Chapter 7

Transcendental Functions

Section 7.1

Inverse Functions and Their Derivatives

DEFINITION A function $f(x)$ is **one-to-one** on a domain D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ in D.

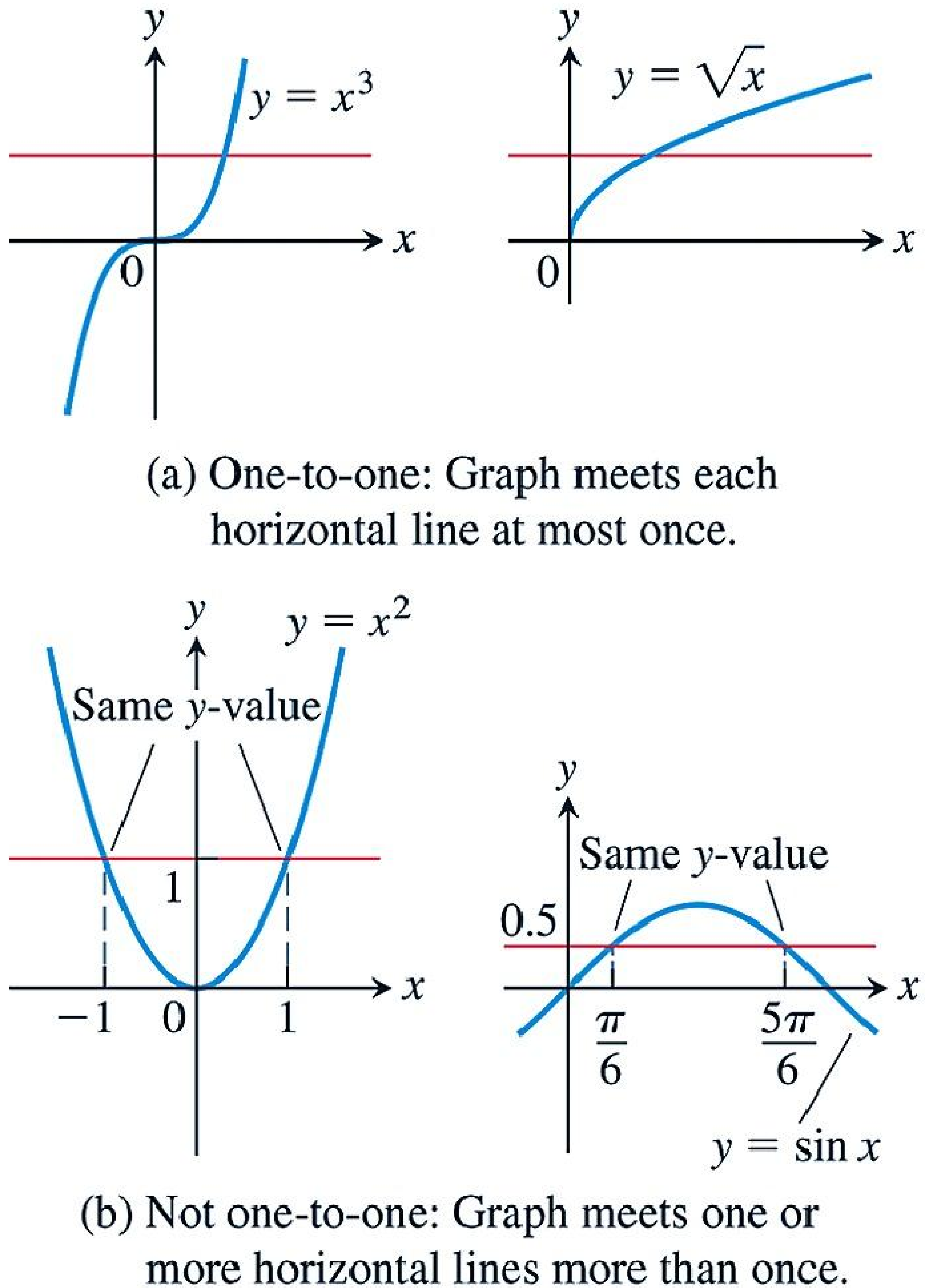


FIGURE 7.1 (a) $y = x^3$ and $y = \sqrt{x}$ are one-to-one on their domains $(-\infty, \infty)$ and $[0, \infty)$. (b) $y = x^2$ and $y = \sin x$ are not one-to-one on their domains $(-\infty, \infty)$.

The Horizontal Line Test for One-to-One Functions

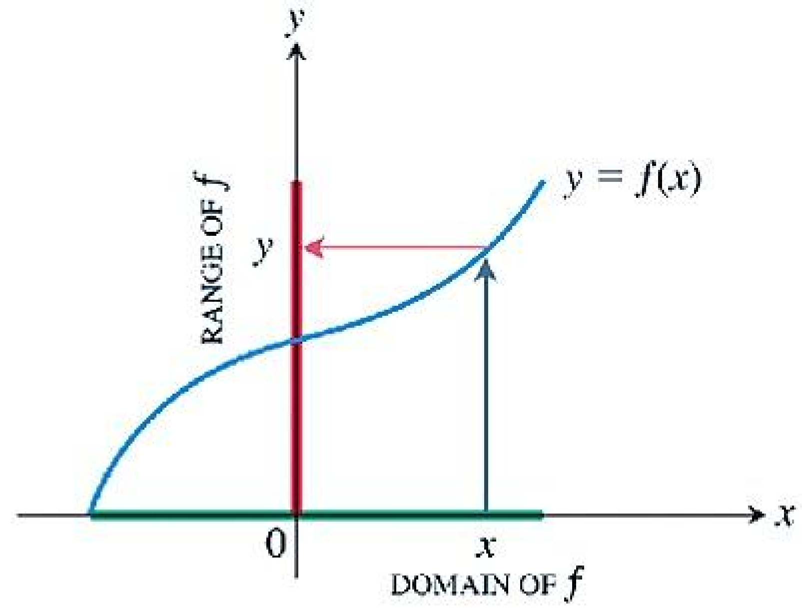
A function $y = f(x)$ is **one-to-one** if and only if its graph intersects each horizontal line at most once.

DEFINITION Suppose that f is a one-to-one function on a domain D with range R. The **inverse function** f^{-1} is defined by

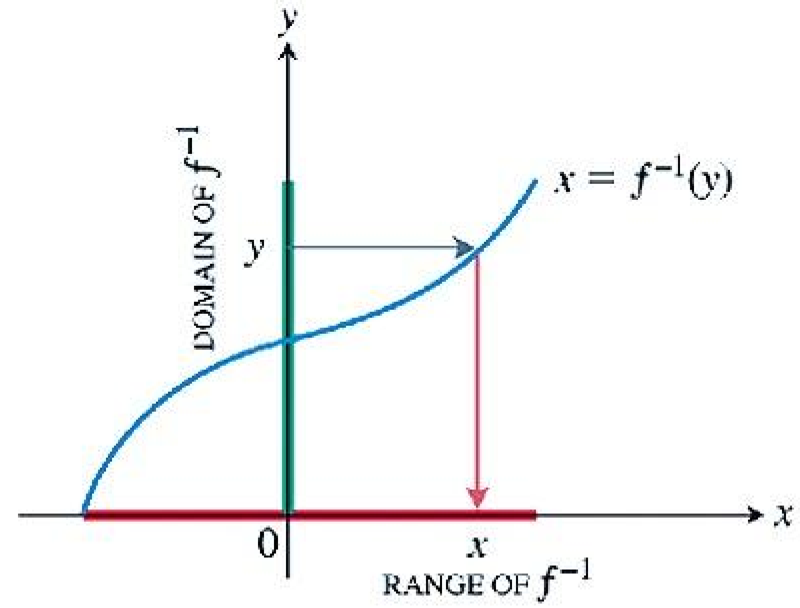
$$f^{-1}(b) = a \quad \text{if} \quad f(a) = b.$$

The domain of f^{-1} is R and the range of f^{-1} is D.

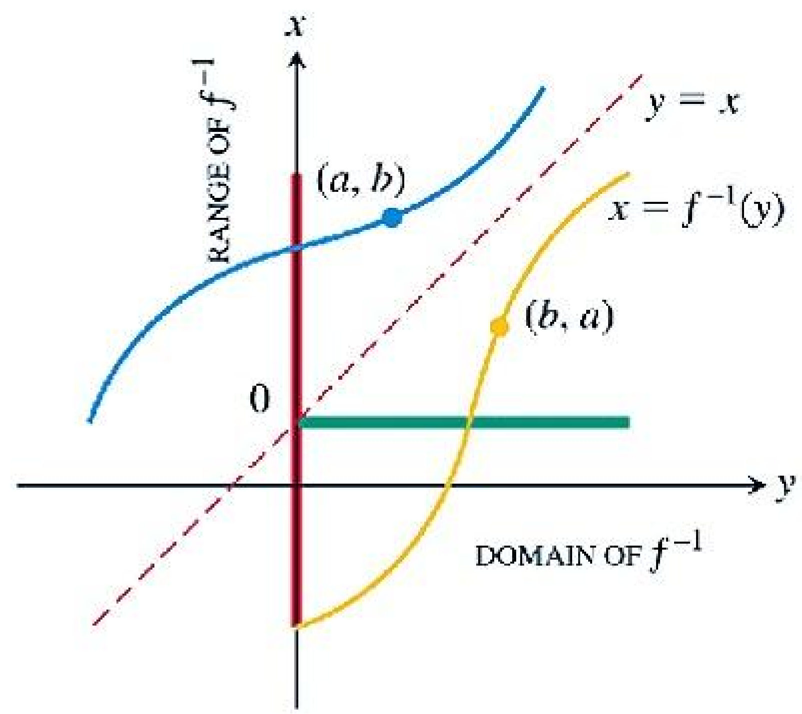
Pearson



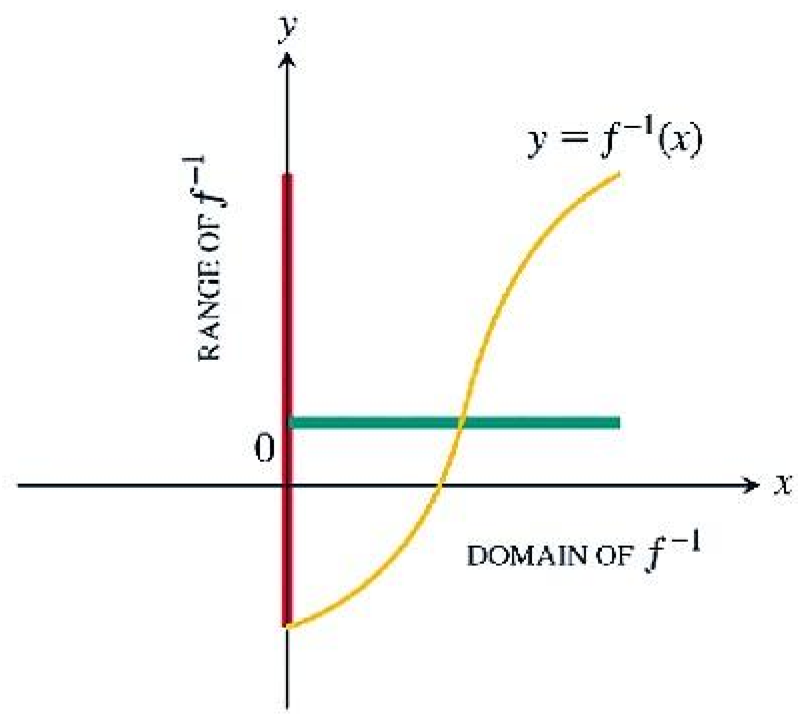
(a) To find the value of f at x, we start at x, go up to the curve, and then over to the y-axis.



(b) The graph of f^{-1} is the graph of f, but with x and y interchanged. To find the x that gave y, we start at y and go over to the curve and down to the x-axis. The domain of f^{-1} is the range of f. The range of f^{-1} is the domain of f.



(c) To draw the graph of f^{-1} in the more usual way, we reflect the system across the line $y = x$.



(d) Then we interchange the letters x and y. We now have a normal-looking graph of f^{-1} as a function of x.

FIGURE 7.2 The graph of $y = f^{-1}(x)$ is obtained by reflecting the graph of $y = f(x)$ about the line $y = x$.

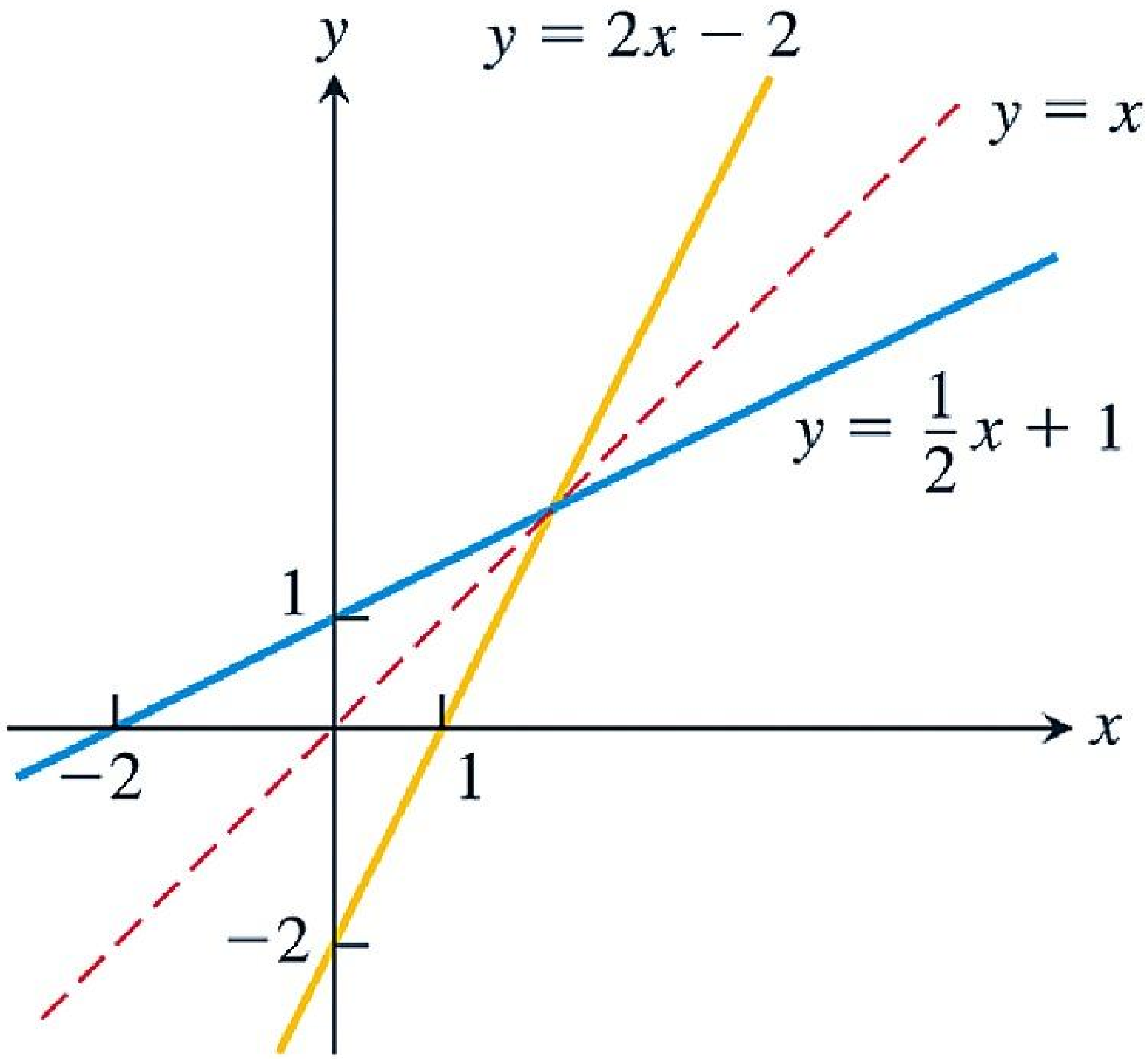


FIGURE 7.3 Graphing the functions $f(x) = (1/2)x + 1$ and $f^{-1}(x) = 2x - 2$ together shows the graphs' symmetry with respect to the line $y = x$ (Example 3).

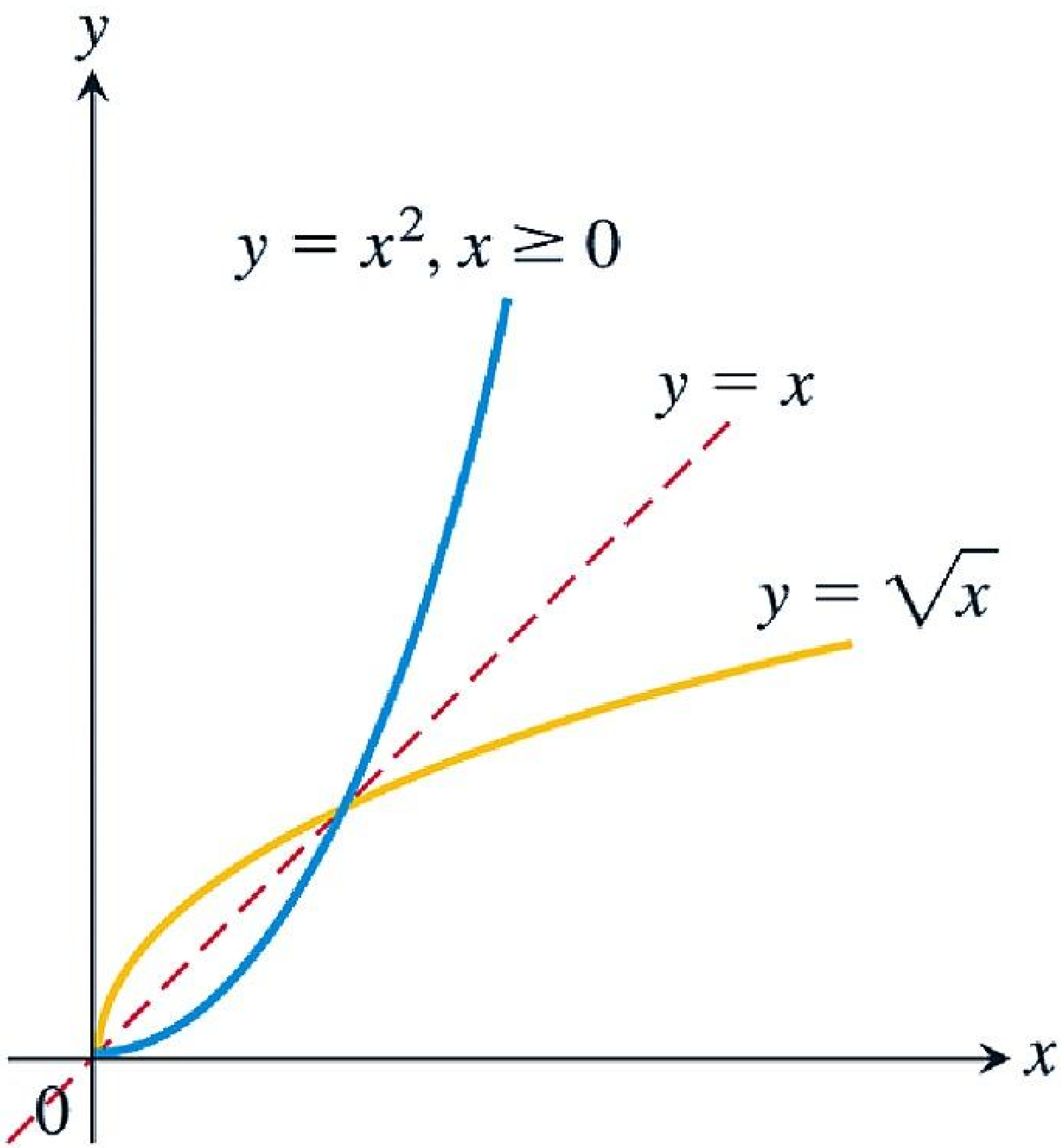
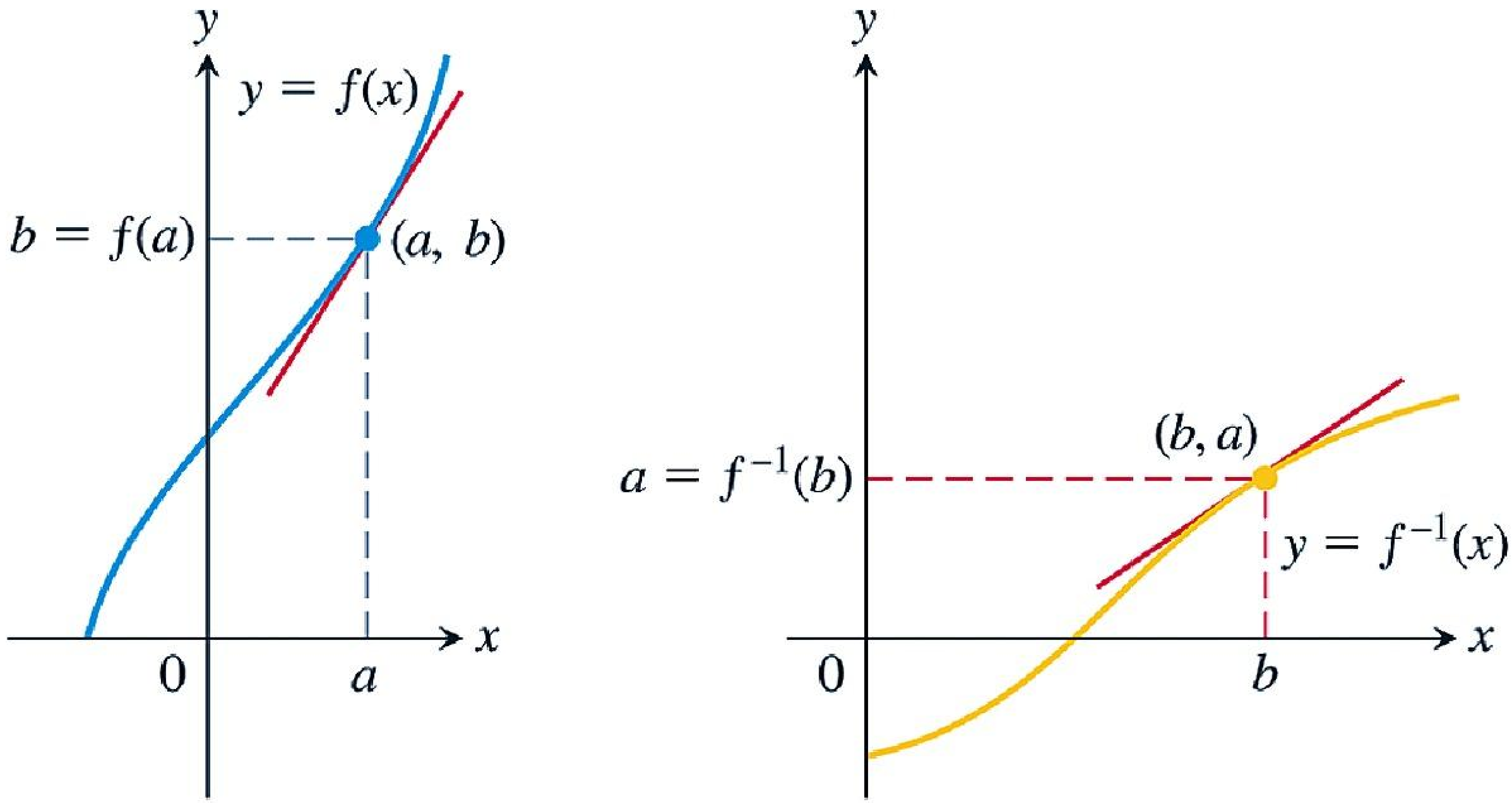


FIGURE 7.4 The functions $y = \sqrt{x}$ and $y = x^2, x \geq 0$, are inverses of one another (Example 4).



