A Modal Logic with Context-Dependent Inference for Non-Monotonic Reasoning

Philippe Besnard* and **Yao-Hua Tan**†

Abstract

Contextual logic CoL is specified through an inference system which takes into account the *context* of reasoning, i.e. all given facts whatsoever. That is to say, conclusions are inferred with respect to the full set of premises. Hence, statements are relative to a context and they are expressed by means of formulas indexed with sets of formulas. A modal extension ECoL is also proposed for which examples are given that illustrate how natural it is to apply such a logic for the formalization of non-monotonic reasoning.

Introduction

In non-monotonic logics (default logic [Reiter 1980] or circumscription [McCarthy 1980] among many others), extra-logical means only are responsible for non-monotonicity to arise. For example, default logic is based on monotonic reasoning as given by classical logic. If it were not for the so-called default extensions, default logic would be monotonic.

Circumscription captures non-monotonic reasoning due to circumscription axioms, but the logic applying to these axioms and the logic by which conclusions are derived is again old, monotonic, classical logic. Here, we present contextual logic, which enjoys a genuine non-monotonic inference system with no extra-logical part to take care of non-monotonic reasoning. We provide contextual logic with a natural deduction system extended to handle indexed formulas (i.e. formulas with a context) and to include a rule that introduces default assumptions in derivations.

Indeed, the basic idea of reasoning with contextual logic is the following. Contexts are sets of formulas that are added as indices to other formulas. An example of a contextual formula is $q_{\{p,p\to q\}}$. Roughly speaking, the context specifies the set of formulas with respect to which a conclusion is to be expressed: in $q_{\{p,p\to q\}}$, the context $\{p, p \to q\}$ indicates that the case under consideration is whether q is a conclusion when exactly the formulas composing the context are given.

Importantly, a set of premises can have, as a conclusion, q_C but not $q_{C'}$ (where $C \neq C'$). Now, contexts are most useful in admitting ***default assumptions***, that are of the form $D\varphi$. For example, in $q_{\{p\to q,Dp\}}$, the context $\{p \to q, Dp\}$ refers to q being a conclusion from the formula $\{p \to q\}$ and the default assumption p. When derived from a set of premises, either of $q_{\{p,p\to q\}}$ and $q_{\{p\to q,Dp\}}$ means that q is concluded but not with the same status (the latter is weaker).

We will discuss several contextual logics in this paper. First, we will introduce the contextual logic CoL, which is the simplest contextual logic. This logic will be used to illustrate the basic ideas that underly contextual logic. Subsequently, we will discuss a modal extension of CoL, the so-called extended contextual logic, which allow for a more relaxed use of the D operator, and hence is more expressive than CoL.

The Syntax of Contextual Logic

We introduce *contextual logic*, abbreviated as CoL, by first giving its language as follows. The syntax, which is propositional for simplicity, is defined in three steps. We first consider a standard propositional language $\mathcal{L}_0$, and we extend $\mathcal{L}_0$ with special formulas called default assumptions. This gives us the language $\mathcal{L}_D$. Finally, we use $\mathcal{L}_D$ to define the syntax of the language $\mathcal{L}$ which consists of propositional formulas indexed with a set of $\mathcal{L}_D$-formulas — the so-called context.

Let $\mathcal{L}_0$ be a standard propositional language. Lowercase letters p, q, r denote atomic propositions. Complex formulas of $\mathcal{L}_0$ are built up in the usual way with the connectives $\neg, \wedge, \vee, \to$ and $\bot$. The Greek letters of the end of the alphabet φ, ψ, χ will denote well-formed formulas of $\mathcal{L}_0$.

We proceed to define the language $\mathcal{L}_D$ which extends $\mathcal{L}_0$ with formulas called default assumptions (see below). $\mathcal{L}_D$ contains all formulas of $\mathcal{L}_0$, and in addition $\mathcal{L}_D$ contains all formulas of $\mathcal{L}_0$ with a default operator D in front. The well-formed formulas of the language

*IRISA, Campus de Beaulieu, 35042 Rennes Cédex, France, besnard@irisa.fr

†EURIDIS, Erasmus University, P.O. Box 1738, 3000 DR Rotterdam, The Netherlands, ytan@euridis.fbk.eur.nl

§This research is supported by the EU via the project Drums-2 number 6156 within the Esprit-III BRA program.

$\mathcal{L}_D$ are defined as follows.

1. If φ is a formula of $\mathcal{L}_0$, then φ is an $\mathcal{L}_D$-formula,
2. If φ is a formula of $\mathcal{L}_0$, then $D\varphi$ is an $\mathcal{L}_D$-formula.

Note that the default operator never occurs in a subformula. In particular, there is no iteration of default operators: e.g. DDp is not an $\mathcal{L}_D$-formula.

If an $\mathcal{L}_D$-formula has the form $D\varphi$, then it is called a *default assumption*.

The Greek letters of the beginning of the alphabet $\alpha, \beta, \gamma, \delta$ will denote well-formed formulas of $\mathcal{L}_D$.

The language $\mathcal{L}$ consists of propositional formulas indexed with a set of $\mathcal{L}_D$-formulas, by virtue of the following single rule.

- If φ is a formula of $\mathcal{L}_0$ and C is a finite set of $\mathcal{L}_D$-formulas, then φ_C is an $\mathcal{L}$-formula.

