A Coupled Optimal Stopping Approach to Pairs Trading over a Finite Horizon

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We study the problem of trading a mean-reverting price spread over a finite horizon with transaction costs and an unbounded number of trades. Modeling the price spread by the Ornstein-Uhlenbeck process, we formulate a coupled optimal stopping problems to determine the optimal timing to switch positions. We analyze the corresponding free-boundary system for the value functions. Our solution approach involves deriving a system of Volterra-type integral equations that uniquely characterize the boundaries associated with the optimal timing decisions. These integral equations are used to numerically compute the optimal boundaries. Numerical examples are provided to illustrate the optimal trading boundaries and examine their sensitivity with respect to model parameters.

1. Introduction

Mean-reverting price spreads are well observed in different markets. Many empirical studies (Vidyamurthy (2004); Gatev et al. (2006); Avellaneda & Lee (2010); Do & Faff (2012); Leung & Li (2016), among others) have shown examples of mean reversion in various markets. They can be generated by taking positions in stocks, ETFs, commodities, cryptocurrencies, futures and other derivatives.

For pairs trading, the Ornstein-Uhlenbeck (OU) model, originally developed by Ornstein & Uhlenbeck (1930), has been widely used to represent the dynamics of the mean-reverting price spreads. We study the problem of trading a mean-reverting price spread over a finite horizon with transaction costs. The trader can potentially trade *infinitely* many times by either taking a position in the spread (state 1) or not (state 0), but all trades must be completed on or before the deadline. As the trader ponders when to trade, the value of future trading opportunities and positions must be included in the evaluation. Therefore, the trader's value functions in two different states are necessarily linked.

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Modeling the price spread by the OU process, we formulate a *coupled* optimal stopping problem to determine the optimal timing to switch positions. We analyze the corresponding free-boundary system for the value functions. We solve perpetual version of this problem analytically. For the finite-horizon trading problem, however, closed-form expressions for the value functions or optimal boundaries are not available. Our solution approach involves deriving a system of Volterra-type integral equations that uniquely characterize the boundaries associated with the optimal timing decisions. These integral equations are used to numerically compute the optimal boundaries. Numerical examples are provided to illustrate the optimal trading boundaries and examine their sensitivity with respect to model parameters, including the speed of mean reversion, volatility, and transaction cost.

To our best knowledge, the formulation of the coupled trading problem along with its representation via Volterra-type integral equations is new. Furthermore, the integral representation lends itself to the numerical computation of the optimal trading boundaries. In particular, we demonstrate that the optimal trading boundaries are time-varying and the trading region changes significantly near the trading deadline. As we extend the finite trading horizon, the boundaries from our approach converge to those from the infinite-horizon case.

Our paper contributes to the growing literature on the optimal stopping approach for mean-reversion trading. In one direction, many studies have considered trading problems with a single entry and single exit. In the perpetual case with an OU price process, closed-form solutions have been found in Leung & Li (2015) and Lipton & López de Prado (2020). In the finite-horizon case, Song et al. (2009) applies a stochastic optimization approach to numerically obtain the optimal strategy while Li (2016) and Leung et al. (2016) numerically solve a series of variational inequalities using finite-difference methods. In comparison, the perpetual version of our trading problem (with unbounded number of trades) also admits a closed-form solution. The finite-horizon case is solved via the Volterra-type integral equations. The use of integral equations for optimal mean-reversion trading with a fixed number of trades can also be found in our companion papers, Kitapbayev & Leung (2017, 2018).

The rest of the paper is structured as follows. In Section 2, we present the finite-horizon trading problem and analyze the associated coupled optimal stopping problem and its free-boundary system. Then, we consider the problem over an infinite horizon in Section 3. In Section 4, we present the numerical solutions of the trading problems along with several illustrative examples of the optimal trading boundaries. In Sections 5 and 6, we illustrate an extension of our approach over the finite and finite horizons. Concluding remarks are included in Section 7.

2. Trading with two states: finite horizon

In the background, we fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathbb{P}$ represents the physical probability measure. We consider a price process, called the pair spread, which represents the price difference between two assets or positions. We assume that the pair spread process X follows an Ornstein-Uhlenbeck (OU) model, and satisfies the stochastic differential equation

$$dX_t = \mu(\theta - X_t)dt + \sigma dW_t \tag{2.1}$$

for $t > 0$, where μ, θ, σ are constant parameters and W is a standard Brownian motion under the measure $\mathbb{P}$.

Now consider a trader who can either take only a long position in X with a fixed size or have no position. When the trader has no position in X, we call it state 0. When the trader has a long position in X, the trader is in state 1. Even with only two possible positions, the trader can potentially trade *infinitely* many times by switching between the two positions. Moreover, all trades must be completed on or before the deadline $T > 0$. In addition, we impose a requirement of zero inventory, which means that the trader's position at time T must be zero.

The assumption of having an unbounded number of trades distinguishes the current paper from the related trading problems studied in the authors' prior work (Kitapbayev & Leung (2017, 2018)), where only single round-trip trades are considered.

We consider a risk-neutral, as opposed to risk-averse or risk-sensitive, approach to evaluate the trading opportunities. There is a subjective discount rate $r > 0$ and a fixed transaction cost $c > 0$ per trade.

Mathematically, the trading problem can be formulated as a system of coupled optimal stopping problems. Specifically, we denote by V_0 (resp. V_1) the value function representing the expected discounted P&L generated from an optimal trading strategy given that the current inventory is 0 (resp. 1). They are defined by the following coupled system

$$V_0(t,x) = \sup_{t \le \tau \le T} \mathbb{E}_{t,x}\left[e^{-r(\tau-t)}(V_1(\tau, X_\tau) - X_\tau - c)^+\right] \tag{2.2}$$

$$V_1(t,x) = \sup_{t \le \tau \le T} \mathbb{E}_{t,x}\left[e^{-r(\tau-t)}(V_0(\tau, X_\tau) + X_\tau - c)\right] \tag{2.3}$$

for $t \in [0,T)$, $x \in \mathbb{R}$, and where $\mathbb{E}_{t,x}$ is the $\mathbb{P}$-expectation given that $X_t = x$ and the supremum is taken over all stopping times with respect to the filtration generated by the spread process X.

The intuition for the coupled system is that if we are currently in state 0, then upon optimal entry time τ^*, the trader pays the amount X_{τ^*} and fixed cost c, but also obtains the value $V_1(\tau^*, X_{\tau^*})$ for switching to state 1. Similarly, if the current state is 1, then upon exit the trader will receive the amount X_{τ^*}, pay fixed cost c, and obtain the value $V_0(\tau^*, X_{\tau^*})$ for switching to state 0. In other words, the coupled system takes into account the discounted values of all the future trading (or switching) opportunities and incorporates these values into the determination of the optimal trading strategy.

Given that the terminal inventory must be zero, we have the terminal condition: $V_0(T,x) = 0$ and $V_1(T,x) = x - c$ for $x \in \mathbb{R}$. This means that in state 0, the trader simply does not trade, and in state 1, the trader will liquidate the long position and receive $x - c$. This also explains the necessity of the positive part in the payoff of V_0.

