Basis for a system to monitor and/or classify Parkinson's movement symptoms

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Abstract

Parkinson's disease, motion symptoms, wavelet triplets, library, classification success, range of different motions dealt with

1 Introduction

The idea is to create a system which can capture the abnormal movements associated with Parkinson's disease (PD) and contrast them with the movements of control subjects. We want to create a library of both types of motion, viewed as paths in three dimensional acceleration space. This library should be representative of the motions encountered, without being so large as to be unwieldy, and we should have a mechanism enabling us to measure the similarity between members of the library and new data.

Our data is in the form of triaxal acceleration readings from a wearable devices or a mobile phone carried by our subjects.

If we were to store the raw accelerometry data in a library, then our data would consist of points in a $3S$-dimensional space, where S is the length of our samples, and we would have to put data with different sample lengths in different spaces. To avoid this, we encode the data in the form of triplets of wavelets, one member of the triplet per acceleration channel. We choose (mother) *wavelets*[1] so that motions executed at slightly different speeds can be represented by the same thing[2].

2 Acceleration space "shapes" encoded as wavelets

We start by working in one acceleration dimension, and then generalise to three.

We define our mother wavelets to be piecewise polynomial functions

$$\psi(x) = \begin{cases} \sum_{i=0}^{n} a_i x^i, & x \in [0,1); \\ 0, & \text{otherwise,} \end{cases} \tag{1}$$

where the wavelet conditions $\int_0^1 \psi(x)\mathrm{d}x = 0$ (zero integral — zi — forcing ψ to be a *wave*) and $\int_0^1 [\psi(x)]^2 \mathrm{d}x = 1$ (unit energy — ue — forcing ψ to be "*small*", i.e., a wave*let*), must hold.

[1] More precisely, wavelets in the continuous wavelet transform paradigm

[2] Actually, actions performed at vastly different speeds would also be represented by the same thing, but this is not needed here

In terms of the vector of coefficients, $\mathbf{a}$, the zi is

$$\mathbf{b}^{\mathrm{T}}\mathbf{a} = 0, \tag{2}$$

for the vector $\mathbf{b}$ given by

$$b_i = \frac{1}{i+1}, \quad i = 0, 1, \ldots, n \tag{3}$$

and the ue is

$$\mathbf{a}^{\mathrm{T}}\mathbf{H}_0\mathbf{a} = 1, \tag{4}$$

for the matrix $\mathbf{H}_0$ given by

$$H_{0,ij} = \frac{1}{i+j+1}, \quad i, j = 0, 1, \ldots, n \tag{5}$$

($\mathbf{H}_0$ is the Hilbert matrix).

If ψ_i, $i = 1, 2$ are two wavelets, with coefficient vectors $\mathbf{a}_i$, then their (L_2) inner product is

$$\langle \psi_1, \psi_2 \rangle_{L_2} = \int_{-\infty}^{\infty} \psi_1(x)\psi_2(x)\mathrm{d}x = \int_0^1 \psi_1(x)\psi_2(x)\mathrm{d}x = \mathbf{a}_1^{\mathrm{T}}\mathbf{H}_0\mathbf{a}_2 \tag{6}$$

and their squared (L_2) distance is

$$\begin{aligned} \|\psi_1 - \psi_2\|_{L_2}^2 &= \|\psi_1\|_{L_2} - 2\langle \psi_1, \psi_2 \rangle_{L_2} + \|\psi_2\|_{L_2}^2 \\ &= \mathbf{a}_1^{\mathrm{T}}\mathbf{H}_0\mathbf{a}_1 - 2\mathbf{a}_1^{\mathrm{T}}\mathbf{H}_0\mathbf{a}_2 + \mathbf{a}_2^{\mathrm{T}}\mathbf{H}_0\mathbf{a}_2 = 2[1 - \mathbf{a}_1^{\mathrm{T}}\mathbf{H}_0\mathbf{a}_2]. \end{aligned} \tag{7}$$

2.1 Wavelet triplets

We first generalise the one-dimensional case of the previous section, and then add some purely three-dimensional machinery.

If our triplet of wavelets is $\boldsymbol{\psi}(x) = [\psi^{(1)}(x), \psi^{(2)}(x), \psi^{(3)}(x)]^{\mathrm{T}}$, where each $\psi^{(j)}$ is defined as in equation (1), but with coefficient vector $\mathbf{a}^{(j)}$, we generalise zi by applying equation (2) to each $\psi^{(j)}$ individually, and ue by

$$\sum_{j=1}^{3} \mathbf{a}^{(j)\mathrm{T}}\mathbf{H}_0\mathbf{a}^{(j)} = 1. \tag{8}$$

In three dimensional acceleration space, the inner product (6) becomes

$$\langle \boldsymbol{\psi}_1, \boldsymbol{\psi}_2 \rangle_{L_2} = \sum_{j=1}^{3} \mathbf{a}_1^{\mathrm{T}(j)}\mathbf{H}_0\mathbf{a}_2^{(j)} \tag{9}$$

and so the squared distance of equation (7) becomes

$$\|\boldsymbol{\psi}_1 - \boldsymbol{\psi}_2\|_{L_2}^2 \quad = \quad 2\left[1 - \sum_{j=1}^{3} \mathbf{a}_1^{\mathrm{T}(j)}\mathbf{H}_0\mathbf{a}_2^{(j)}\right]. \tag{10}$$

2.2 Rotations

We now come to the three dimensional material which is not purely a generalisation of the one dimensional definitions.

As we are primarily interested in “shapes”, we wish to consider shapes which can be rotated or reflected into each other as the same thing.

We do this through equivalence classes: if $\exists \mathbf{O} \in \mathrm{O}(3)$ such that

$$\psi'^{(j)}(x) = \sum_{\ell=1}^{3} O_{j\ell}\psi^{(\ell)} \quad \forall j = 1, 2, 3, \tag{11}$$

then we say $\boldsymbol{\psi}' \sim \boldsymbol{\psi}$, and $\boldsymbol{\psi}'$ is in the Equivalence Class of Wavelet Triplets (ECWT) $[\boldsymbol{\psi}]$ represented by $\boldsymbol{\psi}$.

Assume now that $[\boldsymbol{\psi}]$ and $[\boldsymbol{\psi}']$ are possibly different ECWTs. We can define an inner product on the set of ECWTs:

$$\begin{aligned}
\langle[\boldsymbol{\psi}], [\boldsymbol{\psi}']\rangle_{\mathrm{R}} &= \max_{\boldsymbol{\psi}'' \in [\boldsymbol{\psi}], \boldsymbol{\psi}''' \in [\boldsymbol{\psi}']} \langle \boldsymbol{\psi}'', \boldsymbol{\psi}''' \rangle_{L_2} \\
&= \max_{\mathbf{O} \in \mathrm{O}(3)} \sum_{j=1}^{3}\sum_{k=1}^{3} \left\langle \psi^{(j)}, O_{jk}\psi'^{(k)} \right\rangle_{L_2},
\end{aligned} \tag{12}$$

where the second equality follows from the fact that O(3) is a group.

