Annotated Natural Deduction for Adaptive Reasoning

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Abstract

We present a multi-conclusion natural deduction calculus based on minimal logic extended with a set of rules characterizing the dynamic reasoning typical of adaptive logics.

1 Intro

We present a multi-conclusion natural deduction calculus that mimics the dynamic reasoning at work in adaptive logics ([1]). This is the first attempt to reconstruct the dynamics typical of adaptive logics in a natural deduction setting. The resulting system does not correspond to the usual structure known as the Standard Format for Adaptive Logics: this means that, though we *do not* introduce an adaptive logic proper, we can talk of a natural deduction system for *adaptive reasoning*. We characterize such a way of reasoning as having properties that identify adaptive dynamics. To do so, the standard proof-theoretical procedure of a natural deduction system is enhanced with:

1. a rule-based ability of introducing abnormal formulas of the form $A \wedge \neg A$;[1] the appearance of such formulas on the right-hand side of our derivability sign justifies the claim that our system is extended to a multi-conclusion setting;
2. a rule-based ability of deriving formulas under conditions that some such abnormal formula is not true;
3. the procedural ability of rejecting derivation steps previously obtained by way of marking in view of effectively derived abnormal formulas.

[1] In the current format we focus on inconsistency-adaptive logics, though the generalization to the natural deduction for other adaptive formats seems possible.

These are all properties inspired by the adaptive logics approach. In view of the last property, we need moreover to annotate the derivability relation with a stage counting mechanism to keep track of the steps performed in the derivation tree (thus counting also premises rather than only rules).

Our system is not even close to a standard format for AL. We express the standard triple $\{LLL, \Omega, STRATEGY\}$ in a system where the Lower Limit Logic is extended to include rules both for expressing the abnormal formulas in Ω and to interpret the selection Strategy. In other words, this rule-based approach allows to merge the rules and axioms of a typical LLL and the rules of the AL based on abnormal formulas into a single system of rules.

2 minimalND

We start by defining the type universe for the $\{\neg, \rightarrow, \wedge, \vee\}$ fragment of intuitionistic propositional logic corresponding to minimal logic. We call this logic minimalND and use it as the equivalent of a Lower Limit Logic: we do not explicitly formulate a rule to abort derivations once a proposition of type $\bot$ is derived, as it would be standard in an intuitionistic setting. Instead, we allow a contradiction elimination that corresponds to *ex falso quodlibet*. It will be the role of the adaptive machinery introduced in the next sections to establish how to remove such contradictions.

We start defining the syntax of our language:

Definition 1 (minimalND)**.** *Our starting language for* minimalND *is defined by the following grammar:*

$$\begin{aligned}
&\mathsf{Type} := \mathsf{Prop}\\
&\mathsf{Prop} := A|\bot|\neg\phi \mid \phi_1 \rightarrow \phi_2|\phi_1 \wedge \phi_2|\phi_1 \vee \phi_2\\
&\Gamma := \{\phi_1, \dots, \phi_n\}\\
&\Delta := \{\phi_1, \dots, \phi_n\}
\end{aligned}$$

The type universe of reference is the set of propostions Prop, construed by atomic formulas closed under negation, implication, conjunction, disjunction and allowing $\bot$ to express contradictions. Formula formation rules are given in Figure 1.

Definition 2 (Judgements)**.** *A* minimalND*-judgement is of the form* $\Gamma; \cdot \vdash_{\mathsf{s}} \Delta$*, where:* Γ *is the usual set of assumptions,* Δ *is a set of formulas of the language and* s *is a positive integer.*

