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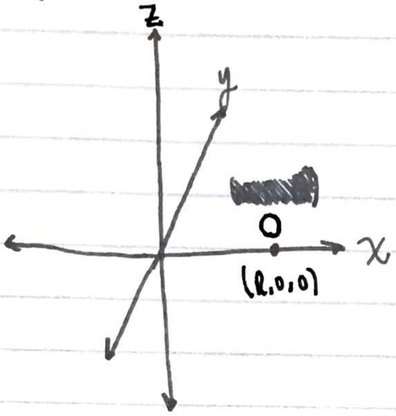
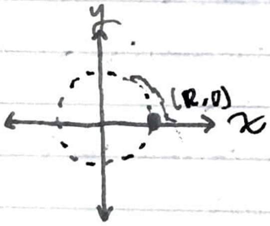
Math 5610: Term Project

Excersise 1:

We're intrested in the trajectory of the point $(0,0,R)$ which has 0 longitude, 0 lattitude, & 0 altitude. Diagram:



Now, z will be fixed, but x, y will change since we're turning the point around the z vector. Since we follow a circular track, we can use the sin, cos functions to pin down where O is @ time t. O will orbit the following track:



We know that (y,x) has a period of 1 Sidereal Day, so we want our Sidereal for t to be $2\pi \Rightarrow P = 1$ Sidereal Day. Our Amp. should be the radius of the Earth R. There is no Phase Shift.

$$x(t) :: R\cos(2\pi t)$$ when t is in Sidereal
$$y(t) :: R\sin(2\pi t)$$

$$O(t) :: (R\cos(2\pi t),\ R\sin(2\pi t),\ 0)$$ in Cartesian

Excersise 2:

The Program will be written up in MATLAB, but we'll detail the pseudocode here:

~~Degrees, Seconds, Radians to R~~

Degrees, Minutes, Seconds to Radians:

$$\text{Deg Total} := \text{Degrees} + \frac{1}{60} \cdot \text{Minutes} + \frac{1}{360} \cdot \text{Seconds} \,;$$

$$\text{Radians} := (\pi \cdot \text{Deg Total}) / 180 \,;$$

Return Radians;

Radians to Degrees, Minutes, Seconds:

$$\text{Deg Total} := (180 \cdot \text{Radians}) / \pi \,;$$
$$\text{Degrees} = \text{floor}(\text{Deg Total}) \,;$$
$$\text{Min Total} = (\text{Deg Total} - \text{Degrees}) \cdot 60 \,;$$
$$\text{Minutes} = \text{floor}(\text{Min Total}) \,;$$
$$\text{Seconds Total} := (\text{Min Total} - \text{Minutes}) \cdot 60 \,;$$
$$\text{Seconds} = \text{Seconds Total} \,;$$

Return Degrees, Minutes, Seconds;

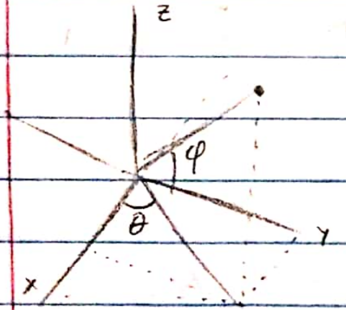
And Hence, we have our pseudocode for Conversions.

3. Given $\circ$ φ_d φ_m φ_s NS λ_d λ_m λ_s EW h we can convert this to Cartesian coordinates with the following process.

First convert (φ_d, φ_m, φ_s NS) into radians using 2) call this latRad

then convert (λ_d, λ_m, λ_s, EW) into radians using 2) call this longRad

Then we can convert our longitude & latitude into XYZ-coordinates.



Then we know z depends only on the latitude. So if φ = latitude,

$$z = R\sin(\text{latRad})$$

Then x & y both depend on the cosine of the latitude.

