1 Continued Fractions

There is something arbitrary (hence unsatisfying) about representing the number π as $3.14159\ldots$. Another way is the continued fraction expansion, where we cut a number into its integer and fractional part and note that since the fractional part is less than one it can be written as the inverse of something greater than one. This quantity greater than one can in turn be written as the sum of integral and fractional parts, and the process is continued in this way.

$$\pi = 3 + 0.14159\ldots = 3 + \frac{1}{7 + .06251\ldots} = 3 + \frac{1}{7 + \frac{1}{15 + .9966\ldots}} = 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1 + .00341723\ldots}}}$$

To save space on the page, this is often written as $\pi = [3; 7, 15, 1, 292, \ldots]$. This continued fraction expansion of π never ends; if it did, π would be rational.

The continued fraction representation of a real number can reveal structure that is not visible in, say, its decimal expansion. Consider the following numbers:

$$\begin{aligned}
\pi &= 3.1415926535\ldots = [3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, \ldots] \\
e &= 2.7182818284\ldots = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, \ldots] \\
\frac{1 + \sqrt{5}}{2} &= 1.6180339887\ldots = [1; 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, \ldots]
\end{aligned}$$

The Golden Ratio has a periodic continued fraction; it turns out that this property characterizes those irrational numbers which are the roots of quadratic equations. The number e is not of this kind, but there is some order to its continued fraction. π, on the other hand, just seems to be doing its own thing. There are interesting continued fractions for π, but they are not *the* continued fraction whose numerators are all equal to one.

2 Khinchin's Constant

Still, it would be nice to have a sense of the behavior of the integers which make up the continued fraction of a "typical" real number. To that end, one might look at the continued fraction of some random number,

$$a = [a_0; a_1, a_2, a_3, \ldots]$$

and seek the average of the integers in its continued fraction expansion:

$$\lim_{n\to\infty} \frac{a_1 + a_2 + a_3 + \cdots + a_n}{n}$$

As one can see using Mathematica with some number such as $\pi/\sqrt{e}$, this quantity diverges, although quite slowly. However, in 1935 the Russian mathematician A. Khinchin proved that if one chooses a real number at random, then with probability 1 the limit of geometric means

$$\lim_{n\to\infty} (a_1 \cdot a_2 \cdot a_3 \cdots a_n)^{1/n}$$

not only exists but is equal to Khinchin's constant $K = 2.68545\ldots$

I find this fact (and the fact that it can be proven) alarming. What kind of thing is the geometric mean of the integers in the continued fraction expansion of a random number? How can such a twisted quantity possibly be calculated? Why should it approach the same value for almost all numbers, and where does this value K come from? Although a rigorous mathematical proof of Khinchin's result requires some mathematical machinery (such as the Birkhoff Ergodic Theorem), it turns out that one can understand this phenomenon using ideas that are not unfamiliar to physicists.

3 The Gauss Map $x \mapsto \{1/x\}$

Look again at the process by which the continued fraction of π was computed above. Instead of focusing on the integers $7, 15, 1, \ldots$, let's consider the sequence of numbers $0.14159\ldots, 0.06251\ldots, 0.9966\ldots, 0.00341\ldots$, each less than one. Starting with the first number $0.14159\ldots$, these are simply generated by iterating the mapping $x \mapsto f(x) \equiv \{1/x\}$, where curly brackets denote the fractional part. The elements $7, 15, 1, \ldots$ of the continued fraction are simply the integral parts that we throw out in evaluating the fractional part.

In a sense, we are viewing this process as a dynamical system. The possible "states" of the system are real numbers $x \in (0, 1)$, and at each "time step" the state changes via the mapping f, which incidentally is called the Gauss map because Gauss used it (surprise!) to investigate continued fractions. Thus we have a sequence of "states" $x_1, x_2, x_3 \ldots \in (0, 1)$ and we may consider "observables" such as $[1/x_n]$, where square brackets denote the integral part. Again, this particular observable gives us the integral elements of the continued fraction expansion, meaning that

$$[1/x_n] = a_n$$

Any observable $A(x)$ can be averaged over the course of a trajectory starting with some state x_1:

$$\langle A \rangle_{x_1} \equiv \frac{1}{n} \sum_{n=1}^{\infty} A(x_n) = \frac{1}{n} \sum_{n=1}^{\infty} A(f^{(n-1)}(x_1))$$

The observable of interest in Khinchin's theorem is $\log[1/x]$ since $[1/x_n] = a_n$ and

$$(a_1 \cdot a_2 \cdot a_3 \cdots a_n)^{1/n} = \exp(\frac{1}{n} \sum_{n=1}^{\infty} \log(a_n))$$

In other words

$$\lim_{n \to \infty} (a_1 \cdot a_2 \cdot a_3 \cdots a_n)^{1/n} = \exp\langle \log[1/x] \rangle_{x_1} \tag{1}$$