The set C in φ_C is called the *context* of φ. In the case where C of φ_C is empty, we usually omit the context C. Formulas with the empty context are used as auxiliary *assumptions* in contextual derivations. If a formula has the form $\varphi_{\{\varphi\}}$, then we say that this formula is a *premise*. The difference between assumptions and premises is at the core of contextual logic as will appear in the sequel, especially when contextual logic is compared with assumption-based truth maintenance systems.

The symbol Σ will be used to denote a set of (well-formed) $\mathcal{L}$-formulas. Most often, it will stand for a set of premises.

We will sometimes need to consider the subset of all default assumptions contained in a context C, written $A(C)$:

$$A(C) = \{\alpha | \alpha \in C \text{ and } \alpha \in \mathcal{L}_D - \mathcal{L}_0\}.$$

In addition, we will sometimes need to consider the "flattened" version of a context C, written $F(C)$, meaning that all occurrences of the default operator in C are deleted:

$$F(C) = \{\varphi | \varphi \in C \text{ and } \varphi \in \mathcal{L}_0\} \cup \{\varphi | D\varphi \in A(C)\}.$$

For example, if $C = \{p, p \wedge r \rightarrow q, Dr\}$ then $F(C) = \{p, p \wedge r \rightarrow q, r\}$.

Contextual Natural Deduction

We now present a non-monotonic inference operator $\mid\!\sim$ that characterizes contextual logic. We do it by specifying *contextual natural deduction*, an inference system for CoL that extends the classical method of natural deduction. The inference operator is a function from $\mathcal{L}$-formulas to $\mathcal{L}$-formulas. We will define the inference operator $\Sigma \mid\!\sim^{\Pi} \varphi$, where Σ is a set of premises (hence a subset of $\mathcal{L}$), Π is a set of default assumptions and φ_C is a contextual formula (an element of $\mathcal{L}$).

CoL has the following inference rules for contextual natural deduction, that define an inference operator $\vdash$.

Default Introduction Rule (ID)

$$\frac{}{\varphi_{\{D\varphi\}}}$$

This rule says that at any stage of the derivation we can simply introduce φ indexed by the default assumption $\{D\varphi\}$.

Context Expansion Rule (CE)

$$\frac{\varphi_{C_1}}{\varphi_{C_1 \cup C_2}}$$

An overall constraint on contexts guarantees that they are not over-expanded (it is the reason for the set Π introduced in the sequel).

Implication Elimination Rule (E→)

$$\frac{\varphi_{C_1} \qquad (\varphi \rightarrow \psi)_{C_2}}{\psi_{C_1 \cup C_2}}$$

Implication Introduction Rule (I→)

$$\begin{array}{ll} \varphi_{\{\}} & \text{assumption [discharged]} \\ \vdots & \\ \dfrac{\psi_C}{(\varphi \rightarrow \psi)_C} & \end{array}$$

Note that the assumption $\varphi_{\{\}}$ always has an empty context. The rationale for this is that assumptions (that will be discharged later!) should never introduce new contexts in the reasoning process.

Conjunction Introduction Rule (I∧)

$$\frac{\varphi_{C_1} \qquad \psi_{C_2}}{(\varphi \wedge \psi)_{C_1 \cup C_2}}$$

Conjunction Elimination Rule (E∧)

$$\frac{(\varphi \wedge \psi)_C}{\varphi_C} \qquad \frac{(\varphi \wedge \psi)_C}{\psi_C}$$

Disjunction Elimination Rule (E∨)

$$\begin{array}{llll} & \varphi_{\{\}} & \psi_{\{\}} & \text{assumptions [discharged]} \\ & \vdots & \vdots & \\ \multicolumn{3}{l}{\dfrac{(\varphi \vee \psi)_{C_1} \quad \chi_{C_2} \quad \chi_{C_3}}{\chi_{C_1 \cup C_2 \cup C_3}}} & \end{array}$$

Disjunction Introduction Rule (I∨)

$$\frac{\varphi_C}{(\varphi \vee \psi)_C} \qquad \frac{\psi_C}{(\varphi \vee \psi)_C}$$

Negation Introduction Rule (I¬)

$$\begin{array}{ll} \varphi_{\{\}} & \text{assumption [discharged]} \\ \vdots & \\ \dfrac{\perp_C}{\neg\varphi_C} & \end{array}$$

Negation Elimination Rule (E¬)

$$\frac{\varphi_{C_1} \qquad \neg\varphi_{C_2}}{\perp_{C_1 \cup C_2}}$$

Double Negation Rule (DNR)

$$\frac{\neg\neg\varphi_C}{\varphi_C}$$

Falsity Elimination Rule (E$\bot$)

$$\frac{\bot_C}{\varphi_C}$$

If Σ is a set of premises (hence $\mathcal{L}$-formulas), then $P(\Sigma)$ denotes the set of propositional formulas (that is, from $\mathcal{L}_0$), which are the formulas from Σ with their context removed.

We define two inference operators $\vdash$ and $\mid\!\sim$. The first operator $\Sigma \vdash \varphi_C$ denotes that the formula φ_C is derived from the set of premises Σ with the rules of contextual natural deduction in the usual way (we just follow [Prawitz 1965] as concerns natural deduction). The operator $\vdash$ is monotonic. The second operator $\Sigma \mid\!\sim^{\Pi} \varphi_C$ is non-monotonic, and is defined in terms of the first operator in the following way.

