Assignment 3

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1. First, put

$$\begin{aligned} x_0 = -1, x_1 = 1, x_2 = 2, \\ y_0 = 2, y_1 = 0, y_2 = 2, \\ y_0' = 1, y_1' = 1, y_2' = 3. \end{aligned}$$

Also, put

$$z_0 = z_1 = -1, z_2 = z_3 = 1, z_4 = z_5 = 2.$$

Now, we can put the Hermite interpolating polynomial $H_5(x)$ by

$$H_5(x) = a_0 + a_1(x - z_0) + \cdots + a_5(x - z_0)\cdots(x - z_5). \tag{1}$$

Then we can make divided difference table as below:

z_0	$f[z_0]$					
z_1	$f[z_1]$	$f[z_0, z_1] = f'(z_1)$				
z_2	$f[z_2]$	$f[z_1, z_2]$	$f[z_0, z_1, z_2]$			
z_3	$f[z_3]$	$f[z_2, z_3] = f'(z_3)$	$f[z_1, z_2, z_3]$	$f[z_0, \cdots, z_3]$		
z_4	$f[z_4]$	$f[z_3, z_4]$	$f[z_2, z_3, z_4]$	$f[z_1, \cdots, z_4]$	$f[z_0, \cdots, z_4]$	
z_5	$f[z_5]$	$f[z_4, z_5] = f'(z_5)$	$f[z_3, z_4, z_5]$	$f[z_2, \cdots, z_5]$	$f[z_1, \cdots, z_5]$	$f[z_0, \cdots, z_5]$

Now, let's compute the entires of the table. Then we get the below table:

-1	2					
-1	2	1				
1	0	-1	-1			
1	0	1	1	1		
2	2	2	1	0	$-\frac{1}{3}$	
2	2	3	1	0	0	$\frac{1}{9}$

(2)

Note that

$$a_n = f[z_0, \cdots, z_n].$$

Hence, we get

$$\begin{aligned} H_5(x) =& 2 + (x+1) - (x+1)^2 + (x+1)^2(x-1) \\ &- \frac{1}{3}(x+1)^2(x-1)^2 + \frac{1}{9}(x+1)^2(x-1)^2(x-2) \end{aligned} \tag{3}$$

2. Below is my code.

```
% Set our function
syms x
f(x) = (x^2 + 1)^(-1);

% Set inputs and outputs
inputs = linspace(-5, 5, 21);
a = f(inputs);
b = zeros(1, 20);
c = zeros(1, 21); % For Step 5&6. c(21) is not output.
d = zeros(1, 20);

% Step 1
h = inputs(2) - inputs(1);

% Step 2
alpha = zeros(1, 20);
for i = 2:20
    alpha(i) = 3/h*(a(i+1)-a(i)) - 3/h*(a(i)-a(i-1));
end

% Step 3
l = zeros(1, 21);
l(1) = 1;
mu = zeros(1, 21);
z = zeros(1, 21);

% Step 4
for i = 2:20
    l(i) = 4*h - h*mu(i-1);
    mu(i) = h/l(i);
    z(i) = (alpha(i) - h*z(i-1)) / l(i);
end

% Step 5
l(21) = 1;
z(21) = 0; % actually z = zeros(1, 21) already implies this
c(21) = 0; % actually c = zeros(1, 21) already implies this

% Step 6
for j = 20:-1:1
    c(j) = z(j) - mu(j)*c(j+1);
    b(j) = (a(j+1) - a(j)) / h - h * (c(j+1) + 2*c(j)) / 3;
    d(j) = (c(j+1) - c(j)) / (3*h);
end

% Step 7: Plotting
X = linspace(-5, 5, 51)
Y = zeros(1, 51);

S(x) = a(1) + b(1)*(x-inputs(1)) + c(1)*(x-inputs(1))^2 + d(1)
    *(x-inputs(1))^3;
j = 1;
for i = 1:51
```

```
    if inputs(j+1) < X(i)
        j = j + 1;
        S(x) = a(j) + b(j)*(x-inputs(j)) + c(j)*(x-inputs(j))^2 + d(j)*(x-inputs(j))^3;
    end
    Y(i) = S(X(i)) - f(X(i));
end
disp(Y);
plot(X, Y);
```

