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Bayesian and Maximum Likelihood Estimation for Kumaraswamy Distribution Based on Ranked Set Sampling

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Abstract In this paper, the estimation of the unknown parameters of the kumaraswamy distribution is considered using both simple random sampling (SRS) and ranked set sampling (RSS) techniques. The estimation is based on maximum likelihood estimation and Bayesian estimation methods. A simulation study is made to compare the resultant estimators in terms of their biases and mean square errors. The efficiency of the estimates made using ranked set sampling are also computed.

Keywords Bias, Maximum likelihood estimators, Mean square error, Kumaraswamy distribution, Ranked set sampling, Simple random sampling

1. Introduction

Making inferences about a population based on a sample of data collected from this population is almost the most important research across most or even all kind of sciences such as agricultural, biological, ecological, engineering, medical, physical, and social sciences. The approach of collecting sample data that are truly representative to the population is an important key to make successful analysis to the scientific questions under investigation. The most common approach to data collection is the simple random sample (SRS) approach. A collection of random variables $X_1,...,X_n$ is said to be a simple random sample (SRS) of size *n* from an underlying probability distribution with probability density function (*pdf*) $f(x)$ and cumulative distribution function (*cdf*) $F(x)$, if each $X_k, k=1,...,n$ has the same probability distribution as the underlying population and the *n* random variables $X_1,...,X_n$ are mutually independent. For finite populations consisting of a total of *N* observations, a collection of *n* sample observations is said to be a simple random sample $X_1,...,X_n$ if each of the ${}^{N}C_n$ possible subsets of *n* observations has the same chance of being selected as the random sample[1]. A serious drawback of the SRS approach is that there is no guarantee that a specific random sample of units selected from the population is truly representative of the population. This specific sample might or might not actually provide good information about the population. Because of that, many attempts and many approaches have been suggested to minimize the effect of this problem. Some of these approaches are systematic sampling, stratified sampling, cluster sampling, and quota sampling. However, none of these approaches uses extra information from specific units in the population to guide their search for a truly representative sample[2].

McIntyre[3] introduced the ranked set sampling (RSS) approach that utilize additional information from individual population units providing a more representative sample from the population under consideration. An important advantage of this approach is that it improves the efficiency of estimators of the population parameters. For example, it improves the efficiency of a sample mean as an estimator of the population mean in situations in which the variable of interest is difficult or expensive to measure, but could be cheaply ranked. Theoretical investigations by Dell and Clutter showed that, regardless of ranking errors, the RSS estimator of a population mean is unbiased and is at least as precise as the SRS estimator with the same number of quantifications[4]. David and Levine investigated the case where ranking is done by a numerical covariate[5]. Furthermore, RSS provides more precise estimators of the variance[6], the cumulative distribution function[7], and the Pearson correlation coefficient[8]. Several authors have used RSS for parametric inference for example, Stokes[9] looked at the maximum likelihood and best linear unbiased estimator of the location-scale parameters in location-scale family of distributions while Yu and co-authors[10] developed an estimator for the population variance of a

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normal distribution based on balanced and unbalanced ranked set samples. On the other hand, several attempts were made to improve the estimation based on RSS. From those, designs for optimal ranked set sampling were constructed for parametric families of distributions[11] and best linear unbiased estimators based on ordered ranked set samples were also developed[12]. A modification of the RSS called moving extremes ranked set sampling (MERSS) was considered for the estimation of the scale parameter of scale distributions[13] and an improved RSS estimator for the population mean was obtained[14]. Ozturk has developed two sampling designs to create artificially stratified samples using RSS[15]. Readers are encouraged to perusal at a historical perspective of the RSS approach, see[16-25].

In order to obtain a random sample of data of size n observations from a population using RSS approach, the following process is applied. (i) Randomly draw m random sets with m elements $X_{i:m}, i=1,\dots,m$ in each sample (m is called the *set size* and is typically small to minimize ranking error). (ii) Allocate the m^2 selected units as randomly as possible into m sets, each of size m. (iii) without yet knowing any values for the variable of interest, rank the units within each set based on a perception of relative values for this variable. (iv) Choose a sample for actual analysis by including the smallest ranked unit in the first set, then the second smallest ranked unit in the second set, continuing until the largest ranked unit is selected in the last set. (v) Randomly draw other m random sets with m elements in each sample with a total of m^2 sample units and repeat steps (ii) through (v) for r cycles until the desired sample $X_{(i::m)j}$; $i=1,\dots,m,\ \ j=1,\dots,r$ of size, $n = m\, r$, is obtained for analysis.