The set Γ on the left-hand side of the derivability sign is to be read conjunctively. Similarly for the semi-colon symbol, which is introduced here but is only used in Section 3 to separate standard assumptions in Γ from conditions (in the adaptive sense). The set Δ and the possible comma on the right-hand side of the derivability sign are both to be read disjunctively. This characterize our calculus as multi-conclusion. Context formation rules, for both left and right-hand side set of formulas are given in Figure 2. Nil establishes the base case of a valid

$$\frac{}{A \in \mathsf{Prop}}\ \textsc{Atom} \qquad \frac{}{\bot \in \mathsf{Prop}}\ \bot \qquad \frac{\phi \in \mathsf{Prop}}{\neg\phi \in \mathsf{Prop}}\ \neg$$

$$\frac{\phi_1 \in \mathsf{Prop} \qquad \phi_2 \in \mathsf{Prop}}{\phi_1 \to \phi_2 \in \mathsf{Prop}}\ \to \qquad \frac{\phi_1 \in \mathsf{Prop} \qquad \phi_2 \in \mathsf{Prop}}{\phi_1 \wedge \phi_2 \in \mathsf{Prop}}\ \wedge$$

$$\frac{\phi_1 \in \mathsf{Prop} \qquad \phi_2 \in \mathsf{Prop}}{\phi_1 \vee \phi_2 \in \mathsf{Prop}}\ \vee$$

Figure 1: Formula Formation Rules

empty context, we use wf as an abbreviation for 'well-formed'; Γ-$\mathsf{Formation}$ allows extension of contexts by propositions; Prem establishes derivability of formulas contained in context (and it defines the equivalent of the adaptive Premise rule); finally, Δ-$\mathsf{Formation}$ allows *disjunctive* extension of derived sets of formulas by well-typed ones.

$$\frac{}{\cdot \vdash_{\mathsf{s}} \mathsf{wf}}\ \textsc{Nil} \qquad \frac{\Gamma; \cdot \vdash_{\mathsf{s}} \mathsf{wf} \qquad \phi \in \mathsf{Prop}}{\Gamma, \phi; \cdot \vdash_{\mathsf{s+1}} \mathsf{wf}}\ \Gamma\textsc{-formation}$$

$$\frac{\Gamma; \cdot \vdash_{\mathsf{s}} \mathsf{wf} \qquad \phi \in \Gamma}{\Gamma; \cdot \vdash_{\mathsf{s+1}} \phi}\ \textsc{Prem} \qquad \frac{\Gamma; \cdot \vdash_{\mathsf{s}} \Delta \qquad \phi \in \mathsf{Prop}}{\Gamma; \cdot \vdash_{\mathsf{s+1}} \Delta, \phi}\ \Delta\textsc{-formation}$$

Figure 2: Context Formation Rules

The derivability sign is enhanced with a signature s that corresponds to a counter of the ordered derivation steps executed to obtain the corresponding ND-formula in a tree. This annotation only comes to use in the next extension of the calculus in Section 3.

The semantics of connectives is given in the standard proof-theoretical way by way of Introduction and Elimination Rules in Figure 3. Introduction of $\to$ corresponds to the Deduction Theorem, while its elimination formalises Modus Ponens. Rules for $\wedge$ are standard; notice that $\vee$-Elimination amounts to formation of a multi-conclusion judgement. $\neg$-Elimination is *ex falso*, while its Introduction is modelled by the translation of positive formulas on the left-hand side of the derivability sign, to their negative counterpart on the right-hand side.

Finally, we introduce a set of rules to enforce structural properties in Figure 4. Wleft is a Weakening on the left-hand side of the judgement: it allows the monotonic extension of assumptions preserving already derivable formulas. Notice that this rule can only work with a strictly empty set of formulas $;\cdot$

$$\frac{\Gamma, \phi_1; \cdot \vdash_{\mathsf{s}} \phi_2}{\Gamma; \cdot \vdash_{\mathsf{s+1}} \phi_1 \rightarrow \phi_2} \rightarrow\mathrm{I} \qquad \frac{\Gamma; \cdot \vdash_{\mathsf{s}} \phi_1 \rightarrow \phi_2 \qquad \Gamma'; \cdot \vdash_{\mathsf{s'}} \phi_1}{\Gamma; \Gamma' \vdash_{\mathsf{max(s,s')+1}} \phi_2} \rightarrow\mathrm{E}$$