Then we get the values of $S(x) - f(x)$ at 51 equally spaced points and the corresponding graph (Fig. 1).

```
0     0.0001     0.0000    -0.0000    -0.0000     0.0000     0.0000
       0.0000    -0.0000    -0.0000    -0.0000    -0.0000     0.0000
      -0.0000    -0.0001     0.0000     0.0001     0.0001    -0.0003
      -0.0009    -0.0000     0.0029     0.0019    -0.0015    -0.0021
       0.0000    -0.0021    -0.0015     0.0019     0.0029    -0.0000
      -0.0009    -0.0003     0.0001     0.0001    -0.0000    -0.0001
      -0.0000     0.0000    -0.0000     0.0000    -0.0000    -0.0000
       0.0000     0.0000     0.0000    -0.0000    -0.0000     0.0000
       0.0001    -0.0000
```

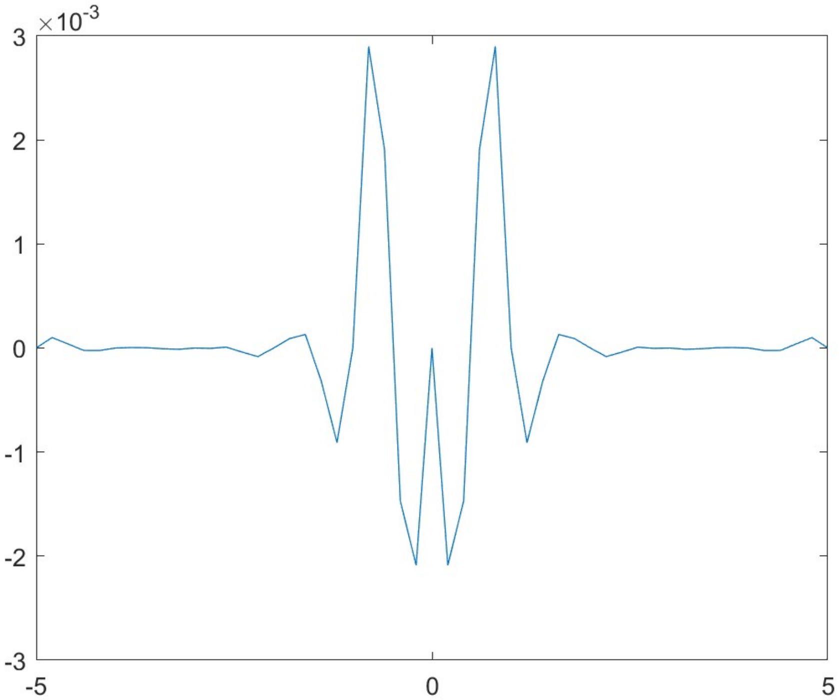

Figure 1: Plotting of $S(x) - f(x)$

Now, compare this problem with the Problem 4 of Assignment 2. In the Problem 4 of Assignment 2, we used Newton interpolating polynomial P interpolating a function f. Since the degree of such polynomial is big, 20, it may oscillates a lot. Actually, we saw that $P(x) - f(x)$

is far from zero near $x = -5, 5$. But for the cubic spline polynomial, since this is a piecewise polynomial with relatively low degree, 3, $S(x) - f(x)$ is near zero in the entire interval $[-5, 5]$. Of course, the value of $P(x) - f(x)$ and $S(x) - f(x)$ are both zero at node points, $x = x_j,\ j = 0, 1, 2, \cdots, 20$.

