Annotated Natural Deduction for Adaptive Reasoning

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Abstract

We present a multi-conclusion natural deduction calculus based on minimal logic extended with a set of rules characterizing the dynamic reasoning typical of adaptive logics.

1 Intro

We present a multi-conclusion natural deduction calculus that mimics the dynamic reasoning at work in adaptive logics ([1]). This is the first attempt to reconstruct the dynamics typical of adaptive logics in a natural deduction setting. The resulting system does not correspond to the usual structure known as the Standard Format for Adaptive Logics: this means that, though we *do not* introduce an adaptive logic proper, we can talk of a natural deduction system for *adaptive reasoning*. We characterize such a way of reasoning as having properties that identify adaptive dynamics. To do so, the standard proof-theoretical procedure of a natural deduction system is enhanced with:

1. a rule-based ability of introducing abnormal formulas of the form $A \wedge \neg A$;[1] the appearance of such formulas on the right-hand side of our derivability sign justifies the claim that our system is extended to a multi-conclusion setting;

2. a rule-based ability of deriving formulas under conditions that some such abnormal formula is not true;

3. the procedural ability of rejecting derivation steps previously obtained by way of marking in view of effectively derived abnormal formulas.

These are all properties inspired by the adaptive logics approach. In view of the last property, we need moreover to annotate the derivability relation with a stage counting mechanism to keep track of the steps performed in the derivation tree (thus counting also premises rather than only rules).

[1] In the current format we focus on inconsistency-adaptive logics, though the generalization to the natural deduction for other adaptive formats seems possible.

Our system is not even close to a standard format for AL. We express the standard triple $\{LLL, \Omega, STRATEGY\}$ in a system where the Lower Limit Logic is extended to include rules both for expressing the abnormal formulas in Ω and to interpret the selection Strategy. In other words, this rule-based approach allows to merge the rules and axioms of a typical LLL and the rules of the AL based on abnormal formulas into a single system of rules.

2 Minimal ND

We start by defining the type universe for the $\{\neg, \rightarrow, \wedge, \vee\}$ fragment of intuitionistic propositional logic corresponding to minimal logic. That is, we do not explicitly formulate a rule to abort derivations once a proposition of type $\perp$ is derived, hence not allowing *ex falso quodlibet*. This minimalND calculus plays the role of our Lower Limit Logic.

Definition 1 (minimalND). *Our starting language for* minimalND *is defined by the following grammar:*

$$
\begin{array}{l}
Type := Prop \\
Prop := A|\perp|\neg\phi \mid \phi_1 \rightarrow \phi_2|\phi_1 \wedge \phi_2|\phi_1 \vee \phi_2 \\
\Gamma := \{\phi_1, \ldots, \phi_n\} \\
\Delta := \{\phi_1, \ldots, \phi_n\}
\end{array}
$$

A minimalND-formula is of the form $\Gamma; \cdot \vdash_{\mathsf{s}} \Delta$, where: Γ is the usual set of assumptions; Δ is a set of formulas of the language; the set *Gamma* and the comma on the left-hand side of the derivability sign are both to be read conjunctively; the set Δ and the comma on the right-hand side of the derivability sign are both to be read disjunctively. The derivability sign is enhanced with a signature s that corresponds to a natural number counting the ordered steps executed to obtain the corresponding ND-formula in a tree. This annotation only comes to use in the next extension of the calculus. Similarly, the semi-colon sign and the $\cdot$ following Γ are useless in the setting of minimalND and come to use only in the next section. We introduce both here for uniformity of notation. We first present formation rules for the proposition, well-formedness of contexts and introduction and elimination rules for the connectives.

$$
\frac{}{A \in \mathsf{Prop}}\ \textsc{Atom} \qquad \frac{}{\perp \in \mathsf{Prop}}\ \perp \qquad \frac{\phi \in \mathsf{Prop}}{\neg\phi \in \mathsf{Prop}}\ \neg
$$

$$
\frac{\phi_1 \in \mathsf{Prop} \qquad \phi_2 \in \mathsf{Prop}}{\phi_1 \rightarrow \phi_2 \in \mathsf{Prop}}\ \rightarrow \qquad \frac{\phi_1 \in \mathsf{Prop} \qquad \phi_2 \in \mathsf{Prop}}{\phi_1 \wedge \phi_2 \in \mathsf{Prop}}\ \wedge
$$

