

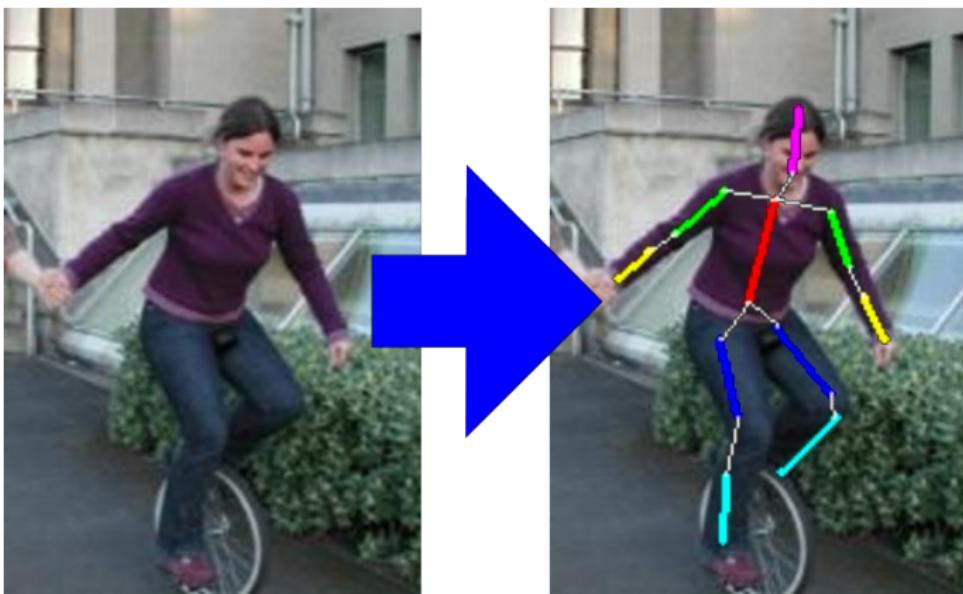
Feature Selection in GPLVM's

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Introduction

Setting

- Observed variables $\mathbf{Y}^{(1)} \in \mathbb{R}^{d_Y^{(1)}}$, $\mathbf{Y}^{(2)} \in \mathbb{R}^{d_Y^{(2)}}$
- Task
 - ▶ Infer $\mathbf{y}_i^{(2)}$ from $\mathbf{y}_i^{(1)}$

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Challenge

- $\mathbf{Y}^{(1)}$ is a high-dimensional, noisy, redundant and sometimes ambiguous representation of $\mathbf{Y}^{(2)}$

Modelling paradigm

Generative

$$p(\mathbf{Y}^{(1)}, \mathbf{Y}^{(2)})$$

- Jointly models all data, uncertainty in “input”
- High dimensional, parametrise $\mathbb{R}^{d_{Y^{(1)}}} \times \mathbb{R}^{d_{Y^{(2)}}}$

Discriminative

$$p(\mathbf{Y}^{(2)} | \mathbf{Y}^{(1)})$$

- Only model “decision” boundary
- Low dimensional “model” $\mathbb{R}^{d_{Y^{(2)}}}$

Computer Vision Challenges

- Pascal VOC Challenge [URL]
 - ▶ Discriminative methods
 - ▶ Lots of feature engineering to achieve generalisation
- ImageNet [URL]
 - ▶ Feature learning through Neural Networks
 - ▶ Representation learning tweaks and tricks to explain away irrelevant variations
- *little success (nor focus) by actual models of images*

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The problem with generative models

Variations

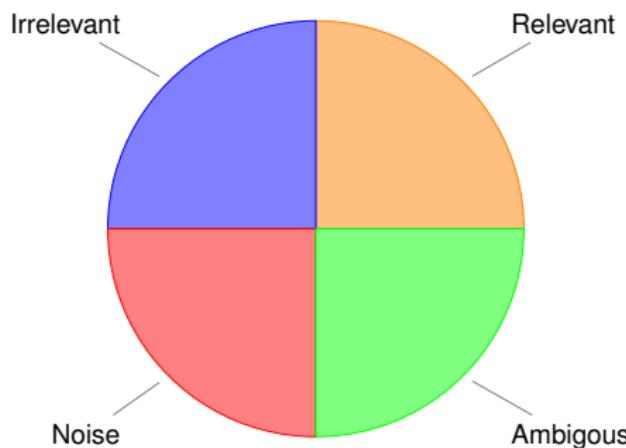
- Signal
 1. $\mathbf{Y}^{(1)}$ informative of $\mathbf{Y}^{(2)}$ (Relevant)
 2. $\mathbf{Y}^{(1)}$ non-informative of $\mathbf{Y}^{(2)}$ (Irrelevant)
 3. $\mathbf{Y}^{(2)}$ non-informative of $\mathbf{Y}^{(1)}$ (Ambiguous)
- Noise in $\mathbf{Y}^{(1)}$ and $\mathbf{Y}^{(2)}$
- *All variations need to be explained in a model of the data*