3. (a) By the two-point forward formula, we get

$$f'(x) = \frac{1}{h}[f(x+h) - f(x)] - \frac{h}{2}f''(\xi)$$

where ξ is between x and $x+h$. Let $\tilde{\ }$ represent the floating number approximation (computed by computer) and e is corresponding round-off error. Then

$$f'(x) = \frac{1}{h}[\tilde{f}(x+h) - \tilde{f}(x)] + \frac{1}{h}[e(x+h) - e(x)] - \frac{h}{2}f''(\xi).$$

Now, assume that the round-off $e(x)$ and $e(x+h)$ are bounded by some number $\epsilon > 0$ and the second derivative of f is bounded by a number $M > 0$, then

$$\left| f'(x) - \frac{1}{h}[\tilde{f}(x+h) - \tilde{f}(x)] \right| \leq \frac{2\epsilon}{h} + \frac{hM}{2}. \tag{4}$$

Here, by the arithmetic mean-geometric mean inequality, we can get the minimum bound for the bound of the error

$$\frac{2\epsilon}{h} + \frac{hM}{2} \geq 2\sqrt{\epsilon M},$$

and the equality holds when $\frac{2\epsilon}{h} = \frac{hM}{2}$, i.e., when $h = 2\sqrt{\epsilon/M}$. We can get ϵ by typing 'eps' in the MATLAB and can simply put $M = 1$ since $f''(x) = -\cos x$. Note that eps value is $2.220446049250313 \times 10^{-16}$. Hence we get the minimum bound for the error when

$$h = 2\sqrt{\frac{\epsilon}{M}} = 2\sqrt{2.220446049250313 \times 10^{-16}} \approx 3 \times 10^{-8}. \tag{5}$$

Until now, we estimate the minimum error using the theorem with what we learned in the class. But actually, round-off error eps is just the upper bound for the round-off error, but not the least upper bound. So the value of h that estimates derivative could be smaller than (5). With this notice, I estimate the derivative using smaller h as well as (5). Below is my code for estimating derivative:

```
format long;

syms x h;
f(x) = cos(x);
N1(x, h) = (f(x+h) - f(x))/h;

h0 = 3 * 10^(-8);
a = -sin(0.25);
b = double(N1(0.25,h0));
e = rel_error(a, b);

fprintf("f'(0.25) is: %.15f\n", a);
fprintf("estimated value is: %.15f\n", b);
fprintf("relative error is: %.15e\n\n", e);

h0 = 10^(-17);
```

```
b = double(N1(0.25,h0));
e = rel_error(a, b);

fprintf("f'(0.25) is: %.15f\n", a);
fprintf("re-estimated value is: %.15f\n", b);
fprintf("relative error is: %.15e\n", e);

function e = rel_error(real_val, estimated_val)
    e = abs((real_val - estimated_val) / estimated_val);
end
```

And the below is the result:

```
f'(0.25) is: -0.247403959254523
estimated value is: -0.247403973788209
relative error is: 5.874475684322718e-08

f'(0.25) is: -0.247403959254523
re-estimated value is: -0.247403959254523
relative error is: 0.000000000000000e+00
```

(b) Put

$$N_1(h) := \frac{1}{h}[f(x+h) - f(x)]. \tag{6}$$

Using Taylor series, we get

$$f'(x) = N_1(h) - \frac{f''(x)}{2}h - \frac{f^{(3)}(x)}{6}h^2 - \frac{f^{(4)}(x)}{24}h^3 - \cdots. \tag{7}$$

Taking $h/2$, we get

$$f'(x) = N_1\left(\frac{h}{2}\right) - \frac{f''(x)}{2}\frac{h}{2} - \frac{f^{(3)}(x)}{6}\frac{h^2}{4} - \frac{f^{(4)}(x)}{24}\frac{h^3}{8} - \cdots.$$

Multiplying it by 2 and subtracting (7), we get

$$f'(x) = 2N_1\left(\frac{h}{2}\right) - N_1(h) + \frac{f^{(3)}(x)}{12}h^2 + \frac{3f^{(4)}(x)}{96}h^3 + \cdots.$$