$$\frac{\Gamma; \cdot \vdash_{\mathsf{s}} \phi_1 \qquad \Gamma'; \cdot \vdash_{\mathsf{s'}} \phi_2}{\Gamma, \Gamma'; \cdot \vdash_{\mathsf{max(s,s')+1}} \phi_1 \wedge \phi_2} \wedge\mathrm{I} \qquad \frac{\Gamma; \cdot \vdash_{\mathsf{s}} \phi_1 \wedge \phi_2}{\Gamma; \cdot \vdash_{\mathsf{s+1}} \phi_{i \in \{1,2\}}} \wedge\mathrm{E}$$

$$\frac{\Gamma; \cdot \vdash_{\mathsf{s}} \phi_1}{\Gamma; \cdot \vdash_{\mathsf{s+1}} \phi_1 \vee \phi_2} \vee\mathrm{I} \qquad \frac{\Gamma; \cdot \vdash_{\mathsf{s}} \phi_2}{\Gamma; \cdot \vdash_{\mathsf{s+1}} \phi_1 \vee \phi_2} \vee\mathrm{I} \qquad \frac{\Gamma; \cdot \vdash_{\mathsf{s}} \phi_1 \vee \phi_2}{\Gamma; \cdot \vdash_{\mathsf{s+1}} \phi_1, \phi_2} \vee\mathrm{E}$$

$$\frac{\Gamma; \cdot \vdash_{\mathsf{s}} \bot}{\Gamma; \cdot \vdash_{\mathsf{s}} \phi} \bot\mathrm{E} \qquad \frac{\Gamma; \phi \vdash_{\mathsf{s}} \psi}{\Gamma; \cdot \vdash_{\mathsf{s+1}} \psi, \neg\phi} \neg\mathrm{I}$$

Figure 3: Rules for I/E of connectives

following Γ: we shall introduce in the next section this as the set of *adaptive conditions*. The reason for this requirement in Wleft is that the set of adaptive conditions strictly depends on the set of assumptions Γ, hence a Weakening of the latter can imply a different formulation of the former. We do not need to formulate a Wright rule for weakening of the set Δ of derivable formulas, as this can be obtained by a detour of $\vee$-Introduction and Elimination. Cleft for Contraction on the left allows elimination of repeted assumptions and Eleft is valid just by set construction, as there is no order. Cright and Eright do a similar job on the right-hand side of the judgement. Finally, Cut (also known as Substitution in some Natural Deduction Caluli) guarantees that derivations can be pasted together, and it in generals requires that there are no clashes of free variables in Γ, Γ'.

$$\frac{\Gamma; \cdot \vdash_{\mathsf{s}} \phi_1}{\Gamma, \phi_2; \cdot \vdash_{\mathsf{s+1}} \phi_1} \textsc{Wleft} \qquad \frac{\Gamma, \phi_1, \phi_1; \cdot \vdash_{\mathsf{s}} \phi_2}{\Gamma, \phi_1; \cdot \vdash_{\mathsf{s+1}} \phi_2} \textsc{Cleft} \qquad \frac{\Gamma, \phi_1, \phi_2; \cdot \vdash_{\mathsf{s}} \phi_3}{\Gamma, \phi_2, \phi_1; \cdot \vdash_{\mathsf{s+1}} \phi_3} \textsc{Eleft}$$

$$\frac{\Gamma; \cdot \vdash_{\mathsf{s}} \phi_1 \qquad \Gamma', \phi_1; \cdot \vdash_{\mathsf{s'}} \phi_2}{\Gamma; \Gamma'; \cdot \vdash_{\mathsf{max(s,s')+1}} \phi_2} \textsc{Cut}$$

$$\frac{\Gamma; \cdot \vdash_{\mathsf{s}} \phi, \phi}{\Gamma; \cdot \vdash_{\mathsf{s+1}} \phi} \textsc{Cright} \qquad \frac{\Gamma; \cdot \vdash_{\mathsf{s}} \phi_1, \phi_2}{\Gamma; \cdot \vdash_{\mathsf{s+1}} \phi_2, \phi_1} \textsc{Eright}$$

Figure 4: Structural Rules

3 AdaptiveND

We now extend minimalND to characterize a new logic called AdaptiveND to allow for inconsistency adaptive reasoning. To this aim one needs:

1. the explicit formulation of an Ω set of propositions of type $\bot$;
2. the formulation of judgements including an *adaptive condition*;
3. the formulation of a rule that allows to derive new formulas independent from such an adaptive condition;
4. the formulation of a rule that allows to derive new formulas dependently from such an adaptive condition.

