Dynamic Proof Theories for Selection Semantics: A Generic Adaptive Logic Framework

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Abstract

We present dynamic proof theories for a huge class of Shoham-like selection semantics. This is achieved by means of a natural generalization of the standard format for adaptive logics. While in the standard format models are selected on the basis of comparing their abnormal parts by means of set-inclusion, we allow in this paper for comparisons with respect to other partial orders and a huge class of selection functions. We demonstrate that a large and important part of the meta-theory of the standard format (soundness, completeness, cumulativity, reassurance, etc.) remains intact when moving to this more general setting.

1 Introduction

One of the classical ideas to strengthen a monotonic core logic **L** is to select a certain subset of its models of a given premise set Γ. Given this selection, a semantic consequence relation is defined in the usual way: A is a consequence of Γ iff it is valid in all selected models.

This idea is an integral part of many formal systems. Variants of it can be found in e.g., Shoham [30, 31], Batens [1], Kraus et. al [12], McCarthy [16], Schlechta [29], etc. Lindström [14] and Makinson [15] offer systematic overviews.

A common way to define the semantic selection is by means of a partial order $\prec$ on the **L**-models and by selecting models that are "good enough" with respect to $\prec$. For instance, in Shoham's approach the $\prec$-minimal models are selected. One way to define the preference order is as follows: we identify a certain structural property of the models and compare the models with respect to this property. For instance in propositional circumscription [13], extended closed world assumption (ECWA) [9, 8],[1] and adaptive logics [4, 24, 33] (henceforth, ALs) a certain set of formulas Ω is deemed as abnormalities. We can then specify the abnormal part of an **L**-model as the set of all abnormalities that are verified by it. Now we can compare models by means of comparing their abnormal parts.

[1]Propositional circumscription and ECWA have been shown to be equivalent [9].

Let us illustrate this approach by means of the semantics of ALs in the standard format. We focus on ALs since these systems are equipped with an adequate dynamic proof theory that explicates the defeasible reasoning processes for which the selection semantics provides meaning. ALs in the standard format are characterized by a triple: (a) the lower limit logic **LLL** which is a monotonic, reflexive, transitive, compact, and has an adequate semantics, (b) a set of abnormalities Ω that is characterized by a (or some) logical form(s), and (c) an adaptive strategy. The latter two elements provide the rationale for the semantic selection and play an analogous role in the proof theory. Let us though first focus on the semantics. This can be spelled out in three steps:

1. The models of the lower limit logic are partially ordered according to their abnormal parts by means of set-inclusion $\subset$.
2. Then models of a given premise set are selected ("preferred") whose abnormal part is lower than a given threshold that is determined by the adaptive strategy. For instance, in case the strategy is Minimal Abnormality the *minimally abnormal models* are chosen (i.e., the models whose abnormal part is minimal with respect to $\subset$). In case of the Reliability strategy, so-called *reliable models* are chosen, i.e., models whose abnormal parts only consist of abnormalities that are also in the abnormal parts of minimally abnormal models.
3. A semantic consequence relation is defined where A follows from Γ iff A is verified by all selected models of Γ.

The probably most interesting and distinguishing feature about ALs is that this semantic consequence relation is associated with an adequate dynamic proof theory. The idea is roughly as follows.[2] Where $\Delta \subseteq \Omega$ is a finite set of abnormalities: whenever $A \vee \bigvee \Delta$ is derivable from Γ by means of **LLL**, then A can be derived in the dynamic proof on the assumption that no abnormality in Δ is true. Of course, not all assumptions are safe and hence the proof theory of ALs comes with a retraction mechanism. Lines whose associated assumption is ill-founded are marked which means that the derivation in this line is considered to be retracted.

Another attractive aspect of the standard format for ALs is its rich meta-theory. The derivability relation of the dynamic proof theory is sound and complete with respect to the selection semantics, and we get other properties such as cumulativity and reassurance (see e.g., [4]).[3]

The selection semantics of the standard format can be readily generalized.

G1 First, instead of ordering **LLL**-models by means of their abnormal parts with respect to $\subset$ we may use another partial order $\prec$.[4]

G2 Second, instead of determining the threshold for the selection in terms of minimally abnormal models or reliable models, we can specify other selections.

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[2]See Section 3.2 for a technically precise formulation of dynamic proofs.

[3]See Section 4 for the definitions of these concepts.

[4]In this paper we will use $\prec$ to denote a *strict* partial order. Of course, one can easily define the corresponding non-strict $\preceq$ by $a \preceq b$ iff $a \prec b$ or $a = b$. Hence, this is a purely conventional choice.

The second point is especially important in cases in which the order that is imposed on the **LLL**-models is not smooth. In these cases there are models M for which there are no models M' with minimally abnormal part that are "better" but instead there are only infinitely descending chains of better and better models than M.

In this paper we will generalized the standard format according to G1 and G2 in a simple and intuitive way. Hence, we provide a sound and complete dynamic proof theory for a huge class of selection semantics. We demonstrate that many properties of the rich meta-theory of the standard format are preserved in this generalization for a large class of selection functions. We show that many ALs that have been considered in the literature fall within this larger class –for instance ALs in the standard format, ALs with counting strategies (see e.g., [21, 22, 20]), lexicographic ALs (see e.g., [25, 26])– and that the characterization of this class offers many possibilities to formulate new logics.

The paper is structured as follows. We start in Section 2 by giving some examples that shall motivate and introduce the reader into various types of selection semantics. In Section 3 we introduce the semantics and the proof theory of our generalized format for ALs. In Section 4 we study its meta-theory. In Section 5 we recapitulate the gained insights by for instance relating them back to our examples from Section 2. Finally, in Section 6 we conclude the paper. In order not to unnecessarily clutter the paper with technical details and to focus on the essential points we present most of the meta-proofs of our results in the Appendix.

2 Some Examples

In this section we will introduce some motivating examples for ALs that fall within the enriched class of logics whose meta-theory is studied in this paper.

There are three ways in which we make the presentation in this section more coherent and free of digressions.

First, all the presented ALs are based on the logic $\mathbf{L}_\circ$ (or simple variants of it). This logic is obtained by adding a "dummy" operator to classical propositional logic **CL**. Where $\mathcal{S}$ is the set of sentential letters, the set of well-formed formulas $\mathcal{W}$ is given by the following grammar:

$$\mathcal{W} := \langle \mathcal{S} \rangle \mid \langle \mathcal{W} \rangle \vee \langle \mathcal{W} \rangle \mid \neg \langle \mathcal{W} \rangle \mid \circ \langle \mathcal{W} \rangle$$

$\mathbf{L}_\circ$ is defined by means of the axioms schemes of **CL** applied to $\mathcal{W}$ and Modus Ponens. The semantics of $\mathbf{L}_\circ$ can be characterized with the help of an additional assignment function $v_\circ$. An $\mathbf{L}_\circ$-model is associated with an assignment $v : \mathcal{S} \to \{0, 1\}$ and an assignment $v_\circ : \mathcal{W} \to \{0, 1\}$. Truth in a model is defined recursively by:

- where $A \in \mathcal{S}$: $M \models A$ iff $v(A) = 1$
- $M \models \neg A$ iff $M \not\models A$
- $M \models A \vee B$ iff $M \models A$ or $M \models B$
- $M \models \circ A$ iff $v_\circ(A) = 1$

Second, the applications we study in this paper are organized around discussive applications inspired e.g. by the research on Rescher-Manor consequence relations (see e.g., [28]).

Third, we will in this section only focus on semantic considerations.

Most of the applications we mention in this section would deserve a deeper discussion were we interested in developing a fully elaborated formal account for them. However, our aim is more modest: it is to make the reader aware of a variety of possibilities to define useful selection semantics. This in turn motivates the generic perspective on ALs which is introduced in the further run of this paper.

2.1 ALs in the standard format

We begin the presentation with ALs in the standard format. Paradigmatically we focus on the minimal abnormality strategy. We discuss reliability in Section 5.2.2.

Suppose we are to logically model a discussion. When some participant states A we represent this by $\circ A$. Hence, given a set of statements Γ we translate it to $\Gamma^\circ = \{\circ A \mid A \in \Gamma\}$. A statement $\circ A$ counts as accepted in a model M of Γ° in case A is valid in M. The idea is to select models in which as many statements as possible are accepted. Note that there are cases in which we cannot accept all given statements since some of these may be conflicting. We will give an example below.

We use the adaptive logic **AL1**. As lower limit logic we use $\mathbf{L}_\circ$, the set of abnormalities is

$$\Omega_\circ = \{\circ A \wedge \neg A \mid A \text{ has no occurrences of `}\circ\text{'}\}$$

Finally, we use the minimal abnormality strategy. The idea is to order the models of the lower limit logic with respect to their abnormal parts and according to $\subset$. Our semantic selection selects all $\mathbf{L}_\circ$-models of Γ whose abnormal part is minimal with respect to $\subset$ and hence in

$$\min_\subset(\mathsf{Ab}^\Gamma_{\mathbf{L}_\circ})$$

where $\mathsf{Ab}^\Gamma_{\mathbf{L}_\circ} =_{\text{df}} \{\mathbf{Ab}(M) \mid M \in \mathcal{M}_{\mathbf{L}_\circ}(\Gamma)\}$ and $\mathbf{Ab}(M) =_{\text{df}} \{A \in \Omega_\circ \mid M \models A\}$. We define the set of models of **AL1** by

$$\mathcal{M}_{\mathbf{AL1}}(\Gamma) =_{\text{df}} \left\{M \in \mathcal{M}_{\mathbf{L}_\circ}(\Gamma) \mid \mathbf{Ab}(M) \in \min_\subset(\mathsf{Ab}^\Gamma_{\mathbf{L}_\circ})\right\}$$

The semantic consequence relation is then defined by

$$\Gamma \Vdash_{\mathbf{AL1}} A \text{ iff } M \models A \text{ for all } M \in \mathcal{M}_{\mathbf{AL1}}(\Gamma) \qquad (\star)$$

Suppose for instance that $\Gamma = \{A \wedge B, \neg A \wedge B, C\}$. Consider the following $\mathbf{L}_\circ$-models of Γ°:

model M	$\mathbf{Ab}(M)$	$M \models$
M_1	$\circ(A \wedge B) \wedge \neg(A \wedge B), \circ C \wedge \neg C$	$\neg A, B, \neg C$
M_2	$\circ(\neg A \wedge B) \wedge \neg(\neg A \wedge B)$	A, B, C
M_3	$\circ(A \wedge B) \wedge \neg(A \wedge B)$	$\neg A, B, C$
M_4	$\circ(A \wedge B) \wedge \neg(A \wedge B), \circ(\neg A \wedge B) \wedge \neg(\neg A \wedge B)$	$A, \neg B, C$

The models are ordered with respect to their abnormal parts according to $\subset$ as follows:

$$\begin{tikzcd}[arrows=dash]
& \text{Ab}(M_4) \arrow[dl] \arrow[dr] & \text{Ab}(M_1) \arrow[d] \\
\text{Ab}(M_2) & & \text{Ab}(M_3)
\end{tikzcd}$$

It is easy to see that all the models in $\mathcal{M}_{\mathbf{AL1}}(\Gamma^{\circ})$ have an abnormal part that is either identical to $\text{Ab}(M_2)$ or identical to $\text{Ab}(M_3)$. Thus $\Gamma^{\circ} \Vdash_{\mathbf{AL1}} B, C$ but for instance $\Gamma^{\circ} \nVdash_{\mathbf{AL1}} A, \neg A$.

Note that there is no conflict concerning C. Hence it is true in all minimal abnormal models. Second, the conflict in our statements concerns a disagreement in A. Since B is derivable from both of the conflicting statements and B is not conflicted otherwise, it is true in both minimally abnormal interpretations of our premises. Hence, it is a consequence.[5]

2.2 Lexicographic ALs

A recent example for ALs that are not in the standard format is the class of *lexicographic ALs* (see [25]). In these logics we work with a structured, i.e., prioritized set of abnormalities: $\Omega = \bigcup_I \Omega_i$ (where $I = \{1, \ldots, n\}$ or $I = \mathbb{N}$). The abnormalities in Ω_1 are considered to be "worst" and hence our priority is to avoid them. Abnormalities in Ω_2 are "second-worst", and so on. Lexicographic ALs have been applied to normative reasoning [26], to belief revision [24], and to reasoning with inconsistencies [35].[6]

For a simple example let us return to our logic $\mathbf{L}_{\circ}$. However, we enhance our expressive powers slightly. We use sequences of $\circ$ in order to indicate the trustworthiness of the information. For instance, $\circ A$ indicates that the information A is provided by a most trust-worthy source. "$\circ\circ$" indicates a less trust-worthy source, etc. This way we can "prioritize" our set of abnormalities $\Omega = \bigcup_{\mathbb{N}} \Omega_i$. Where $\circ^i$ denotes a sequence of i-many $\circ$, we define

$$\Omega_i = \left\{\circ^i A \wedge \neg A \mid A \text{ is a formula without occurrences of '}\circ\text{'}\right\}$$

Now suppose we have the following premises: $\Gamma = \{\circ A, \circ\circ\neg A, \circ B, \circ\circ\circ C\}$. We consider four $\mathbf{L}_{\circ}$-models of Γ:

model M	$\text{Ab}(M)$	$M \models$
M_1	$\circ A \wedge \neg A$	$\neg A, B, C$
M_2	$\circ\circ\neg A \wedge \neg\neg A, \circ\circ\circ C \wedge \neg C$	$A, B, \neg C$
M_3	$\circ\circ\neg A \wedge \neg\neg A, \circ B \wedge \neg B$	$A, \neg B, C$
M_4	$\circ\circ\neg A \wedge \neg\neg A$	A, B, C

[5]It can easily be shown that **AL1** represents the universal Rescher-Manor consequence relation: A is derivable from all maximally consistent subsets of Γ iff $\Gamma^{\circ} \vdash_{\mathbf{AL1}} A$. See [17].

[6]Lexicographic ALs have been compared to sequential combinations of ALs in the standard format and to hierarchical adaptive logics [23] in [27].

Now compare M_1 and M_4. Note that a more trust-worthy source states A than the source that states $\neg A$. Hence we should prefer A over $\neg A$. This makes M_4 preferable to M_1. By a similar argument M_2 is preferable to M_3.

One way of formally realizing this intuition is by means of a lexicographic order.

Definition 2.1 (Lexicographic order on $\wp(\Omega) \times \wp(\Omega)$). Where $\varphi, \psi \subseteq \Omega$ are sets of abnormalities, φ is preferable to ψ, in signs $\varphi \prec_{\mathsf{lex}} \psi$, iff, there is a $n \in \mathbb{N}$ for which

(a) $\varphi \cap \Omega_i = \psi \cap \Omega_i$ for all $i < n$ and

(b) $\varphi \cap \Omega_n \subset \psi \cap \Omega_n$.

This obviously imposes a partial order on the lower limit logic models (resp. on their abnormal parts) since now we can compare models with respect to their abnormal parts and $\prec_{\mathsf{lex}}$. Applying $\prec_{\mathsf{lex}}$ to our four models we get:

$$\mathsf{Ab}(M_4) \prec_{\mathsf{lex}} \mathsf{Ab}(M_2) \prec_{\mathsf{lex}} \mathsf{Ab}(M_3) \prec_{\mathsf{lex}} \mathsf{Ab}(M_1)$$

As a threshold for our selection we use $\min_{\prec_{\mathsf{lex}}}(\mathsf{Ab}^{\Gamma}_{\mathbf{L}_\circ})$ and hence select all the $\prec_{\mathsf{lex}}$-minimally abnormal $\mathbf{L}_\circ$-models of Γ. Thus, the set of selected models of the corresponding AL, we call it **AL2**, is defined by

$$\mathcal{M}_{\mathbf{AL2}}(\Gamma) =_{\mathsf{df}} \left\{M \in \mathcal{M}_{\mathbf{L}_\circ}(\Gamma) \mid \mathsf{Ab}(M) \in \min_{\prec_{\mathsf{lex}}}(\mathsf{Ab}^{\Gamma}_{\mathbf{L}_\circ})\right\}$$

The semantic consequence relation is defined analogous to ($\star$).

In our example all the minimal abnormal models M have the abnormal part $\mathsf{Ab}(M) = \mathsf{Ab}(M_4) = \{\circ\circ\neg A \wedge \neg\neg A\}$. Hence, we get $\Gamma \Vdash_{\mathbf{AL2}} A, B, C$.

Since there is no conflict concerning B and C both are valid in our selected models and hence they are consequences. There is a conflict in A, however the more trustworthy source states A. Hence in our selected models A is the case, as desired.

2.3 Counting Abnormalities

Another class of ALs in which the abnormal parts of **LLL**-models are not compared by means of $\subset$ are ALs with so-called counting strategies (see e.g., [21, 22, 20]), or –more general–, ALs that use quantitative comparisons rather than qualitative ones.

Suppose we have a discussive application where we model possibly conflicting expert opinions. Instead of $\mathbf{L}_\circ$ we use $\mathbf{L}^*_\circ$: instead of $\circ$ we now have a $\circ_i$ for each $i \in \mathbb{N}$. The idea is that each expert gets a number i, and everything she states is preceded by $\circ_i$.

In case some experts' opinions conflict, we do not prioritize between their expertise such as we did in the previous example where we distinguished the "trustworthiness of the source". However, we prefer expert opinions in proportion to how often they have been stated by different experts. E.g., if 6 experts state A and only 2 state $\neg A$, we prioritize A.

This can be realized as follows. We define

$$\Omega_* = \{\circ_i A \wedge \neg A \mid A \text{ is a } \circ\text{-free formula}\}$$

We compare the abnormal parts of $\mathbf{L}_\circ^*$-models by means of the order $\prec_c$:

Definition 2.2 (Counting order on $\wp(\Omega) \times \wp(\Omega)$)**.** Where $\varphi, \psi \subseteq \Omega$ are sets of abnormalities, $\varphi \prec_c \psi$ iff, $|\varphi| < |\psi|$ or $\varphi \subset \psi$ where $|X|$ is the cardinality of X.[7]

So suppose we have the following scenario:

expert	states
1	A, B
2	$\neg A, B$
3	A, B
4	$A, \neg B$

We can translate this into a premise set:

$$\Gamma_c = \{\circ_1 A, \circ_1 B, \circ_2 \neg A, \circ_2 B, \circ_3 A, \circ_3 B, \circ_4 A, \circ_4 \neg B\}$$

Now consider the following $\mathbf{L}_\circ^*$-models of Γ_c:

model	Ab(M)	$\lvert$Ab$(M)\rvert$	$M \models$
M_1	$\circ_2 \neg A \wedge \neg\neg A, \circ_4 \neg B \wedge \neg\neg B$	2	A, B
M_2	$\circ_2 \neg A \wedge \neg\neg A, \circ_1 B \wedge \neg B, \circ_2 B \wedge \neg B, \circ_3 B \wedge \neg B$	4	$A, \neg B$
M_3	$\circ_1 A \wedge \neg A, \circ_3 A \wedge \neg A, \circ_4 A \wedge \neg A, \circ_4 \neg B \wedge \neg\neg B$	4	$\neg A, B$
M_4	$\circ_1 A \wedge \neg A, \circ_3 A \wedge \neg A, \circ_4 A \wedge \neg A, \circ_1 B \wedge \neg B, \circ_2 B \wedge \neg B, \circ_3 B \wedge \neg B$	6	$\neg A, \neg B$

$\prec_c$ imposes the following partial order on our models:

$$\begin{tikzcd}
 & \mathrm{Ab}(M_4) \arrow[dl, dash] \arrow[dr, dash] & \\
\mathrm{Ab}(M_2) \arrow[dr, dash] & & \mathrm{Ab}(M_3) \arrow[dl, dash] \\
 & \mathrm{Ab}(M_1) &
\end{tikzcd}$$

Let **AL3** be the AL that is characterized by the lower limit logic $\mathbf{L}_\circ^*$, the set of abnormalities Ω_*, the order $\prec_c$ on the (abnormal parts) of our $\mathbf{L}_\circ^*$-models, and the threshold $\min_{\prec_c}$ for the semantic selection. Hence,

$$\mathcal{M}_{\mathbf{AL3}}(\Gamma) =_{\mathrm{df}} \left\{ M \in \mathcal{M}_{\mathbf{L}_\circ^*}(\Gamma) \mid \mathrm{Ab}(M) \in \min_{\prec_c}(\mathsf{Ab}^{\Gamma}_{\mathbf{L}_\circ^*}) \right\}$$

The semantic consequence relation is defined analogous to ($\star$).

It is easy to see that in our example all selected models have the abnormal part of M_1. Hence, we get $\Gamma_c \Vdash_{\mathbf{AL3}} A, B$. This is as expected since A was stated by three experts, while $\neg A$ was stated by only one expert. An analogous argument applies to B.

[7] Of course, for finite sets $\varphi \subset \psi$ implies $|\varphi| < |\psi|$. However, for infinite sets the comparison by means of the cardinality does not allow to prefer φ to ψ in case of $\varphi \subset \psi$, although the latter clearly indicates that φ is "better" ("less abnormal") than ψ.

2.4 Colexicographic ALs

Let us stay a bit longer with $\mathbf{L}_\circ^*$. We now focus on a single participant of the discussion and offer a different reading of $\circ_i A$. Namely, A has been uttered by our participant at time point i. We move from time point 1 on forward in time. We are interested in offering an account of the participant's changes of mind: in case she states $\neg A$ at time point 1 but changes her mind later on, say at time point 5 she states A, we expect to derive A. Hence, the consequences of our logic mirror the state of mind of our agent based on what she states at the latest time point she offers a statement.

Take as an example the following premise set

$$\Gamma_{co} = \{\circ_1 A, \circ_2 B, \circ_3 C, \circ_4 \neg B, \circ_5 \neg A\}$$

and consider the following models:

model	Ab(M)	$M \models$
M_1	$\circ_5\neg A \wedge \neg\neg A, \circ_4\neg B \wedge \neg\neg B$	A, B, C
M_2	$\circ_5\neg A \wedge \neg\neg A, \circ_2 B \wedge \neg B$	$A, \neg B, C$
M_3	$\circ_1 A \wedge \neg A, \circ_2 B \wedge \neg B, \circ_3 C \wedge \neg C$	$\neg A, \neg B, \neg C$
M_4	$\circ_1 A \wedge \neg A, \circ_4\neg B \wedge \neg\neg B$	$\neg A, B, C$
M_5	$\circ_1 A \wedge \neg A, \circ_2 B \wedge \neg B$	$\neg A, \neg B, C$

The idea to prioritize later statements over former incompatible ones is realized by means of the following colexicographic order:

Definition 2.3 (Colexicographic order on $\wp(\Omega) \times \wp(\Omega)$). Where $\varphi, \psi \subseteq \Omega$, $\varphi \prec_{\mathsf{co}} \psi$ iff $\varphi \subset \psi$ or there is a $n \in \mathbb{N}$ for which

(a) $\varphi \cap \Omega_i = \psi \cap \Omega_i$ for all $i > n$, and

(b) $\varphi \cap \Omega_n \subset \psi \cap \Omega_n$

If we order our models by means of $\prec_{\mathsf{co}}$ with respect to their abnormal parts which are based on Ω_* we have:

$$\text{Ab}(M_5) \prec_{\mathsf{co}} \text{Ab}(M_3) \prec_{\mathsf{co}} \text{Ab}(M_4) \prec_{\mathsf{co}} \text{Ab}(M_2) \prec_{\mathsf{co}} \text{Ab}(M_1)$$

Indeed, all models M with abnormal part in $\min_{\prec_{\mathsf{co}}}(\mathsf{Ab}^{\Gamma_{co}}_{\mathbf{L}_\circ^*})$ have the same abnormal part as our M_5.

Hence, where **AL4.1** is characterized by the lower limit logic $\mathbf{L}_\circ^*$, the set of abnormalities Ω_*, and by the semantic selection based on $\min_{\prec_{\mathsf{co}}}$, we have $\Gamma \Vdash_{\mathbf{AL4.1}} \neg A, \neg B, C$ (where $\Vdash_{\mathbf{AL4.1}}$ is defined analogous to (*)).[8]

[8]Of course, we could define a logic on the basis of $\mathbf{L}_\circ$ that realizes the same idea. Instead of e.g., $\circ_3 A$ we could use $\circ\circ\circ A$ in order to express that A is stated at time point 3.

2.5 The problem of non-smoothness

Logics such as **AL4.1** are troublesome as soon as the order $\langle \mathsf{Ab}^{\Gamma}_{\mathbf{L}^*_\circ}, \prec_{\mathsf{co}} \rangle$ is not smooth.[9,10] Take for instance the premise set (where A_i, B ($i \in \mathbb{N}$) are distinct propositional letters):

$$\Gamma_a = \{\neg A_i \vee \neg A_j \mid i < j\} \cup \{\circ A_i \mid i \in \mathbb{N}\} \cup \{\circ_5 B\}$$

There is an infinitely descending sequence of "better and better and never best" models that goes as follows (where $\mho = \{\circ A_i \wedge \neg A_i \mid i \in \mathbb{N}\}$):

model	Ab(M)	$M \models$
M_1	$\mho \setminus \{\circ_1 A_1 \wedge \neg A_1\}$	$B, A_1, \neg A_2, \neg A_3, \ldots$
M_2	$\mho \setminus \{\circ_2 A_2 \wedge \neg A_2\}$	$B, \neg A_1, A_2, \neg A_3, \neg A_4, \ldots$
$\vdots$	$\vdots$	$\vdots$
M_n	$\mho \setminus \{\circ_n A_n \wedge \neg A_n\}$	$B, \neg A_1, \ldots, \neg A_{n-1}, A_n, \neg A_{n+1}, \neg A_{n+2}, \ldots$
$\vdots$	$\vdots$	$\vdots$

Note that

$$\ldots \prec_{\mathsf{co}} M_n \prec_{\mathsf{co}} \ldots \prec_{\mathsf{co}} M_2 \prec_{\mathsf{co}} M_1$$

Note further that $\min_{\prec_{\mathsf{co}}}(\mathsf{Ab}^{\Gamma_a}_{\mathbf{L}^*_\circ}) = \emptyset$. This immediately implies that $\Gamma_a \Vdash_{\mathbf{AL4.1}} C$ for any formula C. This is unfortunate since we do not want that our AL trivializes premise sets that are non-trivial in its lower limit logic. This problem can be avoided by slightly adjusting the selection procedure: instead of using as a threshold for our selection the set $\min_{\prec_{\mathsf{co}}}(\mathsf{Ab}^{\Gamma}_{\mathbf{L}^*_\circ})$ we select[11]

$$\Psi_{\prec_{\mathsf{co}}}(\mathsf{Ab}^{\Gamma}_{\mathbf{L}^*_\circ}) =_{\mathsf{df}} \{\mathsf{Ab}(M) \in \mathsf{Ab}^{\Gamma}_{\mathbf{L}^*_\circ} \mid \text{for all } \mathsf{Ab}(M') \in \min_{\prec_{\mathsf{co}}}(\mathsf{Ab}^{\Gamma}_{\mathbf{L}^*_\circ}), \mathsf{Ab}(M') \not\prec_{\mathsf{co}} \mathsf{Ab}(M)\}$$

It follows immediately by the definition that $\min_{\prec_{\mathsf{co}}}(\mathsf{Ab}^{\Gamma}_{\mathbf{L}^*_\circ}) \subseteq \Psi_{\prec_{\mathsf{co}}}(\mathsf{Ab}^{\Gamma}_{\mathbf{L}^*_\circ})$. Hence all minimal models (if there are any) remain selected. The additional selected models are the ones in infinitely descending chains (for which there are no minimal models below).

