

THE ROUTE TO CHAOS IN ROUTING GAMES: POPULATION INCREASE DRIVES PERIOD-DOUBLING INSTABILITY AND CHAOS WITH PRICE OF ANARCHY EQUAL TO ONE

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ABSTRACT. We study a learning dynamic model of routing (congestion) games to explore how an increase in the total demand influences system performance. We focus on non-atomic routing games with two parallel edges of linear cost, where all agents evolve using Multiplicative Weights Updates with a fixed learning rate. Previous game-theoretic equilibrium analysis suggests that system performance is improved in the large population limit, as seen by the Price of Anarchy reduction. In this work, however, we reveal a rather undesirable consequence of non-equilibrium phenomena driven by population increase. As the total demand rises, we prove that the learning dynamics unavoidably become non-equilibrating, typically chaotic. The Price of Anarchy equilibrium predictions of near-optimal system performance thus no longer apply. In fact, the time-average social cost may converge to the worst possible value as population increases.

Every system has a carrying capacity, above which the dynamics is non-equilibrating. If the equilibrium flow is a symmetric 50 – 50% split, the system exhibits one period-doubling bifurcation, so that a single periodic attractor of period two replaces the attracting fixed point when the demand exceeds the carrying capacity. In general, for asymmetric equilibrium flows, increasing the demand destabilizes the system, so that the system eventually becomes Li-Yorke chaotic with positive topological entropy. This demand-driven instability emerges from any pair of linear cost functions. Remarkably, in any non-equilibrating regime, the time-average flows on the edges converge *exactly* to the equilibrium flows, a property akin to no-regret learning in zero-sum games.

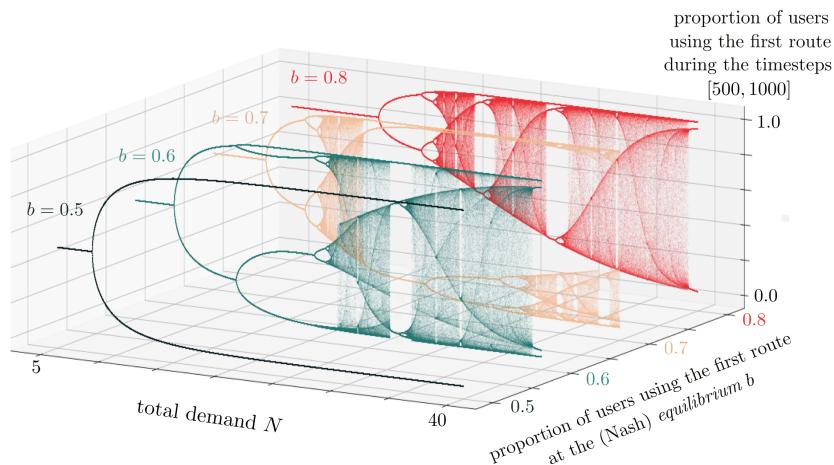


FIGURE 1. If the equilibrium flow is symmetric between the two routes ($b = 0.5$), large total demand N leads to a limit cycle of period 2. In any game with an asymmetric equilibrium split ($b \neq 0.5$), chaos emerges at large N . For detailed discussions, see Figure 2.

1. INTRODUCTION

Congestion games [59] are amongst the most well studied class of games in game theory. Being isomorphic to potential games [48], congestion games are one of the few classes of games in which a variety of learning dynamics are known to converge to Nash equilibria [5, 22, 25, 26, 36, 37]. Proving convergence to equilibria typically exploits the existence of the potential function that acts as a (strong) Lyapunov function for learning dynamics; this function is strictly decreasing when the system is out-of-equilibrium.

Congestion games also play a pivotal role in the study of Price of Anarchy [16, 27, 38, 61]. The Price of Anarchy is defined as the ratio of the social cost of the worst Nash equilibrium to the social cost of the optimal state. A small Price of Anarchy implies that all Nash equilibria are near optimal, and hence any equilibrating learning dynamics suffices to reach approximately optimal system performance. One of the hallmarks of the Price of Anarchy research has been the development of tight Price of Anarchy bounds for congestion games that are independent of the topology of the network or the number of users. Specifically, under the prototypical assumption of linear cost functions, Price of Anarchy in the case of non-atomic agents (in which each agent controls an infinitesimal amount of flow) is at most $4/3$ [61]. In the atomic case (in which each agent controls a discrete unit of flow), Price of Anarchy is at most $5/2$ [16], with small networks sufficing to provide tight lower bounds.

Additionally, congestion games have paved the way for recent developments in Price of Anarchy research, extending our understanding of system performance even for non-equilibrating dynamics. Roughgarden [60] showed that most Price of Anarchy results could be organized in a common framework known as (λ, μ) -smoothness. For classes of games that satisfy this property, such as congestion games, the Price of Anarchy bounds derived for worst case Nash equilibria immediately carry over to worst case instantiations of regret minimizing algorithms. An (online) algorithm is said to minimize regret as long as its time-average performance is roughly as good as that of the best fixed action with hindsight. The most ubiquitous member of this class of algorithms is arguably the Multiplicative Weights Update (MWU) [3]. The aforementioned Price of Anarchy results readily apply to any learning dynamics or any sequence of strategic plays, as long as the algorithms achieve small time-average regret. In the case of congestion games, near optimal performance is guaranteed even when learning does not equilibrate at least asymptotically, see Section 8 for the discussion on convergence time issues.

All these positive results have inspired growing efforts to achieve stronger efficiency guarantees. How close to 1 can the Price of Anarchy get? For example, in linear congestion games, what is the final correct answer after all? Is it $5/2$ as the atomic model suggests, or is the much better non-atomic bound of $4/3$ binding? In the case of polynomial cost functions, the gap between these predictions grows exponentially fast with the degree of the polynomials, making this question even more pressing. In a recent development, [24] argues that even if the agents are atomic, as long as the number of agents N is large, atomic congestion games behave approximately as non-atomic ones; hence, $4/3$ is the correct bound. That is, the much smaller non-atomic bound is the correct one after all. A series of related results has followed [17–19], suggesting strong bounds on Price of Anarchy under the assumption of large demand. What is the connection between atomic and non-atomic congestion games?

Let us consider the simplest congestion game example. A game with two strategies and two agents where the cost/latency of each strategy is equal to its load. The worst Nash equilibrium of this game has both agents choosing a strategy uniformly at random. The

expected cost of each agent is $3/2$, i.e., a cost equal to 1 due to their own load, and an expected extra cost of $1/2$ due to the 50% chance of adopting the same strategy as the other agent. On the other hand, at the optimal state, each agent selects a distinct strategy at a cost of 1. As a result, the Price of Anarchy for this game is $3/2$. Suppose now that we increase the number of agents from 2 to N^1 . The worst equilibrium still has each agent choosing a strategy uniformly at random at an expected cost of $(N - 1)/2 + 1 = (N + 1)/2$. The optimal configuration splits the agents, deterministically and equally to both strategies at a cost of $N/2$ per agent. The Price of Anarchy is $1 + 1/N$, converging to 1 as N grows. Indeed, as the population size grows, the atomic game is more conveniently described by its effective non-atomic counterpart, with a continuum of users, and a unique equilibrium that equidistributes the total demand N between the two strategies. The Price of Anarchy result suggests that traffic networks can surprisingly become more efficient as we overload them. This sounds counterintuitively too good to be true. Have we been oversimplifying certain assumptions?

In this work, we will show that the stronger Price of Anarchy bound can come at a significant cost. That is, increasing the total demand can destabilize the system, leading to an erratic, non-equilibrating behavior. Moreover, even if the agents apply a regret minimizing algorithm, such as MWU with an arbitrarily small learning rate ϵ , increasing the total demand guarantees period-doubling bifurcations will arise and the social cost will increase with non-equilibrium fluctuations from an equilibrium flow, rendering the stronger non-atomic Price of Anarchy bound moot from a practical perspective.

To build an intuition why instability can arise, let us revisit the simple example with two strategies and let us consider a continuum of users N updating their strategies according to a learning dynamic, e.g. MWU with a step-size ϵ . Given any non-equilibrium initial condition, the agents on the over-congested strategy have a strong incentive to migrate to the other strategy. As they all act in unison, if the total demand is sufficiently large, the corrective deviation to the other strategy will be overly aggressive, resulting in the other strategy being over-congested.

With this heuristic consideration, a self-sustaining non-equilibrating behavior, where users bear higher time-average costs than those at the equilibrium flow, may seem reasonable. In this work, we will show rigorously that even a simple network can indeed exhibit robust and rich non-equilibrating dynamics, displaying periodic orbits or chaos with large time-average social costs. Congestion games, despite near-fifty-year investigations, are still far from being fully understood when learning dynamics are taken into account. As we will show, non-equilibrating dynamics can arise even in classic regret-minimizing dynamics with arbitrarily small learning rates ϵ , which arguably encapsulate the learning behavior of rational self-interested agents. Despite the usefulness of (λ, μ) -smoothness, there exist regimes of regret-minimizing dynamics in games whose performance cannot be understood fully using standard equilibrium frameworks. These limitations also apply for learning with decreasing step sizes. Thus, in congestion games with large populations, the Price of Anarchy guarantees from the (λ, μ) -smoothness do not become binding until after very long chaotic histories of possibly arbitrarily bad time-average social cost, see Section 8.

Our results: We study linear non-atomic congestion games with two edges and total demand N . All agents are assumed to evolve their behavior using Multiplicative Weights Updates with an arbitrarily small, fixed learning rate ϵ . In Section 3 we show that every

¹For simplicity, let N be an even number.

such system has a critical threshold, a hidden system capacity, which when exceeded, the system exhibits a bifurcation and no longer converges to its equilibrium flow. If the unique equilibrium flow is the 50–50% split (doubly symmetric game), the system proceeds through exactly one period-doubling bifurcation, where a single attracting periodic orbit of period two replaces the attracting fixed point. In the case where the game possesses an asymmetric equilibrium flow, the bifurcation diagram is much more complex. As the total demand changes, we will see the birth and death of periodic attractors of various periods. All such systems provably exhibit Li-Yorke chaos, given sufficiently large total demand. This implies that there exists an uncountable set of initial conditions such that the set is "scrambled", i.e., given any two initial conditions $x(0), y(0)$ in this set, $\liminf dist(x(t), y(t)) = 0$ while $\limsup dist(x(t), y(t)) > 0$. Our analysis leverages techniques developed for two-agent two-strategy congestion games [15, 52], see Section 7.

Everywhere in the non-equilibrating regime, MWU's time-average behavior is reminiscent of its behavior in zero-sum games. Namely, the time-average flows and costs of the strategies converge *exactly* to their *equilibrium values*. Unlike zero-sum games, however, these non-equilibrium dynamics exhibit large regret (Section 4), and high social costs (Section 5), even when the Price of Anarchy is equal to one. In Section 6, we argue that the system displays another signature of chaotic behavior, positive topological entropy. We provide an intuitive explanation by showing that if we encode three events: A) the system is approximately at equilibrium, B) the first strategy is overly congested, C) the second strategy is overly congested, then the number of possible sequences on the alphabet $\{A, B, C\}$, encapsulating possible system dynamics, grows exponentially with time. Clearly, if the system reached an (approximate) equilibrium, any sequences must terminate with an infinite string of the form $\dots AAA \dots$. Rather, we find that the system can become truly unpredictable, in the sense that small differences in initial conditions lead to considerably distinct sequences of events.

Akin to Drosophila in developmental biology or the hydrogen atom in physics, we believe that two-strategy congestion games with MWU learning dynamics is a minimal yet insightful model of game theory, that helps advance our understanding of the interplay between non-equilibrium dynamics, regret, and Price of Anarchy. The paper ends with discussions and observations that carry over to the Appendices. For instance, we show that, in certain non-equilibrating regime, the system may possess multiple distinct attractors and hence the time-average regret and social cost depend critically on initial conditions. Properties of periodic orbits, the evidence of Feigenbaum's universal route to chaos in our non-unimodal map, as well as extensions to more complex congestion games are also provided in the Appendices.

2. MODEL

We consider a two-strategy *congestion game* (see [59]) with a continuum of players(agents), where all of them apply the *multiplicative weights update* to update their strategies [3]. Each of the players controls an infinitesimal small fraction of the flow. We will assume that the total flow of all the players is equal to N . We will denote the fraction of the players adopting the first strategy at time n as x_n . The second strategy is then chosen by $1 - x_n$ fraction of the players. This model physically encapsulates how a large population of commuters selects between the two alternative paths that connect the initial point to the end point. When a large fraction of the players adopt the same strategy, congestion arises, and the cost of choosing the same strategy increases.

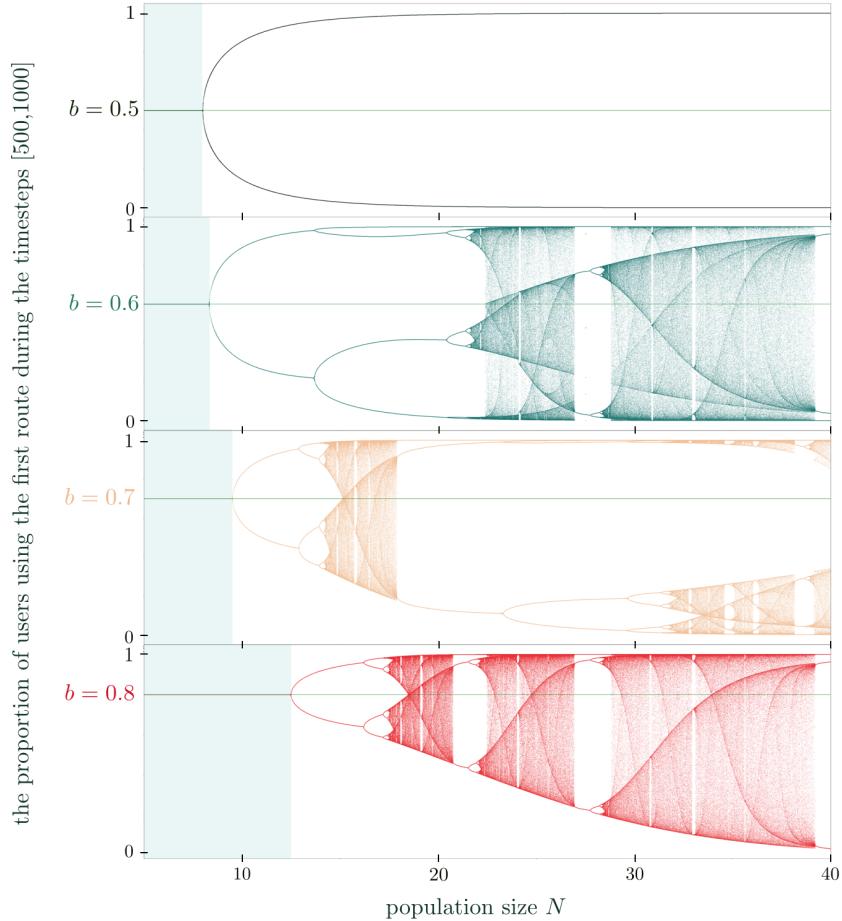


FIGURE 2. These bifurcation diagrams summarize the non-equilibrium phenomena identified in this work. When Multiplicative Weights Update (MWU) learning is applied on a non-atomic linear congestion games with two routes, population increase drives period-doubling instability and chaos. Standard equilibrium analysis only holds at small population sizes, shown in light cyan regions. As population size N (up to a rescaling factor of a fixed learning rate) increases, regret-minimizing MWU algorithm *no longer converges* to the Nash equilibrium flow b , depicted as the green horizontal lines; the proportion of users using the first route deviates significantly from the Nash equilibrium flow. When the equilibrium flow is symmetric between the two routes ($b = 0.5$), large N leads to non-equilibrium dynamics that is attracted toward a limit cycle of period two. For large N , the two periodic points can approach 1 or 0 arbitrarily close, meaning that almost all users will occupy the same route, while simultaneously alternating between the two routes. Thus, the time-average social cost can become as bad as possible. In *any* game with an asymmetric equilibrium flow ($b \neq 0.5$), Li-Yorke chaos is inevitable as N increases. Although the dynamics is non-equilibrating or chaotic, the time-average of the orbits still converges *exactly* to the equilibrium b . This work proves the aforementioned statements, and investigates the implications of non-equilibrium dynamics on the standard Price of Anarchy analysis. Properties of chaotic attractors and of the period-doubling bifurcations, as well as extensions to more complex congestion games are studied in the appendices.

