Annotated Natural Deduction for Adaptive Reasoning

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Abstract

We present a multi-conclusion natural deduction calculus based on minimal logic extended with a set of rules characterizing the dynamic reasoning typical of adaptive logics.

1 Intro

In this paper we outline a multiple-conclusion natural deduction calculus in which the dynamics of standard (Fitch-style) dynamic proofs of Adaptive Logics [1] can be reconstructed. Adaptive logics are a family of logics that can be used to formalise a wide range of defeasible reasoning forms. Their consequence-relations rely on the standard idea of interpreting premises as normally as possible through the selection of models of its premises, but it is only at the level of its proof-theory that its distinctive approach comes to the fore. Adaptive logics, namely, reconstruct defeasible reasoning patterns as dynamic proofs; proofs in which steps performed earlier may later be retracted when the assumptions they were based on no longer hold.

The specific system we describe here is for an inconsistency adaptive logic: this is a logic that captures the paraconsistent reasoning performed to avoid triviality in the face of inconsistency, while trying to make up for its deductive weakness by provisionally applying classical inference-rules when there is no explicit indication that inconsistencies are involved in that inference. This choice brings us closer to the original motivations for the development of adaptive logic [3], but also allows us to engage with current philosophical debates of relevance to Priest's work.

The dynamics of retracting earlier lines in a proof can be captured in a rather natural way in linear proof-formats, including standard axiomatic and Fitch-style natural deduction proofs, but is much less straightforward in a tree-like proof-format. Consider, for instance, the following retraction in an application of *Ex Contradictione Quodlibet*:

(1)	p	Prem	$\emptyset$	
(2)	$p \vee q$	Addition	$\emptyset$	
(3)	$\neg p$	Prem	$\emptyset$	
(4)	q	DS	$\{p\}$	$\checkmark^{5}$
(5)	$p \wedge \neg p$	Adjunction	$\emptyset$	

Here, at line (4) disjunctive syllogism (DS) is applied on the condition that p behaves normally, i.e. that the contradiction $p \wedge \neg p$ hasn't been derived. When this contradiction is effectively derived at line (5), the line (4) is marked and is from then on no longer assumed to be part of the proof. This type of reasoning illustrates the idea of provisional applications of classical inference-rules to paraconsistent logics that reject the disjunctive syllogism, but in which the restricted form $\phi \vee \psi, \neg\phi / \psi \vee (\phi \wedge \neg\phi)$ is retained.

Contrast this, now, with the following attempt to reconstruct a similar reasoning-process in a Prawitz-style proof-tree:

$$\dfrac{\mathrm{DS}^*\,\dfrac{\dfrac{\Gamma \vdash p}{\Gamma \vdash p \vee q}\,\vee\mathrm{I} \qquad \Gamma \vdash \neg p \qquad \Gamma \nvdash p \wedge \neg p}{\Gamma \vdash q} \qquad \dfrac{\Gamma \vdash p \qquad \Gamma \vdash \neg p}{\Gamma \vdash p \wedge \neg p}\,\wedge\mathrm{I}}{?}$$

When in this proof an explicit contradiction is derived in the right-hand branch, the assumption of its invalidity (stated explicitly in the left-hand branch) no longer holds. In this format, however, the order used to construct the proof cannot be read off the proof itself (an issue that could easily be fixed). But also, more importantly, it isn't even clear what it might mean to retract the line where q is derived, since the result of removing that line from the proof is in itself no longer a well-formed proof.

The proof-format we propose solves this problem by making two changes: first, we add indices to judgements to keep track of stages in the construction of a proof; and second, we exploit the fact that judgements that are 'marked' at a certain stage do not have to be removed, because there is simply no need to prevent their implicit re-use since every assumption or premise should explicitly be written down in the place it is used. Instead, it is the derivation of the same judgement at a later stage that is (or may be) blocked, because the original assumption that led to its initial derivation probably no longer holds. We therefore provide, for the first time, an appropriate Natural Deduction translation of adaptive reasoning, whose proofs have been so far always been presented in their linear format.

Because this system uses multiple-conclusion judgements, it also explicitly captures the connection between unconditional derivations of certain disjunctions in the paraconsistent logic and the conditional deductions of one of their disjuncts in the adaptive logic. This formal feature can be used to re-assess certain current debates on how one should best approach the question of *classical recapture* in paraconsistent logics. The latter problem can be summarised as follows. When one adopts a logic that is strictly weaker than classical logic, the question of how one should account for epistemically useful classical inference-forms that are invalidated by one's preferred logic almost immediately arises. In the case of paraconsistent logic, this question is often deemed urgent, as the practical and epistemic usefulness of the inference-forms that are lost, like the disjunctive syllogism, are almost undisputed. Inconsistency-adaptive logics present one possible answer to this challenge under the form of defeasible inference-forms that allow one to use classical inference-steps on the condition that certain assumptions are not violated. It is also a response that Graham has endorsed [4]. His specific proposal on how this should be implemented has, in recent years, become the focus of a renewed interest in the problem of how dialetheists should account for classical recapture. We contend that the combination of a multiple-conclusion calculus with the reconstruction of the defeasible dynamics of adaptive proofs can further clarify this debate.

