INTRODUCTION TO PROBABILITY

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DEFINITION

An experiment is any process that produces an observation or outcome. The set of all possible outcomes of an experiment is called the sample space S.

EXAMPLES

THE GENDER OF A CHILD.

$$S = \{g, b\}$$

FLIPPING TWO COINS

$$S = \{(H,H),(H,T),(T,H),(T,T)\}$$

ORDER OF FINISH IN A RACE AMONG 7 HORSES

$$S = \{\text{all permutations of } 1,2,3,4,5,6,7\}$$

ROLLING TWO SIX-SIDED DICE

$$
\begin{gathered}
S=\{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),\\
(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),\\
(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),\\
(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),\\
(5,1),(5,2),(5,3),(5,4),(5,5),(5,6),\\
(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}
\end{gathered}
$$

DEFINITION

Any set of outcomes of the experiment is called an event. We designate events by the letters A, B, C, and so on. We say that the event A occurs whenever the outcome is contained in A.

EXAMPLES

THE GENDER OF A CHILD.

$$S = \{g, b\}$$

- $A = \{g\}$ is the event that the child is a girl.
- $B = \{b\}$ is the event that the child is a boy.

FLIPPING TWO COINS

$$S = \{(H, H), (H, T), (T, H), (T, T)\}$$

- $A = \{(H, H), (H, T)\}$ is the event that the first coin lands on heads.

ORDER OF FINISH IN A RACE AMONG 7 HORSES

$$S = \{\text{all permutations of } 1,2,3,4,5,6,7\}$$

- $A = \{\text{all outcomes in S starting with } 2\}$ is the event that horse number 2 wins the race.

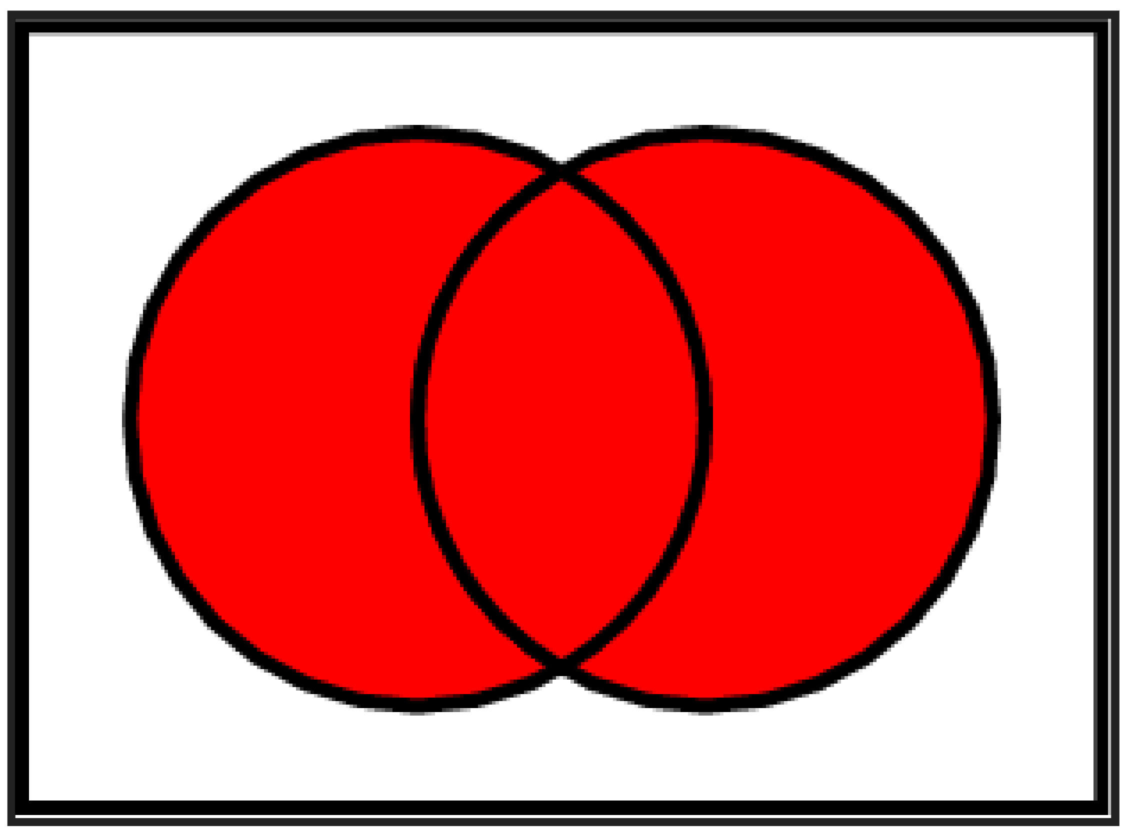
ROLLING TWO SIX-SIDED DICE

$$
\begin{gathered}
S=\{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),\\
(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),\\
(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),\\
(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),\\
(5,1),(5,2),(5,3),(5,4),(5,5),(5,6),\\
(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}
\end{gathered}
$$