Linear routing games: We focus on linear cost functions. Specifically, the cost of each path (link, route, or strategy) here will be assumed proportional to the *load*. By denoting $c(j)$ the cost of selecting the strategy number j , if the coefficients of proportionality are $\alpha, \beta > 0$, we obtain

$$(1) \quad c_1(x) = \alpha Nx, \quad c_2(x) = \beta Nx.$$

Our analysis on the emergence of bifurcations, limit cycles and chaos will carry over immediately to the cost functions of the form $\alpha x + \gamma$. As we will see, the only parameter that is important is the value of the equilibrium split, i.e. the percentage of players using the first strategy at equilibrium. The first advantage of this formulation is that the fraction of agents using each strategy at equilibrium is independent of the flow N . The second advantage is that the Price of Anarchy of these games is exactly 1, independent of α, β , and N . Hence, the model offers the benchmark of Price of Anarchy equilibrium analysis, which typically suggests optimal social cost at large demand N , and our goal is to elucidate how the social cost arising from non-equilibrium learning dynamics can be far from the optimal one.

2.1. Learning in congestion games with multiplicative weights. At time $n + 1$, we assume the players know the cost of the strategies at time n (equivalently, the probabilities $x_n, 1 - x_n$) and update their choices. Since we have a continuum of agents, the realized flow (split) is accurately described by the probabilities $(x_n, 1 - x_n)$. The algorithm for updating the probabilities that we focus on is the *multiplicative weights update* (MWU), the ubiquitous learning algorithm widely employed in Machine Learning, optimization, and game theory [3]. Namely, there is a parameter $\epsilon \in (0, 1)$, which can be treated as the common learning rate of all players, such that each probability gets multiplied by $(1 - \epsilon)$ to the power which is the cost of playing a given strategy by the given player. The numbers obtained in this way usually will not be probabilities, so we have to normalize them. Thus, we get

$$(2) \quad \begin{aligned} x_{n+1} &= \frac{x_n(1 - \epsilon)^{c_1(x_n)}}{x_n(1 - \epsilon)^{c_1(x_n)} + (1 - x_n)(1 - \epsilon)^{c_2(1-x_n)}} \\ &= \frac{x_n}{x_n + (1 - x_n)(1 - \epsilon)^{c_2(1-x_n) - c_1(x_n)}}. \end{aligned}$$

In this way, a large cost at time n will decrease the probability of choosing the same strategy at time $n + 1$.

By substituting into (2) the values of the cost functions from (1) we get:

$$(3) \quad \begin{aligned} x_{n+1} &= \frac{x_n(1 - \epsilon)^{\alpha Nx_n}}{x_n(1 - \epsilon)^{\alpha Nx_n} + (1 - x_n)(1 - \epsilon)^{\beta N(1-x_n)}} \\ &= \frac{x_n}{x_n + (1 - x_n)(1 - \epsilon)^{\beta N - (\alpha + \beta)Nx_n}}. \end{aligned}$$

We introduce the new variables

$$(4) \quad a = (\alpha + \beta)N \ln \left(\frac{1}{1 - \epsilon} \right), \quad b = \frac{\beta}{\alpha + \beta}.$$

In fact, we can assume without loss of generality that $\alpha + \beta = 1$ (i.e., by the transformation $\alpha' = \frac{\alpha}{\alpha + \beta}$, $\beta' = \frac{\beta}{\alpha + \beta}$, and $\epsilon' = 1 - (1 - \epsilon)^{\alpha + \beta}$). Under these assumptions, equations (4) simplify

to

$$(5) \quad a = N \ln \left(\frac{1}{1 - \epsilon} \right), \quad b = \beta.$$

We will thus study the dynamical systems generated by the one-dimensional map:

$$(6) \quad f_{a,b}(x) = \frac{x}{x + (1 - x) \exp(a(x - b))}.$$

As commonly adopted as a standard assumption, the learning rate ϵ can be regarded as a small, fixed constant in the following analysis but the exact value of constant ϵ is not of particular interest as our analysis/results will hold for any fixed choice of ϵ no matter how small. Setting $\epsilon = 1 - 1/e$ such that $\ln(\frac{1}{1-\epsilon}) = 1$ simplifies notation as under this assumption $a = N$. We will then study the effects of the remaining two parameters on system performance, i.e. a , the (*normalized*) *system demand* and b , the (*normalized*) *equilibrium flow*. When $b = 0.5$ fully symmetric game is achieved; whereas, when b is close to 0 or 1, the routing instance becomes close to a Pigou network whose dominant strategy has almost all the agents selecting the same edge at equilibrium.

2.2. Regret, Price of Anarchy and time average social cost. We will now consider this game from the perspective of each agent as an instance of online sequential optimization problem. Consider the set $A = \{1, 2\}$ of 2 actions and a time horizon $T \geq 1$. At each time step $n = 1, 2, \dots, T$: A decision maker picks a probability distribution $\mathbf{x}_n = (x_n, 1 - x_n)$ over her actions A . An adversary picks a cost vector $\mathbf{c}_n : A \rightarrow [-1, 1]$. An action a_n is chosen according to the distribution \mathbf{x}_n , and the decision-maker receives reward $r_n(a_n)$. The decision-maker learns \mathbf{r}_n , the entire reward vector.

An *online decision-making algorithm* such as MWU specifies for each n the probability distribution \mathbf{x}_n , as a function of the cost vectors $\mathbf{c}_1, \dots, \mathbf{c}_{n-1}$ and the realized actions a_1, \dots, a_{n-1} of the first $n - 1$ time steps. An *adversary* for such an algorithm \mathcal{A} specifies for each n the cost vector \mathbf{c}_n , as a function of the probability distributions x_1, \dots, x_n used by \mathcal{A} on the first n days and the realized actions a_1, \dots, a_{n-1} of the first $n - 1$ days. For example, elements of A could represent different investment strategies, different driving routes between home and work, different wireless routers, different strategies of choosing habitats by animals etc.

Rather than typically comparing the expected reward of an algorithm to that of the best action *sequence* in hindsight, we compare it to the reward incurred by the best fixed action in hindsight. Namely, we change our benchmark from $\sum_{n=1}^T \min_{a \in A} c_n(a)$ to $\min_{a \in A} \sum_{n=1}^T c_n(a)$.

Regret: Fix cost vectors $\mathbf{c}_1, \dots, \mathbf{c}_T$. The (expected) *regret* of the (randomized) algorithm \mathcal{A} choosing actions according x_1, \dots, x_T is

$$(7) \quad \underbrace{\sum_{n=1}^T \mathbf{E}_{a_n \sim x_n} c_n(a_n)}_{\text{our algorithm}} - \underbrace{\min_{a \in A} \sum_{n=1}^T c_n(a)}_{\text{best fixed action}},$$

where $\mathbf{E}_{a_n \sim x_n} c_n(a_n)$ expresses the expected cost of the algorithm of period n , when an action $a_n \in A$ is chosen according to the probability distribution x_n .

In the game theory context, the cost vector \mathbf{c}_n of the each player is incurred by playing the (congestion) game with the other players. Formally, the cost vector for each agent at

time n is $\mathbf{c}_n = (\alpha N x_n, \beta N(1 - x_n))$. The expected accumulated cost of the first agent in the time periods $1, \dots, T$ is equal to $\sum_{n=1}^T (\alpha N x_n^2 + \beta N(1 - x_n)^2)$. The expected regret of the algorithm is given by the formula

$$\sum_{n=1}^T (\alpha N x_n^2 + \beta N(1 - x_n)^2) - \min_{a \in A} \sum_{n=1}^T c_n(a),$$

or equivalently,

$$\sum_{n=1}^T (\alpha N x_n^2 + \beta N(1 - x_n)^2) - \min \left\{ \sum_{n=1}^T \alpha N x_n, \sum_{n=1}^T \beta N(1 - x_n) \right\}.$$

Price of Anarchy: The *Price of Anarchy* of a game is the ratio of the supremum of the social cost over all Nash equilibria divided by the social cost of the optimal state, where the social cost of a state is the sum of the costs of all agents. In our case, where the total flow (demand, or population size) is N , and a fraction x of the population adopts the first strategy, the social cost is $SC(x) = \alpha N^2 x^2 + \beta N^2(1 - x)^2$.

In non-atomic congestion games, it is well known that all equilibria have the same social cost. Moreover, for linear cost functions $c_1(x) = \alpha x$, $c_2(x) = \beta x$ it is straightforward to see that the Price of Anarchy is equal to one, as the unique equilibrium flow², β , also is the unique minimizer of the social cost, which attains a value of $N^2 \alpha \beta$.

With above terminologies in mind, we will study the time-average social cost given an initial condition. Since the time average of social cost may not converge to a specific value, research in algorithmic game theory typically focuses on the supremum over all initial conditions over all convergent subsequences. If we only consider time sequences with vanishing time-average regret, their time averages normalized by the social cost of the optimal state is called the *Price of Total Anarchy*, and, for non-atomic congestion games, it is equal to the Price of Anarchy, which is equal to 1 in our case, suggesting optimal system performance [9, 10]. Since MWU is not run with a decreasing step-size, the time-average regret may not vanish and a more careful analysis is needed. Lastly, taking dynamical systems point of view, we will also study typical dynamical trajectories, since simulations suffice to identify the limits of time-averages of these trajectories, which occur for initial conditions with positive Lebesgue measure.

As we will show, as the total demand increases, the system will bifurcate away from the Nash equilibrium; and the normalized time-average social cost will be strictly greater than 1, i.e. the Price of Anarchy result will no longer be predictive of the true system's non-equilibrium behavior. We now state the formula that will be used later. Under the assumption $\alpha + \beta = 1$, the normalized time-average social cost is

$$(8) \quad \frac{\text{Time-average social cost}}{\text{Optimum social cost}} = \frac{\frac{1}{T} \sum_{n=1}^T (\alpha N^2 x_n^2 + \beta N^2(1 - x_n)^2)}{N^2 \alpha \beta} = \frac{\frac{1}{T} \sum_{n=1}^T (x_n^2 - 2\beta x_n + \beta)}{\beta(1 - \beta)}.$$

²This statement is true under the normalization assumption $\alpha + \beta = 1$.

3. LIMIT CYCLES AND CHAOS, WITH TIME-AVERAGE CONVERGENCE TO NASH EQUILIBRIUM

This section discusses the behavior of the one-dimensional map defined by (6), and its remarkable time-average properties, which we will later employ to analyze the time-average regret and the normalized time-average social cost in Sections 4 and 5. The map generated by non-atomic congestion games here reduces to the map studied in [15], in which two-agent linear congestion games are studied. Up to redefinition of the parameters as well as with the symmetric initial conditions, i.e., on the diagonal, the one-dimensional map in the two scenarios are identical. We thus restate the property of such map in this section. For the proofs we refer the reader to [15].

We will start by investigating the dynamics under the map (6) given by

$$(9) \quad f_{a,b}(x) = \frac{x}{x + (1-x)\exp(a(x-b))},$$

with $a > 0$, $b \in (0, 1)$. This map is generally asymmetric, unless $b = 1/2$, see the middle column of Figure 3. It has three fixed points: 0, b and 1. The derivatives at the three fixed points are

$$f'_{a,b}(0) = \exp(ab), \quad f'_{a,b}(1) = \exp(a(1-b)), \quad f'_{a,b}(b) = ab^2 - ab + 1.$$

Hence, the fixed points 0 and 1 are repelling, while b is repelling whenever $a > \frac{2}{b(1-b)}$.

The critical points of $f_{a,b}$ are solutions to $ax^2 - ax + 1 = 0$. Thus, if $0 < a \leq 4$, then $f_{a,b}$ is strictly increasing. If $a > 4$, it has two critical points

$$(10) \quad x_l = \frac{1}{2} \left(1 - \sqrt{1 - \frac{4}{a}} \right), \quad x_r = 1 - x_l = \frac{1}{2} \left(1 + \sqrt{1 - \frac{4}{a}} \right)$$

so $f_{a,b}$ is bimodal.

Let us investigate regularity of $f_{a,b}$. It is clear that it is analytic. However, nice properties of interval maps are guaranteed not by analyticity, but by the negative Schwarzian derivative. Let us recall that the Schwarzian derivative of f is given by the formula

$$Sf = \frac{f'''}{f'} - \frac{3}{2} \left(\frac{f''}{f'} \right)^2.$$

A “metatheorem” states that almost all natural noninvertible interval maps have negative Schwarzian derivative. Note that if $a \leq 4$ then $f_{a,b}$ is a homeomorphism, so we should not expect negative Schwarzian derivative for that case.

Proposition 3.1. *If $a > 4$ then the map $f_{a,b}$ has negative Schwarzian derivative.*

For maps with negative Schwarzian derivative each attracting or neutral periodic orbit has a critical point in its immediate basin of attraction. Thus, we know that if $a > 4$ then $f_{a,b}$ can have at most two attracting or neutral periodic orbits.

3.1. Time-average convergence to Nash equilibrium b . While we know that the fixed point b is often repelling, especially for large values of a , we can show that it is attracting in a time-average sense.