Then if longitude = 0, x = R & y = 0 so

$$x = R\cos(\text{latRad})\cos(\text{longRad}) \;\; \& \;\; y = R\cos(\text{latRad})\sin(\text{longRad})$$

Then to take into account altitude we know it is in the direction of XYZ so we grab the unit vector & apply the altitude as its magnitude:

$$\|(x,y,z)\| = \sqrt{x^2+y^2+z^2} = R \Rightarrow \hat{xyz} = \left\langle \frac{x}{R}, \frac{y}{R}, \frac{z}{R} \right\rangle$$

$\Rightarrow$ With altitude $h \cdot \hat{xyz} = \left\langle \frac{hx}{R}, \frac{hy}{R}, \frac{hz}{R} \right\rangle$

Then we just add that to $\overrightarrow{xyz}$ to get our final result

$$\overrightarrow{xyz} = \langle (R+h)\cos(\text{latRad})\cos(\text{longRad}), (R+h)\cos(\text{latRad})\sin(\text{longRad}), (R+h)\sin(\text{latRad}) \rangle$$

Where $\text{latRad} = \left[\varphi_d + \frac{\varphi_m}{60} + \frac{\varphi_s}{3600}\right] \cdot \frac{\pi}{180} \cdot NS$

$\text{longRad} = \left[\lambda_d + \frac{\lambda_m}{60} + \frac{\lambda_s}{3600}\right] \cdot \frac{\pi}{180} \cdot EW$

4. To add in time we need to shift our x & y accordingly. We can think of longRad as our phase shift.

$\rightarrow$

$z = (R+h)\sin(\text{latRad})$

$\frac{z}{R+h} = \sin(\text{latRad})$

$\sin^{-1}\left(\frac{z}{R+h}\right) = \text{latRad}$

4 con. $\vec{xyz}(t) = \left\langle (R+h)\cos(\text{latRad})\cos\left(\frac{2\pi}{s}t + \text{longRad}\right),\right.$
$(R+h)\cos(\text{latRad})\sin\left(\frac{2\pi}{s}t + \text{longRad}\right),$
$\left.(R+h)\sin(\text{latRad})\right\rangle$

5. If we're given our coordinates in Cartesian Form we should first get our latitude & longitude:

$\varphi = \text{latRad} = \sin^{-1}\left(\frac{z}{R+h}\right)$

$\lambda = \text{longRad} = \tan^{-1}\left(\frac{y}{x}\right)$

Then to get the altitude it's just $\sqrt{x^2+y^2+z^2} - R = h$

Then we just have to convert latRad & longRad to degrees, minutes, seconds using 2.)

Then $\varphi_d = \text{int}\left(\sin^{-1}\left(\frac{z}{R+h}\right)\right)$

$\varphi_m = \text{int}\left[\left(\sin^{-1}\left(\frac{z}{R+h}\right) - \varphi_d\right)\cdot 60\right]$

$\varphi_s = \left(\sin^{-1}\left(\frac{z}{R+h}\right) - \varphi_d - \varphi_m/60\right)\cdot 3600$

$\lambda_d = \text{int}\left(\tan^{-1}\left(\frac{y}{x}\right)\right)$

$\lambda_m = \text{int}\left[\left(\tan^{-1}\left(\frac{y}{x}\right) - \lambda_d\right)\cdot 60\right]$

$\lambda_s = \left(\tan^{-1}\left(\frac{y}{x}\right) - \lambda_d - \frac{\lambda_m}{60}\right)\cdot 3600$

$NS = 1$ if $\sin^{-1}\left(\frac{z}{R+h}\right) \geq 0$; -1 if $\sin^{-1}\left(\frac{z}{R+h}\right) < 0$

$EW = 1$ if $\tan^{-1}\left(\frac{y}{x}\right) \geq 0$; -1 if $\tan^{-1}\left(\frac{y}{x}\right) < 0$

$h = \sqrt{x^2+y^2+z^2} - R$

6. If given coordinates & t, notice that the only thing that changes is λ

Prior $\frac{y}{x} = \tan(\text{longRad})$ so $\text{longRad} = \tan^{-1}\left(\frac{y}{x}\right)$

Now $\frac{y}{x} = \tan\left(\frac{2\pi}{s}t + \text{longRad}\right) \Rightarrow \text{longRad} = \lambda = \left(\tan^{-1}\left(\frac{y}{x}\right) - \frac{2\pi}{s}t\right) \% \ 2\pi$

So $h = \sqrt{x^2+y^2+z^2} - R$

$\varphi_d = \text{int}\left(\sin^{-1}\left(\frac{z}{R}\right)\right)$

$\varphi_m = \text{int}\left[\left(\sin^{-1}\left(\frac{z}{R}\right) - \varphi_d\right)\cdot 60\right]$