The paper is structured as follows. We introduce in Section 2 a basic natural deduction system called minimalND, which acts as the Lower Limit Logic of our adaptive system. In Section 3, we extend the system to account for adaptive reasoning through the definition of an appropriate abnormal form of expressions and appropriate adaptive rules; the new system is called AdaptiveND. In Section 4 we define a marking strategy to identify derivation step that can no longer be assumed to hold in the tree. In Section 5 we define basic meta-theoretical properties. We return to the challenge of classical recapture in Section 6.

2 minimalND

We start by defining the type universe for the $\{\neg, \rightarrow, \wedge, \vee\}$ fragment of intuitionistic propositional logic corresponding to minimal logic. We call this logic minimalND and use it as the equivalent of a Lower Limit Logic—the paraconsistent logic that governs the unconditional steps in a proof. Contrary to what is standard in an intuitionistic setting, we do not allow the deduction of $\perp$ from an explicit contradiction. Whereas $\perp$ can be eliminated via *Ex Falso Quodlibet*, there is no introduction-rule for $\perp$ and this is what makes our base-logic paraconsistent. It is only when the assumption of consistency is introduced that the connection between negation-inconsistency and absolute inconsistency can provisionally be recreated.

We start by defining the syntax of our language:

Definition 1 (minimalND)**.** *Our starting language for* minimalND *is defined by the following grammar:*

$$\begin{aligned}
&\mathsf{Type} := \mathsf{Prop}\\
&\mathsf{Prop} := A|\perp|\neg\phi \mid \phi_1 \rightarrow \phi_2|\phi_1 \wedge \phi_2|\phi_1 \vee \phi_2\\
&\Gamma := \{\phi_1, \ldots, \phi_n\}\\
&\Delta := \{\phi_1, \ldots, \phi_n\}
\end{aligned}$$

The type universe of reference is the set of propostions Prop, construed by atomic formulas closed under negation, implication, conjunction, disjunction and allowing $\perp$ to express absolute contradictions. Formula formation rules are given in Figure 1.

$$\frac{}{A \in \mathsf{Prop}}\ \textsc{Atom} \qquad \frac{}{\perp \in \mathsf{Prop}}\ \perp \qquad \frac{\phi \in \mathsf{Prop}}{\neg\phi \in \mathsf{Prop}}\ \neg$$

$$\frac{\phi_1 \in \mathsf{Prop} \qquad \phi_2 \in \mathsf{Prop}}{\phi_1 \to \phi_2 \in \mathsf{Prop}}\ \to \qquad \frac{\phi_1 \in \mathsf{Prop} \qquad \phi_2 \in \mathsf{Prop}}{\phi_1 \wedge \phi_2 \in \mathsf{Prop}}\ \wedge$$

$$\frac{\phi_1 \in \mathsf{Prop} \qquad \phi_2 \in \mathsf{Prop}}{\phi_1 \vee \phi_2 \in \mathsf{Prop}}\ \vee$$

Figure 1: Formula Formation Rules

Definition 2 (Judgements)**.** *A* minimalND*-judgement is of the form* $\Gamma; \cdot \vdash_{\mathsf{s}} \Delta$*, where:* Γ *is the usual set of assumptions,* Δ *is a set of formulas of the language and* s *is a positive integer.*

The set Γ on the left-hand side of the derivability sign is to be read conjunctively. Similarly for the semi-colon symbol, which is introduced here but is only used in Section 3 to separate standard assumptions in Γ from conditions (in the adaptive sense). The set Δ and the comma (if it occurs) on the right-hand side of the derivability sign are both to be read disjunctively. This characterizes our calculus as multiple-conclusion. Context formation rules, for both left and right-hand side set of formulas are given in Figure 2. Nil establishes the base case of a valid empty context, we use wf as an abbreviation for 'well-formed'; Γ-Formation allows extension of contexts by propositions; Prem establishes derivability of formulas contained in context (and it defines the equivalent of the adaptive Premise rule); finally, Δ-Formation allows *disjunctive* extension of derived sets of formulas by well-typed ones.this is still an open issue. G: IN WHICH WAY?

$$\frac{}{\cdot \vdash_{\mathsf{s}} \mathsf{wf}}\ \textsc{Nil} \qquad \frac{\Gamma; \cdot \vdash_{\mathsf{s}} \mathsf{wf} \qquad \phi \in \mathsf{Prop}}{\Gamma, \phi; \cdot \vdash_{\mathsf{s+1}} \mathsf{wf}}\ \Gamma\textsc{-formation}$$

$$\frac{\Gamma; \cdot \vdash_{\mathsf{s}} \mathsf{wf} \qquad \phi \in \Gamma}{\Gamma; \cdot \vdash_{\mathsf{s+1}} \phi}\ \textsc{Prem} \qquad \frac{\Gamma; \cdot \vdash_{\mathsf{s}} \Delta \cup \{\phi\}}{\Gamma; \cdot \vdash_{\mathsf{s+1}} \Delta, \phi}\ \Delta\textsc{-formation}$$

Figure 2: Context Formation Rules

The derivability sign is enhanced with a signature s that corresponds to a counter of the ordered derivation steps executed to obtain the corresponding

ND-formula in a tree. This annotation only comes to use in the next extension of the calculus in Section 3.