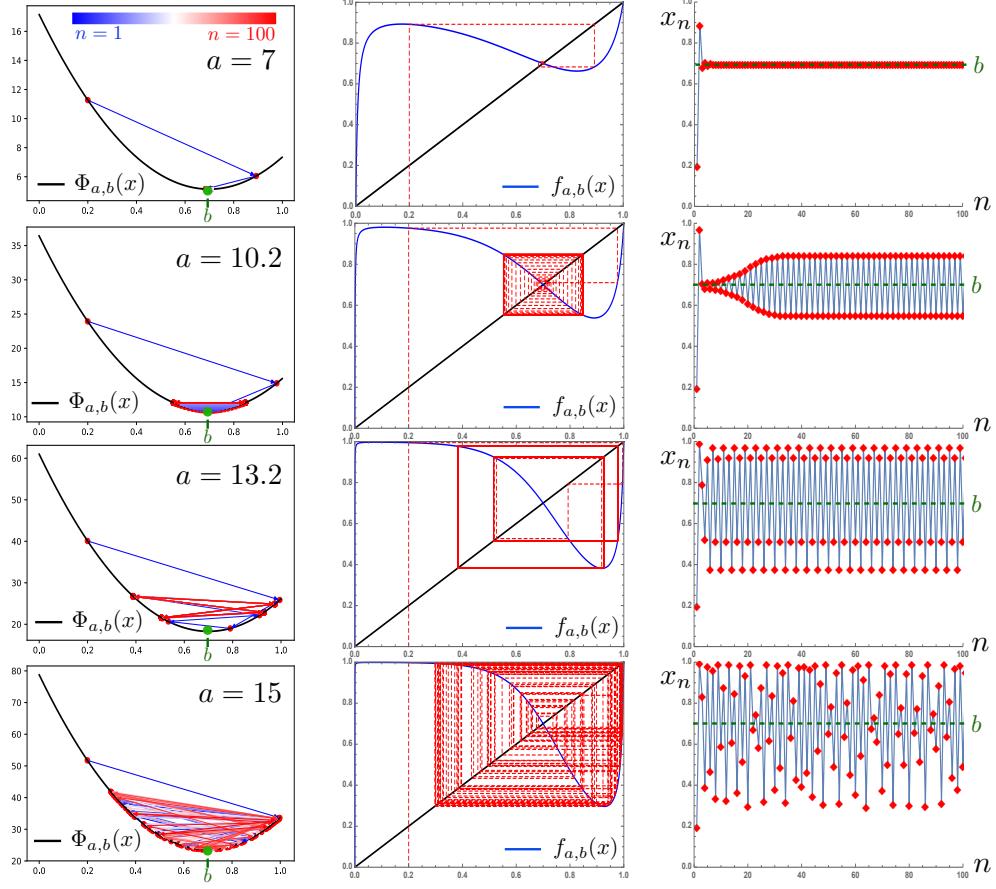


FIGURE 3. Population increase drives period-doubling instability and chaos. Although our congestion game has an associated convex potential function $\Phi_{a,b}(x) = \frac{N^2}{2} (\alpha x^2 + \beta(1-x)^2) = \frac{a^2}{2} ((1-b)x^2 + b(1-x)^2)$ whose unique global minimum is the Nash equilibrium b (without loss of generality, we set $\alpha + \beta = 1$ and $\epsilon = 1 - 1/e$ so that $a = N$ and $b = \beta$), MWU at large a , or equivalently large N with a fixed ϵ , do not converge to the equilibrium, unlike a gradient-like update with a small step size. A line with an arrow connecting $\Phi_{a,b}(x_n)$ to $\Phi_{a,b}(x_{n+1}) = \Phi_{a,b}(f_{a,b}(x_n))$ in the left column is encoded with the color representing the timestep n . Later times are shown in red, while earlier times are shown in blue. Cobweb diagrams of the map $f_{a,b}$ are shown in the middle column, while the dynamics of the map are shown on the right column. From top to bottom, values of a increase while b remains fixed at 0.7, demonstrating population size-driven instability. At these parameter values, the map $f_{a,b}$ is bimodal (blue curves in the middle column). For small a (top row), the dynamics converge to the Nash equilibrium b . As a increases, the dynamics converge to a period 2 attractors (second row), a period 4 attractors (third row), and chaotic attractor (bottom row). As shown in Section 3, however, the time average of these orbits is exactly the Nash equilibrium b , represented by the horizontal green dashed lines on the right column. The initial condition here is set to $x_0 = 0.2$. The bifurcation diagram associated with $b = 0.7$ is shown in Fig. 4

Definition 3.2. For an interval map f a point p is Cesàro attracting if there is a set U such that for every $x \in U$ the averages

$$\frac{1}{T} \sum_{n=0}^{T-1} f^n(x)$$

converge to p .

We can show that b is globally Cesàro attracting. Here by “globally” we mean that the set U from the definition is the interval $(0, 1)$.

Theorem 3.3. For every $a > 0$, $b \in (0, 1)$ and $x \in (0, 1)$ we have

$$(11) \quad \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{n=0}^{T-1} f_{a,b}^n(x) = b.$$

Corollary 3.4. For every periodic orbit $\{x_0, x_1, \dots, x_{T-1}\}$ of $f_{a,b}$ in $(0, 1)$ its center of mass (time average)

$$\frac{x_0 + x_1 + \dots + x_{T-1}}{T}$$

is equal to b .

Applying the Birkhoff Ergodic Theorem (see Section 6), we get a stronger corollary.

Corollary 3.5. For every probability measure μ , invariant for $f_{a,b}$ and such that $\mu(\{0, 1\}) = 0$, we have

$$\int_{[0,1]} x \, d\mu = b.$$

These statements show that the time average of the orbits generated by $f_{a,b}$ converges exactly to the Nash equilibrium b . Next, we discuss what happens as we fix b and increase the total demand by letting a grow large.

3.2. Periodic orbits and chaotic behavior. When $b = 0.5$, the coefficients of the cost functions are identical, i.e. $\alpha = \beta$.

Theorem 3.6. If $0 < a \leq 8$ then $f_{a,0.5}$ -trajectories of all points of $(0, 1)$ converge to the fixed point 0.5 . If $a > 8$ then $f_{a,0.5}$ has a periodic attracting orbit $\{\sigma_a, 1 - \sigma_a\}$, where $0 < \sigma_a < 0.5$. This orbit attracts trajectories of all points of $(0, 1)$, except countably many points, whose trajectories eventually fall into the repelling fixed point 0.5 .

Now we proceed with the case when $b \neq 0.5$, that is, the cost functions differ. As we noticed in Section 3, if $a \leq 4$ then $f_{a,b}$ is strictly increasing and has three fixed points: 0 and 1 repelling and b attracting. Therefore in this case trajectories of all points of $(0, 1)$ converge in a monotone way to b . We know that the fixed point b is repelling if and only if $a > \frac{2}{b(1-b)}$. This is a simple situation, so we turn to the case of large a . We fix $b \in (0, 1) \setminus \{0.5\}$ and let a go to infinity. We will show that if a becomes sufficiently large (but how large, depends on b), then $f_{a,b}$ is Li-Yorke chaotic and has periodic orbits of all periods.

Definition 3.7 (Li-Yorke chaos). *Let (X, f) be a dynamical system and $(x, y) \in X \times X$. We say that (x, y) is a Li-Yorke pair if*

$$\begin{aligned} \liminf_{n \rightarrow \infty} \text{dist}(f^n(x), f^n(y)) &= 0, \\ \limsup_{n \rightarrow \infty} \text{dist}(f^n(x), f^n(y)) &> 0. \end{aligned}$$

A dynamical system (X, f) is Li-Yorke chaotic if there is an uncountable set $S \subset X$ (called scrambled set) such that every pair (x, y) with $x, y \in S$ and $x \neq y$ is a Li-Yorke pair.³

The crucial ingredient of this analysis is the existence of periodic orbit of period 3.

Theorem 3.8. *If $b \in (0, 1) \setminus \{0.5\}$, then there exists a_b such that if $a > a_b$ then $f_{a,b}$ has periodic orbit of period 3.*

By the Sharkovsky Theorem ([66], see also [42]), existence of a periodic orbit of period 3 implies existence of periodic orbits of all periods, and by the result of [42], it implies that the map is Li-Yorke chaotic. Thus, we get the following corollary:

Corollary 3.9. *If $b \in (0, 1) \setminus \{0.5\}$, then there exists a_b such that if $a > a_b$ then $f_{a,b}$ has periodic orbits of all periods and is Li-Yorke chaotic.*

This result has a remarkable implication in non-atomic routing games. Recall that the parameter a expresses the normalized total flow/demand; thus, Corollary 3.9 implies that when the game is asymmetric, i.e. when an interior equilibrium flow is not 50% – 50% split, increasing the total demand of the system will inevitably lead to chaotic behavior, regardless of the form of the cost functions.⁴ In other words, the emergence of chaos at sufficiently large demands is a robust phenomenon.

4. ANALYSIS OF TIME-AVERAGE REGRET

In the previous section, we discussed the time average convergence to Nash equilibrium for the map $f_{a,b}$. We now employ this property to investigate the time-average regret from learning with the MWU.

Theorem 4.1. *The limit of the time-average regret is the total demand N times the limit of the observable $(x - b)^2$ (provided this limit exists). That is*

$$(12) \quad \lim_{T \rightarrow \infty} \frac{R_T}{T} = N \left(\lim_{T \rightarrow \infty} \frac{1}{T} \sum_{n=1}^T (x_n - b)^2 \right).$$

Proof. Recall that the time-average regret is

$$(13) \quad \frac{1}{T} R_T = \frac{1}{T} \sum_{n=1}^T (\alpha N x_n^2 + \beta N (1 - x_n)^2) - \min \left\{ \frac{1}{T} \sum_{n=1}^T \alpha N x_n, \frac{1}{T} \sum_{n=1}^T \beta N (1 - x_n) \right\}$$

Consider

$$(14) \quad \frac{1}{T} \left(\sum_{n=1}^T \alpha x_n - \sum_{n=1}^T \beta (1 - x_n) \right) = \frac{1}{T} \sum_{n=1}^T [(\alpha + \beta)x_n - \beta] = (\alpha + \beta) \left(\frac{1}{T} \sum_{n=1}^T x_n - \frac{\beta}{\alpha + \beta} \right).$$

³Intuition behind this definition as well as other properties of chaotic behavior of dynamical systems are discussed in Section 6.

⁴In fact we can strengthen this result, see Section 6.

The quantity $\frac{\beta}{\alpha+\beta}$ is the system equilibrium b . Without loss of generality we assume as mentioned earlier $\alpha + \beta = 1$. Then, $b = \beta$ and $\lim_{T \rightarrow \infty} \left(\frac{1}{T} \sum_{n=1}^T x_n \right) = b$, by Theorem 3.3.

Therefore, in the limit $T \rightarrow \infty$, two terms in min-term of (13) coincide and we have by substituting $\alpha = 1 - \beta$ and remembering that $\beta = b$

$$\begin{aligned} \lim_{T \rightarrow \infty} \frac{R_T}{T} &= \lim_{T \rightarrow \infty} \frac{N}{T} \sum_{n=1}^T ((1 - \beta)x_n^2 + \beta(1 - x_n)^2 - \beta(1 - \beta)) \\ &= \lim_{T \rightarrow \infty} \frac{N}{T} \sum_{n=1}^T (x_n^2 - 2\beta x_n + \beta^2) = \lim_{T \rightarrow \infty} \frac{N}{T} \sum_{n=1}^T (x_n - b)^2. \end{aligned}$$

□

Observe that if x is a generic point of an ergodic invariant probability measure μ , then the time limit of the observable $(x - b)^2$ is equal to its space average $\int_0^1 (x - b)^2 d\mu(x)$. This quantity is the variance of the random variable identity (we will denote this variable X , so $X(x) = x$) with respect to μ . Typical cases of such a measure μ , for which the set of generic points has positive Lebesgue measure, are when there exists an attracting periodic orbit P and μ is the measure equidistributed on P , and when μ is an ergodic invariant probability measure absolutely continuous with respect to the Lebesgue measure [21]. In analogue to the family of quadratic interval maps [43], we have reasons to expect that for the Lebesgue almost every pair of parameters (a, b) Lebesgue almost every point $x \in (0, 1)$ is generic for a measure of one of those two types.

Upper bound for time-average regret: Let $a > 4$ and $b \in (0, 1)$. Recall from (10) that $f_{a,b}$ has two critical points $x_l = \frac{1}{2} \left(1 - \sqrt{1 - \frac{4}{a}} \right)$ and $x_r = 1 - x_l = \frac{1}{2} \left(1 + \sqrt{1 - \frac{4}{a}} \right)$.

Let

$$y_{\min} = f_{a,b}(x_r), \quad y_{\max} = f_{a,b}(x_l).$$

Lemma 4.2. *For $a > \frac{1}{b(1-b)}$ the interval $I = [y_{\min}, y_{\max}]$ is invariant and globally absorbing on $(0, 1)$.*

Proof. Fix $b \in (0, 1)$. Simple calculations show that $b \in (x_l, x_r)$ if and only if $a > 1/b(1-b)$.⁵ From now we assume that $a > 1/b(1-b)$. Recall that b is a fixed point of $f_{a,b}$ so we have $b = f_{a,b}(b) \in f_{a,b}([x_l, x_r]) = [y_{\min}, y_{\max}] = I$. Therefore $b \in I \cap (x_l, x_r)$ and $I \cap (x_l, x_r) \neq \emptyset$. Because $f_{a,b}$ is decreasing between the critical points, we have $f'_{a,b}(b) = ab^2 - ab + 1 < 0$. The latter and the uniqueness of a fixed point in $(0, 1)$ implies that $f_{a,b}(x) > x$ for $x \in (0, b)$, and $f_{a,b}(x) < x$ for $x \in (b, 1)$.

Obviously, if $x \in I \cap (x_l, x_r)$, then $f_{a,b}(x) \in f_{a,b}([x_l, x_r]) = I$. For $x \in [y_{\min}, x_l]$ we have $f_{a,b}(x) < f_{a,b}(x_l) = y_{\max}$. Suppose that $f_{a,b}(x) < y_{\min}$, then $f_{a,b}(x) < y_{\min} \leq x$ but it is impossible because $f_{a,b}(x) > x$ for $x \in (0, b)$. Thus $f_{a,b}([y_{\min}, x_l]) \subset I$. The same reasoning shows that $f_{a,b}([x_r, y_{\max}]) \subset I$. Thus $f_{a,b}(I) \subset I$.

Now let $x \in (0, x_l)$. Obviously $f_{a,b}(x) < y_{\max}$ and because $x_l < b$ we have $f_{a,b}(x) > x$ for $x < x_l$. To show that the orbit of x falls eventually into I , it is sufficient to show that there exists n such that $f_{a,b}^n(x) > y_{\min}$. Suppose that there is no such n , that is $f_{a,b}^n(x) < y_{\min}$ for all n . The sequence $(f_{a,b}^n(x))_{n \geq 1}$ is increasing and bounded, so it has a limit. Denote the

⁵Therefore $b \in (x_l, x_r)$ when $x = b$ is repelling (for $a > 2/b(1-b)$).

limit by $c \in (0, x_l)$. Then $f_{a,b}(c)$ has to be equal c , but this contradicts the fact that on $(0, x_l)$ there is no fixed point of $f_{a,b}$. Thus, orbits of every point from $(0, x_l)$ will eventually fall into the invariant set I . Similar reasoning will show that orbits of every point from $x \in (x_r, 1)$ will eventually fall into I .

□

Lemma 4.2 implies the following bound for the variance.

Remark 4.3. For $a > \frac{1}{b(1-b)}$ the variance of X is bounded above by

$$\text{Var}(X) = \lim_{T \rightarrow \infty} \left(\frac{1}{T} \sum_{n=1}^T (x_n - b)^2 \right) \leq (y_{\max} - b)(b - y_{\min}).$$

To see how time-average regret changes as one increases the total demand N , we let as usual $\epsilon = 1 - 1/e$, so that $a = N \ln(\frac{1}{1-\epsilon}) = N$.

Theorem 4.4. For $N > \frac{1}{b(1-b)}$ the time-average regret is bounded above by

$$(15) \quad \lim_{T \rightarrow \infty} \frac{R_T}{T} = N \left(\lim_{T \rightarrow \infty} \frac{1}{T} \sum_{n=1}^T (x_n - b)^2 \right) \leq N(y_{\max} - b)(b - y_{\min}).$$

5. ANALYSIS OF TIME-AVERAGE SOCIAL COST

We begin this section with an extreme scenario of the time-average social cost; for a symmetric equilibrium flow ($b = 0.5$), the time-average social cost can be arbitrarily close to its worst possible value! In contrast to the optimal social cost attained at the equilibrium $b = 0.5$, the long-time dynamics alternate between the two periodic points of the limit cycle of period two, which, at large population size, can approach 1 or 0 arbitrarily close, see Figure 2 (top) or Figure 5. This means almost all users will occupy the same route, while simultaneously alternating between the two routes. The time-average social cost thus becomes worsen at larger population size, approaching the worst possible value in the limit of infinite population.

Theorem 5.1. For $b = 0.5$, the time-average social cost can be arbitrarily close to its worst possible value for a sufficiently large a , i.e. for a sufficiently large population size⁶. Formally, for any $\delta > 0$, there exists an a such that, for any initial condition x_1 ,

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \sum_{n=1}^T SC(x_n) > \max_x SC(x) - \delta$$

Proof. For a symmetric equilibrium $b = 0.5$, the two cost functions increase with the loads at the same rate $\alpha = \beta$. Recall from Section 2.2 that the social cost when the fraction x of the population adopts the first strategy is $SC(x) = \alpha N^2 x^2 + \alpha N^2 (1-x)^2$. This strictly convex function attains its minimum at the equilibrium $x = b = 0.5$, and its maximum of αN^2 at $x = 0$ or $x = 1$. To establish that the time-average limit of the social cost can become arbitrarily close to αN^2 , it suffices to show that the distance of the two periodic points (of the unique period-2 limit cycle) to the nearest boundary goes to zero as $a \rightarrow \infty$. By symmetry of the game, the two periodic points of the limit cycle of period 2, which by

⁶Recall from (5) that $a = N \ln(\frac{1}{1-\epsilon})$.