The slopes are reciprocal: $(f^{-1})'(b) = \dfrac{1}{f'(a)}$ or $(f^{-1})'(b) = \dfrac{1}{f'(f^{-1}(b))}$

FIGURE 7.5 The graphs of inverse functions have reciprocal slopes at corresponding points.

THEOREM 1—The Derivative Rule for Inverses

If f has an interval I as domain and $f'(x)$ exists and is never zero on I, then f^{-1} is differentiable at every point in its domain (the range of f). The value of $(f^{-1})'$ at a point b in the domain of f^{-1} is the reciprocal of the value of f' at the point $a = f^{-1}(b)$:

$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))} \tag{1}$$

or

$$\left.\frac{df^{-1}}{dx}\right|_{x=b} = \frac{1}{\left.\dfrac{df}{dx}\right|_{x=f^{-1}(b)}}.$$

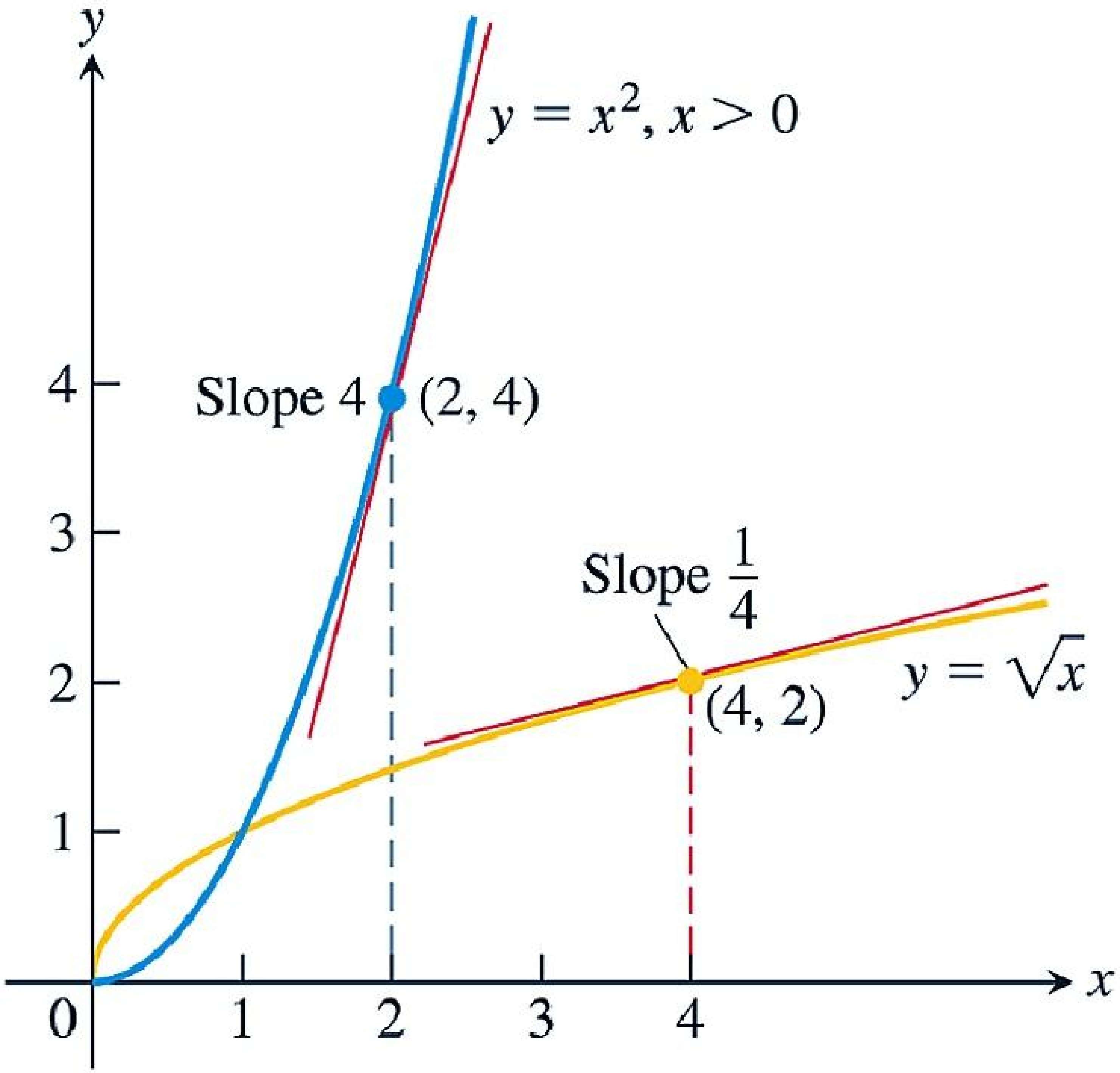


FIGURE 7.6 The derivative of $f^{-1}(x) = \sqrt{x}$ at the point (4, 2) is the reciprocal of the derivative of $f(x) = x^2$ at (2, 4) (Example 5).

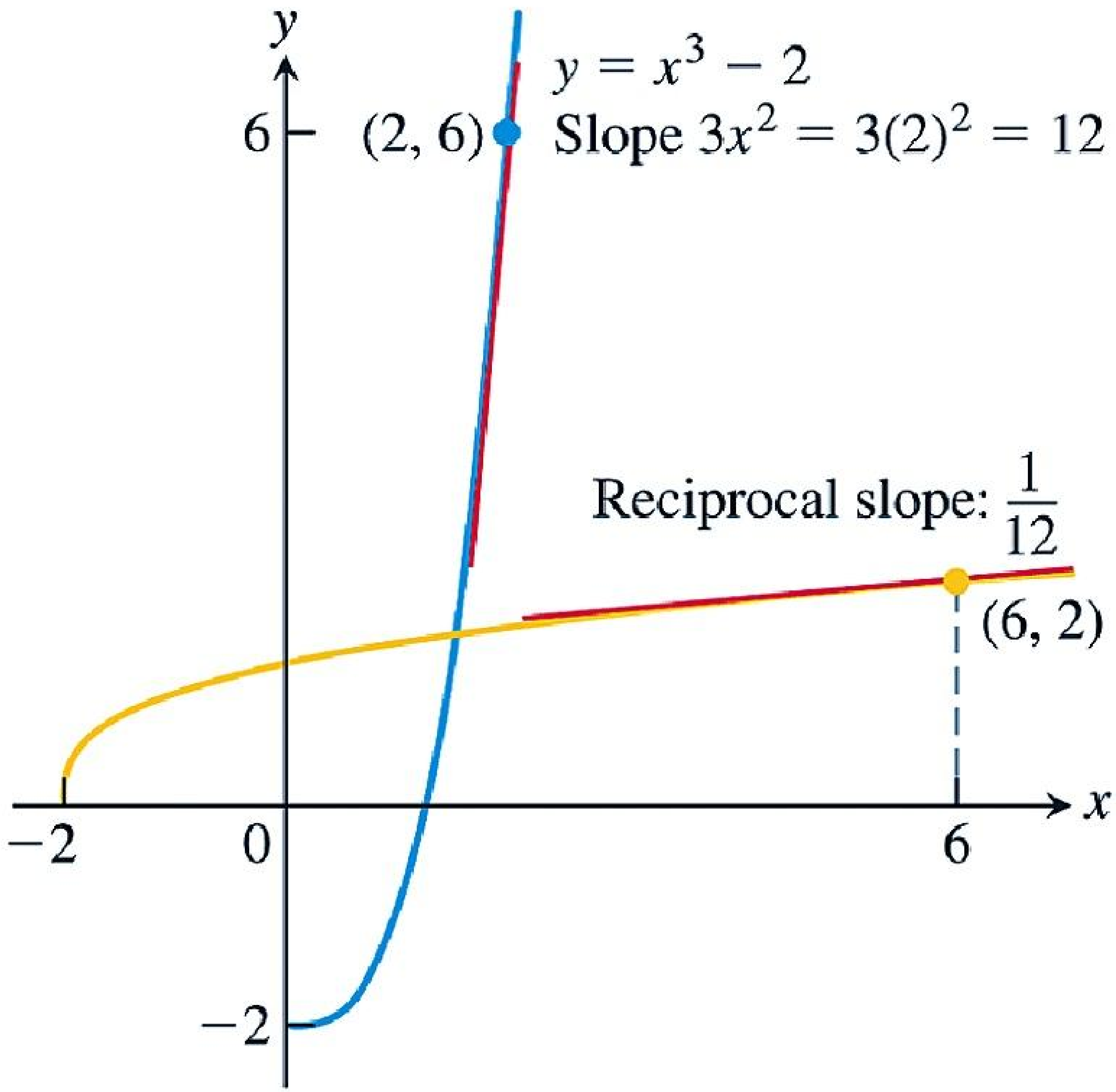


FIGURE 7.7 The derivative of $f(x) = x^3 - 2$ at $x = 2$ tells us the derivative of f^{-1} at $x = 6$ (Example 6).

Section 7.2

Natural Logarithms

DEFINITION The **natural logarithm** is the function given by

$$\ln x = \int_1^x \frac{1}{t}\,dt, \qquad x > 0.$$

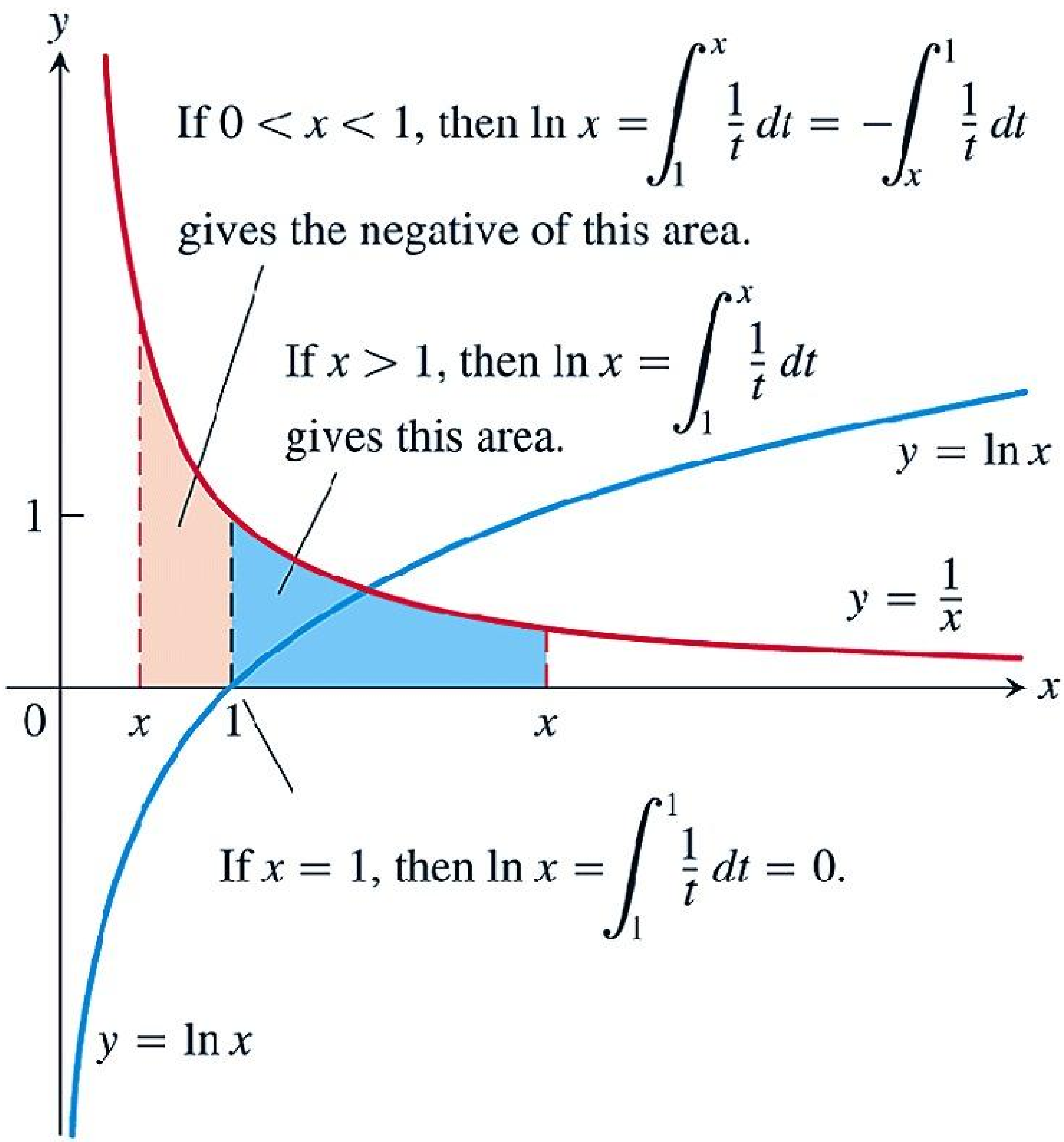


FIGURE 7.8 The graph of $y = \ln x$ and its relation to the function $y = 1/x$, $x > 0$. The graph of the logarithm rises above the x-axis as x moves from 1 to the right, and it falls below the axis as x moves from 1 to the left.

TABLE 7.1 Typical 2-place values of $\ln x$

x	$\ln x$
0	undefined
0.05	-3.00
0.5	-0.69
1	0
2	0.69
3	1.10
4	1.39
10	2.30

DEFINITION The **number *e*** is the number in the domain of the natural logarithm that satisfies

$$\ln(e) = \int_1^e \frac{1}{t}\,dt = 1.$$

$$\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}, \qquad u > 0. \tag{2}$$

THEOREM 2—Algebraic Properties of the Natural Logarithm

For any numbers $b > 0$ and $x > 0$, the natural logarithm satisfies the following rules:

1. *Product Rule:* $\ln bx = \ln b + \ln x$
2. *Quotient Rule:* $\ln \frac{b}{x} = \ln b - \ln x$
3. *Reciprocal Rule:* $\ln \frac{1}{x} = -\ln x$ Rule 2 with $b = 1$
4. *Power Rule:* $\ln x^r = r \ln x$ For r rational

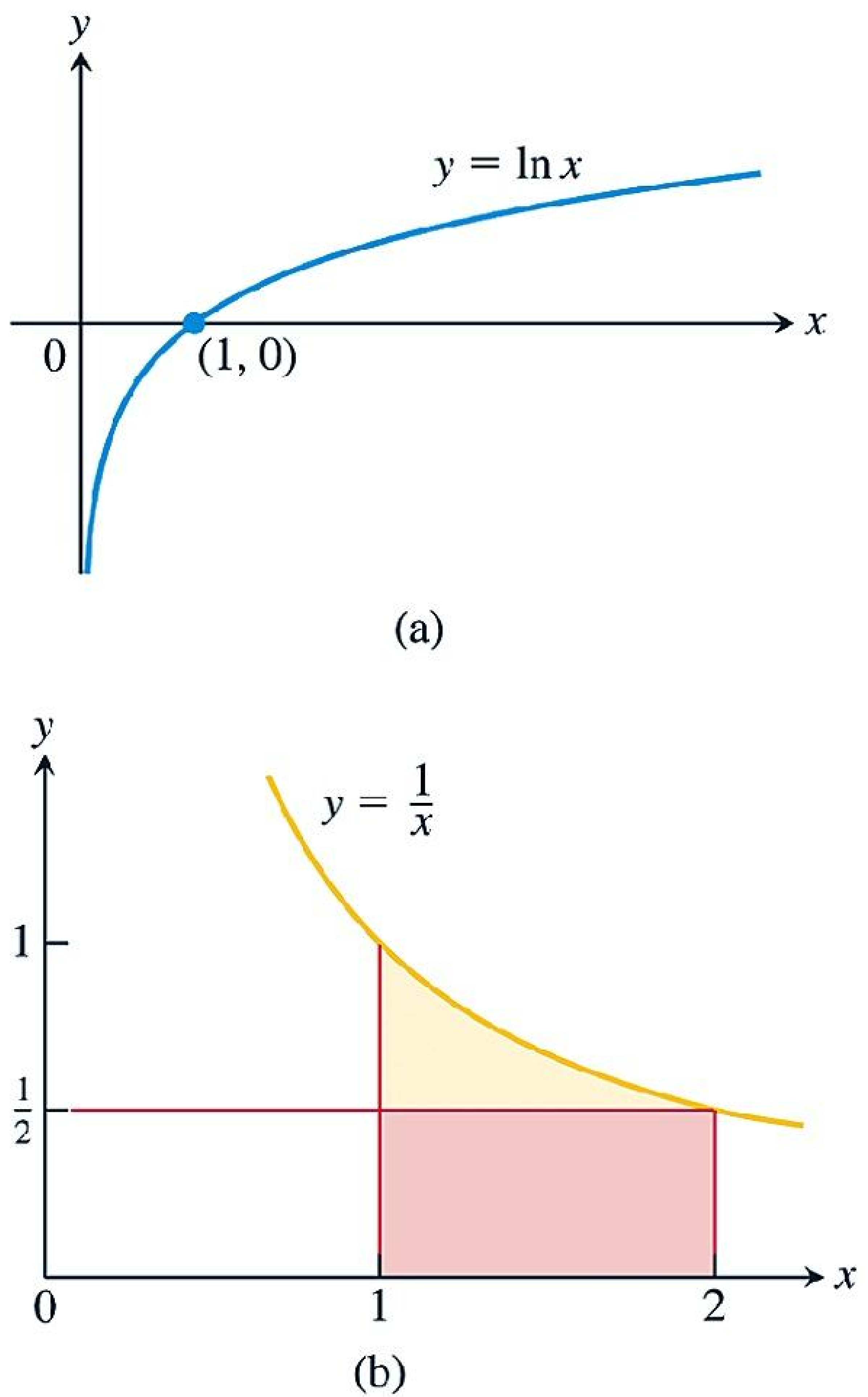


FIGURE 7.9 (a) The graph of the natural logarithm. (b) The rectangle of height $y = 1/2$ fits beneath the graph of $y = 1/x$ for the interval $1 \le x \le 2$.

