

# Near-Optimality in Covering Games by Exposing Global Information

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Mechanism design for distributed systems is fundamentally concerned with aligning individual incentives with social welfare to avoid socially inefficient outcomes that can arise from agents acting autonomously. One simple and natural approach is to centrally broadcast non-binding advice intended to guide the system to a socially near-optimal state while still harnessing the incentives of individual agents. The analytical challenge is proving fast convergence to near optimal states, and in this paper we give the first results that carefully constructed advice vectors yield stronger guarantees.

We apply this approach to a broad family of potential games modeling vertex cover and set cover optimization problems in a distributed setting. This class of problems is interesting because finding exact solutions to their optimization problems is NP-hard yet highly inefficient equilibria exist, so a solution in which agents simply locally optimize is not satisfactory. We show that with an arbitrary advice vector, a set cover game quickly converges to an equilibrium with cost of the same order as the square of the social cost of the advice vector. More interestingly, we show how to efficiently construct an advice vector with a particular structure with cost  $O(\log n)$  times the optimal social cost, and we prove that the system quickly converges to an equilibrium with social cost of this same order.

Categories and Subject Descriptors: F.2 and C.4 [Analysis of Algorithms and Social and Behavior Sciences]

General Terms: Algorithms, Economics, Theory

Additional Key Words and Phrases: Mechanism Design, Algorithmic Game Theory, Price of Anarchy

## 1. INTRODUCTION

It is well understood that socially inefficient outcomes can arise as stable outcomes in decentralized systems from agents pursuing their local goals. The main challenge of mechanism design is to align individual incentives with social welfare so as to avoid such inefficient outcomes. One simple and non-invasive approach is to centrally broadcast non-binding advice intended to guide the system to near-optimal states while keeping the original incentive structure intact.

This paper focuses on a natural set cover game. As a concrete example, say a state's legislature wants to establish a number of subsidized health clinics. Residents in a county that houses such a clinic will enjoy its benefits, but they will also incur additional local taxes to pay for the clinic. Residents in a county without a clinic do not incur additional taxes, but they only receive the benefits of a clinic if there is one in a neighboring county. We would like a particular set of counties to open clinics so as to optimize the aggregate cost-benefit calculation for the state. However, since clinics are locally subsidized, counties individually decide whether to open a clinic, so we cannot centrally dictate a near-optimal distribution of clinics. This paper shows how

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© YYYY ACM 1539-9087/YYYY/01-ARTA \$10.00

DOI 10.1145/0000000.0000000 <http://doi.acm.org/10.1145/0000000.0000000>

to advertise an overall strategy (determined centrally and using global knowledge) so that even if self-interested counties are influenced by their advertised strategies with a small probability and for a short time period, they will eventually reach a socially near-optimal solution in a distributed way.

Set covering problems are important and interesting from a classical optimization point of view, but also as a game-theoretic framework both for analyzing social problems like the one described above, where agents behave autonomously in some natural self-interested way, and for engineering distributed systems in which locally-aware agents can be programmed to behave in this way. In this paper, we model covering problems as games, and we use models from learning theory to describe local decision making by agents in these games. We are interested in demonstrating convergence not to arbitrary local equilibria but to states whose cost is competitive with the global optimum. We accomplish this by incorporating a globally-informed central authority into natural behavior dynamics.

### 1.1. Game Setup, Equilibrium Quality and Dynamic Models

Given a universe of elements with associated costs and a collection of sets of these elements, the minimum weighted set cover optimization problem is to choose the lowest cost subset of elements such that each set is represented by at least one chosen element. While this problem is NP-hard, good approximation algorithms exist; however, such algorithms do not take into account individual incentives.

We analyze a setting in which a central authority knows a subset of elements that approximates an optimal solution to the set cover problem, but elements are modeled as only locally aware agents with cost functions representing a natural distributed game interpretation of the core optimization problem. We generalize the problem by not requiring total coverage, rather the importance of covering a given set is determined by its set weight. Each element  $i$  that chooses to be *on* incurs his own cost  $c_i$ , and each element  $i$  that is *off* pays the sum of the weights of sets he participates in that do not contain any other *on* element. If the element costs are all smaller than the set weights, then the cost-minimizing set of *on* elements is also the optimal set cover. If additionally each set is of size two, then this is the special case of a minimum weighted vertex cover problem.

The healthcare example above illustrates a social network in which agents have inherent costs associated with being *on* (pay for a local clinic) or *off* (hope your neighbor pays for a clinic), which are correlated with the social objective. Another motivation consists of engineering networks in which non-willful distributed agents are programmed to make decisions based on their surroundings. The extensive literature on cooperative control has shown that in this setting many optimization problems can be conveniently solved in a distributed fashion by endowing agents with artificial individual objective functions and cost-minimizing behavior [Shamma(ed) 2008]. As a concrete example, our set cover games include non-cooperative power management models in wireless sensor networks [Campos-Naóez et al. 2008]. The elements are autonomous sensors, and a geographic region is a set consisting of elements corresponding to sensors that could cover that region. A sensor that is *on* is charged some fixed cost, whereas a sensor that is *off* is charged a cost proportional to the number or importance of its adjacent regions that are uncovered by any other sensor.

Much of the work on cooperative control and dynamics-based algorithmic game theory only guarantees that systems converge to an arbitrary equilibrium. Many games, however, have a high Price of Anarchy (PoA), where PoA means the worst case ratio between the social cost in an equilibrium and that of the global optimal configuration. The following special case illustrates that PoA is  $\Omega(n)$  in our set cover game. Suppose  $n$  agents are charged some amount  $c < 1$  when they are *on* and otherwise penalized 1 for

every incident uncovered set. Then a star graph in which vertices are agents and edges are sets has a global optimum with only the center *on*, yielding social cost  $c$ , compared to a low quality Nash Equilibrium in which only the center is *off*, yielding social cost  $c(n - 1)$ .

To circumvent such a high PoA, behavioral models incorporating advertising effects have been recently proposed [Balcan et al. 2009; Balcan et al. 2013]. The models share the common feature that a central authority has knowledge of some joint strategy profile with low social cost, and this authority broadcasts this strategy in the hope that agents will adopt their prescribed strategies. Specifically, the public service advertising model (PSA) of Balcan et al. [2009] assumes that each agent independently has an  $\alpha$  probability of receiving and temporarily adopting the advertising strategy. Those that do not receive and adopt their prescribed strategy behave in a myopic best-response manner. This model is well-suited for an engineering systems setting, where we do not expect all components to receive the central authority's signal. The learning models of Balcan et al. [2013] assume that each agent uses any of a broad class of learning algorithms to continually choose between acting according to their local best-response move and their broadcasted signal. In the learn-then-decide (LTD) model, agents eventually commit to one of these options. In contrast to PSA, LTD is better motivated by a social setting where agents that are only locally aware are interested in exploring the advertising strategy in the hope that it will benefit them personally. In this work, we study both PSA and LTD models.

## 1.2. Our Contribution

The PSA and LTD models share three features that jointly help us give positive results for covering games: advertising seeds the system with a preference for globally efficient behavior, best-response dynamics harnesses the fact that individual and social welfare is aligned and permits potential arguments, and the randomness of which agents receive signals and update orders allows for expected or high probability cost arguments when straightforward structural arguments are not possible. We show indeed that both the PSA and LTD models keep systems out of pathological cover game equilibria. Furthermore, we give the first theoretical results for PSA and LTD models that leverage particular structural aspects of the advice vector  $s^{ad}$ . Specifically, assuming that certain hypergraph parameters defined in Section 2.2 are of constant size we show the following:

- R1:** For the vertex cover games<sup>1</sup> and arbitrary advice  $s^{ad}$ , agents converge to a state of expected cost  $O(\text{cost}(s^{ad}))$  in the PSA and LTD models.
- R2:** For set cover games and arbitrary advice  $s^{ad}$ , agents converge to a state of expected cost  $O(\text{cost}(s^{ad})^2)$  in the PSA and LTD models. We show how to find an advertising strategy whose cost is within a constant factor of the optimum by applying standard LP rounding techniques.
- R3:** For set cover games and some carefully chosen advice  $s^{ad}$ , agents converge to a state of cost  $O(\text{cost}(s^{ad}))$  with high probability in the PSA model. Furthermore, we show how to construct such an advice vector  $s^{ad}$  with cost within an  $O(\log n)$  factor of the optimum in polynomial time.

We emphasize that for each of the above settings, convergence to the low-cost outcome occurs in time polynomial in the number of agents. The core of the analysis underlying each of these results is that additional costs beyond that of the advertised strategy's

<sup>1</sup>As mentioned earlier, a set cover game where each set has size 2 is called a *vertex cover game*, and in such games equilibria have natural connections to vertex covers in the graph induced by the sets (i.e. edges).

can always be attributed to selfish agents that are best-responding, which bounds the inefficiencies in the system relative to the cost of the advice vector.

### 1.3. Related Work

Achieving global coordination in distributed multi-agent systems is a central problem of control theory with multiple real-world applications ([Shamma(ed) 2008] and references therein). More specifically, several papers consider game-theoretic formulations of covering problems which are inspired by practical sensor network problems [Schmid and Wattenhofer 2006; Machado and Tekinay 2007; Sadagopan et al. 2006; Campos-Naóez et al. 2008]. In particular, Campos-Naóez et al. [2008] analyze a distributed model for achieving coverage in sensor networks (in an energy-efficient manner) that can be directly captured by our game. However, Campos-Naóez et al. [2008] and many other control theory papers guarantee only convergence to stable states which are locally optimal. Since these games often have a high Price of Anarchy [Koutsoupias and Papadimitriou 1999; Nisan et al. 2007], the results do not translate to global performance guarantees.

