

Basis for a system to monitor and/or classify Parkinson's movement symptoms

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Abstract

Parkinson's disease, motion symptoms, wavelet triplets, library, classification success, range of different motions dealt with

1 Introduction

The idea is to create a system which can capture the abnormal movements associated with Parkinson's disease (PD) and contrast them with the movements of control subjects. We want to create a library of both types of motion, viewed as paths in three dimensional acceleration space. This library should be representative of the motions encountered, without being so large as to be unwieldy, and we should have a mechanism enabling us to measure the similarity between members of the library and new data.

Our data is in the form of triaxial acceleration readings from a wearable devices or a mobile phone carried by our subjects.

If we were to store the raw accelerometry data in a library, then our data would consist of points in a $3S$ -dimensional space, where S is the length of our samples, and we would have to put data with different sample lengths in different spaces. To avoid this, we encode the data in the form of triplets of wavelets, one member of the triplet per acceleration channel. We choose (mother) *wavelets*¹ so that motions executed at slightly different speeds can be represented by the same thing².

2 Acceleration space “shapes” encoded as wavelets

We start by working in one acceleration dimension, and then generalise to three.

We define our mother wavelets to be piecewise polynomial functions

$$\psi(x) = \begin{cases} \sum_{i=0}^n a_i x^i, & x \in [0, 1); \\ 0, & \text{otherwise,} \end{cases} \quad (1)$$

where the wavelet conditions $\int_0^1 \psi(x) dx = 0$ (zero integral — zi — forcing ψ to be a *wave*) and $\int_0^1 [\psi(x)]^2 dx = 1$ (unit energy — ue — forcing ψ to be “*small*”, i.e., a *wavelet*), must hold.

¹More precisely, wavelets in the continuous wavelet transform paradigm

²Actually, actions performed at vastly different speeds would also be represented by the same thing, but this is not needed here

In terms of the vector of coefficients, \mathbf{a} , the zi is

$$\mathbf{b}^T \mathbf{a} = 0, \quad (2)$$

for the vector \mathbf{b} given by

$$b_i = \frac{1}{i+1}, \quad i = 0, 1, \dots, n \quad (3)$$

and the ue is

$$\mathbf{a}^T \mathbf{H}_0 \mathbf{a} = 1, \quad (4)$$

for the matrix \mathbf{H}_0 given by

$$H_{0,ij} = \frac{1}{i+j+1}, \quad i, j = 0, 1, \dots, n \quad (5)$$

(\mathbf{H}_0 is the Hilbert matrix).

If ψ_i , $i = 1, 2$ are two wavelets, with coefficient vectors \mathbf{a}_i , then their (L_2) inner product is

$$\langle \psi_1, \psi_2 \rangle_{L_2} = \int_{-\infty}^{\infty} \psi_1(x) \psi_2(x) dx = \int_0^1 \psi_1(x) \psi_2(x) dx = \mathbf{a}_1^T \mathbf{H}_0 \mathbf{a}_2 \quad (6)$$

and their squared (L_2) distance is

$$\begin{aligned} \|\psi_1 - \psi_2\|_{L_2}^2 &= \|\psi_1\|_{L_2}^2 - 2 \langle \psi_1, \psi_2 \rangle_{L_2} + \|\psi_2\|_{L_2}^2 \\ &= \mathbf{a}_1^T \mathbf{H}_0 \mathbf{a}_1 - 2 \mathbf{a}_1^T \mathbf{H}_0 \mathbf{a}_2 + \mathbf{a}_2^T \mathbf{H}_0 \mathbf{a}_2 = 2[1 - \mathbf{a}_1^T \mathbf{H}_0 \mathbf{a}_2]. \end{aligned} \quad (7)$$

2.1 Wavelet triplets

We first generalise the one-dimensional case of the previous section, and then add some purely three-dimensional machinery.