But then

$$\begin{aligned}
\langle[\boldsymbol{\psi}], [\boldsymbol{\psi}']\rangle_{\mathrm{R}} &= \max_{\mathbf{O} \in \mathrm{O}(3)} \sum_{j=1}^{3}\sum_{k=1}^{3} O_{jk} \left\langle \psi^{(j)}, \psi'^{(k)} \right\rangle_{L_2} \\
&= \max_{\mathbf{O} \in \mathrm{O}(3)} \sum_{j=1}^{3}\sum_{k=1}^{3} O_{jk}\mathbf{a}^{(j)\mathrm{T}}\mathbf{H}_0\mathbf{a}'^{(k)} \\
&= \max_{\mathbf{O} \in \mathrm{O}(3)} \mathrm{trace}(\mathbf{O}^{\mathrm{T}}\mathbf{K}),
\end{aligned} \tag{13}$$

where $\mathbf{K}$ is defined by

$$K_{jk} = \mathbf{a}^{(j)\mathrm{T}}\mathbf{H}_0\mathbf{a}'^{(k)}. \tag{14}$$

It is known that ([]) $\mathrm{trace}(\mathbf{O}^{\mathrm{T}}\mathbf{K})$ is maximised for fixed $\mathbf{K}$ and $\mathbf{O} \in \mathrm{O}(3)$ by

$$\mathbf{O} = \mathbf{U}\mathbf{V}^{\mathrm{T}}; \text{ where } \mathbf{K} = \mathbf{U}\mathbf{S}\mathbf{V}^{\mathrm{T}} \text{ is the singular value decomposition of } \mathbf{K}. \tag{15}$$

Hence,

$$\begin{aligned}
\langle[\boldsymbol{\psi}], [\boldsymbol{\psi}']\rangle_{\mathrm{R}} &= \mathrm{trace}(\mathbf{V}\mathbf{U}^{\mathrm{T}}\mathbf{U}\mathbf{S}\mathbf{V}^{\mathrm{T}}) = \mathrm{trace}(\mathbf{V}\mathbf{S}\mathbf{V}^{\mathrm{T}}) = \mathrm{trace}(\mathbf{S}) \\
&= \text{sum of the singular values of } \mathbf{K},
\end{aligned} \tag{16}$$

by the properties of orthogonal matrices and the fact that $\mathrm{trace}(\mathbf{ABC}) = \mathrm{trace}(\mathbf{CAB})$ for any matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}$.

There is also the obvious “R” distance between two ECWTs: $\|\boldsymbol{\psi}' - \boldsymbol{\psi}\|_R^2 = 2\left[1 - \langle\boldsymbol{\psi}', \boldsymbol{\psi}\rangle_{\mathrm{R}}\right]$ (of course, $\langle\boldsymbol{\psi}, \boldsymbol{\psi}\rangle_{\mathrm{R}} = \langle\boldsymbol{\psi}, \boldsymbol{\psi}\rangle_{L_2} = 1$).

2.3 Shifts

2.3.1 Shifts in one-dimensional acceleration space

We briefly return to the case of one acceleration dimension.

We can look at the L_2 inner product of a wavelet ψ and a time-shifted (and wrapped) version ψ'_t of another ψ', given by

$$\psi'_t(x) = \begin{cases} \psi'(x+1-t), & x \in [0,t); \\ \psi'(x-t), & x \in [t,1); \\ 0, & \text{otherwise} \end{cases} = \begin{cases} \sum_{k=0}^{n} a'_k (x+1-t)^k, & x \in [0,t); \\ \sum_{k=0}^{n} a'_k (x-t)^k, & x \in [t,1); \\ 0, & \text{otherwise.} \end{cases} \tag{17}$$

Clearly,

$$\begin{aligned}
\langle \psi, \psi'_t \rangle_{L_2} &= \sum_{k,\ell=0}^{n} a_k a'_\ell \left\{ \int_0^t x^k (x+1-t)^\ell \mathrm{d}x + \int_t^1 x^k (x-t)^\ell \mathrm{d}x \right\} \\
&= \sum_{k,\ell=0}^{n} a_k a'_\ell \sum_{r=0}^{\ell} \binom{\ell}{r} \left\{ (1-t)^{\ell-r} \int_0^t x^{k+r} \mathrm{d}x + (-1)^{\ell-r} t^{\ell-r} \int_t^1 x^{k+r} \mathrm{d}x \right\} \\
&= \sum_{k,\ell=0}^{n} a_k a'_\ell \sum_{r=0}^{\ell} \frac{1}{k+r+1} \binom{\ell}{r} \left\{ (1-t)^{\ell-r} t^{k+r+1} + (-1)^{\ell-r} \left[t^{\ell-r} - t^{k+\ell+1} \right] \right\} \\
&= \sum_{k,\ell=0}^{n} a_k a'_\ell \left\{ \sum_{r=0}^{\ell} \sum_{s=0}^{\ell-r} \frac{(-1)^s t^{k+r+s+1}}{k+r+1} \binom{\ell}{r} \binom{\ell-r}{s} + \right. \\
&\qquad\qquad \left. \sum_{r=0}^{\ell} \binom{\ell}{r} \frac{(-1)^r \left[t^r - t^{k+\ell+1} \right]}{k+\ell+1-r} \right\} \\
&= \sum_{k,\ell=0}^{n} a_k a'_\ell \left\{ \sum_{p=0}^{\ell} \sum_{r=0}^{p} \binom{\ell}{r} \binom{\ell-r}{p-r} \frac{(-1)^{p-r} t^{k+p+1}}{k+r+1} + \right. \\
&\qquad\qquad \left. \sum_{r=0}^{\ell} \binom{\ell}{r} \frac{(-1)^r \left[t^r - t^{k+\ell+1} \right]}{k+\ell+1-r} \right\} \\
&= \sum_{k,\ell=0}^{n} a_k a'_\ell \left\{ \sum_{p=0}^{\ell} (-1)^p \binom{\ell}{p} \left[\sum_{r=0}^{p} \frac{(-1)^r}{k+r+1} \binom{p}{r} \right] t^{k+p+1} + \right. \\
&\qquad\qquad \left. \sum_{r=0}^{\ell} \binom{\ell}{r} \frac{(-1)^r t^r}{k+\ell+1-r} - (-1)^\ell \left[\sum_{r=0}^{\ell} \frac{(-1)^r}{k+1+r} \binom{\ell}{r} \right] t^{k+\ell+1} \right\}.
\end{aligned} \tag{18}$$