Then by putting

$$N_2(h) := 2N_1\left(\frac{h}{2}\right) - N_1(h), \tag{8}$$

we can rewrite the equation as below:

$$f'(x) = N_2(h) + \frac{f^{(3)}(x)}{12}h^2 + \frac{3f^{(4)}(x)}{96}h^3 + \cdots. \tag{9}$$

Again, taking $h/2$ on (9), we get

$$f'(x) = N_2\left(\frac{h}{2}\right) + \frac{f^{(3)}(x)}{12}\frac{h^2}{4} + \frac{3f^{(4)}(x)}{96}\frac{h^3}{8} + \cdots.$$

Multiplying it by 4 and subtracting (9), we get

$$3f'(x) = 4N_2\left(\frac{h}{2}\right) - N_2(h) - \frac{3f^{(4)}(x)}{96}\frac{h^3}{2} - \cdots .$$

Then by putting

$$N_3(h) := \frac{1}{3}\left(4N_2\left(\frac{h}{2}\right) - N_2(h)\right), \tag{10}$$

we can rewrite the equation as below:

$$f'(x) = N_3(h) - \frac{f^{(4)}(x)}{192}h^3 - \cdots . \tag{11}$$

Let's see $N_3(h)$ in detail. First,

$$\begin{aligned} N_2(h) &= 2\frac{2}{h}\left[f\left(x+\frac{h}{2}\right) - f(x)\right] - \frac{1}{h}[f(x+h) - f(x)] \\ &= \frac{1}{h}\left[-3f(x) + 4f\left(x+\frac{h}{2}\right) - f(x+h)\right]. \end{aligned}$$

Therefore,

$$\begin{aligned} N_3(h) &= \frac{1}{3}\left(4\frac{2}{h}\left[-3f(x) + 4f\left(x+\frac{h}{4}\right) - f\left(x+\frac{h}{2}\right)\right]\right. \\ &\qquad \left. -\frac{1}{h}\left[-3f(x) + 4f\left(x+\frac{h}{2}\right) - f(x+h)\right]\right) \\ &= \frac{1}{3}\left(\frac{1}{h}\left[-21f(x) + 32f\left(x+\frac{h}{4}\right) - 12f\left(x+\frac{h}{2}\right) + f(x+h)\right]\right), \end{aligned}$$

i.e.,

$$N_3(h) = \frac{1}{3h}\left[-21f(x) + 32f\left(x+\frac{h}{4}\right) - 12f\left(x+\frac{h}{2}\right) + f(x+h)\right]. \tag{12}$$

For the error bound, we can apply similar method of (a). We may write

$$f'(x) = N_3(h) - \frac{h^3}{192}f^{(4)}(\xi),$$

where ξ is between x and $x+h$. Then we get

$$\begin{aligned} &\left|f'(x) - \frac{1}{3h}\left[-21\tilde{f}(x) + 32\tilde{f}\left(x+\frac{h}{4}\right) - 12\tilde{f}\left(x+\frac{h}{2}\right) + \tilde{f}(x+h)\right]\right| \\ &\le \frac{22\epsilon}{h} + \frac{Mh^3}{192}, \end{aligned} \tag{13}$$

where $\epsilon > 0$ is the bound of the round-off error and $M > 0$ is the bound of $f^{(4)}$. We can get ϵ by typing 'eps' in the MATLAB and can simply put $M = 1$ since $f^{(4)}(x) = \cos x$. By calculating h where the derivative of $\frac{22\epsilon}{h} + \frac{Mh^3}{192}$ is zero, we can know that this bound is minimum when

$$h = \left(1408\frac{\epsilon}{M}\right)^{1/4} = \left(1408 \times 2.220446049250313 \times 10^{-16}\right)^{1/4} \approx 7.5 \times 10^{-4}. \tag{14}$$

