Eletrónica de Sistemas de Comunicações
Communication Systems Electronics

Filters

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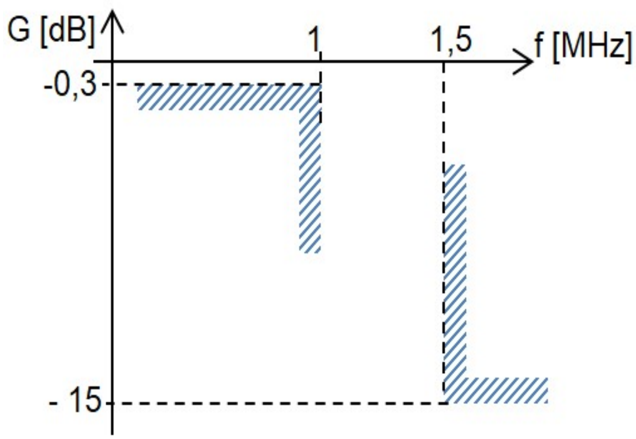
Mestrado em Engenharia Eletrotécnica e de Computadores / Física

Syllabus

- General characteristics and specifications
- Aproximation functions
- Frequency transformations
- Synthesis: passive and active implementations

Summary

- Learning outcomes and competences
  - Know the different filter implementation technologies and be able to design and implement a filter circuit after given specifications.



It is required to implement the filter characterized by the transfer characteristic show in the figure.

- Determine the filter transfer function considering a Chebyshev approximation.

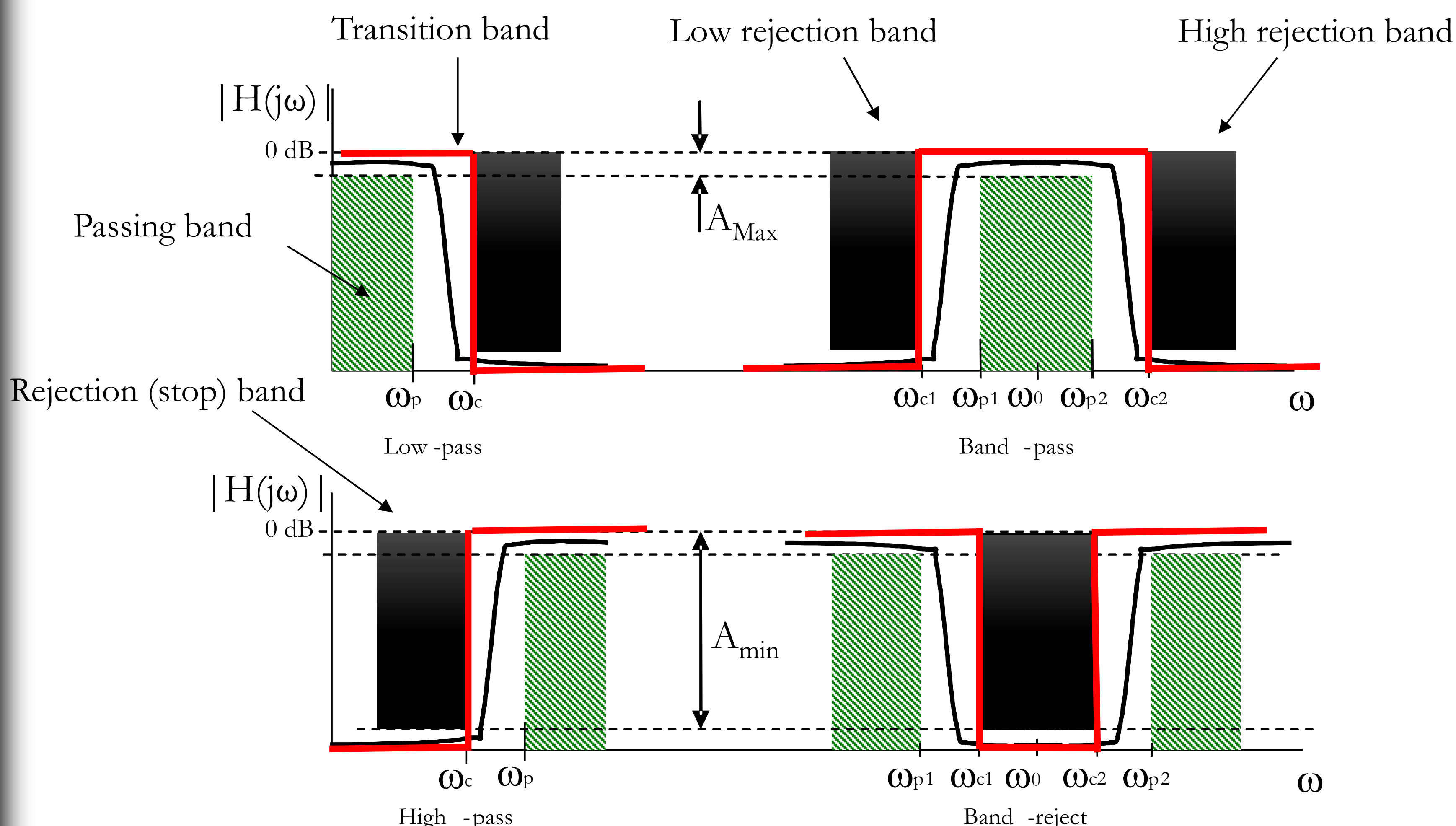
- Design a passive implementation of this filter. The load resistance is $R_L = 250\ \Omega$.

Filters - introduction

- What is a filter?
  - A functional block that processes signals in a form that depends on their frequency content.
  - In general, a filter affects not only the magnitude but also the phase of the signal.
  - The filter performance over frequency is (almost) fully characterized by its transfer function.
  - The frequency response can be represented formally, in a mathematical or graphical form, in magnitude and phase ($|H(j\omega)|$ and $\underline{/H(j\omega)}$)

Filters

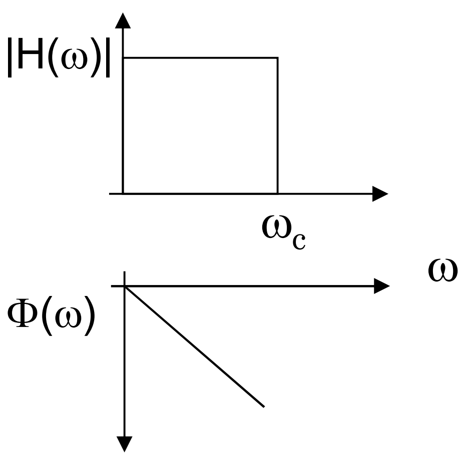
- Specifications using the magnitude response as reference

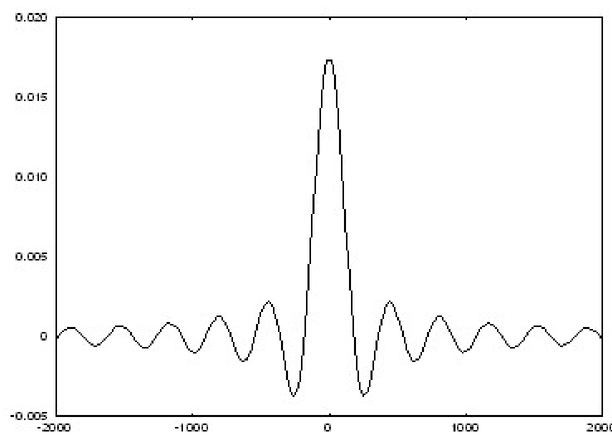


Filters – formal representation

- Ideal filter, an impossibility

$$H(j\omega)=\begin{cases}1 & 0\le|\omega|\le\omega_c\\ 0 & |\omega|>\omega_c\end{cases}\rightarrow h(t)=\mathrm{F}^{-1}\left[H(j\omega)\right]=\begin{cases}\dfrac{\omega_c}{\pi} & t=0\\[2ex] \dfrac{\omega_c}{\pi}\dfrac{\sin(\omega_c t)}{\omega_c t} & t\neq 0\end{cases}$$





h(t) is defined over all time, which is a non-causal realization.

<u>What to do?</u>

Find an H(s) that best fits the specifications:

1 – Feasible

2 – Minimum resources

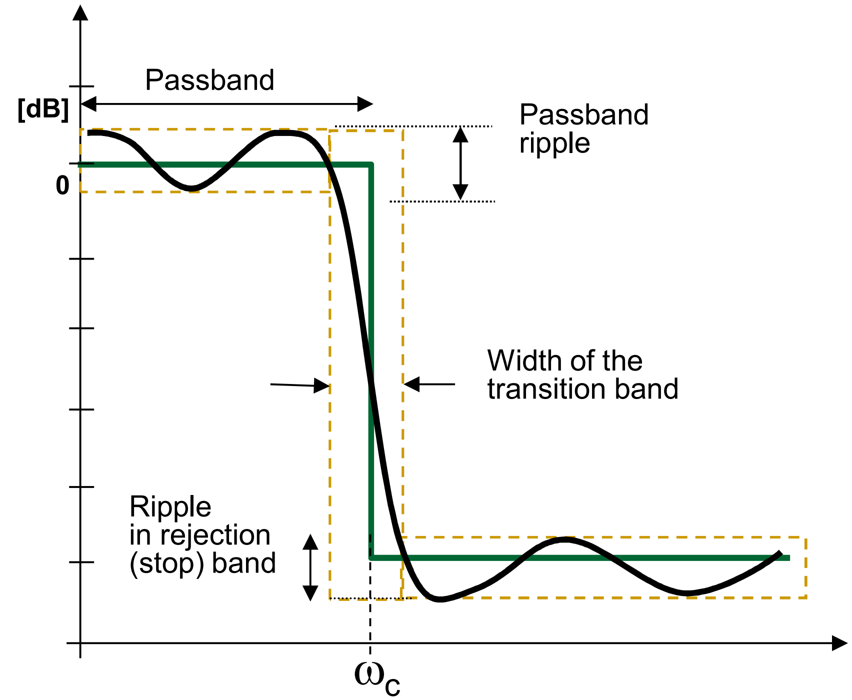
Filters – Specifications

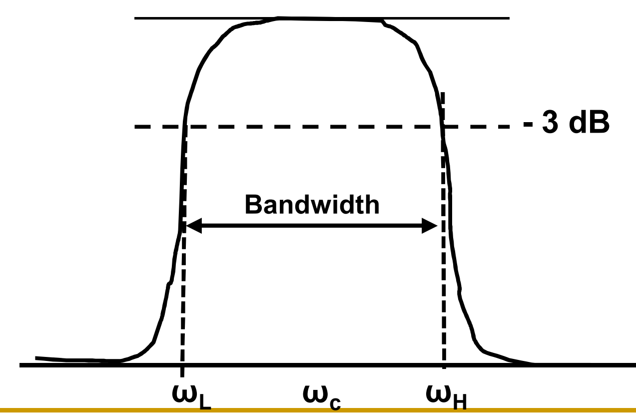
Filters' specifications:

- **Attenuation** *(A)*: Relative output amplitude (in dB) at a given frequency. The reference (0 dB) is usually taken in the point of minimum attenuation, or at a specific frequency value.
- **Frequency bandwidth** (BW)**:** difference between the higher and lower frequencies in the passing band – usually taken at the points of -3 dB gain (or 3 dB attenuation).
- **Central frequency (ω_c)**: arithmetic or geometric mean of -3dB frequencies.

$$f_{c\_ari} = \frac{f_1 + f_2}{2}; \qquad f_{c\_geo} = \sqrt{f_1 \times f_2}$$

- **Quality factor** (Q): central frequency to bandwidth ratio.
- **Ripple** (dB): difference between minimum valley and maximum peak attenuations in the bandwidth under consideration.

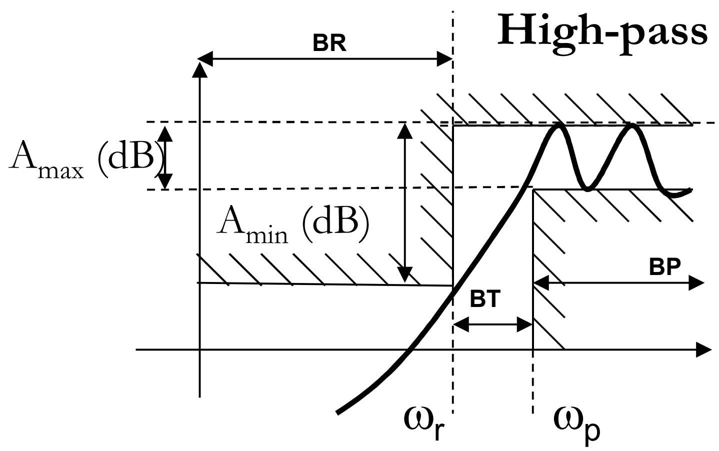


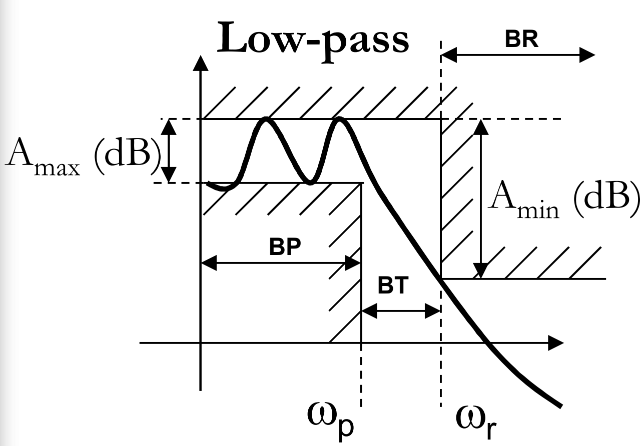


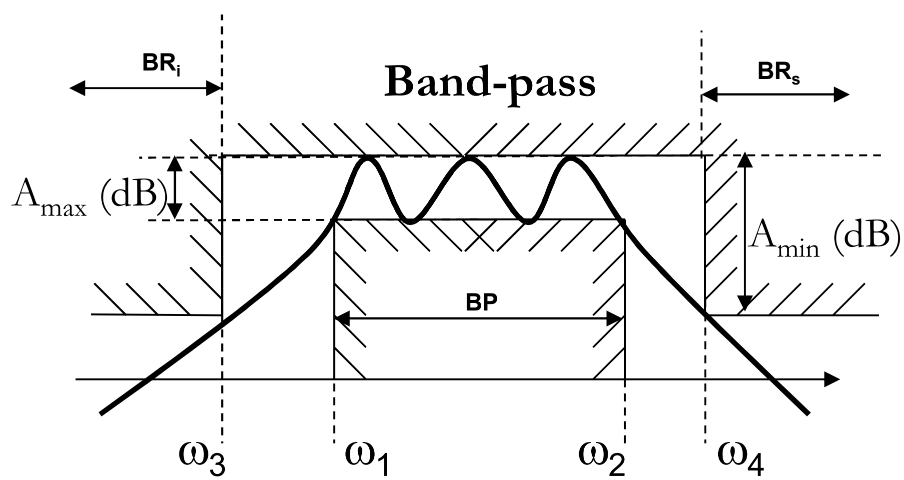
Filters – Specification

- **Transition band** (BT) – width of the band defined by the pass and the rejection (stop) band limits
- **Maximum attenuation in the passing band** (A_{max}, dB)
- **Minimum attenuation in the rejection band** (A_{min}, dB)

  These represent distances between the reference (maximum of $|H(j\omega)|$) and the respective amplitudes at the edge of pass and stop bands.

