PURDUE UNIVERSITY

ECE 68000: MODERN AUTOMATIC CONTROL

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Application of the Lyapunov continuous matrix equation to evaluate performance indices: An example

Evaluating performance indices: An application

- In many control problems a dynamical system model is given in terms of a transfer function rather than in terms of a system of differential equations
- We discuss a method of converting a given rational function into a system of first-order differential equations
- After this process is completed, we use the Lyapunov theory to evaluate a performance index of interest

Converting a rational function into a system of first-order differential equations

- Given a rational function

$$E(s) = \frac{b_{n-1}s^{n-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0},$$

  where $E(s) = \mathcal{L}(e(t))$ is the Laplace transform of $e(t)$
- We show that $e(t)$ can be generated as the solution of the differential equation

$$\frac{d^n e(t)}{dt^n} + a_{n-1}\frac{d^{n-1}e(t)}{dt^{n-1}} + \cdots + a_1\frac{de(t)}{dt} + a_0 e(t) = 0,$$

  subject to a set of initial conditions that we derive next
- Take the Laplace transform of the differential equation and rearrange the resulting terms in an appropriate matrix equation

Taking the Laplace transform

- Recall that

$$\mathcal{L}\left(\frac{d^i e(t)}{dt^i}\right) = s^i E(s) - s^{i-1}e(0) - s^{i-2}\frac{de(0)}{dt} - \cdots - \frac{d^{i-1}e(0)}{dt^{i-1}}$$

- In the s-domain with zero initial conditions,

$$\left(s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0\right) E(s) = 0$$

- Take into account the above relation, and compare coefficients of like powers to obtain

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_{n-1} & 1 \\ a_2 & a_3 & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n-1} & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} e(0) \\ \frac{de(0)}{dt} \\ \vdots \\ \frac{d^{n-2}e(0)}{dt^{n-2}} \\ \frac{d^{n-1}e(0)}{dt^{n-1}} \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-2} \\ b_{n-1} \end{bmatrix}$$

From s-domain to time domain

- The coefficient matrix nonsingular
- Can uniquely determine the initial conditions
- Define the state vector

$$\boldsymbol{x} = \begin{bmatrix} e & \frac{de}{dt} & \cdots & \frac{d^{n-1}e}{dt^{n-1}} \end{bmatrix}^\top,$$

and represent the differential equation as

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x},$$

where

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix}.$$

Initial conditions

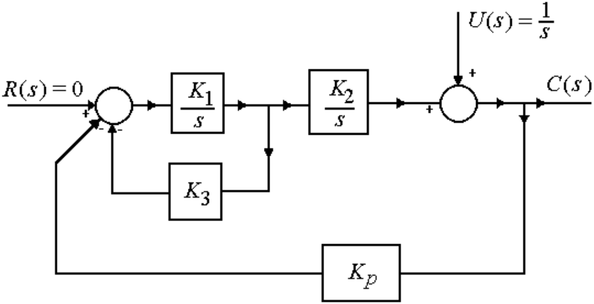
- The vector of initial conditions is

$$\boldsymbol{x}(0) = \begin{bmatrix} e(0) \\ \frac{de(0)}{dt} \\ \vdots \\ \frac{d^{n-2}e(0)}{dt^{n-2}} \\ \frac{d^{n-1}e(0)}{dt^{n-1}} \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & \cdots & a_{n-1} & 1 \\ a_2 & a_3 & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n-1} & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-2} \\ b_{n-1} \end{bmatrix}$$

- Note that $e = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \end{bmatrix} \boldsymbol{x}$
- Observe a relation between the zeros of the given rational function and the initial conditions of the associated state-space realization

Space telescope control system model

Parameters: $K_1 = 0.5$, and $K_1 K_2 K_p = 2.5$



Optimize K_3

- Use the Lyapunov theory to select the gain K_3 that minimizes the effect of the disturbance $u(t) = \mathcal{L}^{-1}(U(s))$ in the sense of the integral of the square error (ISE) criterion
- In other words, find K_3 that leads to minimization of the performance index

$$J_0 = \int_0^{\infty} c(t)^2 dt,$$

where $c(t) = \mathcal{L}^{-1}(C(s))$

- Assume a unit step disturbance and $R(s) = 0$

Finding $C(s) = \mathcal{L}(c(t))$

- From the block diagram

$$C(s) = U(s) - \frac{K_2}{s}\frac{K_1}{s + K_1K_3}K_pC(s)$$

- Hence

$$C(s) = \frac{s + 0.5K_3}{s^2 + 0.5K_3s + 2.5}$$

- Next, determine the state-space representation corresponding to $C(s)$
- Form the differential equation for $c(t)$,

$$\frac{d^2c(t)}{dt^2} + 0.5K_3\frac{dc(t)}{dt} + 2.5c(t) = 0$$

with the equation for the initial conditions

$$\begin{bmatrix} 0.5K_3 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} c(0) \\ \frac{dc(0)}{dt} \end{bmatrix} = \begin{bmatrix} 0.5K_3 \\ 1 \end{bmatrix}$$