Requirement of Maximal Contexts for Default Conclusions

Let Σ be a set of premises and Π be a set of default assumptions. A formula φ_C is an *acceptable* Π-*default conclusion* from Σ, written $\Sigma \mid\!\sim^{\Pi} \varphi_C$, iff

(i) $\Sigma \vdash \varphi_C$,

(ii) $C = P(\Sigma) \cup \Delta$ for some $\Delta \subseteq \Pi$,

(iii) $F(C)$ is consistent wrt $P(\Sigma)$.

By a formula ω consistent with respect to a set of formulas Λ, we mean that $\Lambda \cup \{\omega\}$ must be consistent if Λ is. Of course, extension from ω (in fact, $\{\omega\}$) to the case of a finite set of formulas Ω is unproblematic.

We write $\Sigma \mid\!\sim \varphi_C$ instead of $\Sigma \mid\!\sim^{\Pi} \varphi_C$ when it does not matter what exact stock of default assumptions Π is given.

Intuitively, Π provides the list of potential default assumptions from which we can freely draw for derivations. We presumably do not use all elements in Π (especially if Π contains contradictory default assumptions $D\varphi$ and $D\neg\varphi$).

If $\Sigma \mid\!\sim^{\Pi} \varphi_C$, we also say that φ_C is a *final conclusion* from Σ wrt Π. If simply $\Sigma \vdash \varphi_C$, we say that φ_C is an *intermediate conclusion* from Σ.

Note that if a context C is default-free, then the intermediate and final conclusions are the same. This distinction between intermediate and final conclusions might look strange in classical logic, where deduction works as an any-time algorithm, but it is quite natural in common sense reasoning. For example, the difference between these two types of conclusion is analogous to final verdict and intermediate arguments in legal reasoning. A judge is of course not supposed to draw any conclusions before both parties have made their full arguments.

Some Examples of Reasoning with CoL

From now on, we call *facts* statements such as "Tweety is a bird" and "Nixon is a Quaker" and we call *rules* statements such as "Birds fly" and "Quakers are pacifist".

Example 1 (Bird-Penguin Triangle)
Given the set $\Sigma = \{b_{\{b\}}, (b \wedge \neg p \rightarrow f)_{\{b\wedge\neg p\rightarrow f\}}\}$, the formula $f_{\{b, b\wedge\neg p\rightarrow f, D\neg p\}}$ is an acceptable default conclusion.

$$\dfrac{\dfrac{b_{\{b\}} \quad \dfrac{}{(\neg p)_{\{D\neg p\}}}\,(ID)}{(b \wedge \neg p)_{\{b, D\neg p\}}}\,(I\wedge) \qquad (b \wedge \neg p \rightarrow f)_{\{b\wedge\neg p\rightarrow f\}}}{f_{\{b, b\wedge\neg p\rightarrow f, D\neg p\}}}\,(E\rightarrow)$$

So, $\Sigma \mid\!\sim^{\Pi} f_C$ whenever $A(C) \subseteq \Pi$ (where C is an abbreviation for $\{b, b \wedge \neg p \rightarrow f, D\neg p\}$). Note that $C - A(C) = \{b, b \wedge \neg p \rightarrow f\} = P(\Sigma)$. Note also that $(f)_{\{b, b\wedge\neg p\rightarrow f, D\neg p\}}$ is no longer an acceptable default conclusion if we add either $p_{\{p\}}$ or $(\neg f)_{\{\neg f\}}$ to Σ. But it is still acceptable if we add the unrelated premise $r_{\{r\}}$ to Σ. Also, given the premise set $\Sigma \cup \{p_{\{p\}}\}$ we have that the context of the formula $f_{\{b, b\wedge\neg p\rightarrow f, D\neg p\}}$ does not satisfy condition (iii), hence it is not acceptable. And given the premise set $\Sigma \cup \{(\neg f)_{\{\neg f\}}\}$ we have that the context of the formula $f_{\{b, b\wedge\neg p\rightarrow f, D\neg p\}}$ does not satisfy condition (iii), hence it is not acceptable either.

We now discuss some more examples to give a further impression of the inferential behaviour of contextual logic.

Example 2 (Quaker-Republican Diamond)
Given the set $\Sigma = \{q \rightarrow p_{\{q\rightarrow p\}}, r \rightarrow \neg p_{\{r\rightarrow\neg p\}}\}$, contextual logic yields:

(a) $\Sigma \mid\!\sim p_{\{q\rightarrow p, r\rightarrow\neg p, Dq\}}$

(b) $\Sigma \mid\!\sim \neg p_{\{q\rightarrow p, r\rightarrow\neg p, Dr\}}$

(c) $\Sigma \not\mid\!\sim p_{\{q\rightarrow p, r\rightarrow\neg p, Dq, Dr\}}$

(d) $\Sigma \not\mid\!\sim \neg p_{\{q\rightarrow p, r\rightarrow\neg p, Dq, Dr\}}$

The contexts in the conclusions of (a) and (b) are reminiscent of default extensions in default logic.

In view of the context $\{q \rightarrow p, r \rightarrow \neg p, Dq\}$ the formula p is true whereas in view of the other context $\{q \rightarrow p, r \rightarrow \neg p, Dr\}$ the formula $\neg p$ is true. We could say that within the context that is induced by the default assumption Dq the conclusion p is true, whereas in the other context induced by the default assumption Dr the conclusion $\neg p$ is true. This is an example of a general property comparable to the so-called *orthogonality* of default extensions, i.e. the phenomenon that the union of two default extensions is always inconsistent in default logic.

