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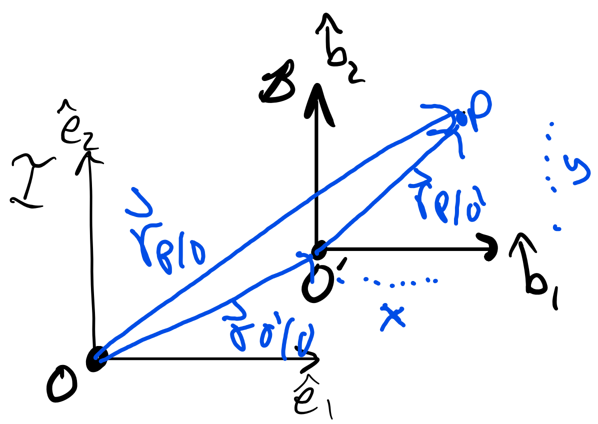
Introduction to Relative Motion · 3.6 KP

For now consider translation only ($\mathcal{B}$ non rotating)

E.O.M. in $(x,y)_{\mathcal{B}}$



<u>Body frame Kinematics</u>

$$\vec{r}_{P/O'} = x\hat{b}_1 + y\hat{b}_2$$

$$^{\mathcal{B}}\vec{v}_{P/O'} = \frac{^{\mathcal{B}}d}{dt}\left(x\hat{b}_1 + y\hat{b}_2\right) = \dot{x}\hat{b}_1 + \dot{y}\hat{b}_2$$

$$^{\mathcal{B}}\vec{a}_{P/O'} = \ddot{x}\hat{b}_1 + \ddot{y}\hat{b}_2$$

What is the inertial velocity vector for $\vec{r}_{P/O'}$?

$$^{\mathcal{I}}\vec{v}_{P/O'} = \frac{^{\mathcal{I}}d}{dt}\left(x\hat{b}_1 + y\hat{b}_2\right)$$

$$= \dot{x}\hat{b}_1 + x\frac{^{\mathcal{I}}d}{dt}(\hat{b}_1) + \dot{y}\hat{b}_2 + y\frac{^{\mathcal{I}}d}{dt}(\hat{b}_2)$$

But $\frac{^{\mathcal{I}}d}{dt}(\hat{b}_1) = \underbrace{{}^{\mathcal{I}}\vec{\omega}^{\mathcal{B}}}_{0} \times \hat{b}_1$ Similarly, $\frac{^{\mathcal{I}}d}{dt}(\hat{b}_2)$

So, ${}^{I}\vec{v}_{P/O'} = \dot{x}\hat{b}_1 + \dot{y}\hat{b}_2 = {}^{B}\vec{v}_{P/O'}$

(Translation Only Case)

Similarly,

$${}^{I}\vec{a}_{P/O'} = \ddot{x}\hat{b}_1 + \ddot{y}\hat{b}_2 = {}^{B}\vec{a}_{P/O'}$$

Let's reference the kinematics back to point O of the $\mathcal{I}$ frame to see what happens:

$$\vec{r}_{P/O} = \vec{r}_{P/O'} + \vec{r}_{O'/O}$$

$${}^{I}\vec{v}_{P/O} = {}^{I}\vec{v}_{P/O'} + {}^{I}\vec{v}_{O'/O}$$

$${}^{I}\vec{a}_{P/O} = {}^{I}\vec{a}_{P/O'} + {}^{I}\vec{a}_{O'/O}$$

Newton's 2nd Law:

$$\vec{F}_P = m_P\, {}^{I}\vec{a}_{P/O} = m_P\left({}^{I}\vec{a}_{P/O'} + {}^{I}\vec{a}_{O'/O}\right)$$

If the frame is translating only,

$$= m_P\left({}^{B}\vec{a}_{P/O'} + {}^{I}\vec{a}_{O'/O}\right)$$

Rearranging:

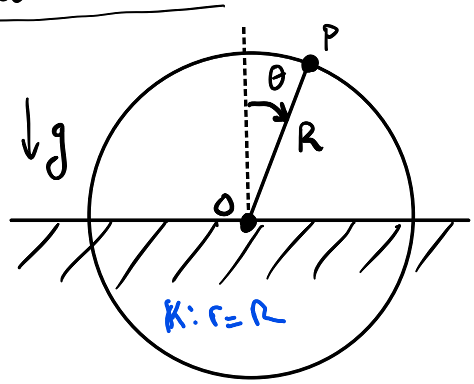
$$\underbrace{\vec{F}_P - m_P\, {}^{I}\vec{a}_{O'/O}}_{\text{effect of frame acceleration}} = \underbrace{m_P\, {}^{B}\vec{a}_{P/O'}}_{\text{useful if sensors are in body/translating frame}}$$

If the frame is not accelerating,

$$\vec{F}_P = m_P\, {}^{B}\vec{a}_{P/O'} \Rightarrow \text{N2L holds in this frame}$$

(I.e. frame is inertial)

Tutorial 3.3 Find the E.O.M. for a particle sliding on a hemisphere.