The semantics of connectives is given in the standard proof-theoretical way by way of Introduction and Elimination Rules in Figure 3. Introduction of $\rightarrow$ corresponds to conditional proof, while its elimination formalises Modus Ponens. Rules for $\wedge$ are standard; notice that $\vee$-Elimination makes the disjunctive reading of the comma on the right hand-side of the turnstile explicit. $\perp$ can be eliminated by *Ex Falso*, but cannot be introduced. Dually, our paraconsistent negation $\neg$ can be introduced, but not eliminated.

$$\frac{\Gamma, \phi_1; \cdot \vdash_{\mathsf{s}} \Delta, \phi_2}{\Gamma; \cdot \vdash_{\mathsf{s}+1} \Delta, \phi_1 \rightarrow \phi_2} \rightarrow\mathrm{I} \qquad \frac{\Gamma; \cdot \vdash_{\mathsf{s}} \Delta, \phi_1 \rightarrow \phi_2 \qquad \Gamma'; \cdot \vdash_{\mathsf{s'}} \Delta', \phi_1}{\Gamma; \Gamma' \vdash_{\mathsf{max(s,s')}+1} \Delta, \Delta', \phi_2} \rightarrow\mathrm{E}$$

$$\frac{\Gamma; \cdot \vdash_{\mathsf{s}} \Delta, \phi_1 \qquad \Gamma'; \cdot \vdash_{\mathsf{s'}} \Delta', \phi_2}{\Gamma, \Gamma'; \cdot \vdash_{\mathsf{max(s,s')}+1} \Delta, \Delta', \phi_1 \wedge \phi_2} \wedge\mathrm{I} \qquad \frac{\Gamma; \cdot \vdash_{\mathsf{s}} \Delta, \phi_1 \wedge \phi_2}{\Gamma; \cdot \vdash_{\mathsf{s}+1} \Delta, \phi_{i \in \{1,2\}}} \wedge\mathrm{E}$$

$$\frac{\Gamma; \cdot \vdash_{\mathsf{s}} \Delta, \phi_1}{\Gamma; \cdot \vdash_{\mathsf{s}+1} \Delta, \phi_1 \vee \phi_2} \vee\mathrm{I} \qquad \frac{\Gamma; \cdot \vdash_{\mathsf{s}} \Delta, \phi_2}{\Gamma; \cdot \vdash_{\mathsf{s}+1} \Delta, \phi_1 \vee \phi_2} \vee\mathrm{I} \qquad \frac{\Gamma; \cdot \vdash_{\mathsf{s}} \Delta, \phi_1 \vee \phi_2}{\Gamma; \cdot \vdash_{\mathsf{s}+1} \Delta, \phi_1, \phi_2} \vee\mathrm{E}$$

$$\frac{\Gamma; \cdot \vdash_{\mathsf{s}} \Delta, \perp}{\Gamma; \cdot \vdash_{\mathsf{s}} \Delta, \phi} \perp\mathrm{E} \qquad \frac{\Gamma; \phi \vdash_{\mathsf{s}} \Delta, \psi}{\Gamma; \cdot \vdash_{\mathsf{s}+1} \Delta, \psi, \neg\phi} \neg\mathrm{I}$$

Figure 3: Rules for I/E of connectives

Finally, we introduce in Figure 4 a set of rules to enforce structural properties. Wleft is a Weakening on the left-hand side of the judgement: it allows the monotonic extension of assumptions preserving already derivable formulas. Notice that this rule can only work with a strictly empty set of formulas $;\cdot$ following Γ: we shall introduce in the next section this as the set of *adaptive conditions*. The reason for this requirement in Wleft is that the set of adaptive conditions strictly depends on the set of assumptions Γ, hence a Weakening of the latter can imply a different formulation of the former. We do not need to formulate a Wright rule for weakening of the set Δ of derivable formulas, as this can be obtained by a detour of $\vee$-Introduction and Elimination. Cleft for Contraction on the left allows elimination of repeated assumptions and Eleft for Exchange on the left is valid just by set construction, as there is no order. Cright and Eright do a similar job on the right-hand side of the judgement. Finally, Cut (also known as Substitution in some Natural Deduction Caluli) guarantees that derivations can be pasted together, and it in generals requires that there

$$\frac{\Gamma; \cdot \vdash_{\mathsf{s}} \Delta, \phi_1}{\Gamma, \phi_2; \cdot \vdash_{\mathsf{s+1}} \Delta, \phi_1}\ \textsc{Wleft} \qquad \frac{\Gamma, \phi_1, \phi_1; \cdot \vdash_{\mathsf{s}} \Delta, \phi_2}{\Gamma, \phi_1; \cdot \vdash_{\mathsf{s+1}} \Delta, \phi_2}\ \textsc{Cleft}$$

$$\frac{\Gamma, \phi_1, \phi_2; \cdot \vdash_{\mathsf{s}} \Delta, \phi_3}{\Gamma, \phi_2, \phi_1; \cdot \vdash_{\mathsf{s+1}} \Delta, \phi_3}\ \textsc{Eleft}$$

$$\frac{\Gamma; \cdot \vdash_{\mathsf{s}} \Delta, \phi_1 \qquad \Gamma', \phi_1; \cdot \vdash_{\mathsf{s'}} \Delta', \phi_2}{\Gamma; \Gamma'; \cdot \vdash_{\mathsf{max(s,s')+1}} \Delta, \Delta', \phi_2}\ \textsc{Cut}$$

$$\frac{\Gamma; \cdot \vdash_{\mathsf{s}} \Delta, \phi, \phi}{\Gamma; \cdot \vdash_{\mathsf{s+1}} \Delta, \phi}\ \textsc{Cright} \qquad \frac{\Gamma; \cdot \vdash_{\mathsf{s}} \Delta, \phi_1, \phi_2}{\Gamma; \cdot \vdash_{\mathsf{s+1}} \Delta, \phi_2, \phi_1}\ \textsc{Eright}$$

Figure 4: Structural Rules

are no clashes of free variables in Γ, Γ'.