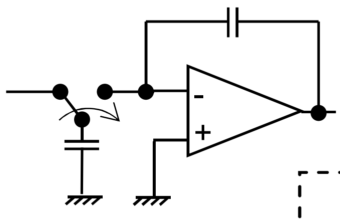


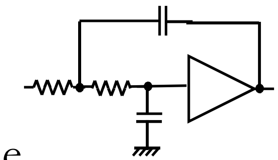




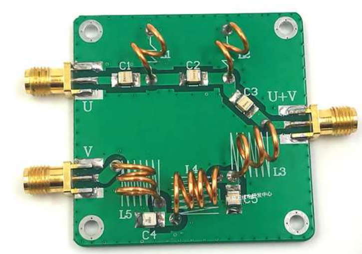
Filters - implementation

- Signal processing





V segment DC-185MHZ
U segment 350-560MHZ

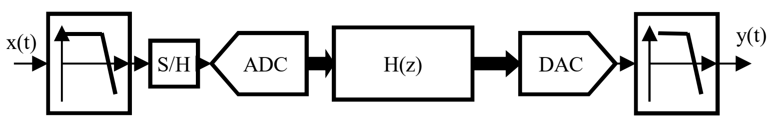


Continuous amplitude

Clocked analogue Switched-capacitor or switched-current filters	**Analogue** Analogue filters Amplifiers
Digital Filters implemented with microprocessors or reconfigurable logic	**Analogue** Comparators Asynchronous digital Communication systems Action potentials

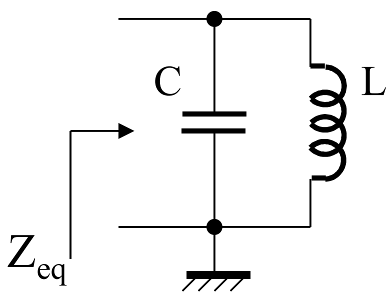
Discrete time ← → Continuous time

Discrete amplitude



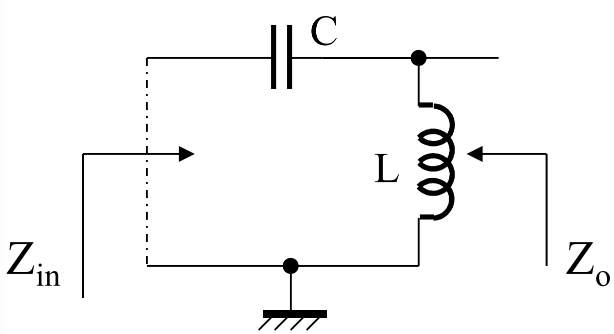
Filters – 2nd order resonance circuits

- LC resonance



$$Z_{eq}(s) = \frac{s/C}{s^2+\omega_o^2}\bigg|_{s=j\omega} = \frac{j\omega/C}{-\omega^2+\omega_o^2}\bigg|_{\omega=\omega_o} \rightarrow Z_{eq}(j\omega_o) = \infty$$

$$\omega_o = \frac{1}{\sqrt{LC}} \longrightarrow \textbf{Resonance frequency}$$

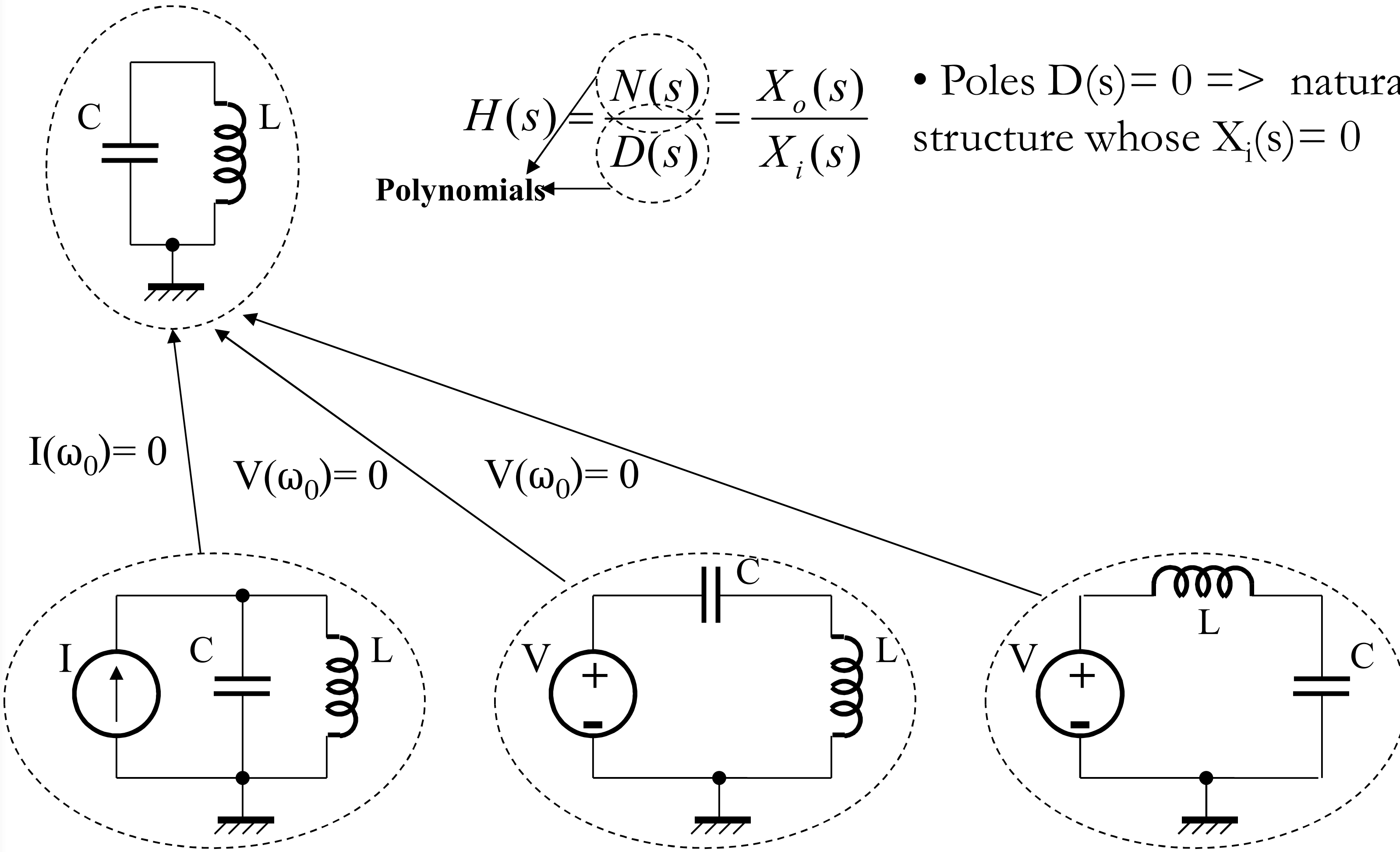


$$Z_{in}(j\omega) = \frac{-\omega^2+\omega_o^2}{j\omega/L}\bigg|_{\omega=\omega_o} \longrightarrow Z_{in}(j\omega_o) = 0$$

$$Z_o(j\omega) = Z_{eq}(j\omega)\Big|_{\omega=\omega_o} \longrightarrow Z_o(j\omega_o) = \infty$$

Filters – 2nd order resonance circuits

- LC resonance



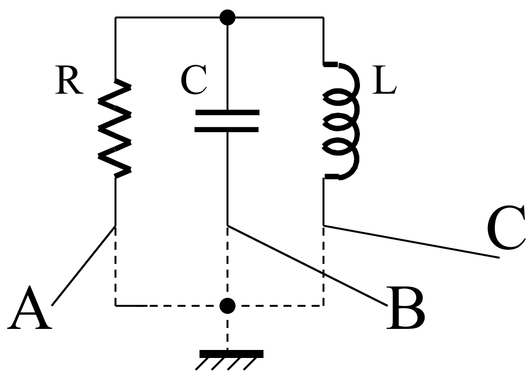
$$H(s) = \frac{N(s)}{D(s)} = \frac{X_o(s)}{X_i(s)}$$

Polynomials

- Poles D(s)= 0 => natural structure whose $X_i(s)$= 0

Filters – 2nd order resonance circuits

- RLC resonance

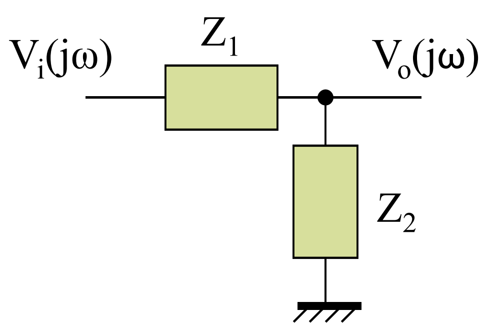


$$D(s) = s^2 + s\frac{1}{RC} + \frac{1}{LC} = s^2 + s\frac{\omega_0}{Q} + \omega_o^2$$

$$\omega_o = \frac{1}{\sqrt{LC}}$$

$$Q = \omega_o RC$$

- Synthesis of LP, HP, BP and BR after voltage dividers:

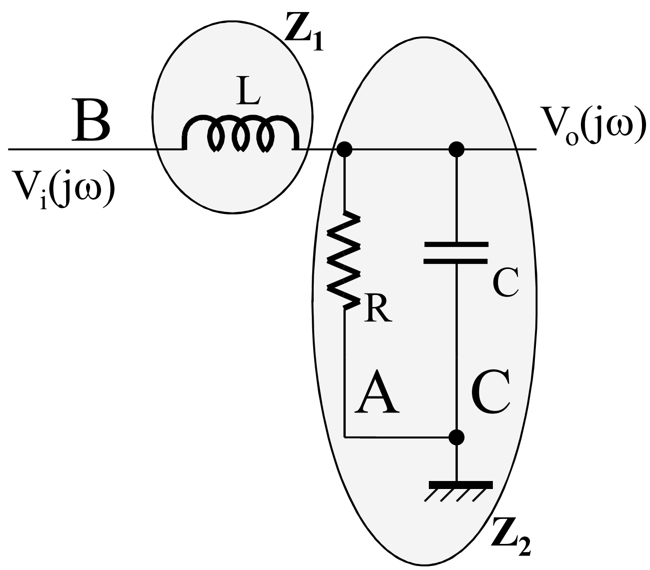


$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{Z_2}{Z_1 + Z_2}$$

Filters – passive

- Low-pass filter without “notch” frequency – two zeros at infinite frequency.

The zeros are $Z_1 \xrightarrow{X_L} \infty \wedge Z_2 \xrightarrow{X_C} 0$ when $\boxed{s \to \infty}$.
These are the only zeros.



$$H(s) = \frac{\omega_o^2}{s^2 + \frac{\omega_o}{Q}s + \omega_o^2} = \frac{\frac{1}{LC}}{s^2 + \frac{1}{RC}s + \frac{1}{LC}}$$

Exercise: Obtain the expression for Q

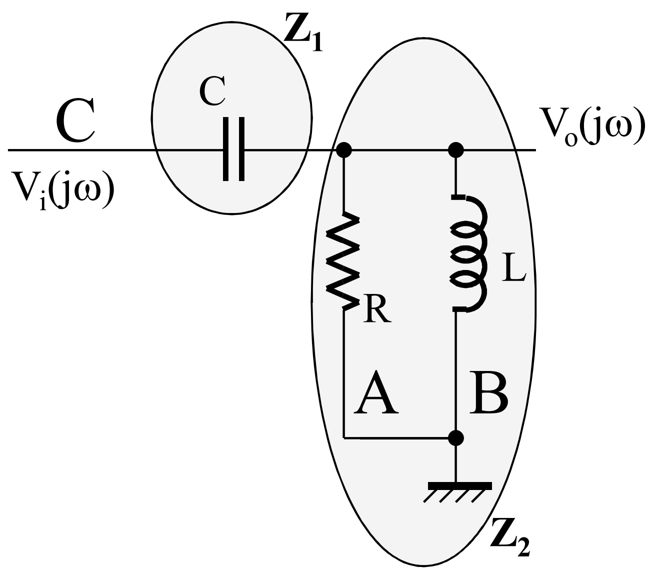
$$\omega_{-3dB} = \omega_o \left[1 - \frac{1}{2Q^2} + \sqrt{\frac{1}{4Q^4} - \frac{1}{Q^2} + 2}\right]^{1/2}$$

$$\omega_{-3dB}|_{Q=0,7} = \omega_o;\ \omega_{-3dB}|_{Q=1} = 0,64\omega_o;$$

Filters – passive

- High-pass filter without "notch" frequency - <u>2 zeros</u> at the origin.

The zeros are $\underset{\nearrow \mathbf{X_C}}{Z_1} \to \infty \wedge \underset{\nearrow \mathbf{X_L}}{Z_2} \to 0$ when $\boxed{s \to 0}$.
These are the only zeros.



$$H(s) = \frac{s^2}{s^2 + \frac{\omega_o}{Q}s + \omega_o^2} =$$

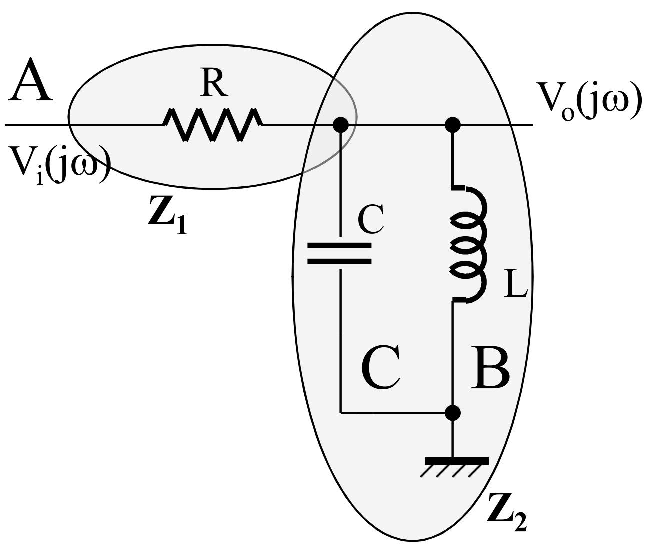
$$= \frac{s^2}{s^2 + \frac{1}{RC}s + \frac{1}{LC}}$$

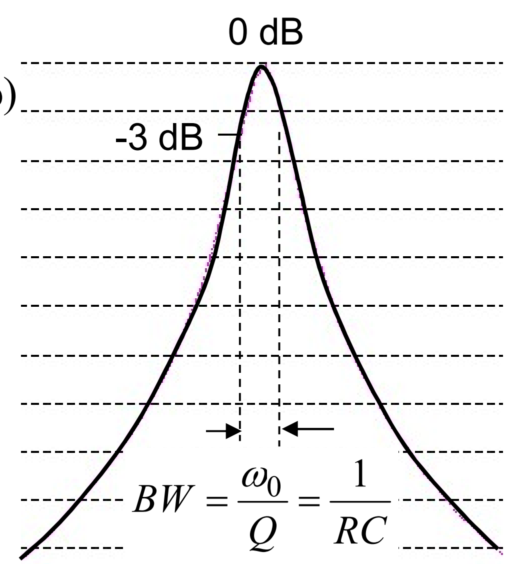
Exercise:
Find the expression for Q

Filters – passive

- Band-pass filter – 1 finite zero (origin) and another one in the infinite.

Two zeros given by Z_2, $\boxed{s \to 0 \Rightarrow L; \quad s \to \infty \Rightarrow C}$



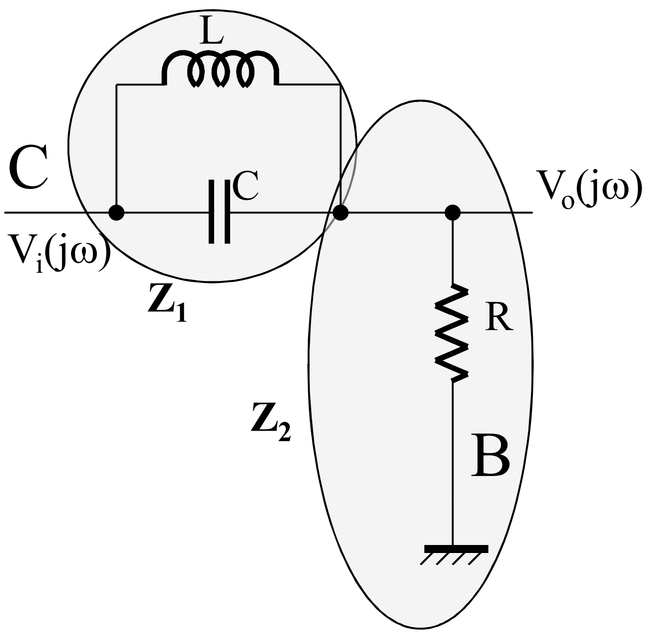


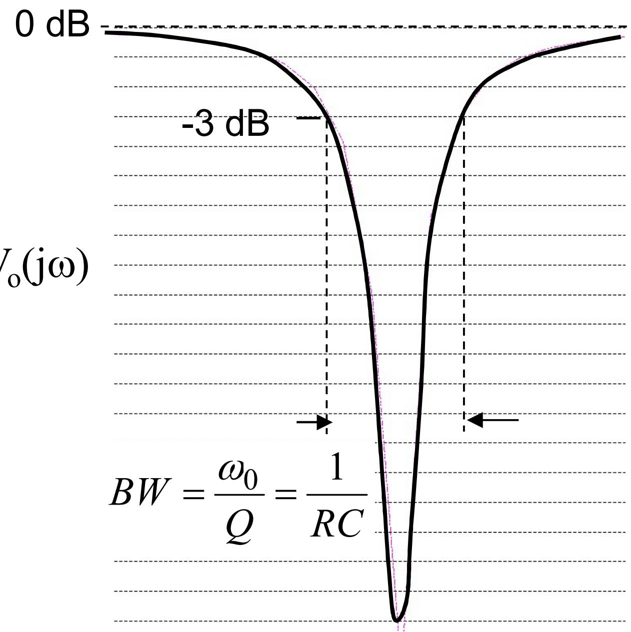
$$H(s) = \frac{\frac{\omega_o}{Q} s}{s^2 + \frac{\omega_o}{Q} s + \omega_o^2} = \frac{1}{RC} \frac{s}{s^2 + \frac{1}{RC} s + \frac{1}{LC}}$$

Filters – passive

- Band-reject filter – 2 finite zeros at $\pm j\omega_o$.

