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Computability Issues for Adaptive Logics in Expanded Standard Format

Abstract. In a rather general setting, we prove a number of fundamental theorems concerning computational complexity of derivability in adaptive logics. For that setting, the so-called standard format of adaptive logics is suitably extended, and the corresponding completeness results are established in a very uniform way.

Keywords: adaptive logics, dynamic reasoning, standard format, reliability strategy, minimal abnormality strategy, computational complexity, expressiveness.

Introduction and overview

Adaptive logic is a well-developed approach to non-monotonic (and, in effect, dynamic) argumentation, which may be viewed as a unifying tool for capturing the idea of default reasoning (cf. [1]). Naturally, since the logics under consideration are non-monotonic, their consequence relations tend to be rather complicated—this, of course, raises the task of measuring their computational complexity. Surprisingly, there still exist just a few works devoted to issues of computability in adaptive logics. But an even more surprising fact is that one of the first articles in this direction [6] (restring attention to inconsistency-adaptive logics) was aimed at criticizing the importance of this branch of logic for applications.

In their work [6], L. Horsten and P. Welsh were interested in complexity of the consequences in the adaptive logic $CLuN^{\mathbf{r}}$ (with the propositional weak paraconsistent logic $CLuN$ being its lower limit logic, and supplied with the reliability strategy). They proved that there is an infinite computable set Γ of premisses s. t. the collection of all its $CLuN^{\mathbf{r}}$-consequences is Σ^0_3-hard, and this estimation turns out to be exact (viz. we have Σ^0_3-completeness). Later, P. Verdee [10] showed the adaptive logic $CLuN^{\mathbf{m}}$ (based on the same logic $CLuN$, but supplied the minimal abnormality strategy) is significantly much more complex—he suggested a computable set of premisses for which the collection of $CLuN^{\mathbf{m}}$-consequences is Π^1_1-hard and, thus, even not arithmetical (note that such a complexity is typical for different kinds of model-theoretic

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logics). The fact that Π^1_1 can be achieved at the propositional level is quite remarkable, and the expressive power of $CLuN^{\mathbf{m}}$ and other propositional adaptive logics based on the minimal abnormality strategy worth paying special attention in subsequent research.

Previously, the algorithmic properties of inconsistency-adaptive logics $CLuN^{\mathbf{r}}$ and $CLuN^{\mathbf{m}}$ were investigated in [7]. Namely, we provided very simple (and straightforward) proofs for decidability of the finitary consequence relations of both logics, and established general connections between the complexity of a set of premisses and that of its $CLuN^{\mathbf{r}}$- and $CLuN^{\mathbf{m}}$-consequences. E. g., we proved that, whenever there are only finitely many formulas unreliable w. r. t. a set of premisses Γ, then one can essentially reduce the complexity bound for its $CLuN^{\mathbf{m}}$-consequences—viz. they form a set computably enumerable relative to Γ (as an oracle). The main goal of what follows is to obtain similar results in a much broader context, for the class of adaptive logics being as wide as possible.

The rest of the paper is organized as follows. Section 1 contains preliminary material on computability, including description of the arithmetical hierarchy and its connections to the first-order definability. In Section 2, we present an expanded version of the standard format for adaptive logics (cf. [2])—particularly, our extension allows to define adaptive logics by means of multi-consequence relations and also avoids the traditional requirement concerning the presence of classically treated connectives in the lower limit logics. Here, starting with a lower limit logic **LLL** that satisfies the strong completeness property, we establish the completeness theorems for the adaptive logics $\mathbf{LLL}^{\mathbf{r}}$ and $\mathbf{LLL}^{\mathbf{m}}$, i. e., **LLL** supplied with the reliability strategy and the minimal abnormality strategy, respectively. Section 3 is devoted to providing complexity upper bounds for derivability in adaptive logics and related notions. For instance, we prove that both $\mathbf{LLL}^{\mathbf{r}}$ and $\mathbf{LLL}^{\mathbf{m}}$ are finitely decidable, whenever **LLL** is finitely decidable and satisfies the so-called property of local abnormalities. Also, letting **LLL** be finitely enumerable and assuming various restrictions on a set of premisses Γ, the estimations for computational complexity of $\mathbf{LLL}^{\mathbf{r}}$- and $\mathbf{LLL}^{\mathbf{m}}$-consequences of Γ are given. In particular, in a general form, we answer the question of Christian Straßer about the complexity of $\mathbf{LLL}^{\mathbf{m}}$-consequences for Γ with $\Phi(\Gamma)$ consisting of finite sets only (see Section 2)— originally, the question was about $CLuN^{\mathbf{m}}$. We conclude with a survey of results on complexity lower bounds in Section 4—there it is shown that the principal estimations obtained earlier in Section 3 appear to be exact and, indeed, can be achieved by considering $CLuN^{\mathbf{r}}$ and $CLuN^{\mathbf{m}}$, probably most basic and the simplest (inconsistency-)adaptive logics.

1. Preliminaries on computability

We assume the reader is familiar with the basics of computability theory—cf. [5, 9]. Still, it is reasonable to outline the definition of the *arithmetical hierarchy*. Say that an n-ary relation R on the set of natural numbers ω is Σ^0_1 (or is *in* Σ^0_1) iff it can be obtained as a projection of a $(n+1)$-ary computable relation, i. e.,

$$R \;=\; \{(m_1,\ldots,m_n) \mid \exists x\,((m_1,\ldots,m_n,x) \in S)\}$$

for some computable $S \subseteq \omega^{n+1}$. Then, $R \subseteq \omega^n$ is (in) Π^0_1 iff its complement $\overline{R} := \omega^n \setminus R$ appears to be Σ^0_1. Next, Σ^0_{k+1} consists precisely of all projections of Π^0_k-relations, and the elements of Π^0_{k+1} are just complements of those in Σ^0_{k+1}. In fact, taking into account that the class of Σ^0_k-relations is closed under projections, one can easily prove that any relation

$$\{\overline{m} \mid (\exists x_1 \ldots \exists x_{n_1})\,(\forall y_1 \ldots \forall y_{n_2}) \ldots R\,(x_1,\ldots,x_{n_1},y_1,\ldots,y_{n_2},\ldots,\overline{m})\}$$

defined via a computable R preceded by the prefix with k alternations of (blocks of) quantifiers, and starting with an existential quantifier ($\exists$), is in Σ^0_{k+1}. Clearly, in a similar situation when the prefix starts with a universal quantifier ($\forall$), we arrive at Π^0_{k+1}-relations. The two families

$$\left\{\Sigma^0_{k+1} \mid k \in \omega\right\} \quad \text{and} \quad \left\{\Pi^0_{k+1} \mid k \in \omega\right\}\,,$$

together with the class $\Sigma^0_0 = \Pi^0_0$ of all computable relations, form the so-called arithmetical hierarchy. Let us denote $\Sigma^0_k \cap \Pi^0_k$ by Δ^0_k. Remark that Σ^0_1 coincides with the class of all *computably enumerable* (*c. e.*, for short) relations, while $\Delta^0_1 = \Sigma^0_0 = \Pi^0_0$ (due to Post's theorem).

Now if one begins with the class of all sets computable with respect to (w. r. t.) a fixed *oracle* $X \subseteq \omega$ (e. g., see [5, Chapter 10] or [9, Chapter 9]), instead of the computable sets, it results in the definition of the *relativised w. r. t. X arithmetical hierarchy* consisting of the families

$$\left\{\Sigma^{0,X}_k \mid k \in \omega\right\} \quad \text{and} \quad \left\{\Pi^{0,X}_k \mid k \in \omega\right\}.$$

A set which belongs to one of the classes in the (relativised w. r. t. X) arithmetical hierarchy is called *arithmetical* (*w. r. t. X*).

Here is a well-known presentation of arithmetical sets: $R \subseteq \omega^n$ is Σ^0_k (Π^0_k) iff there is an arithmetical $\Sigma_k(\Pi_k)$-formula $\Psi\,(x_1,\ldots,x_n)$ s. t.

$$R \;=\; \{(m_1,\ldots,m_n) \mid \mathfrak{N} \Vdash_{\mathrm{FOL}} \Psi\,(m_1,\ldots,m_n)\}\,,$$

where $\mathfrak{N} := \langle \omega; +, \times, 0, 1, \leqslant \rangle$ is the *standard model of arithmetic*. Thus, the arithmetical sets are those definable via arithmetical first-order formulae.

A relation $R \subseteq \omega^n$ is called Π^1_1 iff there is a second-order arithmetical formula $\Psi(x_1, \ldots, x_n, P)$ with the only predicate variable P (which is free, and no set quantifiers occur in Ψ) s. t.

$$S \;=\; \{(m_1, \ldots, m_n) \mid \mathfrak{N} \Vdash_{\mathrm{SOL}} \forall P \, \Psi(m_1, \ldots, m_n, P)\}$$

(here P ranges over all subsets of natural numbers).

Similarly, R is $\Pi^{1,X}_1$ (i. e., is Π^1_1-definable w. r. t. X) iff

$$R \;=\; \{(m_1, \ldots, m_n) \mid \mathfrak{N} \Vdash_{\mathrm{SOL}} \forall P \, \Psi(m_1, \ldots, m_n, P, Q)\,[Q/X]\},$$

where $\Psi(x_1, \ldots, x_n, P, Q)$ is a second-order arithmetical formula with only two predicate variables P and Q, and Q is interpreted by X in $\mathfrak{N}$ (so X plays the role of a second-order parameter).

Henceforth by *bounded quantifies* we mean all expressions of the sorts

$$\exists x \leqslant y\,, \quad \forall x \leqslant y\,, \quad \exists x < y \quad \text{and} \quad \forall x < y\,.$$

For $\alpha(x, y) \in \{x \leqslant y\,,\, x < y\}$ and an arithmetical (first- or second-order) formula Ψ, let $(\exists \alpha(x, y))\,\Psi$ and $(\exists \alpha(x, y))\,\Psi$ abbreviate the formulas

$$\exists x\,(\alpha(x, y) \wedge \Psi) \quad \text{and} \quad \forall x\,(\neg \alpha(x, y) \vee \Psi)\,,$$

respectively.[1] Recall that the classes in the (relativised w. r. t. X) arithmetical hierarchy are closed under bounded quantification, and even computable terms may be used in place of 'y'. Namely, for every arithmetical $\Sigma_k(\Pi_k)$-formula $\Psi(x, \overline{y}, \overline{z})$, where $\overline{y} := (y_1, \ldots, y_n)$ and $\overline{z} := (z_1, \ldots, z_l)$, and every computable function f of n arguments, the sets

$$\{(\overline{m}, \overline{s}) \mid \mathfrak{N} \Vdash_{\mathrm{FOL}} (\exists \alpha(x, f(\overline{m})))\,\Psi(x, \overline{m}, \overline{s})\}\,,$$
$$\{(\overline{m}, \overline{s}) \mid \mathfrak{N} \Vdash_{\mathrm{FOL}} (\forall \alpha(x, f(\overline{m})))\,\Psi(x, \overline{m}, \overline{s})\}\,,$$

belong to the same class Σ^0_k (Π^0_k) in the arithmetical hierarchy; and the analogous result holds for the relativised arithmetical hierarchy.