In this article, the unknown parameters of the Kumaraswamy (Kw) distribution will be estimated under both SRS and RSS approaches. The estimation is made using maximum likelihood (ML) estimation and Bayesian estimation methods. The Kumaraswamy's double bounded (Kw) distribution is a family of continuous probability distributions defined on the interval[0,1] differing in the values of their two non-negative shape parameters, α and β[26]. It is similar to the Beta distribution, but much simpler to use especially in simulation studies due to the simple closed form of both its probability density function and cumulative distribution function. The Kw distribution *pdf* and *cdf* are given by

$$f(x)=\alpha\,\beta\,x^{\alpha-1}(1-x^{\alpha})^{\beta-1}, \tag{1}$$

$$F(x)=1-(1-x^{\alpha})^{\beta}, \tag{2}$$

respectively, where $0<x<1$ and the shape parameters $\alpha,\ \beta>0$.

The rest of the article is organized as follows. In Section 2, ML and Bayesian methods of estimation of unknown parameters are discussed under SRS. In Section 3, the same methods of estimation are discussed based on RSS. Simulation studies are carried out to illustrate theoretical results in Section 4. Finally, conclusions are presented in Section 5.

2. Estimation Using SRS Approach

2.1. Maximum Likelihood Estimation

Let $X_1, X_2, \dots, X_n$ be a random sample of size n drawn from the Kw distribution with shape parameters α and β. The likelihood function of α and β for the observed samples is

$$L_s(data;\alpha,\beta)=\alpha^n\,\beta^n\prod_{k=1}^{n}x_k^{\alpha-1}\prod_{k=1}^{n}(1-x_k^{\alpha})^{\beta-1} \tag{3}$$

Therefore, the log-likelihood function of α and β will be

$$\log L_s=n\,\log\alpha+n\log\beta+(\alpha-1)\sum_{k=1}^{n}\log x_k$$

$$+(\beta-1)\sum_{k=1}^{n}\log[1-x_k^{\alpha}]\,. \tag{4}$$

The estimators $\hat{\alpha}_{ml,s}$ and $\hat{\beta}_{ml,s}$ of the parameters α and β respectively can be obtained as the solution of the likelihood equations

$$\frac{n}{\alpha}+\sum_{k=1}^{n}\log x_k-(\beta-1)\sum_{k=1}^{n}\frac{x_k^{\alpha}\log\alpha}{(1-x_k^{\alpha})}=0 \tag{5}$$

$$\frac{n}{\beta}+\sum_{k=1}^{n}\log[1-x_k^{\alpha}]=0\,. \tag{6}$$

From Equations (5) and (6) we have

$$\hat{\beta}_{ml,s}=\frac{-n}{\sum_{k=1}^{n}\log[1-x_k^{\hat{\alpha}_{ml,s}}]}, \tag{7}$$

where $\hat{\alpha}_{ml,s}$ is the solution of the nonlinear equation

$$\frac{n}{\hat{\alpha}_{ml,s}}+\sum_{k=1}^{n}\log x_k-(\hat{\beta}_{ml,s}-1)\sum_{k=1}^{n}\frac{x_k^{\hat{\alpha}_{ml,s}}\log\hat{\alpha}_{ml,s}}{(1-x_k^{\hat{\alpha}_{ml,s}})}=0\,. \tag{8}$$

The ML estimators $\hat{\alpha}_{ml,s}$ and $\hat{\beta}_{ml,s}$ are the solution of the two nonlinear Equations (7) and (8). These estimators cannot be obtained in closed form, therefore, numerical analysis is used to study their properties.

2.2. Bayesian Estimation

In this section, the Bayes estimators of shape parameters α

and β denoted by $\hat{\alpha}_{Bs,s}$, $\hat{\beta}_{Bs,s}$ respectively, are obtained under the assumption that α and β are independent random variables with prior distributions Gamma(a_1,b_1) and Gamma(a_2,b_2) respectively with *pdf's*

$$\pi_1(\alpha) = \frac{b_1^{a_1}}{\Gamma(a_1)}\, \alpha^{a_1-1}\, e^{-b_1\,\alpha}; \tag{9}$$

and

$$\pi_2(\beta) = \frac{b_2^{a_2}}{\Gamma(a_2)}\, \beta^{a_2-1}\, e^{-b_2\,\chi}; \tag{10}$$