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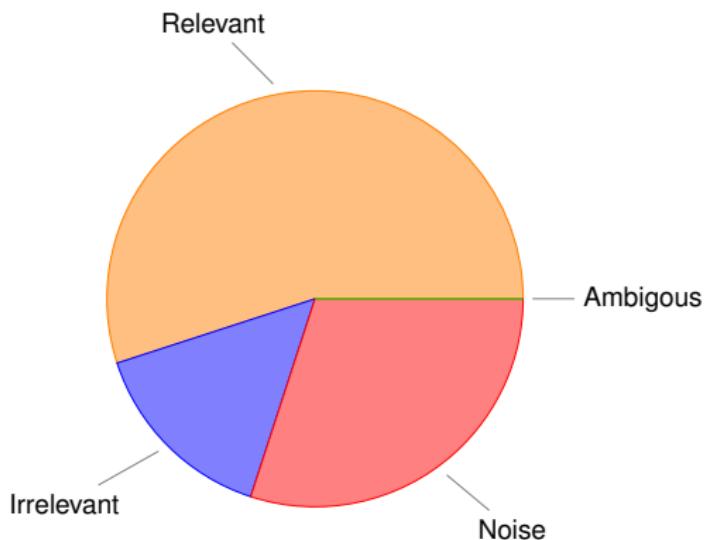
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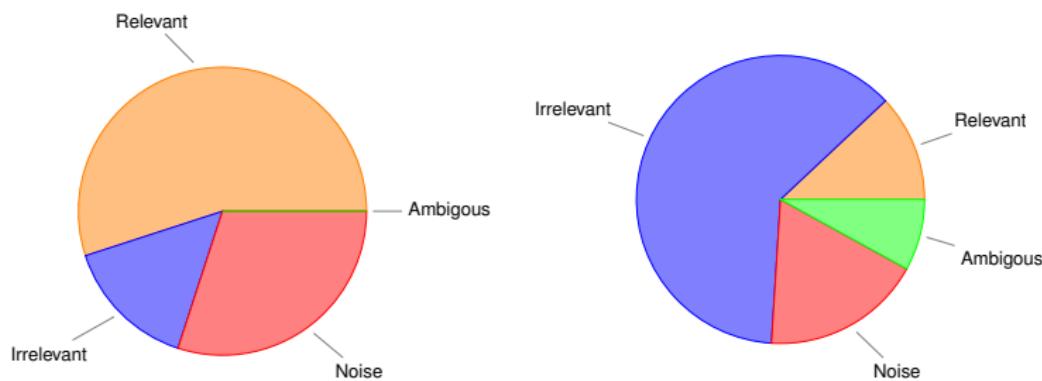
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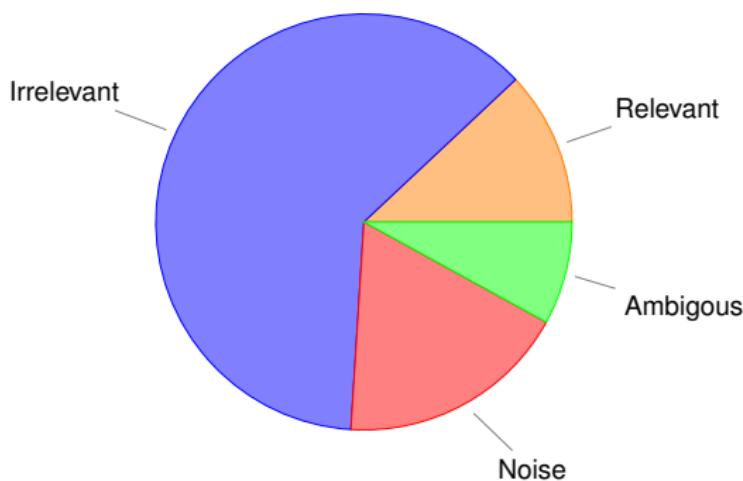
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The problem with generative models



The problem with generative models

Approaches

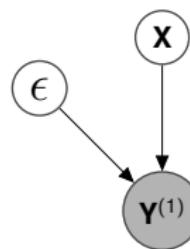
- Heuristics
 - ▶ Remove non-informative and noise by “hand” from data (pre-processing)
- Pseudo-heuristics
 - ▶ similarity engineering
- Full model
 - ▶ Factorise variations

This Talk

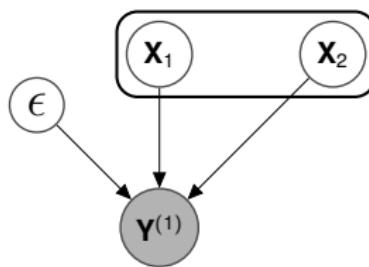
Factorised representation learning as a means of performing *feature selection* in a generative model.