With the same reasoning in (a), I estimate the derivative using smaller h as well as (14). Below is my code for estimating derivative:

```
format long;

syms x h;
f(x) = cos(x);
N3(x, h) = 1 / (3 * h) * (-21*f(x) + 32*f(x+h/4) - 12*f(x+h/2)
    + f(x+h));

h0 = 7.5 * 10^(-4);
a = -sin(0.25);
b = double(N3(0.25,h0));
e = rel_error(a, b);

fprintf("f'(0.25) is: %.15f\n", a);
fprintf("estimated value is: %.15f\n", b);
fprintf("relative error is: %.15e\n\n", e);

h0 = 10^(-5);
b = double(N3(0.25,h0));
e = rel_error(a, b);

fprintf("f'(0.25) is: %.15f\n", a);
fprintf("re-estimated value is: %.15f\n", b);
fprintf("relative error is: %.15e\n", e);

function e = rel_error(real_val, estimated_val)
    e = abs((real_val - estimated_val) / estimated_val);
end
```

And the below is the result:

```
f'(0.25) is: -0.247403959254523
estimated value is: -0.247403959252394
relative error is: 8.604651682115397e-12

f'(0.25) is: -0.247403959254523
re-estimated value is: -0.247403959254523
relative error is: 0.000000000000000e+00
```

4. Let's consider a function $f \in C^4[x_0, x_3]$ and the Lagrange interpolating function $P_3(x)$ interpolating $f(x)$ at four points: x_0, x_1, x_2, x_3. Here, we assume that these points are equally spaced with distance h, i.e., $h = x_i - x_{i-1}$. By the theorem for the Lagrange interpolation function, for any $x \in [x_0, x_3]$ we can choose $\xi(x) \in [x_0, x_3]$ satisfying

$$f(x) = P_3(x) + \frac{f^{(4)}(\xi(x))}{4!}(x - x_0)(x - x_1)(x - x_2)(x - x_3). \tag{15}$$

Now let's see $P_3(x)$ in detail. Let

$$P_3(x) = \sum_{i=0}^{3} a_i (x - x^*)^i,$$

where $x^* = (x_0 + x_3)/2$. Then by plugging in $x = x_k,\ k = 0, 1, 2, 3$, we get

$$\begin{aligned}
f(x_0) = P_3(x_0) &= a_0 - \frac{3}{2}ha_1 + \frac{9}{4}h^2 a_2 - \frac{27}{8}h^3 a_3, \\
f(x_1) = P_3(x_1) &= a_0 - \frac{1}{2}ha_1 + \frac{1}{4}h^2 a_2 - \frac{1}{8}h^3 a_3, \\
f(x_2) = P_3(x_2) &= a_0 + \frac{1}{2}ha_1 + \frac{1}{4}h^2 a_2 + \frac{1}{8}h^3 a_3, \\
f(x_3) = P_3(x_3) &= a_0 + \frac{3}{2}ha_1 + \frac{9}{4}h^2 a_2 + \frac{27}{8}h^3 a_3.
\end{aligned}$$

Here,

$$\begin{aligned}
f(x_0) + f(x_3) &= 2a_0 + \frac{9}{2}h^2 a_2, \\
f(x_1) + f(x_2) &= 2a_0 + \frac{1}{2}h^2 a_2.
\end{aligned}$$

Hence, we get

$$\begin{aligned}
a_0 &= \frac{1}{16}[-f(x_0) + 9f(x_1) + 9f(x_2) - f(x_3)], \\
a_2 &= \frac{1}{4h^2}[f(x_0) - f(x_1) - f(x_2) + f(x_3)].
\end{aligned}$$