If our triplet of wavelets is $\boldsymbol{\psi}(x) = [\psi^{(1)}(x), \psi^{(2)}(x), \psi^{(3)}(x)]^T$, where each $\psi^{(j)}$ is defined as in equation (1), but with coefficient vector $\mathbf{a}^{(j)}$, we generalise zi by applying equation (2) to each $\psi^{(j)}$ individually, and ue by

$$\sum_{j=1}^3 \mathbf{a}^{(j)T} \mathbf{H}_0 \mathbf{a}^{(j)} = 1. \quad (8)$$

In three dimensional acceleration space, the inner product (6) becomes

$$\langle \boldsymbol{\psi}_1, \boldsymbol{\psi}_2 \rangle_{L_2} = \sum_{j=1}^3 \mathbf{a}_1^{T(j)} \mathbf{H}_0 \mathbf{a}_2^{(j)} \quad (9)$$

and so the squared distance of equation (7) becomes

$$\|\boldsymbol{\psi}_1 - \boldsymbol{\psi}_2\|_{L_2}^2 = 2 \left[1 - \sum_{j=1}^3 \mathbf{a}_1^{T(j)} \mathbf{H}_0 \mathbf{a}_2^{(j)} \right]. \quad (10)$$

2.2 Rotations

We now come to the three dimensional material which is not purely a generalisation of the one dimensional definitions.

As we are primarily interested in “shapes”, we wish to consider shapes which can be rotated or reflected into each other as the same thing.

We do this through equivalence classes: if $\exists \mathbf{O} \in \text{O}(3)$ such that

$$\psi'^{(j)}(x) = \sum_{\ell=1}^3 O_{j\ell} \psi^{(\ell)} \quad \forall j = 1, 2, 3, \quad (11)$$

then we say $\psi' \sim \psi$, and ψ' is in the Equivalence Class of Wavelet Triplets (ECWT) $[\psi]$ represented by ψ .

Assume now that $[\psi]$ and $[\psi']$ are possibly different ECWTs. We can define an inner product on the set of ECWTs:

$$\begin{aligned} \langle [\psi], [\psi'] \rangle_R &= \max_{\psi'' \in [\psi], \psi''' \in [\psi']} \langle \psi'', \psi''' \rangle_{L_2} \\ &= \max_{\mathbf{O} \in \text{O}(3)} \sum_{j=1}^3 \sum_{k=1}^3 \left\langle \psi^{(j)}, O_{jk} \psi'^{(k)} \right\rangle_{L_2}, \end{aligned} \quad (12)$$

where the second equality follows from the fact that $\text{O}(3)$ is a group.

But then

$$\begin{aligned} \langle [\psi], [\psi'] \rangle_R &= \max_{\mathbf{O} \in \text{O}(3)} \sum_{j=1}^3 \sum_{k=1}^3 O_{jk} \left\langle \psi^{(j)}, \psi'^{(k)} \right\rangle_{L_2} \\ &= \max_{\mathbf{O} \in \text{O}(3)} \sum_{j=1}^3 \sum_{k=1}^3 O_{jk} \mathbf{a}^{(j)\text{T}} \mathbf{H}_0 \mathbf{a}'^{(k)} \\ &= \max_{\mathbf{O} \in \text{O}(3)} \text{trace}(\mathbf{O}^T \mathbf{K}), \end{aligned} \quad (13)$$

where \mathbf{K} is defined by

$$K_{jk} = \mathbf{a}^{(j)\text{T}} \mathbf{H}_0 \mathbf{a}'^{(k)}. \quad (14)$$

It is known that $(\text{trace}(\mathbf{O}^T \mathbf{K}))$ is maximised for fixed \mathbf{K} and $\mathbf{O} \in \text{O}(3)$ by

$$\mathbf{O} = \mathbf{U} \mathbf{V}^T; \text{ where } \mathbf{K} = \mathbf{U} \mathbf{S} \mathbf{V}^T \text{ is the singular value decomposition of } \mathbf{K}. \quad (15)$$

Hence,

$$\begin{aligned} \langle [\psi], [\psi'] \rangle_R &= \text{trace}(\mathbf{V} \mathbf{U}^T \mathbf{U} \mathbf{S} \mathbf{V}^T) = \text{trace}(\mathbf{V} \mathbf{S} \mathbf{V}^T) = \text{trace}(\mathbf{S}) \\ &= \text{sum of the singular values of } \mathbf{K}, \end{aligned} \quad (16)$$

by the properties of orthogonal matrices and the fact that $\text{trace}(\mathbf{A} \mathbf{B} \mathbf{C}) = \text{trace}(\mathbf{C} \mathbf{A} \mathbf{B})$ for any matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}$.

