12.1 PDE.

(Superposition)

Ⓗ u_1, u_2 : two solutions of a homogeneous linear PDE in some region D

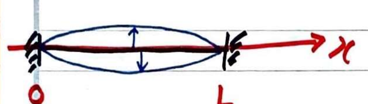
Ⓒ For any $C_1, C_2 \in \mathbb{R}$
$u(x) = C_1 u_1(x) + C_2 u_2(x)$ is also a solution of the PDE in D.

12.2 Wave equation

12.3 Wave equation: Separation of variables

$u_{tt} - c^2 u_{xx} = 0$:

$$\left(\begin{array}{l}\text{Notation: } u_t = \dfrac{\partial u}{\partial t}, \quad u_x = \dfrac{\partial u}{\partial x} \\ u_{tt} = \dfrac{\partial^2 u}{\partial t^2}, \quad u_{xx} = \dfrac{\partial^2 u}{\partial x^2}\end{array}\right)$$



$u(x,t)$ = the displacement of the string

$c > 0$; the wave speed.

Initial Boundary value problem (IBVP)

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, & 0 < x < L,\ t > 0 \\ u(0,t) = 0, \quad u(L,t) = 0 \\ u(x,0) = f(x), \quad u_t(x,0) = g(x), & 0 < x < L \end{cases}$$

(Separation of variables)

Let $u(x,t) = F(x)\,G(t)$

$u_{tt} = F(x)\,G''(t), \quad u_{xx} = F''(x)\,G(t)$

$u_{tt} = c^2 u_{xx}$ iff $F(x)\,\underline{G''(t)} = c^2 \underline{F''(x)}\,G(t)$

Assume $u(x,t) \neq 0$

iff $f(x) \neq 0$ or $g(x) \neq 0$

$$\frac{\cancel{F(x)}G''(t)}{\cancel{F(x)}G(t)} = \frac{c^2 F''(x)\cancel{G(t)}}{F(x)\cancel{G(t)}}$$

$$\frac{G''(t)}{c^2 G(t)} = \frac{F''(x)}{F(x)} = \text{constant} \overset{\text{let}}{=} k.$$

(1) $F'' = kF$ (2) $G'' = kc^2 G$

BC: $u(0,t) = F(0)G(t) = 0$ for any $t > 0$

$\therefore F(0) = 0$

$u(L,t) = F(L)G(t) = 0$ for any $t > 0$

$\therefore F(L) = 0$

(1) $F'' - kF = 0, \quad F(0) = 0, \; F(L) = 0$

(SL)

$r^2 - k = 0$: $r = \pm\sqrt{k}$

① $k > 0$: $F(x) = C_1 e^{\sqrt{k}x} + C_2 e^{-\sqrt{k}x}$

$F(0) = C_1 + C_2 = 0$ ↙ BC : $C_2 = -C_1$

$F(L) = C_1 e^{\sqrt{k}L} + \underset{\text{''}-C_1}{C_2} e^{-\sqrt{k}L} = 0$ ↙ BC

$C_1(\underbrace{e^{\sqrt{k}L} - e^{-\sqrt{k}L}}_{\neq 0}) = 0$: $C_1 = 0, \; C_2 = 0$

$F(x) \equiv 0$ for any $x \in \mathbb{R}$

identically zero

No eigenfunctions.

② $k = 0$: $F'' = 0$ $F(x) = C_1 x + C_2$

$F(0) = C_2 = 0$ ↙ BC, $F(x) = C_1 x$.

$F(L) = C_1 L = 0$ ↙ : $C_1 = 0$,

No eigenfunctions.

③ $k < 0$: Let $h = -k$ iff $k = -h$

$r = \pm\sqrt{k} = \pm\sqrt{-h} = \pm\sqrt{h}\, i$

$F(x) = C_1 \cos(\sqrt{h}x) + C_2 \sin(\sqrt{h}x).$

$F(0) = C_1 = 0 \checkmark BC$: $F(x) = C_2 \sin(\sqrt{h}\,x)$

$F(L) = C_2 \sin(\sqrt{h}\,L) = 0$: $C_2 \neq 0$

$\underline{\sin(\sqrt{h}\,L) = 0}$: $\sqrt{h}\,L = n\pi > 0$

$(n = 1, 2, 3, \cdots)$

$\sqrt{h} = \frac{n\pi}{L}$: $h = \left(\frac{n\pi}{L}\right)^2$: $k = -\left(\frac{n\pi}{L}\right)^2$