If u is a differentiable function that is never zero, then

$$\int \frac{1}{u}\, du = \ln |u| + C. \tag{3}$$

Integrals of the tangent, cotangent, secant, and cosecant functions

$$\int \tan u \, du = \ln|\sec u| + C \qquad \int \sec u \, du = \ln|\sec u + \tan u| + C$$

$$\int \cot u \, du = \ln|\sin u| + C \qquad \int \csc u \, du = -\ln|\csc u + \cot u| + C$$

Section 7.3

Exponential Functions

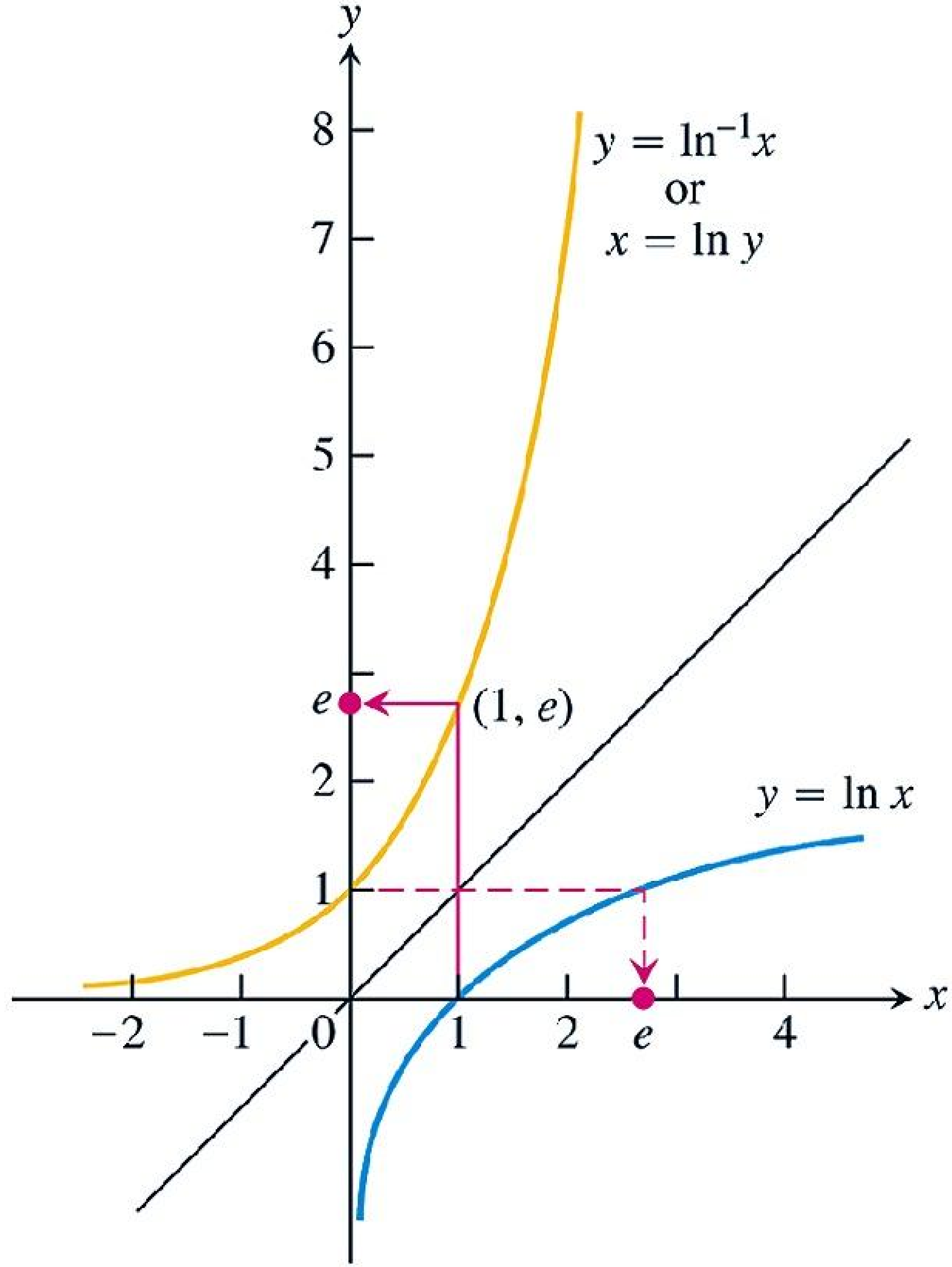


FIGURE 7.10 The graphs of $y = \ln x$ and $y = \ln^{-1} x = \exp x$. The number e is $\ln^{-1} 1 = \exp(1)$.

DEFINITION For every real number x, we define the **natural exponential function** to be $e^x = \exp x$.

Typical values of e^x

x	e^x (rounded)
-1	0.37
0	1
1	2.72
2	7.39
10	22026
100	2.6881×10^{43}

Inverse Equations for e^x and $\ln x$

$$e^{\ln x} = x \qquad \text{(all } x > 0\text{)}$$

$$\ln (e^x) = x \qquad \text{(all } x\text{)}$$

Pearson

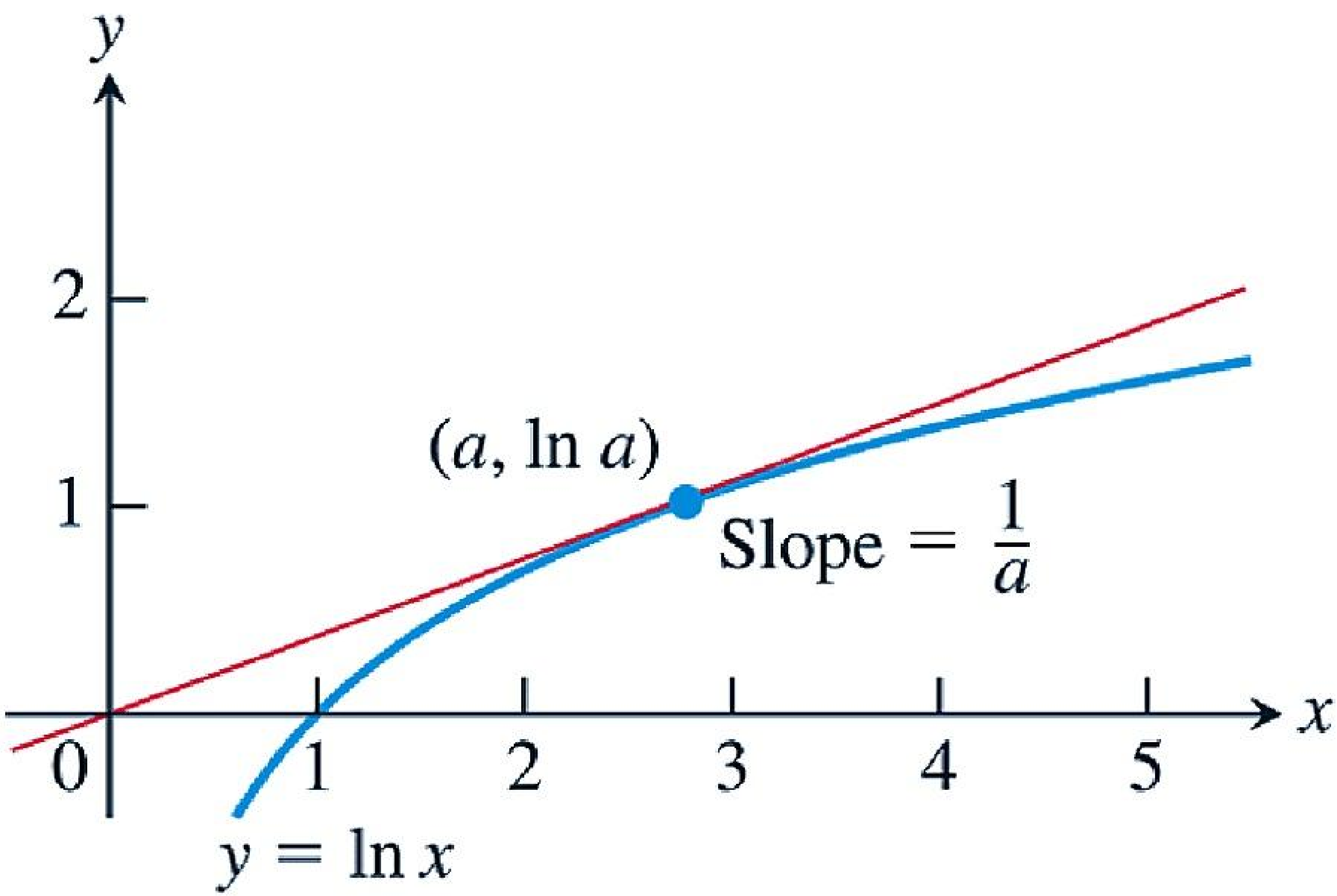


FIGURE 7.11 The tangent line intersects the curve at some point $(a, \ln a)$, where the slope of the curve is $1/a$ (Example 2).

If u is any differentiable function of x, then

$$\frac{d}{dx}e^u = e^u \frac{du}{dx}. \tag{2}$$

The general antiderivative of the exponential function

$$\int e^u \, du = e^u + C$$

THEOREM 3 For all numbers x, x_1, and x_2, the natural exponential e^x obeys the following laws:

1. $e^{x_1} e^{x_2} = e^{x_1 + x_2}$

2. $e^{-x} = \dfrac{1}{e^x}$

3. $\dfrac{e^{x_1}}{e^{x_2}} = e^{x_1 - x_2}$

4. $(e^{x_1})^r = e^{r x_1}$, if r is rational

DEFINITION For any numbers $a > 0$ and x, the **exponential function with base a** is

$$a^x = e^{x \ln a}.$$

DEFINITION For any $x > 0$ and for any real number n,

$$x^n = e^{n \ln x}.$$

General Power Rule for Derivatives

For $x > 0$ and any real number n,

$$\frac{d}{dx}x^n = nx^{n-1}.$$

If $x \leq 0$, then the formula holds whenever the derivative, x^n, and x^{n-1} all exist.

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THEOREM 4—The Number e as a Limit

The number e can be calculated as the limit

$$e = \lim_{x \to 0} (1 + x)^{1/x}.$$

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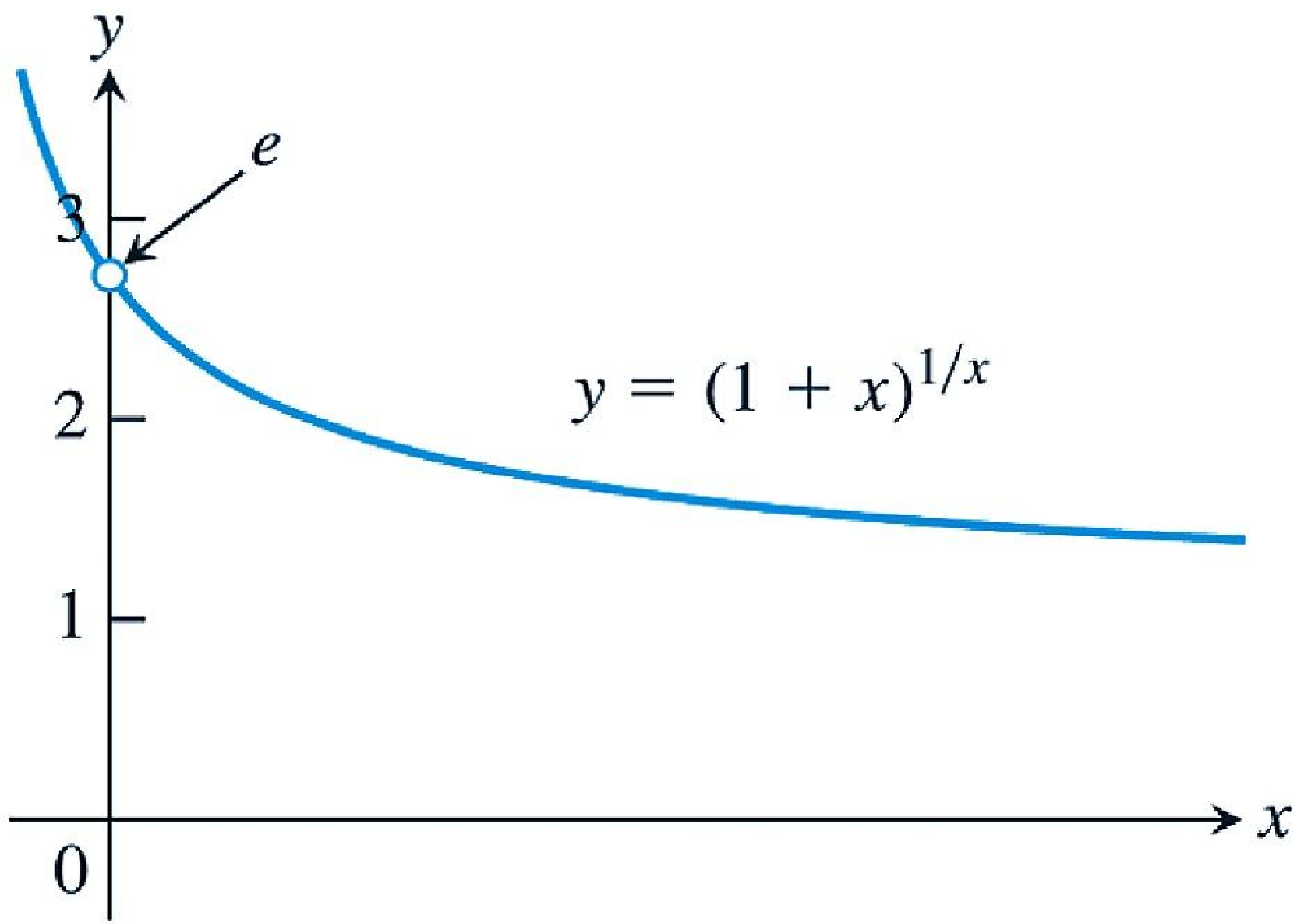


FIGURE 7.12 The number e is the limit of the function graphed here as $x \to 0$.

If $a > 0$ and u is a differentiable function of x, then a^u is a differentiable function of x and

$$\frac{d}{dx}a^u = a^u \ln a \frac{du}{dx}. \qquad (3)$$

The integral equivalent of this last result gives the general antiderivative

$$\int a^u \, du = \frac{a^u}{\ln a} + C. \qquad (4)$$

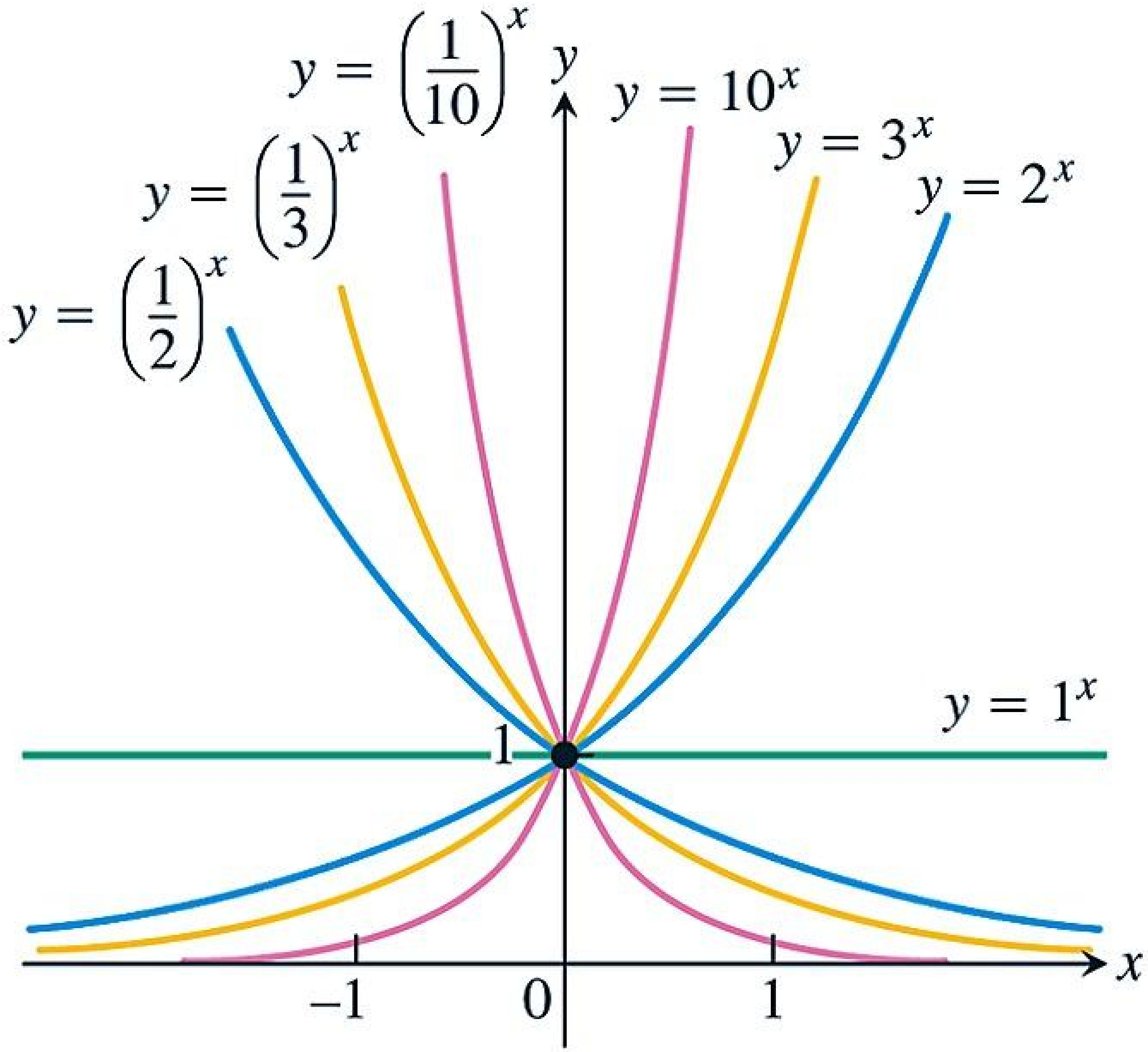


FIGURE 7.13 Exponential functions decrease if $0 < a < 1$ and increase if $a > 1$. As $x \to \infty$, we have $a^x \to 0$ if $0 < a < 1$ and $a^x \to \infty$ if $a > 1$. As $x \to -\infty$, we have $a^x \to \infty$ if $0 < a < 1$ and $a^x \to 0$ if $a > 1$.

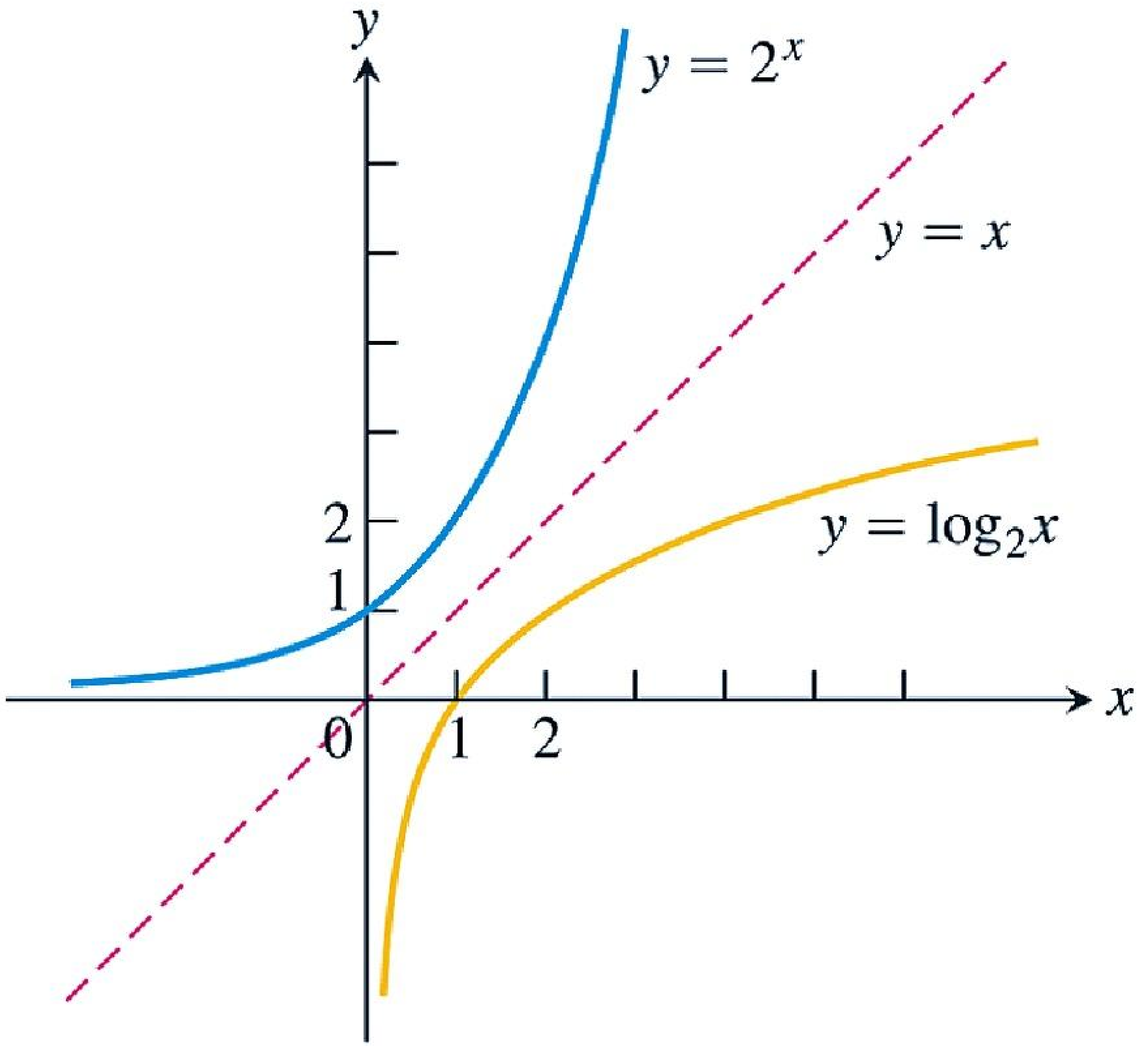


FIGURE 7.14 The graph of 2^x and its inverse, $\log_2 x$.

DEFINITION For any positive number $a \neq 1$,

$\log_a x$ is the inverse function of a^x.