Covering problems have also been studied extensively from the perspective of implementation theory as well as distributed mechanism design. Similar to our approach, preexisting set cover games in Buchbinder et al. [2010] and Escoffier et al. [2010] have also been designed as congestion games [Rosenthal 1973]. However, in these models the agents are the hyperedges that choose a vertex to cover them, and the cost of the vertex is divided among them according to some rule. Buchbinder et al. [2010] investigates the influence of a central authority that can influence choices by taxes and subsidies in a best-response dynamics. On the other hand, Escoffier et al. [2010] focuses on the effect of different cost sharing rules of the vertices. In contrast, our work does not require either the use of taxation nor cost sharing to achieve good performance guarantees.

Cardinal and Hoefer [2010] define a general class of covering games, including the vertex cover game [Cardinal and Hoefer 2006], as well as the selfish network design game by Anshelevich et al. [2008]. These game are based on covering problems defined via a linear integer program. Variables represent resources, and the agents correspond to certain sets of constraints they have to satisfy. Our games can be thought abstractly to belong to the dual space of these games. Further generalizations of such models were studied by Hoefer [2011], and by Harks and Peis [2012], investigating settings where the price of each resource may depend on the number of agents using it.

In terms of circumventing bad price of anarchy results a number of different approaches have been explored. Sharma and Williamson. [2007] assume that the authorities enjoy complete control over some fraction of the agents. Similarly, Kempe et al. [2003] and Kempe et al. [2005] focus on the problem of identifying and controlling the influential nodes of a network. In contrast, we incorporate strategic behavior for all agents. Another line of research offers stronger performance guarantees using specific learning algorithms that employ equilibrium selection (e.g. in distributed load balancing [Kleinberg et al. 2011a], in general congestion games [Kleinberg et al. 2009], in evolutionary linguistic systems [Fox et al. 2012]). The importance of such results that go beyond the analysis of performance of Nash equilibria has also been stressed in recent work [Kleinberg et al. 2011b; Piliouras and Shamma 2014; Immorlica et al. 2010; Nadav and Piliouras 2010] where, *e.g.*, it has been shown that even in simple and well studied games, the performance of natural learning dynamics can be arbitrarily better than the social welfare of even the best Nash equilibria. A mini review of this literature can be found in Ligett and Piliouras [2011]. Unfortunately, these techniques do not yield guarantees of fast convergence to good states in our class of games. Recently, other advertising behavioral models less sophisticated than the PSA and LTD models

have been used to circumvent bad price of anarchy results in network creation games [Demaine and Zadimoghaddam 2010; Fabrikant et al. 2003], but these results are not applicable to the cover game that is the focus of this paper.

By using the PSA and LTD models of Balcan et al. [2009] and Balcan et al. [2013], we do not have to make the hard choice between enforcing top-down solutions (which may be infeasible in both engineering systems and social settings) and poor performance guarantees. We build on the works of Balcan et al. [2009] and Balcan et al. [2013], which provide quality guarantees for particular games including fair cost-sharing and party affiliation games, and show that for a broad class of covering and packing problems, incorporating mild influence from a weak central authority guides the system into a near-optimal state when agents are only optimizing locally.

The results we have covered so far focus on noncooperative approaches to covering games. Within mechanism design, different lines of work focus on cooperation and specifically cost sharing mechanisms, e.g. [Deng et al. 1999; Devanur et al. 2005; Immorlica et al. 2008; Fang and Kong 2007; Li et al. 2005; Li et al. 2010].

*Subsequent work.* In a recent followup to our conference paper [Balcan et al. 2012], Piliouras et al. [2012] introduce a novel class of covering games whose Price of Anarchy matches the integrality gap of the corresponding centralized optimization problems for linear and also for submodular costs. For example, the resulting vertex cover game has a Price of Anarchy of two. In the case of linear costs there exist sequences of best-response moves that converge in linear time. However, not all best-response sequences converge and therefore some coordination is needed in terms of determining the order with which agents get to move. Furthermore, the structure of these games is significantly more complicated than ours since they exhibit a continuum of strategies. The utility functions are also intricate and encode financial transactions amongst the vertices of the graph. In contrast, our games are as simple as possible. Each agent has only two strategies: *on* and *off*. Our approach is better suited to capture the realities of many network applications and is easier to implement in practice. In another follow-up work, instead of set cover, Jin et al. [2013] consider an Ising game for modeling the diffusion of social opinions. The authors show that, as in our scenario, advertising global information results in quick convergence a low-cost Nash equilibrium, although global information is dispensed in a manner different from our setting.

## 2. PRELIMINARIES

### 2.1. Background on General Games

We represent a general game as a triple  $\mathcal{G} = \langle N, (S_i), (\text{cost}_i) \rangle$ , where  $N$  is a set of  $n$  agents,  $S_i$  is the finite action space of agent  $i \in N$ , and  $\text{cost}_i$  denotes the cost function of agent  $i$ . The joint action space of the agents is  $S = S_1 \times \dots \times S_n$ . For a joint action  $s \in S$ , we denote by  $s_{-i}$  the actions of all agents  $j \neq i$ . Agents' cost functions map joint actions to non-negative real numbers, i.e.  $\text{cost}_i : S \rightarrow \mathbb{R}^+$  for all  $i \in N$ . In this paper, we define a social cost function,  $\text{cost} : S \rightarrow \mathbb{R}$ , simply as the summation of individual agents' costs. The optimal social cost is denoted by

$$OPT = \min_{s \in S} \text{cost}(s).$$

Given a joint action  $s$ , the *best-response set* of agent  $i$  is the set of actions that minimizes agent  $i$ 's cost subject to the other agents' fixed actions  $s_{-i}$ , i.e.

$$BR_i(s_{-i}) = \arg \min_{a \in S_i} \text{cost}_i(a, s_{-i}).$$

*Best-response dynamics* is a process in which at each time step, an arbitrary agent not already playing a best-response move updates his action to one in his current best-

response set. A joint action  $s \in S$  is a *pure Nash equilibrium* if no agent  $i \in N$  can benefit from deviating to another action, namely,  $s_i \in BR_i(s_{-i})$  for every  $i \in N$ .

A game  $\mathcal{G}$  is called an *exact potential game* [Monderer and Shapley 1996] if there exists a potential function  $\Phi : S \rightarrow \mathbb{R}$  such that

$$\text{cost}_i(a', s_{-i}) - \text{cost}_i(a, s_{-i}) = \Phi(a', s_{-i}) - \Phi(a, s_{-i}),$$

for all  $i \in N$ ,  $s_{-i} \in S_{-i}$ , and  $a, a' \in S_i$ . For general potential games, only the signs of both sides of these equations must be equal. While general games are not guaranteed to have a pure Nash equilibrium, all finite potential games do, and furthermore best-response dynamics in such games always converges to a pure Nash equilibrium [Monderer and Shapley 1996; Nisan et al. 2007]. However, the convergence time can be exponentially large in terms of the number of agents in general.

Two well known concepts for quantifying the inefficiency of equilibria relative to non-equilibria are *Price of Anarchy* and *Price of Stability*. If  $OPT$  corresponds to the cost of the socially optimal state and  $\mathcal{N}(\mathcal{G})$  to the set of pure Nash equilibria of game  $\mathcal{G}$ , Price of Anarchy (PoA) and Price of Stability (PoS) are defined as

$$\text{PoA} = \max_{s \in \mathcal{N}(\mathcal{G})} \frac{\text{cost}(s)}{OPT} \quad \text{PoS} = \min_{s \in \mathcal{N}(\mathcal{G})} \frac{\text{cost}(s)}{OPT}.$$

## 2.2. Covering Game

A cover game  $\mathcal{G} = \langle [n], (S_i), (\text{cost}_i) \rangle$  is specified by a collection of sets  $\mathcal{F} \subseteq 2^{[n]}$ , costs  $c_i$  for  $i \in [n]$ , and weights  $w_\sigma$  for  $\sigma \in \mathcal{F}$ . Each agent has action space  $S_i = \{\text{on}, \text{off}\}$ . For a joint strategy  $s \in S$ , we use  $L(s)$  to denote the agents that are *on* in  $s$  and  $R(s)$  to denote the agents that are *off*, dropping the  $s$  when clear from context<sup>2</sup>. Those sets with elements all in  $R$  are referred to as *uncovered* and denoted by  $\mathcal{F}_R$ . An agent pays either his cost of being *on* or the weights of all uncovered sets he participates in:

$$\text{cost}_i(s) = \begin{cases} c_i & \text{if } s_i = \text{on} \\ \sum_{\sigma \in \mathcal{F}_R : i \in \sigma} w_\sigma & \text{if } s_i = \text{off}. \end{cases} \quad (1)$$

For  $\sigma \subseteq [n]$ ,  $\mathcal{F}' \subseteq \mathcal{F}$ , we define for shorthand  $c(\sigma) := \sum_{i \in \sigma} c_i$  and  $w(\mathcal{F}') := \sum_{\sigma \in \mathcal{F}'} w_\sigma$ . Then the social cost has the following simple form:

$$\text{cost}(s) = \sum_{i \in [n]} \text{cost}_i(s) = c(L) + \sum_{\sigma \in \mathcal{F}_R} |\sigma| \cdot w_\sigma. \quad (2)$$

*Parametrizing our results.* Denote  $c_{\max} := \max_{i \in [n]} c_i$ ,  $c_{\min} := \min_{i \in [n]} c_i$ ,  $w_{\max} := \max_{\sigma \in \mathcal{F}} w_\sigma$ , and  $w_{\min} := \min_{\sigma \in \mathcal{F}} w_\sigma$ . For expository simplicity, we consider costs and weights which are bounded above and below by constants, i.e.,

$$c_{\max}, c_{\min}, w_{\max}, w_{\min} = \Theta(1),$$

although we can push these quantities through the analysis to give results for general costs and weights, as shown in (6). We also define

$$F_{\max} := \max_{\sigma \in \mathcal{F}} |\sigma|.$$

Note that when  $F_{\max} = 2$ , the game can be specified by a simple graph with vertex costs and edge weights, where an *on* vertex covers its incident edges. Our results