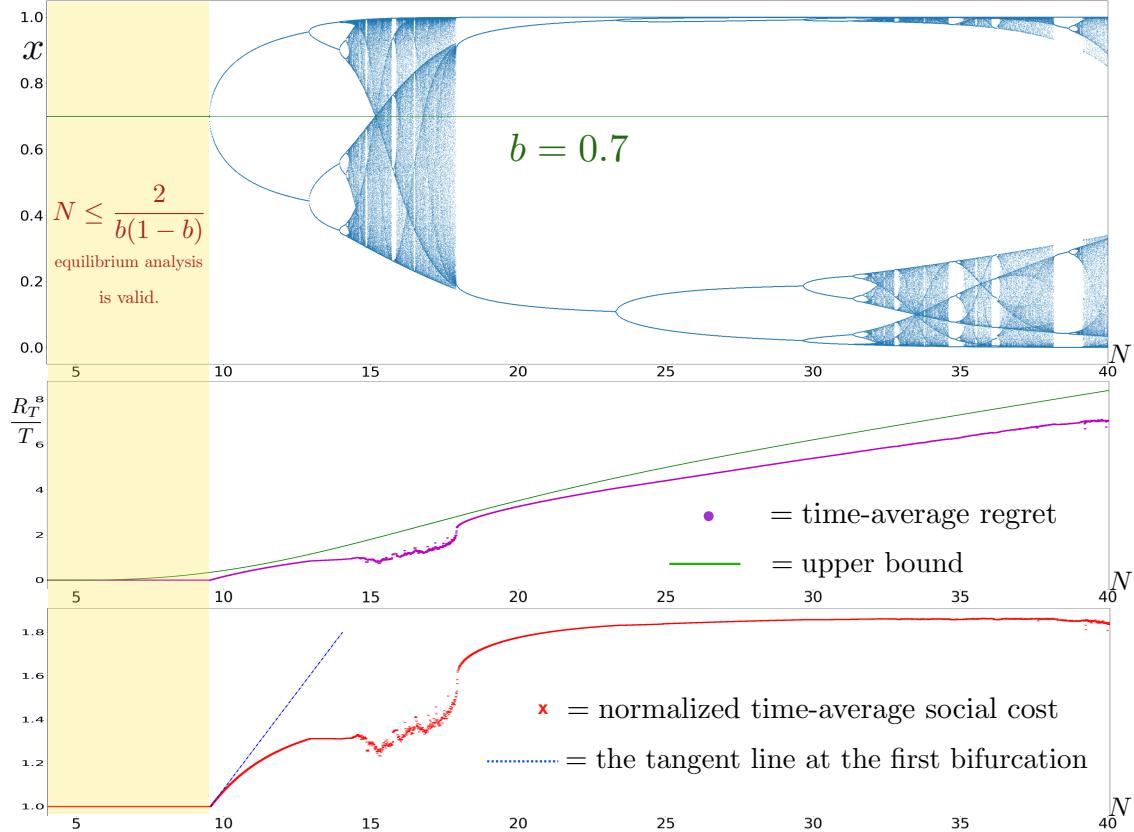


FIGURE 4. Bifurcation diagram (Top) demonstrates instability of the routing game driven by the increase in total demand N . The Nash equilibrium here is set to $b = 0.7$. As usual, we fix the learning rate $\epsilon = 1 - 1/e$ so that $a = N$ for simplicity. At small N , the dynamics converges toward the fixed point b which is the Nash equilibrium. However, as N exceeds the carrying capacity $N_b^* = 2/b(1-b)$, the period-doubling route to chaos arises, and the long-time dynamics no longer equilibrate the the Nash equilibrium. Remarkably, the time-average orbits are exactly b , as is evident from the green line that tracks the center of masses of the blue orbits. (Middle) The time-average regret $\frac{1}{T}R_T$, shown in the purple circle symbol, suddenly become strictly positive at the first bifurcation, consistent with the prediction of (12) which states that the time-average regret is proportional to the fluctuations from the Nash equilibrium. These time-average regrets also lie below the upper bound of Equation (15) as expected, shown in the green line. (Bottom) normalized time-average social cost also *suddenly* becomes greater than 1 at the first bifurcation, consistent with the prediction of Equation (8). Hence, even though the classic Price of Anarchy (PoA) is reasonable to describe the equilibrium dynamics that takes place before the first bifurcation arises, the non-equilibrating dynamics has the PoA exceeding unity by a factor proportional to the fluctuations away from the Nash equilibrium. The rate at which the normalized time-average social cost increases above unity at the first bifurcation is depicted by the tangent line (dashed blue), which is calculated from Equation (18).

Theorem 3.6 exists for $a > 8$, are symmetric around the equilibrium $b = 0.5$. Denote the two periodic points by σ_a and $1 - \sigma_a$ where $\sigma_a \in (0, 0.5)$. Then, it suffices to show that given any $\delta > 0$, there exist an a such that $\sigma_a < \delta$.

For brevity, we denote the map $f_{a,0.5}$ by f_a . Then, since σ_a is a periodic point of a limit cycle of period 2, we have $f_a^2(\sigma_a) = \sigma_a$. The last equality implies $f_a(\sigma_a) = 1 - \sigma_a$, which, after simple calculations, implies

$$\left(\frac{\sigma_a}{1 - \sigma_a} \right)^2 = \exp[a(\sigma_a - 0.5)].$$

Consider the function $\phi(x) = \left(\frac{x}{1-x}\right)^2 - \exp[a(x-0.5)]$, then $\phi(0) = -\exp[-0.5a] < 0$. On the other hand, for any $\delta \in (0, 0.5)$, $\phi(\delta) = \left(\frac{\delta}{1-\delta}\right)^2 - \exp[a(\delta-0.5)] > \frac{1}{2}\left(\frac{\delta}{1-\delta}\right)^2 > 0$ for a sufficiently large $a > 0$. The intermediate value theorem implies $\sigma_a \in (0, \delta)$, and the theorem follows. \square

More generally, when the equilibrium flow is asymmetric ($b \neq 0.5$), we can relate the normalized time-average social cost to the non-equilibrium fluctuations from the equilibrium flow. From (8), we obtain

$$(16) \quad \text{normalized time-average social cost} = \frac{\frac{1}{T} \sum_{n=1}^T (x_n^2 - 2\beta x_n + \beta)}{\beta(1-\beta)} = 1 + \frac{\text{Var}(X)}{\beta(1-\beta)}.$$

If the dynamics converges to the fixed point b , the variance vanishes and the normalized time-average social cost coincides with the Price of Total Anarchy which is 1. However, as the total demand N increases, the system suddenly bifurcates at $N = N_b^* \equiv 2/b(1-b)$, which is the carrying capacity of the network. Above the carrying capacity N_b^* , the system is non-equilibrating, and the variance becomes positive. As a result, the normalized time-average social cost becomes greater than 1. How fast the Price of Anarchy metric fails depend on how fast the variance *suddenly* increases at the first bifurcation point, which we now analyze.

5.1. Analysis of variance spreading at the first period-doubling bifurcation. We study the behavior of the variance as a crosses the period doubling bifurcation point. We first consider the model situation, where the map is $g(x) = (\gamma_1 - 1)x + \gamma_2 x^2 + \gamma_3 x^3$. Note that γ_1 and γ_2 here are not the coefficients of the cost functions. In this model situation, the bifurcation occurs at $\gamma_1 = 0$. If $\gamma_1 > 0$ then the fixed point $x = 0$ is attracting, and as $\gamma_1 < 0$ then it is repelling, but under some conditions on the coefficients there is an attracting periodic orbit of period 2. We are interested only at the limit behavior as γ_1 goes to zero, in a small neighborhood of $x = 0$. Therefore we may ignore all powers of x larger than 3 and all powers of γ_1 larger than 1. Period 2 points are non-zero solutions of the equation

$$x = (\gamma_1 - 1)[(\gamma_1 - 1)x + \gamma_2 x^2 + \gamma_3 x^3] + \gamma_2[(\gamma_1 - 1)x + \gamma_2 x^2 + \gamma_3 x^3]^2 + \gamma_3[(\gamma_1 - 1)x + \gamma_2 x^2 + \gamma_3 x^3]^3.$$

Ignoring higher order terms and dividing by x , we get the equation

$$(2\gamma_2^2\gamma_1 - 2\gamma_2^2 + 4\gamma_1\gamma_3 - 2\gamma_3)x^2 - \gamma_1\gamma_2x - 2\gamma_1 = 0.$$

Its discriminant is (after ignoring higher order terms in γ_1)

$$\Delta = -16\gamma_1(\gamma_2^2 + \gamma_3),$$

so

$$x = \frac{\gamma_1\gamma_2 \pm 4\sqrt{-\gamma_1(\gamma_2^2 + \gamma_3)}}{4(\gamma_2^2\gamma_1 - \gamma_2^2 + 2\gamma_1\gamma_3 - \gamma_3)}.$$

Assume now that $\gamma_2^2 + \gamma_3 > 0$. This is equivalent to $Sg(0) < 0$, so we will be able to apply it to our map (see Proposition 3.1). If γ_1 is close to zero, in the numerator γ_1 is negligible

compared to $\sqrt{\gamma_1}$, and in the denominator γ_1 is negligible compared to a constant. Thus, approximately we have

$$x = \pm \sqrt{\frac{-\gamma_1}{\gamma_2^2 + \gamma_3}}.$$

Therefore, the variance is $\text{Var}(X) = \frac{-\gamma_1}{\gamma_2^2 + \gamma_3}$.

After Taylor expanding the map of Equation (6) around the fixed point b and comparing the coefficients to those of g , we obtain $\gamma_1 = 2 + ab(b - 1)$, $\gamma_2 = a(b - \frac{1}{2})(1 + ab(b - 1))$ and $\gamma_3 = a(1 + a(\frac{1}{6} + b(b - 1))(3 + ab(b - 1)))$. Recalling the first bifurcation occurs at $a_b^* = 2/b(1 - b)$, we thus deduce the derivative of the variance with respect to a at the first period-doubling bifurcation:

$$(17) \quad \frac{d\text{Var}(X)}{da} \Big|_{a=a_b^*} = -\frac{d\left(\frac{\gamma_1}{\gamma_2^2 + \gamma_3}\right)}{da} \Big|_{a=a_b^*} = \frac{3b^3(1 - b)^3}{2 - 6b(1 - b)},$$

which is a unimodal function in the interval $[0, 1]$ that is symmetric around $b = 0.5$, at which the maximum 0.09375 is attained.

This allows us to deduce how fast the normalized time-average social cost increases at the first bifurcation, signaling how equilibrium Price of Anarchy metric fails as we increase a , or equivalently increase N . Namely, from Equation (8), one finds that the derivative of the normalized time-average social cost with respect to a reads

$$(18) \quad \frac{d}{da} (\text{normalized time-average social cost}) \Big|_{a=a_b^*} = \frac{1}{b(1 - b)} \frac{d\text{Var}(X)}{da} \Big|_{a=a_b^*} = \frac{3b^2(1 - b)^2}{2 - 6b(1 - b)}.$$

When $a < 2/b(1 - b)$, the system equilibrates and the normalized time-average social cost is unity. However, when a exceeds $2/b(1 - b)$, the system is out of equilibrium, and normalized time-average social cost *suddenly* increases with a finite rate, given by Equation (18). At the first period-doubling bifurcation, the second-derivative with respect to a becomes *discontinuous*, akin to the second order phase transition phenomena in statistical physics. Fig. 4 confirms the prediction of Equation (18).

Likewise, as the variance becomes positive, the time-average regret also becomes non-zero. At the first period-doubling bifurcation, the time-average regret given by Equation (12) *suddenly* increases with a at the rate

$$(19) \quad \frac{d}{da} (\text{time-average regret}) \Big|_{a=a_b^*} = \frac{d(a\text{Var}(X))}{da} \Big|_{a=a_b^*} = \frac{3b^2(1 - b)^2}{1 - 3b(1 - b)}.$$

Therefore, at the first period-doubling bifurcation, where the equilibrium analysis begins to breakdown, the following equality holds

$$(20) \quad \frac{d}{da} (\text{time-average regret}) \Big|_{a=a_b^*} = 2 \frac{d}{da} (\text{normalized time-average social cost}) \Big|_{a=a_b^*}.$$

6. CHAOS AND BIRKHOFF ERGODIC THEOREM

This section familiarizes the reader with key concepts from dynamical systems necessary for this work, e.g., chaotic behavior, absolutely continuous invariant measures, topological entropy, and ergodic theorem.

It seems that there is no universally accepted definition of chaotic behavior of a dynamical system. Most definitions of chaos concern one of the following aspects:

- complex behavior of trajectories, such as Li-Yorke chaos;
- fast growth of the number of distinguishable orbits of length n , such as having positive topological entropy;
- existence of absolutely continuous invariant measures;
- sensitive dependence on initial conditions, such as Devaney or Auslander-Yorke chaos;
- recurrence properties, such as transitivity or mixing.

In this article, the first three are crucial. Also, in the presence of chaos, studying precise single orbit dynamics can be intractable; we study the average behavior of trajectories instead. Thus, it is important to know whether the average converges. This is when ergodic theorems come into play.

6.1. Li-Yorke chaos and topological entropy. The origin of the definition of Li-Yorke chaos (see Definition 3.7) is in the seminal Li and Yorke's article [42]. Intuitively orbits of two points from the scrambled set have to gather themselves arbitrarily close and spring aside infinitely many times but (if X is compact) it cannot happen simultaneously for each pair of points. Why should a system with this property be chaotic? Obviously existence of the scrambled set implies that orbits of points behave in unpredictable, complex way. More arguments come from the theory of interval transformations, in view of which it was introduced. For such maps the existence of one Li-Yorke pair implies the existence of an uncountable scrambled set [39] and it is not very far from implying all other properties that have been called chaotic in this context, see e.g. [62]. In general, Li-Yorke chaos has been proved to be a necessary condition for many other 'chaotic' properties to hold. A nice survey of properties of Li-Yorke chaotic systems can be found in [8].

Crucial feature of the chaotic behavior of a dynamical system is exponential growth of the number of distinguishable orbits. This happens if and only if the topological entropy of the system is positive. In fact positivity of topological entropy turned out to be an essential criterion of chaos [31]. This choice comes from the fact that the future of a deterministic (zero entropy) dynamical system can be predicted if its past is known (see [73, Chapter 7]) and positive entropy is related to randomness and chaos.

For every dynamical system over compact phase space, we can define a number $h(f) \in [0, \infty]$ called the topological entropy of transformation f . This quantity was first introduced by Adler, Konheim and McAndrew [1] as the topological counterpart of metric (and Shannon) entropy.

For a given positive integer n we define the n -th Bowen-Dinaburg metric on X , ρ_n^f as

$$\rho_n^f(x, y) = \max_{0 \leq i < n} \text{dist}(f^i(x), f^i(y)).$$

We say that the set E is (n, ε) -separated if $\rho_n^f(x, y) > \varepsilon$ for any distinct $x, y \in E$ and by $s(n, \varepsilon, f)$ we denote the cardinality of the most numerous (n, ε) -separated set for (X, f) .