2. Expanded standard format for adaptive logics

We begin with presenting the so-called *standard format of adaptive logics*—however, this will be done in a more general setting than in [2], omitting

[1] Note that $x < y$ is, in turn, a shorthand for $(x \leqslant y \wedge x \neq y)$.

several commonly used assumptions about the underlying language. E. g., we suppose that the *lower limit logic* is characterized in terms of a multi-consequence relation, i. e., a binary relation between the sets of formulas. Thus, one may avoid mentioning the disjunction connective ($\vee$) when describing various adaptive logics. Still, if $\vee$ is already in the language, then, using a (one-)consequence relation $\vdash$ (viz. between the sets of formulas and the formulas), the multi-consequence can be defined as:

$$\Gamma \vdash \Delta \quad \text{iff} \quad \Gamma \vdash A_1 \vee \ldots \vee A_n \ \text{ for some } \{A_1, \ldots, A_n\} \subseteq \Delta\,,$$

where Γ and Δ are sets of formulae. This definition of multi-consequence relation assumes that the set of consequences Δ is non-empty. We will assume that all considered multi-consequence relations relation satisfy that restriction. Of course, the presence of classically treated connectives (say, the classical negation) is also not mandatory.

Fix a language $\mathcal{L}$, with the set of $\mathcal{L}$-formulas denoted by $For_{\mathcal{L}}$. Let **LLL** be a *lower limit logic* in $\mathcal{L}$, which is a monotonic logic (in $\mathcal{L}$) supplied with a multi-consequence relation $\vdash_{\mathbf{LLL}}$ (between the sets of $\mathcal{L}$-formulae), a suitable class of **LLL**-models $\mathcal{K}_{\mathbf{LLL}}$, and a satisfiability relation $\Vdash_{\mathbf{LLL}}$ (between the **LLL**-models and the $\mathcal{L}$-formulas). Here we require that $\vdash_{\mathbf{LLL}}$ satisfies certain standard properties, namely

- $\Gamma \vdash_{\mathbf{LLL}} \Delta \implies \Delta \neq \varnothing$ (*non-empty consequence*)
- $A \in \Gamma \implies \Gamma \vdash_{\mathbf{LLL}} \{A\}$ (*reflexivity*);
- $\Gamma \vdash_{\mathbf{LLL}} \{A\} \cup \Delta$ for all $A \in \Gamma'$, and $\Gamma' \vdash_{\mathbf{LLL}} \Delta' \implies \Gamma \vdash_{\mathbf{LLL}} \Delta \cup \Delta'$ (*transitivity*);
- $\Gamma \vdash_{\mathbf{LLL}} \Delta$, $\Gamma \subseteq \Gamma'$, and $\Delta \subseteq \Delta' \implies \Gamma' \vdash_{\mathbf{LLL}} \Delta'$ (*monotonicity*);
- $\Gamma \vdash_{\mathbf{LLL}} \Delta \implies \Gamma' \vdash_{\mathbf{LLL}} \Delta$ for some finite $\Gamma' \subseteq \Gamma$ (*left-compactness*);
- $\Gamma \vdash_{\mathbf{LLL}} \Delta \implies \Gamma \vdash_{\mathbf{LLL}} \Delta'$ for some finite $\Delta' \subseteq \Delta$ (*right-compactness*).

Further, the *semantical consequence relation* $\vDash_{\mathbf{LLL}}$: $\Gamma \vDash_{\mathbf{LLL}} \Delta$ holds iff for every $\mathcal{M} \in \mathcal{K}_{\mathbf{LLL}}$, if $\mathcal{M} \Vdash_{\mathbf{LLL}} \Gamma$, then $\mathcal{M} \Vdash_{\mathbf{LLL}} A$ for some $A \in \Delta$.

Finally, assume that the above two (syntactical and semantical) consequence relations coincide, i. e., we have the (strong) completeness theorem: for any $\Gamma \cup \Delta \subseteq For_{\mathcal{L}}$ with $\Delta \neq \varnothing$,

$$\Gamma \vdash_{\mathbf{LLL}} \Delta \quad \Longleftrightarrow \quad \Gamma \vDash_{\mathbf{LLL}} \Delta\,.$$ [2]

[2] In effect, $\vdash_{\mathbf{LLL}} = \vDash_{\mathbf{LLL}}$ will be a subrelation of the associated adaptive consequence relations we are aiming to define.

Remark that there is nothing extraordinary in this property, since, whenever the completeness result for a single-consequence relation is established by means of the canonical model method, very often the whole construction may be easily extended to the associated multi-consequence relation. Many basic examples of completeness results of this kind for monotonic non-classical logics, both modal and non-modal ones, can be found in Part I of [4]. In what follows, an expression $\mathcal{M} \Vdash_{\mathbf{LLL}} \Gamma$ means that $\mathcal{M} \Vdash_{\mathbf{LLL}} A$ for all $A \in \Gamma$, and (at times) we will write $\Gamma \vdash_{\mathbf{LLL}} A$ and $\Gamma \vDash_{\mathbf{LLL}} A$ instead of $\Gamma \vdash_{\mathbf{LLL}} \{A\}$ and $\Gamma \vDash_{\mathbf{LLL}} \{A\}$, correspondingly.

Let us fix a collection $\Omega \subseteq For_{\mathcal{L}}$ the elements of which will be called *abnormalities*. Since it is commonly assumed that these are distinguished by their syntactical form (for instance, Ω may consist of all $\mathcal{L}$-formulas of the sort $A \wedge \neg A$), Ω is supposed to be decidable.

For $\Delta \cup \Gamma \subseteq For_{\mathcal{L}}$, let $\Delta \subseteq_{fin} \Gamma$ be a shorthand for 'Δ is a finite subset of Γ'. A non-empty $\Delta \subseteq_{fin} \Omega$ is a *minimal Ab-consequence of* Γ iff $\Gamma \vdash_{\mathbf{LLL}} \Delta$ and there is no $\Delta' \subset \Delta$ s. t. $\Gamma \vdash_{\mathbf{LLL}} \Delta'$.[3] We employ the following notation:

$$\begin{aligned} \Sigma(\Gamma) &:= \{\Delta \mid \Delta \text{ is a minimal } Ab\text{-consequence of } \Gamma\}, \\ U(\Gamma) &:= \{A \in For_{\mathcal{L}} \mid A \in \Delta \text{ for some } \Delta \in \Sigma(\Gamma)\} \end{aligned}$$

(the elements of the latter are said to be *unreliable with respect to* Γ).

Take $Ab(\mathcal{M})$ to be $\{A \in \Omega \mid \mathcal{M} \Vdash_{\mathbf{LLL}} A\}$, for each **LLL**-model $\mathcal{M}$. An **LLL**-model $\mathcal{M}$ of Γ (viz. $\mathcal{M} \Vdash_{\mathbf{LLL}} \Gamma$ holds) is *reliable* iff $Ab(\mathcal{M}) \subseteq U(\Gamma)$, and is *minimally abnormal* iff there is no (other) **LLL**-model $\mathcal{M}'$ of this Γ with $Ab(\mathcal{M}') \subset Ab(\mathcal{M})$. Now we are ready to define semantically the two associated adaptive multi-consequence relations: for $\Gamma \cup \Delta \subseteq For_{\mathcal{L}}$ with $\Delta \neq \varnothing$, $\Gamma \vDash_{\mathbf{LLL^r}} \Delta$ ($\Gamma \vDash_{\mathbf{LLL^m}} \Delta$) iff for every reliable (minimally abnormal, respectively) model $\mathcal{M}$ of Γ, there exists $A \in \Delta$ s. t. $\mathcal{M} \Vdash_{\mathbf{LLL}} A$.

In this way, $\vDash_{\mathbf{LLL^r}}$ provides the semantics for the adaptive logic $\mathbf{LLL^r}$ based on the lower limit logic **LLL**, the collection of abnormalities Ω, and augmented by the *reliability strategy*. Similarly, $\vDash_{\mathbf{LLL^m}}$ corresponds to the adaptive logic $\mathbf{LLL^m}$ which is based on the same lower limit logic and abnormalities, but exploits a different strategy of handling the abnormalities involved, namely the *minimal abnormality strategy*.

The next criterion for the semantical $\mathbf{LLL^r}$-consequence generalizes an analogous particular result for the adaptive logic $CLuN^{\mathbf{r}}$ (where $\mathbf{LLL} := CLuN$ and $\Omega := \{A \wedge \neg A \mid A \in For_{CL}\}$) established previously in [3].

[3] Here and below '$S_1 \subset S_2$' always stands for '$S_1 \subseteq S_2$ and $S_1 \neq S_2$'.

THEOREM 2.1. *For any* $\Gamma \cup \Delta \subseteq For_{\mathcal{L}}$ *with* $\Delta \neq \varnothing$, $\Gamma \vDash_{\mathbf{LLL}^{\mathrm{r}}} \Delta$ *iff there exists* $\Theta \subseteq_{fin} \Omega$ *s. t.* $\Gamma \vdash_{\mathbf{LLL}} \Delta \cup \Theta$ *and* $\Theta \cap U(\Gamma) = \varnothing$.

PROOF. $\boxed{\Rightarrow}$ Assume that $\Gamma \vdash_{\mathbf{LLL}} \Delta \cup \Theta$ for $\Theta \subseteq_{fin} \Omega \setminus U(\Gamma)$. The strong completeness implies $\Gamma \vDash_{\mathbf{LLL}} \Delta \cup \Theta$. Let $\mathcal{M}$ be a reliable model of Γ. Then $Ab(\mathcal{M}) \subseteq U(\Gamma)$ and $\mathcal{M} \Vdash_{\mathbf{LLL}} A$, where $A \in \Delta \cup \Theta$. Since $\Theta \cap U(\Gamma) = \varnothing$, we have $\Theta \cap Ab(\mathcal{M}) = \varnothing$. Consequently, $\mathcal{M} \Vdash_{\mathbf{LLL}} A$ for some $A \in \Delta$. We have thus proved $\Gamma \vDash_{\mathbf{LLL}^{\mathrm{r}}} \Delta$.