where $\alpha, \beta > 0$ and the hyper-parameters $a_1, a_2 > 0,$ and $b_1, b_2 > 0$ are assumed to be known. Based on the above assumptions and the likelihood function presented in Equation (3), the joint density of the data, α and β can be obtained as

$$L_s(data,\alpha,\beta) = L(data;\alpha,\beta)\ \pi(\alpha)\ \pi(\beta)\ = K_1\ \alpha^{n+a_1-1}\ \beta^{n+a_2-1}\ e^{-b_1\,\alpha} e^{-b_2\,\beta} \prod_{k=1}^{n} x_k^{\alpha-1} \prod_{k=1}^{n} (1-x_k^{\alpha})^{\beta-1}$$

$$= K_1\ \Psi\,. \tag{11}$$

where K_1 is constant and

$$\Psi = \alpha^{n+a_1-1}\, \beta^{n+a_2-1}\, e^{-b_1\alpha - b_2\beta + (\alpha-1)\sum\limits_{k=1}^{n} \log[x_k] + (\beta-1)\sum\limits_{k=1}^{n} \log[1-x_k^{\alpha}]}. \tag{12}$$

Therefore, the joint posterior density of the data, α and β given the data can be obtained as

$$\pi_s(\alpha,\beta\,/\,data) = \frac{L(data,\alpha,\beta)}{\int_0^\infty\int_0^\infty L(data,\alpha,\beta)\,d\alpha\,d\beta},\ = \frac{\Psi}{\int_0^\infty\int_0^\infty \Psi\,d\alpha\,d\beta}, \tag{13}$$

According to that, the posterior *pdf*'s of α and β are

$$\pi_{\alpha,s}(\alpha\,/\,data) = \frac{\int_0^\infty \Psi\ d\beta}{\int_0^\infty\int_0^\infty \Psi\ d\alpha\ d\beta}, \tag{14}$$

and

$$\pi_{\beta,s}(\beta\,/\,data) = \frac{\int_0^\infty \Psi\ d\alpha}{\int_0^\infty\int_0^\infty \Psi\ d\alpha\,d\beta}. \tag{15}$$

respectively. Therefore, the Bayes estimators for the parameters α and β denoted by $\hat{\alpha}_{Bs,s}$, $\hat{\beta}_{Bs,s}$ under squared error loss function are defined, respectively, as

$$\hat{\alpha}_{BS,s} = E(\alpha\,/\,data) = \frac{\int_0^\infty\int_0^\infty \alpha\ \Psi\ d\beta\ d\alpha}{\int_0^\infty\int_0^\infty \Psi\ d\alpha\,d\beta}, \tag{16}$$

and

$$\hat{\beta}_{BS,s} = E(\beta / data) = \frac{\int_0^\infty \int_0^\infty \beta\, \Psi \; d\alpha\, d\beta}{\int_0^\infty \int_0^\infty \Psi \; d\alpha\, d\beta}. \tag{17}$$

These estimators cannot be obtained in closed form. Thus, the properties of these estimators will be discussed using simulation studies.

3. Estimation Using RSS Approach

3.1. Maximum Likelihood Estimation

Assume that $X_{(i:m)j}$; $0 < X_{(i:m)j} < 1,\ i = 1,...,m$ and $j = 1,...,r$ is a ranked set sample with sample size $n = m\,r$ from the *Kw* distribution, where m is the set size and r is the number of cycles. For simplification purposes, $X_{(i:m)j}$ will be denoted as X_{ij}. The *pdf* of the random variables X_{ij} is given by

$$g(X_{ij}) = \frac{m!}{(i-1)!(m-i)!} f(X_{ij})[F(X_{ij})]^{i-1}[1-F(X_{ij})]^{m-i};$$

which in the case of the *Kw* distribution will be

$$g(X_{ij}) = \frac{m!}{(i-1)!(m-i)!} \alpha\, \beta\, X_{ij}^{\alpha-1}(1-X_{ij}^{\alpha})^{\beta(m-i+1)-1} \times [1-(1-X_{ij}^{\alpha})^{\beta}]^{i-1}. \tag{18}$$

The likelihood function of α and β for the observed sample is given by

$$L_r(data;\alpha,\beta) = K_2 \prod_{j=1}^{r}\prod_{i=1}^{m}(\alpha\,\beta\, X_{ij}^{\alpha-1}(1-X_{ij}^{\alpha})^{\beta(m-i+1)-1}[1-(1-X_{ij}^{\alpha})^{\beta}]^{i-1} \times [1-(1-X_{ij}^{\alpha})^{\beta}]^{i-1}). \tag{19}$$