There is also the obvious “R” distance between two ECWTs: $\|\psi' - \psi\|_R^2 = 2 [1 - \langle \psi', \psi \rangle_R]$ (of course, $\langle \psi, \psi \rangle_R = \langle \psi, \psi \rangle_{L_2} = 1$).

2.3 Shifts

2.3.1 Shifts in one-dimensional acceleration space

We briefly return to the case of one acceleration dimension.

We can look at the L_2 inner product of a wavelet ψ and a time-shifted (and wrapped) version ψ'_t of another ψ' , given by

$$\psi'_t(x) = \begin{cases} \psi'(x+1-t), & x \in [0, t); \\ \psi'(x-t), & x \in [t, 1); \\ 0, & \text{otherwise} \end{cases} = \begin{cases} \sum_{k=0}^n a'_k (x+1-t)^k, & x \in [0, t); \\ \sum_{k=0}^n a'_k (x-t)^k, & x \in [t, 1); \\ 0, & \text{otherwise.} \end{cases} \quad (17)$$

Clearly,

$$\begin{aligned} \langle \psi, \psi'_t \rangle_{L_2} &= \sum_{k, \ell=0}^n a_k a'_\ell \left\{ \int_0^t x^k (x+1-t)^\ell dx + \int_t^1 x^k (x-t)^\ell dx \right\} \\ &= \sum_{k, \ell=0}^n a_k a'_\ell \sum_{r=0}^{\ell} \binom{\ell}{r} \left\{ (1-t)^{\ell-r} \int_0^t x^{k+r} dx + (-1)^{\ell-r} t^{\ell-r} \int_t^1 x^{k+r} dx \right\} \\ &= \sum_{k, \ell=0}^n a_k a'_\ell \sum_{r=0}^{\ell} \frac{1}{k+r+1} \binom{\ell}{r} \left\{ (1-t)^{\ell-r} t^{k+r+1} + (-1)^{\ell-r} [t^{\ell-r} - t^{k+\ell+1}] \right\} \\ &= \sum_{k, \ell=0}^n a_k a'_\ell \left\{ \sum_{r=0}^{\ell} \sum_{s=0}^{\ell-r} \frac{(-1)^s t^{k+r+s+1}}{k+r+1} \binom{\ell}{r} \binom{\ell-r}{s} + \right. \\ &\quad \left. \sum_{r=0}^{\ell} \binom{\ell}{r} \frac{(-1)^r [t^r - t^{k+\ell+1}]}{k+\ell+1-r} \right\} \\ &= \sum_{k, \ell=0}^n a_k a'_\ell \left\{ \sum_{p=0}^{\ell} \sum_{r=0}^p \binom{\ell}{r} \binom{\ell-r}{p-r} \frac{(-1)^{p-r} t^{k+p+1}}{k+r+1} + \right. \\ &\quad \left. \sum_{r=0}^{\ell} \binom{\ell}{r} \frac{(-1)^r [t^r - t^{k+\ell+1}]}{k+\ell+1-r} \right\} \\ &= \sum_{k, \ell=0}^n a_k a'_\ell \left\{ \sum_{p=0}^{\ell} (-1)^p \binom{\ell}{p} \left[\sum_{r=0}^p \frac{(-1)^r}{k+r+1} \binom{p}{r} \right] t^{k+p+1} + \right. \\ &\quad \left. \sum_{r=0}^{\ell} \binom{\ell}{r} \frac{(-1)^r t^r}{k+\ell+1-r} - (-1)^\ell \left[\sum_{r=0}^{\ell} \frac{(-1)^r}{k+1+r} \binom{\ell}{r} \right] t^{k+\ell+1} \right\}. \end{aligned} \quad (18)$$

But $\sum_{r=0}^{\ell} \frac{(-1)^r}{k+1+r} \binom{\ell}{r} = \int_0^1 x^k \sum_{r=0}^{\ell} (-1)^r \binom{\ell}{r} x^r dx = \int_0^1 x^k (1-x)^\ell dx$, and, recalling the definition

of the beta function, this is $B(k+1, \ell+1) = \frac{k!\ell!}{(k+\ell+1)!}$, so equation (18) becomes