- $A=\{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$ is the event that the sum of the dice is 7

UNION OF EVENTS

$$A \cup B$$



EXAMPLES

THE GENDER OF A CHILD.

$$S = \{g, b\}$$

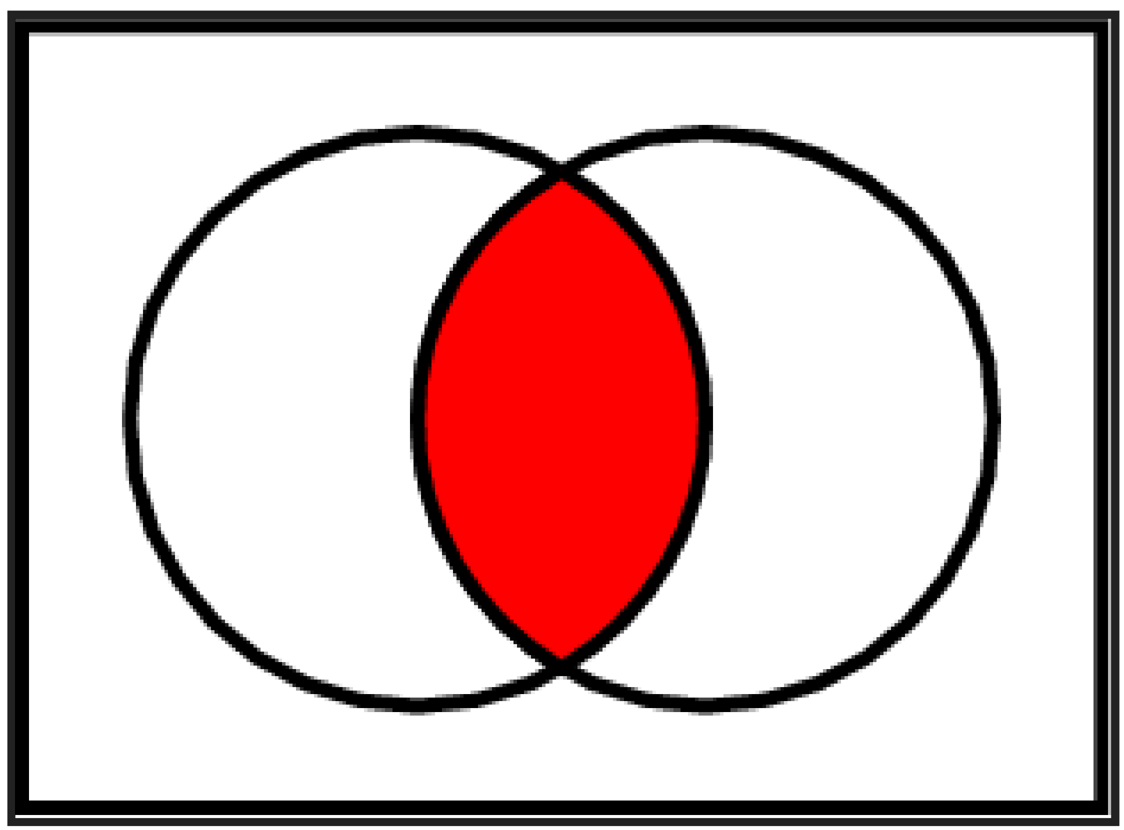
- $A = \{g\}$ is the event that the child is a girl.
- $B = \{b\}$ is the event that the child is a boy.
- $A \cup B = \{g, b\} = S$

ORDER OF FINISH IN A RACE AMONG 7 HORSES

- $A = \{\text{all outcomes starting with 4}\}$
- $B = \{\text{all outcomes in with seconf element equal to 2}\}$
- $A \cup B$ is the event that either the number 4 horse wins or the number 2 horse comes in second or both.