Two zeros introduced by Z_1, $\quad s \rightarrow \pm j\omega_0 \Rightarrow L//C$





$$BW = \frac{\omega_0}{Q} = \frac{1}{RC}$$

$$H(s) = \frac{s^2 + \omega_o^2}{s^2 + \frac{\omega_o}{Q}s + \omega_o^2} = $$

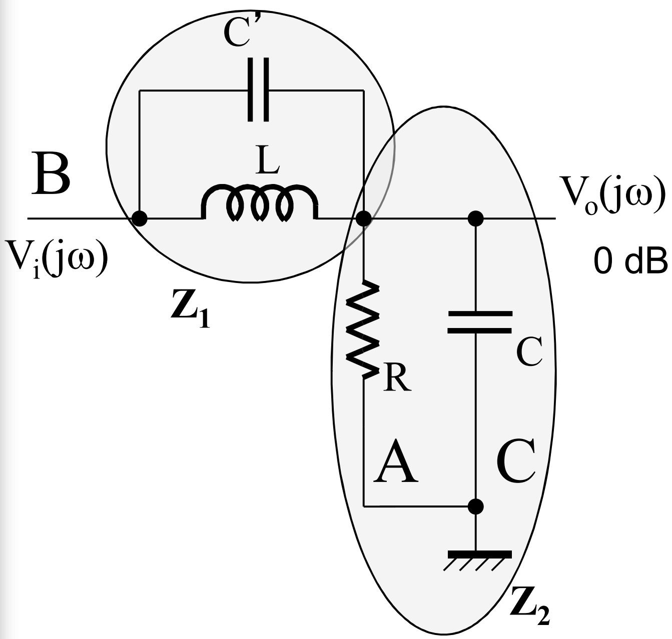
$$= \frac{s^2 + \frac{1}{LC}}{s^2 + \frac{1}{RC}s + \frac{1}{LC}}$$

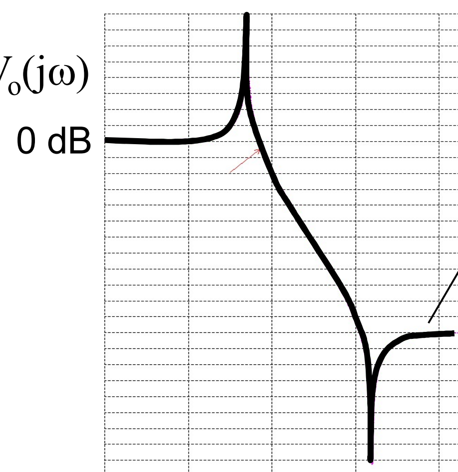
Filters – passive

- Low-pass filter with "notch" frequency – 2 finite zeros in the jω axis.

For the parallel LC $\Rightarrow Z_{eq} = \infty; \quad \omega = \omega_o$ and $Z_1 \to \infty$ for

$$\boxed{\begin{array}{l} s_{zero} = \pm j\omega_n \\ \omega_n > \omega_o \end{array}}$$





L//C'

$$H(s) = \boxed{\frac{C'}{C'+C}} \frac{s^2 + \omega_n^2}{s^2 + \frac{\omega_o}{Q}s + \omega_o^2}$$

$$\begin{cases} \omega_o = \dfrac{1}{\sqrt{(C+C')\times L}} \\ \omega_n = \dfrac{1}{\sqrt{L \times C'}} \end{cases} \qquad \omega_n > \omega_o$$

Filters – passive

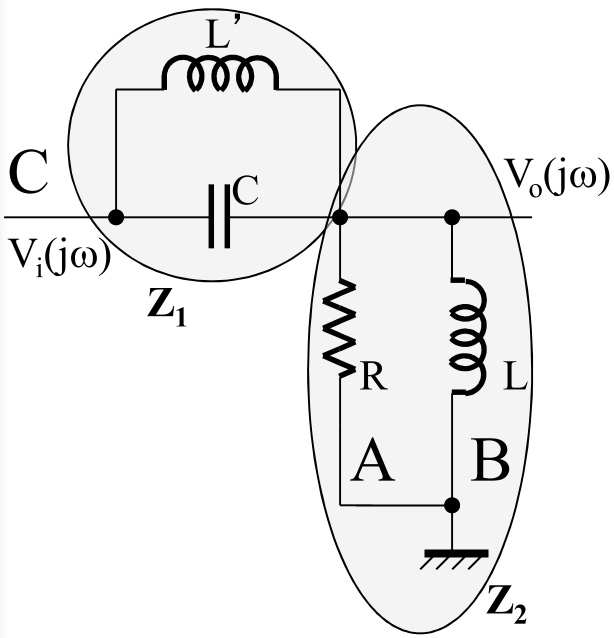
- High-pass filter with "notch" frequency – 2 finite zeros.

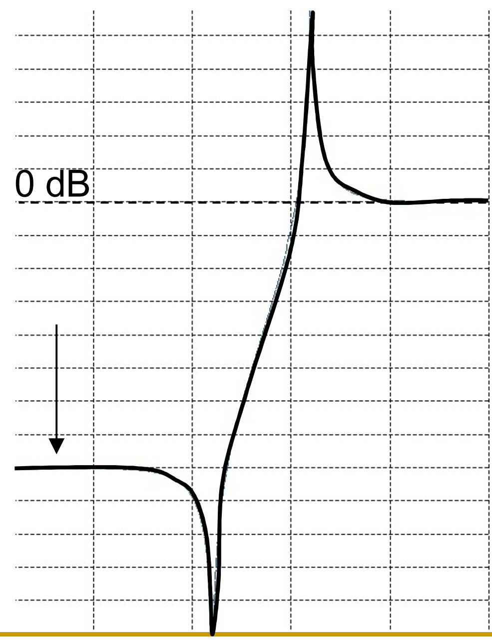
For the parallel LC $\Rightarrow Z_{eq} = \infty; \quad \omega = \omega_o$ and $\quad Z_1 \to \infty$ for

L' //C

$$\boxed{\begin{array}{l} s_{zero} = \pm j\omega_n \\ \omega_n < \omega_o \end{array}}$$

$$H(s) = \frac{L}{L+L'} \frac{s^2 + \omega_n^2}{s^2 + \frac{\omega_o}{Q}s + \omega_o^2}$$

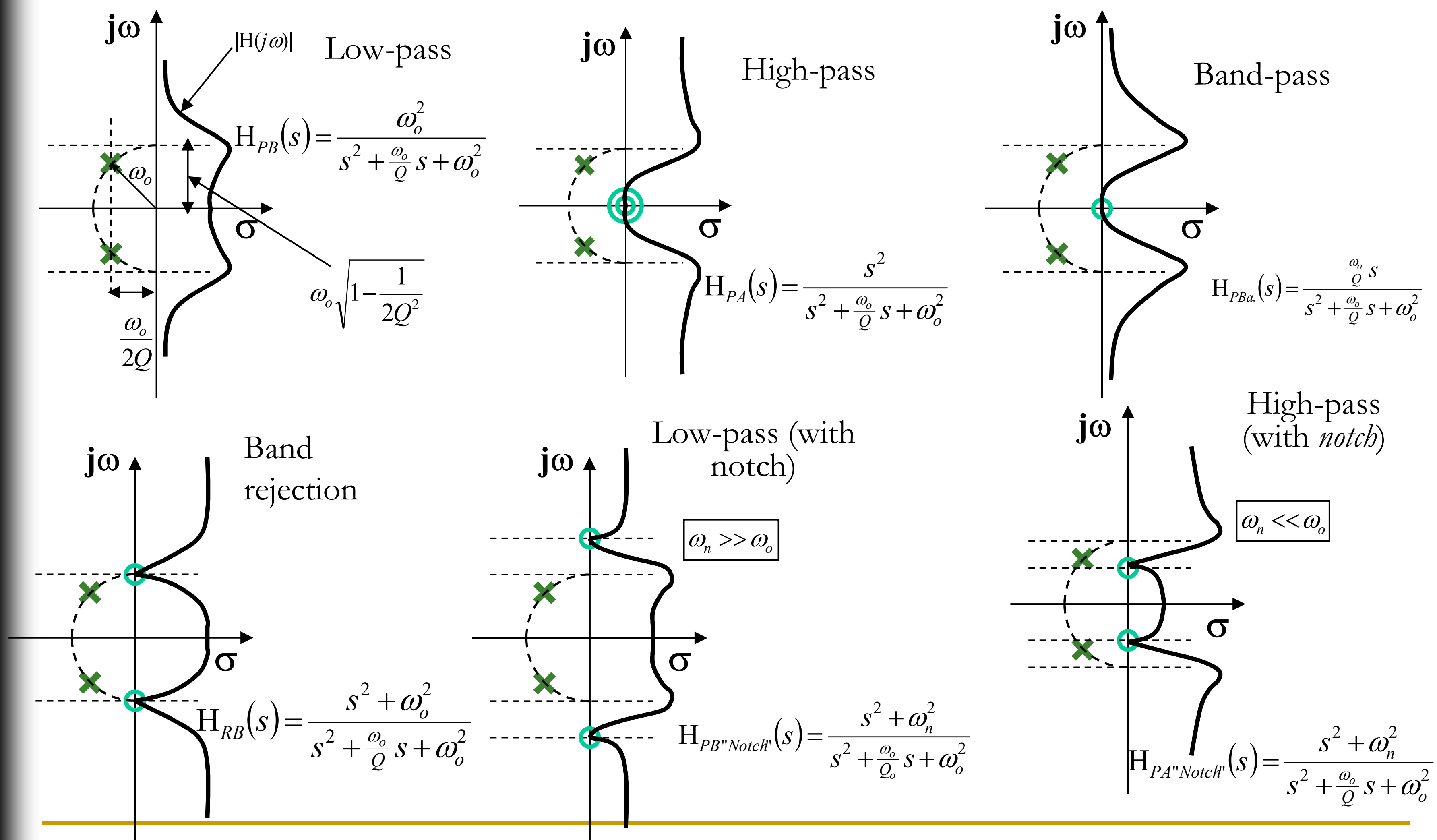




$$\left\{\begin{array}{l} \omega_o = \dfrac{1}{\sqrt{L // L' \times C}} \\ \omega_n = \dfrac{1}{\sqrt{L' \times C}} \end{array}\right. \qquad \omega_n < \omega_o$$

Filters – typical pole/zero locations: 2nd order

A sketch of the magnitude response measured in s=jω (along the imaginary axys, logarithmic scales) is shown as well.



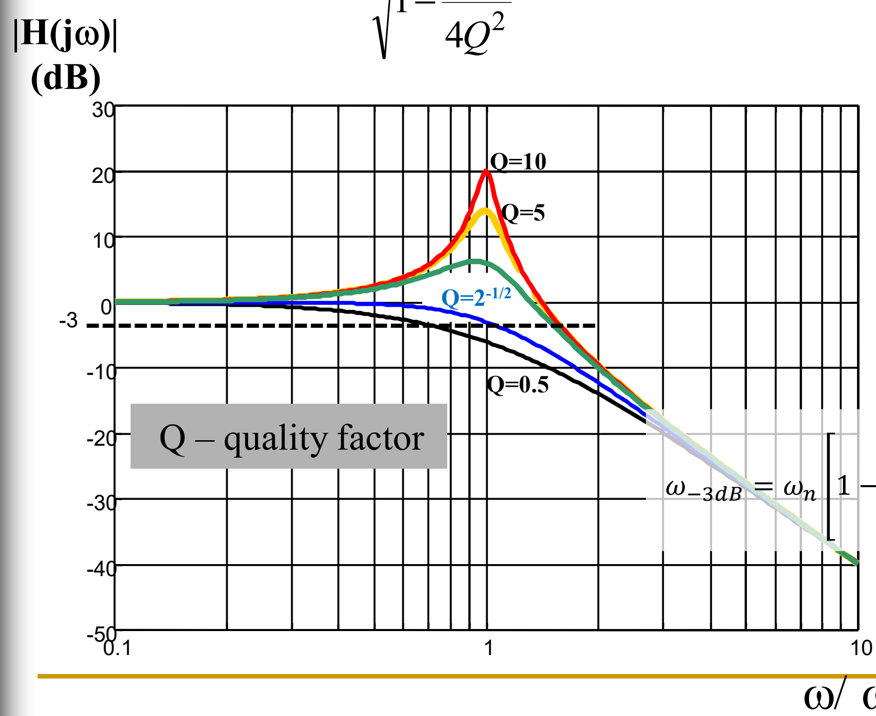
Filters – low-pass, 2nd order

$$H(s)=\frac{\omega_0^2}{D(s)}=\frac{\omega_0^2}{s^2+s\frac{\omega_0}{Q}+\omega_0^2}$$

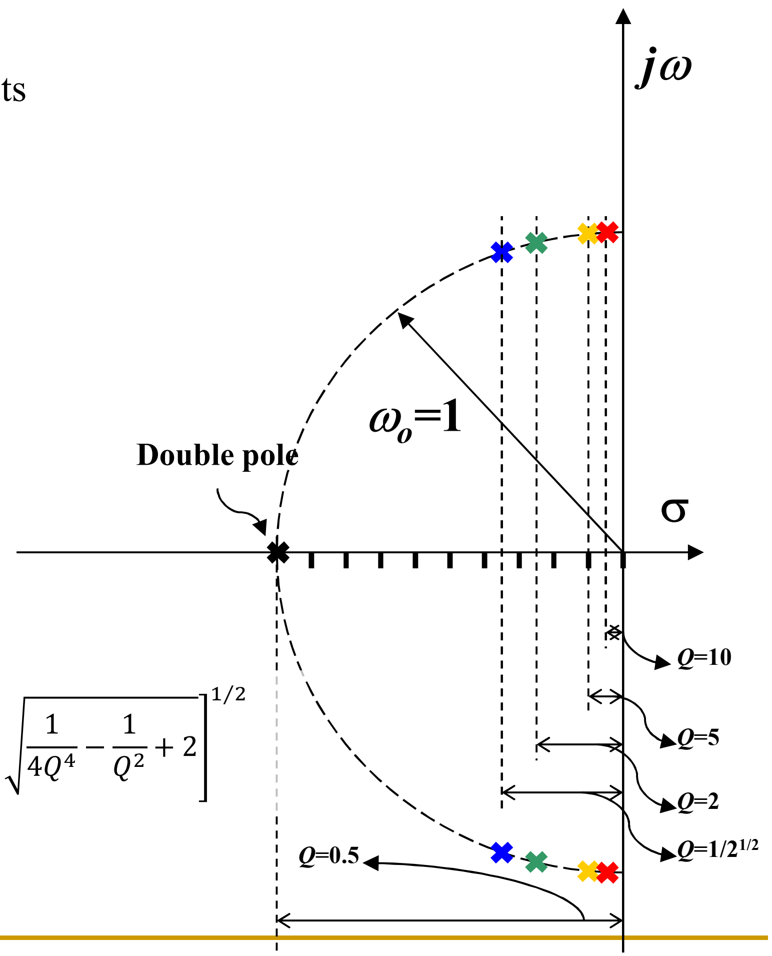
$$p_1,p_2=\alpha\pm j\beta=-\frac{\omega_o}{2Q}\pm j\omega_o\sqrt{1-\frac{1}{4Q^2}}$$

$$\omega_0=\sqrt{\alpha^2+\beta^2};\ Q=\frac{\sqrt{\alpha^2+\beta^2}}{2\alpha};\ Q>1/2\rightarrow\text{complex roots}$$

$$\left|H(j\omega_{\max})\right|_{\max}=\frac{Q}{\sqrt{1-\frac{1}{4Q^2}}};\quad \omega_{\max}=\omega_o\sqrt{1-\frac{1}{2Q^2}}$$



$$\omega_{-3dB}=\omega_n\left[1-\frac{1}{2Q^2}+\sqrt{\frac{1}{4Q^4}-\frac{1}{Q^2}+2}\right]^{1/2}$$



Filters

- In real applications Q varies between 0.5 and 100, being values around 1 the commonest ones.