Inverse Equations for a^x and $\log_a x$

$$a^{\log_a x} = x \qquad (x > 0)$$

$$\log_a(a^x) = x \qquad (\text{all } x)$$

TABLE 7.2 Rules for base a logarithms

For any numbers $x > 0$ and $y > 0$,

1. *Product Rule*:
$$\log_a xy = \log_a x + \log_a y$$
2. *Quotient Rule*:
$$\log_a \frac{x}{y} = \log_a x - \log_a y$$
3. *Reciprocal Rule*:
$$\log_a \frac{1}{y} = -\log_a y$$
4. *Power Rule*:
$$\log_a x^y = y \log_a x$$

$$\log_a x = \frac{\ln x}{\ln a}. \tag{5}$$

$$\boxed{\frac{d}{dx}(\log_a u) = \frac{1}{\ln a} \cdot \frac{1}{u}\frac{du}{dx}}$$

Pearson

Section 7.4

Exponential Change and Separable Differential Equations

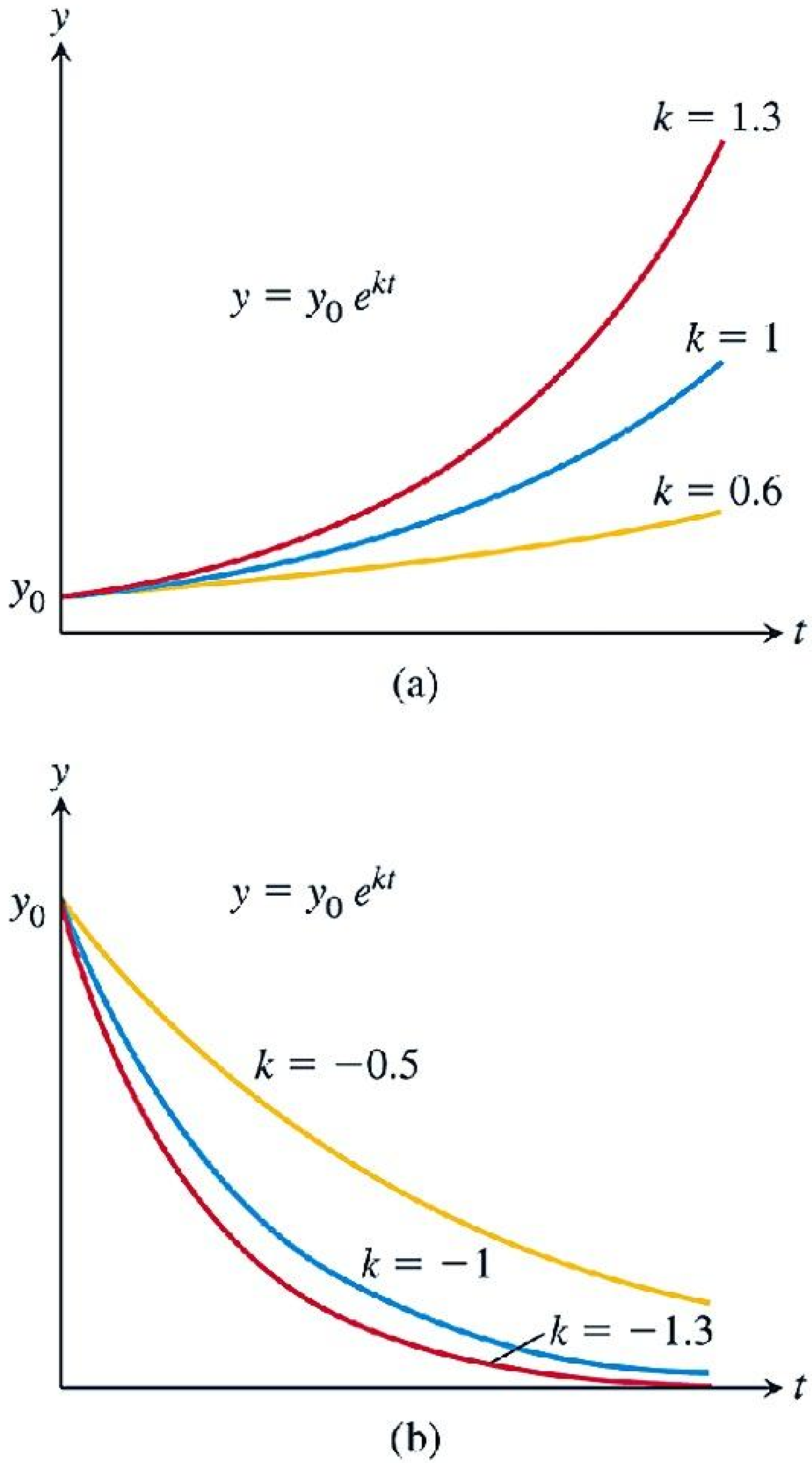


FIGURE 7.15 Graphs of (a) exponential growth and (b) exponential decay. As $|k|$ increases, the growth ($k > 0$) or decay ($k < 0$) intensifies.

The solution of the initial value problem

$$\frac{dy}{dt} = ky, \qquad y(0) = y_0$$

is

$$y = y_0 e^{kt}. \tag{2}$$

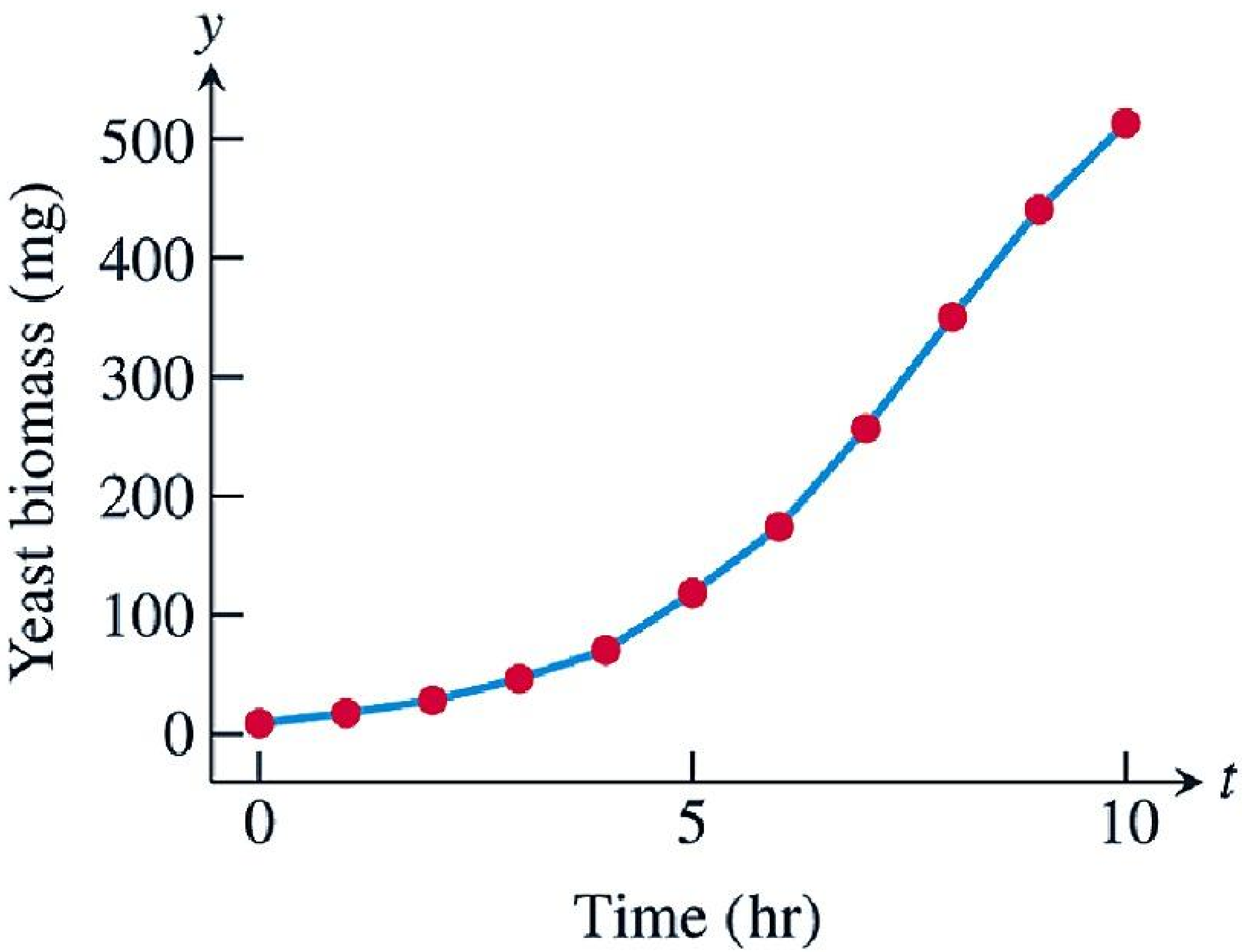


FIGURE 7.16 Graph of the growth of a yeast population over a 10-hour period, based on the data in Example 3.

Time (hr)	Yeast biomass (mg)
0	9.6
1	18.3
2	29.0
3	47.2
4	71.1
5	119.1
6	174.6
7	257.3
8	350.7
9	441.0
10	513.3

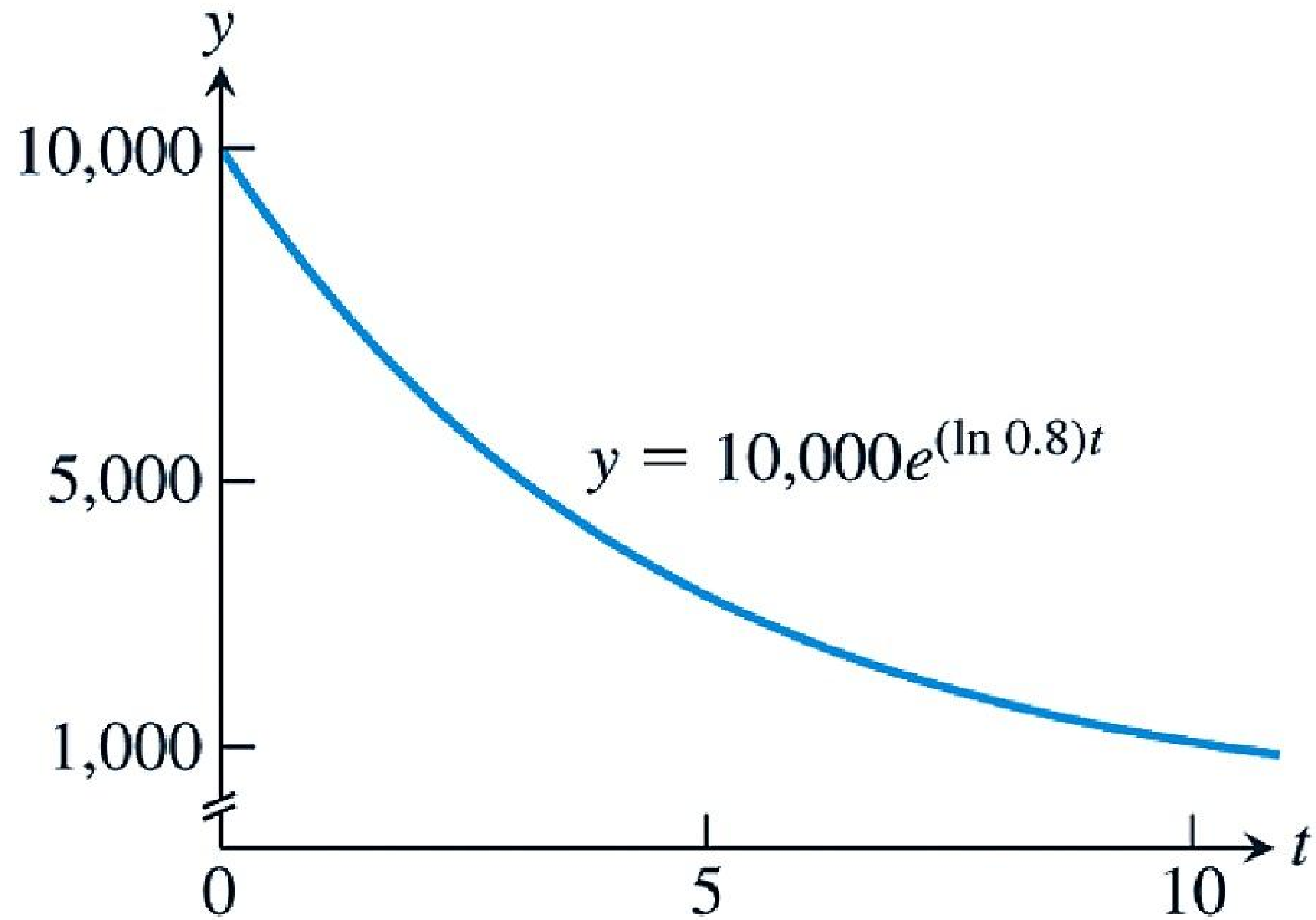


FIGURE 7.17 A graph of the number of people infected by a disease exhibits exponential decay (Example 4).

$$\text{Half-life} = \frac{\ln 2}{k} \tag{7}$$

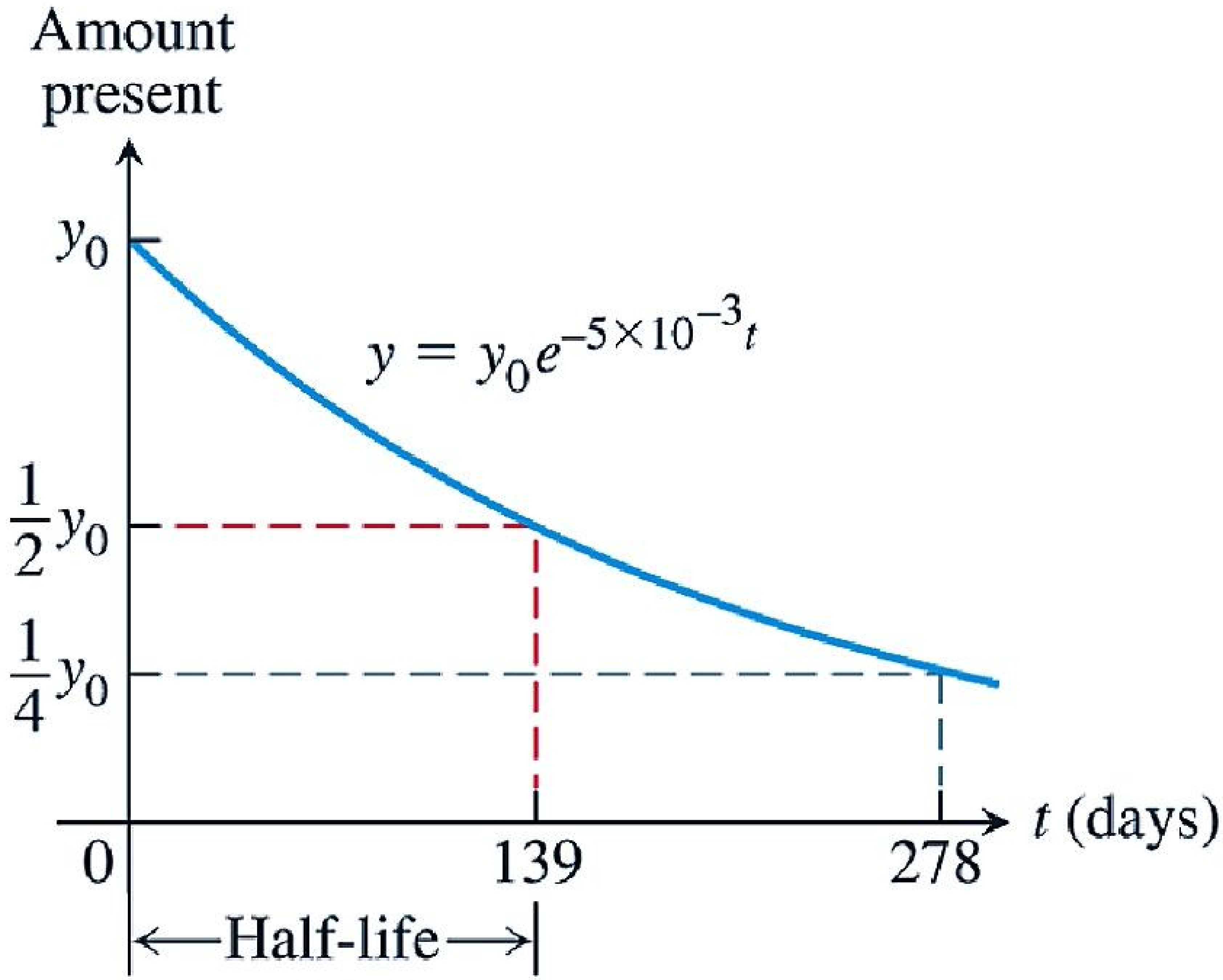


FIGURE 7.18 Amount of polonium-210 present at time t, where y_0 represents the number of radioactive atoms initially present.

Section 7.5

Indeterminate Forms and L′ Hôpital's Rule

THEOREM 5—L'Hôpital's Rule

Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

assuming that the limit on the right side of this equation exists.

Using L'Hôpital's Rule

To find

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

by l'Hôpital's Rule, continue to differentiate f and g, so long as we still get the form $0/0$ at $x = a$. But as soon as one or the other of these derivatives is different from zero at $x = a$ we stop differentiating. L'Hôpital's Rule does not apply when either the numerator or denominator has a finite nonzero limit.

If $\lim_{x \to a} \ln f(x) = L$, then

$$\lim_{x \to a} f(x) = \lim_{x \to a} e^{\ln f(x)} = e^L.$$

Here a may be either finite or infinite.

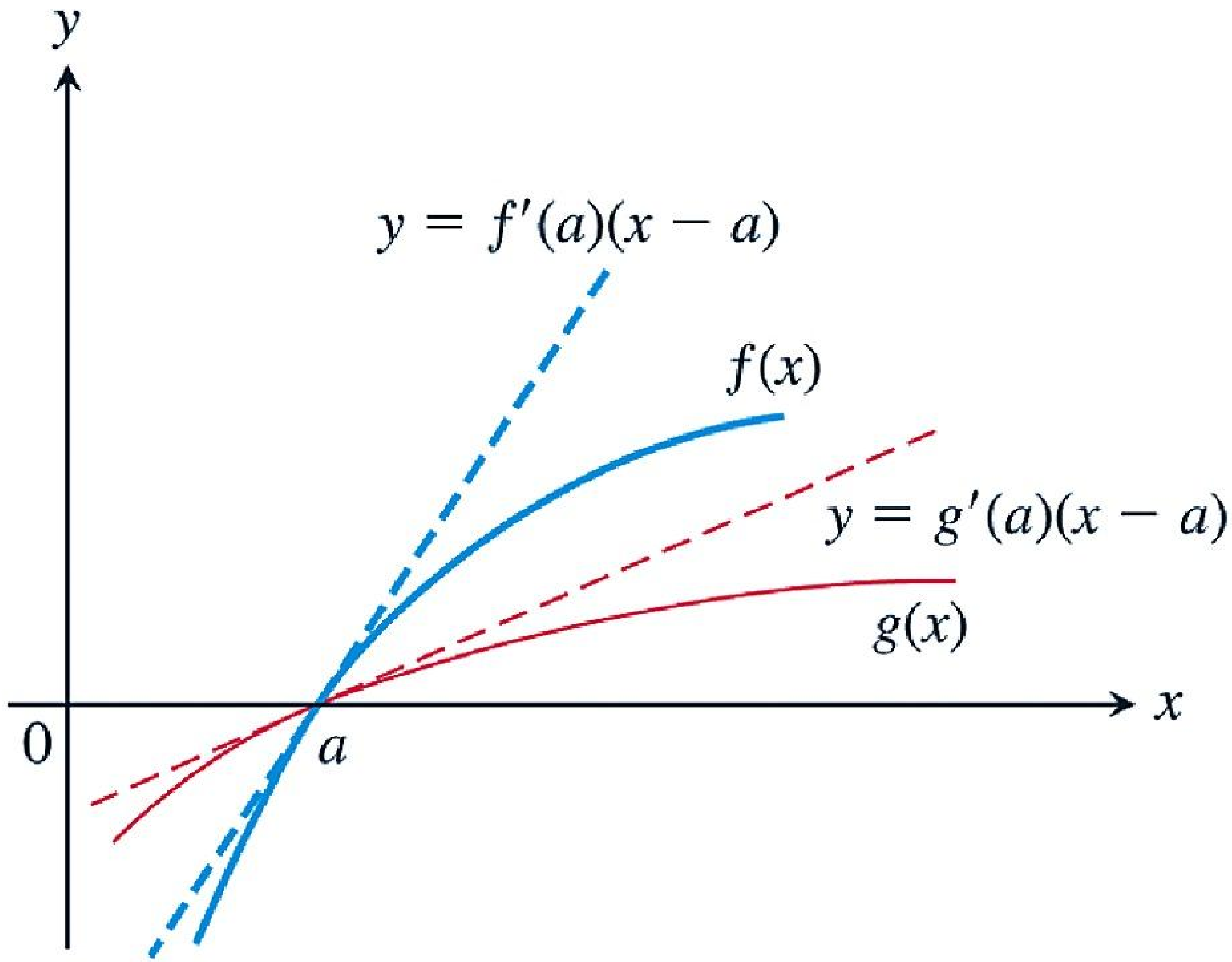


FIGURE 7.19 The two functions in l'Hôpital's Rule, graphed with their linear approximations at $x = a$.