<sup>2</sup>A helpful mental image/mnemonic rule is that we partition the vertices into two sets, placing the *on* vertices on the Left of an imaginary axis, whereas we put the *off* ones to the Right.

when  $F_{\max} = 2$  are stronger than in the general case (see Theorems 3.1 and 4.1). For  $F_{\max} > 2$ , a given pair of elements may appear in multiple sets. Our results depend on the maximum number of sets containing any given pair of elements, so we define the following parameter capturing this quantity:

$$\Delta := \max |\{\sigma \in \mathcal{F} : i, j \in \sigma, i \neq j\}|.$$

In this paper, we primarily focus on the case when  $F_{\max} = O(1)$ . We note that this holds in many practical applications of interest. In current wireless sensor technology, for example, the maximum sensing range is around a hundred meters [Zennaro et al. 2010], while the size of sensors has a lower bound. Hence, the number of sensors that can cover a given geographical area is bounded above. Furthermore, a ‘good’ sensor network ideally requires less overlapping in sensing areas, and even  $F_{\max} = 2$  can be achieved by carefully designing locations of sensors [Zalyubovskiy et al. 2009]. Examples of such near optimal tilings using uniform sensors appears in figure 1.



Fig. 1. Examples of “tight” plane tiling with low sensor overlap (i.e. low  $F_{\max}$ )

*Notation for our proofs.* Given some joint strategy, sometimes it is useful to consider sets that are covered by a unique element  $\ell \in L$ . For any  $\ell \in L$  we define

$$\mathcal{F}_\ell^* := \{\sigma \in \mathcal{F} : \ell \in \sigma, \sigma \setminus \{\ell\} \subseteq R\}.$$

We will show that of particular interest are advice vectors in which each *on* element uniquely covers many sets. We define the *core minimum* of such a strategy as

$$\delta^* := \min_{\ell \in L} |\mathcal{F}_\ell^*|.$$

*Equilibria of the game and interpretation as a packing problem.* Observe that  $c_i$  expresses how costly it is for agent  $i$  to cover the sets that contain him. For example, if  $c_{\max} < w_{\min}$ , it will always be cheaper for an agent to be *on* than to participate in any uncovered sets, so every set will be covered in equilibrium. The socially optimal equilibrium will necessarily be the minimum cost set cover (or a minimum vertex cover when  $F_{\max} = 2$ ). We note that if we simply redefine the costs so that  $i$  pays  $c_i$  if he is *off* and pays the sum of the weights of the *fully-covered* sets he participates in if he is *on*, this game is a packing analog of the original cover game. The equilibria when  $c_{\max} < w_{\min}$  are configurations in which no set is fully covered; in particular, when  $F_{\max} = 2$ , the sets of *on* agents in any equilibrium is a maximal independent set.

Recall that best-response dynamics converge to pure Nash equilibria for potential games, and observe that the cover game is an exact potential game with potential function

$$\Phi(s) = c(L) + w(\mathcal{F}_R). \quad (3)$$

Combining this with the social cost formula, we have that for any  $s \in S$

$$\Phi(s) \leq \text{cost}(s) \leq F_{\max} \cdot \Phi(s). \quad (4)$$

### 2.3. Optimization and Equilibrium Quality

The star graph example from the introduction reveals that PoA in the cover game can be  $\Omega(n)$ . This motivates the need for efficient dynamics with better guarantees than convergence to arbitrary equilibria.

As a step in that direction, we observe that a centralized, poly-time LP-rounding algorithm can find a low-cost configuration  $s^{ad}$  for the cover game. Specifically, let  $x^* = \arg \min_x \{ \sum_{i=1}^n c_i \cdot x_i \mid \text{s.t. } \sum_{i \in \sigma} x_i \geq 1 \ \forall \sigma \in \mathcal{F}, \ x_i \in [0, 1] \}$ , and then for all  $i \in [n]$ , set  $s_i^{ad}$  to *on* if  $x_i^* \geq 1/F_{\max}$  and *off* otherwise.

**FACT 1.** *The configuration  $s^{ad}$  obtained from poly-time LP-rounding has*

$$\text{cost}(s^{ad}) \leq F_{\max} \lceil c_{\max}/w_{\min} \rceil \cdot \text{OPT}.$$

**PROOF.** Let  $s^*$  be some joint strategy that achieves optimal social cost, and let  $s'$  be the joint strategy obtained by turning *on* an arbitrary element in each set  $\sigma$  that is uncovered in  $s^*$ . We have as follows:

$$\begin{aligned} \text{cost}(s^{ad}) &\leq F_{\max} \cdot \sum_i c_i \cdot x_i^* \leq F_{\max} \cdot \text{cost}(s') \\ &\leq F_{\max} \lceil c_{\max}/w_{\min} \rceil \cdot \text{cost}(s^*) = F_{\max} \lceil c_{\max}/w_{\min} \rceil \cdot \text{OPT}. \end{aligned}$$

□

### 3. PUBLIC SERVICE ADVERTISING

In this section and the following one, we show that the low system performance suggested by the price of anarchy can be avoided in cover games even when using best response-inspired dynamics as long as these dynamics incorporate some form of suggestion from a weak central authority that is aware of a high quality equilibrium.

The first model we study in this paper is the public service advertising (PSA) model in [Balcan et al. 2009] in which a central authority broadcasts a strategy for each agent, which some agents receive and temporarily follow. Agent behavior is described in two phases:

- 1:** Play begins in an arbitrary state, and a central authority advertises joint action  $s^{ad} \in S$ . Each agent receives the proposed strategy independently with probability  $\alpha \in (0, 1)$ . Agents that receive this signal are called *receptive*. They play their advertising strategies throughout Phase 1, and non-receptive agents undergo best-response dynamics to settle on a joint strategy that is a Nash equilibrium given the fixed behavior of receptive agents. We call this joint strategy  $s'$ .
- 2:** All agents participate in best-response dynamics until convergence to some Nash equilibrium  $s''$ .

Since our cover game is a potential game and all potential games eventually converge to a Nash equilibrium under best-response dynamics, both phases are guaranteed to terminate. Furthermore, convergence occurs in poly-time with respect to parameters  $\{n, \{c_i\}, \{w_\sigma\}\}$ .<sup>3</sup>

<sup>3</sup>This is because  $\Phi$  is bounded above and below by functions of these parameters and decreases under best-response dynamics.



### 3.1. Effect of Advertising in PSA

In this section we show that advertising helps significantly in cover games. In particular, we show that if the advertising strategy  $s^{ad}$  has low social cost, then the cost of the resulting equilibrium is low even if only a small constant  $\alpha$  fraction of the agents receive and respond to the signal.

Theorem 3.1 formalizes the general result of this section, relating the expected cost of the PSA outcome in vertex cover ( $F_{\max} = 2$ ) and set cover games to that of an arbitrary advertised strategy and other game-specific parameters. At a high level, this theorem's proof associates the costs of the outcome of PSA with the cost of the advertised strategy by leveraging the simplicity of best-response dynamics to charge each component of the final cost to some component of the advertised cost. The subtlest part of the analysis leverages the fact that agents receive the signal with independent probability, and this probabilistic reasoning forces an expected cost bound. The bottleneck reflected in our final bound, however, is due to a crude use of our structural assumption that at most  $\Delta$  sets contain a given pair of elements. Theorem 3.3 gives a stronger, high probability guarantee for the general set cover game in the setting that the advertised strategy is not only low-cost but efficient in that each *on* element uniquely covers many sets.

We first give formal statements of these two theorems and simple corollaries, and then we prove the two theorems in Sections 3.2 and 3.3, where we introduce notation used in later proofs.

**THEOREM 3.1.** *For any advertising strategy  $s^{ad}$ , expected cost at the end of PSA is*

$$E[\text{cost}(s'')] \leq \begin{cases} O(1) \cdot \text{cost}(s^{ad}) & \text{if } F_{\max} = 2 \\ O(\Delta) \cdot \text{cost}(s^{ad})^2 & \text{if } F_{\max} = O(1) \\ O\left(\frac{\Delta F_{\max}}{\alpha^2 F_{\max}}\right) \cdot \text{cost}(s^{ad})^2 & \text{otherwise.} \end{cases} \quad (5)$$

Recall that the above bounds assume  $\Theta(1)$  costs and weights. This assumption allows for simplifications such as  $c(L) = O(|L|)$  in the proof of the theorem in Section 3.2. By following this proof exactly (using bounds such as  $c(L) \leq c_{\max} \cdot |L|$ ), we can obtain the following results for arbitrary costs and weights. The calculation is routine and omitted for brevity.

$$E[\text{cost}(s'')] \leq \begin{cases} O\left(\left\lceil \frac{c_{\max}}{w_{\min}} \right\rceil \frac{c_{\max}}{c_{\min}}\right) \text{cost}(s^{ad}) & \text{if } F_{\max} = 2 \\ O\left(\Delta \left\lceil \frac{c_{\max}}{w_{\min}} \right\rceil \frac{c_{\max}}{c_{\min}^3}\right) \text{cost}(s^{ad})^2 & \text{if } F_{\max} = O(1) \\ O\left(\frac{\Delta F_{\max}}{\alpha^2 F_{\max}} \left\lceil \frac{c_{\max}}{w_{\min}} \right\rceil \frac{c_{\max}}{c_{\min}^3}\right) \text{cost}(s^{ad})^2 & \text{otherwise.} \end{cases} \quad (6)$$

If  $s^{ad}$  is obtained from the LP-rounding  $O(F_{\max})$ -approximation algorithm described in Section 2.3, the following corollary is immediate from Theorem 3.1.