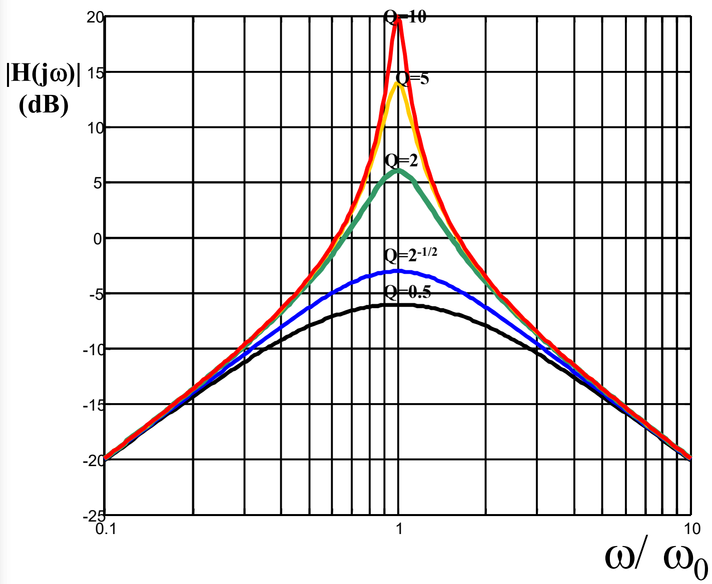
- In particular $Q = {}^{1}/_{\sqrt{2}} = 0{,}707$ corresponds to the case of maximum constant gain in the pass-band (*maximum flat*). For $Q > {}^{1}/_{\sqrt{2}}$ (poles close to the imaginary axis) the frequency response shows resonant peaks (under-damped time response).

- Filters with $Q > {}^{1}/_{\sqrt{2}}$ are found in the synthesis of formants to simulate the vocal tract (or musical instruments), as well as to generate electronic effects in music.

- The same is applicable to high-pass filters.

Filters – bandpass

$H(j\omega)_{PBand}$ shows a peak at the natural frequency $\omega = \omega_o$, the resonant frequency

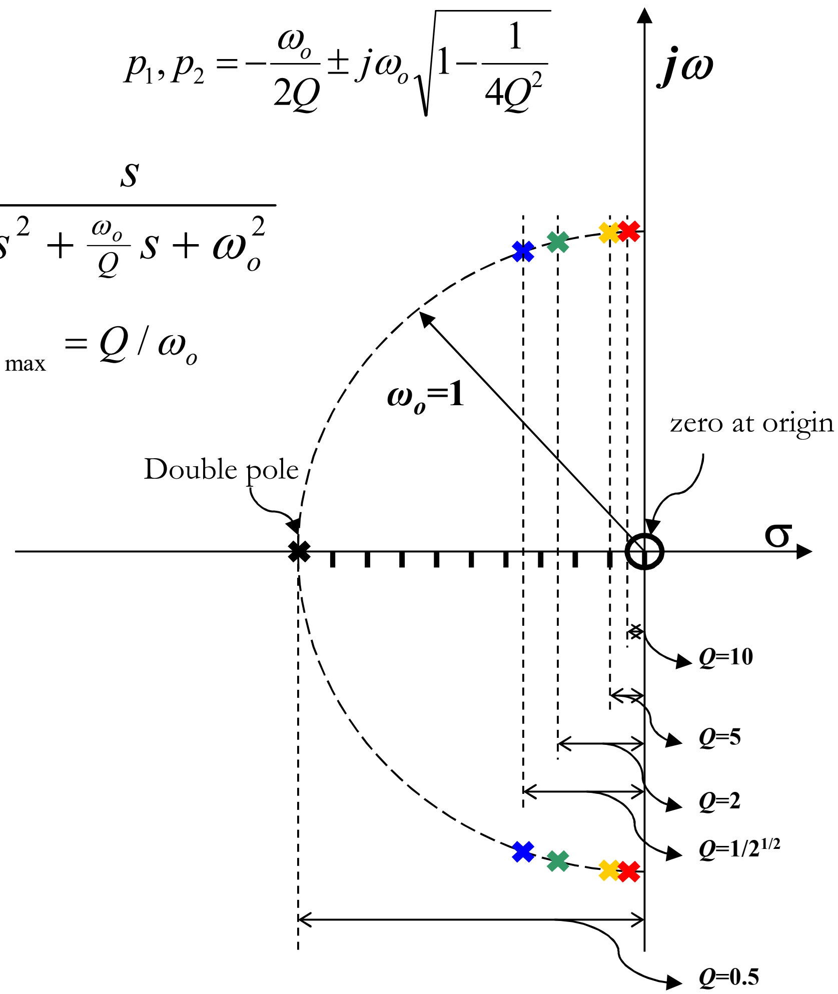


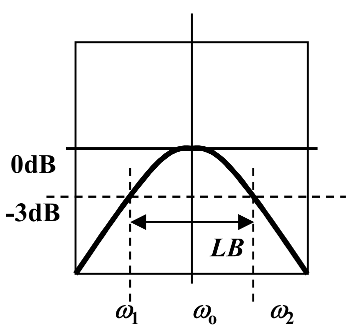
$$p_1, p_2 = -\frac{\omega_o}{2Q} \pm j\omega_o\sqrt{1-\frac{1}{4Q^2}}$$

$$\mathrm{H}(s) = \frac{s}{s^2 + \frac{\omega_o}{Q}s + \omega_o^2}$$

$$\left|\mathrm{H}(j\omega_{\max})\right|_{\max} = Q/\omega_o$$

$$\omega_{\max} = \omega_o$$





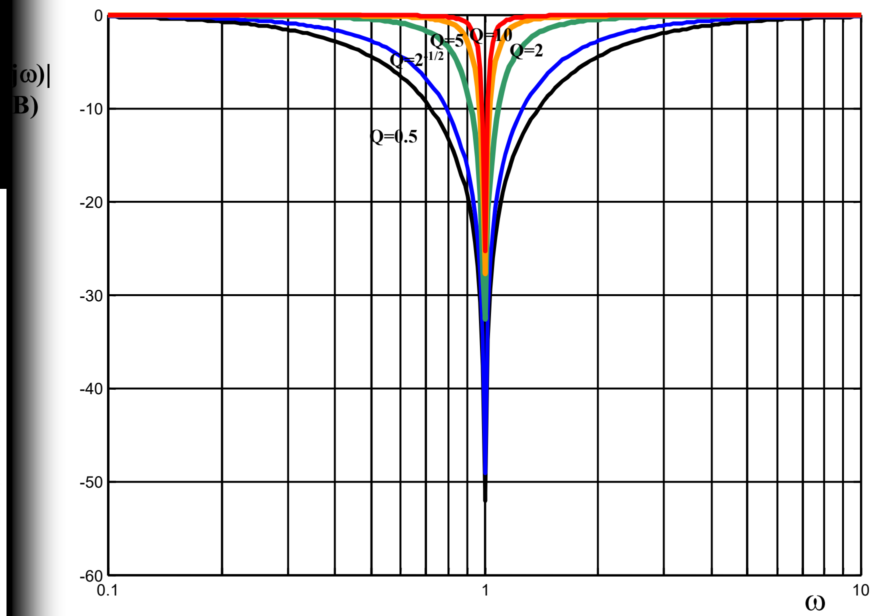
$$BW = \omega_2 - \omega_1$$

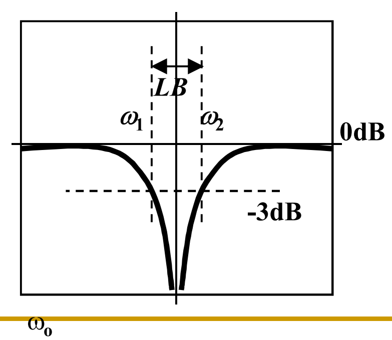
$$\omega_o = \sqrt{\omega_1 \times \omega_2}$$

$$Q = \frac{\omega_o}{BW}$$

Filters – band rejection

$\omega = \omega_o$, notch frequency





$$BW = \omega_2 - \omega_1$$

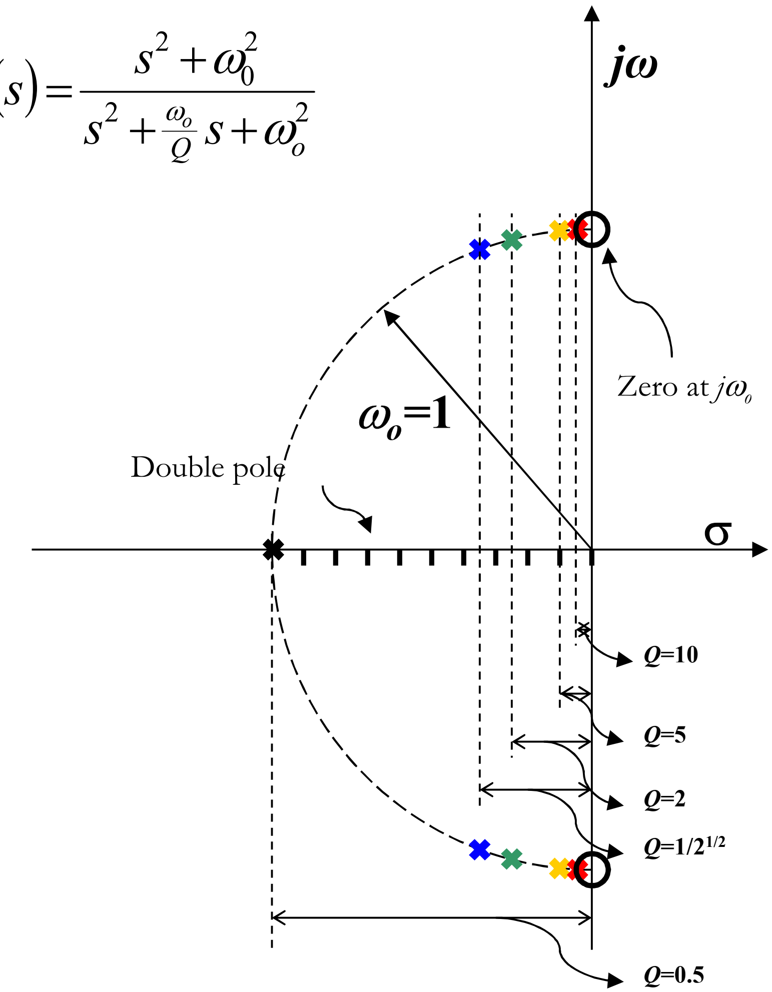
$$\omega_o = \sqrt{\omega_1 \times \omega_2}$$

$$Q = \frac{\omega_o}{BW}$$

Note that:

$$\boxed{H(s)_{RB} = H(s)_{LP} + H(s)_{HP} = 1 - H(s)_{PB}}$$

$$\mathrm{H}_{SBa}(s) = \frac{s^2 + \omega_0^2}{s^2 + \frac{\omega_o}{Q}s + \omega_o^2}$$



Second-order filters

- Second-order filters are important blocks. A combination of them allows to implement higher order filters relatively easily.

$$H(s) = \frac{\boldsymbol{k_1} s^2 + \boldsymbol{k_2} s + \boldsymbol{k_3}}{s^2 + \frac{\omega_o}{Q} s + \omega_o^2}$$

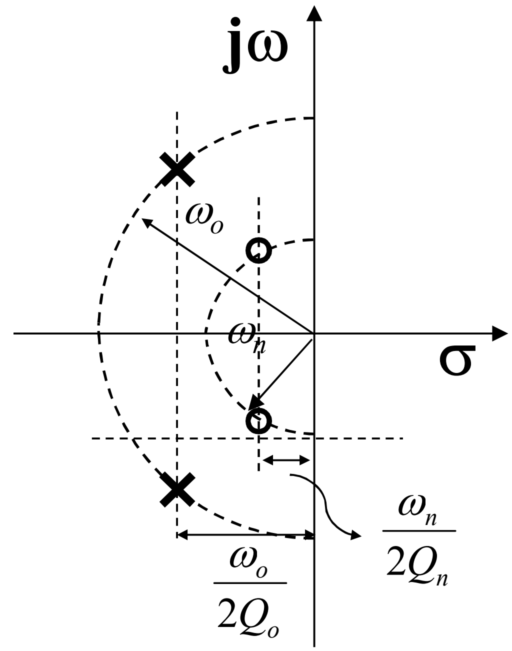
- Low-pass – $k_1 = k_2 = 0$
- High-pass – $k_2 = k_3 = 0$
- Band-pass – $k_3 = k_1 = 0$
- Reject-band – $k_2 = 0$
- All-pass – $k_1 = 1;\ k_2 = -\frac{\omega_0}{Q};\ k_3 = \omega_0^2$

Filters – 2nd order behaviour, conclusions

- General form (bi-quadratic equation)

$$H(s) = K\frac{(s-z)(s-z^*)}{(s-p)(s-p^*)}$$

$$\Leftrightarrow H(s) = K\frac{s^2 + \frac{\omega_n}{Q_n}s + \omega_n^2}{s^2 + \frac{\omega_o}{Q_o}s + \omega_o^2}$$



- |H(jω)| is maximum for $\omega \approx \omega_0$
  - The maximum width narrows with an increasing Q_0
- |H(jω)| is minimum for $\omega \approx \omega_n$
  - The minimum width narrows with an increasing Q_n

Filters – 2nd order behaviour, group delay - τ

- General form:

$$H(s)=K\frac{s^2+\frac{\omega_n}{Q_n}s+\omega_n^2}{s^2+\frac{\omega_o}{Q_o}s+\omega_o^2}$$

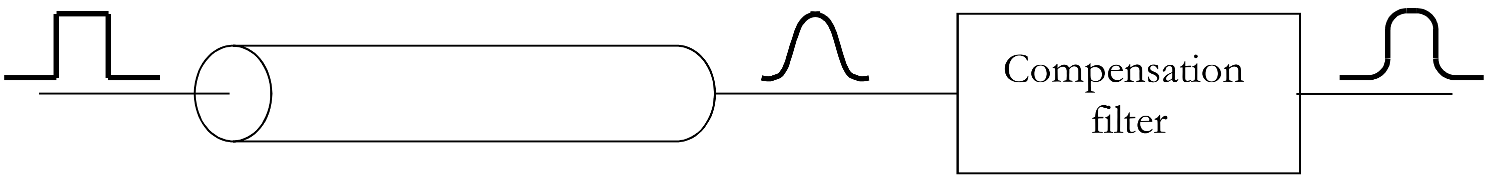
$$\Phi(j\omega)=\arctan\left(\frac{\frac{\omega_n}{Q_n}\omega}{\omega_n^2-\omega^2}\right)-\arctan\left(\frac{\frac{\omega_o}{Q_o}\omega}{\omega_o^2-\omega^2}\right)$$

$$\tau=\frac{d(-\Phi(j\omega))}{d\omega}=-\frac{\frac{\omega_n}{Q_n}\left(\omega_n^2+\omega^2\right)}{\left(\omega_n^2-\omega^2\right)^2+\left(\frac{\omega_n}{Q_n}\omega\right)^2}+\frac{\frac{\omega_o}{Q_o}\left(\omega_o^2+\omega^2\right)}{\left(\omega_o^2-\omega^2\right)^2+\left(\frac{\omega_o}{Q_o}\omega\right)^2}$$

Group Delay is used as a criterion to evaluate phase nonlinearity. A linear phase variation with frequency (over a band of frequencies) implies a constant Group Delay and no phase distortion in that frequency band. In order to preserve the integrity of a pulse through a system, it is mandatory that the system's <u>group delay is constant</u> up to the pulse's maximum frequency component.