We offer accordingly new definitions for the syntax of this new logic and the related form of judgement.

Definition 3 (AdaptiveND)**.** *The language of* AdaptiveND *is as follows:*

$$\begin{array}{l}
\mathsf{Type} := \mathsf{Prop} \\
\mathsf{Prop} := A|\bot|\neg\phi|\phi_1 \to \phi_2|\phi_1 \wedge \phi_2|\phi_1 \vee \phi_2 \\
\Gamma := \{\phi_1, \dots, \phi_n\} \\
\Delta := \{\phi_1, \dots, \phi_n\} \\
\Omega := \{\phi \wedge \neg\phi \mid \phi \in Prop\}
\end{array}$$

Definition 4 (Judgements)**.** *An* AdaptiveND*-judgement is of the form* $\Gamma; \phi^- \vdash_s \Delta$*, where:*

1. *the left-hand side of* $\vdash_\mathsf{s}$ *has* Γ *as in* minimalND*;*
2. *the semicolon sign on the left-hand side of* $\vdash_\mathsf{s}$ *is conjunctive;*
3. ϕ *refers to a formula in* Ω*, i.e. with a specific inconsistent logical form; below we introduce an appropriate* Ω*-formation rule;*[2]
4. *the last place of the left-hand side context is always reserved to negated formulas of the* Ω *form; we shall use* ϕ^- *to refer to the negation of* ϕ*, for all* $\phi \in \Omega$*;*
5. *the right-hand side is in disjunctive form.*

When Ω is empty on the left-hand side of $\vdash$, we shall write $\Gamma; \cdot \vdash$, thus reducing to the form of a minimalND-formula. Moreover, in AdaptiveND, the annotation on the proof stage s is optionally followed by one of the following two marks: $\boxtimes$ to mark that at the current stage some previously derived formula is retracted; $\checkmark$ to mark that at the current stage some previously derived formulas

[2]As mentioned above, the current setting of AdaptiveND is specified for an inconsistency-adaptive logic.

is now stable, i.e. will no longer marked by $\boxtimes$. These symbols will be formally introduced in Sections 4 and 5 respectively.

We now introduce the rules for $\mathsf{AdaptiveND}$. In Figure 5, we illustrate the formation and use of formulas $\phi \in \Omega$. By Ω-$\mathsf{Formation}$, given a proposition ϕ (possibly non-atomic), a contradiction with $\neg\phi$ is a formula of the Ω type; following the Adaptive tradition ϕ can also be called an *abnormality* or an *abnormal fomrula*. By $\mathsf{Adaptive\ Condition\ Formation}$, given a valid context Γ and a formula ϕ of the Ω type, a context Γ followed by the adaptive Condition stating that ϕ *is false*, is awell-formed context. This corresponds to the use of conditions as additional elements of proof line in the standard Adaptive proof theort (Fitch-style).

$$\frac{\phi \in \mathsf{Prop}}{(\phi \wedge \neg\phi) \in \Omega}\ \Omega\text{-FORMATION}$$

$$\frac{\Gamma; \cdot \vdash_{\mathsf{s}} \mathsf{wf} \qquad \phi \in \Omega}{\Gamma; \phi^{-} \vdash_{\mathsf{s+1}} \mathsf{wf}}\ \text{ADAPTIVE CONDITION-FORMATION}$$

