Basis for a system to monitor and/or classify Parkinson's movement symptoms

Garry Hollier Chris Bailey Jim Austin

February 12, 2022

Abstract

Parkinson's disease, motion symptoms, wavelet triplets, library, classification success, range of different motions dealt with

1 Introduction

The idea is to create a system which can capture the abnormal movements associated with Parkinson's disease (PD) and contrast them with the movements of control subjects. We want to create a library of both types of motion, viewed as paths in three dimensional acceleration space. This library should be representative of the motions encountered, without being so large as to be unwieldy, and we should have a mechanism enabling us to measure the similarity between members of the library and new data.

Our data is in the form of triaxal acceleration readings from a wearable devices or a mobile phone carried by our subjects.

If we were to store the raw accelerometry data in a library, then our data would consist of points in a $3S$-dimensional space, where S is the length of our samples, and we would have to put data with different sample lengths in different spaces. To avoid this, we encode the data in the form of triplets of wavelets, one member of the triplet per acceleration channel. We choose (mother) *wavelets*[1] so that motions executed at slightly different speeds can be represented by the same thing[2].

2 Acceleration space "shapes" encoded as wavelets

We start by working in one acceleration dimension, and then generalise to three.

We define our mother wavelets to be piecewise polynomial functions

$$\psi(x) = \begin{cases} \sum_{i=0}^{n} a_i x^i, & x \in [0,1); \\ 0, & \text{otherwise}, \end{cases} \tag{1}$$

where the wavelet conditions $\int_0^1 \psi(x)\mathrm{d}x = 0$ (zero integral — zi — forcing ψ to be a *wave*) and $\int_0^1 [\psi(x)]^2 \mathrm{d}x = 1$ (unit energy — ue — forcing ψ to be "*small*", i.e., a wave*let*), must hold.

[1]More precisely, wavelets in the continuous wavelet transform paradigm

[2]Actually, actions performed at vastly different speeds would also be represented by the same thing, but this is not needed here

In terms of the vector of coefficients, $\mathbf{a}$, the zi is

$$\mathbf{b}^{\mathrm{T}}\mathbf{a} = 0, \tag{2}$$

for the vector $\mathbf{b}$ given by

$$b_i = \frac{1}{i+1}, \quad i = 0, 1, \ldots, n \tag{3}$$

and the ue is

$$\mathbf{a}^{\mathrm{T}}\mathbf{H}_0\mathbf{a} = 1, \tag{4}$$

for the matrix $\mathbf{H}_0$ given by

$$H_{0,ij} = \frac{1}{i+j+1}, \quad i, j = 0, 1, \ldots, n \tag{5}$$

($\mathbf{H}_0$ is the Hilbert matrix).

If ψ_i, $i = 1, 2$ are two wavelets, with coefficient vectors $\mathbf{a}_i$, then their (L_2) inner product is

$$\langle \psi_1, \psi_2 \rangle_{L_2} = \int_{-\infty}^{\infty} \psi_1(x)\psi_2(x)\mathrm{d}x = \int_0^1 \psi_1(x)\psi_2(x)\mathrm{d}x = \mathbf{a}_1^{\mathrm{T}}\mathbf{H}_0\mathbf{a}_2 \tag{6}$$

and their squared (L_2) distance is

$$\begin{aligned} \|\psi_1 - \psi_2\|_{L_2}^2 &= \|\psi_1\|_{L_2} - 2\langle \psi_1, \psi_2 \rangle_{L_2} + \|\psi_2\|_{L_2}^2 \\ &= \mathbf{a}_1^{\mathrm{T}}\mathbf{H}_0\mathbf{a}_1 - 2\mathbf{a}_1^{\mathrm{T}}\mathbf{H}_0\mathbf{a}_2 + \mathbf{a}_2^{\mathrm{T}}\mathbf{H}_0\mathbf{a}_2 = 2[1 - \mathbf{a}_1^{\mathrm{T}}\mathbf{H}_0\mathbf{a}_2]. \end{aligned} \tag{7}$$

2.1 Wavelet triplets

We first generalise the one-dimensional case of the previous section, and then add some purely three-dimensional machinery.

If our triplet of wavelets is $\boldsymbol{\psi}(x) = [\psi^{(1)}(x), \psi^{(2)}(x), \psi^{(3)}(x)]^{\mathrm{T}}$, where each $\psi^{(j)}$ is defined as in equation (1), but with coefficient vector $\mathbf{a}^{(j)}$, we generalise zi by applying equation (2) to each $\psi^{(j)}$ individually, and ue by

$$\sum_{j=1}^{3} \mathbf{a}^{(j)\mathrm{T}}\mathbf{H}_0\mathbf{a}^{(j)} = 1. \tag{8}$$

In three dimensional acceleration space, the inner product (6) becomes

$$\langle \boldsymbol{\psi}_1, \boldsymbol{\psi}_2 \rangle_{L_2} = \sum_{j=1}^{3} \mathbf{a}_1^{\mathrm{T}(j)}\mathbf{H}_0\mathbf{a}_2^{(j)} \tag{9}$$

and so the squared distance of equation (7) becomes

$$\|\boldsymbol{\psi}_1 - \boldsymbol{\psi}_2\|_{L_2}^2 = 2\left[1 - \sum_{j=1}^{3} \mathbf{a}_1^{\mathrm{T}(j)}\mathbf{H}_0\mathbf{a}_2^{(j)}\right]. \tag{10}$$

2.2 Rotations

We now come to the three dimensional material which is not purely a generalisation of the one dimensional definitions.

