Continuous Random Variables

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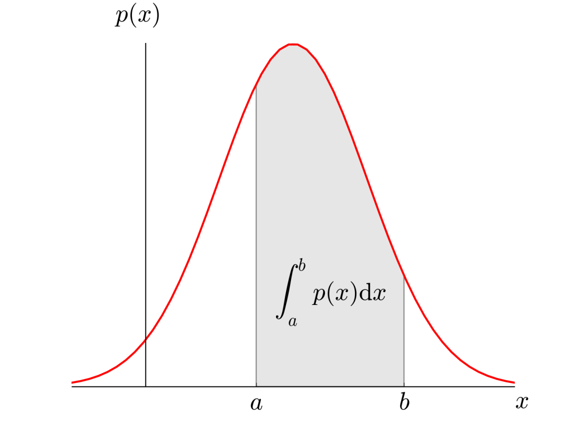
Definition: Continuous Random Variable

A random variable X is continuous if it can take on an uncountable infinite number of possible outcomes.

- The probability of X taking any definite value is exactly zero, otherwise the sum of all probabilities would diverge.
- The probability that the possible values lie in some fixed interval $[a, b]$ is the probability we are interested in.

The probability density function

$$P(a \leq X \leq b) = \int_a^b f(x)dx$$



Properties of $f(x)$

- $f(x) > 0$ for all x
- $\int_{-\infty}^{\infty} f(x)dx = 1$
- $f(a)$ is not a probability, it can be interpreted as a (relative) measure of how likely it is that X will be near a.

The distribution function

$$F(a) = P(X \leq a)$$

The relation between the probability density function f and the distribution function F:

$$F(b) = \int_{-\infty}^{b} f(x)dx$$
$$f(x) = \frac{d}{dx}F(x)$$

Also, for both discrete and continuous random variables:

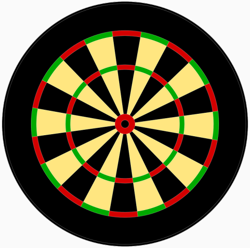
$$P(a < X \leq b) = P(X \leq b) - P(X \leq a) = F(b) - F(a)$$

The darts example

Let X be the distance between the hitting point and the center of the disc. Find $P(0 < X \leq r/2)$ and $P(r/2 < X \leq r)$

$$F(b) = P(X \leq b) = \frac{\pi b^2}{\pi r^2} = \frac{b^2}{r^2} \quad \text{for } 0 \leq b \leq r$$

$$f(x) = \frac{d}{dx} F(x) = \frac{1}{r^2} \frac{d}{dx} x^2 = \frac{2x}{r^2}$$



Expectation and variance

$$E[X] = \int_{-\infty}^{\infty} xf(x)dx'$$

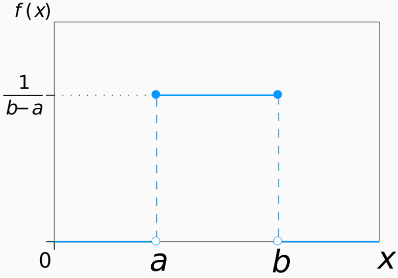
$$Var(X) = E[(X - E[X])^2] = E[X^2] - E[X]^2$$

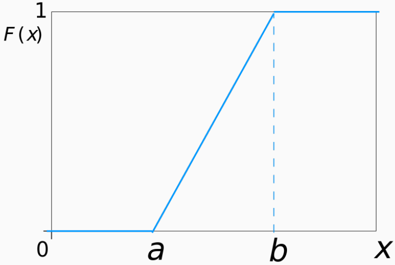
The same properties as with discrete random variables apply.

The uniform distribution $U(a, b)$

$$f(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

$$F(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \leq b \\ 1 & x > b \end{cases}$$





$$\mathrm{E}[X] = \frac{a+b}{2} \qquad \mathrm{Var}(X) = \frac{(b-a)^2}{12}$$

The uniform distribution $U(a, b)$. Example

Buses arrive at a specified stop at 15-minute intervals starting at 7 A.M. That is, they arrive at 7, 7:15, 7:30, 7:45, and so on. If a passenger arrives at the stop at a time that is uniformly distributed between 7 and 7:30, find the probability that he waits (a) less than 5 minutes for a bus; (b) at least 12 minutes for a bus.

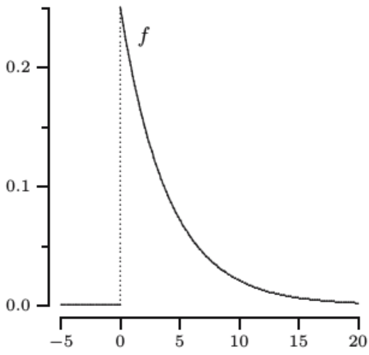
The exponential distribution $Exp(\lambda)$

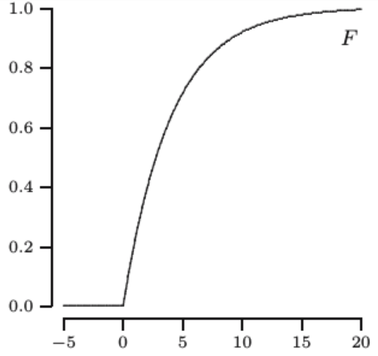
The exponential distribution describes the time for a continuous process to change state.