Definition 6.1. *The topological entropy of f is defined as*

$$h(f) = \lim_{\varepsilon \searrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log s(n, \varepsilon, f).$$

We begin with the intuitive explanation of the idea. Let us assume that we observe the dynamical system with the precision $\varepsilon > 0$, that is, we can distinguish any two points, only if they are apart by at least ε . Then, we are able to see only as many points as is the cardinality of the biggest $(1, \varepsilon)$ -separated set. Therefore, after n iterations we will see

at most $s(n, \varepsilon, f)$ different orbits. If transformation f is mixing points, then $s(n, \varepsilon, f)$ will grow. Taking upper limit over n will give us the exponential ratio of asymptotic growth of number of (distinguishable) orbits, and going with ε to zero will give us the quantity which can be treated as a measure of exponential speed, with which the number of orbits grow (with n).

Both positive topological entropy and Li-Yorke chaos are local properties; in fact, entropy depends only on a specific subset of the phase space and is concentrated on the set of so-called nonwandering points [11]. The question whether positive topological entropy implies Li-Yorke chaos remained open for some time, but eventually it was shown to be true; see [7]. On the other hand, there are Li-Yorke chaotic interval maps with zero topological entropy (as was shown independently by Smítal [67] and Xiong [74]). For deeper discussion of these matters we refer the reader to the excellent surveys by Blanchard [6], Glasner and Ye [32], Li and Ye [41] and Ruette's book [62].

Entropy of $f_{a,b}$ is positive: After this discussion we can show how entropy behaves for $f_{a,b}$. For any interval map, we have the following:

Theorem 6.2 ([46]). *For an interval map f , the following assertions are equivalent:*

- i) f has a periodic point whose period is not a power of 2,
- ii) the topological entropy of f is positive.

Thus, Theorem 6.2 combined with Corollary 3.9 strengthen the latter.

Corollary 6.3. *If $b \in (0, 1) \setminus \{1/2\}$, then there exists a_b such that if $a > a_b$ then $f_{a,b}$ has periodic orbits of all periods, positive topological entropy and is Li-Yorke chaotic.*

Calculating entropy: In general, computing the entropy is not an easy task. However, in the context of interval maps, topological entropy can be computed quite straightforwardly — it is equal to the exponential growth rate of the minimal number of monotone subintervals for f^n .

Theorem 6.4 ([47]). *Let f be a piecewise monotone interval map and, for all $n \geq 1$, let c_n be the minimal cardinality of a monotone partition for f^n . Then*

$$h(f) = \lim_{n \rightarrow \infty} \frac{1}{n} \log c_n = \inf_{n \geq 1} \frac{1}{n} \log c_n.$$

Moreover, for piecewise monotone interval maps, the entropy computed with any partition into intervals, on which the map is monotone, is the *topological entropy* [2, Prop. 4.2.3]. This gives us a way to understand what positive entropy of $f_{a,b}$ means from a game-theoretic perspective. For $a > \frac{1}{b(1-b)}$ the map $f_{a,b}$ is a bimodal map with two critical points x_l, x_r (defined in (10)) and a (unique in $(0, 1)$) equilibrium $b \in (x_l, x_r)$. Because x is the probability of choosing the first strategy, we can say that if $x < x_l$ or $x > x_r$, then one of the strategies is overused and if x is close to b , $x \in [x_l, x_r]$, then the system is approximately at equilibrium. Now, we can take a partition $\{[0, x_l], [x_l, x_r], (x_r, 1]\}$ into three intervals on which $f_{a,b}$ is monotone. For every $x \in [0, 1]$ and for every $n \geq 1$ we encode three events for the n -th iteration of x : $\mathbf{x}[n] = A$ if the system is approximately at equilibrium, that is if $f_{a,b}^n(x) \in [x_l, x_r]$; $\mathbf{x}[n] = B$ if the second strategy is overused, that is when $f_{a,b}^n(x) \in [0, x_l]$ and $\mathbf{x}[n] = C$ if the first strategy is overused, $f_{a,b}^n(x) \in (x_r, 1]$. This way for every $x \in [0, 1]$ we get an infinite sequence \mathbf{x} on the alphabet $\{A, B, C\}$. Now, the fact that $h(f_{a,b}) > 0$ implies that the number of different blocks of length n , which we can observe looking at strings, will grow exponentially.

6.2. Invariant measures and ergodic theorem. We can also discuss a discrete dynamical system in terms of a measure preserving transformation defined on the probability space. This approach can handle not only purely mathematical concepts but also physical phenomena in nature. This subsection is devoted to invariant measures, absolutely continuous measures and the most fundamental idea in ergodic theory — the Birkhoff Ergodic Theorem, which states that with probability one the average of a function along an orbit of an ergodic transformation is equal to the integral of the given function.

Definitions. Let (X, \mathcal{B}, μ) be a probability space and $f: X \mapsto X$ be a measurable map. The measure μ is f -invariant (a map f is μ -invariant) if $\mu(f^{-1}E) = \mu(E)$ for every $E \in \mathcal{B}$. For f -invariant measure μ we say that μ is ergodic (f is ergodic) if $E \in \mathcal{B}$ satisfies $f^{-1}E = E$ if and only if $\mu(E) = 0$ or 1 . A measure μ is absolutely continuous with respect to Lebesgue measure if and only if for every set $E \in \mathcal{B}$ of zero Lebesgue measure $\mu(E) = 0$.

We can now state ergodic theorem.

Theorem 6.5 (Birkhoff Ergodic Theorem). *Let (X, \mathcal{B}, μ) be a probability space. If f is μ -invariant and g is integrable, then*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} g(f^k(x)) = g^*(x)$$

for some $g^* \in L^1(X, \mu)$ with $g^*(f(x)) = g^*(x)$ for almost every x . Furthermore if f is ergodic, then g^* is constant and

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} g(f^k(x)) = \int_X g \, d\mu$$

for almost every x .

Lastly, why absolutely continuous invariant measures matter? Computer-based investigations are widely used to gain insights into the dynamics of chaotic phenomena. However, one must exercise caution in the interpretation of computer simulations. Often, chaotic systems exhibit multiple ergodic invariant measures [30]; it is thus important to distinguish between the measure exhibited by an actual orbit, and the measure of the orbit obtained from computer simulations, which may differ due to accumulated computational round-off errors. But if the absolutely continuous measure with respect to Lebesgue measure exists, then computer simulations will yield the measure that we expect [12]. Thus, the theoretical measure and the computational measure coincide in this work.

7. RELATED WORK

The study of learning dynamics in game theory has a long history, dating back to the work of Brown [13] and Robinson [58] on fictitious play in zero-sum games, which shortly followed von Neumann's seminal work on zero-sum games [71, 72]. Excellent reference books are the following: Cesa-Bianchi and Lugosi [14], Fudenberg and Levine [28], Hofbauer and Sigmund [33], Young [75] and Sandholm [64].

Main precursors: Palaipanlai et al. [52] put forward non-equilibrium dynamics arising from Multiplicative Weights Update (MWU) learning in congestion games. They establish the existence of an attracting limit cycle of period two and of Li-Yorke chaos for MWU dynamics in *atomic* congestion games with two agents and two links with linear cost functions. Symmetry of the game (i.e. the existence of a symmetric equilibrium where both

agents select each path with probability 0.5) results in a limit cycle of period two, whereas a *specific instance* of a game with asymmetric equilibrium leads to Li-Yorke chaos, provided that agents adapt the strategies with a sufficiently large learning rate (step size) ϵ (equivalently, if agents use a fixed learning rate ϵ but their costs are scaled up sufficiently large). Shortly afterwards, Chotibut et al. [15] established that Li-Yorke chaos is prevalent in *any* two-agent *atomic* congestion games with two parallel links and linear cost functions, provided the equilibrium is asymmetric. Namely, in *any* game with an asymmetric equilibrium, Li-Yorke chaos emerges as the cost functions grow sufficiently large, and only if the initial condition is symmetric, i.e., both agents start with the same initial conditions. For the first time, [15] also establishes that, despite periodic or chaotic behaviors, the time-average strategies of both agents *always* converge *exactly* to a Nash equilibrium. While this work leverages extensively techniques from [15], it also investigates other definitions of chaos, e.g., positive topological entropy, studies *non-atomic* congestion games, and relates the results to the Price of Anarchy analysis. Lastly, in the appendices, we provide preliminary results for learning dynamics in larger and more complex congestion games.

Chaos in game theory: Under the assumption of perfect rationality, it is not surprising that Nash equilibria are central concepts in game theory. However, in reality, players do not typically play a game following Nash equilibrium strategy. The seminal work of Sato et al. [65] showed analytically by computing the Lyapunov exponents of the system that even in a simple two-player game of rock-paper-scissor, replicator dynamics (the continuous-time analogue of MWU) can lead to chaos, rendering the equilibrium strategy inaccessible. For two-player games with a large number of available strategies (complicated games), Galla and Farmer [29] argue that experienced weighted attraction (EWA) learning, a behavioral economics model of learning dynamics, exhibits also chaotic behaviors in a large parameter space. The prevalence of these chaotic dynamics also persists in games with many players, as shown in the recent follow-up work [63]. Thus, careful examinations suggest a complex behavioral landscape in many games (small or large) for which no single theoretical framework currently applies. Recently, chaotic dynamics that arise from models of discrete-time learning in games have also been reported. Sparrow et al. [68] and van Strien and Sparrow [70] prove that fictitious play learning dynamics for a class of 3x3 games, including the Shapley's game and zero-sum dynamics, possess rich periodic and chaotic behavior. [?] prove that many online learning algorithms, including MWU, with a constant step size is Lyapunov chaotic when applied to zero-sum games. Palaiopoulos et al. [52], Chotibut et al. [15], and this work also belong to the category of chaos from discrete-time learning dynamics.

Other non-equilibrium phenomena in game theory: In recent years, the algorithmic game theory community has produced several interesting non-equilibrium results. Daskalakis et al. [20] show that MWU does not converge even in a time-average sense in a specific 3x3 game. Kleinberg et al. [35] establish non-convergent dynamics for a continuous-time variant of MWU, known as the replicator dynamics, in a 2x2x2 game and show as a result that the system social welfare converges to states that dominate all Nash equilibria. [51] analyzes continuous-time fictitious play in a 3x3 game and shows similarly that the dynamics dominate in performance Nash equilibria. Our results add a new chapter in this direction, providing detailed understanding of the non-equilibrium phenomena arising from MWU in *non-atomic* congestion games, as well as their important implications on regret and social costs. In evolutionary game theory contexts, which typically study continuous-time variant of MWU (replicator dynamics), numerous non-convergence results are known but again are commonly

restricted to small games [64]. Piliouras and Shamma [57] and Piliouras et al. [55] show that replicator dynamics exhibit a specific type of near periodic behavior, known as Poincaré recurrence. Recently, Mertikopoulos et al. [45] show that Poincaré recurrence also shows up in a more general class of continuous-time dynamics known as Follow-the-Regularized-Leader (FTRL). Mai et al. [44] show that these arguments can also be applied to dynamically evolving games. Perfect periodic or cyclic behaviors can also arise in team competition [56] as well as in network competition [49]. Works in this category combine interesting arguments such as volume preservation and the existence of constants of motions (“conservation of energy”) to show cyclic behaviors. Recently, Bailey and Piliouras [4] establishes a connection between game theory, online optimization and a ubiquitous system in classical physics known as Hamiltonian system, which naturally possess conservation laws. Ostrovski and van Strien [50] consider a class of piecewise affine Hamiltonian vector fields whose orbits are piecewise straight lines and develop connections with best-reply dynamics. In the case of discrete-time dynamics, such as MWU or gradient descent, the system trajectories are first order approximations of the continuous dynamics, and conservation laws may no longer hold. So far, it is not clear to what extent the connections to Hamiltonian dynamics can be generalized. Finally, Papadimitriou and Piliouras [53] put forward a program for linking game theory to topology of dynamical systems.

8. DISCUSSION ON FIXED VERSUS SHRINKING STEP SIZES AND REGRET

What about shrinking step sizes and vanishing regret? In this paper, we examine MWU with a fixed step size ϵ that has a non-vanishing regret. Can our results be disregarded if the agents leverage shrinking step sizes algorithms, e.g. at a rate $\epsilon = 1/\sqrt{T}$, which results in a vanishing regret of $O(1/\sqrt{T})$? Can shrinking step sizes algorithm be a quick and painless fix? *No. The answer lies inside the big O notation.*

When agents implement a no-regret strategy and the cost in each time step is $c_n : S \rightarrow \mathbb{R}$, with $c_n(s) \in [0, M]$, we can show

$$\mathbb{E}\left[\sum_{n=1}^T c_k(s_n)\right] < \mathbb{E}\left[\min_{s \in S} \sum_{n=1}^T c_n(s)\right] + (M+1)\sqrt{T \log(S)}$$

where S is the number of strategies available to the agents (see also [14] [Sections 2.6, 2.8, Remark 2.2] for the discussion on why the term $O(M\sqrt{T \log(S)})$ cannot be further improved in a general case). Hence, the time-average regret is $\frac{(M+1)\sqrt{\log(S)}}{\sqrt{T}}$, which vanishes as $T \rightarrow \infty$. However, for large enough M the amount of time T for the regret to become negligibly small can be impractically large.

What is the value of M in our setting of congestion games? It is the worst possible cost $M = N \max\{\alpha, \beta\}$. So, for a large population size N , even for MWU with shrinking ϵ , the wait until the regret is negligibly small can be impractically long. For any meaningful time horizon, the regret of the agents can still be so large that the (λ, μ) -robustness type of results [60] cannot be applied. A new theoretical framework to study these long transient periods with large regrets is needed.

Succinctly put, no-regret algorithms are rather inefficient in finding coarse correlated equilibria in large games, exactly because their regrets vanish at a slow rate. This is why different centralized algorithms have been developed to compute correlated equilibria in games with many agents that necessitate compact descriptions [34, 54]. Quoting from [54], “(No-regret)

learning methods require an exponential number of iterations to converge, proportional to $(1/\epsilon)^k$ for a constant $k > 1$. Our ellipsoid-based algorithm, on the other hand is polynomial in $\log(\frac{1}{\epsilon})$.” Also from [54], “Intractability of an equilibrium concept would make it implausible as a model of behavior. In the words of Kamal Jain: “If your PC cannot find it, then neither can the market.”” Thus, regret-minimizing algorithms are an unreasonably slow method for converging to correlated equilibria. This algorithmic inefficiency necessitated the development of complex centralized ellipsoid methods algorithms. Our analysis explains the algorithmic behavior along this long *transient* epoch, showing that the behavior can be rather different and much more inefficient than the standard asymptotic equilibrium analysis suggests. Of course, this paper focuses on small networks (two parallel edges) and linear cost functions, but we expect these gaps to only be amplified when we consider larger networks and more general (e.g. polynomial) cost functions.

Finally, a continuously shrinking learning step size is rather an artificial model of human learning behavior; a sufficiently small step size means learning almost does not happen. More plausibly, to exclude the situation with unrealistic no-learning behavior, a lower bound on the learning rate ϵ shall be imposed. And it is this exact parameter that we shall adopt as the fixed learning rate of our MWU model. The fact that we are using a fixed (but arbitrarily small) instead of continuously shrinking step size with the limit value of zero, is a feature of our model that does not artificially curtail agent adaptivity merely to enable theoretical results that only become binding after unreasonably long time horizons.

9. CONCLUSION

We explore the Multiplicative Weight Update algorithm in two-strategy non-atomic congestion games. We find that standard game-theoretic equilibrium analysis, such as the Price of Anarchy, fails to capture extremely rich non-equilibrium phenomena of our simple model. Even when the Price of Anarchy is equal to 1, the system can be dynamically unstable when the total demand N increases, with long-time dynamics converging toward attracting orbits. Every system has a carrying capacity, above which the dynamics become non-equilibrating. In fact, they become chaotic when N is sufficiently large, provided the equilibrium flow is asymmetric.