INTERSECTION OF EVENTS

$$A \cap B$$



EXAMPLES

FLIPPING TWO COINS

$$S = \{(H,H),(H,T),(T,H),(T,T)\}$$

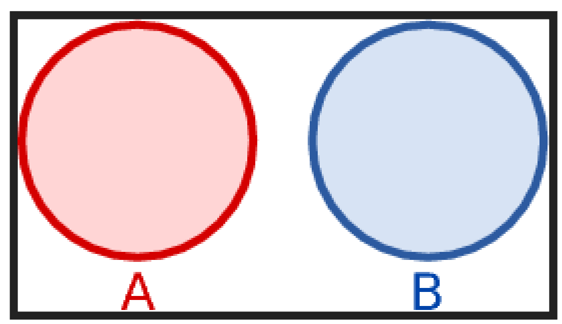
- $A = \{(H,H),(H,T)\}$ is the event that the first coin lands on heads.
- $B = \{(H,T,(T,T)\}$ is the event that the second coin lands on tails.
- $A \cap B = \{H,T\}$

ORDER OF FINISH IN A RACE AMONG 7 HORSES

- $A = \{\text{all outcomes starting with } 2\}$
- $B = \{\text{all outcomes starting with } 3\}$
- $A \cap B = \emptyset$ is the null event or empty set.

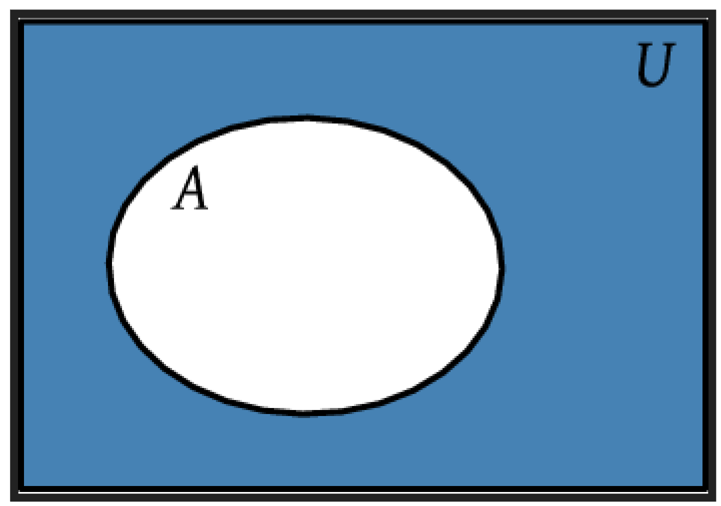
DISJOINT OR MUTUALLY EXCLUSIVE EVENTS

$$A \cap B = \emptyset$$



COMPLEMENT

All outcomes in the sample space that are not in A.



$$A^c$$

$$S^c = \emptyset$$

EXAMPLE: GENDER OF A CHILD.

$$S = \{g, b\}$$

- $A = \{g\}$ is the event that the child is a girl.
- $A^c = \{b\}$

PROPERTIES OF PROBABILITY

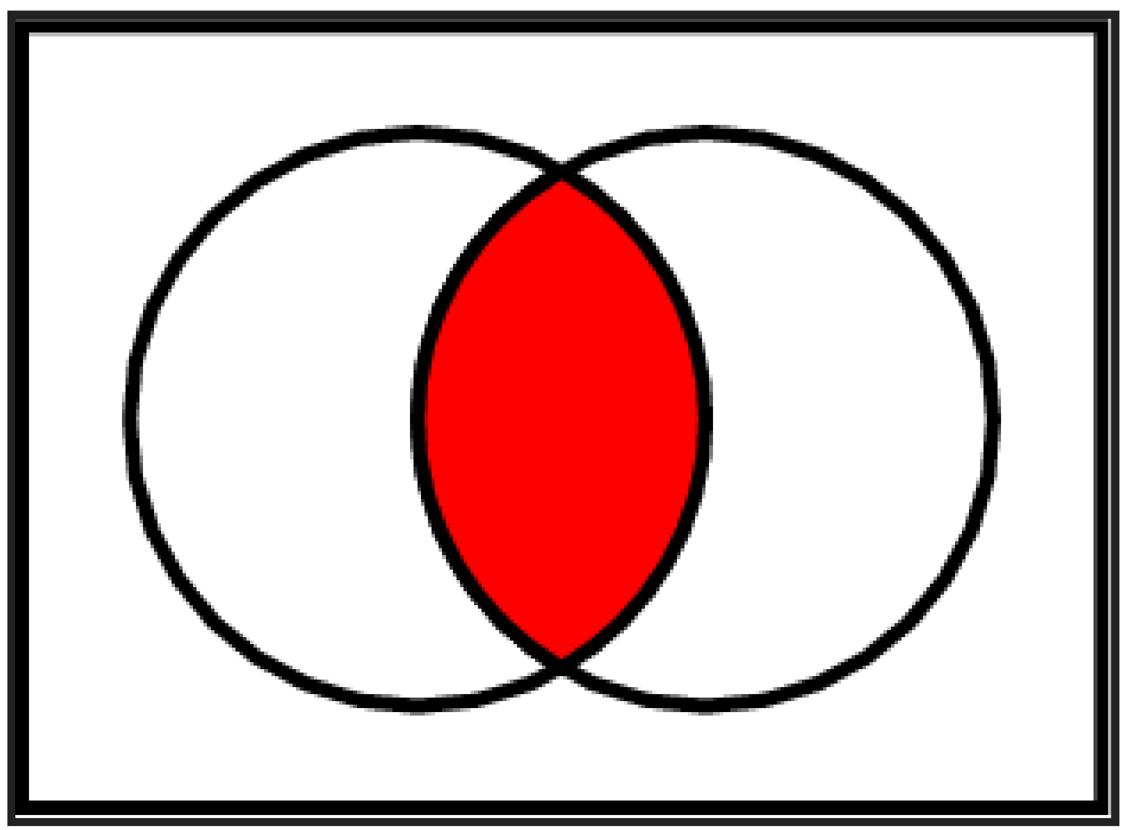
For each event A there is a number, denoted $P(A)$ and called the probability of event A.