But $\sum_{r=0}^{\ell} \frac{(-1)^r}{k+1+r} \binom{\ell}{r} = \int_0^1 x^k \sum_{r=0}^{\ell} (-1)^r \binom{\ell}{r} x^r \mathrm{d}x = \int_0^1 x^k (1-x)^\ell \mathrm{d}x$, and, recalling the definition of the beta function, this is $B(k+1,\ell+1) = \frac{k!\ell!}{(k+\ell+1)!}$, so equation (18) becomes

$$\begin{aligned}
\langle\psi,\psi'_t\rangle_{L_2} &= \sum_{k,\ell=0}^{n} a_k a'_\ell \left\{ \sum_{p=0}^{\ell}\binom{\ell}{p}\frac{(-1)^p k!p!t^{k+p+1}}{(k+p+1)!} + \right. \\
&\qquad\qquad \left. \sum_{r=0}^{\ell}\binom{\ell}{r}\frac{(-1)^r t^r}{k+\ell+1-r} - \frac{(-1)^\ell k!\ell!t^{k+\ell+1}}{(k+\ell+1)!} \right\} \\
&= \sum_{k,\ell=0}^{n} a_k a'_\ell \left\{ \sum_{p=0}^{\ell-1}(-1)^p \frac{k!\ell!}{(\ell-p)!(k+p+1)!}t^{k+p+1} + \sum_{r=0}^{\ell}\frac{(-1)^r}{k+\ell+1-r}\binom{\ell}{r}t^r \right\} \\
&= \sum_{k,\ell=0}^{n} a_k a'_\ell \left\{ \sum_{s=0}^{\ell}\frac{(-1)^s}{k+\ell+1-s}\binom{\ell}{s}t^s + \sum_{s=k+1}^{k+\ell}(-1)^{k+1+s}\frac{k!\ell!}{s!(k+\ell+1-s)!}t^s \right\} \\
&= \sum_{s=0}^{2n}\mathbf{a}^{\mathrm{T}}\mathbf{H}^{(s)}\mathbf{a}'t^s, \qquad (19)
\end{aligned}$$

where $\mathbf{H}^{(s)} = \mathbf{F}^{(s)} + \mathbf{G}^{(s)}$ for

$$F^{(s)}_{k\ell} = \begin{cases} \frac{(-1)^s}{k+\ell+1-s}\binom{\ell}{s}, & s \le \ell \le n; \\ 0, & \text{otherwise}, \end{cases} \qquad (20)$$

and, if $1 \le s \le n+1$,

$$G^{(s)}_{k\ell} = \begin{cases} \frac{(-1)^{k+1+s}k!\ell!}{s!(k+\ell+1-s)!}, & 1 \le \ell \le s, s-\ell \le k \le s-1; \\ & \text{or} \quad s \le \ell \le n, 0 \le k \le s-1; \\ 0, & \text{otherwise}; \end{cases} \qquad (21)$$

if $n+1 \le s \le 2n$,

$$G^{(s)}_{k\ell} = \begin{cases} \frac{(-1)^{k+1+s}k!\ell!}{s!(k+\ell+1-s)!}, & s-n \le \ell \le s, s-\ell \le k \le n; \\ 0, & \text{otherwise}. \end{cases} \qquad (22)$$

Note that $\mathbf{F}^{(0)} = \mathbf{H}_0$, $\mathbf{G}^{(0)} = 0$, so $\mathbf{H}^{(0)} = \mathbf{H}_0$, and $\mathbf{F}^{(s)} = 0$ for $s > n$. Also, $F^{(s)}_{k0} = G^{(s)}_{k0} = 0$ and $F^{(s)}_{0\ell} = -G^{(s)}_{0\ell}$ if $s \ge 1$, so we need not calculate elements of the first row or column of $\mathbf{F}^{(s)}$ and $\mathbf{G}^{(s)}$ as the corresponding elements of $\mathbf{H}^{(s)}$ are known to be zero if $s \ge 1$.

We can now define an inner product on our wavelets through

$$\langle\psi,\psi'\rangle_S = \max_{t\in[0,1]}\langle\psi,\psi'_t\rangle_{L_2} \qquad (23)$$

(we include $t = 1$ in the values of t considered so that maximum-finding algorithms on compact sets actually work — doing so is trivial as $\lim_{t\to 1}\psi_t = \psi_0 = \psi$). As equation (19) shows, $\langle\psi,\psi'_t\rangle_{L_2}$ is the restriction to $[0,1)$ of a polynomial p, say, of degree at most $2n$. This means the derivative of p has at most $2n-1$ real roots, and, as the maxima of a polynomial are separated by its minima, p has at most n maxima on $(-\infty,\infty)$. However, we are not particularly interested in finding the best shift in one-dimensional acceleration space, and content ourselves with observing that $\langle,\rangle_S$ is symmetric, as it should be, as $\langle\psi,\psi'_t\rangle_{L_2} = \langle\psi',\psi_{1-t}\rangle_{L_2}$.

2.3.2 Shifts in three-dimensional acceleration space

To simplify the notation, we will drop the square brackets around the ECWT $[\boldsymbol{\psi}]$ represented by $\boldsymbol{\psi}$, and allow the latter to stand for its entire equivalence class. Moreover, we will extend the notion of the equivalence class to $\boldsymbol{\psi}_t$, given by

$$\psi_t^{(j)}(x) = \begin{cases} \psi^{(j)}(x+1-t), & x \in [0,t); \\ \psi^{(j)}(x-t), & x \in [t,1); \qquad j = 1,2,3, \\ 0, & \text{otherwise}, \end{cases} \tag{24}$$

in the obvious way — two shifted triplets are in the same class if one can be rotated into the other.

We can now define a new inner product on the set of ECWTs:

$$\langle \boldsymbol{\psi}, \boldsymbol{\psi}' \rangle_{\mathrm{SR}} = \max_{t\in[0,1)} \langle \boldsymbol{\psi}, \boldsymbol{\psi}'_t \rangle_{\mathrm{R}}. \tag{25}$$

By our previous working, its obvious that

$$\langle \boldsymbol{\psi}, \boldsymbol{\psi}' \rangle_{\mathrm{SR}} = \max_{t\in[0,1)} \sum \{\text{singular values of } \mathbf{K}(t)\}, \tag{26}$$

where

$$K_{k\ell}(t) = {\mathbf{a}^{(k)}}^{\mathrm{T}} \mathbf{H}(t) {\mathbf{a}'}^{(\ell)}. \tag{27}$$

