Chapter 14

Multiple Integrals

Pearson

Section 15.1

Double and Iterated Integrals over Rectangles

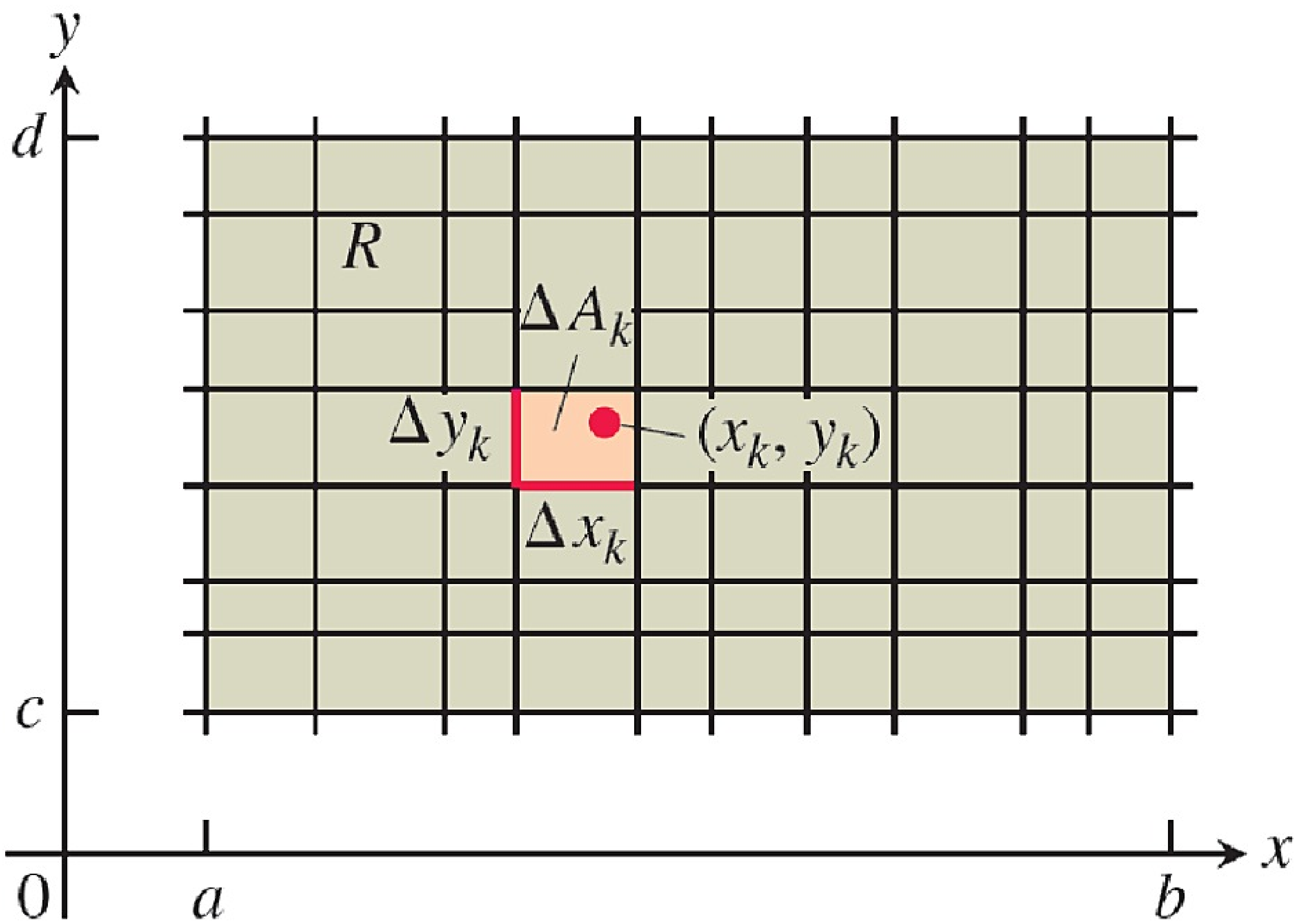


FIGURE 14.1 Rectangular grid partitioning the region R into small rectangles of area $\Delta A_k = \Delta x_k \, \Delta y_k$.

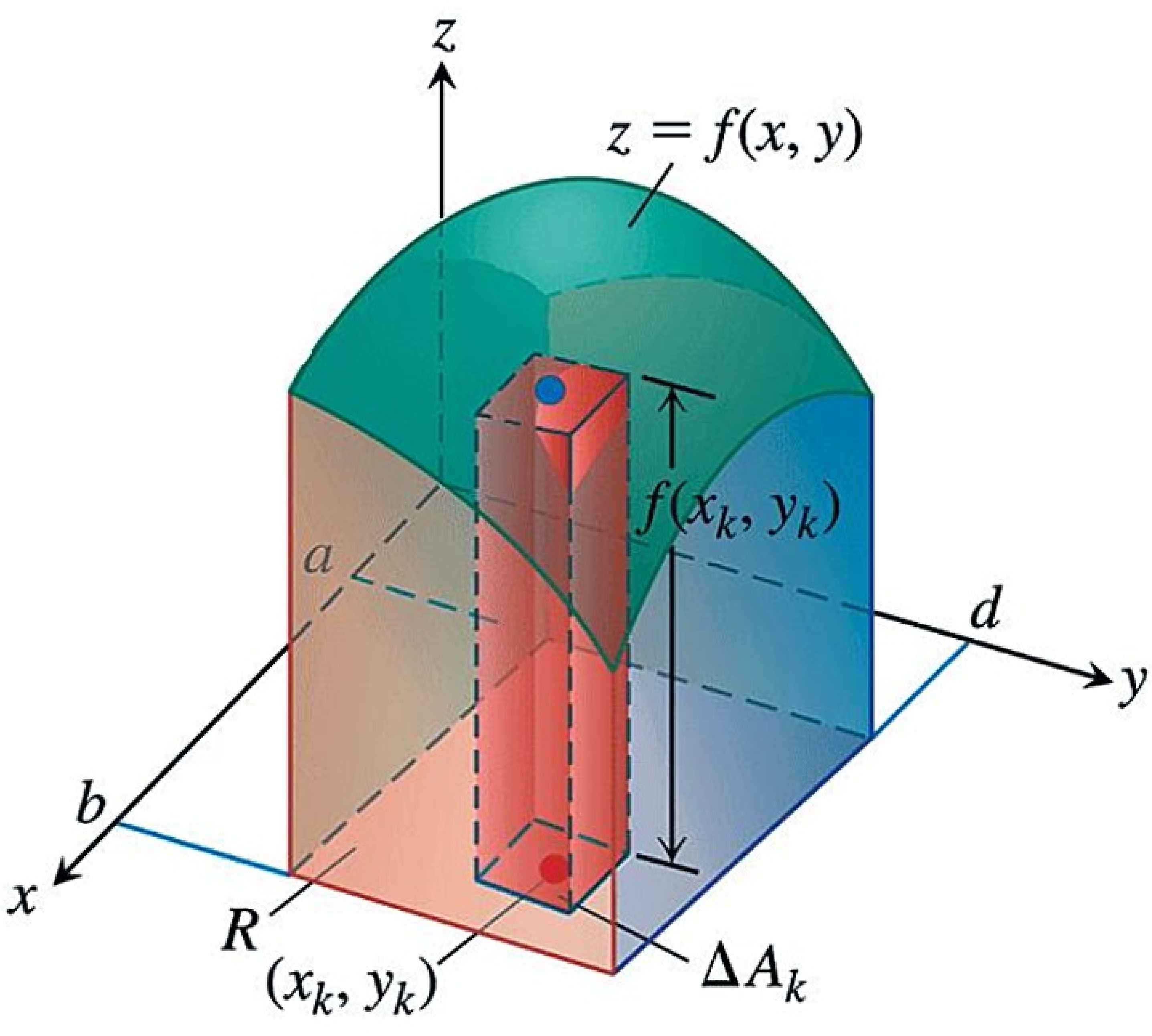


FIGURE 14.2 Approximating solids with rectangular boxes leads us to define the volumes of more general solids as double integrals. The volume of the solid shown here is the double integral of $f(x, y)$ over the base region R.

$$\text{Volume} = \lim_{n \to \infty} S_n = \iint_R f(x, y)\, dA,$$

where $\Delta A_k \to 0$ as $n \to \infty$.

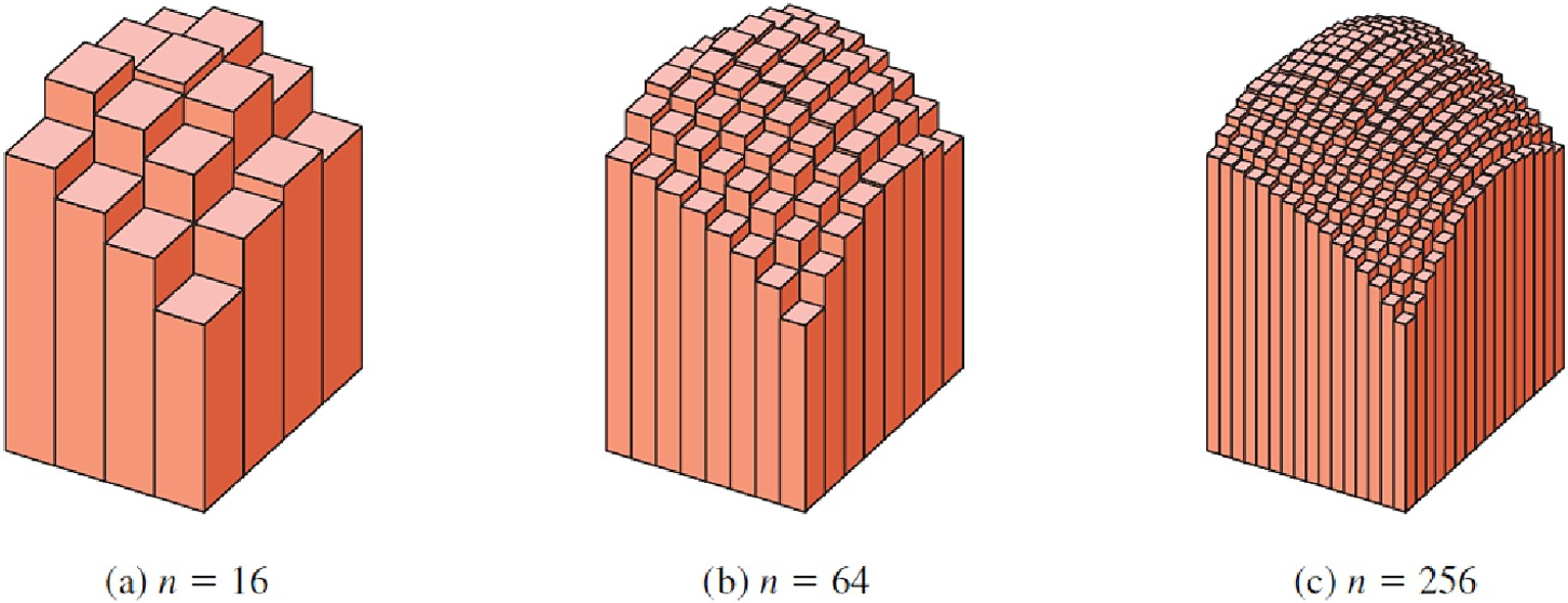


FIGURE 14.3 As n increases, the Riemann sum approximations approach the total volume of the solid shown in Figure 14.2.

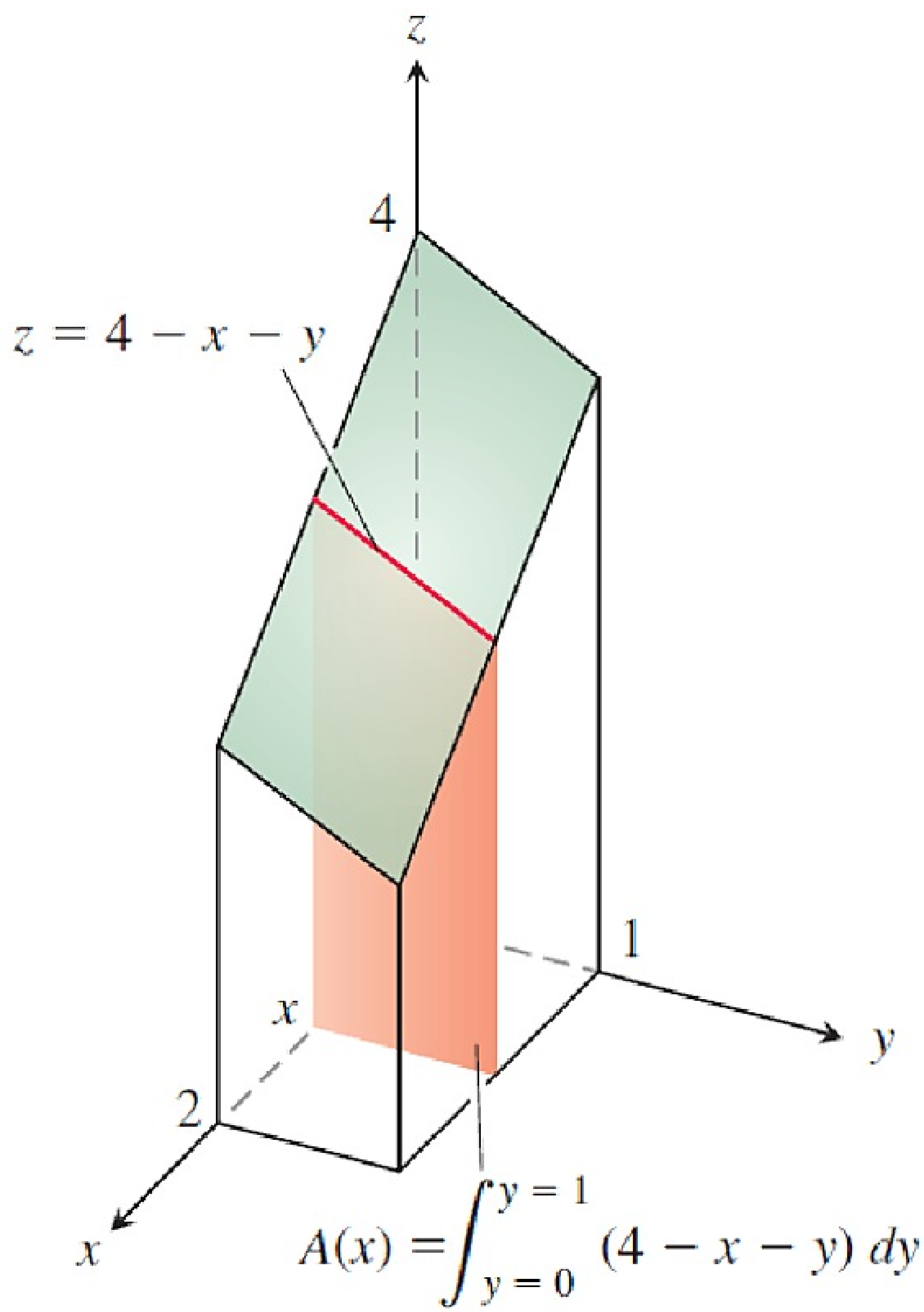


FIGURE 14.4 To obtain the cross-sectional area $A(x)$, we hold x fixed and integrate with respect to y.