As we are primarily interested in "shapes", we wish to consider shapes which can be rotated or reflected into each other as the same thing.

We do this through equivalence classes: if $\exists \mathbf{O} \in \mathrm{O}(3)$ such that

$$\psi'^{(j)}(x) = \sum_{\ell=1}^{3} O_{j\ell}\psi^{(\ell)} \quad \forall j = 1, 2, 3, \tag{11}$$

then we say $\boldsymbol{\psi}' \sim \boldsymbol{\psi}$, and $\boldsymbol{\psi}'$ is in the Equivalence Class of Wavelet Triplets (ECWT) $[\boldsymbol{\psi}]$ represented by $\boldsymbol{\psi}$.

Assume now that $[\boldsymbol{\psi}]$ and $[\boldsymbol{\psi}']$ are possibly different ECWTs. We can define an inner product on the set of ECWTs:

$$\begin{aligned} \langle[\boldsymbol{\psi}], [\boldsymbol{\psi}']\rangle_R &= \max_{\boldsymbol{\psi}'' \in [\boldsymbol{\psi}], \boldsymbol{\psi}''' \in [\boldsymbol{\psi}']} \langle \boldsymbol{\psi}'', \boldsymbol{\psi}''' \rangle_{L_2} \\ &= \max_{\mathbf{O} \in \mathrm{O}(3)} \sum_{j=1}^{3} \sum_{k=1}^{3} \left\langle \psi^{(j)}, O_{jk}\psi'^{(k)} \right\rangle_{L_2}, \end{aligned} \tag{12}$$

where the second equality follows from the fact that $\mathrm{O}(3)$ is a group.

But then

$$\begin{aligned} \langle[\boldsymbol{\psi}], [\boldsymbol{\psi}']\rangle_R &= \max_{\mathbf{O} \in \mathrm{O}(3)} \sum_{j=1}^{3} \sum_{k=1}^{3} O_{jk} \left\langle \psi^{(j)}, \psi'^{(k)} \right\rangle_{L_2} \\ &= \max_{\mathbf{O} \in \mathrm{O}(3)} \sum_{j=1}^{3} \sum_{k=1}^{3} O_{jk} \mathbf{a}^{(j)\mathrm{T}} \mathbf{H}_0 \mathbf{a}'^{(k)} \\ &= \max_{\mathbf{O} \in \mathrm{O}(3)} \mathrm{trace}(\mathbf{O}^{\mathrm{T}}\mathbf{K}), \end{aligned} \tag{13}$$

where $\mathbf{K}$ is defined by

$$K_{jk} = \mathbf{a}^{(j)\mathrm{T}} \mathbf{H}_0 \mathbf{a}'^{(k)}. \tag{14}$$

It is known that ([]) $\mathrm{trace}(\mathbf{O}^{\mathrm{T}}\mathbf{K})$ is maximised for fixed $\mathbf{K}$ and $\mathbf{O} \in \mathrm{O}(3)$ by

$$\mathbf{O} = \mathbf{U}\mathbf{V}^{\mathrm{T}}; \text{ where } \mathbf{K} = \mathbf{U}\mathbf{S}\mathbf{V}^{\mathrm{T}} \text{ is the singular value decomposition of } \mathbf{K}. \tag{15}$$

Hence,

$$\begin{aligned} \langle[\boldsymbol{\psi}], [\boldsymbol{\psi}']\rangle_R &= \mathrm{trace}(\mathbf{V}\mathbf{U}^{\mathrm{T}}\mathbf{U}\mathbf{S}\mathbf{V}^{\mathrm{T}}) = \mathrm{trace}(\mathbf{V}\mathbf{S}\mathbf{V}^{\mathrm{T}}) = \mathrm{trace}(\mathbf{S}) \\ &= \text{sum of the singular values of } \mathbf{K}, \end{aligned} \tag{16}$$

by the properties of orthogonal matrices and the fact that $\mathrm{trace}(\mathbf{ABC}) = \mathrm{trace}(\mathbf{CAB})$ for any matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}$.

There is also the obvious "R" distance between two ECWTs: $\|\boldsymbol{\psi}' - \boldsymbol{\psi}\|_R^2 = 2\left[1 - \langle \boldsymbol{\psi}', \boldsymbol{\psi} \rangle_R\right]$ (of course, $\langle \boldsymbol{\psi}, \boldsymbol{\psi} \rangle_R = \langle \boldsymbol{\psi}, \boldsymbol{\psi} \rangle_{L_2} = 1$).

2.3 Shifts

2.3.1 Shifts in one-dimensional acceleration space

We briefly return to the case of one acceleration dimension.