- Factor Analysis
- Multiview learning
- GP formulation

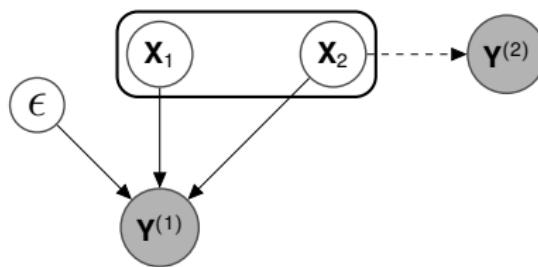
Factorised Representation Learning



Factorised Representation Learning



Factorised Representation Learning



Factor Analysis

$$\mathbf{y}_i = \mathbf{Ax}_i + \epsilon$$

$$p(\mathbf{y}|\mathbf{x}) = \mathcal{N}(\mathbf{y}|\mathbf{Ax}, \Sigma)$$

- **A** - factor loadings
- **X** - latent representation
- Solution not identifiable
- Introduce additional information

Factor Analysis

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Factor Analysis

FA according to Carl

- Structure of factor loadings

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & a_{13} & a_{14} & 0 & a_{16} & 0 \\ a_{21} & a_{22} & 0 & 0 & a_{25} & a_{26} & a_{27} \\ a_{31} & 0 & a_{33} & a_{34} & 0 & a_{36} & a_{37} \end{bmatrix}$$

- Column space structure of loadings

Factor Analysis

$$\mathbf{y}_i = \mathbf{Ax}_i + \epsilon$$

$$p(\mathbf{y}|\mathbf{x}) = \mathcal{N}(\mathbf{y}|\mathbf{Ax}, \Sigma)$$

Covariance

- Isotropic covariance implies PCA/MDS
- Full covariance plus diagonal implies “traditional” factor analysis

Factor Analysis

$$\mathbf{y}_i = \mathbf{Ax}_i + \epsilon$$

$$p(\mathbf{y}|\mathbf{x}) = \mathcal{N}(\mathbf{y}|\mathbf{Ax}, \Sigma)$$

Latent Variable

- Gaussian distribution for PCA and FA
- Non Gaussian for ICA

Factor Analysis

$$\mathbf{y}_i = \mathbf{Ax}_i + \epsilon$$
$$p(\mathbf{y}|\mathbf{x}) = \mathcal{N}(\mathbf{y}|\mathbf{Ax}, \Sigma)$$

Mapping

- Introduce general mapping f

$$p(\mathbf{y}|\mathbf{f}, \mathbf{x}) = p(\mathbf{y}|\mathbf{f})p(\mathbf{f}|\mathbf{x})$$

- Gaussian Process prior on mapping

Place a GP-prior over the mapping and get GP-LVM

Motivation

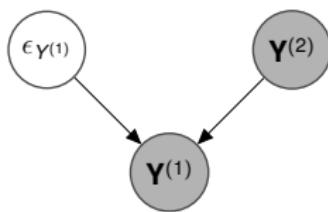
Introduction

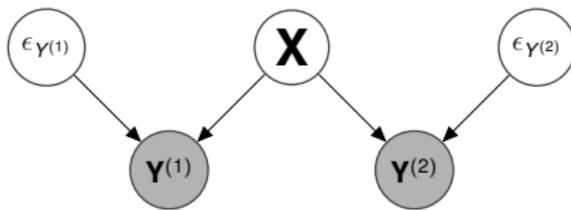
Supervised Factorised Representation Learning

Experiments

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- Noise in $\mathbf{Y}^{(1)}$ and $\mathbf{Y}^{(2)}$ (irrelevant)





Multiview Factor Analysis

Cannonical Correlation Analysis (Hotelling 1936)

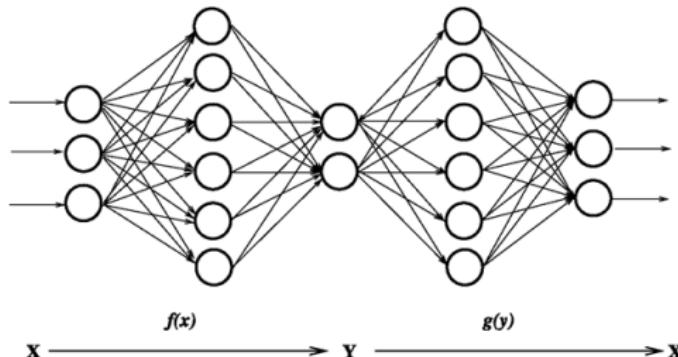
$$\{\hat{\mathbf{u}}, \hat{\mathbf{v}}\} = \underset{\mathbf{u}, \mathbf{v}}{\operatorname{argmax}} \rho(\mathbf{u}^T \mathbf{X}, \mathbf{v}^T \mathbf{Y})$$

- Correlation

$$\rho(\mathbf{X}, \mathbf{Y}) = \frac{\mathbb{E} [(\mathbf{X} - \mu_X)(\mathbf{Y} - \mu_Y)^\top]}{\sqrt{\mathbb{E}[\mathbf{X} - \mu_X] \mathbb{E}[\mathbf{Y} - \mu_Y]}}$$

- Learn a project of the data

Multiview Factor Analysis



Hybrid models

- Neuroscale (Lowe and Tipping 1997)
- Bottleneck networks (Hinton and Salakhutdinov 2006)
- De-noising Auto-encoders (Vincent *et al.* 2008)