Figure 1: Formula Formation Rules

$$\frac{}{\cdot \vdash_{\mathsf{s}} \mathsf{wf}}\ \textsc{Nil} \qquad \frac{\Gamma; \cdot \vdash_{\mathsf{s}} \mathsf{wf} \qquad \phi \in \mathsf{Prop}}{\Gamma, \phi; \cdot \vdash_{\mathsf{s+1}} \mathsf{wf}}\ \Gamma\textsc{-formation}$$

$$\frac{\Gamma; \cdot \vdash_{\mathsf{s}} \mathsf{wf} \qquad \phi \in \Gamma}{\Gamma; \cdot \vdash_{\mathsf{s+1}} \phi}\ \textsc{Prem} \qquad \frac{\Gamma; \cdot \vdash_{\mathsf{s}} \Delta \qquad \phi \in \mathsf{Prop}}{\Gamma; \cdot \vdash_{\mathsf{s+1}} \Delta, \phi}\ \Delta\textsc{-formation}$$

Figure 2: Context Formation Rules

$$\frac{\Gamma, \phi_1; \cdot \vdash_{\mathsf{s}} \phi_2}{\Gamma; \cdot \vdash_{\mathsf{s+1}} \phi_1 \to \phi_2} \to\mathrm{I} \qquad \frac{\Gamma; \cdot \vdash_{\mathsf{s}} \phi_1 \to \phi_2 \qquad \Gamma'; \cdot \vdash_{\mathsf{s'}} \phi_1}{\Gamma; \Gamma' \vdash_{\mathsf{max(s,s')+1}} \phi_2} \to\mathrm{E}$$

$$\frac{\Gamma; \cdot \vdash_{\mathsf{s}} \phi_1 \qquad \Gamma'; \cdot \vdash_{\mathsf{s'}} \phi_2}{\Gamma, \Gamma'; \cdot \vdash_{\mathsf{max(s,s')+1}} \phi_1 \wedge \phi_2} \wedge\mathrm{I} \qquad \frac{\Gamma; \cdot \vdash_{\mathsf{s}} \phi_1 \wedge \phi_2}{\Gamma; \cdot \vdash_{\mathsf{s+1}} \phi_{i \in \{1,2\}}} \wedge\mathrm{E}$$

$$\frac{\Gamma; \cdot \vdash_{\mathsf{s}} \phi_1}{\Gamma; \cdot \vdash_{\mathsf{s+1}} \phi_1 \vee \phi_2} \vee\mathrm{I} \qquad \frac{\Gamma; \cdot \vdash_{\mathsf{s}} \phi_2}{\Gamma; \cdot \vdash_{\mathsf{s+1}} \phi_1 \vee \phi_2} \vee\mathrm{I} \qquad \frac{\Gamma; \cdot \vdash_{\mathsf{s}} \phi_1 \vee \phi_2}{\Gamma; \cdot \vdash_{\mathsf{s+1}} \phi_1, \phi_2} \vee\mathrm{E}$$

$$\frac{\Gamma; \cdot \vdash_{\mathsf{s}} \bot}{\Gamma; \cdot \vdash_{\mathsf{s}} \phi} \bot\mathrm{E} \qquad \frac{\Gamma; \phi \vdash_{\mathsf{s}} \psi}{\Gamma; \cdot \vdash_{\mathsf{s+1}} \psi, \neg\phi} \neg\mathrm{I}$$