Therefore, the log-likelihood function of α and β will be

$$\begin{aligned} Log\, L_r =& \log K_2 + m\,r \log\alpha + m\,r\log\beta + (\alpha-1)\sum_{j=1}^{r}\sum_{i=1}^{m}\log X_{ij} \\ &+(\beta(m-i+1)-1)\sum_{j=1}^{r}\sum_{i=1}^{m}\log[1-X_{ij}^{\alpha}] + (i-1)\sum_{j=1}^{r}\sum_{i=1}^{m}\log[1-(1-X_{ij}^{\alpha})^{\beta}], \end{aligned} \tag{20}$$

where K_2 is constant. This implies that

$$\frac{m\,r}{\alpha} + \sum_{j=1}^{r}\sum_{i=1}^{m}\log X_{ij} + (\beta(m-i+1)-1)\sum_{j=1}^{r}\sum_{i=1}^{m}\frac{X_{ij}^{\alpha}\log[X_{ij}]}{1-X_{ij}^{\alpha}} + (i-1)\sum_{j=1}^{r}\sum_{i=1}^{m}\frac{X_{ij}^{\alpha}(1-X_{ij}^{\alpha})^{\beta-1}\log[X_{ij}]}{1-(1-X_{ij}^{\alpha})^{\beta}} = 0, \tag{21}$$

and

$$\frac{m\,r}{\beta} + (m-i+1)\sum_{j=1}^{r}\sum_{i=1}^{m}\log[1-X_{ij}^{\alpha}] + (i-1)\sum_{j=1}^{r}\sum_{i=1}^{m}\frac{(1-X_{ij}^{\alpha})^{\beta}\log[1-X_{ij}^{\alpha}]}{1-(1-X_{ij}^{\alpha})^{\beta}} = 0. \tag{22}$$

$\hat{\alpha}_{ml,r}$ and $\hat{\beta}_{ml,r}$ are the solution of the two nonlinear Equations (21) and (22), and numerical analysis is used to study their properties.

3.2. Bayesian Estimation

The Bayes estimators of the shape parameters α and β denoted by $\hat{\alpha}_{Bs,r}$ and $\hat{\beta}_{Bs,r}$ respectively, are obtained similar to the procedure used in section (2.2). Let α and β be independent random variables with prior distributions given in Equations (9) and (10). Based on these assumptions and the likelihood function presented in Equation (19), the joint density of the data, α and β can be obtained as

$$L_r(data,\alpha,\beta) = L(data;\alpha,\beta)\,\pi(\alpha)\,\pi(\beta) = K_3\,\alpha^{mr+a_1-1}\,\beta^{mr+a_2-1}\,e^{-b_1\alpha-b_2\beta+(\alpha-1)\sum\limits_{j=1}^{r}\sum\limits_{i=1}^{m}\log[X_{ij}]}$$
$$\times e^{(\beta(m-i+1)-1)\sum\limits_{j=1}^{r}\sum\limits_{i=1}^{m}\log[1-X_{ij}^{\alpha}]+(i-1)\sum\limits_{j=1}^{r}\sum\limits_{i=1}^{m}\log[1-(1-X_{ij}^{\alpha})^{\beta}]} \tag{23}$$

Therefore, the joint posterior density of the data, α and β given the data can be obtained as

$$\pi_r(\alpha,\beta/data) = \frac{L(data,\alpha,\beta)}{\int_0^\infty\int_0^\infty L(data,\alpha,\beta)\,d\alpha\,d\beta}, = \frac{\Lambda}{\int_0^\infty\int_0^\infty \Lambda\,d\alpha\,d\beta}. \tag{24}$$

where

$$\Lambda = \alpha^{mr+a_1-1}\,\beta^{mr+a_2-1}\,e^{-b_1\alpha-b_2\beta+(\alpha-1)\sum\limits_{j=1}^{r}\sum\limits_{i=1}^{m}\log[X_{ij}]}$$
$$\times e^{(\beta(m-i+1)-1)\sum\limits_{j=1}^{r}\sum\limits_{i=1}^{m}\log[1-X_{ij}^{\alpha}]+(i-1)\sum\limits_{j=1}^{r}\sum\limits_{i=1}^{m}\log[1-(1-X_{ij}^{\alpha})^{\beta}]} \tag{25}$$