In order to solve the system (2.2)-(2.3), we begin by postulating that there are two trading boundaries $b_0 < b_1$ such that we enter into the long position when $x \le b_0(t)$ (the spread is too low) and exit when $x \ge b_1(t)$ (the spread is too high). A few important points regarding these boundaries:

- Both boundaries are functions of time t defined on $[0,T)$, which makes it difficult to compute them, unlike in the case of infinite horizon problems (see Section 3 below), where these boundaries are constant;

- When in state 0 (resp. 1), the region above b_0 (resp. below b_1) represents the no-trade region. Figure 1 illustrates a sample path of X and corresponding entry/exit times;
- As standard in optimal stopping theory for Markov processes, the boundaries do not depend on the initial value of the process X_0. They depend only on the model parameters. Once we determine them numerically, we follow a trajectory of X and trade when it hits the corresponding boundaries;

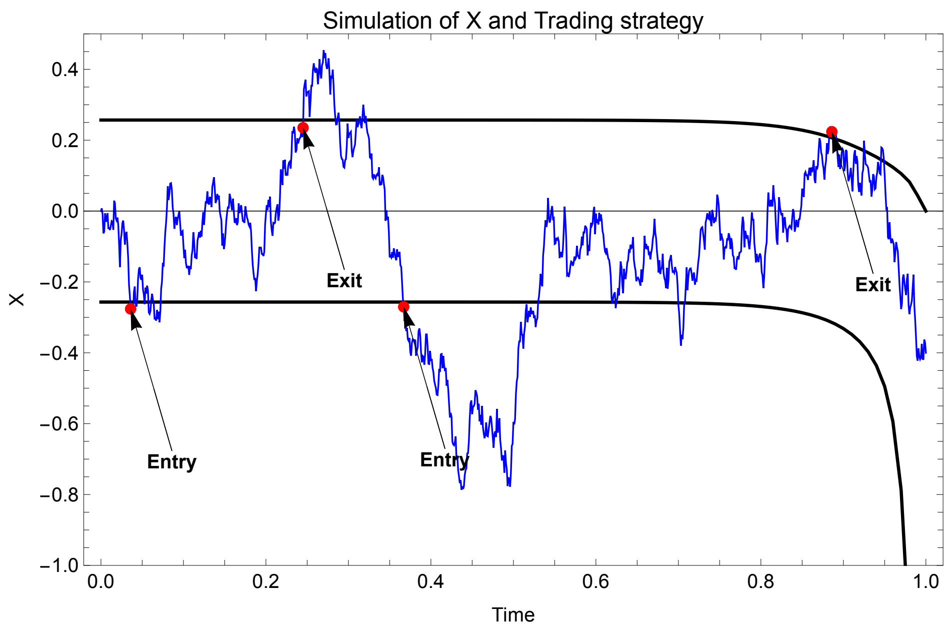


Figure 1: This figure displays the simulation of the strategy given trajectory of X (blue) and trading boundaries, b_0 (lower black) and b_1 (upper black). The red dots label the corresponding entry and exit times. The parameters are $\mu = 5$, $\theta = 0$, $\sigma = 1$, $r = 0.01$, $c = 0.05$, $T = 1$.

These optimal trading boundaries, along with the two value functions V_0 and V_1, form the free boundary PDE system described below. The fact that the value functions satisfy PDEs is common to options pricing (Black-Scholes PDE) and optimal control (HJB equation) problems. To write down the PDE system, we first define the infinitesimal generator of X by

$$\mathbb{L} = \mu(\theta - x)\frac{\partial}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}. \tag{2.4}$$

Using that X is a strong Markov process and standard arguments from optimal stopping theory (see, e.g., Peskir & Shiryaev (2006)), we can show that in the waiting region of state 0, the value function V_0 satisfies the following PDE

$$\frac{\partial}{\partial t}V_0(t,x) + \mathbb{L}V_0(t,x) = rV_0(t,x) \quad \text{for } x > b_0(t). \tag{2.5}$$

Similarly, in the waiting region of state 1, the value function V_1 satisfies

$$\frac{\partial}{\partial t}V_1(t,x) + \mathbb{L}V_1(t,x) = rV_1(t,x) \quad \text{for } x < b_1(t). \tag{2.6}$$

Furthermore, continuous and smooth pasting conditions are satisfied at $x = b_0(t)$ and $x = b_1(t)$, respectively,

$$V_0(t,b_0(t)) = V_1(t,b_0(t)) - b_0(t) - c \qquad \text{for } t \in [0,T) \tag{2.7}$$

$$\frac{\partial V_0}{\partial x}(t,b_0(t)) = \frac{\partial V_1}{\partial x}(t,b_0(t)) - 1 \qquad \text{for } t \in [0,T) \tag{2.8}$$

$$V_1(t,b_1(t)) = V_0(t,b_1(t)) + b_1(t) - c \qquad \text{for } t \in [0,T) \tag{2.9}$$

$$\frac{\partial V_1}{\partial x}(t,b_1(t)) = \frac{\partial V_0}{\partial x}(t,b_1(t)) + 1 \qquad \text{for } t \in [0,T). \tag{2.10}$$

The conditions (2.7) and (2.9) imply that the value functions are continuous at the trading boundaries. The smooth pasting conditions (2.8) and (2.10) are imposed due to the optimality of trading boundaries b_0 and b_1, respectively. These conditions are standard for optimal stopping problems, e.g. pricing of American options. Let us define the function $f(t,x) := V_1(t,x) - V_0(t,x)$, which is the difference of two value functions. Then jointly with (b_0,b_1) it satisfies the following free-boundary system

$$\frac{\partial}{\partial t}f(t,x) + \mathbb{L}f(t,x) = rf(t,x) \qquad \text{for } b_0(t) < x < b_1(t) \tag{2.11}$$

$$f(t,b_0(t)) = b_0(t) + c \qquad \text{for } t \in [0,T) \tag{2.12}$$

$$\frac{\partial f}{\partial x}(t,b_0(t)) = 1 \qquad \text{for } t \in [0,T) \tag{2.13}$$

$$f(t,b_1(t)) = b_1(t) - c \qquad \text{for } t \in [0,T) \tag{2.14}$$

$$\frac{\partial f}{\partial x}(t,b_1(t)) = 1 \qquad \text{for } t \in [0,T) \tag{2.15}$$

$$f(T,x) = x - c \qquad \text{for } x \in \mathbb{R}. \tag{2.16}$$

We note that in contrast to the system (2.5)-(2.10) with two unknown value functions, (V_0,V_1), the system (2.11)-(2.16) has a single unknown function, i.e., f. Both systems have two free boundaries, b_0 and b_1.

We can also deduce that $b_0(T-) = -\infty$ as there is no reason to take the long position near T due to the lack of time to make a profit on movements of X and pay the fixed fee c twice (recall that we assumed that the position must be liquidated at T). Now let us obtain the terminal value of $b_1(T-)$ at T. In the state 1, the immediate payoff is $V_0(t,x) + x - c$ and as $V_0(t,x) \to 0$ when t goes T, it converges to $x - c$. Let us choose t sufficiently close to T with $x > b_0(t)$ and follow optimal stopping time τ, i.e., $X_u > b_0(u)$ for $u \in [t,\tau]$. Then by applying Ito's formula for the discounted payoff at τ, taking expectation and using optional sampling theorem, we have

$$\mathbb{E}_{t,x}\left[e^{-r(\tau-t)}(V_0(\tau,X_\tau)+X_\tau-c)\right] = V_0(t,x) + x - c + \mathbb{E}_{t,x}\left[\int_t^\tau e^{-r(u-t)}H_1(X_u)du\right] \tag{2.17}$$

where we used that $(V_0)_t + \mathbb{L}V_0 - rV_0 = 0$ for $x > b_0(t)$ and we defined

$$H_1(x) := -(\mu + r)x + \mu\theta + rc. \tag{2.18}$$

Therefore, the value of $b_1(T-)$ can be obtained as the root of the equation $H_1(x) = 0$. In other words,

$$b_1(T-) = \frac{\mu\theta + rc}{\mu + r}. \tag{2.19}$$

We will also make use of the following function

$$H_0(x) := \frac{\partial}{\partial t}V_0(t,x) + \mathbb{L}V_0(t,x) - rV_0(t,x) = (\mu + r)x - \mu\theta + rc \quad \text{for } x \le b_0(t) \tag{2.20}$$

where we used that $\partial V_1/\partial t + \mathbb{L}V_1 - rV_1 = 0$ for $x \le b_0(t) < b_1(t)$.