Figure 3: Rules for I/E of connectives

$$\frac{\Gamma; \cdot \vdash_{\mathsf{s}} \phi_1}{\Gamma, \phi_2; \cdot \vdash_{\mathsf{s+1}} \phi_1}\ \textsc{Wleft} \qquad \frac{\Gamma, \phi_1, \phi_1; \cdot \vdash_{\mathsf{s}} \phi_2}{\Gamma, \phi_1; \cdot \vdash_{\mathsf{s+1}} \phi_2}\ \textsc{Cleft} \qquad \frac{\Gamma, \phi_1, \phi_2; \cdot \vdash_{\mathsf{s}} \phi_3}{\Gamma, \phi_2, \phi_1; \cdot \vdash_{\mathsf{s+1}} \phi_3}\ \textsc{Eleft}$$

$$\frac{\Gamma; \cdot \vdash_{\mathsf{s}} \phi_1 \qquad \Gamma', \phi_1; \cdot \vdash_{\mathsf{s'}} \phi_2}{\Gamma; \Gamma'; \cdot \vdash_{\mathsf{max(s,s')+1}} \phi_2}\ \textsc{Cut}$$

$$\frac{\Gamma; \cdot \vdash_{\mathsf{s}} \phi, \phi}{\Gamma; \cdot \vdash_{\mathsf{s+1}} \phi}\ \textsc{Cright} \qquad \frac{\Gamma; \cdot \vdash_{\mathsf{s}} \phi_1, \phi_2}{\Gamma; \cdot \vdash_{\mathsf{s+1}} \phi_2, \phi_1}\ \textsc{Eright}$$

Figure 4: Structural Rules

The classical negation in this fragment is expressed by implication to $\bot$. Notice that this fragment does not contain a standard rule for aborting a derivation after obtaining $\bot$, which expresses absolute inconsistency. A paraconsistent negation $\neg$ is used for inconsistencies; its elimination is left to the calculus introduced in the next section by way of an adaptive mechanism. The Prem rule defines the equivalent of the adaptive Premise rule. [HERE MORE EXPLANATION ON I/E RULE: FOR STRUCTURAL RULES ADD EXPLANATION ON HOW THE WRIGHT can be obtained by disjunction introduction followed by disjunction elimination.

Notice moreover that WLEFT holds only with an empty set of abnormalities in context, i.e. the requriement $\Gamma; \cdot$ in its premise must be understood as strict. The reason for this is that as long as the context is of the form $\Gamma; \phi^-$, the abnormal formula is based on Γ and this set would be changed by an application of the Weakening rule.

We should also remember that this Minimal Logic fragment verify

$$\frac{\Gamma; \cdot \vdash_{\mathsf{s}} \phi \to \psi}{\Gamma; \cdot \vdash_{\mathsf{s+1}} \neg\phi, \psi}$$

but it does not verify

$$\frac{\Gamma; \cdot \vdash_{\mathsf{s}} \neg\phi \to \psi}{\Gamma; \cdot \vdash_{\mathsf{s+1}} \phi, \psi}$$

hence is not fully CLuN equivalent.]

3 Adaptive ND

Definition 2 ($\mathsf{AdaptiveND}$)**.** *The language of* $\mathsf{AdaptiveND}$ *is as follows:*

$$\begin{aligned}
&Type := Prop \\
&Prop := A|\bot|\neg\phi|\phi_1 \to \phi_2|\phi_1 \wedge \phi_2|\phi_1 \vee \phi_2 \\
&\Gamma := \{\phi_1, \dots, \phi_n\} \\
&\Delta := \{\phi_1, \dots, \phi_n\} \\
&\Omega := \{\phi \wedge \neg\phi \mid \phi \in Prop\}
\end{aligned}$$

We refer to an $\mathsf{AdaptiveND}$-formula as an extension of a $\mathsf{minimalND}$-formula of the form $\Gamma; \phi^- \vdash_s \psi$, where:

1. the left-hand side of $\vdash_{\mathsf{s}}$ has Γ as in $\mathsf{minimalND}$;

2. the semicolon sign on the left-hand side of $\vdash_{\mathsf{s}}$ is conjunctive;

3. ϕ refers to a formula with a specific inconsistent logical form, obtained by way of the Ω-formation rule presented below;[2]

4. the last place of the left-hand side context is always reserved to negated formulas of the Ω form; we shall use ϕ^- to refer to the negation of ϕ, for all $\phi \in \Omega$;

5. the right-hand side is in disjunctive form.