$$\begin{aligned}
\langle \psi, \psi'_t \rangle_{L_2} &= \sum_{k, \ell=0}^n a_k a'_\ell \left\{ \sum_{p=0}^{\ell} \binom{\ell}{p} \frac{(-1)^p k! p! t^{k+p+1}}{(k+p+1)!} + \right. \\
&\quad \left. \sum_{r=0}^{\ell} \binom{\ell}{r} \frac{(-1)^r t^r}{k+\ell+1-r} - \frac{(-1)^\ell k!\ell! t^{k+\ell+1}}{(k+\ell+1)!} \right\} \\
&= \sum_{k, \ell=0}^n a_k a'_\ell \left\{ \sum_{p=0}^{\ell-1} (-1)^p \frac{k!\ell!}{(\ell-p)!(k+p+1)!} t^{k+p+1} + \sum_{r=0}^{\ell} \frac{(-1)^r}{k+\ell+1-r} \binom{\ell}{r} t^r \right\} \\
&= \sum_{k, \ell=0}^n a_k a'_\ell \left\{ \sum_{s=0}^{\ell} \frac{(-1)^s}{k+\ell+1-s} \binom{\ell}{s} t^s + \sum_{s=k+1}^{k+\ell} (-1)^{k+1+s} \frac{k!\ell!}{s!(k+\ell+1-s)!} t^s \right\} \\
&= \sum_{s=0}^{2n} \mathbf{a}^T \mathbf{H}^{(s)} \mathbf{a}' t^s, \tag{19}
\end{aligned}$$

where $\mathbf{H}^{(s)} = \mathbf{F}^{(s)} + \mathbf{G}^{(s)}$ for

$$F_{k\ell}^{(s)} = \begin{cases} \frac{(-1)^s}{k+\ell+1-s} \binom{\ell}{s}, & s \leq \ell \leq n; \\ 0, & \text{otherwise,} \end{cases} \tag{20}$$

and, if $1 \leq s \leq n+1$,

$$G_{k\ell}^{(s)} = \begin{cases} \frac{(-1)^{k+1+s} k!\ell!}{s!(k+\ell+1-s)!}, & \text{or } 1 \leq \ell \leq s, s-\ell \leq k \leq s-1; \\ 0, & \text{otherwise;} \end{cases} \tag{21}$$

if $n+1 \leq s \leq 2n$,

$$G_{k\ell}^{(s)} = \begin{cases} \frac{(-1)^{k+1+s} k!\ell!}{s!(k+\ell+1-s)!}, & s-n \leq \ell \leq s, s-\ell \leq k \leq n; \\ 0, & \text{otherwise.} \end{cases} \tag{22}$$

Note that $\mathbf{F}^{(0)} = \mathbf{H}_0$, $\mathbf{G}^{(0)} = 0$, so $\mathbf{H}^{(0)} = \mathbf{H}_0$, and $\mathbf{F}^{(s)} = 0$ for $s > n$. Also, $F_{k0}^{(s)} = G_{k0}^{(s)} = 0$ and $F_{0\ell}^{(s)} = -G_{0\ell}^{(s)}$ if $s \geq 1$, so we need not calculate elements of the first row or column of $\mathbf{F}^{(s)}$ and $\mathbf{G}^{(s)}$ as the corresponding elements of $\mathbf{H}^{(s)}$ are known to be zero if $s \geq 1$.

We can now define an inner product on our wavelets through

$$\langle \psi, \psi' \rangle_S = \max_{t \in [0,1]} \langle \psi, \psi'_t \rangle_{L_2} \tag{23}$$

(we include $t = 1$ in the values of t considered so that maximum-finding algorithms on compact sets actually work — doing so is trivial as $\lim_{t \rightarrow 1} \psi_t = \psi_0 = \psi$). As equation (19) shows, $\langle \psi, \psi'_t \rangle_{L_2}$ is the restriction to $[0, 1]$ of a polynomial p , say, of degree at most $2n$. This means the derivative of p has at most $2n - 1$ real roots, and, as the maxima of a polynomial are separated by its minima, p has at most n maxima on $(-\infty, \infty)$. However, we are not particularly interested in finding the best shift in one-dimensional acceleration space, and content ourselves with observing that \langle, \rangle_S is symmetric, as it should be, as $\langle \psi, \psi'_t \rangle_{L_2} = \langle \psi', \psi_{1-t} \rangle_{L_2}$.

