DiscreteMathematics

NAME - Prashant Gupta

ROLL NO.- 220803

ASSIGNMENT-1

1

Problem 1

Prove that there is no positive integer n such that $n^2 + n^3 = 100$.

SOLUTION - To prove this, assume that there exists a positive integer n such that $n^2 + n^3 = 100$.

$$n^2 + n^3 = 100$$

Factor out n^2:

$$n^2(1 + n) = 100$$

Now, consider the possible integer values of n:

- **For $n = 1$:**

$$1^2(1 + 1) = 2 \neq 100$$

- **For $n = 2$:**

$$2^2(1 + 2) = 4 \cdot 3 = 12 \neq 100$$

- **For $n = 3$:**

$$3^2(1 + 3) = 9 \cdot 4 = 36 \neq 100$$

- **For $n = 4$:**

$$4^2(1 + 4) = 16 \cdot 5 = 80 \neq 100$$

- **For $n = 5$:**

$$5^2(1 + 5) = 25 \cdot 6 = 150 \neq 100$$

For $n \geq 5$, $n^2(1 + n)$ increases further and does not equal 100. Therefore, there is no positive integer n such that $n^2 + n^3 = 100$.

Problem 2

Prove that $n^2 + 1 \geq 2n$ when n is a positive integer with $1 \leq n \leq 4$.

SOLUTION - Consider the inequality:

$$n^2 + 1 \geq 2n$$

Rewrite it as:

$$n^2 - 2n + 1 \geq 0$$

Factor it:

$$(n - 1)^2 \geq 0$$

Since the square of any real number is non-negative, $(n - 1)^2 \geq 0$ is always true. Hence, the inequality holds for all n. Specifically, for $n = 1, 2, 3, and 4$:

- **$n = 1$:**

$$1^2 + 1 = 2 \quad and \quad 2 \cdot 1 = 2 \quad \Rightarrow \quad 2 \geq 2$$

- $n = 2$:

$$2^2 + 1 = 5 \quad and \quad 2 \cdot 2 = 4 \quad \Rightarrow \quad 5 \geq 4$$

- $n = 3$:

$$3^2 + 1 = 10 \quad and \quad 2 \cdot 3 = 6 \quad \Rightarrow \quad 10 \geq 6$$

- $n = 4$:

$$4^2 + 1 = 17 \quad and \quad 2 \cdot 4 = 8 \quad \Rightarrow \quad 17 \geq 8$$

Thus, $n^2 + 1 \geq 2n$ **for** $1 \leq n \leq 4$**.**

Problem 3

Find a compound proposition involving the propositional variables $p, q, r,$ **and** s **that is true when exactly three of these propositional variables are true and is false otherwise.**

SOLUTION - One possible compound proposition is:

$$(p \wedge q \wedge r \wedge \neg s) \vee (p \wedge q \wedge \neg r \wedge s) \vee (p \wedge \neg q \wedge r \wedge s) \vee (\neg p \wedge q \wedge r \wedge s)$$

This proposition is true when exactly three of p, q, r, s **are true and false otherwise.**

Problem 4

Let $P(x)$ **and** $Q(x)$ **be propositional functions. Show that** $\exists x (P(x) \to Q(x))$ **and** $\forall x P(x) \to \exists x Q(x)$ **always have the same truth value.**

SOLUTION - 1. Consider $\exists x(P(x) \to Q(x))$: - This statement is true if there exists some x for which $P(x) \to Q(x)$ is true. - $P(x) \to Q(x)$ is true when $P(x)$ is false or $Q(x)$ is true.

2. Consider $\forall x P(x) \to \exists x Q(x)$: - This statement is true if for all x, $P(x)$ implies $\exists x Q(x)$. - $P(x)$ implies $\exists x Q(x)$ means if $P(x)$ is true for any x, there must exist some x for which $Q(x)$ is true.

To show these statements are equivalent, consider:

- If $\exists x(P(x) \to Q(x))$ is true, then there is some x for which $P(x) \to Q(x)$ holds. If $P(x)$ is false for this x, $P(x) \to Q(x)$ is true. If $P(x)$ is true, then $Q(x)$ must be true for this x. Hence, $\exists x Q(x)$ is true.

- If $\exists x Q(x)$ is true, then there is some x for which $Q(x)$ is true. For all x, if $P(x)$ is true, there must be some x for which $Q(x)$ is true, hence $\forall x P(x) \to \exists x Q(x)$ is true.

Therefore, $\exists x(P(x) \to Q(x))$ and $\forall x P(x) \to \exists x Q(x)$ have the same truth value.

Problem 5

Suppose that A and B are sets such that the power set of A is a subset of the power set of B. Does it follow that $A \subseteq B$?

SOLUTION - Yes, it follows that $A \subseteq B$.

- **Given $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, every subset of A is also a subset of B.**

- **The set A itself is a subset of A, so $A \in \mathcal{P}(A)$.**
- **Since $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, A must be in $\mathcal{P}(B)$.**
- **Therefore, $A \subseteq B$.**

Problem 6

Let A and B be sets. Show that $A \subseteq B$ if and only if $A \cap B = A$.

SOLUTION - 1. $\Rightarrow$: Assume $A \subseteq B$.

- **If $x \in A$, then $x \in B$.**
- **Therefore, $x \in A \cap B$.**
- **Since all elements of A are in B, $A \subseteq A \cap B$.**
- **Because $A \subseteq B$, $A \cap B \subseteq A$.**
- **Hence, $A = A \cap B$.**

2. $\Leftarrow$: Assume $A = A \cap B$.

- **If $x \in A$, then $x \in A \cap B$, which implies $x \in B$.**
- **Hence, $A \subseteq B$.**

Thus, $A \subseteq B$ if and only if $A \cap B = A$.