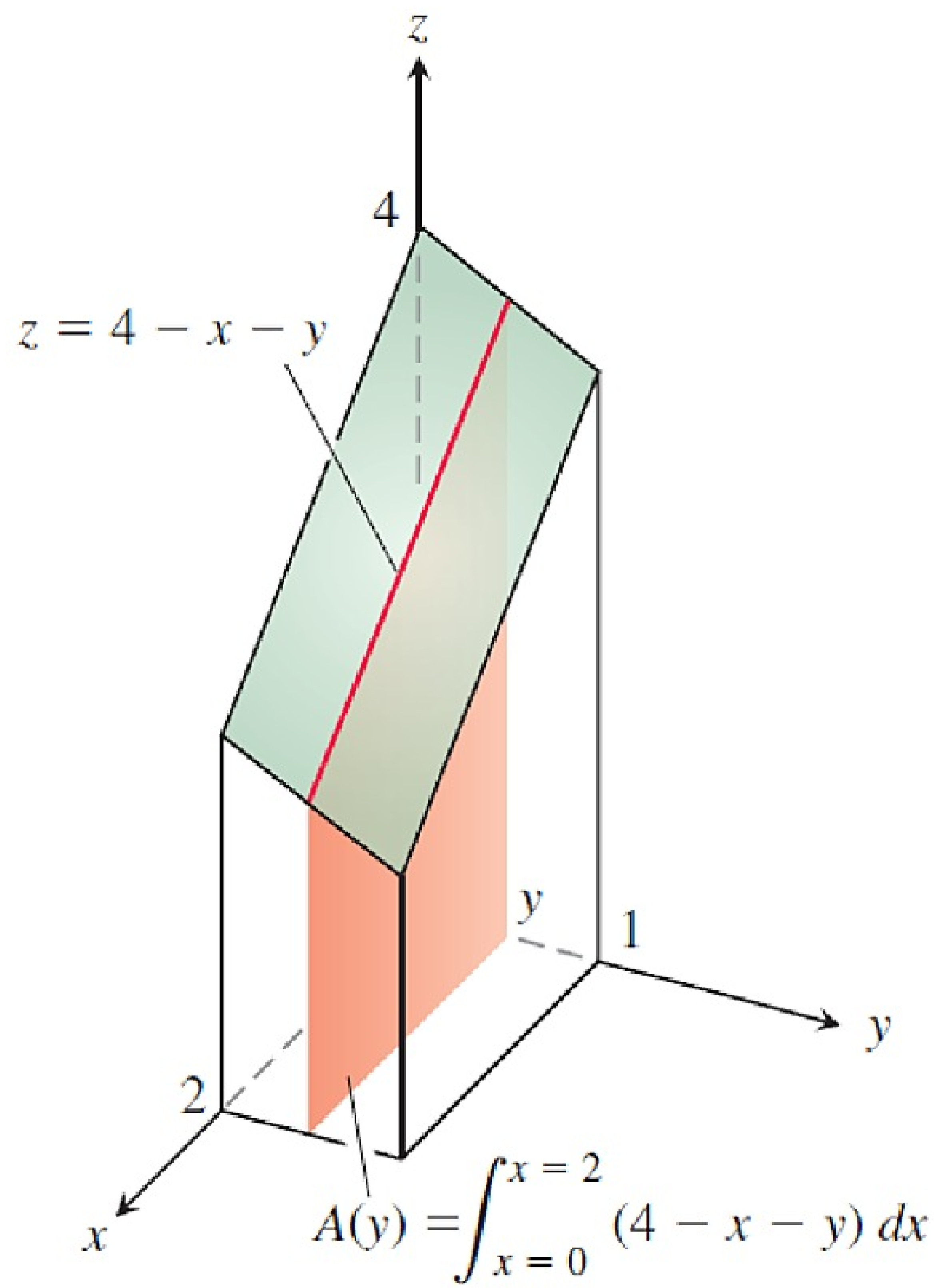


FIGURE 14.5 To obtain the cross-sectional area $A(y)$, we hold y fixed and integrate with respect to x.

THEOREM 1—Fubini's Theorem (First Form) If $f(x, y)$ is continuous throughout the rectangular region R: $a \leq x \leq b$, $c \leq y \leq d$, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_a^b f(x, y)\, dx\, dy = \int_a^b \int_c^d f(x, y)\, dy\, dx.$$

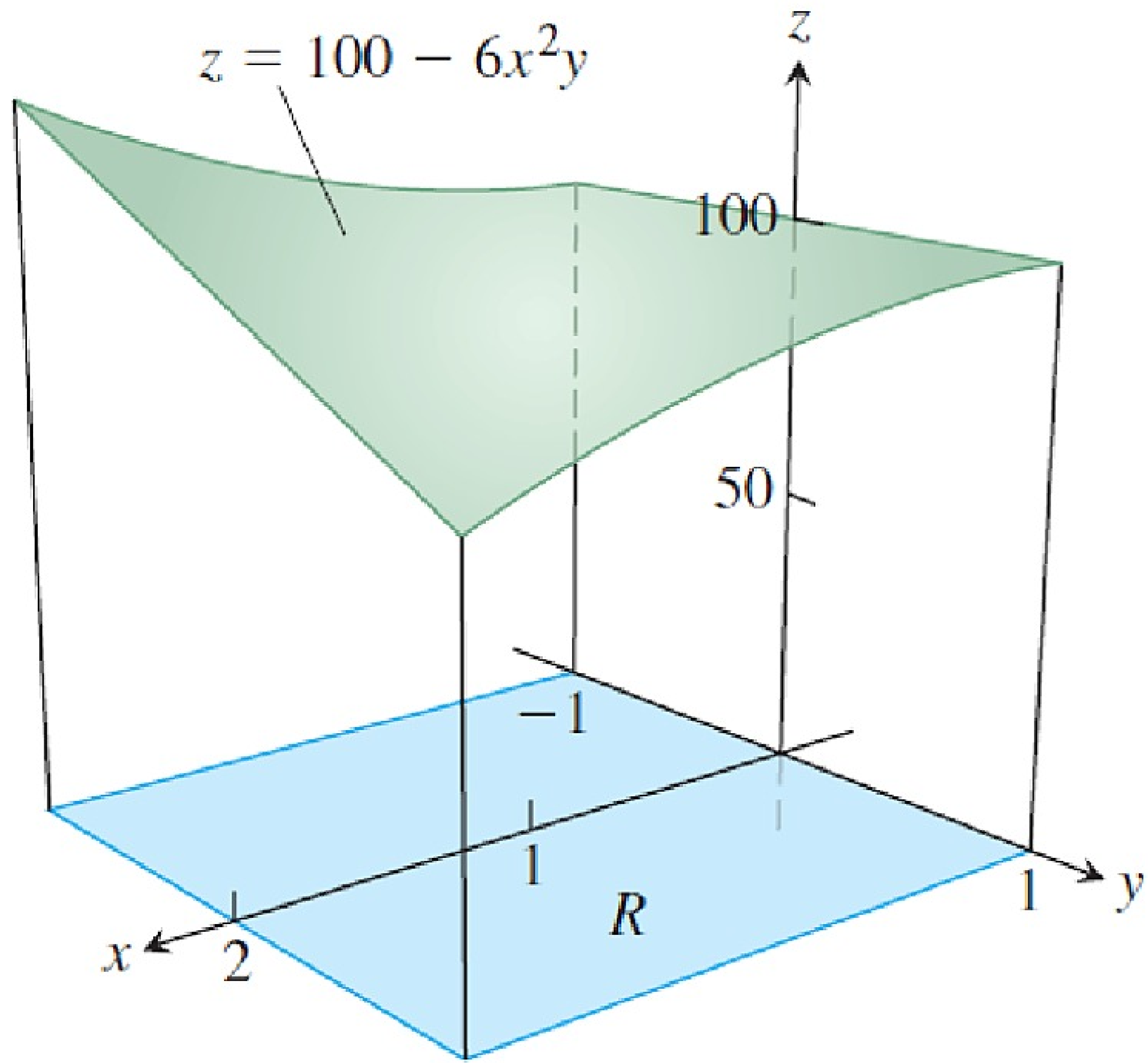


FIGURE 14.6 The double integral $\iint_R f(x, y)\, dA$ gives the volume under this surface over the rectangular region R (Example 1).

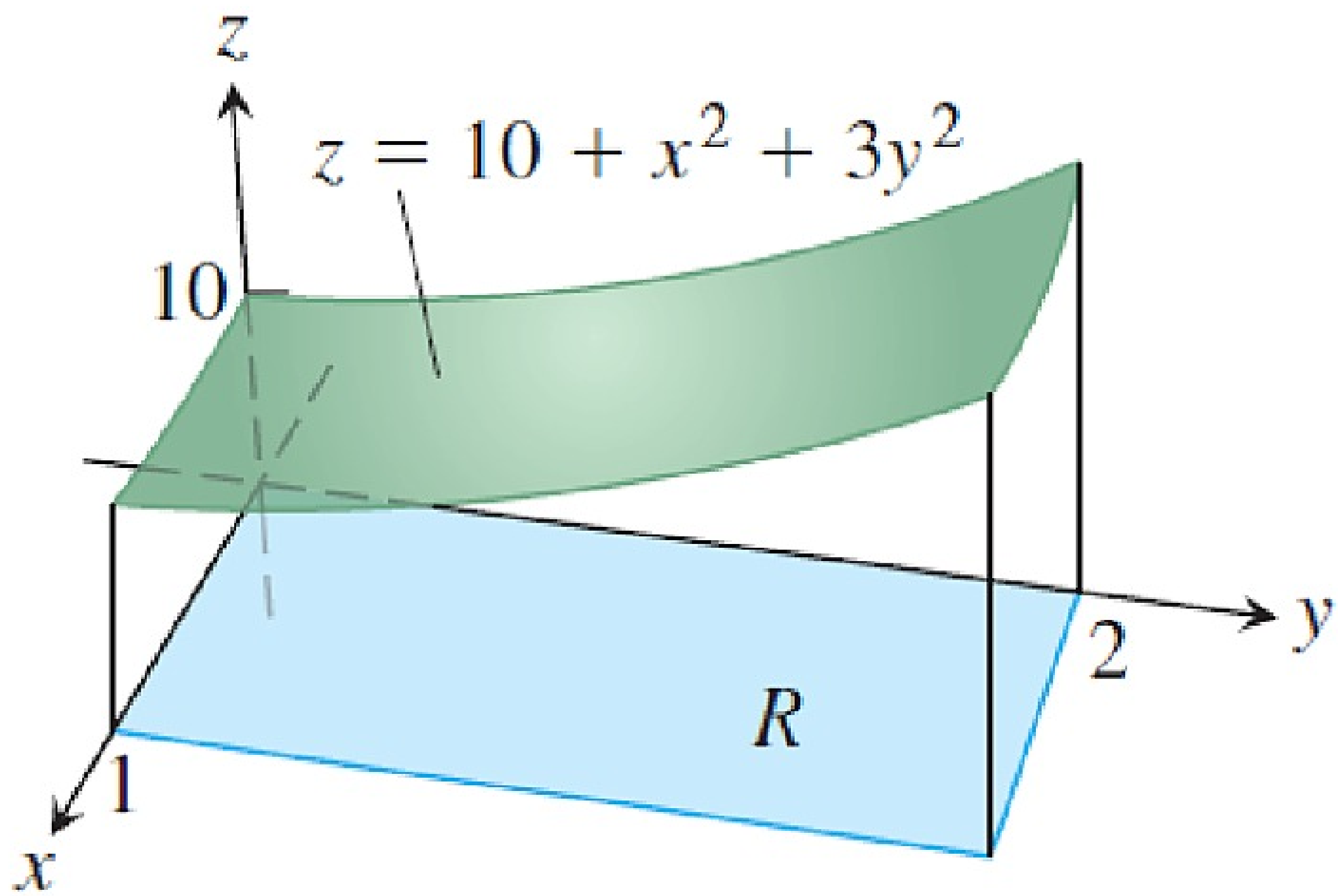


FIGURE 14.7 The double integral $\iint_R f(x, y)\, dA$ gives the volume under this surface over the rectangular region R (Example 2).

Section 14.2

Double Integrals over General Regions

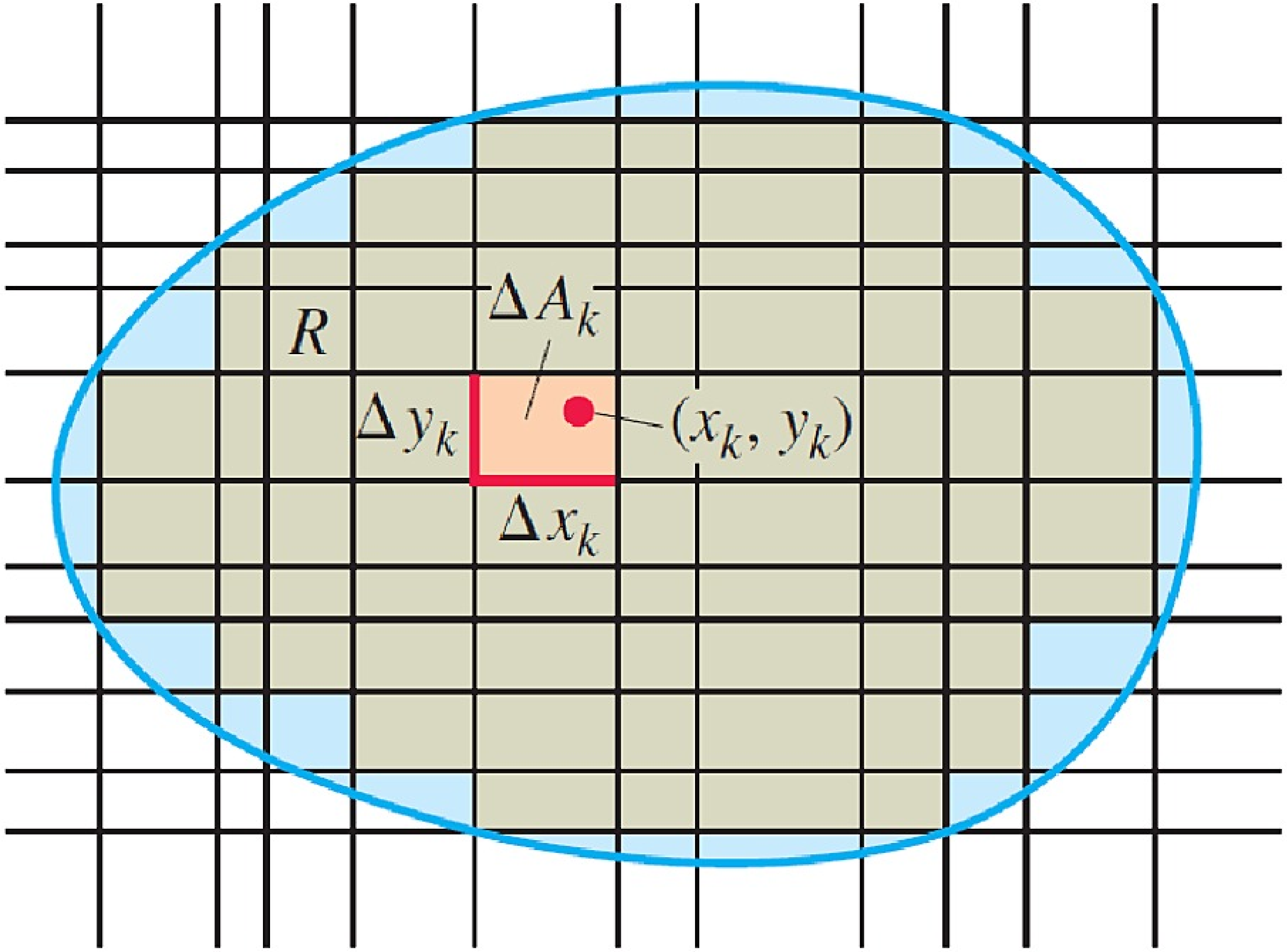


FIGURE 14.8 A rectangular grid partitioning a bounded, nonrectangular region into rectangular cells.

If $f(x, y)$ is positive and continuous over R, we define the volume of the solid region between R and the surface $z = f(x, y)$ to be $\iint_R f(x, y)\, dA$, as before (Figure 14.9).