Now we derive the following result that resembles an early exercise premium representation in American options pricing literature (see e.g. Carr et al. (1992)). In our context, the valuation formulas can be called the early entry/exit premium representations.

Theorem 2.1. *The value functions V_0 and V_1 can be characterized as*

$$V_0(t,x) = -\mathbb{E}_{t,x}\left[\int_t^T e^{-r(u-t)} H_0(X_u) I(X_u \le b_0(u)) du\right] \tag{2.21}$$

$$V_1(t,x) = \mathbb{E}_{t,x}\left[e^{-r(T-t)}(X_T - c)\right] - \mathbb{E}_{t,x}\left[\int_t^T e^{-r(u-t)} H_1(X_u) I(X_u \ge b_1(u)) du\right] \tag{2.22}$$

for $t \in [0,T)$ and $x \in \mathbb{R}$.

The proof is given in Appendix A. The intuition behind the valuation formulas is the following: if we start from state 0 at time t, one particular suboptimal strategy is to wait until T without trading at all, and clearly it has zero value. However there is an early entry premium term given by the right-hand side of (2.21). The benefits of optimal early entry are $-H_0(X_u)$ and are taken into account only if $X_u \le b_0(u)$. Then these benefits are discounted by $e^{-r(u-t)}$, integrated over $[t,T]$, and the expected value is taken.

Similarly, the first term on the right-hand side of (2.22) represents the value of naive strategy of waiting until T to liquidate the position when starting from state 1. This simple policy has the expected value $\mathbb{E}_{t,x}\left[e^{-r(T-t)}(X_T - c)\right]$. As it is a suboptimal strategy, the second term corresponds to an early liquidation premium and can be explained in a similar way as in (2.21).

The formulas (2.21)-(2.22) do not provide complete characterization as they still depend on unknown trading boundaries b_0 and b_1. To characterize them, we first note that the function $f = V_1 - V_0$ can be characterized as

$$f(t,x) = \mathbb{E}_{t,x}\left[e^{-r(T-t)}(X_T - c)\right] - \mathbb{E}_{t,x}\left[\int_t^T e^{-r(u-t)} H_1(X_u) I(X_u \ge b_1(u)) du\right] \tag{2.23}$$

$$+\mathbb{E}_{t,x}\left[\int_t^T e^{-r(u-t)}H_0(X_u)I(X_u \le b_0(u))du\right]$$

for $t \in [0,T)$ and $x \in \mathbb{R}$. Then, by recalling (2.12) and (2.14), and inserting $x = b_0(t)$ and $x = b_1(t)$, respectively, into (2.23), we immediately get the system of coupled integral equations of Volterra type that we present below.

Proposition 2.2. *The pair of the optimal trading boundaries (b_0, b_1) satisfies the system of coupled Volterra-type nonlinear integral equations*

$$\begin{aligned} b_0(t)+c =& \mathbb{E}_{t,b_0(t)}\left[e^{-r(T-t)}(X_T-c)\right] - \mathbb{E}_{t,b_0(t)}\left[\int_t^T e^{-r(u-t)}H_1(X_u)I(X_u \ge b_1(u))du\right] \\ &+ \mathbb{E}_{t,b_0(t)}\left[\int_t^T e^{-r(u-t)}H_0(X_u)I(X_u \le b_0(u))du\right] \end{aligned} \tag{2.24}$$

$$\begin{aligned} b_1(t)-c =& \mathbb{E}_{t,b_1(t)}\left[e^{-r(T-t)}(X_T-c)\right] - \mathbb{E}_{t,b_1(t)}\left[\int_t^T e^{-r(u-t)}H_1(X_u)I(X_u \ge b_1(u))du\right] \\ &+ \mathbb{E}_{t,b_1(t)}\left[\int_t^T e^{-r(u-t)}H_0(X_u)I(X_u \le b_0(u))du\right] \end{aligned} \tag{2.25}$$

for $t \in [0,T)$ with $b_0(T-) = -\infty$ and $b_1(T-) = (\mu\theta + rc)/(\mu + r)$.

Volterra equations commonly arise to determine numerically the optimal stopping boundaries in an efficient way (see Carr et al. (1992) for American options and Detemple (2005), Peskir & Shiryaev (2006) for more examples). The main feature of Volterra equations (versus Fredholm-type) is that it has a recursive nature, in our case, in the backward direction. In other words, the integrals at time t depend on the values of b_0 and b_1 at future times $u \in [t,T]$.

We now show how to compute some expressions in (2.24)-(2.25). Using that X has normal marginals, we have

$$\begin{aligned} G(t,x) &:= \mathbb{E}_{t,x}\left[e^{-r(T-t)}(X_T-c)\right] \\ &= xe^{-(\mu+r)(T-t)} + e^{-r(T-t)}\theta(1-e^{-\mu(T-t)}) - e^{-r(T-t)}c \end{aligned} \tag{2.26}$$

$$\begin{aligned} L_0(t,u,x,z) &:= \mathbb{E}_{t,x}\left[H_0(X_u)I(X_u \le z)\right] \\ &= (\mu+r)\mathbb{E}_{t,x}\left[X_u I(X_u \le z)\right] - (\mu\theta - rc)\mathbb{P}_{t,x}(X_u \le z) \\ &= (\mu+r)\left(m(u-t,x)\Phi(\bar{z}) - std(u-t)\phi(\bar{z})\right) - (\mu\theta - rc)\Phi\left(\bar{z}\right) \end{aligned} \tag{2.27}$$

$$\begin{aligned} L_1(t,u,x,z) &:= \mathbb{E}_{t,x}\left[H_1(X_u)I(X_u \ge z)\right] \\ &= -(\mu+r)\mathbb{E}_{t,x}\left[X_u I(X_u \ge z)\right] + (\mu\theta + rc)\mathbb{P}_{t,x}(X_u \ge z) \\ &= -(\mu+r)\left(m(u-t,x)\Phi\left(-\bar{z}\right) + std(u-t)\phi(\bar{z})\right) + (\mu\theta + rc)\Phi\left(-\bar{z}\right) \end{aligned} \tag{2.28}$$

$$L(t,u,x,z_0,z_1) := L_1(t,u,x,z_1) - L_0(t,u,x,z_0) \tag{2.29}$$

for $t < u$ and $x, z, z_0, z_1 > 0$, where

$$\bar{z} = \frac{z - m(u-t,x)}{std(u-t,x)} \tag{2.30}$$

N	16	32	64	128	256	512
V_0	0.3822	0.3773	0.3752	0.3744	0.3742	0.3741

Table 1: This table illustrates the convergence of the numerical scheme described in Section 2 based on (2.24)-(2.25) and (2.21)-(2.22). The first row represents the number of time discretization steps N, the second row corresponds to the value function V_0 in state 0 at $t = 0$ for $X_0 = 0$.