Although this function would not define an inner product on the space of equivalence classes of all sets of triplets of L_2-integrable functions which can be rotated into one another, as $\|[\mathbf{f}]-[\mathbf{g}]\|_{\mathrm{SR}} = 0$ does not imply $[\mathbf{f}] = [\mathbf{g}]$, it does form one when $\mathbf{f}$ and $\mathbf{g}$ are triplets of polynomials[3] inside $[0,1)$ and zero outside. In that case, if $\|[\mathbf{f}]-[\mathbf{g}]\|_{\mathrm{SR}} = 0$ and $[\mathbf{f}] \neq [\mathbf{g}]$, $\exists \mathbf{O} \in \mathrm{O}(3), t \in (0,1)$ such that ${g'}^{(j)} = \sum_{k=1}^{3} O_{jk} g^{(k)}$ and $\mathbf{f} = \mathbf{g}'_t$. But the components of $\mathbf{f}$ are polynomials inside $[0,1)$ and zero outside, while the nonzero components of $\mathbf{g}'_t$ are not polynomials inside $[0,1)$, as $t \notin \{0,1\}$. This can be seen as at least one of the derivatives of the nonzero components of $\mathbf{g}'_t$ is discontinuous at t.

3 Library of "shapes"

3.1 Tailoring triplets to data

We now need to create an initial library of triplets from data.

Given a window into the data of length N samples of three-dimensional acceleration, represented by the matrix $\mathbf{F} \in \mathbb{R}^{N\times 3}$, we model the data by the piecewise constant functions

$$f^{(j)}(x) = \begin{cases} F_{1j}, & x \in \left[0, \frac{1}{2}\Delta\right); \\ F_{kj}, & x \in \left[\frac{2k-3}{2}\Delta, \frac{2k-1}{2}\Delta\right), \quad k = 2,3,\ldots,N-1; \\ F_{Nj}, & x \in \left[1-\frac{1}{2}\Delta, 1\right); \\ 0, & \text{otherwise}, \end{cases} \tag{28}$$

where $\Delta = \frac{1}{N-1}$.

[3]It is possible to be more precise here, by introducing equivalence classes $[[\mathbf{f}]]$ of equivalence classes $[\mathbf{f}]$, where $[\mathbf{g}] \in [[\mathbf{f}]]$ iff $\|[\mathbf{g}]-[\mathbf{f}]\|_{\mathrm{SR}} = 0$, but this would take us too far down a tangent

The appropriate inner product for matching wavelet triplets to the data is $\langle , \rangle_{L_2}$, as the matching process automatically selects the best rotation[4].

As

$$
\begin{aligned}
\langle \mathbf{f}, \boldsymbol{\psi} \rangle_{L_2} &= \sum_{j=1}^{3} \sum_{\ell=0}^{n} a_\ell^{(j)} \left\{ F_{1j} \int_0^{\frac{1}{2}\Delta} x^\ell \mathrm{d}x + \sum_{k=2}^{N-1} F_{kj} \int_{\frac{2k-3}{2}\Delta}^{\frac{2k-1}{2}\Delta} x^\ell \mathrm{d}x + F_{Nj} \int_{1-\frac{1}{2}\Delta}^{1} x^\ell \mathrm{d}x \right\} \\
&= \sum_{j=1}^{3} \sum_{\ell=0}^{n} \frac{a_\ell^{(j)}}{\ell+1} \left\{ \left[F_{kj} + \sum_{k=2}^{N-1} f_k^{(j)} \left[(2k-1)^{\ell+1} - (2k-3)^{\ell+1} \right] \right] \left(\frac{\Delta}{2} \right)^{\ell+1} + \right. \\
&\qquad \left. F_{Nj} \left[1 - \left(1 - \frac{\Delta}{2} \right)^{\ell+1} \right] \right\} \\
&= \sum_{j=1}^{3} \mathbf{a}^{(j)\mathrm{T}} \mathbf{K} \mathbf{F} \mathbf{e}_j^{(3)}, \qquad (29)
\end{aligned}
$$

where $\mathbf{K}$ is given by

$$
K_{\ell k} = \begin{cases} \frac{1}{1+\ell} \left(\frac{1}{2}\Delta \right)^{\ell+1}, & k = 1; \\ \frac{1}{1+\ell} \left(\frac{1}{2}\Delta \right)^{\ell+1} \left[(2k-1)^{\ell+1} - (2k-3)^{\ell+1} \right], & 2 \le k \le N-1; \\ \frac{1}{1+\ell} \left[1 - \left(1 - \frac{1}{2}\Delta \right)^{\ell+1} \right], & k = N, \end{cases} \qquad (30)
$$

where we need to minimise $\|\boldsymbol{\psi} - \mathbf{f}\|_{L_2}^2 = \|\boldsymbol{\psi}\|_{L_2}^2 - 2\langle \boldsymbol{\psi}, \mathbf{f} \rangle_{L_2} + \|\mathbf{f}\|_{L_2}^2 = \sum_{j=1}^{3} [\mathbf{a}^{(j)\mathrm{T}} \mathbf{H}_0 \mathbf{a}^{(j)} - 2\mathbf{a}^{(j)\mathrm{T}} \mathbf{K}\mathbf{F}\mathbf{e}_j^{(3)}] + \|\mathbf{f}\|_{L_2}^2$, subject to $\mathbf{b}^{\mathrm{T}} \mathbf{a}^{(j)} = 0$, $j = 1, 2, 3$.

Dropping a constant term and using a vector, $\boldsymbol{\lambda}$, of Lagrange multipliers, we form the Lagrangian

$$
Q = \sum_{j=1}^{3} \left[\mathbf{a}^{(j)\mathrm{T}} \mathbf{H}_0 \mathbf{a}^{(j)} - 2\mathbf{a}^{(j)\mathrm{T}} \mathbf{K}\mathbf{F}\mathbf{e}_j^{(3)} + \lambda_j \mathbf{b}^{\mathrm{T}} \mathbf{a}^{(j)} \right], \qquad (31)
$$

and saddle points of Q will be solutions of the constrained problem.