Let $\underline{F_n(x) = \sin\left(\frac{n\pi x}{L}\right)}$, $k_n \overset{let}{=} -\left(\frac{n\pi}{L}\right)^2$

(2) $G'' = kc^2 G = -\left(\frac{n\pi}{L}\right)^2 c^2 G$

$G'' + \left(\frac{cn\pi}{L}\right)^2 G = 0$

$r^2 + \left(\frac{cn\pi}{L}\right)^2 = 0$: $r = \pm \frac{cn\pi}{L} i$

$G_n(t) \overset{let}{=} A_n \cos\left(\frac{cn\pi}{L}t\right) + B_n \sin\left(\frac{cn\pi t}{L}\right)$

$n = 1, 2, 3, \cdots$

IC :

Let $u_n(x,t) = F_n(x)\, G_n(t)$,

$u_n(x,t) = \sin\left(\frac{n\pi x}{L}\right)\left[A_n \cos\left(\frac{cn\pi t}{L}\right) + B_n \sin\left(\frac{cn\pi t}{L}\right)\right]$

$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$

$= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[A_n \cos\left(\frac{cn\pi t}{L}\right) + B_n \sin\left(\frac{cn\pi t}{L}\right)\right]$

$A_n = ?$ $B_n = ?$

$u(x,0) = \underline{\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) = f(x)} \checkmark IC$

$(0 < x < L)$

: the Fourier sine series of $f(x)$

$\underline{A_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx}$

$u_t(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[A_n\left(-\frac{cn\pi}{L}\right) \sin\left(\frac{cn\pi t}{L}\right) + B_n\left(\frac{cn\pi}{L}\right)\cos\left(\frac{cn\pi t}{L}\right)\right]$

$u_t(x,0) = \underline{\sum_{n=1}^{\infty} B_n\left(\frac{cn\pi}{L}\right)\sin\left(\frac{n\pi x}{L}\right) = g(x)}$ IC

$B_n\left(\frac{cn\pi}{\cancel{L}}\right) = \frac{2}{\cancel{L}}\int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx$

$$B_n = \frac{2}{cn\pi}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Remark: $g(x) \equiv 0$ i.e. $u_t(x,0) = 0,\ 0<x<L$

$B_n = 0,\ n = 1, 2, \cdots$

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{cn\pi t}{L}\right).$$

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

(d'Alembert formula) $\underline{u_t(x,0) = g(x) \equiv 0}$

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right]$$

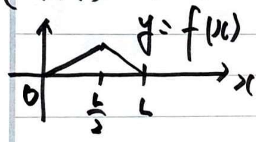
(pf) $\sin(\alpha)\cos(\beta) = \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$

$$u(x,t) = \sum_{n=1}^{\infty}\frac{A_n}{2}\left[\sin\left(\frac{n\pi x}{L} + \frac{cn\pi t}{L}\right) + \sin\left(\frac{n\pi x}{L} - \frac{cn\pi t}{L}\right)\right]$$

$$= \frac{1}{2}\sum_{n=1}^{\infty} A_n\left[\sin\left(\left(\frac{n\pi}{L}\right)(x+ct)\right) + \sin\left(\left(\frac{n\pi}{L}\right)(x-ct)\right)\right]$$

$$= \frac{1}{2}\Bigg[\underbrace{\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}\right)(x+ct)}_{= f(x+ct)} + \underbrace{\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}\right)(x-ct)}_{= f(x-ct).}\Bigg]$$

(Ex 1) P551 $g(x) = 0,\ 0<x<L$



$$u(x,t) = \frac{f(x+ct) + f(x-ct)}{2}$$

$$= \frac{1}{2}\left[f(x - \underline{(-ct)}) + f(x - \underline{ct})\right]$$

translation of $f_0(x)$

$$f(x) = \begin{cases} \frac{2}{L}x, & 0<x<\frac{L}{2} \\ \frac{2}{L}(L-x), & \frac{L}{2}<x<L \end{cases}$$