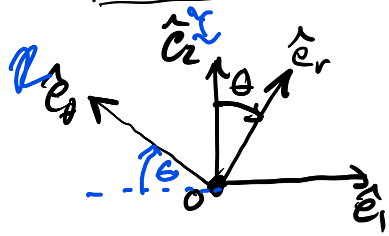
DOF?

$M = 2N - K$

$M = 2(1) - 1$

$M = 1$

Reference frames



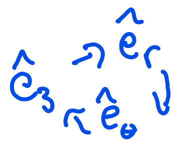
	$\hat{e}_r$	$\hat{e}_\theta$
$\hat{e}_1$	$s\theta$	$-c\theta$
$\hat{e}_2$	$c\theta$	$s\theta$

Kinematics

$\vec{r}_{P/O} = R\hat{e}_r$

Note: ${}^{I}\frac{d}{dt}(\hat{e}_r) = {}^{I}\vec{\omega}^{0} \times \hat{e}_r = -\dot{\theta}\hat{e}_3 \times \hat{e}_r = -\dot{\theta}\hat{e}_\theta$

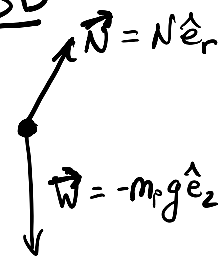
${}^{I}\vec{V}_{P/O} = -R\dot{\theta}\hat{e}_\theta$



$${}^{I}\vec{a}_{P/O} = -R\ddot{\theta}\hat{e}_\theta - R\dot{\theta}\underbrace{{}^{I}\frac{d}{dt}(\hat{e}_\theta)}_{= -\dot{\theta}\hat{e}_3 \times \hat{e}_\theta = \dot{\theta}\hat{e}_r}$$

$${}^{I}\vec{a}_{P/O} = -R\ddot{\theta}\hat{e}_\theta - R\dot{\theta}^2\hat{e}_r$$

FBD



$\hat{e}_2 = c\theta\,\hat{e}_r + s\theta\,\hat{e}_\theta$

Newton's 2nd Law

$\vec{F}_p = m_p\,{}^{I}\vec{a}_{P/O}$

$$N\hat{e}_r - m_p g\hat{e}_2 = m_p(-R\ddot{\theta}\hat{e}_\theta - R\dot{\theta}^2\hat{e}_r)$$

E.O.M.'s

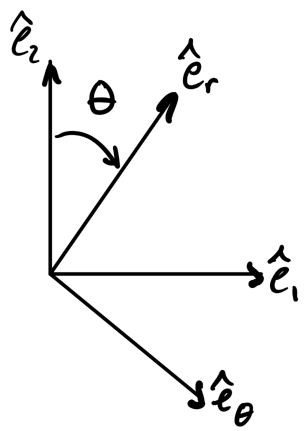
$$\hat{e}_r: \quad N - m_p g\cos\theta = -m_p R\dot{\theta}^2 \qquad \underbrace{N = m_p(g\cos\theta - R\dot{\theta}^2)}_{\text{Constraint Force}}$$

$$\hat{e}_\theta: \quad -\cancel{m_p} g\sin\theta = -\cancel{m_p} R\ddot{\theta}$$

$$\Rightarrow \boxed{\ddot{\theta} = \frac{\sin\theta\, g}{R}}$$

See the remainder of the tutorial to solve for t^* and θ^* when the particle falls off.

What if we had defined our reference frames differently?



Their $\hat{e}_3$ vectors are not aligned if they are both right-handed frames.

	$\hat{e}_r$	$\hat{e}_\theta$
$\hat{e}_1$	$S\theta$	$C\theta$
$\hat{e}_2$	$C\theta$	$-S\theta$

${}^I C^R$

$\det({}^I C^R) = -1$ (Not rotation Matrix)

Ch 4: Linear and Angular Momentum of a Particle

Momentum is a tool for solving problems that can sometimes make find trajectories or properties of trajectories easier. Can $\therefore$ make finding EoM's easier, example is 4.4 on pendulum

N2L: $\vec{F}_p = m_p \, {}^I\vec{a}_{P/O}$

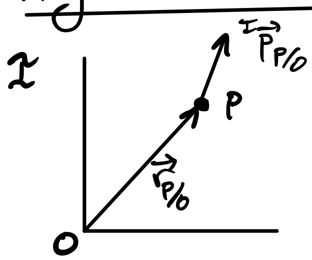
Linear Momentum $= {}^I\vec{p}_{P/O} = m_p \, {}^I\vec{v}_{P/O}$

$= m_p \, \frac{{}^I d}{dt}\left({}^I\vec{v}_{P/O}\right) = \frac{{}^I d}{dt}\left({}^I\vec{p}_{P/O}\right)$

Conservation?