Filters – delay equalizers

- Delay equalizers compensate for linear distortion due to systems' nonlinear phase response.



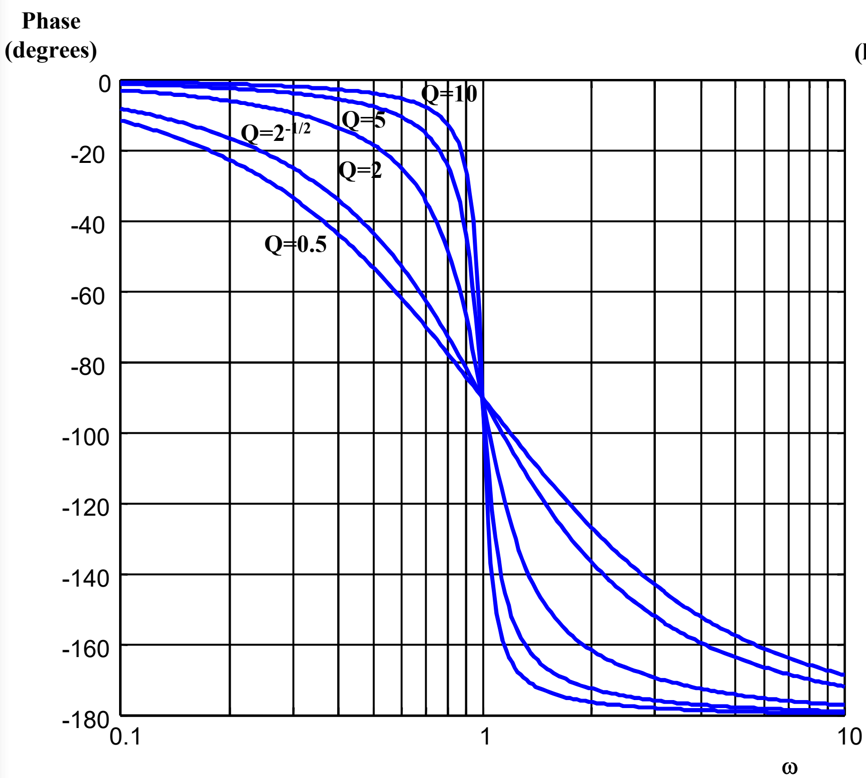
- Ideal phase (linear) characteristic ($\angle H(j\omega) = -\omega\tau_0$) results in a constant time delay ($\tau_0 \Rightarrow x_o(t) = x(t-\tau_0)$)

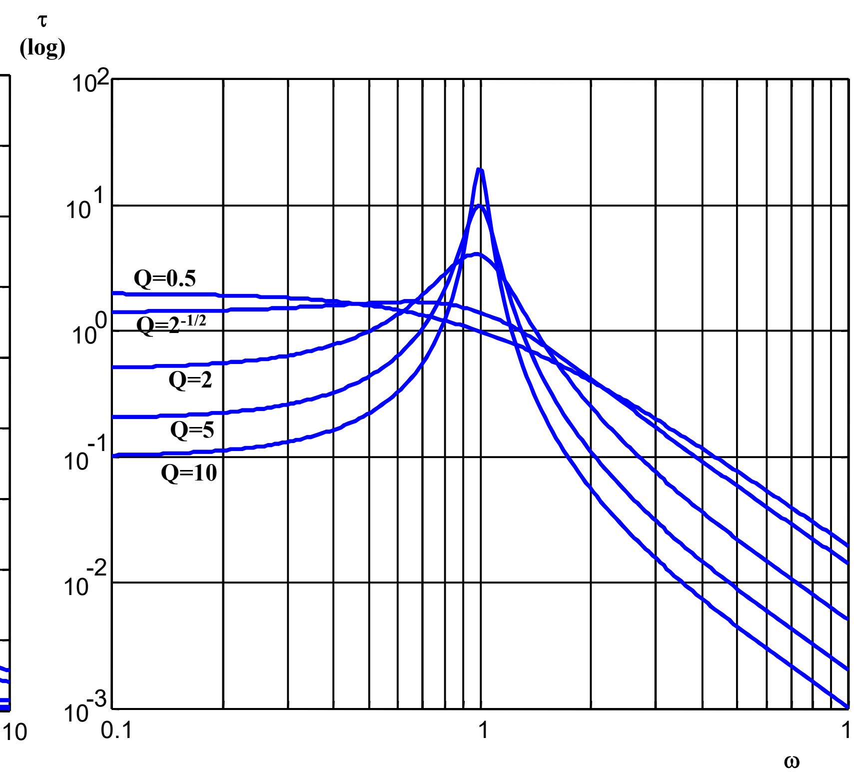
- Group delay (or simply delay) – parameter used to analyse systems' phase linearity, is defined as:

$$\tau = \frac{d\left(-\Phi(\omega)\right)}{d\omega}; \qquad \Phi(\omega) = \angle H(j\omega)$$

Filters – 2nd order behaviour, group delay

- Phase and group delay of a 2nd order low-pass filter

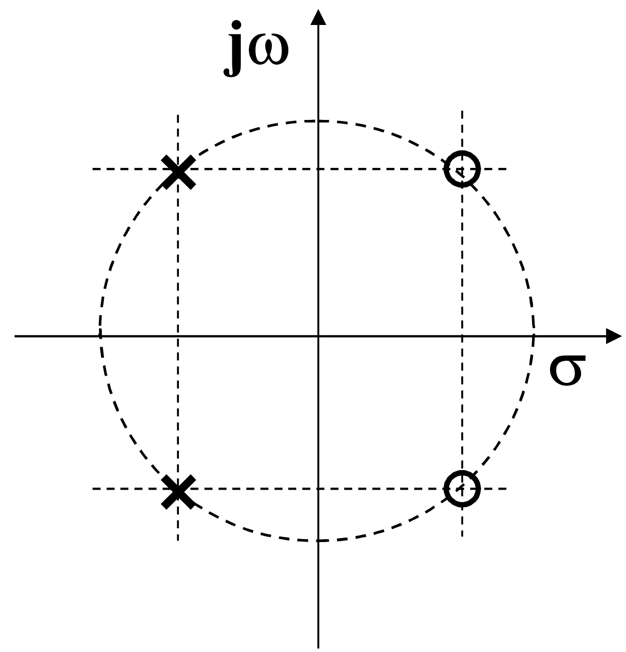




Filters – group delay compensation

- Use an extra filter, flat in the magnitude response, but with a phase delay adequate for compensating the group delay in the bandwidth of interest.

- The solution is an all-pass filter.



$$H(s) = K\frac{s^2 - \frac{\omega_o}{Q_o}s + \omega_o^2}{s^2 + \frac{\omega_o}{Q_o}s + \omega_o^2}$$

$$\Phi(j\omega) = -2\arctan\left(\frac{\frac{\omega_o}{Q_o}\omega}{\omega_o^2 - \omega^2}\right)$$

$$\tau = \frac{d(-\Phi(j\omega))}{d\omega} = 2\frac{\frac{\omega_o}{Q_o}\left(\omega_o^2 + \omega^2\right)}{\left(\omega_o^2 - \omega^2\right)^2 + \left(\frac{\omega_o}{Q_o}\omega\right)^2}$$

Filters – design

- Steps

Specs ⇒ Approximating rational function ⇒ Realizable transfer function ⇒ Synthesis ⇒ Filter circuit

Feasibility conditions ⇒ Approximating rational function

Filters – finding H(s)

- H(s) is a function defined by a set of poles and zeros, strategically placed in the s plane, such that the frequency response (magnitude and eventually also phase) respects the specifications.

$$H(s)=\frac{\mathrm{N}(s)}{\mathrm{D}(s)};\quad \mathrm{O}[\mathrm{N}(s)]\leq \mathrm{O}[D(s)]\Rightarrow$$ Causality and stability

- All H(s) poles have a negative real part
  - ☞ Stability
- Complex poles and zeros occur as conjugate pairs
  - ☞ Real coefficients

Filters – finding H(s)

- The transfer function of any analogue filter (whether active or passive) can be expressed as the ratio of two polynomials – rational function :

$$H(s) = K\frac{N(s)}{D(s)} = K\frac{\prod_{m=0}^{M-1}(s - z_m)}{\prod_{n=0}^{N-1}(s - p_n)}$$

Special case when M=0, all-pole response :

$$H(s) = \frac{K}{D(s)} = \frac{K}{\prod_{n=0}^{N-1}(s - p_n)}$$

Filters – finding H(s)

- The problem consists of finding a function whose characteristics fit the filtering characteristics required by the specifications.

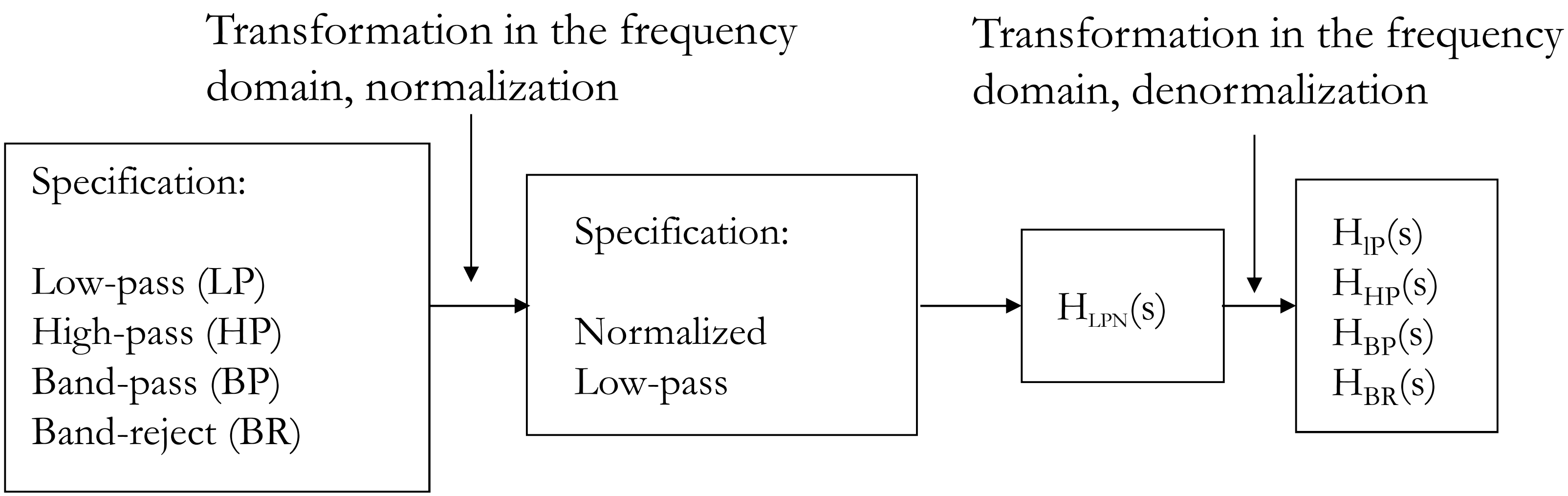
- For that, one resorts to rational functions whose roots are well known.

$$|\mathrm{Attenuation}(j\omega)|^2 = (|\mathrm{H}(j\omega)|^2)^{-1} = 1 + \left|\frac{P(j\omega)}{N(j\omega)}\right|^2 = 1 + |\mathrm{P}(j\omega)|^2\Big|_{|N(j\omega)|^2=1}$$

- In our study: $\mathrm{P}(s) = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$

- The most popular functions are: Butterworth, Chebyshev, Elliptic (Cauer) and Bessel.

Filters – finding H(s)

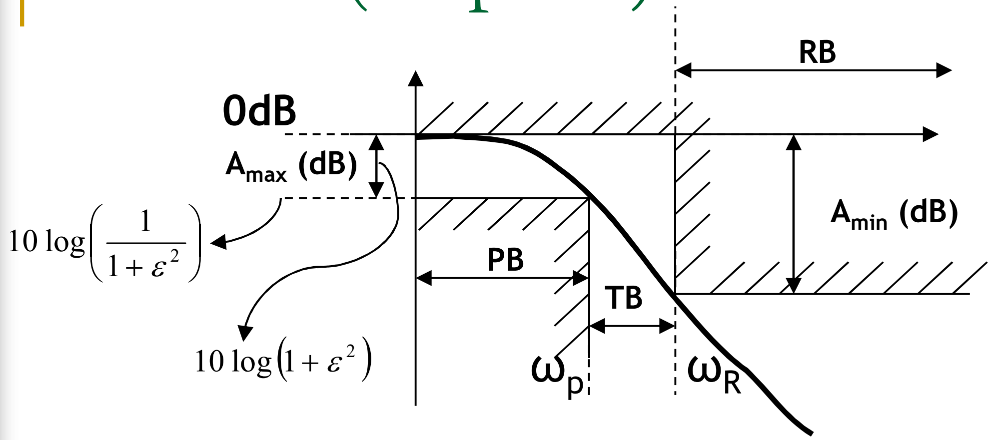


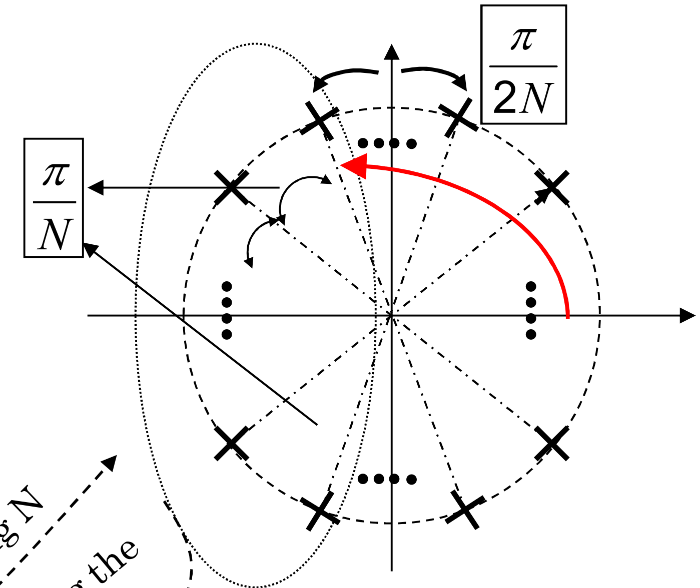
Once the normalized low-pass filter is obtained, the required filter transfer function is obtained after performing the inverse transformation.

All the study of approximation functions for the different filters' topologies, is thus reduced to the study of the normalized low-pass filter.

Normalization stands for a maximum gain in the pass-band of 0 dB, and a unitary (ω_n=1) high cut-off frequency.