The resulting system is equivalent to the propositional fragment of **CLuN**, the logic obtained by adding excluded middle to the positive fragment of classical logic. This is a very weak paraconsistent (but not paracomplete) logic that does not validate any of the usual De Morgan rules [2], and has been used as the Lower Limit Logic of one of the first adaptive logics.

Theorem 1. minimalND *is sound and complete w.r.t. to the propositional fragment of* ***CLuN***.

Proof. Soundness can be verified as usual, with the key step verifying that ($\neg$I) is sound in view of the completeness-clause for negation (if ϕ is False, then $\neg\phi$ is True). Completeness follows from the provability of all **CLuN**-axioms. Below, we only give the proofs for excluded middle and Peirce's Law.

$$\textsc{Cright}\ \cfrac{\cfrac{\cfrac{\cfrac{\textsc{Axiom}\ \cfrac{}{p; \cdot \vdash_1 p}}{\emptyset; \cdot \vdash_2 p, \neg p}\ \neg\mathrm{I}}{\emptyset; \cdot \vdash_3 p \vee \neg p, \neg p}\ \vee\mathrm{I}}{\emptyset; \cdot \vdash_4 p \vee \neg p, p \vee \neg p}\ \vee\mathrm{I}}{\emptyset; \cdot \vdash_5 p \vee \neg p}$$

$$\cfrac{\cfrac{\cfrac{\to\mathrm{I}\ \cfrac{\vee\mathrm{I}\ \vee\mathrm{E}\ \cfrac{p \vdash_1 p}{p \vdash_2 p, q}}{\vdash_3 p, p \to q} \qquad \cfrac{}{(p \to q) \to p \vdash_4 (p \to q) \to p}\ \textsc{Prem}}{(p \to q) \to p \vdash_5 p, p}\ \to\mathrm{E}}{(p \to q) \to p \vdash_6 p}\ \textsc{Cright}}{\vdash_7 ((p \to q) \to p) \to p}\ \to\mathrm{I}$$

□

3 AdaptiveND

We now extend minimalND to characterize a new logic called AdaptiveND to allow for inconsistency adaptive reasoning. To this aim one needs:

1. the explicit formulation of an Ω set of propositions;
2. the formulation of judgements including an *adaptive condition*;
3. the formulation of a rule that allows to derive new formulas independent from such an adaptive condition;
4. the formulation of a rule that allows to derive new formulas that depend from such an adaptive condition.

We offer accordingly new definitions for the syntax of this logic and the related form of judgements.

Definition 3 (AdaptiveND)**.** *The language of* AdaptiveND *is as follows:*

$$\begin{array}{l} \mathsf{Type} := \mathsf{Prop} \\ \mathsf{Prop} := A|\bot|\neg\phi|\phi_1 \rightarrow \phi_2|\phi_1 \wedge \phi_2|\phi_1 \vee \phi_2 \\ \Gamma := \{\phi_1, \ldots, \phi_n\} \\ \Delta := \{\phi_1, \ldots, \phi_n\} \\ \Omega := \{\phi \wedge \neg\phi \mid \phi \in Prop\} \end{array}$$

Definition 4 (Judgements)**.** *An* AdaptiveND*-judgement is of the form* $\Gamma; \Theta^- \vdash_s \Delta$*, where:*

1. *the left-hand side of* $\vdash_\mathsf{s}$ *has* Γ *as in* minimalND*;*
2. *the semicolon sign on the left-hand side of* $\vdash_\mathsf{s}$ *is conjunctive;*
3. Θ *refers to a finite subset of* Ω*, i.e. a set of formulas of a specific inconsistent logical form; we write* ϕ *instead of* $\{\phi\}$ *when* Θ *is the singleton* $\{\phi\}$*; below we introduce an appropriate* Ω*-formation rule;*[1]
4. *the last place of the left-hand side context is always reserved to negated formulas of type* Ω*; we shall use* ϕ^- *to refer to the negation of* ϕ*, and* Θ^- *for* $\{\phi^- \mid \phi \in \Theta\}$*;*
5. *the right-hand side is in disjunctive form.*

When the second place on the left-hand side of $\vdash$ is empty, we shall write $\Gamma; \cdot \vdash$, thus reducing to the form of a minimalND-judgement. Moreover, in AdaptiveND, the annotation on the proof stage s is optionally followed by one of the following two marks: $\boxtimes$ to mark that at the current stage some previously derived formula is retracted; $\checkmark$ to mark that at the current stage some previously

[1] As mentioned above, the current setting of AdaptiveND is specified for an inconsistency-adaptive logic.

derived formulas is now stable, i.e. will no longer be marked by $\boxtimes$. These symbols will be formally introduced in Sections 4 and 5 respectively.

