ECE 68000: MODERN AUTOMATIC CONTROL

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Observation error stability test in terms of a linear matrix inequality

Observation error stability test

$$\boldsymbol{e}[k+1] = (\boldsymbol{A}_1 - \boldsymbol{L}\boldsymbol{C})\boldsymbol{e}[k] - \boldsymbol{L}\boldsymbol{D}\boldsymbol{v}[k] := \boldsymbol{E}\boldsymbol{e}[k] + \boldsymbol{N}\boldsymbol{v}[k]$$

Theorem

If there exist matrices $\boldsymbol{P} = \boldsymbol{P}^\top \succ 0$ *and* $\boldsymbol{L}$, *and* $\alpha \in (0,1)$ *such that*

$$\begin{bmatrix} \boldsymbol{E}^\top \boldsymbol{P}\boldsymbol{E} - (1-\alpha)\boldsymbol{P} & * \\ \boldsymbol{N}^\top \boldsymbol{P}\boldsymbol{E} & \boldsymbol{N}^\top \boldsymbol{P}\boldsymbol{N} - \alpha \boldsymbol{I} \end{bmatrix} \preceq 0$$

then the state observation error is l_∞*-stable with performance level* $\gamma = 1/\sqrt{\lambda_{\min}(\boldsymbol{P})}$

Stability of the error dynamics lemma

Observation error: $\boldsymbol{e}[k+1] = (\boldsymbol{A}_1 - \boldsymbol{LC})\boldsymbol{e}[k] - \boldsymbol{LDv}[k]$

Lemma

Suppose there exists a function $V : \mathbb{R}^n \to \mathbb{R}$ *and scalars* $\delta \in (0, 1)$, $\beta_1, \beta_2 > 0$ *and* $\mu_1, \mu_2 \geq 0$ *such that*

$$\beta_1 \|\boldsymbol{e}[k]\|^2 \leq V(\boldsymbol{e}[k]) \leq \beta_2 \|\boldsymbol{e}[k]\|^2,$$

$$\Delta V[k] \leq -\delta(V(\boldsymbol{e}[k]) - \mu_1 \|\boldsymbol{v}[k]\|^2)$$

for all $k \geq 0$. *Then, the observation error is globally uniformly* l_∞*-stable with performance level* $\gamma = \sqrt{\mu_1 \mu_2}$ *with respect to the output disturbance* $\boldsymbol{v}[k]$, *where* $\|\boldsymbol{e}[k]\|^2 \leq \mu_2 V(\boldsymbol{e}[k])$ *with* $\mu_2 = 1/\beta_1$

Error stability test proof

- Since $P = P^\top \succ 0$, conditions of the lemma are satisfied with $\beta_1 = \lambda_{\min}(P)$, $\beta_2 = \lambda_{\max}(P)$, and $\mu_2 = 1/\lambda_{\min}(P)$
- Let $V[k] = \boldsymbol{e}[k]^\top \boldsymbol{P}\boldsymbol{e}[k]$ be a Lyapunov function candidate for the estimation error dynamics
- Evaluate the first forward difference

$$\Delta V[k] = V[k+1] - V[k]$$

on the trajectories of the error dynamics

$$\begin{aligned} \Delta V[k] &= \boldsymbol{e}[k]^\top(\boldsymbol{E}^\top\boldsymbol{P}\boldsymbol{E} - \boldsymbol{P})\boldsymbol{e}[k] + 2\boldsymbol{e}[k]^\top\boldsymbol{E}^\top\boldsymbol{P}\boldsymbol{N}\boldsymbol{v}[k] \\ &\quad + \boldsymbol{v}[k]^\top\boldsymbol{N}^\top\boldsymbol{P}\boldsymbol{N}\boldsymbol{v}[k] \end{aligned}$$

Error stability test proof—Contd.

- Let $\boldsymbol{\zeta} = \left[\begin{array}{cc} \boldsymbol{e}[k]^\top & \boldsymbol{v}[k]^\top \end{array}\right]^\top$
- Pre-multiplying and post-multiplying

$$\left[\begin{array}{cc} \boldsymbol{E}^\top \boldsymbol{P}\boldsymbol{E} - (1-\alpha)\boldsymbol{P} & * \\ \boldsymbol{N}^\top \boldsymbol{P}\boldsymbol{E} & \boldsymbol{N}^\top \boldsymbol{P}\boldsymbol{N} - \alpha \boldsymbol{I} \end{array}\right] \preceq 0$$