**COROLLARY 3.2.** *There exists a poly-time algorithm to find an advertising strategy  $s^{ad}$  such that the expected cost at the end of PSA is*

$$E[\text{cost}(s'')] \leq \begin{cases} O(1) \cdot \text{OPT} & \text{if } F_{\max} = 2 \\ O(\Delta) \cdot \text{OPT}^2 & \text{if } F_{\max} = O(1) \\ O(\Delta F_{\max}^3 / \alpha^{2F_{\max}}) \cdot \text{OPT}^2 & \text{otherwise.} \end{cases}$$

For improved performance guarantees, we look for strategies that are efficient in a particular sense. Recall that for a given strategy profile, the core minimum degree  $\delta^*$  of *on* elements is the minimum number of sets uniquely covered by any particular *on* element. We say that an advertising strategy  $s^{ad}$  satisfies Condition  $(\star)$  if for all  $x \geq \frac{\delta^*}{\Delta(F_{\max}-1)}$ ,

$$\left( \left\lfloor \frac{c_{\max}}{w_{\min}} \right\rfloor + 1 \right) x^{\lfloor c_{\max}/w_{\min} \rfloor} (1 - \alpha^{F_{\max}})^{x - \lfloor c_{\max}/w_{\min} \rfloor} \leq \frac{1}{n^2}. \quad (\star)$$

Intuitively, this condition ensures that each element that is *on* in  $s^{ad}$  is in many sets in which it is the unique element that is *on* in  $s^{ad}$ . When this is the case, it is very likely that in Phase 1, some of these sets will have *each* element (except perhaps the single element *on* in  $s^{ad}$ ) receptive to advertising. This unique *on* element will turn *on* in Phase 1, and every set will be covered. We achieve the precise condition by reverse engineering our analysis starting with this goal.

In polynomial time, we can construct a low-cost advertising strategy satisfying Condition  $(\star)$  under the natural  $F_{\max} = O(1)$  assumption (in addition to the usual assumption of constant costs and weights). First observe that any joint strategy  $s$  with  $\delta^*(s) \geq B\Delta \log n$  for a large enough constant  $B$  (depending on constants  $c_{\max}/w_{\min}$ ,  $\alpha$ ,  $F_{\max}$ ) satisfies Condition  $(\star)$ .<sup>4</sup> Then let  $s^*$  be the strategy with social cost  $O(1) \cdot OPT$  obtained by LP-rounding (Fact 1). By greedily turning *off* every agent that is the unique *on* element in fewer than  $B\Delta \log n$  sets in  $s^*$ , we can construct  $s^{ad}$  satisfying Condition  $(\star)$  with social cost  $O(\Delta \log n) \cdot OPT$ .<sup>5</sup>

Theorem 3.3 formalizes a high probability and stronger version of Theorem 3.1 for the general set cover game when  $s^{ad}$  satisfies Condition  $(\star)$ , and Corollary 3.4 follows immediately by using the low-cost advertising strategy produced by the greedy algorithm described above. Note that this result requires no assumptions on the costs, weights, or  $F_{\max}$  of the hypergraph.

**THEOREM 3.3.** *For any advertising strategy  $s^{ad}$  satisfying Condition  $(\star)$ , with probability at least  $1 - 1/n$  the cost at the end of PSA is*

$$\text{cost}(s'') = O(1) \cdot \text{cost}(s^{ad}).$$

Using the greedily constructed advertising strategy described above, we have the following immediate corollary in the case that costs, weights, and  $F_{\max}$  are constant:

**COROLLARY 3.4.** *There exists a poly-time algorithm when  $F_{\max} = \Theta(1)$  to find an advertising strategy  $s^{ad}$  such that with probability at least  $1 - 1/n$  the cost at the end of PSA is*

$$\text{cost}(s'') = O(\Delta \log n) \cdot OPT.$$

### 3.2. Proof of Theorem 3.1

We begin with an overview and high level observations that reduce the proof to the two lemmas that follow. Our first key observation is that since Phase 2 is simple best-response dynamics, the cost of the final equilibrium is at most a constant factor greater than the cost at the end of Phase 1 by (4). Therefore it suffices to bound the expected cost at the end of Phase 1 relative to that of the advertised strategy. The only social cost additional to that of the advertised strategy at the end of Phase 1 is due to sets

<sup>4</sup>Note  $(1 - \alpha^{F_{\max}})^{x - \lfloor c_{\max}/w_{\min} \rfloor} = O(1/n^d)$  for arbitrarily large constant  $d$  and  $x^{\lfloor c_{\max}/w_{\min} \rfloor} = \tilde{O}(1)$  when  $x \geq \delta^*/(\Delta(F_{\max} - 1)) \geq B \log n / (F_{\max} - 1)$  for sufficiently large constant  $B$ .

<sup>5</sup>To see why, note that at most  $OPT/c_{\max}$  agents are *on* in  $s^*$ , and turning any one *off* results in at most  $F_{\max} B \Delta \log n$  sets becoming uncovered, so the total cost increases by  $O(\Delta \log n) \cdot OPT$ .

that are uncovered in  $s'$  but covered in  $s^{ad}$  and elements that are on in  $s'$  but not in  $s^{ad}$ . Lemma 3.5 simply bounds the weights of sets uncovered in  $s'$  but covered in  $s^{ad}$  by associating each one with an agent that is *on* in  $s^{ad}$  and therefore contributing to the cost of the advertised strategy. Lemma 3.6 carefully bounds the cost of agents that are *on* in  $s'$  but not in  $s^{ad}$  by partitioning such agents into three types and treating each induced set separately in bounding their costs relative to the cost of the advertised strategy. In particular, our bottleneck for  $F_{\max} > 2$  is in associating some of these costly on agents with sets containing a pair of agents that are supposed to be *on* in the advertised strategy but are in fact *off* at the end of Phase 1. This causes our dependency on both the square of the cost of  $s^{ad}$  and our hypergraph parameter  $\Delta$ .

In these and subsequent proofs, we let  $L$  and  $R$  denote the sets of elements that are *on* and *off*, respectively, in  $s^{ad}$ , and we further define  $R_{on}$  to be the set of elements that are *on* in  $s'$  but not in  $s^{ad}$  and  $L_{off}$  to be the set of elements that are *off* in  $s'$  but not in  $s^{ad}$  (see figure 2).  $\mathcal{F}_{bad}$  denotes the sets uncovered in  $s'$  but covered in  $s^{ad}$ . From (2), the constant cost and weight assumption, we have

$$\begin{aligned} \mathbb{E}[\text{cost}(s')] &\leq \text{cost}(s^{ad}) + \mathbb{E}[c(R_{on})] + F_{\max} \cdot \mathbb{E}[w(\mathcal{F}_{bad})] \\ &= \text{cost}(s^{ad}) + O(\mathbb{E}[|R_{on}|]) + F_{\max} \cdot \mathbb{E}[w(\mathcal{F}_{bad})]. \end{aligned}$$

Since  $\text{cost}(s^{ad}) \geq c(L) + w(\mathcal{F}_R) = \Theta(|L|) + \Theta(|\mathcal{F}_R|)$ , the following two lemmas give the desired bound on  $\text{cost}(s')$ , completing the proof of Theorem 3.1.

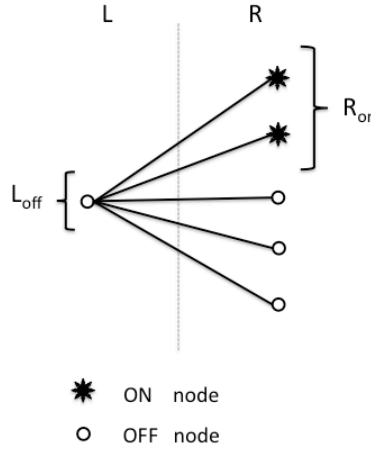


Fig. 2. Notational example of state  $s'$  of star graph with optimal advice vector  $s^{ad}$ . The optimal advice vector  $s^{ad}$  sets the center node *on*, so this is the sole node in the  $L$  set, and all other nodes *off*, which constitute the  $R$  set.  $R_{on}$  and  $L_{off}$  capture the elements that do not follow the advice vector at the end of Phase 1 in  $s'$ .  $R_{on}$  is the set of elements that are *on* in  $s'$  but not in  $s^{ad}$  and  $L_{off}$  is the set of elements that are *off* in  $s'$  but not in  $s^{ad}$ .

LEMMA 3.5.  $w(\mathcal{F}_{bad}) \leq c(L)$ .

PROOF. Note that each set in  $\mathcal{F}_{bad}$  must contain a best-responding element in  $L_{off}$ , so  $w(\mathcal{F}_{bad}) \leq \sum_{\ell \in L_{off}} \sum_{\sigma \in \mathcal{F}_{bad}: \ell \in \sigma} w_\sigma \leq \sum_{\ell \in L_{off}} c_\ell \leq c(L)$ .  $\square$

LEMMA 3.6.

$$E[|R_{on}|] \leq \begin{cases} |\mathcal{F}_R| + O(|L|) & \text{if } F_{\max} = 2 \\ |\mathcal{F}_R| + \Delta|L|^2 + O(\Delta|L|) & \text{if } F_{\max} = O(1) \\ |\mathcal{F}_R| + \Delta|L|^2 + O\left(\frac{\Delta F_{\max}}{\alpha^2 F_{\max}}\right) \cdot |L| & \text{otherwise.} \end{cases}$$

PROOF. We consider three types of sets at the end of Phase 1 such that each  $r \in R_{on}$  must be in one of these types of sets, and we bound the number of each type of set. These bounds are based on structural arguments, including the fact that our parameter  $\Delta$  bounds the maximum number of sets that can contain any given pair of elements. The last of these bounds is given in expectation over the set of agents receptive to advertising. While the second term is the bottleneck for bounding  $E[|R_{on}|]$  in terms of  $s^{ad}$ , the third term is the one given in expectation and requires much more sophisticated analysis including algorithmic arguments that allow us to decouple related events.