$\varphi_s = \left(\sin^{-1}\left(\frac{z}{R}\right) - \varphi_d - \frac{\varphi_m}{60}\right)\cdot 3600$

$NS = 1$ if $\sin^{-1}\left(\frac{z}{R}\right) \geq 0$; -1 if $\sin^{-1}\left(\frac{z}{R}\right) < 0$

$\lambda_d = \text{int}\left(\tan^{-1}\left(\frac{y}{x}\right) - \frac{2\pi}{s}t\right)$

$\lambda_m = \text{int}\left[\left(\tan^{-1}\left(\frac{y}{x}\right) - \frac{2\pi}{s}t - \lambda_d\right)\cdot 60\right]$

$\lambda_s = \left(\tan^{-1}\left(\frac{y}{x}\right) - \frac{2\pi}{s}t - \lambda_d - \frac{\lambda_m}{60}\right)\cdot 3600$

$EW = 1$ if $\tan^{-1}\left(\frac{y}{x}\right) - \frac{2\pi}{s}t \geq 0$; -1 otherwise

7. Using 4.) & values: t 40 45 55.0 | " 50 58.0 ÷ 1372.00

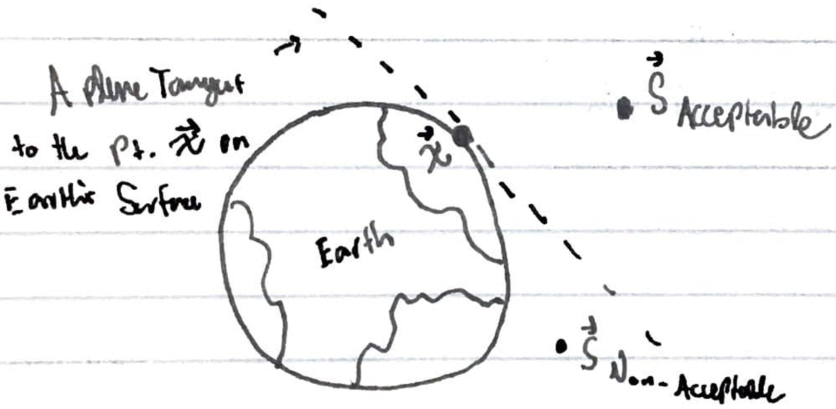
We find $\overrightarrow{xyz}(t) = \langle 4823683.697\cos\left(\frac{2\pi}{5}t - 0.2068\right),$

$4823683.697\sin\left(\frac{2\pi}{5}t - 0.2068\right),$

$4158593.416\rangle$

Excersise 8:

We'll use some technical knowledge from Calc III. At any given time, an individual resides on Earth at point $\vec{x}$ & a satellite resides at point $\vec{s}$. Geometrically, the individual will only be capable of being seen if $\vec{s}$ is above some plane. We can describe the situation with a diagram seen below:



So any point above the plane, or Horizon, will allow $\vec{x}$ to be visible. Otherwise, $\vec{x}$ can't be seen. Thus we need to construct a plane tangent to point $\vec{x}$ - $\overrightarrow{\text{center of Earth}} = \vec{x}$, since center of Earth $\equiv \vec{0}$. To do this, we the following:

w/ $\vec{x} = \begin{bmatrix} x_E \\ y_E \\ z_E \end{bmatrix}$ & $\vec{x} - \vec{0} = \begin{bmatrix} x_E \\ y_E \\ z_E \end{bmatrix}$

We dot $\vec{x}-0$ w/ the vector w/ arbitrary x, y, z. $\begin{pmatrix} x-x_E \\ y-y_E \\ z-z_E \end{pmatrix}$ & we set that equal to 0 to get a fixed plane eq'n tangent to $\vec{x}$.

We get the plane: $x_E(x-x_E)+y_E(y-y_E)+z_E(z-z_E)=0$

Exercise 9:

We're given the Values x_v, t_v for the equation $\|x_v - x_s\| = c(t_v - t_s)$. Now we're going to find x_s but to get x_s, we need a value for t_s. We'll find t_s using Newton's method & we'll then plug that into equation 20 for $x_s(t_s)$.

