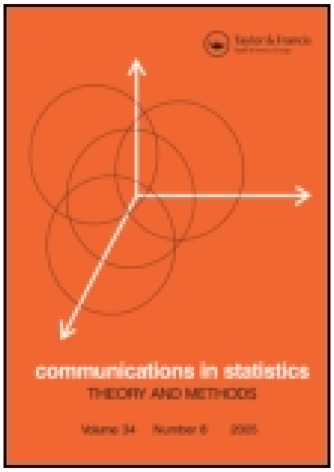

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New Properties of the Kumaraswamy Distribution

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New Properties of the Kumaraswamy Distribution

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The Kumaraswamy distribution is very similar to the Beta distribution but has the key advantage of a closed-form cumulative distribution function. This makes it much better suited than the Beta distribution for computation-intensive activities like simulation modeling and the estimation of models by simulation-based methods. However, in spite of the fact that the Kumaraswamy distribution was introduced in 1980, further theoretical research on the distribution was not developed until very recently (Garg, 2008; Jones, 2009; Mitnik, 2009; Nadarajah, 2008). This article contributes to this recent research and: (a) shows that Kumaraswamy variables exhibit closeness under exponentiation and under linear transformation; (b) derives an expression for the moments of the general form of the distribution; (c) specifies some of the distribution's limiting distributions; and (d) introduces an analytical expression for the mean absolute deviation around the median as a function of the parameters of the distribution, and establishes some bounds for this dispersion measure and for the variance.

Keywords Beta distribution; Closeness properties; Dispersion bounds; Kumaraswamy distribution; Limiting distributions; Mean absolute deviation around the median; Moments.

Mathematics Subject Classification Primary 60E05; Secondary 62E99.

1. Introduction

The Kumaraswamy distribution is a continuous probability distribution with double-bounded support. It is very similar to the Beta distribution, and can thus assume a strikingly large variety of shapes and be used to model many random processes and uncertainties. One key difference between the Kumaraswamy and Beta distributions is the availability for the former, but not for the latter, of an invertible closed-form cumulative distribution function. This makes it much better suited than the Beta distribution for computation-intensive activities like simulation modeling (e.g., Banks, 1998; Rennard, 2006) and the estimation of models by simulation-based methods (e.g., the method of simulated moments and

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indirect inference; see, for instance, Gouriéroux and Monfort, 1996). In these activities, which have become increasingly important in the last 15 years, using the Kumaraswamy rather than the Beta distribution would be much more efficient and easier to implement from a computational point of view. This is of paramount importance in settings like these, where computer-power constraints are often binding. In addition, the Kumaraswamy distribution can be fruitfully employed in the context of the quantile modeling approach (Jones, 2009, p. 71) and, in particular, to model conditional quantiles parametrically (Mitnik, 2009, Sec. 4).

The Kumaraswamy distribution was originally conceived to model hydrological phenomena (Kumaraswamy, 1980), and has been used for this but also for other purposes (for examples see Courard-Hauri, 2007; Fletcher and Ponnambalam, 1996; Ganji et al., 2006; Sanchez et al., 2007; Seifi et al., 2000; Sundar and Subbiah, 1989). However, in spite of the advantages that the availability of a closed-form distribution function entails, and although Poondi Kumaraswamy introduced the double-bounded distribution that bears his name almost three decades ago, further research on the distribution itself was not developed until very recently (Garg, 2008; Nadarajah, 2008; Mitnik, 2009; and, most notably, Jones, 2009). The results presented here in—a by-product of on-going research on queuing matching models of the labor market and on parametric quantile regression, in which the Kumaraswamy distribution is employed intensively—contribute to this recent research by uncovering several new properties of this distribution.

The article is organized as follows. Section 2 presents the features of the Kumaraswamy distribution relevant for the rest of the article—all of which were essentially contained in Kumaraswamy's 1980 article—and very briefly reviews the recent literature on this distribution. Section 3 introduces two very simple but previously unreported properties of Kumaraswamy variables: closeness under linear transformation and under exponentiation. Section 4 derives an expression for the moments of the general form of the distribution. Section 5 specifies some of the distribution's limiting distributions. Lastly, Sec. 6 introduces an analytical expression for the mean absolute deviation around the median as a function of the parameters of the distribution, and establishes some bounds both for this dispersion measure and for the variance.

2. The Kumaraswamy Distribution: Definition and Known Properties

In its general form, the probability density function of the continuous part of the distribution Kumaraswamy introduced in his 1980 article can be written as

$$f_Z(z) = \frac{1}{(b-c)} pq \left(\frac{z-c}{b-c}\right)^{p-1} \left[1 - \left(\frac{z-c}{b-c}\right)^{p}\right]^{q-1}, \quad c < z < b, \tag{1}$$

with shape parameters $p > 0$ and $q > 0$, and boundary parameters c and b. The general form of the distribution will be denoted by $\mathrm{K}(p, q, c, b)$. Making the transformation $X = \frac{Z-c}{b-c}$ and using the change of variable theorem, we obtain the standard form of the Kumaraswamy density function

$$f_X(x) = pqx^{p-1}(1 - x^p)^{q-1}, \quad 0 < x < 1, \tag{2}$$

which will be denoted by $\mathrm{K}(p, q) \equiv \mathrm{K}(p, q, 0, 1)$. In what follows the standard form of the distribution will be employed unless otherwise indicated.

