Note on Giuseppe's note on adaptive natural deduction

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1 Abnormality handling in terms of discharge-manageme

One of the similarities between natural deduction (in short, ND) and adaptive logic proofs is that both involve assumptions, albeit of different kinds.

In ND we infer from assumptions and there are ways to discharge certain previous assumptions. One can define a consequence relation by: $\Gamma \vdash A$ iff A is derivable by the inference and deduction rules of ND from charged assumptions in Γ.

In adaptive logics (in short, ALs) we usually speak about Γ as being the set of premises and reserve the term "assumption" for something else. Namely, we derive formulas on conditions Δ. These conditions are finite sets of members of a set of so-called abnormalities that are associated with each adaptive logic. The fact that A is derived on the condition Δ in an adaptive proof means that A is derived on the *assumption* that each abnormality in Δ is false.

The link between ALs and ND via the notion of assumption suggests to design ND systems for ALs along the following lines. We allow for the introduction of what we may call *abnormality-assumptions*: namely, an assumption that a certain (finite) set of abnormalities is false. Recall that usually assumptions in ND are charged when introduced and may be later discharged in view of deduction rules. We see that for instance in the following derivation tree (I put discharged assumptions in brackets):

$$\dfrac{\dfrac{\dfrac{(A) \quad A \supset B}{B} \qquad \dfrac{(A) \quad A \supset C}{C}}{B \wedge C}}{A \supset (B \wedge C)}$$

The handling of abnormality-assumptions is inverse: they are discharged when introduced but may get charged when we reason on. An example (I indicate abnormalities by "!" and suppose that $\vee$ and $\neg$ have a classical meaning):

$$\dfrac{(\neg !B) \quad A \vee !B}{A} \quad \rightsquigarrow \quad \dfrac{\dfrac{\neg !B \quad A \vee !B}{A} \qquad !B \vee !C}{A \wedge (!B \vee !C)}$$

On the left the abnormality assumption is discharged (by default). However, in the extension of the derivation tree on the right, it gets charged due to the presence of $!B \vee !C$.

Note that the handling of the charging of assumptions is usually implemented by means of deduction rules (see Prawitz p. 23). For the handling of the charging of abnormality-assumptions we need an extra rule which is applied whenever a disjunction of abnormalities is derived. Let me demonstrate this for the modeling of the so-called reliability strategy in terms of ND.

First, it will be useful to formally distinguish between abnormality assumptions and usual assumptions. Hence, I will put a bracket over normality assumptions. E.g.,

$$\frac{(\overbracket{\neg !B}) \quad A \vee !B}{A} \quad \rightsquigarrow \quad \frac{\dfrac{\overbracket{\neg !B} \quad A \vee !B}{A} \qquad !B \vee !C}{A \wedge (!B \vee !C)} \tag{1}$$

Note the difference to the following derivation tree sequence where $\neg !B$ is introduced as a usual assumption:

$$\frac{\neg !B \quad A \vee !B}{A} \quad \rightsquigarrow \quad \frac{\dfrac{\neg !B \quad A \vee !B}{A} \qquad !B \vee !C}{A \wedge (!B \vee !C)}$$

So far, I only allowed for abnormality assumptions of the form $\neg !B$. We can also assume many abnormalities to be false at once: $\neg !B_1 \wedge \neg !B_2 \wedge \ldots \wedge \neg !B_n$. Where $\Delta = \{B_1, \ldots, B_n\}$ we write $\Delta^{\neg}$ for $\neg !B_1 \wedge \neg !B_2 \wedge \ldots \wedge \neg !B_n$. Hence, all abnormality assumptions have the form $\Delta^{\neg}$ for some finite set of abnormalities Δ.

Let τ be a derivation tree: let $\mathrm{top}(\tau)$ be the set of all usual assumptions in τ, $\mathrm{top}*(\tau)$ the set of all charged assumptions in $\mathrm{top}(\tau)$, $\overbracket{\mathrm{top}}(\tau)$ the set of all abnormality assumptions in τ and $\overbracket{\mathrm{top}}*(\tau)$ the set of all charged abnormality assumptions in $\overbracket{\mathrm{top}}(\tau)$.

For instance in (1) we have on the left: $\mathrm{top}(\tau) = \mathrm{top}*(\tau) = \{A \vee !B\}$ and $\overbracket{\mathrm{top}}(\tau) = \{\neg !B\}$ while $\overbracket{\mathrm{top}}*(\tau) = \emptyset$. On the right the situation changes since $\overbracket{\mathrm{top}}*(\tau) = \{\neg !B\}$.