As $\frac{\partial Q}{\partial \mathbf{a}^{(j)\mathrm{T}}} = 2\mathbf{H}_0 \mathbf{a}^{(j)} - 2\mathbf{K}\mathbf{F}\mathbf{e}_j^{(3)} + \lambda_j \mathbf{b}$, $\frac{\partial Q}{\partial \mathbf{a}^{(j)\mathrm{T}}} = 0$ implies

$$
\mathbf{a}^{(j)} = \mathbf{H}_0^{-1} \left[\mathbf{K}\mathbf{F}\mathbf{e}_j^{(3)} - \frac{1}{2}\lambda_j \mathbf{b} \right], \qquad (32)
$$

and, at these values of $\mathbf{a}^{(j)}$,

$$
Q = -\sum_{j=1}^{3} \left[\mathbf{K}\mathbf{F}\mathbf{e}_j^{(3)} - \frac{1}{2}\lambda_j \mathbf{b} \right]^{\mathrm{T}} \mathbf{H}_0^{-1} \left[\mathbf{K}\mathbf{F}\mathbf{e}_j^{(3)} - \frac{1}{2}\lambda_j \mathbf{b} \right] \qquad (33)
$$

[4]a) we wish to match a *multiple* of a triplet of *daughter* wavelets (obtained from the mother wavelet triplet by shifting and scaling its argument), and thereby selecting the mother wavelet. This most easily done by scaling the sample time and shifting the origin of the data, as we done in equation (28), and then we obtain the mother wavelet directly;
b) we can take care of the multiple, which we do not require explicitly, by dropping the unit energy condition on the wavelet during fitting, and reimposing it at the end

so $\frac{\partial Q}{\partial \lambda_j} = \mathbf{b}^{\mathrm{T}}\mathbf{H}_0^{-1}\left[\mathbf{K}\mathbf{F}\mathbf{e}_j^{(3)} - \frac{1}{2}\lambda_j\mathbf{b}\right]$, and $\frac{\partial Q}{\partial \lambda_j} = 0$ implies

$$\lambda_j = 2\frac{\mathbf{b}^{\mathrm{T}}\mathbf{H}_0^{-1}\mathbf{K}\mathbf{F}\mathbf{e}_j^{(3)}}{\mathbf{b}^{\mathrm{T}}\mathbf{H}_0^{-1}\mathbf{b}} \tag{34}$$

and

$$\begin{aligned}
\mathbf{a}^{(j)} &= \mathbf{H}_0^{-1}\left[\mathbf{I} - \frac{\mathbf{b}\mathbf{b}^{\mathrm{T}}\mathbf{H}_0^{-1}}{\mathbf{b}^{\mathrm{T}}\mathbf{H}_0^{-1}\mathbf{b}}\right]\mathbf{K}\mathbf{F}\mathbf{e}_j^{(3)} = \left[\mathbf{I} - \frac{\mathbf{H}_0^{-1}\mathbf{b}\mathbf{b}^{\mathrm{T}}}{\mathbf{b}^{\mathrm{T}}\mathbf{H}_0^{-1}\mathbf{b}}\right]\mathbf{H}_0^{-1}\mathbf{K}\mathbf{F}\mathbf{e}_j^{(3)} \\
&= \left[\mathbf{H}_0^{-1} - \mathbf{e}_0^{(n+1)}\mathbf{e}_0^{(n+1)^{\mathrm{T}}}\right]\mathbf{K}\mathbf{F}\mathbf{e}_j^{(3)},
\end{aligned} \tag{35}$$

where the final equality comes from the fact that $\mathbf{b}$ is the first column of $\mathbf{H}_0$, so $\mathbf{H}_0\mathbf{b} = \mathbf{e}_0^{(n+1)}$, where we define $\mathbf{e}_j^{(p)} \in \mathbb{R}^p$ to be the vector with zeros in all but the jth place, where it has a 1.

3.2 Goodness of Fit

As the zi condition holds for all wavelets in a triplet, if $\mathbf{F} \to \mathbf{F} - \mathbf{e}^{(N)}\boldsymbol{\alpha}^T$, where $\mathbf{e}^{(N)} = [1, 1, \ldots, 1]^{\mathrm{T}} \in \mathbb{R}^N$ and $\boldsymbol{\alpha} \in \mathbb{R}^3$ (so $\mathbf{f} \to \mathbf{f} - \boldsymbol{\alpha}$), we have that $\langle\boldsymbol{\psi}, \mathbf{f}\rangle_{L_2} \to \langle\boldsymbol{\psi}, \mathbf{f}\rangle_{L_2}$, and then the square of cosine of the angle between $\boldsymbol{\psi}$ and $\mathbf{f}$, $\frac{\langle\boldsymbol{\psi},\mathbf{f}\rangle_{L_2}^2}{\|\boldsymbol{\psi}\|_{L_2}^2\|\mathbf{f}\|_{L_2}^2} \to \frac{\langle\boldsymbol{\psi},\mathbf{f}\rangle_{L_2}^2}{\|\boldsymbol{\psi}\|_{L_2}^2\|\mathbf{f}-\boldsymbol{\alpha}\|_{L_2}^2}$. Now, $\|\mathbf{f}\|_{L_2}^2 = \sum_{j=1}^3\left[F_{1j}^2\int_0^{\frac{1}{2}\Delta}\mathrm{d}x + \sum_{\ell=2}^{N-1}F_{\ell j}^2\int_{\frac{\ell-3}{2}\Delta}^{\frac{\ell-1}{2}\Delta}\mathrm{d}x + F_{Nj}^2\int_{1-\frac{1}{2}\Delta}^{1}\mathrm{d}x\right] = \sum_{j=1}^3\Delta\left[\frac{1}{2}F_{1j}^2 + \sum_{\ell=2}^{N-1}F_{\ell j}^2 + \frac{1}{2}F_{Nj}^2\right] = \sum_{j=1}^3\mathbf{e}_j^{(3)^{\mathrm{T}}}\mathbf{F}^{\mathrm{T}}\mathbf{J}\mathbf{F}\mathbf{e}_j^{(3)} = \mathrm{trace}(\mathbf{F}^{\mathrm{T}}\mathbf{J}\mathbf{F})$, where $\mathbf{J} = \Delta\mathrm{diag}\left(\left[\frac{1}{2}, 1, 1, \ldots, 1, \frac{1}{2}\right]^{\mathrm{T}}\right) \in \mathbb{R}^N$, $\langle\mathbf{f}, \boldsymbol{\alpha}\rangle_{L_2} = \sum_{j=1}^3\alpha_j\left[F_{1j}\int_0^{\frac{1}{2}\Delta}\mathrm{d}x + \sum_{\ell=2}^{N-1}F_{\ell j}\int_{\frac{\ell-3}{2}\Delta}^{\frac{\ell-1}{2}\Delta}\mathrm{d}x + F_{Nj}\int_{1-\frac{1}{2}\Delta}^{1}\mathrm{d}x\right] = \boldsymbol{\alpha}^{\mathrm{T}}\bar{\mathbf{f}}$, where $\bar{\mathbf{f}} = \left[\overline{f^{(1)}}, \overline{f^{(2)}}, \overline{f^{(1)}}\right]^{\mathrm{T}}$ for $\overline{f^{(j)}} = \mathbf{e}^{(N)^{\mathrm{T}}}\mathbf{J}\mathbf{F}\mathbf{e}_j^{(3)}$, and $\|\boldsymbol{\alpha}\|_{L_2}^2 = \|\boldsymbol{\alpha}\|^2$, the squared Euclidean norm of $\boldsymbol{\alpha}$ in $\mathbb{R}^3$.