Since each element  $R_{on}$  plays a best-response move in  $s'$ , we can associate each  $r \in R_{on}$  with a distinct set in which  $r$  is the only *on* element. Each such set is contained in one of the following:

$$\begin{aligned} \mathcal{F}^{(=0)} &:= \{\sigma \in \mathcal{F} : |\sigma \cap R| > 0, |\sigma \cap L| = 0\} \\ \mathcal{F}^{(=1)} &:= \{\sigma \in \mathcal{F} : |\sigma \cap R| > 0, |\sigma \cap L| = 1, \sigma \cap L \subseteq L_{off}\} \\ \mathcal{F}^{(>1)} &:= \{\sigma \in \mathcal{F} : |\sigma \cap R| > 0, |\sigma \cap L| > 1, \sigma \cap L \subseteq L_{off}\} \end{aligned}$$

Observe that by definition,

$$|\mathcal{F}^{(=0)}| = |\mathcal{F}_R|. \quad (7)$$

Since each  $\sigma \in \mathcal{F}^{(>1)}$  contains at least three elements, there are no such sets when  $F_{\max} = 2$ . In general, we use the definition of  $\Delta$  and the fact that there are  $\binom{|L|}{2} \leq L^2$  pairs of agents in  $L$  to achieve the following bound,

$$|\mathcal{F}^{(>1)}| \leq \begin{cases} 0 & \text{if } F_{\max} = 2 \\ \Delta \cdot |L|^2 & \text{otherwise.} \end{cases} \quad (8)$$

We now bound the expected number of sets  $\mathcal{F}^{(=1)}$  containing a unique element in  $L$ , which is *off*. Recall that  $\mathcal{F}_\ell^*$  is all sets with  $\ell$  the unique element in  $L$ . Then

$$E[|\mathcal{F}^{(=1)}|] \leq \sum_{\ell \in L} |\mathcal{F}_\ell^*| \cdot \Pr[\ell \in L_{off}].$$

Observe that  $\ell \in \sigma_\ell \in \mathcal{F}_\ell^*$  will never be *off* in  $s'$  if it participates more than  $c_{\max}/w_{\min}$  sets where it is the unique  $L$  element and all other elements are *off* in  $s'$ . We use this fact to bound the probability that  $\ell \in L_{off}$  by bounding the probability that many of the sets in  $\mathcal{F}_\ell^*$  contain only other elements that are following their advertised strategy of *off*. However, there may be overlap in the  $R$  nodes of the sets in  $\mathcal{F}_\ell^*$ , so these probabilities are dependent. To circumvent this, we take some subset  $\hat{\mathcal{F}}_\ell^* \subseteq \mathcal{F}_\ell^*$  such that no

pair of sets in  $\hat{\mathcal{F}}_\ell^*$  have common elements in  $R$ . Then, it follows that

$$\begin{aligned}
 \Pr[\ell \in L_{\text{off}}] &\leq \Pr[|\{\rho \in \mathcal{F}_\ell^* : \text{all } \rho \setminus \{\ell\} \text{ are off}\}| \leq c_{\max}/w_{\min}] \\
 &\leq \Pr[|\{\rho \in \mathcal{F}_\ell^* : \text{all } \rho \setminus \{\ell\} \text{ are receptive}\}| \leq c_{\max}/w_{\min}] \\
 &\leq \Pr[|\{\rho \in \hat{\mathcal{F}}_\ell^* : \text{all } \rho \setminus \{\ell\} \text{ are receptive}\}| \leq c_{\max}/w_{\min}] \\
 &\leq \Pr \left[ \sum_{i=1}^{|\hat{\mathcal{F}}_\ell^*|} X_i \leq c_{\max}/w_{\min} \right],
 \end{aligned} \tag{9}$$

where  $X_i \in \{0, 1\}$  denotes the random variable indicating the event that for the  $i$ -th set  $\rho \in \hat{\mathcal{F}}_\ell^*$ , all elements in  $\rho \setminus \{\ell\}$  are receptive. Note that the  $\{X_i\}$  are independent random variables with  $\Pr[X_i = 1] \geq \alpha^{F_{\max}}$ . Let  $\{Y_i\}$  for  $i = 1, \dots, |\hat{\mathcal{F}}_\ell^*|$  be independent and identically distributed random variables with  $\Pr[Y_i = 1] = \alpha^{F_{\max}}$  and  $\Pr[Y_i = 0] = 1 - \alpha^{F_{\max}}$ . Then assuming  $|\hat{\mathcal{F}}_\ell^*| \geq \lfloor c_{\max}/w_{\min} \rfloor$ , we have:

$$\begin{aligned}
 \Pr \left[ \sum_{i=1}^{|\hat{\mathcal{F}}_\ell^*|} X_i \leq c_{\max}/w_{\min} \right] &\leq \Pr \left[ \sum_{i=1}^{|\hat{\mathcal{F}}_\ell^*|} Y_i \leq c_{\max}/w_{\min} \right] \\
 &\leq \sum_{i=0}^{\lfloor \frac{c_{\max}}{w_{\min}} \rfloor} \binom{|\hat{\mathcal{F}}_\ell^*|}{i} (1 - \alpha^{F_{\max}})^{|\hat{\mathcal{F}}_\ell^*| - i} (\alpha^{F_{\max}})^i.
 \end{aligned} \tag{10}$$

Note that by definition of  $F_{\max}$  and  $\Delta$ , we can choose some  $R$ -disjoint  $\hat{\mathcal{F}}_\ell^* \subseteq \mathcal{F}_\ell^*$  with  $|\mathcal{F}_\ell^*| \leq \Delta(F_{\max} - 1)|\hat{\mathcal{F}}_\ell^*|$ . We bound  $\mathbb{E}[|\mathcal{F}^{(=1)}|]$ , again assuming  $|\hat{\mathcal{F}}_\ell^*| \geq \lfloor c_{\max}/w_{\min} \rfloor$ , with the help of Proposition A.1, stated and proven in Appendix A.

$$\begin{aligned}
 \mathbb{E}[|\mathcal{F}^{(=1)}|] &\leq \sum_{\ell \in L} |\mathcal{F}_\ell^*| \sum_{i=0}^{\lfloor \frac{c_{\max}}{w_{\min}} \rfloor} \binom{|\hat{\mathcal{F}}_\ell^*|}{i} (1 - \alpha^{F_{\max}})^{|\hat{\mathcal{F}}_\ell^*| - i} (\alpha^{F_{\max}})^i \\
 &\leq (F_{\max} - 1)\Delta \sum_{\ell \in L} \sum_{i=0}^{\lfloor \frac{c_{\max}}{w_{\min}} \rfloor} |\hat{\mathcal{F}}_\ell^*| \binom{|\hat{\mathcal{F}}_\ell^*|}{i} (1 - \alpha^{F_{\max}})^{|\hat{\mathcal{F}}_\ell^*| - i} (\alpha^{F_{\max}})^i \\
 &= O \left( \frac{\Delta F_{\max}}{\alpha^{2F_{\max}}} \right) \cdot |L|,
 \end{aligned} \tag{11}$$

When  $|\hat{\mathcal{F}}_\ell^*| < \lfloor c_{\max}/w_{\min} \rfloor$ , we have  $\mathbb{E}[|\mathcal{F}^{(=1)}|] \leq \sum_{\ell \in L} |\mathcal{F}_\ell^*| \leq (F_{\max} - 1)\Delta \sum_{\ell \in L} |\hat{\mathcal{F}}_\ell^*|$ , which is dominated by the above expression. Finally, since  $|R_{\text{on}}| \leq |\mathcal{F}^{(=0)}| + |\mathcal{F}^{(>1)}| + |\mathcal{F}^{(=1)}|$  by construction, (7), (8) and (11) together give the desired conclusion of Lemma 3.6, noting that  $\Delta = 1$  when  $F_{\max} = 2$ .  $\square$

### 3.3. Proof of Theorem 3.3

We will use the same notation presented in the proof of Theorem 3.1. As with Theorem 3.1, it suffices to bound the cost of  $s'$ . In particular, we will prove that  $\text{cost}(s') = O(\text{cost}(s^{ad}))$  with all but at most  $1/n$  probability.

Recall that  $\text{cost}(s^{ad}) \geq c(L) + w(\mathcal{F}_R)$ . The following lemma proves that for  $s^{ad}$  satisfying Condition  $(\star)$ , all agents in  $L$  turn on in Phase 1 with probability at least  $1 - 1/n$

and under this event, the cost of agents in  $R_{on}$  is bounded by  $w(\mathcal{F}_R) \leq \text{cost}(s^{ad})$ , establishing the desired conclusion of Theorem 3.3.