THEOREM 6—Cauchy's Mean Value Theorem

Suppose functions f and g are continuous on $[a, b]$ and differentiable throughout (a, b) and also suppose $g'(x) \neq 0$ throughout (a, b). Then there exists a number c in (a, b) at which

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

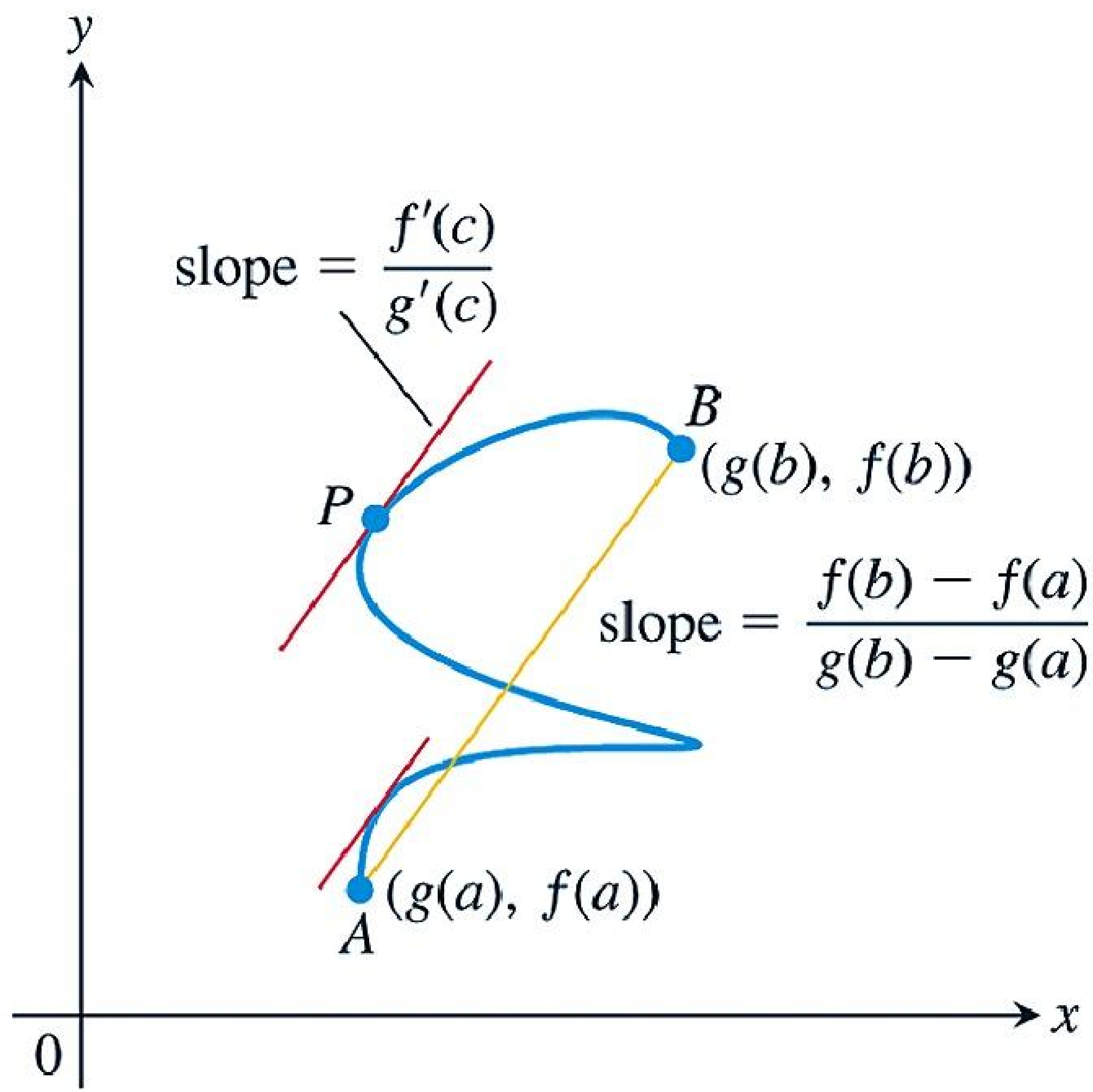


FIGURE 7.20 There is at least one point P on the curve C for which the slope of the tangent line to the curve at P is the same as the slope of the secant line joining the points $A(g(a), f(a))$ and $B(g(b), f(b))$.

Section 7.6

Inverse Trigonometric Functions

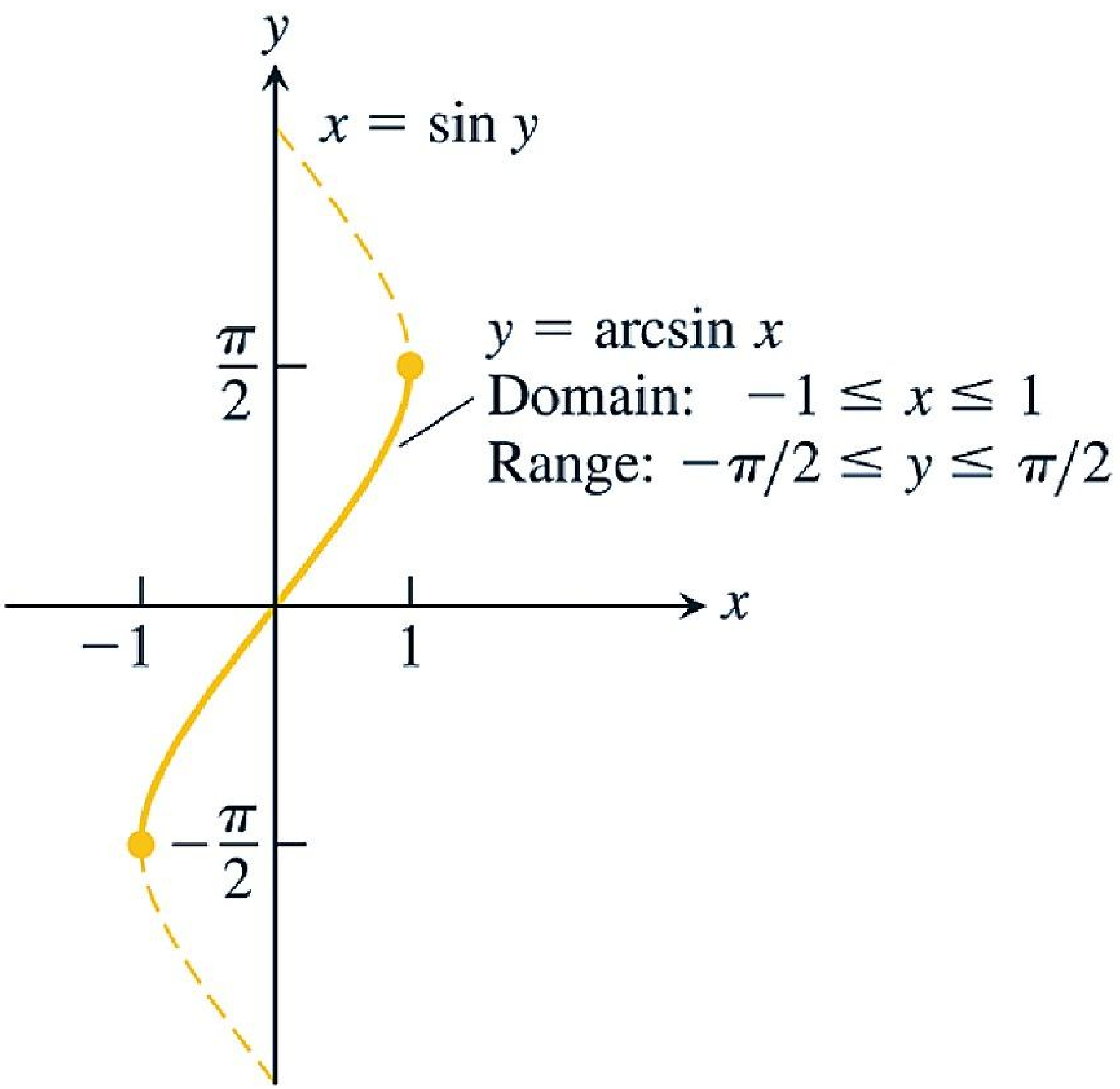
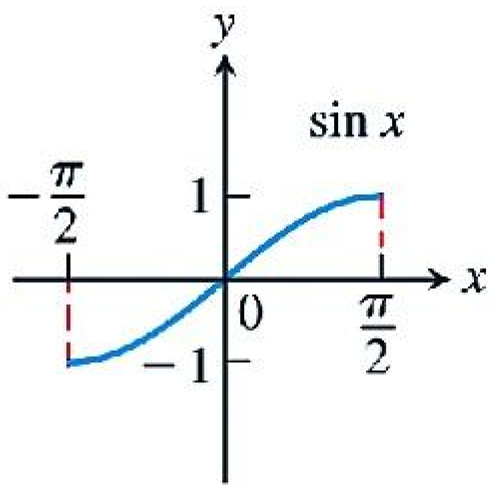
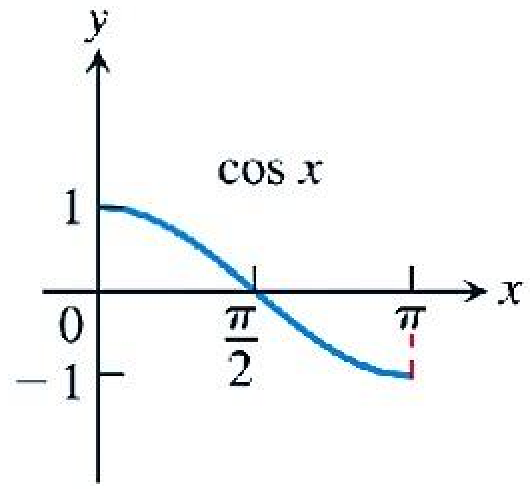


FIGURE 7.21 The graph of $y = \arcsin x$.

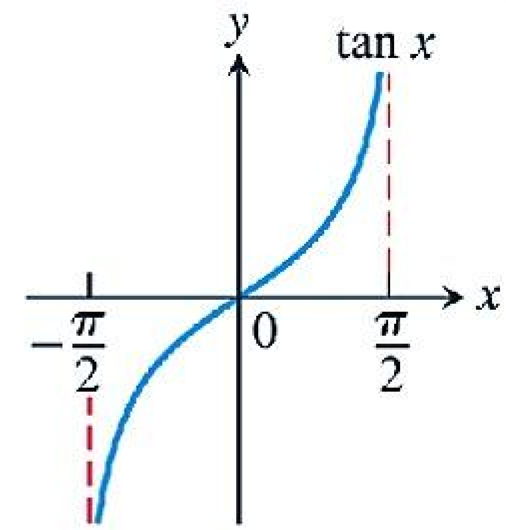
Domain restrictions that make the trigonometric functions one-to-one



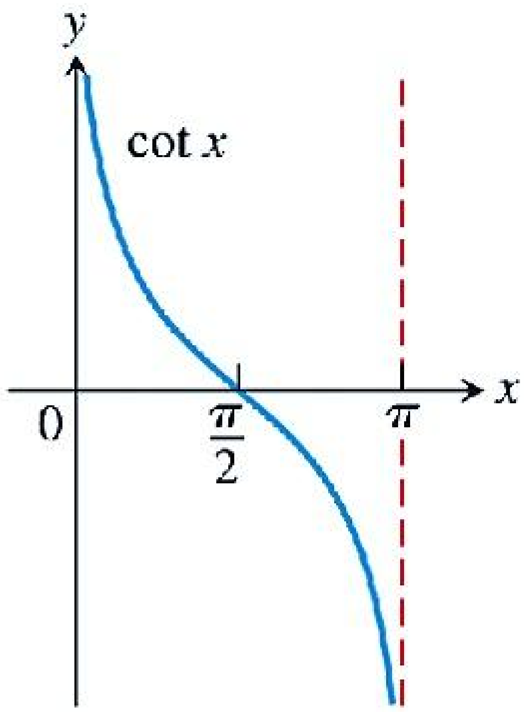
$y = \sin x$
Domain: $[-\pi/2, \pi/2]$
Range: $[-1, 1]$



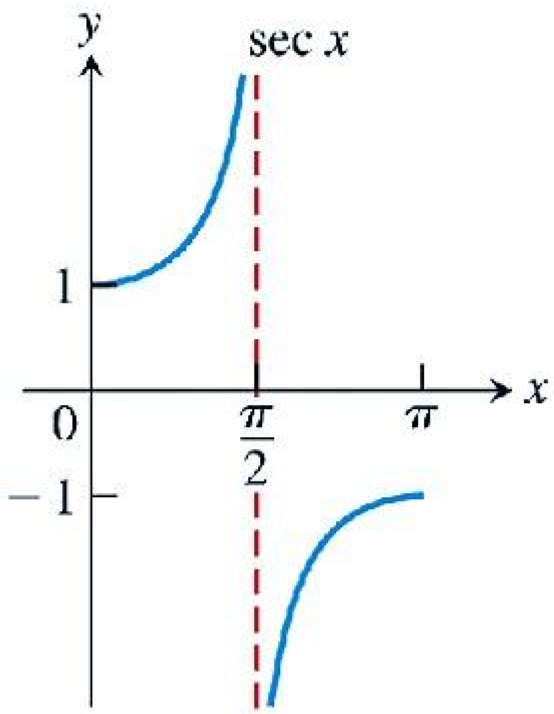
$y = \cos x$
Domain: $[0, \pi]$
Range: $[-1, 1]$



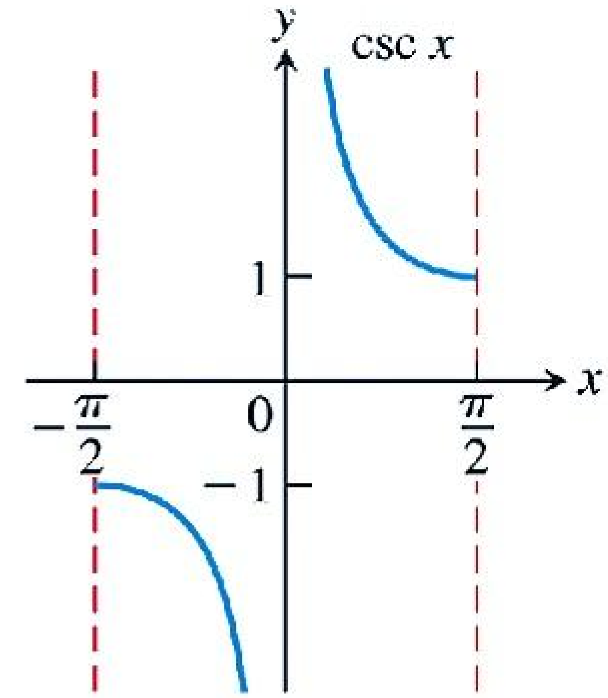
$y = \tan x$
Domain: $(-\pi/2, \pi/2)$
Range: $(-\infty, \infty)$



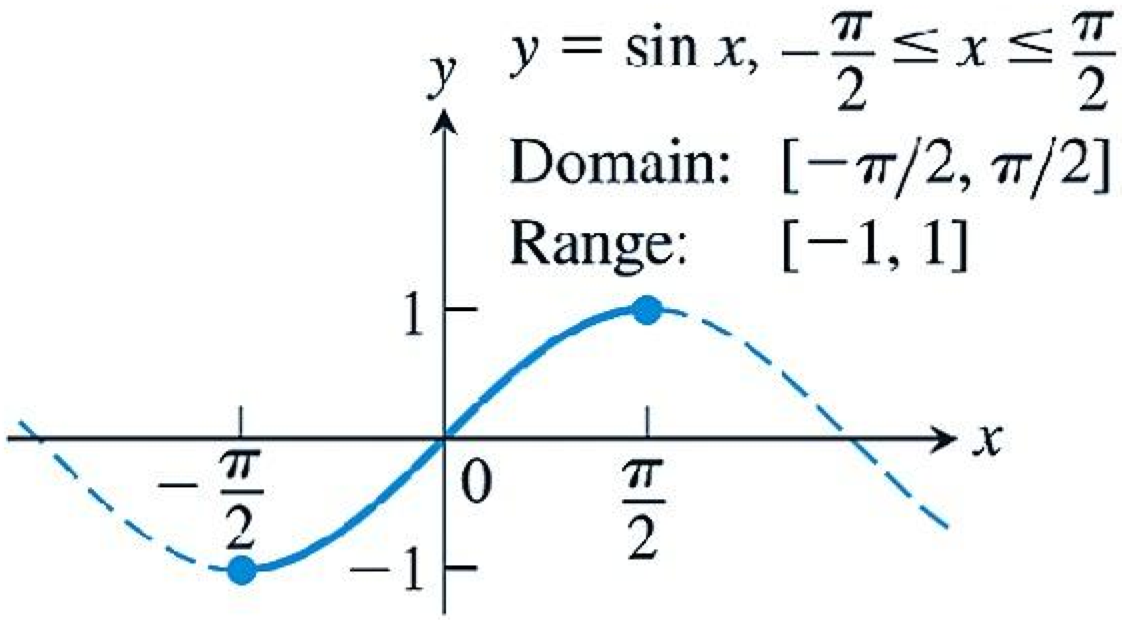
$y = \cot x$
Domain: $(0, \pi)$
Range: $(-\infty, \infty)$



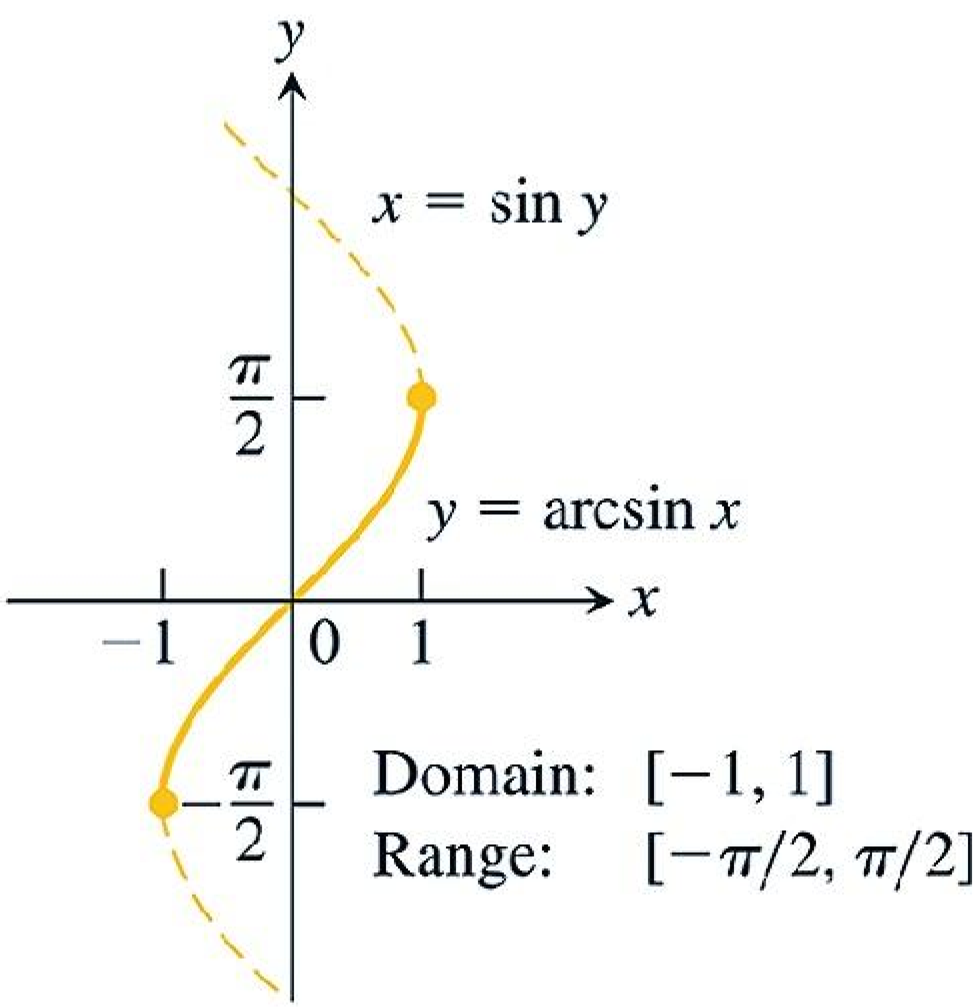
$y = \sec x$
Domain: $[0, \pi/2) \cup (\pi/2, \pi]$
Range: $(-\infty, -1] \cup [1, \infty)$



$y = \csc x$
Domain: $[-\pi/2, 0) \cup (0, \pi/2]$
Range: $(-\infty, -1] \cup [1, \infty)$



(a)



(b)

FIGURE 7.22 The graphs of (a) $y = \sin x, -\pi/2 \le x \le \pi/2$, and (b) its inverse, $y = \arcsin x$. The graph of arcsin x, obtained by reflection across the line $y = x$, is a portion of the curve $x = \sin y$.

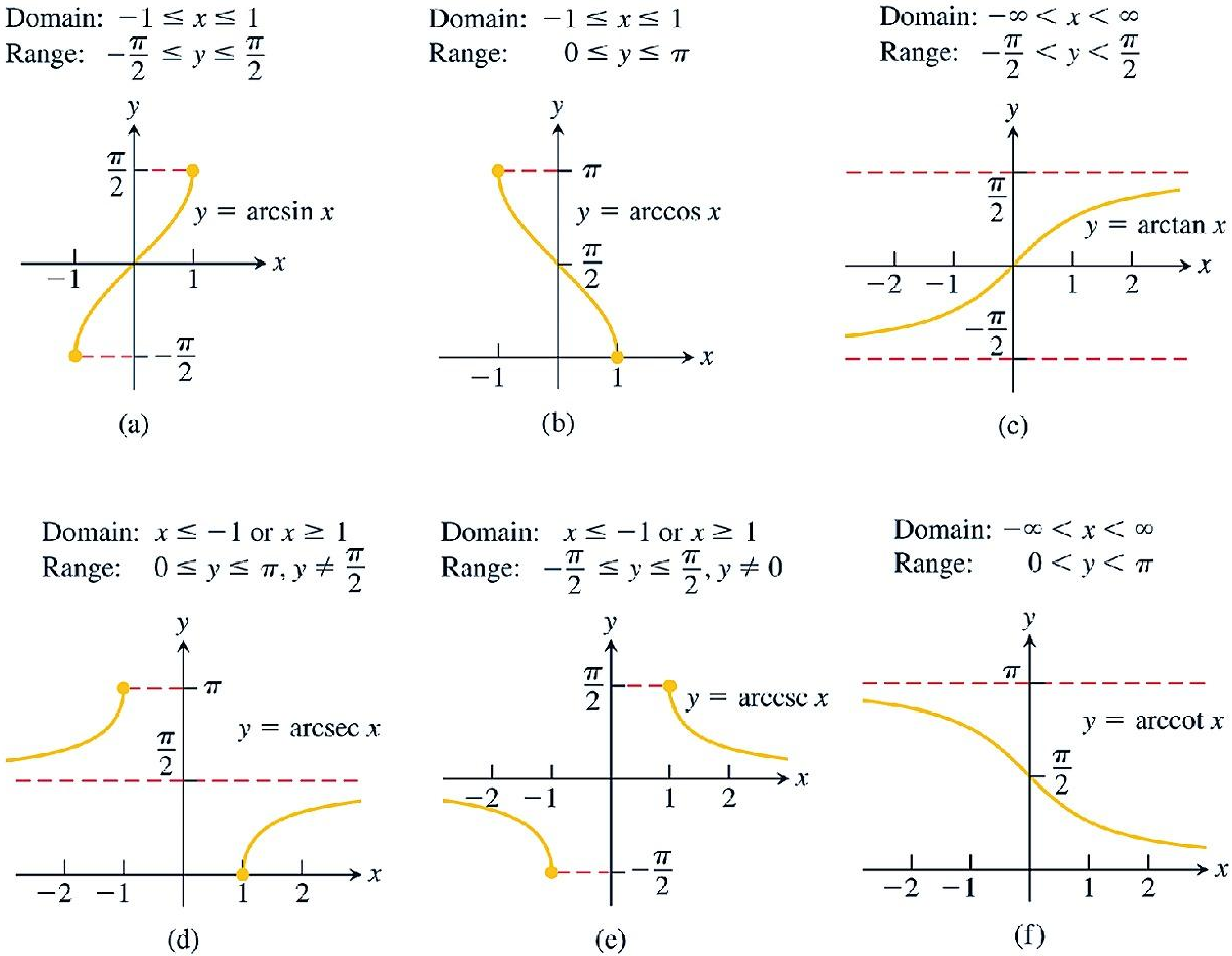


FIGURE 7.23 Graphs of the six basic inverse trigonometric functions.