Definition 1 (Derivability in τ)**.** Given a derivation tree τ we say that A *is derived at a node* n *in* τ *(from* Γ*) [on the abnormality assumption* Δ*]* iff there is a subtree τ' of τ such that n is in τ', $\overbracket{\mathrm{top}}*(\tau') = \emptyset$, $(\mathrm{top}*(\tau') \subseteq \Gamma)$ [and $\overbracket{\mathrm{top}}(\tau') = \Delta$].

For instance, in (1), on the left A is derived from $\{A \vee !B\}$ on the abnormality assumption $\emptyset$, while on the right it is not. The reason is that on the left the abnormality assumption $\neg !B$ is discharged, while on the right it is charged (and note that there is no subtree where A is derived such that all abnormality assumptions are discharged).

We say that $\mathsf{Dab}(\Delta)$ is a minimal Dab-formula in τ iff $\mathsf{Dab}(\Delta)$ is derived in τ on the abnormality assumption $\emptyset$ and for all $\Delta' \subset \Delta$, $\mathsf{Dab}(\Delta')$ is not derived in τ on the abnormality assumption $\emptyset$.

Where $\mathsf{Dab}(\Delta_1), \mathsf{Dab}(\Delta_2), \ldots$ are the minimal Dab-formulas in τ, let $U(\tau) = \Delta_1 \cup \Delta_2 \cup \ldots$

Definition 2 (Charging of abnormality-assumptions, Reliability)**.** An abnormality assumption $\Delta^{\neg}$ is charged in τ iff $U(\tau) \cap \Delta \neq \emptyset$.

Let us extend our example in (1) a bit more:

$$\dfrac{(\neg !B) \quad A \vee !B}{A} \quad \rightsquigarrow \quad \dfrac{\dfrac{\neg !B \quad A \vee !B}{A} \qquad !B \vee !C}{A \wedge (!B \vee !C)} \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\dfrac{(\neg !B) \quad A \vee !B}{A} \qquad !B \vee !C}{A \wedge (!B \vee !C)}}{A} \qquad !C}{A \wedge !C} \tag{2}$$

The reason why on the right side our abnormality assumption is discharged again is that $!B \vee !C$ ceases to be a minimal Dab-formula due to the introduction of $!C$.

As our example demonstrates, derivability is a dynamic matter. We now define a static notion of derivability, so-called final derivability.

We call an extension τ' of τ a Γ-extension of τ iff $\mathrm{top} * (\tau') \subseteq \Gamma$.

Definition 3 (Final derivability)**.** A is finally derived in τ from Γ iff (i) it is derived at a node n in τ from Γ and (ii) whenever A is not derived in an Γ-extension τ' of τ at node n then there is a Γ-extension τ'' of τ' such that A is derived at node n in τ''.

For instance, A is finally derived from $\Gamma = \{A \vee !B, !B \vee !C, !C\}$ at the left tree in (2) at some node, let's call it n. Although it is not derived in the Γ-extension of the tree in the middle, on the right side the tree is further Γ-extended and A is derived again at node n. Note that there is no Γ-extension τ' of the tree on the right such that A at node n is not derived in τ'.

Definition 4. $\Gamma \vdash_{\mathbf{AdND}} A$ iff A is finally derived from Γ in some tree τ.

2 Formula-Condition pairs in ND

In this chapter I propose a possible simplification of the adaptive natural deduction system proposed in the last note.

For the sake of simplicity I will demonstrate the approach on the basis of minimal logic. It should generalize for any LLL though.

Prawitz proposes the following inference rules for minimal logic:[1]

$$\frac{A \quad B}{A \wedge B}\ \&I \qquad\qquad \frac{A \quad B}{A}\ \&E \quad \frac{A \quad B}{B}\ \&E$$

$$\frac{A}{A \vee B}\ \vee I \quad \frac{B}{A \vee B}\ \vee I \qquad\qquad \frac{A \vee B \quad \begin{matrix}(A)\\ C\end{matrix} \quad \begin{matrix}(B)\\ C\end{matrix}}{C}\ \vee E$$

$$\frac{\begin{matrix}(A)\\ B\end{matrix}}{A \supset B}\ \supset I \qquad\qquad \frac{A \quad A \supset B}{B}\ \supset E$$

In the adaptive treatment we need to represent conditions in which the (defeasible) assumptions are stored with the help of which formulas are derived. Hence, instead of only listing formulas in the antecedent spots and the conclusion spot of rules, we list formula-condition pairs $A.\Delta$ where Δ is a finite set of abnormalities.