- $0 \leq P(A) \leq 1$
- $P(S) = 1$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A)$ is the long run frequency of event A.

ADDITION RULE OF PROBABILITY

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



PROBABILITY OF THE COMPLEMENT

$$P(A^c) = 1 - P(A)$$

EXAMPLES

A certain retail establishment accepts either the American Express or the VISA credit card. A total of 22 percent of its customers carry an American Express card, 58 percent carry a VISA credit card, and 14 percent carry both. What is the probability that a customer will have at least one of these cards?

TOSSING A COIN AND ROLL OF A DIE

An R simulation.

EQUIPROBABILITY

Some processes are expected to have equally likely outcomes (coin, dice, etc.).

$$S = \{1, 2, \ldots, N\}$$

$$P(\{1\} = P(\{2\}) = \cdots = P(\{N\})$$

THEN

$$P(A) = \frac{\text{number of outcoms in S that are in A}}{N}$$

EXAMPLES

In a survey of 420 members of a retirement center, it was found that 144 are smokers and 276 are not. If a member is selected in such a way that each of the members is equally likely to be the one selected, what is the probability that person is a smoker?

Suppose that when two dice are rolled, each of the 36 possible outcomes is equally likely. Find the probability that the sum of the dice is 6 and that it is 7.

One man and one woman are to be selected from a group that consists of 10 married couples. If all possible selections are equally likely, what is the probability that the woman and man selected are married to each other?

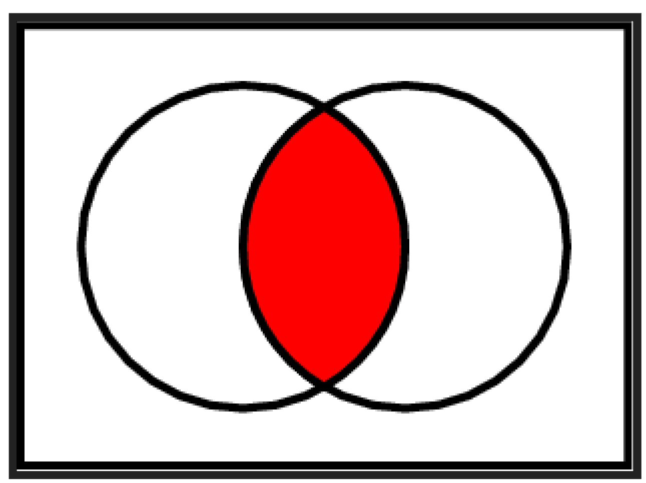
An elementary school is offering two optional language classes, one in French and the other in Spanish. These classes are open to any of the 120 upper-grade students in the school. Suppose there are 32 students in the French class, 36 in the Spanish class, and a total of 8 who are in both classes. If an upper-grade student is randomly chosen, what is the probability that this student is enrolled in at least one of these classes?

CONDITIONAL PROBABILITY

If some partial information concerning the outcome of the experiment is available we have conditional probabilities.

The conditional probability of B given that A has occurred:

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$



EXAMPLES

As a further check of the preceding formula for the conditional probability, use it to compute the conditional probability that the sum of a pair of rolled dice is 10, given that the first die lands on 4.

The organization that employs Jacobi is organizing a parent-daughter dinner for those employees having at least one daughter. Each of these employees is asked to attend along with one of his or her daughters. If Jacobi is known to have two children, what is the conditional probability that they are both girls given that Jacobi is invited to the dinner? Assume the sample space S is given by

$$S = \{(g, g), (g, b), (b, g), (b, b)\}$$

and that all these outcomes are equally likely, where the outcome (g, b) means, for instance, that Jacobi's oldest child is a girl and youngest is a boy.