Figure 5: Ω Formation rules

Next, the calculus is extended by introducing two rules, see Figure 6. *RU* is called the unconditional rule: it says that if a formula ϕ_1 is derivable in $\mathsf{AdaptiveND}$, and another formula ϕ_2 is derivable from ϕ_1 without additional assumptions or adaptive conditions, then ϕ_2 is derivable from ϕ_1 and the context $\Gamma; \phi^{-} \vdash$ in which the latter holds. *RC* is called the conditional rule: it says that if a disjunction ψ, ϕ is derivable from Γ, with ϕ an abnormal formula, then ψ can also be derived alone under Γ and the Adaptive Condition that ϕ be false.

$$\frac{\Gamma; \phi^{-} \vdash_{\mathsf{s}} \phi_1 \qquad \phi_1; \cdot \vdash_{\mathsf{s+1}} \phi_2}{\Gamma, \phi_1; \phi^{-} \vdash_{\mathsf{s+2}} \phi_2}\ \mathrm{RU} \qquad\qquad \frac{\Gamma; \cdot \vdash_{\mathsf{s}} \psi, \phi \qquad \phi \in \Omega}{\Gamma; \phi^{-} \vdash_{\mathsf{s+1}} \psi}\ \mathrm{RC}$$

Figure 6: Adaptive Rules

The Adaptive strategy developed in the next Section has the aim of establishing which abnormal formulas can no longer be safely considered as conclusions of an $\mathsf{Adaptive\ Condition\ Formation}$ Rule, thereby requiring a retraction of the formulas that are derivable from it. To this aim, it is essential to establish minimal disjunctions of such formulas, denoted by $\bigvee(\Delta^{min})$, with $\Delta \in \Omega$. The rule in Figure 7 establishes the construction of such minimal disjunctions. It says that given a derivable disjunctive formula of the Ω type at some stage s of a derivation, that can be considered minimal at stage s' if at no previous stage $\mathsf{t} < \mathsf{s}'$ a shorter one can be derived in the same context Γ.

$$\frac{\Gamma; \cdot \vdash_{\mathsf{s}} \Delta \qquad \Delta \subset \Omega \qquad \text{with no } \Delta' \subseteq \Delta \in \Omega, \text{ s.t. } \Gamma; \cdot \vdash_{\mathsf{t<s'}} \Delta'}{\Gamma; \cdot \vdash_{\mathsf{s'}} \Delta^{min}} \textsc{MinDab}$$

Figure 7: Minimal Abnormal Formulas Rule

The derivation of minimal disjunction of abnormalities is a process that occurs along with the development of the proof-tree. This means that the following procedure to mark formulas depend on the possible derivation of certain such formulas.

3.1 A simple example

We present here a simple derivation in AdaptiveND, where $\Gamma = \{(\neg p \vee q), p, (p \rightarrow q)\}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma; \cdot \vdash_{\mathsf{1}} (\neg p \vee q)}\text{PREM}}{\Gamma; \cdot \vdash_{\mathsf{2}} \neg p, q}\vee\text{E} \qquad \dfrac{}{\Gamma; \cdot \vdash_{\mathsf{3}} p}\text{PREM}}{\Gamma; \cdot \vdash_{\mathsf{4}} (p \wedge \neg p), q}\wedge\text{I} \qquad (p \wedge \neg p) \in \Omega}{\Gamma; (p \wedge \neg p)^{-} \vdash_{\mathsf{5}} q}\text{RC}$$

The derivation above up to stage 4 is a obtained by MinimalND rules. Stage 5 derives a formula on condition of the abnormality $(p \wedge \neg p)$ being false. This corresponds to changing a multiple conclusion judgement at stage 4 into a single conclusion one at stage 5 by turning one of the conclusions into an adaptive condition. This move is justified by the syntactical form of the abnormality declared by the RC rule.

4 Rules for Marking

In standard Adaptive Logics, one introduces strategies to tell, given some judgement deriving a Minimal Disjunction of Abnormalities, which one of the disjunct can be assumed to be false, i.e. for which one a RC rule can be applied; and which one has to be accepted. Accordingly, formulas derived under the former can be considered valid, formulas previously derived by assuming the latter false have to be retracted. Adaptive Logics come with marking mechanisms that allow such retractions, according to different possible strategies. The most well-known strategies and their rationale are:[3]

[3] Some reference here?