$$m(t,x) = \mathbb{E}_x[X_t] = xe^{-\mu t} + \theta(1 - e^{-\mu t}) \tag{2.31}$$

$$std^2(t,x) = \mathbb{E}_x\left[X_t^2\right] - (\mathbb{E}_x[X_t])^2 = \frac{\sigma^2}{2\mu}\left(1 - e^{-2\mu t}\right). \tag{2.32}$$

Before we solve the system of equations (2.24)-(2.25) numerically, let us rewrite it in a more compact form

$$G(t, b_0(t)) - b_0(t) - c = \int_t^T e^{-r(u-t)} L(t, u, b_0(t), b_0(u), b_1(u))du \tag{2.33}$$

$$G(t, b_1(t)) - b_1(t) + c = \int_t^T e^{-r(u-t)} L(t, u, b_1(t), b_0(u), b_1(u))du \tag{2.34}$$

for $t \in [0,T)$ subject to $b_0(T-) = -\infty$ and $b_1(T-) = (\mu\theta + rc)/(\mu + r)$. Now we can apply standard numerical procedures using the recursive (i.e., Volterra) type of equations. First, we discretize time interval $[0,T]$ as $0 = t_0 < t_1 < \ldots < t_N = T$ with $h = t_i - t_{i-1} = T/N$, $i = 1, \ldots, N$. Then we approximate integral terms using a quadrature scheme so that integral equations can be approximated as

$$G(t_i, b_0(t_i)) - b_0(t_i) - c = h \sum_{j=i+1}^{N} e^{-r(t_j - t_i)} L(t_i, t_j, b_0(t_i), b_0(t_j), b_1(t_j)) \tag{2.35}$$

$$G(t_i, b_1(t_i)) - b_1(t_i) + c = h \sum_{j=i+1}^{N} e^{-r(t_j - t_i)} L(t_i, t_j, b_1(t_i), b_0(t_j), b_1(t_j)) \tag{2.36}$$

for $i = N-1, N-2, \ldots, 0$. Starting with $b_0(t_N) = -\infty$ and $b_1(t_N) = (\mu\theta + rc)/(\mu + r)$, we proceed using backward induction, and find sequentially $(b_0(t_{N-1}), b_1(t_{N-1}))$, $(b_0(t_{N-2}), b_1(t_{N-2})), \ldots$, etc., as the solutions to algebraic equations (2.35)-(2.36) for given i.

In Figure 2, we illustrate the optimal trading boundaries computed from this method. As we can see, the two trading boundaries flatten as time t goes backward from T to time 0. With still much time before the deadline, the optimal trading strategy resembles that from the perpetual case (dashed lines). Once we recover the optimal trading boundaries numerically, we compute the value functions V_0 and V_1 using (2.21)-(2.22). Table 1 illustrates the convergence of the numerical scheme, i.e., dependence of the estimated value function $V_0(0, X_0)$ on step size $h = T/N$. We note that we have used a standard integration scheme and a uniform grid. The convergence could be improved by employing more tailored schemes.

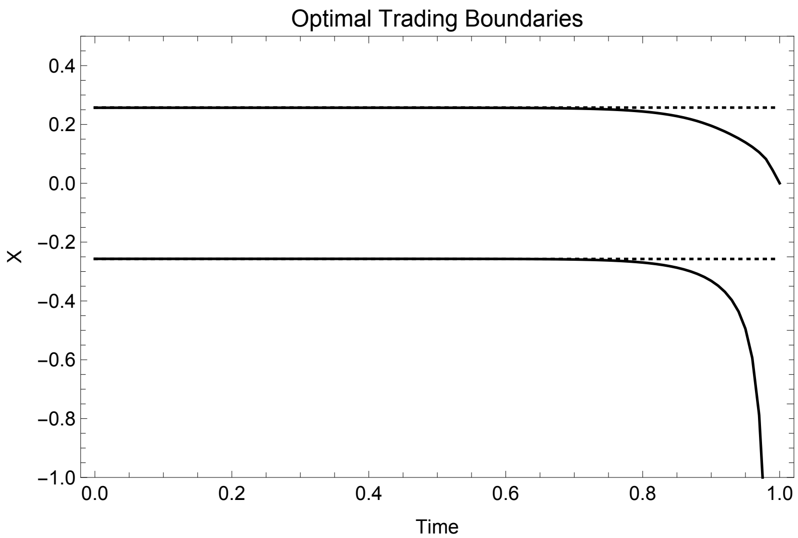


Figure 2: This figure displays the optimal trading boundaries, b_0 (lower solid) and b_1 (upper solid) given as the solutions to (2.24)-(2.25). The dashed constant lines represent optimal trading thresholds for the infinite horizon case. The parameters are $\mu = 5$, $\theta = 0$, $\sigma = 1$, $r = 0.01$, $c = 0.05$, $T = 1$.

3. Trading with two states: infinite horizon

In this section, we briefly go over a similar but simpler problem with an infinite trading horizon. The result here will be useful for comparison (see Figure 2 for example). This perpetual case has been studied by Leung et al. (2014) for the Cox-Ingersoll-Ross process and Leung et al. (2015) for the exponential OU model. Below, we solve the infinite-horizon analogue of the problem from the previous section for the OU model. A related but different infinite horizon optimal switching problem under the OU model has been studied by Song & Zhang (2013).

As in the previous section, we denote by V_0 (resp. V_1) the value function representing the maximized discounted expected value that can be generated from trading given that the current inventory is 0 (resp. 1). They are defined through the coupled optimal stopping problems

$$V_0(x) = \sup_{\tau \geq 0} \mathbb{E}_x \left[e^{-r\tau} (V_1(X_\tau) - X_\tau - c) \right] \tag{3.1}$$

$$V_1(x) = \sup_{\tau \geq 0} \mathbb{E}_x \left[e^{-r\tau} (V_0(X_\tau) + X_\tau - c) \right] \tag{3.2}$$

for $x \in \mathbb{R}$. Here, $\mathbb{E}_x$ is the $\mathbb{P}$-expectation given that $X_0 = x$. By removing the finite trading deadline, the problem becomes time-homogeneous, and the trader only needs to keep track of the current spread level. Therefore, we seek to derive the associated system of ordinary differential equations (ODEs) for the value functions.

The optimal trading boundaries are constants representing the critical price levels that trigger the trades. We denote them by b_0^∞ and b_1^∞, which are determined along with the value functions $V_0(x)$ and $V_1(x)$.