Hence, $\|\mathbf{f}-\boldsymbol{\alpha}\|_{L_2}^2 = \sum_{j=1}^3\mathbf{e}_j^{(3)^{\mathrm{T}}}\mathbf{F}^{\mathrm{T}}\mathbf{J}\mathbf{F}\mathbf{e}_j^{(3)} - 2\boldsymbol{\alpha}^{\mathrm{T}}\bar{\mathbf{f}} + \boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{\alpha}$. Obviously, $\|\tilde{\mathbf{f}}-\boldsymbol{\alpha}\|_{L_2}^2$ is minimised with respect to $\boldsymbol{\alpha}$ if $\alpha_j = \overline{f^{(j)}}$, and, the square of the cosine of the angle between $\boldsymbol{\psi}$ and the centralised data $\mathbf{F} - \mathbf{e}^{(N)}\bar{\mathbf{f}}^{\mathrm{T}}$ maximises $\frac{\langle\boldsymbol{\psi},\mathbf{f}\rangle_{L_2}^2}{\|\boldsymbol{\psi}\|_{L_2}^2\|\mathbf{f}-\boldsymbol{\alpha}\|_{L_2}^2}$ at $\frac{\langle\boldsymbol{\psi},\mathbf{f}\rangle_{L_2}^2}{\|\boldsymbol{\psi}\|_{L_2}^2\|\mathbf{f}-\bar{\mathbf{f}}\|_{L_2}^2}$. As $\boldsymbol{\psi}$ fits the centralised data as much as it fits the data itself, we may define the Goodness of Fit as

$$\mathrm{GoF} = \frac{\langle\boldsymbol{\psi}, \mathbf{f}\rangle_{L_2}^2}{\|\boldsymbol{\psi}\|_{L_2}^2\|\mathbf{f} - \bar{\mathbf{f}}\|_{L_2}^2} = \frac{\sum_{j=1}^3\mathbf{f}^{(j)^{\mathrm{T}}}\mathbf{K}^{\mathrm{T}}\left(\mathbf{H}_0^{-1} - \mathbf{e}_0^{(n+1)}\mathbf{e}_0^{(n+1)^{\mathrm{T}}}\right)\mathbf{K}\mathbf{f}^{(j)}}{\sum_{j=1}^3\left(\mathbf{f}^{(j)} - \overline{\mathbf{f}^{(j)}}\right)^{\mathrm{T}}\mathbf{J}\left(\mathbf{f}^{(j)} - \overline{\mathbf{f}^{(j)}}\right)}, \tag{36}$$

by equation (35) for $\mathbf{a}^{(j)}$.

By assuming that $\mathbf{f}$ is already centralised and noting that the GoF given by equation (36) is unaffected by rescaling $\mathbf{f}$, we can maximise it by maximising $\sum_{j=1}^3\mathbf{f}^{(j)^{\mathrm{T}}}\mathbf{K}^{\mathrm{T}}\left(\mathbf{H}_0^{-1} - \mathbf{e}_0^{(n+1)}\mathbf{e}_0^{(n+1)^{\mathrm{T}}}\right)\mathbf{K}\mathbf{f}^{(j)}$ subject to $\sum_{j=1}^3\mathbf{f}^{(j)^{\mathrm{T}}}\mathbf{J}\mathbf{f}^{(j)} = 1$. By symmetry, we can set $\mathbf{f}^{(j)} = \mathbf{f}_0, j = 1, 2, 3$, and then need to maximise $\mathbf{f}_0^{\mathrm{T}}\mathbf{K}^{\mathrm{T}}\left(\mathbf{H}_0^{-1} - \mathbf{e}_0^{(n+1)}\mathbf{e}_0^{(n+1)^{\mathrm{T}}}\right)\mathbf{K}\mathbf{f}_0$ subject to $\mathbf{f}_0^{\mathrm{T}}\mathbf{J}\mathbf{f}_0 = 1$. If we write $\mathbf{f}_0 = \mathbf{J}^{-\frac{1}{2}}\mathbf{g}$, where $\mathbf{J}^{-\frac{1}{2}}$ is a diagonal matrix whose diagonal elements are the reciprocals of the positive square roots of the diagonal elements of $\mathbf{J}$, then we need to maximise $\mathbf{g}^{\mathrm{T}}\mathbf{J}^{-\frac{1}{2}}\mathbf{K}^{\mathrm{T}}\left(\mathbf{H}_0^{-1}-\mathbf{e}_0^{(n+1)}\mathbf{e}_0^{(n+1)^{\mathrm{T}}}\right)\mathbf{K}\mathbf{J}^{-\frac{1}{2}}\mathbf{g}$ subject to $\mathbf{g}^{\mathrm{T}}\mathbf{g}=1$. But this is achieved by choosing $\mathbf{g}$ to be an eigenvector of $\mathbf{J}^{-\frac{1}{2}}\mathbf{K}^{\mathrm{T}}\left(\mathbf{H}_0^{-1}-\mathbf{e}_0^{(n+1)}\mathbf{e}_0^{(n+1)^{\mathrm{T}}}\right)\mathbf{K}\mathbf{J}^{-\frac{1}{2}}$ with the maximum eigenvalue, and then the GoF will be that eigenvalue.

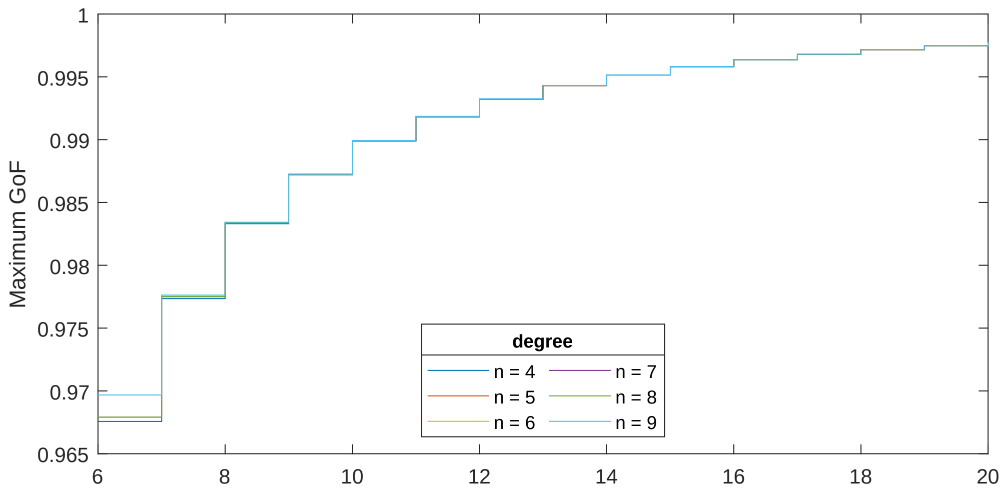

Figure 1: *Maximum GoF against sample size for piecewise polynomial wavelet triplets of varying degrees*

In Figure 1 we show the maximum GoFs for various n and N. As expected, the maximum increases with N, as the length of the flat segments to be matched by polynomials decreases, and with n (albeit marginally), i.e., as the number of free parameters and the flexibility of corresponding functions increases.