DEFINITION

$\boldsymbol{y} = \textbf{arcsin}\, \boldsymbol{x}$ is the number in $[-\pi/2, \pi/2]$ for which $\sin y = x$.

$\boldsymbol{y} = \textbf{arccos}\, \boldsymbol{x}$ is the number in $[0, \pi]$ for which $\cos y = x$.

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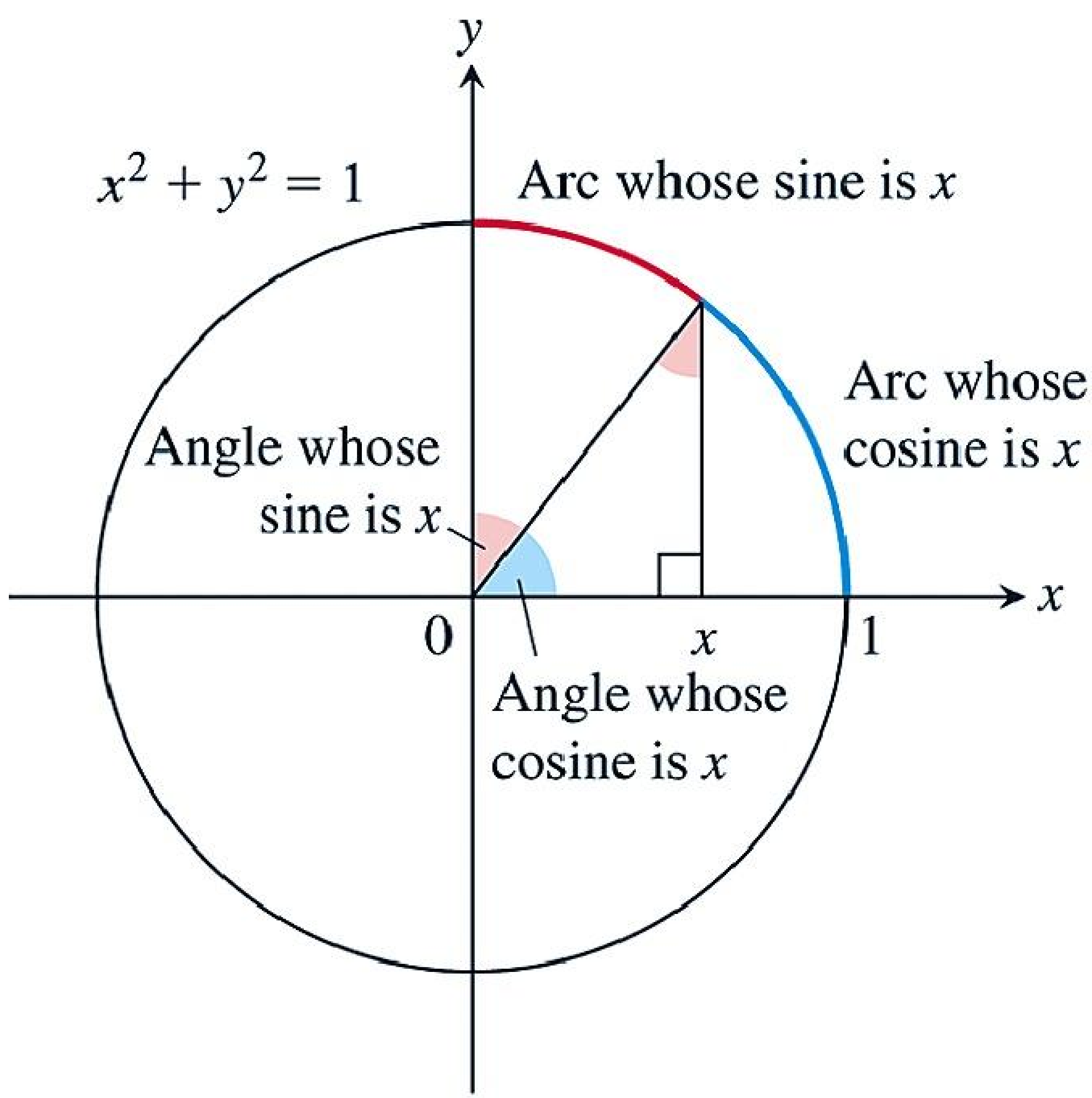
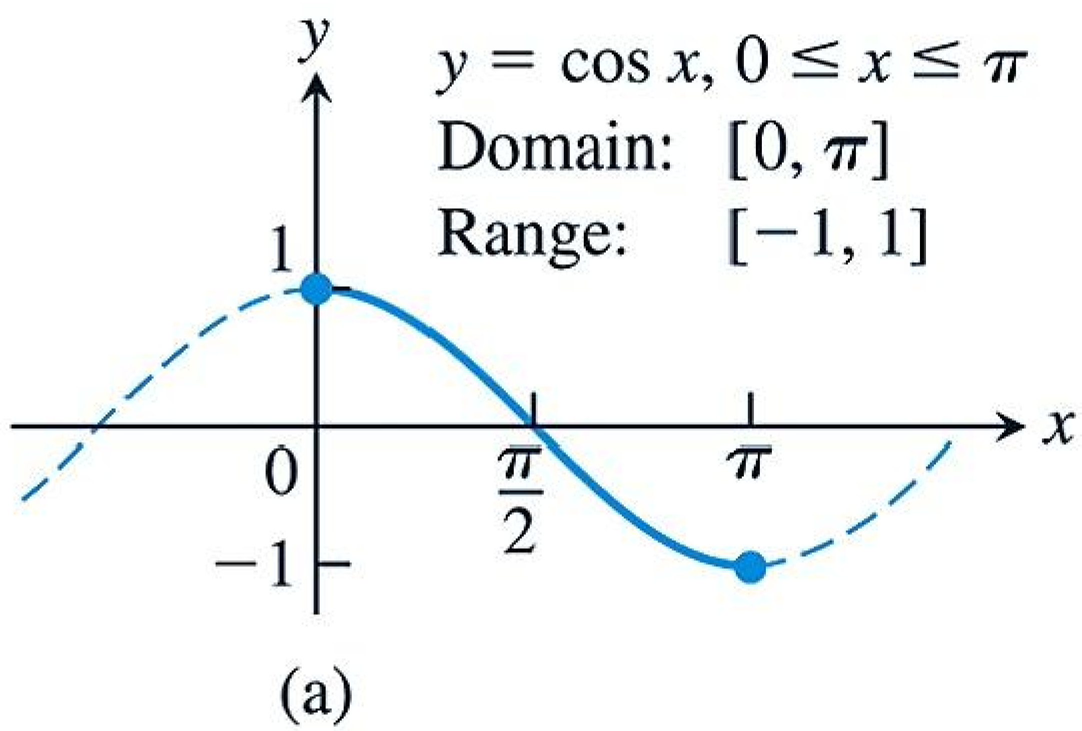
The "Arc" in Arcsine and Arccosine

For a unit circle and radian angles, the arc length equation $s = r\theta$ becomes $s = \theta$, so central angles and the arcs they subtend have the same measure. If $x = \sin y$, then, in addition to being the angle whose sine is x, y is also the length of the arc on the unit circle that subtends an angle whose sine is x. So we call y "the arc whose sine is x."





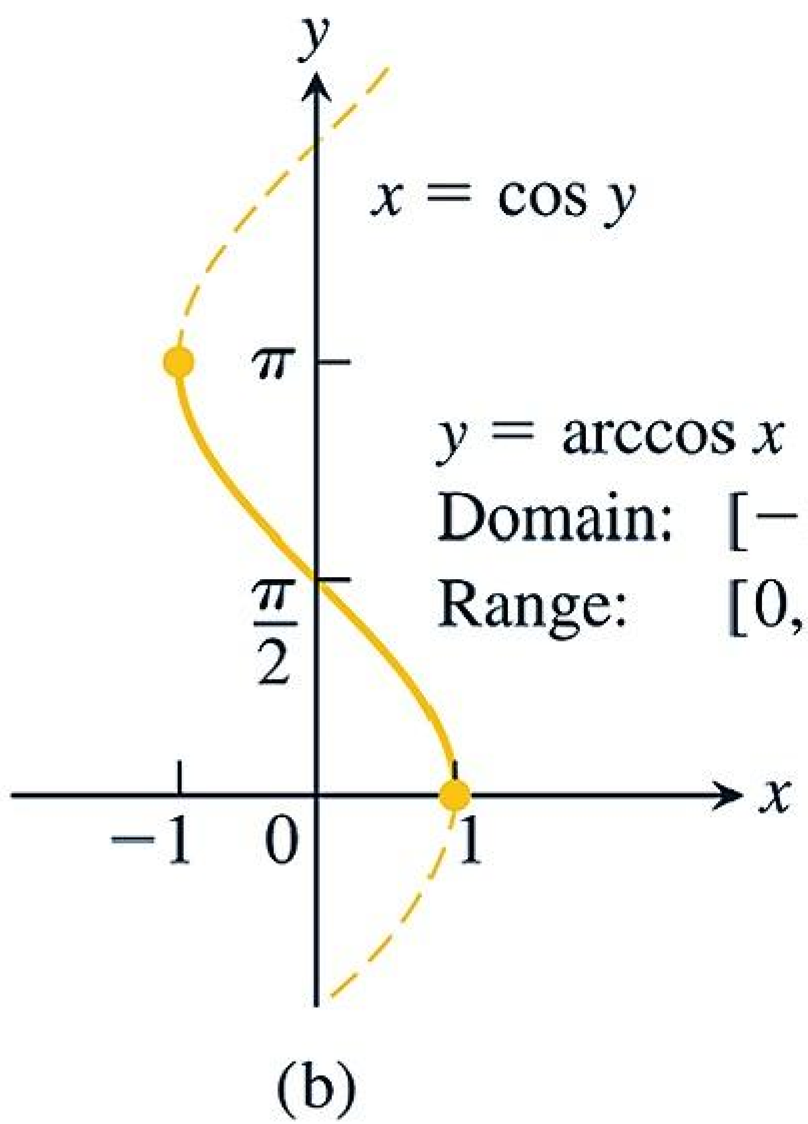


FIGURE 7.24 The graphs of (a) $y = \cos x$, $0 \le x \le \pi$, and (b) its inverse, $y = \arccos x$. The graph of $\arccos x$, obtained by reflection across the line $y = x$, is a portion of the curve $x = \cos y$.

x	$\arcsin x$	$\arccos x$
$\sqrt{3}/2$	$\pi/3$	$\pi/6$
$\sqrt{2}/2$	$\pi/4$	$\pi/4$
$1/2$	$\pi/6$	$\pi/3$
$-1/2$	$-\pi/6$	$2\pi/3$
$-\sqrt{2}/2$	$-\pi/4$	$3\pi/4$
$-\sqrt{3}/2$	$-\pi/3$	$5\pi/6$

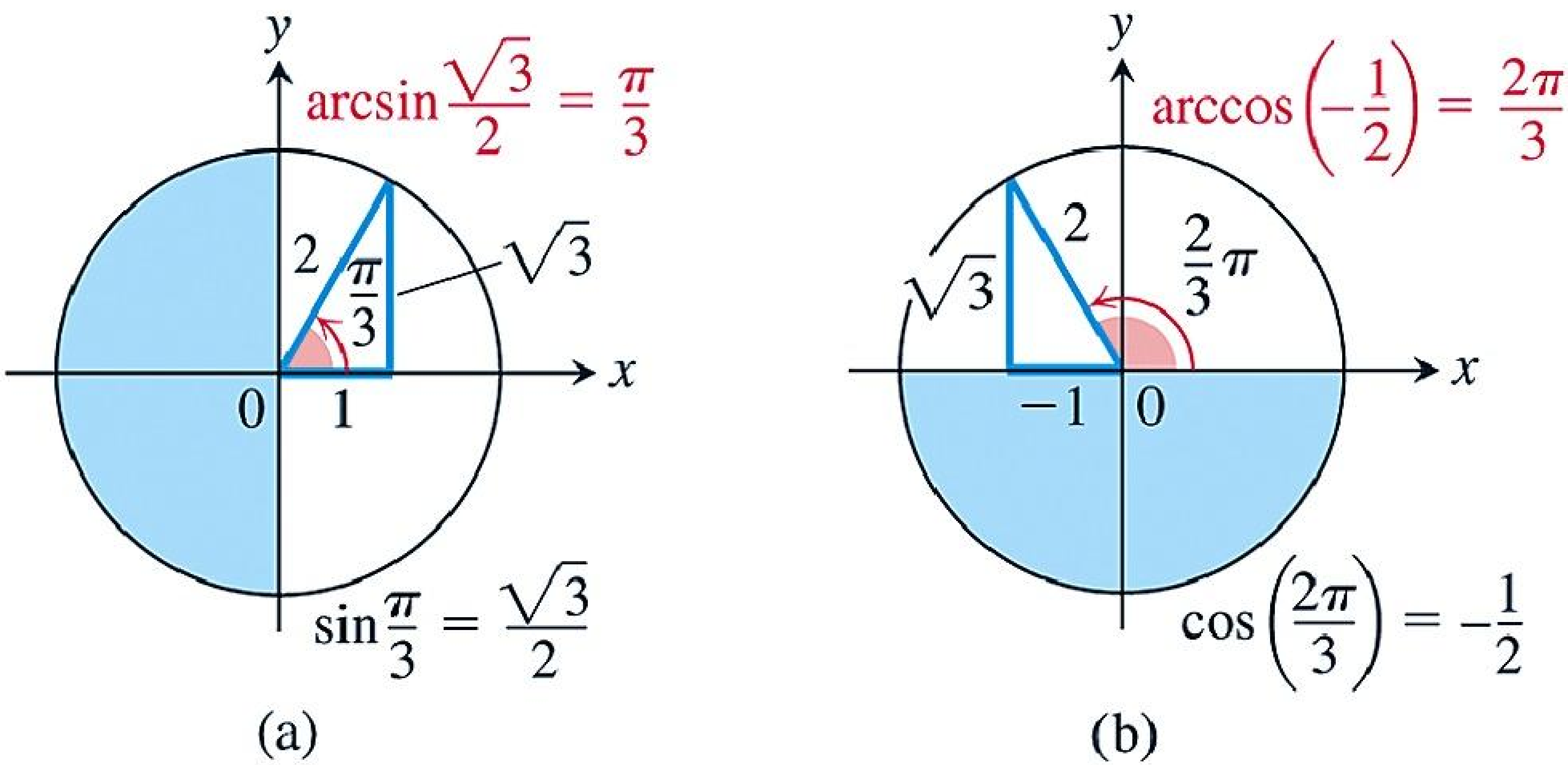


FIGURE 7.25 Values of the arcsine and arccosine functions (Example 1).

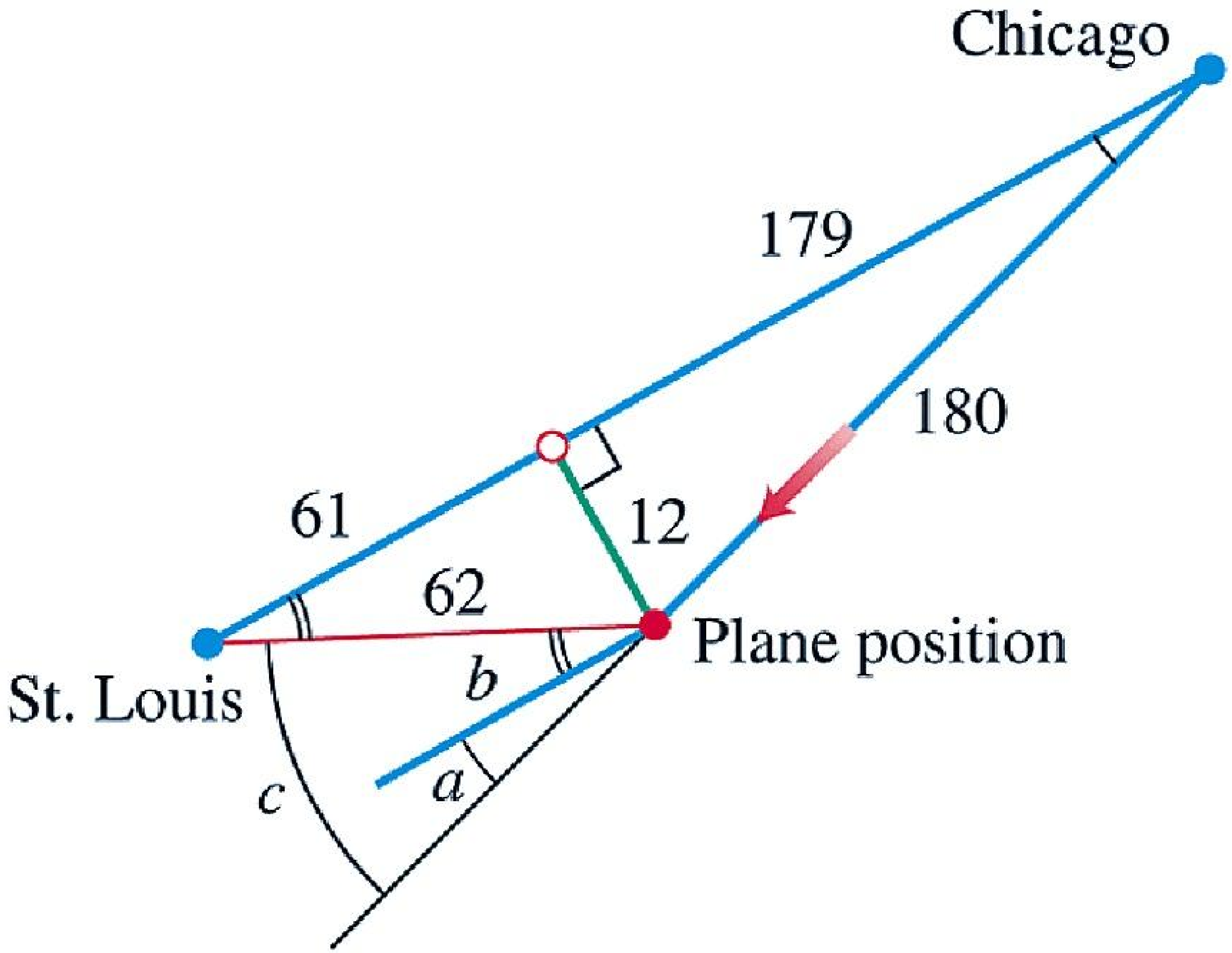


FIGURE 7.26 Diagram for drift correction (Example 2), with distances surrounded to the nearest mile (drawing not to scale).

