ECE 602: LUMPED LINEAR SYSTEMS

Professor Jianghai Hu

Quadratic Forms

Symmetric and Skew Symmetric Matrices

$A \in \mathbb{R}^{n \times n}$ is symmetric if $A^T = A$

- All eigenvalues are real; and eigenvectors for distinct eigenvalues are orthogonal
- Equivalently, diagonalizable by an orthogonal matrix $Q \in \mathbb{R}^{n \times n}$ (i.e., $Q^{-1} = Q^T$):

$$Q^{-1} A Q = Q^T A Q = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$$

A is skew symmetric if $A^T = -A$

- All eigenvalues are purely imaginary
- If n is odd, then A is singular
- Diagonalizable by a (complex) unitary matrix $U \in \mathbb{C}^{n \times n}$ (i.e., $U^{-1} = U^*$)

(Skew) Symmetric Decomposition of Matrices

An arbitrary $A \in \mathbb{R}^{n \times n}$ can be uniquely decomposed as

$$A = \underbrace{(A + A^T)/2}_{A_{\mathrm{sym}}} + \underbrace{(A - A^T)/2}_{A_{\mathrm{skew}}}$$

Geometric Interpretation:

- For $X, Y \in \mathbb{R}^{m \times n}$, define their inner product as

$$\langle X, Y \rangle := \sum_{i=1,\ldots,m,\ j=1,\ldots,n} X_{ij} Y_{ij} = \operatorname{tr}(X^T Y)$$

- For $X, Y \in \mathbb{R}^{n \times n}$ with X symmetric and Y skew symmetric, $\langle X, Y \rangle = 0$
- Sets of n-by-n symmetric and skew symmetric matrices are two subspaces of $\mathbb{R}^{n \times n}$ that are orthogonal complement of each other
- A_{sym} and A_{skew} are the orthogonal projections of A onto the two subspaces

Quadratic Forms

Quadratic form defined by a **symmetric** matrix $A \in \mathbb{R}^{n \times n}$:

$$\varphi_A(x) := \langle x, Ax \rangle = x^T A x, \quad \forall x \in \mathbb{R}^n$$

- Uniqueness of matrix representation: for symmetric matrices $A, A' \in \mathbb{R}^{n \times n}$

$$\varphi_A(x) = \varphi_{A'}(x),\ \forall x \in \mathbb{R}^n \quad \Leftrightarrow \quad A = A'$$

- If $A \in \mathbb{R}^{n \times n}$ is not symmetric, then $\varphi_A = \varphi_{A_{\mathrm{sym}}}$ since $\varphi_{A_{\mathrm{skew}}} \equiv 0$

Example:

1. $\varphi(x) = \|Bx\|^2$ for $B \in \mathbb{R}^{m \times n}$
2. $\varphi(x) = x_1^2 - x_1 x_2 + 5 x_1 x_3 + x_2^2 - x_3^2$
3. $\varphi(x) = (x_1 - x_2)^2 + \cdots + (x_{n-1} - x_n)^2$

Bounds of Quadratic Forms

Suppose a symmetric matrix $A \in \mathbb{R}^{n \times n}$ has the eigenvalues $\lambda_1, \ldots, \lambda_n$

- Let $\lambda_{\min}(A) := \min\{\lambda_1, \ldots, \lambda_n\}$
- Let $\lambda_{\max}(A) := \max\{\lambda_1, \ldots, \lambda_n\}$

The quadratic form defined by A is bounded by

$$\lambda_{\min}\|x\|^2 \leq x^T A x \leq \lambda_{\max}\|x\|^2, \quad \forall x \in \mathbb{R}^n$$

Positive and Negative (Semi)definite Matrices

A symmetric matrix $A \in \mathbb{R}^{n \times n}$ is called

- positive semidefinite if $x^T A x \geq 0$ for all $x \in \mathbb{R}^n$
  - Denoted by $A \succeq 0$
- positive definite if $x^T A x > 0$ for all nonzero $x \in \mathbb{R}^n$
  - Denoted by $A \succ 0$
- negative semidefinite if $-A$ is positive semidefinite
  - Denoted by $A \preceq 0$
- negative definite if $-A$ is positive definite
  - Denoted by $A \prec 0$

Conditions for Positive Definite Matrices

$A = A^T$ being positive semidefinite is equivalent to each of the following:

1. All eigenvalues of A are nonnegative, i.e., $\lambda_{\min}(A) \geq 0$
2. *Sylvester's criterion*: All principal minors of A are nonnegative
3. $A = B^T B$ for some $B \in \mathbb{R}^{m \times n}$, i.e., $\varphi_A(x)$ is "sum of squares"

$A = A^T$ being positive definite is equivalent to each of the following:

1. All eigenvalues of A are positive, i.e., $\lambda_{\min}(A) > 0$
2. *Sylvester's criterion*: All leading principal minors of A are positive
3. $A = B^T B$ for some $B \in \mathbb{R}^{m \times n}$ that is 1-to-1

Positive (Semi)definite Cones

Define the following subsets of $\mathbb{R}^{n\times n}$:

1. $\mathbb{S}^n$: set of all n-by-n symmetric matrices
2. $\mathbb{S}^n_+$: set of all n-by-n positive semidefinite matrices
3. $\mathbb{S}^n_{++}$: set of all n-by-n positive definite matrices

Geometric properties of $\mathbb{S}^n_+$ and $\mathbb{S}^n_{++}$:

- Both are cones, e.g., $X \in \mathbb{S}^n_+$ and $\alpha > 0$ imply $\alpha X \in \mathbb{S}^n_+$
- $\mathbb{S}^n_+$ is a closed cone and $\mathbb{S}^n_{++}$ is an open cone
- Both are convex, e.g., $\lambda X + (1-\lambda)Y \in \mathbb{S}^n_+$, $\forall\, X, Y \in \mathbb{S}^n_+$, $\forall\, \lambda \in [0,1]$
- Both are acute cones (under the inner product $\langle X, Y\rangle := \mathrm{tr}\,(X^T Y)$)

Comparison of Symmetric Matrices

For two symmetric matrices $A, B \in \mathbb{R}^{n\times n}$, write $A \succeq B$ if $A - B \succeq 0$

- Equivalently, $\varphi_A(x) \geq \varphi_B(x)$ for all $x \in \mathbb{R}^n$

Similarly, write

- $A \succ B$ if $A - B \succ 0$
  - Equivalently, $\varphi_A(x) > \varphi_B(x)$ for all nonzero $x \in \mathbb{R}^n$
- $A \preceq B$ if $A - B \preceq 0$
- $A \prec B$ if $A - B \prec 0$

Example: $\lambda_{\min}(A) \cdot I_n \preceq A \preceq \lambda_{\max}(A) \cdot I_n$ for $A \in \mathbb{S}^n$

The above relations define a partial order on $\mathbb{S}^n$

- **Transitivity**: if $A \succeq B$ and $B \succeq C$, then $A \succeq C$
- **Incomparability**: there exist $A, B \in \mathbb{S}^n$ such that neither $A \succeq B$ nor $B \succeq A$ holds