However, there is still a problem. In our previous example each M_i (where $i \in \mathbb{N}$) is selected according to this new selection. Consider a M_i^B such that $\mathsf{Ab}(M_i^B) = \mathsf{Ab}(M_i) \cup \{\circ_5 B \wedge \neg B\}$. This M_i^B is also selected, since there is no M' for which $\mathsf{Ab}(M') \in \min_{\prec_{\mathsf{co}}}(\mathsf{Ab}^{\Gamma_a}_{\mathbf{L}^*_\circ})$ such that $\mathsf{Ab}(M') \prec_{\mathsf{co}} \mathsf{Ab}(M_i^B)$. This seems clearly counter-intuitive.

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[9] $\langle X, \prec \rangle$ is *smooth* iff for all $x \in X$ there is a $y \in \min_{\prec}(X)$ such that $y \preceq x$.

[10] Non-smooth configurations similar to the following example have been discussed in the literature. See for instance Batens' discussion of Priest's $\mathbf{LP^m}$ in [2], or an example in the context of Circumscription discussed by Bossu and Siegel in [6].

[11] Batens mentioned this idea in the context of inconsistency-tolerant logics in [2]. Here it is applied in a more generic setting and we systematically investigate its meta-theory.

It is worthwhile to pinpoint why this is counter-intuitive and to try to translate this insight into our selection procedure. Note first that although according to $\prec_{\mathsf{co}}$ the model M_i^B is worse than all the models M_j (where $j \geq i$), $\prec_{\mathsf{co}}$ doesn't offer a demarcation principle or rationale by means of which we would select some M_j but not M_i^B. According to $\prec_{\mathsf{co}}$, M_i^B –just like M_j– is just another model in an infinitely descending chain for which there is no minimal model M' that is equal or better. There is just as much reason for (de-)selecting M_i^B as for M_j. "Wait", one may say, "but $\mathrm{Ab}(M_j) \prec_{\mathsf{co}} \mathrm{Ab}(M_i^B)$. Isn't that a reason to select M_j and to deselect M_i^B?" However, analogously $\mathrm{Ab}(M_{j+1}) \prec_{\mathsf{co}} \mathrm{Ab}(M_j)$ and $\mathrm{Ab}(M_{j+2}) \prec_{\mathsf{co}} \mathrm{Ab}(M_{j+1})$ and so forth. Hence, by applying this line of reasoning symmetrically, none of our M_j's would be selected. Altogether, we have an all-or-nothing choice: either we select all models in the infinitely descending chains or none. Every other choice would be asymmetric and hence ad hoc with respect to $\prec_{\mathsf{co}}$.

So, given that $\prec_{\mathsf{co}}$ isn't doing any work in demarcating M_i^B from the M_j's, is our intuition that we should rather de-select M_i^B based on a confusion? Rather, we suggest, is it based on a second-order principle that we use on top of comparing models by means of $\prec_{\mathsf{co}}$. The second order principle offers additional means of qualitatively demarcating some models from others. It helps us to express that some models that defer with respect to $\prec_{\mathsf{co}}$ defer in a more significant sense than others. E.g., M_i and M_i^B defer more significantly than M_{i+1} and M_i, although according to $\prec_{\mathsf{co}}$ the situation is symmetric ($\mathrm{Ab}(M_{i+1}) \prec_{\mathsf{co}} \mathrm{Ab}(M_i)$ and $\mathrm{Ab}(M_i) \prec_{\mathsf{co}} \mathrm{Ab}(M_i^B)$). In this case the second order principle says: M_i and M_i^B defer significantly because $\mathrm{Ab}(M_i) \subset \mathrm{Ab}(M_i^B)$, while $\mathrm{Ab}(M_{i+1}) \not\subset \mathrm{Ab}(M_i)$ and hence these two models do not defer significantly.

Altogether, what we are interested in is another partial order $\prec'$ which emphasizes certain distinctions made within $\prec_{\mathsf{co}}$ while neglecting others as less important. $\prec'$ should not introduce new distinctions that were not made by means of $\prec_{\mathsf{co}}$ already. Hence, $\prec' \subset \prec_{\mathsf{co}}$. Our discussion above motivates to use $\subset$ in order to implement our second order principle. The selection can then proceed by means of:[12]

$$\Psi_{[\prec_{\mathsf{co}},\subset]}\big(\mathsf{Ab}^{\Gamma}_{\mathbf{L}^*_{\circ}}\big) =_{\mathrm{df}} \Psi_{\subset}\big(\Psi_{\prec_{\mathsf{co}}}\big(\mathsf{Ab}^{\Gamma}_{\mathbf{L}^*_{\circ}}\big)\big)$$

According to this selection only the models M_i are selected, while any model that validates $\neg B$ (such as M_i^B) is de-selected. Hence, where **AL4.2** is the AL based on the semantic selection offered by $\Psi_{[\prec_{\mathsf{co}},\subset]}$, we get $\Gamma_a \Vdash_{\mathbf{AL4.2}} B$.

2.6 More refined quantitative examples

Let us come back to our application in the example of Section 2.3. There, models were selected in which statements that are offered the most frequent by our experts are validated. Now suppose we are not only interested in the models that validate the most frequently stated statements, but also some others which are "good

[12]This is not the same as simply defining another partial order $\prec'_{\mathsf{co}}$ by "$\mathrm{Ab}(M) \prec'_{\mathsf{co}} \mathrm{Ab}(M')$ iff $\mathrm{Ab}(M) \prec_{\mathsf{co}} \mathrm{Ab}(M')$ or $\mathrm{Ab}(M) \subset \mathrm{Ab}(M')$" and then to use $\Psi_{\prec'_{\mathsf{co}}}$. Note that since $\subset \subseteq \prec_{\mathsf{co}}$, also $\prec'_{\mathsf{co}} = \prec_{\mathsf{co}}$.

enough". One way to do so would be by introducing a threshold value τ: instead of selecting the models whose abnormal parts are in $\min_{\prec_c}(\mathsf{Ab}^{\Gamma}_{\mathbf{L}^*_\circ})$ we select the models whose abnormal part is in

$$\Lambda^1_c(\mathsf{Ab}^{\Gamma}_{\mathbf{L}^*_\circ}) =_{\mathrm{df}} \{\mathsf{Ab}(M) \in \mathsf{Ab}^{\Gamma}_{\mathbf{L}^*_\circ} \mid |\mathsf{Ab}(M)| - \tau \leq |\varphi|\}$$

where φ is an arbitrary element in $\min_{\prec_c}(\mathsf{Ab}^{\Gamma}_{\mathbf{L}^*_\circ})$.

For instance, suppose $\tau = 3$. For our example based on Γ_c, both M_2 and M_3 would now be selected, while M_4 is still not selected. The value τ hence introduces some error tolerance: in this case we allow the majority to be mistaken about A (model M_3) or to be mistaken about B (model M_2) but not to be mistaken about both, A and B (model M_4 is not selected).

Another option would be to use the following selection:[13]

$$\Lambda^2_c(\mathsf{Ab}^{\Gamma}_{\mathbf{L}^*_\circ}) =_{\mathrm{df}} \left\{ \mathsf{Ab}(M) \in \mathsf{Ab}^{\Gamma}_{\mathbf{L}^*_\circ} \middle| |\mathsf{Ab}(M)| \leq \frac{\sum_{\mathsf{Ab}(M') \in \min_{\subset}(\mathsf{Ab}^{\Gamma}_{\mathbf{L}^*_\circ})} |\mathsf{Ab}(M')|}{|\min_{\subset}(\mathsf{Ab}^{\Gamma}_{\mathbf{L}^*_\circ})|} \right\}$$

Take for instance

$$\Gamma^2_c = \{\circ_i(A \wedge B), \circ_j(A \wedge \neg B), \circ_k A, \circ_{11}\neg A \mid i \in \{1,2,3\}, j \in \{4,5,6,7\}, k \in \{8,9,10\}\}$$

We have three types of models M with $\mathsf{Ab}(M) \in \min_{\subset}(\mathsf{Ab}^{\Gamma^2_c}_{\mathbf{L}^*_\circ})$:

model	$\mathsf{Ab}(M)$	$\lvert\mathsf{Ab}(M)\rvert$	$M \models$
M_1	$\circ_{11}\neg A \wedge \neg\neg A, \circ_i(A \wedge B) \wedge \neg(A \wedge B)$ ($i \in \{1,2,3\}$)	4	$A, \neg B$
M_2	$\circ_{11}\neg A \wedge \neg\neg A, \circ_j(A \wedge \neg B) \wedge \neg(A \wedge \neg B)$ ($j \in \{4,5,6,7\}$)	5	A, B
M_3	$\circ_i(A \wedge B) \wedge \neg(A \wedge B)$ ($i \in \{1,2,3\}$), $\circ_j(A \wedge \neg B) \wedge \neg(A \wedge \neg B)$, ($j \in \{4,5,6,7\}$), $\circ_k A \wedge \neg A$ ($k \in \{8,9,10\}$)	10	$\neg A, B$

A model M is selected if its abnormal part $\mathsf{Ab}(M) \in \Lambda^2_c(\mathsf{Ab}^{\Gamma^2_c}_{\mathbf{L}^*_\circ})$, i.e., if $|\mathsf{Ab}(M)| \leq \frac{4+5+10}{3}$. Hence, M_1 and M_2 are selected, while M_3 is not selected.

This selection is very contextual: a median value is calculated on the basis of how many statements are conflicted in the maximally consistent sets of statements. The models that conflict less than this median value are accepted. Suppose the distribution of conflicts in the example above would be different: instead of 4-5-10 ($|\mathsf{Ab}(M_1)|$, $|\mathsf{Ab}(M_2)$, $|\mathsf{Ab}(M_3)$) we could have for instance 2-7-9. Then we would get the median value 6. In this case only M_1 would be selected.

[13]Where $n \in \mathbb{N}^{\infty}$, we define $\frac{\infty}{n} =_{\mathrm{df}} \infty$.

2.7 Summary

We have seen various examples for ALs that use orders different from set-inclusion in order to select models. Let us abstract away from the concrete examples and identify the formats of ALs we have discussed above. We characterize ALs in the following by triples: $\langle \mathbf{LLL}, \Omega, \Lambda \rangle$ where **LLL** is the lower limit logic, Ω is the set of abnormalities, and Λ determines the threshold for the selection of the models.[14]

We started with ALs in the standard format characterized by:

$$\langle \mathbf{LLL}, \Omega, \min_{\subset} \rangle \tag{1}$$

Selected are the **LLL**-models M of a premise set Γ whose abnormal part is in

$$\min_{\subset}(\mathsf{Ab}^{\Gamma}_{\mathbf{LLL}}) = \{\mathsf{Ab}(M) \mid M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma), \\ \text{for all } M' \in \mathcal{M}_{\mathbf{LLL}}(\Gamma), \mathsf{Ab}(M') \not\subset \mathsf{Ab}(M)\}$$

Then we moved on to ALs with other partial orders $\prec$, but presupposing that

(†) for all premise sets Γ, $\langle \mathsf{Ab}^{\Gamma}_{\mathbf{LLL}}, \prec \rangle$ is smooth.

These logics can be characterized by:

$$\langle \mathbf{LLL}, \Omega, \min_{\prec} \rangle \tag{2}$$

The selected models of a premise set Γ are the ones whose abnormal part is in:

$$\min_{\prec}(\mathsf{Ab}^{\Gamma}_{\mathbf{LLL}}) = \{\mathsf{Ab}(M) \mid M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma), \\ \text{for all } M' \in \mathcal{M}_{\mathbf{LLL}}(\Gamma), \mathsf{Ab}(M') \not\prec \mathsf{Ab}(M)\}$$

Dropping the smoothness requirement (†) for $\prec$ we first proposed the following format:

$$\langle \mathbf{LLL}, \Omega, \Psi_{\prec} \rangle \tag{3}$$

The selected models of a premise set Γ are the ones whose abnormal part is in:

$$\Psi_{\prec}(\mathsf{Ab}^{\Gamma}_{\mathbf{LLL}}) = \{\mathsf{Ab}(M) \mid M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma), \\ \text{there is no } \mathsf{Ab}(M') \in \min_{\prec}(\mathsf{Ab}^{\Gamma}_{\mathbf{LLL}}) \text{ such that } \mathsf{Ab}(M') \prec \mathsf{Ab}(M)\}$$

Since

Fact 2.1. *If* $\langle \mathsf{Ab}^{\Gamma}_{\mathbf{LLL}}, \prec \rangle$ *is smooth, then* $\Psi^{\prec}(\mathsf{Ab}^{\Gamma}_{\mathbf{LLL}}) = \min_{\prec}(\mathsf{Ab}^{\Gamma}_{\mathbf{LLL}})$.

[14] We still focus on the semantic aspect of ALs in order to not open more doors than necessary at this point of the discussion. However, this should not distract from the fact that all these semantic features have a syntactic counter-part. We will investigate also the syntax of ALs beginning with the next section.

and given the presupposition (†) we used for the format (2), the semantic selection characterized by $\langle \mathbf{LLL}, \Omega, \min_{\prec} \rangle$ is identical to the one characterized by $\langle \mathbf{LLL}, \Omega, \Psi_{\prec} \rangle$.

As the example in Section 2.5 motivates, sometimes we are interested in basing our selection on more refined selections. On way to do so was by means of an order $\prec'$ where $\prec' \subseteq \prec$:

$$\langle \mathbf{LLL}, \Omega, \Psi_{[\prec, \prec']} \rangle \tag{4}$$

Here, the selection is refined by making use of the order $\prec'$ so that the selected models for a premise set Γ are the ones whose abnormal part is in

$$\Psi_{[\prec, \prec']}(\mathsf{Ab}^{\Gamma}_{\mathbf{LLL}}) = \Psi_{\prec'}(\Psi_{\prec}(\mathsf{Ab}^{\Gamma}_{\mathbf{LLL}}))$$

Of course, one may generalize this approach to a sequence of orders.

Definition 2.4. We call a $\prec'$ an *abstraction order of* $\prec$ iff $\prec' \subseteq \prec$. A sequence of partial orders $\langle \prec_1, \prec_2, \ldots, \prec_n \rangle$ is an *abstraction sequence* iff for each $i < n$, $\prec_{i+1}$ is an abstraction order of $\prec_i$.

Where $\langle \prec_1, \prec_2, \ldots, \prec_n \rangle$ is an abstraction sequence we define

$$\Psi_{[\prec_1, \ldots, \prec_n]}(\mathsf{Ab}^{\Gamma}_{\mathbf{LLL}}) =_{\mathsf{df}} \Psi_{\prec_n}(\Psi_{\prec_{n-1}}(\ldots \Psi_{\prec_1}(\mathsf{Ab}^{\Gamma}_{\mathbf{LLL}})))$$

The most abstract representation can be obtained when we abstract from the concrete way the threshold is obtained by an order $\prec$ and just focus on the threshold function itself.

$$\langle \mathbf{LLL}, \Omega, \Lambda \rangle \tag{5}$$

Obviously, all the meta-theory that is available for the most generic format (5) straight-forwardly applies to the other formats as well.

In the next section we characterize the proof theory and semantics of this format. Then, in Section 4, we study the meta-theory of this rich class of ALs. In Section 5 we will wrap things up, also by relating the meta-theoretic insights back to the examples from the current section.

3 The Generic Adaptive Logic $\mathbf{AL}_{\boldsymbol{\Lambda}}$

We will now define adaptive logics $\mathbf{AL}_{\boldsymbol{\Lambda}}$ characterized by our most generic presentation $\langle \mathbf{LLL}, \Omega, \Lambda \rangle$ where

- the *Lower Limit Logic* **LLL** has an adequate semantics, is a Tarski logic and thus has the following properties:

$$\Gamma \subseteq Cn_{\mathbf{LLL}}(\Gamma) \tag{Reflexivity}$$
$$Cn_{\mathbf{LLL}}(\Gamma) \subseteq Cn_{\mathbf{LLL}}(\Gamma \cup \Gamma') \tag{Monotonicity}$$
$$\text{Where } \Gamma' \subseteq Cn_{\mathbf{LLL}}(\Gamma),\ Cn_{\mathbf{LLL}}(\Gamma') \subseteq Cn_{\mathbf{LLL}}(\Gamma) \tag{Transitivity}$$

$$\text{If } A \in Cn_{\mathbf{LLL}}(\Gamma) \text{ then there is a finite } \Gamma' \subseteq \Gamma \text{ such that } A \in Cn_{\mathbf{LLL}}(\Gamma'). \qquad \text{(Compactness)}$$

Furthermore we will presuppose that **LLL** is supraclassical and has at least a classical negation $\neg$ and a classical disjunction $\vee$.[15]

- the *set of abnormalities* Ω is characterized by a (or many) logical form(s),[16]
- $\Lambda : \wp(\wp(\Omega)) \to \wp(\wp(\Omega))$ is a *threshold function*, where

Definition 3.1. A *threshold function* $\Lambda : \wp(\wp(\Omega)) \to \wp(\wp(\Omega))$ satisfies the following:[17]

T1 inclusive ($\Lambda(X) \subseteq X$)

T2 $\Lambda(X)$ is a $\subset$-lower set[18] of X

T3 if $\emptyset \in X$, $\emptyset \in \Lambda(X)$.

Definition 3.2. Where $\prec$ is a partial order on Y: X is a $\prec$*-lower set of* Y iff for all $x \in X$ and all $y \in Y$, if $y \prec x$ then $y \in X$.

Of course, $\Lambda(X)$ shall select sets in X and not introduce new sets of abnormalities. Hence, **T1**. **T2** expresses that if some set φ of abnormalities is selected and hence deemed "sufficiently normal" according to Λ, then any $\psi \in X$ for which $\psi \subset \varphi$ shall also be selected: after all ψ contains even less abnormalities than φ. Moreover, **T2** helps to significantly simplify the meta-theory. **T3** expresses that the abnormality-free empty set should always be selected: there is indeed no reason to deem it not "sufficiently normal".

In the following we will first present the semantics, then the dynamic proof theory of $\mathbf{AL}_{\boldsymbol{\Lambda}}$. It will become evident that we hereby generalize the standard format. Hence, all the meta-theory for $\mathbf{AL}_{\boldsymbol{\Lambda}}$ immediately applies to the standard format as well (see also the discussion in Section 5.2).

3.1 The Semantics

As in the standard format we use a selection semantics. The threshold function Λ defines which models we select from $\mathcal{M}_{\mathbf{LLL}}(\Gamma)$.

Definition 3.3. $\mathcal{M}_{\mathbf{AL}_{\boldsymbol{\Lambda}}}(\Gamma) =_{\text{df}} \{M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma) \mid \mathsf{Ab}(M) \in \Lambda(\mathsf{Ab}^{\Gamma}_{\mathbf{LLL}})\}$

---

[15] In case a logic **L** that satisfies our four requirements, reflexivity, monotonicity, transitivity, and compactness, is not supraclassical, $\neg$ and $\vee$ can be superimposed. We describe the procedure in Appendix H. Note that in the standard format $\neg$ and $\vee$ are superimposed even in case **L** has already a classical negation and a classical disjunction. This is not required in the approach presented in this paper. We also discuss this further in Appendix H. In this paper we also deviate from the standard format in other aspects (see Section 3.2.4 and Appendix I and J). It will always be indicated. In [32] $\mathbf{AL}_{\boldsymbol{\Lambda}}$ is characterized in a structurally much more conservative way with respect to the standard format.

[16] The logical form F is supposed to be **LLL**-contingent, i.e., neither $\vdash_{\mathbf{LLL}} \mathsf{F}$ nor $\vdash_{\mathbf{LLL}} \neg\mathsf{F}$ holds.

[17] Functions only satisfying **T1** are sometimes called *selection functions* REF-FRE.

[18] See the Definition 3.2 below.

Definition 3.4. $\Gamma \Vdash_{\mathbf{AL}_\Lambda} A$ iff for all $M \in \mathcal{M}_{\mathbf{AL}_\Lambda}(\Gamma)$, $M \models A$.

Remark 3.1. Note that where $\Lambda = \min_\subset$ we get exactly the semantics of the minimal abnormality strategy of the standard format. Where $\Lambda(X) = \{\varphi \in X \mid \varphi \subseteq \bigcup \min_\subset(X)\}$ we get the semantics of the reliability strategy of the standard format. We will come back to this in Section 5.2.

3.2 The Dynamic Proof Theory

Traditional proofs are strictly accumulative. Whenever a formula is derived at some line from some premise set Γ, it has the status of a consequence of Γ from that point on: i.e., no successive derivation can change anything about that. We may call such proofs *static*.

When using formal proofs in order to explicate defeasible reasoning this static feature is a serious drawback. In defeasible reasoning processes we jump to conclusions –often in view of certain normality/default/etc. assumptions–, some of which may be retracted in view of newly gained insights into our premises. The classic example in default reasoning is the bird Tweety: we may at some point in a proof derive that Tweety flies in view of our knowledge that birds usually fly and on the normality assumption that Tweety is not exceptional in this case. However, analyzing our premises further we may be able to derive that Tweety is a penguin, which indicates that our previous assumption is ill-founded. In this case we want to retract the previous derivation that Tweety flies. This dynamic aspect of defeasible reasoning has been noted by various authors, e.g., by Pollock who speaks about "diachronic defeasibility" [19] and Batens who speaks about the "internal dynamics" of defeasible reasoning [3].

In *dynamic* proofs we do justice to the internal dynamics of defeasible reasoning. These proofs are non-static and non-accumulative because a once derived formula may loose its status of being derived at a later point in the proof. This is realized by means of two additional elements that distinguish dynamic proofs from static proofs.

On the one hand, formulas can be derived in view of normality assumptions. These assumptions are represented by (finite) sets of abnormalities which are associated with each proof line. An *annotated proof line* consists of: optionally a mark, a line number l, a formula A, a justification that may call upon a number of lines $l_1, \ldots, l_n$ (where $l_i < l$ for all $i \leq n$) and that calls upon a rule R, and a condition Δ which is a finite set of abnormalities. Schematically an unmarked line looks as follows:

$$l \quad A \qquad\qquad l_1, \ldots, l_n; R \quad \Delta$$

The same line marked:

$$\checkmark\, l \quad A \qquad\qquad l_1, \ldots, l_n; R \quad \Delta$$

That a formula A is derived on the condition Δ means that A is derived on the assumption that each abnormality in Δ is false. In case $\Delta = \emptyset$ we say that A is derived unconditionally. This means that no normality assumption was used for the derivation of A.

On the other hand, there is a marking mechanism which checks for each line whether the associated assumption is acceptable. In case it is not, the line is marked. The formula of a line counts only as being admissible at some point in the proof in case the line is not marked. We will describe the marking definition in detail below.

A static proof of some logic **L** is the result of a (finite) sequence of *derivation steps*: each step consists of adding a new line at the end of a list of lines in a way that is licensed by the derivation rules of **L**.

As noted above, a dynamic proof is not only the result of derivation steps but also lines are *marked* as determined by the marking definition. This has the effect that after a new line l has been derived and before a new line $l+1$ is added by means of a derivation step, lines 1 to l are marked. In this view a dynamic proof is the result of a finite sequence of derivation steps producing lines 1–l, completed by the application of the marking definition to the lines 1 to l.

An *extension* of a dynamic proof is obtained by adding a number derivation steps, resulting in an extended list of annotated lines, and by the application of the marking definition the this list.[19]

It will be useful for the meta-level reasoning that is necessary to define a static notion of derivability (see Section 3.2.3) to also allow for limit cases in which dynamic proofs are extended infinitely. This concerns the hypothetical case in which an infinite amount of derivation steps is applied and to the resulting (infinite) list of lines the marking definition is applied.[20]

Finally, it will often be useful to speak about *stages* of (extended) dynamic proofs. Each (extended) dynamic proof has a history which corresponds to the order in which the derivation steps are applied. For each initial sequence of derivation steps the corresponding stage represents the associated list of proof lines to which the marking definition is applied. We will customary refer to finite stages with the last line number in the list of lines they represent.

3.2.1 The Generic Derivation Rules

The derivation step is governed by three generic rules: PREM, RU, and RC. The rule PREM allows to introduce premises from the premise set Γ on the empty condition.

$$\text{Where } A \in \Gamma: \qquad \frac{\begin{matrix}\vdots & \quad \vdots\end{matrix}}{\begin{matrix}A & \quad \emptyset\end{matrix}} \qquad \text{(PREM)}$$

[19]Opposite to the standard format there is no need to allow for the possibility of inserting lines: hence, the additional derivation steps produce lines that are added at the end of the list. See also our discussion in Section 3.2.4 and Appendix I.

[20]For us the resulting objects, infinite lists of annotated lines, are not themselves dynamic proofs viz. finite lists of annotated lines that are the result of actual object-level reasoning. However, nothing in the mechanism of the framework we describe depends on this philosophical question.

The RU (Rule Unconditional) makes the whole derivational strength of the lower limit logic available. If B is derivable from $A_1, \ldots, A_n$ in **LLL** then it is also derivable in the dynamic proof.

$$\text{Where } A_1, \ldots, A_n \vdash_{\mathbf{LLL}} B: \qquad \begin{array}{ll} A_1 & \Delta_1 \\ \vdots & \vdots \\ A_n & \Delta_n \\ \hline B & \Delta_1 \cup \ldots \cup \Delta_n \end{array} \qquad \text{(RU)}$$

Note that the (possibly empty) conditions on which the A_i's are derived are carried forward and merged in the line at which B is derived.