The cumulative distribution function of the Kumaraswamy distribution has a closed form expression, namely

$$F(x) = 1 - (1 - x^p)^q, \quad 0 < x < 1. \tag{3}$$

From (3), it immediately follows that the quantile function $F^{-1}(u)$ is also available in closed-form:

$$x = \left[1 - (1 - u)^{\frac{1}{q}}\right]^{\frac{1}{p}}, \quad 0 < u < 1. \tag{4}$$

In particular, the median of the Kumaraswamy distribution can be written as

$$md(X) = \omega = (1 - 0.5^{\frac{1}{q}})^{\frac{1}{p}}. \tag{5}$$

If the random variable X is distributed $\mathrm{K}(p, q)$, its moments around zero can be expressed as

$$\mu_r'(X) = qB\left(1 + \frac{r}{p}, q\right), \tag{6}$$

where $B(\alpha, \beta) = \int_0^1 s^{\alpha-1}(1-s)^{\beta-1}ds = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$ is the Beta function and $\Gamma(v) = \int_0^\infty t^{v-1}e^{-t}dt$ is the Gamma function. Thus, the expectation and variance of X are

$$E(X) = \mu = \mu_1'(X) = qB\left(1 + \frac{1}{p}, q\right) \tag{7}$$

$$Var(X) = \mu_2 = \mu_2'(X) - \mu^2 = qB\left(1 + \frac{2}{p}, q\right) - \left[qB\left(1 + \frac{1}{p}, q\right)\right]^2. \tag{8}$$

Table 1 shows the possible shapes of the Kumaraswamy distribution and its behavior at the boundaries of its support, as a function of the values of its parameters; the table also identifies the distribution's special cases. Notably, as also pointed out by Jones (2009, p. 74), the relationships between the features of the Kumaraswamy distribution and the values of its parameters summarized in Table 1 also obtain, without any exception, in the case of the Beta distribution (for the Beta distribution, see Johnson et al., 1995, p. 219).

As anticipated in the Introduction, recent research has substantially extended our knowledge of the Kumaraswamy distribution's properties and of its relationships with other distributions. Nadarajah (2008; see also Jones, 2009, p. 72) observed that, like the Beta distribution, the Kumaraswamy distribution is a special case of McDonald's generalized Beta of the first kind distribution (McDonald, 1984; see also McDonald and Richards, 1987). Although Naradajah only considered the case with $c = 0$, his observation also applies to the general case of $c < b$ if we slightly generalize McDonald's formulation by adding a lower-bound parameter to his density function. Garg (2008) derived the distribution of single order statistics, the joint distribution of two order statistics, and the distribution of the product and quotient of two order statistics when the random variables are independent and identically Kumaraswamy-distributed. Mitnik (2009) introduced a median-dispersion re-parameterization of the general Kumaraswamy

Table 1
Characteristics of the Kumaraswamy distribution for different parameter values

	$q < 1$	$q = 1$	$q > 1$
$p < 1$	Distribution with one antimode	Monotonically decreasing (power-function) distribution	Monotonically decreasing distribution
	$\lim_{x\to 0} f(x) = \infty$	$\lim_{x\to 0} f(x) = \infty$	$\lim_{x\to 0} f(x) = \infty$
	$\lim_{x\to 1} f(x) = \infty$	$\lim_{x\to 1} f(x) = p$	$\lim_{x\to 1} f(x) = 0$
$p = 1$	Monotonically increasing distribution	Uniform distribution	Monotonically decreasing distribution
	$\lim_{x\to 0} f(x) = q$	$\lim_{x\to 0} f(x) = \lim_{x\to 1} f(x) = 1$	$\lim_{x\to 0} f(x) = q$
	$\lim_{x\to 1} f(x) = \infty$		$\lim_{x\to 1} f(x) = 0$
$p > 1$	Monotonically increasing distribution	Monotonically increasing (power-function) distribution	Distribution with one mode
	$\lim_{x\to 0} f(x) = 0$	$\lim_{x\to 0} f(x) = 0$	$\lim_{x\to 0} f(x) = 0$
	$\lim_{x\to 1} f(x) = \infty$	$\lim_{x\to 1} f(x) = p$	$\lim_{x\to 1} f(x) = 0$

distribution aimed at facilitating its use in regression models and in latent-variable and other models estimated through simulation-based methods. Lastly, Jones (2009) offered a characterization of the Kumaraswamy distribution as the distribution of the minimum of all the maxima in samples of uniform size drawn from a uniform distribution; derived general formulas for L-moments and for the moments of order statistics of the distribution; studied the distribution's skewness and kurtosis properties, some limiting distributions, and some distributions related by transformation; investigated inference by maximum likelihood for the distribution; and compared the Beta and the Kumaraswamy distributions from several points of view, emphasizing the tractability advantages of the latter.

3. Relationships Between Kumaraswamy Variables

Similarly to what is the case with Beta-distributed variables, any linear transformation of a Kumaraswamy-distributed variable is also Kumaraswamy-distributed, and its distribution has the same shape parameters as the original distribution. Unlike Beta-distributed variables, however, Kumaraswamy variables are also closed under exponentiation—any positive power of a Kumaraswamy random variable also has a Kumaraswamy distribution. These two properties can be more precisely formulated as follows.

Proposition 3.1. *Closeness under linear transformation: If* $Y = \beta + \alpha X, \alpha \neq 0$, *then* $X \sim \mathrm{K}(p, q, c, b) \leftrightarrow Y \sim \mathrm{K}(p, q, \beta + \alpha c, \beta + \alpha b)$. *One corollary is that if* $X \sim \mathrm{K}(p, q)$ *and* Y *is the standardization of* X, *that is,* $Y = \frac{X-\mu(X)}{[Var(X)]^{\frac{1}{2}}}$, *then* $Y \sim \mathrm{K}(p, q, -\frac{\mu(X)}{[Var(X)]^{\frac{1}{2}}}, \frac{1-\mu(X)}{[Var(X)]^{\frac{1}{2}}})$. *Another corollary is that if* $Y = d - X$, *with* d *any real number, then* $X \sim \mathrm{K}(p, q, c, b) \leftrightarrow Y \sim \mathrm{K}(p, q, d - c, d - b)$.

Proposition 3.2. *Closeness under exponentiation: If* $Y = X^m$ *and* $m > 0$, *then* $X \sim \mathrm{K}(p, q) \leftrightarrow Y \sim \mathrm{K}(\frac{p}{m}, q)$. *As is clear from the Kumaraswamy and Beta probability density functions that for any* $p > 0$ *and* $q > 0$, $X \sim \mathrm{K}(1, q) \leftrightarrow X \sim \mathrm{Beta}(1, q)$, *it follows from this proposition that if* $Y = X^p$, *then* $X \sim \mathrm{K}(p, q) \leftrightarrow Y \sim \mathrm{Beta}(1, q)$.

