Question 1

$$A=\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \Rightarrow A^{-1}=\begin{bmatrix} (A_{11}-A_{12}A_{22}^{-1}A_{21})^{-1} & -(A_{11}-A_{12}A_{22}^{-1}A_{21})^{-1}A_{12}A_{22}^{-1} \\ -(A_{22}-A_{21}A_{11}^{-1}A_{12})^{-1}A_{21}A_{11}^{-1} & (A_{22}-A_{21}A_{11}^{-1}A_{12})^{-1} \end{bmatrix}$$

$$y=X\beta_x+Z\beta_z+e$$

To solve for β_x and β_z we can minimize the sum of squared errors

$$\underset{\beta_x,\beta_z}{\text{Min}} \sum_{i=1}^{n} e_i^2 = \underset{\beta_x,\beta_z}{\text{Min}}\, e'e = \underset{\beta_x,\beta_z}{\text{Min}} (y-X\beta_x-Z\beta_z)'(y-X\beta_x-Z\beta_z)$$

$$\Rightarrow \frac{\partial}{\partial\beta_x}(y-X\beta_x-Z\beta_z)'(y-X\beta_x-Z\beta_z)=0 \Rightarrow -2X'(y-X\beta_x-Z\beta_z)=0$$

$$\text{and } \frac{\partial}{\partial\beta_z}(y-X\beta_x-Z\beta_z)'(y-X\beta_x-Z\beta_z)=0 \Rightarrow -2Z'(y-X\beta_x-Z\beta_z)=0$$

$$\Rightarrow X'y=X'X\hat{\beta}_x+X'Z\hat{\beta}_z \text{ and } Z'y=Z'X\hat{\beta}_x+Z'Z\hat{\beta}_z$$

$$\Rightarrow \begin{pmatrix} X'X & X'Z \\ Z'X & Z'Z \end{pmatrix}\begin{pmatrix} \hat{\beta}_x \\ \hat{\beta}_z \end{pmatrix}=\begin{pmatrix} X'y \\ Z'y \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \hat{\beta}_x \\ \hat{\beta}_z \end{pmatrix}=\begin{pmatrix} X'X & X'Z \\ Z'X & Z'Z \end{pmatrix}^{-1}\begin{pmatrix} X'y \\ Z'y \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \hat{\beta}_x \\ \hat{\beta}_z \end{pmatrix}=\begin{bmatrix} (X'X-X'Z(Z'Z)^{-1}Z'X)^{-1} & -(X'X-X'Z(Z'Z)^{-1}Z'X)^{-1}X'Z(Z'Z)^{-1} \\ -(Z'Z-Z'X(X'X)^{-1}X'Z)^{-1}Z'X(X'X)^{-1} & (Z'Z-Z'X(X'X)^{-1}X'Z)^{-1} \end{bmatrix}\begin{bmatrix} X'y \\ Z'y \end{bmatrix}$$

$$\Rightarrow \begin{pmatrix} \hat{\beta}_x \\ \hat{\beta}_z \end{pmatrix}=\begin{pmatrix} (X'X-X'Z(Z'Z)^{-1}Z'X)^{-1}X'y-(X'X-X'Z(Z'Z)^{-1}Z'X)^{-1}X'Z(Z'Z)^{-1}Z'y \\ (Z'Z-Z'X(X'X)^{-1}X'Z)^{-1}Z'y-(Z'Z-Z'X(X'X)^{-1}X'Z)^{-1}Z'X(X'X)^{-1}X'y \end{pmatrix}$$

$$\Rightarrow \hat{\beta}_x=(X'X-X'Z(Z'Z)^{-1}Z'X)^{-1}(X'y-X'Z(Z'Z)^{-1}Z'y)$$

$$\text{and } \hat{\beta}_z=(Z'Z-Z'X(X'X)^{-1}X'Z)^{-1}(Z'y-Z'X(X'X)^{-1}X'y)$$

$$\Rightarrow \hat{\beta}_x=(X'X-X'P_zX)^{-1}(X'y-X'P_zy), \text{ let } P_z=Z(Z'Z)^{-1}Z'$$

$$\text{and } \hat{\beta}_z=(Z'Z-Z'P_xZ)^{-1}(Z'y-Z'P_xy), \text{ let } P_x=X(X'X)^{-1}X'$$