All default assumptions found in the above examples were facts, i.e. in a default assumption $D\varphi$ the formula φ is an atomic proposition. However, that is not necessary. We could also model the Quaker-Republican Diamond in such a way that the default assumptions are rules instead of facts.

Example 3 (Quaker-Republican Diamond: another representation)
Given the premise set $\Sigma = \{q_{\{q\}}, r_{\{r\}}\}$ we have the following inferences in contextual logic.

(a) $\Sigma \mid\!\sim p_{\{q,r,D(q\rightarrow p)\}}$
(b) $\Sigma \mid\!\sim \neg p_{\{q,r,D(r\rightarrow\neg p)\}}$
(c) $\Sigma \not\mid\!\sim p_{\{q,r,D(q\rightarrow p),D(r\rightarrow\neg p)\}}$
(d) $\Sigma \not\mid\!\sim \neg p_{\{q,r,D(q\rightarrow p),D(r\rightarrow\neg p)\}}$

Note that in examples 2 and 3, adding two default assumptions does not yield any acceptable default conclusion. More generally, a set of premises Σ such that $P(\Sigma)$ is consistent yields no conclusion whose context involves contradictory default assumptions.

A Glimpse at Expressiveness

Let us look at CoL in terms of connectives, mainly disjunction, implication and negation.

Example 4
Consider $\Sigma = \{(p \vee q)_{\{p\vee q\}}\}$ and $\Pi = \{D(p \rightarrow r), D(q \rightarrow r)\}$. Clearly,

$\Sigma \mid\!\sim r_{\{p\vee q, D(p\rightarrow r), D(q\rightarrow r)\}}$

That is, reasoning by cases is captured by contextual logic.

Example 5
Let $\Sigma = \{(\neg q)_{\{\neg q\}}\}$ and $\Pi = \{D(p \rightarrow q)\}$. Then, $\Sigma \mid\!\sim (\neg p)_{\{\neg q, D(p\rightarrow q)\}}$.

This shows that contextual logic also accounts for contraposition.

Example 6
Consider the premise $p_{\{p\}}$ with the default assumptions $D(p \rightarrow q)$ and $D(q \rightarrow r)$. Now,

$p_{\{p\}} \mid\!\sim r_{\{p, D(p\rightarrow q), D(q\rightarrow r)\}}$

So, implication chaining is unproblematic in CoL. We can even interleave implications as premises and default assumptions, e.g. removing $D(q \rightarrow r)$ and adding the premise $(q \rightarrow r)_{\{q\rightarrow r\}}$ still leaves r as a conclusion ($\{p, q \rightarrow r, D(p \rightarrow q)\}$ being the context).

A more general perspective tells us that many subtle distinctions can be explicitly expressed in contextual logic. E.g., the meaning of the conclusion $q_{\{Dp, p\rightarrow q\}}$ differs from that of the conclusion $q_{\{p, D(p\rightarrow q)\}}$.

Extended Contextual Logic (ECoL)

Up to now, we could view CoL as a mere proof system for a variant of default logic similar to Poole's *Theorist* system [Poole 1988]. This is due to the fact that we have presented so far only a simplified version of CoL. However, there is more to come. In this section we extend the expressiveness of contextual logic by allowing a more relaxed use of the modal operator D. We thus get the so-called *Extended Contextual Logic ECoL.*

In CoL, we are a bit restricted in expressing conditional rules "if p then q" that fail to have the strength of the material implication $p \rightarrow q$. We have only one way to do it, and that is to make the corresponding formula $p \rightarrow q$ a default assumption $D(p \rightarrow q)$. Stated otherwise, if we do not want to include $(p \rightarrow q)_{\{p\rightarrow q\}}$ in Σ then we have only one possibility left, and that is to include $D(p \rightarrow q)$ in Π.

The expression $Dp \rightarrow q$, if allowed, would still have a different meaning than $p \rightarrow q$ being taken into account as a premise or a default assumption. In particular, $D(p \rightarrow q)$ is weak enough to have no inconsistency arising when $\Sigma = \{p_{\{p\}}, (\neg q)_{\{\neg q\}}\}$. On the contrary, $Dp \rightarrow q$ would yield an inconsistency when $\Sigma = \{p_{\{p\}}, (\neg q)_{\{\neg q\}}\}$. This would be due to having Dp as a consequence of p, which is one of various schemata that should hold in order to deal with the extended syntax. Another theorem would be distributivity of D over implication, that is $D(\varphi \rightarrow \psi) \rightarrow (D\varphi \rightarrow D\psi)$.

An extended contextual logic (ECoL) that allows for explicit D operators in the premises can be defined as follows. First, we adapt the definition of the language in the obvious way. The languages $\mathcal{L}_0$ and $\mathcal{L}_D$ in CoL are combined into one language $\mathcal{L}_{ED}$, which is defined analogously to $\mathcal{L}_0$ except that it has the following extra formation rule:

If φ is a formula of $\mathcal{L}_{ED}$, so is $D\varphi$.

Examples of formulas which are allowed in $\mathcal{L}_{ED}$ but not in $\mathcal{L}_D$ are:

(1) $\varphi \wedge D\psi \rightarrow \chi$,
(2a) $\varphi \rightarrow D\psi$,
(2b) $\varphi \rightarrow \neg D\psi$,
(3) $D\varphi \rightarrow D\psi$,

Formula (1) expresses that only $D\psi$ can be assumed by default, and not for example φ.