The proofs are very simple, so in each case only proof of the implication in one direction is given. For Proposition 3.1, we have:

$$f_{(Y)}(y) = f_X\left(\frac{y-\beta}{\alpha}\right)\left|\frac{dx}{dy}\right| = \frac{1}{(b-c)}pq\left[\frac{\left(\frac{y-\beta}{\alpha}\right)-c}{b-c}\right]^{p-1}\left\{1-\left[\frac{\left(\frac{y-\beta}{\alpha}\right)-c}{b-c}\right]^{p}\right\}^{q-1}\frac{1}{\alpha}$$

$$= \frac{1}{(\beta+\alpha b)-(\beta+\alpha c)}pq\left[\frac{y-(\beta+\alpha c)}{(\beta+\alpha b)-(\beta+\alpha c)}\right]^{p-1}$$

$$\times\left\{1-\left[\frac{y-(\beta+\alpha c)}{(\beta+\alpha b)-(\beta+\alpha c)}\right]^{p}\right\}^{q-1}.$$

Likewise, for Proposition 3.2:

$$f_Y(y) = f_X\left(y^{\frac{1}{m}}\right)\left|\frac{dx}{dy}\right| = pq\left(y^{\frac{1}{m}}\right)^{p-1}\left[1-\left(y^{\frac{1}{m}}\right)^{p}\right]^{q-1}\frac{1}{m}y^{\frac{1}{m}-1} = \frac{p}{m}qy^{\frac{p}{m}-1}\left(1-y^{\frac{p}{m}}\right)^{q-1}.$$

4. Moments of the General Form of the Distribution

Many stochastic phenomena involve random variables measured in scales different from (0, 1), and which cannot be transformed to this scale for modeling purposes. If the Kumaraswamy distribution is used to model these phenomena, researchers will often need to calculate the moments of the general form of the distribution as a function of its parameters. These moments can be derived using (6), closeness under linear transformation, and known relationships among moments. The result is the following.

Proposition 4.1. *If* $Y \sim \mathrm{K}(p, q, c, b)$, $c < y < b$, *then*

$$\mu'_r(Y) = (b-c)^r\sum_{j=0}^{r}\binom{r}{j}qB\left(1+\frac{r-j}{p}, q\right)\left(\frac{c}{b-c}\right)^j.$$

For $c = 0$, *this expression becomes a much simpler one,* $\mu'_r(Y) = b^r qB(1 + \frac{r}{p}, q)$, *which can also be derived from McDonald's general expression for the moments of the generalized Beta of the first kind distribution (see McDonald, 1984, p. 650); and, of course, the expression reduces to (6) when* $c = 0$ *and* $b = 1$.

To prove Proposition 4.1, observe that from the definition of raw moments we have

$$\mu'_r(Y) = \int_c^b y^r \frac{1}{b-c} pq \left(\frac{y-c}{b-c}\right)^p \left[1 - \left(\frac{y-c}{b-c}\right)^p\right]^{q-1} dy. \tag{9}$$

Given closeness under linear transformation, the variable $Z = \frac{Y-c}{b-c}$ is distributed $\mathrm{K}(p, q)$. Doing the change of variable $Z = \frac{Y-c}{b-c}$ in (9), we obtain

$$\mu'_r(Y) = \int_0^1 [z(b-c)+c]^r \frac{1}{b-c} pqz^p (1-z^p)^{q-1} (b-c)dz$$

$$\mu'_r(Y) = (b-c)^r \int_0^1 \left(z + \frac{c}{b-c}\right)^r f_Z(Z)dz$$

$$\mu'_r(Y) = (b-c)^r \mu'_r\left(Z; -\frac{c}{b-c}\right), \tag{10}$$

where $\mu'_r\left(Z; -\frac{c}{b-c}\right)$ denotes the rth moment of Z around $-\frac{c}{b-c}$. As the moments of any distribution around an arbitrary value α can be expressed as functions of the distribution's raw moments by using $\mu'_r(X; \alpha) = \sum_{j=0}^{r} \binom{r}{j} \mu'_{r-j}(X)(-\alpha)^j$ (Stuart and Ord, 1987, p. 73), we can first substitute this formula for the moments of Z in (10), and then use (6) to express the moments of Y around zero as follows:

$$\mu'_r(Y) = (b-c)^r \sum_{j=0}^{r} \binom{r}{j} \mu'_{r-j}(Z) \left(\frac{c}{b-c}\right)^j$$

$$\mu'_r(Y) = (b-c)^r \sum_{j=0}^{r} \binom{r}{j} qB\left(1 + \frac{r-j}{p}, q\right) \left(\frac{c}{b-c}\right)^j,$$

which concludes the proof.

5. Limiting Distributions

Jones (2009) specified the following three limiting distributions. Given $X \sim \mathrm{K}(p, q)$, the density of $Y = q^{\frac{1}{p}} X \sim \mathrm{K}(p, q, 0, q^{\frac{1}{p}})$ tends to the density of the Weibull distribution with parameter p as $q \to \infty$; the density of $Z = p(1-X) \sim \mathrm{K}(p, q, 0, p)$ tends to the density of minus the logarithm of the power function distribution with parameter q as $p \to \infty$; and the density of $W = p(1 - q^{\frac{1}{p}} X) \sim \mathrm{K}(p, q, p - pq^{\frac{1}{p}}, p)$ tends to the density of the standard extreme value distribution of the first type as both $p \to \infty$ and $q \to \infty$. In these three cases, at least one boundary parameter is a function of one or more shape parameters that tend to infinity. In contrast, the following six propositions specify limiting distributions in which the support of the distribution is not affected by the values of its shape parameters.

Proposition 5.1. *When $p \to \infty$ or $q \to 0$, the standard Kumaraswamy distribution tends to the degenerate distribution with parameter $\rho = 1$.*

Proposition 5.2. *When $p \to 0$ or $q \to \infty$, the standard Kumaraswamy distribution tends to the degenerate distribution with parameter $\rho = 0$.*

Let's define the set $\theta_{\mu_0} = \left\{(p, q) \mid \mu = qB\left(1 + \frac{1}{p}, q\right) = \mu_0\right\}$. Then we have the following.