We now introduce the rules for $\mathsf{AdaptiveND}$. In Figure 5, we describe the formation and use of formulas $\phi \in \Omega$. By Ω-$\mathsf{Formation}$, the explicit contradiction $\phi \wedge \neg\phi$, with ϕ any proposition, is a formula of the Ω type. In the Adaptive tradition formulas of type Ω are called an *abnormality* or *abnormal formula*. By $\mathsf{Adaptive\ Condition\ Formation}$, given a valid context Γ and a formula ϕ of the Ω type, a context Γ followed by the Adaptive Condition that expresses the defeasible assumption that ϕ *is false*, is a well-formed context. This corresponds to the use of conditions as additional elements of proof line in the standard linear proof format of adaptive logic. By $\mathsf{Adaptive\ Condition\ Extension}$, a newly derived formula of type Ω can be added to an existing non-empty Adaptive Condition.

$$\frac{\phi \in \mathsf{Prop}}{(\phi \wedge \neg\phi) \in \Omega}\ \Omega\text{-FORMATION}$$

$$\frac{\Gamma; \cdot \vdash_{\mathsf{s}} \mathsf{wf} \qquad \phi \in \Omega}{\Gamma; \phi^- \vdash_{\mathsf{s+1}} \mathsf{wf}}\ \text{ADAPTIVE CONDITION-FORMATION}$$

$$\frac{\Gamma; \Theta^- \vdash_{\mathsf{s}} \mathsf{wf} \qquad \phi \in \Omega}{\Gamma; \Theta^-, \phi^- \vdash_{\mathsf{s+1}} \mathsf{wf}}\ \text{ADAPTIVE CONDITION-EXTENSION}$$

Figure 5: Ω Formation rules

Next, the calculus is extended by introducing the conditional rule *RC* (Figure 6), which states that if a disjunction ψ, ϕ is derivable from Γ, with ϕ an abnormal formula, then ψ can also be derived alone under Γ and the Adaptive Condition that ϕ be false. Because the application of *RC* can be delayed by keeping formulae of type Ω on the right hand-side of the turnstile, the role of the unconditional rules of the standard calculus is subsumed under the usual cut-rule. The single and multi-premise versions of the unconditional rules displayed in Figure 7 can thus be treated as derived rules as shown in Figure 8.

$$\frac{\Gamma; \cdot \vdash_{\mathsf{s}} \psi, \phi \qquad \phi \in \Omega}{\Gamma; \phi^- \vdash_{\mathsf{s+1}} \psi}\ \text{RC}$$

Figure 6: Conditional Rule

The Adaptive strategy developed in the next Section has the aim of establishing which abnormal formulas can no longer be safely considered as conclusions of an $\mathsf{Adaptive\ Condition\ Formation}$ Rule, thereby requiring a retraction of the formulas that are derivable from it. To this aim, it is essential to establish min-

$$\frac{\Gamma;\phi^- \vdash_{\mathsf{s}} \phi_1 \qquad \phi_1;\cdot \vdash_{\mathsf{s+1}} \phi_2}{\Gamma,\phi_1;\phi^- \vdash_{\mathsf{s+2}} \phi_2}\ \mathrm{RU}$$

$$\frac{\Gamma;\phi^- \vdash_{\mathsf{s}} \phi_1 \qquad \Gamma';\phi'^- \vdash_{\mathsf{s+1}} \phi_2 \qquad \phi_1,\phi_2;\cdot \vdash_{\mathsf{s+2}} \phi_3}{\Gamma,\Gamma';(\phi\cup\phi')^- \vdash_{\mathsf{s+3}} \phi_3}\ \mathrm{RU2}$$

Figure 7: Unconditional Rules

$$\frac{\dfrac{\Gamma;\cdot \vdash_1 \phi_1,\phi \qquad \phi_1;\cdot \vdash_2 \phi_2}{\Gamma,\phi_1;\cdot \vdash_3 \phi_2,\phi}\ \textsc{Cut} \qquad \phi\in\Omega}{\Gamma,\phi_1;\phi^- \vdash_4 \phi_2}\ \mathrm{RC}$$

$$\frac{\dfrac{\Gamma;\cdot \vdash_1 \phi_1,\phi \qquad \Gamma';\cdot \vdash_2 \phi_2,\phi' \qquad \phi_1,\phi_2;\cdot \vdash_3 \phi_3}{\Gamma;\cdot \vdash_4 \phi_1,\phi \qquad \Gamma',\phi_1;\cdot \vdash_5 \phi_3,\phi'}\ \textsc{Cut}}{\dfrac{\dfrac{\Gamma,\Gamma';\cdot \vdash_6 \phi_3,\phi,\phi' \qquad \phi\in\Omega}{\Gamma,\Gamma';\phi^- \vdash_7 \phi_3,\phi' \qquad \phi'\in\Omega}\ \mathrm{RC}}{\Gamma,\Gamma';(\phi\cup\phi')^- \vdash_8 \phi_3}\ \mathrm{RC}}\ \textsc{Cut}$$

Figure 8: Redundancy of unconditional rules

imal disjunctions of such formulas, denoted by $\bigvee(\Delta^{min})$, with $\Delta \in \Omega$. The rule in Figure 9 establishes the construction of such minimal disjunctions. It says that a disjunctive formula of the Ω type derived at some stage s of a derivation can be considered minimal at stage $\mathsf{s'}$ if at no previous stage $\mathsf{t} < \mathsf{s'}$ a shorter one can be derived in the same context Γ.