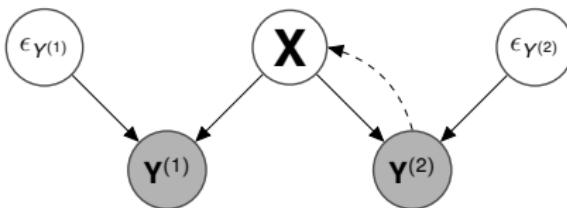
Multiview Factor Analysis

$$\begin{aligned} p(\mathbf{y}|\mathbf{f}, \mathbf{x}) &= p(\mathbf{y}|\mathbf{f})p(\mathbf{f}|\mathbf{x}) \\ \mathbf{x} &= g(\mathbf{y}) \end{aligned}$$

BC GP-LVM (Lawrence and Quiñonero-Candela 2006)

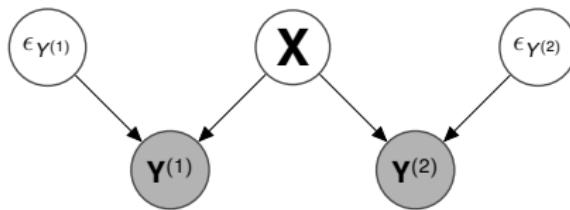
- Constrain latent space to reflect similarity in input
- Multi-view constrained (Ek *et al.* 2007, Snoek *et al.* 2012)
- Constrain latent space to only represent variation in input space that exist in output

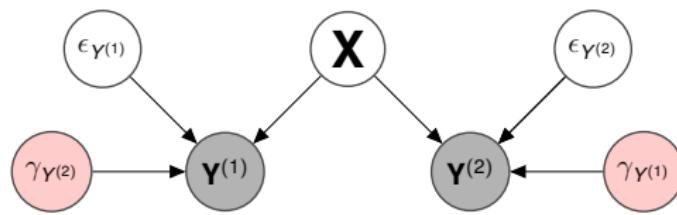
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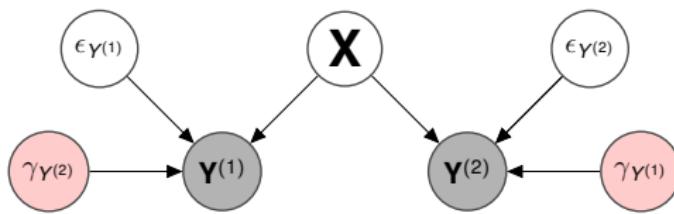


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Variations

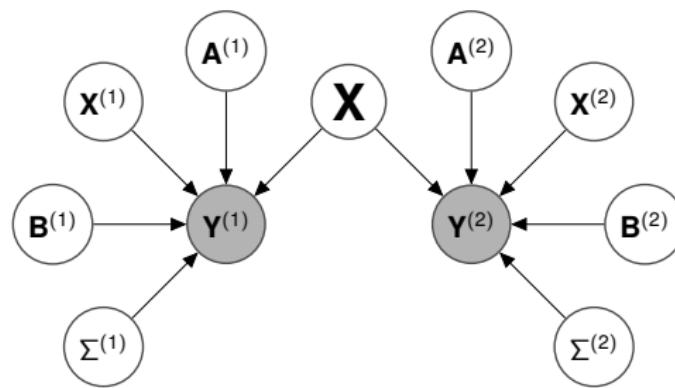
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- Noise in $\mathbf{Y}^{(1)}$ and $\mathbf{Y}^{(2)}$ (irrelevant)

Inter-Battery Factor Analysis¹

$$\begin{aligned}\mathbf{y}^{(m)} &\sim \mathcal{N}(\mathbf{A}^{(m)}\mathbf{x} + \mathbf{B}^{(m)}\mathbf{x}^{(m)}, \Sigma^{(m)}) \\ \mathbf{x} &\sim \mathcal{N}(\mathbf{0}, \mathbf{I}) \\ \mathbf{x}^{(m)} &\sim \mathcal{N}(\mathbf{0}, \mathbf{I})\end{aligned}$$

¹Tucker 1958.

Inter-Battery Factor Analysis¹



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- Explain away *both* structured and unstructured noise
- Specific model of ambiguities
 - Even more unidentifiable
 - ▶ Rank preserving transformations
 - ▶ Allocations of factors

¹Tucker 1958.

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$$\mathbf{x}^{(m)} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$$

- Marginalise view dependent latent variable

$$\mathbf{y}^{(m)} \sim \mathcal{N}(\mathbf{A}^{(m)}\mathbf{x}, \mathbf{B}^{(m)}(\mathbf{B}^{(m)})^T + \Sigma^{(m)})$$

- Full covariance (Bach and Jordan 2005)

¹Tucker 1958.

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$$\mathbf{x} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$$

$$\mathbf{x}^{(m)} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$$

- Do not want to “explain away” the view dependent variations

$$p(\mathbf{x}, \mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \mathbf{A}^{(1)}, \mathbf{A}^{(2)}, \mathbf{B}^{(1)}, \mathbf{B}^{(2)}, \Sigma^{(1)}, \Sigma^{(2)} | \mathbf{Y}^{(1)}, \mathbf{Y}^{(2)})$$

¹Tucker 1958.