Proposition 5.3. *When $p \to 0, q \to 0$, and $(p, q) \in \theta_{\mu_0}$, the standard Kumaraswamy distribution tends to the Bernoulli distribution with parameter $s = \mu_0$.*

Proposition 5.4. *When $p \to \infty, q \to \infty$, and $(p, q) \in \theta_{\mu_0}$, the standard Kumaraswamy distribution tends to the degenerate distribution with parameter $\rho = \mu_0$.*

Let's define the set $\tau_{\omega_0} = \left\{(p, q) \mid \omega = (1 - 0.5^{\frac{1}{q}})^{\frac{1}{p}} = \omega_0\right\}$. Then we have the following.

Proposition 5.5. *When $p \to 0$, $q \to 0$, and $(p, q) \in \tau_{\omega_0}$, the standard Kumaraswamy distribution tends to the Bernoulli distribution with parameter $s = 0.5$, regardless of the particular value of ω_0.*

Proposition 5.6. *When $p \to \infty, q \to \infty$, and $(p, q) \in \tau_{\omega_0}$, the standard Kumaraswamy distribution tends to the degenerate distribution with parameter $\rho = \omega_0$.*

Let $X \sim \mathrm{K}(p, q)$, with expected value $\mu = \mu_0$. Proposition 5.1 can be proved by taking limit in (6) for $p \to \infty$ and for $q \to 0$:

$$\lim_{p\to\infty} \mu'_r(X) = \lim_{p\to\infty} qB\left(1 + \frac{r}{p}, q\right) = \frac{q}{q} = 1.$$

$$\lim_{q\to 0} \mu'_r(X) = \lim_{q\to 0} qB\left(1 + \frac{r}{p}, q\right) = \lim_{q\to 0} \frac{q\Gamma\left(1 + \frac{r}{p}\right)\Gamma(q)}{\Gamma\left(1 + \frac{r}{p} + q\right)}$$

$$= \lim_{q\to 0} \frac{\Gamma\left(1 + \frac{r}{p}\right)\Gamma(q+1)}{\Gamma\left(1 + \frac{r}{p} + q\right)} = 1,$$

where the well-known properties $B(1, \beta) = \beta^{-1}$, $\alpha\Gamma(\alpha) = \Gamma(\alpha + 1)$, and $\Gamma(1) = 1$ are used. As the moments of the degenerate distribution with parameter ρ are $\mu'_r = E(X^r) = \rho^r$, when $p \to \infty$ or $q \to 0$ the standard Kumaraswamy distribution tends to the degenerate distribution with parameter $\rho = 1$, as stated.

To prove Proposition 5.2, observe that using (6) and a series representation of the Beta function, $B(\alpha, \beta) = \frac{1}{\beta}\sum_{k=0}^{\infty}(-1)^k \beta \frac{(\beta-1)\dots(\beta-k)}{k!(\alpha+k)}$ with $\beta > 0$ (Gradshteyn and Ryzhik, 2007, p. 909), the moments around zero of the Kumaraswamy distribution can be written as

$$\mu'_r = qB\left(1 + \frac{r}{p}, q\right) = \sum_{k=0}^{\infty}(-1)^k q \frac{(q-1)\dots(q-k)}{k!\left(1 + \frac{r}{p} + k\right)}. \tag{11}$$

Likewise, using (6) and a product representation of the Beta function, $B(\alpha, \beta) = \frac{1}{\alpha+\beta-1}\prod_{k=1}^{\infty}\frac{k(\alpha+\beta+k-2)}{(\alpha+k-1)(\beta+k-1)}$ with $\beta \neq -1, -2, \dots$. (Gradshteyn and Ryzhik, 2007,

p. 909), these moments can be expressed as

$$\mu_r' = qB\left(1+\frac{r}{p},q\right) = \frac{q}{\frac{r}{p}+q}\prod_{k=1}^{\infty}\frac{k(\frac{r}{p}+q+k-1)}{\left(\frac{r}{p}+q\right)(q+k-1)}. \tag{12}$$

Taking limit for $p \to 0$ in (11) and for $q \to \infty$ in (12), we obtain:

$$\lim_{p\to 0}\mu_r' = \lim_{p\to 0}\sum_{k=0}^{\infty}(-1)^k q\frac{(q-1)\dots(q-k)}{k!\left(1+\frac{r}{p}+k\right)} = 0.$$

$$\begin{aligned}
\lim_{q\to\infty}\mu_r' &= \lim_{q\to\infty}\frac{q}{\frac{r}{p}+q}\prod_{k=1}^{\infty}\frac{k\left(\frac{r}{p}+q+k-1\right)}{\left(\frac{r}{p}+q\right)(q+k-1)}\\
&= \lim_{q\to\infty}\frac{1}{\frac{\frac{r}{p}}{q}+1}\prod_{k=1}^{\infty}\frac{k\,(q+k-1)+k\frac{r}{p}}{\left(\frac{r}{p}+q\right)(q+k-1)}\\
&= \lim_{q\to\infty}\frac{1}{\frac{\frac{r}{p}}{q}+1}\prod_{k=1}^{\infty}\frac{k}{\left(\frac{r}{p}+q\right)}+\frac{k\frac{r}{p}}{\left(\frac{r}{p}+q\right)(q+k-1)} = 0.
\end{aligned}$$

Therefore, when $p \to 0$ or $q \to \infty$ the standard Kumaraswamy distribution tends to the degenerate distribution with parameter $\rho = 0$, as stated.