$\boxed{\Leftarrow}$ Let $\Gamma \nvdash_{\mathbf{LLL}} \Delta \cup \Theta$ for all $\Theta \subseteq_{fin} \Omega \setminus U(\Gamma)$. The right compactness and the monotonicity of $\vdash_{\mathbf{LLL}}$ imply that $\Gamma \nvdash_{\mathbf{LLL}} \Delta \cup (\Omega \setminus U(\Gamma))$. By strong completeness we have $\Gamma \nvDash_{\mathbf{LLL}} \Delta \cup (\Omega \setminus U(\Gamma))$. Consequently, there is a model $\mathcal{M}$ of Γ such that $\mathcal{M} \nVdash_{\mathbf{LLL}} A$ for all $A \in \Delta \cup (\Omega \setminus U(\Gamma))$. This implies, in particular, that $Ab(\mathcal{M}) \subseteq U(\Gamma)$. We have thus found out a reliable model of Γ that refutes all $A \in \Delta$, i. e., $\Gamma \nvDash_{\mathbf{LLL}^{\mathrm{r}}} \Delta$. ■

Before moving on, to a semantical criterion for the minimal abnormality strategy, we need to say a few words about so-called ‘choice sets’. Taking Σ to be a family of sets, a set Δ is a *choice set for* Σ iff for any $\varphi \in \Sigma$, $\Delta \cap \varphi \neq \varnothing$. Further, such a choice set Δ is *minimal* (for Σ) iff there is no other choice set Δ' for Σ with $\Delta' \subset \Delta$.

It is well-known that for an arbitrary family of finite sets, there exists a minimal choice set for it—see, e.g., [1, Fact 5.1.2]. The following fact is an obvious strengthening of [1, Fact 5.1.3].

PROPOSITION 2.2. *Let* Σ *be a family of sets. A choice set* Δ *for* Σ *is minimal iff for each* $a \in \Delta$, *there exists* $\varphi \in \Sigma$ *s. t.* $\Delta \cap \varphi = \{a\}$.

For every $\Gamma \subseteq For_{\mathcal{L}}$, we denote the collection of all minimal choice sets for the family $\Sigma(\Gamma)$ by $\Phi(\Gamma)$. It turns out that the elements of $\Phi(\Gamma)$ are precisely those sets that can be represented as a set of abnormalities true in some minimally abnormal model of Γ.

PROPOSITION 2.3. *Let* $\Gamma \subseteq For_{\mathcal{L}}$. *Then*

$$\Phi(\Gamma) \;=\; \{Ab(\mathcal{M}) \mid \mathcal{M} \text{ is a minimally abnormal } \mathbf{LLL}\text{-model of } \Gamma\}\,.$$

PROOF. First we notice that $Ab(\mathcal{M})$ is a choice set for $\Sigma(\Gamma)$ for every $\mathcal{M} \in \mathcal{K}_{\mathbf{LLL}}$ such that $\mathcal{M} \Vdash_{\mathbf{LLL}} \Gamma$. Indeed, if $\Delta \in \Sigma(\Gamma)$, then $\Gamma \vdash_{\mathbf{LLL}} \Delta$ and by strong completeness $\Gamma \vDash_{\mathbf{LLL}} \Delta$. Consequently, $\mathcal{M} \Vdash_{\mathbf{LLL}} A$ for some $A \in \Delta$, i. e., $Ab(\mathcal{M}) \cap \Delta \neq \varnothing$.

Let $\varphi \in \Phi(\Gamma)$. We show that there is a minimally abnormal model $\mathcal{M}$ of Γ with $Ab(\mathcal{M}) = \varphi$.

Prove that $\Gamma \nvdash_{\mathbf{LLL}} \Omega \setminus \varphi$. If it does not hold, then by the right compactness there is a non-empty $\Delta \subseteq_{fin} \Omega$ such that $\Gamma \vdash_{\mathbf{LLL}} \Delta$ and $\Delta \cap \varphi = \varnothing$. Then there is a $\Delta' \subseteq \Delta$ such that $\Gamma \vdash_{\mathbf{LLL}} \Delta'$ and $\Delta' \in \Sigma(\Gamma)$. In this case $\Delta' \cap \varphi = \varnothing$, which conflicts with the fact that φ is a choice set for $\Sigma(\Gamma)$. The obtained contradiction proves $\Gamma \nvdash_{\mathbf{LLL}} \Omega \setminus \varphi$.

By strong completeness there is a model $\mathcal{M}$ of Γ such that $\mathcal{M} \nVdash_{\mathbf{LLL}} A$ for all $A \in \Omega \setminus \varphi$. Consequently, $Ab(\mathcal{M}) \subseteq \varphi$. Since $Ab(\mathcal{M})$ is a choice set for $\Sigma(\Gamma)$ and φ is a minimal choice set, we conclude that $\varphi = Ab(\mathcal{M})$ and $\mathcal{M}$ is a minimally abnormal model of Γ.

Now we take a minimally abnormal model $\mathcal{M}$ of Γ and prove that $Ab(\mathcal{M}) \in \Phi(\Gamma)$. We know that $Ab(\mathcal{M})$ is a choice set for $\Sigma(\Gamma)$. If this choice set is not minimal, then there is $\varphi \in \Sigma(\Gamma)$ such that $\varphi \subset Ab(\mathcal{M})$. It was proved above that $\varphi = Ab(\mathcal{M}')$ for some minimally abnormal model $\mathcal{M}'$ of Γ, which contradicts to the minimal abnormality of $\mathcal{M}$. ∎

Corollary 2.4. *Every minimally abnormal model of* $\Gamma \subseteq For_{\mathcal{L}}$ *is reliable.*

Proof. If $\mathcal{M}$ is a minimally abnormal model of Γ, then by the previous proposition $Ab(\mathcal{M})$ is a minimal choice set for $\Sigma(\Gamma)$. In particular, $Ab(\mathcal{M}) \subseteq \bigcup \Sigma(\Gamma)$. By definition $U(\Gamma) = \bigcup \Sigma(\Gamma)$, consequently, $\mathcal{M}$ is a reliable model of Γ. ∎

Finally, it's time for the semantical criterion of $\mathbf{LLL^m}$-consequence—for which, the partial case of $CLuN^{\mathbf{m}}$ (in turn) was proved in [3].

Theorem 2.5. *For any* $\Gamma \cup \Delta \subseteq For_{\mathcal{L}}$ *with* $\Delta \neq \varnothing$, $\Gamma \vDash_{\mathbf{LLL^m}} \Delta$ *iff for each* $\varphi \in \Phi(\Gamma)$, *there exists* $\Theta \subseteq_{fin} \Omega$ *s. t.* $\Gamma \vdash_{\mathbf{LLL}} \Delta \cup \Theta$ *and* $\Theta \cap \varphi = \varnothing$.

Proof. $\boxed{\Rightarrow}$ Suppose there exists $\varphi \in \Phi(\Gamma)$ such that for every $\Theta \subseteq_{fin} \Omega \setminus \varphi$, we have $\Gamma \nvdash_{\mathbf{LLL}} \Delta \cup \Theta$. By the right compactness of $\vdash_{\mathbf{LLL}}$ we conclude that $\Gamma \nvdash_{\mathbf{LLL}} \Delta \cup (\Omega \setminus \varphi)$. Hence, due to the strong completeness, there is a model $\mathcal{M}$ of Γ that refutes all elements of $\Delta \cup (\Omega \setminus \varphi)$. Particularly, $Ab(\mathcal{M}) \subseteq \varphi$. From Proposition 2.3 it follows that $Ab(\mathcal{M}) = \varphi$ and $\mathcal{M}$ is a minimally abnormal model of Γ. Since $\mathcal{M} \nVdash_{\mathbf{LLL}} A$ for all $A \in \Delta$ we proved that $\Gamma \nvDash_{\mathbf{LLL^m}} \Delta$.

$\boxed{\Leftarrow}$ Assume that for every $\varphi \in \Phi(\Gamma)$, there exists $\Theta \subseteq_{fin} \Omega$ with the property: $\Gamma \vdash_{\mathbf{LLL}} \Delta \cup \Theta$ and $\Theta \cap \varphi = \varnothing$. If there is a minimally abnormal model $\mathcal{M}$ of Γ such that $\mathcal{M} \nVdash_{\mathbf{LLL}} A$ for all $A \in \Delta$, then $Ab(\mathcal{M}) \in \Phi(\Gamma)$ and so $\Gamma \vdash_{\mathbf{LLL}} \Delta \cup \Theta$ for some $\Theta \subseteq_{fin} \Omega$ with $\Theta \cap Ab(\mathcal{M}) = \varnothing$. By the strong completeness we have $\Gamma \vDash_{\mathbf{LLL}} \Delta \cup \Theta$. Since $\mathcal{M} \nVdash_{\mathbf{LLL}} A$ for all $A \in \Delta$, we obtain $\mathcal{M} \Vdash_{\mathbf{LLL}} B$ for $B \in \Theta$ which conflicts with $\Theta \cap Ab(\mathcal{M}) = \varnothing$. ∎

Further, we describe proof procedures for the adaptive logics $\mathbf{LLL^r}$ and $\mathbf{LLL^m}$. A crucial notion here is that of a 'stage of proof' (from a given set of premisses). Namely, for every $\Gamma \subseteq For_{\mathcal{L}}$, a *stage of proof from* Γ is represented by a sequence s (finite or infinite) of lines, each of which is a tuple consisting of the *five components*:

i. its number (that is, a natural number);

ii. head (a non-empty finite set of $\mathcal{L}$-formulas);

iii. line numbers for premisses (a string of natural numbers);

iv. name of adaptive rule (PREM, RU, or RC);

v. condition (a finite subset of abnormalities—those from Ω), —

and, moreover, any such line must be of one of the following *three types*:

$$\text{(i) } n\,,\ \ \text{(ii) } \{A\}\,,\ \ \text{(iii) } \text{—}\,,\ \ \text{(iv) PREM}\,,\ \ \text{(v) } \varnothing \tag{PREM}$$

where n is its number in the sequence s, and A belongs to Γ;

$$\text{(i) } n\,,\ \ \text{(ii) } \Delta,\ \ \text{(iii) } i_1,\ldots,i_k\,,\ \ \text{(iv) RU}\,,\ \ \text{(v) } \Theta_1 \cup \cdots \cup \Theta_k \tag{RU}$$

where n is its number in s, the heads and conditions of lines numbered by $i_1,\ldots,i_k < n$ in s are $\{A_1\},\ldots,\{A_k\}$ and $\Delta_1,\ldots,\Delta_k$, respectively, and $\{A_1,\ldots,A_k\} \vdash_{\mathbf{LLL}} \Delta$, with $\varnothing \neq \Delta \subseteq_{fin} For_{\mathcal{L}}$;

$$\text{(i) } n\,,\ \ \text{(ii) } \Delta,\ \ \text{(iii) } i_1,\ldots,i_k\,,\ \ \text{(iv) RC}\,,\ \ \text{(v) } \Theta_1 \cup \cdots \cup \Theta_k \cup \Theta \tag{RC}$$

where n is its number in s, the heads and conditions of lines numbered by $i_1,\ldots,i_k < n$ in s are $\{A_1\},\ldots,\{A_k\}$ and $\Delta_1,\ldots,\Delta_k$, respectively, and $\{A_1,\ldots,A_k\} \vdash_{\mathbf{LLL}} \Delta \cup \Theta$, with $\varnothing \neq \Delta \subseteq_{fin} For_{\mathcal{L}}$ and $\Theta \subseteq_{fin} \Omega$.[4]

In case a stage of proof s (for Γ fixed) contains a line numbered i with a head Δ and a condition Θ, we say that Δ *is derived in s at line i under condition* Θ. A stage of proof s' is called an *extension of* s iff the sequence of lines of s forms a subsequence of that of s' (whenever all the (i)-st and (iii)-rd components of lines in s are appropriately renumbered).