This demand-driven instability is a universal phenomenon that holds for arbitrary linear cost functions, provided the game is non-trivial, i.e., dominant solvable (Pigou network). The dynamics we study here also exhibit a remarkable time-average property; namely, in any non-equilibrating regime driven by large total demand, the time-average flows of the paths converge *exactly* to the Nash equilibrium value, a property of zero-sum games. In potential games, such as congestion games studied here, our analysis shows that the time-average convergence property *does not* guarantee small regret nor low social costs, even when the Price of Anarchy is equal to 1. In fact, time-average regret and time-average social costs increase with fluctuations from the equilibrium values, which can be large in the non-equilibrating regime. In the case of a symmetric equilibrium flow, fluctuations arise from an extreme swing; almost all users will take the same route and simultaneously alternate between the two routes, when the population size is large. The time-average social cost in this situation can thus be as bad as it can get.

Our benign-looking learning in games model is full of surprises and puzzles. The dynamical system approach provides a useful framework to investigate the unusual connections between non-equilibrating dynamics, and the classic game-theoretic (equilibrium) metrics

such as regret, and Price of Anarchy. For instance, we show in the Appendix that, in certain non-equilibrating regime, the system may have multiple distinct attractors and hence the time-average regret and social cost depend critically on initial conditions. Also in the Appendix, we report other interesting observations, discuss future directions and even include extensions to settings with heterogeneous users and games with multiple strategies. Notable properties of periodic orbits, such as the coexistence of two attracting periodic orbits, and the Feigenbaum's period-doubling bifurcation route to chaos are also presented.

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Appendices

APPENDIX A. PROPERTIES OF ATTRACTING ORBITS

In this section, we investigate the properties of the attracting periodic orbits associated with the interval map $f_{a,b} : [0, 1] \rightarrow [0, 1]$

$$(21) \quad f_{a,b}(x) = \frac{x}{x + (1 - x) \exp(a(x - b))}.$$

We've argued in the main text that, when $b = 0.5$, the dynamics will converge toward the fixed point $b = 0.5$ whenever $a < 8$. And for any $a \geq 8$, the long-time dynamics will converge toward the attracting periodic orbits of period 2 located at $\{\sigma_a, 1 - \sigma_a\}$. The bifurcation diagram is thus symmetric around $b = 0.5$ as shown in the top picture of Fig. 5. In this case, the time-average regret is well-approximated by its upper bound, and the normalized time-average social cost asymptotes to the maximum value of 2.

When b differs from 0.5, we have argued in the main text that the emergence of chaos is inevitable, provided a is sufficiently large. The period-doubling bifurcations route to chaos is guaranteed to arise. Fig. 2. of the main text shows chaotic bifurcation diagrams when $b = 0.7$. In this asymmetric case, standard equilibrium analysis only applies when the fixed point b is stable, which is when $|f'_{a,b}(b)| \leq 1$, or equivalently when $a \leq 2/b(1 - b)$.

Feigenbaum's universal route to chaos: The period diagrams as a function of the two free-parameters a and b are shown in Fig. 6. It's interesting to report numerical observations of Feigenbaum's route to chaos for our bimodal map $f_{a,b}$. Although Feigenbaum's universality is known to apply among a one-dimensional *unimodal* interval map with a quadratic maximum [23, 40, 69], we also observe the Feigenbaum's period-doubling route to chaos for our *bimodal* interval map. Specifically, by fixing a and varying b , we numerically measure the ratios

$$(22) \quad \delta_n \equiv \frac{b_{n+1} - b_n}{b_{n+2} - b_{n+1}}, \quad \alpha_n \equiv \frac{d_n}{d_{n+1}},$$

where b_n denotes the value at which a period 2^n -orbits appears and d_n denotes the value of the nearest orbits' element to 0 in the 2^n -orbits. As n grows large (we truncate our observation at $n = 12$), we find

$$(23) \quad \delta_{n=12} \approx 4.669 \dots, \quad \alpha_{n=12} \approx -2.502 \dots,$$

which agree, to 4 digits, with the Feigenbaum's universal constants, $\delta = 4.669201609102990\dots$ and $\alpha = -2.502907875\dots$, that appear, for example, in the period-doubling route to chaos in the logistic map.

Coexistence of two attracting periodic orbits and non-uniqueness of regret and social cost: The map $f_{a,b}$ has a negative Schwartzian derivative when $a > 4$, thus the long-time dynamics can live on one of the two coexisting attracting or neutral periodic orbits. Although the time-average of every periodic orbits converges *exactly* to the Nash equilibrium b , the variance $\lim_{T \rightarrow \infty} \frac{1}{T} \sum_{n=1}^T (x_n - b)^2$ of the coexisting periodic orbits can differ. Thus, the normalized time-average social cost and the time-average regret, which depend on the variance, can be multi-value. Which value is attained depends on the variance of the attracting periodic orbits that the dynamics asymptotically reaches, which depend on the initial conditions x_0 . Period diagrams of Fig. 7 reveal how the two coexisting initial condition-dependent attracting periodic orbits are intertwined, and Fig. 8 reports

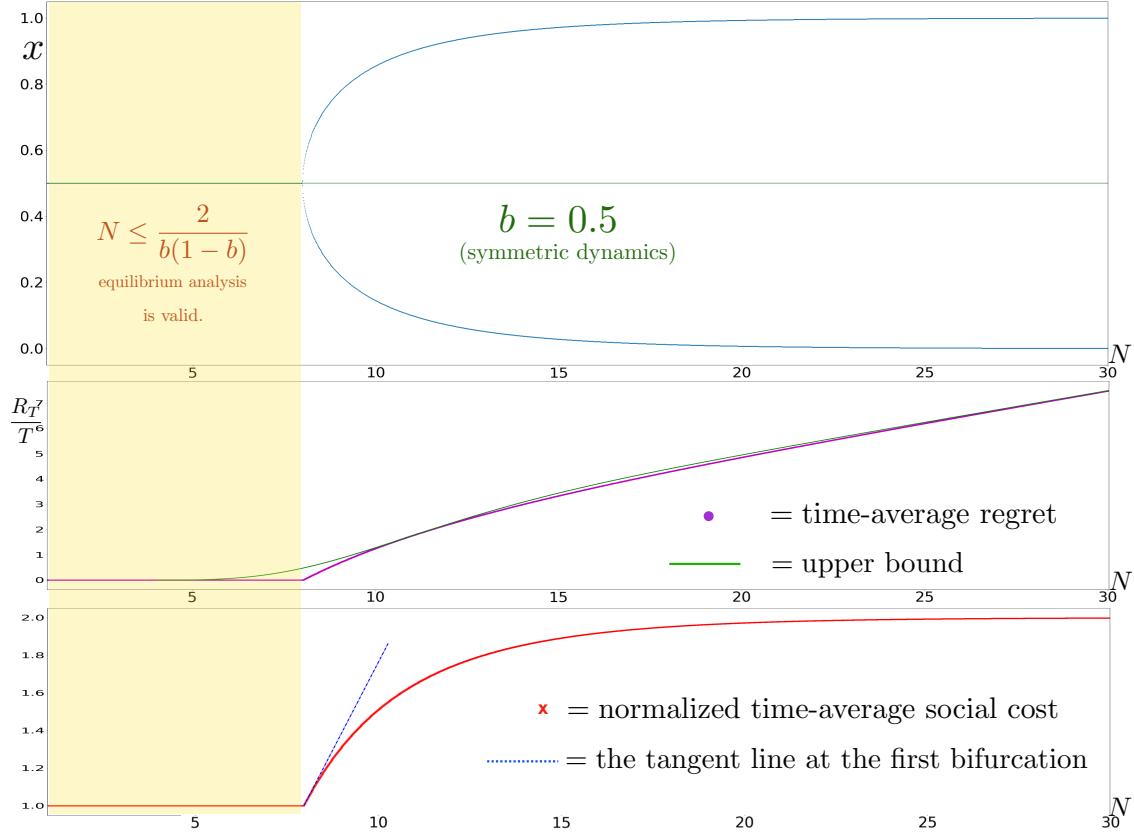


FIGURE 5. Even in the symmetric case of $b = 0.5$, bifurcation diagram (Top) demonstrates instability of the routing game driven by the increase in total demand N . Here, $\epsilon = 1 - 1/e$ so $a = N$ as usual. In this symmetric case, the capacity of the network, under which long-time dynamics equilibrate, is $N_b^* = 8$. Above the capacity, attracting periodic orbits of period 2 emerges. (Middle) The time-average regret, shown in the purple circle symbol, suddenly becomes strictly positive at the bifurcation. The regret bound also well approximates the actual values. (Bottom) normalized time-average social cost also *suddenly* becomes greater than 1 at the bifurcation. Even in the symmetric case, the classic Price of Anarchy metric fails for $N > N_b^* = 8$.

the evidence of two coexisting periodic orbits whose variances differ, leading to multi-valued time-average regret and social cost.

Stability of the orbits:

In addition to the period diagrams, we investigate the stability of the attracting orbits by considering the Lyapunov exponents $\langle \log |f'_{a,b}| \rangle$, where $\langle \cdot \rangle$ denotes time-average. Fig. 9 (bottom) shows the Lyapunov exponents associated with different attracting orbits, revealing that extended-leg structures arise from the situations when the orbits become superstable, that is when one of the two critical points is an element of the orbits⁷. Within the regime of the same period (same color), there are situations when the two superstable extended-leg

⁷Recall that the orbit is superstable if one of the critical points is an element of the orbits, so that $f'_{a,b}(x_c) = 0$. This means the Lyapunov exponents in principle is $-\infty$, visualized as a white bright color.

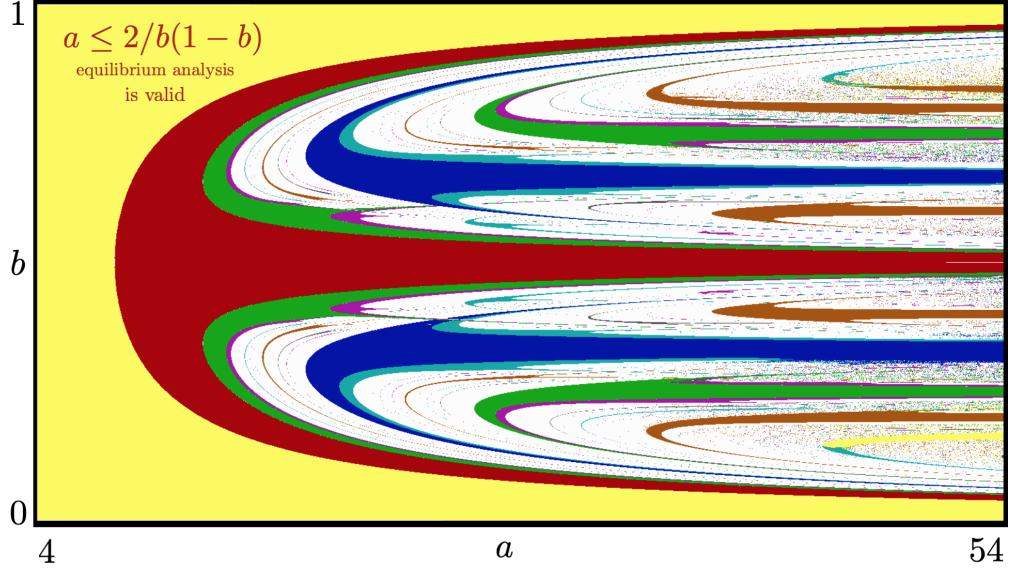


FIGURE 6. Period diagrams of the small-period attracting periodic orbits associated with the map (21). The colors encode the periods of attracting periodic orbits as follows: period 1 (fixed point) = **yellow**, period 2 = **red**, period 3 = **blue**, period 4 = **green**, period 5 = **brown**, period 6 = **cyan**, period 7 = **darkgray**, period 8 = **magenta**, and period larger than 8 = **white**. The equilibrium analysis is only viable when the fixed point b is stable, i.e. when $a \leq 2/b(1 - b)$. In other region of the phase-space, non-equilibrating dynamics arise and system proceeds through the period-doubling bifurcation route to chaos in the white region. The picture is generated from the following algorithm: 20000 preliminary iterations are burned. Then a point is considered periodic of period n if $|f^n(x) - x| < 0.0000000001$ and it is not periodic of any period smaller than n . Slight asymmetry is caused by the fact that the starting point is the left critical point $x_l = 1/2 - \sqrt{1/4 - 1/a}$. In addition, for a fixed a , as we vary b and penetrate into the chaotic regimes (white) from the outer layers, we numerically observe Feigenbaum's universal route to chaos as discussed below.

curves intersect. These scenarios happen when *both* critical points are elements of periodic orbits.

Also, note Fig. 6 reveals that the qualitatively similar extended-leg structures in the period diagrams appear in layers, with a chaotic regime sandwiched between two layers. Notice also that the consecutive layers have periods differ by 1. To understand why these layers with increasing periods appear, we investigate superstable periodic orbits in these layers and found that, all elements of the orbits, except for the left critical points $x_l = 1/2 - \sqrt{1/4 - 1/a}$ and its image $f_{a,b}(x_l)$, are approximately 0, independent of the period of the orbits. With this observation, we now approximate one of the superstable regions within each layer, using the time-average convergence to the Nash equilibrium property of Corollary 3.4. Namely, let x_l be an element of a periodic orbit of period p such that only x_l and $f_{a,b}(x_l)$ are significantly larger than 0, then from Corollary 3.4 we have

$$(24) \quad x_l + f_{a,b}(x_l) + \underbrace{\{f_{a,b}^2(x_l) + \cdots + f_{a,b}^{p-1}(x_l)\}}_{\approx 0} = pb.$$

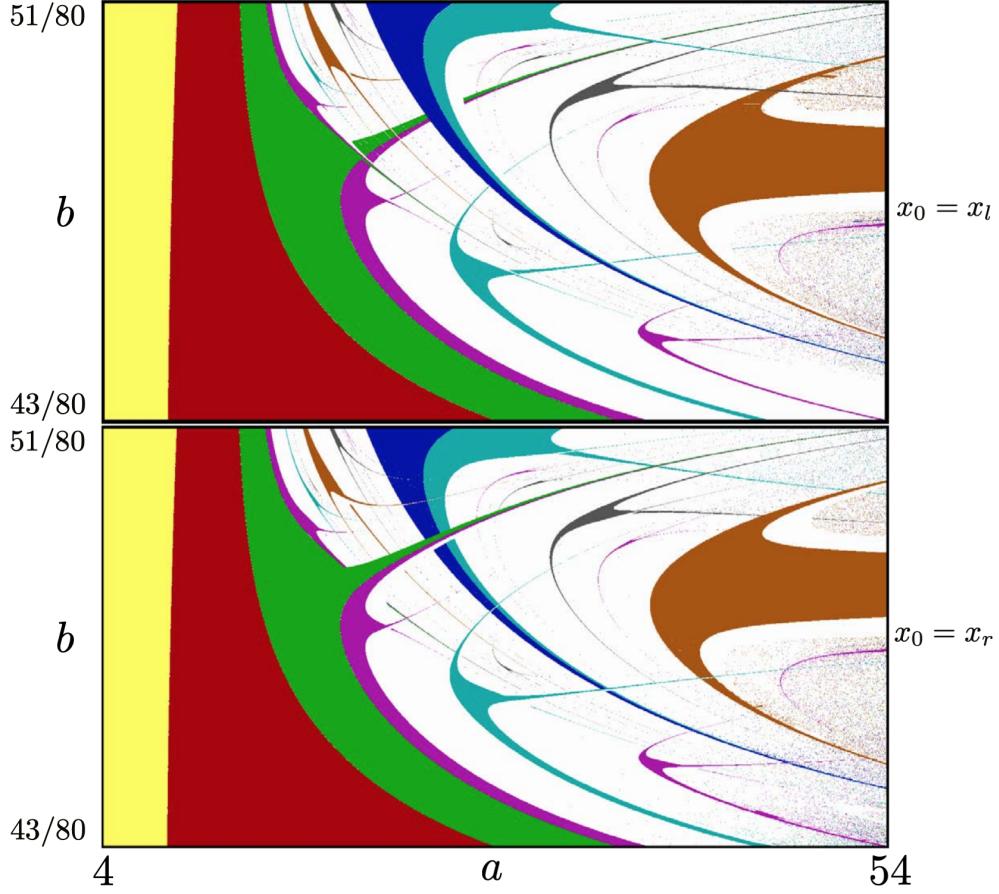


FIGURE 7. Coexistence of two initial condition-dependent attracting periodic orbits. The pictures are generated from the same procedure as explained in Fig.6, except that here the initial conditions for the top and the bottom pictures are located at the left and the right critical points, respectively. Also, $b \in [43/80, 51/80]$ and $a \in [4, 54]$. The color schemes are the same as those of Fig. 6 : period 1 (fixed point) = yellow, period 2 = red, period 3 = blue, period 4 = green, period 5 = brown, period 6 = cyan, period 7 = darkgray, period 8 = magenta, and period larger than 8 = white.