In order to prove Propositions 5.3 and 5.4 observe, first, that first partial derivatives of moments with respect to the distribution parameters can be calculated using (6) and $\frac{\partial}{\partial j}B(\alpha,\beta) = B(\alpha,\beta)[\psi(j)-\psi(\alpha+\beta)]$, $j=\alpha,\beta$, where $\psi(z) = \frac{\partial \ln\Gamma(z)}{\partial z} = \frac{1}{\Gamma(z)}\frac{\partial\Gamma(z)}{\partial z}$ is the Digamma function. We thus have:

$$\frac{\partial\mu_r'}{\partial p} = -p^{-2}qrB\left(1+\frac{r}{p},q\right)\left[\psi\left(1+\frac{r}{p}\right)-\psi\left(1+q+\frac{r}{p}\right)\right] > 0 \tag{13}$$

$$\begin{aligned}
\frac{\partial\mu_r'}{\partial q} &= B\left(1+\frac{r}{p},q\right)\left\{1+q\left[\psi\left(q\right)-\psi\left(1+q+\frac{r}{p}\right)\right]\right\}\\
&= B\left(1+\frac{r}{p},q\right)\left[1+q\psi\left(q+1\right)-q\frac{1}{q}-q\psi\left(1+q+\frac{r}{p}\right)\right]\\
&= B\left(1+\frac{r}{p},q\right)q\left[\psi\left(1+q\right)-\psi\left(1+q+\frac{r}{p}\right)\right] < 0,
\end{aligned} \tag{14}$$

where the property $\psi(z+1) = \psi(z)+\frac{1}{z}$ is used in deriving (14), and $\frac{\partial\psi(z)}{\partial z} > 0$ in determining the signs of (13) and (14).

Second, as Propositions 5.3 and 5.4 assume that $(p,q) \in \theta_{\mu_0}$, we can define a function $q = Q(p,\mu_0)$ associated to the set θ_{μ_0}, which maps the first element in every pair $(p,q) \in \theta_{\mu_0}$ into the second element of the pair. Although this function is unknown, its derivative with respect to p can be calculated using the implicit function theorem. Substituting the function Q in (6), the raw moments of X become $\mu_r'(X) = Q(p,\mu_0)B(1+\frac{r}{p},Q(p,\mu_0))$. Leaving the arguments of Q unstated, we thus have:

$$\mu_0 = QB\left(1+\frac{1}{p}, Q\right). \tag{15}$$

Differentiating both sides of (15) with respect to p and isolating $\frac{\partial Q}{\partial p}$,

$$0 = \frac{\partial QB\left(1+\frac{1}{p}, Q\right)}{\partial p} + \frac{\partial QB\left(1+\frac{1}{p}, Q\right)}{\partial Q}\frac{\partial Q}{\partial p}$$

$$\frac{\partial Q}{\partial p} = -\frac{\partial QB\left(1+\frac{1}{p}, Q\right)}{\partial p}\left[\frac{\partial QB\left(1+\frac{1}{p}, Q\right)}{\partial Q}\right]^{-1}. \tag{16}$$

Substituting (13) and (14) in (16),

$$\frac{\partial Q}{\partial p} = -\frac{-p^{-2}QB\left(1+\frac{1}{p}, Q\right)\left[\left(\psi\left(1+\frac{1}{p}\right)-\psi\left(1+Q+\frac{1}{p}\right)\right)\right]}{QB\left(1+\frac{1}{p}, Q\right)\left[\psi(1+Q)-\psi\left(1+Q+\frac{1}{p}\right)\right]}$$

$$\frac{\partial Q}{\partial p} = \frac{\left[\psi\left(1+\frac{1}{p}\right)-\psi\left(1+Q+\frac{1}{p}\right)\right]}{p^2\left[\psi(1+Q)-\psi\left(1+Q+\frac{1}{p}\right)\right]} > 0, \tag{17}$$

where $\frac{\partial\psi(z)}{\partial z} > 0$ is used again in determining the sign of (17).

When q changes in response to a change in p in order to keep $\mu = \mu_0$, higher-order moments are affected by these changes in q and p. The total effect on these moments is summarized by the following equation:

$$\frac{d\mu_r'(\mu \underset{q}{\leftarrow} \mu_0)}{dp} = \frac{\partial\mu_r'}{\partial p} + \frac{\partial\mu_r'}{\partial Q}\frac{\partial Q}{\partial p}, \tag{18}$$

where the first term on the right-hand side of (18) is the direct effect of a change of p on μ_r', the second term is the indirect effect that operates through the induced change in q, and the left arrow indicates that μ is kept fixed at μ_0 by compensatory changes in the value of q in response to changes in p. Substituting (13), (14), and (17) in (18), we get:

$$\begin{aligned}\frac{d\mu_r'(\mu \underset{q}{\leftarrow} \mu_0)}{dp} &= p^{-2}QB\left(1+\frac{r}{p}, Q\right)\left\{-r\left[\psi\left(1+\frac{r}{p}\right)-\psi\left(1+Q+\frac{r}{p}\right)\right]\right.\\ &\quad \left.+\frac{\psi(1+Q)-\psi\left(1+Q+\frac{r}{p}\right)}{\psi(1+Q)-\psi\left(1+Q+\frac{1}{p}\right)}\left[\psi\left(1+\frac{1}{p}\right)-\psi\left(1+Q+\frac{1}{p}\right)\right]\right\}.\end{aligned} \tag{19}$$

When $r = 1$, the terms between tall brackets in (19) cancel out and $\frac{d\mu_r'(\mu \underset{q}{\leftarrow} \mu_0)}{dp} = 0$, as expected. But for $r > 1$, all moments but μ move in the opposite direction of p when changes in p are compensated by changes in q that keep μ fixed at μ_0.