**LEMMA 3.7.** *If the advertising strategy  $s^{ad}$  satisfies Condition  $(\star)$ , then  $\mathcal{F}_{bad} = \emptyset$  and  $c(R_{on}) \leq w(\mathcal{F}_R)$  with probability at least  $1 - 1/n$ .*

**PROOF.** As in the proof of Lemma 3.6 (and using the same notation), for any  $\ell \in L$  there is some subset  $\hat{\mathcal{F}}_\ell^* \subseteq \mathcal{F}_\ell^*$  such that no pair of sets in  $\hat{\mathcal{F}}_\ell^*$  have common elements in  $R$  and  $|\hat{\mathcal{F}}_\ell^*| \geq \frac{|\mathcal{F}_\ell^*|}{\Delta(F_{\max}-1)} \geq \frac{\delta^*}{\Delta(F_{\max}-1)}$ . Applying the bound on  $\Pr[\ell \in L_{off}]$  derived in the proof Lemma 3.6 as a starting point,

$$\begin{aligned} \Pr[\ell \in L_{off}] &\leq \sum_{i=0}^{\lfloor \frac{c_{\max}}{w_{\min}} \rfloor} \binom{|\hat{\mathcal{F}}_\ell^*|}{i} (1 - \alpha^{F_{\max}})^{|\hat{\mathcal{F}}_\ell^*|-i} (\alpha^{F_{\max}})^i \\ &\leq \sum_{i=0}^{\lfloor \frac{c_{\max}}{w_{\min}} \rfloor} |\hat{\mathcal{F}}_\ell^*|^i (1 - \alpha^{F_{\max}})^{|\hat{\mathcal{F}}_\ell^*|-i} \\ &\leq \left( \left\lfloor \frac{c_{\max}}{w_{\min}} \right\rfloor + 1 \right) |\hat{\mathcal{F}}_\ell^*|^{\lfloor \frac{c_{\max}}{w_{\min}} \rfloor} (1 - \alpha^{F_{\max}})^{|\hat{\mathcal{F}}_\ell^*| - \lfloor \frac{c_{\max}}{w_{\min}} \rfloor}, \end{aligned}$$

and by assumption that  $s^{ad}$  satisfies  $(\star)$ , the above expression is at most  $1/n^2$ . By union bound,  $\Pr[L_{off} = \emptyset] \geq 1 - 1/n$  and hence  $\mathcal{F}_{bad} = \emptyset$  at the end of Phase 1 with at least this probability.

Assume this event, and observe that for each best-responding  $r \in R_{on}$ ,  $c_r$  is no greater than the total weight of all sets containing  $r$  as the unique *on* agent. Since we assume all nodes in  $L$  are *on*, these sets are a subset of  $\mathcal{F}_R$ . Further, since there is no overlap in these sets between different agents in  $R_{on}$ , we can sum over all  $r \in R_{on}$  to derive  $c(R_{on}) \leq w(\mathcal{F}_R)$ . This completes the proof of Lemma 3.7.  $\square$

#### 4. LEARN-THEN-DECIDE

We study the set cover game in the learn-then-decide (LTD) model of [Balcan et al. 2013]. In contrast to PSA, agents in LTD are neither strictly receptive nor strictly best-responders in the initial exploration phase, but they choose one of these options for the final exploitation phase. The PSA model is appropriate for an engineering setting such as sensor networks, where devices may be programmed to respond in Phase 1 to a signal that only reaches some devices due to technical constraints. On the other hand, the LTD model is better for a social setting, where agents may be skeptical of the central authority and experiment in Phase 1, sometimes following the advertised strategy and other times applying a best-response strategy.

- 1:** Play begins in an arbitrary state, and joint action  $s^{ad} \in S$  is advertised by a central authority. Agent  $i$  is associated with fixed probability  $p_i \geq \beta \in (0, 1)$ , where  $\beta$  is constant. Agents are chosen to update uniformly at random for each of  $T^*$  time steps. When  $i$  updates, he plays  $s_i^{ad}$  with probability  $p_i$  or a best-response move with probability  $1 - p_i$ . The state at time  $T^*$  is denoted  $s'$ .
- 2:** At time  $T^*$ , all agents in random order individually commit arbitrarily to  $s_i^{ad}$  or the best-response strategy. Finally, agents take turns in random order playing their chosen strategy until best-responders reach a Nash equilibrium  $s''$  given the fixed behavior of  $s^{ad}$  followers.

#### 4.1. Effect of Advertising in LTD

Theorem 4.1 relates the cost of the outcome of LTD to that of the advertised strategy and is analogous to Theorem 3.1 in the PSA model. At a high level, we bound the cost at the end of Phase 1 by associating it with the costs of the advertised strategy using techniques similar to those of Lemmas 3.5 and 3.6 in Section 3.2 since in some sense Phase 1 of LTD is very similar to Phase 1 of PSA. However, it is more challenging to prove that cost does not increase too much in Phase 2 in LTD. To accomplish this, we assume that Phase 2 lasts long enough that all agents can move several times; this allows us to make more precise claims about the results of the best-response dynamics. We show that our required sequence of moves occurs with high probability for a polynomially small number of steps. We again assume for expository simplicity that costs and weights are constant.

**THEOREM 4.1.** *There exists a  $T^* \in \text{poly}(n)$  such that for any advertising strategy  $s^{ad}$ , the expected cost at the end of LTD is*

$$E[\text{cost}(s'')] \leq \begin{cases} O(1) \cdot \text{cost}(s^{ad}) & \text{if } F_{\max} = 2 \\ O(\Delta^2) \cdot \text{cost}(s^{ad})^2 & \text{if } F_{\max} = O(1). \end{cases} \quad (12)$$

Theorem 4.1 implies that if  $s^{ad}$  is obtained from the  $O(F_{\max})$ -approximation poly-time algorithm described in Section 2.2, the following corollary holds.

**COROLLARY 4.2.** *There exists a  $T^* \in \text{poly}(n)$  and a poly-time algorithm to find an advertising strategy  $s^{ad}$  such that the expected cost at the end of LTD is*

$$E[\text{cost}(s'')] \leq \begin{cases} O(1) \cdot \text{OPT} & \text{if } F_{\max} = 2 \\ O(\Delta^2) \cdot \text{OPT}^2 & \text{if } F_{\max} = O(1). \end{cases}$$

Note that analogs of the weak results for general  $F_{\max}$  presented in Theorem 3.1 and Corollary 3.2 for the PSA model are possible in LTD as well. However, we believe that our analysis is far from tight for non-constant weights, costs, and  $F_{\max}$ , where in particular the price of anarchy (PoA) grows exponentially with respect to  $F_{\max}$  (See (6) and Corollary 3.2). For this reason, we do not make tedious and routine efforts to obtain such bounds in the LTD model.

#### 4.2. Proof of Theorem 4.1

To begin with, we note that while LTD differs from PSA in both phases, we can analyze Phase 1 of LTD in a manner similar to Phase 1 of PSA by defining an event that occurs with high probability in Phase 1 of LTD and then modifying the techniques of Theorem 3.1 to bound the cost of the state at the end of Phase 1 relative to that of the advertised strategy. However, showing that the cost stays low in Phase 2 imposes new challenges that we circumvent using the fact that update order is random, and this causes us to lose an additional  $\Delta$  factor.

We will use the same notation as in the proof of Theorem 3.1, and in Lemma 4.4 we will additionally use  $L_{on} := L \setminus L_{off}$  and  $R_{off} := R \setminus R_{on}$ , recalling that  $L_{off}, R_{on}$  refer to the strategies of elements at the end of Phase 1. We first define  $\mathcal{E} = \mathcal{E}(T', T^*)$  for  $1 < T' < T^*$  as the event that every element in  $L$  updates at least once before time  $T'$  after every element in  $R$  has updated at least once, and then every element in  $R$  again updates at least once at some time  $t \in [T', T^*]$ . (The complement is denoted  $\mathcal{E}^c$ .) Clearly there exist  $T', T^* \in \text{poly}(n)$  such that  $\Pr[\mathcal{E}] \geq 1 - 1/n^{F_{\max}}$ . Using the fact that

social cost is always at most  $c_{\max} \cdot n + w_{\max} \cdot F_{\max} \cdot |\mathcal{F}| = O(n^{F_{\max}})$ , we have

$$\begin{aligned} \mathbb{E}[\text{cost}(s'')] &= \Pr[\mathcal{E}] \cdot \mathbb{E}[\text{cost}(s'') \mid \mathcal{E}] + \Pr[\mathcal{E}^c] \cdot \mathbb{E}[\text{cost}(s'') \mid \mathcal{E}^c] \\ &\leq \mathbb{E}[\text{cost}(s'') \mid \mathcal{E}] + \frac{1}{n^{F_{\max}}} \cdot O(n^{F_{\max}}) \\ &= \mathbb{E}[\text{cost}(s'') \mid \mathcal{E}] + O(1), \end{aligned} \quad (13)$$

so it suffices to bound  $\mathbb{E}[\text{cost}(s'') \mid \mathcal{E}]$ . Lemma 4.3 first bounds the expected social cost at the end of Phase 1 under the event  $\mathcal{E}$ , and then we bound the increase in social cost in Phase 2 with Lemma 4.4. Together, these lemmas imply Theorem 4.1.

LEMMA 4.3.

$$\mathbb{E}[\text{cost}(s') \mid \mathcal{E}] \leq \begin{cases} O(1) \cdot \text{cost}(s^{ad}) & \text{if } F_{\max} = 2 \\ O(\Delta) \cdot \text{cost}(s^{ad})^2 & \text{if } F_{\max} = O(1). \end{cases}$$

PROOF. Recall that as in the proof of Theorem 3.1,

$$\begin{aligned} \text{cost}(s') &= \text{cost}(s^{ad}) + O(|R_{on}|) + O(w(\mathcal{F}_{bad})) \\ \text{cost}(s^{ad}) &= \Theta(|L|) + \Theta(|\mathcal{F}_R|), \end{aligned} \quad (14)$$

so it suffices to bound  $w(\mathcal{F}_{bad})$  and  $|R_{on}|$  in terms of  $|L|$  and  $|\mathcal{F}_R|$ .