We also note that the maximum GoF for $N=6$ is bounded below by 0.965, equivalent to an angle between $\boldsymbol{\psi}$ and $\mathbf{f}$ of about 11°, the bound increasing quite quickly to 0.9975, equivalent to about 3°, as N increases.

3.3 Normalisation

Finally, we replace the $\mathbf{a}^{(j)}$ by their normalised versions

$$\mathbf{a}^{(j)}=\frac{\left[\mathbf{H}_0^{-1}-\mathbf{e}_0^{n+1}\mathbf{e}_0^{n+1^{\mathrm{T}}}\right]\mathbf{K}\mathbf{f}^{(j)}}{\sqrt{\sum_{\ell=1}^{3}\mathbf{f}^{(\ell)^{\mathrm{T}}}\mathbf{K}^{\mathrm{T}}\left[\mathbf{H}_0^{-1}-\mathbf{e}_0^{n+1}\mathbf{e}_0^{n+1^{\mathrm{T}}}\right]\mathbf{K}\mathbf{f}^{(\ell)}}},\tag{37}$$

to obtain a proper representative of an ECWT.

We note that the GoF of equation (36) is equivalent to the square of the $\langle\cdot,\cdot\rangle_{L_2}$ inner product of the normalised $\boldsymbol{\psi}$ and $\mathbf{f}$, divided by the "signal energy" $\|\mathbf{f}-\bar{\mathbf{f}}\|^2$ of $\mathbf{f}$.

3.4 Selection of a base library

Now we can find a wavelet triplet for each sufficiently long window into the data, we can build a library, $\mathcal{L}$, of such triplets/shapes. We do this by choosing a library size, $S_{\mathcal{L}}$, a threshold Θ_{GoF} and a set of window lengths, $\mathcal{W} = \{W_1, W_2, \ldots, W_{N_W}\}$.[5] In turn, we choose all the windows of length W_1 into the data (i.e., data points 1 to W_1, then 2 to $W_1 + 1$, and so on, until the last point in the window is the last point in the data), then all the windows of length W_2 into the data, ending with all windows of length W_{N_W}. For each of these windows, we find the wavelet triplet best fitted to the window's data, as shown above. If the triplet's GoF exceeds Θ_{GoF}, we add it to $\mathcal{L}$. (We use Θ_{GoF} to avoid filling up our library with poorly-fitted wavelet triplets.)

If we ever have $S_{\mathcal{L}}$ members in $\mathcal{L}$, we order it into a heap[] by increasing GoF. From this point onwards, we compare the GoF of a triplet derived from a new window with the smallest GoF of a triplet already in $\mathcal{L}$, and, if the new GoF is larger, we replace the triplet with the smallest GoF by the new one, re-ordering $\mathcal{L}$ as a heap. This is a computationally cheap (*how cheap?*[]) way of maintaining a collection of the triplets with the highest GoFs encountered so far.

3.5 Pruning $\mathcal{L}$

Following the procedure of the previous section is capable of producing a very large $\mathcal{L}$ (if Θ_{GoF} is small enough — and, of course, $S_{\mathcal{L}}$ is large enough) from relatively little data.

The advantage of a large $\mathcal{L}$ is that it is more likely that every motion characterising PD and control subjects will be captured, insofar as such motions *can* be captured by our techniques.

However, if the library is larger, it is also more likely that some of its members will be redundant. To remove this redundancy, we use a fine clustering to select representative members of $\mathcal{L}$, and then use these representatives in place of the larger library.

As we wish to have representative *members* of $\mathcal{L}$, we cannot use k-means clustering. Avoiding k-means clustering also enables us to avoid the situation where purported representatives sit in the middle of hollows in the data.

However, we *could* use *k-medoids* clustering, which chooses the centres of clusters from the data. As the computational complexity of k-medoids here would be $kS_{\mathcal{L}}^2T$ ([]), where T is the number of iterations, and $S_{\mathcal{L}}$ (large library) and k (fine clustering) would be fairly large, we will in fact use divide and rule and a version of k-medoids which admits weighting of the things to be clustered. As an approximation of k-medoids will deliver fairly representative cluster centres, we can use a fairly small T.

Instead of just using k-medoids on the original library $\mathcal{L}$, we start by choosing members of the library from Euclidean neighbourhoods, and then we use a variant, *weighted* k-medoids, on these members, weighted by the number of members of $\mathcal{L}$ in their neighbourhoods. This variant is also of complexity kS^2T, where S is the cardinality of the set being clustered.

We will use the distance induced by the inner product $\langle\cdot,\cdot\rangle_{\text{SR}}$ within our weighted k-medoids algorithm.

If $\mathcal{L}'$ is our eventual smaller library, we note that the use of $\langle\cdot,\cdot\rangle_{\text{SR}}$ to generate it implicitly makes it a library of equivalence classes represented by a subset of the triplets in $\mathcal{L}$. Also, if $S_{\mathcal{L}'}$ is the size of $\mathcal{L}'$, we let $RS_{\mathcal{L}'}$ be the number of representatives from the neighbourhoods of the first stage of our procedure. Then, the weighted k-medoids stage will be of complexity $R^2S_{\mathcal{L}'}^3T$.

[5] If the sample rate of the data in seconds is S_R, then windows of length $S_R/7$ correspond to phenomena with a base frequency of 7Hz

3.5.1 Selecting a subset of $\mathcal{L}$ from neighbourhoods

Our method for doing this[6] is based on the stacked coefficient representation, $\mathbf{a}^{\text{st}} = \left[\mathbf{a}^{(1)\text{T}}, \mathbf{a}^{(2)\text{T}}, \mathbf{a}^{(3)\text{T}}\right]^{\text{T}}$, of our wavelet triplets. Let B be a "box", $[c_0, d_0) \times [c_1, d_1) \times \cdots \times [c_{3n+2}, d_{3n+2}) \subset \mathbb{R}^{(3n+3)\times(3n+3)}$, $c_k < d_k \forall k$, in the space of this representation, large enough to contain the $\mathbf{a}^{\text{st}}$ of all triplets in $\mathcal{L}$, and let $|B|$ be the number of $\mathbf{a}^{\text{st}}$s in B.

