ECE 602: LUMPED LINEAR SYSTEMS

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Input-Output Stability of LTI Systems

Input-Output Stability of LTI Systems

- **Objective**: Investigate the stability properties of the zero-state response of linear time-invariant (LTI) systems
- The response of a linear time-invariant (LTI) system = (the zero-state response) + (the zero-input response)
- We discussed methods that can be used to investigate the stability properties of the zero-input response of linear systems
- Here, we devote ourselves to the stbility of zero-state response
- To proceed, we need a definition of a bounded signal

Definition

A signal $u(t)$ is bounded if there exists a constant $B < \infty$ such that

$$|u(t)| \leq B \quad \text{for all} \quad t \geq 0$$

Bounded-input bounded-output (BIBO) stability of LTI Systems

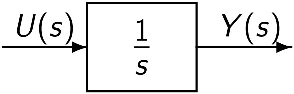
- BIBO stability is concerned with a particular property of the zero-state response of the system

Definition

A system is BIBO stable if every bounded input produces a bounded output

- Example—an integrator

$$\frac{Y(s)}{U(s)} = G(s) = \frac{1}{s}$$



An integrator is not BIBO stable system

- Indeed, suppose that the system's input is the unit step, that is, $u(t) = 1(t)$
- The Laplace transform of the unit step is $U(s) = 1/s$
- Then the Laplace transform of the system's output is

$$Y(s) = \frac{1}{s^2}$$

- Therefore

$$y(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2}\right) = t1(t),$$

  which is an unbounded signal

BIBO stability test

Theorem

A single-input single-output (SISO) continuous LTI causal system is BIBO stable if and only if its impulse response, $g(t)$, is absolutely integrable on $[0, \infty)$, that is,

$$\int_0^\infty |g(t)| dt \leq M < \infty$$

for some positive constant M

- We prove that the absolute integrability of the impulse response is sufficient for BIBO stability
- The absolute integrability of the impulse response is also necessary for BIBO stability

BIBO stability test proof

- Assume that $g(t)$ is absolutely integrable and let the system's input $u(t)$ be bounded on $[0, \infty)$
- Use the formula for an LTI system response to obtain

$$
\begin{aligned}
|y(t)| &= \left| \int_0^t g(\tau)u(t-\tau)d\tau \right| \\
&\leq \int_0^t |g(\tau)||u(t-\tau)|d\tau \\
&\leq B\int_0^t |g(\tau)|d\tau \\
&\leq BM,
\end{aligned}
$$

which means that the output $y(t)$ is bounded for any bounded input $u(t)$

Example of BIBO stable system

$$U(s) \longrightarrow \boxed{\frac{2}{s+1}} \longrightarrow Y(s)$$

- The system's impulse response is

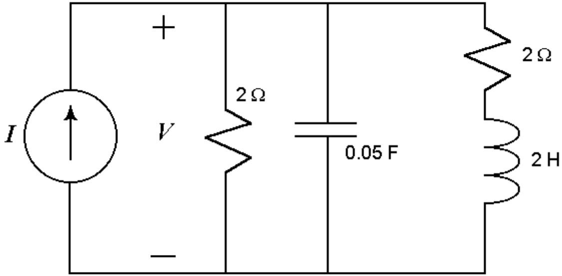
$$\begin{aligned} g(t) &= \mathcal{L}^{-1}(G(s)) \\ &= \mathcal{L}^{-1}\left(\frac{2}{s+1}\right) \\ &= 2e^{-t}1(t) \end{aligned}$$

- The impulse response is absolutely integrable because

$$\int_0^\infty \left|2e^{-t}\right| dt \leq 2 < \infty.$$

- Therefore, the system is BIBO stable

Another example: The input is the source current and the output is the voltage across the source terminals



Circuit analysis

- Draw a block diagram of the circuit;
- Find the circuit transfer function;
- Construct a state-space realization;
- Find the impulse response and determine if the circuit is BIBO stable. If not, then find a bounded input, I, that produces an unbounded output, V