- *Reliability*: once a $\bigvee(\Delta^{min})$ is derived at some stage s, *every* formula assuming at some stage $\mathsf{s\text{-}i}$ a $\phi \in \bigvee(\Delta^{min})$ at s to be false, needs to be retracted;

- *Minimal Abnormality*: once a $\bigvee(\Delta^{min})$ is derived at some stage s, *every* formula assuming at some stage $\mathsf{s\text{-}i}$ a $\phi \in \bigvee(\Delta^{min})$ at s to be false and such that ϕ is in a minimal set of Δ, needs to be retracted.

In the first case, one considers all possible abnormal formulas to be invalid; in the second case, one tries to minimize the number of such unavoidable contradictions. In this section, we provide rules that extend AdaptiveND in view of the Reliability strategy, providing a proof-theoretical equivalent of the standard marking condition. We leave the definition of a proof-theoretical Minimal Abnormality strategy to a later stage.

4.1 Marking Rule for Reliablity

Reliability is the derivability strategy that takes the most cautious interpretation of abnormalities: any formula that in view of the premises might behave abnormally, because it occurs in a minimal disjunction of abnormalities, is deemed unreliable and should not be assumed to behave normally. This means in practice that at any proof stage where a formula ψ is derived using some $\phi^{-} \in \Omega$ in context by an instance of the corresponding formation rule, is 'marked'. Here marking means to reject the formula ψ, or invalidate it. In the following we shall introduce a new derivability rule that internalizes this process in AdaptiveND.

We define a new derivability rule $\boxtimes$R that depends on the formulation of the union set of all minimal $\bigvee(\Delta^{min})$ obtained by instances of the MinDab rule above.

$$\frac{\Gamma; \cdot \vdash_{\mathsf{s}} \Delta^{min} \qquad \Gamma; \phi^{-} \vdash_{\mathsf{s'}} \psi \qquad \phi \in \Delta^{min}}{\Gamma \vdash_{\mathsf{max(s,s')+1}\boxtimes\mathsf{R}} \psi} \boxtimes\mathrm{R}$$

The meaning of $\boxtimes$R is the following: if at stage s a minimal disjunction of abnormalities Δ^{min} is derived for Γ, and at a later stage $\mathsf{s'}$ a formula ψ is derived from the same premise set by assuming a component of Δ^{min} false by an Adaptive Condition Formation rule, then at a next stage ψ is marked as retracted.

4.2 Extending the example

Let us now extend the example from Section 3.1 with a new branch to illustrate the derivation step obtained by the Marking Rule $\boxtimes$R. Let us call $\mathbb{D}$ the derivation already shown, which had as a conclusion at stage $\mathsf{5}$ the derivation of q in context Γ and under the adaptive condition that $(p \wedge \neg p)$ is false. We extend it now as follows:

$$\dfrac{\dfrac{\mathbb{D}}{\Gamma; (p \wedge \neg p)^- \vdash_5 q} \qquad \dfrac{\dfrac{\Gamma; \cdot \vdash_6 p \qquad \Gamma; \cdot \vdash_7 p \to \neg p}{\Gamma; \cdot \vdash_8 \neg p} \to \mathrm{E} \qquad \Gamma; \cdot \vdash_9 p}{\Gamma; \cdot \vdash_{10} p \wedge \neg p} \wedge \mathrm{I}}{\Gamma; \cdot \vdash_{11 \boxtimes \mathrm{R}} q} \boxtimes \mathrm{R}$$

In this derivation a new abnormality is derived at stage 10, namely the same that is assumed to be false at stage 5. Notice that it is essential that this abnormality be derived under an empty condition, i.e. under context $\Gamma; \cdot$, as explained above for the required strict condition on WLeft. Moreover, a difference between the Fitch-style proofs standard for Adaptive Logics and the Natural Deduction derivation style becomes here evident. In the former, a marking rule implies the need to proceed backwards on the derivation, to mark all previous occurrences of the marked formula which can no longer be considered derived. In the latter, on the other hand, there is no need to remove formulas because the result obtained at stage 5 cannot be reused in an extension of this proof. Instead a new derivation step is performed (stage 11), where the conclusion q is marked. Moreover, if we were ever to get again $\Gamma; (p \wedge \neg p)^- \vdash_{\mathsf{i}} q$, that would be obtained by some new derivation $\mathbb{D}'$ and therefore result as a conclusion at some stage $\mathsf{i} > 5$.