When Ω is empty on the left-hand side of $\vdash$, we shall write $\Gamma; \cdot \vdash$, thus reducing to the form of a $\mathsf{minimalND}$-formula. Moreover, in $\mathsf{AdaptiveND}$, the annotation on the proof stage s is optionally followed by one of the following two marks: $\boxtimes$ to mark that at the current stage some previously derived formula is retracted; $\checkmark$ to mark that at the current stage some previously derived formulas is now stable, i.e. will no longer marked by $\boxtimes$.

A feature of $\mathsf{AdaptiveND}$ that we will use below is that it requires to have *classical disjunctions of formulas generated under the* Ω*-rule*. To be more precise, whenever we derive more than a formula in the set Ω, we might want to establish which of those is unavoidable, and thus extend Γ with a set of disjunctions of ω's which is classical (as we cannot infer it from any already verified ω). This means that $\mathsf{AdaptiveND}$ needs a new formation rule for $\vee_{\mathsf{CL}}$, which is in fact restricted to ϕ's that are in Ω. We shall also give a formation rule for Ω and one for $\Gamma; \Omega^-$. Furthermore, two additional rules are introduced for deriving formulas, simply or on conditions.

Marking definitions will be also described below. The mechanism enforced by the marking definitions (either for retracting or for stabilizing a derived content) makes a proof a sequence of derivation steps. The dynamic nature of this form of reasoning is implemented in our ND calculus by allowing extensions by derivation steps appended at the end of the current proof-tree.[3] For obtaining the stable notion of final derivability, it will be useful to allow *infinite* extensions of a proof-tree.

3.1 Rules of AdaptiveND

The set of rules of IPC is extended by the following new set to obtain $\mathsf{AdaptiveND}$:

$$\frac{\phi \in Prop}{(\phi \wedge \neg\phi) \in \Omega}\ \Omega\text{-FORMATION} \qquad\qquad \frac{\Gamma; \cdot \vdash_{\mathsf{s}} \mathsf{wf} \qquad \phi \in \Omega}{\Gamma; \phi^- \vdash_{\mathsf{s+1}} \mathsf{wf}}\ \Gamma; \phi^-\text{-FORMATION}$$

Figure 5: Ω Formation rules

[2]As mentioned above, the current setting of $\mathsf{AdaptiveND}$ is specified for an inconsistency-adaptive logic.

[3]That is, we will avoid the more complex approach that allows for line insertions in a given derivation.

$$\frac{\Gamma; \phi^- \vdash_{\mathsf{s}} \phi_1 \qquad \phi_1; \cdot \vdash_{\mathsf{s+1}} \phi_2}{\Gamma, \phi_1; \phi^- \vdash_{\mathsf{s+2}} \phi_2}\text{RU} \qquad \frac{\Gamma; \cdot \vdash_{\mathsf{s}} \psi, \phi \qquad \phi \in \Omega}{\Gamma; \phi^- \vdash_{\mathsf{s+1}} \psi}\text{RC}$$

Figure 6: Adaptive Rules

We call $\phi \in \Omega$, following the adaptive tradition, an abnormal formula. How the system reacts to such a case, is no longer by the $\bot$I rule (which is thus now meant to be dropped), but rather by the additional rules of AdaptiveND. First we admit that the complement of an Ω-set can be used to extend a context Γ: $\Omega^- = \{\neg\phi \mid \forall \phi \in \Omega\}$. Next, the calculus is extended by introducing two rules. *RU* is called the unconditional rule: it says that if a formula ϕ_1 is derivable from Γ and Ω^-, and if ϕ_2 is derivable from ϕ_1, then ϕ_2 is derivable from ϕ_1 inheriting the whole context Ω^-. *RC* is called the conditional rule: it applies the $\Gamma\Omega^-$-formation rule by saying that if a formula ϕ or a set Ω (possibly a singleton) is derivable under context Γ, then ϕ is derivable from Γ and Ω^-.

$$\frac{\Gamma; \cdot \vdash_{\mathsf{s}} \Delta \qquad \Delta \subset \Omega \qquad \text{with no } \Delta' \in \Omega, \text{ s.t. } \Gamma; \cdot \vdash_{t<s'} \Delta'}{\Gamma; \cdot \vdash_{\mathsf{s'}} \Delta^{min}}\textsc{MinDab}$$

Figure 7: Minimal Abnormal Formulas Rule

Notice that the derivation of minimal disjunction of abnormalities is a process that occurs along with the development of the proof-tree. This means that the following procedure to mark formulas depend on the possible derivation of certain such formulas.