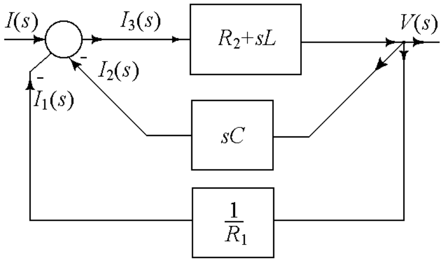
Drawing the circuit's block diagram

- Apply the Kirchhoff's current law (KCL) at the upper node

$$I(s) = I_1(s) + I_2(s) + I_3(s)$$

where $I_1(s) = \frac{V(s)}{R_1}$, $\quad I_2(s) = sCV(s)$, $\quad$ and $\quad I_3(s) = \frac{V(s)}{sL+R_2}$

- Re-write as $I_3(s) = I(s) - I_1(s) - I_2(s)$



Transfer function of the circuit

- Use the block diagram to obtain

$$\frac{V(s)}{I(s)} = G(s) = \frac{G_1(s)}{1 + \frac{G_1(s)}{R_1}}$$

where

$$G_1(s) = \frac{sL + R_2}{1 + sC(sL + R_2)}$$

- Manipulate

$$\begin{aligned}
\frac{V(s)}{I(s)} &= \frac{LR_1 s + R_1 R_2}{CLR_1 s^2 + (L + R_1 R_2 C)s + R_1 + R_2} \\
&= \frac{4s + 4}{0.2s^2 + 2.2s + 4} \\
&= \frac{20s + 20}{s^2 + 11s + 20}
\end{aligned}$$

- The transfer function is strictly proper so $d = 0$

Circuit's transfer function to state-space

- Possible state-space realization

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \begin{bmatrix} 0 & 1 \\ -20 & -11 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} i(t) \\ v(t) &= \begin{bmatrix} 20 & 20 \end{bmatrix} \boldsymbol{x}(t)\end{aligned}$$

- The impulse response: take the inverse Laplace transform of the transfer function $G(s)$
- Use MATLAB's commands to obtain the partial fraction expansion of $G(s)$

```
syms s; G=(20*s+20)/(s^2+11*s+20);
g=diff(int(G));g=vpa(g,5);pretty(g)
```

- The impulse response is

$$\begin{aligned}g(t) &= \mathcal{L}^{-1}(G(s)) \\ &= \mathcal{L}^{-1}\left(\frac{24.056}{s+8.7016} - \frac{4.056}{s+2.2984}\right) \\ &= 24.056e^{-8.7016t} - 4.056e^{-2.2984t}\end{aligned}$$

Circuit's transfer function is BIBO stable

- The system is BIBO stable because the poles of the transfer function $G(s)$ are in open left-half complex plane
- Another test for the BIBO stability is the absolute integrability of the impulse response
- Check that the impulse response is absolutely integrable

$$
\begin{aligned}
\int_0^\infty |g(t)|dt &= \int_0^\infty \left|24.056e^{-8.7016t} - 4.056e^{-2.2984t}\right| dt \\
&\leq \int_0^\infty \left|24.056e^{-8.7016t}\right| dt + \int_0^\infty \left|4.056e^{-2.2984t}\right| dt \\
&= 2.7645 + 1.7647 \\
&= 4.5293
\end{aligned}
$$

- Can verify calculations using the MATLAB commands

```
Int1=double(int(abs(24.056*exp(-8.7016*t)),t,0,inf));
Int2=double(int(abs(4.056*exp(-2.2984*t)),t,0,inf));
Int1+Int2
```

BIBO stability test for DT LTI systems

- Output of a DT SISO LTI causal system relaxed at $k = 0$

$$y[k] = \sum_{m=0}^{k} g[k-m]u[m] = \sum_{m=0}^{k} g[m]u[k-m],$$

where $g[k]$ is the impulse response sequence.