Finally, there is RC (Rule Conditional). If $B \vee \mathsf{Dab}(\Delta)$ is derivable in **LLL** we can derive B on the assumption that no abnormality in Δ is true.

$$\text{Where } A_1, \ldots, A_n \vdash_{\mathbf{LLL}} B \vee \mathsf{Dab}(\Delta): \qquad \begin{array}{ll} A_1 & \Delta_1 \\ \vdots & \vdots \\ A_n & \Delta_n \\ \hline B & \Delta_1 \cup \ldots \cup \Delta_n \cup \Delta \end{array} \qquad \text{(RC)}$$

Let us come back to our example from Section 2.4 in order to demonstrate the generic rules. First we introduce our first three premises:

1	$\circ_1 A$	PREM	$\emptyset$
2	$\circ_2 B$	PREM	$\emptyset$
3	$\circ_3 C$	PREM	$\emptyset$

From this we can conditionally derive the following:

4	A	1; RC	$\{\circ_1 A \wedge \neg A\}$
5	B	2; RC	$\{\circ_2 B \wedge \neg B\}$
6	C	3; RC	$\{\circ_3 C \wedge \neg C\}$

The proof gets more interesting when we introduce our fourth premise $\circ_4 \neg B$ since this leads to a conflict:

7	$\circ_4 \neg B$	PREM	$\emptyset$
8	$(\circ_2 B \wedge \neg B) \vee (\circ_4 \neg B \wedge \neg\neg B)$	2,7; RU	$\emptyset$
9	$\neg B$	7; RC	$\{\circ_4 \neg B \wedge \neg\neg B\}$

Note that at this point we have an obvious conflict between the formula B derived at line 5 and the formula $\neg B$ derived at line 9. This brings us to our next point: the retraction mechanism of dynamic proofs.

3.2.2 The Marking of Lines

In view of the formula $(\circ_2 B \wedge \neg B) \vee (\circ_4 \neg B \wedge \neg\neg B)$ derived at line 8 one of the assumptions on lines 5 and 9 has to be false. Note moreover, that according to $\prec_{\mathsf{co}}$, the abnormality in the condition of line 9 is more severe than the one in the condition of line 5 since $\{\circ_2 B \wedge \neg B\} \prec_{\mathsf{co}} \{\circ_4 \neg B \wedge \neg\neg B\}$. Hence, the rationale is to prioritize the statement that was voiced later out of two conflicting statements.

The previous paragraph highlights two central ideas behind the adaptive marking.

First, the question whether the assumption of a line can be upheld is essentially informed by the minimal disjunctions of abnormalities that are derived at a given stage of the proof on the empty condition.

Notation 3.1. $\Sigma_s(\Gamma) = \{\Delta \mid \mathsf{Dab}(\Delta)$ is derived on the empty condition at stage s and there is no $\Delta' \subset \Delta$ such that $\mathsf{Dab}(\Delta')$ is derived on the empty condition at stage $s\}$

We know that for each $\Delta \in \Sigma_s(\Gamma)$ at least one abnormality $A \in \Delta$ has to be true. Hence, the choice sets[21] over the elements of $\Sigma_s(\Gamma)$ offer possible interpretations of the disjunctions of abnormalities that are derived at stage s.

Notation 3.2. $\Xi_s(\Gamma)$ is the set of choice sets in $\wp(\Omega)$ of $\Sigma_s(\Gamma)$. For the special case in which $\Sigma_s(\Gamma) = \emptyset$ we define $\Xi_s(\Gamma) =_{\mathsf{df}} \{\emptyset\}$.

For instance, $\{\circ_2 B \wedge \neg B\}$, $\{\circ_4 \neg B \wedge \neg\neg B\}$ and $\{\circ_2 B \wedge \neg B, \circ_4 \neg B \wedge \neg\neg B\}$ are elements in $\Xi_9(\Gamma)$.

The marking will proceed in view of these choice sets. Note that in case the condition Δ of a line l has a non-empty intersection with some choice set $\varphi \in \Sigma_s(\Gamma)$ ($\varphi \cap \Delta \neq \emptyset$), the assumption of line l does not hold with respect to the interpretation offered by φ.

The second central idea is that not all choice sets are treated equally. After all, $\prec_{\mathsf{co}}$ gives us a rationale according to which the choice sets are ordered: it imposes a partial order on $\Xi_s(\Gamma)$. In our case we have for instance:

$$\{\circ_2 B \wedge \neg B\} \prec_{\mathsf{co}} \{\circ_4 \neg B \wedge \neg\neg B\} \prec_{\mathsf{co}} \{\circ_2 B \wedge \neg B, \circ_4 \neg B \wedge \neg\neg B\}$$

The threshold function Λ associated with our adaptive logic tells us which of these choice sets in $\Xi_s(\Gamma)$ offer a sufficiently normal interpretation. For instance for $\Lambda = \min_{\prec_{\mathsf{co}}}$ or $\Lambda = \Psi_{\prec_{\mathsf{co}}}$ we get $\Lambda(\Xi_9(\Gamma)) = \{\{\circ_2 B \wedge \neg B\}\}$.

Altogether the idea is that, ideally, a formula A is derived in such a way that for each sufficiently normal interpretation of the Dab-formulas that are derived on the condition $\emptyset$ at stage s, A is derived on an assumption that is safe. Expressed more formally: for each $\varphi \in \Lambda(\Xi_s(\Gamma))$, A is derived on a condition Δ such that $\varphi \cap \Delta = \emptyset$. This means that in each sufficiently normal interpretation of our Dab-formulas, A is valid. This immediately motivates a marking definition.

Definition 3.5 (Marking). Line l with formula A and condition Δ is *marked at stage* s iff, $\Lambda(\Xi_s(\Gamma)) \neq \emptyset$ and

(i) there is no $\varphi \in \Lambda(\Xi_s(\Gamma))$ such that $\varphi \cap \Delta = \emptyset$, or

(ii) for a $\varphi \in \Lambda(\Xi_s(\Gamma))$ there is no line l' at stage s with formula A and a condition Θ such that $\Theta \cap \varphi = \emptyset$.

[21] X is a *choice set* of a set of sets Σ iff for each $\Delta \in \Sigma$, $X \cap \Delta \neq \emptyset$.

Remark 3.2. This means that, inversely, a line l with condition Δ and formula A is *not* marked at stage s iff, $\Lambda(\Xi_s(\Gamma)) = \emptyset$ or the following two conditions holds:

(i) there is a $\varphi \in \Lambda(\Xi_s(\Gamma))$ such that $\Delta \cap \varphi = \emptyset$, and

(ii) for all $\varphi \in \Lambda(\Xi_s(\Gamma))$ there is a line l' with formula A and a condition Θ for which $\Theta \cap \varphi = \emptyset$.

Item (i) says that Δ has to be safe with respect to at least some of the sufficiently normal interpretations of our Dab-formulas at stage s.

Item (ii) expresses an additional requirement: for each sufficiently normal interpretation φ of our Dab-formulas, A should be derived on an assumption that holds with respect to φ.

This indicates that an assumption is only safe for A (in the sense that the corresponding line is not marked) in case it is part of a set of safe assumptions that are robust with respect to all the sufficiently normal interpretations of the Dab-formulas.[22]

Remark 3.3. Note that there is no way that a line with the condition $\emptyset$ is marked.

Let us proceed with our example in order to illustrate the marking in more detail. At stage 9, line 5 is marked since its condition intersects with the only choice set in $\Lambda(\Xi_9(\Gamma))$, namely $\{\circ_2 B \wedge \neg B\}$. Moreover, line 9 is unmarked. This is the expected outcome since at time point 4 our agent changed her mind and stated $\neg B$ and it is the rationale of our logic to prefer later statements over earlier conflicting ones.

Let us continue and introduce our last premise:

10	$\circ_5 \neg A$	PREM	$\emptyset$
11	$(\circ_1 A \wedge \neg A) \vee (\circ_5 \neg A \wedge \neg\neg A)$	1, 10; RU	$\emptyset$
12	$\neg A$	10; RC	$\{\circ_5 \neg A \wedge \neg\neg A\}$

At this point $\Xi_{12}(\Gamma)$ contains for instance the following choice sets:

$$\varphi_1 = \{\circ_2 B \wedge \neg B, \circ_1 A \wedge \neg A\}$$
$$\varphi_2 = \{\circ_2 B \wedge \neg B, \circ_5 \neg A \wedge \neg\neg A\}$$
$$\varphi_3 = \{\circ_4 \neg B \wedge \neg\neg B, \circ_1 A \wedge \neg A\}$$
$$\varphi_4 = \{\circ_4 \neg B \wedge \neg\neg B, \circ_5 \neg A \wedge \neg\neg A\}$$

Note that $\varphi_1 \prec_{co} \varphi_3 \prec_{co} \varphi_2 \prec_{co} \varphi_4$. Indeed, as is easy to check, $\Lambda(\Xi_{12}(\Gamma_{co})) = \{\varphi_1\}$. Hence, lines 4 and 5 are marked while lines 9 and 12 are unmarked.

Note that markings may also disappear again at a later stage of the proof. Suppose for instance that at time point 8 our agent once more changes her mind and states A. Hence, suppose $\circ_8 A \in \Gamma_{co}$. We can extend the proof further:

13	$\circ_8 A$	PREM	$\emptyset$
14	A	13; RC	$\{\circ_8 A \wedge \neg A\}$
15	$(\circ_5 \neg A \wedge \neg\neg A) \vee (\circ_8 A \wedge \neg A)$	10, 13; RU	$\emptyset$

[22] There is an obvious link to the notion of admissibility in abstract argumentation which will be further explored in future research.

It is easy to see that now $\Lambda(\Xi_{15}(\Gamma)) = \{\{\circ_2 B \wedge \neg B, \circ_5 \neg A \wedge \neg\neg A\}\}$. As a consequence, line 12 is marked while lines 4 and 14 are unmarked at stage 15.

Based on the marking we can define a dynamic notion of derivability resp. of the acceptance of derived formulas for $\mathbf{AL}_{\mathbf{\Lambda}}$-proofs:

Definition 3.6. A formula A is *admissible at stage* s of a dynamic proof iff there is a line l with formula A that is not marked at stage s.[23]

For instance, while A is admissible at stage 4, it is not admissible at stage 11, and it is admissible again at stage 14 and 15.

3.2.3 Final Admissibility

In order to define the consequence relation of $\mathbf{AL}_{\mathbf{\Lambda}}$ we make use of a non-dynamic notion of derivability resp. of the acceptance of arguments. There is an important conceptual difference between admissibility at a stage and the static notion of admissibility that is going to be introduced in this section. Obviously both notions are normative in the sense that they license some derived A to be called "admissible". However, the former notion is relative to the degree in which the given premises have been analyzed at a given stage of a proof. By further analyzing the premises one may retract from some derived formulas the status of being admissible. In contrast, the latter notion concerns the (usually) hypothetical stage in which a reasoner would have exhaustively analyzed all premises at least with respect to what is relevant with respect to A. If then A can still be deemed "admissible" then (some of) the assumptions under which A is argued for are indeed robust with respect to our premises. Hence, A can be counted as a consequence, since further analyzing the premises will not change our judgment.

Naively one may request that A is *finally admissible*[24] at a stage s on line l of a dynamic proof in case in every extension of the given proof, l remains unmarked. However, there are simple examples that demonstrate that for some semantic consequences A, whenever A is derived at some finite stage s on a line l, there is always a way of extending the stage in a way such that line l is marked.

A better solution is as follows: one may request that line l on which A is derived in a dynamic proof is unmarked in the (possibly infinite) extension of the proof in which the premise set is analyzed exhaustively with respect to all arguments that can be produced for A and with respect to all information that is available about abnormalities. This is a stage in which all minimal Dab-formulas are derived on the empty condition, and A is derived on all (minimal) conditions. Since such a stage may be infinite and hence does not correspond to any dynamic proof, the

[23]What we call "admissible at stage s" is called "derivable at stage s" in the context of the standard format. However, we noticed that this nomenclature sometimes creates difficulties and misunderstandings (in some papers even ambiguities) in communicating that a formula A is "derived" but marked(!) at stage s. Hence, for us "A is derived (at stage s) at line l" simply means that A is the second element of a line l (at stage s) independent whether the line is marked or not.

[24]This notion is called "finally derivable" in the standard format. See Footnote 23.

question whether A is finally admissible at a stage of a proof requests meta-level reasoning. We may need to engage in the abstract meta-level reasoning about an infinite extension of a proof which itself cannot be concretely produced on the object-level.

Definition 3.7. We call an extension of a stage s *A-complete* iff,

(i) if $\mathsf{Dab}(\Delta)$ is derivable on the condition $\emptyset$ then $\mathsf{Dab}(\Delta')$ is derived on the condition $\emptyset$ for some $\Delta' \subseteq \Delta$

(ii) if A is derivable on the condition Δ then A is derived on a condition $\Delta' \subseteq \Delta$.

In an A-complete stage nothing remarkable can happen anymore with respect to the markings: the premises are analyzed exhaustively. All the information about abnormalities that is relevant for the marking is derived (item (i)), moreover A is derived on all conditions that are relevant for the marking (item (ii)).

In Appendix A we show that for any finite proof an A-complete stage exists.

Definition 3.8. A is *finally admissible at a line* l at a finite stage s iff line l is unmarked in an A-complete extension of s.

Of course, in case s is already A-complete and line l is unmarked, this suffices for A being finally derivable at stage s.

We show in the Appendix that due to the exhaustive nature of an A-complete stage s, the marking of a line l with formula A is independent of the concrete nature of the A-complete stage: if l is marked in some A-complete extension, then it is marked in any A-complete extension.

The notion of final admissibility allows us to introduce both, a derivability relation and a consequence relation for $\mathbf{AL}_{\mathbf{\Lambda}}$.

Definition 3.9. $\Gamma \vdash_{\mathbf{AL}_{\mathbf{\Lambda}}} A$ iff A is finally admissible in a proof from Γ.

Definition 3.10. $A \in Cn_{\mathbf{AL}_{\mathbf{\Lambda}}}(\Gamma)$ iff $\Gamma \vdash_{\mathbf{AL}_{\mathbf{\Lambda}}} A$

3.2.4 Final Admissibility, Finite Stages, and the Standard Format

One interesting question concerning final admissibility is whether every A that is finally admissible (and hence every adaptive consequence) can be derived at a finite stage on an unmarked line (and hence is admissible at a finite stage). In general, the answer is negative (see **I1** below).

This once more underlines the conceptual difference between the two notions of admissibility: admissibility at a stage of a dynamic proof and final admissibility. In general these are incomparable notions, since

I1. there are cases in which a finally admissible A is not admissible at any finite stage of a proof[25]

[25] Examples can be found in in [33], Chapter 2.8, and in FRE-REF, Chapter REF.

I2. there are cases in which any dynamic proof can be finitely extended such that A is admissible, although A is not finally admissible.[26]

This demonstrates that meta-level reasoning and object-level reasoning do not coincide for dynamic proofs: there are meta-arguments that reveal insights that are decisive for the acceptance of a defeasible argument that essentially rely on an analysis of the premises that cannot be obtained by the finitary means of the object-level reasoning that is explicated in dynamic proofs.

Of course, **I1** and **I2** do not hold for a huge class of premise sets and ALs (e.g., logics based on a finite language). And there are well-known ways to avoid **I1**.

In the standard format, the ALs are restricted to premise sets that do not contain the classical disjunction used for expressing Dab-formulas. However, if there is a classical disjunction in the object language of the lower limit logic, an additional disjunction needs to be superimposed in order to express Dab-formulas. Of course, this strategy can also be adopted in our approach in order to avoid **I1**.

Another way to avoid **I1** is to work with multi-consequence relations [18]. Here, Dab-formulas are simply sets of abnormalities. We didn't adopt this approach in this paper since in the many cases in which the lower limit logic already contains a disjunction, working with multi-consequences introduces more clutter. However, we think that especially in cases in which the lower limit logic does not feature a classical disjunction the multi-consequence approach is very elegant since one need not go through the enterprise of superimposing classical connectives on the lower limit logic. Note though that it is easy to see that the consequence relations of multi-consequence ALs can straight-forwardly be represented in our format (after all, they are equivalent to the corresponding logics in the standard format).

Finally it should be pointed out that although we deviate in our definition of final admissibility from the corresponding definition of final derivability in the standard format, replacing our definition with the latter leads to the same results. See Appendix I for a more detailed discussion.

4 Some Metatheory of $\mathbf{AL}_\Lambda$

In this section we will establish some crucial meta-theoretic properties of $\mathbf{AL}_\Lambda$. We start off by simple properties concerning derivability in $\mathbf{AL}_\Lambda$ in Section 4.1.

When it comes to properties such as soundness, completeness and cumulativity we need to restrict our focus to specific threshold functions Λ that guarantee these properties. In Section 4.2 we present the stratagem which we follow in this paper to prove soundness and completeness. In view of this it will be useful to introduce various restrictions on Λ in Section 4.3. Then we will focus on soundness and completeness in Section 4.4, on cumulativity in Section 4.5, and on reassurance in Section 4.6. It is the general result of our study that we are able to characterize a

[26] One example with the lower limit logic $\mathbf{L}^*_\circ$ is $\Gamma = \{(\circ_i a_i \wedge \neg a_i) \vee (\circ_j a_j \wedge \neg a_j) \mid i,j \in \mathbb{N}, i \neq j\} \cup \{b \vee (\circ a_i \wedge \neg a_i) \mid i \geq 2\}$. Any dynamic proof can be finitely extended to a stage in which b is admissible, still b is not finally admissible.

huge class of threshold functions Λ for which we get a rich meta-theory (soundness, completeness, cumulativity, and reassurance).

4.1 Derivability in $\mathbf{AL}_{\Lambda}$

In the following we use the convention:

Notation 4.1. Where $\Delta = \emptyset$, "$\vee\mathsf{Dab}(\Delta)$" denotes the empty string.

Our first result is not surprising but nevertheless important:

Lemma 4.1. $\Gamma \vdash_{\mathbf{LLL}} A \vee \mathsf{Dab}(\Delta)$ *iff A is derivable on the condition Δ.*

It shows that we can derive A on the assumption that no abnormality in Δ is true iff $A \vee \mathsf{Dab}(\Delta)$ is derivable in **LLL**.

Another interesting property is:

Theorem 4.1 (Proof Invariance). *If A is an $\mathbf{AL}_{\Lambda}$-consequence of Γ then any dynamic proof from Γ can be finitely extended in such a way that A is finally admissible in it.*

Finally, the derivability relation (and hence also the consequence relation) of $\mathbf{AL}_{\Lambda}$ can be characterized in terms of the derivability relation of **LLL**.

Notation 4.2. $\Theta^{\neg} =_{\text{df}} \{\neg A \mid A \in \Theta\}$

Notation 4.3. $\Sigma(\Gamma) =_{\text{df}} \{\Delta \mid \Gamma \vdash_{\mathbf{LLL}} \mathsf{Dab}(\Delta)$ and for all $\Delta' \subset \Delta, \Gamma \nvdash_{\mathbf{LLL}} \mathsf{Dab}(\Delta')\}$

Notation 4.4. $\Xi(\Gamma)$ is the set of all choice sets in $\wp(\Omega)$ of $\Sigma(\Gamma)$. For the special case in which $\Sigma(\Gamma) = \emptyset$ we define $\Xi(\Gamma) =_{\text{df}} \{\emptyset\}$.

Theorem 4.2. $\Gamma \vdash_{\mathbf{AL}_{\Lambda}} A$ *iff for every $\varphi \in \Lambda(\Xi(\Gamma))$ there is a $\Delta \subseteq \Omega \setminus \varphi$ for which* $\Gamma \vdash_{\mathbf{LLL}} A \vee \mathsf{Dab}(\Delta)$.

In an equivalent formulation, we have:

Theorem 4.3. $\Gamma \vdash_{\mathbf{AL}_{\Lambda}} A$ *iff for every $\varphi \in \Lambda(\Xi(\Gamma))$, $\Gamma \cup (\Omega \setminus \varphi)^{\neg} \vdash_{\mathbf{LLL}} A$.*

The $\mathbf{AL}_{\Lambda}$ consequence relation is closed under **LLL**:

Theorem 4.4. $Cn_{\mathbf{LLL}}(Cn_{\mathbf{AL}_{\Lambda}}(\Gamma)) = Cn_{\mathbf{AL}_{\Lambda}}(\Gamma)$

Similarly, it follows immediately by Remark 3.3 that:

Theorem 4.5. *(i)* $Cn_{\mathbf{LLL}}(\Gamma) \subseteq Cn_{\mathbf{AL}_{\Lambda}}(\Gamma)$, *(ii)* $Cn_{\mathbf{AL}_{\Lambda}}(Cn_{\mathbf{LLL}}(\Gamma)) = Cn_{\mathbf{AL}_{\Lambda}}(\Gamma)$.

4.2 A Stratagem for Generic Soundness and Completeness Proofs

In this section we show that the representational result for $\vdash_{\mathbf{AL}_{\Lambda}}$ in Theorem 4.3 can be further strengthened by only taking into account a subset of the choice sets that are selected by means of the threshold function Λ. This will help us to make a bridge to a similar representational result for $\Vdash_{\mathbf{AL}_{\Lambda}}$ and provide us with a stratagem in order to proof soundness and completeness for $\mathbf{AL}_{\Lambda}$ generically (for a specific subclass of threshold functions).

The following set will be very useful:

Notation 4.5. $\Xi^{\perp}(\Gamma) = \{\varphi \in \Xi(\Gamma) \mid$ there are $\varphi' \subseteq \varphi$ and $\Delta \subseteq \Omega \setminus \varphi$ for which $\Gamma \vdash_{\mathbf{LLL}} \neg \bigwedge \varphi' \vee \mathsf{Dab}(\Delta)\}$.[27]

$\Xi^{\perp}(\Gamma)$ are the choice sets φ in $\Xi(\Gamma)$ such that $\Gamma \cup \varphi \cup (\Gamma \setminus \varphi)^{\neg}$ is **LLL**-trivial.[28] Expressed in semantical terms:[29]

Lemma 4.2. $\Xi(\Gamma) \setminus \Xi^{\perp}(\Gamma) = \mathsf{Ab}^{\Gamma}_{\mathbf{LLL}}$

The following result strengthens Theorem 4.3:

Theorem 4.6. $\Gamma \vdash_{\mathbf{AL}_{\Lambda}} A$ *iff for every* $\varphi \in \Lambda(\Xi(\Gamma)) \setminus \Xi^{\perp}(\Gamma), \Gamma \cup (\Omega \setminus \varphi)^{\neg} \vdash_{\mathbf{LLL}} A$.

The representational results for $\vdash_{\mathbf{AL}_{\Lambda}}$ have a semantic analogon:

Theorem 4.7. $\Gamma \Vdash_{\mathbf{AL}_{\Lambda}} A$ *iff for all* $\varphi \in \Lambda(\mathsf{Ab}^{\Gamma}_{\mathbf{LLL}})$, $\Gamma \cup (\Omega \setminus \varphi)^{\neg} \Vdash_{\mathbf{LLL}} A$.

Hence, by the compactness of **LLL**, we get:

Corollary 4.1. $\Gamma \Vdash_{\mathbf{AL}_{\Lambda}} A$ *iff for all* $\varphi \in \Lambda(\mathsf{Ab}^{\Gamma}_{\mathbf{LLL}})$ *there is a* $\Delta \subseteq \Omega \setminus \varphi$ *for which* $\Gamma \Vdash_{\mathbf{LLL}} A \vee \mathsf{Dab}(\Delta)$.

Note that by Lemma 4.2:

Corollary 4.2. $\Lambda(\mathsf{Ab}^{\Gamma}_{\mathbf{LLL}}) = \Lambda(\Xi(\Gamma) \setminus \Xi^{\perp}(\Gamma))$

The following corollary follows immediately by Corollary 4.2, Theorem 4.7, and Theorem 4.6:

Corollary 4.3. *If* $\Lambda(\Xi(\Gamma)) \setminus \Xi^{\perp}(\Gamma) = \Lambda(\Xi(\Gamma) \setminus \Xi^{\perp}(\Gamma))$ *then:* $\Gamma \vdash_{\mathbf{AL}_{\Lambda}} A$ *iff* $\Gamma \Vdash_{\mathbf{AL}_{\Lambda}} A$.

This equips us with a stratagem for soundness and completeness results. All we need is to ensure that

$$\Lambda(\Xi(\Gamma)) \setminus \Xi^{\perp}(\Gamma) = \Lambda(\Xi(\Gamma) \setminus \Xi^{\perp}(\Gamma)) \tag{S}$$

If our threshold function Λ has properties which ensure (S), we are safe: soundness and completeness is guaranteed.

4.3 Some Criteria for the Threshold Function Λ

In this section we will introduce some criteria for the threshold function Λ. The idea is to show that, given Λ satisfies certain criteria it follows that $\mathbf{AL}_{\Lambda}$ has certain meta-theoretic properties.

Instead of defining the criteria on the whole domain $\wp(\Omega)$ of Λ it is sufficient to restrict the focus to the following subset of $\wp(\Omega)$:

Notation 4.6. Υ is the set of all $X \subseteq \wp(\Omega)$ for which $\langle X, \subset \rangle$ is smooth.

[27] Whenever we use $\wedge$ in this paper, it is an abbreviation: $A \wedge B =_{\mathrm{df}} \neg(A \vee B)$.
[28] Γ is **LLL**-*trivial* iff $\Gamma \vdash_{\mathbf{LLL}} A$ for every $A \in \mathcal{W}$. Otherwise it is **LLL**-*non-trivial*.
[29] Proofs for the results of this section can be found in Appendix B.

We will present two groups of criteria. The first group of criteria concerns properties of Λ in relation to $\subset$. Set inclusion plays a rather central role which is highlighted by criterion **T2** for threshold functions. Moreover, it is the partial order according to which choice sets and abnormal parts of models are compared in the standard format for ALs. Another reason which makes it meta-theoretically very interesting are the following two theorems:[30]

Theorem 4.8. *Where* $\Gamma \subseteq \mathcal{W}$*:* $\langle \Xi(\Gamma), \subset \rangle$ *is smooth.*

Theorem 4.9. *Where* $\Gamma \subseteq \mathcal{W}$*:* $\langle \mathsf{Ab}^{\Gamma}_{\mathbf{LLL}}, \subset \rangle$ *is smooth.*

Note that due to the two Theorems, $\{\Xi(\Gamma) \mid \Gamma \subseteq \mathcal{W}\}$ and $\{\mathsf{Ab}^{\Gamma}_{\mathbf{LLL}} \mid \Gamma \subseteq \mathcal{W}\}$ are subsets of Υ.