2.3.2 Shifts in three-dimensional acceleration space

To simplify the notation, we will drop the square brackets around the ECWT $[\psi]$ represented by ψ , and allow the latter to stand for its entire equivalence class. Moreover, we will extend the notion of the equivalence class to ψ_t , given by

$$\psi_t^{(j)}(x) = \begin{cases} \psi^{(j)}(x+1-t), & x \in [0, t); \\ \psi^{(j)}(x-t), & x \in [t, 1); \\ 0, & \text{otherwise,} \end{cases} \quad j = 1, 2, 3, \quad (24)$$

in the obvious way — two shifted triplets are in the same class if one can be rotated into the other.

We can now define a new inner product on the set of ECWTs:

$$\langle \psi, \psi' \rangle_{SR} = \max_{t \in [0, 1)} \langle \psi, \psi'_t \rangle_R. \quad (25)$$

By our previous working, its obvious that

$$\langle \psi, \psi' \rangle_{SR} = \max_{t \in [0, 1)} \sum \{\text{singular values of } \mathbf{K}(t)\}, \quad (26)$$

where

$$K_{kt}(t) = \mathbf{a}^{(k)\top} \mathbf{H}(t) \mathbf{a}'^{(\ell)}. \quad (27)$$

Although this function would not define an inner product on the space of equivalence classes of all sets of triplets of L_2 -integrable functions which can be rotated into one another, as $\|[\mathbf{f}] - [\mathbf{g}]\|_{SR} = 0$ does not imply $[\mathbf{f}] = [\mathbf{g}]$, it does form one when \mathbf{f} and \mathbf{g} are triplets of polynomials³ inside $[0, 1)$ and zero outside. In that case, if $\|[\mathbf{f}] - [\mathbf{g}]\|_{SR} = 0$ and $[\mathbf{f}] \neq [\mathbf{g}]$, $\exists \mathbf{O} \in \mathbf{O}(3), t \in (0, 1)$ such that $g'^{(j)} = \sum_{k=1}^3 O_{jk} g^{(k)}$ and $\mathbf{f} = \mathbf{g}'_t$. But the components of \mathbf{f} are polynomials inside $[0, 1)$ and zero outside, while the nonzero components of \mathbf{g}'_t are not polynomials inside $[0, 1)$, as $t \notin \{0, 1\}$. This can be seen as at least one of the derivatives of the nonzero components of \mathbf{g}'_t is discontinuous at t .

3 Library of “shapes”

3.1 Tailoring wavelets to data

We now need to create an initial library of “shapes” from data.

Given a window into the data of length N samples of three dimensional acceleration data, $\{\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_N\}$, where $\mathbf{f}_k = [f_k^{(1)}, f_k^{(2)}, f_k^{(3)}]^\top$, we model the data by the piecewise constant functions

$$f^{(j)}(x) = \begin{cases} f_1^{(j)}, & x \in [0, \frac{1}{2}\Delta); \\ f_k^{(j)}, & x \in [\frac{2k-3}{2}\Delta, \frac{2k-1}{2}\Delta), \quad k = 2, 3, \dots, N-1; \\ f_N^{(j)}, & x \in [1 - \frac{1}{2}\Delta, 1); \\ 0, & \text{otherwise,} \end{cases} \quad (28)$$

where $\Delta = \frac{1}{N-1}$.

The appropriate inner product for matching wavelet triplets to the data is \langle, \rangle_{L_2} , as the matching process automatically selects the best rotation, and the window stepping through the data is equivalent to time-shifting the ECWTs.

As

³It is possible to be more precise here, by introducing equivalence classes $[[\mathbf{f}]]$ of equivalence classes $[\mathbf{f}]$, where $[\mathbf{g}] \in [[\mathbf{f}]]$ iff $\|[\mathbf{g}] - [\mathbf{f}]\|_{SR} = 0$, but this would take us too far down a tangent

3.2 Selection of library

4 Results

4.1 Data

4.2 Numbers

4.3 Interpretation

The thesis established, at least for the data set considered, that the method yields results which are competitive with measuring the excess signal energy associated with the frequencies classically considered characteristic with Parkinson's. However, also showed that a richer characterisation of Parkinson's through motion symptoms is at least possible.

5 Conclusion

Thesis showed basic feasibility, but produced for scant data. Further, more exhaustive studies are required to properly establish feasibility, and, if this is done, the interpretation of the results.