Formulas (2a)–(2b) express dependencies of default assumptions on factual information. For example, the formula $p \rightarrow \neg Df$ might be used to express that the fact of being a penguin blocks the use of the default of flying. Hence, this type of formulas is important to express the so-called principle of specificity among default rules.

Formula (3) can be used to express dependencies between default assumptions. For example, $Df \rightarrow Dw$ can be used to express that if one assumes by default that something can fly, one could as well assume by default that it has wings.

Similarly to $\mathcal{L}$ in CoL, the language $\mathcal{L}_E$ in ECoL is defined by the single clause:

- If φ is an $\mathcal{L}_{ED}$-formula and C a finite set of $\mathcal{L}_{ED}$-formulas, then φ_C is an $\mathcal{L}_E$-formula.

The following rule governs the behaviour of the D operator in ECoL.

Modal Rule (MR)

$$\dfrac{\begin{array}{ccc} [D\varphi_{C_1}] & \ldots & [D\psi_{C_n}] \\ [\varphi_{C_1}] & \ldots & [\psi_{C_n}] \\ & \vdots & \\ & \chi_C & \end{array}}{D\chi_C} \qquad \text{assumptions [discharged]}$$

Application of the modal rule is as follows. If we have a deduction from $\varphi, \ldots, \psi$ to χ, then we can discharge any of $\varphi, \ldots, \psi$ and replace them by the assumptions $D\varphi, \ldots, D\psi$, also introducing $D\chi$ as the conclusion.

Note that, although this inference rule can be "upward growing", it can be so only with default assumptions.

Presumably, the most illustrative inference and the most illustrative proof is that of φ from $D\varphi$ and $\neg D\bot$ (the contexts have been arbitrarily chosen so that the reader can also see how contexts propagate in the course of a proof where the modal rule is applied):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}(D\varphi)_{\{\neg\bot\}}\\ {}^{(1)}\varphi_{\{\neg\bot\}} \quad {}^{(2)}(\neg\varphi)_{\{\top\}}\end{array}}{\bot_{\{\top,\neg\bot\}}}\,{\scriptstyle (E\neg)}}{(D\bot)_{\{\top,\neg\bot\}}}\,{\scriptstyle (1)\ (MR)} \qquad (\neg D\bot)_{\{\neg D\bot\}}}{\bot_{\{\top,\neg\bot,\neg D\bot\}}}\,{\scriptstyle (E\neg)}}{(\neg\neg\varphi)_{\{\top,\neg\bot,\neg D\bot\}}}\,{\scriptstyle (2)\ (I\neg)}}{\varphi_{\{\top,\neg\bot,\neg D\bot\}}}\,{\scriptstyle (DNR)}$$

This proof starts with applying the modal rule to the deduction of $\bot_{\{\top,\neg\bot\}}$ from the two assumptions $\varphi_{\{\neg\bot\}}$ and $(\neg\varphi)_{\{\top\}}$. The modal rule is applied in such a way that $(\neg\varphi)_{\{\top\}}$ is left as an assumption (remember that discharging any assumption is *optional*) whereas $\varphi_{\{\neg\bot\}}$ is discharged and replaced by $(D\varphi)_{\{\neg\bot\}}$. Also, the conclusion $\bot_{\{\top,\neg\bot\}}$ gives rise to the new conclusion $(D\bot)_{\{\top,\neg\bot\}}$.

In the above proof, the fact that φ is discharged when applying the modal rule is indicated by (1) next to the (MR) bar and by (1) as a superscript in front of φ itself. Another discharge of an assumption is indicated by (2) when applying the I$\neg$ rule: Of course, discharges are numbered consecutively as they occur in the proof.

The simplest proof involving the modal rule is certainly the following one (from now on, the empty context is omitted in order to improve readability as indicated above):

$$\dfrac{\dfrac{{}^{(1)}\varphi}{D\varphi}\,{\scriptstyle (MR)}}{\varphi \rightarrow D\varphi}\,{\scriptstyle (1)\ (I\rightarrow)}$$

Here, we apply the rule for the case where no assumption is taken into account, i.e. the only assumption φ is not discharged. It is later discharged when the I$\rightarrow$ rule is applied.

More generally, the modal rule admits all degenerate cases (for instance, discharge with respect to irrelevant assumptions) as happens with other rules in natural deduction. The modal rule behaves similarly to the rules defined by [Prawitz 1965] for S4 and S5. In particular, the modal rule induces the same phenomenon that Prawitz described for his modal rules, namely the existence of non-trivial proofs with maximal formulas (a formula is maximal when it results from applying an introduction rule and is subject to the corresponding elimination rule). However, this topic will not be discussed further here because it is not central to the matter of non-monotonic reasoning.

A proof is best read by starting with the lowest node in a branch where a formula occurs which fails to have a bar on top of it. A clear example is given by a proof of $DD\varphi \rightarrow DDD\varphi$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}{}^{(3)}DD\varphi\\ {}^{(2)}D\varphi\\ {}^{(1)}\varphi\end{array}}{D\varphi}\,{\scriptstyle (MR)}}{DD\varphi}\,{\scriptstyle (1)\ (MR)}}{DDD\varphi}\,{\scriptstyle (2)\ (MR)}}{DD\varphi \rightarrow DDD\varphi}\,{\scriptstyle (3)\ (I\rightarrow)}$$

Some formulas of particular interest that have a proof using the modal rule are:

A1. $D(\varphi \rightarrow \psi) \rightarrow (D\varphi \rightarrow D\psi)$

A2. $(D\varphi \vee D\psi) \rightarrow D(\varphi \vee \psi)$

A3. $D(\varphi \wedge \psi) \leftrightarrow (D\varphi \wedge D\psi)$

A4. $\varphi \rightarrow D\varphi$

A proof for A4 was given above, so we only need to provide proofs for A1-A3.