- The time until a radioactive particle decays, or the time between clicks of a geiger counter.
- The time it takes before your next telephone call.
- Distance between roadkills on a given road.

The exponential distribution $Exp(\lambda)$

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases} \qquad F(x) = \begin{cases} 1 - e^{-\lambda x} & x \geq 0 \\ 0 & x < b \end{cases}$$





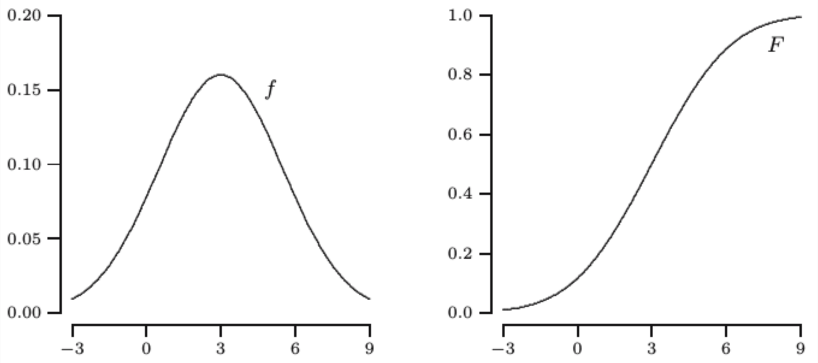
$$\mathrm{E}[X] = \lambda^{-1} \qquad \mathrm{Var}(X) = \lambda^{-2}$$

Example

A study of the response time of a certain computer system yields that the response time in seconds has an exponentially distributed time with parameter 0.25. What is the probability that the response time exceeds 5 seconds?

The normal distribution $N(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \qquad F(a) = \int_{-\infty}^{a} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$



$$\mathrm{E}[X] = \mu \qquad \mathrm{Var}(X) = \sigma^2$$

The normal distribution $N(\mu, \sigma^2)$

Some examples of this behavior are the height of a person, the velocity in any direction of a molecule in gas, and the error made in measuring a physical quantity.

If X is normal with mean μ and variance σ^2, then for any constants a and b, $b \neq 0$, the random variable $Y = a + bX$ is also a normal random variable with parameters

$$\begin{aligned} \mathrm{E}[Y] &= a + b\mu \\ \mathrm{Var}(Y) &= b^2\sigma^2 \end{aligned}$$

The normal distribution $N(\mu, \sigma^2)$

Any $N(\mu, \sigma^2)$ distributed random variable can be turned into an $N(0, 1)$ distributed random variable by a simple transformation:

$$Z = \frac{X - \mu}{\sigma}$$

This yields the standard normal distribution ϕ:

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

with an associated distribution function:

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy$$

The normal distribution $N(\mu, \sigma^2)$

$$
\begin{aligned}
P(X < b) =& \Phi\left(\frac{b-\mu}{\sigma}\right) \\
P(a < X < b) =& \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right) \\
\Phi(-x) =& 1 - \Phi(x)
\end{aligned}
$$

The normal distribution $N(\mu, \sigma^2)$. Example

Let the random variable Z have a standard normal distribution. Use R(pnorm) to find $P(Z \leq 0.75)$. How do you know, without doing any calculations, that the answer should be larger than 0.5?

The normal distribution $N(\mu, \sigma^2)$. Example

If X is a normal random variable with mean $\mu = 3$ and variance $\sigma^2 = 16$, find (a) $P(X < 11)$; (b) $P(X > -1)$; (c) $P(2 < X < 7)$.

The normal distribution $N(\mu, \sigma^2)$

The sum of independent normal random variables is also a normal random variable.

$\sum_{i=1}^{n} X_i$ is normal with mean $\sum_{i=1}^{n} \mu_i$ and variance $\sum_{i=1}^{n} \sigma_i^2$.

The normal distribution $N(\mu, \sigma^2)$. Example

Data from the National Oceanic and Atmospheric Administration indicate that the yearly precipitation in Los Angeles is a normal random variable with a mean of 12.08 inches and a standard deviation of 3.1 inches. (a) Find the probability that the total precipitation during the next 2 years will exceed 25 inches. (b) Find the probability that next year's precipitation will exceed that of the following year by more than 3 inches.

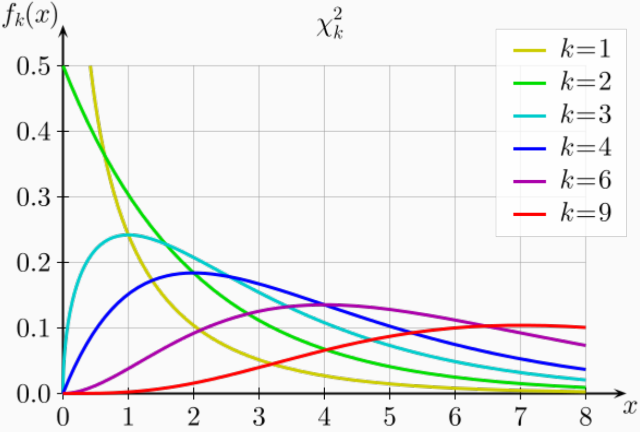
The Chi-Square Distribution

If $Z_1, Z_2, \ldots, Z_n$ are independent standard normal random variables, then

$$X = Z_1^2 + Z_2^2 + \cdots + Z_n^2$$

is said to have a chi-square distribution with n degrees of freedom. We will use the notation $X \sim \chi_n^2$ to signify that X has a chi-square distribution with n degrees of freedom.