by $\boldsymbol{\zeta}^\top$ and $\boldsymbol{\zeta}$, respectively to obtain

$$\Delta V[k] + \alpha(V[k] - \|\boldsymbol{v}[k]\|^2) \leq 0$$

- Indeed

$$\left[\begin{array}{c} \boldsymbol{e}[k] \\ \boldsymbol{v}[k] \end{array}\right]^\top \left[\begin{array}{cc} \boldsymbol{E}^\top \boldsymbol{P}\boldsymbol{E} - (1-\alpha)\boldsymbol{P} & * \\ \boldsymbol{N}^\top \boldsymbol{P}\boldsymbol{E} & \boldsymbol{N}^\top \boldsymbol{P}\boldsymbol{N} - \alpha \boldsymbol{I} \end{array}\right] \left[\begin{array}{c} \boldsymbol{e}[k] \\ \boldsymbol{v}[k] \end{array}\right] \leq 0$$

Error stability test proof—manipulations

- We have

$$
\begin{aligned}
&\Big[\boldsymbol{e}[k]^\top \Big(\boldsymbol{E}^\top \boldsymbol{P}\boldsymbol{E} - (1-\alpha)\boldsymbol{P}\Big) + \boldsymbol{v}[k]^\top \boldsymbol{N}^\top \boldsymbol{P}\boldsymbol{E} \\
&\qquad \boldsymbol{e}[k]^\top \boldsymbol{E}^\top \boldsymbol{P}\boldsymbol{N} + \boldsymbol{v}[k]^\top \Big(\boldsymbol{N}^\top \boldsymbol{P}\boldsymbol{N} - \alpha\boldsymbol{I}\Big)\Big] \begin{bmatrix} \boldsymbol{e}[k] \\ \boldsymbol{v}[k] \end{bmatrix} \\
&= \boldsymbol{e}[k]^\top \Big(\boldsymbol{E}^\top \boldsymbol{P}\boldsymbol{E} - (1-\alpha)\boldsymbol{P}\Big)\boldsymbol{e}[k] + \boldsymbol{v}[k]^\top \boldsymbol{N}^\top \boldsymbol{P}\boldsymbol{E}\boldsymbol{e}[k] \\
&\qquad \boldsymbol{e}[k]^\top \boldsymbol{E}^\top \boldsymbol{P}\boldsymbol{N}\boldsymbol{v}[k] + \boldsymbol{v}[k]^\top \Big(\boldsymbol{N}^\top \boldsymbol{P}\boldsymbol{N} - \alpha\boldsymbol{I}\Big)\boldsymbol{v}[k] \\
&= \boldsymbol{e}[k]^\top \Big(\boldsymbol{E}^\top \boldsymbol{P}\boldsymbol{E} - \boldsymbol{P}\Big)\boldsymbol{e}[k] + 2\boldsymbol{e}[k]^\top \boldsymbol{E}^\top \boldsymbol{P}\boldsymbol{N}\boldsymbol{v}[k] \\
&\qquad + \boldsymbol{v}[k]^\top \boldsymbol{N}^\top \boldsymbol{P}\boldsymbol{N}\boldsymbol{v}[k] + \alpha\left(\boldsymbol{e}[k]^\top \boldsymbol{P}\boldsymbol{e}[k] - \|\boldsymbol{v}[k]\|^2\right) \\
&= \Delta V[k] + \alpha(V[k] - \|\boldsymbol{v}[k]\|^2) \\
&\leq 0
\end{aligned}
$$

Error stability test proof—Contd.

- Condition of the lemma holds with $\mu_1 = 1$
- The observer error satisfies

$$\limsup_{k\to\infty} \|e[k]\| \leq \gamma \limsup_{k\to\infty} \|v[k]\|_\infty$$

  where

$$\gamma = 1/\sqrt{\lambda_{\min}(P)}$$

- In summary, the state error dynamics are ℓ_∞-stable with performance level γ

From a matrix inequality to an LMI

- Let $\boldsymbol{Z} = \boldsymbol{P}\boldsymbol{L}$, then solving the matrix inequality

  $$\begin{bmatrix} \boldsymbol{E}^\top \boldsymbol{P}\boldsymbol{E} - (1-\alpha)\boldsymbol{P} & * \\ \boldsymbol{N}^\top \boldsymbol{P}\boldsymbol{E} & \boldsymbol{N}^\top \boldsymbol{P}\boldsymbol{N} - \alpha \boldsymbol{I} \end{bmatrix} \preceq 0$$

  is equivalent to solving the LMI

  $$\begin{bmatrix} -\boldsymbol{P} & * \\ \boldsymbol{\Omega_{21}} & \boldsymbol{\Omega_{22}} \end{bmatrix} \preceq 0,$$