(1)

Firstly, we have the equation $\|x_v - x_s\| = c(t_v - t_s)$, where we'll let $\Delta t = t_v - t_s$. Also, by a taylor Series expansion, we have that

$x_s(t_s) \approx x_s(t_v) + x_s'(t_v)\Delta t$. Substituting this into our first equation;

(2) $\|x_v - x_s(t_v) - x_s'(t_v)\Delta t\| = c\Delta t$. Now, t_s is the only parameter in this equation & can be solved for. Hence, equation 2 gives us an initial approximation for t_s. We'll call this $t_{s_0} \approx t_s$. This is great, However, we want a better approximation than t_{s_0}. Consider the following,

$\|x_v - x_s\| = c(t_v - t_s) \Leftrightarrow \frac{\|x_v - x_s(t_s)\|}{c} + t_v = t_s$. An equation of the form $f(t_s) = t_s$, which can be solved using fixed point iteration. Better yet, we'll apply newton's method to:

(3) $0 = -t_s + \frac{\|x_v - x_s(t_s)\|}{c} + t_v$, which satisfies form $f(t_s) = 0$. To apply newton's we need a derivative, which we'll compute numerically for efficiency. Also, we'll use t_{s_0} as our first guess since already, $t_{s_0} \approx t_s$.

Newton's Method will give us t_s which we can then plug into equation 20 to get $x_s(t_s)$. Thereby, we have our solution. In Summary:

1. Get t_{s_0} from a Taylor Series Expansion.
2. Numerically compute a derivative for (3)
3. Iterate Newton's Method on (3) using data from 1. & 2.
4. Use t_s & eqn 20) to get $x_s(t_s)$

Excersise 10:

As asked, we'll write down the 4 equations:

$$\begin{cases} (1) \quad \|x_v - x_{s1}\| = c(t_v - t_{s1}) \\ (2) \quad \|x_v - x_{s2}\| = c(t_v - t_{s2}) \\ (3) \quad \|x_v - x_{s3}\| = c(t_v - t_{s3}) \\ (4) \quad \|x_v - x_{s4}\| = c(t_v - t_{s4}) \end{cases}$$

Again, all of these eqn's are satisfied with solution:

$x_v = x_{s1} = x_{s2} = x_{s3} = x_{s4}$, $t_v = t_{s1} = t_{s2} = t_{s3} = t_{s4}$. Since we got that $\vec{0} = \vec{0}$

Excersise 11:

We start with

$$F(x) = \begin{pmatrix} \|x_v - x_{s1}\| - c(t_v - t_{s1}) \\ \|x_v - x_{s2}\| - c(t_v - t_{s2}) \\ \|x_v - x_{s3}\| - c(t_v - t_{s3}) \\ \|x_v - x_{s4}\| - c(t_v - t_{s4}) \end{pmatrix} = \vec{0}$$

~~Which we know does not have a solution since we can't have our times t_{s1-4} exactly equal to~~

Which we will find a best fit solution for using the least squares Algorithm:

$$F(x) = 0 \iff F^T(x)F(x) = 0 \iff \begin{pmatrix} \|x_v - x_{s1}\| - c(t_v - t_{s1}) \\ \|x_v - x_{s2}\| - c(t_v - t_{s2}) \\ \|x_v - x_{s3}\| - c(t_v - t_{s3}) \\ \|x_v - x_{s4}\| - c(t_v - t_{s4}) \end{pmatrix} \Big(\|x_v - x_{s1}\| - c(t_v - t_{s1}),\ \|x_v - x_{s2}\| - c(t_v - t_{s2}),\ \|x_v - x_{s3}\| - c(t_v - t_{s3}),\ \|x_v - x_{s4}\| - c(t_v - t_{s4}) \Big) = 0$$

$$\iff \left(\|x_v - x_{s1}\| - c(t_v - t_{s1})\right)^2 + \left(\|x_v - x_{s2}\| - c(t_v - t_{s2})\right)^2 + \left(\|x_v - x_{s3}\| - c(t_v - t_{s3})\right)^2 + \left(\|x_v - x_{s4}\| - c(t_v - t_{s4})\right)^2 = 0$$