Proposition 2.6. *Let $\Gamma \subseteq For_{\mathcal{L}}$, $\varnothing \neq \Delta \subseteq_{fin} For_{\mathcal{L}}$, and $\Theta \subseteq_{fin} \Omega$. Then $\Gamma \vdash_{\mathbf{LLL}} \Delta \cup \Theta$ iff there exists a finite stage of proof from Γ s. t. Δ is derived in this stage at some line under condition Θ.*

[4]Remark that $k \in \omega$, so the tuple $(i_1,\ldots,i_k)$ may be empty.

PROOF. Suppose $\Gamma \vdash_{\mathbf{LLL}} \Delta \cup \Theta$. Since the relation $\vdash_{\mathbf{LLL}}$ is left-compact there are formulas $\{A_1, \ldots, A_n\} \subseteq \Gamma$ such that $\{A_1, \ldots, A_n\} \vdash_{\mathbf{LLL}} \Delta \cup \Theta$. We may start a stage of proof with lines: i, $\{A_i\}$, —, PREM, $\varnothing$; where $i \in \{1, \ldots, n\}$. Then we add the following line:

$$n + 1\,,\ \Delta\,,\ \langle 1, \ldots, n \rangle\,,\ \mathrm{RC}\,,\ \Theta\,.$$

We have thus constructed the stage s of proof from Γ such that Δ is derived at line $n + 1$ of s under condition Θ.

Now we assume that s is a stage of proof from Γ such that Δ is derived at line i of this stage under condition Θ. The fact that $\Gamma \vdash_{\mathbf{LLL}} \Delta \cup \Theta$ can be proved by induction on i using the reflexivity and the transitivity of $\vdash_{\mathbf{LLL}}$. ■

Remark that the notion of a stage of proof does not depend on the strategy of handling abnormalities. Rather, the two strategies are involved (in the adaptive proof theory) in the form of 'marking definitions'.

We start with the reliability strategy. Suppose s is a stage of proof from $\Gamma \subseteq For_{\mathcal{L}}$. A non-empty $\Delta \subseteq_{fin} \Omega$ is called a *minimal Ab-consequence at s* iff it is derived at some line in s under the empty condition, and no proper subset $\Delta' \subset \Delta$ has that property (i. e., is derived at some line in s under the empty condition). Take

$$\begin{aligned} \Sigma_s \ &:= \ \{\Delta \mid \Delta \text{ is a minimal } Ab\text{-consequence at } s\}\,; \\ U_s \ &:= \ \{A \in For_{\mathcal{L}} \mid A \in \Delta \text{ for some } \Delta \in \Sigma_s\} \end{aligned}$$

(the $\mathcal{L}$-formulae from U_s are said to be *unreliable at s*).[5] Henceforth, in a context where no confusion may arise, lines (of a given stage of proof s) are named by their numbers, at times.

DEFINITION 2.7. An i-th line of a finite stage s of proof from Γ is $\mathbf{r}$-*marked* (or *marked according to the reliability strategy*) *in s* iff $\Delta \cap U_s \neq \varnothing$, where Δ is the condition for the i-th line. Whenever i is not $\mathbf{r}$-marked in s, the term $\mathbf{r}$-*unmarked* will also be used, for convenience.

DEFINITION 2.8. A non-empty $\Delta \subseteq_{fin} For_{\mathcal{L}}$ is *finally* $\mathbf{LLL^r}$*-derived in a finite stage s of proof from* Γ iff Δ is derived at some line i of s, and the following requirements are satisfied:

[5]Here, we write U_s instead of $U_s(\Gamma)$, which is more widely used, to emphasize that this collection is determined solely by the stage s, while the full set of premisses Γ is not indeed required. Similarly, we employ the notation Φ_s instead of $\Phi_s(\Gamma)$ below.

- the i-th line (or simply 'the line i') is not **r**-marked in s;
- any finite extension of s, in which i becomes **r**-marked, may be further finitely extended so that this line will turn out to be **r**-unmarked again (after suitable renumberings are made, of course).

DEFINITION 2.9. A non-empty $\Delta \subseteq_{fin} For_{\mathcal{L}}$ is *finally* $\mathbf{LLL^r}$*-derivable from* Γ (we denote this by $\Gamma \vdash_{\mathbf{LLL^r}} \Delta$) iff it can be finally $\mathbf{LLL^r}$-derived in some finite stage s of proof from Γ.

Now a syntactical variant of Theorem 2.1 (i. e., a criterion for the final $\mathbf{LLL^r}$-derivability) can be established.

THEOREM 2.10. *For any* $\Gamma \subseteq For_{\mathcal{L}}$ *and non-empty* $\Delta \subseteq_{fin} For_{\mathcal{L}}$*, we have* $\Gamma \vdash_{\mathbf{LLL^r}} \Delta$ *iff there exists* $\Theta \subseteq_{fin} \Omega$ *s. t.* $\Gamma \vdash_{\mathbf{LLL}} \Delta \cup \Theta$ *and* $\Theta \cap U(\Gamma) = \varnothing$*.*

PROOF. Assume that $\Gamma \vdash_{\mathbf{LLL}} \Delta \cup \Theta$ and $\Theta \cap U(\Gamma) = \varnothing$. Naturally, we may assume $\Delta \cap \Theta = \varnothing$. Otherwise, we take $\Theta \setminus \Delta$ instead of Θ. By compactness we have $\{A_1, \ldots, A_n\} \vdash_{\mathbf{LLL}} \Delta \cup \Theta$ for $\{A_1, \ldots, A_n\} \subseteq \Gamma$. As in the proof of Proposition 2.6 we construct a stage of proof consisting of $n+1$ lines such that Δ is derived under condition Θ at line $n+1$ of this stage. If this line is **r**-marked at this stage, this means that some formula A_i is an abnormality and belongs to Θ or that $\Delta \subseteq \Omega$. The former is impossible, since in this case $A_i \in U(\Gamma)$, where as $\Theta \cap U(\Gamma) = \varnothing$. If $\Delta \subseteq \Omega$ and $A_i \notin \Delta$, $1 \leq i \leq n$, then $\Delta \subseteq U(\Gamma)$, but $\Delta \cap \Theta = \varnothing$. We have thus proved that line $n+1$ is not **r**-marked.

Assume that some extension t of s is such that the line, where Δ is derived under condition Θ, becomes **r**-marked. Let $\Theta_1, \ldots, \Theta_k$ be all elements of Σ_t such that $\Theta \cap \Theta_i \neq \varnothing$. Since $\Theta \cap U(\Gamma) = \varnothing$, for each Θ_i there is $\Theta_i' \in \Sigma(\Gamma)$ such that $\Theta_i' \subset \Theta_i$. Acting as in the proof of Proposition 2.6 we extend t to the stage u such that all Θ_i' are derived under the empty conditions at some lines of u. It is obvious that $(\Sigma_t \setminus \{\Theta_1, \ldots, \Theta_k\}) \cup \{\Theta_1', \ldots, \Theta_k'\} \subseteq \Sigma_u$. Additionally, Σ_u may contain singletons $\{A\}$, in which case $A \in U(\Gamma)$ and $A \notin \Theta$. In this way, Δ is derived in u at a line, which is not **r**-marked. We have thus proved that $\Gamma \vdash_{\mathbf{LLL^r}} \Delta$.

Now we assume that $\Gamma \vdash_{\mathbf{LLL}} \Delta \cup \Theta$ implies $\Theta \cap U(\Gamma) \neq \varnothing$. Let s be a stage of proof from Γ such that Δ is derived at some line of this stage under condition Θ. By Proposition 2.6 we have $\Gamma \vdash_{\mathbf{LLL}} \Delta \cup \Theta$ and so $\Theta \cap U(\Gamma) \neq \varnothing$. Let $\Theta_1 \in \Sigma(\Gamma)$ be such that $\Theta \cap \Theta_1 \neq \varnothing$. If we extend s to a stage t such that Θ_1 is derived at some line of t under the empty condition, then $\Theta_1 \subseteq U_t$ and $\Theta_1 \subseteq U_v$ for any extension v of t. Thus, the line, where Δ is derived under condition Θ, received an **r**-mark in t, which can not be removed in any further extension of t. We have proved that $\Gamma \nvdash_{\mathbf{LLL^r}} \Delta$. ■

Thus, by combining Theorems 2.1 and 2.10, we immediately obtain the (strong) completeness for the adaptive logic $\mathbf{LLL^r}$.

COROLLARY 2.11. *For any* $\Gamma \subseteq For_{\mathcal{L}}$ *and non-empty* $\Delta \subseteq_{fin} For_{\mathcal{L}}$,

$$\Gamma \vdash_{\mathbf{LLL^r}} \Delta \quad \Longleftrightarrow \quad \Gamma \vDash_{\mathbf{LLL^r}} \Delta\,.$$

Next, let us consider the minimal abnormality strategy, where infinite stages of proofs play an important role. Suppose s is a stage of proof from $\Gamma \subseteq For_{\mathcal{L}}$. Let Φ_s denote the collection of all minimal choice sets for the family Σ_s (introduced above).

DEFINITION 2.12. An i-th line of a stage s of proof from Γ is **m**-*marked* (or *marked according to the minimal abnormality strategy*) *in* s iff for the head Δ and the condition Θ of the i-th line, one of the following requirements is satisfied:

- there is no $\varphi \in \Phi_s$ with $\varphi \cap \Theta = \varnothing$;
- for some $\varphi \in \Phi_s$, there is no line in s at which Δ is derived under condition Θ' with $\varphi \cap \Theta' = \varnothing$.

At times, the phrase 'not **m**-marked' will be replaced by '**m**-*unmarked*' (cf. also Definition 2.7), for convenience.