If $\vec{F}_p = 0$, $\therefore$ ${}^I\vec{p}_{P/O}$ is conserved

Angular momentum relative to an inertially fixed point O



$${}^I\vec{h}_{P/O} \triangleq \vec{r}_{P/O} \times {}^I\vec{p}_{P/O} = \vec{r}_{P/O} \times m_p \, {}^I\vec{v}_{P/O}$$

How does angular momentum change with time?

$$\frac{{}^I d}{dt}\left({}^I\vec{h}_{P/O}\right) = \left(\frac{{}^I d}{dt}\left(\vec{r}_{P/O}\right) \times {}^I\vec{p}_{P/O}\right) + \left(\vec{r}_{P/O} \times \frac{{}^I d}{dt}\left(m_p \, {}^I\vec{v}_{P/O}\right)\right)$$

$$= \underbrace{\left({}^I\vec{v}_{P/O} \times m_p \vec{v}_{P/O}\right)}_{0} + \left(\vec{r}_{P/O} \times \underbrace{m_p \, {}^I\vec{a}_{P/O}}_{F_p}\right)$$

$$\frac{{}^{I}d}{dt}\left({}^{I}\vec{h}_{P/O}\right) = \underbrace{\vec{r}_{P/O} \times \vec{F}_P}_{\triangleq \vec{M}_{P/O}}$$

$$\boxed{\frac{{}^{I}d}{dt}\left({}^{I}\vec{h}_{P/O}\right) = \vec{M}_{P/O}}$$

Rotational version of N2L
(Angular momentum form of N_2L)

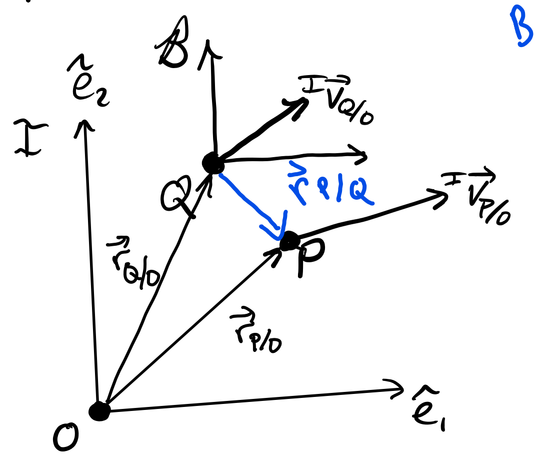
Conservation?

If $\vec{M}_{P/O} = 0$, ${}^{I}\vec{h}_{P/O}$ is conserved

Angular momentum about an arbitrary point

Useful when you want to use angular momentum in an accelerating frame.

$\mathcal{B}$ is translating with acceleration



$$\vec{r}_{P/Q} = \vec{r}_{P/O} - \vec{r}_{Q/O}$$

$$ {}^{I}\vec{h}_{P/Q} = \vec{r}_{P/Q} \times m_P\, {}^{I}\vec{v}_{P/Q}$$

$$= (\vec{r}_{P/O} - \vec{r}_{Q/O}) \times m_P \left({}^{I}\vec{v}_{P/O} - {}^{I}\vec{v}_{Q/O}\right)$$

$$\frac{{}^{I}d}{dt}\left({}^{I}\vec{h}_{P/Q}\right) = \left({}^{I}\vec{v}_{P/O} - {}^{I}\vec{v}_{Q/O}\right) \times m_P\left({}^{I}\vec{v}_{P/O} - {}^{I}\vec{v}_{Q/O}\right)$$

$$+ \underbrace{(\vec{r}_{P/O} - \vec{r}_{Q/O})}_{\vec{r}_{P/Q}} \times \underbrace{m_P\left({}^{I}\vec{a}_{P/O} - {}^{I}\vec{a}_{Q/O}\right)}_{\vec{F}_P}$$

$$\frac{{}^I d}{dt}\left({}^I\vec{h}_{P/Q}\right) = \underbrace{\vec{r}_{P/Q} \times \vec{F}_P}_{\triangleq \vec{M}_{P/Q}} - \vec{r}_{P/Q} \times m_P\, {}^I\vec{a}_{Q/O}$$

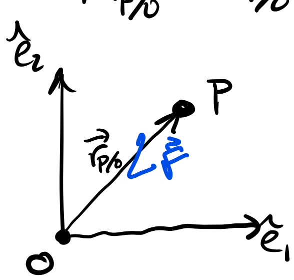
$$\boxed{\frac{{}^I d}{dt}\left({}^I\vec{h}_{P/Q}\right) = \vec{M}_{P/Q} - \underbrace{\vec{r}_{P/Q} \times m_P\, {}^I\vec{a}_{Q/O}}_{\text{Inertial moment frame effect}}}$$

Rotational form of N2L about an arbitrary point Q.