Inter-Battery Factor Analysis¹

$$\mathbf{X} = [\mathbf{x}, \mathbf{x}^{(1)}, \dots, \mathbf{x}^{(N)}], \mathbf{Y} = [\mathbf{y}, \mathbf{y}^{(1)}, \dots, \mathbf{y}^{(N)}]$$

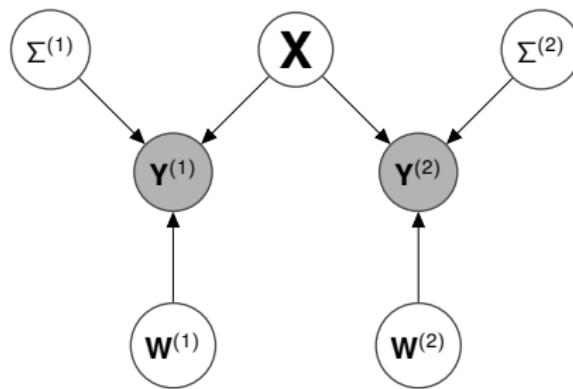
$$\mathbf{W} = \begin{bmatrix} \mathbf{A}^{(1)} & \mathbf{B}^{(1)} & 0 & \dots & 0 \\ \mathbf{A}^{(2)} & 0 & \mathbf{B}^{(2)} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{A}^{(N)} & 0 & 0 & 0 & \mathbf{B}^{(N)} \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} \Sigma^{(1)} & 0 & \dots \\ 0 & \ddots & 0 \\ \vdots & 0 & \Sigma^{(N)} \end{bmatrix}$$

$$\mathbf{Y} \sim \mathcal{N}(\mathbf{WX}, \Sigma)$$

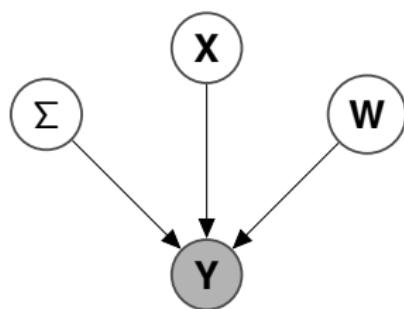
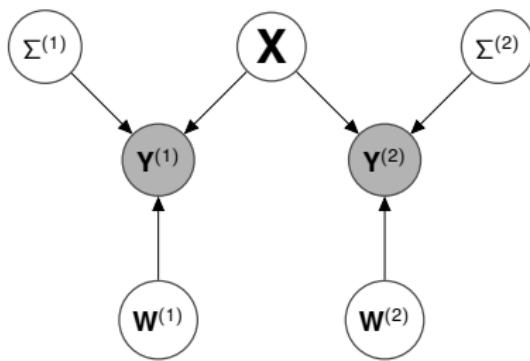
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Other models

- Concatenate model reduces to FA with specific structure of \mathbf{W}
- Bayesian FA^a: ignore structure of \mathbf{W}
- PPCA^b: spherical Σ

^aGhahramani and Beal 1999.

^bTipping and Bishop 1999.

¹Tucker 1958.

Bayesian IBFA²

$$\Sigma \sim IW(\mathbf{S}_0, v_0)$$

$$p(\mathbf{W}) = \prod_{m=1}^2 p(\mathbf{W}^{(m)} | \alpha_0, \beta_0)$$

$$p(\mathbf{W}^{(m)} | \alpha_0, \beta_0) = \prod_{k=1}^K p(w_k^{(m)} | \alpha_k^{(m)}) p(\alpha_k^{(m)} | \alpha_0, \beta_0)$$

$$p(\alpha_k^{(m)} | \alpha_0, \beta_0) \sim \Gamma(\alpha_0, \beta_0)$$

$$p(w_k^{(m)} | \alpha_k^{(m)}) = \mathcal{N}\left(\mathbf{0}, \left(\alpha_k^{(m)}\right)^{-1} \mathbf{I}\right)$$

²Klami *et al.* 2013.

Bayesian IBFA²

Factorisation

- Prior on \mathbf{W} induces group row-wise sparsity
- Jointly encourages *shared* representation (columns)
- Variational inference of parameters
- Linear generative mapping

²Klami *et al.* 2013.

Bayesian IBFA²

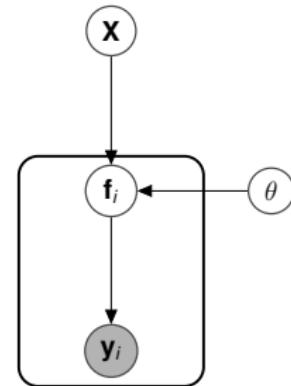
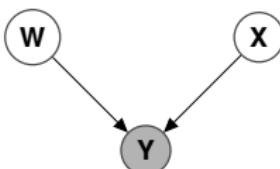
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Non-parametric IBFA³

$$\mathbf{X} = \mathbf{WY}$$

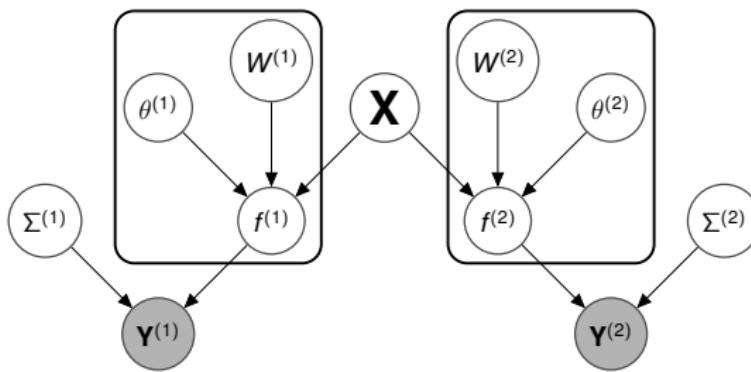


Next step

- History repeats itself
 - ▶ MDS/PCA \Rightarrow linear probabilistic \Rightarrow non-linear probabilistic
- IBFA with nonparametric mapping allows for non-linearities

³Damianou *et al.* 2012.

