ECE 602: LUMPED LINEAR SYSTEMS

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Local Stability and Linearized Dynamics of Nonlinear Systems at Equilibrium Points

Lumped Nonlinear Systems

Lumped continuous-time nonlinear system:

$$\dot{x}(t) = f(x, u, t), \qquad y(t) = g(x, u, t)$$

- $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^n, \quad g : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^p$

Lumped discrete-time nonlinear system:

$$x[k+1] = f(x[k], u[k], k), \qquad y[k] = g(x[k], u[k], k)$$

- $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{Z} \to \mathbb{R}^n, \quad g : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{Z} \to \mathbb{R}^p$

Time-Invariant Autonomous Nonlinear Systems

Given a time-invariant autonomous nonlinear system $\dot{x} = f(x)$, it has an equilibrium point x_e if $f(x_e) = 0$

- $x(t) = x_e$ for all t is a solution of the system

Definition (Local Asymptotic Stability)

$\dot{x} = f(x)$ is locally asymptotically stable at the equilibrium point x_e if there exists some $r > 0$ such that

$$\|x(0) - x_\mathrm{e}\| < r \quad \Rightarrow \quad \lim_{t \to \infty} \|x(t) - x_\mathrm{e}\| \to 0$$

- All solutions starting in a ball of radius r around x_e converge to it
- If r can be chosen to be ∞, we get global asymptotic stability
- Can similarly define local exponential stability

Linearized Dynamics at Equilibrium Points

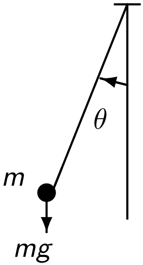
Suppose $\dot{x} = f(x)$ has an equilibrium point x_{e} and f is differentiable

The linearized dynamics of the nonlinear system at x_e is

$$\dot{z}(t) = Df(x_{\mathrm{e}})\, z(t)$$

- $Df(x) = \left[\frac{\partial f_i}{\partial x_j}(x)\right]_{i,j=1,\ldots,n}$ is the Jacobian matrix of $f : \mathbb{R}^n \to \mathbb{R}^n$
- $x(t) \approx x_{\mathrm{e}} + z(t)$ if $\|x(t) - x_{\mathrm{e}}\|$ is sufficiently small

Example: Simple Pendulum



Dynamics: $\ddot{\theta} = -mg\ell \sin\theta - \eta\dot{\theta}$

- $\eta > 0$ is damping coefficient

State $x = \begin{bmatrix}\theta & \dot{\theta}\end{bmatrix}^T$ has dynamics

$$\frac{d}{dt}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} = f(x) = \begin{bmatrix}x_2 \\ -mg\ell \sin x_1 - \eta x_2\end{bmatrix}$$

Two equilibrium points $x_{\mathrm{e}1} = \begin{bmatrix}0 \\ 0\end{bmatrix}$, $x_{\mathrm{e}2} = \begin{bmatrix}\pi \\ 0\end{bmatrix}$, with linearized dynamics:

$$\frac{d}{dt}z(t) = \underbrace{\begin{bmatrix}0 & 1 \\ -mg\ell & -\eta\end{bmatrix}}_{Df(x_{\mathrm{e}1})} z(t), \qquad \frac{d}{dt}z(t) = \underbrace{\begin{bmatrix}0 & 1 \\ mg\ell & -\eta\end{bmatrix}}_{Df(x_{\mathrm{e}2})} z(t)$$