The value functions satisfy ODEs in the waiting regions

$$\mathbb{L}V_0(x) = rV_0(x) \quad \text{for } x > b_0^\infty \tag{3.3}$$

$$\mathbb{L}V_1(x) = rV_1(x) \quad \text{for } x < b_1^\infty \tag{3.4}$$

and corresponding solutions are given by

$$V_0(x) = A_0 F(x) + B_0 \widehat{F}(x) \quad \text{for } x > b_0^\infty \tag{3.5}$$

$$V_1(x) = A_1 F(x) + B_1 \widehat{F}(x) \quad \text{for } x < b_1^\infty \tag{3.6}$$

where A_0, B_0, A_1, B_1 are unknown constants to be determined, while F and $\widehat{F}$ are special functions that solve ODE

$$F(x) = \int_0^\infty u^{r/\mu - 1} e^{\sqrt{2\mu/\sigma^2}(x-\theta)u - u^2/2} du \tag{3.7}$$

$$\widehat{F}(x) = \int_0^\infty u^{r/\mu - 1} e^{\sqrt{2\mu/\sigma^2}(\theta - x)u - u^2/2} du \tag{3.8}$$

for $x \in \mathbb{R}$ (see, e.g., Borodin & Salminen (2002)). As $V_0(x)$ goes to 0 when $x \to +\infty$, we deduce that $A_0 = 0$. Similarly, $B_1 = 0$. The value matching and smooth pasting conditions at $x = b_0^\infty$ and $x = b_1^\infty$ imply the system of four nonlinear equations with four unknowns $(b_0^\infty, b_1^\infty, A_1, B_0)$

$$B_0 \widehat{F}(b_0^\infty) = A_1 F(b_0^\infty) - b_0^\infty - c \tag{3.9}$$

$$B_0 \widehat{F}'(b_0^\infty) = A_1 F'(b_0^\infty) - 1 \tag{3.10}$$

$$A_1 F(b_1^\infty) = B_0 \widehat{F}(b_1^\infty) + b_1^\infty - c \tag{3.11}$$

$$A_1 F'(b_1^\infty) = B_0 \widehat{F}'(b_1^\infty) + 1 \tag{3.12}$$

which are to be solved using standard numerical routines.

4. Numerical results

In this section, we explore the effect of model parameters on the optimal trading boundaries and the value function V_0. In particular, we are interested in the effects of μ, σ, c. The results are shown in Figure 3. We also compare the performances of our optimal strategy with time-dependent boundaries and of the benchmark strategies with constant entry/exit thresholds.

On panel (a) of Figure 3, we illustrate optimal upper and lower trading boundaries for three different speeds of mean reversion $\mu \in \{3, 5, 7\}$. As μ increases, the upper boundary moves lower while the lower boundary moves higher. As a result, the two boundaries become closer to each other. The intuition is that, when the price spread has a faster mean reversion, the trader seeks to act faster to open and close positions by having closer trading boundaries so that the spread process X will more likely hit both boundaries more often. This makes sense as the deviation of the spread from its equilibrium disappears faster, and one needs to enter and exit earlier to exploit the opportunity.

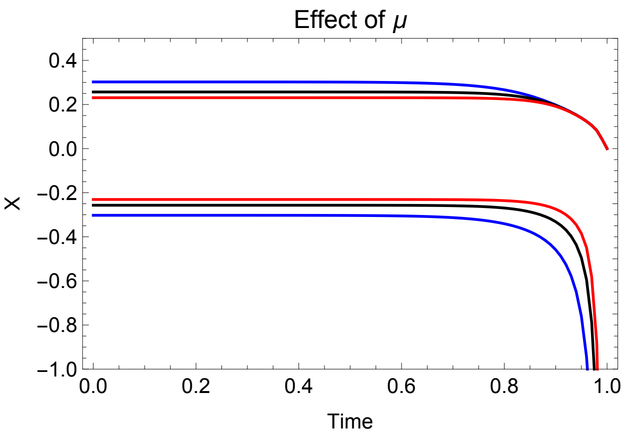
Turning to panel (b) of Figure 3, we plot the optimal upper and lower trading boundaries for three different values of volatility $\sigma \in \{0.7, 1, 1.3\}$. A higher σ means that the upper boundary moves higher while the lower boundary moves lower. This means that the two boundaries become further apart. This effect is intuitive. The more volatile the spread, the less certain the trader is that deviations from the equilibrium will revert. As a consequence, the trader imposes a wider trade region and trades only when the spread is sufficiently far from its mean.

Lastly, we observe the effect of the transaction cost c by inspecting panel (c) of Figure 3. The optimal upper and lower trading boundaries move higher and lower, respectively, as the transaction cost c increases from 0.02 to 0.08. The critical price levels at which traders are triggered move further away from the equilibrium. This suggests that the trader demands a higher spread when establishing positions. Indeed, if each trade is more costly, then the trader is incentivized to wait longer before entering and exiting to ensure that the trade is more likely to be profitable on average.

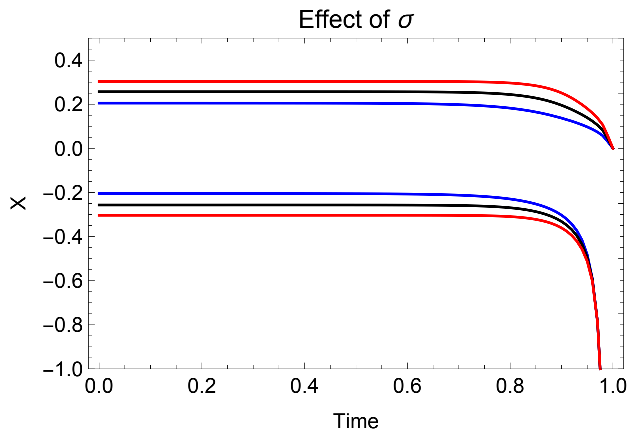
In Figure 4, we examine the effects of μ and σ on the value function V_0. In panel (a), the value function, plotted as a function of spread x, is shifted lower as μ increases. This is a consequence of the trading boundaries moving closer to each other as seen in panel (a) of Figure 3. Although a higher speed of mean reversion potentially leads to more trades, each trade will result in a smaller profit. The aggregate effect is that the value function decreases as μ increases. The opposite effect is observed for the volatility σ. As σ increases, the value function V_0 increases in value. The intuition is that a higher σ broadens the trading regions, leading to more profit for each trade, and also increases the chance of hitting both boundaries since the spread X is more volatile.

Next, we evaluate the performance of our optimal strategy given the boundaries $b_0(t)$ and $b_1(t)$ computed as solutions to the Volterra integral equations (2.24)-(2.25). For this, we compare the value function $V_0(0, X_0)$ of the state 0 with (i) the value function $V_0^{CT,\infty}(0, X_0)$ that corresponds to the strategy that allows for the unbounded number of trades over $[0, T]$ and follows constant thresholds b_0^{CT} and b_1^{CT}; (ii) the value function $V_0^{CT,1}(0, X_0)$ of the strategy that allows for at most one round-trip trade on $[0, T]$ and follows constant thresholds b_0^{CT} and b_1^{CT}. The latter two value functions are estimated using Monte-Carlo simulation with a large number of trajectories. We consider the range of pairs of constant thresholds (b_0^{CT}, b_1^{CT}) that are symmetric relative to θ. To compute the value function V_0 of the optimal strategy, we use the representation (2.21).

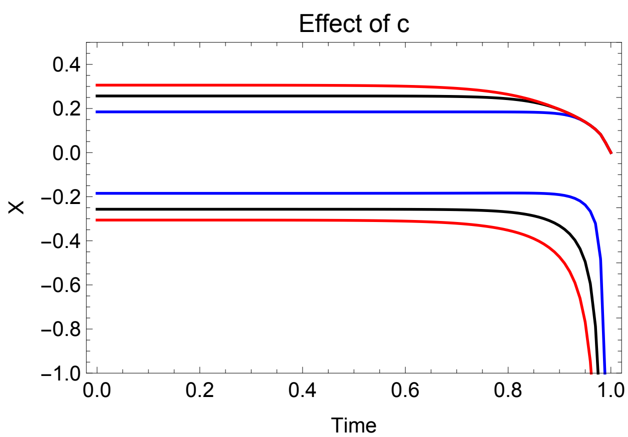
Table 2 summarizes the results of this comparison for a set of parameters used in the figures above. We can observe that $V_0^{CT,1}$ is maximized at $(b_0, b_1) = (-0.3, 0.3)$ and $V_0^{CT,\infty}$ has maximum value at $(b_0, b_1) = (-0.22, 0.22)$. This is intuitive since the unbounded number of trades suggests that $V_0^{CT,\infty}$ should favor frequent trades of small gains, and thus a smaller waiting region. For all pairs, $V_0^{CT,\infty}$ is greater than $V_0^{CT,1}$ due to the unbounded number of trades. As expected from a theoretical point of view, the value function V_0 dominates both $V_0^{CT,\infty} = V_0^{CT,\infty}(b_0^{CT}, b_1^{CT})$ and $V_0^{CT,1} = V_0^{CT,1}(b_0^{CT}, b_1^{CT})$ for all boundary pairs (b_0^{CT}, b_1^{CT}). This comparison can give an idea about the added value when trading under optimal time-dependent boundaries.