DEFINITION 2.13. A non-empty $\Delta \subseteq_{fin} For_{\mathcal{L}}$ is *finally* $\mathbf{LLL^m}$*-derived in a stage* s *of proof from* Γ iff Δ is derived at some line i of s, and the following requirements are satisfied:

- the i-th line is not **m**-marked in s;
- any extension of s, in which i becomes **m**-marked, may be further extended so that this line will turn out to be **m**-unmarked again (after suitable renumberings are made).

DEFINITION 2.14. A non-empty $\Delta \subseteq_{fin} For_{\mathcal{L}}$ is *finally* $\mathbf{LLL^m}$*-derivable from* Γ (we denote this by $\Gamma \vdash_{\mathbf{LLL^r}} \Delta$) iff it can be finally $\mathbf{LLL^r}$-derived in some stage s of proof from Γ.

In effect, it is not hard to substantially refine the last definition by restricting attention to a special class of stages of proofs.

Proposition 2.15. *A non-empty* $\Delta \subseteq_{fin} For_{\mathcal{L}}$ *is finally* $\mathbf{LLL^m}$*-derivable from* Γ *iff there exists a stage* s *of proof from* Γ *possessing the properties:*

- Σ_s *coincides with* $\Sigma(\Gamma)$*;*
- *for every* $\varphi \in \Phi(\Gamma)$*, there is some line* i *in* s *s. t.* Δ *is derived at this line under condition* Θ_i *with* $\varphi \cap \Theta_i = \varnothing$*.*

Proof. Assume that s is a stage of proof from Γ such that $\Sigma_s = \Sigma(\Gamma)$ and for every $\varphi \in \Phi(\Gamma)$, there is a line i of s such that Δ is derived at this line under condition Θ_i with $\varphi \cap \Theta_i = \varnothing$. Choose some $\varphi_0 \in \Phi(\Gamma)$ and the respective line i_0. Clearly, $\Phi_s = \Phi(\Gamma)$ and the line i_0 is not $\mathbf{m}$-marked in s. Any extension t of s will obviously satisfy the properties $\Sigma_t = \Sigma(\Gamma)$ and $\Phi_t = \Phi(\Gamma)$. Thus, line i_0 remains unmarked in any extension of s. This proves that Δ is finally $\mathbf{LLL^m}$-derivable from Γ.

Let $\Gamma \vdash_{\mathbf{LLL^m}} \Delta$ and a stage s of proof from Γ confirm this fact. We can extend S to a stage of proof t such that $\Sigma_t = \Sigma(\Gamma)$. If Δ is not derived in t at $\mathbf{m}$-unmarked line, then there is an extension u of t, where Δ is derived at an $\mathbf{m}$-unmarked line. Since $\Sigma_t = \Sigma(\Gamma)$, we also have $\Sigma_u = \Sigma(\Gamma)$ and $\Phi_u = \Phi(\Gamma)$. Since Δ is derived at an $\mathbf{m}$-unmarked line of u, for every $\varphi \in \Phi(\Gamma)$, there is a line i of u such that Δ is derived at this line under condition Θ_i with $\varphi \cap \Theta_i = \varnothing$. ■

Using this observation, a syntactical variant of Theorem 2.5 (i. e., a criterion for the final $\mathbf{LLL^m}$-derivability) is obtained.

Theorem 2.16. *For any* $\Gamma \subseteq For_{\mathcal{L}}$ *and non-empty* $\Delta \subseteq_{fin} For_{\mathcal{L}}$*, we have* $\Gamma \vdash_{\mathbf{LLL^m}} \Delta$ *iff for each* $\varphi \in \Phi(\Gamma)$*, there exists* $\Theta \subseteq_{fin} \Omega$ *s. t.* $\Gamma \vdash_{\mathbf{LLL}} \Delta \cup \Theta$ *and* $\Theta \cap \varphi = \varnothing$*.*

Proof. The implication from left to right immediately follows from Propositions 2.6 and 2.15.

Assume that for each $\varphi \in \Phi(\Gamma)$, there exists $\Theta \subseteq_{fin} \Omega$ such that $\Gamma \vdash_{\mathbf{LLL}} \Delta \cup \Theta$ and $\Theta \cap \varphi = \varnothing$. We can construct a stage s of proof from Γ such that for every $\Theta \in \Sigma(\Gamma)$, Θ is derived at some line of s under empty condition, and for every $\varphi \in \Phi(\Gamma)$, Δ is derived at some line of s under condition Θ with $\Theta \cap \varphi = \varnothing$. It is clear that s satisfies the right-hand side condition of Proposition 2.15. ■

In this way, it is straightforward that Theorems 2.5 and 2.16 together imply the (strong) completeness for the adaptive logic $\mathbf{LLL^m}$.

Corollary 2.17. *For any* $\Gamma \subseteq For_{\mathcal{L}}$ *and non-empty* $\Delta \subseteq_{fin} For_{\mathcal{L}}$*,*

$$\Gamma \vdash_{\mathbf{LLL^m}} \Delta \quad \Longleftrightarrow \quad \Gamma \vDash_{\mathbf{LLL^m}} \Delta\,.$$

3. Complexity upper bounds

Fix a Gödel numbering γ for $For_{\mathcal{L}}$, i. e., γ is an effective one-to-one mapping from $For_{\mathcal{L}}$ onto ω that additionally satisfies the condition:

$$A \text{ is a proper subformula of } B \quad \Longrightarrow \quad \gamma(A) < \gamma(B)\,.$$

Having such a numbering (for $\mathcal{L}$-formulae) allows us to provide an effective coding for more complex syntactical objects (cf. [8]), like finite sequences of $\mathcal{L}$-formulas, lines of stages of proofs, finite stages of proofs, finite sets of $\mathcal{L}$-formulas, finite sets of finite sets of $\mathcal{L}$-formulae, etc. In this way, one may speak, e. g., of Gödel numbers $\gamma(\Gamma)$ for $\Gamma \subseteq_{fin} For_{\mathcal{L}}$ (and, intuitively, even identify these with their codes).[6]

The lower limit logic **LLL** is called *finitely decidable* (*finitely enumerable*) iff the set of pairs

$$\{(\gamma(\Gamma), \gamma(\Delta)) \mid \Gamma \cup \Delta \subseteq_{fin} For_{\mathcal{L}} \text{ and } \Gamma \vdash_{\mathbf{LLL}} \Delta\}$$

is computable (computably enumerable, respectively). Due to the above remarks, we will not mention Gödel numbers explicitly in this and similar situations, and so (instead) will speak, e. g., of decidability or enumerability of the collection

$$\{(\Gamma, \Delta) \mid \Gamma \cup \Delta \subseteq_{fin} For_{\mathcal{L}} \text{ and } \Gamma \vdash_{\mathbf{LLL}} \Delta\}\,.$$

Further, **LLL** has *property of local abnormalities* iff for any $\Gamma \cup \Delta \subseteq_{fin} For_{\mathcal{L}}$, we can effectively construct $\Omega_{\Gamma,\Delta} \subseteq_{fin} \Omega$ s. t. for every $\Theta \subseteq \Omega$,

$$\Gamma \vdash_{\mathbf{LLL}} \Delta \cup \Theta \quad \Longrightarrow \quad \Gamma \vdash_{\mathbf{LLL}} \Delta \cup (\Theta \cap \Omega_{\Gamma,\Delta})\,.$$

Here the expression 'effectively constructed' means that there is a (total) computable function f of two arguments transforming each $(\gamma(\Gamma), \gamma(\Delta))$ into $\gamma(\Omega_{\Gamma,\Delta})$. In this context, we write Ω_{Γ} instead of $\Omega_{\Gamma,\varnothing}$. For instance, it was shown in [7] that in case of $CLuN$ the set $\Omega_{\Gamma,\Delta}$ can be defined as

$$\Omega_{\Gamma,\Delta} \;:=\; \{A \wedge \neg A \mid \neg A \in Subf(\Gamma \cup \Delta)\}\,,$$

where $Subf(\Gamma \cup \Delta)$ is the set of all subformulas of formulas in $\Gamma \cup \Delta$.

Thus, from Theorems 2.10 and 2.16, we immediately obtain

[6]Notice that all such numberings are presupposed to satisfy the corresponding natural analogs of the above monotonicity requirement.

PROPOSITION 3.1. *Suppose* **LLL** *has property of local abnormalities. Then, for any* $\Gamma \cup \Delta \subseteq_{fin} For_{\mathcal{L}}$ *with* $\Delta \neq \varnothing$*, the following hold:*

1. $\Gamma \vdash_{\mathbf{LLL^r}} \Delta$ *iff there exists* $\Theta \subseteq \Omega_{\Gamma,\Delta}$ *s. t.* $\Gamma \vdash_{\mathbf{LLL}} \Delta \cup \Theta$ *and* $\Theta \cap U(\Gamma) = \varnothing$*;*
2. $\Gamma \vdash_{\mathbf{LLL^m}} \Delta$ *iff for each* $\varphi \in \Phi(\Gamma)$*, there exists* $\Theta \subseteq_{fin} \Omega_{\Gamma,\Delta}$ *s. t.* $\Gamma \vdash_{\mathbf{LLL}} \Delta \cup \Theta$ *and* $\Theta \cap \varphi = \varnothing$*.*

Also, restricting attention to finite sets of $\mathcal{L}$-formulae, we get

PROPOSITION 3.2. *Suppose* **LLL** *has property of local abnormalities, and let* $\Gamma \subseteq_{fin} For_{\mathcal{L}}$*. Then the following hold:*

1. $U(\Gamma)$ *is a finite set of* $\mathcal{L}$*-formulae, while* $\Sigma(\Gamma)$ *and* $\Phi(\Gamma)$ *are both finite families of finite sets of* $\mathcal{L}$*-formulae;*
2. *if* **LLL** *is finitely decidable, then the functions*

$$\lambda_U : \Gamma \subseteq_{fin} For_{\mathcal{L}} \mapsto U(\Gamma), \quad \lambda_\Sigma : \Gamma \subseteq_{fin} For_{\mathcal{L}} \mapsto \Sigma(\Gamma), \quad \text{and} \quad \lambda_\Phi : \Gamma \subseteq_{fin} For_{\mathcal{L}} \mapsto \Phi(\Gamma)$$

are all computable.

PROOF. $\boxed{1}$ If $\Gamma \vdash_{\mathbf{LLL}} \Theta$ and $\Theta \subseteq_{fin} \Omega$, then $\Gamma \vdash_{\mathbf{LLL}} \Theta \cap \Omega_\Gamma$ by the property of local abnormalities. Thus, since $\Sigma(\Gamma)$ consists precisely of minimal *Ab*-consequences of Γ, $\Theta \in \Sigma(\Gamma)$ implies $\Theta \subseteq \Omega_\Gamma$. But Ω_Γ is finite, whence $\Sigma(\Gamma)$ is a finite family of finite sets. Next, $U(\Gamma)$ is the union of $\Sigma(\Gamma)$ and so, too, appears to be finite. Finally, $\Phi(\Gamma)$ is the family of all minimal choice sets for $\Sigma(\Gamma)$, and $\varphi \in \Phi(\Gamma)$ entails $\varphi \subseteq U(\Gamma)$. Therefore, $\Phi(\Gamma)$ itself, as well as all its members, are finite.