3.2 A simple example

We present here a derivation, where $\Gamma = \{(\neg p \vee q), p, (p \to q)\}$:

$$\cfrac{\cfrac{\cfrac{\cfrac{}{\Gamma; \cdot \vdash_1 (\neg p \vee q)}\text{PREM}}{\Gamma; \cdot \vdash_2 \neg p, q}\vee\text{E} \qquad \cfrac{}{\Gamma; \cdot \vdash_3 p}\text{PREM}}{\Gamma; \cdot \vdash_4 (p \wedge \neg p), q}\wedge\text{I} \qquad \cfrac{}{(p \wedge \neg p) \in \Omega}}{\Gamma; (p \wedge \neg p)^- \vdash_5 q}\text{CR}$$

The derivation above up to stage 4 is a obtained by MinimalND rules. Stage 5 derives a formula on condition of an abnormality being false. This corresponds to changing a multiple conclusion sequent at stage 4 into a single conclusion one at stage 5 by turning one of the conclusions into a condition. This move is justified by the syntactical form of the abnormality declared by the RC rule.

4 Rules for Marking

In standard Adaptive Logics, one introduces strategies to tell, given some Minimal Disjunctions of Abnormalities is derived, which one of those can be disregard, and which one has to be accepted. Adaptive Logics come with marking mechanisms that allow such retractions, according to different possible strategies. The most well-known strategies and their rationale are:

- *Reliability*: once a $Dab(\Delta)^{min}_{s}$ is derived, every formula that at some stage $s-i$ assumes a $\phi \in Dab(\Delta)^{min}_{s}$ to be false, needs to be retracted;
- *Minimal Abnormality*: once a $Dab(\Delta)^{min}_{s}$ is derived, every formula that at some stage $s-i$ assumes a ϕ to be false, such that ϕ is in a minimal set of Δ, needs to be retracted.

In the first case, one considers all possible abnormal formulas to be invalid; in the second case, one tries to minimize the number of such unavoidable contradictions. In this section, we provide rules that extend AdaptiveND in view of the Reliability strategy, providing a proof-theoretical equivalent of the standard marking condition. We leave the definition of a proof-theoretical Minimal Abnormality strategy to a later stage.

4.1 Marking Rule for Reliablity

Reliability is the derivability strategy that takes the most cautious interpretation of abnormalities: any formula that in view of the premises might behave abnormally, because it occurs in a minimal disjunction of abnormalities, is deemed unreliable and should not be asusmed to behave normally. This means in practice that at any proof stage where a formula ψ is derived using some $\phi^{-} \in \Omega$ in context by an instance of the corresponding formation rule, is 'marked'. Here marking means to reject the formula ψ, or invalidate it. In the following we shall introduce a new derivability rule that internalizes this process in AdaptiveND.

We define a new derivability rule $\boxtimes$R that depends on the formulation of the union set of all minimal $Dab(\Delta)^{min}_{i}$ obtained by instances of the MinDab rule above.

$$\frac{\Gamma; \cdot \vdash_{\mathsf{s}} \Delta^{min} \qquad \Gamma; \phi^{-} \vdash_{\mathsf{s'}} \psi \qquad \phi \in \Delta^{min}}{\Gamma \vdash_{\mathsf{max(s,s')+1\boxtimes R}} \psi} \boxtimes\mathrm{R}$$

The meaning of $\boxtimes$R is the following: if at stage s a minimal disjunction of abnormalities Δ is derived for Γ, and at a following stage a formula ψ is derived from the same premise set by assuming a component of Δ^{min}, then at a later stage ψ is marked as retracted.