Note: If ${}^I\vec{a}_{Q/O} = 0$, the rotational form of N2L holds.

Dfn: A <u>central force</u> is a nonzero force that ... That is, for $\vec{F}_P \neq 0$ a central force satisfies

$$\vec{M}_{P/O} = \vec{r}_{P/O} \times \vec{F}_P = 0 \qquad \frac{{}^I d}{dt}\left({}^I\vec{h}_{P/O}\right) = 0$$



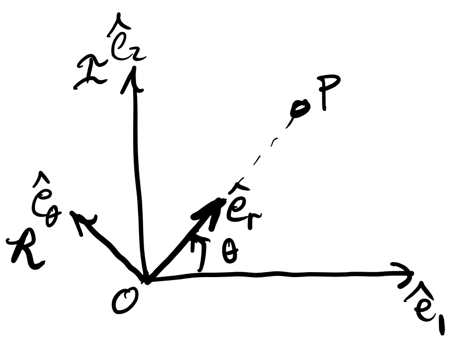
Note: for a central force, ${}^I\vec{h}_{P/O}$ is conserved.

Particles motion is confined to plane ⊥ to ${}^I\vec{h}_{P/O}$ vector.

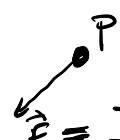
3D → 2D

Ex. 4.9 Simple Satellite

Tutorial 4.2 Orbit Equation



FBD



$\vec{F}_P = \dfrac{-G m_O m_P}{r^2}\hat{e}_r$ gravitational force

Kinematics

$$\vec{r}_{P/O} = r\hat{e}_r$$
$$^{\mathcal{I}}\vec{v}_{P/O} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$$
$$^{\mathcal{I}}\vec{a}_{P/O} = (\qquad)\hat{e}_r + (\qquad\qquad)\hat{e}_\theta$$

N2L

$$\vec{F}_P = m_P\,^{\mathcal{I}}\vec{a}_{P/O}$$
$$= m_P\left[(\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta\right]$$

$$\ddot{r} = r\dot{\theta}^2 - \frac{G m_O}{r^2}$$
$$\ddot{\theta} = \frac{-2\dot{r}\dot{\theta}}{r}$$

$\vec{F}_P$ is a central force in this problem

$$\vec{M}_{P/O} = \vec{r}_{P/O} \times \vec{F}_P = 0 \Rightarrow$$

$$^{I}\vec{h}_{P/O} = \vec{r}_{P/O} \times m_P\, ^{I}\vec{v}_{P/O}$$

$$= r\hat{e}_r \times m_p(\qquad\qquad)$$

$$= \qquad \hat{e}_3$$

Let $\mu = Gm_0$

$$\Rightarrow \ddot{r} = \frac{h_0^2}{r^3} - \frac{\mu}{r^2} \qquad \ddot{\theta} = \frac{-2\dot{r}h_0}{r^3}$$

Solving for $r(t)$ and $\theta(t)$ in this problem is known as "Kepler's Problem" (see Goldstein)

Instead, let's focus on a parametric solution $r(\theta)$

Changing variables $y = \frac{1}{r} \Rightarrow h_0 = \frac{\dot{\theta}}{y^2}$

$$\dot{r} = \frac{d}{dt}\left(\frac{1}{y}\right) = \frac{-1}{y^2}\dot{y} =$$

$$\ddot{r} = -h_0 \frac{d^2y}{d\theta^2}\dot{\theta} =$$

Equating with the other $\ddot{r}$ expression, and substituting

$$-h_0^2 y^2 \frac{d^2y}{d\theta^2} = h_0^2 y^3 - \mu y^2$$

$$\boxed{\frac{d^2y}{d\theta^2} = \qquad\qquad}$$ Parametric equation of motion

Solution:

$$y(\theta) = \frac{\mu}{h_0^2}\left(1 + e\cos(\theta - \theta_0)\right)$$

where e, θ_0 are constants

$$r(\theta) = \frac{h_0^2/\mu}{1 + e\cos(\theta - \theta_0)}$$