$\boxed{2}$ Assume **LLL** is finitely decidable. It means, particularly, that for any $\Gamma \subseteq_{fin} For_{\mathcal{L}}$ and $\Theta \subseteq \Omega_\Gamma$, we can computably check whether $\Gamma \vdash_{\mathbf{LLL}} \Theta$. This allows us to effectively (and uniformly in Γ) construct both $\Sigma(\Gamma)$ and $U(\Gamma)$. Given these two, for each $\varphi \subseteq U(\Gamma)$, we can also decide (again, effectively) whether φ is a choice set for $\Sigma(\Gamma)$—as a result, we distinguish all the minimal choice sets for $\Sigma(\Gamma)$, and eventually obtain $\Phi(\Gamma)$. ■

THEOREM 3.3. *If* **LLL** *has property of local abnormalities and is finitely decidable, then the adaptive consequence relations* $\vdash_{\mathbf{LLL^r}}$ *and* $\vdash_{\mathbf{LLL^m}}$ *are also finitely decidable, i. e., the two collections*

$$\{(\Gamma, \Delta) \mid \Gamma \cup \Delta \subseteq_{fin} For_{\mathcal{L}} \text{ and } \Gamma \vdash_{\mathbf{LLL^r}} \Delta\},$$
$$\{(\Gamma, \Delta) \mid \Gamma \cup \Delta \subseteq_{fin} For_{\mathcal{L}} \text{ and } \Gamma \vdash_{\mathbf{LLL^m}} \Delta\}$$

are decidable.

Proof. Suppose that $\Gamma \cup \Delta \subseteq_{fin} For_{\mathcal{L}}$ and $\Delta \neq \varnothing$. According to Item 1 of Proposition 3.1, to decide whether $\Gamma \vdash_{AL^{\mathbf{r}}} \Delta$ or not, we need to check if $\Gamma \vdash_{\mathbf{LLL}} \Delta \cup \Theta$ for all $\Theta \subseteq \Omega_{\Gamma,\Delta} \setminus U(\Gamma)$. While $\Omega_{\Gamma,\Delta}$ (which is finite) can be computed from Γ and Δ by the property of local abnormalities, $U(\Gamma)$ is finite and may be effectively constructed from Γ by Proposition 3.2. So we only have to check whether $\Gamma \vdash_{\mathbf{LLL}} \Delta \cup \Theta$ or not for the finite number of known Θ. Since **LLL** is finitely decidable, the latter can be carried out, again, in an computable way.

For the second condition (namely $\Gamma \vdash_{\mathbf{LLL^m}} \Delta$), the proof is analogous (and employs Item 2 of Proposition 3.1). ■

Therefore, both adaptive consequence relations are finitely decidable, provided that **LLL** is finitely decidable and has property of local abnormalities. How does the situation changes when **LLL** is finitely enumerable? Let us start with considering the relation 'to be a minimal *Ab*-consequence' between finite sets of $\mathcal{L}$-formulae (for a finitely enumerable **LLL**).

Proposition 3.4. *If* **LLL** *is finitely enumerable, then the collection*

$$\{(\Gamma, \Delta) \mid \Gamma \subseteq_{fin} For_{\mathcal{L}} \text{ and } \Delta \in \Sigma(\Gamma)\}$$

is Δ^0_2*. And if, in addition,* **LLL** *has property of local abnormalities, then the collections*

$$\{(\Gamma, A) \mid \Gamma \subseteq_{fin} For_{\mathcal{L}} \text{ and } A \in U(\Gamma)\},$$
$$\{(\Gamma, \varphi) \mid \Gamma \subseteq_{fin} For_{\mathcal{L}},\ \varphi \subseteq_{fin} \Omega \text{ and } \varphi \in \Phi(\Gamma)\}$$

are Δ^0_2 *as well.*

Proof. Assume henceforth that Γ and Δ range over finite subsets of $For_{\mathcal{L}}$.

Clearly, one is able to check effectively whether a given finite set of $\mathcal{L}$-formulas consists of abnormalities, i. e., that it is a subset of Ω. The relation $\Delta \in \Sigma(\Gamma)$ may be expressed as

$$(\varnothing \neq \Delta \subseteq \Omega) \wedge (\Gamma \vdash_{\mathbf{LLL}} \Delta) \wedge \left(\forall \Delta' \subset \Delta\right) \left(\Gamma \nvdash_{\mathbf{LLL}} \Delta'\right).$$

Since **LLL** is finitely enumerable, the condition $\Gamma \vdash_{\mathbf{LLL}} \Delta$ is definable (in $\mathfrak{N}$) by an arithmetical Σ_1-formula, and so $\Gamma \nvdash \Delta'$ is, in turn, definable via an Π_1-formula. Moreover, $(\forall \Delta' \subset \Delta)\,(\Gamma \nvdash_{\mathbf{LLL}} \Delta')$ is, too, expressible by a Π_1-formula, because '$\forall \Delta' \subset \Delta$' can be viewed as a kind of bounded quantifier (remember that all Gödel numberings employed are monotone, in a natural sense). Thus, the (binary) relation $\Delta \in \Sigma(\Gamma)$ is definable by means of a conjunction of a Σ^0_1- and a Π^0_1-formula, whence it is at worst Δ^0_2.

Suppose **LLL** also has property of local abnormalities. As we've already mentioned in the proof of Proposition 3.2, $\Delta \in \Sigma(\Gamma)$ implies $\Delta \subseteq \Omega_\Gamma$, so $A \in U(\Gamma)$ may be expressed as

$$(\exists \Delta \subseteq \Omega_\Gamma)\,(\Delta \in \Sigma(\Gamma) \wedge A \in \Delta)\,,$$

which is a Δ^0_2-relation preceded by a bounded quantifier (here Ω_Γ should be replaced by an effective mapping sending each $\Gamma \subseteq_{fin} For_{\mathcal{L}}$ to Ω_Γ). Consequently, the (binary) relation $A \in U(\Gamma)$ is Δ^0_2 as well.

Recall that $\Phi(\Gamma)$ is the family of all minimal choice sets for $\Sigma(\Gamma)$. Thus, taking into account Proposition 2.2, we obtain the following presentation for the condition $\varphi \in \Phi(\Gamma)$:

$$\begin{aligned}&(\forall \Delta \subseteq \Omega_\Gamma)\,(\Delta \notin \Sigma(\Gamma) \vee (\varphi \cap \Delta \neq \varnothing)) \wedge \\ &\qquad\qquad (\forall A \in \varphi)\,(\exists \Delta \subseteq \Omega_\Gamma)\,(\Delta \in \Sigma(\Gamma) \wedge (\varphi \cap \Delta = \{A\}))\,.\end{aligned}$$

Clearly, since $\Delta \in \Sigma(\Gamma)$ is Δ^0_2, its negation is also Δ^0_2. In this way, the (binary) relation $\varphi \in \Phi(\Gamma)$ may be eventually defined (notice, we used only bounded quantifiers) as an intersection of two Δ^0_2-sets, whence it is Δ^0_2. ■

Proposition 3.5. *If* **LLL** *has property of local abnormalities and is finitely enumerable, then the two collections*

$$\begin{aligned}&\{(\Gamma,\Delta) \mid \Gamma \cup \Delta \subseteq_{fin} For_{\mathcal{L}} \ \textit{and}\ \Gamma \vdash_{\mathbf{LLL^r}} \Delta\}\,,\\ &\{(\Gamma,\Delta) \mid \Gamma \cup \Delta \subseteq_{fin} For_{\mathcal{L}} \ \textit{and}\ \Gamma \vdash_{\mathbf{LLL^m}} \Delta\}\end{aligned}$$

appear to be Δ^0_2.

Proof. Here assume that Γ and Δ range over finite subsets of $For_{\mathcal{L}}$.

Due to Item 1 of Proposition 3.1, we may express $\Gamma \vdash_{\mathbf{LLL^r}} \Delta$ as

$$(\Delta \neq \varnothing) \wedge (\exists \Theta \subseteq \Omega_{\Gamma,\Delta})\,(\Gamma \vdash_{\mathbf{LLL}} \Delta \cup \Theta \wedge (\forall A \in \Theta)\,(A \notin U(\Gamma)))\,.$$

By the previous proposition, the relation $A \in U(\Gamma)$ is Δ^0_2, whence its negation and also $(\forall A \in \Theta)\,(A \notin U(\Gamma))$ are Δ^0_2. On the other hand, the condition $\Gamma \vdash_{\mathbf{LLL}} \Delta \cup \Theta$ is definable via a Σ_1-formula. Since an intersection of a Σ^0_1- and a Δ^0_2-set gives a Δ^0_2-set, while adding a bounded quantifier does not change the complexity class, we eventually get Δ^0_2, as desired.

As we have already seen, $\varphi \in \Phi(\Gamma)$ entails $\varphi \subseteq U(\Gamma)$, and $U(\Gamma) \subseteq \Omega_\Gamma$. Thus, due to Item 2 of Proposition 3.1, $\Gamma \vdash_{\mathbf{LLL^m}} \Delta$ can be presented as

$$(\forall \varphi \subseteq \Omega_\Gamma)\,(\exists \Theta \subseteq \Omega_{\Gamma,\Delta})\,(\varphi \notin \Phi(\Gamma) \vee (\Gamma \vdash_{\mathbf{LLL}} \Delta \cup \Theta \wedge (\varphi \cap \Theta = \varnothing)))$$

Again, by the previous proposition, the relations $\varphi \in \Phi(\Gamma)$ and $A \in U(\Gamma)$ are Δ^0_2 (while $\Gamma \vdash_{\mathbf{LLL}} \Delta \cup \Theta$ is even Σ_1). Consequently, as long as only bounded quantifiers are involved, we arrive at Δ^0_2 again. ■

Now we turn to (adaptive) consequences of infinite sets of $\mathcal{L}$-formulae. For an arbitrary $\Gamma \subseteq For_{\mathcal{L}}$, denote

$$\begin{aligned} Cn_{\mathbf{LLL^r}}(\Gamma) &:= \{\Delta \subseteq_{fin} For_{\mathcal{L}} \mid \Gamma \vdash_{\mathbf{LLL^r}} \Delta\}, \\ Cn_{\mathbf{LLL^m}}(\Gamma) &:= \{\Delta \subseteq_{fin} For_{\mathcal{L}} \mid \Gamma \vdash_{\mathbf{LLL^m}} \Delta\}. \end{aligned}$$

In the sequel, the subscripts $\mathbf{LLL^r}$ and $\mathbf{LLL^m}$ will be replaced with $\mathbf{r}$ and $\mathbf{m}$, respectively, when there is no risk of confusion.