As regards A1, we have the proof below:

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}{}^{(3)}D(\varphi\rightarrow\psi) \quad {}^{(2)}D\varphi\\ {}^{(1)}\varphi\rightarrow\psi \qquad {}^{(1)}\varphi\end{array}}{\psi}\,{\scriptstyle (E\rightarrow)}}{D\psi}\,{\scriptstyle (1)\ (MR)}}{D\varphi \rightarrow D\psi}\,{\scriptstyle (2)\ (I\rightarrow)}}{D(\varphi\rightarrow\psi) \rightarrow (D\varphi \rightarrow D\psi)}\,{\scriptstyle (3)\ (I\rightarrow)}$$

Next, A2 admits the following proof:

$$\dfrac{\dfrac{{}^{(4)}D\varphi \vee D\psi \quad \dfrac{\dfrac{\begin{array}{c}{}^{(3)}D\varphi\\ {}^{(1)}\varphi\end{array}}{\varphi\vee\psi}\,{\scriptstyle (I\vee)}}{D(\varphi\vee\psi)}\,{\scriptstyle (1)\ (MR)} \quad \dfrac{\dfrac{\begin{array}{c}{}^{(3)}D\psi\\ {}^{(2)}\psi\end{array}}{\varphi\vee\psi}\,{\scriptstyle (I\vee)}}{D(\varphi\vee\psi)}\,{\scriptstyle (2)\ (MR)}}{D(\varphi\vee\psi)}\,{\scriptstyle (3)\ (E\vee)}}{(D\varphi \vee D\psi) \rightarrow D(\varphi\vee\psi)}\,{\scriptstyle (4)\ (I\rightarrow)}$$

As regards A3, we have first

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}{}^{(3)}D(\varphi\wedge\psi)\\ {}^{(1)}\varphi\wedge\psi\end{array}}{\varphi}\,{\scriptstyle (E\wedge)}}{D\varphi}\,{\scriptstyle (1)\ (MR)} \quad \dfrac{\dfrac{\begin{array}{c}{}^{(3)}D(\varphi\wedge\psi)\\ {}^{(2)}\varphi\wedge\psi\end{array}}{\psi}\,{\scriptstyle (E\wedge)}}{D\psi}\,{\scriptstyle (2)\ (MR)}}{D\varphi \wedge D\psi}\,{\scriptstyle (I\wedge)}}{D(\varphi\wedge\psi) \rightarrow (D\varphi \wedge D\psi)}\,{\scriptstyle (3)\ (I\rightarrow)}$$

and the other part of A3 is obtained by

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}{}^{(3)}D\varphi \quad {}^{(2)}D\psi\\ {}^{(1)}\varphi \qquad {}^{(1)}\psi\end{array}}{\varphi\wedge\psi}\,{\scriptstyle (I\wedge)}}{D(\varphi\wedge\psi)}\,{\scriptstyle (1)\;(MR)}}{D\psi\to D(\varphi\wedge\psi)}\,{\scriptstyle (2)\;(I\to)}}{D\varphi\to(D\psi\to D(\varphi\wedge\psi))}\,{\scriptstyle (3)\;(I\to)} \quad \dfrac{{}^{(4)}D\varphi\wedge D\psi}{D\varphi}\,{\scriptstyle (E\wedge)}}{D\psi\to(D\varphi\wedge\psi)}\,{\scriptstyle (E\to)} \quad \dfrac{{}^{(4)}D\varphi\wedge D\psi}{D\psi}\,{\scriptstyle (E\wedge)}}{D\varphi\wedge D\psi}\,{\scriptstyle (E\to)}}{(D\varphi\wedge D\psi)\to D(\varphi\wedge\psi)}\,{\scriptstyle (4)\;(I\to)}
$$

The above list of formulas that have a proof is now significant enough: We can say a few words about semantics. The crucial point is that $\neg D\bot \to (\varphi \leftrightarrow D\varphi)$ is a theorem. (Note, however, that $\neg D\bot$ itself is not a theorem!) This clearly indicates that, in a Kripke model, if there exists at all a world which is accessible from the actual world, then it can only be the actual world itself. Accordingly, the model theory for the logic axiomatized by the modal rule is defined by the class of all Kripke models in which the accessibility relation is at most reflexive.

Returning to proof-theoretic considerations, we replace the default introduction rule of CoL by the following rule.

Extended Default Introduction Rule (EID)

$$\dfrac{}{D\varphi_{\{D\varphi\}}}$$

All the other inference rules of contextual natural deduction remain unchanged. In ECoL, we again make a distinction between two inference operators $\vdash_E$ and $\mathrel{|\!\sim}_E$. The first operator $\Sigma \vdash_E \varphi_C$ denotes that the formula φ_C is derived from the set of premises Σ with the rules of contextual natural deduction (including the modal rule) in the usual way, and the default introduction rule ID is replaced by the extended default introduction rule EID. The operator $\vdash_E$ is monotonic. The second operator $\Sigma \mathrel{|\!\sim}^{\Pi}_{E} \varphi_C$ is non-monotonic, and is again defined in terms of the first operator:

Requirement of Maximal Contexts for Default Conclusions in ECoL
Let Σ be a set of $\mathcal{L}_E$-formulas and Π be a set of $\mathcal{L}_{ED}$-formulas. Then, $\Sigma \mathrel{|\!\sim}^{\Pi}_{E} \varphi_C$ iff

(i) $\Sigma \vdash_E \varphi_C$,

(ii) $C = P(\Sigma) \cup \Delta$ for some $\Delta \subseteq \Pi$,

(iii) if $P(\Sigma) \nvdash^*_E D\bot$ then $P(\Sigma) \cup C \nvdash^*_E D\bot$.

where $\vdash^*_E$ indicates provability without the EID rule.