We can look at the L_2 inner product of a wavelet ψ and a time-shifted (and wrapped) version ψ'_t of another ψ', given by

$$\psi'_t(x) = \begin{cases} \psi'(x+1-t), & x \in [0,t); \\ \psi'(x-t), & x \in [t,1); \\ 0, & \text{otherwise} \end{cases} = \begin{cases} \sum_{k=0}^n a'_k (x+1-t)^k, & x \in [0,t); \\ \sum_{k=0}^n a'_k (x-t)^k, & x \in [t,1); \\ 0, & \text{otherwise.} \end{cases} \tag{17}$$

Clearly,

$$\begin{aligned}
\langle \psi, \psi'_t \rangle_{L_2} &= \sum_{k,\ell=0}^n a_k a'_\ell \left\{ \int_0^t x^k (x+1-t)^\ell \mathrm{d}x + \int_t^1 x^k (x-t)^\ell \mathrm{d}x \right\} \\
&= \sum_{k,\ell=0}^n a_k a'_\ell \sum_{r=0}^\ell \binom{\ell}{r} \left\{ (1-t)^{\ell-r} \int_0^t x^{k+r} \mathrm{d}x + (-1)^{\ell-r} t^{\ell-r} \int_t^1 x^{k+r} \mathrm{d}x \right\} \\
&= \sum_{k,\ell=0}^n a_k a'_\ell \sum_{r=0}^\ell \frac{1}{k+r+1} \binom{\ell}{r} \left\{ (1-t)^{\ell-r} t^{k+r+1} + (-1)^{\ell-r} \left[t^{\ell-r} - t^{k+\ell+1} \right] \right\} \\
&= \sum_{k,\ell=0}^n a_k a'_\ell \left\{ \sum_{r=0}^\ell \sum_{s=0}^{\ell-r} \frac{(-1)^s t^{k+r+s+1}}{k+r+1} \binom{\ell}{r} \binom{\ell-r}{s} + \right. \\
&\qquad\qquad \left. \sum_{r=0}^\ell \binom{\ell}{r} \frac{(-1)^r \left[t^r - t^{k+\ell+1} \right]}{k+\ell+1-r} \right\} \\
&= \sum_{k,\ell=0}^n a_k a'_\ell \left\{ \sum_{p=0}^\ell \sum_{r=0}^p \binom{\ell}{r} \binom{\ell-r}{p-r} \frac{(-1)^{p-r} t^{k+p+1}}{k+r+1} + \right. \\
&\qquad\qquad \left. \sum_{r=0}^\ell \binom{\ell}{r} \frac{(-1)^r \left[t^r - t^{k+\ell+1} \right]}{k+\ell+1-r} \right\} \\
&= \sum_{k,\ell=0}^n a_k a'_\ell \left\{ \sum_{p=0}^\ell (-1)^p \binom{\ell}{p} \left[\sum_{r=0}^p \frac{(-1)^r}{k+r+1} \binom{p}{r} \right] t^{k+p+1} + \right. \\
&\qquad\qquad \left. \sum_{r=0}^\ell \binom{\ell}{r} \frac{(-1)^r t^r}{k+\ell+1-r} - (-1)^\ell \left[\sum_{r=0}^\ell \frac{(-1)^r}{k+1+r} \binom{\ell}{r} \right] t^{k+\ell+1} \right\}.
\end{aligned} \tag{18}$$

But $\sum_{r=0}^\ell \frac{(-1)^r}{k+1+r} \binom{\ell}{r} = \int_0^1 x^k \sum_{r=0}^\ell (-1)^r \binom{\ell}{r} x^r \mathrm{d}x = \int_0^1 x^k (1-x)^\ell \mathrm{d}x$, and, recalling the definition of the beta function, this is $B(k+1,\ell+1) = \frac{k!\ell!}{(k+\ell+1)!}$, so equation (18) becomes

$$\begin{aligned}
\langle \psi, \psi'_t \rangle_{L_2} &= \sum_{k,\ell=0}^{n} a_k a'_\ell \left\{ \sum_{p=0}^{\ell} \binom{\ell}{p} \frac{(-1)^p k! p! t^{k+p+1}}{(k+p+1)!} + \right. \\
&\qquad \left. \sum_{r=0}^{\ell} \binom{\ell}{r} \frac{(-1)^r t^r}{k+\ell+1-r} - \frac{(-1)^\ell k!\ell! t^{k+\ell+1}}{(k+\ell+1)!} \right\} \\
&= \sum_{k,\ell=0}^{n} a_k a'_\ell \left\{ \sum_{p=0}^{\ell-1} (-1)^p \frac{k!\ell!}{(\ell-p)!(k+p+1)!} t^{k+p+1} + \sum_{r=0}^{\ell} \frac{(-1)^r}{k+\ell+1-r} \binom{\ell}{r} t^r \right\} \\
&= \sum_{k,\ell=0}^{n} a_k a'_\ell \left\{ \sum_{s=0}^{\ell} \frac{(-1)^s}{k+\ell+1-s} \binom{\ell}{s} t^s + \sum_{s=k+1}^{k+\ell} (-1)^{k+1+s} \frac{k!\ell!}{s!(k+\ell+1-s)!} t^s \right\} \\
&= \sum_{s=0}^{2n} \mathbf{a}^{\mathrm{T}} \mathbf{H}^{(s)} \mathbf{a}' t^s, \qquad (19)
\end{aligned}$$

where $\mathbf{H}^{(s)} = \mathbf{F}^{(s)} + \mathbf{G}^{(s)}$ for

$$F_{k\ell}^{(s)} = \begin{cases} \frac{(-1)^s}{k+\ell+1-s}\binom{\ell}{s}, & s \le \ell \le n; \\ 0, & \text{otherwise}, \end{cases} \qquad (20)$$