  for $\boldsymbol{P}$ and $\boldsymbol{Z}$, where

  $$\boldsymbol{\Omega_{21}}^\top = \begin{bmatrix} \boldsymbol{P}\boldsymbol{A}_1 - \boldsymbol{Z}\boldsymbol{C} & -\boldsymbol{Z}\boldsymbol{D} \end{bmatrix}$$

  and

  $$\boldsymbol{\Omega_{22}} = \begin{bmatrix} -(1-\alpha)\boldsymbol{P} & \boldsymbol{O}_{n\times m_2} \\ \boldsymbol{O}_{m_2\times n} & -\alpha \boldsymbol{I} \end{bmatrix}$$

- Take the Schur complement

  $$\boldsymbol{\Omega_{22}} + \boldsymbol{\Omega_{21}}\boldsymbol{P}^{-1}\boldsymbol{\Omega_{21}}^\top \preceq 0$$

  which yields the matrix inequality

Verification

Recall that $\boldsymbol{e}[k+1] = (\boldsymbol{A}_1 - \boldsymbol{LC})\boldsymbol{e}[k] - \boldsymbol{LDv}[k] := \boldsymbol{Ee}[k] + \boldsymbol{Nv}[k]$ and $\boldsymbol{Z} = \boldsymbol{PL}$.
Then

$$
\begin{aligned}
&\boldsymbol{\Omega}_{22} + \boldsymbol{\Omega}_{21}\boldsymbol{P}^{-1}\boldsymbol{\Omega}_{21}^\top \\
&= \begin{bmatrix} -(1-\alpha)\boldsymbol{P} & \boldsymbol{O}_{n\times m_2} \\ \boldsymbol{O}_{m_2\times n} & -\alpha\boldsymbol{I} \end{bmatrix} + \begin{bmatrix} \boldsymbol{A}_1^\top - \boldsymbol{C}^\top\boldsymbol{L}^\top \\ -\boldsymbol{D}^\top\boldsymbol{L}^\top \end{bmatrix} \begin{bmatrix} \boldsymbol{PA}_1 - \boldsymbol{ZC} & -\boldsymbol{ZD} \end{bmatrix} \\
&= \begin{bmatrix} -(1-\alpha)\boldsymbol{P} & \boldsymbol{O}_{n\times m_2} \\ \boldsymbol{O}_{m_2\times n} & -\alpha\boldsymbol{I} \end{bmatrix} + \begin{bmatrix} \boldsymbol{E}^\top \\ \boldsymbol{N}^\top \end{bmatrix} \begin{bmatrix} \boldsymbol{PE} & \boldsymbol{PN} \end{bmatrix} \\
&= \begin{bmatrix} -(1-\alpha)\boldsymbol{P} & \boldsymbol{O}_{n\times m_2} \\ \boldsymbol{O}_{m_2\times n} & -\alpha\boldsymbol{I} \end{bmatrix} + \begin{bmatrix} \boldsymbol{E}^\top\boldsymbol{PE} & \boldsymbol{E}^\top\boldsymbol{PN} \\ \boldsymbol{N}^\top\boldsymbol{PE} & \boldsymbol{N}^\top\boldsymbol{PN} \end{bmatrix} \\
&= \begin{bmatrix} \boldsymbol{E}^\top\boldsymbol{PE} - (1-\alpha)\boldsymbol{P} & \boldsymbol{E}^\top\boldsymbol{PN} \\ \boldsymbol{N}^\top\boldsymbol{PE} & \boldsymbol{N}^\top\boldsymbol{PN} - \alpha\boldsymbol{I} \end{bmatrix} \\
&\preceq 0
\end{aligned}
$$

Sufficient condition for UIO existence

Theorem

The UIO exists if

1. *there exists* $\boldsymbol{M}$ *such that*

$$(\boldsymbol{I}_n - \boldsymbol{M}\boldsymbol{C})\boldsymbol{B}_2 = \boldsymbol{O}_{n\times m_2} \quad \textit{and} \quad \boldsymbol{M}\boldsymbol{D} = \boldsymbol{O}_{n\times r}$$

2. *the pair* $(\boldsymbol{A}_1, \boldsymbol{C})$ *is detectable*

If $(\boldsymbol{A}_1, \boldsymbol{C})$ detectable, then we can find the observer gain matrix $\boldsymbol{L}$ such that $(\boldsymbol{A}_1 - \boldsymbol{L}\boldsymbol{C})$ is Schur stable