That is, we use essentially the same definition of Requirement of Maximal Contexts for $\mathrel{|\!\sim}^{\Pi}_{E}$ as we did for $\mathrel{|\!\sim}^{\Pi}$ in CoL. Of course, $P(\Sigma)$ is redefined in the obvious way: If Σ is a set of $\mathcal{L}_E$-formulas, then $P(\Sigma)$ denotes the set of $\mathcal{L}_{ED}$-formulas which are the formulas from Σ with their context removed.

A more significant difference is that

$$F(C) \text{ is consistent wrt } P(\Sigma)$$

which can be written

$$\text{if } P(\Sigma) \nvdash \bot \text{ then } P(\Sigma) \cup F(C) \nvdash \bot$$

is actually a special case of the new modal constraint (iii). Indeed, $\mathcal{L}$-formulas (i.e. the ones that are dealt with in CoL), are such that, clearly, $P(\Sigma) \vdash^*_E \bot$ iff $P(\Sigma) \vdash^*_E D\bot$. Moreover, $F(C)$ would be adapted so that we delete not only single occurrences of the D operator in front of formulas in the context C, but we delete also sequences of D operators. In fact, we consider that $\varphi \leftrightarrow D\varphi$ holds and to apply it, we simply need to postulate $\neg D\bot$ (cf above). Accordingly, $P(\Sigma) \cup F(C) \vdash \bot$ would become $P(\Sigma) \cup C \cup \{\neg D\bot\} \vdash^*_E \bot$ and then, $P(\Sigma) \cup C \vdash^*_E D\bot$.

Example 7
Given $\Sigma = \{b_{\{b\}}, b \wedge D\neg p \to f_{\{b\wedge D\neg p \to f\}}\}$ we have the following inferences in ECoL.

(a) $\Sigma \mathrel{|\!\sim}_E f_{\{b, b\wedge D\neg p\to f, D\neg p\}}$

(b) $\Sigma, p \not\mathrel{|\!\sim}_E f_{\{b, b\wedge D\neg p\to f, D\neg p\}}$

(c) $\Sigma, \neg D\neg p \not\mathrel{|\!\sim}_E f_{\{b, b\wedge D\neg p\to f, D\neg p\}}$

ECoL enjoys considerable expressiveness as is now to be discussed. In particular, dependencies between default assumptions can be neatly handled in ECoL as opposed to Poole's Theorist System and even default logic. Consider, for example, the dependency that if one assumes by default that an "air" animal can fly, then one can as well assume by default that it has feathers. In (normal) default logic, this can be expressed in two different ways:
Either by

(1) the normal default rule *airanimal* : *fly*/*fly* and the strict rule *fly* $\to$ *feathers*,

or by

(2) the two normal default rules *airanimal* : *fly*/*fly* and *fly* : *feathers*/*feathers*.

Knowing of a particular air animal that it has no feathers (as expressed by the formula $\neg$*feathers*), then

(1) makes the conclusion $\neg fly$ to follow by modus tollens, whereas (2) does not allow for a similar application of modus tollens, and hence one does not go beyond the conclusion fly. In the first case, "default contraposition" occurs, whereas in the second case there is no such contraposition.

Default contraposition is appropriate in some cases but not all cases. As for penguins, the fact that they have no feathers is the reason that they cannot fly. Hence, default contraposition is desirable in the case of penguins. However, bats do not have feathers either, but still they can fly. Default contraposition should not hold in the case of bats. The problem with default logic is that there is no way to accommodate at the same time cases with default contraposition and cases without default contraposition. Default theory (1) causes default contraposition to always apply, and default theory (2) causes default contraposition to never apply. There is no representation in default logic that enables us to apply default contraposition selectively for some objects, but not for some others.

In ECoL, such a distinction can be made by expressing simply the intended default dependency with the formula $Dfly \rightarrow Dfeathers$. The featherlessness of penguins has a default contraposition effect, motivating the formula $penguin \rightarrow \neg Dfeathers$, which also blocks the default of flying. The featherlessness of bats should not have this default contraposition effect, hence the formula $bat \rightarrow \neg feathers$.

By virtue of A4, $\neg feathers$ implies $D\neg feathers$, but $\neg Dfeathers$ is not implied (in the logic underlying ECoL, the logic with language $\mathcal{L}_{ED}$ and axiomatized by the modal rule), and neither is $\neg Dfly$ implied despite the contraposition of $Dfly \rightarrow Dfeathers$. This is related to the fact that $\neg(D\varphi \wedge D\neg\varphi)$ is not provable in the underlying logic. Allowing for $D\neg\varphi \wedge D\varphi$ to be consistent is harmless, because even if $D\varphi$ and $D\neg\varphi$ are true at the same time, the requirement of maximal contexts will prevent that both default assumptions are used in the same context in ECoL. Hence, the entailment definition of ECoL filters away, so to say, the unintuitive usages of $D\varphi \wedge D\neg\varphi$.