and, if $1 \le s \le n+1$,

$$G_{k\ell}^{(s)} = \begin{cases} \frac{(-1)^{k+1+s} k!\ell!}{s!(k+\ell+1-s)!}, & 1 \le \ell \le s, s-\ell \le k \le s-1; \\ & \text{or} \quad s \le \ell \le n, 0 \le k \le s-1; \\ 0, & \text{otherwise}; \end{cases} \qquad (21)$$

if $n+1 \le s \le 2n$,

$$G_{k\ell}^{(s)} = \begin{cases} \frac{(-1)^{k+1+s} k!\ell!}{s!(k+\ell+1-s)!}, & s-n \le \ell \le s, s-\ell \le k \le n; \\ 0, & \text{otherwise}. \end{cases} \qquad (22)$$

Note that $\mathbf{F}^{(0)} = \mathbf{H}_0$, $\mathbf{G}^{(0)} = 0$, so $\mathbf{H}^{(0)} = \mathbf{H}_0$, and $\mathbf{F}^{(s)} = 0$ for $s > n$. Also, $F_{k0}^{(s)} = G_{k0}^{(s)} = 0$ and $F_{0\ell}^{(s)} = -G_{0\ell}^{(s)}$ if $s \ge 1$, so we need not calculate elements of the first row or column of $\mathbf{F}^{(s)}$ and $\mathbf{G}^{(s)}$ as the corresponding elements of $\mathbf{H}^{(s)}$ are known to be zero if $s \ge 1$.

We can now define an inner product on our wavelets through

$$\langle \psi, \psi' \rangle_S = \max_{t \in [0,1]} \langle \psi, \psi'_t \rangle_{L_2} \qquad (23)$$

(we include $t = 1$ in the values of t considered so that maximum-finding algorithms on compact sets actually work — doing so is trivial as $\lim_{t \to 1} \psi_t = \psi_0 = \psi$). As equation (19) shows, $\langle \psi, \psi'_t \rangle_{L_2}$ is the restriction to $[0,1)$ of a polynomial p, say, of degree at most $2n$. This means the derivative of p has at most $2n-1$ real roots, and, as the maxima of a polynomial are separated by its minima, p has at most n maxima on $(-\infty, \infty)$. However, we are not particularly interested in finding the best shift in one-dimensional acceleration space, and content ourselves with observing that $\langle , \rangle_S$ is symmetric, as it should be, as $\langle \psi, \psi'_t \rangle_{L_2} = \langle \psi', \psi_{1-t} \rangle_{L_2}$.

2.3.2 Shifts in three-dimensional acceleration space

To simplify the notation, we will drop the square brackets around the ECWT $[\boldsymbol{\psi}]$ represented by $\boldsymbol{\psi}$, and allow the latter to stand for its entire equivalence class. Moreover, we will extend the notion of the equivalence class to $\boldsymbol{\psi}_t$, given by

$$\psi_t^{(j)}(x) = \begin{cases} \psi^{(j)}(x+1-t), & x \in [0,t); \\ \psi^{(j)}(x-t), & x \in [t,1); \quad\quad j = 1,2,3, \\ 0, & \text{otherwise}, \end{cases} \tag{24}$$

in the obvious way — two shifted triplets are in the same class if one can be rotated into the other.

We can now define a new inner product on the set of ECWTs:

$$\langle \boldsymbol{\psi}, \boldsymbol{\psi}' \rangle_{SR} = \max_{t\in[0,1)} \langle \boldsymbol{\psi}, \boldsymbol{\psi}'_t \rangle_R. \tag{25}$$

By our previous working, its obvious that

$$\langle \boldsymbol{\psi}, \boldsymbol{\psi}' \rangle_{SR} = \max_{t\in[0,1)} \sum \{\text{singular values of } \mathbf{K}(t)\}, \tag{26}$$

where

$$K_{k\ell}(t) = {\mathbf{a}^{(k)}}^{\mathrm{T}} \mathbf{H}(t) {\mathbf{a}'}^{(\ell)}. \tag{27}$$

Although this function would not define an inner product on the space of equivalence classes of all sets of triplets of L_2-integrable functions which can be rotated into one another, as $\|[\mathbf{f}]-[\mathbf{g}]\|_{SR} = 0$ does not imply $[\mathbf{f}] = [\mathbf{g}]$, it does form one when $\mathbf{f}$ and $\mathbf{g}$ are triplets of polynomials[3] inside $[0,1)$ and zero outside. In that case, if $\|[\mathbf{f}]-[\mathbf{g}]\|_{SR} = 0$ and $[\mathbf{f}] \neq [\mathbf{g}]$, $\exists \mathbf{O} \in \mathrm{O}(3), t \in (0,1)$ such that ${g'}^{(j)} = \sum_{k=1}^{3} O_{jk} g^{(k)}$ and $\mathbf{f} = \mathbf{g}'_t$. But the components of $\mathbf{f}$ are polynomials inside $[0,1)$ and zero outside, while the nonzero components of $\mathbf{g}'_t$ are not polynomials inside $[0,1)$, as $t \notin \{0,1\}$. This can be seen as at least one of the derivatives of the nonzero components of $\mathbf{g}'_t$ is discontinuous at t.