Now, we fix the gradient $= 0$ to get the following system:

$$\begin{cases} \frac{\partial f}{\partial x} F^T(x)F(x) = 0 \\ \frac{\partial f}{\partial y} F^T(x)F(x) = 0 \\ \frac{\partial f}{\partial z} F^T(x)F(x) = 0 \\ \frac{\partial f}{\partial t} F^T(x)F(x) = 0 \end{cases}$$

We can then solve the new system for $\langle x_v, y_v, z_v, t_v \rangle$. This then gives us $\vec{x}_v$, so we can plug $\vec{x}_v$ into $F(x)$ & solve for ~~[illegible]~~ t_v.

knowing $\vec{X}_r$ begs $\vec{t}_r$ & Hence we get both $\vec{x}_r$ & $\vec{t}_r$

Excersise 12:

Neglecting $h(t)$, since this cannot be determined for a sattelite traversing an undetermined track, we can use $\vec{X}_s(t)$ to determine the ground track of the Sattellite.

Eqn (20) Says:

$$X_s(t) := (R+h)\left[U\cos\left(\frac{2\pi t}{P}+\theta\right)+V\sin\left(\frac{2\pi t}{P}+\theta\right)\right]$$ where

t = seconds, R = Radius of Earth, h: Altitude of Sattelite, U is a unit Vect, V is an orthogonal unit vector to U, P is 1/2 a SR Day, θ is a phase of Orbit.

We're Specifically intrested in S_1, so let's determine the ground track for that Sattelite according to the appendix:

$$X_s(t) := R\left[\begin{pmatrix}1\\0\\0\end{pmatrix}\cos\left(\frac{2\pi t}{43082.1}+43,082.044999\right)+\begin{pmatrix}0\\.573576\\.819152\end{pmatrix}\sin\left(\frac{2\pi t}{43082.1}+43,082.044999\right)\right]$$

Better Stated: $x_s(t) :: \langle \cos\left(\frac{2\pi t}{43082.1}+43,082.044999\right), .573576\sin\left(\frac{2\pi t}{43082.1}+43,082.044999\right), .819152\sin\left(\frac{2\pi t}{43082.1}+43,082.044999\right)\rangle$

With Regards to the period being 1/2 a Siderial Day:

One big advantage from a design perspective is that the Sattelites orbit the Earth twice in one day, & they do not follow just one section of the Earth; Instead, the Sattelite groups track varying locations over the Earth's surface on any given day. Because of this the System is redundant.

That is, if any group of Sattelites fails to accurately track some section of the earth, or something goes wrong in their data collection, then another group of Sattelites can makeup for that disparity in performance since another group of Sattelites will examine the mispurported section at some point in the day. Hence, the sattelites periodicity & positioning allows for corrections in mispurported data & the system will be redundant.

Exersice 13:

In this case, we have the system:

$$F(x) := \begin{bmatrix} \|x_v - x_{s1}\| - c(t_v - t_s) \\ \|x_v - x_{s2}\| - c(t_v - t_{s2}) \\ \|x_v - x_{s3}\| - c(t_v - t_{s3}) \\ \|x_r - x_{s4}\| - c(t_v - t_{s4}) \end{bmatrix} = 0$$

Newton's Method is described by

$$x^{(k+1)} = \delta x^k + x^k \qquad \text{where,}$$

$$J(x^k)\,\delta x^k = -F(x^k).$$

Essentially, we begin w/ an initial guess x^0 then we solve $J(x^0)\delta x^0 = -F(x^0)$ for δx^0. Then we sum $\delta x^0 + x^0$ to get x^1 & we repeat the process until we either max out our iterations or get within our tolerance. Of course, to carry out this process, we need J defined.

The Jacobian:

$$J = \begin{bmatrix} \frac{\partial F(1,1)}{\partial x} & \frac{\partial F(1,1)}{\partial y} & \frac{\partial F(1,1)}{\partial z} & \frac{\partial F(1,1)}{\partial t} \\ \frac{\partial F(2,1)}{\partial x} & \frac{\partial F(2,1)}{\partial y} & \frac{\partial F(2,1)}{\partial z} & \frac{\partial F(2,1)}{\partial t} \\ \frac{\partial F(3,1)}{\partial x} & \frac{\partial F(3,1)}{\partial y} & \frac{\partial F(3,1)}{\partial z} & \frac{\partial F(3,1)}{\partial t} \\ \frac{\partial F(4,1)}{\partial x} & \frac{\partial F(4,1)}{\partial y} & \frac{\partial F(4,1)}{\partial z} & \frac{\partial F(4,1)}{\partial t} \end{bmatrix}$$