Filters – (Stephen) Butterworth approximation





$$H(s)=\frac{\omega_o^N}{(S-p_1)(S-p_2)\ldots(S-p_N)}\approx H(s)=\frac{1}{1+\varepsilon\left(\frac{s}{\omega_p}\right)^N}$$

$$\left|H(s)\right|^2\Big|_{G_0=1}=\frac{1}{1+\varepsilon^2\left(\frac{\omega}{\omega_p}\right)^{2N}}\Rightarrow\left|H(j\omega)\right|=\frac{1}{\sqrt{1+\varepsilon^2\left(\frac{\omega}{\omega_p}\right)^{2N}}}$$

knowing N

after determining the N natural modes

$$p_k=e^{\left[\frac{j\pi}{2}\left(\frac{2k+N-1}{N}\right)\right]}$$
$$k=1,2,\cdots,2N$$

$$\boxed{\varepsilon=\sqrt{10^{0,1A_{\max}}-1}}$$

$$\boxed{A_{\min}=10\log\left(1+\varepsilon^2\left[\frac{\omega_R}{\omega_p}\right]^{2N}\right)}$$

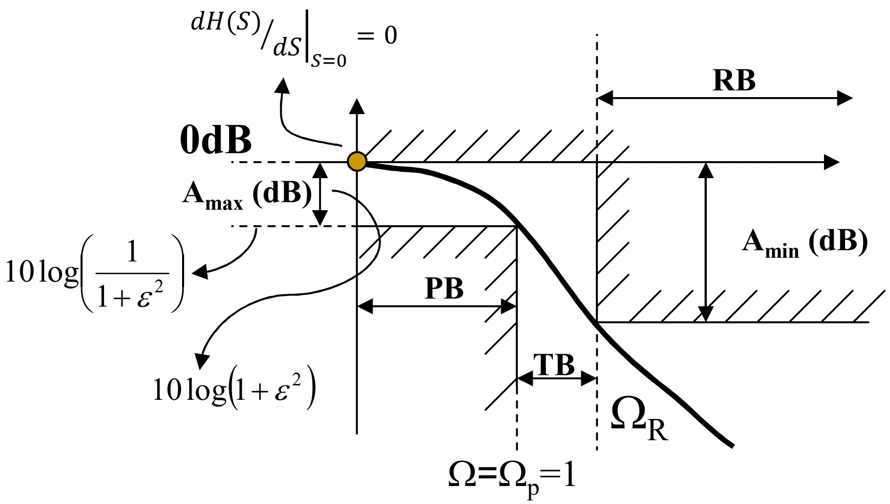
$$\boxed{H(s)_{NORMALIZED}=\frac{1}{\prod_k(S-p_k)}}$$

$$\boxed{S\rightarrow s\left(\frac{\varepsilon^{\frac{1}{N}}}{\omega_p}\right)}$$

$$\boxed{H(s)}$$

Filters – Butterworth approximation

- Butterworth



$$A_{max} = 20\,log\sqrt{1+\varepsilon^2}$$

$\varepsilon = 1$:

$$\Omega = 1,\quad 20\,log\,H_N(\Omega_p) = 20\,log\frac{1}{\sqrt{2}} = -3\;dB$$

$$\Omega < 1,\quad 20\log H_N(\Omega) > 20\log\frac{1}{\sqrt{2}} = -3\;dB$$

$$\Omega > \Omega_r,\quad 20\log H_N(\Omega_r) \le A_{\min}$$

In the stopband:

$$A_{min} \ge 20\log\frac{1}{\sqrt{1+\varepsilon^2\Omega^{2n}}} \rightarrow 10\log(1+\varepsilon^2\Omega^{2n}) \ge A_{min}\ ,\ \varepsilon^2\Omega^{2n} \ge 10^{A_{min}/10} - 1 \rightarrow$$

$$N \ge \frac{\log\left(\frac{10^{\frac{A\min}{10}}-1}{\varepsilon^2}\right)}{2\log\Omega} = \frac{\log\left(\frac{10^{\frac{A\min}{10}}-1}{\varepsilon^2}\right)}{2\log\frac{\omega}{\omega_P}}$$

$$H_N(S_b) = \frac{1}{S_b + a^{n-1}S_{n-1} + \ldots + 1} \quad or \quad |H_N(\Omega_b)| = \frac{1}{\sqrt{1+\varepsilon^2\Omega^{2n}}}$$

Filters – Butterworth approximation

Order, n	$P(S_b)$ Butterworth polynomials (denominator of the transfer function).
1	$1+S_b$
2	$1+\sqrt{2}S_b+{S_b}^2$
3	$(1+S_b)(1+S_b+{S_b}^2)=1+2S_b+2{S_b}^2+{S_b}^3$
4	$({S_b}^2+0{,}7653S_b+1)({S_b}^2+1{,}848S_b+1)=1+2{,}613S_b+3{,}414{S_b}^2+2{,}613{S_b}^3+{S_b}^4$
5	$(1+S_b)({S_b}^2+0{,}618S_b+1)({S_b}^2+1{,}618S_b+1)=1+3{,}236S_b+5{,}236{S_b}^2+5{,}236{S_b}^3+3{,}236{S_b}^4+{S_b}^5$
6	$({S_b}^2+0{,}517S_b+1)({S_b}^2+\sqrt{2}S_b+1)({S_b}^2+1{,}932S_b+1)$

Butterworth poles (normalized low-pass)

Ordem, n	Real part, α	Imaginary part, jβ
2	0,7071	0,7071
3	0,5; 1,0	0,866
4	0,9239; 0,3827	0,3827; 0,9239
5	0,809; 0,309; 1,0	0,5878; 0,9511
6	0,9659; 0,7071; 0,2588	0,2588; 0,7071; 0,9659

Filters – Butterworth approximation

$$H(S_b) = \frac{1}{S_b^{\ n} + a_{n-1}S_b^{\ n-1} + a_{n-2}S_b^{\ n-2} + a_{n-3}S_b^{\ n-3} + \ldots + a_1 S_b + 1}$$

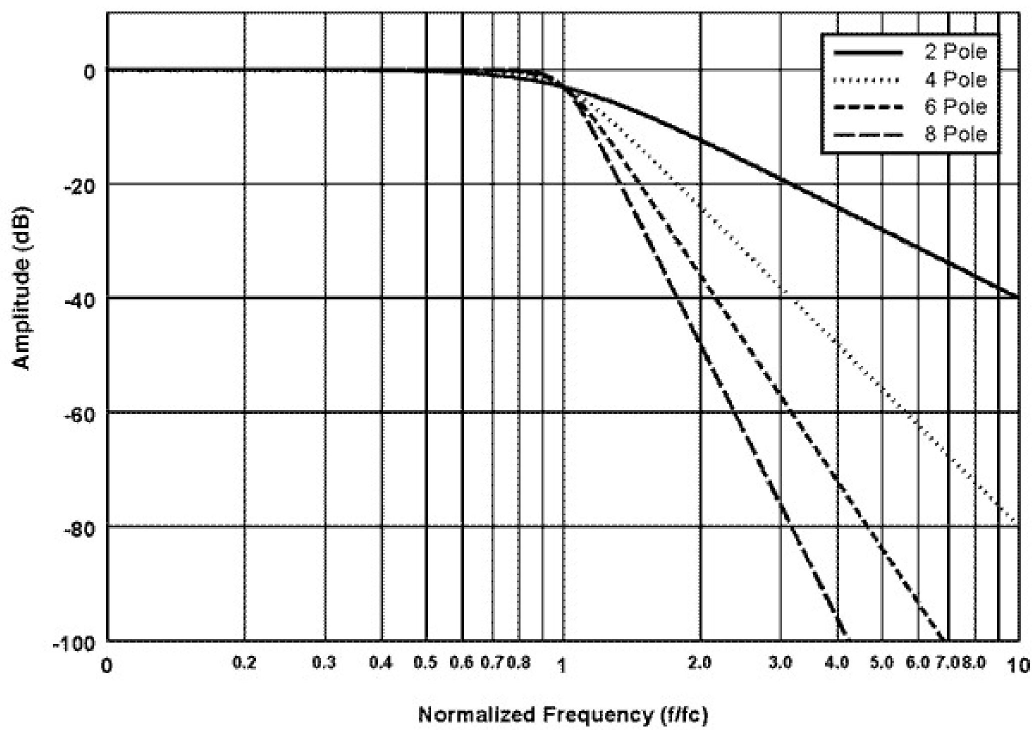
Polynomial coefficients of a Butterworth transfer function denominator.

Order, n	a1	a2	a3	a4	a5	a6	a7	a8	a9
2	1.414214								
3	2,0	2,0							
4	2.613126	3.414214	2.613126						
5	3.236068	5.236068	5.236068	3.236068					
6	3.863703	7.464102	9.141620	7.464102	3.863703				
7	4.493959	10.097835	14.591794	14.591794	10.097835	4.493959			
8	5.125831	13.137071	21.846151	25.688356	21.846151	13.137071	5.125831		
9	5.758770	16.581719	31.163437	41.986386	41.986386	31.163437	16.581719	5.758770	
10	6.392453	20.431729	42.802061	64.882396	74.233429	64.882396	42.802061	20.431729	6.392453

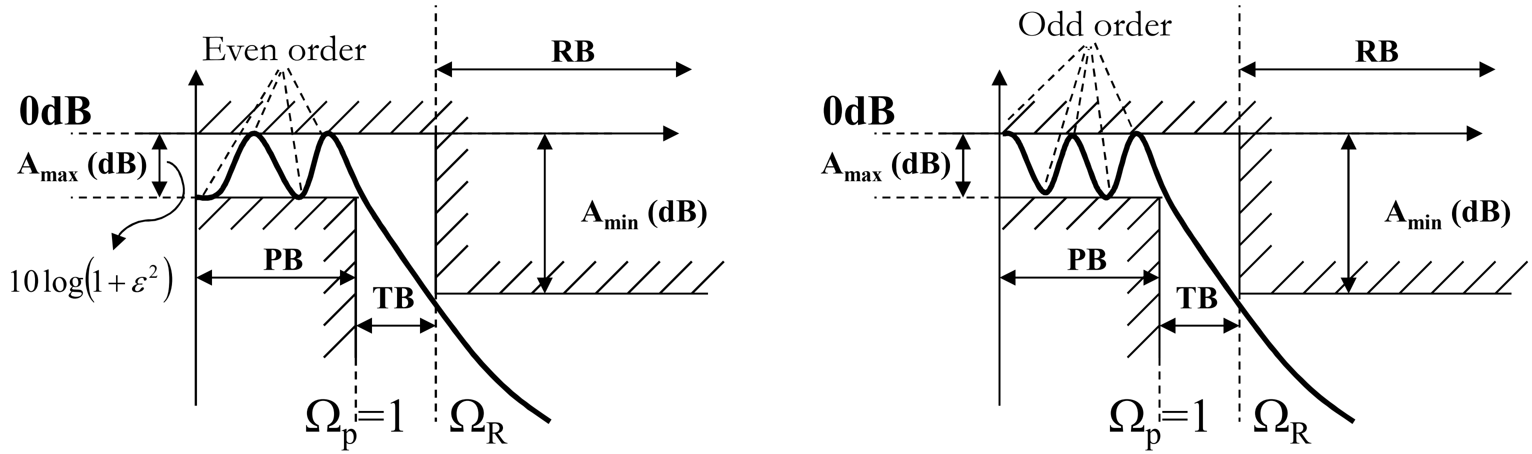
Filters – Butterworth approximation

- The Butterworth filter shows maximally flat amplitude response in the pass-band.
- Moderate roll-off in transition band
- Not good phase linearity.

$$n \times 20 \, {}^{dB}\!/_{d\acute{e}c}$$



Filters – (Pafnuty) Chebyshev approximation



$$H(s)=\frac{G_o\Omega_p^N}{\varepsilon 2^{N-1}(S_b-p_1)\dots(S_b-p_N)}\Rightarrow\frac{1}{1+\varepsilon\cdot C_N(\omega/\omega_p)}\Rightarrow|H(j\omega)|=\frac{1}{\sqrt{1+\varepsilon^2\cdot C_N^2(\omega/\omega_p)}}$$

Recurrent expression

$$C_N(\omega/\omega_p)=\begin{cases}\cos(N\arccos(\omega/\omega_p)) & |\omega/\omega_p|\le 1\\ \cosh(N\,\mathrm{arccosh}(\omega/\omega_p)) & |\omega/\omega_p|>1\end{cases}\rightarrow\begin{cases}C_0(\omega/\omega_p)=1;\quad C_1(\omega/\omega_p)=\dfrac{\omega}{\omega_p}\\ C_{N+1}(\omega/\omega_p)=2\dfrac{\omega}{\omega_p}C_N(\omega/\omega_p)-C_{n-1}(\omega/\omega_p)\end{cases}$$

$$\begin{cases}C_N(-\omega)=C_N(\omega)\quad N\ even,\quad C_N(-\omega)=-C_N(\omega)\quad N\ odd\\ C_N(0)=(-1)^{N/2}\quad N\ even,\quad C_N(0)=0\quad N\ odd\\ C_N(\pm 1)=1\quad N\ even,\quad C_N(\pm 1)=\pm 1\quad N\ odd\end{cases}$$

Filters – Chebyshev approximation

- Whatever the filter order

  $20\log|H(1)| = -10\log(1+\varepsilon^2)$

$$\Omega_{cutoff} = \cosh\left(\frac{1}{N}\cosh^{-1}\left(\frac{1}{\varepsilon}\right)\right) \qquad (\varepsilon \le 1)$$

- In the pass band ($\omega < \omega_p$), $20\log|H(j\omega)|$ evolves according to $\cos[N\mathrm{arcos}(\omega)]$ and the response magnitude varies between 0 and -10 log $(1+\varepsilon^2)$ dB.

- For $\omega > \omega_p$

$$|H(\Omega)| = \frac{1}{\sqrt{1+\frac{\varepsilon^2}{4}(2\Omega)^{2N}}} \approx \frac{1}{\frac{\varepsilon}{2}(2\Omega)^N} = \frac{1}{\left[\frac{2\omega}{\omega_p}\left(\frac{\varepsilon}{2}\right)^{\frac{1}{N}}\right]^N}$$

Filters – Chebyshev approximation

$$\varepsilon = \sqrt{10^{0{,}1 A_{\max}} - 1}$$

$$\downarrow$$

$$A_{\min} = 10\log\left(1 + \varepsilon^2 \cosh^2\left[N.\cosh^{-1}\left(\omega_R/\omega_p\right)\right]\right)$$

$$\Rightarrow N \geq \frac{\cosh^{-1}\left[\dfrac{\sqrt{10^{0{,}1 A_{\min}} - 1}}{\varepsilon}\right]}{\cosh^{-1}\left(\omega_R/\omega_p\right)}$$

$$p_k = -\sin\left(\frac{\pi}{2}\left[\frac{2k-1}{N}\right]\right)\sinh\left(\frac{1}{N}\sinh^{-1}\left[\frac{1}{\varepsilon}\right]\right) + j\cos\left(\frac{\pi}{2}\left[\frac{2k-1}{N}\right]\right)\cosh\left(\frac{1}{N}\sinh^{-1}\left[\frac{1}{\varepsilon}\right]\right)$$

$k = 1,2,\ldots,2N$

$$H(j\Omega)_{\text{Normalized}} = \frac{K}{\prod_k (s - p_k)}$$

$$s \to s\left(\frac{1}{\omega_p}\right) \longrightarrow H(j\omega)$$

- The poles are distributed in an ellipse in the s plane

Filters – Chebyshev approximation

$$\cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right); \quad \sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right); \quad x \geq 1$$

- Slope of the magnitude characteristic at cut-off frequency

$$\left.\frac{d\left|H(j\Omega)\right|}{d\Omega}\right|_{\Omega=1} = -\frac{\varepsilon^2}{\left(1+\varepsilon^2\right)^{3/2}} N^2$$

<u>How does it compare to the slope of the Butterworth characteristic?</u>

- Phase linearity and magnitude flatness in the passing band are weaker than those of Butterworth's
- In the stop-band the attenuation is given by

$$A_{min} \approx (6N-1) + 20 \log \varepsilon + 20 N \log \Omega \qquad \Omega >> 1$$

<u>How does it compare to the attenuation of the Butterworth characteristic?</u>

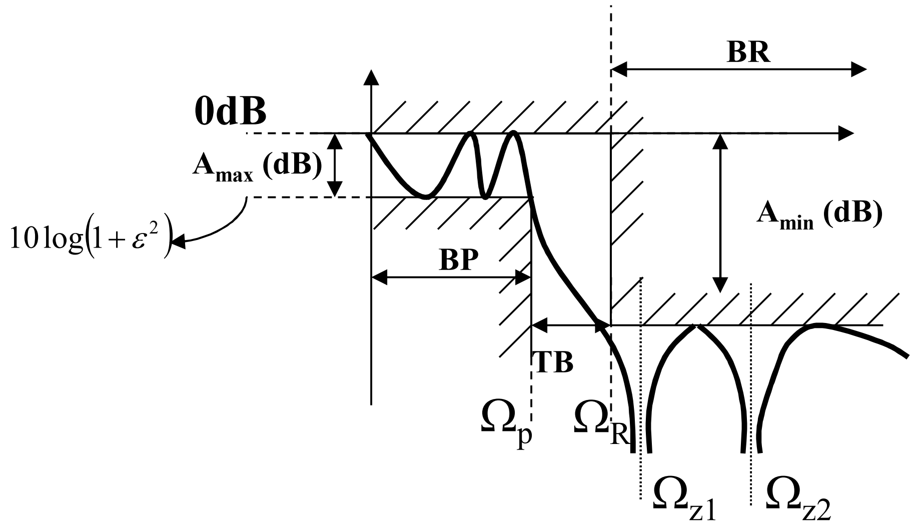
Filters – Chebyshev approximation

- Normalized Chebyshev Polynomials for 3-dB Pass-band Ripple.