3 Library of "shapes"

3.1 Tailoring ECWTs to data

We now need to create an initial library of "shapes" from data.

Given a window into the data of length N samples of three-dimensional acceleration, represented by the matrix $\mathbf{F} \in \mathbb{R}^{N\times 3}$, we model the data by the piecewise constant functions

$$f^{(j)}(x) = \begin{cases} F_{1j}, & x \in \left[0, \frac{1}{2}\Delta\right); \\ F_{kj}, & x \in \left[\frac{2k-3}{2}\Delta, \frac{2k-1}{2}\Delta\right), \quad k = 2,3,\ldots,N-1; \\ F_{Nj}, & x \in \left[1-\frac{1}{2}\Delta, 1\right); \\ 0, & \text{otherwise}, \end{cases} \tag{28}$$

where $\Delta = \frac{1}{N-1}$.

[3]It is possible to be more precise here, by introducing equivalence classes $[[\mathbf{f}]]$ of equivalence classes $[\mathbf{f}]$, where $[\mathbf{g}] \in [[\mathbf{f}]]$ iff $\|[\mathbf{g}]-[\mathbf{f}]\|_{SR} = 0$, but this would take us too far down a tangent

The appropriate inner product for matching wavelet triplets to the data is $\langle , \rangle_{L_2}$, as the matching process automatically selects the best rotation[4], and the window stepping through the data corresponds to time-shifting the ECWTs.

As

$$\begin{aligned}\langle \mathbf{f}, \boldsymbol{\psi} \rangle_{L_2} &= \sum_{j=1}^{3} \sum_{\ell=0}^{n} a_\ell^{(j)} \left\{ F_{1j} \int_0^{\frac{1}{2}\Delta} x^\ell \mathrm{d}x + \sum_{k=2}^{N-1} F_{kj} \int_{\frac{2k-3}{2}\Delta}^{\frac{2k-1}{2}\Delta} x^\ell \mathrm{d}x + F_{Nj} \int_{1-\frac{1}{2}\Delta}^{1} x^\ell \mathrm{d}x \right\} \\ &= \sum_{j=1}^{3} \sum_{\ell=0}^{n} \frac{a_\ell^{(j)}}{\ell+1} \left\{ \left[F_{kj} + \sum_{k=2}^{N-1} f_k^{(j)} \left[(2k-1)^{\ell+1} - (2k-3)^{\ell+1} \right] \right] \left(\frac{\Delta}{2} \right)^{\ell+1} + \right. \\ &\qquad \left. F_{Nj} \left[1 - \left(1 - \frac{\Delta}{2} \right)^{\ell+1} \right] \right\} \\ &= \sum_{j=1}^{3} \mathbf{a}^{(j)^{\mathrm{T}}} \mathbf{K} \mathbf{F} \mathbf{e}_j^{(3)}, \end{aligned} \tag{29}$$

where $\mathbf{K}$ is given by

$$K_{\ell k} = \begin{cases} \frac{1}{1+\ell} \left(\frac{1}{2}\Delta \right)^{\ell+1}, & k = 1; \\ \frac{1}{1+\ell} \left(\frac{1}{2}\Delta \right)^{\ell+1} \left[(2k-1)^{\ell+1} - (2k-3)^{\ell+1} \right], & 2 \le k \le N-1; \\ \frac{1}{1+\ell} \left[1 - \left(1 - \frac{1}{2}\Delta \right)^{\ell+1} \right], & k = N, \end{cases} \tag{30}$$

where we need to minimise $\|\boldsymbol{\psi} - \mathbf{f}\|_{L_2}^2 = \|\boldsymbol{\psi}\|_{L_2}^2 - 2\langle \boldsymbol{\psi}, \mathbf{f} \rangle_{L_2} + \|\mathbf{f}\|_{L_2}^2 = \sum_{j=1}^{3} [\mathbf{a}^{(j)^{\mathrm{T}}} \mathbf{H}_0 \mathbf{a}^{(j)} - 2\mathbf{a}^{(j)^{\mathrm{T}}} \mathbf{K} \mathbf{F} \mathbf{e}_j^{(3)}] + \|\mathbf{f}\|_{L_2}^2$, subject to $\mathbf{b}^{\mathrm{T}} \mathbf{a}^{(j)} = 0$, $j = 1, 2, 3$.

Dropping a constant term and using a vector, $\boldsymbol{\lambda}$, of Lagrange multipliers, we form the Lagrangian

$$Q = \sum_{j=1}^{3} \left[\mathbf{a}^{(j)^{\mathrm{T}}} \mathbf{H}_0 \mathbf{a}^{(j)} - 2\mathbf{a}^{(j)^{\mathrm{T}}} \mathbf{K} \mathbf{F} \mathbf{e}_j^{(3)} + \lambda_j \mathbf{b}^{\mathrm{T}} \mathbf{a}^{(j)} \right], \tag{31}$$

and saddle points of Q will be solutions of the constrained problem.