Non-parametric IBFA³



³Damianou *et al.* 2012.

Non-parametric IBFA³

Manifold Relevance Determination

- Factorisation inside mapping prior

$$k^Y(\mathbf{x}_i, \mathbf{x}_j) = (\sigma_{ard}^Y)^2 e^{-\frac{1}{2} \sum_{q=1}^Q w_q^Y (x_{i,q} - x_{j,q})^2}$$

- Requires bayesian treatment^a
 - ▶ Encourages reduction of (dimensions of) latent space
 - ▶ ARD parameters facilitates “turning dimensions off”
- Probabilistic non-linear IBFA

^aTitsias and Lawrence 2010.

³Damianou *et al.* 2012.

Summary

- Feature *learning* in a generative model can be viewed as factor analysis
- Feature *selection* in a generative model can be viewed as multiview factor analysis or inter battery factor analysis
- GP/GP-LVM framework allows for non-parametric formulation of inter battery factor analysis

Introduction

Supervised Factorised Representation Learning

Experiments

Experiments

Yale Faces

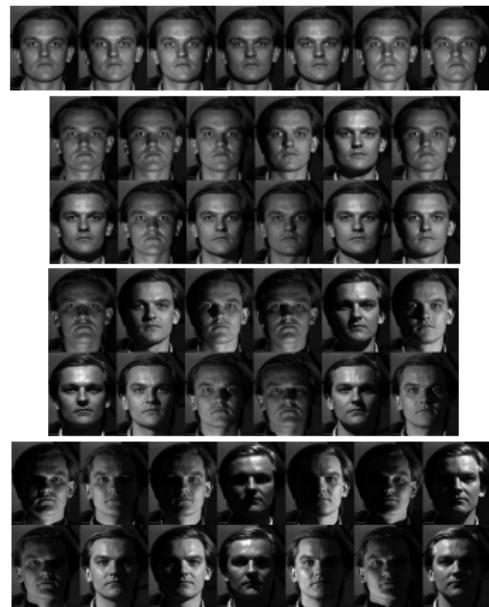
- Three faces
- 64 illuminations
- $\mathbf{y}_i \in \mathbb{R}^{192 \times 168}$
- Light alignment



Experiments

Yale Faces

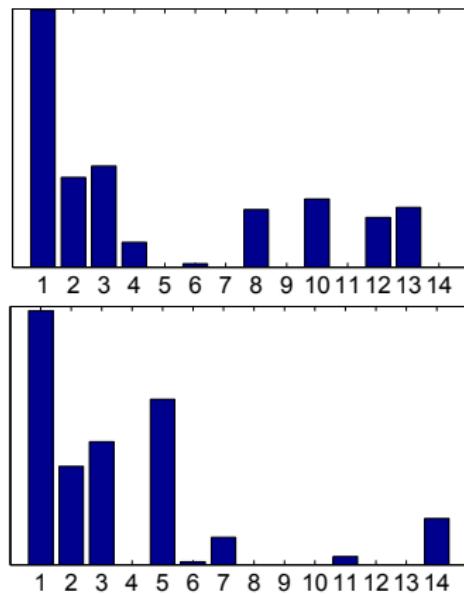
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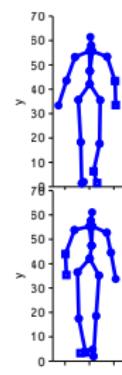
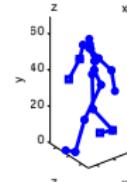
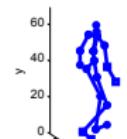
Loading video

Experiments

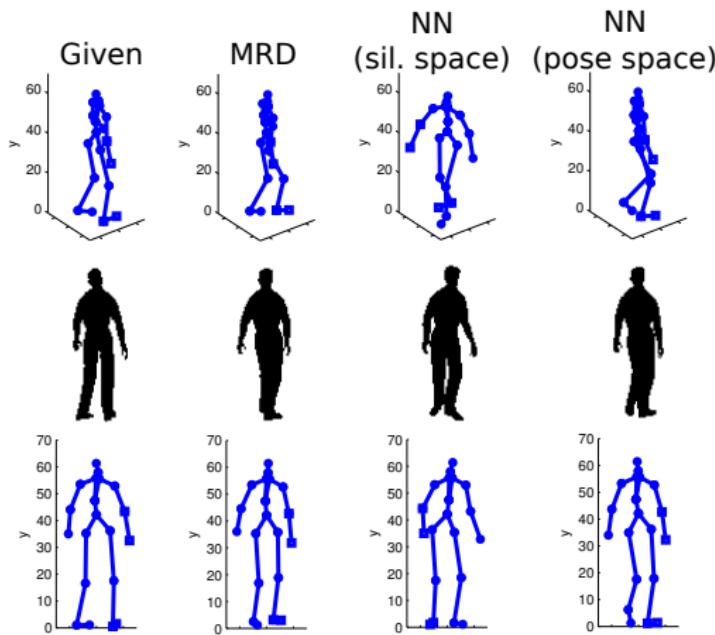
Pose Estimation (^a)

^aAgarwal and Triggs 2003.