Filter Order	Chebyshev	Butterworth
1	$1.00s + 1$	$s + 1$
2	$1.41s^2 + 0.911s + 1$	$s^2 + 1.41s + 1$
3	$3.98s^3 + 2.38s^2 + 3.70s + 1$	$s^3 + 2.00s^2 + 2.00s + 1$
4	$5.65s^4 + 3.29s^3 + 6.60s^2 + 2.29s + 1$	$s^4 + 2.61s^3 + 3.41s^2 + 2.61s + 1$
5	$15.9s^5 + 9.11s^4 + 22.5s^3 + 8.71s^2 + 6.48s + 1$	$s^5 + 3.24s^4 + 5.24s^3 + 5.24s^2 + 3.24s + 1$

All polynomials have value 1 for $s = 0$. The transfer function gain for $\omega = 0$ rad/s is 1. At $\omega = \omega_p$ rad/s, all polynomials yield a gain 3-dB lower than the *maximum* gain for $\omega < \omega_p$ rad/s. The maximum gain of the Butterworth filter is 1, and occurs at $\omega = 0$ rad/s. The maximum gain of the 3-dB Chebyshev filter is $\sqrt{2}$ (3 dB) and occurs at one or more values of ω between 0 rad/s and ω_p rad/s.

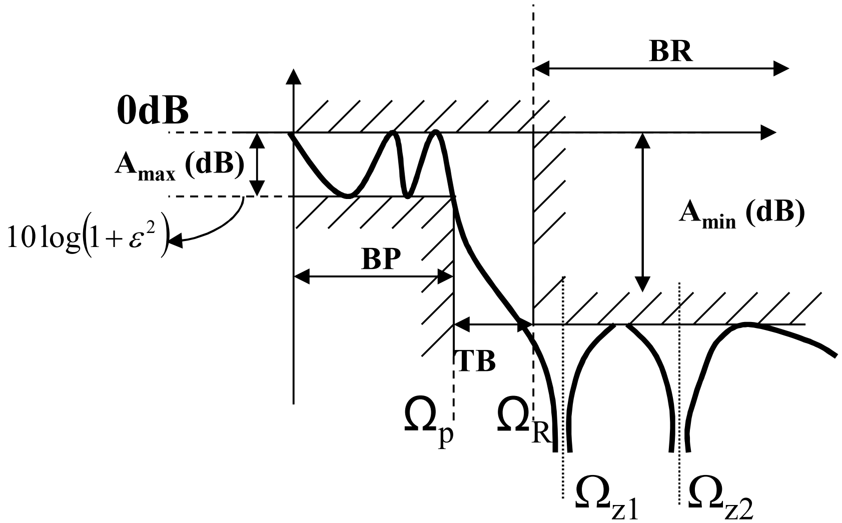
Filters – Elliptic approximation (Wilhelm) Cauer



• The attenuation of both Chebyshev and Butterworth filters, tends to Nx20 dB/dec beyond Ω_R.

• The Cauer filter has zeros in the stop-band → rational function with poles and zeros (Chebyshev and Butterworth present poles only).

Filters – Elliptic approximation (Cauer)



$$\left|H(j\omega)\right| = \frac{1}{\sqrt{1+\varepsilon^2 \cdot \Re_n^2\left(\xi, \Omega/\Omega_p\right)}}$$

ε – ripple factor

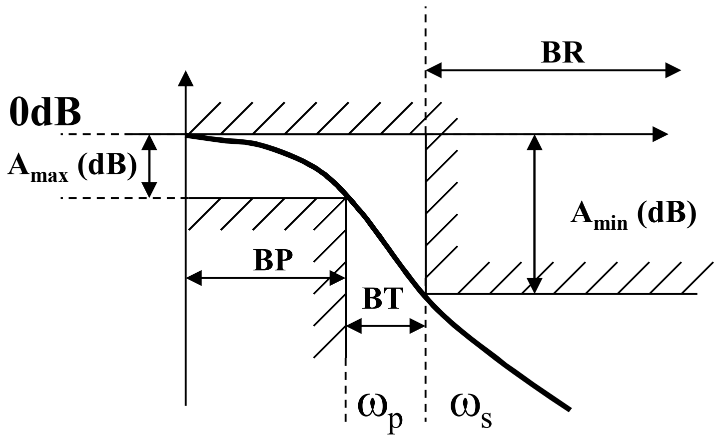
ξ - selectivity factor

- Ω_{z1} is placed in the imaginary axis close to Ω_R , what increases the slope in the transition band
- $\rightarrow$ This allows for lower filter orders, in general.

- The Cauer mathematical approximation is complex and requires the knowledge of elliptic functions.
- Poor phase linearity

Filters – Bessel approximation

- Bessel – the maximally linear phase approximation



- The Bessel polynomial approximates the transfer function to an ideal delay: $H(s) = e^{-s\tau}$

$$H_n(s) = \frac{B_n(0)}{B_n(s)} \qquad B_n(s) = (2n-1)B_{n-1}(s) + s^2 B_{n-2}(s)$$

$$B_0(s) = 1$$

$$B_1(s) = s + 1$$

$$s \rightarrow s\left(\frac{1}{\omega_p}\right) \longrightarrow H(s)$$

Filters – Bessel approximation

$$H_N(s)=\frac{1}{1+a_1 s+a_2 s^2+\ldots+a_N s^N}$$

$$\frac{a_i+1}{a_i}=\frac{2(N-i)}{(2N-i)(i+1)} \qquad (i=1,2,\ldots,N-1)$$

The a_x coefficients are adjusted to make the phase as linear as possible at ω=0. The obtained function is also called maximally flat group-delay approximation.

$$H_N(s)=\frac{1}{1+s+\frac{N-1}{2N-1}\left\{s^2+\frac{2(N-2)}{3(2N-2)}\left[s^3+\frac{2(N-3)}{4(2N-3)}\left(s^4+\ldots+\frac{2s^N}{N(N+1)}\right)\right]\right\}}$$

$$H_1(s)=\frac{1}{s+1};\ H_2(s)=\frac{3}{s^2+3s+3};\ H_3(s)=\frac{15}{s^3+6s^2+15s+15}$$

- The group delay is the flattest in the pass-band.
- The specification of Bessel filters is usually done in terms of phase linearity.

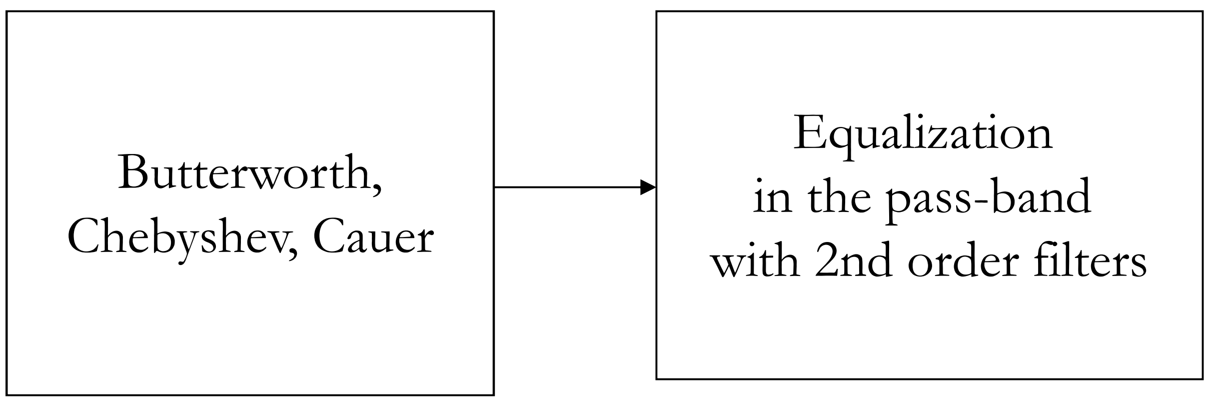
Filters – Bessel approximation

- Prototype value real and imaginary pole locations (ω=1 at the 3 dB cutoff point).

Order (N)	Re Part ($-\sigma$)	Im Part ($\pm j\omega$)
1	1.0000	
2	1.1030	0.6368
3	1.0509 1.3270	1.0025
4	1.3596 0.9877	0.4071 1.2476
5	1.3851 0.9606 1.5069	0.7201 1.4756
6	1.5735 1.3836 0.9318	0.3213 0.9727 1.6640
7	1.6130 1.3797 0.9104 1.6853	0.5896 1.1923 1.8375

Filters – phase linearity

- If phase linearity is important the use of Butterworth, Chebyshev or Cauer filters, followed by an all-pass filter is usually more efficient.

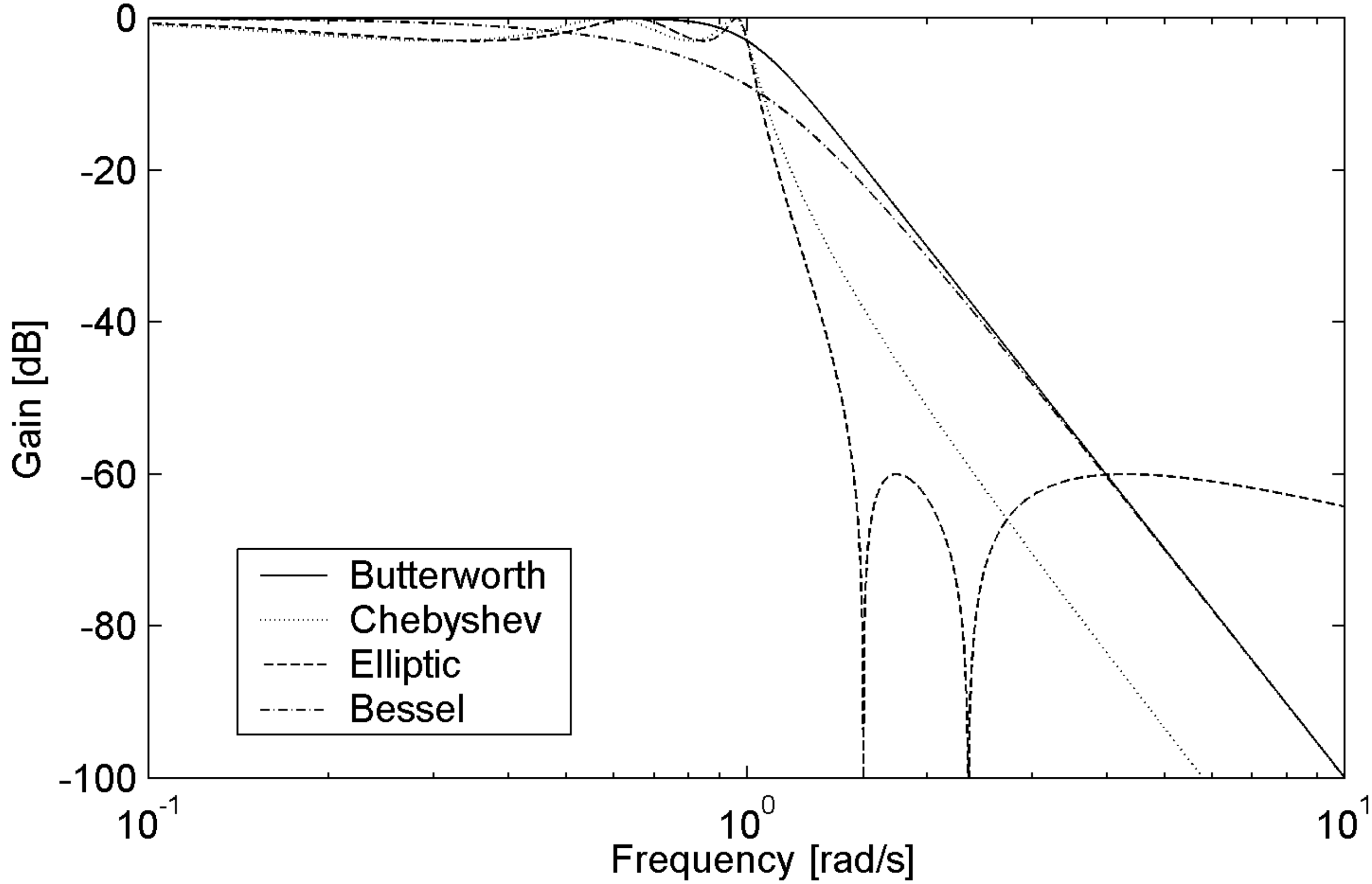


Filters – Comparison

All responses satisfy a determined specification
$(\Omega_p, A_{max}), (\Omega_R, A_{min})$

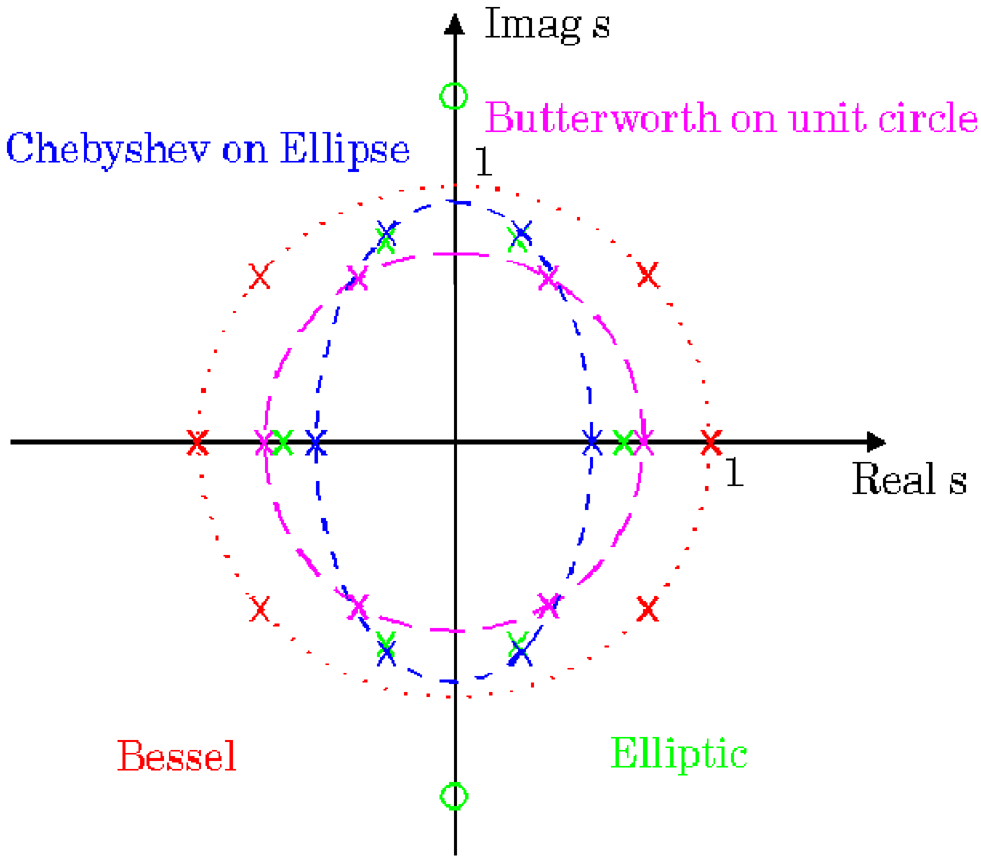
- The elliptic filter shows the lowest order, comparing to Chebyshev, Butterworth and Bessel.
- Butterworth is maximally flat in magnitude in the passing band.
- Chebyshev filters show the worst group delay variation, Butterworth come next and then the elliptic. Bessel are the best.
- Cauer functions have finite poles and zeros. The other ones have zeros in infinite.