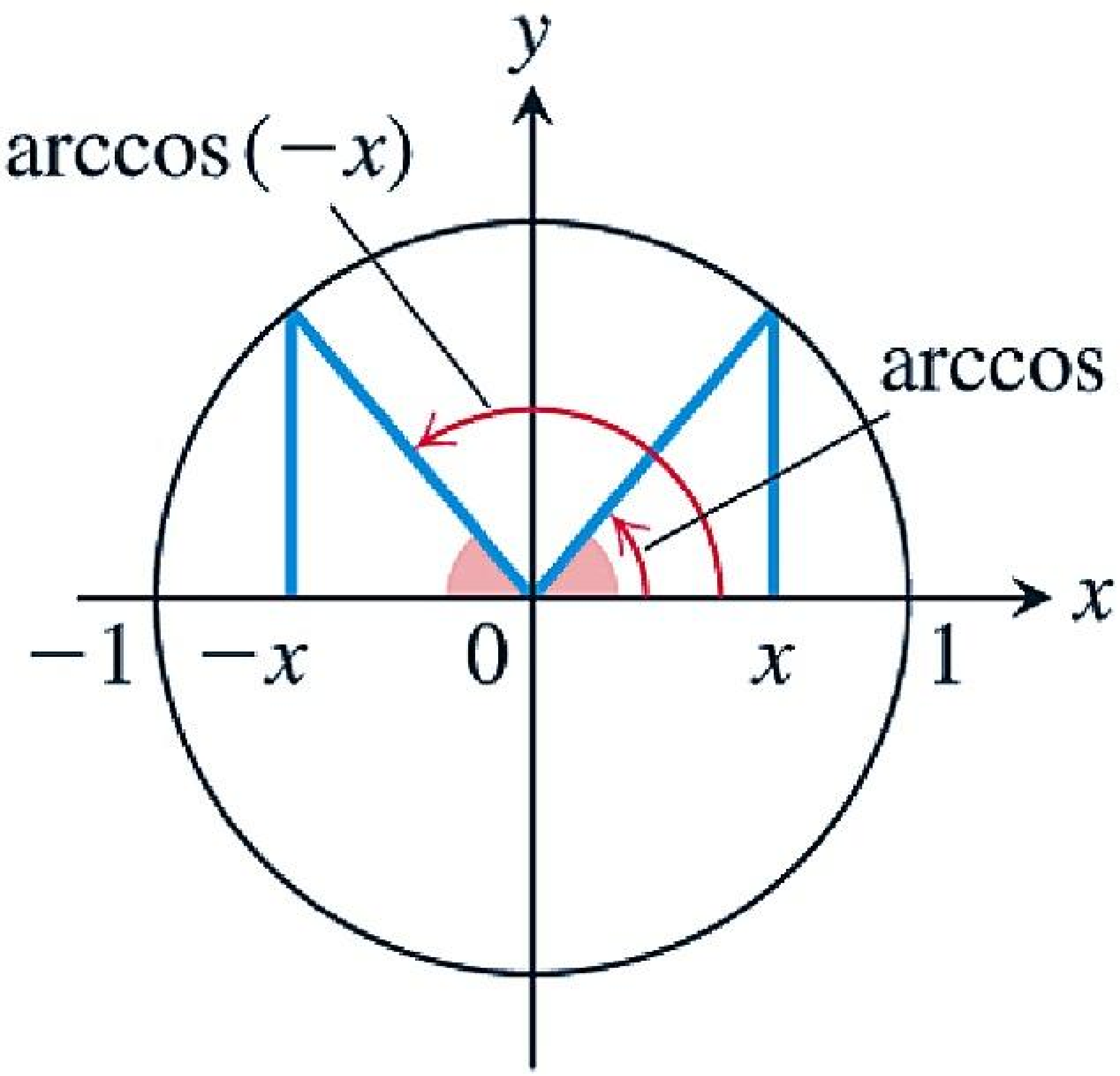


FIGURE 7.27 $\arccos x$ and $\arccos(-x)$ are supplementary angles (so their sum is π).

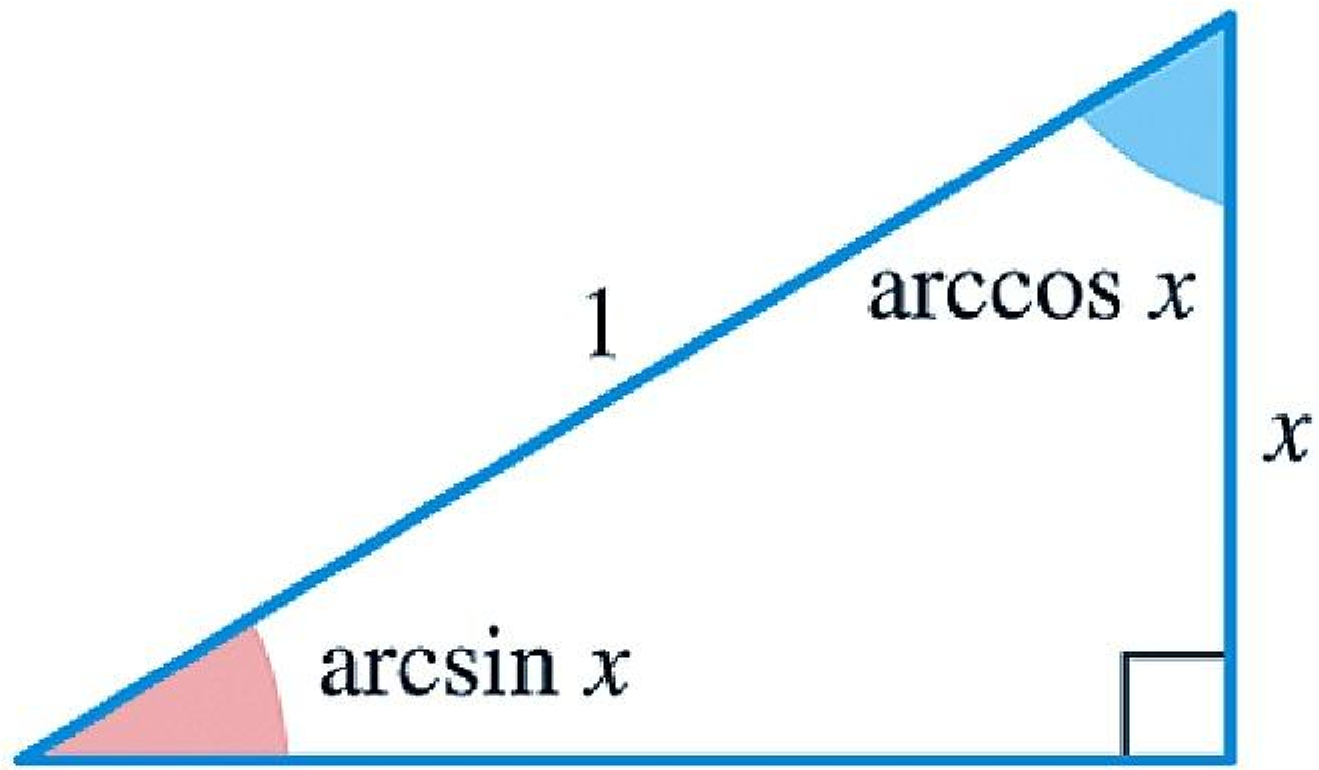


FIGURE 7.28 arcsin x and arccos x are complementary angles (so their sum is $\pi/2$).

DEFINITIONS

y = **arctan** $\boldsymbol{x}$ is the number in $(-\pi/2, \pi/2)$ for which $\tan y = x$.

y = **arccot** $\boldsymbol{x}$ is the number in $(0, \pi)$ for which $\cot y = x$.

y = **arcsec** $\boldsymbol{x}$ is the number in $[0, \pi/2) \cup (\pi/2, \pi]$ for which $\sec y = x$.

y = **arccsc** $\boldsymbol{x}$ is the number in $[-\pi/2, 0) \cup (0, \pi/2]$ for which $\csc y = x$.

Domain: $|x| \geq 1$
Range: $0 \leq y \leq \pi, y \neq \frac{\pi}{2}$

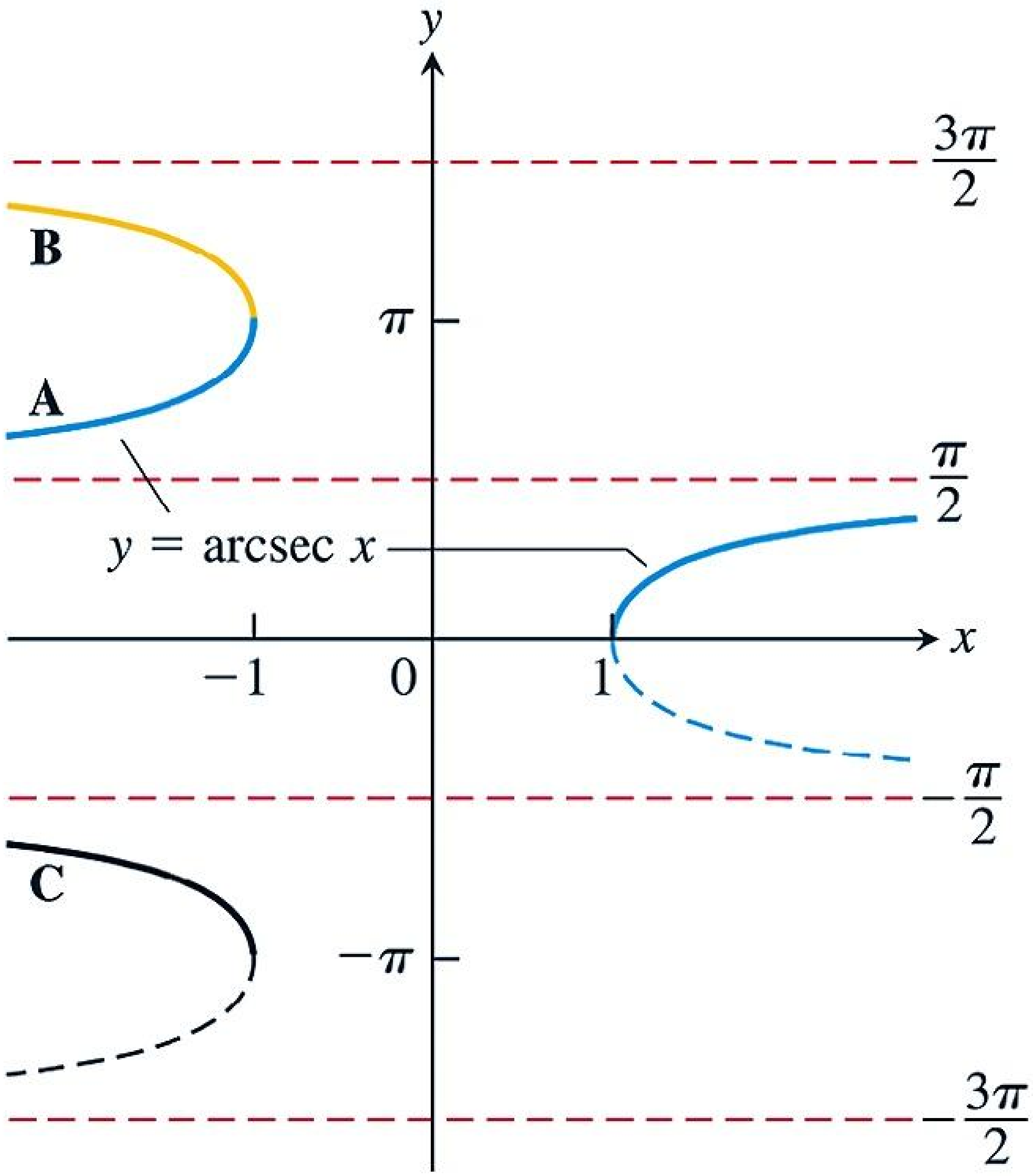


FIGURE 7.29 There are several logical choices for the left-hand branch of $y = \text{arcsec } x$. With choice **A**, $\text{arcsec } x = \arccos(1/x)$, a useful identity employed by many calculators.

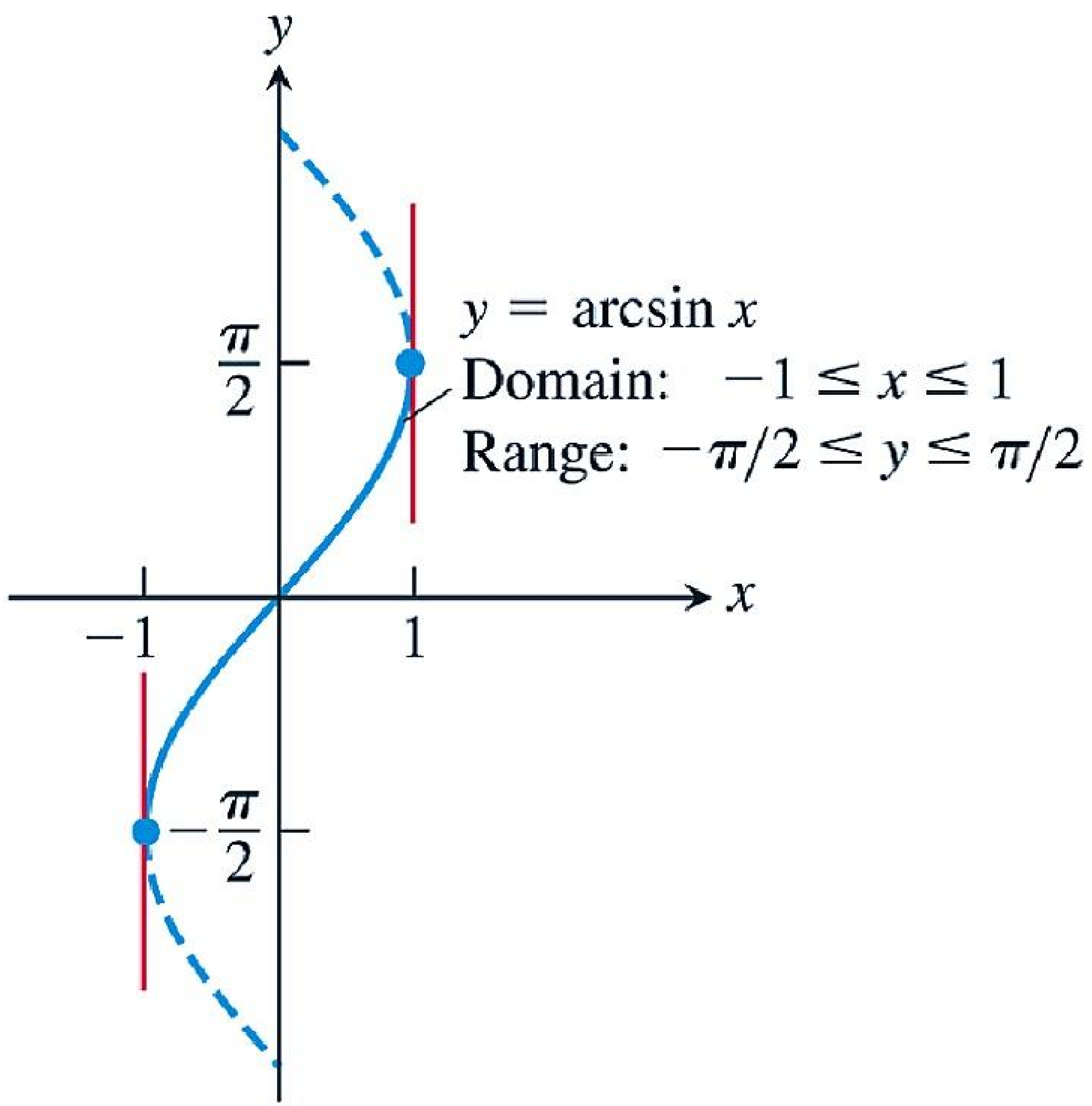


FIGURE 7.30 The graph of $y = \arcsin x$ has vertical tangents at $x = -1$ and $x = 1$.

$$\frac{d}{dx}(\arcsin u) = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}, \qquad |u| < 1.$$

$$\boxed{\frac{d}{dx}(\arctan u) = \frac{1}{1 + u^2}\frac{du}{dx}.}$$

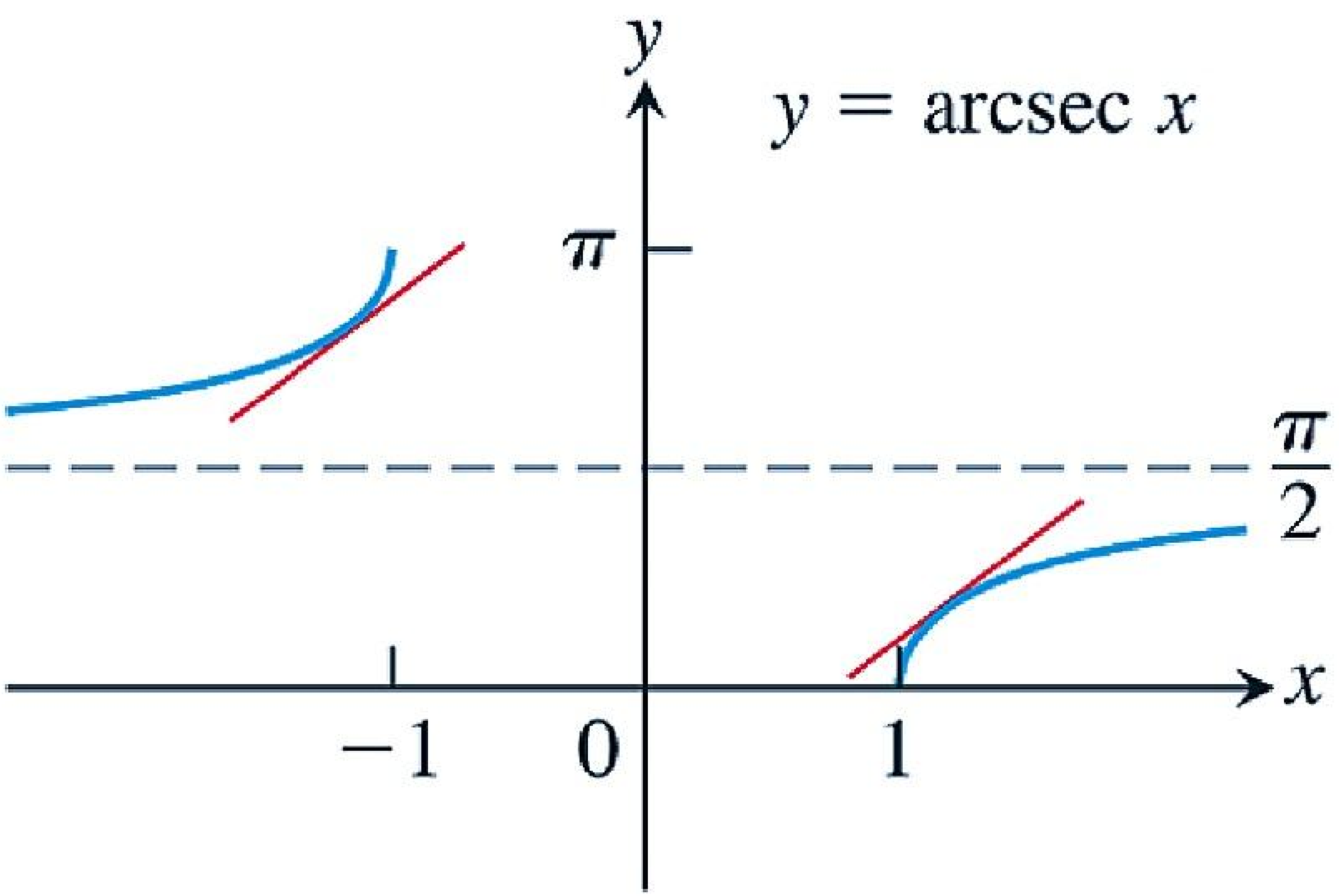


FIGURE 7.31 The slope of the curve $y = \operatorname{arcsec} x$ is positive for both $x < -1$ and $x > 1$.

Inverse Function–Inverse Cofunction Identities

$$\arccos x = \pi/2 - \arcsin x$$
$$\operatorname{arccot} x = \pi/2 - \arctan x$$
$$\operatorname{arccsc} x = \pi/2 - \operatorname{arcsec} x$$

TABLE 7.3 Derivatives of the inverse trigonometric functions

1. $\dfrac{d(\arcsin u)}{dx} = \dfrac{1}{\sqrt{1-u^2}}\dfrac{du}{dx}, \quad |u| < 1$

2. $\dfrac{d(\arccos u)}{dx} = -\dfrac{1}{\sqrt{1-u^2}}\dfrac{du}{dx}, \quad |u| < 1$

3. $\dfrac{d(\arctan u)}{dx} = \dfrac{1}{1+u^2}\dfrac{du}{dx}$

4. $\dfrac{d(\operatorname{arccot} u)}{dx} = -\dfrac{1}{1+u^2}\dfrac{du}{dx}$

5. $\dfrac{d(\operatorname{arcsec} u)}{dx} = \dfrac{1}{|u|\sqrt{u^2-1}}\dfrac{du}{dx}, \quad |u| > 1$

6. $\dfrac{d(\operatorname{arccsc} u)}{dx} = -\dfrac{1}{|u|\sqrt{u^2-1}}\dfrac{du}{dx}, \quad |u| > 1$

TABLE 7.4 **Integrals evaluated with inverse trigonometric functions**

The following formulas hold for any constant $a > 0$.