The second group of criteria concerns properties of Λ with respect to any partial order $\prec \subseteq \wp(\Omega) \times \wp(\Omega)$ where $\subset \subseteq \prec$. As we have already pointed out, Theorem 4.8 does not hold for any $\prec$. Hence, this group for instance features the case where $\Lambda = \Psi_{\prec}$.

The following notions will be useful in what follows:

Definition 4.1. Where $\prec$ is a partial order on Y and $X \subseteq Y$: X is a $\prec$*-dense in* Y iff for all $y \in Y$ either $y \in X$ or there is a $x \in X$ for which $x \prec y$.

Definition 4.2. Where $\Lambda : \wp(\wp(Z)) \to \wp(\wp(Z))$, $\Upsilon_Z \subseteq \wp(\wp(Z))$ and $\prec \subseteq \wp(Z) \times \wp(Z)$: We say that Λ is $\prec$*-density invariant on* Υ_Z iff for all $X, Y \in \Upsilon_Z$, if X is $\prec$-dense in Y, $\Lambda(X) = \Lambda(Y) \cap X$.

The two most generic criteria in our first group are: Where $X, Y \in \Upsilon$,

C1 where $\min_{\subset}(X) = \min_{\subset}(Y)$: $\Lambda(X) \cap Y = \Lambda(Y) \cap X$

C2 Λ is $\subset$-density invariant on Υ

As we show in the Appendix D, due to the smoothness of $\langle \Xi(\Gamma), \subset \rangle$ these two criteria are equivalent. Obviously, where

C5 $\Lambda(X) = \min_{\subset}(X)$

C1 and **C2** hold. The following criterion is similar but more generic:

C3 $\Lambda(X) = \Lambda'(\min_{\subset}(X))$ where Λ' is a threshold function

It is equivalent to:

C4 where $\min_{\subset}(X) = \min_{\subset}(Y)$: $\Lambda(X) = \Lambda(Y)$

Sometimes the following criterion is useful as well (see Section 5.2.2):

C6 $\Lambda(X) \supseteq \min_{\subset}(X)$

The following Lemma summarizes the relationships of these criteria. For the proofs see Appendix E. For an illustration see Figure 1.

[30] Proofs for these Theorems can be found in Appendix D.

Fact 4.1. *(i)* **C3** *is equivalent to* **C4***; (ii)* **C4** *implies* **C1***; (iii)* **C5** *implies* **C3***; (iv)* **C1** *implies* **C2***; (v)* **C2** *implies* **C1***; (vi)* **C5** *implies* **C6**.

We now introduce the second group of criteria. This time we suppose that Λ is defined on the basis of or related to a partial order $\prec \subseteq \wp(\Omega) \times \wp(\Omega)$. We presuppose that $\subset \subseteq \prec$. The rationale is the same as the one behind **T2**: if $\varphi \subset \psi$ then the set of abnormalities φ is less abnormal than ψ and hence should be preferred.

Notation 4.7. We write, as usual, $x \preceq y$ iff $x \prec y$ or $x = y$.

Where $X \in \Upsilon$ we state the following two most generic conditions:

$\mathbf{C}_1^{\prec}$ Λ is $\prec$-density invariant on Υ

$\mathbf{C}_2^{\prec}$ $\Lambda(X)$ is $\prec$-dense in X

The next two criteria concern the case in which Λ is defined by means of a Ψ-function as demonstrated in Section 2.5:

$\mathbf{C}_3^{\prec}$ $\Lambda(X) = \Psi_{\prec}(X)$

$\mathbf{C}_4^{\prec}$ $\Lambda(X) = \Psi_{[\prec_1, \prec_2, \ldots, \prec_n]}(X)$ where[31] $\prec = \prec_1$

Finally, Λ may be defined in terms of $\min_{\prec}$:

$\mathbf{C}_5^{\prec}$ $\Lambda(X) = \min_{\prec}(X)$

For the relationship between the different criteria see Figure 1 and Fact 4.2 below. It is easy to see that $\min_{\prec}$, $\Psi_{\prec}$, and $\Psi_{[\prec_1, \ldots, \prec_n]}$ are threshold functions.[32]

Fact 4.2. *(i)* $\mathbf{C}_3^{\prec}$ *implies* $\mathbf{C}_1^{\prec}$*; (ii)* $\mathbf{C}_3^{\prec}$ *implies* $\mathbf{C}_4^{\prec}$*; (iii)* $\mathbf{C}_4^{\prec}$ *implies* $\mathbf{C}_2^{\prec}$*; (iv)* $\mathbf{C}_5^{\prec}$ *implies* **C3***; (v)* **C5** *implies* $\mathbf{C}_3^{\prec}$*; (vi)* **C1** *implies* $\mathbf{C}_1^{\prec}$*; (vii)* **C6** *implies* $\mathbf{C}_2^{\prec}$.

4.4 Soundness and Completeness

In the Appendix F we prove the following two central lemmas:

Lemma 4.3. *Where* Γ *is* **LLL***-non-trivial: If* Λ *satisfies* **C1** *or* **C2** *or* $\mathbf{C}_1^{\prec}$*, then* $\Lambda(\Xi(\Gamma) \setminus \Xi^{\perp}(\Gamma)) = \Lambda(\Xi(\Gamma)) \setminus \Xi^{\perp}(\Gamma)$.

Lemma 4.4. *Where* Γ *is* **LLL***-non-trivial:*

$$\Psi_{[\prec_1, \ldots, \prec_n]}(\Xi(\Gamma) \setminus \Xi^{\perp}(\Gamma)) = \Psi_{[\prec_1, \ldots, \prec_n]}(\Xi(\Gamma)) \setminus \Xi^{\perp}(\Gamma)$$

Theorem 4.10 (Soundness and Completeness)**.** *Where* Λ *satisfies some of following criteria* **C1** *or* **C2** *or* $\mathbf{C}_1^{\prec}$ *or* $\mathbf{C}_4^{\prec}$*:* $\Gamma \vdash_{\mathbf{AL}_{\mathbf{\Lambda}}} A$ *iff* $\Gamma \Vdash_{\mathbf{AL}_{\mathbf{\Lambda}}} A$.

Proof. This follows immediately by Corollary 4.3, Lemma 4.3, and Lemma 4.4. □

Let us shortly consider the case $\Lambda = \min_{\prec}$ where smoothness is guaranteed. We have:

[31]This only serves the purpose to make the comparison with the other criteria easier.

[32]See Fact E.3 and Fact E.12 in Appendix E.

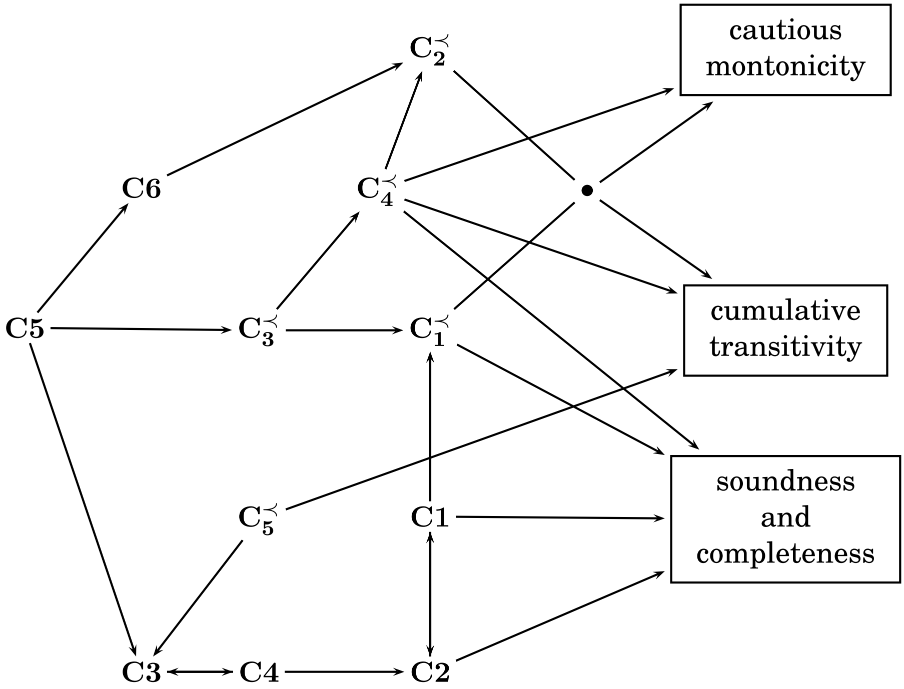

Figure 1: The criteria on the threshold function Λ

Lemma 4.5. *If $\langle \Xi(\Gamma), \prec\rangle$ is smooth, then (i) $\min_{\prec}(\Xi(\Gamma)) = \Psi_{\prec}(\Xi(\Gamma))$, (ii) $\min_{\prec}(\mathsf{Ab}^{\Gamma}_{\mathbf{LLL}}) = \Psi_{\prec}(\mathsf{Ab}^{\Gamma}_{\mathbf{LLL}})$, (iii) $\langle \mathsf{Ab}^{\Gamma}_{\mathbf{LLL}}, \prec\rangle$ is smooth.*

Corollary 4.4. *Where $\langle \Xi(\Theta), \prec\rangle$ is smooth for all $\Theta \subseteq \mathcal{W}$ and $\Gamma \subseteq \mathcal{W}$:*

(i) $\Gamma \vdash_{\mathbf{AL}_{\min_{\prec}}} A$ *iff* $\Gamma \vdash_{\mathbf{AL}_{\mathbf{\Psi}_{\prec}}} A$

(ii) $\Gamma \Vdash_{\mathbf{AL}_{\min_{\prec}}} A$ *iff* $\Gamma \Vdash_{\mathbf{AL}_{\mathbf{\Psi}_{\prec}}} A$.

Corollary 4.5. *Where $\langle \Xi(\Theta), \prec\rangle$ is smooth for all $\Theta \subseteq \mathcal{W}$: $\mathbf{AL}_{\min_{\prec}}$ is sound and complete.*

Remark 4.1. Of course, given our discussion in Section 4.2 it is clear that we get soundness and completeness for all threshold functions Λ for which $\Lambda(\Xi(\Gamma)) = \Lambda(\Xi(\Gamma) \setminus \Xi^{\perp}(\Gamma))$ for all $\Gamma \subseteq \mathcal{W}$. Obviously then by **T1**, $\Lambda(\Xi(\Gamma)) = \Lambda(\Xi(\Gamma) \setminus \Xi^{\perp}(\Gamma)) = \Lambda(\Xi(\Gamma)) \setminus \Xi^{\perp}(\Gamma)$ and the rest follows by Corollary 4.3.

Alternatively one could substitute $\Xi_s(\Gamma)$ in the marking definition with $\Xi_s(\Gamma) \setminus \Xi_s^{\perp}(\Gamma)$ where $\Xi_s^{\perp}(\Gamma) =_{\mathrm{df}} \{\varphi \in \Xi_s(\Gamma) \mid$ there are $\varphi' \subseteq \varphi$ and $\Delta \subseteq \Omega \setminus \varphi$ for which $\neg \bigwedge \varphi'$ is derived on the condition Δ at stage $s\}$. If moreover the definition of an A-complete stage is supplemented by: "(iii) for each $\varphi \in \Xi^{\perp}(\Gamma)$ there is a $\varphi' \subseteq \varphi$ and a $\Delta \subseteq \Omega \setminus \varphi$ such that $\neg \bigwedge \varphi'$ is derived on the condition Δ", then it is easy to see that this also warrants soundness and completeness for $\mathbf{AL}_{\mathbf{\Lambda}}$ where Λ is arbitrary.

4.5 Cumulativity

The following two criteria play a crucial role in what follows:[33]

CT where $\Lambda(X) \subseteq Y \subseteq X$: $\Lambda(X) \subseteq \Lambda(Y)$

CM where $\Lambda(X) \subseteq Y \subseteq X$: $\Lambda(X) \supseteq \Lambda(Y)$

Remark 4.2. For the following syntactic results it is enough to restrict **CT** and **CM** to $X, Y \in \{\Xi(\Gamma) \mid \Gamma \subseteq \mathcal{W}\}$. Likewise, for the semantic results they can be restricted to $X, Y \in \{\mathsf{Ab}^{\Gamma}_{\mathbf{LLL}} \mid \Gamma \subseteq \mathcal{W}\}$.

Lemma 4.6. *(i) ($\mathbf{C}^{\prec}_1$ and $\mathbf{C}^{\prec}_2$) imply* **CT** *and* **CM***; (ii) $\mathbf{C}^{\prec}_4$ implies* **CT** *and* **CM***; (iii) $\mathbf{C}^{\prec}_5$ implies* **CT**.

Lemma 4.7. *Where $\Gamma, \Gamma' \subseteq \mathcal{W}$: $\Xi(\Gamma \cup \Gamma') \subseteq \Xi(\Gamma)$.*

Lemma 4.8. *Where $\Gamma, \Gamma' \subseteq \mathcal{W}$ and $\Gamma' \subseteq Cn_{\mathbf{AL}_\Lambda}(\Gamma)$: $\Lambda(\Xi(\Gamma)) \subseteq \Xi(\Gamma \cup \Gamma')$.*

Theorem 4.11 (Cautious Monotonicity)**.** *Where Λ satisfies* **CM** *and $\Gamma, \Gamma' \subseteq \mathcal{W}$: If $\Gamma' \subseteq Cn_{\mathbf{AL}_\Lambda}(\Gamma)$ then $Cn_{\mathbf{AL}_\Lambda}(\Gamma) \subseteq Cn_{\mathbf{AL}_\Lambda}(\Gamma \cup \Gamma')$.*

Let us demonstrate this proof:

Proof. By Lemma 4.7 and Lemma 4.8, $\Lambda(\Xi(\Gamma)) \subseteq \Xi(\Gamma \cup \Gamma') \subseteq \Xi(\Gamma)$. By **CM**, (1) $\Lambda(\Xi(\Gamma)) \supseteq \Lambda(\Xi(\Gamma \cup \Gamma'))$. Suppose $\Gamma \vdash_{\mathbf{AL}_\Lambda} A$. By Theorem 4.3, for all $\varphi \in \Lambda(\Xi(\Gamma))$, $\Gamma \cup (\Omega \setminus \varphi)^{\neg} \vdash_{\mathbf{LLL}} A$. Let $\varphi \in \Lambda(\Xi(\Gamma \cup \Gamma'))$. By (1), $\varphi \in \Lambda(\Xi(\Gamma))$. Hence, by the monotonicity of **LLL**, $\Gamma \cup \Gamma' \cup (\Omega \setminus \varphi)^{\neg} \vdash_{\mathbf{LLL}} A$. Since φ was arbitrary in $\Lambda(\Xi(\Gamma \cup \Gamma'))$, by Theorem 4.3, $\Gamma \cup \Gamma' \vdash_{\mathbf{AL}_\Lambda} A$. □

Theorem 4.12 (Cumulative Transitivity)**.** *Where Λ satisfies* **CT** *and $\Gamma, \Gamma' \subseteq \mathcal{W}$: If $\Gamma' \subseteq Cn_{\mathbf{AL}_\Lambda}(\Gamma)$ then $Cn_{\mathbf{AL}_\Lambda}(\Gamma \cup \Gamma') \subseteq Cn_{\mathbf{AL}_\Lambda}(\Gamma)$.*

The proof is similar to the proof for Cautious Monotonicity and can be found in Appendix G.

Corollary 4.6 (Fixed Point)**.** *Where $\Gamma \subseteq \mathcal{W}$ and Λ satisfies* **CT***: $Cn_{\mathbf{AL}_\Lambda}(Cn_{\mathbf{AL}_\Lambda}(\Gamma)) = Cn_{\mathbf{AL}_\Lambda}(\Gamma)$.*

Note that the right-to-left direction follows immediately by the reflexivity of $\vdash_{\mathbf{AL}_\Lambda}$. The left-to-right direction follows by the cumulative transitivity.

By Theorem 4.11, Theorem 4.12, and Lemma 4.6:

Corollary 4.7 (Cumulativity)**.** *Where $\Gamma, \Gamma' \subseteq \mathcal{W}$ and Λ satisfies ($\mathbf{C}^{\prec}_1$ and $\mathbf{C}^{\prec}_2$) or $\mathbf{C}^{\prec}_4$: If $\Gamma' \subseteq Cn_{\mathbf{AL}_\Lambda}(\Gamma)$ then $Cn_{\mathbf{AL}_\Lambda}(\Gamma) = Cn_{\mathbf{AL}_\Lambda}(\Gamma \cup \Gamma')$.*

Corollary 4.8 (Fixed Point)**.** *Where $\Gamma, \Gamma' \subseteq \mathcal{W}$ and Λ satisfies ($\mathbf{C}^{\prec}_1$ and $\mathbf{C}^{\prec}_2$) or $\mathbf{C}^{\prec}_4$: $Cn_{\mathbf{AL}_\Lambda}(Cn_{\mathbf{AL}_\Lambda}(\Gamma)) = Cn_{\mathbf{AL}_\Lambda}(\Gamma)$.*

[33] REF-FRE

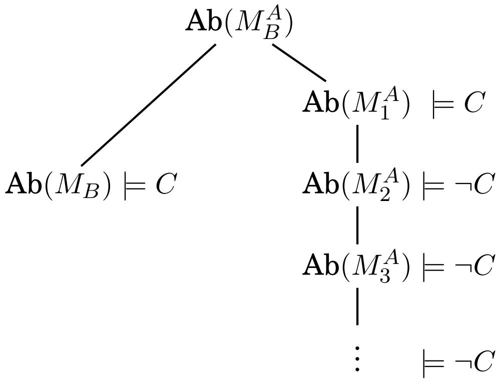

Figure 2: Illustration with an infinite chain of models

Since both, $\mathbf{C}_1^{\prec}$ and $\mathbf{C}_4^{\prec}$, ensure soundness and completeness, we also immediately get a corresponding corollary that states cumulativity for the semantic consequence relation of $\mathbf{AL}_{\mathbf{\Lambda}}$.

Note that $\mathbf{C}_5^{\prec}$ does not guarantee cautious monotonicity. In other words, ALs with a threshold function $\min_{\prec}$ do not in general validate cautious monotonicity. Of course, as soon as $\langle \Xi(\Gamma), \prec \rangle$ is smooth for all $\Gamma \subseteq \mathcal{W}$, the consequence set of the logic $\mathbf{AL}_{\min_{\prec}}$ is identical to the consequence set of the logic $\mathbf{AL}_{\mathbf{\Psi}_{\prec}}$ (Corollary 4.4) and hence in that case cumulativity is satisfied.

Example 4.1. Let us give an example where cautious monotonicity does not hold. We use the logic $\mathbf{AL4.1}$ from the example from Section 2.4. Let the premise set be

$$\Gamma = \big\{!^1 B_1 \vee (!^i A_i \vee !^j A_j) \mid i, j \in \mathbb{N}, i < j\big\} \cup \\ \big\{!^i B_i \supset !^{i+1} B_{i+1} \mid i \in \mathbb{N}\big\} \cup \big\{!^1 B_1 \supset C, \neg !^1 A_1 \supset C, !^1 A_1 \supset \neg C\big\}$$

where $!^i A =_{\text{df}} \circ_i A \wedge \neg A$ and the formulas A_i, B_j and C are all different propositional letters (where $i, j \in \mathbb{N}$). Figure 2 represents an excerpt of the order on the (abnormal parts of the) $\mathbf{L}_{\circ}^{*}$-models of Γ by means of $\prec_{\mathsf{co}}$ where $\mathsf{Ab}(M_B) = \{!^i B_i \mid i \in \mathbb{N}\}$, $\mathsf{Ab}(M_i^A) = \{!^j A_j \mid j \neq i\}$ and $\mathsf{Ab}(M_B^A) = \mathsf{Ab}(M_B) \cup \{!^i A_i \mid i \in \mathbb{N}\}$. It is not difficult to see that all models M in $\min_{\prec_{\mathsf{co}}}\big(\mathsf{Ab}^{\Gamma}_{\mathbf{LLL}}\big)$ have the same abnormal part as M_B. Note that for all these models $M \models C$ (due to the premise $!^1 B_1 \supset C$). Similarly, as indicated in the figure, $M_1^A \models C$ while $M_i^A \models \neg C$ for all $i > 1$. Altogether, since all selected models in $\min_{\prec_{\mathsf{co}}}(\Gamma)$ validate C and $!^1 B_1$ we have $\Gamma \Vdash_{\mathbf{AL4.1}} C$ and $\Gamma \Vdash_{\mathbf{AL4.1}} !^1 B_1$.

Let now $\Gamma' = \{C\} \subseteq Cn_{\mathbf{AL4.1}}(\Gamma)$. Again, it can easily be seen that models with the abnormal part of M_1^A are also selected by $\min_{\prec_{\mathsf{co}}}\big(\mathsf{Ab}^{\Gamma \cup \Gamma'}_{\mathbf{LLL}}\big)$. Hence, $\Gamma \cup \Gamma' \nVdash_{\mathbf{AL4.1}} !^1 B_1$ and thus, $Cn_{\mathbf{AL4.1}}(\Gamma) \nsubseteq Cn_{\mathbf{AL4.1}}(\Gamma \cup \Gamma')$.

Of course, we can state the corresponding semantic results to Theorems 4.11 and 4.12.[34]

[34] FRE: refer to the literature

Theorem 4.13. *Where $\Gamma \Vdash_{\mathbf{AL_\Lambda}} B$ for all $B \in \Gamma'$:*

(i) if Λ satisfies **CM** *then: $\Gamma \Vdash_{\mathbf{AL_\Lambda}} A$ implies $\Gamma \cup \Gamma' \Vdash_{\mathbf{AL_\Lambda}} A$.*

(ii) if Λ satisfies **CT** *then: $\Gamma \cup \Gamma' \Vdash_{\mathbf{AL_\Lambda}} A$ implies $\Gamma \Vdash_{\mathbf{AL_\Lambda}} A$.*

4.6 Reassurance

Reassurance means that whenever Γ is **LLL**-non-trivial, Γ is also $\mathbf{AL_\Lambda}$-non-trivial. In semantic terms: if $\mathcal{M}_{\mathbf{LLL}}(\Gamma) \neq \emptyset$ then $\mathcal{M}_{\mathbf{AL_\Lambda}}(\Gamma) \neq \emptyset$.

The following criterion is important in this context: for all $X \in \{\Xi(\Gamma), \mathsf{Ab}^{\Gamma}_{\mathbf{LLL}} \mid \Gamma \subseteq \mathcal{W}$ is **LLL**-non-trivial$\}$,

RA $\Lambda(X) \neq \emptyset$.

Theorem 4.14. *Where Λ satisfies* **RA***: if Γ is* **LLL***-non-trivial then Γ is* $\mathbf{AL_\Lambda}$*-non-trivial.*

Note that for all $X \neq \emptyset$, $\Psi_{[\prec_1,\ldots,\prec_n]}(X) \neq \emptyset$. Hence,

Lemma 4.9. $\mathbf{C}_4^{\prec}$ *implies* **RA***.*

Remark 4.3. **RA** is guaranteed for $\Lambda = \min_{\prec}$ in case $\langle \Xi(\Gamma), \prec \rangle$ is smooth for all $\Gamma \subseteq \mathcal{W}$ (this follows immediately by Corollary 4.4 and Lemma 4.9.

Sometimes adaptive logicians also speak about *Strong Reassurance*. In the context of the standard format for ALs where the abnormal parts of the **LLL**-models are ordered by $\subset$ this is defined as follows: for every $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$ there is a $M' \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$ such that $\mathsf{Ab}(M') \subseteq \mathsf{Ab}(M)$ and $\mathsf{Ab}(M') \in \min_{\subset}(\mathsf{Ab}^{\Gamma}_{\mathbf{LLL}})$.

Let us generalize the rationale behind this criterion to a partial order $\prec$.

One way to read it is: $\langle \mathsf{Ab}^{\Gamma}_{\mathbf{LLL}}, \prec \rangle$ is smooth.

Another, less strict reading, is: for every $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$ there is a M' such that $\mathsf{Ab}(M') \preceq \mathsf{Ab}(M)$ and $M' \in \Lambda(\mathsf{Ab}^{\Gamma}_{\mathbf{LLL}})$. Informally this expresses that for each model M there should be a better model M' (in view of $\prec$) that is selected.

One way to motivate this is by requiring that whenever a model M is not selected we need a justification for not selecting M. This justification has the form of pointing to a better model M' that is selected. We will henceforth adopt this reading.

Theorem 4.15. *Where Λ satisfies* $\mathbf{C}_2^{\prec}$*: Strong Reassurance (with respect to $\prec$) holds for* $\mathbf{AL_\Lambda}$*.*

5 Wrapping Things Up

5.1 Smoothness and $\Psi_{\prec}$

The traditional perspective behind preference semantics in the vein of Shoham is to select $\prec$-minimal models (of a given premise set) with respect to a partial order $\prec$. We have already seen that adaptive logics $\mathbf{AL_\Lambda}$ with $\Lambda = \min_{\prec}$ enjoy a rich meta-theory.

- Since $\mathbf{C}_5^{\prec}$ trivially holds for $\min_{\prec}$, we also get **C3** by Fact 4.2 and hence **C2** by Fact 4.1. By Theorem 4.10 we get soundness and completeness.
- By Lemma 4.6 we get **CT** which, by Theorem 4.12 ensures cumulative transitivity. In Example 4.1 we have seen that cautious monotonicity does not hold in general.

One of the most serious problems behind this approach is the possibility of non-smoothness. Some models may be such that there is no $\prec$-minimal, and hence no selected model below. Hence, strong reassurance may fail. In the worst case there are no $\prec$-minimal models and we loose the reassurance property (see the example in Section 2.4). Sometimes we may encounter asymmetric situations where for some models there are minimal models below them, while for other models there are not. Only selecting the minimal models in such cases can lead to rather counter-intuitive results. Suppose for instance that (a) in all minimal models C holds, and (b) in each infinitely descending chain of models there is a M such that in all models below M, C doesn't hold. See Figure 2 and Example 4.1 for an illustration. Since our selection by means of $\min_{\prec}$ ignores the models in these infinitely descending chains we still derive C.