Filters - comparison



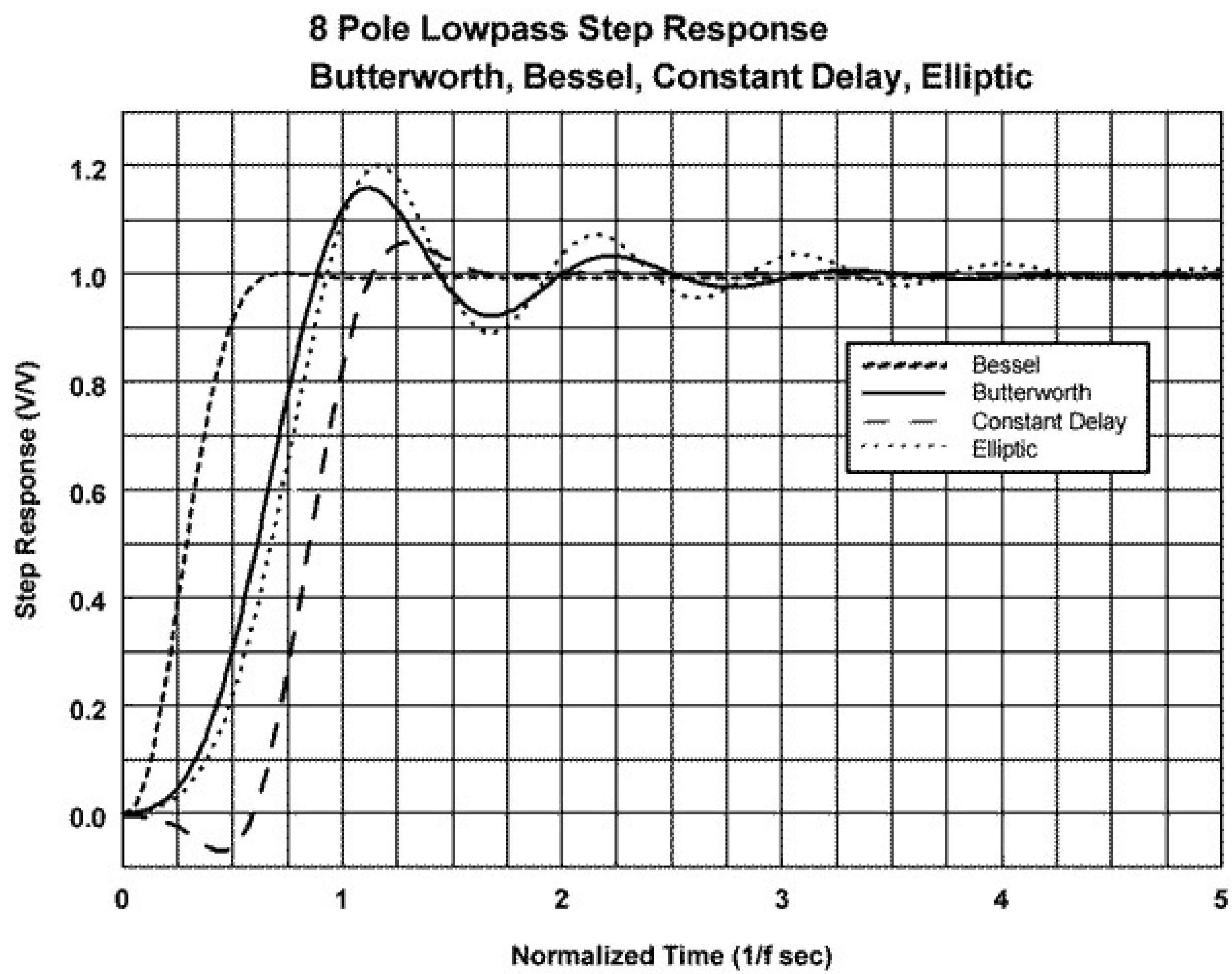
Filters – comparison

- Root locus



Filters - comparison

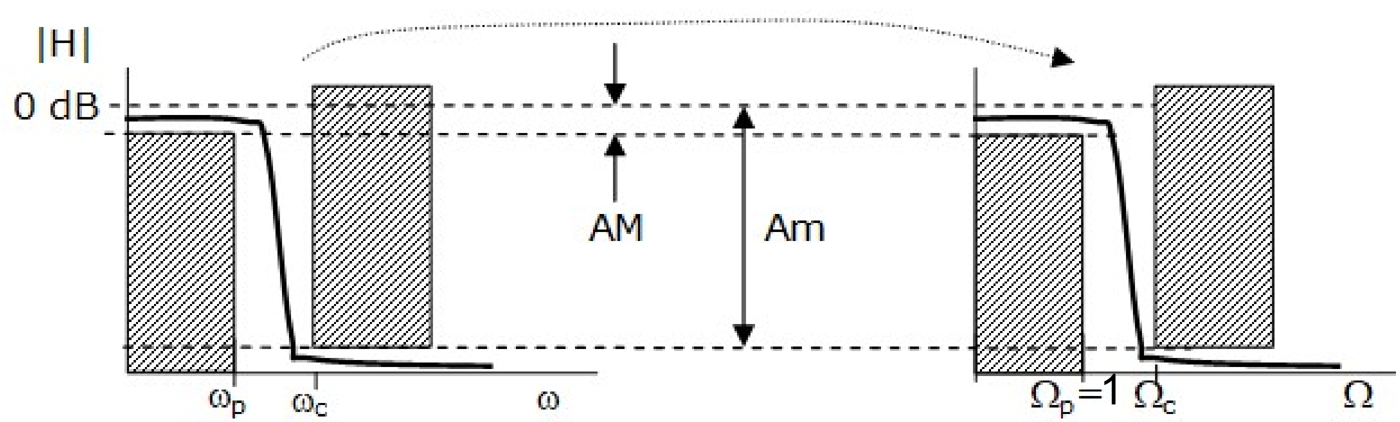
- Step response



Filters – Frequency normalization process

- Low-pass

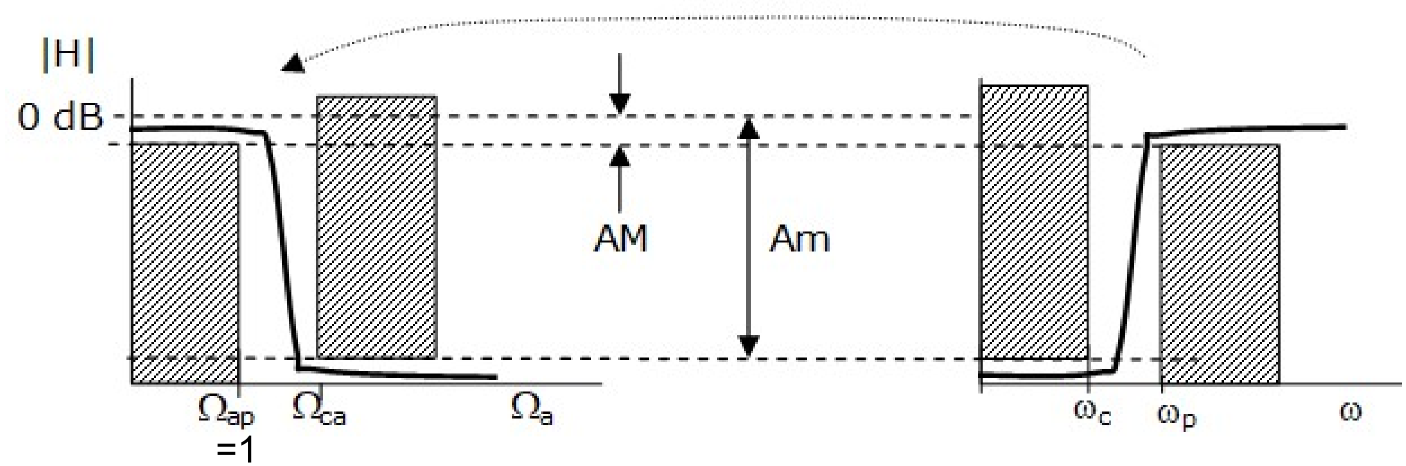
Low-pass	Normalized low-pass
$s(\omega)$	$S_b = {}^{s}\!/_{\omega_p}, \quad \Omega_b = {}^{\omega}\!/_{\omega_p}$
ω_p	$\Omega_{bp} = 1$
ω_C	$\Omega_{bC} = {}^{\omega_C}\!/_{\omega_p}$



Filters – Frequency normalization process

- High-pass

High-pass	Normalized high-pass
$s(\omega)$	$S_h = {}^{\omega_p}\!/_{s}\ , \quad X = {}^{\omega_p}\!/_{\omega}$
ω_p	$\Omega_{ap} = 1$
ω_c	$\Omega_{aC} = {}^{\omega_p}\!/_{\omega_c}$



Filters – Frequency normalization process

- Band-pass

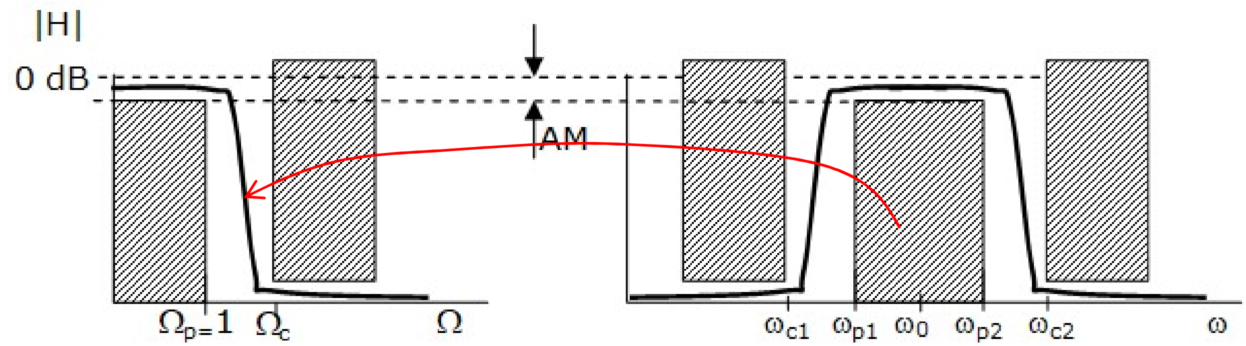
Band-pass	Normalized band-pass
$s(\omega)$	$S_{bp} = \frac{s^2+\omega_o^2}{s.B} = \frac{1}{\Delta x}\left(S_b + \frac{1}{S_b}\right), \quad \Omega_{bp} = \frac{\lvert\omega^2-\omega_0^2\rvert}{\omega(\omega_2-\omega_1)}$
ω_{p1}, ω_{p2}	$\Omega_{bp} = 1$
ω_{C1}, ω_{C2}	$\Omega_C = \frac{\omega_{c2}-\omega_{c1}}{\omega_{p2}-\omega_{p1}}$

In case of simetry

$$\omega_0 = \sqrt{\omega_{p1}\omega_{p2}} = \sqrt{\omega_{c1}\omega_{c2}}$$

$$s \to S_{pb} = \frac{s^2+\omega_o^2}{s.B} = \frac{1}{\Delta x}\left(S_b + \frac{1}{S_b}\right) = \frac{1}{\Delta x}\left(j\frac{f}{f_0} + \frac{f_0}{jf}\right), \quad with \quad B = f_2 - f_1,$$

$$\Delta x = \frac{f_2-f_1}{f_0}, and \quad \Omega_{pb} = \frac{f_0}{f_2-f_1}\left|\frac{f}{f_0} - \frac{f_0}{f}\right| = \frac{\lvert f^2 - f_0^2\rvert}{f(f_2-f_1)}$$



Filters – Frequency normalization process

- Rejection-band

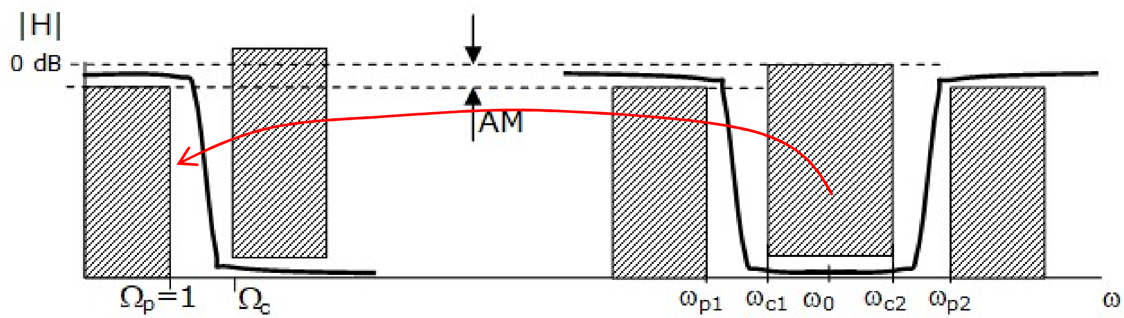
Rejection-band	Normalized rejection-band
$s(\omega)$	$S_{rb} = \frac{s.B}{s^2+\omega_o^2} = \Delta x \frac{1}{\left(S_b + \frac{1}{S_b}\right)}, \quad \Omega_{rb} = \frac{\omega(\omega_2-\omega_1)}{\lvert\omega^2-\omega_0^2\rvert}$
ω_{p1}, ω_{p2}	$\Omega_{rb} = 1$
ω_{C1}, ω_{C2}	$\Omega_C = \frac{\omega_{c2}-\omega_{c1}}{\omega_{p2}-\omega_{p1}}$

In case of simetry

$$\omega_0 = \sqrt{\omega_{p1}\omega_{p2}} = \sqrt{\omega_{c1}\omega_{c2}}$$

$$s \to S_{rb} = \frac{s.B}{s^2+\omega_o^2} = \Delta x \frac{1}{\left(S_b+\frac{1}{S_b}\right)} = \Delta x \frac{1}{\left(j\frac{f}{f_0}+\frac{f_0}{jf}\right)}, \quad with \quad B = f_2 - f_1, \quad \Delta x = \frac{f_2-f_1}{f_0}, and$$

$$\Omega_{rb} = \frac{f_2-f_1}{f_0}\frac{1}{\left|\frac{f}{f_0}-\frac{f_0}{f}\right|} = f(f_2-f_1)\frac{1}{\lvert f^2-f_0^2\rvert}$$



Frequency Transformations

Initial specification ➔	Conversion to the normalized low-pass version ➔	Denormalization to obtain the final transfer function
High-pass	Low-pass (normalized)	High-pass
$(\omega_p; A_{max})$ $(\omega_R; A_{min})$	$H_N(S)$ $(1; A_{max})$ $\left(\frac{\omega_p}{\omega_R}; A_{min}\right)$	$H(s) = H_N\left(\frac{\omega_p}{s}\right)$ $S = \frac{\omega_p}{s}$
Band-pass	Low-pass (normalized)	Band-pass
$(\omega_1, \omega_2; A_{max})$ $(\omega_3, \omega_4; A_{min})$ $B = \omega_2 - \omega_1$ $\omega_0 = \sqrt{\omega_1 \times \omega_2}$	$H_N(S)$ $(1; A_{max})$ $\left(\frac{\omega_4 - \omega_3}{\omega_2 - \omega_1}; A_{min}\right)$	$H(s) = H_N\left(\frac{s^2 + \omega_o^2}{Bs}\right)$ $S = \frac{s^2 + \omega_o^2}{Bs}$
Band-reject	Low-pass (normalized)	Band-reject
$(\omega_1, \omega_2; A_{min})$ $(\omega_3, \omega_4; A_{max})$ $B = \omega_2 - \omega_1$ $\omega_0 = \sqrt{\omega_1 \times \omega_2}$	$H_N(S)$ $(1; A_{max})$ $\left(\frac{\omega_2 - \omega_1}{\omega_4 - \omega_3}; A_{min}\right)$	$H(s) = H_N\left(\frac{Bs}{s^2 + \omega_o^2}\right)$ $S = \frac{Bs}{s^2 + \omega_o^2}$

Note: For band-pass and band-reject, symmetry is assumed, i.e:

$$\omega_0 = \sqrt{\omega_1\omega_2} = \sqrt{\omega_3\omega_4}$$

If this is not verified, change the frequency values to a symmetric situation that guarantees the specifications

Exercises

1- A 4th-order low-pass filter shows an attenuation of 2 dB at 6,28 Mrad/s.
Calculate the atennuation at 22 Mrad/s.

2- Consider the following specifications:

A_{max}= 1 dB; A_{min}=35 dB; f_p=10 kHz, f_r=35 kHz

a) Draw the graphic of specifications.
b) Determine the corresponding Butterworth aproximation function.
c) The location of the respective poles.

3- Consider the following specifications :

A_{max}= 0,5 dB; A_{min}= 20 dB; ω_p=30 krad/s, ω_r=10 krad/s

a) Draw the graphic of specifications.
b) Determine the corresponding Chebyshev aproximation function.

Filters

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