The Bayes estimators for parameters α and β denoted by $\hat{\alpha}_{Bs,r}$, $\hat{\beta}_{Bs,r}$ under squared error loss function are defined, respectively, as

$$\hat{\alpha}_{BS,r} = E(\alpha/data) = \frac{\int_0^\infty\int_0^\infty \alpha\,\Lambda\,d\beta\,d\alpha}{\int_0^\infty\int_0^\infty \Lambda\,d\alpha\,d\beta}, \tag{26}$$

and

$$\hat{\beta}_{BS,r} = E(\beta/data) = \frac{\int_0^\infty\int_0^\infty \beta\,\Lambda\,d\alpha\,d\beta}{\int_0^\infty\int_0^\infty \Lambda\,d\alpha\,d\beta}. \tag{27}$$

4. Simulation Study

Numerical solutions are used to obtain the ML and Bayes estimators of the unknown parameters of the kumaraswamy distribution and to compare the performance of these estimators based on RSS and SRS approaches. Monte Carlo simulation study is made using MATHEMATICA software and is based on 10,000 replications. The simulations are made for several combinations of the parameters n, m, r, and β values while the value of the shape parameter α is equal to one. The comparison is carried out through biases, MSEs of the estimators $\hat{\alpha}_{ml,s}$, $\hat{\beta}_{ml,s}$, $\hat{\alpha}_{Bs,s}$, $\hat{\beta}_{Bs,s}$, $\hat{\alpha}_{ml,r}$, $\hat{\beta}_{ml,r}$, $\hat{\alpha}_{Bs,r}$ and $\hat{\beta}_{Bs,r}$. Also, the efficiency of the estimators that are derived using RSS with respect to those using SRS are computed, where the efficiency of an estimator $\hat{\theta}_2$ with respect to an estimator $\hat{\theta}_1$ is given by

$$eff(\hat{\theta}_2) = \frac{MSE(\hat{\theta}_1)}{MSE(\hat{\theta}_2)} \tag{30}$$

The larger the efficiency the better is $\hat{\theta}_2$ in terms of MSE. The results are reported in Tables 1 – 3 (Appendix A). One can conclude from these results that the estimates of α and β based on RSS have smaller biases than the corresponding estimates using SRS. Biases and MSEs of the estimates made by both methods decrease as set sizes increase. It is also noted that biases and MSEs of the shape parameter β increases when its population value increases. Also, almost in all cases the biases and MSEs for the Bayes estimates of both parameters α and β are smaller than the corresponding values for the ML estimates of α and β respectively. As a result, and from table 3, the ML and Bayes estimators of both parameters α and β derived using RSS are more efficient of the corresponding estimators derived using SRS.

5. Conclusions

In this article, estimation problem of unknown parameters of the kumaraswamy distribution based on RSS was considered. ML and Bayesian methods of estimation are used where Bayes estimates were obtained under squared error loss function. Based on the simulation study, it is observed that the Bayes estimators perform better than ML estimators relative to their biased and MSE's. Furthermore, biases and MSEs of the estimates for the shape parameter β under RSS approach are smaller than the corresponding estimates computed under the SRS approach. This indicates that estimation under the RSS approach is more efficient than estimation under the SRS approach.

Appendix A

Table 1. Biases of the estimators of the *Kw* distribution for population parameter $\alpha = 1$ and the prior hyper-parameter $(a_1, a_2, b_1, b_2) = (1, 1, 3, 3)$

n	β	$\hat{\alpha}_{ml,s}$	$\hat{\alpha}_{Bs,s}$	$\hat{\beta}_{ml,s}$	$\hat{\beta}_{Bs,s}$	m,r	$\hat{\alpha}_{ml,r}$	$\hat{\alpha}_{Bs,r}$	$\hat{\beta}_{ml,r}$	$\hat{\beta}_{Bs,r}$
		0.2168	0.1528	0.1574	0.1109	2,10	0.1453	0.1132	0.0955	0.0790
20	0.5					4,5	0.1077	0.0839	0.0707	0.0585
						5,4	0.0798	0.0621	0.0524	0.0434
		0.1641	0.1222	0.3039	0.2263	2,10	0.1100	0.0819	0.2037	0.1517
	1					4,5	0.0815	0.0607	0.1509	0.1124
						5,4	0.0604	0.0450	0.1118	0.0833
		0.1128	0.0836	0.4974	0.3685	2,10	0.0756	0.0560	0.3334	0.2470
	3					4,5	0.0560	0.0415	0.2470	0.1830
						5,4	0.0415	0.0308	0.1830	0.1356
		0.1174	0.0961	0.1694	0.1268	2,15	0.0787	0.0644	0.1136	0.0850
30	0.5					3,10	0.0583	0.0477	0.0841	0.0630
						5,6	0.0432	0.0354	0.0623	0.0466
		0.1004	0.0798	0.2207	0.1635	2,15	0.0673	0.0535	0.1479	0.1096
	1					3,10	0.0499	0.0396	0.1096	0.0812
						5,6	0.0369	0.0294	0.0812	0.0601
		0.0873	0.0666	0.2579	0.1969	2,15	0.0585	0.0446	0.1729	0.1320
	3					3,10	0.0434	0.0331	0.1281	0.0978
						5,6	0.0321	0.0245	0.0949	0.0724