Numerical results show that the approximation that every elements of the periodic orbits except x_l and $f_{a,b}(x_l)$ are close to 0 becomes better and better for periodic orbits with larger periods; hence, we're interested in the limit of $a \gg 1$. To leading order in $\frac{1}{a}$, $x_l \approx \frac{1}{a}$ and $f_{a,b}(x_l) \approx \frac{1}{1+ae^{1-ab}}$ so that (24) gives $\frac{1}{a} + \frac{1}{1+ae^{(1-ab)}} \approx pb$. Defining

$$(25) \quad S(a, b) = \frac{1}{ab} + \frac{1}{b + (ab)e^{(1-ab)}},$$

we obtain the condition

$$(26) \quad S(a, b) \approx p,$$

that should become more accurate as $a \gg 1$, for x_l to be on the periodic orbit of period p with the aforementioned property. Fig. 10 reveals that the level sets of $S(a, b)$ for $p = 2, 3, \dots, 10$ accurately tracks the extended-leg structures with increasing periods, showing that these superstable orbits are the skeletons of the extended-leg layers shown in Fig. 6.

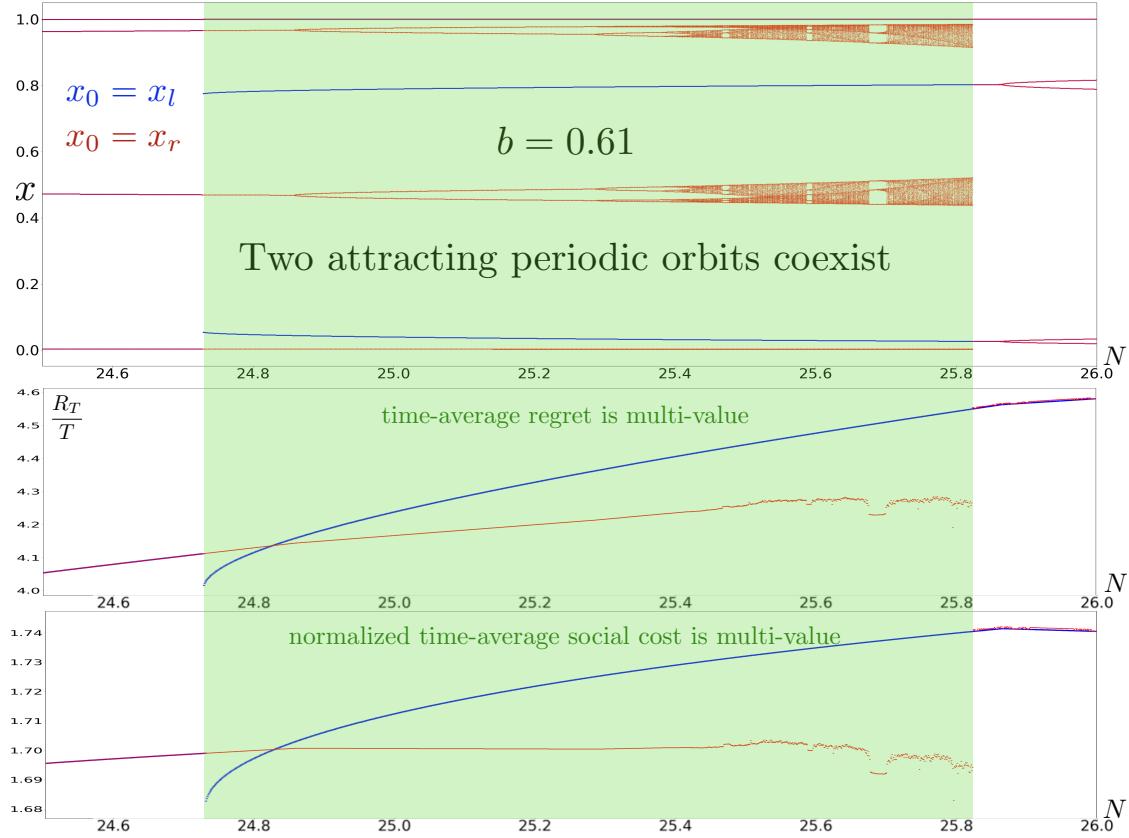


FIGURE 8. Coexistence of two attracting periodic orbits at $b = 0.61$ with two different variances implies *non-uniqueness* of time-average regret and normalized time-average social cost. As usual, we set $\epsilon = 1 - 1/e$ so that $N = a$. (Top) The range of N in the shaded green region show coexistence of two attracting periodic orbits. The blue (red) periodic orbits is selected if the initial condition is the left (right) critical point x_l (x_r). There are at most 2 coexisting attracting periodic orbits, as guaranteed by the negative Schwartzian derivative for our bimodal map $f_{a,b}$. The variance of the two periodic orbits are clearly different; thus, the time-average regret (middle) and the normalized time-average social cost (bottom) which depend on the variance are multi-value. Which values are attained depend on initial conditions.

In addition, we can approximate the condition when *both* critical points x_l and x_r become the elements of these superstable periodic orbits. In these specific permutations of the orbits, we require $x_r = f_{a,b}(x_l)$. And from (24) we obtain $x_l + f_{a,b}(x_l) \approx pb$. Since $x_l + x_r = 1$, we conclude that both critical points will be on the periodic orbit of period p with the aforementioned property when

$$(27) \quad b \approx \frac{1}{p}, \quad \text{and} \quad \frac{1}{a} [2 \ln(a-1) + 1] \approx \frac{1}{p},$$

where the condition on a follows from (26) and $b \approx \frac{1}{p}$. Therefore, if we plot the relationship $b = \frac{1}{a} [2 \ln(a-1) + 1]$, the graph will encompass the situations when both critical points are on the periodic orbits with the aforementioned property. This is illustrated by the dashed

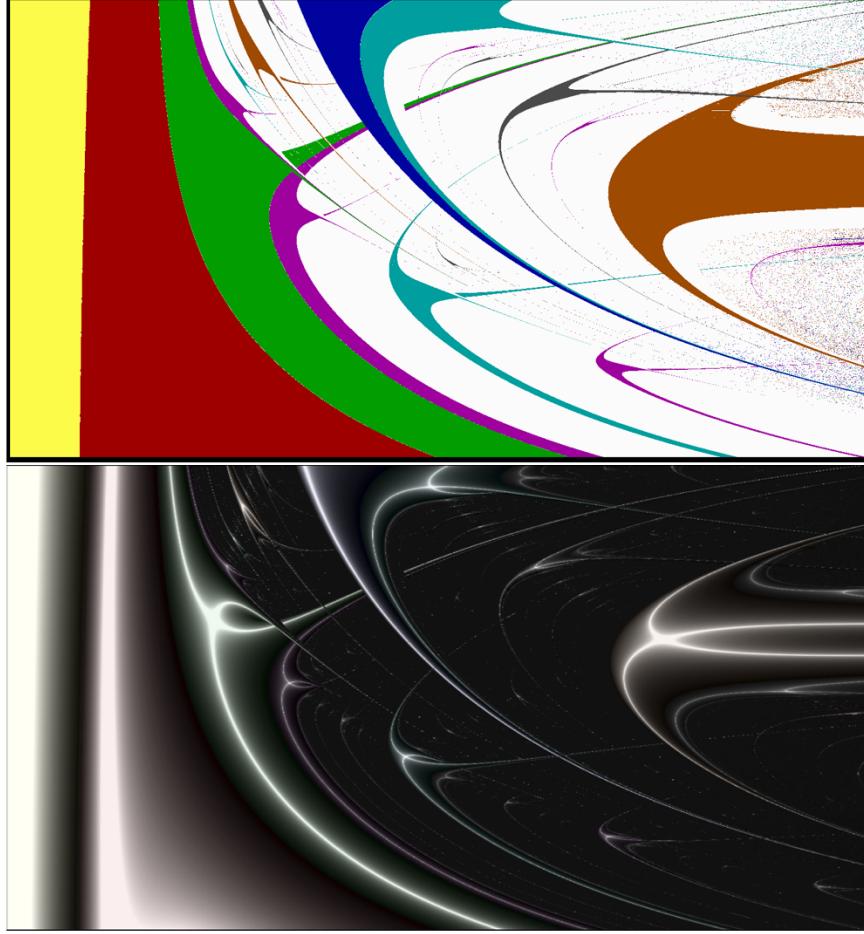


FIGURE 9. (Bottom) Lyapunov exponents $\langle \log |f'_{a,b}| \rangle$ numerically approximated by $\frac{1}{T} \sum_{n=1}^T \log |f'_{a,b}(x_n)|$ with $T = 2000$, shown in gray scale, superposed on the period diagrams (Top) adopted from Fig. 7 (Top). The color scheme of the Lyapunov exponents is such that $\langle \log |f'_{a,b}| \rangle < -1.5$ is shown in white (very stable orbits) and $\langle \log |f'_{a,b}| \rangle > 0$ is shown in black (unstable or chaotic). One can clearly see that the extended-leg structures arise from having superstable orbits as the skeleton of each attracting periodic orbits regime. When both critical points are elements of the attracting orbit, the two extended legs intersect. As expected, close to the bifurcation boundaries and in the chaotic regime, the orbits becomes unstable, as represented by the black color.

olive green line of Fig. 10 that passes through the intersections between two superstable curves within each period- p region.

APPENDIX B. EXTENSIONS TO MORE COMPLEX CONGESTION GAMES

B.1. Games with more than two strategies. We will consider a m -strategy *congestion game* with a continuum of players/agents, where all of them use *multiplicative weights update*. Each of the players controls an infinitesimally small fraction of the flow. We will assume that the total flow of all the agents is equal to N . We will denote the fraction of the players using i one of the m strategies at time n as $x_i(n)$ where $i \in \{1, \dots, m\}$. Intuitively, this

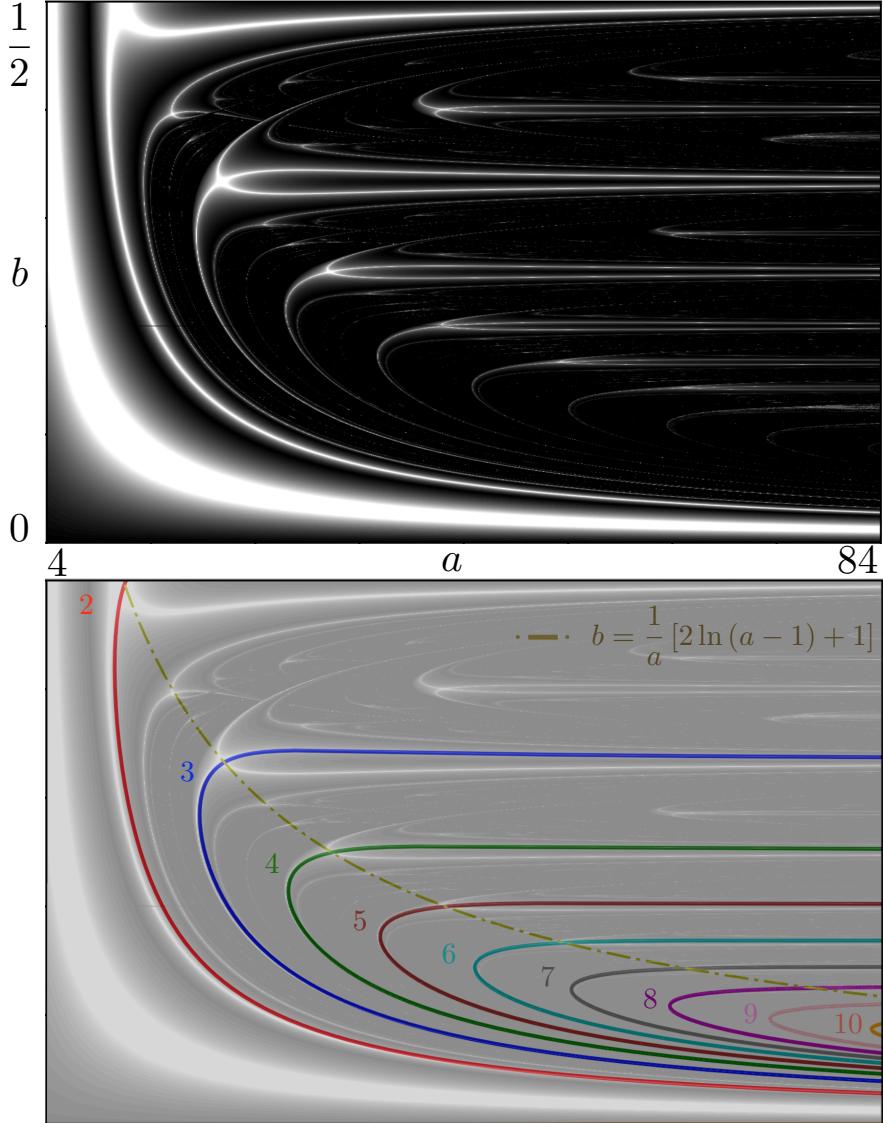


FIGURE 10. Layers of extended-leg structures with increasing periods arise from specific permutations of superstable periodic orbits. As argued in the stability of the orbits section, $S(a, b) = p$ defines the superstable periodic orbits of period p with the property that only x_l and $f(x_l)$ are the only two elements of the periodic orbits that are not near 0. The level sets of $S(a, b)$ at $p = 2, 3, \dots, 10$ are displayed in different colors (bottom), which accurately track extended-leg curves with large negative Lyapunov exponents (top). The color scheme of the Lyapunov exponents is such that $\langle \log |f'_{a,b}| \rangle < -1.5$ is shown in white (very stable orbits) and $\langle \log |f'_{a,b}| \rangle > 0$ is shown in black (unstable or chaotic). The dashed olive green curve $b = \frac{1}{a} [2 \ln(a-1) + 1]$ obtained from (27) encompasses the situations when *both* critical points are the only two non-near-zero elements of the periodic orbits, i.e. when the two superstable curves in each region of the same period p intersect. These results provide a reasonable answer to why layers of extended-leg structures with increasing periods appear in the period diagram of Fig. 6.

model captures how a large population of cars chooses between multiple alternative, parallel paths for going from point A to point B . If a large fraction of the players choose the same strategy, this leads to the congestion, and the cost increases. We will assume that the cost is proportional to the *load*. If we denote by $c(i)$ the cost of any player playing strategy number i , and the coefficients of proportionality are α_i , then we get

$$(28) \quad c_i = \alpha_i N x_i \quad \forall i \in \{1, \dots, m\}.$$

At time $n + 1$ the players know already the cost of the strategies at time n and update their choices. Since we have a continuum of agents we will assume that the fractions of users using the first, second, m -th path are respectively equal to the probabilities $x_1(n), \dots, x_m(n)$. Once again we will update the probabilities using MWU. The update rule in the case of m strategies is as follows:

$$(29) \quad x_i(n+1) = x_i(n) \frac{(1-\epsilon)^{c_i(n)}}{\sum_{j \in \{1, \dots, m\}} x_j(n)(1-\epsilon)^{c_j(n)}},$$