As $\frac{\partial Q}{\partial \mathbf{a}^{(j)^{\mathrm{T}}}} = 2\mathbf{H}_0 \mathbf{a}^{(j)} - 2\mathbf{K} \mathbf{F} \mathbf{e}_j^{(3)} + \lambda_j \mathbf{b}$, $\frac{\partial Q}{\partial \mathbf{a}^{(j)^{\mathrm{T}}}} = 0$ implies

$$\mathbf{a}^{(j)} = \mathbf{H}_0^{-1} \left[\mathbf{K} \mathbf{F} \mathbf{e}_j^{(3)} - \frac{1}{2} \lambda_j \mathbf{b} \right], \tag{32}$$

and, at these values of $\mathbf{a}^{(j)}$,

$$Q = -\sum_{j=1}^{3} \left[\mathbf{K} \mathbf{F} \mathbf{e}_j^{(3)} - \frac{1}{2} \lambda_j \mathbf{b} \right]^{\mathrm{T}} \mathbf{H}_0^{-1} \left[\mathbf{K} \mathbf{F} \mathbf{e}_j^{(3)} - \frac{1}{2} \lambda_j \mathbf{b} \right] \tag{33}$$

[4]a) The matched triplet will be considered as a representative of its ECWT;
b) we wish to match a *multiple* of a triple of *daughter* wavelets (obtained from the mother wavelet triplet by shifting and scaling its argument), and thereby selecting the mother wavelet. This most easily done by scaling the sample time and shifting the origin of the data, as we done in equation (28), and then we obtain the mother wavelet directly;
c) we can take care of the multiple, which we do not require explicitly, by dropping the unit energy condition on the wavelet during fitting, and reimposing it at the end

so $\frac{\partial Q}{\partial \lambda_j} = \mathbf{b}^{\mathrm{T}}\mathbf{H}_0^{-1}\left[\mathbf{KFe}_j^{(3)} - \frac{1}{2}\lambda_j\mathbf{b}\right]$, and $\frac{\partial Q}{\partial \lambda_j} = 0$ implies

$$\lambda_j = 2\frac{\mathbf{b}^{\mathrm{T}}\mathbf{H}_0^{-1}\mathbf{KFe}_j^{(3)}}{\mathbf{b}^{\mathrm{T}}\mathbf{H}_0^{-1}\mathbf{b}} \tag{34}$$

and

$$\begin{aligned}\mathbf{a}^{(j)} &= \mathbf{H}_0^{-1}\left[\mathbf{I} - \frac{\mathbf{bb}^{\mathrm{T}}\mathbf{H}_0^{-1}}{\mathbf{b}^{\mathrm{T}}\mathbf{H}_0^{-1}\mathbf{b}}\right]\mathbf{KFe}_j^{(3)} = \left[\mathbf{I} - \frac{\mathbf{H}_0^{-1}\mathbf{bb}^{\mathrm{T}}}{\mathbf{b}^{\mathrm{T}}\mathbf{H}_0^{-1}\mathbf{b}}\right]\mathbf{H}_0^{-1}\mathbf{KFe}_j^{(3)} \\ &= \left[\mathbf{H}_0^{-1} - \mathbf{e}_0^{(n+1)}\mathbf{e}_0^{(n+1)^{\mathrm{T}}}\right]\mathbf{KFe}_j^{(3)},\end{aligned} \tag{35}$$

where the final equality comes from the fact that $\mathbf{b}$ is the first column of $\mathbf{H}_0$, so $\mathbf{H}_0\mathbf{b} = \mathbf{e}_0^{(n+1)}$, where we define $\mathbf{e}_j^{(p)} \in \mathbb{R}^p$ to be the vector with zeros in all but the jth place, where it has a 1.

3.2 Goodness of Fit

As the zi condition holds for all wavelets in a triplet, if $\mathbf{F} \to \mathbf{F} - \mathbf{e}^{(N)}\boldsymbol{\alpha}^T$, where $\mathbf{e}^{(N)} = [1, 1, \ldots, 1]^{\mathrm{T}} \in \mathbb{R}^N$ and $\boldsymbol{\alpha} \in \mathbb{R}^3$ (so $\mathbf{f} \to \mathbf{f} - \boldsymbol{\alpha}$), we have that $\langle \boldsymbol{\psi}, \mathbf{f}\rangle_{L_2} \to \langle \boldsymbol{\psi}, \mathbf{f}\rangle_{L_2}$, and then the square of cosine of the angle between $\boldsymbol{\psi}$ and $\mathbf{f}$, $\frac{\langle \boldsymbol{\psi},\mathbf{f}\rangle_{L_2}^2}{\|\boldsymbol{\psi}\|_{L_2}^2\|\mathbf{f}\|_{L_2}^2} \to \frac{\langle \boldsymbol{\psi},\mathbf{f}\rangle_{L_2}^2}{\|\boldsymbol{\psi}\|_{L_2}^2\|\mathbf{f}-\boldsymbol{\alpha}\|_{L_2}^2}$. Now, $\|\mathbf{f}\|_{L_2}^2 = \sum_{j=1}^3 \left[F_{1j}^2 \int_0^{\frac{1}{2}\Delta} \mathrm{d}x + \sum_{\ell=2}^{N-1} F_{\ell j}^2 \int_{\frac{\ell-3}{2}\Delta}^{\frac{\ell-1}{2}\Delta} \mathrm{d}x + F_{Nj}^2 \int_{1-\frac{1}{2}\Delta}^1 \mathrm{d}x\right] = \sum_{j=1}^3 \Delta\left[\frac{1}{2}F_{1j}^2 + \sum_{\ell=2}^{N-1} F_{\ell j}^2 + \frac{1}{2}F_{Nj}^2\right] = \sum_{j=1}^3 \mathbf{e}_j^{(3)^{\mathrm{T}}}\mathbf{F}^{\mathrm{T}}\mathbf{JFe}_j^{(3)} = \mathrm{trace}(\mathbf{F}^{\mathrm{T}}\mathbf{JF})$, where $\mathbf{J} = \Delta\mathrm{diag}\left(\left[\frac{1}{2}, 1, 1, \ldots, 1, \frac{1}{2}\right]^{\mathrm{T}}\right) \in \mathbb{R}^N$, $\langle \mathbf{f}, \boldsymbol{\alpha}\rangle_{L_2} = \sum_{j=1}^3 \alpha_j\left[F_{1j}\int_0^{\frac{1}{2}\Delta} \mathrm{d}x + \sum_{\ell=2}^{N-1} F_{\ell j}\int_{\frac{\ell-3}{2}\Delta}^{\frac{\ell-1}{2}\Delta} \mathrm{d}x + F_{Nj}\int_{1-\frac{1}{2}\Delta}^1 \mathrm{d}x\right] = \boldsymbol{\alpha}^{\mathrm{T}}\bar{\mathbf{f}}$, where $\bar{\mathbf{f}} = \left[\overline{f^{(1)}}, \overline{f^{(2)}}, \overline{f^{(1)}}\right]^{\mathrm{T}}$ for $\overline{f^{(j)}} = \mathbf{e}^{(N)^{\mathrm{T}}}\mathbf{JFe}_j^{(3)}$, and $\|\boldsymbol{\alpha}\|_{L_2}^2 = \|\boldsymbol{\alpha}\|^2$, the squared Euclidean norm of $\boldsymbol{\alpha}$ in $\mathbb{R}^3$.