We have presented a way to avoid such problems. The idea is to use $\Psi_{\prec}$ (or a more refined $\Psi_{[\prec_1,\ldots,\prec_n]}$) instead of $\min_{\prec}$. Since $\min_{\prec}(\Xi(\Gamma)) = \Psi_{\prec}(\Xi(\Gamma)) = \Psi_{[\prec_1,\ldots,\prec_n]}(\Xi(\Gamma))$ whenever $\langle \Xi(\Gamma), \prec \rangle$ is smooth, these logics are equivalent for all "non-problematic" Γ. Hence, whenever $\langle \Xi(\Gamma), \prec \rangle$ is smooth, the logics $\mathbf{AL}_{\min_{\prec}}$, $\mathbf{AL}_{\mathbf{\Psi}_{\prec}}$ and $\mathbf{AL}_{\mathbf{\Psi}_{[\prec_1,\ldots,\prec_n]}}$ lead to an identical consequence relation. However, in $\mathbf{AL}_{\mathbf{\Psi}_{\prec}}$ and $\mathbf{AL}_{\mathbf{\Psi}_{[\prec_1,\ldots,\prec_n]}}$ the problems pointed out above for non-smooth cases are avoided.

Note also that at any finite stage s of a dynamic proof the marking definitions for $\min_{\prec}$, $\Psi_{\prec}$, and $\Psi_{[\prec_1,\ldots,\prec_n]}$ are identical since then $\Sigma_s(\Gamma)$ is finite and hence trivially $\langle \Xi_s(\Gamma), \prec \rangle$ is smooth.

This underlines the fact that non-smoothness is a rather meta-theoretic problem concerning limit-cases. Although such cases may hardly ever appear in real applications, in meta-theoretic considerations they cannot be ignored. It is hence good news that the technical solution to avoid meta-theoretic shortcomings need not concern the user of the logic since the adaptive proofs proceed nearly identical irrespective whether we choose the threshold $\min_{\prec}$ or $\Psi_{\prec}$. The only difference concerns the case when we consider an A-complete extension of a dynamic proof in order to check whether a formula is finally admissible. However, also for this check, most premise sets in practical applications will be such that smoothness is guaranteed even in the limit.

Finally, for the threshold functions $\Psi_{\prec}$ and $\Psi_{[\prec_1,\ldots,\prec_n]}$ we get an even richer meta-theory than for $\min_{\prec}$:

- Since we have $\mathbf{C}_4^{\prec}$ we get by Theorem 4.10 soundness and completeness.
- By Corollary 4.7 we get cumulativity.
- By Lemma 4.9 and Theorem 4.14 we get reassurance.

- By Fact 4.2 we have $\mathbf{C}_2^{\prec}$ and hence by Theorem 4.15 we get strong reassurance.

In view of Lemma 4.5 and Corollary 4.4 this immediately shows that, where $\langle \Xi(\Gamma), \prec \rangle$ is smooth for all $\Gamma \subseteq \mathcal{W}$, we get the same rich meta-theory for $\mathbf{AL}_{\min_{\prec}}$.

5.2 The Standard Format

5.2.1 Minimal Abnormality

ALs with the minimal abnormality strategy are representable in our format by means of the threshold function $\Lambda = \min_{\subset}$. Note that by Theorem 4.8, $\langle \Xi(\Gamma), \subset \rangle$ is smooth for all $\Gamma \subseteq \mathcal{W}$. Hence, as was argued in Section 5.1, we get:

- soundness and completeness
- cumulativity
- reassurance and strong reassurance

5.2.2 Reliability

In order to represent the reliability strategy let

$$\Lambda_U(X) = \left\{ \varphi \in X \mid \varphi \subseteq \bigcup \min_{\subset}(X) \right\}$$

Obviously Λ_U is a threshold function since it satisfies **T1** and **T2**. We get the full meta-theory:

- By the definition Λ_U satisfies **C1**. Hence, we get soundness and completeness by Theorem 4.10.
- Since we obviously have **C6** we get by Fact 4.2 $\mathbf{C}_2^{\prec}$ and since we have **C1** we get by the same Lemma $\mathbf{C}_1^{\prec}$. By Corollary 4.7 we get cumulativity.
- Since $\langle \Xi(\Gamma), \subset \rangle$ is smooth by Theorem 4.8, also $\Lambda_U(\Xi(\Gamma)) \neq \emptyset$ for all $\Gamma \subseteq \mathcal{W}$. Hence, Λ_U satisfies **RA**. This guarantees reassurance by Theorem 4.14.
- Since we have $\mathbf{C}_2^{\prec}$ we get strong reassurance by Theorem 4.15.

5.3 (Co)-Lexicographic ALs

Lexicographic ALs as introduced in [25] are represented in our generic format by $\Lambda_{\mathsf{lex}} = \min_{\prec_{\mathsf{lex}}}$ where $\prec_{\mathsf{lex}}$ is defined as in Definition 2.1 on the basis of a structured set of abnormalities $\Omega = \bigcup_I \Omega_i$. Note that

Fact 5.1. $\subset \subseteq \prec_{\mathsf{lex}}$

Hence Λ_{lex} satisfies $\mathbf{C}_5^{\prec}$. In the appendix of [25] we have shown that $\langle \Xi(\Gamma), \prec_{\mathsf{lex}} \rangle$ is smooth. Hence, as was argued in Section 5.1, we get:

- soundness and completeness
- cumulativity

- reassurance and strong reassurance

Our more generic perspective in this paper gives rise to other lexicographic ALs which do not fall within the scope of the format presented in [25]. One may for instance define $\prec'_{\mathsf{lex}} \subseteq \wp(\Omega) \times \wp(\Omega)$ as follows on the basis of partial orders $\prec_i \subseteq \wp(\Omega_i) \times \wp(\Omega_i)$ (where $\subset \subseteq \prec_i$):

Definition 5.1. Where $\varphi, \psi \subseteq \Omega$ are sets of abnormalities, φ is preferable to ψ, in signs $\varphi \prec'_{\mathsf{lex}} \psi$, iff, there is an $n \in \mathbb{N}$ for which

(a) $\varphi \cap \Omega_i = \psi \cap \Omega_i$ for all $i < n$ and

(b) $\varphi \cap \Omega_n \prec_n \psi \cap \Omega_n$.

If we are able to guarantee the smoothness of $\langle \Xi(\Gamma), \prec'_{\mathsf{lex}} \rangle$ we can define Λ'_{lex} as before by $\min_{\prec'_{\mathsf{lex}}}$. Otherwise we define Λ'_{lex} by $\Psi_{\prec'_{\mathsf{lex}}}$. As pointed out in Section 5.1, we get the full meta-theory (soundness, completeness, cumulativity, reassurance, strong reassurance) for the resulting logic.[35]

In the example from Section 2.4 we have seen another variant: instead of a lexicographic order we used a colexicographic order. We have seen that, unlike for $\prec_{\mathsf{lex}}$, colexicographic orders sometimes give rise to non-smoothness. Hence, we used the selection function $\Psi_{\prec_{\mathsf{co}}}$ or the refined variant $\Psi_{[\prec_{\mathsf{co}}, \subset]}$. As discussed in Section 5.1, we get the full meta-theory for these logics (soundness, completeness, cumulativity and reassurance, strong reassurance).

5.4 More on Quantitative Variants

5.4.1 Counting Strategies

In the example from Section 2.3 we have introduced the order $\prec_c$. Note that

Fact 5.2. *$\langle \Xi(\Gamma), \prec_c \rangle$ is smooth.*

Proof. 1. case: $\Xi(\Gamma)$ only contains infinite sets. In this case $\min_{\prec_c}(\Xi(\Gamma)) = \min_{\subset}(\Xi(\Gamma))$ (recall that $\subset \subseteq \prec_c$). Hence we get smoothness by Theorem 4.8. *2. case:* $\Xi(\Gamma)$ contains finite sets. Let $n = \min_{<}(\{|\varphi| \mid \varphi \in \Xi(\Gamma)\})$. Note that $\min_{\prec_c}(\Xi(\Gamma)) = \{\varphi \in \Xi(\Gamma) \mid |\varphi| = n\} \neq \emptyset$. Let $\psi \in \Xi(\Gamma) \setminus \min_{\prec_c}(\Xi(\Gamma))$ and $\varphi \in \min_{\prec_c}(\Xi(\Gamma))$. Hence, $|\varphi| < |\psi|$ and thus $\varphi \prec_c \psi$. □

Hence, as pointed out in Section 5.1 we have the full meta-theory (soundness, completeness, cumulativity, reassurance, strong reassurance) for ALs based on the threshold function $\min_{\prec_c}$.

Of course, one may for instance use more refined partial orders for quantitative comparisons, such as

[35] A further generalizations does not just allow for more and more severe abnormalities, but also for less and less severe abnormalities. Hence, instead of the index set $\mathbb{N}$ we may use the integers $\mathbb{N}^- \cup \{0\} \cup \mathbb{N}$ and define $\varphi \prec^*_{\mathsf{lex}} \psi$ iff (i) there is a $n \in \mathbb{N}^- \cup \{0\} \cup \mathbb{N}$ such that (i.a) for all $j < n$, $\varphi \cap \Omega_j = \psi \cap \Omega_j$, and (i.b) $\varphi \cap \Omega_n \prec_n \psi \cap \Omega_n$; or (ii) for all $i \in \mathbb{N}^- \cup \{0\} \cup \mathbb{N}$, $\varphi \cap \Omega_i \preceq_i \psi \cap \Omega_i$.

Definition 5.2. $\varphi \prec'_c \psi$ iff $|\varphi \setminus \psi| < |\psi \setminus \varphi|$.

It is easy to see that $\prec_c \subseteq \prec'_c$. Note that $\prec'_c$ allows also to compare infinite sets that are not comparable by $\prec_c$. As an easy example apply the two partial orders to $\wp(\mathbb{N})$. Let $\varphi = E \cup \{1, \ldots, 10\}$ and $\psi = E \cup \{21, \ldots, 40\}$ where E is the set of even numbers in $\mathbb{N}$. Note that $\varphi \prec'_c \psi$ since $|\varphi \setminus \psi| = |\{1, 3, \ldots, 9\}| = 5 < 10 = |\{21, 23, \ldots, 39\}| = |\psi \setminus \varphi|$. However, φ and ψ are not comparable with respect to $\prec_c$.

Since there are premise sets Γ for which $\langle \Xi(\Gamma), \prec'_c \rangle$ is non-smooth,[36] $\Lambda = \min_{\prec'_c}$ does not suffice to give the full meta-theory for this variant. However, $\Lambda = \Psi_{\prec'_c}$ is sufficient.

5.4.2 More Involved Quantitative Approaches

We have already seen more involved examples that make use of a selection of models based on quantitative considerations in the examples in Section 2.6. Note that both threshold functions,

Fact 5.3. *Λ^1_c and Λ^2_c satisfy* **C1**.

Proof. Suppose $\min_\subset(X) = \min_\subset(Y)$. By Fact E.6, $\min_{\prec_c}(X) = \min_{\prec_c}(Y)$. The case $X = \emptyset$ is trivial. Let $X \neq \emptyset$. By Fact 5.2, $\min_{\prec_c}(X) \neq \emptyset$. Let $\varphi \in \min_{\prec_c}(X)$ and $k = |\varphi| \in \mathbb{N}^\infty$. Hence, $\Lambda^1_c(X) \cap Y = \{x \in X \cap Y \mid |x| - \tau \leq k\} = \Lambda^1_c(Y) \cap X$. The proof for Λ^2_c is similar and left to the reader. □

Hence, by Theorem 4.10 we get soundness and completeness for both logics. Moreover, it can be easily shown that Λ^1_c satisfies both **CT** and **CM** and hence we get cumulativity, while cumulative transitivity does not hold for the logic based on Λ^2_c. We leave the proofs and examples to the interested reader and focus instead on yet another interesting quantitative approach that can be characterized in terms of a logic $\mathbf{AL}_\mathbf{\Lambda}$.

The driving idea behind prioritized ALs can be characterized by an iterative procedure:[37]

- First we pick out models of the premises in which as less abnormalities in Ω_1 are validated as possible.
- Second, we refine the given selection from step 1 in such a way that models are selected that validate as less abnormalities in Ω_2 as possible.
- etc.

In some applications this procedure may be suboptimal. Take for instance default logics that are able to express the specificity order among defaults. Although in most cases it is more intuitive to prefer interpretations of a given set of defaults that violate more general defaults over interpretations that violate more specific

[36] Due to space restrictions we skip the rather technical examples for non-smoothness.

[37] Indeed, this procedure amounts to an adequate procedural selection semantics for lexicographic ALs (see e.g., [27]).

defaults. For instance, given we know of Tweety that it is a penguin, and the following defaults, “Bird(X), then fly(X).”, “Penguin(X), then not-fly(X)” we are usually inclined to choose an interpretation that violates the former default and we hence conclude that Tweety does not fly.

However, consider a case where we have the choice between violating one more specific default and, say, 20 slightly less specific defaults. In such cases, as has been pointed out by Goldszmidt, Morris and Pearl [10], it may be better to choose an interpretation that violates the more specific default. There are many weighing functions which can be employed for this. We will demonstrate the point with a very simple one.

Suppose that $\Omega = \Omega_1 \cup \ldots \cup \Omega_n$ where $\Omega_i = \{\circ_i A \wedge \neg A \mid A \text{ is } \circ\text{-free}\}$. Let $\mu(\Delta) = \sum_{i=1}^{n} |\Delta \cap \Omega_i|/i$ if Δ is finite and $\mu(\Delta) = \infty$ otherwise. Define $\Delta \prec_\mu \Delta'$ iff $\mu(\Delta) < \mu(\Delta')$ or $\Delta \subset \Delta'$. We denote $\circ_i A \wedge \neg A$ by $!^i A$. In a similar way as in Fact 5.2 we can show that $\langle \Xi(\Gamma), \prec_\mu \rangle$ is smooth for all $\Gamma \subseteq \mathcal{W}$. Hence, for $\mathbf{AL}_{\min_{\prec_\mu}}$ we get the full meta-theory (soundness, completeness, cumulativity, reassurance, strong reassurance) as pointed out in Section 5.1.

Example 5.1. Suppose we have the situation that an expert of highest expertise states A. However, this is in conflict with the statements B, C, D, E and F which are stated by other experts of slightly less expertise. Hence, suppose our premise set is

$$\Gamma = \{\circ_1 A, \circ_3 B, \circ_2 C, \circ_3 D, \circ_3 E, \circ_3 F, \\ \neg A \vee \neg B, \neg A \vee \neg C, \neg A \vee \neg D, \neg A \vee \neg E, \neg A \vee \neg F\}$$

Hence, we have

$$\Sigma(\Gamma) = \{\{!^1 A, !^3 B\}, \{!^1 A, !^2 C\}, \{!^1 A, !^3 D\}, \{!^1 A, !^3 E\}, \{!^1 A, !^3 F\}\}$$

which gives to the following $\subset$-minimal choice sets in $\Xi(\Gamma)$: $\varphi = \{!^1 A\}$ and $\psi = \{!^3 B, !^2 C, !^3 D, !^3 E, !^3 F\}$. We have:

$$\mu(\varphi) = \tfrac{1}{1} = 1 < \mu(\psi) = 0 + \tfrac{1}{2} + \tfrac{4}{3} = 1\tfrac{5}{6}$$

and hence, $\varphi <_\mu \psi$. This means that with the AL characterized by the triple $\langle \mathbf{L}^*_\circ, \Omega_*, \min_{\prec_\mu} \rangle$, B, C, D and E are finally admissible while we don't get A. Hence, the expert opinions of the group of less expertise opposing our number one expert is prioritized by the logic.

In contrast, in a prioritized AL such as the one characterized by $\langle \mathbf{L}^*_\circ, \Omega_*, \min_{\prec_{\mathsf{lex}}} \rangle$ the situation would be inverse: $\psi \prec_{\mathsf{lex}} \varphi$ and hence we get the consequence A while B, C, D and F are not finally derivable. The reason is that the order $\prec_{\mathsf{lex}}$ proceeds strictly stepwise: since $\psi \cap \Omega_1 \subset \varphi \cap \Omega_1$ the order $\prec_{\mathsf{lex}}$ doesn't take into account any differences between the two choice sets concerning abnormalities of higher levels. The situation is different for $\prec_\mu$ since here the fact that on the higher levels 2 and 3 ψ is less normal than φ is the reason that $\varphi \prec_\mu \psi$ despite that fact that ψ fares better with respect to Ω_1.

Of course, in many cases both approaches will agree. Suppose for instance that our number one expert who states A is only opposed by one expert of lesser expertise who states $\neg A$. Hence, let $\Gamma' = \{\circ_1 A, \circ_2 \neg A\}$. Now we get $\Xi(\Gamma') \supseteq \{\{!^1 A\}, \{!^2 \neg A\}\}$. We have $\mu(\{!^1 A\}) = 1 > \mu(\{!^2 \neg A\}) = 1/2$ and hence $\{!^2 \neg A\} \prec_\mu \{!^1 A\}$. We also have $\{!^2 \neg A\} \prec_{\mathsf{lex}} \{!^1 A\}$. Hence, in both logics, A is a consequence.

6 Conclusion

In this paper it was demonstrated that the standard format for ALs can be generalized in a natural way. Semantically speaking, in the standard format the models are compared with respect to the set of abnormalities they verify and set inclusion. The generalization allows for comparisons with respect to other partial orders $\prec$. Also, we allow for a rich class of threshold functions Λ that select the models of the AL out of the models of the lower limit logic. We have shown that a huge class of ALs based on Λ has a strong meta-theory (soundness, completeness, cumulativity, reassurance, strong reassurance).

The generalization is natural since the main mechanisms of the standard format remain intact:

- As in the standard format we still have a selection semantics in which models are selected by virtue of their abnormal parts
- The dynamic proof format is the same as in the standard format. Formulas are derived conditionally where the conditions are sets of abnormalities. Derivations are executed by means of the familiar three generic rules PREM, RU, and RC. The marking is structurally analogous to the marking of the minimal abnormality strategy. The only difference is that instead of using the $\subset$-minimal choice sets with respect to the derived disjunctions of abnormalities, we use choice sets selected by the threshold function Λ.

We have given some special attention to the problem of non-smoothness. In the standard format where the abnormal parts of models are compared with respect to $\subset$ this problem does not appear: e.g., the set of abnormal parts of the lower limit logic models of some premise set Γ is always guaranteed to be smooth with respect to $\subset$. However, as soon as we introduce other partial orders $\prec$ instead of $\subset$, this may not hold anymore. We have offered a way to deal with such situations: instead of using the threshold function $\min_\prec$ we introduced the threshold function $\Psi_\prec$. In this way the problems connected to non-smoothness are avoided while for premise sets in which smoothness is guaranteed we get equivalent consequence sets (compared to the logic based on $\min_\prec$).

The current study opens various research paths. We intend to investigate complexity issues. Excellent groundwork for this investigation has been laid by investigations into the complexity of ALs in the standard format ([36, 18]).

In this paper only one marking definition is studied. It is structural analogous to the marking definition of the minimal abnormality strategy of the standard format.

There are other marking definitions which we are going to study in the new format such as variants of the marking for reliability or the normal selection strategy (see e.g., [34]). Concerning the former, it is well-known that this marking definition leads to lower complexity bounds (see [36, 18]). Finally, there are other types of semantic selections than the ones in the vein of Shoham studied in this paper. We intend to device dynamic proof theories for the limit variant presented in [29] which had previously been applied in the context of circumscription logic in [6]. The basic idea is that, given some ordering $\prec$ on the models of some logic $\mathbf{L}$, A is a consequence of Γ iff for each descending chain (w.r.t. $\prec$) of models of Γ there is a point from which on A is verified by each model.[38]

Finally, one may argue that generalizations as the one offered in this paper are useless formal overkills, it is shooting at flies with cannon-balls. For our defense let us point out the following.

First, we hope to have convinced the reader that there are many applications that are intuitively represented with orders different from $\subset$. One class of applications concerns situations in which we have a structured set of abnormalities (e.g., they may be prioritized as in the lexicographic and the colexicographic format). The recently developed format of lexicographic ALs gave already rise to various useful formal systems. Similar developments can be expected for the many other possibilities that are offered within the new format. Another class of applications concerns quantitative approaches, that is logics in which we compare the abnormal parts of models by means of quantitative considerations (rather than qualitative ones such as in the standard format).

Once the reader agrees that there are application contexts in which a departure from the standard format is useful (or even needed), it is not difficult to further convince her of the usefulness of having a generic format with a rich associated meta-theory. On the one hand this reduces labor: once a logic is devised in this format there is no need anymore to check and prove many of the interesting meta-theoretic properties. We get them for free based on the research offered in this paper. On the other hand, generic formats are fruitful for unification. Logics formulated in the same format can be compared more easily, techniques used for one application context can be easier transferred to other application contexts, etc.

Finally, some may argue or at least conjecture that all ALs can be translated into the standard format. However, even if that were true, it is very likely that many of the translations will turn out to be technically very cumbersome and/or artificial on an intuitive level. Since one of the goals of ALs is to explicate reasoning processes, this seems counter-productive to this goal. In many cases switching to different orderings of the models (e.g., ones based on quantitative considerations) and to different threshold functions may offer a more natural and intuitive explication of reasoning processes than the best possible translations into the standard format can offer.

[38] Where $\prec$ is transitive, this can also be expressed as follows: $\Gamma \Vdash A$ iff for every $M \in \mathcal{M}_{\mathbf{L}}(\Gamma)$ there is a $M' \in \mathcal{M}_{\mathbf{L}}(\Gamma)$ such that $M' \preceq M$ and for all $M'' \in \mathcal{M}_{\mathbf{L}}(\Gamma)$ for which $M'' \prec M'$, $M'' \models A$. This formulation can be found in [7] and the same idea has been used in the context of deontic logic in [11].

In this sense we hope that the research offered in this paper proves useful and fruitful for one of the main goals behind the adaptive logic program: the intuitive and natural explication of defeasible reasoning processes.

Appendix

A The A-Complete Stage of an Adaptive Proof

In this section we demonstrate in a constructive manner that an A-complete stage exists for any dynamic proof.

Let $\Gamma \subseteq \mathcal{W}$ and $A \in \mathcal{W}$. Note that each well-formed formula has a Gödel-number. From this it follows immediately that $Cn_{\mathbf{LLL}}(\Gamma)$ is enumerable. We are interested in the subset of $Cn_{\mathbf{LLL}}(\Gamma)$ that contains all the minimal Dab-consequences and all formulas $A \vee \mathsf{Dab}(\Delta) \in Cn_{\mathbf{LLL}}(\Gamma)$ where for all $\Delta' \subset \Delta$, $A \vee \mathsf{Dab}(\Delta') \notin Cn_{\mathbf{LLL}}(\Gamma)$. Let $\{B_1, B_2, \ldots\}$ be an enumeration of these formulas. Due to the compactness of **LLL**, for each B_i there are some $A_1, \ldots, A_{m(i)}$ such that $A_1, \ldots, A_{m(i)} \vdash_{\mathbf{LLL}} B_i$. Hence, for each B_i we have the following proof $\mathcal{P}_i$:

l_1^i	A_1	PREM	$\emptyset$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$l_{m(i)}^i$	$A_{m(i)}$	PREM	$\emptyset$
$l_{m(i)+1}^i$	B_i	$l_1^i, \ldots, l_{m(i)}^i$; RU	$\emptyset$

In case B_i is of the form $A \vee \mathsf{Dab}(\Delta)$ we add a further line.

$l_{m(i)+2}^i$	A	$l_{m(i)+1}^i$; RC	Δ

Where $\mathcal{P}$ contains the lines $l_1^0, l_2^0, \ldots, l_n^0$, we now combine the proofs $\mathcal{P}, \mathcal{P}_1, \mathcal{P}_2, \ldots$ to an extension of $\mathcal{P}$ by means of listing the respective lines as follows (and by renumbering the lines accordingly):

$$l_1^0, \ldots, l_n^0, l_1^1, \ldots, l_{m(1)}^1, \ldots, \ldots,$$

Obviously, the corresponding stage is A-complete.

Fact A.1. *Where $\mathcal{P}$ and $\mathcal{P}'$ are dynamic proofs from Γ: If a line with formula A and condition Δ is marked in an A-complete extension of $\mathcal{P}$ then a line with formula A and condition Δ is also marked in all A-complete extensions of $\mathcal{P}'$.*

Note that the marking at a stage depends on the minimal Dab-formulas derived at this stage and on the arguments produced for A. Hence, the fact holds since in an A-complete stage every minimal Dab-formula is derived on the empty condition and A is derived on some condition Δ whenever A can be derived on a condition $\Delta' \supseteq \Delta$.

The following fact holds for the extension of a dynamic proof $\mathcal{P}$ to an A-complete stage s:

Fact A.2. $\Sigma_s(\Gamma) = \Sigma(\Gamma)$ *and* $\Xi_s(\Gamma) = \Xi(\Gamma)$

B Representational Results for $\vdash_{\mathbf{AL}_{\mathbf{\Lambda}}}$ and $\Vdash_{\mathbf{AL}_{\mathbf{\Lambda}}}$

Lemma B.1. $\Gamma \vdash_{\mathbf{LLL}} A \vee \mathsf{Dab}(\Delta)$ *iff any dynamic proof can be finitely extended in such a way that* A *is derived on the condition* Δ.

Proof. "Left-to-Right": Let $\mathcal{P}$ be a dynamic proof consisting of lines $l_1, \ldots, l_m$. Suppose $\Gamma \vdash_{\mathbf{LLL}} A \vee \mathsf{Dab}(\Delta)$. By the compactness of **LLL**, there is a finite $\{A_1, \ldots, A_n\}$ such that $\{A_1, \ldots, A_n\} \vdash_{\mathbf{LLL}} A \vee \mathsf{Dab}(\Delta)$. Hence, we can introduce each A_i at line l_{m+i} by means of PREM. Then, we can derive A on the condition Δ by means of the justification "$l_{m+1}, \ldots, l_{m+n}$; RC" at line l_{m+n+1}.

"Right-to-left": We show this by an induction on the number of times i the rule RC has been used in order to derive A on the condition Δ.

"$i = 1$": A has been derived from $A_1, \ldots, A_n$ on the condition Δ. Thus, (1) $\{A_1, \ldots, A_n\} \vdash_{\mathbf{LLL}} A \vee \mathsf{Dab}(\Delta)$. Since this is the first time that RC is used, each A_i is derived on the condition $\emptyset$. Hence, (2) for each A_i, $\Gamma \vdash_{\mathbf{LLL}} A_i$. By the transitivity of **LLL**, $\Gamma \vdash_{\mathbf{LLL}} A \vee \mathsf{Dab}(\Delta)$.