1. $\displaystyle\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$ (Valid for $u^2 < a^2$)

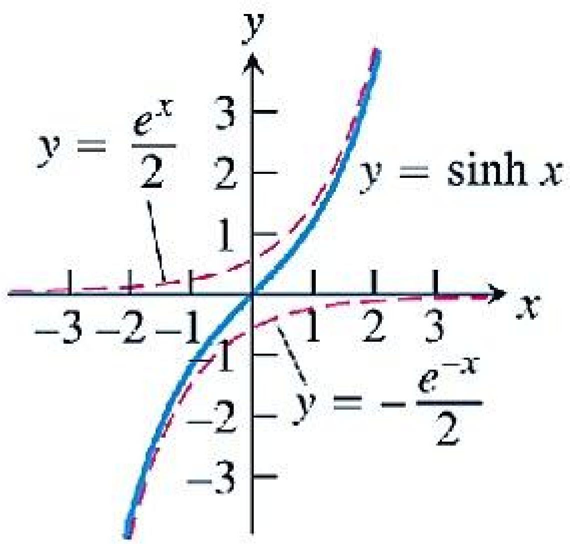
2. $\displaystyle\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$ (Valid for all u)

3. $\displaystyle\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$ (Valid for $|u| > a > 0$)

Section 7.7

Hyperbolic Functions

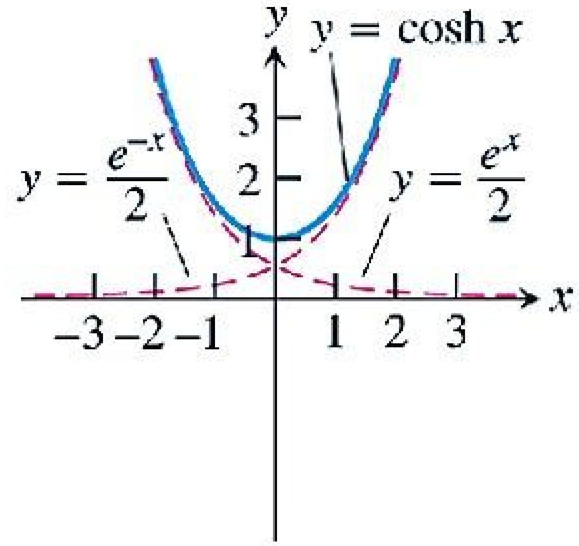
TABLE 7.5 The six basic hyperbolic functions



(a)

Hyperbolic sine:

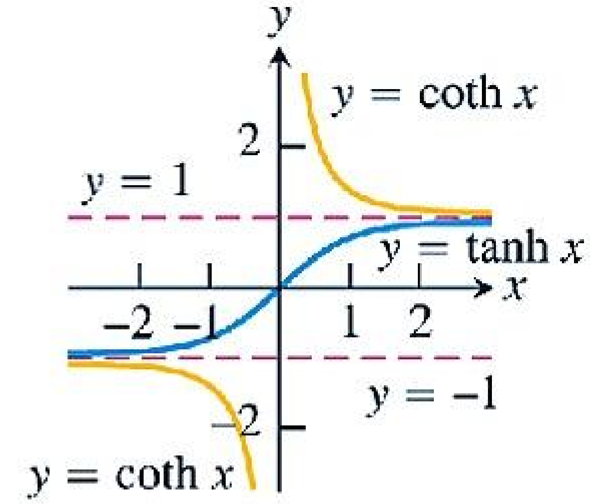
$$\sinh x = \frac{e^x - e^{-x}}{2}$$



(b)

Hyperbolic cosine:

$$\cosh x = \frac{e^x + e^{-x}}{2}$$



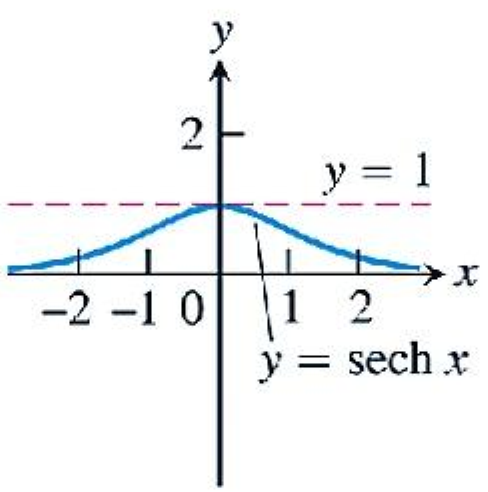
(c)

Hyperbolic tangent:

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Hyperbolic cotangent:

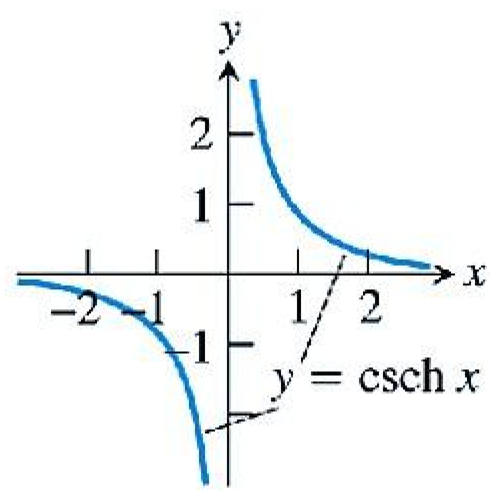
$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$



(d)

Hyperbolic secant:

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$



(e)

Hyperbolic cosecant:

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

TABLE 7.6 Identities for hyperbolic functions

$$\cosh^2 x - \sinh^2 x = 1$$

$$\sinh 2x = 2 \sinh x \cosh x$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

$$\cosh^2 x = \frac{\cosh 2x + 1}{2}$$

$$\sinh^2 x = \frac{\cosh 2x - 1}{2}$$

$$\tanh^2 x = 1 - \operatorname{sech}^2 x$$

$$\coth^2 x = 1 + \operatorname{csch}^2 x$$

TABLE 7.7 Derivatives of hyperbolic functions

$$\frac{d}{dx}(\sinh u) = \cosh u \frac{du}{dx}$$

$$\frac{d}{dx}(\cosh u) = \sinh u \frac{du}{dx}$$

$$\frac{d}{dx}(\tanh u) = \operatorname{sech}^2 u \frac{du}{dx}$$

$$\frac{d}{dx}(\coth u) = -\operatorname{csch}^2 u \frac{du}{dx}$$

$$\frac{d}{dx}(\operatorname{sech} u) = -\operatorname{sech} u \tanh u \frac{du}{dx}$$

$$\frac{d}{dx}(\operatorname{csch} u) = -\operatorname{csch} u \coth u \frac{du}{dx}$$

TABLE 7.8 **Integral formulas for hyperbolic functions**

$$\int \sinh u\, du = \cosh u + C$$

$$\int \cosh u\, du = \sinh u + C$$

$$\int \operatorname{sech}^2 u\, du = \tanh u + C$$

$$\int \operatorname{csch}^2 u\, du = -\coth u + C$$

$$\int \operatorname{sech} u \tanh u\, du = -\operatorname{sech} u + C$$

$$\int \operatorname{csch} u \coth u\, du = -\operatorname{csch} u + C$$

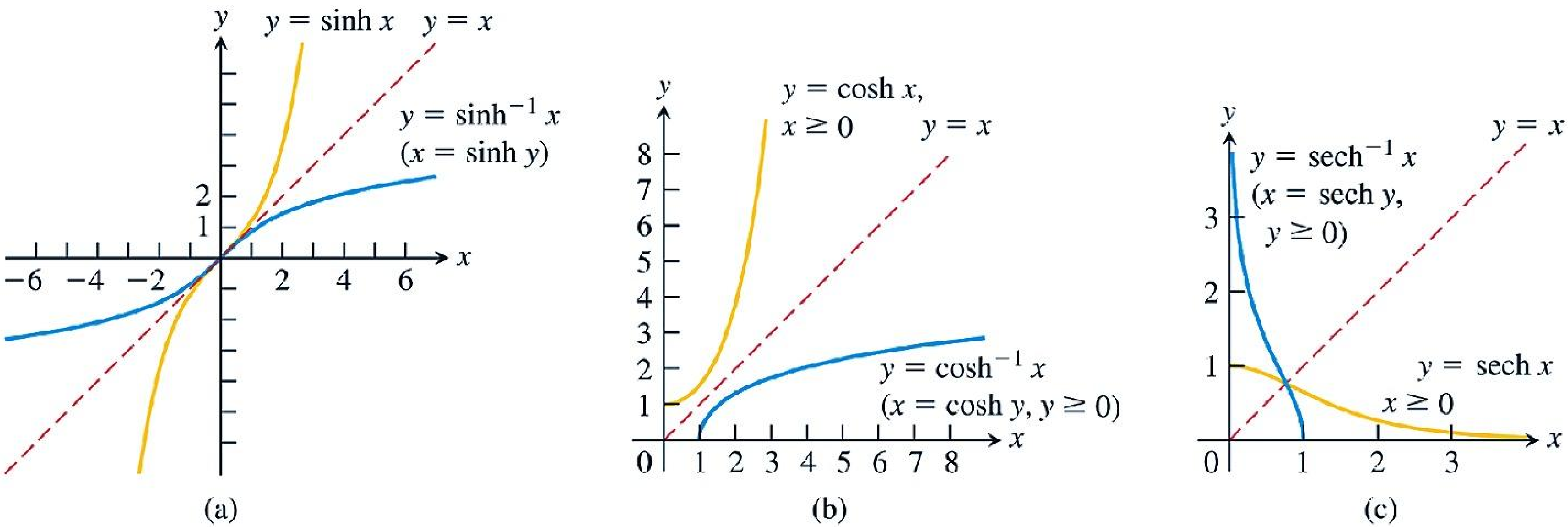


FIGURE 7.32 The graphs of the inverse hyperbolic sine, cosine, and secant of x. Notice the symmetries about the line $y = x$.

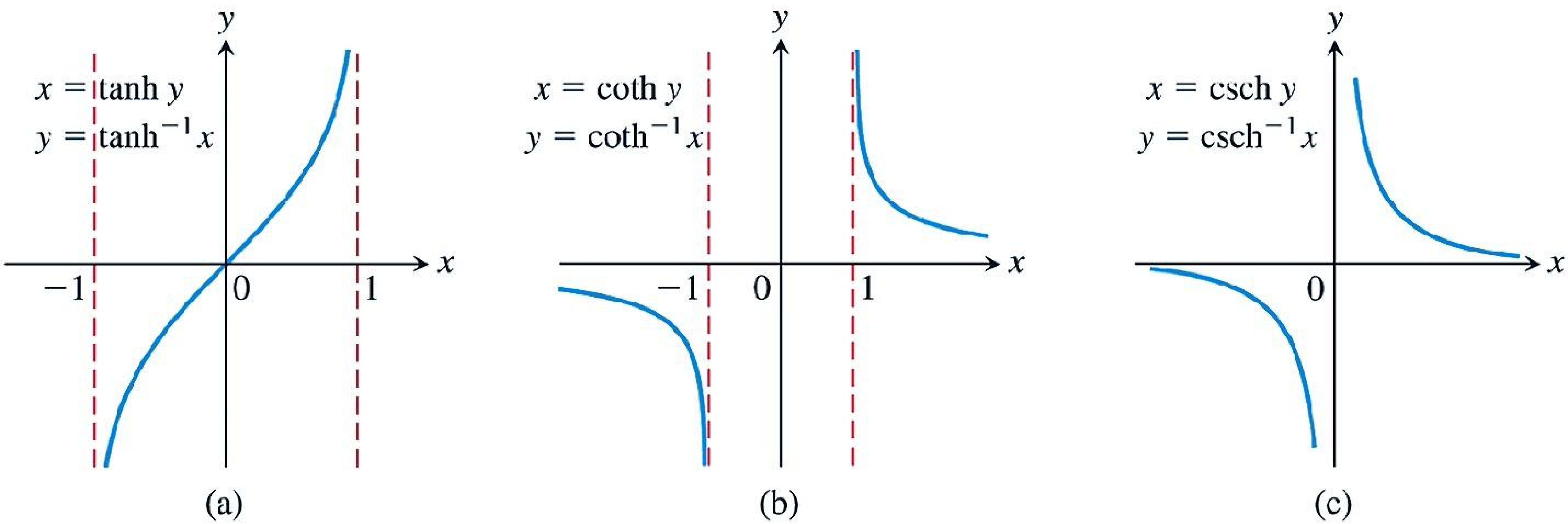


FIGURE 7.33 The graphs of the inverse hyperbolic tangent, cotangent, and cosecant of x.

TABLE 7.9 Identities for inverse hyperbolic functions

$$\operatorname{sech}^{-1} x = \cosh^{-1} \frac{1}{x}$$

$$\operatorname{csch}^{-1} x = \sinh^{-1} \frac{1}{x}$$

$$\coth^{-1} x = \tanh^{-1} \frac{1}{x}$$

TABLE 7.10 Derivatives of inverse hyperbolic functions

$$\frac{d(\sinh^{-1} u)}{dx} = \frac{1}{\sqrt{1 + u^2}} \frac{du}{dx}$$

$$\frac{d(\cosh^{-1} u)}{dx} = \frac{1}{\sqrt{u^2 - 1}} \frac{du}{dx}, \qquad u > 1$$

$$\frac{d(\tanh^{-1} u)}{dx} = \frac{1}{1 - u^2} \frac{du}{dx}, \qquad |u| < 1$$

$$\frac{d(\coth^{-1} u)}{dx} = \frac{1}{1 - u^2} \frac{du}{dx}, \qquad |u| > 1$$

$$\frac{d(\operatorname{sech}^{-1} u)}{dx} = -\frac{1}{u\sqrt{1 - u^2}} \frac{du}{dx}, \qquad 0 < u < 1$$

$$\frac{d(\operatorname{csch}^{-1} u)}{dx} = -\frac{1}{|u|\sqrt{1 + u^2}} \frac{du}{dx}, \qquad u \neq 0$$

TABLE 7.11 Integrals leading to inverse hyperbolic functions

1. $\displaystyle\int \frac{du}{\sqrt{a^2+u^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + C, \qquad a > 0$

2. $\displaystyle\int \frac{du}{\sqrt{u^2-a^2}} = \cosh^{-1}\left(\frac{u}{a}\right) + C, \qquad u > a > 0$

3. $\displaystyle\int \frac{du}{a^2-u^2} = \begin{cases} \dfrac{1}{a}\tanh^{-1}\left(\dfrac{u}{a}\right) + C, & u^2 < a^2 \\ \dfrac{1}{a}\coth^{-1}\left(\dfrac{u}{a}\right) + C, & u^2 > a^2 \end{cases}$

4. $\displaystyle\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\operatorname{sech}^{-1}\left(\frac{u}{a}\right) + C, \qquad 0 < u < a$

5. $\displaystyle\int \frac{du}{u\sqrt{a^2+u^2}} = -\frac{1}{a}\operatorname{csch}^{-1}\left|\frac{u}{a}\right| + C, \qquad u \neq 0 \text{ and } a > 0$

Section 7.8

Relative Rates of Growth

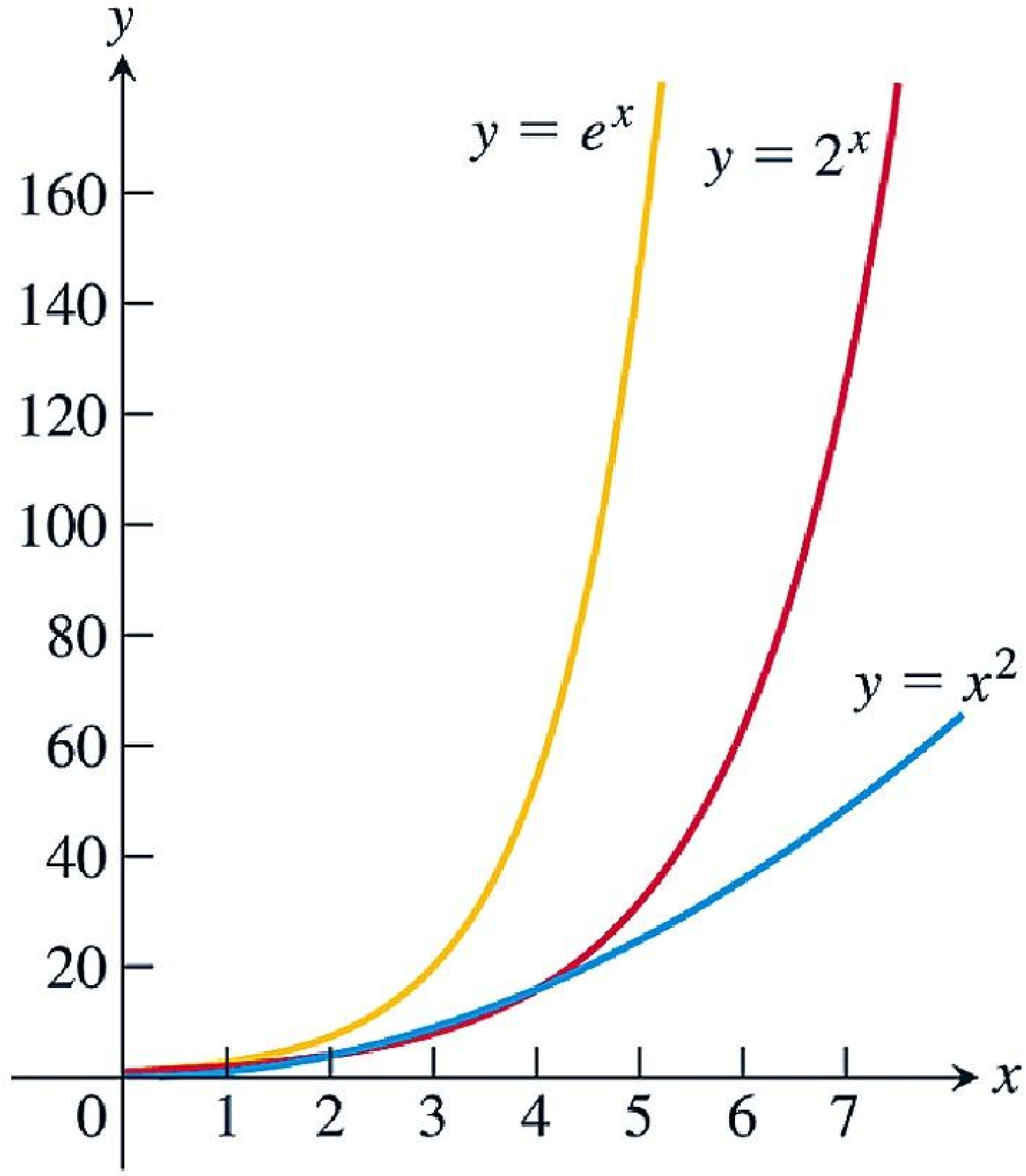


FIGURE 7.34 The graphs of e^x, 2^x, and x^2.

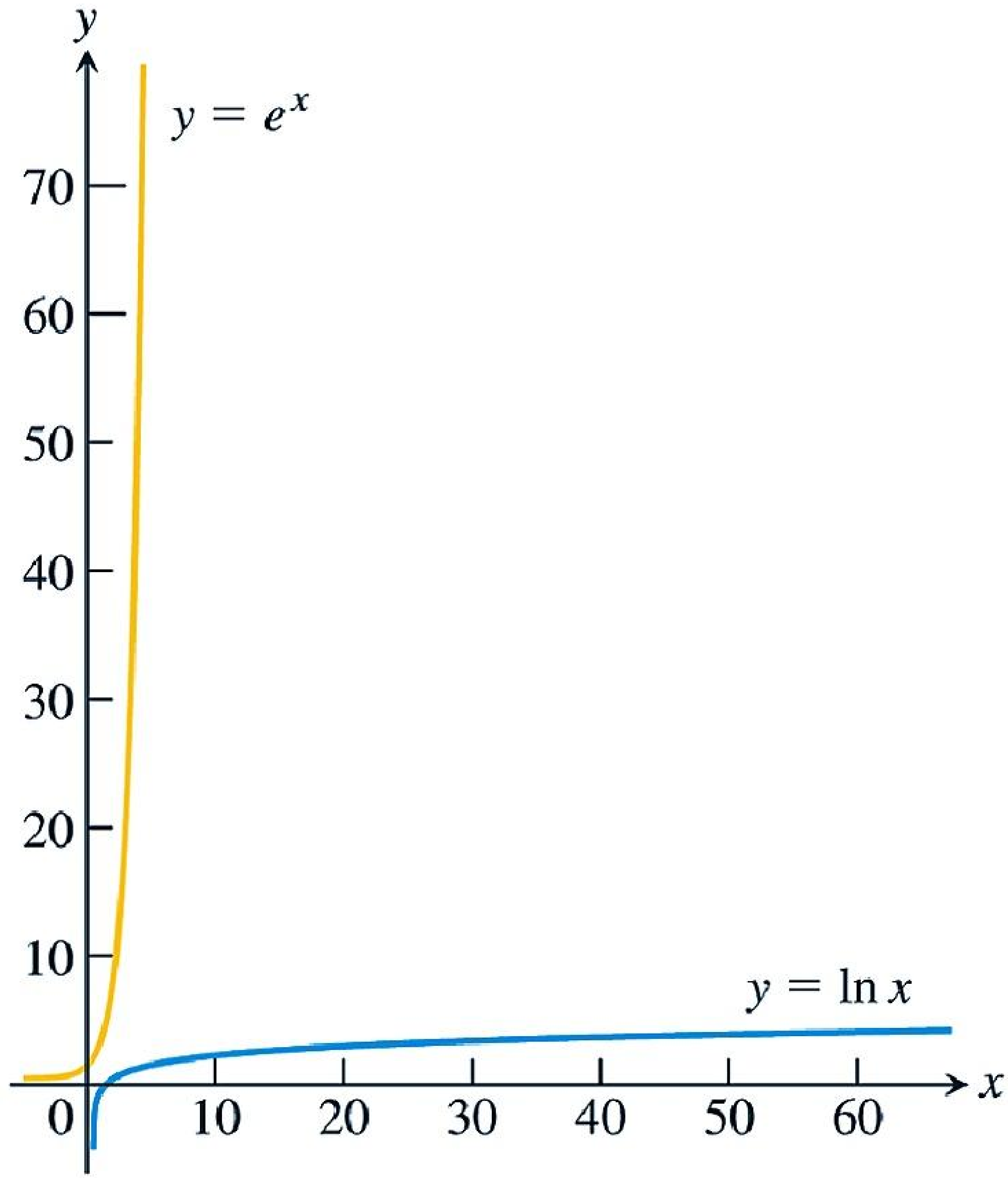


FIGURE 7.35 Scale drawings of the graphs of e^x and $\ln x$.

DEFINITION Let $f(x)$ and $g(x)$ be positive for x sufficiently large.

1. f **grows faster than** g as $x \to \infty$ if

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \infty$$

or, equivalently, if

$$\lim_{x \to \infty} \frac{g(x)}{f(x)} = 0.$$

We also say that g **grows slower than** f as $x \to \infty$.

2. f and g **grow at the same rate** as $x \to \infty$ if

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = L$$

where L is finite and positive.

DEFINITION A function f is **of smaller order than** g as $x \to \infty$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$. We indicate this by writing $\boldsymbol{f = o(g)}$ ("f is little-oh of g").

DEFINITION Let $f(x)$ and $g(x)$ be positive for x sufficiently large. Then f is **of at most the order of** g as $x \to \infty$ if there is a positive integer M for which

$$\frac{f(x)}{g(x)} \leq M,$$

for x sufficiently large. We indicate this by writing $\boldsymbol{f = O(g)}$ ("f is big-oh of g").