Hence, $\|\mathbf{f}-\boldsymbol{\alpha}\|_{L_2}^2 = \sum_{j=1}^3 \mathbf{e}_j^{(3)^{\mathrm{T}}}\mathbf{F}^{\mathrm{T}}\mathbf{JFe}_j^{(3)} - 2\boldsymbol{\alpha}^{\mathrm{T}}\bar{\mathbf{f}} + \boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{\alpha}$. Obviously, $\|\tilde{\mathbf{f}}-\boldsymbol{\alpha}\|_{L_2}^2$ is minimised with respect to $\boldsymbol{\alpha}$ if $\alpha_j = \overline{f^{(j)}}$, and, the square of the cosine of the angle between $\boldsymbol{\psi}$ and the centralised data $\mathbf{F} - \mathbf{e}^{(N)}\bar{\mathbf{f}}^{\mathrm{T}}$ maximises $\frac{\langle \boldsymbol{\psi},\mathbf{f}\rangle_{L_2}^2}{\|\boldsymbol{\psi}\|_{L_2}^2\|\mathbf{f}-\boldsymbol{\alpha}\|_{L_2}^2}$ at $\frac{\langle \boldsymbol{\psi},\mathbf{f}\rangle_{L_2}^2}{\|\boldsymbol{\psi}\|_{L_2}^2\|\mathbf{f}-\bar{\mathbf{f}}\|_{L_2}^2}$. As $\boldsymbol{\psi}$ fits the centralised data as much as it fits the data itself, we may define the Goodness of Fit as

$$\mathrm{GoF} = \frac{\langle \boldsymbol{\psi}, \tilde{\mathbf{f}}\rangle_{L_2}^2}{\|\boldsymbol{\psi}\|_{L_2}^2\|\tilde{\mathbf{f}} - \bar{\mathbf{f}}\|_{L_2}^2} = \frac{\sum_{j=1}^3 \mathbf{f}^{(j)^{\mathrm{T}}}\mathbf{K}^{\mathrm{T}}\left(\mathbf{H}_0^{-1} - \mathbf{e}_0^{(n+1)}\mathbf{e}_0^{(n+1)^{\mathrm{T}}}\right)\mathbf{Kf}^{(j)}}{\sum_{j=1}^3\left(\mathbf{f}^{(j)} - \overline{\mathbf{f}^{(j)}}\right)^{\mathrm{T}}\mathbf{J}\left(\mathbf{f}^{(j)} - \overline{\mathbf{f}^{(j)}}\right)}, \tag{36}$$

by equation (35) for $\mathbf{a}^{(j)}$.