(a) Effect of μ

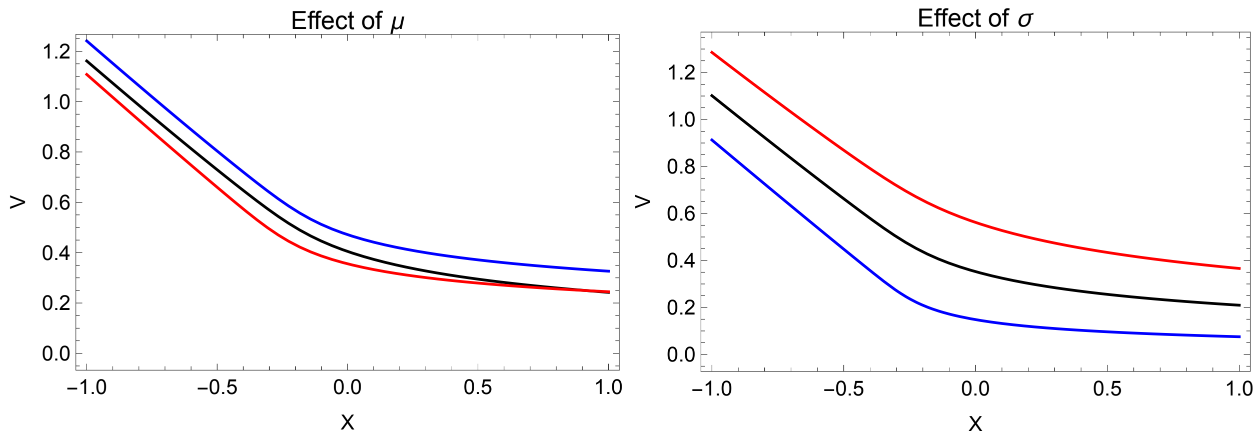


(b) Effect of σ



(c) Effect of c

Figure 3: The plots illustrate the effects of mean reversion speed μ, σ, c on the optimal trading boundaries, b_0 (lower) and b_1 (upper). Panel (a): $\mu = 3$ (blue), $\mu = 5$ (black), $\mu = 7$ (red). Panel (b): $\sigma = 0.7$ (blue), $\sigma = 1$ (black), $\sigma = 1.3$ (red). Panel (c): $c = 0.02$ (blue), $c = 0.05$ (black), $c = 0.08$ (red). The central parameter values are $\mu = 5$, $\theta = 0$, $\sigma = 1$, $r = 0.01$, $c = 0.05$, $T = 1$.



(a) Effect of μ

(b) Effect of σ

Figure 4: The plots illustrate the effect of mean reversion speed μ and σ on the value function $V_0(0, X_0)$ as the function of X_0. Panel (a): $\mu = 3$ (blue), $\mu = 5$ (black), $\mu = 7$ (red). Panel (b): $\sigma = 0.7$ (blue), $\sigma = 1$ (black), $\sigma = 1.3$ (red). The central parameter values are $\mu = 5$, $\theta = 0$, $\sigma = 1$, $r = 0.01$, $c = 0.05$, $T = 1$.

(b_0, b_1)	V_0	$V_0^{CT,\infty}$	$V_0^{CT,1}$
optimal	0.3741		
(-0.20, 0.20)		0.3689	0.2808
(-0.21, 0.21)		0.3699	0.2885
(-0.22, 0.22)		0.3705	0.2956
(-0.23, 0.23)		0.3703	0.3018
(-0.24, 0.24)		0.3696	0.3071
(-0.25, 0.25)		0.3684	0.3116
(-0.26, 0.26)		0.3667	0.3153
(-0.27, 0.27)		0.3644	0.3181
(-0.28, 0.28)		0.3619	0.3202
(-0.29, 0.29)		0.3590	0.3217
(-0.30, 0.30)		0.3556	0.3224
(-0.31, 0.31)		0.3519	0.3224
(-0.32, 0.32)		0.3480	0.3219
(-0.33, 0.33)		0.3436	0.3206

Table 2: This table compares the following value functions: 1) V_0 under the optimal trading boundaries from Section 2 (column 2); 2) $V_0^{CT,\infty}$ under constant thresholds and unbounded number of trades (column 3); 3) $V_0^{CT,1}$ under constant thresholds and a single round-trip trade (column 4). Column 1 indicates a choice for the pair of trading boundaries. For all strategies, we assume zero initial inventory in X, i.e., state 0. All strategies are considered in the case of a finite horizon $T = 1$. Parameter values are $\mu = 5$, $\theta = 0$, $\sigma = 1$, $r = 0.01$, $c = 0.05$. To estimate $V_0^{CT,\infty}$ and $V_0^{CT,1}$, we used 10^6 trajectories..

5. Trading with three states: finite horizon

In this section, we have a similar setting as in Section 2 but assume now that the trader can also open a short position in X. The trading problem is formulated by a system of three optimal stopping problems. Specifically, we denote by V_0, V_1, and V_{-1} the value functions to represent the maximized expected value that the trader can generate from trading given that the current inventory is 0, 1, and -1, respectively. Together, they solve the system

$$V_0(t,x) = \sup_{t \le \tau \le T} \mathbb{E}_{t,x}\left[e^{-r(\tau-t)} \max\{V_1(\tau, X_\tau) - X_\tau - c, V_{-1}(\tau, X_\tau) + X_\tau - c, 0)\}\right] \tag{5.1}$$

$$V_1(t,x) = \sup_{t \le \tau \le T} \mathbb{E}_{t,x}\left[e^{-r(\tau-t)}(V_0(\tau, X_\tau) + X_\tau - c)\right] \tag{5.2}$$

$$V_{-1}(t,x) = \sup_{t \le \tau \le T} \mathbb{E}_{t,x}\left[e^{-r(\tau-t)}(V_0(\tau, X_\tau) - X_\tau - c)\right] \tag{5.3}$$

for $t \in [0,T)$, $x \in \mathbb{R}$. As before, we assume that the terminal inventory must be zero, hence the terminal condition: $V_0(T,x) = 0$, $V_1(T,x) = x - c$ and $V_{-1}(T,x) = -x - c$ for $x \in \mathbb{R}$.

Given that the trader does not currently have a position, the trading regions can be described by two (lower and upper) trading boundaries $(b_{0,1}, b_{0,-1})$. The intuition is that, while in state 0, the trader starts the long position when the spread is sufficiently low ($x \le b_{0,1}(t)$) or the short position when the spread is sufficiently high ($x \ge b_{0,-1}(t)$).