If R is a region like the one shown in the xy-plane in Figure 14.10, bounded "above" and "below" by the curves $y = g_2(x)$ and $y = g_1(x)$ and on the sides by the lines $x = a, x = b$, we may again calculate the volume by the method of slicing. We first calculate the cross-sectional area

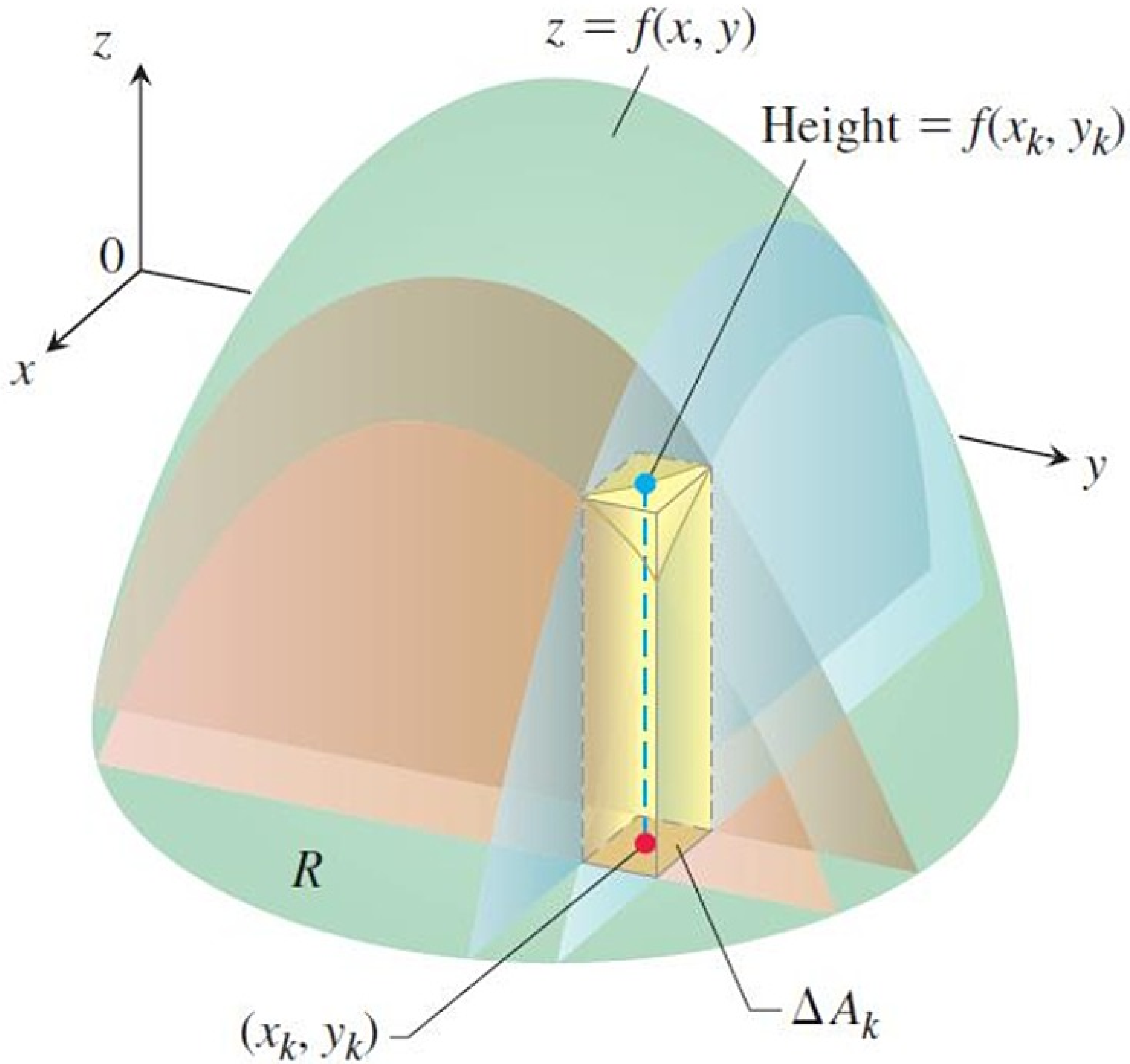
$$A(x) = \int_{y=g_1(x)}^{y=g_2(x)} f(x, y)\, dy$$

and then integrate $A(x)$ from $x = a$ to $x = b$ to get the volume as an iterated integral:

$$V = \int_a^b A(x)\, dx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx. \tag{1}$$

Similarly, if R is a region like the one shown in Figure 14.11, bounded by the curves $x = h_2(y)$ and $x = h_1(y)$ and the lines $y = c$ and $y = d$, then the volume calculated by slicing is given by the iterated integral

$$\text{Volume} = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy. \tag{2}$$



$$\text{Volume} = \lim \sum f(x_k, y_k)\, \Delta A_k = \iint_R f(x, y)\, dA$$

FIGURE 14.9 We define the volumes of solids with curved bases as a limit of approximating rectangular boxes.

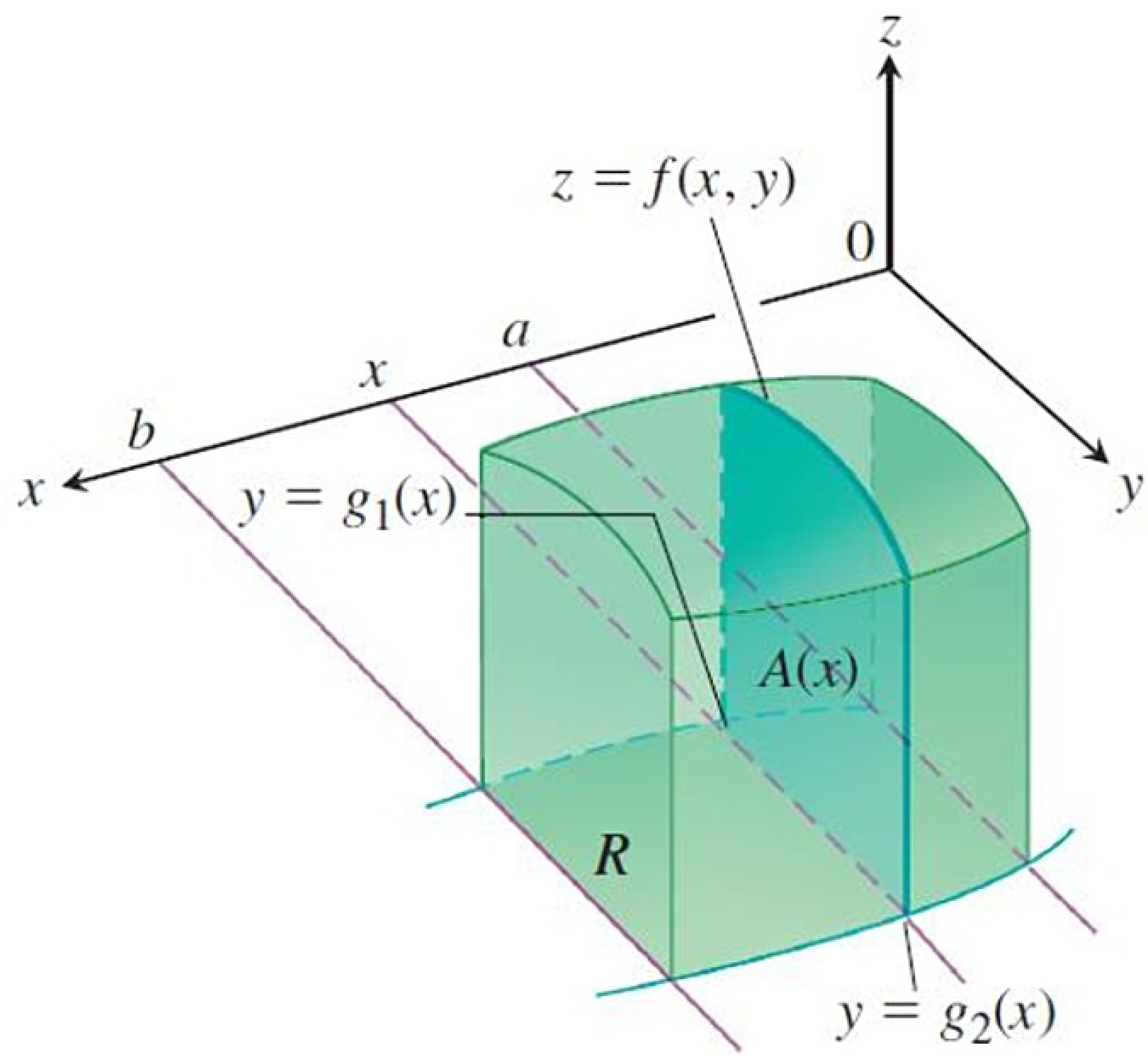


FIGURE 14.10 The area of the vertical slice shown here is $A(x)$. To calculate the volume of the solid, we integrate this area from $x = a$ to $x = b$:

$$\int_a^b A(x)\,dx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\,dy\,dx.$$

Pearson

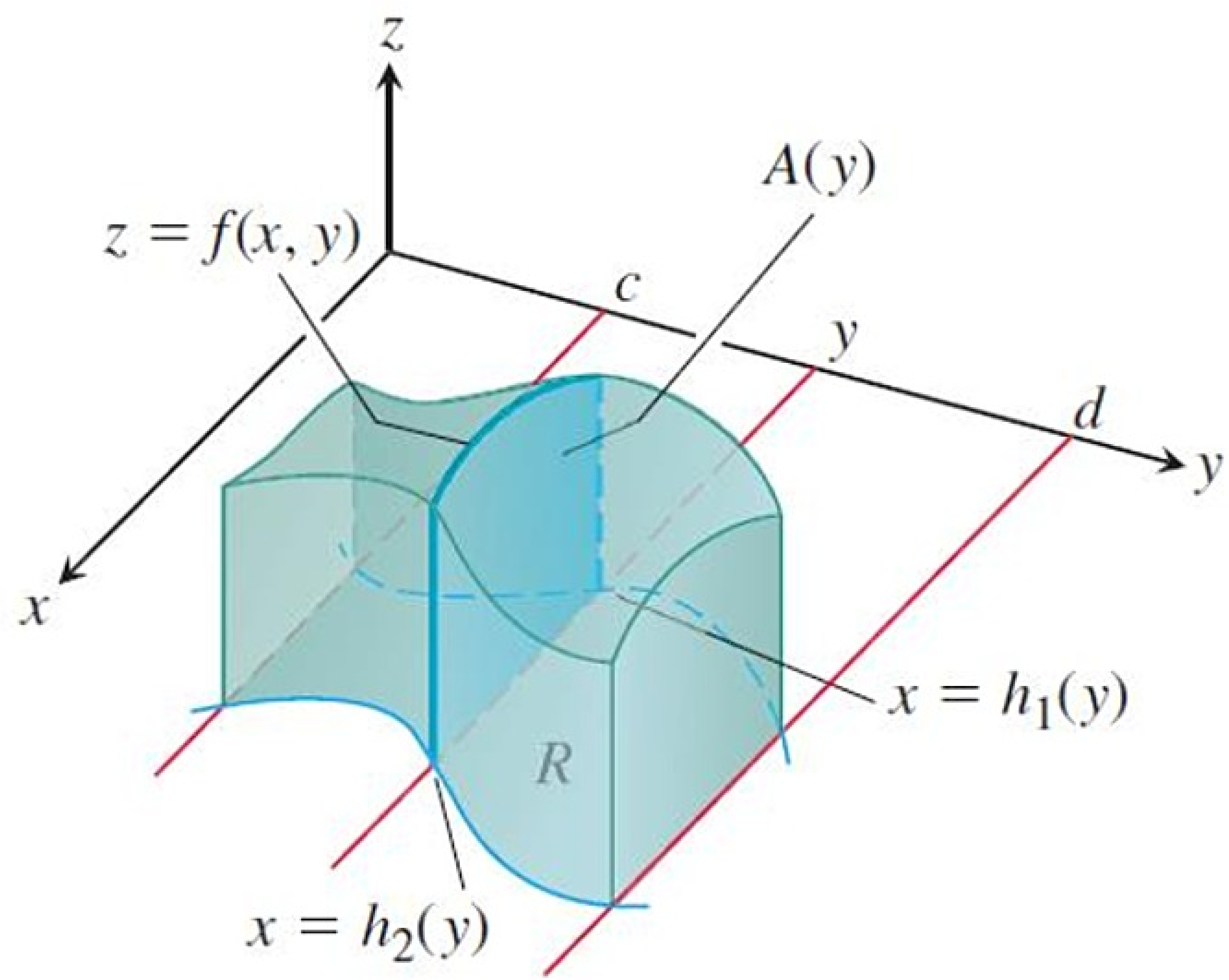


FIGURE 14.11 The volume of the solid shown here is

$$\int_c^d A(y)\,dy = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\,dx\,dy.$$

For a given solid, Theorem 2 says we can calculate the volume as in Figure 14.10, or in the way shown here. Both calculations have the same result.

THEOREM 2—Fubini's Theorem (Stronger Form) Let $f(x, y)$ be continuous on a region R.

1. If R is defined by $a \leq x \leq b$, $g_1(x) \leq y \leq g_2(x)$, with g_1 and g_2 continuous on $[a, b]$, then

$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$

2. If R is defined by $c \leq y \leq d$, $h_1(y) \leq x \leq h_2(y)$, with h_1 and h_2 continuous on $[c, d]$, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$

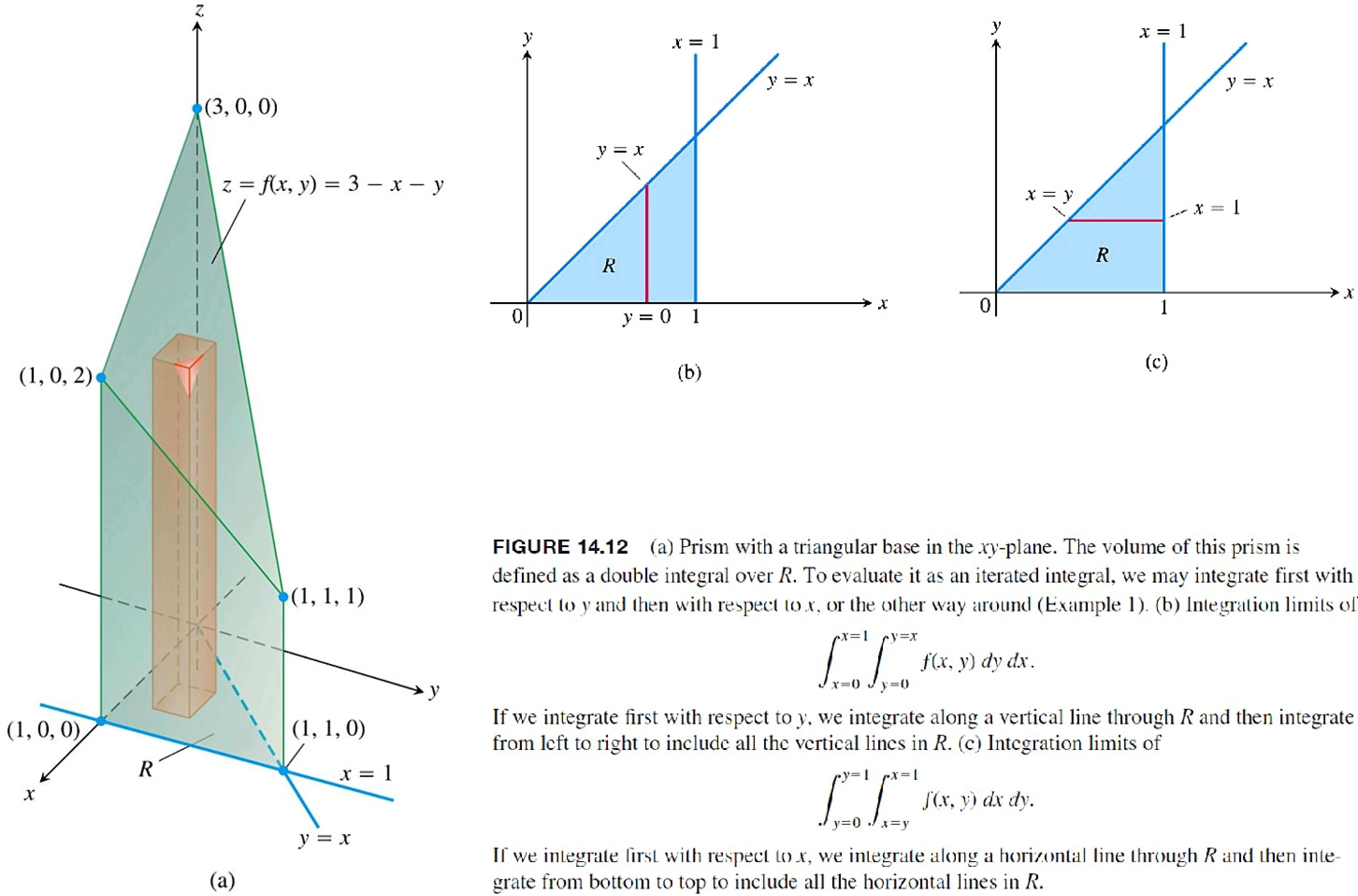


FIGURE 14.12 (a) Prism with a triangular base in the xy-plane. The volume of this prism is defined as a double integral over R. To evaluate it as an iterated integral, we may integrate first with respect to y and then with respect to x, or the other way around (Example 1). (b) Integration limits of

$$\int_{x=0}^{x=1} \int_{y=0}^{y=x} f(x, y)\, dy\, dx.$$

If we integrate first with respect to y, we integrate along a vertical line through R and then integrate from left to right to include all the vertical lines in R. (c) Integration limits of

$$\int_{y=0}^{y=1} \int_{x=y}^{x=1} f(x, y)\, dx\, dy.$$

If we integrate first with respect to x, we integrate along a horizontal line through R and then integrate from bottom to top to include all the horizontal lines in R.