- Silhouette images
- Image features
- Estimate 3D pose
- Highly Ambiguous



Experiments



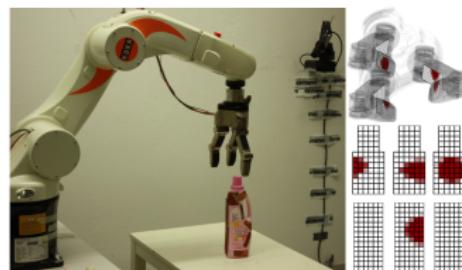
Experiments

	Error
Mean Training Pose	6.16
Linear Regression	5.86
GP Regression	4.27
Nearest Neighbour (sil. space)	4.88
Nearest Neighbour with sequences (sil. space)	4.04
Nearest Neighbour (pose space)	2.08
Shared GP-LVM	5.13
MRD without Dynamics	4.67
MRD with Dynamics	2.94

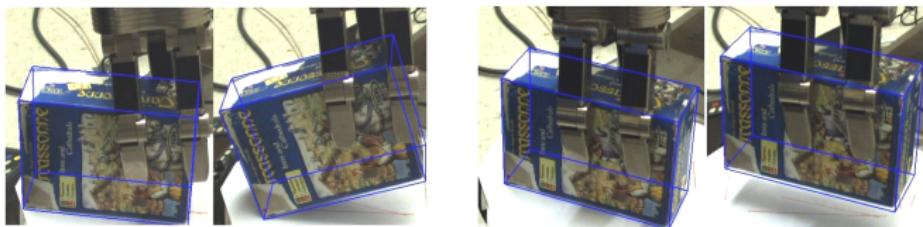
Experiments

Robotic Grasping

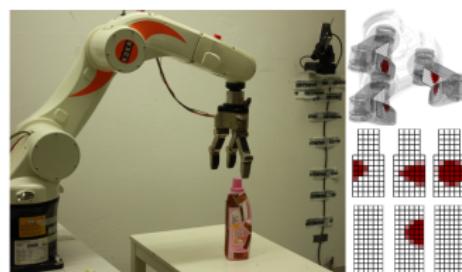
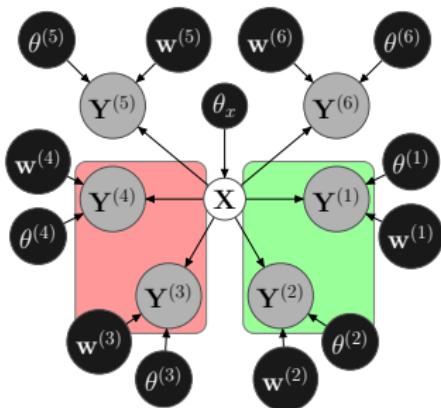
- Gripper pose
- Tactile sensor
- Object pose and identity