$$\frac{\Gamma;\cdot \vdash_{\mathsf{s}} \Delta \qquad \Delta\subset\Omega \qquad \text{with no } \Delta' \subseteq \Delta \in \Omega,\ \text{s.t. } \Gamma;\cdot \vdash_{\mathsf{t<s'}} \Delta'}{\Gamma;\cdot \vdash_{\mathsf{s'}} \Delta^{min}}\ \textsc{MinDab}$$

Figure 9: Minimal Abnormal Formulas Rule

The derivation of minimal disjunction of abnormalities is a process that occurs along with the development of the proof-tree. This means that the following procedure to mark formulas depend on the possible derivation of certain such formulas.

3.1 A simple example

We present here a simple derivation in AdaptiveND, where $\Gamma = \{(\neg p \vee q), p, (p \rightarrow q)\}$:

$$\dfrac{\dfrac{\dfrac{}{\Gamma; \cdot \vdash_1 (\neg p \vee q)}\text{PREM}}{\Gamma; \cdot \vdash_2 \neg p, q}\vee\text{E} \qquad \dfrac{}{\Gamma; \cdot \vdash_3 p}\text{PREM}}{\Gamma; \cdot \vdash_4 (p \wedge \neg p), q}\wedge\text{I} \qquad (p \wedge \neg p) \in \Omega \over \Gamma; (p \wedge \neg p)^- \vdash_5 q} \text{RC}$$

The derivation above up to stage 4 is a obtained by MinimalND rules. Stage 5 derives a formula on condition of the abnormality $(p \wedge \neg p)$ being false. This corresponds to changing a multiple conclusion judgement at stage 4 into a single conclusion one at stage 5 by turning one of the conclusions into an adaptive condition. This move is justified by the syntactical form of the abnormality declared by the RC rule.

4 Rules for Marking

In standard Adaptive Logics, one introduces strategies to tell, given some judgement deriving a Minimal Disjunction of Abnormalities, which one of the disjunct can be assumed to be false, i.e. for which one a RC rule can be applied; and which one has to be accepted. Accordingly, formulas derived under the former can be considered valid, formulas previously derived by assuming the latter false have to be retracted. Adaptive Logics come with marking mechanisms that allow such retractions, according to different possible strategies. The most well-known strategies and their rationale are:[2]

- *Reliability*: once a $\bigvee(\Delta^{min})$ is derived at some stage s, *every* formula assuming at some stage $\mathsf{s\text{-}i}$ a $\phi \in \bigvee(\Delta^{min})$ to be false, needs to be retracted;
- *Minimal Abnormality*: once a $\bigvee(\Delta^{min})$ is derived at some stage s, *every* formula assuming at some stage $\mathsf{s\text{-}i}$ a $\phi \in \bigvee(\Delta^{min})$ to be false and such that ϕ is in a minimal set of Δ, needs to be retracted.

In the first case, one considers all possible abnormal formulas to be invalid; in the second case, one tries to minimize the number of such unavoidable contradictions. In this section, we provide rules that extend AdaptiveND in view of the Reliability strategy, providing a proof-theoretical equivalent of the standard marking condition. We leave the definition of a proof-theoretical Minimal Abnormality strategy to a later stage.

[2]Some reference here?

4.1 Marking Rule for Reliablity

Reliability is the derivability strategy that takes the most cautious interpretation of abnormalities: any formula that in view of the premises might behave abnormally, because it occurs in a minimal disjunction of abnormalities, is deemed unreliable and should not be assumed to behave normally. This means in practice that at any proof stage where a formula ψ is derived using some $\phi^- \in \Omega$ in context by an instance of the corresponding formation rule, is 'marked'. Here marking means to reject the formula ψ, or invalidate it. In the following we shall introduce a new derivability rule that internalizes this process in AdaptiveND.

We define a new derivability rule $\boxtimes$R that depends on the formulation of the union set of all minimal $\bigvee(\Delta^{min})$ obtained by instances of the MinDab rule above.

$$\frac{\Gamma; \cdot \vdash_{\mathsf{s}} \Delta^{min} \qquad \Gamma; \phi^- \vdash_{\mathsf{s'}} \psi \qquad \phi \in \Delta^{min}}{\Gamma \vdash_{\mathsf{max(s,s')+1\boxtimes R}} \psi} \boxtimes\mathrm{R}$$

The meaning of $\boxtimes$R is the following: if at stage s a minimal disjunction of abnormalities Δ^{min} is derived for Γ, and at a later stage $\mathsf{s'}$ a formula ψ is derived from the same premise set by assuming a component of Δ^{min} false by an Adaptive Condition Formation rule, then at a next stage ψ is marked as retracted.