"$i \Rightarrow i+1$": Suppose A is derived on the condition $\Delta_1 \cup \ldots \cup \Delta_n \cup \Delta$ from $A_1, \ldots, A_n$ which are derived on the conditions $\Delta_1, \ldots, \Delta_n$ respectively. Hence, (3) $\{A_1, \ldots, A_n\} \vdash_{\mathbf{LLL}} A \vee \mathsf{Dab}(\Delta)$. By the induction hypothesis, (4) $\Gamma \vdash_{\mathbf{LLL}} A_i \vee \mathsf{Dab}(\Delta_i)$ for each $i \leq n$. It follows immediately by the transitivity of **LLL** and properties of $\vee$ that $\Gamma \vdash_{\mathbf{LLL}} A \vee \mathsf{Dab}(\Delta_1 \cup \ldots \cup \Delta_n \cup \Delta)$. □

Lemma 4.1 follows immediately.

Lemma B.2. *If* $\Gamma \vdash_{\mathbf{AL}_{\mathbf{\Lambda}}} A$ *then for every* $\varphi \in \Lambda(\Xi(\Gamma))$ *there is a finite* $\Delta \subseteq \Omega \setminus \varphi$ *for which* $\Gamma \vdash_{\mathbf{LLL}} A \vee \mathsf{Dab}(\Delta)$.

Proof. Suppose $\Gamma \vdash_{\mathbf{AL}_{\mathbf{\Lambda}}} A$ and $\Lambda(\Xi(\Gamma)) \neq \emptyset$. Hence, there is a finite $\mathbf{AL}_{\mathbf{\Lambda}}$-proof $\mathcal{P}$ from Γ and a stage s such that A is derived in this proof at stage s at line l with a condition Δ, and in an A-complete extension s' of this proof l is unmarked. Hence, there is a $\varphi \in \Lambda(\Xi_{s'}(\Gamma))$ such that $\Delta \cap \varphi = \emptyset$, and for each $\psi \in \Lambda(\Xi_{s'}(\Gamma))$, A is derived on a condition $\Delta_\psi \subseteq \Omega \setminus \psi$. The rest follows by Fact A.2 and Lemma 4.1. □

Lemma B.3. *Where* $\Gamma \vdash_{\mathbf{AL}_{\mathbf{\Lambda}}} A$, s *is a finite stage of a dynamic proof from* Γ, *and* A *is derived at line* l *on the condition* Δ: A *is finally admissible at stage* s *at line* l *iff there is a* $\varphi \in \Lambda(\Xi(\Gamma))$ *for which* $\varphi \cap \Delta = \emptyset$ *or* $\Lambda(\Xi(\Gamma)) = \emptyset$.

Proof. Suppose the antecedent holds. "$\Rightarrow$": Since A is finally admissible at stage s at line l on the condition Δ, line l is unmarked in an A-complete extension stage s'. Suppose $\Lambda(\Xi(\Gamma)) \neq \emptyset$. By item (i) of the marking definition, there is a $\varphi \in \Lambda(\Xi_{s'}(\Gamma))$ for which $\Delta \cap \varphi = \emptyset$. By Fact A.2, there is a $\varphi \in \Lambda(\Xi(\Gamma))$ for which $\Delta \cap \varphi = \emptyset$. "$\Leftarrow$": If $\Lambda(\Xi(\Gamma)) = \emptyset$ then line l is unmarked in an A-complete extension of the proof since by Fact A.2, $\Lambda(\Xi_s(\Gamma)) = \emptyset$. Suppose $\Lambda(\Xi(\Gamma)) \neq \emptyset$ and there is a $\varphi \in \Lambda(\Xi(\Gamma))$ for which $\varphi \cap \Delta = \emptyset$. By Lemma B.2, for every $\psi \in \Lambda(\Xi(\Gamma))$ there is a $\Delta_\psi \subseteq \Omega \setminus \psi$ for which $\Gamma \vdash_{\mathbf{LLL}} A \vee \mathsf{Dab}(\Delta_\psi)$. Hence, in the A-complete extension stage s of the proof there is a $\Delta'_\psi \subseteq \Delta_\psi$ for each $\psi \in \Lambda(\Xi(\Gamma))$ such that A is derived on the condition Δ'_ψ. Since by Fact A.2, $\Xi(\Gamma) = \Xi_{s'}(\Gamma)$, line l is unmarked at stage s'. □

Theorem 4.1 (Proof Invariance)**.** *If A is an $\mathbf{AL}_{\mathbf{\Lambda}}$-consequence of Γ then any dynamic proof can be finitely extended in such a way that A is finally admissible in it.*

Proof. Let $\Gamma \vdash_{\mathbf{AL}_{\mathbf{\Lambda}}} A$. Let A be derived in a dynamic proof at stage s on the condition Δ. 1. case: $\Lambda(\Xi(\Gamma)) \neq \emptyset$. Let $\varphi \in \Lambda(\Xi(\Gamma))$ arbitrary. By Lemma B.2 there is a $\Delta \subseteq \Omega \setminus \varphi$ such that $\Gamma \vdash_{\mathbf{LLL}} A \vee \mathsf{Dab}(\Delta)$. By Lemma B.1 there is a finite extension of stage s in which A is derived on the condition Δ at a stage s'. By Lemma B.3, A is finally admissible at stage s'. 2. case: $\Lambda(\Xi(\Gamma)) = \emptyset$. By Lemma B.3, A is finally admissible at stage s. □

Lemma B.4. *If $\Lambda(\Xi(\Gamma)) = \emptyset$ then $\Gamma \vdash_{\mathbf{AL}_{\mathbf{\Lambda}}} A$ for all $A \in \mathcal{W}$.*

Proof. Suppose $\Lambda(\Xi(\Gamma)) = \emptyset$. Assume $\Sigma(\Gamma) = \emptyset$. Then, by definition $\Xi(\Gamma) = \{\emptyset\}$ and by **T3**, $\emptyset \in \Lambda(\Xi(\Gamma))$,—a contradiction. Let $\Delta \in \Sigma(\Gamma)$. Hence, $\Gamma \vdash_{\mathbf{LLL}} \mathsf{Dab}(\Delta)$. Hence, by the compactness of **LLL**, there are $\{A_1, \ldots, A_n\} \subseteq \Gamma$ such that $\{A_1, \ldots, A_n\} \vdash_{\mathbf{LLL}} \mathsf{Dab}(\Delta)$. We construct a dynamic proof as follows: on line 1 we introduce A_1 by PREM, …, on line n we introduce A_n by PREM, on line $n{+}1$ we derive $\mathsf{Dab}(\Delta)$ by RU on the empty condition. On line $n{+}2$ we derive $\neg\mathsf{Dab}(\Delta)$ by RC on the condition Δ. On line $n{+}3$ we derive A on the justification "$n{+}1, n{+}2$; RU" on the condition Δ. Since $\Lambda(\Xi(\Gamma)) = \emptyset$, by Lemma B.3, A is finally admissible at this stage. □

Lemma B.5. *If for every $\varphi \in \Lambda(\Xi(\Gamma))$ there is a finite $\Delta_\varphi \subseteq \Omega \setminus \varphi$ such that $\Gamma \vdash_{\mathbf{LLL}} A \vee \mathsf{Dab}(\Delta_\varphi)$, then $\Gamma \vdash_{\mathbf{AL}_{\mathbf{\Lambda}}} A$.*

Proof. The case $\Lambda(\Xi(\Gamma)) = \emptyset$ is covered by Lemma B.4. Let $\Lambda(\Xi(\Gamma)) \neq \emptyset$. By Lemma 4.1, for every Δ_φ there is a finite $\mathbf{AL}_{\mathbf{\Lambda}}$-proof from Γ and a stage s_φ in which A is derived on the condition Δ_φ at line l. Let $\mathcal{P}$ be any such proof. We extend the proof further to an A-complete stage. Note that for each $\varphi \in \Lambda(\Xi(\Gamma))$ A is derived on a condition $\Delta'_\varphi \subseteq \Delta_\varphi$ at this stage. By the supposition, Fact A.2, and Definition 3.2, line l is unmarked at this stage. □

The following representational theorem characterizes the consequence relation of $\mathbf{AL}_{\mathbf{\Lambda}}$ entirely by means of the consequence relation of **LLL** and the members of $\Lambda(\Xi(\Gamma))$. By Lemma B.2 and Lemma B.5 we immediately get:

Theorem 4.2. $\Gamma \vdash_{\mathbf{AL}_{\mathbf{\Lambda}}} A$ *iff for every* $\varphi \in \Lambda(\Xi(\Gamma))$ *there is a* $\Delta \subseteq \Omega \setminus \varphi$ *for which* $\Gamma \vdash_{\mathbf{LLL}} A \vee \mathsf{Dab}(\Delta)$.

Due to the compactness of **LLL**, Theorem 4.2 can be alternatively expressed by:

Theorem 4.3. $\Gamma \vdash_{\mathbf{AL}_{\mathbf{\Lambda}}} A$ *iff for every* $\varphi \in \Lambda(\Xi(\Gamma))$, $\Gamma \cup (\Omega \setminus \varphi)^\neg \vdash_{\mathbf{LLL}} A$.

Proof. Suppose for every $\varphi \in \Lambda(\Xi(\Gamma))$, $\Gamma \cup (\Omega \setminus \varphi)^\neg \vdash_{\mathbf{LLL}} A$. Let $\varphi \in \Lambda(\Xi(\Gamma))$. Hence, $\Gamma \cup (\Omega \setminus \varphi)^\neg \vdash_{\mathbf{LLL}} A$. By the compactness of **LLL**, there is a finite $\Delta \subseteq \Omega \setminus \varphi$ such that $\Gamma \cup \Delta^\neg \vdash_{\mathbf{LLL}} A$. By the deduction theorem, $\Gamma \vdash_{\mathbf{LLL}} A \vee \mathsf{Dab}(\Delta)$. Hence, by Lemma B.5, $\Gamma \vdash_{\mathbf{AL}_{\mathbf{\Lambda}}} A$.

Let $\Gamma \vdash_{\mathbf{AL}_{\mathbf{\Lambda}}} A$. By Lemma B.2, for every $\varphi \in \Lambda(\Xi(\Gamma))$ there is a $\Delta \subseteq \Omega \setminus \varphi$ for which $\Gamma \vdash_{\mathbf{LLL}} A \vee \mathsf{Dab}(\Delta)$. Assume for some $\varphi \in \Lambda(\Xi(\Gamma))$, $\Gamma \cup (\Omega \setminus \varphi)^\neg \nvdash_{\mathbf{LLL}} A$. By the monotonicity of **LLL**, there is no $\Delta \subseteq \Omega \setminus \varphi$ such that $\Gamma \cup \Delta^{\neg} \vdash_{\mathbf{LLL}} A$. Hence, there is no $\Delta \subseteq \Omega \setminus \varphi$ such that $\Gamma \vdash_{\mathbf{LLL}} A \vee \mathsf{Dab}(\Delta)$,—a contradiction. $\square$

Theorem 4.4. $Cn_{\mathbf{LLL}}(Cn_{\mathbf{AL}_{\Lambda}}(\Gamma)) = Cn_{\mathbf{AL}_{\Lambda}}(\Gamma)$

Proof. "$\subseteq$": Suppose $Cn_{\mathbf{AL}_{\Lambda}}(\Gamma) \vdash_{\mathbf{LLL}} A$. Hence, by the compactness of **LLL**, there are $A_1, \ldots, A_n \in Cn_{\mathbf{AL}_{\Lambda}}(\Gamma)$ for which (1) $\{A_1, \ldots, A_n\} \vdash_{\mathbf{LLL}} A$. By Theorem 4.2, (2) for each $i \leq n$ and for all $\varphi \in \Lambda(\Xi(\Gamma))$, $\Gamma \cup (\Omega \setminus \varphi)^{\neg} \vdash_{\mathbf{LLL}} A_i$. Hence, by (1), (2), and the transitivity of **LLL**, for all $\varphi \in \Lambda(\Xi(\Gamma))$, $\Gamma \cup (\Omega \setminus \varphi)^{\neg} \vdash_{\mathbf{LLL}} A$. By Theorem 4.2, $\Gamma \vdash_{\mathbf{AL}_{\Lambda}} A$. "$\supseteq$": Follows due to the reflexivity of **LLL**. $\square$

We now show that this representational result can be further strengthened.

Lemma 4.2. $\Xi(\Gamma) \setminus \Xi^{\perp}(\Gamma) = \mathsf{Ab}^{\Gamma}_{\mathbf{LLL}}$

Proof. Let $\varphi \in \Xi(\Gamma) \setminus \Xi^{\perp}(\Gamma)$. Assume there is no $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$ such that $\mathrm{Ab}(M) = \varphi$. Hence, $\mathcal{M}_{\mathbf{LLL}}(\Gamma \cup (\Omega \setminus \varphi)^{\neg} \cup \varphi) = \emptyset$. By the compactness of **LLL**, there are finite $\varphi' \subseteq \varphi$ and $\Delta \subseteq \Omega \setminus \varphi$ such that $\mathcal{M}_{\mathbf{LLL}}(\Gamma \cup \varphi' \cup \Delta^{\neg}) = \emptyset$. Hence, $\Gamma \Vdash_{\mathbf{LLL}} \neg \bigwedge \varphi' \vee \mathsf{Dab}(\Delta)$. By the completeness of **LLL**, $\Gamma \vdash_{\mathbf{LLL}} \neg \bigwedge \varphi' \vee \mathsf{Dab}(\Delta)$,—a contradiction since then $\varphi \in \Xi^{\perp}(\Gamma)$.

Let $\varphi \in \mathsf{Ab}^{\Gamma}_{\mathbf{LLL}}$. Let $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$ such that $\mathrm{Ab}(M) = \varphi$, and $\Theta \in \Sigma(\Gamma)$. By the soundness of **LLL**, $\Gamma \Vdash_{\mathbf{LLL}} \mathsf{Dab}(\Theta)$. Hence, $M \models \mathsf{Dab}(\Theta)$ and hence there is a $A \in \Theta$ such that $M \models A$. Thus, $\mathrm{Ab}(M) \cap \Theta \neq \emptyset$. Hence, $\varphi \in \Xi(\Gamma)$. Assume now that $\varphi \in \Xi^{\perp}(\Gamma)$ and hence that there are finite $\varphi' \subseteq \varphi$ and $\Delta \subseteq \Omega \setminus \varphi$ such that $\neg \bigwedge \varphi' \vee \mathsf{Dab}(\Delta) \in Cn_{\mathbf{LLL}}(\Gamma)$. By the soundness of **LLL**, $\Gamma \Vdash_{\mathbf{LLL}} \neg \bigwedge \varphi' \vee \mathsf{Dab}(\Delta)$ and hence $\mathcal{M}_{\mathbf{LLL}}(\Gamma \cup (\Omega \setminus \varphi)^{\neg} \cup \varphi) = \emptyset$,—a contradiction. $\square$

The following result strengthens Theorem 4.3:

Theorem 4.6. $\Gamma \vdash_{\mathbf{AL}_{\Lambda}} A$ *iff for every* $\varphi \in \Lambda(\Xi(\Gamma)) \setminus \Xi^{\perp}(\Gamma), \Gamma \cup (\Omega \setminus \varphi)^{\neg} \vdash_{\mathbf{LLL}} A$.

Proof. Let $\varphi \in \Lambda(\Xi(\Gamma)) \cap \Xi^{\perp}(\Gamma)$. $\Gamma \cup (\Omega \setminus \varphi)^{\neg} \vdash_{\mathbf{LLL}} A$ iff [by the soundness and completeness of **LLL**] $\Gamma \cup (\Omega \setminus \varphi)^{\neg} \Vdash_{\mathbf{LLL}} A$ iff for all $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma \cup (\Omega \setminus \varphi)^{\neg})$, $M \models A$ iff for all $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$ for which $\mathrm{Ab}(M) \subseteq \varphi$, $M \models A$ iff $\Gamma \cup (\Omega \setminus \psi)^{\neg} \Vdash_{\mathbf{LLL}} A$ where $\psi \in \{\psi \in \mathsf{Ab}^{\Gamma}_{\mathbf{LLL}} \mid \psi \subseteq \varphi\}$ iff [by Lemma 4.2 and by the soundness and completeness of **LLL**] $\Gamma \cup (\Omega \setminus \psi)^{\neg} \vdash_{\mathbf{LLL}} A$ where $\psi \in \{\psi \in \Xi(\Gamma) \setminus \Xi^{\perp}(\Gamma) \mid \psi \subset \varphi\}$ iff [by **T2** and since $\varphi \in \Lambda(\Xi(\Gamma))$] $\Gamma \cup (\Omega \setminus \psi)^{\neg} \vdash_{\mathbf{LLL}} A$ where $\psi \in X_{\varphi}$ and $X_{\varphi} =_{\mathrm{df}} \{\psi \in \Lambda(\Xi(\Gamma)) \setminus \Xi^{\perp}(\Gamma) \mid \psi \subset \varphi\}$. Note that $X_{\varphi} \subseteq \Lambda(\Xi(\Gamma)) \setminus \Xi^{\perp}(\Gamma)$.

Hence, for each $\varphi \in \Lambda(\Xi(\Gamma)) \cap \Xi^{\perp}(\Gamma)$ there is a $X_{\varphi} \subseteq \Lambda(\Xi(\Gamma)) \setminus \Xi^{\perp}(\Gamma)$ such that (†) $Cn_{\mathbf{LLL}}(\Gamma \cup (\Omega \setminus \varphi)^{\neg}) = \bigcap_{\psi \in X_{\varphi}} Cn_{\mathbf{LLL}}(\Gamma \cup (\Omega \setminus \psi)^{\neg})$.

By Theorem 4.3, (1) $Cn_{\mathbf{AL}_{\Lambda}}(\Gamma) = \bigcap_{\varphi \in \Lambda(\Xi(\Gamma))} Cn_{\mathbf{LLL}}(\Gamma \cup (\Omega \setminus \varphi)^{\neg})$. Obviously,

$$\bigcap_{\varphi \in \Lambda(\Xi(\Gamma))} Cn_{\mathbf{LLL}}(\Gamma \cup (\Omega \setminus \varphi)^{\neg}) =$$

$$\bigcap_{\varphi \in \Lambda(\Xi(\Gamma)) \setminus \Xi^{\perp}(\Gamma)} Cn_{\mathbf{LLL}}(\Gamma \cup (\Omega \setminus \varphi)^{\neg}) \cap \bigcap_{\varphi \in \Lambda(\Xi(\Gamma)) \cap \Xi^{\perp}(\Gamma)} Cn_{\mathbf{LLL}}(\Gamma \cup (\Omega \setminus \varphi)^{\neg}) \quad (2)$$

By (†), $\bigcap_{\varphi \in \Lambda(\Xi(\Gamma)) \cap \Xi^{\perp}(\Gamma)} Cn_{\mathbf{LLL}}(\Gamma \cup (\Omega \setminus \varphi)^{\neg}) =$

$$\bigcap_{\varphi \in \Lambda(\Xi(\Gamma)) \cap \Xi^{\perp}(\Gamma)} \bigcap_{\psi \in X_{\varphi}} Cn_{\mathbf{LLL}}(\Gamma \cup (\Omega \setminus \psi)^{\neg}) \supseteq$$

$$\bigcap_{\varphi \in \Lambda(\Xi(\Gamma)) \setminus \Xi^{\perp}(\Gamma)} Cn_{\mathbf{LLL}}(\Gamma \cup (\Omega \setminus \varphi)^{\neg}) \quad (3)$$

Hence, by (1), (2) and (3):

$$Cn_{\mathbf{AL}_{\mathbf{\Lambda}}}(\Gamma) = \bigcap_{\varphi \in \Lambda(\Xi(\Gamma)) \setminus \Xi^{\perp}(\Gamma)} Cn_{\mathbf{LLL}}(\Gamma \cup (\Omega \setminus \varphi)^{\neg}) \qquad \square$$

Theorem 4.7. *$\Gamma \Vdash_{\mathbf{AL}_{\mathbf{\Lambda}}} A$ iff for all $\varphi \in \Lambda(\mathsf{Ab}^{\Gamma}_{\mathbf{LLL}})$, $\Gamma \cup (\Omega \setminus \varphi)^{\neg} \Vdash_{\mathbf{LLL}} A$.*

Proof. $\Gamma \Vdash_{\mathbf{AL}_{\mathbf{\Lambda}}} A$ iff $M \models A$ for all M for which $\mathrm{Ab}(M) \in \Lambda(\mathsf{Ab}^{\Gamma}_{\mathbf{LLL}})$ iff $\Gamma \cup \mathrm{Ab}(M) \cup (\Omega \setminus \mathrm{Ab}(M))^{\neg} \Vdash_{\mathbf{LLL}} A$ for all $\mathrm{Ab}(M) \in \Lambda(\mathsf{Ab}^{\Gamma}_{\mathbf{LLL}})$ iff [by **T2**] $\Gamma \cup (\Omega \setminus \mathrm{Ab}(M))^{\neg} \Vdash_{\mathbf{LLL}} A$ for all $\mathrm{Ab}(M) \in \Lambda(\mathsf{Ab}^{\Gamma}_{\mathbf{LLL}})$. □

C Some Insights Concerning Choice Sets

In this section, let X be an enumerable set, Σ be a set of finite subsets of X, and

Notation C.1. Where Ψ is a set of sets we write $\Psi_{\min}$ for the set of $\subset$-minimal sets in Ψ.

Notation C.2. CS is the function that returns the choice sets (restricted to elements in X) of a set of sets.

The following Lemma will be essential to prove the smoothness of $\langle \Xi(\Gamma), \subset \rangle$ and the smoothness of $\langle \mathsf{Ab}^{\Gamma}_{\mathbf{LLL}}, \subset \rangle$ in the next section.

Lemma C.1. *Where $Y = \{a_i \mid i \in I\} \in \mathsf{CS}(\Sigma)$, let $\hat{Y} = \bigcap_{i \in I} Y_i$ where $Y_1 = Y$ and*

$$Y_{i+1} = \begin{cases} Y_i & \textit{if there is a } \Delta \in \Sigma \textit{ such that } Y_i \cap \Delta = \{a_i\} \\ Y_i \setminus \{a_i\} & \textit{else} \end{cases}$$

we have: $\hat{Y} \in \mathsf{CS}_{\min}(\Sigma)$ and $\hat{Y} \subseteq Y$.

Proof. By the construction, (†) $Y_i \supseteq Y_{i+1}$ for all $i, i+1 \in I$. Note that, also by the construction, ($\star$) Y_i is a choice set of Σ, i.e., $Y_i \in \mathsf{CS}(\Sigma)$ for each $i \in I$.

We now show that $\hat{Y} \in \mathsf{CS}(\Sigma)$. Assume otherwise and hence that there is a $\Delta \in \Sigma$ such that $\hat{Y} \cap \Delta = \emptyset$. Since Δ is finite and by ($\star$), $\Delta \cap Y_1 = \{b_1, \ldots, b_n\}$ for some $n \in \mathbb{N}$. By our assumption there is no $j \leq n$ such that $b_j \in Y_i \cap \Delta$ for all $i \in I$. Hence, for all $j \leq n$ there is a lowest $i_j \in I$ such that $b_j \notin Y_{i_j} \cap \Delta$. Take $k = \max(\{i_j \mid 1 \leq j \leq n\})$. Then, since due to (†) $\{b_1, \ldots, b_n\} \supseteq Y_i \cap \Delta \supseteq Y_{i+1} \cap \Delta$, also $b_j \notin Y_k \cap \Delta$ for all $j \leq n$ and thus $Y_k \cap \Delta = \emptyset$. However, this is a contradiction, since by ($\star$) $Y_k \in \mathsf{CS}(\Sigma)$. Thus, there is a $j \leq n$ such that $b_j \in \hat{Y}$. Hence, $\hat{Y} \in \mathsf{CS}(\Sigma)$.

Now assume $\hat{Y}$ is not minimal. Hence, there is a $a_i \in \hat{Y}$ (where $i \in I$) such that $\hat{Y} \setminus \{a_i\} \in \mathsf{CS}(\Sigma)$. By the construction, there is a $\Delta \in \Sigma$ such that $Y_i \cap \Delta = \{a_i\}$. By (†) and since $\hat{Y} \in \mathsf{CS}(\Sigma)$, $\hat{Y} \cap \Delta = \{a_i\}$,—a contradiction since then $\hat{Y} \setminus \{a_i\}$ cannot be a choice set of Σ. Hence, $\hat{Y} \in \mathsf{CS}_{\min}(\Sigma)$. □

D The smoothness of $\langle \Xi(\Gamma), \subset \rangle$ and of $\langle \mathsf{Ab}^{\Gamma}_{\mathbf{LLL}}, \subset \rangle$

Note that Ω is an enumerable set and $\Sigma(\Gamma)$ is a set of finite subsets of Ω. Hence, by Lemma C.1 immediately Theorem 4.8 follows.

Lemma D.1. *Where Γ is* **LLL***-non-trivial and $\varphi \in \Xi(\Gamma)$: $\Gamma \cup (\Omega \setminus \varphi)^{\neg}$ is* **LLL***-non-trivial.*

Proof. Assume $\Gamma \cup (\Omega \setminus \varphi)^{\neg}$ is **LLL**-trivial. Hence, by the compactness of **LLL**, there is a finite $\Delta \subseteq \Omega \setminus \varphi$ such that $\Gamma \vdash_{\mathbf{LLL}} \mathsf{Dab}(\Delta)$. Hence, $\Delta' \in \Sigma(\Gamma)$ for some $\Delta' \subseteq \Delta$. Since φ is a choice set of $\Sigma(\Gamma)$, $\varphi \cap \Delta \neq \emptyset$,—a contradiction. □

Lemma D.2. *Where Γ is* **LLL***-non-trivial:* $\min_{\subset}(\Xi(\Gamma)) \subseteq \Xi(\Gamma) \setminus \Xi^{\perp}(\Gamma)$.

Proof. Let $\varphi \in \min_{\subset}(\Xi(\Gamma))$. By Lemma D.1, $\mathcal{M}_{\mathbf{LLL}}(\Gamma \cup (\Omega \setminus \varphi)^{\neg}) \neq \emptyset$. Hence, there is a $\psi \subseteq \varphi$ such that $\psi \in \mathsf{Ab}^{\Gamma}_{\mathbf{LLL}}$. Thus, by Lemma 4.2 there is a $\psi \subseteq \varphi$ such that $\psi \in \Xi(\Gamma) \setminus \Xi^{\perp}(\Gamma)$. Hence, $\varphi = \psi$. □

Lemma D.3. *Where Γ is* **LLL***-non-trivial: $\Xi(\Gamma) \setminus \Xi^{\perp}(\Gamma)$ is $\subset$-dense in $\Xi(\Gamma)$.*

Proof. This follows immediately by Theorem 4.8 and Lemma D.2. □

Fact D.1. *Where $\prec$ is a partial order on Y and X is $\prec$-dense in Y:* $\min_{\prec}(X) = \min_{\prec}(Y)$.