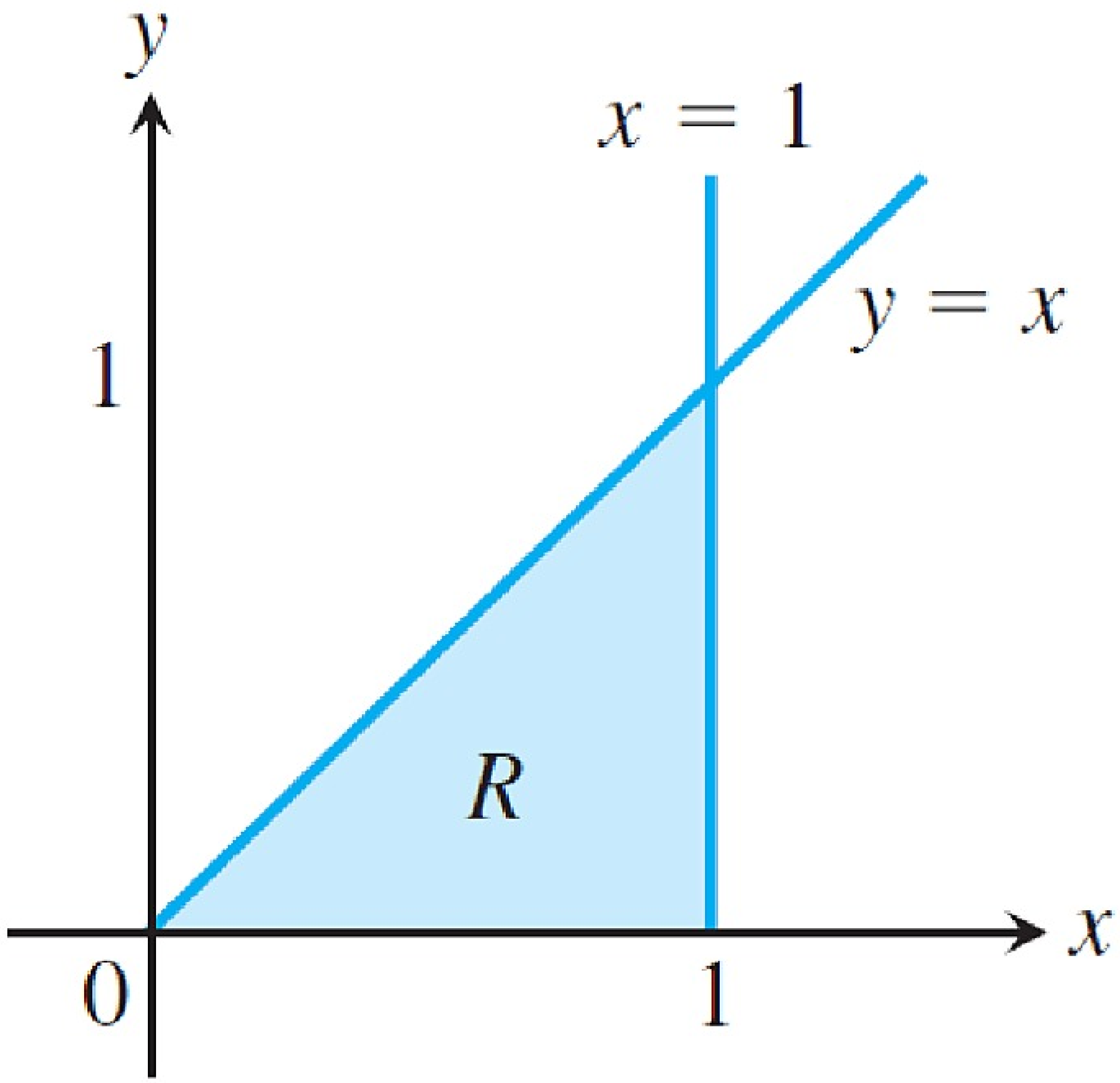
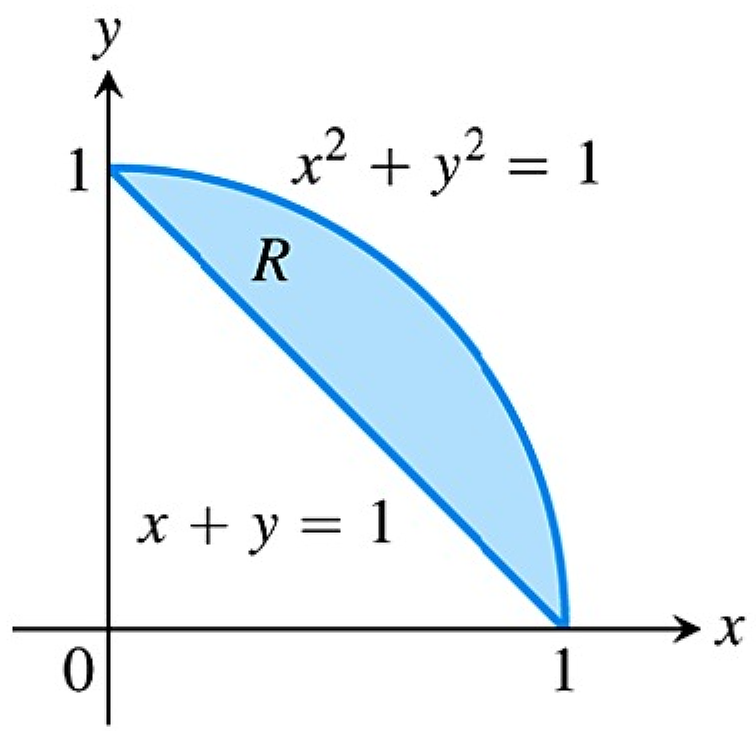
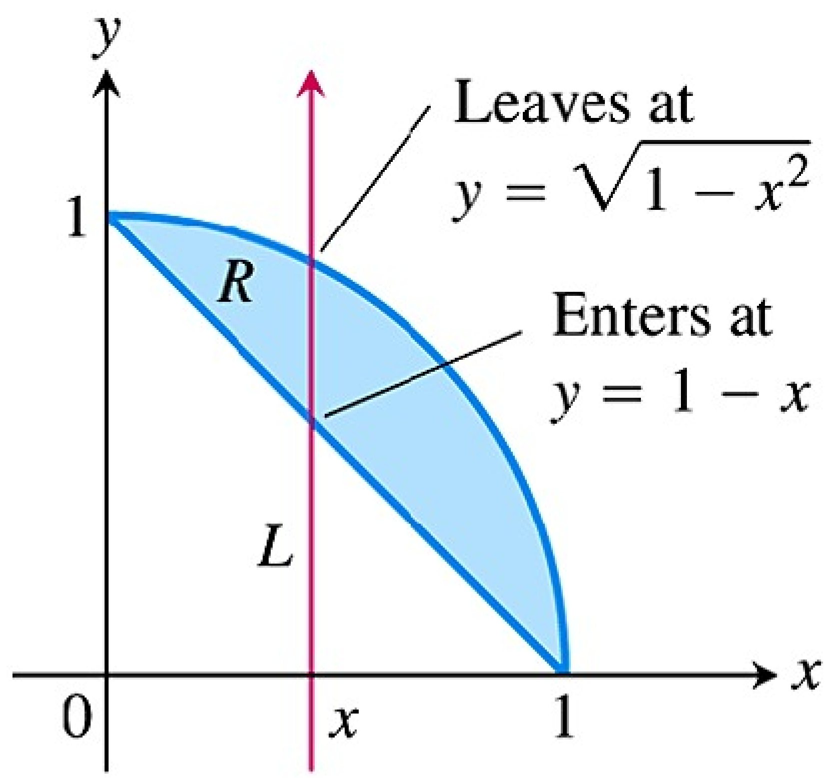


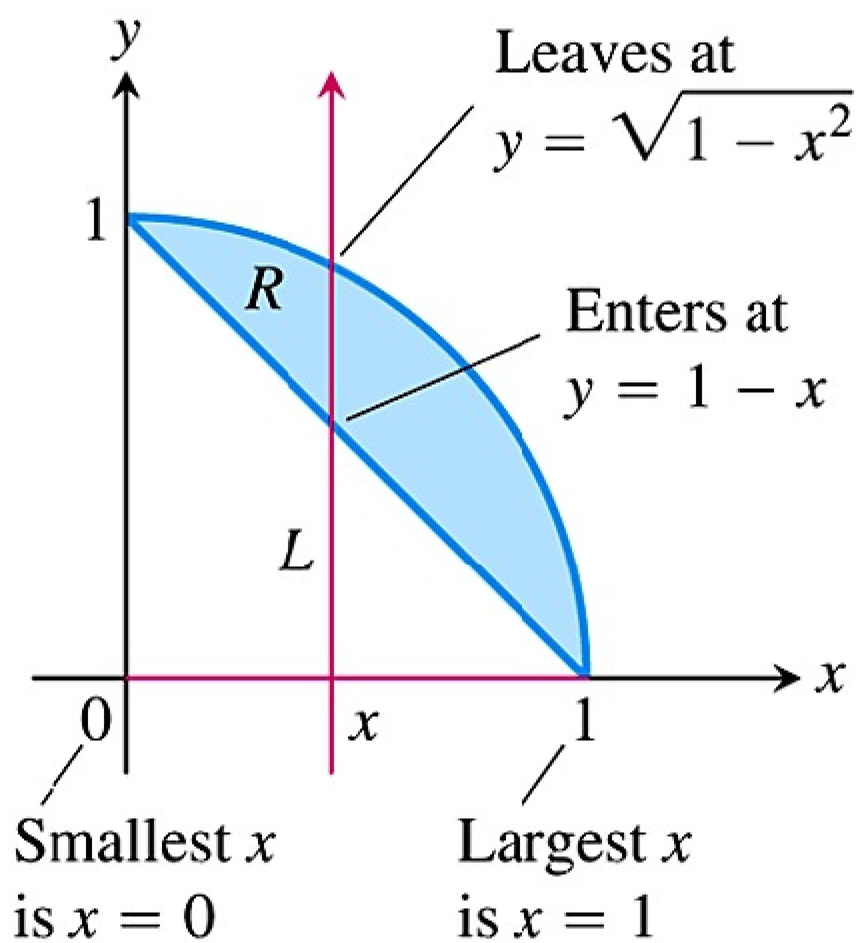
FIGURE 14.13 The region of integration in Example 2.



(a)



(b)



(c)

FIGURE 14.14 Finding the limits of integration when integrating first with respect to y and then with respect to x.

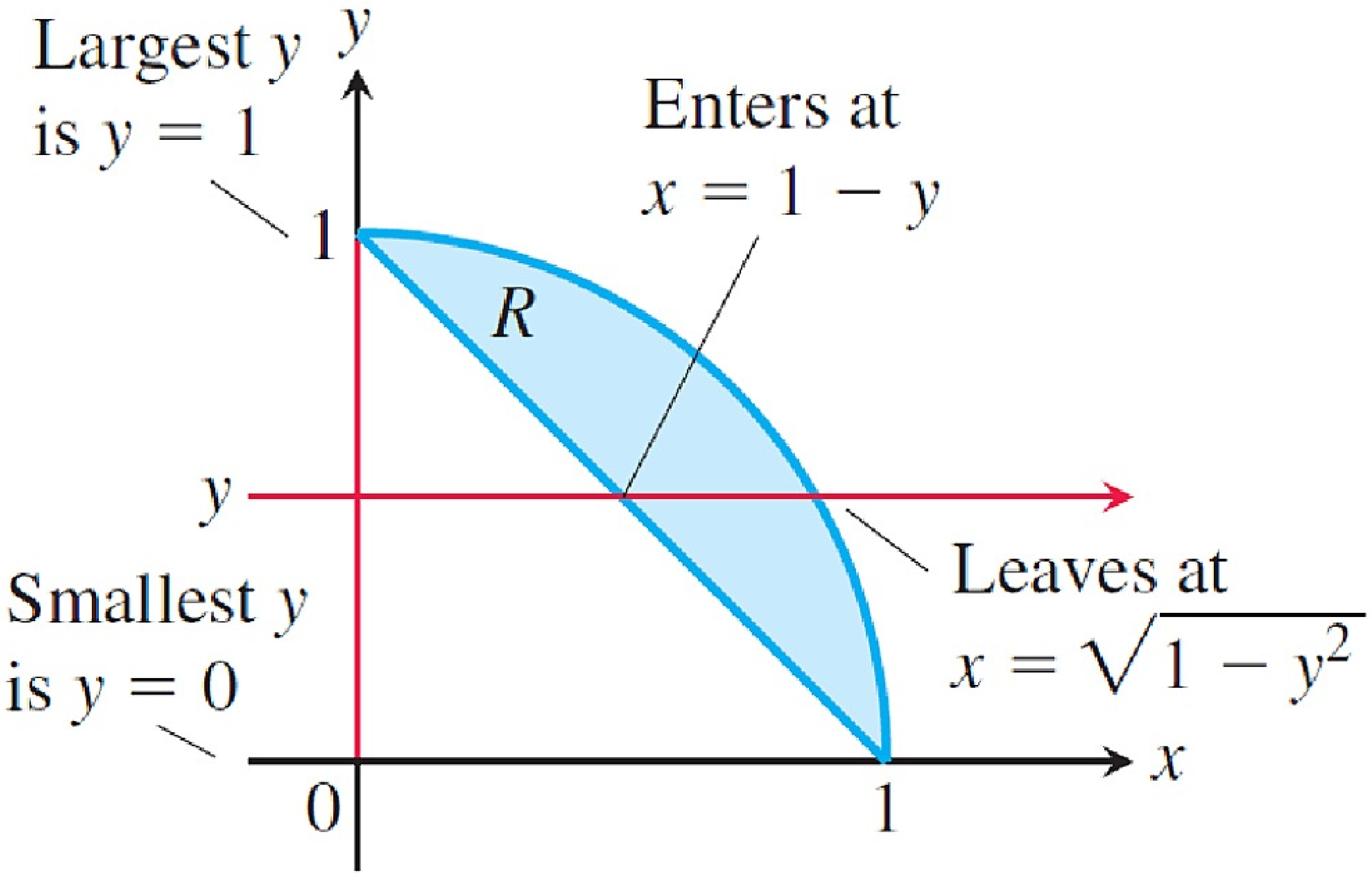


FIGURE 14.15 Finding the limits of integration when integrating first with respect to x and then with respect to y.

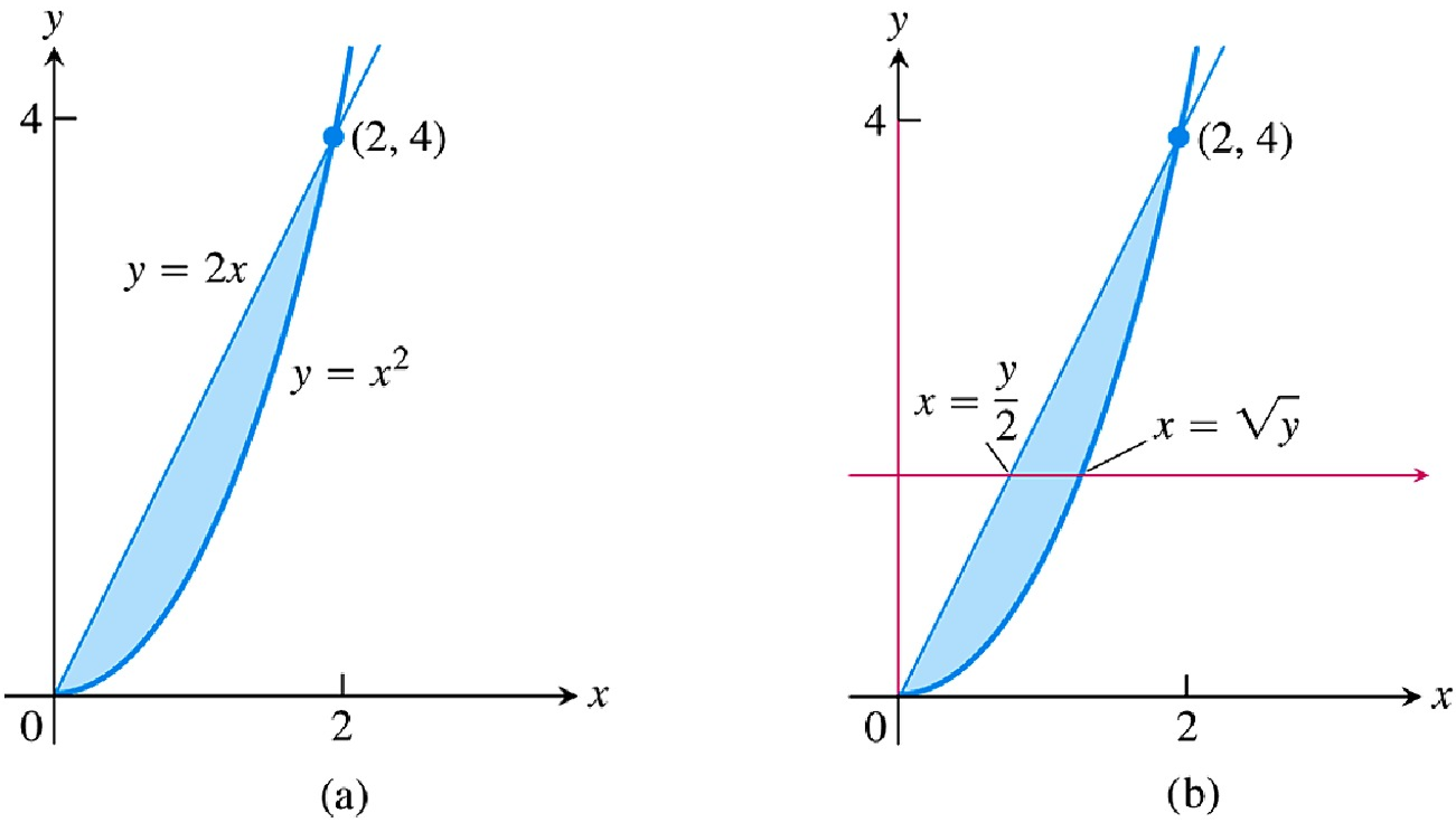


FIGURE 14.16 Region of integration for Example 3.