Theorem B.1. *Given any non-atomic congestion game with m actions as described by model (28), (29), there exists a total system demand N_0 such that for if $N \geq N_0$ in each time period n , there exists at least one pair of strategies i, j such that the relative flow in that pair of strategies (i.e. $x_i/(x_i+x_j), x_j/(x_i+x_j)$) follows exactly the MWU dynamics of a two strategy non-atomic congestion game in its non-equilibrating regime.*

Proof. The critical observation is that we can compare the relative probabilities/loads of two strategies/paths. Specifically let $i, j \in \{1, \dots, m\}$. Define variable $y_{i,j} \stackrel{\text{def}}{=} \frac{x_i}{x_i+x_j}$. Then by substituting in equation (29) we have that:

$$\begin{aligned} y_{i,j}(n+1) &= \frac{x_i(n+1)}{x_i(n+1) + x_j(n+1)} \\ &= \frac{x_i(n)(1-\epsilon)^{c_i(n)}}{x_i(n)(1-\epsilon)^{c_i(n)} + x_j(n)(1-\epsilon)^{c_j(n)}} \\ &= \frac{x_i(n)(1-\epsilon)^{\alpha_i N x_i(n)}}{x_i(n)(1-\epsilon)^{\alpha_i N x_i(n)} + x_j(n)(1-\epsilon)^{\alpha_j N x_j(n)}} \\ &= \frac{y_{i,j}(n)(1-\epsilon)^{\alpha_i N x_i(n)}}{y_{i,j}(n)(1-\epsilon)^{\alpha_i N x_i(n)} + (1-y_{i,j}(n))(1-\epsilon)^{\alpha_j N x_j(n)}} \\ &= \frac{y_{i,j}(n)(1-\epsilon)^{\alpha_i N(x_i(n)+x_j(n))y_{i,j}(n)}}{y_{i,j}(n)(1-\epsilon)^{\alpha_i N(x_i(n)+x_j(n))y_{i,j}(n)} + (1-y_{i,j}(n))(1-\epsilon)^{\alpha_j N(x_i(n)+x_j(n))(1-y_{i,j}(n))}} \end{aligned}$$

If we define a time-dependent $\epsilon(n)$ such that $1 - \epsilon(n) = (1 - \epsilon)^{(x_i(n)+x_j(n))}$ then the above equation results in:

$$y_{i,j}(n+1) = \frac{y_{i,j}(n)(1 - \epsilon(n))^{\alpha_i N y_{i,j}(n)}}{y_{i,j}(n)(1 - \epsilon(n))^{\alpha_i N y_{i,j}(n)} + (1 - y_{i,j}(n))(1 - \epsilon(n))^{\alpha_j N(1-y_{i,j}(n))}}$$

That is, the dynamics of the ratio of probabilities when comparing strategies i, j are identical to that of a game with two strategies but where the learning rate ϵ changes over time and where $\epsilon(n) = 1 - (1 - \epsilon)^{x_i(n)+x_j(n)}$. The $y_{i,j}$ dynamics on any pair of edges (i, j)

exactly follow the dynamics of the two strategy game with cost functions $a_i N(n)x, a_j N(n)x$ where the effective demand is $N(n) = N(x_i + x_j)$.

We know that given any pair of edges with constants a_i, a_j there exists a lower bound for $N(a_i, a_j)$ after which the system bifurcates and is no longer convergent (Theorems 3.6, 3.8 and Corollary 3.9). If $N_0 = \binom{m}{2} \max_{i \neq j} N(a_i, a_j)$, this trivially suffices to imply that on each time period n , there exists at least one pair of strategies i, j (e.g. the most heavily congested pair) such that the relative flow in that pair exceeds its effective capacity $N(a_i, a_j)$, and is thus in its non-equilibrating regime. \square

B.2. Games with heterogeneous users. This is the model for the case of heterogeneous population. We will start with the simplest possible case where there are only two subpopulations. We will consider a two-strategy *congestion game* with two continuums of players/agents, where all of them use *multiplicative weights update*. Each of the players controls an infinitesimal small fraction of the flow. Out of the total flow/demand N of the first population has size $N\eta_1$ whereas the total flow of the second population is $N\eta_2$. A canonical example would be $\eta_1 = \eta_2 = 0.5$.

We will denote the fraction of the players of the first (resp. second) population using the first strategy at time n as x_n (resp. y_n). The second strategy is chosen by $1 - x_n$ (resp. $1 - y_n$) fraction of the players. Intuitively, this model captures how two large population of cars (e.g. taxis versus normal cars) chooses between two alternative, parallel paths for going from point A to point B . If a large fraction of the players choose the same strategy, this leads to the congestion, and the cost increases. We will assume that the cost is proportional to the *load*. If we denote by $c(j)$ the cost of the player playing the strategy number j , and the coefficients of proportionality are α, β , then we get

$$(30) \quad c(1) = \alpha N(\eta_1 x + \eta_2 y), \quad c(2) = \beta N(\eta_1 + \eta_2 - \eta_1 x - \eta_2 y)$$

For *multiplicative weights update* (MWU), for the first (resp. second), there is a parameter $\epsilon_1 \in (0, 1)$, (resp. $\epsilon_2 \in (0, 1)$) which can be treated as the common learning rate of all players of that population. Thus, we get

$$(31) \quad \begin{aligned} x_{n+1} &= \frac{x_n(1 - \epsilon_1)^{c(1)}}{x_n(1 - \epsilon_1)^{c(1)} + (1 - x_n)(1 - \epsilon_1)^{c(2)}}, \\ y_{n+1} &= \frac{y_n(1 - \epsilon_2)^{c(1)}}{y_n(1 - \epsilon_2)^{c(1)} + (1 - y_n)(1 - \epsilon_2)^{c(2)}}. \end{aligned}$$

By combining equations (30) and (31),

$$(32) \quad \begin{aligned} x_{n+1} &= \frac{x_n}{x_n + (1 - x_n)(1 - \epsilon_1)^{N(\beta(\eta_1 + \eta_2) - (\alpha + \beta)(\eta_1 x_n + \eta_2 y_n))}}, \\ y_{n+1} &= \frac{y_n}{y_n + (1 - y_n)(1 - \epsilon_2)^{N(\beta(\eta_1 + \eta_2) - (\alpha + \beta)(\eta_1 x_n + \eta_2 y_n))}}. \end{aligned}$$

After a similar change of variables as in the homogeneous case formula (32) becomes

$$(33) \quad \begin{aligned} x_{n+1} &= \frac{x_n}{x_n + (1 - x_n) \exp(a_1(\eta_1 x_n + \eta_2 y_n - b))}, \\ y_{n+1} &= \frac{y_n}{y_n + (1 - y_n) \exp(a_2(\eta_1 x_n + \eta_2 y_n - b))}. \end{aligned}$$

In the simplest case of the equal shares/mixtures (i.e. $\eta_1 = \eta_2 = 0.5$) we have:

$$(34) \quad \begin{aligned} x_{n+1} &= \frac{x_n}{x_n + (1 - x_n) \exp(a_1(0.5(x_n + y_n) - b))}, \\ y_{n+1} &= \frac{y_n}{y_n + (1 - y_n) \exp(a_2(0.5(x_n + y_n) - b))}. \end{aligned}$$

Dimensional reduction: Although the heterogeneous model contains more independent variables than the homogeneous case, the dynamics are constrained in a lower-dimensional manifold. That is, we will show that the function $I(x, y) = \frac{(1-x)^{a_2} y^{a_1}}{(1-y)^{a_1} x^{a_2}}$ is an invariant function for population mixtures. This means that the curves $I(x, y) = c$ are invariant for any time step n , where c parametrizes the family of invariant curves.

Lemma B.2. *The function $I(x, y) = \frac{(1-x)^{a_2} y^{a_1}}{(1-y)^{a_1} x^{a_2}}$ is an invariant function (first integral) of the dynamics.*

Proof. It is easy to check that the set of equations (33) is equivalent to

$$(35) \quad \begin{aligned} \frac{x_{n+1}}{1 - x_{n+1}} &= \frac{x_n}{(1 - x_n) \exp(a_1(\eta_1 x_n + \eta_2 y_n - b))}, \\ \frac{y_{n+1}}{1 - y_{n+1}} &= \frac{y_n}{(1 - y_n) \exp(a_2(\eta_1 x_n + \eta_2 y_n - b))}. \end{aligned}$$

By raising the first equation to power a_2 and the second equation to power a_1 and dividing them we derive that:

$$\frac{x_{n+1}^{a_2} (1 - y_{n+1})^{a_1}}{(1 - x_{n+1})^{a_2} y_{n+1}^{a_1}} = \frac{x_n^{a_2} (1 - y_n)^{a_1}}{(1 - x_n)^{a_2} y_n^{a_1}}$$

That is the function $I(x, y) = \frac{(1-x)^{a_2} y^{a_1}}{(1-y)^{a_1} x^{a_2}}$ is an invariant function (first integral) of the dynamics. \square

Time-average convergence of the mixture to Nash equilibrium b . for the considered model we can show result similar to Theorem 3.3 for the homogeneous population, that is that b is Cesàro attracting mixture of trajectories.

Theorem B.3. *For every $a_1, a_2 > 0$, $b \in (0, 1)$ and $(x_0, y_0) \in (0, 1)^2$ we have*

$$(36) \quad \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{n=0}^{T-1} (\eta_1 x_n + \eta_2 y_n) = b.$$

Proof. Let $f(x_n, y_n) = (x_{n+1}, y_{n+1})$ be defined by (33) where $\eta_1, \eta_2 \in (0, 1)$ and $\eta_1 + \eta_2 = 1$.

The map $t \mapsto \frac{t}{1-t}$ is a homeomorphism of $(0, 1)$ onto $(0, \infty)$, and its inverse is given by $t \mapsto \frac{t}{1+t}$. Thus, we can introduce new variables, $z = \frac{x}{1-x}$ and $w = \frac{y}{1-y}$. In these variables our map will be $g : (0, \infty)^2 \rightarrow (0, \infty)^2$, and if $g(z_n, w_n) = (z_{n+1}, w_{n+1})$, then

$$(37) \quad \begin{aligned} z_{n+1} &= z_n \exp\left(-a_1\left(\eta_1 \frac{z_n}{1+z_n} + \eta_2 \frac{w_n}{1+w_n} - b\right)\right), \\ w_{n+1} &= w_n \exp\left(-a_2\left(\eta_1 \frac{z_n}{1+z_n} + \eta_2 \frac{w_n}{1+w_n} - b\right)\right). \end{aligned}$$

If $w_n = cz_n^{a_2/a_1}$ then $w_{n+1} = cz_{n+1}^{a_2/a_1}$. This shows that if z_n is close to 0 then also w_n is close to 0, and by (37) we get $z_{n+1} > z_n$. Similarly, if z_n is close to infinity, then also w_n is close to infinity, and by (37) we get $z_{n+1} < z_n$. Together with another inequality obtained from (37),

$$z_n \exp(-a_1(1-b)) < z_{n+1} < z_n \exp(a_1 b),$$

this proves that if $z_0, w_0 \in (0, \infty)$ then $\inf_{n \geq 0} z_n > 0$ and $\sup_{n \geq 0} z_n < \infty$.

The first equation of (37) can be rewritten as

$$z_{n+1} = z_n \exp(a_1(\eta_1 x_n + \eta_2 y_n - b)),$$

so by induction we get

$$z_T = z_0 \exp \left(a_1 \left(\sum_{n=0}^{T-1} (\eta_1 x_n + \eta_2 y_n) - Tb \right) \right).$$

Therefore there exists a real constant M (depending on the parameters and the initial point (x_0, y_0)), such that

$$\left| \sum_{n=0}^{T-1} (\eta_1 x_n + \eta_2 y_n) - Tb \right| \leq M$$

for every T . Dividing by T and passing to the limit, we get

$$(38) \quad \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{n=0}^{T-1} (\eta_1 x_n + \eta_2 y_n) = b.$$

□

We end here with numerical results to demonstrate that, perhaps not surprisingly, this class of games not only possesses complex non-equilibrium behavior, but also allows for an immediate generalization to a more realistic, larger dimensional system, in which new and even more complex non-equilibrium phenomena can arise. Developing a more complete theoretical understanding of these issues, will likely require the introduction of new tools and techniques.

Figures (11) and (12) show attracting orbits generated from the map (34) (with $\eta_1 = \eta_2 = 0.5$) for fixed values of a_1, a_2, b . There, 5000 random starting points are initialized. To approximate where the attractors lie, the first 1000 iterates were made without plotting; the next 200 were visualized.

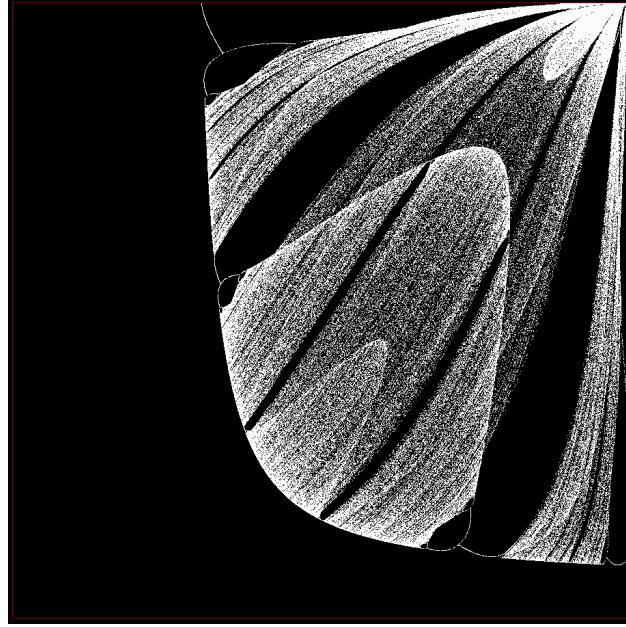


FIGURE 11. Attractor for the map (34) in the two-subpopulation model with $a_1 = 20, a_2 = 30, b = 0.8$. The white dots are the coordinates (x, y) generated from initializing 5000 (x_0, y_0) 's at random from the unit square domain, iterating them with (34) 1000 times, then visualizing the next 200 iterates.

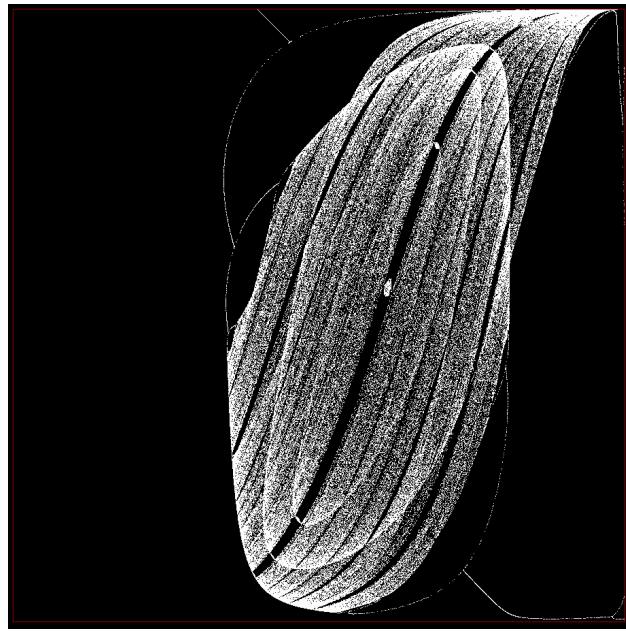


FIGURE 12. Attractor for the map (34) in the two-subpopulation model with $a_1 = 10, a_2 = 30, b = 0.7$. The white dots are the coordinates (x, y) generated from initializing 5000 (x_0, y_0) 's at random from the unit square domain, iterating them with (34) 1000 times, then visualizing the next 200 iterates.

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