Proof. Assume $x \in \min_{\prec}(X) \setminus \min_{\prec}(Y)$. Since $X \subseteq Y$, $x \in Y$. Hence, there is a $y \in Y \setminus X$ for which $y \prec x$. Since X is $\prec$-dense in Y, there is a $z \in X$ for which $z \prec y$. By the transitivity of $\prec$, $z \prec x$,—a contradiction. Suppose $x \in \min_{\prec}(Y)$. Hence, there is no $y \in Y$ for which $y \prec x$. Since X is $\prec$-dense in Y, $x \in X$ and hence $x \in \min_{\prec}(X)$. □

Theorem 4.9. *Where $\Gamma \subseteq \mathcal{W}$: $\langle \mathsf{Ab}^{\Gamma}_{\mathbf{LLL}}, \subset \rangle$ is smooth.*

Proof. By Fact D.1, Lemma D.3, and Lemma 4.2, $\min_{\subset}(\mathsf{Ab}^{\Gamma}_{\mathbf{LLL}}) = \min_{\subset}(\Xi(\Gamma) \setminus \Xi^{\perp}(\Gamma)) = \min_{\subset}(\Xi(\Gamma))$. The rest follows immediately with Theorem 4.8. □

E The Relationship Between the Criteria for Λ

Recall that we presuppose that for all partial orders $\prec$ and $\prec_i$ ($i \in \mathbb{N}$): $\subset \subseteq \prec$ and $\subset \subseteq \prec_i$.

Fact E.1. *Where X and Y are arbitrary sets for which $\langle X, \subset \rangle$ and $\langle Y, \subset \rangle$ are smooth and* $\min_{\subset}(X) = \min_{\subset}(Y)$: *$\langle X \cap Y, \subset \rangle$ is smooth.*

Fact 4.1. *(i)* **C3** *is equivalent to* **C4***; (ii)* **C4** *implies* **C1***; (iii)* **C5** *implies* **C3***; (iv)* **C1** *implies* **C2***; (v)* **C2** *implies* **C1***; (vi)* **C5** *implies* **C6***.*

Proof. *Ad (i):* **C4** follows trivially. Suppose **C4** holds. Since $X \in \Upsilon$, $\langle X, \subset \rangle$ is smooth and hence $\min_{\subset}(X)$ is $\subset$-dense in X. By Fact D.1, $\min_{\subset}(X) = \min_{\subset}(\min_{\subset}(X))$. Hence, by **C4**, $\Lambda(X) = \Lambda(\min_{\subset}(X))$ which implies **C3**. *Ad (ii):* **C1** follows immediately due to **T1**. *Ad (iii):* Let Λ' be the identity function. *Ad (iv)*: Let X be $\subset$-dense in Y. By Fact D.1, $\min_{\subset}(X) = \min_{\subset}(Y)$. Hence, by **C1**, $\Lambda(X) \cap Y = \Lambda(Y) \cap X$. By **T1** and since $X \subseteq Y$, $\Lambda(X) = \Lambda(Y) \cap X$. Hence **C2** holds. *Ad (v):* Suppose $\min_{\subset}(X) = \min_{\subset}(Y)$. Note that by Fact E.1, $X \cap Y \in \Upsilon$. We first show that $X \cap Y$ is $\subset$-dense in X and in Y. Without loss of generality we show the case for X. Let $x \in X$. By the smoothness of $\langle X, \subset \rangle$ there is a $y \in \min_{\subset}(X)$ such that $y \subseteq x$. Since $\min_{\subset}(X) = \min_{\subset}(Y)$, $y \in X \cap Y$. Hence, $X \cap Y$ is $\subset$-dense in X. Thus, by **C2**, $\Lambda(X \cap Y) = \Lambda(X) \cap X \cap Y$ and by **T1**, $\Lambda(X \cap Y) = \Lambda(X) \cap Y$. By an analogous argument, $\Lambda(X \cap Y) = \Lambda(Y) \cap X$. Hence, $\Lambda(X) \cap Y = \Lambda(Y) \cap X$. *Ad (vi):* Trivial. □

In the following, where not indicated different, X and Y are arbitrary sets (they need not be in Υ).

Fact E.2. $\min_{\prec}(X) \subseteq \min_{\subset}(X)$

Proof. Let $x \in X \setminus \min_{\subset}(X)$. Hence, there is a $y \in X$ such that $y \subset x$. Hence, since $\subset \subseteq \prec$, $y \prec x$ and thus, $x \in X \setminus \min_{\prec}(X)$. □

Fact E.3. $\min_{\prec}$ *is a threshold function.*

Proof. **T1** and **T3** hold trivially. **T2** holds due to Fact E.2. □

Also the next two facts are trivial:

Fact E.4. $\min_{\prec}(X) \subseteq \Psi_{\prec}(X)$

Fact E.5. *Where* $\langle X, \prec \rangle$ *is smooth,* $\Psi_{\prec}(X) = \min_{\prec}(X)$*.*

Fact E.6. *Where* $X \in \Upsilon$*:* $\min_{\prec}(X) = \min_{\prec}(\min_{\subset}(X))$*.*

Proof. Let $x \in \min_{\prec}(X)$. By Fact E.2, $x \in \min_{\subset}(X)$. Assume now that $x \notin \min_{\prec}(\min_{\subset}(X))$. Hence, there is a $y \in \min_{\subset}(X)$ such that $y \prec x$,—a contradiction to the minimality of x.

Let $x \in \min_{\prec}(\min_{\subset}(X))$. Assume $x \notin \min_{\prec}(X)$. Hence, there is a $y \in X \setminus \min_{\subset}(X)$ such that $y \prec x$. Since $\langle X, \subset \rangle$ is smooth, there is a $z \in \min_{\subset}(X)$ such that $z \subset y$. Hence, $z \prec y$ and by the transitivity of $\prec$, $z \prec x$,—a contradiction to the minimality of x. □

Fact E.7. *Where* X *is* $\prec$*-dense in* Y*:* $\Psi_{\prec}(X) = \Psi_{\prec}(Y) \cap X$*.*

Proof. Let $x \in \Psi_{\prec}(X)$. Assume $x \notin \Psi_{\prec}(Y) \cap X$. Since $\Psi_{\prec}$ is inclusive, $x \in X$ and thus $x \notin \Psi_{\prec}(Y)$. Hence, there is a $y \in \min_{\prec}(Y)$ for which $y \prec x$. However, by Fact D.1, $y \in \min_{\prec}(X)$ and hence $x \notin \Psi_{\prec}(X)$,—a contradiction.

Let $x \in \Psi_{\prec}(Y) \cap X$. Assume $x \notin \Psi_{\prec}(X)$. Since $x \in X$, there is a $y \in \min_{\prec}(X)$ for which $y \prec x$. However, by Fact D.1, $y \in \min_{\prec}(Y)$ and thus $x \notin \Psi_{\prec}(Y)$,—a contradiction. □

Fact E.8. *$\Psi_{\prec}(X)$ is $\prec$-dense in X.*

Proof. Let $x \in X \setminus \Psi_{\prec}(X)$. Hence, there is a $y \in \min_{\prec}(X)$ such that $y \prec x$. The rest follows by Fact E.4. □

Fact E.9. *$\Psi_{\prec}(X)$ is a $\prec$-lower set of X.*

Proof. Let $x \in \Psi_{\prec}(X)$ and $y \in X$ such that $y \prec x$. Hence, $x \notin \min_{\prec}(X)$. Hence, there is no $z \in \min_{\prec}(X)$ such that $z \prec x$. Hence, by the transitivity of $\prec$, there is no $z \in \min_{\prec}(X)$ such that $z \prec y$. Thus, $y \in \Psi_{\prec}(X)$. □

Fact E.10. *$\Psi_{[\prec_1,\ldots,\prec_n]}(X)$ is a $\prec_n$-lower set of X.*

Proof. We show this by induction on n. The case $n = 1$ is shown in Fact E.9. "$n \Rightarrow n+1$": Let $x \in \Psi_{[\prec_1,\ldots,\prec_{n+1}]}(X)$ and $y \in X$ such that $y \prec_{n+1} x$. Hence, $x \in \Psi_{[\prec_1,\ldots,\prec_n]}(X)$ and, since $\prec_{n+1} \subseteq \prec_n$, $y \prec_n x$. Hence, $y \in \Psi_{[\prec_1,\ldots,\prec_n]}(X)$ by the induction hypothesis. The rest follows by Fact E.9. □

Fact E.11. *Where $\prec_2 \subseteq \prec_1$: If X is a $\prec_1$-lower set of Y, then X is a $\prec_2$-lower set of Y.*

Proof. Let $x \in X$ and $y \in Y$ such that $y \prec_2 x$. Hence, $y \prec_1 X$ and thus, $y \in X$ since X is a $\prec_1$-lower set of Y. □

Fact E.12. *$\Psi_{[\prec_1,\ldots,\prec_n]}$ (and hence also $\Psi_{\prec}$) is a threshold function.*

Proof. **T1** and **T3** are immediate. **T2** follows by Fact E.10, Fact E.11 and since $\subset \subseteq \prec_n$. □

The following related facts E.13–E.17 are especially useful in Section 4.5.

Fact E.13. *Where $\min_{\prec}(X) \subseteq Y \subseteq X$: $\min_{\prec}(X) \subseteq \min_{\prec}(Y)$.*

Proof. Let $x \in \min_{\prec}(X)$. Assume $x \notin \min_{\prec}(Y)$. Hence, there is a $y \in Y$ such that $y \prec x$. Since $Y \subseteq X$ this is a contradiction to the minimality of x. □

Fact E.14. *Where $\Psi_{\prec}(X) \subseteq Y \subseteq X$: $\Psi_{\prec}(X) = \Psi_{\prec}(Y)$.*

Proof. Since by Fact E.8 $\Psi_{\prec}(X)$ is $\prec$-dense in X and since $\Psi_{\prec}(X) \subseteq Y$, also Y is $\prec$-dense in X. Hence, by Fact E.7 and since $\Psi_{\prec}$ is inclusive, $\Psi_{\prec}(Y) = \Psi_{\prec}(X) \cap Y = \Psi_{\prec}(X)$. □

Fact E.15. *Where* $\langle \prec_1, \ldots, \prec_n \rangle$ *is an abstraction sequence:*

$$\min_{\prec_1}(X) \subseteq \min_{\prec_2}(\Psi_{\prec_1}(X)) \subseteq \ldots \subseteq \min_{\prec_n}(\Psi_{[\prec_1,\ldots,\prec_{n-1}]}(X))$$

Proof. We show this by induction. "$i = 1$": Let $x \in \min_{\prec_1}(X)$. Assume $x \notin \min_{\prec_2}(\Psi_{\prec_1}(X))$. Note that $\min_{\prec_1}(X) \subseteq \Psi_{\prec_1}(X)$ and hence $x \in \Psi_{\prec_1}(X)$. Thus, there is a $y \in \Psi_{\prec_1}(X)$ such that $y \prec_2 x$. But then $y \prec_1 x$,—a contradiction to the minimality of x.

"$i - 1 \Rightarrow i$": Let $x \in \min_{\prec_i}(\Psi_{[\prec_1,\ldots,\prec_{i-1}]}(X))$. Hence, $x \in \Psi_{[\prec_1,\ldots,\prec_i]}(X)$. Assume there is a $y \in \Psi_{[\prec_1,\ldots,\prec_i]}(X)$ such that $y \prec_{i+1} x$. Then, $y \prec_i x$ and $y \in \Psi_{[\prec_1,\ldots,\prec_{i-1}]}(X)$,—a contradiction to the minimality of x. □

Fact E.16. *Where* $\langle \prec_1, \ldots, \prec_n \rangle$ *is an abstraction sequence,* $x \in \Psi_{[\prec_1,\ldots,\prec_i]}(X)$ *and* $0 \leq i < j \leq n$:[39] *either* $x \in \Psi_{[\prec_1,\ldots,\prec_j]}(X)$ *or there is a* $y \in \Psi_{[\prec_1,\ldots,\prec_j]}(X)$ *for which* $y \prec_{i+1} x$.

Proof. Let $0 \leq i < n$. We show the fact by induction for all j such that $i < j \leq n$.

"$j = i + 1$": Suppose $x \notin \Psi_{[\prec_1,\ldots,\prec_j]}(X)$. Since $x \in \Psi_{[\prec_1,\ldots,\prec_i]}(X)$ there is a $y \in \min_{\prec_j}(\Psi_{[\prec_1,\ldots,\prec_i]}(X)) \subseteq \Psi_{[\prec_1,\ldots,\prec_j]}(X)$ such that $y \prec_j x$.

"$j \Rightarrow j + 1$": By the induction hypothesis there is a $x' \preceq_{i+1} x$ such that $x' \in \Psi_{[\prec_1,\ldots,\prec_j]}(X)$. Suppose $x' \notin \Psi_{[\prec_1,\ldots,\prec_{j+1}]}(X)$. Hence, there is a $y \in \min_{\prec_{j+1}}(\Psi_{[\prec_1,\ldots,\prec_j]}(X))$ such that $y \prec_{j+1} x'$. Hence, $y \prec_{i+1} x'$ and by the transitivity of $\prec_{i+1}$ also $y \prec_{i+1} x$. □

Fact E.17. *Where* $\Psi_{[\prec_1,\ldots,\prec_n]}(X) \subseteq Y \subseteq X$: $\Psi_{[\prec_1,\ldots,\prec_n]}(X) = \Psi_{[\prec_1,\ldots,\prec_n]}(Y)$.

Proof. "$\supseteq$": Let $x \in \Psi_{[\prec_1,\ldots,\prec_n]}(Y)$. Note that (†) $x \in \Psi_{[\prec_1,\ldots,\prec_i]}(Y)$ for all $i \leq n$ and $x \in Y$. We show by induction that $x \in \Psi_{[\prec_1,\ldots,\prec_i]}(X)$ for every $i \leq n$.

"$i = 1$": Assume $x \notin \Psi_{\prec_1}(X)$. Hence, there is a $y \in \min_{\prec_1}(X)$ such that $y \prec_1 x$. By Fact E.15, $y \in \Psi_{[\prec_1,\ldots,\prec_n]}(X)$ and hence $y \in Y$. Since $x \in \Psi_{\prec_1}(Y)$, $y \notin \min_{\prec_1}(Y)$. Hence, there is a $z \in Y$ such that $z \prec_1 y$,—a contradiction to the $\prec_1$-minimality of y in X.

"$i \Rightarrow i + 1$": By the induction hypothesis, $x \in \Psi_{[\prec_1,\ldots,\prec_i]}(X)$. Assume $x \notin \Psi_{[\prec_1,\ldots,\prec_{i+1}]}(X)$. Hence, there is a $y \in \min_{\prec_{i+1}}(\Psi_{[\prec_1,\ldots,\prec_i]}(X))$ such that $y \prec_{i+1} x$. Hence, by Fact E.15, $y \in \Psi_{[\prec_1,\ldots,\prec_n]}(X)$ and thus $y \in Y$. Since $\Psi_{[\prec_1,\ldots,\prec_{i+1}]}(Y)$ is a $\prec_{i+1}$-lower set of Y by Fact E.10 and by (†), $y \in \Psi_{[\prec_1,\ldots,\prec_{i+1}]}(Y)$. Hence, since $x, y \in \Psi_{[\prec_1,\ldots,\prec_{i+1}]}(Y)$ and $y \prec_{i+1} x$ there is a $z \in \Psi_{[\prec_1,\ldots,\prec_{i+1}]}(Y)$ such that $z \prec_{i+1} y$ (otherwise $y \in \min_{\prec_{i+1}}(\Psi_{[\prec_1,\ldots,\prec_i]}(Y))$ and hence $x \notin \Psi_{[\prec_1,\ldots,\prec_{i+1}]}(Y)$). Since $y \in \Psi_{[\prec_1,\ldots,\prec_{i+1}]}(X)$ and $\Psi_{[\prec_1,\ldots,\prec_{i+1}]}(X)$ is by Fact E.10 a $\prec_{i+1}$-lower set of X, also $z \in \Psi_{[\prec_1,\ldots,\prec_{i+1}]}(X)$,—a contradiction to the $\prec_{i+1}$-minimality of y in $\Psi_{[\prec_1,\ldots,\prec_{i+1}]}(X)$.

"$\subseteq$": We show by induction that $x \in \Psi_{[\prec_1,\ldots,\prec_i]}(Y)$ for all $i \leq n$ and all $x \in \Psi_{[\prec_1,\ldots,\prec_n]}(X)$. Note that $x \in Y$ and $x \in \Psi_{[\prec_1,\ldots,\prec_i]}(X)$ for all $i \leq n$.

"$i = 1$": Assume $x \notin \Psi_{\prec_1}(Y)$. Hence, there is a $y \in \min_{\prec_1}(Y)$ such that $y \prec_1 x$. Hence, there is a $z \in X$ such that $z \prec_1 y$ (otherwise $x \notin \Psi_{\prec_1}(X)$). By Fact E.16,

[39] For the special case $i = 0$ let $\Psi_{[\prec_1,\ldots,\prec_0]}(X)$ denote X.

there is a $z' \in \Psi_{[\prec_1,\dots,\prec_n]}(X)$ such that $z' \preceq_1 z$. Since $z' \in Y$ and $z' \prec_1 y$ this is a contradiction to the $\prec_1$-minimality of y in Y.

"$i \Rightarrow i+1$": By the induction hypothesis $x \in \Psi_{[\prec_1,\dots,\prec_j]}(Y)$ for every $j \leq i$ and every $x \in \Psi_{[\prec_1,\dots,\prec_n]}(X)$. Let $x \in \Psi_{[\prec_1,\dots,\prec_n]}(X)$. Assume $x \notin \Psi_{[\prec_1,\dots,\prec_{i+1}]}(Y)$. Since $x \in \Psi_{[\prec_1,\dots,\prec_i]}(Y)$, there is a $y \in \min_{\prec_{i+1}}(\Psi_{[\prec_1,\dots,\prec_i]}(Y))$ such that $y \prec_{i+1} x$. By Fact E.15, $y \in \Psi_{[\prec_1,\dots,\prec_n]}(Y)$ and hence by "$\supseteq$", $y \in \Psi_{[\prec_1,\dots,\prec_n]}(X)$. Hence, $y \in \Psi_{[\prec_1,\dots,\prec_i]}(X)$. Since $y \prec_{i+1} x$ and $x \in \Psi_{[\prec_1,\dots,\prec_i]}(X)$, there is a $z \in \Psi_{[\prec_1,\dots,\prec_i]}(X)$ such that $z \prec_{i+1} y$ (otherwise $y \in \min_{\prec_{i+1}}(\Psi_{[\prec_1,\dots,\prec_i]}(X))$ and hence $x \notin \Psi_{[\prec_1,\dots,\prec_{i+1}]}(X)$). We know by Fact E.16 that there is a $z' \in \Psi_{[\prec_1,\dots,\prec_n]}(X)$ for which $z' \preceq_{i+1} z$ and hence $z' \prec_{i+1} y$. By the induction hypothesis, $z' \in \Psi_{[\prec_1,\dots,\prec_i]}(Y)$,—a contradiction to the $\prec_{i+1}$-minimality of y in $\Psi_{[\prec_1,\dots,\prec_i]}(Y)$. □

Fact E.18. *Where $\prec_2 \subseteq \prec_1$: If X is $\prec_2$-dense in Y, then it is $\prec_1$-dense in Y.*

Proof. Let $x \in Y$. Hence, there is a $y \in X$ for which $y \preceq_2 x$. Since $\prec_2 \subseteq \prec_1$, also $y \preceq_1 x$. □

Fact E.19. *$\Psi_{[\prec_1,\dots,\prec_n]}(X)$ is $\prec_1$-dense in X.*

Proof. Follows by Fact E.16 (let $j = n$). □

Fact 4.2. *(i) $\mathbf{C}_3^{\prec}$ implies $\mathbf{C}_1^{\prec}$; (ii) $\mathbf{C}_3^{\prec}$ implies $\mathbf{C}_4^{\prec}$; (iii) $\mathbf{C}_4^{\prec}$ implies $\mathbf{C}_2^{\prec}$; (iv) $\mathbf{C}_5^{\prec}$ implies* **C3***; (v)* **C5** *implies $\mathbf{C}_3^{\prec}$; (vi)* **C1** *implies $\mathbf{C}_1^{\prec}$; (vii)* **C6** *implies $\mathbf{C}_2^{\prec}$.*

Proof. Ad (i): $\mathbf{C}_1^{\prec}$ follows by Fact E.7. *Ad (ii):* Trivial. *Ad (iii):* $\mathbf{C}_2^{\prec}$ follows by Fact E.19. *Ad (iv):* This follows by Fact E.3 and Fact E.6. *Ad (v):* Let $\prec = \subset$. This follows by the definition of Υ and Fact E.5. *Ad (vi):* By Fact 4.1 **C1** implies **C2**. Now let $\prec = \subset$. *Ad (vii):* Let $\prec = \subset$. Due to the smoothness of $\langle X, \subset \rangle$, $\min_{\subset}(X)$ is $\subset$-dense in X. Hence, also $\Lambda(X) \supseteq \min_{\subset}(X)$ is $\subset$-dense in X. □

F Soundness and Completeness

Before we come to our central result concerning soundness and completeness (Theorem 4.10), we give a few more useful insights into $\Xi(\Gamma)$ and $\Xi^{\perp}(\Gamma)$. By Fact E.18, Lemma D.3, and since $\subset \subseteq \prec$,

Lemma F.1. *Where Γ is* **LLL***-non-trivial: $\Xi(\Gamma) \setminus \Xi^{\perp}(\Gamma)$ is $\prec$-dense in $\Xi(\Gamma)$*

And hence, by Fact D.1,

Lemma F.2. *Where Γ is* **LLL***-non-trivial: $\min_{\prec}(\Xi(\Gamma) \setminus \Xi^{\perp}(\Gamma)) = \min_{\prec}(\Xi(\Gamma))$*

By Theorem 4.8, Theorem 4.9, and Lemma 4.2 we have:

Corollary F.1. *Where $\Gamma \subseteq \mathcal{W}$: $\Xi(\Gamma) \in \Upsilon$, $\Xi(\Gamma) \setminus \Xi^{\perp}(\Gamma) \in \Upsilon$, and $\mathsf{Ab}^{\Gamma}_{\mathbf{LLL}} \in \Upsilon$.*

Lemma 4.3. *Where Γ is* **LLL***-non-trivial: If Λ satisfies* **C1** *or* **C2** *or $\mathbf{C}_1^{\prec}$, then $\Lambda(\Xi(\Gamma) \setminus \Xi^{\perp}(\Gamma)) = \Lambda(\Xi(\Gamma)) \setminus \Xi^{\perp}(\Gamma)$.*

Proof. Ad $\mathbf{C}_1^{\prec}$*:* this follows immediately with Lemma F.1. *Ad* **C1***:* This follows immediately with Lemma F.2 (where $\prec = \subset$) and **T1**. *Ad* **C2***:* This follows immediately with Lemma F.1 (where $\prec = \subset$). ☐

Lemma F.3. *Where* Γ *is* **LLL***-non-trivial and* $\langle \prec_1, \ldots, \prec_{n+1} \rangle$ *is an abstraction sequence:* $\Psi_{[\prec_1,\ldots,\prec_n]}(\Xi(\Gamma)) \setminus \Xi^{\perp}(\Gamma)$ *is* $\prec_{n+1}$*-dense in* $\Psi_{[\prec_1,\ldots,\prec_n]}(\Xi(\Gamma))$.

Proof. Let $\varphi \in \Psi_{[\prec_1,\ldots,\prec_n]}(\Xi(\Gamma)) \cap \Xi^{\perp}(\Gamma)$. By Lemma F.1, there is a $\psi \in \Xi(\Gamma) \setminus \Xi^{\perp}(\Gamma)$ for which $\psi \prec_{n+1} \varphi$. Since $\prec_{n+1} \subseteq \prec_n$, also $\psi \prec_n \varphi$. By Fact E.10, $\psi \in \Psi_{[\prec_1,\ldots,\prec_n]}(\Xi(\Gamma)) \setminus \Xi^{\perp}(\Gamma)$. ☐

Lemma 4.4. *Where* Γ *is* **LLL***-non-trivial:*

$$\Psi_{[\prec_1,\ldots,\prec_n]}(\Xi(\Gamma) \setminus \Xi^{\perp}(\Gamma)) = \Psi_{[\prec_1,\ldots,\prec_n]}(\Xi(\Gamma)) \setminus \Xi^{\perp}(\Gamma)$$

Proof. We show this by induction.

"$n = 1$": Since by Lemma F.1 $\Xi(\Gamma) \setminus \Xi^{\perp}(\Gamma)$ is $\prec_1$-dense in $\Xi(\Gamma)$, and by Fact E.7, $\Psi_{\prec_1}(\Xi(\Gamma) \setminus \Xi^{\perp}(\Gamma)) = \Psi_{\prec_1}(\Xi(\Gamma)) \cap (\Xi(\Gamma) \setminus \Xi^{\perp}(\Gamma)) = \Psi_{\prec_1}(\Xi(\Gamma)) \setminus \Xi^{\perp}(\Gamma)$.

"$n \Rightarrow n+1$": By the induction hypothesis, (†) $\Psi_{[\prec_1,\ldots,\prec_n]}(\Xi(\Gamma) \setminus \Xi^{\perp}(\Gamma)) = \Psi_{[\prec_1,\ldots,\prec_n]}(\Xi(\Gamma)) \setminus \Xi^{\perp}(\Gamma)$.