Proving this result requires some additional work. A necessary and sufficient condition for (19) to be negative is that

$$-r\left[\psi\left(1+\frac{r}{p}\right)-\psi\left(1+Q+\frac{r}{p}\right)\right]$$
$$+\frac{\psi\left(1+Q\right)-\psi\left(1+Q+\frac{r}{p}\right)}{\psi\left(1+Q\right)-\psi\left(1+Q+\frac{1}{p}\right)}\left[\psi\left(1+\frac{1}{p}\right)-\psi\left(1+Q+\frac{1}{p}\right)\right]<0,$$

which can be more conveniently written as

$$\left[\psi\left(1+Q+\frac{r}{p}\right)-\psi\left(1+Q\right)\right]\left[\psi\left(1+Q+\frac{1}{p}\right)-\psi\left(1+\frac{1}{p}\right)\right]$$
$$>r\left[\psi\left(1+Q+\frac{1}{p}\right)-\psi\left(1+Q\right)\right]\left[\psi\left(1+Q+\frac{r}{p}\right)-\psi\left(1+\frac{r}{p}\right)\right]. \quad (20)$$

Now, the series representation of the Digamma function $\psi(z)=\sum_{k=1}^{\infty}\frac{1}{k}-\frac{1}{k+z-1}-\gamma$ (Arfken and Weber, 2001, p. 643) can be employed to provide a series representation of the difference between digamma functions:

$$\psi(x)-\psi(y)=\sum_{k=1}^{\infty}\frac{1}{k}-\frac{1}{k+x-1}-\gamma-\sum_{k=1}^{\infty}\frac{1}{k}-\frac{1}{k+y-1}+\gamma$$
$$=\sum_{k=1}^{\infty}\frac{x-y}{\left(k+y-1\right)\left(k+x-1\right)}. \quad (21)$$

Substituting (21) in (20), we get

$$\sum_{k=1}^{\infty}\frac{\frac{r}{p}}{\left(k+Q\right)\left(k+Q+\frac{r}{p}\right)}\sum_{m=1}^{\infty}\frac{Q}{\left(m+\frac{1}{p}\right)\left(m+Q+\frac{1}{p}\right)}$$
$$>r\sum_{k=1}^{\infty}\frac{\frac{1}{p}}{\left(k+Q\right)\left(k+Q+\frac{1}{p}\right)}\sum_{m=1}^{\infty}\frac{Q}{\left(m+\frac{r}{p}\right)\left(m+Q+\frac{r}{p}\right)}$$

$$\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{\left(k+Q\right)\left(k+Q+\frac{r}{p}\right)\left(m+\frac{1}{p}\right)\left(m+Q+\frac{1}{p}\right)}$$
$$>\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{\left(k+Q\right)\left(k+Q+\frac{1}{p}\right)\left(m+\frac{r}{p}\right)\left(m+Q+\frac{r}{p}\right)}$$

$$\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{\left(k+Q\right)\left(m+\frac{1}{p}\right)\left(k+Q+\frac{r}{p}\right)\left(m+Q+\frac{1}{p}\right)}$$
$$>\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{\begin{array}{c}\left(k+Q\right)\left(m+\frac{1}{p}\right)\left(k+Q+\frac{1}{p}\right)\left(m+Q+\frac{r}{p}\right)\\ +\left(k+Q\right)\left(\frac{r-1}{p}\right)\left(k+Q+\frac{1}{p}\right)\left(m+Q+\frac{r}{p}\right)\end{array}}.$$

This last inequality can be expressed as

$$\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{F_1(k,m,r)} > \sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{F_2(k,m,r)+G(k,m,r)}, \tag{22}$$

with $F_1(k,m,r) = (k+Q)(m+\frac{1}{p})(k+Q+\frac{r}{p})(m+Q+\frac{1}{p})$, $F_2(k,m,r) = (k+Q)(m+\frac{1}{p})(k+Q+\frac{1}{p})(m+Q+\frac{r}{p})$, and $G(k,m,r) = (k+Q)(\frac{r-1}{p})(k+Q+\frac{1}{p})(m+Q+\frac{r}{p})$. Given that m and k have exactly the same range, it follows that $\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{F_1(k,m,r)} = \sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{F_2(k,m,r)}$ for any value of r. Then, as $G(k,m,r) > 0$, it follows that(22) obtains, therefore proving that

$$\frac{d\mu_r'\left(\mu \underset{q}{\leftarrow} \mu_0\right)}{dp} < 0. \tag{23}$$

From the very definition of raw moments as $\mu_r' = \int_a^b x^r f_x dx$, the moments of any distribution with support in $(0,1)$, including the moments of the standard Kumaraswamy distribution, are by necessity decreasing in r; that is,

$$\mu \equiv \mu_1' > \mu_2' > \mu_3' \ldots. \tag{24}$$

Hence, from (23), when p moves towards zero the values of q that keep $\mu = \mu_0$ are such that all higher-order moments increase their value. As p gets closer to zero, higher-order moments get larger; but given (24), they can never be larger than μ. Therefore:

$$\lim_{\substack{p\to 0, q\to 0\\ (p,q)\in\theta_{\mu_0}}} \mu_r'(X) = \lim_{p\to 0} Q(p,\mu_0)\, B\left(1+\frac{r}{p}, Q(p,\mu_0)\right) = \mu_0. \tag{25}$$

As the moments of the Bernoulli distribution are $\mu_r' = E(X^r) = s$, $0 < s < 1$, it follows from (25) that when $p \to 0, q \to 0$, and $(p,q) \in \theta_{\mu_0}$ the standard Kumaraswamy distribution tends to the Bernoulli distribution with parameter equal to the Kumaraswamy distribution's mean, as stated in Proposition 5.3.

Likewise, when p moves towards infinity the values of q that keep $\mu = \mu_0$ are such that all higher-order moments decrease in value. As p gets larger, higher-order moments get smaller; however, given that $Var(X) = \mu_2' - \mu^2 \geq 0$, it must be the case that $\mu_2' \geq \mu_0^2$. Therefore:

$$\lim_{\substack{p\to\infty, q\to\infty\\ (p,q)\in\theta_{\mu_0}}} \mu_2'(X) = \mu_0^2. \tag{26}$$

As (26) entails that $Var(X) = 0$, it follows that when $p \to \infty$, $q \to \infty$, and $(p,q) \in \theta_{\mu_0}$, the standard Kumaraswamy distribution tends to the degenerate distribution with parameter equal to the Kumaraswamy distribution's mean, as stated in Proposition 5.4.