If $f(x, y)$ and $g(x, y)$ are continuous on the bounded region R, then the following properties hold.

1. *Constant Multiple:* $\displaystyle\iint_R cf(x, y)\, dA = c\iint_R f(x, y)\, dA \quad \text{(any number } c\text{)}$

2. *Sum and Difference:*

$$\iint_R (f(x, y) \pm g(x, y))\, dA = \iint_R f(x, y)\, dA \pm \iint_R g(x, y)\, dA$$

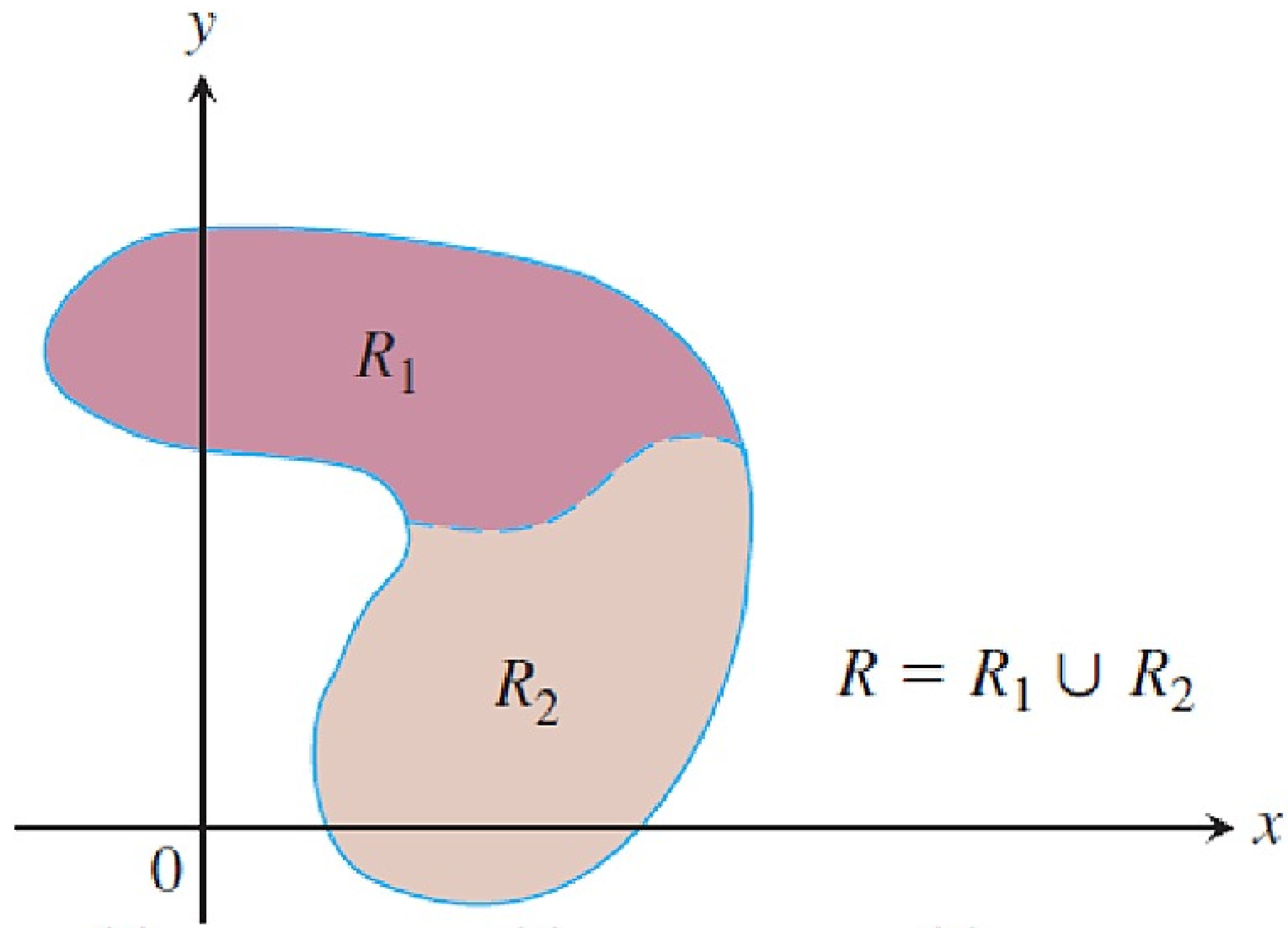
3. *Domination:*

(a) $\displaystyle\iint_R f(x, y)\, dA \geq 0 \quad \text{if} \quad f(x, y) \geq 0 \text{ on } R$

(b) $\displaystyle\iint_R f(x, y)\, dA \geq \iint_R g(x, y)\, dA \quad \text{if} \quad f(x, y) \geq g(x, y) \text{ on } R$

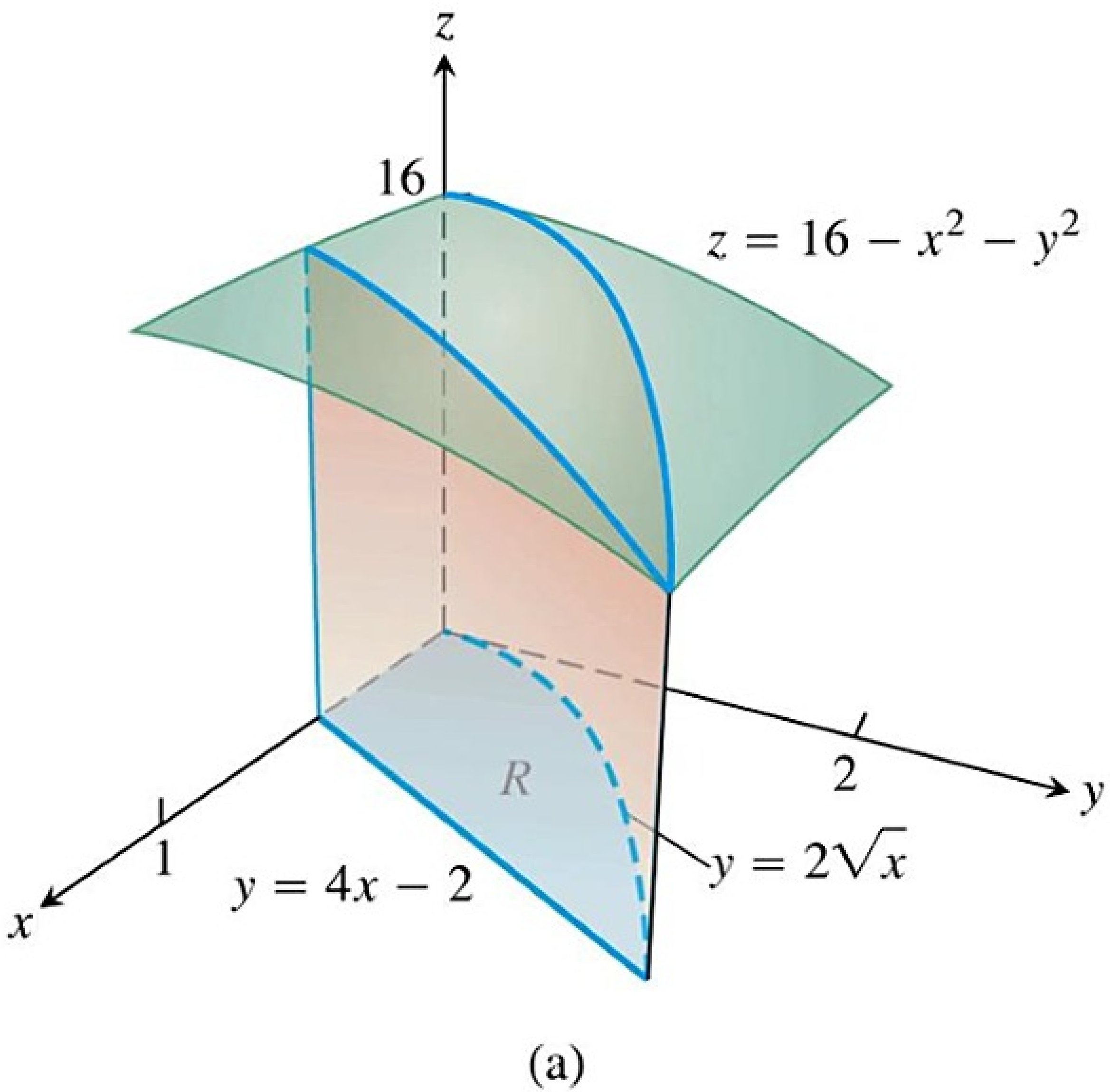
4. *Additivity:* $\displaystyle\iint_R f(x, y)\, dA = \iint_{R_1} f(x, y)\, dA + \iint_{R_2} f(x, y)\, dA$

if R is the union of two nonoverlapping regions R_1 and R_2



$$\iint_R f(x, y)\, dA = \iint_{R_1} f(x, y)\, dA + \iint_{R_2} f(x, y)\, dA$$

FIGURE 14.17 The Additivity Property for rectangular regions holds for regions bounded by smooth curves.



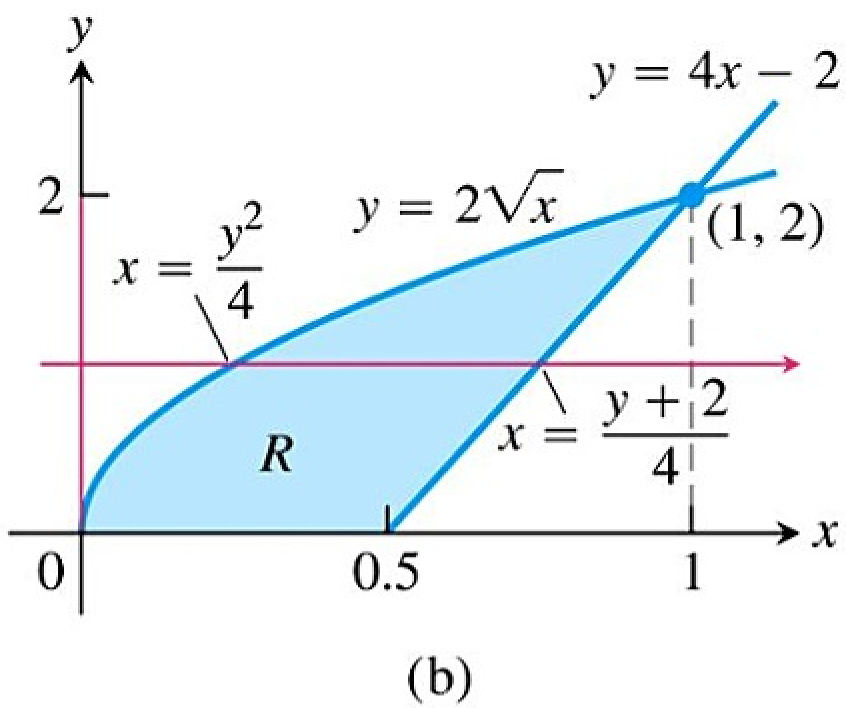


FIGURE 14.18 (a) The solid "wedge-like" region whose volume is found in Example 4. (b) The region of integration R showing the order $dx\,dy$.

Section 14.3

Area by Double Integration

DEFINITION The **area** of a closed, bounded plane region R is

$$A = \iint_R dA.$$

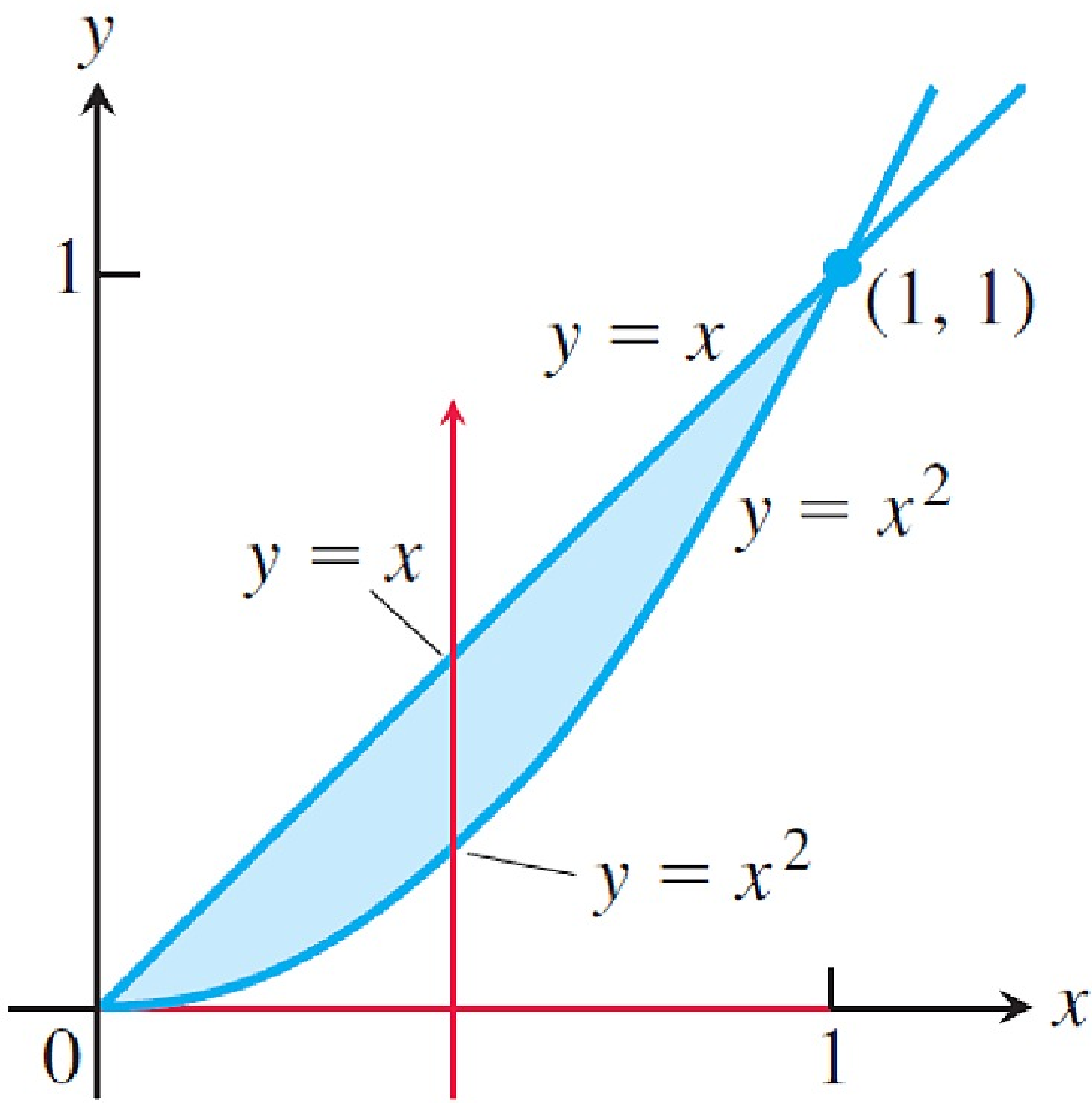
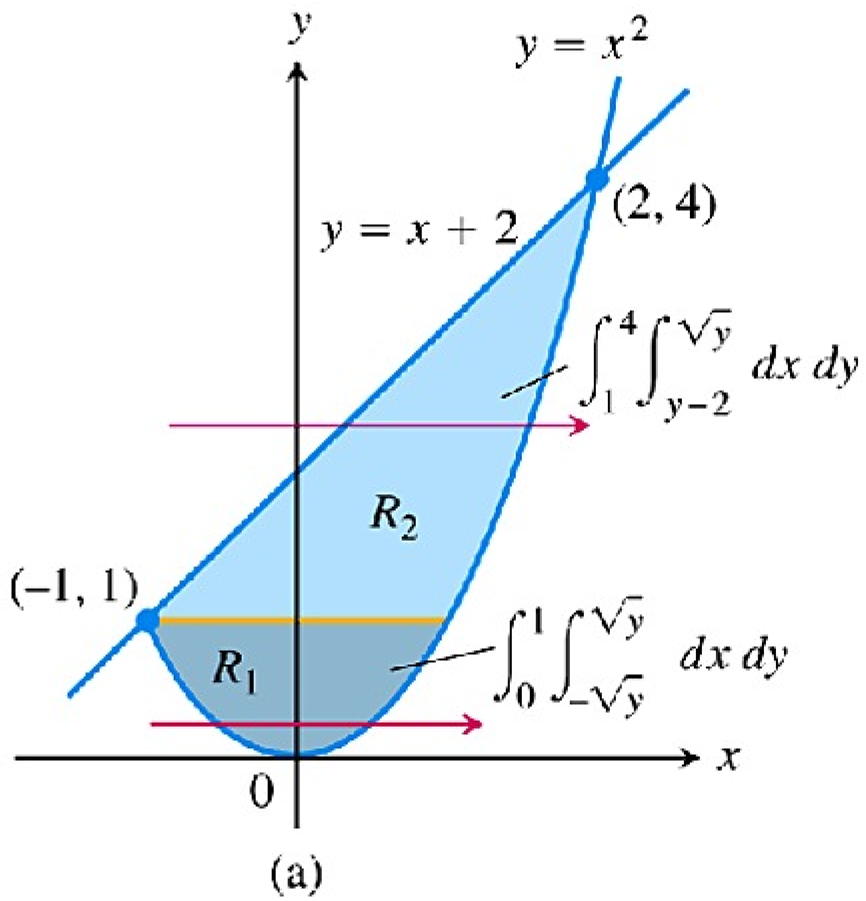