By integrating P_3 on $[x_0, x_3]$, we get

$$\begin{aligned}
\int_{x_0}^{x_3} P_3(x)dx &= \sum_{i=0}^{3} a_i \int_{x_0}^{x_3} (x - x^*)^i \\
&= a_0 \int_{x_0}^{x_3} 1dx + a_2 \int_{x_0}^{x_3} (x - x^*)^2 dx \\
&= a_0 3h + a_2 2 \int_0^{\frac{3}{2}h} x^2 dx \\
&= 3a_0 h + \frac{9}{4}a_2 h^3 \\
&= \frac{3h}{16}[-f(x_0) + 9f(x_1) + 9f(x_2) - f(x_3)] \\
&\quad + \frac{9h}{16}[f(x_0) - f(x_1) - f(x_2) + f(x_3)] \\
&= \frac{3h}{8}[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)].
\end{aligned} \tag{16}$$

Note that I used a similarity at $x = x^*$ when intergrating it.

Now let's see error term in detail. Here, by using Weighted MVT for integral, we can choose $\xi_i \in [x_i, x_{i+1}],\ i = 0, 1, 2$ satisfying

$$\begin{aligned}
&\int_{x_i}^{x_{i+1}} \frac{f^{(4)}(\xi(x))}{4!}(x - x_0)(x - x_1)(x - x_2)(x - x_3)dx \\
&= \frac{f^{(4)}(\xi_i)}{4!} \int_{x_i}^{x_{i+1}} (x - x_0)(x - x_1)(x - x_2)(x - x_3)dx.
\end{aligned}$$

. Note that

$$\begin{aligned}
\int_{x_0}^{x_1} (x-x_0)(x-x_1)(x-x_2)(x-x_3)dx &= \int_0^h x(x-h)(x-2h)(x-3h)dx \\
&= -\frac{19}{30}h^5, \\
\int_{x_1}^{x_2} (x-x_0)(x-x_1)(x-x_2)(x-x_3)dx &= \int_h^{2h} x(x-h)(x-2h)(x-3h)dx \\
&= \frac{11}{30}h^5, \\
\int_{x_2}^{x_3} (x-x_0)(x-x_1)(x-x_2)(x-x_3)dx &= \int_{2h}^{3h} x(x-h)(x-2h)(x-3h)dx \\
&= -\frac{19}{30}h^5.
\end{aligned}$$

Then we get

$$\begin{aligned}
&\int_{x_0}^{x_3} \frac{f^{(4)}(\xi(x))}{4!}(x-x_0)(x-x_1)(x-x_2)(x-x_3)dx \\
&= \frac{h^5}{720}(-19f^{(4)}(\xi_0) + 11f^{(4)}(\xi_1) - 19f^{(4)}(\xi_2)).
\end{aligned}$$

I want to claim

$$-19f^{(4)}(\xi_0) + 11f^{(4)}(\xi_1) - 19f^{(4)}(\xi_2) = (-19+11-19)f^{(4)}(\xi),$$

where $\xi \in (x_0, x_3)$ using IVT, but it is hard to do that because it is not internal dividing point, but external dividing point. So, I'll just use the Closed Newton-Cotes Formulas for the error term directly. Then we get the error term:

$$\begin{aligned}
\frac{h^5 f^{(4)}(\xi)}{4!}\int_0^3 t(t-1)(t-2)(t-3)dt &= \frac{h^5 f^{(4)}(\xi)}{24}\int_0^3 t^4 - 6t^3 + 11t^2 - 6t\ dt \\
&= \frac{h^5 f^{(4)}(\xi)}{24}\left[\frac{t^5}{5} - \frac{3t^4}{2} + \frac{11t^3}{3} - 3t^2\right]_0^3 \\
&= \frac{h^5 f^{(4)}(\xi)}{24}\left(-\frac{9}{10}\right) \\
&= -\frac{3h^5}{80}f^{(4)}(\xi),
\end{aligned} \tag{17}$$

where $\xi \in (x_0, x_3)$. Hence, we get the following Simpson't three-eights rule

$$\int_{x_0}^{x_3} f(x)dx = \frac{3h}{8}[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)] - \frac{3h^5}{80}f^{(4)}(\xi). \tag{18}$$