Let $\mathcal{B} = B_0, B_1, B_2, ..., B_{M-1}$, be a sequence of M boxes, with $|B_0| \geq |B_1| \geq |B_2| \geq \cdots \geq |B_{M-1}|$. If the cardinality $\overline{\overline{\mathcal{B}}}$ of $\mathcal{B}$ is less than $RS_{\mathcal{L}'}$, we split B_0 into two smaller boxes.

If $B_0 = [c_0, d_0) \times [c_1, d_1) \times \cdots \times [c_{3n+2}, d_{3n+2})$, we first find $k_0 = \arg\max_{0\leq k\leq 3n+2} \min_{\mathbf{a}^{\text{st}}\in\mathcal{L}^{\text{st}}\cap B_0} \{|a_k^{\text{st}} - g_k|\}$, where $g_k = \frac{1}{2}(c_k + d_k)$ and $\mathcal{L}^{\text{st}}$ is the set of stacked coefficient vectors corresponding to triplets in $\mathcal{L}$. We then set $B_0' = [c_0, d_0) \times [c_1, d_1) \times \cdots \times [c_{k_0}, g_{k_0}) \times \cdots \times [c_{3n+2}, d_{3n+2})$, $B_0'' = [c_0, d_0) \times [c_1, d_1) \times \cdots \times [g_{k_0}, d_{k_0}) \times \cdots \times [c_{3n+2}, d_{3n+2})$.

Then, if $|B_0''| > |B_0'|$, interchange the labelling of B_0' and B_0'', so that $|B_0'| \geq |B_0''|$ in all cases.

If $|B_0''| = 0$, then $|B_0'| = |B_0|$, and we simply replace B_0 by B_0' in $\mathcal{B}$, and drop B_0''. $\overline{\overline{\mathcal{B}}}$ is unchanged by these actions.

On the other hand, if $|B_0''| > 0$, and we drop B_0 in $\mathcal{B}$. Then we find the first $\ell \in \{1, 2, \ldots, M-1\}$, ℓ_0, say, such that $|B_0'| > |B_\ell|$, if it exists, and insert B_0' immediately before B_{ℓ_0} in the sequence $B_1, B_2, ..., B_{M-1}$. If ℓ_0 does not exist, we insert B_0', and then B_0'' at the end of the sequence.

If ℓ_0 does exist, we find the first $\ell \in \{\ell_0, \ell_0+1, \ldots, M-1\}$, ℓ_1, say, if it exists, such that $|B_0''| > |B_\ell|$ and insert B_0'' immediately before B_{ℓ_1} in the sequence $B_1, B_2, \ldots, B_{\ell_0}, \ldots, B_{M-1}$. If ℓ_1 does not exist, insert B_0'' at the end of the sequence. Finally, relabel the sequence $\mathcal{B} = B_0, B_1, B_2, ..., B_M$. $\overline{\overline{\mathcal{B}}}$ is increased by 1 by these actions.

Obviously, although the length, M, of the sequence of boxes need not increase at every step, it will eventually increase so long as $M < S_{\mathcal{L}}$, so it will eventually reach a length of $RS_{\mathcal{L}'}$, provided $RS_{\mathcal{L}'} < S_{\mathcal{L}}$.

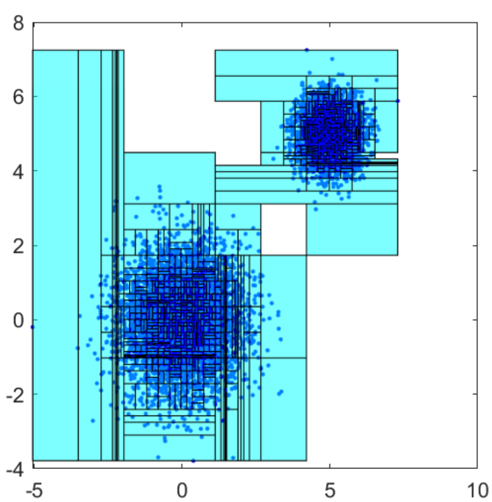

Figure 2: *The "boxes" algorithm applied to 10^4 points drawn from two 2d Gaussians, with the target number of boxes* 10^3

[6]We assume here that the kth components of all the coefficient vectors of all triplets in $\mathcal{L}$ are distinct $\forall k$. If this is not the case, trivial changes to our procedure may be necessary

It is also clear that the "boxes" algorithm can be applied to any point set. In Figure 2 we show an example.

Returning to our method for selecting a subset of $\mathcal{L}$ to apply weighted k-medoids to, for each box B, we simply pick the point in $B \cap \mathcal{L}$ with the greatest GoF, and weight this point with the cardinality $\overline{\overline{B \cap \mathcal{L}}}$.

3.5.2 Weighted k-medoids

Ordinary k-medoids takes a point set $X = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_M\} \subset \mathbb{R}^D$, and tries to find $Y = \{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_k\} \subset X$ which minimises $\sum_{\ell=1}^{k} \sum_{\mathbf{x} \in C_\ell} d(\mathbf{x}, \mathbf{y}_\ell)$, where $C_\ell = \{\mathbf{x} : d(\mathbf{x}, \mathbf{y}_\ell) < d(\mathbf{x}, \mathbf{y}_j) \forall j : 1 \leq j \leq k, j \neq \ell\}$ are the clusters. $d(\cdot, \cdot)$ is a distance on $\mathbb{R}^D$, or possibly a squared distance thereon.

Weighted k-medoids in addition takes a weight vector $\mathbf{w} = [w_1, w_2, \ldots, w_M]^{\mathrm{T}} \in \mathbb{R}^M$, and tries to find $Y \subset X$ which minimises $\sum_{\ell=1}^{k} \sum_{\mathbf{x}_p \in C_\ell} w_p d(\mathbf{x}_p, \mathbf{y}_\ell)$, where $C_\ell = \{\mathbf{x} : d(\mathbf{x}, \mathbf{y}_\ell) < d(\mathbf{x}, \mathbf{y}_j) \forall j : 1 \leq j \leq k, j \neq \ell\}$ are the clusters. In the case where $\mathbf{w}$ is a vector of positive integers, weighted k-medoids is identical to ordinary k-medoids when each $\mathbf{x}_p$ has w_p duplicates; however, the computational complexity is $M^2 k$, where M is the number of *distinct* $\mathbf{x}_\ell$, and not $\left[\sum_{\ell=1}^{M} w_\ell\right]^2 k$.

4 Results

4.1 Data

4.2 Numbers

4.3 Interpretation

The thesis established, at least for the data set considered, that the method yields results which are competitive with measuring the excess signal energy associated with the frequencies classically considered characteristic with Parkinson's. However, also showed that a richer characterisation of Parkinson's through motion symptoms is at least possible.

5 Conclusion

Thesis showed basic feasibility, but produced for scant data. Further, more exhaustive studies are required to properly establish feasibility, and, if this is done, the interpretation of the results.