FIGURE 14.19 The region in Example 1.



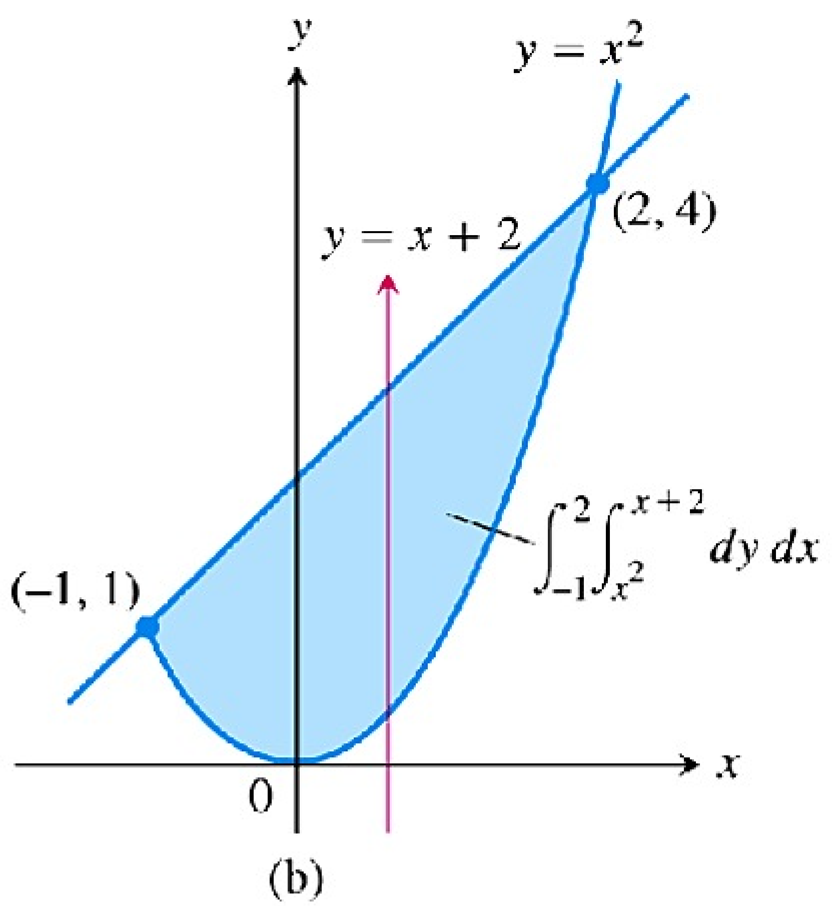


FIGURE 14.20 Calculating this area takes (a) two double integrals if the first integration is with respect to x, but (b) only one if the first integration is with respect to y (Example 2).

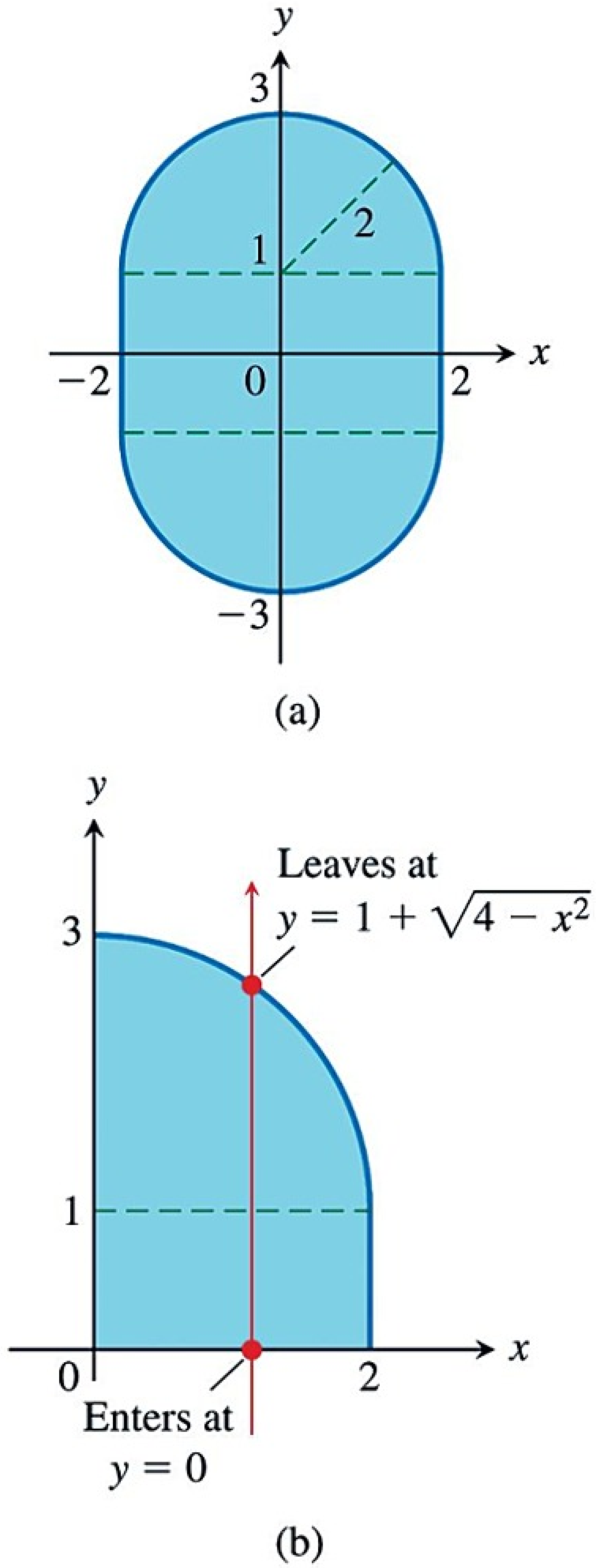


FIGURE 14.21 (a) The playing field described by the region R in Example 3. (b) First quadrant of the playing field.

$$\textbf{Average value} \text{ of } f \text{ over } R = \frac{1}{\text{area of } R} \iint\limits_R f \, dA. \qquad (3)$$

Section 14.4

Double Integrals in Polar Form

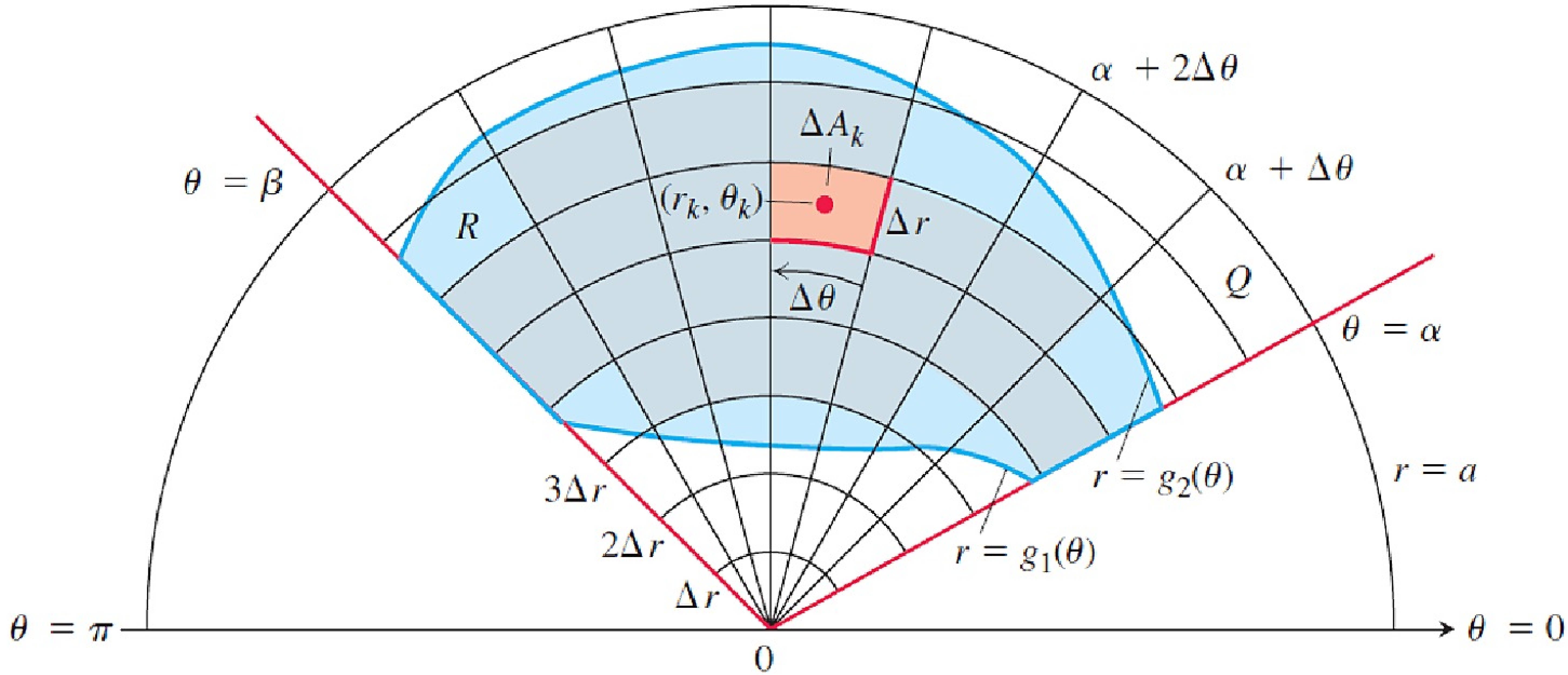


FIGURE 14.22 The region R: $g_1(\theta) \leq r \leq g_2(\theta)$, $\alpha \leq \theta \leq \beta$, is contained in the fan-shaped region Q: $0 \leq r \leq a$, $\alpha \leq \theta \leq \beta$, where $0 \leq \beta - \alpha \leq 2\pi$. The partition of Q by circular arcs and rays induces a partition of R.