As Propositions 5.5 and 5.6 assume that $(p,q) \in \Gamma_{\omega_0}$, it is the case that $q = \frac{\ln 0.5}{\ln(1-\omega_0^p)}$ and that $(p \to 0) \leftrightarrow (q \to 0)$ and $(p \to \infty) \leftrightarrow (q \to \infty)$, which will be used in what follows. An easy way to prove Proposition 5.5 is by examining the behavior of the cumulative distribution function when p and q tend to their limit values (a substantially longer proof showing that when $p \to 0$ the moments of the Kumaraswamy distribution and the moments of the Bernoulli distribution coincide is available from the author). Substituting the above expression for q in (3) we obtain $F(x) = 1 - (1 - x^p)^{\frac{\ln 0.5}{\ln(1-\omega_0^p)}}$, which can be rewritten as $\ln[1 - F(x)] = \ln 0.5 \ln[1 - x^p][\ln(1 - \omega_0^p)]^{-1}$. Using the series representation $\ln z = 2\sum_{k=1}^{\infty} \frac{1}{2k-1}(\frac{z-1}{z+1})^{2k-1}$ with $|z| > 0$ (Gradshteyn and Ryzhik, 2007, p. 53), this can be re-expressed as

$$\ln[1 - F(x)] = \ln 0.5 \sum_{k=1}^{\infty} \frac{1}{2k-1}\left(\frac{x^p}{2 - x^p}\right)^{2k-1} \left[\sum_{k=1}^{\infty} \frac{1}{2k-1}\left(\frac{\omega_0^p}{2 - \omega_0^p}\right)^{2k-1}\right]^{-1}.$$

Taking limit for $p \to 0$, we have:

$$\begin{aligned}
\ln\left[1 - \lim_{p\to 0} F(x)\right] &= \ln 0.5 \sum_{k=1}^{\infty} \frac{1}{2k-1}\left(\frac{x^0}{2 - x^0}\right)^{2k-1} \left[\sum_{k=1}^{\infty} \frac{1}{2k-1}\left(\frac{\omega_0^0}{2 - \omega_0^0}\right)^{2k-1}\right]^{-1} \\
&= \ln 0.5 \sum_{k=1}^{\infty} \frac{1}{2k-1}\left(\frac{1}{1}\right)^{2k-1} \left[\sum_{k=1}^{\infty} \frac{1}{2k-1}\left(\frac{1}{1}\right)^{2k-1}\right]^{-1} \\
\lim_{p\to 0} F(x) &= 0.5.
\end{aligned}$$

It follows that when $p \to 0$ (and thus $q \to 0$), half of the probability mass concentrates in 0 while the other half concentrates in 1. Hence, when $p \to 0$, $q \to 0$, and $(p, q) \in \tau_{\omega_0}$, the standard Kumaraswamy distribution converges, regardless of the value of its median, to the Bernoulli distribution with parameter $s = 0.5$, as stated in Proposition 5.5.

Lastly, to prove Proposition 5.6, it is possible to use the fact that the mean absolute deviation around the median of a variable can be written as

$$\delta_2(X) = \int_0^1 |x - \omega| f(x)dx = \int_0^{\frac{1}{2}} [F^{-1}(1 - u) - F^{-1}(u)]du \tag{27}$$

(see, e.g., Pham-Gia and Hung, 2001, p. 924). Substituting (4) in (27), the mean absolute deviation around the median of $X \sim \mathrm{K}(p, q)$ is

$$\delta_2(X) = \int_0^{1/2} IQR(u; p, q)du, \tag{28}$$

where $IQR(u; p, q) = (1 - u^{\frac{1}{q}})^{\frac{1}{p}} - [1 - (1 - u)^{\frac{1}{q}}]^{\frac{1}{p}}$ is the inter-quantile range associated to u. Replacing q by its expression in terms of p and then taking limit for $p \to \infty$, we get:

$$\begin{aligned}
\lim_{p\to\infty} IQR(u; p, q) &= \lim_{p\to\infty} \left[1 - u^{\frac{\ln(1-\omega_0^p)}{\ln 0.5}}\right]^{\frac{1}{p}} - \left[1 - (1 - u)^{\frac{\ln(1-\omega_0^p)}{\ln 0.5}}\right]^{\frac{1}{p}} \\
&= \left[1 - u^0\right]^0 - \left[1 - (1 - u)^0\right]^0 = 1 - 1 = 0.
\end{aligned}$$

Given that $IQR(u; p, q) \to 0$ for any $0 < u \leq 0.5$ and that $IQR(u; p, q)$ is bounded, it follows from (28), using Lebesgue's dominated convergence theorem, that $\delta_2(X) = 0$ when $p \to \infty$. As the degenerate distribution is the only distribution with $\delta_2(X) = 0$, when $p \to \infty$, $q \to \infty$, and $(p, q) \in \tau_{\omega_0}$ the standard Kumaraswamy distribution tends to the degenerate distribution with location parameter equal to the Kumaraswmay distribution's median, as stated in Proposition 5.6.

6. Dispersion Measures and Bounds

Here, the following results for $X \sim \mathrm{K}(p, q)$ are proven.

Proposition 6.1. *The mean absolute deviation of X around its median can be expressed as* $\delta_2(X) = 2qB(2^{-\frac{1}{q}}, q, 1 + \frac{1}{p}) - qB(q, 1 + \frac{1}{p})$, *where* $B(z, \alpha, \beta) = \int_0^z s^{\alpha-1}(1 - s)^{\beta-1} ds$ *is the incomplete Beta function.*

Proposition 6.2. *Similarly to the standard Beta distribution* (*Pham-Gia*, 1994, *pp.* 2178–2179), $Var(X) < \frac{1}{4}$; *this bound is approached when the Kumaraswamy distribution converges to the Bernoulli distribution with parameter* $s = \frac{1}{2}$. *Other bounds for* $Var(X)$ *are partially different from those for the Beta distribution.*

Proposition 6.3. $\delta_2(X) < \frac{1}{2}$, *and this bound is approached when the Kumaraswamy distribution converges to the Bernoulli distribution with parameter* $s = \frac{1}{2}$.