In addition, not having $\neg(D\varphi \wedge D\neg\varphi)$ as a theorem has the consequence that not only $D\varphi \wedge D\neg\varphi$ but also $\neg\varphi \wedge D\varphi$ is consistent. For reasons similar to the one just discussed, the consistency of $\neg\varphi \wedge D\varphi$ is harmless.

There is also a technical reason not to have $\neg(D\varphi \wedge D\neg\varphi)$ as a theorem of the underlying logic. The reason is that if this formula were to be a theorem like A1–A2, then $\varphi \rightarrow \psi$ would be equivalent with $D\varphi \rightarrow D\psi$, i.e. $(\varphi \rightarrow \psi) \leftrightarrow (D\varphi \rightarrow D\psi)$ would hold. This would mean that the D operator collapses in the case of implications. The left to right direction of this equivalence comes from A1 and A4. The right to left direction can be shown as follows. Assuming φ, then A4 gives $D\varphi$. Now, $D\psi$ follows due to $D\varphi \rightarrow D\psi$. If we accept $\neg(D\varphi \wedge D\neg\varphi)$ as a theorem, then $D\psi$ implies $\neg D\neg\psi$, and by A4, we get ψ.

Another formula that seems intuitive as a theorem for the D operator is $\neg D\varphi \leftrightarrow D\neg\varphi$. However, should we accept it, then we get the collapse $\varphi \leftrightarrow D\varphi$. So, $D\varphi$ could always be replaced simply by the formula φ, which means that the D operator would be useless. The left to right direction follows immediately from A4. The right to left direction follows from A4 and $\neg D\varphi \leftrightarrow D\neg\varphi$. Here are the details. By virtue of A4, $\neg\varphi \rightarrow D\neg\varphi$ holds. Then, $\neg D\varphi \leftrightarrow D\neg\varphi$ yields $\neg\varphi \rightarrow \neg D\varphi$, hence $D\varphi \rightarrow \varphi$.

Related Work

In addition to non-monotonic logics (most notably default logic), two categories of work relate to contextual logic. One is about contexts of course, whereas the other is about deductive dependencies. In the former, there are various frameworks about contexts [McCarthy 1993] [Nait-Abdallah 1994]. In the latter, are ATMS [de Kleer 1986], LDS [Gabbay 1995] and also proof systems close to contextual natural deduction as proposed by [Batens 1991] [Gabbay and Hunter 1993].

The first issue is the relationship between CoL and LDS [Gabbay 1995]. Plainly, contextual natural deduction is a labelled deduction system. It is a matter of notation and focus, because LDS is a conceptual framework of which contextual natural deduction is an instance worth investigating.

Second, what about CoL and ATMS [de Kleer 1986]? The difference comes from the notion of hypotheses. In the ATMS approach, there is only one kind of hypotheses. This is the reason why an ATMS is not a logic, not even a proof system, in that it only performs bookkeeping: it just records deductive dependencies. There are two kinds of hypotheses in contextual natural deduction: auxiliary assumptions and premises, the latter corresponding to the hypotheses found in the ATMS approach. In reasoning, handling assumptions is essential (in particular, discharging assumptions). Auxiliary assumptions (formulas with the empty context) are the reason why contextual natural deduction does more than bookkeeping, it actually defines a logic.

Contextual logic bears some similarity to dynamic dialectical logics [Batens 1991] in the way it requires a global use of premises. Nevertheless, contextual natural deduction seems more economical than the proof systems defined for dynamic dialectical logics that refer to maximal "extensions" of subproofs in order to validate a proof.

Contextual natural deduction has apparent connections with the proof systems proposed by [Gabbay and Hunter 1993] according to the idea of restricting the access for rules to formulas. In particular, the mechanism for propagating labels (or contexts) is the same for restricted access logics and contextual logic. However, contextual logic is less constrained because it does not prevent rules to apply to any formulas. Also, it is more expressive in that there is nothing like default assumptions in restricted access logics.

In this respect, the ionic logic approach proposed in [Nait-Abdallah 1994] suffers from no limitations. Up to the point that this becomes problematic. Indeed, a number of axioms are introduced to govern the behaviour of the special starred expressions (roughly corresponding to default assumptions). As a result, the notion of an inference in the ionic logic approach is much more involved than it is in contextual logic.

Contextual logic only makes use of contexts. This is a major difference with the work on contexts undertaken by [McCarthy 1993] where contexts are central to the enterprise. For instance, we do not express links between contexts. On the positive side, by sticking to a simple account of contexts, we were able to give a rather satisfactory inference system.

We now turn to a comparison between default logic and contextual logic. First, the language of default logic does not allows us to distinguish between valid conclusions and conclusions drawn using default rules. In contrast, such a distinction is made explicit in contexts by CoL. Also, in default logic, there is no representation for the default assumptions that generate a default extension (the so-called "generating defaults"). A fortiori, there is no representation of the possible dependence of a derived formula on specific defaults in the set of generating defaults. All corresponding possibilities are given by contexts, as should be clear by now to the reader.

Finally, no incremental approach to computing default extensions is possible in the sense that the space of all deductions from the premises must be explored for a conclusion to be formally derived by default. Such a burden on default logic does not extend to contextual logic which is the first nonmonotonic logic with an incremental inference system: any derivation has a value on its own, that value being given by its context (a conclusion with an inconsistent context is worthless).

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