4.2 Extending the example

We now extend the example from Section 3.2 with a new branch to illustrate the derivation step obtained by a Marking Rule. Let us call $\mathbb{D}$ the derivation already shown. We extend it now as follows:

$$\dfrac{\dfrac{\mathbb{D}}{\Gamma; (p \wedge \neg p)^- \vdash_5 q} \qquad \dfrac{\dfrac{\Gamma; \cdot \vdash_6 p \qquad \Gamma; \cdot \vdash_7 p \to \neg p}{\Gamma; \cdot \vdash_8 \neg p} \to \mathrm{E} \qquad \Gamma; \cdot \vdash_9 p}{\Gamma; \cdot \vdash_{10} p \wedge \neg p} \wedge \mathrm{I}}{\Gamma; \cdot \vdash_{11 \boxtimes \mathsf{R}} q} \boxtimes$$

Notice that it is essential at step 10 that the abnormality be derived under an empty condition, i.e. under context $\Gamma; \cdot$. Moreover, a difference between the Fitch-style proofs standard for Adaptive Logics and the Natural Deduction derivation style becomes here evident. In the former, a marking rule implies the need to proceed backwards on the derivation, to mark all previous occurrences of the marked formula which can no longer be considered derived. In the latter, on the other hand, there is no need to remove formulas because the result obtained at stage 5 cannot be reused in an extension of this proof. Moreover, if we were ever to get again $\Gamma; (p \wedge \neg p)^- \vdash_i q$, that would be obtained by some new derivation $\mathbb{D}'$ and therefore result as a conclusion at some stage $i \neq 5$.

4.3 An example with Dab-selection

Let us consider now a slightly more complex example. We want to show a situation in which two abnormalities can be derived, and accordingly a formula might be marked, or not. Let us start with a premise set $\Gamma = \{(p \vee r), \neg p, (p \vee q), \neg q, (\neg p \to q)\}$. Now consider the following derivation, dubbed $\mathbb{D}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma; \cdot \vdash_1 (p \vee r)}\,\mathrm{PREM}}{\Gamma; \cdot \vdash_2 p, r} \vee\mathrm{E} \qquad \dfrac{}{\Gamma; \cdot \vdash_3 \neg p}\,\mathrm{PREM}}{\Gamma; \cdot \vdash_4 (p \wedge \neg p), r} \wedge\mathrm{I} \qquad \dfrac{}{(p \wedge \neg p) \in \Omega}}{\Gamma; (p \wedge \neg p)^- \vdash_5 r}\,\mathrm{CR}$$

Let us now consider the following derivation, dubbed $\mathbb{D}^{'}$

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma; \cdot \vdash_{6} (p \vee q)}\,\text{PREM}}{\Gamma; \cdot \vdash_{7} p, q}\,\vee\text{E} \qquad \dfrac{}{\Gamma; \cdot \vdash_{8} \neg p}\,\text{PREM}}{\Gamma; \cdot \vdash_{9} (p \wedge \neg p), r}\,\wedge\text{I} \qquad \dfrac{}{\Gamma; \cdot \vdash_{10} \neg q}}{\Gamma; \cdot \vdash_{11} (p \wedge \neg p), (q \wedge \neg q)}\,\text{CR}
$$

If we join now the two branches $\mathbb{D}, \mathbb{D}'$ to form $\mathbb{E}$, we are allowed a marking step:

$$
\dfrac{\dfrac{\mathbb{D}}{\Gamma; (p \wedge \neg p)^{-} \vdash_{5} r} \qquad \dfrac{\mathbb{D}'}{\Gamma; \cdot \vdash_{11} (p \wedge \neg p), (q \wedge \neg q)} \qquad (p \wedge \neg p) \in \Delta^{min}}{\Gamma; \cdot \vdash_{12 \boxtimes \mathsf{R}} r}\,\boxtimes
$$