By assuming that $\mathbf{f}$ is already centralised and noting that the GoF given by equation (36) is unaffected by rescaling $\mathbf{f}$, we can maximise it by maximising $\sum_{j=1}^3 \mathbf{f}^{(j)^{\mathrm{T}}}\mathbf{K}^{\mathrm{T}}\left(\mathbf{H}_0^{-1} - \mathbf{e}_0^{(n+1)}\mathbf{e}_0^{(n+1)^{\mathrm{T}}}\right)\mathbf{Kf}^{(j)}$ subject to $\sum_{j=1}^3 \mathbf{f}^{(j)^{\mathrm{T}}}\mathbf{Jf}^{(j)} = 1$. By symmetry, we can set $\mathbf{f}^{(j)} = \mathbf{f}_0, j = 1, 2, 3$, and then need to maximise $\mathbf{f}_0^{\mathrm{T}}\mathbf{K}^{\mathrm{T}}\left(\mathbf{H}_0^{-1} - \mathbf{e}_0^{(n+1)}\mathbf{e}_0^{(n+1)^{\mathrm{T}}}\right)\mathbf{Kf}_0$ subject to $\mathbf{f}_0^{\mathrm{T}}\mathbf{Jf}_0 = 1$. If we write $\mathbf{f}_0 = \mathbf{J}^{-\frac{1}{2}}\mathbf{g}$, where $\mathbf{J}^{-\frac{1}{2}}$ is a diagonal matrix whose diagonal elements are the reciprocals of the positive square roots of the diagonal elements of $\mathbf{J}$, then we need to maximise $\mathbf{g}^{\mathrm{T}}\mathbf{J}^{-\frac{1}{2}}\mathbf{K}^{\mathrm{T}}\left(\mathbf{H}_0^{-1}-\mathbf{e}_0^{(n+1)}\mathbf{e}_0^{(n+1)\mathrm{T}}\right)\mathbf{K}\mathbf{J}^{-\frac{1}{2}}\mathbf{g}$ subject to $\mathbf{g}^{\mathrm{T}}\mathbf{g}=1$. But this is achieved by choosing $\mathbf{g}$ to be an eigenvector of $\mathbf{J}^{-\frac{1}{2}}\mathbf{K}^{\mathrm{T}}\left(\mathbf{H}_0^{-1}-\mathbf{e}_0^{(n+1)}\mathbf{e}_0^{(n+1)\mathrm{T}}\right)\mathbf{K}\mathbf{J}^{-\frac{1}{2}}$ with the maximum eigenvalue, and then the GoF will be that eigenvalue.

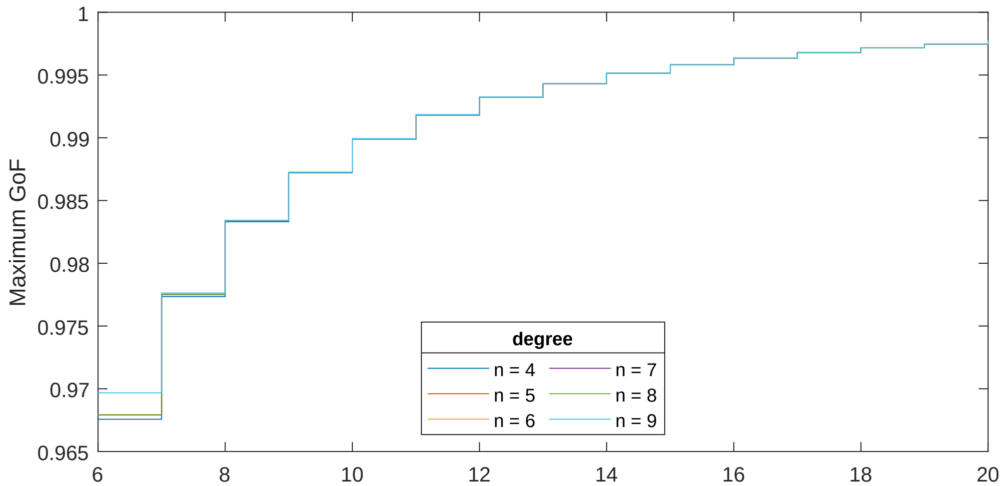

Figure 1: *Maximum GoF against sample size for piecewise polynomial wavelet triplets of varying degrees*

In Figure 1 we show the maximum GoFs for various n and N. As expected, the maximum increases with N, as the length of the flat segments to be matched by polynomials decreases, and with n (albeit marginally), i.e., as the number of free parameters and the flexibility of corresponding functions increases.

3.3 Normalisation

Finally, we replace the $\mathbf{a}^{(j)}$ by their normalised versions

$$\mathbf{a}^{(j)}=\frac{\left[\mathbf{H}_0^{-1}-\mathbf{e}_0^{n+1}\mathbf{e}_0^{n+1\mathrm{T}}\right]\mathbf{K}\mathbf{f}^{(j)}}{\sqrt{\sum_{\ell=1}^{3}\mathbf{f}^{(\ell)\mathrm{T}}\mathbf{K}^{\mathrm{T}}\left[\mathbf{H}_0^{-1}-\mathbf{e}_0^{n+1}\mathbf{e}_0^{n+1\mathrm{T}}\right]\mathbf{K}\mathbf{f}^{(\ell)}}}, \tag{37}$$

to obtain a proper representative of an ECWT.

3.4 Selection of a base library

Now we can find a wavelet triplet for each sufficiently long window into the data, we can build a library of such triplets. We do this by choosing a threshold Θ_{GoF} and a set of window lengths, $\mathcal{W}=\{W_1,W_2,\ldots,W_{N_W}\}$. In turn, we choose all the windows of length W_1 into the data (i.e.,

4 Results

4.1 Data

4.2 Numbers

4.3 Interpretation

The thesis established, at least for the data set considered, that the method yields results which are competitive with measuring the excess signal energy associated with the frequencies classically considered characteristic with Parkinson's. However, also showed that a richer characterisation of Parkinson's through motion symptoms is at least possible.

5 Conclusion

Thesis showed basic feasibility, but produced for scant data. Further, more exhaustive studies are required to properly establish feasibility, and, if this is done, the interpretation of the results.