The Chi-Square Distribution



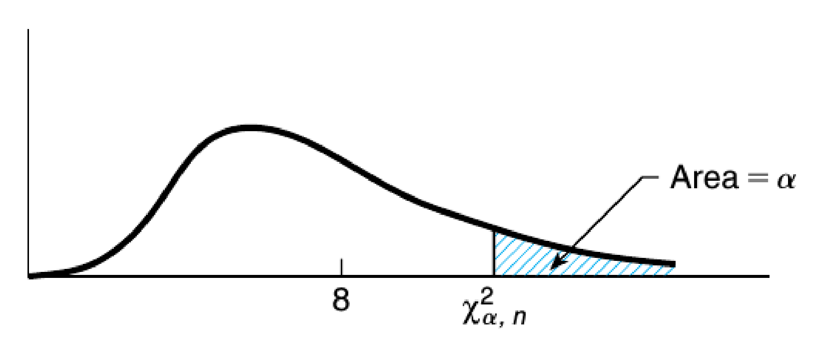
The Chi-Square Distribution

If X_1 and X_2 are independent chi-square random variables with n_1 and n_2 degrees of freedom, respectively, then $X_1 + X_2$ is chi-square with $n_1 + n_2$ degrees of freedom.

The Chi-Square Distribution

If X is a chi-square random variable with n degrees of freedom, then for any $\alpha \in (0,1)$, the quantity $\chi^2_{\alpha,n}$ is defined to be such that

$$P\left(X \geq \chi^2_{\alpha,n}\right) = \alpha$$



The Chi-Square Distribution. Example

Determine $P\left(\chi^2_{26} \leq 30\right) = 0.732$ (Use the R function pchisq)

Find $\chi^2_{0.05,15} = 24.996$ (Use the R function qchisq)

Suppose that we are attempting to locate a target in three-dimensional space, and that the three coordinate errors (in meters) of the point chosen are independent normal random variables with mean 0 and standard deviation 2. Find the probability that the distance between the point chosen and the target exceeds 3 meters.

The t-Distribution

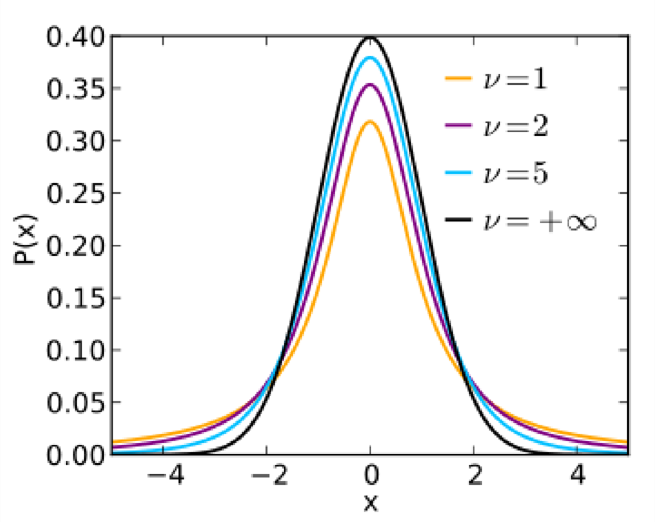
If Z and χ_n^2 are independent random variables, with Z having a standard normal distribution and χ_n^2 having a chi-square distribution with n degrees of freedom, then the random variable T_n defined by

$$T_n = \frac{Z}{\sqrt{\chi_n^2/n}}$$

is said to have a t-distribution with n degrees of freedom.

For let $t_{\alpha,n}$ be such that $P\left(T_n \geq t_{\alpha,n}\right) = \alpha$

The t-Distribution



$$E[T_n] = 0 \quad \mathrm{Var}(T_n) = \frac{n}{n-2}$$

The t-Distribution. Example

Find (a) $P\,(T_{12} \leq 1.4) = 0.9066$ and (b) $t_{0.025,9} = 2.2621$. (Use R functions pt and qt)

The F-Distribution

If χ_n^2 and χ_m^2 are independent chi-square random variables with n and m degrees of freedom, respectively, then the random variable $F_{n,m}$ defined by

$$F_{n,m} = \frac{\chi_n^2/n}{\chi_m^2/m}$$

is said to have an F-distribution with n and m degrees of freedom.

Let $F_{\alpha,n,m}$ be such that $P\left(F_{n,m} > F\alpha, n, m\right) = \alpha$.

Example: Determine $P\left(F_{6,14} \leq 1.5\right) = 0.7518$ (Use the R function pf).