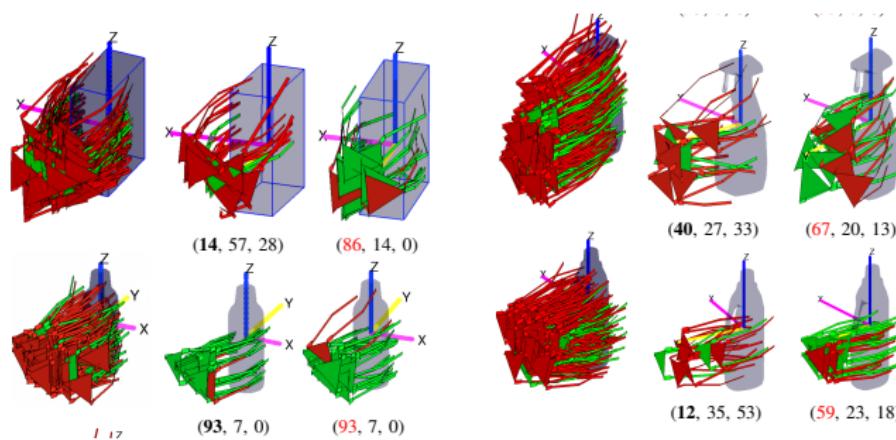
Experiments



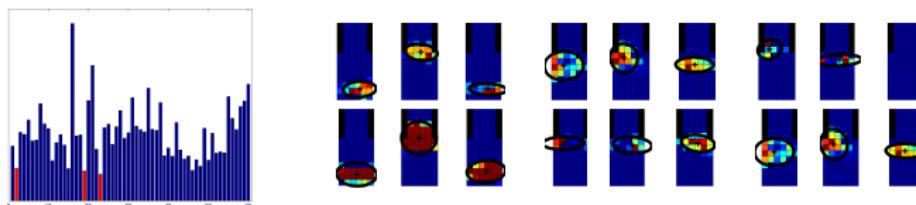
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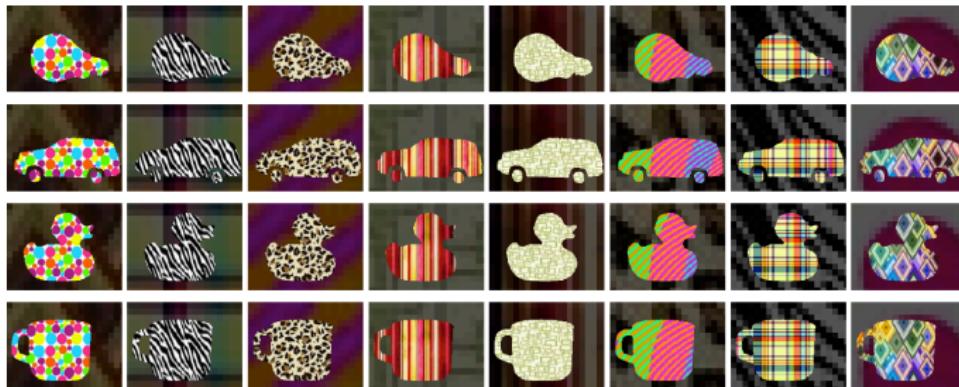
Experiments



Experiments

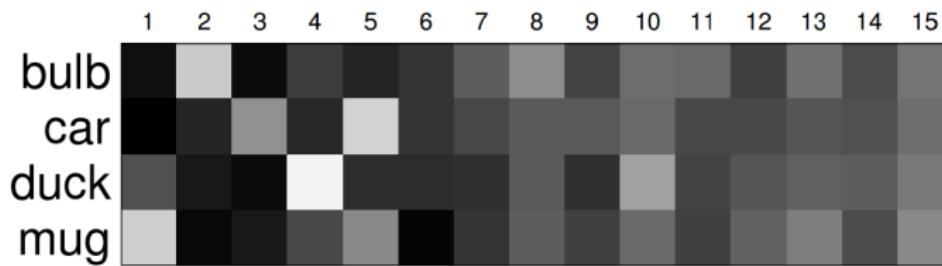


Bonus: Topic Modelling⁴



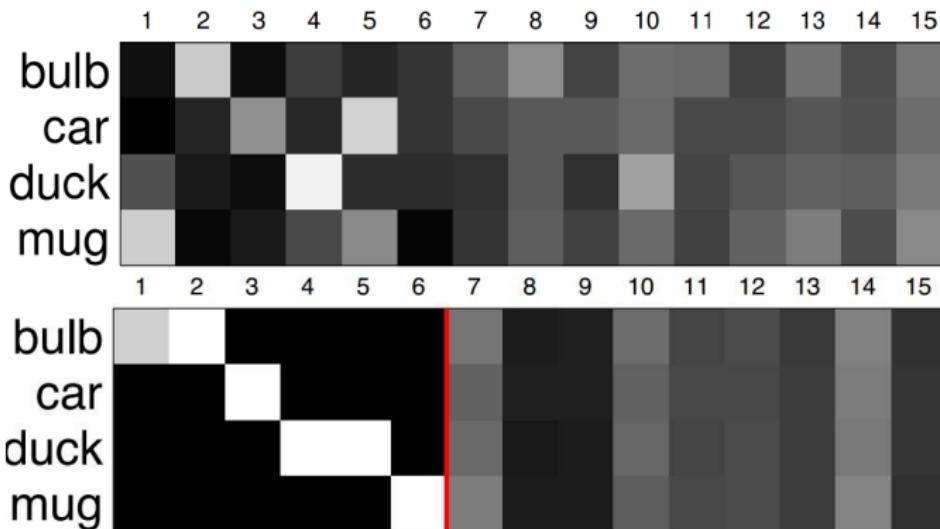
⁴Zhang *et al.* 2013.

Bonus: Topic Modelling⁴



⁴Zhang *et al.* 2013.

Bonus: Topic Modelling⁴



⁴Zhang *et al.* 2013.

Bonus: Topic Modelling⁴

0.25	0.13	0.5	0.13	0.88	0	0.13	0
0.13	0.5	0.25	0.13	0.25	0.75	0	0
0.25	0.13	0.5	0.13	0	0.13	0.88	0
0.13	0.25	0.5	0.13	0.13	0	0.13	0.75

⁴Zhang *et al.* 2013.

Future Work

- Approximate marginalisation of latent space
 - ▶ interesting priors
 - ▶ auto-encoders
 - ▶ deep models
- Bigger data-sets
- Automatic alignment

e.o.f.

References I

-  **H Hotelling.** “Relations between two sets of variates”. In: *Biometrika* 28.3/4 (1936), pp. 321–377.
-  **D. Lowe and Michael E Tipping.** “NeuroScale: Novel topographic feature extraction using RBF networks”. In: *Advances in Neural Information Processing Systems* (1997), pp. 543–549.
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