We'll say. $F(j,1)$ where $1 \leq j \leq 4$, $j \in \mathbb{N}$:

$$\left.\begin{aligned} \frac{\partial F(j,1)}{\partial x} &= \frac{x_v - x_{s_j}}{\|\bar{x}_v - \bar{x}_{s_j}\|} \\ \frac{\partial F(j,1)}{\partial y} &= \frac{y_v - y_{s_j}}{\|\bar{x}_v - \bar{x}_{s_j}\|} \\ \frac{\partial F(j,1)}{\partial z} &= \frac{z_v - z_{s_j}}{\|\bar{x}_v - \bar{x}_{s_j}\|} \end{aligned}\right\}$$ where $\overline{x_{s_j}}$ is the parameter

$$\frac{\partial F(j,1)}{\partial t} = -c = -\text{speed of light}$$

Thereby. Defining our Jacobian so we can execute Newton's for 4 eqns.

Hence we have Newton's described out.

Exercise 14:

We'll define:

$$E = f(\vec{x}) = (\|x_v - x_{s1}\| - \|x_v - x_{s2}\| - c(t_{s2} - t_{s1}))^2 + \dots$$
$$+ (\|x_v - x_{sm-1}\| - \|x_v - x_{sm}\| - c(t_{sm} - t_{sm-1}))^2$$

Now, we're going to apply Newton's Method to the system:

System to solve
$$\begin{cases} \frac{\partial}{\partial x}(E) = 0 \\ \frac{\partial}{\partial y}(E) = 0 \\ \frac{\partial}{\partial z}(E) = 0 \\ \frac{\partial}{\partial t}(E) = 0 \end{cases}$$

Doing so we'll give us $\vec{x}_v, t_v$. To apply Newton's, we're going to need the Jacobian which is defined as follows:

$$J = \begin{bmatrix} \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}E\right) & \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}E\right) & \frac{\partial}{\partial z}\left(\frac{\partial}{\partial x}E\right) & \frac{\partial}{\partial t}\left(\frac{\partial}{\partial x}E\right) \\ \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}E\right) & \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}E\right) & \frac{\partial}{\partial z}\left(\frac{\partial}{\partial y}E\right) & \frac{\partial}{\partial t}\left(\frac{\partial}{\partial y}E\right) \\ \frac{\partial}{\partial x}\left(\frac{\partial}{\partial z}E\right) & \frac{\partial}{\partial y}\left(\frac{\partial}{\partial z}E\right) & \frac{\partial}{\partial z}\left(\frac{\partial}{\partial z}E\right) & \frac{\partial}{\partial t}\left(\frac{\partial}{\partial z}E\right) \\ \frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}E\right) & \frac{\partial}{\partial y}\left(\frac{\partial}{\partial t}E\right) & \frac{\partial}{\partial z}\left(\frac{\partial}{\partial t}E\right) & \frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}E\right) \end{bmatrix}$$

Of course, explicitly determining the Jacobian is awful to do computationally, so instead we'll use a finite difference approximation to calculate the Jacobian.

Thus, we can apply Newton's Method iteratively where

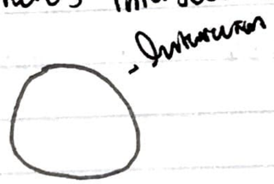
$$x^{k+1} := \delta x^k + x^k \quad \text{and} \quad J(x^k)\,\delta x^k = -F(x^k).$$

Thereby, we've detailed how we'll apply Newton's Method to a possibly overdetermined system.

Exercise 15:

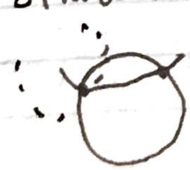
We'll examine the problem geometrically, as our document suggests. Firstly, we'll assume that none of our satellites share the exact same position since that is physically impossible. That implies that no two satellites can share the exact same spherical domain. Now consider the following:

2 Spheres intersect in a circle:



3 Spheres intersect in 2 points:

4 Spheres intersect @ 1 point:



Since the circle can only go through one point or the other.

Hence, 4 sattelites will yield a unique solution as will any number of sattelites above 4.