For skewed distributions, analytic expressions for the distribution's mean absolute deviation around the median as a function of the distribution's parameters are seldom available (Pham-Gia and Hung, 2001, p. 924). However, although the Kumaraswamy distribution is never fully symmetrical (Jones, 2009, pp. 74–75) and can be very skewed, it is possible to derive for it such analytic expression. Applying the distributive property of definite integrals to (28) we obtain:

$$\begin{aligned}\delta_2(X) &= \int_0^{\frac{1}{2}} \left\{1 - [1 - (1 - u)]^{\frac{1}{q}}\right\}^{\frac{1}{p}} - \left[1 - (1 - u)^{\frac{1}{q}}\right]^{\frac{1}{p}} du \\ &= \int_0^{\frac{1}{2}} \left(1 - u^{\frac{1}{q}}\right)^{\frac{1}{p}} du - \int_0^{\frac{1}{2}} \left[1 - (1 - u)^{\frac{1}{q}}\right]^{\frac{1}{p}} du. \qquad (29)\end{aligned}$$

Doing the change of variables $s = u^{\frac{1}{q}}$ in the first integral of (29) and $s = (1 - u)^{\frac{1}{q}}$ in the second, and applying a few well-known properties of definite integrals, we obtain:

$$\begin{aligned}\delta_2(X) &= q\int_0^{2^{-\frac{1}{q}}} s^{q-1} (1 - s)^{\frac{1}{p}} ds - q\int_{2^{-\frac{1}{q}}}^1 s^{q-1} (1 - s)^{\frac{1}{p}} ds \\ &= q\int_0^{2^{-\frac{1}{q}}} s^{q-1} (1 - s)^{\frac{1}{p}} ds - q\left[\int_0^1 s^{q-1} (1 - s)^{\frac{1}{p}} ds - \int_0^{2^{-\frac{1}{q}}} s^{q-1} (1 - s)^{\frac{1}{p}} ds\right] \\ &= 2q\int_0^{2^{-\frac{1}{q}}} s^{q-1} (1 - s)^{\frac{1}{p}} ds - q\int_0^1 s^{q-1} (1 - s)^{\frac{1}{p}} ds \\ &= 2qB\left(2^{-\frac{1}{q}}, q, 1 + \frac{1}{p}\right) - qB\left(q, 1 + \frac{1}{p}\right), \qquad (30)\end{aligned}$$

as stated in Proposition 6.1.

Given that Kumaraswamy-distributed variables are bounded, all measures of their dispersion must have upper bounds. Propositions 6.2 and 6.3 specify upper bounds for $Var(X)$ and $\delta_2(X)$. To prove Proposition 6.2, the following lemma is given, which follows very directly from (8), (17), and (23):

Lemma 6.1. *Let's denote the variance of $X \sim \mathrm{K}(p, q)$ by $Var_{(p,q)}(X)$. If $(p', q') \in \theta_{\mu_0}$, $(p'', q'') \in \theta_{\mu_0}$, and $p'' < p'$ or, equivalently, $q'' < q'$, then $Var_{(p',q')}(X) < Var_{(p'',q'')}(X)$.*

From this lemma, for any $\mu = \mu_0$, if $(p, q) \in \theta_{\mu_0}$ then $Var(X)$ grows monotonically as p and q get smaller. Therefore, the upper bound of $Var(X)$ conditional on the value of μ is approached when p and q tend simultaneously to zero in such a way that μ remains fixed. This conditional upper bound can be calculated by taking limit for $p \to 0$, $q \to 0$ and $(p, q) \in \theta_{\mu_0}$ in (8), and using (25):

$$\begin{aligned} Max[Var(X)|\mu = \mu_0] &= \lim_{\substack{p\to 0, q\to 0,\\ (p,q)\in\theta_{\mu_0}}} Var(X) \\ &= \lim_{p\to 0} Q(p, \mu_0)\, B\left(1 + \frac{2}{p}, Q(p, \mu_0)\right) \\ &\quad - \left[Q(p, \mu_0)\, B\left(1 + \frac{1}{p}, Q(p, \mu_0)\right)\right]^2 \\ &= \mu_0 - \mu_0^2 = \mu_0(1 - \mu_0). \end{aligned} \tag{31}$$

The unconditional upper bound of $Var(X)$ is the maximum possible value of the conditional upper bound. Given that $0 < \mu_0 < 1$, it follows immediately from (31) that the unconditional upper bound corresponds to $\mu_0 = \frac{1}{2}$. Hence, $Var(X) < \frac{1}{4}$ and, from Proposition 5.3, this bound is approached when $p \to 0$, $q \to 0$ and $(p, q) \in \theta_{\frac{1}{2}}$ – that is, when the Kumaraswamy distribution converges to the Bernoulli distribution with parameter $s = \frac{1}{2}$, as stated in Proposition 6.3.

Numerical experimentation has revealed other interesting bounds, partially different from those of the variance of the Beta distribution (see Pham-Gia, 1994, pp. 2178–2179 for the latter bounds). First, like in the case of the Beta distribution, if the Kumaraswamy distribution is unimodal ($p > 1$ and $q > 1$), then its variance is smaller than the variance of the standard uniform distribution, that is, $Var(X) < \frac{1}{12}$. Second, and differently from the Beta distribution, the converse is not true. The distribution may be U-shaped ($p < 1$ and $q < 1$) and still $Var(X) < \frac{1}{12}$. (For instance, for $p = 0.5$ and $q = 0.1$, $Var(X) = 0.0678\cdots < \frac{1}{12}$.) Lastly, if $p \geq 1.1444\ldots$ ($q \geq 1.1333\ldots$), then $Var(X) \leq \frac{1}{12}$, regardless of the value of q (p).

To prove Proposition 6.3, observe that for any distribution it is always the case that $\delta_2(Z) \leq \sigma_Z = [\mathrm{Var}(Z)]^{\frac{1}{2}}$ (Pham-Gia and Hung, 2001, p. 923). As by Proposition 6.2 $Var(X) < \frac{1}{4}$, it follows immediately that $\delta_2(X) < \frac{1}{2}$. Moreover, from the very definition of $\delta_2(X)$ it is easy to see that this bound is approached when the Kumaraswamy distribution approaches symmetry—so that $md(X) \to \frac{1}{2}$ and $\mu \to \frac{1}{2}$—and the probability density is concentrated near 0 and near 1, that is, when the distribution converges to the Bernoulli distribution with parameter $s = \frac{1}{2}$.

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