Question 1

1. Define a function which takes the sample mean of a vector

Ans. $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i = \frac{1}{n}\sum_{i=1}^{n} x_i \cdot 1 = \frac{1}{n}\langle x, 1\rangle$

2. Define a function which takes a vector and outputs a difference between a vector and its associated mean

Ans. $$x - \bar{x} = \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix}}_{n\times 1 \text{ vector}} - \underbrace{\begin{pmatrix} \bar{x} \\ \bar{x} \\ \bar{x} \\ \vdots \end{pmatrix}}_{n \text{ times}} = x - \underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{pmatrix}}_{n\times 1 \text{ vector}} \cdot \underbrace{\begin{pmatrix} \bar{x} \end{pmatrix}}_{1\times 1 \text{ vector}}$$

$$= x - \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \cdot \left(\frac{1}{n}\langle x, 1\rangle\right)$$

3. Define the sample variance and sample covariance functions

Ans. sample variance $= \dfrac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1} = \frac{1}{n-1}\langle x - \bar{x}, x - \bar{x}\rangle$

Let $A = x_i - \bar{x}$

$$A = \underbrace{\begin{pmatrix} x_1 - \bar{x} \\ x_2 - \bar{x} \\ \vdots \\ x_n - \bar{x} \end{pmatrix}}_{n\times 1 \text{ vector}} \Rightarrow A' = \underbrace{\begin{pmatrix} x_1 - \bar{x} & x_2 - \bar{x} & \cdots & x_n - \bar{x} \end{pmatrix}}_{1\times n \text{ vector}}$$

$$A' \cdot A = \left((x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \cdots (x_n - \bar{x})^2\right)$$

$$\Rightarrow \frac{1}{n-1} \cdot A'A = \frac{1}{n-1} \cdot \sum_{i=1}^{n}(x_i - \bar{x})^2$$

If $B = y_i - \bar{y}$ Similarly, we can show that $\frac{1}{n-1} A' \cdot B = \frac{1}{n-1} \cdot \sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})$

$$= \frac{1}{n-1}\langle x - \bar{x}, y - \bar{y}\rangle$$

Question 2. If $A \cdot A = A$ then $A$ is called idempotent.

$$P = X(X'X)^{-1}X', \quad M = I - P, \quad y = X\beta + e$$

1. $P$ is idempotent. $P \cdot P = X\cancel{(X'X)^{-1}X'X}(X'X)^{-1}X' = X(X'X)^{-1}X'$
$= P$

2. $M$ is idempotent. $M \cdot M = (I-P)(I-P) = I - IP - PI + P \cdot P$
$= I - P - \cancel{P} + \cancel{P} = I - P = M$

3. $\hat{y} = Py$, from OLS we know $y = X\hat{\beta} + \hat{e}$
and $\hat{y} = X\hat{\beta}$
and $\hat{\beta} = (X'X)^{-1}X'y$
$\Rightarrow \hat{y} = X(X'X)^{-1}X'y = Py$.

4. $\hat{e} = My$, from OLS we know $\hat{e} = y - \hat{y}$
$\Rightarrow \hat{e} = y - Py$
$\Rightarrow \hat{e} = (I-P)y = My$

5. $y = Py + My$, from OLS we know $y = \hat{y} + \hat{e}$
$\Rightarrow y = Py + My$

6. $P = X(X'X)^{-1}X' \Rightarrow P' = \left[X(X'X)^{-1}X'\right]' = X((X'X)')^{-1}X' = X(X'X)^{-1}X' = P$
$\Rightarrow P' = P$

$\hat{y} \perp \hat{e} \Rightarrow \langle \hat{y}, \hat{e} \rangle = 0$
$\Rightarrow \langle Py, My \rangle = 0$
$\Rightarrow (Py)'My = 0$
$\Rightarrow y'P'My = 0$
$\Rightarrow y'P \cdot (I-P)y = 0$
$\Rightarrow y'(P - P \cdot P)y = 0$
$\Rightarrow y'(P-P)y = 0$
$\Rightarrow 0 = 0$ //

So, $\hat{y}$ and $\hat{e}$ are orthogonal.