- An input sequence $u[k]$ is bounded if for some constant $B < \infty$

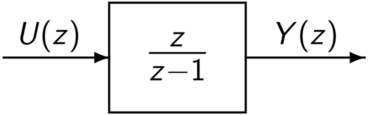
$$|u[k]| \leq B \quad \text{for all} \quad k = 0, 1, 2, \ldots$$

Theorem

A SISO DT LTI causal system is BIBO stable if and only if its impulse response, $g[k]$, is absolutely summable, that is,

$$\sum_{k=0}^{\infty} |g[k]| \leq M < \infty \text{ for some constant } M > 0$$

DT LTI not BIBO stable system



- The system's impulse response is

$$g[k] = 1 \quad \text{for} \quad k = 0, 1, 2, \ldots$$

- The impulse response is not absolutely summable because

$$\sum_{k=0}^{\infty} |g[k]| = \sum_{k=0}^{\infty} 1 = \infty$$

- Therefore, the system is not BIBO stable

Another DT LTI BIBO stable system

- The system with the impulse response

$$g[k] = (-0.5)^k, \quad k = 0, 1, 2, \ldots$$

  is BIBO stable because this impulse response is summable
- Indeed

$$\sum_{k=0}^{\infty} |g[k]| = \sum_{k=0}^{\infty} (-0.5)^k = \frac{2}{3}$$

- The above tests for checking if a given system is BIBO stable or not are not particularly convenient
- We next provide tests for BIBO stability that are very easy to use

LTI systems BIBO stablity tests

Theorem

A CT SISO LTI system with a proper rational transfer function is BIBO stable if and only if every pole of the transfer function has a negative real part

Theorem

A DT SISO LTI system with a proper rational transfer function is BIBO stable if and only if every pole of the transfer function has a magnitude less than 1. Equivalently, all poles of the transfer function are located in the open unit disc in the z-plane

Example

- The system,

$$
\begin{aligned}
\boldsymbol{x}[k+1] &= \begin{bmatrix} -0.8 & 0 & 0 \\ 0.4 & 0 & 0.4 \\ 0 & -0.8 & -0.8 \end{bmatrix} \boldsymbol{x}[k] + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u[k] \\
&= \boldsymbol{A}\boldsymbol{x}[k] + \boldsymbol{b}u[k] \\
y[k] &= \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \boldsymbol{x}[k] - 2u[k] \\
&= \boldsymbol{c}\boldsymbol{x}[k] + du[k]
\end{aligned}
$$

- The system is BIBO stable
- Why?

Is the example system BIBO stable

- Asymptotic stability of a discrete-time LTI state-space model, where d is constant, implies asymptotic stability of the system's transfer function which, in turn, is necessary and sufficient for BIBO stability of the given system
- The eigenvalues of the $\boldsymbol{A}$ matrix are

  $$\{-0.4 + 0.4j, -0.4 - 0.4j, -0.8\}$$

  Their magnitudes are

  $$0.5657, 0.5657, 0.8000,$$

  respectively, that is, their magnitudes are are less than 1, which means that the eigenvalues are all inside of the open unit circle
- Thus the system is BIBO stable if the poles of the system's transfer function are all in the open unit disc

Alternative BIBO stablity test

- The transfer function is

$$
\begin{aligned}
G(z) &= \boldsymbol{c}[z\boldsymbol{I}_3 - \boldsymbol{A}]^{-1}\boldsymbol{b} + d \\
&= \frac{z^2 - 0.32}{z^3 + 1.6z^2 + 0.96z + 0.256} - 2
\end{aligned}
$$

- The poles of the above transfer function are the same as the eigenvalues of the matrix $\boldsymbol{A}$ above
- Therefore the transfer function is BIBO stable

CT LTI MIMO system BIBO stablity test

Theorem

A CT MIMO LTI system with a proper rational transfer function matrix, $\boldsymbol{G}(s)$, is BIBO stable if and only if every pole of $G_{ij}(s)$ has a negative real part, where

$$\boldsymbol{G}(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) & \cdots & G_{1m}(s) \\ \vdots & \vdots & \ddots & \vdots \\ G_{p1}(s) & G_{p2}(s) & \cdots & G_{pm}(s) \end{bmatrix}$$

DT LTI MIMO system BIBO stablity test

Theorem

A discrete MIMO LTI system with a proper rational transfer function matrix, $\boldsymbol{G}(z)$, is BIBO stable if and only if every pole of $G_{ij}(z)$ is located in the open unit disc in the z-plane, where

$$\boldsymbol{G}(z) = \begin{bmatrix} G_{11}(z) & G_{12}(z) & \cdots & G_{1m}(z) \\ \vdots & \vdots & \ddots & \vdots \\ G_{p1}(z) & G_{p2}(z) & \cdots & G_{pm}(z) \end{bmatrix}$$