By Lemma F.3 and (†), $\Psi_{[\prec_1,\ldots,\prec_n]}(\Xi(\Gamma) \setminus \Xi^{\perp}(\Gamma))$ is $\prec_{n+1}$-dense in $\Psi_{[\prec_1,\ldots,\prec_n]}(\Xi(\Gamma))$. Hence, by Fact E.7,

$$\Psi_{[\prec_1,\ldots,\prec_{n+1}]}(\Xi(\Gamma) \setminus \Xi^{\perp}(\Gamma)) = \Psi_{[\prec_1,\ldots,\prec_{n+1}]}(\Xi(\Gamma)) \cap \Psi_{[\prec_1,\ldots,\prec_n]}(\Xi(\Gamma) \setminus \Xi^{\perp}(\Gamma)) \quad (6)$$

By (†), $$\Psi_{[\prec_1,\ldots,\prec_{n+1}]}(\Xi(\Gamma)) \cap \Psi_{[\prec_1,\ldots,\prec_n]}(\Xi(\Gamma) \setminus \Xi^{\perp}(\Gamma)) = \Psi_{[\prec_1,\ldots,\prec_{n+1}]}(\Xi(\Gamma)) \cap \left(\Psi_{[\prec_1,\ldots,\prec_n]}(\Xi(\Gamma)) \setminus \Xi^{\perp}(\Gamma)\right) \quad (7)$$

Since $\Psi_{[\prec_1,\ldots,\prec_{n+1}]}(\Xi(\Gamma)) \subseteq \Psi_{[\prec_1,\ldots,\prec_n]}(\Xi(\Gamma))$,

$$\Psi_{[\prec_1,\ldots,\prec_{n+1}]}(\Xi(\Gamma)) \cap \left(\Psi_{[\prec_1,\ldots,\prec_n]}(\Xi(\Gamma)) \setminus \Xi^{\perp}(\Gamma)\right) = \Psi_{[\prec_1,\ldots,\prec_{n+1}]}(\Xi(\Gamma)) \setminus \Xi^{\perp}(\Gamma) \quad (8)$$

By (6), (7), and (8),

$$\Psi_{[\prec_1,\ldots,\prec_{n+1}]}(\Xi(\Gamma) \setminus \Xi^{\perp}(\Gamma)) = \Psi_{[\prec_1,\ldots,\prec_{n+1}]}(\Xi(\Gamma)) \setminus \Xi^{\perp}(\Gamma)$$ ☐

Lemma 4.5. *If* $\langle \Xi(\Gamma), \prec \rangle$ *is smooth, then (i)* $\min_{\prec}(\Xi(\Gamma)) = \Psi_{\prec}(\Xi(\Gamma))$*, (ii)* $\min_{\prec}(\mathsf{Ab}^{\Gamma}_{\mathbf{LLL}}) = \Psi_{\prec}(\mathsf{Ab}^{\Gamma}_{\mathbf{LLL}})$*, (iii)* $\langle \mathsf{Ab}^{\Gamma}_{\mathbf{LLL}}, \prec \rangle$ *is smooth.*

Proof. Ad (i): trivial. *Ad (iii):* Where Γ is **LLL**-trivial, $\mathsf{Ab}^{\Gamma}_{\mathbf{LLL}} = \emptyset$ and there is nothing to show. Suppose Γ is **LLL**-non-trivial. Let $\varphi \in \mathsf{Ab}^{\Gamma}_{\mathbf{LLL}}$. By Lemma 4.2, $\varphi \in \Xi(\Gamma) \setminus \Xi^{\perp}(\Gamma)$. By the smoothness of $\langle \Xi(\Gamma), \prec \rangle$ there is a $\psi \preceq \varphi$ such that $\psi \in \min_{\prec}(\Xi(\Gamma))$. By Lemma F.2, $\psi \in \min_{\prec}(\Xi(\Gamma) \setminus \Xi^{\perp}(\Gamma))$ and hence by Lemma 4.2, $\psi \in \min_{\prec}(\mathsf{Ab}^{\Gamma}_{\mathbf{LLL}})$. *Ad (ii):* Given (iii) this is trivial. ☐

G Cumulativity and Reassurance

Lemma 4.6. *(i) ($\mathbf{C}_1^{\prec}$ and $\mathbf{C}_2^{\prec}$) imply* **CT** *and* **CM***; (ii) $\mathbf{C}_4^{\prec}$ implies* **CT** *and* **CM***; (iii) $\mathbf{C}_5^{\prec}$ implies* **CT**.

Proof. Let $\Lambda(X) \subseteq Y \subseteq X$. *Ad (i):* Since by $\mathbf{C}_2^{\prec}$ $\Lambda(X)$ is $\prec$-dense in X and $\Lambda(X) \subseteq Y$, also Y is $\prec$-dense in X. Hence, by $\mathbf{C}_1^{\prec}$ and **T1**, $\Lambda(Y) = \Lambda(X) \cap Y = \Lambda(X)$. *Ad (ii):* This follows by Fact E.17. *Ad (iii):* This follows by Fact E.13. □

Lemma 4.7. *Where $\Gamma, \Gamma' \subseteq \mathcal{W}$: $\Xi(\Gamma \cup \Gamma') \subseteq \Xi(\Gamma)$.*

Proof. Let $\varphi \in \Xi(\Gamma \cup \Gamma')$. Let $\Delta \in \Sigma(\Gamma)$. By the monotonicity of **LLL**, $\Delta \in \Sigma(\Gamma \cup \Gamma')$ for some $\Delta' \subseteq \Delta$. Thus, $\varphi \cap \Delta \neq \emptyset$. □

Lemma 4.8. *Where $\Gamma, \Gamma' \subseteq \mathcal{W}$ and $\Gamma' \subseteq Cn_{\mathbf{AL}_\Lambda}(\Gamma)$: $\Lambda(\Xi(\Gamma)) \subseteq \Xi(\Gamma \cup \Gamma')$.*

Proof. In case Γ is **LLL**-trivial, $\Xi(\Gamma) = \Xi(\Gamma \cup \Gamma') = \{\Omega\}$ and hence by **T1**, $\Lambda(\Xi(\Gamma)) \subseteq \Xi(\Gamma \cup \Gamma')$. Suppose now that Γ is not **LLL**-trivial.

Let $\Delta \in \Sigma(\Gamma \cup \Gamma')$. Hence, $\Gamma \cup \Gamma' \vdash_{\mathbf{LLL}} \mathsf{Dab}(\Delta)$. By the supposition and the reflexivity of $\mathbf{AL}_\Lambda$, $\Gamma \cup \Gamma' \subseteq Cn_{\mathbf{AL}_\Lambda}(\Gamma)$. Hence, by the monotonicity of **LLL**, $Cn_{\mathbf{AL}_\Lambda}(\Gamma) \vdash_{\mathbf{LLL}} \mathsf{Dab}(\Delta)$. By Theorem 4.4, $\mathsf{Dab}(\Delta) \in Cn_{\mathbf{AL}_\Lambda}(\Gamma)$. Let $\varphi \in \Lambda(\Xi(\Gamma))$. By Theorem 4.3, $\Gamma \cup (\Omega \setminus \varphi)^\neg \vdash_{\mathbf{LLL}} \mathsf{Dab}(\Delta)$. By Lemma D.1, $\Gamma \cup (\Omega \setminus \varphi)^\neg$ is **LLL**-non-trivial. Hence, $\Delta \cap \varphi \neq \emptyset$. Since Δ was arbitrary in $\Sigma(\Gamma \cup \Gamma')$, $\varphi \in \Xi(\Gamma \cup \Gamma')$. □

Theorem 4.12 (Cumulative Transitivity)**.** *Where Λ satisfies* **CT** *and $\Gamma, \Gamma' \subseteq \mathcal{W}$: If $\Gamma' \subseteq Cn_{\mathbf{AL}_\Lambda}(\Gamma)$ then $Cn_{\mathbf{AL}_\Lambda}(\Gamma \cup \Gamma') \subseteq Cn_{\mathbf{AL}_\Lambda}(\Gamma)$.*

Proof. By Lemma 4.7 and Lemma 4.8, $\Lambda(\Xi(\Gamma)) \subseteq \Xi(\Gamma \cup \Gamma') \subseteq \Xi(\Gamma)$. By **CT**, $\Lambda(\Xi(\Gamma)) \subseteq \Lambda(\Xi(\Gamma \cup \Gamma'))$. Suppose $\Gamma \cup \Gamma' \vdash_{\mathbf{AL}_\Lambda} A$. Hence, by Theorem 4.3, for all $\varphi \in \Lambda(\Xi(\Gamma \cup \Gamma'))$, $\Gamma \cup \Gamma' \cup (\Omega \setminus \varphi)^\neg \vdash_{\mathbf{LLL}} A$. Let $\varphi \in \Lambda(\Xi(\Gamma))$. Since $\Lambda(\Xi(\Gamma)) \subseteq \Lambda(\Xi(\Gamma \cup \Gamma'))$, $\Gamma \cup \Gamma' \cup (\Omega \setminus \varphi)^\neg \vdash_{\mathbf{LLL}} A$. By the compactness and the deduction theorem, there is a finite $\Theta \subseteq \Gamma'$ such that (1) $\Gamma \cup (\Omega \setminus \varphi)^\neg \vdash_{\mathbf{LLL}} \neg \bigwedge \Theta \vee A$. Since $\Gamma' \subseteq Cn_{\mathbf{AL}_\Lambda}(\Gamma)$, by Theorem 4.3, $\Gamma \cup (\Omega \setminus \varphi)^\neg \vdash_{\mathbf{LLL}} B$ for all $B \in \Gamma'$. Hence, (2) $\Gamma \cup (\Omega \setminus \varphi)^\neg \vdash_{\mathbf{LLL}} \bigwedge \Theta$. By (1), (2) and Modus Ponens, $\Gamma \cup (\Omega \setminus \varphi)^\neg \vdash_{\mathbf{LLL}} A$. Since φ was arbitrary in $\Lambda(\Xi(\Gamma))$, $\Gamma \vdash_{\mathbf{AL}_\Lambda} A$ by Theorem 4.3. □

Theorem 4.13. *Where $\Gamma \Vdash_{\mathbf{AL}_\Lambda} B$ for all $B \in \Gamma'$:*

(i) *if Λ satisfies* **CM** *then: $\Gamma \Vdash_{\mathbf{AL}_\Lambda} A$ implies $\Gamma \cup \Gamma' \Vdash_{\mathbf{AL}_\Lambda} A$.*

(ii) *if Λ satisfies* **CT** *then: $\Gamma \cup \Gamma' \Vdash_{\mathbf{AL}_\Lambda} A$ implies $\Gamma \Vdash_{\mathbf{AL}_\Lambda} A$.*

Proof. Since $\Gamma \Vdash_{\mathbf{AL}_\Lambda} B$ for all $B \in \Gamma'$, (1) $\mathcal{M}_{\mathbf{AL}_\Lambda}(\Gamma) \subseteq \mathcal{M}_{\mathbf{LLL}}(\Gamma \cup \Gamma')$ and hence $\Lambda(\mathsf{Ab}^{\Gamma}_{\mathbf{LLL}}) \subseteq \mathsf{Ab}^{\Gamma \cup \Gamma'}_{\mathbf{LLL}}$. Since by the monotonicity of **LLL**, $\mathcal{M}_{\mathbf{LLL}}(\Gamma \cup \Gamma') \subseteq \mathcal{M}_{\mathbf{LLL}}(\Gamma)$ also (2) $\mathsf{Ab}^{\Gamma \cup \Gamma'}_{\mathbf{LLL}} \subseteq \mathsf{Ab}^{\Gamma}_{\mathbf{LLL}}$.

Ad (i): By **CM**, (1) and (2): $\Lambda(\mathsf{Ab}^{\Gamma}_{\mathbf{LLL}}) \supseteq \Lambda(\mathsf{Ab}^{\Gamma \cup \Gamma'}_{\mathbf{LLL}})$ and hence $\mathcal{M}_{\mathbf{AL}_\Lambda}(\Gamma) \supseteq \mathcal{M}_{\mathbf{AL}_\Lambda}(\Gamma \cup \Gamma')$. The rest follows immediately.

Ad (ii): By **CT**, (1) and (2): $\Lambda(\mathsf{Ab}^{\Gamma}_{\mathbf{LLL}}) \subseteq \Lambda(\mathsf{Ab}^{\Gamma \cup \Gamma'}_{\mathbf{LLL}})$ and hence $\mathcal{M}_{\mathbf{AL}_\Lambda}(\Gamma) \subseteq \mathcal{M}_{\mathbf{AL}_\Lambda}(\Gamma \cup \Gamma')$. The rest follows immediately. □

Theorem 4.14. *Where* Λ *satisfies* **RA***: if* Γ *is* **LLL***-non-trivial then* Γ *is* $\mathbf{AL}_{\mathbf{\Lambda}}$*-non-trivial.*

Proof. Suppose Γ is **LLL**-non-trivial. Assume Γ is $\mathbf{AL}_{\mathbf{\Lambda}}$-trivial. Hence $\Gamma \vdash_{\mathbf{AL}_{\mathbf{\Lambda}}} A \wedge \neg A$. By **RA**, $\Lambda(\Xi(\Gamma)) \neq \emptyset$. Let $\varphi \in \Lambda(\Xi(\Gamma))$. By Theorem 4.3, $\Gamma \cup (\Omega \setminus \varphi)^{\neg} \vdash_{\mathbf{LLL}} A \wedge \neg A$. This is a contradiction by Lemma D.1. □

H Superimposing ¬ and ∨

Suppose the reflexive, monotonic, transitive, and compact logic **L** does not have a classical $\neg$ and a classical $\vee$ and neither are they definable in **L**. In this case we can superimpose these connectives on **L** in order to transform **L** into a lower limit logic. Where $\mathcal{W}'$ are the well-formed formulas of **L** we define $\mathcal{W}$ by the following grammar:

$$\mathcal{W} := \langle \mathcal{W}' \rangle \mid \neg\langle \mathcal{W} \rangle \mid \langle \mathcal{W} \rangle \vee \langle \mathcal{W} \rangle$$

We enrich the definition of truth in a **L**-model M as follows:

- $M \models \neg A$ iff $M \not\models A$
- $M \models A \vee B$ iff $M \models A$ or $M \models B$

Axiomatically the procedure is similar.

In the standard format for ALs it is necessary to superimpose a classical negation $\neg$ and a classical disjunction $\vee$ even in cases in which the lower limit logic has already a classical negation and a classical disjunction in its object language. Moreover, (most parts of) the meta-theory is restricted to premise and (sometimes also) consequence sets without the superimposed connectives. One reason is as follows: it has been shown that, given $\neg$ and $\vee$ are not superimposed on the language of **LLL**, there are premise sets for which an adaptive semantic consequence A is not finally admissible, i.e., whenever A is derived at a finite stage of the proof it is already marked and can only be unmarked in an infinite extension of the proof (this is **I1** in Section 3.2.4).[40]

Of course, in some application contexts one may simply not be interested in premises and consequences with occurrences of the superimposed connectives which motivates the respective restrictions on the premise and consequence sets.

In our format there is no need to superimpose $\neg$ and $\vee$ in cases in which the lower limit logic already has a classical negation and a classical disjunction, or in cases in which these connectives are definable. The reason is simply that we bite the bullet: as discussed in Section 3.2.4 **I1** cannot be avoided for all premise sets. Note though, **I1** can be avoided in our format exactly in the same way as in the standard format: by restricting the premise set to formulas without the classical disjunction that is used in order to express Dab-formulas.

[40]This has been first noted by Batens and Verdée and is documented e.g., in [33], Chapter 2.8 for minimal abnormality and in FRE-REF for reliability. Due to space restrictions we do not reproduce the rather technical examples here.

The possibility of formulating ALs without the need of superimposing $\neg$ and $\vee$ in case they are definable in the lower limit logic offers the following advantages: (i) logical connectives need not be replicated, (ii) we need not tinker around with enriching the lower limit logic by adding new connectives. It should also be remarked that the problem **I1** is a rather artificial one: for most premise sets Γ in practical applications finally admissible formulas will be admissible at a finite stage of any dynamic proof from Γ.

One may argue that this leads to a piecemeal approach in which, for every specific lower limit logic we have to decide whether to add superimposed connectives and if so, which. For those who mind that we recommend to just adopt the approach of the standard format and always superimpose all classical connectives: there is nothing which prevents this approach in the format presented in this paper. For all others, who may be interested in minimizing the used language in view of conciseness (and there may be different ways to do so): in the format presented in this paper they have the freedom to do so.

Finally note that we need no restrictions on premise sets for our meta-theoretic results (such as in the standard format where many meta-theoretic insights only apply to the fragment of the language without the superimposed connectives): our meta-theory applies to the full language (irrespective of whether and which classical connectives are superimposed on the language).

One may be worried that since our meta-theory is not formulated for the restricted language such as in the standard format, this means that many ALs in the standard format cannot be represented in our format. In Appendix J we show that there is no reason to worry: all consequence relations of ALs in the standard format are representable in our format.

I Final Derivability in the Standard Format

Final derivability in the standard format of adaptive logics is defined as follows:

Definition I.1. A is *finally admissible*[41] *at a line* l at a finite stage s iff (i) A is derived at line l at stage s, (ii) for every further extension of the proof to a stage in which line l is marked there is yet another extension in which line l is unmarked.

One may understand this definition in terms of a two-person-game (see e.g., [5]). The proponent produces A in a finite proof at stage s at line l. The opponent is now allowed to extend to proof in any way in order to mark line l. Then our proponent is allowed to response and extend the proof once more. She has a winning strategy in case she is able to extend any extension by the opponent in such a way that l is unmarked.

It has been pointed out that, at least for the minimal abnormality strategy, this definition is only adequate in case we allow for both(!) extensions to be infinite (see e.g., [5, 4]). Thus, we quantify over all infinite extensions of a given proof,

[41] In the nomenclature of the standard format it is called "finally derived". See Footnote 23.

and then even over all infinite extensions of former infinite extensions. Note that we need here a notion of extending an infinite proof. Either we allow proofs to be represented by transfinite lists, or we allow for the insertion of lines at a point when the proof already consists of infinitely many lines. Hence, we speak about a derivation step which is performed after infinitely many derivation steps have been performed. Hence, even if the list of lines is not transfinite, the sequence of derivation steps is. At this point the comparison with the 2-player-game gets odd since the players cannot produce their arguments/moves in the game in a concrete way since the latter may be infinite objects.

In contrast, proofs in our paper are finitary and linear: new lines are always added at the end of the proof. We see proofs as products of object-level reasoning and as such they are entities that can concretely be constructed. Hence, each derivation step is performed at a finite point. The only point where infinity enters the picture in our account is when we check whether a line is marked in an A-complete extension of a dynamic proof in item (ii) of our definition for final admissibility. The A-complete stage may be infinite. Note, that in this case it does not constitute a dynamic proof since the latter is by definition a list of derivation steps. One may think about the corresponding list of lines as a limit case of a proof:[42] after all each initial sub-sequence is finite and hence corresponds to a proper proof and each derivation step takes place at a finite point (opposite to the transfinite proofs that are called for in Definition I.1).

Altogether, we identify dynamic proofs themselves with object-level reasoning, while the definition for final admissibility calls also for meta-level reasoning (such as in item (ii) of our definition). In the definition of final admissibility in the standard format there seems to be no clear conceptual demarcation line between dynamic proofs and meta-level reasoning since dynamic proofs may themselves be infinite entities and even the product of a transfinite sequence of steps which of course cannot be constructed by the object-level reasoning of finite beings. It is due to this conceptual points that we changed the definition of final admissibility.

Of course, an A-complete stage may often be reached at a finite stage and hence in a dynamic proof. However, to check whether the given stage is indeed A-complete requires meta-level reasoning. Hence, on both approaches meta-level reasoning is unavoidable when establishing final admissibility.

Note finally that we could have also worked with the standard format's definition of final admissibility. Like in the standard format we would allow for derivation steps (by insertion of lines) to be performed after an infinite sequence of steps has been performed before and would hence enhance the notion of (extended) dynamic proofs to lists of lines that are the product of infinite and even transfinite sequence of steps. Indeed, the meta-theoretic results would be equivalent as can easily be demonstrated.

Definition I.2. A is *finally admissible* at a finite stage s iff (i) A is derived at some line l at stage s, and (ii) every extension of stage s to a stage s' in which l is marked

[42] While we would not call this limit case a proof, others may have no objections to that.

there is yet another extension s'' of s' in which l is unmarked.

Theorem I.1. *Where $\mathcal{P}$ is a dynamic proof and s a finite stage of this proof: A is finally admissible at stage s with respect to Definition 3.8 iff A is finally admissible at stage s with respect to Definition I.2.*

Proof. "$\Rightarrow$": Let $\mathcal{P}_\omega$ be an A-complete extension of the empty proof consisting of the lines $l_1^\omega, l_2^\omega, \ldots$. Suppose l is marked in some extension of stage s to stage s'. 1. case: s' is a finite stage. We append lines $l_1^\omega, l_2^\omega, \ldots$ to the finite proof. This results in an A-complete stage and hence line l is unmarked. 2. case: s' is infinite and consists of the lines $l_1, l_2, \ldots$. We now extend stage s' by inserting lines as follows: $l_1, l_1^\omega, l_2, l_2^\omega, \ldots$. This results in an A-complete proof and hence line l is unmarked.

"$\Leftarrow$": We extend the proof $\mathcal{P}$ to an A-complete stage. Assume line l is marked. Since any further extension of the proof is also an A-complete stage, by Fact A.1, line l is marked in any further extension,—a contradiction to the final admissibility of A on line l according to Definition I.2. $\square$

J A Representation Theorem for ALs in the Standard Format

Where **LLL** is a reflexive, transitive, monotonic and compact logic, let $\mathbf{LLL}^+$ be the result of superimposing classical connectives on the set of wffs $\mathcal{W}$ of **LLL** (at least a disjunction $\check{\vee}$ and a negation $\check{\neg}$).[43] The consequence relations of an AL in the standard format is usually taken to be $Cn^{\mathcal{W}}_{\mathbf{AL}}(\Gamma) =_{\rm df} \{A \in \mathcal{W} \mid \Gamma \vdash_{\mathbf{AL}} A\}$ where $\Gamma \subseteq \mathcal{W}$. The following representation theorem is well-known for the adaptive logics $\mathbf{AL^m}$ characterized by the minimal abnormality strategy, the lower limit logic **LLL**, and the set of abnormalities Ω (see [4]):

Theorem J.1. *Where $\Gamma \subseteq \mathcal{W}$: $A \in Cn^{\mathcal{W}}_{\mathbf{AL^m}}(\Gamma)$ iff for every $\varphi \in \min_{\subset}(\Xi_{\mathbf{LLL}^+}(\Gamma))$, $\Gamma \cup (\Omega \setminus \varphi)^{\check{\neg}} \vdash_{\mathbf{LLL}^+} A$, where $\Xi_{\mathbf{LLL}^+(\Gamma)}$ is the set of choice sets over $\Sigma_{\mathbf{LLL}^+}(\Gamma) = \{\Delta \mid \Gamma \vdash_{\mathbf{LLL}^+} \mathsf{Dab}(\Delta)$ and for all $\Delta' \subset \Delta, \Gamma \nvdash_{\mathbf{LLL}^+} \mathsf{Dab}(\Delta')\}$.*

Let $\mathbf{AL}_{\min_\subset}$ be characterized by the triple $\langle \mathbf{LLL}^+, \Omega, \min_\subset \rangle$ where Dab-formulas are expressed by means of $\check{\vee}$. The following Corollary follows immediately by Theorem 4.3 and Theorem J.1:

Corollary J.1. *Where $\Gamma \subseteq \mathcal{W}$: $Cn^{\mathcal{W}}_{\mathbf{AL^m}}(\Gamma) = Cn_{\mathbf{AL}_{\min_\subset}}(\Gamma) \cap \mathcal{W}$.*

The following representation theorem is well-known for the adaptive logics $\mathbf{AL^r}$ characterized by the reliability strategy, the lower limit logic **LLL**, and the set of abnormalities Ω (see [4]):

Theorem J.2. *Where $\Gamma \subseteq \mathcal{W}$: $A \in Cn^{\mathcal{W}}_{\mathbf{AL^r}}(\Gamma)$ iff there is a $\Delta \subseteq \Omega$ such that $\Gamma \vdash_{\mathbf{LLL}^+} A \check{\vee} \mathsf{Dab}(\Delta)$ and $\Delta \cap \bigcup \min_\subset(\Xi_{\mathbf{LLL}^+}(\Gamma)) = \emptyset$.*

[43] Compare Appendix H.

Let $\mathbf{AL}_{\mathbf{\Lambda_U}}$ be characterized by the triple $\langle \mathbf{LLL}^+, \Omega, \Lambda_U \rangle$ where Λ_U is defined as in Section 5.2.2 and Dab-formulas are expressed by means of $\check{\vee}$.

Corollary J.2. *Where* $\Gamma \subseteq \mathcal{W}$*:* $Cn^{\mathcal{W}}_{\mathbf{AL^r}}(\Gamma) = Cn_{\mathbf{AL}_{\mathbf{\Lambda_U}}}(\Gamma) \cap \mathcal{W}$.

Proof. Let $A \in Cn^{\mathcal{W}}_{\mathbf{AL^r}}(\Gamma)$. Thus, there is a $\Delta \subseteq \Omega$, such that $\Delta \cap \bigcup \min_{\subset}(\Xi_{\mathbf{LLL}^+}(\Gamma)) = \emptyset$ and $\Gamma \vdash_{\mathbf{LLL}^+} A \check{\vee} \mathsf{Dab}(\Delta)$. Hence, for all $\varphi \in \Lambda_U(\Xi_{\mathbf{LLL}^+}(\Gamma))$, $\Delta \cap \varphi = \emptyset$. Hence $A \in Cn_{\mathbf{AL}_{\mathbf{\Lambda_U}}}(\Gamma) \cap \mathcal{W}$ by Theorem 4.3.

Suppose $A \notin Cn^{\mathcal{W}}_{\mathbf{AL^r}}(\Gamma)$. Hence, there is no $\Delta \subseteq \Omega$ such that $\Gamma \vdash_{\mathbf{LLL}^+} A \check{\vee} \mathsf{Dab}(\Delta)$ and $\Delta \cap \bigcup \min_{\subset}(\Xi_{\mathbf{LLL}^+}(\Gamma)) = \emptyset$. Hence, (1) for all $\Delta \subseteq \Omega$ for which $\Gamma \vdash_{\mathbf{LLL}^+} A \check{\vee} \mathsf{Dab}(\Delta)$, $\Delta \cap \bigcup \min_{\subset}(\Xi_{\mathbf{LLL}^+}(\Gamma)) \neq \emptyset$. Note that (2) $\bigcup \min_{\subset}(\Xi_{\mathbf{LLL}^+}(\Gamma)) \in \Lambda_U(\Xi_{\mathbf{LLL}^+}(\Gamma))$. By (1), (2) and Theorem 4.3, $A \notin Cn_{\mathbf{AL}_{\mathbf{\Lambda_U}}}(\Gamma) \cap \mathcal{W}$. □

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