Remark that for any $\Gamma \subseteq For_{\mathcal{L}}$, $\Sigma(\Gamma)$ is a family of finite sets of abnormalities, and $U(\Gamma)$ is a set of abnormalities. Therefore, we can always view their elements as appropriately encoded by natural numbers, and it makes sense to talk about algorithmic complexity of $\Sigma(\Gamma)$ and $U(\Gamma)$ in a general case. However, $\Phi(\Gamma)$ may easily contain infinite sets (of abnormalities), so the situation is more difficult and certain restrictions are needed.

Proposition 3.6. *Let* **LLL** *be finitely enumerable. For an arbitrary set of $\mathcal{L}$-formulae Γ, $\Sigma(\Gamma)$ is $\Delta^{0,\Gamma}_2$, while $U(\Gamma)$ is $\Sigma^{0,\Gamma}_2$. And if, in addition, all elements of $\Phi(\Gamma)$ turn out to be finite, then $\Phi(\Gamma)$ is $\Delta^{0,\Gamma}_3$.*

Proof. In what follows, all unbounded quantifiers are assumed to range over finite subsets of $For_{\mathcal{L}}$ (or, rather, over their Gödel codes).

Trivially, the unary relation $\Theta \subseteq \Gamma$ on the finite subsets Θ of $For_{\mathcal{L}}$ is presented by $(\forall A \in \Theta)(A \in \Gamma)$, and hence is computable w. r. t. (the oracle) Γ. Then, using the left-compactness of $\vdash_{\mathbf{LLL}}$, we may express the condition $\Delta \in \Sigma(\Gamma)$ (for Δ finite, like before) as

$$(\varnothing \neq \Delta \subseteq \Omega) \wedge \exists\Theta\,(\Theta \subseteq \Gamma \wedge \Theta \vdash_{\mathbf{LLL}} \Delta) \wedge \\ \forall\Theta'\left(\Theta' \not\subseteq \Gamma \vee \left(\forall\Delta' \subset \Delta\right)\left(\Theta' \nvdash_{\mathbf{LLL}} \Delta'\right)\right).$$

Due to the finite enumerability of $\vdash_{\mathbf{LLL}}$, $\Theta \vdash_{\mathbf{LLL}} \Delta$ and $\Theta' \nvdash_{\mathbf{LLL}} \Delta'$ are definable (in $\mathfrak{N}$) via a Σ_1- and a Π_1-formula, respectively. Thus, employing some standard transformations, it is straightforward to get both Σ_2- and Π_2-definability w. r. t. Γ, whence $\Sigma(\Gamma)$ is $\Delta^{0,\Gamma}_2$. The condition $A \in U(\Gamma)$ means that $\exists\Delta\,(\Delta \in \Sigma(\Gamma) \wedge A \in \Delta)$, so $U(\Gamma)$ is $\Sigma^{0,\Gamma}_2$.

Suppose all elements of $\Phi(\Gamma)$ are finite. Now $\varphi \in \Phi(\Gamma)$ is presented by

$$\forall\Delta\,(\Delta \notin \Sigma(\Gamma) \vee (\varphi \cap \Delta \neq \varnothing)) \wedge \\ \forall A \in \varphi\, \exists\Delta\,(\Delta \in \Sigma(\Gamma) \wedge (\varphi \cap \Delta = \{A\})).$$

Since $\Delta \in \Sigma(\Gamma)$ is $\Sigma_2^{0,\Gamma}$ and, therefore, $\Delta \notin \Sigma(\Gamma)$ is $\Pi_2^{0,\Gamma}$, we arrive at an intersection of a $\Sigma_2^{0,\Gamma}$- and a $\Pi_2^{0,\Gamma}$-relation, hence $\Phi(\Gamma)$ is $\Delta_3^{0,\Gamma}$. ■

Notice, the quantifiers over finite sets in the above proof, unlike in the proof of Proposition 3.4, are not bounded. However, the advantage here is that we no longer need the property of local abnormalities for **LLL**.

In many situations, we already know (the upper bound for) the complexity of Γ—that allows us to improve the estimations given.

COROLLARY 3.7. *Let* **LLL** *be finitely enumerable. For each* $\Gamma \subseteq For_{\mathcal{L}}$, *if* Γ *is* Σ_{m+1}^0, *then* $\Sigma(\Gamma)$ *is* Δ_{m+2}^0, *and* $U(\Gamma)$ *is* Σ_{m+2}^0. *And if, in addition, all elements of* $\Phi(\Gamma)$ *turn out to be finite, then* $\Phi(\Gamma)$ *is* Δ_{m+3}^0.

PROOF. Remark that if Γ is Σ_{m+1}^0, the condition $\Theta \subseteq \Gamma$ (from the proof of Proposition 3.4) appears to be Σ_{m+1}^0, and $\Theta' \nsubseteq \Gamma$ is Π_{m+1}^0. The rest is straightforward. ■

THEOREM 3.8. *Suppose that* **LLL** *is finitely enumerable, and* $\Gamma \subseteq For_{\mathcal{L}}$. *Then* $Cn^{\mathbf{r}}(\Gamma)$ *is* $\Sigma_3^{0,\Gamma}$. *Moreover, the following implications hold:*

1. *if all elements of* $\Phi(\Gamma)$ *are finite, then* $Cn^{\mathbf{m}}(\Gamma)$ *is* $\Pi_3^{0,\Gamma}$;
2. *if* $U(\Gamma)$ *is finite, then both* $Cn^{\mathbf{r}}(\Gamma)$ *and* $Cn^{\mathbf{m}}(\Gamma)$ *are* $\Sigma_1^{0,\Gamma}$.

PROOF. Just as before, all unbounded quantifiers are assumed to range over finite subsets of $For_{\mathcal{L}}$. Fix some set Γ of premisses.

According to Theorem 2.10, the unary relation $\Gamma \vdash_{\mathbf{LLL}^{\mathbf{r}}} \Delta$ on the finite subsets Δ of $For_{\mathcal{L}}$ can be specified by

$$(\Delta \neq \varnothing) \wedge \exists\Theta\, \exists\Gamma' \big(\Theta \subseteq \Omega \wedge \Gamma' \subseteq \Gamma \wedge \Gamma' \vdash_{\mathbf{LLL}} \Delta \cup \Theta \wedge (\forall A \in \Theta)\, (A \notin U(\Gamma))\big) \quad (\star)$$

Since **LLL** is finitely enumerable, the condition $\Gamma' \vdash_{\mathbf{LLL}} \Delta \cup \Theta$ is Σ_1-definable (in $\mathfrak{N}$), while $U(\Gamma)$ is $\Sigma_2^{0,\Gamma}$ by Proposition 3.6 and, consequently, its complement is $\Pi_2^{0,\Gamma}$. As a result, we obtain a $\Pi_2^{0,\Gamma}$-relation preceded by two existential quantifiers. In this way, $Cn^{\mathbf{r}}(\Gamma)$ turns out to be $\Sigma_3^{0,\Gamma}$.

$\boxed{1}$ Assume that $\Phi(\Gamma)$ consists of finite sets of $\mathcal{L}$-formulas. So $\Phi(\Gamma)$ is $\Delta_3^{0,\Gamma}$ by Proposition 3.6. At the same time, due to Theorem 2.16, the unary relation $\Gamma \vdash_{\mathbf{LLL}^{\mathbf{m}}} \Delta$ (for $\Delta \subseteq_{fin} For_{\mathcal{L}}$) may be presented as

$$(\Delta \neq \varnothing) \wedge \forall\varphi\, (\varphi \notin \Phi(\Gamma) \vee \exists\Theta\, \exists\Gamma' \big(\Theta \subseteq \Omega \wedge \Gamma' \subseteq \Gamma \wedge \Gamma' \vdash_{\mathbf{LLL}} \Delta \cup \Theta \wedge (\varphi \cap \Theta = \varnothing)\big)) \quad (\dagger)$$

Here, the condition $\varphi \notin \Phi(\Gamma)$ is $\Delta_3^{0,\Gamma}$, and so we arrive at a $\Pi_3^{0,\Gamma}$-relation (because $\Delta_3^{0,\Gamma} \subseteq \Pi_3^{0,\Gamma}$) preceded by a $\forall$-quantifier, which eventually entails (after gluing together universal quantifiers) that $Cn^{\mathbf{m}}(\Gamma)$ is $\Pi_3^{0,\Gamma}$.

$\boxed{2}$ If $U(\Gamma)$ is finite, then the unary relation $\Theta \cap U(\Gamma) = \varnothing$, that is, $(\forall A \in \Theta)(A \notin U(\Gamma))$ (for Θ finite), is computable. In such a case, the relation represented by $(\star)$ is easily seen to be Σ_1-definable w. r. t. (the oracle) Γ, whence $Cn^{\mathbf{r}}(\Gamma)$ appears to be $\Sigma_1^{0,\Gamma}$.

Since all elements of $\Phi(\Gamma)$ are subsets of $U(\Gamma)$, $\Phi(\Gamma)$ is finite, too, and hence computable. The expression $(\dagger)$ can then be rewritten as

$$(\Delta \neq \varnothing) \wedge (\forall \varphi \subseteq U(\Gamma))(\exists \Theta)\left(\exists \Gamma'\right)(\varphi \notin \Phi(\Gamma) \vee \\ \left(\Theta \subseteq \Omega \wedge \Gamma' \subseteq \Gamma \wedge \Gamma' \vdash_{\mathbf{LLL}} \Delta \cup \Theta \wedge (\varphi \cap \Theta = \varnothing)\right)) \quad (\ddagger)$$

which clearly defines a $\Sigma_1^{0,\Gamma}$-set, namely $Cn^{\mathbf{m}}(\Gamma)$. ■

Corollary 3.9. *Suppose* **LLL** *is finitely enumerable, and* $\Gamma \subseteq For_{\mathcal{L}}$ *is* Σ_{m+1}^0. *Then* $Cn^{\mathbf{r}}(\Gamma)$ *is* Σ_{m+3}^0. *Moreover, the following implications hold:*

1. *if all elements of* $\Phi(\Gamma)$ *are finite, then* $Cn^{\mathbf{m}}(\Gamma)$ *is* Π_{m+3}^0*;*
2. *if* $U(\Gamma)$ *is finite, then both* $Cn^{\mathbf{r}}(\Gamma)$ *and* $Cn^{\mathbf{m}}(\Gamma)$ *are* Σ_{m+1}^0.