4 Ergodicity

Calculating the average of an observable over a trajectory in the space of states is familiar to us. It is similar to doing a simulation starting with some state of a system, applying a rule at each step and keeping an average of some quantity. Of course, we hope that the average which we calculate is independent of the initial state, but rather reflects some kind of average over the space of all states with some equilibrium probability distribution. If this is true for "almost" any initial state, then the dynamical system is called "ergodic."
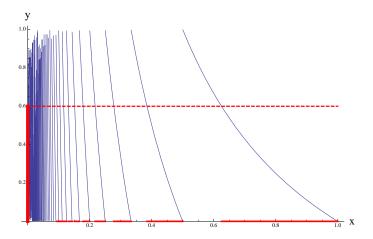
When physicists do Monte Carlo simulations they never prove that the simulation will lead to the equilibrium Boltzmann distribution. It is enough that the Monte Carlo step is designed to preserve the Boltzmann distribution. The probability distribution will keep changing until it stops changing. From then on, the average over the trajectory reflects the equilibrium distribution and is independent of the initial state. This generally works well, so let's be physicists and assume that as long as we can find an "equilibrium" probability distribution $\rho(x)$ over the interval $(0,1)$ of states, an average $\langle A \rangle_{x_1}$ over a trajectory will "almost always" be equal to an average $\langle A \rangle \equiv \int_0^1 A(x)\rho(x)dx$. The mathematicians call ρ an invariant measure and use the Birkhoff Ergodic Theorem to prove what we are assuming.

5   Equilibrium Distribution

Gauss himself found the probability distribution $\rho(x)$ which is preserved by
the Gauss map $f = x \mapsto \{1/x\}$

$$\rho(x) = \frac{1}{\log 2} \cdot \frac{1}{x+1}$$

To show that it is indeed preserved by the Gauss map, consider $y = f(x)$
where x is drawn at random using the distribution $\rho(x)$. What is the prob-
ability distribution of y? The easiest way to find out is to calculate the
cumulative probability distribution of y, or in other words the probability
that $y \leq a$. Now $y = \{1/x\}$ so there are many ways for y to end up in the
interval $(0, a)$, as illustrated in the figure below:

$$P(y \leq a) = \sum_{n=1}^{\infty} P(\frac{1}{n+a} \leq x \leq \frac{1}{n}) = \sum_{n=1}^{\infty} \int_{(n+a)^{-1}}^{n^{-1}} \rho(x)dx$$

$$= \frac{1}{\log 2} \sum_{n=1}^{\infty} \int_{(n+a)^{-1}}^{n^{-1}} \frac{1}{x+1}dx = \frac{1}{\log 2} \sum_{n=1}^{\infty} (\log \frac{n+1}{n} - \log \frac{n+1+a}{n+a})$$

$$= \frac{1}{\log 2} \sum_{n=1}^{\infty} (\log \frac{n+a}{n} - \log \frac{n+1+a}{n+1}) = \frac{1}{\log 2} \lim_{N\to\infty} \left[(\log(1+a) - \log(\frac{N+1+a}{N+1})) \right]$$

$$= \frac{1}{\log 2} \cdot \log(1+a)$$

The probability distribution function of the variable $y = f(x)$ is therefore

$$\rho_y(a) = \frac{d}{da}P(y \le a) = \frac{1}{\log 2} \cdot \frac{1}{1+a} = \rho(a)$$

Therefore the probability distribution $\rho(x)$ is left invariant by the Gauss map!

6 Putting it all Together

Recall that we are assuming that the Gauss map is ergodic, so that for almost every initial state x_1,

$$\langle \log[1/x] \rangle_{x_1} = \langle \log[1/x] \rangle \equiv \int_0^1 \log[1/x]\,\rho(x)dx$$

$$= \sum_{n=1}^{\infty} \int_{(n+1)^{-1}}^{n^{-1}} \log(n)\rho(x)dx = \frac{1}{\log 2} \cdot \sum_{n=1}^{\infty} \log(n) \cdot \int_{(n+1)^{-1}}^{n^{-1}} \frac{1}{x+1}dx$$

$$= \sum_{n=1}^{\infty} \frac{\log n}{\log 2} \cdot \left[\log(\frac{n+1}{n}) - \log(\frac{n+2}{n+1})\right]$$

$$= \sum_{n=1}^{\infty} \log_2 n \cdot \log\left[1 + \frac{1}{n(n+2)}\right]$$

Finally, using properties of logarithms and equation (1), we have Khinchin's result that for almost every real number, the continued fraction $[a_0; a_1, a_2, a_3 \ldots]$ satisfies

$$\boxed{\lim_{n \to \infty} (a_1 \cdot a_2 \cdot a_3 \cdots a_n)^{1/n} = \prod_{n=1}^{\infty} \left[1 + \frac{1}{n(n+2)}\right]^{\log_2 n}}$$

7 Reference

I read about all this in a book by Geon Ho Choe called *Computational Ergodic Theory*, published by Springer.