Some key concepts explained

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Use a distribution table to compute a probability

Let $X \sim \mathrm{N}(\mu, \sigma)$ with μ, σ known and $a, b \in \mathrm{R}$.

❒ **Question:** Compute $P(a \leqslant X \leqslant b)$.

❒ **Step 1 — Standardize** X

We introduce Z, such that

$$Z = \frac{X - \mu}{\sigma} \sim \mathrm{N}(0, 1)$$

❒ **Step 2 — Express the probability in terms of** Z

We have:

$$P(a \leqslant X \leqslant b) = P(\frac{a - \mu}{\sigma} \leqslant \underbrace{\frac{X - \mu}{\sigma}}_{Z} \leqslant \frac{b - \mu}{\sigma}) = P\left(Z \leqslant \frac{b - \mu}{\sigma}\right) - P\left(Z \leqslant \frac{a - \mu}{\sigma}\right)$$

❒ **Step 3 — Find each term using the distribution table**

Given that the values of $\frac{a-\mu}{\sigma}$ and $\frac{b-\mu}{\sigma}$ are known, we just have to look them up in a distribution table similar to this one.

Summing up We just computed the value of the probability by standardizing the normal variable to be able to look up the values in a standard normal distribution table.

Confidence intervals

Compute the confidence interval for μ

Note: the example below is specific to the case where the variance is known and n is large. The following reasoning can be reproduced for other cases in a similar fashion.

Let $X_1, ..., X_n$ be a random sample with mean μ and standard deviation σ where **only** σ is **known**, and let $\alpha \in [0, 1]$.

❐ **Question:** Compute a confidence interval on μ with confidence level $1 - \alpha$, that we note $CI_{1-\alpha}$.

❐ Step 1 — Write in mathematical terms what we are seaching for

We want to find a confidence interval $CI_{1-\alpha}$ of confidence level $1 - \alpha$ for μ:

$$P(\mu \in CI_{1-\alpha}) = 1 - \alpha$$

❐ Step 2 — Consider the sample mean of X

We consider $\overline{X}$, which is such that:

$$\overline{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$$

❒ Step 3 — Standardize $\overline{X}$

We introduce Z, such that:

$$Z = \frac{\overline{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \underset{n \gg 1}{\sim} \mathrm{N}(0, 1)$$

In general, this relationship is valid for large n but it is always true in the particular case when the X_i are normal.

❒ Step 4 — Use Z to find the quantiles

We can find the quantiles of Z which are such that:

$$P\left(-z_{\frac{\alpha}{2}} \leqslant Z \leqslant z_{\frac{\alpha}{2}}\right) = 1 - \alpha$$

Given that Z follows a standard normal distribution, the quantity $z_{\frac{\alpha}{2}}$ can be found in the distribution table.

❒ Step 5 — Re-write Z in terms of $\overline{X}$

Knowing that $Z = \frac{\overline{X}-\mu}{\frac{\sigma}{\sqrt{n}}}$, we can re-write the previous expression:

$$P\left(\overline{X} - z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} \leqslant \mu \leqslant \overline{X} + z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}\right) = 1 - \alpha$$

❒ **Step 6 — Deduce the confidence interval**

By taking into account steps 1 and 5, we can now deduce the confidence interval for μ:

$$\boxed{CI_{1-\alpha} = [\overline{X} - z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}, \overline{X} + z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}]}$$

Compute the confidence interval for σ^2

Let $X_1, ..., X_n$ be a random sample with mean μ and standard deviation σ where σ is **unknown**, and let $\alpha \in [0, 1]$.

❒ **Question:** Compute a confidence interval on σ^2 with confidence level $1 - \alpha$, that we note $CI_{1-\alpha}$.

❒ **Step 1 — Write in mathematical terms what we are seaching for**

We want to find a confidence interval $CI_{1-\alpha}$ of confidence level $1 - \alpha$ for σ^2:

$$P(\sigma^2 \in CI_{1-\alpha}) = 1 - \alpha$$

❒ **Step 2 — Consider the sample variance of X**

We consider s^2, which is such that:

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$$

❐ Step 3 — Standardize s^2

We introduce K, such that:

$$K = \frac{s^2(n-1)}{\sigma^2} \sim \chi^2_{n-1}$$

Here, K follows a χ^2 distribution with $n-1$ degrees of freedom.

❐ Step 4 — Use K to find the quantiles

We can find the quantiles χ^2_1, χ^2_2 of K which are such that:

$$P(\chi^2_1 \leqslant K \leqslant \chi^2_2) = 1 - \alpha$$

Given that K follows a χ^2 distribution with $n-1$ degrees of freedom, the quantiles can be found in the distribution table.

❐ Step 5 — Re-write K in terms of s^2

Knowing that $K = \frac{s^2(n-1)}{\sigma^2}$, we can re-write the previous expression:

$$P\left(\frac{s^2(n-1)}{\chi^2_2} \leqslant \sigma^2 \leqslant \frac{s^2(n-1)}{\chi^2_1}\right) = 1 - \alpha$$

❐ Step 6 — Deduce the confidence interval

By taking into account steps 1 and 5, we can now deduce the confidence interval for σ^2:

$$\boxed{CI_{1-\alpha} = [\frac{s^2(n-1)}{\chi_2^2}, \frac{s^2(n-1)}{\chi_1^2}]}$$