When in state 1, the trader has already established a position in X, so it is optimal to close the long position when the spread is sufficiently high. This is described by a single liquidation boundary $b_1(t)$, and the trader exits when $x \ge b_1(t)$. Similarly, in state -1, there is a single liquidation boundary $b_{-1}(t)$. That is, it is optimal to close the short position when $x \le b_{-1}(t)$.

The lower and upper trading boundaries satisfy $b_{0,1}(t) < b_{0,-1}(t)$ for $t \in [0,T)$. We can show that in the waiting regions, the value functions satisfy the following system of PDEs

$$\frac{\partial}{\partial t} V_0(t,x) + \mathbb{L} V_0(t,x) = r V_0(t,x) \quad \text{for } b_{0,1}(t) < x < b_{0,-1}(t) \tag{5.4}$$

$$\frac{\partial}{\partial t} V_1(t,x) + \mathbb{L} V_1(t,x) = r V_1(t,x) \quad \text{for } x < b_1(t) \tag{5.5}$$

$$\frac{\partial}{\partial t} V_{-1}(t,x) + \mathbb{L} V_{-1}(t,x) = r V_{-1}(t,x) \quad \text{for } x > b_{-1}(t) \tag{5.6}$$

subject to continuous and smooth pasting conditions at the optimal trading boundaries

$$V_0(t, b_{0,1}(t)) = V_1(t, b_{0,1}(t)) - b_{0,1}(t) - c \qquad \text{for } t \in [0,T) \tag{5.7}$$

$$\frac{\partial V_0}{\partial x}(t, b_{0,1}(t)) = \frac{\partial V_1}{\partial x}(t, b_{0,1}(t)) - 1 \qquad \text{for } t \in [0,T) \tag{5.8}$$

$$V_0(t, b_{0,-1}(t)) = V_{-1}(t, b_{0,-1}(t)) + b_{0,-1}(t) - c \qquad \text{for } t \in [0,T) \tag{5.9}$$

$$\frac{\partial V_0}{\partial x}(t, b_{0,-1}(t)) = \frac{\partial V_1}{\partial x}(t, b_{0,-1}(t)) + 1 \qquad \text{for } t \in [0,T) \tag{5.10}$$

$$V_1(t, b_1(t)) = V_0(t, b_1(t)) + b_1(t) - c \qquad \text{for } t \in [0,T) \tag{5.11}$$

$$\frac{\partial V_1}{\partial x}(t, b_1(t)) = \frac{\partial V_0}{\partial x}(t, b_0(t)) + 1 \qquad \text{for } t \in [0,T) \tag{5.12}$$

$$V_{-1}(t,b_{-1}(t)) = V_0(t,b_{-1}(t)) - b_{-1}(t) - c \qquad \text{for } t\in[0,T) \tag{5.13}$$

$$\frac{\partial V_{-1}}{\partial x}(t,b_{-1}(t)) = \frac{\partial V_0}{\partial x}(t,b_{-1}(t)) - 1 \qquad \text{for } t\in[0,T). \tag{5.14}$$

Since there is no reason to enter into a position near T due to the lack of time to make profit and the need to pay the fixed fee c twice to ensure zero inventory at T, it follows that $b_{0,1}(T-) = -\infty$ and $b_{0,-1}(T-) = +\infty$. We can also deduce that

$$b_1(T-) = \frac{\mu\theta + rc}{\mu + r} \tag{5.15}$$

$$b_{-1}(T-) = \frac{\mu\theta - rc}{\mu + r}. \tag{5.16}$$

Now applying Ito's calculus for $e^{-rt}V_i(t,X_t)$, $i = 0,1,-1$ we derive the integral representations.

Proposition 5.1. *The value functions V_0 and V_1 can be characterized as*

$$\begin{aligned} V_0(t,x) &= -\mathbb{E}_{t,x}\left[\int_t^T e^{-r(u-t)}H_0(X_u)I(X_u\le b_{0,1}(u))du\right] \\ &\quad -\mathbb{E}_{t,x}\left[\int_t^T e^{-r(u-t)}H_1(X_u)I(X_u\ge b_{0,-1}(u))du\right] \end{aligned} \tag{5.17}$$

$$V_1(t,x) = \mathbb{E}_{t,x}\left[e^{-r(T-t)}(X_T - c)\right] - \mathbb{E}_{t,x}\left[\int_t^T e^{-r(u-t)}H_1(X_u)I(X_u\ge b_1(u))du\right] \tag{5.18}$$

$$V_{-1}(t,x) = \mathbb{E}_{t,x}\left[e^{-r(T-t)}(-X_T - c)\right] - \mathbb{E}_{t,x}\left[\int_t^T e^{-r(u-t)}H_0(X_u)I(X_u\le b_{-1}(u))du\right] \tag{5.19}$$

for $t\in[0,T)$ and $x\in\mathbb{R}$.

The formulas (5.17)-(5.19) do not provide complete characterization as they depend on still unknown trading boundaries. As in the Section 2, the main argument is to evaluate the value functions at these boundaries and make use of the value matching conditions to obtain the system of integral equations

$$\begin{aligned} V_1(t,b_{0,1}(t)) - b_{0,1}(t) - c &= -\mathbb{E}_{t,b_{0,1}(t)}\left[\int_t^T e^{-r(u-t)}H_0(X_u)I(X_u\le b_{0,1}(u))du\right] \\ &\quad -\mathbb{E}_{t,b_{0,1}(t)}\left[\int_t^T e^{-r(u-t)}H_1(X_u)I(X_u\ge b_{0,-1}(u))du\right] \end{aligned} \tag{5.20}$$

$$\begin{aligned} V_{-1}(t,b_{0,-1}(t)) + b_{0,-1}(t) - c &= -\mathbb{E}_{t,b_{0,-1}(t)}\left[\int_t^T e^{-r(u-t)}H_0(X_u)I(X_u\le b_{0,1}(u))du\right] \\ &\quad -\mathbb{E}_{t,b_{0,-1}(t)}\left[\int_t^T e^{-r(u-t)}H_1(X_u)I(X_u\ge b_{0,-1}(u))du\right] \end{aligned} \tag{5.21}$$

$$V_0(t,b_1(t)) + b_1(t) - c = \mathbb{E}_{t,b_1(t)}\left[e^{-r(T-t)}(X_T - c)\right] \tag{5.22}$$

$$-\mathbb{E}_{t,b_1(t)}\left[\int_t^T e^{-r(u-t)}H_1(X_u)I(X_u \geq b_1(u))du\right]$$

$$V_0(t,b_{-1}(t)) - b_{-1}(t) - c \;=\; \mathbb{E}_{t,b_{-1}(t)}\left[e^{-r(T-t)}(-X_T - c)\right] - \mathbb{E}_{t,b_{-1}(t)}\left[\int_t^T e^{-r(u-t)}H_0(X_u)I(X_u \leq b_{-1}(u))du\right] \tag{5.23}$$

for $t \in [0,T)$ subject to $b_0(T-) = -\infty$, $b_{0,-1}(T-) = +\infty$, $b_1(T-) = (\mu\theta + rc)/(\mu + r)$ and $b_{-1}(T-) = (\mu\theta - rc)/(\mu + r)$.

As in Section 2 we have unknown value functions on the left-hand sides but we can employ the integral representations (5.17)-(5.19) for the value functions V_0, V_1, V_{-1} and rewrite (5.20)-(5.23) as the system of four nonlinear integral equations for $(b_{0,1}, b_{0,-1}, b_1, b_{-1})$. The same numerical approach can be used as in Section 2. We discretize the time interval $[0,T]$ and approximate integral terms using a quadrature scheme. Then, we proceed to apply a backward induction given that the boundaries are known at T.