Table 2. MSEs of the estimators of the *Kw* distribution for population parameter $\alpha = 1$ and the prior hyper-parameter $(a_1, a_2, b_1, b_2) = (1, 1, 3, 3)$

n	β	$\hat{\alpha}_{ml,s}$	$\hat{\alpha}_{Bs,s}$	$\hat{\beta}_{ml,s}$	$\hat{\beta}_{Bs,s}$	m,r	$\hat{\alpha}_{ml,r}$	$\hat{\alpha}_{Bs,r}$	$\hat{\beta}_{ml,r}$	$\hat{\beta}_{Bs,r}$
		0.5186	0.4629	0.1853	0.1591	2,10	0.4039	0.3395	0.1532	0.1266
20	0.5					4,5	0.3407	0.2780	0.1293	0.1046
						5,4	0.3053	0.2540	0.1182	0.0996
		0.3348	0.2986	4.1984	2.4461	2,10	0.2451	0.2045	3.6136	1.9730
	1					4,5	0.2034	0.1700	2.9586	1.6988
						5,4	0.1933	0.1611	2.6970	1.5533
		0.3029	0.2583	6.5648	4.6987	2,10	0.2251	0.1945	5.9401	3.8470
	3					4,5	0.1934	0.1617	5.3748	3.4809
						5,4	0.1839	0.1533	4.8633	3.1496
		0.4694	0.4247	0.1259	0.1096	2,15	0.3340	0.2808	0.0994	0.0849
30	0.5					3,10	0.3114	0.2540	0.0814	0.0695
						5,6	0.2790	0.2322	0.0774	0.0661
		0.2639	0.2553	3.0833	2.6385	2,15	0.1803	0.1691	2.5765	1.8856
	1					3,10	0.1586	0.1434	1.9939	1.6282
						5,6	0.1477	0.1358	1.8967	1.5309
		0.2182	0.1947	4.1609	3.4519	2,15	0.1633	0.1372	3.6101	2.5829
	3					3,10	0.1553	0.1305	3.4340	2.4570
						5,6	0.1398	0.1193	3.2665	2.3371

Table 3. Efficiency of the estimators of the Kw distribution for population parameters α = 1 and the prior hyper-parameter (a_1, a_2, b_1, b_2) = (1, 1, 3, 3)

n	β	m,r	$eff(\hat{\alpha}_{ml,r})$	$eff(\hat{\alpha}_{Bs,r})$	$eff(\hat{\beta}_{ml,r})$	$eff(\hat{\beta}_{Bs,r})$
		2,10	1.2840	1.3634	1.2092	1.2562
20	0.5	4,5	1.5220	1.6653	1.4333	1.5212
		5,4	1.6989	1.8221	1.5672	1.5981
		2,10	1.3662	1.4603	1.1618	1.2398
	1	4,5	1.6463	1.7563	1.4191	1.4399
		5,4	1.7316	1.8534	1.5567	1.5747
		2,10	1.3457	1.3280	1.1052	1.2214
	3	4,5	1.5658	1.5972	1.2214	1.3499
		5,4	1.6469	1.6855	1.3499	1.4918
		2,15	1.4054	1.5127	1.2660	1.2910
30	0.5	3,10	1.5073	1.6718	1.5464	1.5768
		5,6	1.6826	1.8292	1.6256	1.6577
		2,15	1.4640	1.5098	1.1967	1.3993
	1	3,10	1.6640	1.7805	1.5464	1.6205
		5,6	1.7867	1.8796	1.6256	1.7235
		2,15	1.3364	1.4191	1.1526	1.3364
	3	3,10	1.4049	1.4918	1.2117	1.4049
		5,6	1.5608	1.6320	1.2738	1.4770

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