We now translate the rules of the given LLL by means of the following scheme:

$$\frac{A_1 \quad \dots \quad A_n}{B} \quad \rightsquigarrow \quad \frac{A_1.\Delta_1 \quad \dots \quad A_n.\Delta_n}{B.\bigcup_{i=1}^{n} \Delta_i}$$

In the specific case of minimal logic we end up with:

$$\frac{A.\Delta_A \quad B.\Delta_B}{A \wedge B.\Delta_A \cup \Delta_B}\ \&IU \qquad \frac{A.\Delta_A \quad B.\Delta_B}{A.\Delta_A \cup \Delta_B}\ \&EU \quad \frac{A.\Delta_A \quad B.\Delta_B}{B.\Delta_A \cup \Delta_B}\ \&EU$$

$$\frac{A.\Delta}{A \vee B.\Delta}\ \vee IU \quad \frac{B.\Delta}{A \vee B.\Delta}\ \vee IU \qquad \frac{A \vee B.\Delta \quad \begin{matrix}(A)\\ C.\Delta_A\end{matrix} \quad \begin{matrix}(B)\\ C.\Delta_B\end{matrix}}{C.\Delta \cup \Delta_A \cup \Delta_B}\ \vee EU$$

$$\frac{\begin{matrix}(A)\\ B.\Delta\end{matrix}}{A \supset B.\Delta}\ \supset IU \qquad\qquad \frac{A.\Delta \quad A \supset B.\Delta'}{B.\Delta \cup \Delta'}\ \supset EU$$

We add the following conditional rule:

$$\frac{A \vee \mathsf{Dab}(\Delta).\Delta'}{A.\Delta \cup \Delta'}\ RC$$

Let in the following τ be a derivation tree constructed by the given rules.

[1] Some are later reformulated in terms of deduction rules so to give a precise account of the discharging of assumptions. This could be done analogously here, but for the sake of simplicity I stick to the simple representation for sketching this idea.

In the derivation trees we allow only for top-nodes of the form $A.\emptyset$.

In order to define the marking of nodes in derivation trees we need some more terminology.

Let $\mathsf{Dab}(\Delta_1).\emptyset, \mathsf{Dab}(\Delta_2).\emptyset, \ldots$ be all nodes of the form $\mathsf{Dab}(\Delta).\emptyset$ in τ. We say $\mathsf{Dab}(\Delta)$ is a minimal Dab-formula in τ iff there is no $\Delta_i \subset \Delta$ and there is a $\Delta_j = \Delta$. Let $U(\tau) = \Delta_1 \cup \Delta_2 \cup \ldots$.

Definition 5 (Marking of nodes in τ, Reliability)**.** A node $A.\Delta$ is marked in τ iff $\Delta \cap U(\tau) \neq \emptyset$.

Let's take a look at our example from the previous section:

$$\dfrac{A \vee !B.\emptyset}{A.\{!B\}} \rightsquigarrow \dfrac{\dfrac{A \vee !B.\emptyset}{\checkmark\ A.\{!B\}} \quad !B \vee !C.\emptyset}{\checkmark\ A \wedge (!B \vee !C).\{!B\}} \rightsquigarrow \dfrac{\dfrac{\dfrac{A \vee !B.\emptyset}{A.\{!B\}} \quad !B \vee !C.\emptyset}{A \wedge (!B \vee !C).\{!B\}} \quad !C.\emptyset}{A \wedge !C.\{!B\}}$$

Again, we're interested in a stable notion of derivability:

Let $A_1.\emptyset, A_2.\emptyset, \ldots$ be the charged top-nodes of τ. We say τ is an Γ-tree iff $\{A_1, A_2, \ldots$ Γ. Similarly, we call an extension τ' of τ a Γ-extension iff τ' is a Γ-tree.

Definition 6 (Final derivability in τ)**.** *A is finally derived in τ from Γ* iff (i) τ is a Γ-tree, (ii) there is an unmarked node n with $A.\Delta$, (iii) for every Γ-extension τ' of τ in which n is marked there is a further Γ-extension τ'' such that n is unmarked.

Definition 7. $\Gamma \vdash_{\mathbf{AdND}} A$ iff A is finally derived from Γ in some tree τ.