6. Trading with three states: infinite horizon

In this section, we discuss the same problem as in Section 5 but with an infinite horizon. The methodology is similar to the one in Section 3. We denote by V_0, V_1, and V_{-1} the value functions representing the maximized expected value that the trader can generate from trading given that the current inventory is 0,1, and -1, respectively. We formulate the triple of optimal stopping problems

$$V_0(t,x) = \sup_{\tau \geq 0} \mathbb{E}_{t,x}\left[e^{-r\tau}\max\{V_1(\tau,X_\tau) - X_\tau - c, V_{-1}(\tau,X_\tau) + X_\tau - c)\}\right] \tag{6.1}$$

$$V_1(t,x) = \sup_{\tau \geq 0} \mathbb{E}_{t,x}\left[e^{-r\tau}(V_0(\tau,X_\tau) + X_\tau - c)\right] \tag{6.2}$$

$$V_{-1}(t,x) = \sup_{\tau \geq 0} \mathbb{E}_{t,x}\left[e^{-r\tau}(V_0(\tau,X_\tau) - X_\tau - c)\right] \tag{6.3}$$

for $x \in \mathbb{R}$.

There are four optimal trading thresholds, $b_{0,1}^\infty$, $b_{0,-1}^\infty$, b_1^∞ and b_{-1}^∞ that need to be found. The value functions satisfy the ODEs in the waiting regions

$$\mathbb{L}V_0(t,x) = rV_0(t,x) \quad \text{for } b_{0,1}^\infty < x < b_{0,-1}^\infty \tag{6.4}$$

$$\mathbb{L}V_1(t,x) = rV_1(t,x) \quad \text{for } x < b_1^\infty \tag{6.5}$$

$$\mathbb{L}V_{-1}(t,x) = rV_{-1}(t,x) \quad \text{for } x > b_{-1}^\infty \tag{6.6}$$

and corresponding solutions are given by

$$V_0(x) = A_0F(x) + B_0\widehat{F}(x) \quad \text{for } b_{0,1}^\infty < x < b_{0,-1}^\infty \tag{6.7}$$

$$V_1(x) = A_1F(x) + B_1\widehat{F}(x) \quad \text{for } x < b_1^\infty \tag{6.8}$$

$$V_{-1}(x) = A_{-1}F(x) + B_{-1}\widehat{F}(x) \quad \text{for } x > b_{-1}^\infty \tag{6.9}$$

where $(A_0, B_0, A_1, B_1, A_{-1}, B_{-1})$ are unknown constants to be determined, and F and $\widehat{F}$ we given in Section 3. Using that $V_1(-\infty) = 0$ and $V_{-1}(\infty) = 0$, we can deduce that $B_1 = 0$ and $A_{-1} = 0$. The value matching and smooth pasting conditions at the optimal trading boundaries imply the system of eight algebraic equations with eight unknowns $(b^{\infty}_{0,1}, b^{\infty}_{0,-1}, b^{\infty}_{1}, b^{\infty}_{-1}, A_0, B_0, A_1, B_{-1})$

$$A_0 F(b^{\infty}_{0,1}) + B_0 \widehat{F}(b^{\infty}_{0,1}) = A_1 F(b^{\infty}_{0,1}) - b^{\infty}_{0,1} - c \tag{6.10}$$

$$A_0 F'(b^{\infty}_{0,1}) + B_0 \widehat{F}'(b^{\infty}_{0,1}) = A_1 F(b^{\infty}_{0,1}) - 1 \tag{6.11}$$

$$A_0 F(b^{\infty}_{0,-1}) + B_0 \widehat{F}(b^{\infty}_{0,-1}) = B_{-1} \widehat{F}(b^{\infty}_{0,-1}) + b^{\infty}_{0,-1} - c \tag{6.12}$$

$$A_0 F'(b^{\infty}_{0,-1}) + B_0 \widehat{F}'(b^{\infty}_{0,-1}) = B_{-1} \widehat{F}(b^{\infty}_{0,-1}) + 1 \tag{6.13}$$

$$A_1 F(b^{\infty}_{1}) = A_0 F(b^{\infty}_{1}) + B_0 \widehat{F}(b^{\infty}_{1}) + b^{\infty}_{1} - c \tag{6.14}$$

$$A_1 F'(b^{\infty}_{1}) = A_0 F'(b^{\infty}_{1}) + B_0 \widehat{F}'(b^{\infty}_{1}) + 1 \tag{6.15}$$

$$B_{-1} \widehat{F}(b^{\infty}_{-1}) = A_0 F(b^{\infty}_{-1}) + B_0 \widehat{F}(b^{\infty}_{-1}) - b^{\infty}_{-1} - c \tag{6.16}$$

$$B_{-1} \widehat{F}'(b^{\infty}_{-1}) = A_0 F'(b^{\infty}_{-1}) + B_0 \widehat{F}'(b^{\infty}_{-1}) - 1 \tag{6.17}$$

which can be solved numerically.

7. Conclusions

We have presented a coupled optimal stopping approach to trading a mean-reverting price process over a finite time horizon. Our approach determines the values of trading opportunities depending on the trader's current position and accounting for future trading opportunities up to a deadline. The coupled optimal stopping problem leads to the analytical study of the associated free-boundary system. The trader's optimal trading strategies are described by the corresponding optimal trading boundaries, which are obtained by solving the system of nonlinear Volterra equations numerically. Our empirical experiments have shown the time-varying property of the optimal trading boundaries and the effects of different parameters. Finally, we illustrated the added value of our strategy compared to the benchmark policy that employs the constant thresholds for entry and exit signals.

While the OU model is used as the underlying stochastic model in this paper, one could consider the same trading problem using a different stochastic process and apply our integral equations approach to study the coupled optimal stopping system. Other future research directions include incorporating additional risk controls into the trading problem, or introducing multiple price processes for the purposes of portfolio diversification or asset selection.

Appendix

A. Sketch of the proof of Theorem 2.1

We apply the local time-space formula (Peskir (2005)) for $\mathrm{e}^{-r(s-t)}V_0(s,X_s)$ along with (2.5), the definition (2.20) of H_0, the smooth-fit property (2.8) to get

$$\text{(A.1)} \qquad \begin{aligned} \mathrm{e}^{-r(s-t)}V_0(s,X_s) &= V_0(t,x) + M_s + \int_t^s \mathrm{e}^{-r(u-t)}\left((V_0)_t + \mathbb{L}_X V_0 - rV_0\right)(u,X_u)du \\ &= V_0(t,x) + M_s + \int_t^s \mathrm{e}^{-r(u-t)} H_0(X_u) I(X_u \le b_0(u))du \end{aligned}$$

where $M = (M_s)_{s\ge t}$ is the martingale part with $M_t = 0$. Now letting $s = T$, taking the expectation $\mathbb{E}_{t,x}$, using the optional sampling theorem, rearranging terms and noting that $V_0(T,\cdot) = 0$, we obtain the early entry premium formula

$$\text{(A.2)} \qquad V_0(t,x) = -\mathbb{E}_{t,x}\left[\int_t^T e^{-r(u-t)} H_0(X_u) I(X_u \le b_0(u))du\right]$$

for $t \in [0,T)$ and $x \in \mathbb{R}$. In the analogous way we prove the early exit premium formula for V_1.

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Compliance with Ethical Standards

The author declares that there are no conflicts of interest, research involving Human Participants and/or Animals, or informed consent.

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