Kumaraswamy distribution

pdf, $f(x) = ab x^{a-1}(1-x^a)^{b-1}$, $0 < x < 1$

cdf, $F(x) = 1-(1-x^a)^b$

Likelihood function, $\mathcal{L} = a^n b^n \prod_{i=1}^{n} x_i^{a-1} \prod_{i=1}^{n} (1-x_i^a)^{b-1}$

$$\ln \mathcal{L} = n\ln a + n\ln b + (a-1)\sum_{i=1}^{n} \ln x_i + (b-1)\sum_{i=1}^{n} \ln(1-x_i^a)$$

$$\frac{\partial \ln \mathcal{L}}{\partial a} = \frac{n}{a} + \sum_{i=1}^{n} \ln x_i - (b-1)\sum_{i=1}^{n} \frac{x_i^a \ln x_i}{1-x_i^a}$$

$$= \frac{n}{a} - b\sum_{i=1}^{n} \frac{x_i^a \ln x_i}{1-x_i^a} + \sum_{i=1}^{n} \frac{\ln x_i}{1-x_i^a}$$

$$= \frac{n}{a} + \sum_{i=1}^{n} \frac{(1-bx_i^a)\ln x_i}{1-x_i^a}$$

$$\frac{\partial \ln \mathcal{L}}{\partial b} = \frac{n}{b} + \sum_{i=1}^{n} \ln(1-x_i^a)$$

Maximum likelihood, $\frac{\partial \ln \mathcal{L}}{\partial a} = 0$ and $\frac{\partial \ln \mathcal{L}}{\partial b} = 0$

$$\Rightarrow \quad b = \frac{-n}{\sum_{i=1}^{n} \ln(1-x_i^a)} \quad \text{and} \quad a = \frac{-n}{\sum_{i=1}^{n} \frac{(1-bx_i^a)\ln x_i}{1-x_i^a}}$$

ordered statistics

$$f_{X(j)}(x) = \frac{n!}{(j-1)!(n-j)!} f(x)\left[F(x)\right]^{j-1}\left[1-F(x)\right]^{n-j}$$

$$f_{X(1)}(x) = nf(x)\left[1-F(x)\right]^{n-1}, \quad f_{X(n)}(x) = nf(x)\left[F(x)\right]^{n-1}$$

$$f_{X(1)}(x) = nab\, x_{min}^{a-1}\left(1-x_{min}^a\right)^{bn-1}, \quad f_{X(n)}(x) = nab\, x_{max}^{a-1}\left(1-x_{max}^a\right)^{b-1} \cdot \left[1-(1-x_{max})^b\right]^{n-1}$$

$E[x_{(1)}]$ $\quad$ $E[x_{(n)}]$

$$\cancel{E[LB]} = x_{min} \cdot f_{X(1)}(x), \quad \cancel{E[UB]} = x_{max} \cdot f_{X(n)}(x)$$

$$\frac{Z_{(1)}-LB}{UB-LB} = x_{(1)}, \quad \frac{Z_{(n)}-LB}{UB-LB} = x_{(n)} \quad \Rightarrow \quad \begin{aligned}(1-x_{(1)})LB + x_{(1)}(UB) &= Z_{(1)} \\ (1-x_{(n)})LB + x_{(n)}(UB) &= Z_{(n)}\end{aligned}$$