4.2 Extending the example

Let us now extend the example from Section 3.1 with a new branch to illustrate the derivation step obtained by the Marking Rule $\boxtimes$R. Let us call $\mathbb{D}$ the derivation already shown, which had as a conclusion at stage $\mathsf{5}$ the derivation of q in context Γ and under the adaptive condition that $(p \wedge \neg p)$ is false. We extend it now as follows:

$$\frac{\dfrac{\mathbb{D}}{\Gamma; (p \wedge \neg p)^- \vdash_{\mathsf{5}} q} \qquad \dfrac{\dfrac{\Gamma; \cdot \vdash_{\mathsf{6}} p \qquad \Gamma; \cdot \vdash_{\mathsf{7}} p \to \neg p}{\Gamma; \cdot \vdash_{\mathsf{8}} \neg p} \to \mathrm{E} \qquad \Gamma; \cdot \vdash_{\mathsf{9}} p}{\Gamma; \cdot \vdash_{\mathsf{10}} p \wedge \neg p} \wedge \mathrm{I}}{\Gamma; \cdot \vdash_{\mathsf{11\boxtimes R}} q} \boxtimes\mathrm{R}$$

In this derivation a new abnormality is derived at stage $\mathsf{10}$, namely the same that is assumed to be false at stage $\mathsf{5}$. Notice that it is essential that this abnormality be derived under an empty condition, i.e. under context $\Gamma; \cdot$, as explained above for the required strict condition on WLeft. Moreover, a difference between the Fitch-style proofs standard for Adaptive Logics and the Natural Deduction derivation style becomes here evident. In the former, a marking rule implies the need to proceed backwards on the derivation, to mark all previous occurrences of the marked formula which can no longer be considered derived.

In the latter, on the other hand, there is no need to remove formulas because the result obtained at stage 5 cannot be reused in an extension of this proof. Instead a new derivation step is performed (stage 11), where the conclusion q is marked. Moreover, if we were ever to get again $\Gamma; (p \wedge \neg p)^{-} \vdash_{\mathsf{i}} q$, that would be obtained by some new derivation $\mathbb{D}'$ and therefore result as a conclusion at some stage $\mathsf{i} > 11$.

4.3 An example with $\bigvee(\Delta^{min})$-selection

The previous example is rather simple, in that it shows a formula that is first derived under an adaptive condition (referring to an abnormal formula assumed to be false), and then retracted after that condition is validated again.

Let us consider now a slightly more complex example. We want to show a situation in which a disjunction of two abnormalities can be derived: accordingly, there might be more than one formula to be marked. Let us start with a premise set $\Gamma = \{(p \vee r), \neg p, (p \vee q), \neg q, (\neg p \to q)\}$. Now consider the following derivation, dubbed $\mathbb{D}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma; \cdot \vdash_{\mathsf{1}} (p \vee r)}\ \text{PREM}}{\Gamma; \cdot \vdash_{\mathsf{2}} p, r}\ \vee\text{E} \qquad \dfrac{}{\Gamma; \cdot \vdash_{\mathsf{3}} \neg p}\ \text{PREM}}{\Gamma; \cdot \vdash_{\mathsf{4}} (p \wedge \neg p), r}\ \wedge\text{I} \qquad (p \wedge \neg p) \in \Omega}{\Gamma; (p \wedge \neg p)^{-} \vdash_{\mathsf{5}} r}\ \text{RC}$$

At stage 4 a disjunction of an abnormality with r is derived, and by RC at stage 6 the formula r is derived alone, assuming the relevant abnormality false. Let us now consider the following derivation, dubbed $\mathbb{D}^{'}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma; \cdot \vdash_{\mathsf{6}} (p \vee q)}\ \text{PREM}}{\Gamma; \cdot \vdash_{\mathsf{7}} p, q}\ \vee\text{E} \qquad \dfrac{}{\Gamma; \cdot \vdash_{\mathsf{8}} \neg p}\ \text{PREM}}{\Gamma; \cdot \vdash_{\mathsf{9}} (p \wedge \neg p), r}\ \wedge\text{I} \qquad \dfrac{}{\Gamma; \cdot \vdash_{\mathsf{10}} \neg q}}{\Gamma; \cdot \vdash_{\mathsf{11}} (p \wedge \neg p), (q \wedge \neg q)}\ \text{RC}$$

Here the previously derived abnormality $(p \wedge \neg p)$ is derived in disjunctive form with a new abnormality $(q \wedge \neg q)$ at stage 11, where the latter is obtained by $\wedge$I from stages 7,9. If we join now the two branches $\mathbb{D}, \mathbb{D}'$ to form $\mathbb{E}$, we are allowed a marking step:

$$\cfrac{\cfrac{\mathbb{D}}{\Gamma;(p\wedge\neg p)^{-}\vdash_{5} r} \qquad \cfrac{\mathbb{D}'}{\Gamma;\cdot\vdash_{11}(p\wedge\neg p),(q\wedge\neg q)} \qquad (p\wedge\neg p)\in\Delta^{min}}{\Gamma;\cdot\vdash_{12\boxtimes\mathsf{R}} r}\ \boxtimes$$

At stage 12 the formula r is no longer valid, because its adaptive condition is a minimal abnormal formula of a derived disjunction of abnormalities. Now we can provide a further extension of this derivation dubbed $\mathbb{D}''$:

$$\cfrac{\cfrac{\cfrac{}{\Gamma;\cdot\vdash_{13}\neg p}\ \text{PREM} \qquad \cfrac{}{\Gamma;\cdot\vdash_{14}\neg p\to q}\ \text{PREM}}{\Gamma;\cdot\vdash_{15} q}\ \to\text{I} \qquad \cfrac{}{\Gamma;\cdot\vdash_{16}\neg q}\ \text{PREM}}{\Gamma;\cdot\vdash_{17} q\wedge\neg q}\ \wedge\text{I}$$

