt$ $\quad A = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$ $\quad x(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$y = [\sqrt{2} \quad 0]\, x = Cx \qquad y^2 = y^T y = (Cx)^T Cx = x^T C^T C x$$

$$J_0 = \int_0^\infty x^T \underbrace{C^T C}_{Q} x \, dt = \int_0^\infty x^T \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix} [\sqrt{2} \quad 0]\, x \, dt$$

$$Q = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

$$A^T P + PA = -Q$$

$$\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{pmatrix} + \begin{pmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} -P_{12} & -P_{22} \\ P_{11}-P_{12} & P_{12}-P_{22} \end{pmatrix} + \begin{pmatrix} -P_{12} & P_{11}-P_{12} \\ -P_{22} & P_{12}-P_{22} \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix}$$

$$-2P_{12} = -2 \quad \Rightarrow \quad P_{12} = 1$$
$$-P_{22} + P_{11} - P_{12} = 0 \quad \Rightarrow \quad P_{11} = 2$$
$$2P_{12} - 2P_{22} = 0 \quad \Rightarrow \quad P_{22} = 1$$

$$P = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$J_0 = x^T(0)\, P\, x(0) = (1 \quad -1)\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = (1 \quad 0)\begin{pmatrix} 1 \\ -1 \end{pmatrix} = 1$$

$$\boxed{J_0 = 1}$$

#7) $J_1 = \int_0^{\infty} 4t \, \|x\|^2 \, dt \qquad A = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \qquad x(0) = \begin{pmatrix} 1 \\ 8 \end{pmatrix}$

$\|x\|^2 = x^T x = (Ix)^T (Ix) = x^T I^T I x = x^T \underbrace{I}_{Q} x$

$Q = I_{2\times 2}$

$A^T P + PA = -Q$

$\begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} + \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

$\begin{pmatrix} -p_{11} & -p_{12} \\ -2p_{12} & -2p_{22} \end{pmatrix} + \begin{pmatrix} -p_{11} & -2p_{12} \\ -p_{12} & -2p_{22} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

$-2p_{11} = -1 \Rightarrow p_{11} = \frac{1}{2}$

$-3p_{12} = 0 \Rightarrow p_{12} = 0$

$-4p_{22} = -1 \Rightarrow p_{22} = \frac{1}{4}$

$P = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/4 \end{pmatrix} \qquad P_1 = \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix}$

$A^T P_1 + P_1 A = -P = \begin{pmatrix} -1/2 & 0 \\ 0 & -1/4 \end{pmatrix}$

$-2p_{11} = -\frac{1}{2} \Rightarrow p_{11} = 1/4$

$-3p_{12} = 0 \Rightarrow p_{12} = 0$

$-4p_{22} = -\frac{1}{4} \Rightarrow p_{22} = 1/16$

$P_1 = \begin{pmatrix} 1/4 & 0 \\ 0 & 1/16 \end{pmatrix} \qquad J_1 = x^T(0) P_1 x(0) = \begin{pmatrix} 1 & 8 \end{pmatrix} \begin{pmatrix} 1/4 & 0 \\ 0 & 1/16 \end{pmatrix} \begin{pmatrix} 1 \\ 8 \end{pmatrix}$

$= \begin{pmatrix} \frac{1}{4} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 8 \end{pmatrix} = \frac{17}{4}$

$\boxed{J_1 = \frac{17}{4}}$

#8) $J = \int_0^\infty u^2 dt = \int_0^\infty x^T Q x + u^T R u \, dt$ $\quad x(0) = 1$

$Q = 0$, $R = 1$ $\qquad u = -Kx$

$K = R^{-1} B^T P$

$A = 1$ $\quad B = 2$

CARE: $A^T P + PA + Q - PBR^{-1}B^T P = 0$

$(1)P + (P)(1) + 0 - (P)(2)(1)^{-1}(2)^T P = 0$

$2P - 4P^2 = 0$

$P(2 - 4P) = 0$

$P \neq 0$ (reject)

$P = \frac{1}{2}$

Check: $(1)(\frac{1}{2}) + (\frac{1}{2})(1) + 0 - (\frac{1}{2})(2)(1)(2)(\frac{1}{2}) = 0$

$1 - 1 = 0$ ✓

$K = R^{-1} B^T P = (1)(2)(\frac{1}{2}) = 1$

$\boxed{u = -Kx = -x}$ $\quad K = 1$

$J = x(0)^T P x(0) = (1)(\frac{1}{2})(1)$

$\boxed{J = \frac{1}{2}}$ ← Optimal cost of J