Now we can provide a further extension of this derivation dubbed $\mathbb{D}''$:

$$
\dfrac{\dfrac{\dfrac{}{\Gamma; \cdot \vdash_{13} \neg p}\,\text{PREM} \qquad \dfrac{}{\Gamma; \cdot \vdash_{14} \neg p \to q}\,\text{PREM}}{\Gamma; \cdot \vdash_{15} q}\,\to \text{I} \qquad \dfrac{}{\Gamma; \cdot \vdash_{16} \neg q}\,\text{PREM}}{\Gamma; \cdot \vdash_{17} q \wedge \neg q}\,\wedge \text{I}
$$

At this stage the abnormality at stage 17 is a minimal one. This also means that if we derive a copy of derivation $\mathbb{D}$, where each step is re-numbered consecutively, and join it to $\mathbb{D}''$ and $\mathbb{E}$, it is possible to derive again the judgement that was marked at stage 12, as follows:

$$
\dfrac{\dfrac{\mathbb{E}}{\Gamma; \cdot \vdash_{12 \boxtimes \mathsf{R}} r} \qquad \dfrac{\mathbb{D}''}{\Gamma; \cdot \vdash_{17} q \wedge \neg q} \qquad \dfrac{\mathbb{D}}{\Gamma; \cdot \vdash_{18} (p \wedge \neg p), r}}{\Gamma; (p \wedge \neg p)^{-} \vdash_{19} r}\,\text{RC}^{*}
$$

where $*$ is the side condition that $(p \wedge \neg p) \in \Omega$.

5 Derivability

The marking condition establishes a dynamic derivability relation. Whenever a certain formula is derived on some $\phi \in \Delta^{min}$ condition in the premises, it might still be marked afterwards according to $\boxtimes R$. This gives us a notion of derivability at stage:

Definition 3 (Derivability at stage). $\Gamma; \phi^- \vdash_{\mathsf{s}} \psi$ *iff at* s *it is not the case that* $\Gamma \vdash_{\mathsf{s}\boxtimes} \psi$.

A more stable notion of derivability holds when marking is no longer possible. To this aim, one requires that the stage s at which a formula ϕ is derived remains unmarked in the possibly infinite extension of the derivation tree, given that all minimal Dab-formulas are derived on the empty condition and ϕ is derived on all (minimal) conditions. We introduce therefore an extension of a standard proof-tree:

Definition 4. $Sf(\Gamma) = \bigcup_{\phi \in \Gamma} \{\psi \mid \psi \textit{ is a subformula of } \phi\}$

Definition 5. $\Omega(\Gamma) = \{\psi \wedge \neg\psi \mid \psi \in Sf(\Gamma)\}$

Definition 6. *Let P be a proof-tree of* $\mathsf{AdaptiveND}$*. We say that P is complete relative to $\Omega(\Gamma)$ at stage* s *if* $\Gamma; \cdot \vdash_{\mathsf{s}'} Dab(\Delta)$*, for every derivable $Dab(\Delta)$ with $\Delta \subseteq \Omega(\Gamma)$ and some $s' < s$.*

Now we can derive our notion of final derivability:

Definition 7 (Final Derivability). *A formula ψ is finally derived $\Gamma; \phi^- \vdash_{\checkmark} \psi$ iff there is a stage s in some complete proof-tree P such that $\Gamma; \phi^- \vdash_{\mathsf{s}} \psi$and $\phi \notin \Delta$, where Δ is the conclusion obtained at some stage of P by an instance of the* MinDab *rule.*

Notice that the second requirement might not be satisfied at any finite stage s, hence it might be guaranteed only at meta-theoretical level.

Definition 8. $\Gamma \vdash_{\mathsf{AdaptiveND}} \phi$ *iff* $\Gamma; \Omega^- \vdash_{\checkmark} \phi$

Notice that not every finally derivable formula of $\mathsf{AdaptiveND}$ is derivable at stage. In particular, for the case of $\mathsf{AdaptiveND}$, where the disjunction used to construct *Dab*-formulas is explicitly admitted in the premise set by the extension of a given Γ with $Dab(\Delta)^-$ on the left-hand side of $\vdash$, one can conceive of cases in which a finally derivable ϕ is not derivable at any finite stage of a proof-tree.[4]

References

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[4]For an example see Strasser PHD, ch.2.8.

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