$\mathbb{D}''$ has the effect of producing a new minimal abnormality at stage 17. This also means that if we obtain a copy of derivation $\mathbb{D}$, where each step is renumbered consecutively, and join it to $\mathbb{D}''$ and $\mathbb{E}$, it is possible to establish again $(p\wedge\neg p)$ as an adaptive condition and accoridngly derive again the judgement that was marked at stage 12, as follows:

$$\cfrac{\cfrac{\mathbb{E}}{\Gamma;\cdot\vdash_{12\boxtimes\mathsf{R}} r} \qquad \cfrac{\mathbb{D}''}{\Gamma;\cdot\vdash_{17} q\wedge\neg q} \qquad \cfrac{\mathbb{D}}{\Gamma;\cdot\vdash_{18}(p\wedge\neg p),r}}{\Gamma;(p\wedge\neg p)^{-}\vdash_{19} r}\ \text{RC*}$$

where $*$ is the side condition that $(p\wedge\neg p)\in\Omega$. If the derivation is no longer extended, the formula r can be considered finally derived. In the next section we complete our system with the required meta-theoretical analysis needed to define derivability at stage and final derivability.

5 Derivability

In the example from the previous section we have illustrated how the marking condition establishes a dynamic derivability relation, which allows to derive formulas and retract them. Whenever a certain formula is derived on some $\phi\in\Delta^{min}$ adaptive condition, it might still be marked afterwards according to $\boxtimes R$. This gives us a notion of derivability at stage:

Definition 5 (Derivability at stage)**.** *A formula ψ is derived at stage* s *iff* $\Gamma; \phi^- \vdash_{\mathsf{s}} \psi$ *and it is not the case that* $\Gamma; \cdot \vdash_{\mathsf{s}\boxtimes} \psi$*.*

A more stable notion of derivability holds when marking is no longer possible. To this aim, one requires that the stage s at which a formula ϕ is derived remains unmarked in all the extensions of the derivation tree which can be obtained by using all *relevant* abnormalities as adaptive conditions. This relevance criterion is essential if one wants to guarantee finite surveyability of the proof tree to establish whether a formula is never marked (again). We define therefore a set of *abnormalities relevant to* Γ. To do so we first identify the union set of all subformulas of the premise set Γ:

Definition 6 (Subformulas of the premise set)**.** $Sf(\Gamma) = \bigcup_{\phi \in \Gamma}\{\psi \mid \psi$ *is a subformula of* $\phi\}$*.*

From $Sf(\Gamma)$ we the construe all the possible abnormalities which can be obtained by its members:

Definition 7 (Abnormalities relevant to the premise set)**.** $\Omega(\Gamma) = \{\psi \wedge \neg\psi \mid \psi \in Sf(\Gamma)\}$*.*

Given this preparatory work, we now introduce the notion of a *complete proof-tree* with respect to derivable relevant disjunction of abnormalities:

Definition 8 (Completeness relative to relevant abnormalities)**.** *Let P be a proof-tree of* AdaptiveND*. We say that P is complete relative to $\Omega(\Gamma)$ at stage* s *if* $\Gamma; \cdot \vdash_{\mathsf{s'}} \bigvee(\Delta)$*, for every derivable $\bigvee(\Delta)$ with $\Delta \subseteq \Omega(\Gamma)$ and some* $\mathsf{s'}<\mathsf{s}$*.*

We can now formulate our notion of final derivability:

Definition 9 (Final Derivability)**.** *A formula ψ is finally derived* $\Gamma; \phi^- \vdash_{\checkmark} \psi$ *iff there is a stage* s *in some complete proof-tree P such that* $\Gamma; \phi^- \vdash_{\mathsf{s}} \psi$ *and* $\phi \notin \Delta^{min}$*.*

The definition guarantees final derivability for any derived formula whose adaptive condition is not minimal.

6 Classical recapture: logical options and defeasible inferences

When one adopts a logic that is strictly weaker than classical logic, the question of how one should account for epistemically useful classical inference-forms that are invalidated by one's preferred logic almost immediately arises. In the case of paraconsistent logic this question is often deemed quite urgent, as the practical and epistemic usefulness of the inference-forms that are lost, like the disjunctive syllogism, are almost undisputed. This is the problem of *classical recapture*. Inconsistency-adaptive logics present one possible answer to this challenge under the form of defeasible inference-forms that allow one to use classical inference-steps on the condition that certain assumptions are not violated. The class of adaptive logics that have been formulated since the earliest formulation of this paradigm as a response to the problem of classical recapture generalise this idea, and provide a general framework in which many types of defeasible inference-forms can be rigourously formalised.

In several papers Graham Priest has explicitly endorsed this type of approach to the classical recapture problem of the paraconsistent logic **LP**, and has defended his own preferred formalisation and motivation for what he calls *Minimally Inconsistent LP*. When compared to the systems developed in the adaptive logic tradition, this system stands out because it is based on a strong paraconsistent logic that nevertheless lacks a detachable implication (which makes the problem of classical recapture even more stringent), but also because it is put forward as a potential universal inference engine that can be applied in all contexts. At the level of its formal characterisation, minimally inconsistent LP differs from its closest neighbour ACLuNs^m

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