The LTD model does not allow us to use the clean analysis of Lemma 3.5 to bound all the sets in  $\mathcal{F}_{bad}$  since elements in  $L_{off} \cap \mathcal{F}_{bad}$  are not necessarily best-responding at the end of Phase 1, so we separately analyze the weights of two types of sets in  $\mathcal{F}_{bad}$  and are able to use the analysis in Lemma 3.6 bounding  $\mathbb{E}[|R_{on}|]$  in PSA to bound the weight of the harder to analyze of the  $\mathcal{F}_{bad}$  sets in LTD.

First consider a set in  $\mathcal{F}_{bad} \cap 2^L$ , which consists only of elements in  $L_{off}$ . Attribute the weight of such a set to its element  $\ell$  that updated most recently before the end of Phase 1. Because  $\ell \in L_{off}$  played best-response most recently, the weight of all sets in  $\mathcal{F}_{bad} \cap 2^L$  attributed to  $\ell$  is at most  $c_\ell$ . Summing over all  $\ell \in L_{off} \subseteq L$  gives

$$w(\mathcal{F}_{bad} \cap 2^L) \leq c(L) = O(|L|). \quad (15)$$

Now consider a set in  $\mathcal{F}_{bad} \setminus 2^L$ , which has elements in both  $L$  and  $R$  and all of them are *off* in  $s'$ . Recall the definitions of  $\mathcal{F}^{(>1)}$  and  $\mathcal{F}^{(=1)}$  in the proof of Lemma 3.6 and observe that  $\mathcal{F}_{bad} \setminus 2^L \subseteq \mathcal{F}^{(>1)} \cup \mathcal{F}^{(=1)}$ . Assuming event  $\mathcal{E}$ , we can modify the analysis of  $|\mathcal{F}^{(=1)}|$  in Lemma 3.6 so that (10) with the  $\beta$  in place of  $\alpha$  bounds that probability that some  $\ell \in \mathcal{F}_\ell^*$  is *off* when it last updates in Phase 1. To do this, interpret “all  $r \in \rho \cap R$  are *off*” as referring to the time  $\ell$  last updates in Phase 1 and “ $r \in \rho \cap R$  receptive” as the event that  $r$  played  $s^{ad}$  with probability  $p_r \geq \beta$  at the most recent time step that it updated before the last update of  $\ell$  last updates in Phase 1. As before, (8) bounds  $|\mathcal{F}^{(>1)}|$ . Then we have

$$\mathbb{E}[|\mathcal{F}_{bad} \setminus 2^L| \mid \mathcal{E}] \leq \begin{cases} O(|L|) & \text{if } F_{\max} = 2 \\ \Delta|L|^2 + O(\Delta|L|) & \text{if } F_{\max} = O(1). \end{cases} \quad (16)$$

From (15) and (16), it follows that

$$\mathbb{E}[w(\mathcal{F}_{bad}) \mid \mathcal{E}] = \begin{cases} O(|L|) & \text{if } F_{\max} = 2 \\ O(\Delta|L|^2) + O(\Delta|L|) & \text{if } F_{\max} = O(1). \end{cases} \quad (17)$$



This modification to the proof of Lemma 3.6 also works for bounding  $|R_{on}|$  in the LTD model assuming event  $\mathcal{E}$ , so we have

$$E[|R_{on}| \mid \mathcal{E}] = \begin{cases} |\mathcal{F}_R| + O(|L|) & \text{if } F_{\max} = 2 \\ |\mathcal{F}_R| + \Delta|L|^2 + O(\Delta|L|) & \text{if } F_{\max} = O(1). \end{cases} \quad (18)$$

Together, (14), (17) and (18) give Lemma 4.3.  $\square$

We bound the cost increase in Phase 2 assuming  $\mathcal{E}$ . From (4) and  $F_{\max} = O(1)$  it suffices to provide a bound on the expected increase in the potential function throughout Phase 2, i.e.

$$\begin{aligned} \text{cost}(s'') &\leq F_{\max} \cdot \Phi(s'') + F_{\max} \cdot (-\Phi(s') + \text{cost}(s')) \\ &= O(\Phi(s'') - \Phi(s')) + O(\text{cost}(s')). \end{aligned} \quad (19)$$

The following lemma bounds the expected potential increase  $\Phi(s'') - \Phi(s')$  assuming event  $\mathcal{E}$ . As with Lemma 4.3, we can reuse some of the analysis in Lemma 3.6, but we additionally employ new probabilistic reasoning leveraging agents' update order to bound the number of a certain type of sets associated with moves that increase potential in Phase 2. The last step in this reasoning decouples dependent events by creating an  $R$ -disjoint subset of sets as in Lemma 3.6, and this is responsible for the extra  $\Delta$  term compared to the bounds for the PSA model.

LEMMA 4.4.

$$E[\Phi(s'') - \Phi(s') \mid \mathcal{E}] \leq \begin{cases} O(1) \cdot \text{cost}(s^{ad}) & \text{if } F_{\max} = 2 \\ O(\Delta^2) \cdot \text{cost}(s^{ad})^2 & \text{if } F_{\max} = O(1). \end{cases}$$

PROOF. Since best-response moves do not increase the potential function  $\Phi$ , we only consider updates of agents following the advertising strategy  $s^{ad}$  in Phase 2. Since each such agent changes strategies at most once in Phase 2, it suffices to consider a single *off-on* move for each agent in  $L$  (i.e., the agent changes her strategy from *off* to *on*) and a single *on-off* move for each agent in  $R_{on}$ . For each  $\ell \in L$ , an *off-on* move increases potential by at most  $c_\ell$ , so

$$\text{off-on moves increase potential by } \leq c(L) = O(|L|). \quad (20)$$

Now for each  $r \in R_{on}$  that first turns *off* at time  $t \geq T^*$  in Phase 2, let  $\mathcal{F}_r^*$  be the collection of sets containing  $r$  such that all of their other elements are *off* at time  $t$ . The potential increases by at most  $w(\mathcal{F}_r^*) = O(|\mathcal{F}_r^*|)$  at time  $t$ . Hence, we have

$$\text{on-off moves increase potential by } \leq O\left(\sum_{r \in R_{on}} |\mathcal{F}_r^*|\right) = O(|\cup_{r \in R_{on}} \mathcal{F}_r^*|), \quad (21)$$

where the equality follows from the fact that  $\mathcal{F}_{r_1}^* \cap \mathcal{F}_{r_2}^* = \emptyset$  if  $r_1 \neq r_2$ .

To bound the expectation of  $|\cup_{r \in R_{on}} \mathcal{F}_r^*|$ , we consider a partitioning  $\cup_{r \in R_{on}} \mathcal{F}_r^* = \mathcal{F}^{(L_{off})} \cup \mathcal{F}^{(L_{on})}$ , where

$$\mathcal{F}^{(L_{off})} := \{\sigma \in \cup_{r \in R_{on}} \mathcal{F}_r^* : \sigma \cap L \subseteq L_{off}\}$$

$$\mathcal{F}^{(L_{on})} := \{\sigma \in \cup_{r \in R_{on}} \mathcal{F}_r^* : \sigma \cap L_{on} \neq \emptyset\}.$$

Recall the definitions of  $\mathcal{F}^{(>1)}$ ,  $\mathcal{F}^{(=1)}$  from Lemma 3.6. Note that  $\mathcal{F}^{(L_{off})} \setminus \mathcal{F}_R \subseteq \mathcal{F}^{(>1)} \cup \mathcal{F}^{(=1)}$  since such sets have an element in  $R$  and at least one element in  $L$ , all of which are *off* in  $s'$ . As done to justify inequality (16) in Lemma 4.3, assuming event  $\mathcal{E}$ , we can modify the analysis of  $|\mathcal{F}^{(=1)}|$  in Lemma 3.6 so that (10) with the  $\beta$  in place of  $\alpha$  bounds that probability that some  $\ell \in \mathcal{F}_\ell^*$  is *off* when it last updates in Phase 1. To do

this, interpret “all  $r \in \rho \cap R$  are *off*” as referring to the time  $\ell$  last updates in Phase 1 and “ $r \in \rho \cap R$  receptive” as the event that  $r$  played  $s^{ad}$  with probability  $p_r \geq \beta$  at the most recent time step that it updated before the last update of  $\ell$  last updates in Phase 1. As before, (8) bounds  $|\mathcal{F}^{(>1)}|$ . Then we have:

$$\mathbb{E}[|\mathcal{F}^{(L_{off})}| \mid \mathcal{E}] \leq |\mathcal{F}_R| + \begin{cases} O(|L|) & \text{if } F_{\max} = 2 \\ \Delta|L|^2 + O(\Delta|L|) & \text{if } F_{\max} = O(1). \end{cases} \quad (22)$$

We now show that random updates in Phase 2 limits the expected number of sets in  $\mathcal{F}^{(L_{on})}$ . Our key observation is that for some  $\sigma$  containing some  $r_\sigma \in R_{on}$  and  $\ell_\sigma \in L_{on}$  to indeed be uncovered by the *on-off* move of  $r_\sigma$ , the ‘gatekeeper’ element  $\ell_\sigma$  must have turned *off* before the first update of  $r_\sigma$ . Further, observe that  $\ell_\sigma$  can turn *off* only if doing so uncovers at most  $c_{\max}/w_{\min}$  sets in which  $\ell_\sigma$  participates. Hence, all but at most  $c_{\max}/w_{\min}$  sets of the following type must have an element in  $R_{off}$  that updates (and, in particular, turns *on*) before  $\ell_\sigma$  updates, and therefore also before  $r_\sigma$  updates:

$$\mathcal{F}_{\ell_\sigma}^{(R_{off})} = \{\rho \in \mathcal{F} : \ell_\sigma \in \rho, \rho \cap R \subset R_{off}\}.$$