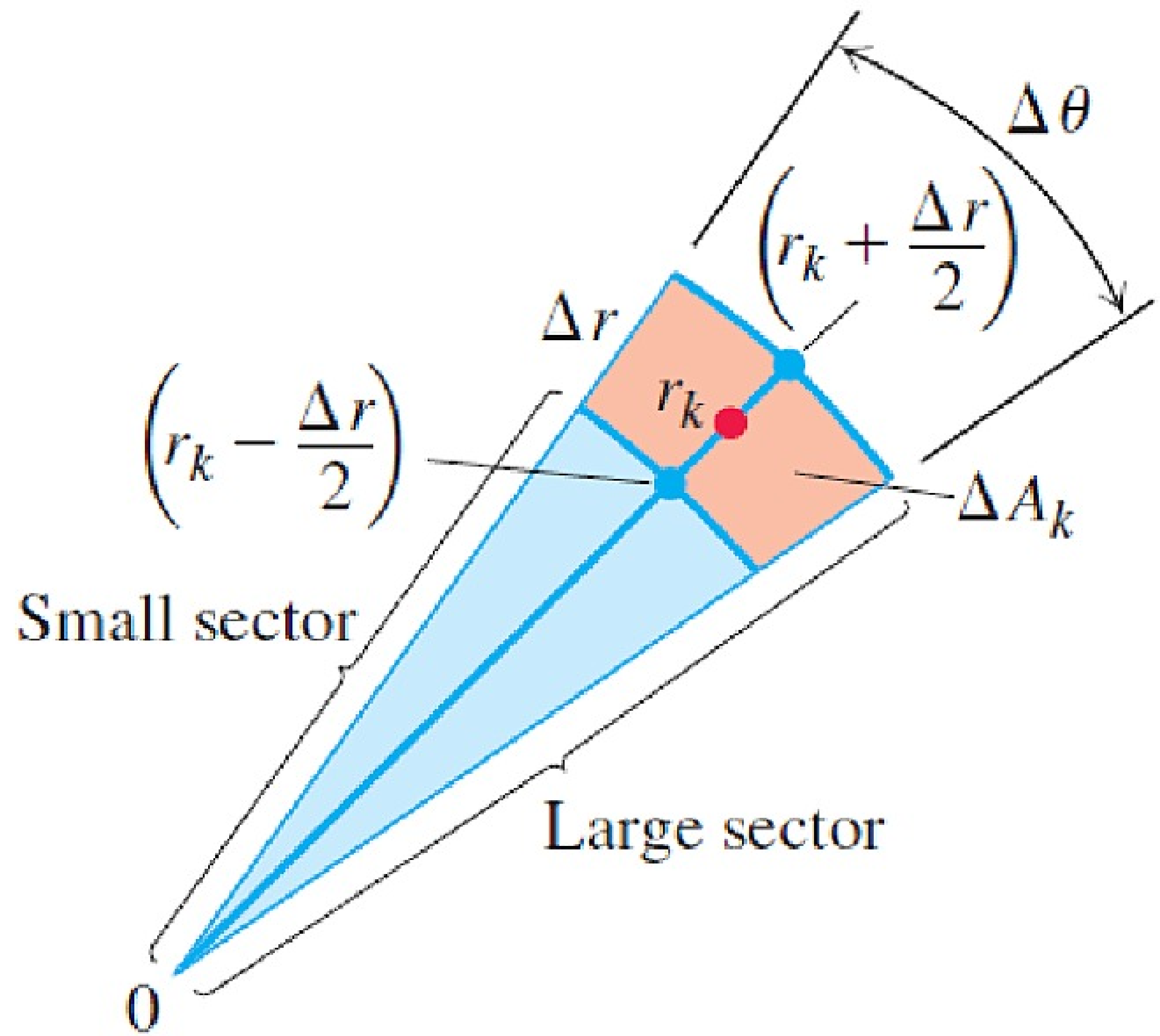
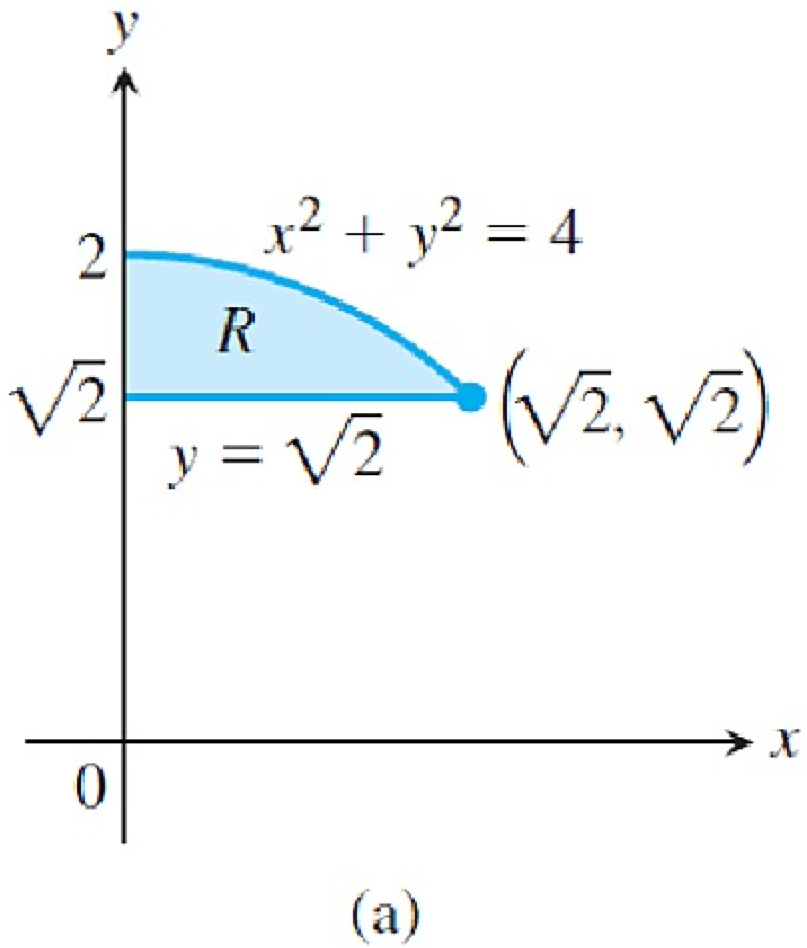
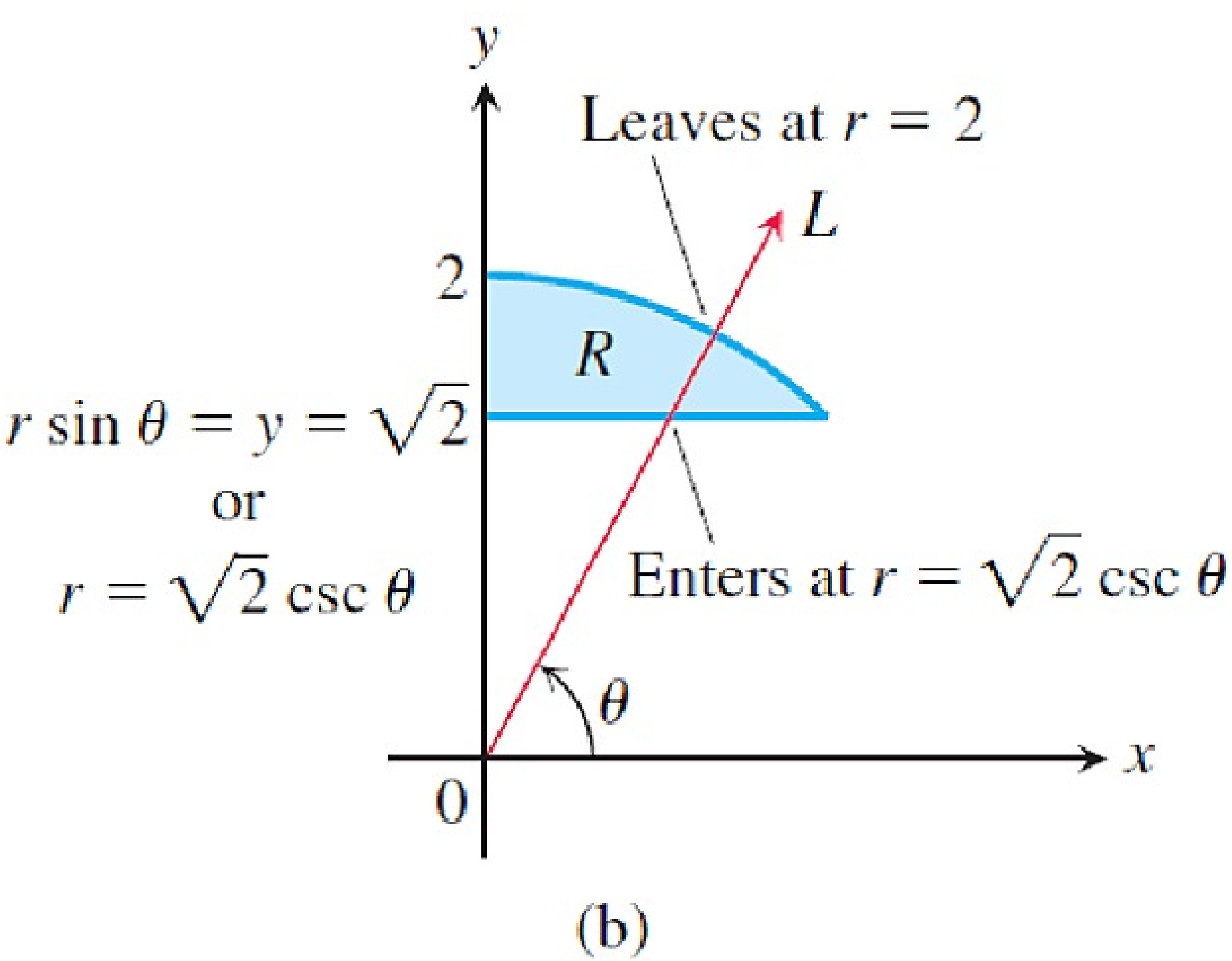


FIGURE 14.23 The observation that

$$\Delta A_k = \begin{pmatrix} \text{area of} \\ \text{large sector} \end{pmatrix} - \begin{pmatrix} \text{area of} \\ \text{small sector} \end{pmatrix}$$

leads to the formula $\Delta A_k = r_k \, \Delta r \, \Delta\theta$.





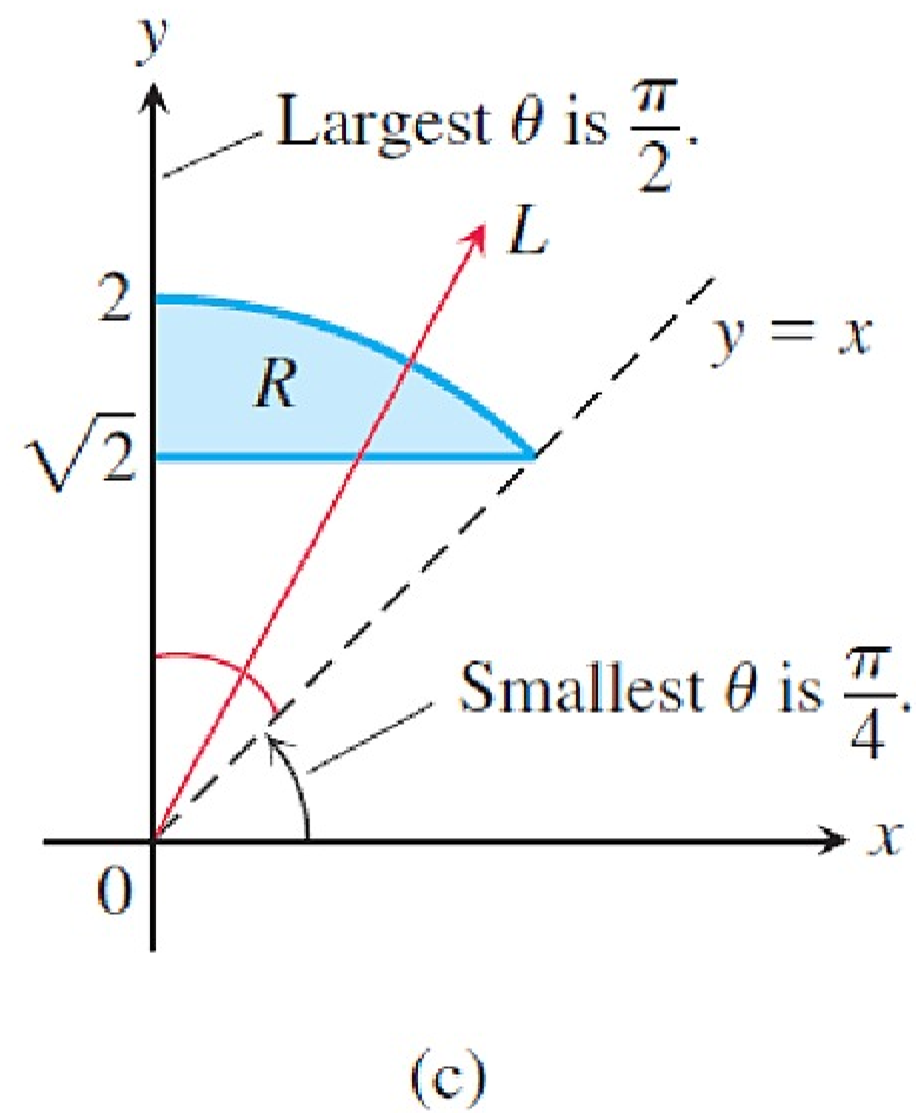


FIGURE 14.24 Finding the limits of integration in polar coordinates.

Area in Polar Coordinates

The area of a closed and bounded region R in the polar coordinate plane is

$$A = \iint_R r\, dr\, d\theta.$$

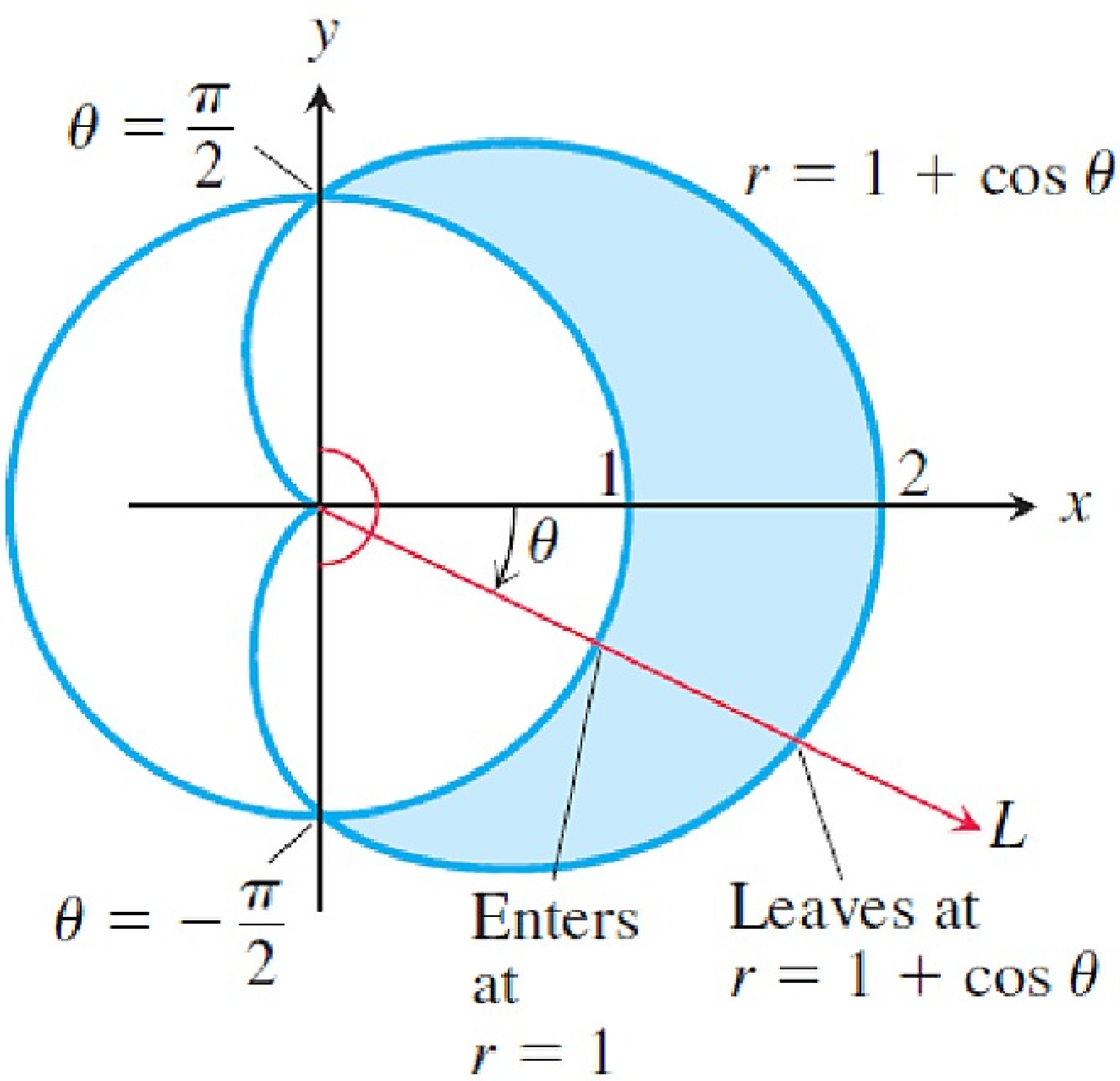


FIGURE 14.25 Finding the limits of integration in polar coordinates for the region in Example 1.

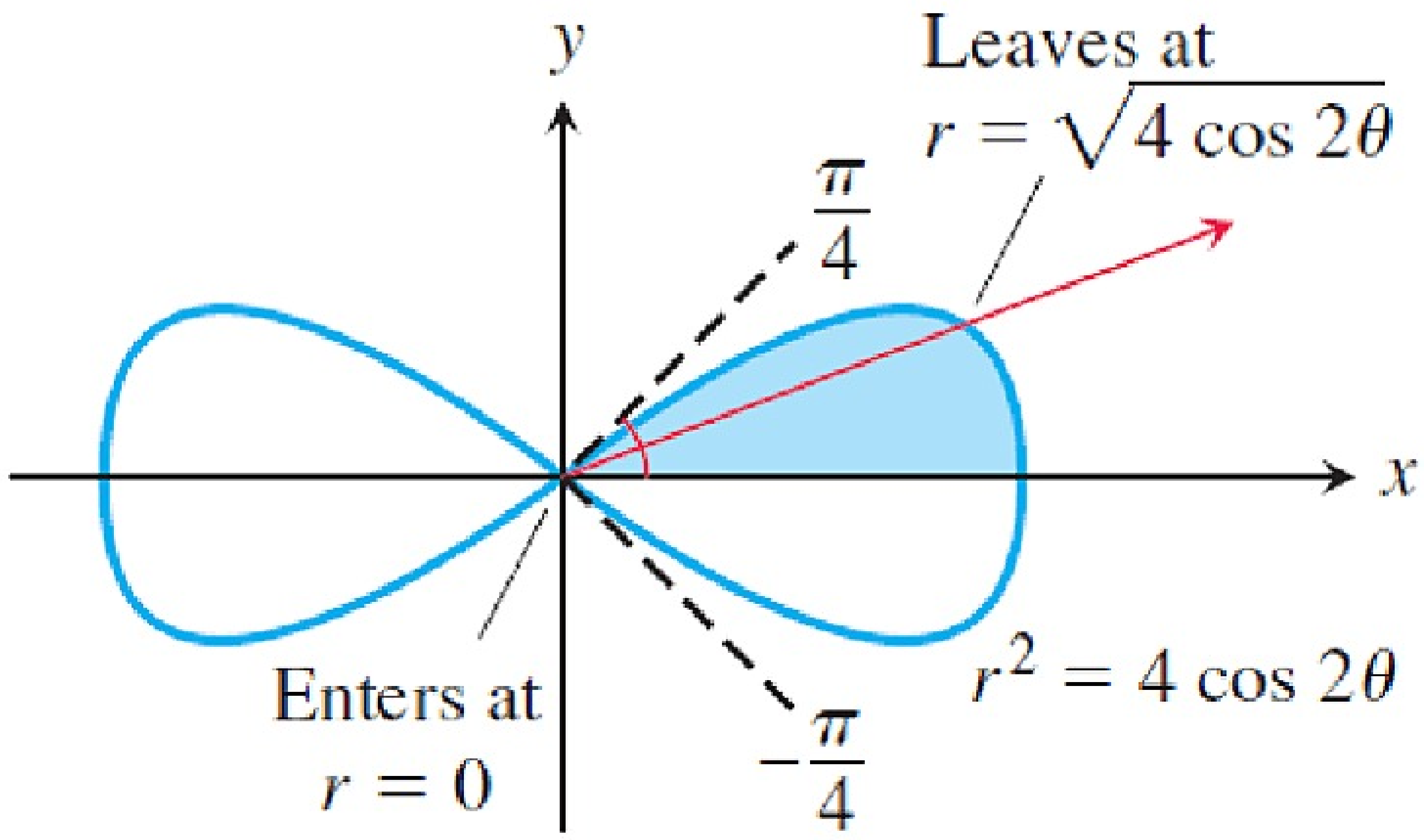


FIGURE 14.26 To integrate over the shaded region, we run r from 0 to $\sqrt{4 \cos 2\theta}$ and θ from 0 to $\pi/4$ (Example 2).