MULTIPLICATION RULE

$$P(A \cap B) = P(A)P(B|A)$$

EXAMPLES

Suppose that two people are randomly chosen from a group of 4 women and 6 men.

1. What is the probability that both are women?
2. What is the probability that one is a woman and the other a man?

INDEPENDENT EVENTS

Events A and B are independent if

$$P(A \cap B) = P(A)P(B)$$

Note that this means

$$P(B|A) = P(B)$$

EXAMPLES

Suppose that we roll a pair of fair dice, so each of the 36 possible outcomes is equally likely. Let A denote the event that the first die lands on 3, let B be the event that the sum of the dice is 8, and let C be the event that the sum of the dice is 7.

1. Are A and B independent?
2. Are A and C independent?

A couple is planning on having three children. Assuming that each child is equally likely to be of either sex and that the sexes of the children are independent, find the probability that

1. All three children will be girls.
2. At least one child will be a girl.

BAYES' THEOREM

H is the event a hypothesis is true. E is evidence concerning the hypothesis.

$$P(H|E) = \frac{P(E|H)P(H)}{P(E|H)P(H) + P(E|H^c)P(H^c)}$$

PROOF

Consider the following equation

$$P(A) = P(A|B)P(B) + P(A|B^c)P(B^c)$$

By the definition of conditional probability

$$P(H|E) = \frac{P(H \cap E)}{P(E)} = \frac{P(E|H)P(H)}{P(E)}$$

Combining both equations gives us Bayes' theorem.

EXAMPLES

An insurance company believes that people can be divided into two classes— those who are prone to have accidents and those who are not. The data indicate that an accident-prone person will have an accident in a 1-year period with probability 0.1; the probability for all others is 0.05. Suppose that the probability is 0.2 that a new policyholder is accident-prone.

1. What is the probability that a new policyholder will have an accident in the first year?
2. If a new policyholder has an accident in the first year, what is the probability that he or she is accident-prone?

A blood test is 99 percent effective in detecting a certain disease when the dis- ease is present. However, the test also yields a false-positive result for 2 percent of the healthy patients tested. (That is, if a healthy person is tested, then with probability 0.02 the test will say that this person has the disease.) Suppose 0.5 percent of the population has the disease. Find the conditional probability that a randomly tested individual actually has the disease given that his or her test result is positive.

COUNTING PRINCIPLES

Basic principle of counting:

Suppose an experiment consists of two parts. If part 1 can result in any of n possible outcomes and if for each outcome of part 1 there are m possible outcomes of part 2, then there is a total of nm possible outcomes of the experiment.

EXAMPLES

One man and one woman are to be selected from a group consisting of 12 women and 8 men. How many different choices are possible?

Two people are to be selected from a group that consists of 10 married couples. How many different choices are possible? If each choice is equally likely, what is the probability that the two people selected are married to each other?

GENERALIZED BASIC PRINCIPLE OF COUNTING

Suppose an experiment consists of r parts. Suppose there are n_1 possible outcomes of part 1 and then n_2 possible outcomes of part 2 and then n_3 possible outcomes of part 3, and so on. Then there is a total of $n_1 \cdot n_2 \cdots n_r$ possible outcomes of the experiment.

PERMUTATION OF N OBJECTS

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1$$

EXAMPLES

If four people are in a room, what is the probability that no two of them celebrate their birthday on the same day of the year?

COMBINATION OF N OBJECTS TAKEN R AT A TIME

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}$$

EXAMPLES

1. How many different groups of size 2 can be selected from the items a, b, c?
2. How many different groups of size 2 can be chosen from a set of 6 people?
3. How many different groups of size 3 can be chosen from a set of 6 people?

A random sample of size 3 is to be selected from a set of 10 items. What is the probability that a prespecified item will be selected?

A committee of 4 people is to be selected from a group of 5 men and 7 women. If the selection is made randomly, what is the probability the committee will consist of 2 men and 2 women?

SYMMETRY

$$\binom{n}{r} = \frac{n!}{(n-r)!r!} = \binom{n}{n-r}$$

EXAMPLES

Compare $\binom{8}{5}$ and $\binom{12}{10}$.

Suppose that n + m digits, n of which are equal to 1 and m of which are equal to 0, are to be arranged in a linear order. How many different arrangements are possible? For instance, if n = 2 and m = 1, then there are 3 possible arrangements:

$$1, 1, 0 \quad 1, 0, 1 \quad 0, 1, 1$$