State-space model

- Hence

$$\left[\begin{array}{c} c(0) \\ \frac{dc(0)}{dt} \end{array}\right] = \left[\begin{array}{cc} 0.5K_3 & 1 \\ 1 & 0 \end{array}\right]^{-1} \left[\begin{array}{c} 0.5K_3 \\ 1 \end{array}\right] = \left[\begin{array}{c} 1 \\ 0 \end{array}\right]$$

- Let $x_1 = c$, $x_2 = \frac{dc}{dt}$
- The differential equation in vector-matrix form corresponding to $C(s)$

$$\left.\begin{array}{rcl} \dot{\boldsymbol{x}}(t) & = & \left[\begin{array}{cc} 0 & 1 \\ -2.5 & -0.5K_3 \end{array}\right] \boldsymbol{x}(t), \quad \boldsymbol{x}(0) = \left[\begin{array}{c} 1 \\ 0 \end{array}\right] \\ c(t) & = & \boldsymbol{c}\boldsymbol{x}(t), \end{array}\right\}$$

where $\boldsymbol{c} = \left[\begin{array}{cc} 1 & 0 \end{array}\right]$

Finding optimal K_3

- Evaluate J_0,

$$J_0 = \int_0^\infty c(t)^2 dt = \int_0^\infty \boldsymbol{x}(t)^\top \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \boldsymbol{x}(t) dt$$
$$= \int_0^\infty \boldsymbol{x}(t)^\top \boldsymbol{c}^\top \boldsymbol{c}\, \boldsymbol{x}(t) dt$$

- $(\boldsymbol{A}, \boldsymbol{c})$ observable; can solve the Lyapunov equation

$$\boldsymbol{A}^\top \boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} = -\boldsymbol{c}^\top \boldsymbol{c}$$

for $\boldsymbol{P}$ to obtain

$$\boldsymbol{P} = \begin{bmatrix} \frac{K_3}{10} + \frac{1}{K_3} & \frac{1}{5} \\ \frac{1}{5} & \frac{2}{5K_3} \end{bmatrix}$$

- Hence, $J_0 = \boldsymbol{x}(0)^\top \boldsymbol{P}\boldsymbol{x}(0) = \frac{K_3}{10} + \frac{1}{K_3}$

Optimal $K_3 = \sqrt{10}$

- First-derivative test—differentiate $J_0 = \frac{K_3}{10} + \frac{1}{K_3}$ and equate to zero
$$\frac{dJ_0}{dK_3} = \frac{1}{10} - \frac{1}{K_3^2} = 0.$$
- Select
$$K_3 = \sqrt{10}$$
giving a positive definite $\boldsymbol{P}$
- Note that because
$$\left.\frac{d^2 J_0}{dK_3^2}\right|_{K_3=\sqrt{10}} = \left.\frac{2}{K_3^3}\right|_{K_3=\sqrt{10}} > 0$$
that is, second-order sufficiency condition for the minimum satisfied
- Hence $K_3 = \sqrt{10} \approx 3.16$ minimizes J_0

The integral of time multiplied by the squared error (ITSE)

- Determine the gain K_3 that minimizes the effect of the disturbance $u(t)$ in the sense of the integral of time multiplied by the squared error (ITSE) criterion
- That is, select K_3 to minimize the performance index

$$J_1 = \int_0^\infty t c(t)^2 dt.$$

- First solve the Lyapunov equation,

$$\boldsymbol{A}^\top \boldsymbol{P}_1 + \boldsymbol{P}_1 \boldsymbol{A} = -\boldsymbol{P},$$

to get

$$\boldsymbol{P}_1 = \begin{bmatrix} \frac{2}{K_3^2} + \frac{K_3^2}{100} & \frac{1}{5}\left(\frac{1}{K_3} + \frac{K_3}{10}\right) \\ \frac{1}{5}\left(\frac{1}{K_3} + \frac{K_3}{10}\right) & \frac{4}{5K_3^2} + \frac{1}{25} \end{bmatrix}$$

Minimization of ITSE

- Hence

$$J_1 = \boldsymbol{x}(0)^\top \boldsymbol{P}_1 \boldsymbol{x}(0) = \frac{2}{K_3^2} + \frac{K_3^2}{100}.$$

- First-derivative test—differentiate J_1 and equate to zero

$$\frac{dJ_1}{dK_3} = -\frac{4}{K_3^3} + \frac{K_3}{50} = 0.$$

- Select $K_3 = \sqrt[4]{200}$ to keep $\boldsymbol{P}_1$ positive definite
- For this choice of K_3, the second order sufficiency condition for the minimum of J_1 is satisfied
- Thus, $K_3 \approx 3.76$ minimizes J_1