4.3 An example with $\bigvee(\Delta^{min})$-selection

The previous example is rather simple, in that it simply indicates a formula that is first derived under an adaptive condition (referring to an abnormal formula assumed to be false), and then retracted after that condition is validated again.

Let us consider now a slightly more complex example. We want to show a situation in which a disjunction of two abnormalities can be derived: accordingly, there might be more than one formula to be marked. Let us start with a premise set $\Gamma = \{(p \vee r), \neg p, (p \vee q), \neg q, (\neg p \to q)\}$. Now consider the following derivation, dubbed $\mathbb{D}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma; \cdot \vdash_1 (p \vee r)}\mathrm{PREM}}{\Gamma; \cdot \vdash_2 p, r} \vee \mathrm{E} \qquad \dfrac{}{\Gamma; \cdot \vdash_3 \neg p}\mathrm{PREM}}{\Gamma; \cdot \vdash_4 (p \wedge \neg p), r} \wedge \mathrm{I} \qquad (p \wedge \neg p) \in \Omega}{\Gamma; (p \wedge \neg p)^- \vdash_5 r} \mathrm{RC}$$

At stage 4 a disjunction of an abnormality with r is derived, and by RC at stage 6 the formula r is derived alone, assuming the relevant abnormality false. Let us now consider the following derivation, dubbed $\mathbb{D}^{'}$:

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma;\cdot\vdash_{6} (p\vee q)}\,\text{PREM}}{\Gamma;\cdot\vdash_{7} p,q}\,\vee\text{E} \qquad \dfrac{}{\Gamma;\cdot\vdash_{8} \neg p}\,\text{PREM}}{\Gamma;\cdot\vdash_{9} (p\wedge\neg p), r}\,\wedge\text{I} \qquad \dfrac{}{\Gamma;\cdot\vdash_{10} \neg q}}{\Gamma;\cdot\vdash_{11} (p\wedge\neg p),(q\wedge\neg q)}\,\text{RC}
$$

Here the previously derived abnormality $(p\wedge\neg p)$ is derived in disjunctive form with a new abnormality $(q\wedge\neg q)$ at stage 11, where the latter is obtained by $\wedge$I from stages 7,9. If we join now the two branches $\mathbb{D},\mathbb{D}'$ to form $\mathbb{E}$, we are allowed a marking step:

$$
\dfrac{\dfrac{\mathbb{D}}{\Gamma;(p\wedge\neg p)^{-}\vdash_{5} r} \qquad \dfrac{\mathbb{D}'}{\Gamma;\cdot\vdash_{11} (p\wedge\neg p),(q\wedge\neg q)} \qquad (p\wedge\neg p)\in\Delta^{min}}{\Gamma;\cdot\vdash_{12\boxtimes\mathsf{R}} r}\,\boxtimes
$$

At stage 12 the formula r is no longer valid, because its adaptive condition is a minimal abnormal formula of a derived disjunction of abnormalities. Now we can provide a further extension of this derivation dubbed $\mathbb{D}''$:

$$
\dfrac{\dfrac{\dfrac{}{\Gamma;\cdot\vdash_{13} \neg p}\,\text{PREM} \qquad \dfrac{}{\Gamma;\cdot\vdash_{14} \neg p\to q}\,\text{PREM}}{\Gamma;\cdot\vdash_{15} q}\,\to\text{I} \qquad \dfrac{}{\Gamma;\cdot\vdash_{16} \neg q}\,\text{PREM}}{\Gamma;\cdot\vdash_{17} q\wedge\neg q}\,\wedge\text{I}
$$