Proof. Again, if Γ is Σ_{m+1}^0, the condition $\Gamma' \subseteq \Gamma$ (from the proof of Theorem 3.8) has the complexity Σ_{m+1}^0 as well. Thus, using Corollary 3.7, it is easy to show that $(\star)$ is equivalent to a Σ_{m+3}^0-formula.

$\boxed{1}$ Analogously, by transforming $(\dagger)$ into a Π_{m+3}^0-form.

$\boxed{2}$ By a similar argument, $(\ddagger)$ is reduced to a Σ_{m+1}^0-form. ■

Note that the above statement can be reformulated in a uniform way for certain classes of premiss sets—cf. [7, Corollary 3.11], for example.

Finally, we consider the algorithmic complexity for $\mathbf{LLL}^{\mathbf{m}}$-consequences in the general case—which, involving infinite stages of proofs, forces us to pass to a more abstract framework of second-order arithmetic.

Theorem 3.10. *Suppose that* **LLL** *is finitely enumerable, and* $\Gamma \subseteq For_{\mathcal{L}}$. *Then* $Cn^{\mathbf{m}}(\Gamma)$ *is* $\Pi_1^{1,\Gamma}$.

Proof. As usual, (first-order) variables in expressions are: Δ, Θ and Γ' ranging over finite subsets of $For_{\mathcal{L}}$, and A ranging over $\mathcal{L}$-formulas.

Let P be an unary predicate variable (that, intuitively, will range over arbitrary subsets of ω—or rather over all subsets of $For_{\mathcal{L}}$, modulo a Gödel numbering in hand). Fix some Γ. In view of Proposition 2.2, the property '$P \in \Phi(\Gamma)$', i. e., 'P is a minimal choice set for $\Sigma(\Gamma)$' may be expressed as

$$\begin{aligned}\Phi^{\Gamma}(P) \;:=\; & \forall\Delta\,(\Delta \notin \Sigma(\Gamma) \vee (P \cap \Delta \neq \varnothing)) \wedge \\ & \qquad \forall A\,(A \notin P \vee \exists\Delta\,(\Delta \in \Sigma(\Gamma) \wedge (P \cap \Delta = \{A\})))\end{aligned}$$

Taking into account Proposition 3.6, this can be specified by a second-order arithmetical formula without set quantifiers (viz. over predicates), with parameter Γ (as an oracle), and the only free variable P.

Next, by employing Theorem 2.16, $\Gamma \vdash_{\mathbf{LLL}^{\mathbf{m}}} \Delta$ is presented as

$$\begin{aligned}&(\Delta \neq \varnothing) \wedge \forall P\left(\neg\Phi^{\Gamma}(P) \vee \right. \\ &\qquad \left. \exists\Theta\,\exists\Gamma'\left(\Theta \subseteq \Omega \wedge \Gamma' \subseteq \Gamma \wedge \Gamma' \vdash_{\mathbf{LLL}} \Delta \cup \Theta \wedge (P \cap \Theta = \varnothing)\right)\right).\end{aligned}$$

Obviously, it implies that $Cn^{\mathbf{m}}(\Gamma)$ is definable (in $\mathfrak{N}$) by means of a Π^1_1-formula with parameter Γ, as desired. ■

COROLLARY 3.11. *Suppose* **LLL** *is finitely enumerable, and* $\Gamma \subseteq For_{\mathcal{L}}$. *If* Γ *is arithmetical, then* $Cn^{\mathbf{m}}(\Gamma)$ *is* Π^1_1.

4. Complexity lower bounds: examples

In effect, many of the estimations provided in the previous section turn out to be exact for particular adaptive logics. Possibly the most prominent examples of inconsistency-adaptive logics are the *adaptive CLuN's* [3], so we've chosen them to be our 'model logics'.

Let For_{CL} be the collection of all propositional formulae build up from the propositional symbols *Prop* using logical connectives $\wedge$, $\vee$, $\rightarrow$ and $\neg$. In our setting, the lower limit logic **LLL** is the propositional weak paraconsistent logic *CLuN*—which may be viewed as the smallest subset of For_{CL} containing the axioms of propositional classical positive logic, plus $p \vee \neg p$, and closed under the rules of substitution and 'modus ponens'.

The consequence relation $\vdash_{CLuN}$ (associated with *CLuN*) is defined as follows: for $\Gamma \cup \Delta \subseteq For_{CL}$, $\Gamma \vdash_{CLuN} \Delta$ iff there exist $\{A_1, \ldots, A_n\} \subseteq \Delta$ s. t. $A_1 \vee \cdots \vee A_n$ can be derived (in a finite number of steps) from the elements of $CLuN \cup \Gamma$ by means of 'modus ponens' only. Clearly, $\vdash_{CLuN}$ satisfies all the requirements (on $\vdash_{\mathbf{LLL}}$) from Section 2.

The models for *CLuN* are simply valuations $v : For_{CL} \rightarrow \{0, 1\}$ possessing the properties:

1. $v(A \wedge B) = 1 \iff v(A) = 1$ and $v(B) = 1$;

2. $v(A \vee B) = 1 \iff v(A) = 1$ or $v(B) = 1$;
3. $v(A \to B) = 1 \iff v(A) = 0$ or $v(B) = 1$;
4. $v(A) = 0 \implies v(\neg A) = 1$.

If $\varepsilon \in \{0, 1\}$, then $v(\Gamma) = \varepsilon$ abbreviates '$v(A) = \varepsilon$ for all $A \in \Gamma$'. Now, assuming $\Gamma \cup \Delta \subseteq For_{CL}$, $\Gamma \vDash_{CLuN} \Delta$ means that for every $CLuN$-valuation v, either $v(\Delta) \neq 0$, or $v(\Gamma) \neq 1$.[7]

It is well-known that $CLuN$ is strongly complete w. r. t. the semantics just described, i. e.,

$$\Gamma \vdash_{CLuN} \Delta \iff \Gamma \vDash_{CLuN} \Delta\,.$$

In addition, since any $v(\Gamma)$ is completely determined by how v acts on the subformulas of formulas in Γ, the $\vdash_{CLuN}$-relation restricted to finite sets (for both premisses and conclusions) is decidable.

Now, taking

$$\Omega \;:=\; \{A \wedge \neg A \mid A \in For_{CL}\}\,,$$

it is straightforward to define the adaptive logics $CLuN^{\mathbf{r}}$ and $CLuN^{\mathbf{m}}$ (viz. $CLuN$ supplied with the reliability strategy and the minimal abnormality strategy, respectively), according to the presentation from Section 2.

To avoid confusion with the general case, for each $\Gamma \subseteq For_{CL}$, let

$$\mathtt{Cn}^{\mathbf{r}}(\Gamma) \;:=\; Cn_{CLuN^{\mathbf{r}}}(\Gamma)\,.$$

Then, the Σ^0_3 lower bound proof (for $\mathtt{Cn}^{\mathbf{r}}(\Gamma)$, with Γ computable) from [6] can be easily adapted to derive

Proposition 4.1 (see [7]). *For every $m \in \omega$, there exists a $\Pi^0_m(\Sigma^0_{m+1})$-set $\Gamma \subseteq For_{CL}$ s. t. $\mathtt{Cn}^{\mathbf{r}}(\Gamma)$ is Σ^0_{m+3}-hard.*

Together with Corollary 3.9, it implies Σ^0_{m+3}-completeness of $\mathtt{Cn}^{\mathbf{r}}(\Gamma)$ for certain $\Pi^0_m(\Sigma^0_{m+1})$-sets of premisses Γ.

Next, let us write $\mathtt{U}(\Gamma)$ (where $\Gamma \subseteq For_{CL}$) for $U(\Gamma)$ in case of $CLuN$ with the abnormalities Ω as above.

Proposition 4.2. *For every $m \in \omega$, there is a Σ^0_{m+1}-set $\Gamma \subseteq For_{CL}$ s. t. $\mathtt{U}(\Gamma)$ is Σ^0_{m+2}-hard.*

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[7] Imagine that Γ and Δ stand for the (possibly infinite) conjunction and disjunction of their elements, correspondingly.

Proof. Take a Σ^0_{m+2}-complete subset S of natural numbers. Certainly, it is definable in $\mathfrak{N}$ by an arithmetical formula of the sort $\exists i\,\Psi(i,n)$, for some Π^0_{m+1}-formula $\Psi(i,n)$. Assume Γ consists of:

- $(p_n \wedge \neg p_n) \vee \left(q^i_n \wedge \neg q^i_n\right)$ for all i and n in $\mathbb{N}$;
- $\left(q^i_n \wedge \neg q^i_n\right)$ for any i and n with $\neg\Psi(i,n)$.

The resulting Γ is obviously Σ^0_{m+1}, and it is not hard to show that

$$p_n \wedge \neg p_n \in \mathtt{U}(\Gamma) \quad \Longleftrightarrow \quad \mathfrak{N} \Vdash_{\mathrm{FOL}} \exists i\,\Psi(i,n)\,,$$

whence $\mathtt{U}(\Gamma)$ appears to be at least Σ^0_{m+2}-hard. ■

Thus, the estimation from Corollary 3.7 also turns out to be exact for certain Σ^0_{m+1}-sets of premisses.

Moreover, at times there is a room for further improvements. Particularly, though $\mathtt{U}(\Gamma)$ ($\mathtt{Cn}^{\mathtt{r}}(\Gamma)$) is $\Sigma^0_2(\Sigma^0_3)$-complete for some c. e. set Γ (due to the above), even computable set will suffice here, since Γ is, in fact, **LLL**-equivalent to a computable Γ'—see [7, Proposition 3.12].

Take $CLuN_\perp$ to be $CLuN$ augmented with the constant $\perp$ interpreted as 'always false formula' (consequently, the classical negation of A is definable as $A \to \perp$), here $\mathcal{L} := \{\wedge, \vee, \to, \neg, \perp\}$. Then, the upper bound from Corollary 3.11 is exact, because, as was shown earlier in [10], there exists a computable $\Gamma \subseteq For_{\mathcal{L}}$ s. t. $Cn_{CLuN^{\mathtt{m}}_\perp}(\Gamma)$ is Π^1_1-hard.

Finally, remark that the Δ-like bounds from Section 4 cannot be indeed totally precise w. r. t. the m-reducibility, since no Δ^0_k-universal sets (here $k \in \omega$) are possible in the arithmetical hierarchy, and it may well be that a thiner classification is needed for this situation. However, Σ^0_i-, Π^0_i- and Δ^0_i-sets with $i \in \{0,1,2,3\}$ are the most studied in the hierarchy (cf. [5, Section 10.5] and [9, § 14.8] for the details) and, e. g., there is an interesting intuition behind Δ^0_2-sets based on computable approximations.

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