We use these observations to bound the probability that  $\sigma$  containing some  $r_\sigma \in R_{on}$  and  $\ell_\sigma \in L_{on}$  is uncovered by  $r_\sigma$ , where randomness is taken over the location of  $r_\sigma$  in an arbitrary fixed update order of the other agents in  $R$ . There are at least  $|\mathcal{F}_{\ell_\sigma}^{(R_{off})}|/\Delta$  elements that are the first updating agent in some set  $\rho \in \mathcal{F}_{\ell_\sigma}^{(R_{off})}$  and  $r_\sigma$  can update before at most  $c_{\max}/w_{\min}$  of them in order for  $\ell_\sigma$  to have a chance to turn *off*, so we can bound the probability that  $\sigma$  is uncovered by  $r_\sigma$  by:

$$\Pr[\sigma \in \mathcal{F}_{r_\sigma}^* \mid \mathcal{E}] \leq \frac{c_{\max}/w_{\min} + 1}{|\mathcal{F}_{\ell_\sigma}^{(R_{off})}|/\Delta + 1},$$

Hence, by union bound over all  $r \in \sigma \cap R_{on}$  that could uncover  $\sigma$  with an *on-off* move,

$$\Pr[\sigma \in \cup_{r \in (\sigma \cap R_{on})} \mathcal{F}_r^* \mid \mathcal{E}] \leq (F_{\max} - 1) \cdot \frac{c_{\max}/w_{\min} + 1}{|\mathcal{F}_{\ell_\sigma}^{(R_{off})}|/\Delta + 1} = O\left(\frac{c_{\max}/w_{\min} + 1}{|\mathcal{F}_{\ell_\sigma}^{(R_{off})}|/\Delta + 1}\right).$$

Now let  $\mathcal{F}_\ell^{(R)} \subseteq \mathcal{F}$  be the sets containing  $\ell \in L$  and at least one element of  $R$ . Note that  $\beta^{F_{\max}}$  is a lower bound on the probability for any  $\sigma$  that all  $\sigma \cap R$  followed  $s^{ad}$  and turned *off* in their last update in Phase 1, and there is a subset of  $\mathcal{F}_\ell^{(R)}$  of size at least  $\frac{|\mathcal{F}_\ell^{(R)}|}{\Delta(F_{\max}-1)}$  with disjoint elements in  $R$ . Thus we can argue that given  $\mathcal{E}$ , the random variable  $|\mathcal{F}_\ell^{(R_{off})}|$  has (first-order) dominance over the binomial random variable  $X \sim$

$B\left(\frac{|\mathcal{F}_\ell^{(R)}|}{\Delta(F_{\max}-1)}, \beta^{F_{\max}}\right)$ . Using this, we have

$$\begin{aligned}
\mathbb{E}[|\mathcal{F}^{(L_{on})}| \mid \mathcal{E}] &\leq \sum_{\ell \in L} \sum_{\sigma \in \mathcal{F}_\ell^{(R)}} \Pr[\sigma \in \cup_{r \in (\sigma \cap R_{on})} \mathcal{F}_r^* \mid \mathcal{E}] \\
&\leq \sum_{\ell \in L} \sum_{\sigma \in \mathcal{F}_\ell^{(R)}} O\left(\mathbb{E}\left[\frac{\frac{c_{\max}}{w_{\min}} + 1}{|\mathcal{F}_\ell^{(R_{off})}|/\Delta + 1}\right]\right) \\
&\leq \sum_{\ell \in L} \sum_{\sigma \in \mathcal{F}_\ell^{(R)}} O(\Delta) \cdot \mathbb{E}\left[\frac{1}{X+1}\right] \\
&\leq \sum_{\ell \in L} \sum_{\sigma \in \mathcal{F}_\ell^{(R)}} O(\Delta) \cdot O\left(\frac{\Delta(F_{\max}-1)}{|\mathcal{F}_\ell^{(R)}|}\right) \\
&= O(\Delta^2 \cdot |L|), \tag{23}
\end{aligned}$$

where the third inequality uses  $\beta = \Theta(1)$  and the fact that  $\mathbb{E}[1/(1+Y)] \leq \frac{1}{np}$  for binomial random variable  $Y \sim B(n, p)$ . Combining this fact with (20) and (22) we derive the desired conclusion.  $\square$

## 5. DISCUSSION

In recent years, game-theoretic frameworks have provided informative models for analyzing the outcomes of games among autonomous agents or components programmed as autonomous agents. However, many games, including those studied in this paper, often suffer from high Price of Anarchy, meaning that without a central authority it is hard to induce a state with low social cost. In this paper we study how weak broadcasting signals from a central authority are enough to induce states with low social cost in a general class of covering problems, and we show that even stronger guarantees are possible for carefully chosen broadcasting signals.

In the case of the vertex cover setting, where all sets are of size two, our results are essentially tight. Furthermore, such a setting arises in practical wireless sensing networks [Zalyubovskiy et al. 2009], as we mentioned earlier. In the more general set cover setting, we still get strong results assuming constant size sets, although our results may not be tight. An additional benefit of our constant set size assumption, i.e.,  $F_{\max} = O(1)$ , is that it allows us to give a poly-time procedure for both computing a good advice strategy and then letting the dynamics converge to an equilibrium that is within the  $O(\log n)$  factor of optimal. We remark that were it not for the constant set size assumption, this result would be optimal, since Raz and Safra [1997] show that finding an  $o(\log n)$ -approximation of the general set cover problem is NP-hard. Since there exists a poly-time algorithm for  $O(\log n)$ -approximation of the general set cover problem [Chvatal 1979], it is conceivable that different analysis permitting arbitrary set sizes and possibly using a different characterization of a good advice strategy could give this optimal result. This is indeed an interesting open question: for arbitrary  $F_{\max}$ , does the proposed dynamics in this paper converge to an equilibrium that is within the  $O(\log n)$  factor of optimal? If the answer is yes, it can lead to an alternative  $O(\log n)$ -approximation algorithm for the set cover problem. Even if the answer is no, we believe that our results are still valuable since our dynamics are simple, distributed, and provide useful insights on how advertising can circumvent a bad price of anarchy.

Related work subsequent to the conference version of this paper has analyzed similar settings with variants on the dynamics and games studied here (see, e.g., [Piliouras et al. 2012; Jin et al. 2013]). Our techniques may continue to be of broader interest for analyzing other classic optimization problems in a distributed fashion.

## 6. ACKNOWLEDGMENTS

This work was supported in part by NSF grants CCF-1101215 and CCF-0953192, ONR grant N00014-09-1-0751, by AFOSR grant FA9550-09-1-0538, and by FA9550-09-1-0420. We would like to thank the anonymous referees for their helpful comments.

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**A. TECHNICAL PROPOSITION**

PROPOSITION A.1. *For any  $a \in (0, 1)$  and  $0 < c \leq d$ ,*

$$\sum_{i=0}^{\lfloor c \rfloor} d \binom{d}{i} (1-a)^{d-i} a^i = O\left(\frac{\lceil c \rceil}{a^2(1-a)^2}\right).$$

PROOF. This is immediate in the case that  $c < 1$  because  $d(1-a)^d = O(1/a)$  for all  $d \geq 0$  as long as  $a \in (0, 1)$ . Hence, assume  $c \geq 1$ . Let  $\bar{a} = \max(a, 1-a)$  and define  $\xi \in (0, 1)$  to be the largest real number satisfying

$$(e/\xi)^\xi < \sqrt{1/\bar{a}},$$

where it is not hard to check that  $\xi = \Omega((1-\bar{a})^2)$ . For the case with  $d \leq c/\xi$ ,  $c \leq d$  gives

$$\sum_{i=0}^{\lfloor c \rfloor} d \binom{d}{i} (1-a)^{d-i} a^i \leq d \sum_{i=0}^d \binom{d}{i} (1-a)^{d-i} a^i = d \leq c/\xi = O(c/(1-\bar{a})^2).$$

Now consider when  $d > c/\xi$ . Observe that

$$d \sum_{i=0}^{\lfloor c \rfloor} \binom{d}{i} (1-a)^{d-i} a^i \leq d \cdot \bar{a}^d \sum_{i=0}^{\lfloor c \rfloor} \binom{d}{i} \leq d \cdot \bar{a}^d \sum_{i=0}^{\lfloor c \rfloor} \frac{d^i}{i!}$$

Further, we have

$$\begin{aligned} d \cdot \bar{a}^d \sum_{i=0}^{\lfloor c \rfloor} \frac{d^i}{i!} &= O(1/(1-\bar{a})) \cdot \bar{a}^{d/2} \sum_{i=0}^{\lfloor c \rfloor} \frac{d^i}{i!} \\ &= O(c/(1-\bar{a})) \cdot \bar{a}^{d/2} \cdot \frac{d^{\lfloor c \rfloor}}{\lfloor c \rfloor!} \\ &= O(c/(1-\bar{a})) \cdot \bar{a}^{d/2} \left(\frac{d \cdot e}{\lfloor c \rfloor}\right)^{\lfloor c \rfloor} \\ &= O(c/(1-\bar{a})) \cdot \bar{a}^{d/2} \left(\frac{d \cdot e}{\xi \cdot d}\right)^{\xi \cdot d} \\ &= O(c/(1-\bar{a})) \cdot \bar{a}^{d/2} \cdot \bar{a}^{-d/2} \\ &= O(c/(1-\bar{a})), \end{aligned}$$

where we use (a)  $d \cdot \bar{a}^{d/2}$  is  $O(1/(1-\bar{a}))$ , (b)  $d^i/i!$  is increasing with respect to  $i$  for  $i < c < d$ , (c)  $x! = \Omega((x/e)^x)$ , (d)  $c < \xi \cdot d$  and (e) the definition of  $\xi$ . This completes the proof of Proposition A.1.  $\square$