$\mathbb{D}''$ has the effect of producing a new minimal abnormality at stage 17. This also means that if we derive a copy of derivation $\mathbb{D}$, where each step is renumbered consecutively, and join it to $\mathbb{D}''$ and $\mathbb{E}$, it is possible to establish again $(p\wedge\neg p)$ as an adaptive condition and accoridngly derive again the judgement that was marked at stage 12, as follows:

$$
\dfrac{\dfrac{\mathbb{E}}{\Gamma;\cdot\vdash_{12\boxtimes\mathsf{R}} r} \qquad \dfrac{\mathbb{D}''}{\Gamma;\cdot\vdash_{17} q\wedge\neg q} \qquad \dfrac{\mathbb{D}}{\Gamma;\cdot\vdash_{18} (p\wedge\neg p), r}}{\Gamma;(p\wedge\neg p)^{-}\vdash_{19} r}\,\text{RC*}
$$

where $*$ is the side condition that $(p \wedge \neg p) \in \Omega$. If the derivation is no longer extended, the formula r can be considered finally derived. In the next section we complete our system with the required meta-theoretical analysis needed to define derivability at stage and final derivability.

5 Derivability

In the example from the previous section we have illustrated how the marking condition establishes a dynamic derivability relation, which allows to derive formulas and retract them. Whenever a certain formula is derived on some $\phi \in \Delta^{min}$ adaptive condition, it might still be marked afterwards according to $\boxtimes R$. This gives us a notion of derivability at stage:

Definition 5 (Derivability at stage)**.** *A formula ψ is derived at stage* s *iff* $\Gamma; \phi^{-} \vdash_{\mathsf{s}} \psi$ *and it is not the case that* $\Gamma; \cdot \vdash_{\mathsf{s}\boxtimes} \psi$*.*

A more stable notion of derivability holds when marking is no longer possible. To this aim, one requires that the stage s at which a formula ϕ is derived remains unmarked in all the extensions of the derivation tree which can be obtained by using all *relevant* abnormalities as adaptive conditions. This relevance criterion is essential if one wants to guarantee finiteness of the proof tree to be surveyed in order to establish whetehr a formula is never marked (again), We define therefore a set of *abnormalities relevant to* Γ. to do so we first identify the union set of all subformulas of the premise set Γ:

Definition 6 (Subformulas of the premise set)**.** $Sf(\Gamma) = \bigcup_{\phi \in \Gamma}\{\psi \mid \psi$ *is a subformula of* $\phi\}$

From $Sf(\Gamma)$ we the construe all the possible abnormalities which can be obtained by its members:

Definition 7 (Abnormalities relevant to the premise set)**.** $\Omega(\Gamma) = \{\psi \wedge \neg\psi \mid \psi \in Sf(\Gamma)\}$

Given this preparatory work, we now introduce the notion of a *complete proof-tree* with respect to derivable relevant disjunction of abnormalities:

Definition 8 (Completeness relative to relevant abnormalities)**.** *Let P be a proof-tree of* $\mathsf{AdaptiveND}$*. We say that P is complete relative to $\Omega(\Gamma)$ at stage* s *if* $\Gamma; \cdot \vdash_{\mathsf{s'}} \bigvee(\Delta)$*, for every derivable* $\bigvee(\Delta)$ *with* $\Delta \subseteq \Omega(\Gamma)$ *and some* $\mathsf{s'}<\mathsf{s}$.

We can now formulate our notion of final derivability:

Definition 9 (Final Derivability)**.** *A formula ψ is finally derived* $\Gamma; \phi^{-} \vdash_{\checkmark} \psi$ *iff there is a stage* s *in some complete proof-tree P such that* $\Gamma; \phi^{-} \vdash_{\mathsf{s}} \psi$ *and* $\phi \notin \Delta^{min}$*.*

The definition guarantees final derivability for any derived formula whose adaptive condition is not minimal. Notice that the second requirement might not be satisfied at any finite stage s, hence it might be guaranteed only at meta-theoretical level.

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