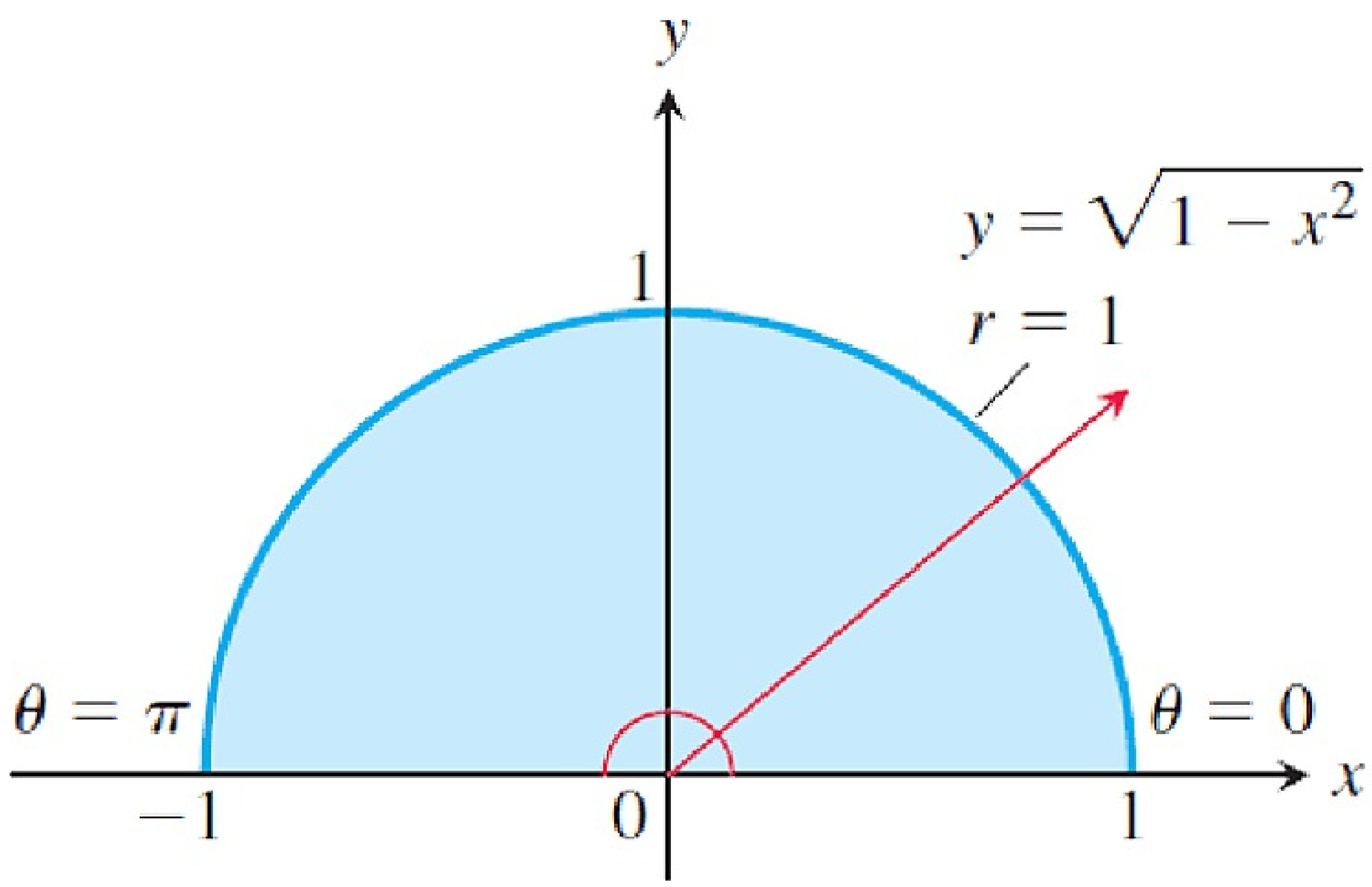


FIGURE 14.27 The semicircular region in Example 3 is the region

$$0 \leq r \leq 1, \qquad 0 \leq \theta \leq \pi.$$

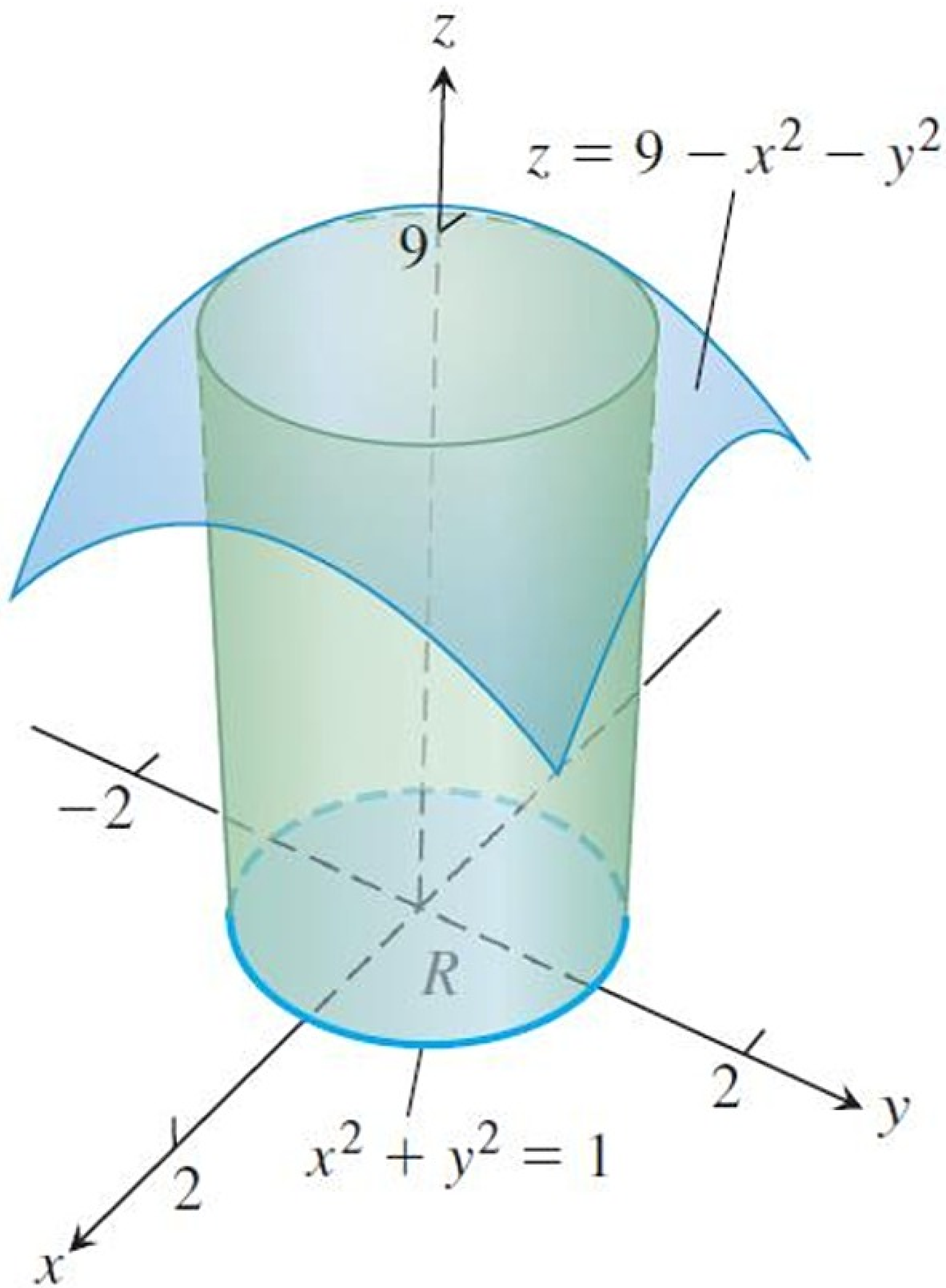


FIGURE 14.28 The solid region in Example 5.

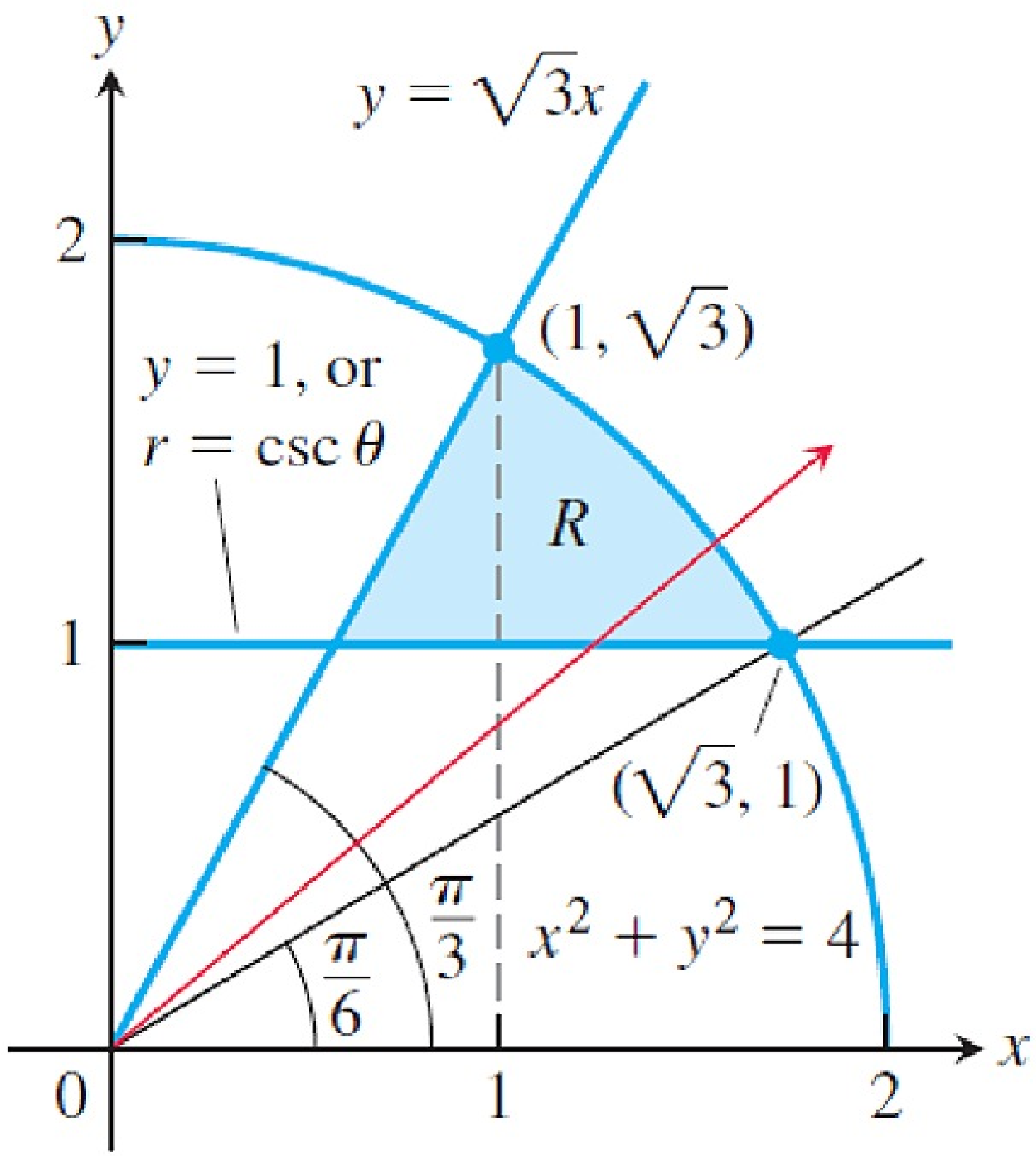


FIGURE 14.29 The region R in Example 6.

Section 14.5

Triple Integrals in Rectangular Coordinates

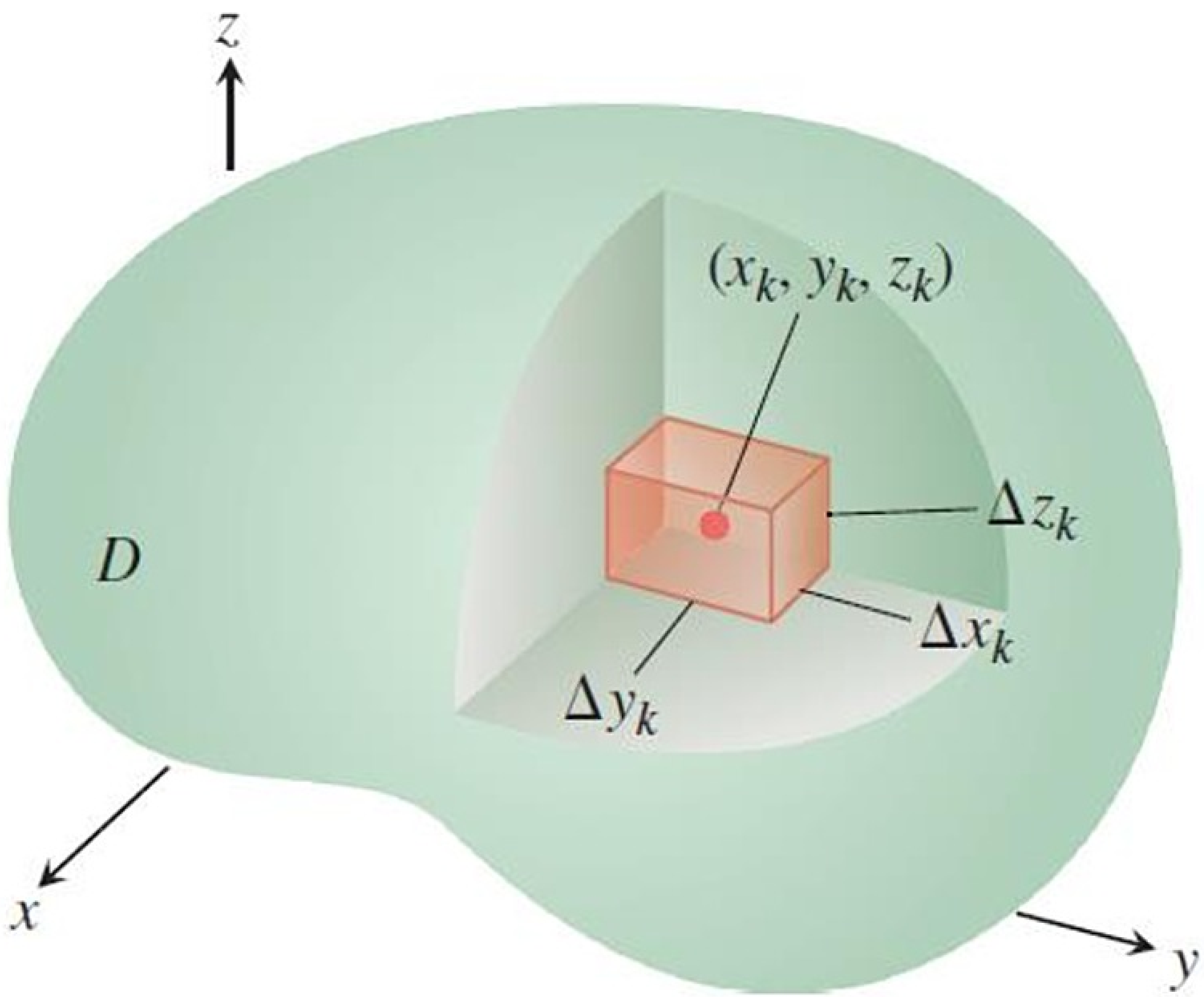


FIGURE 14.30 Partitioning a solid with rectangular cells of volume ΔV_k.

DEFINITION The **volume** of a closed, bounded region D in space is

$$V = \iiint_D dV.$$

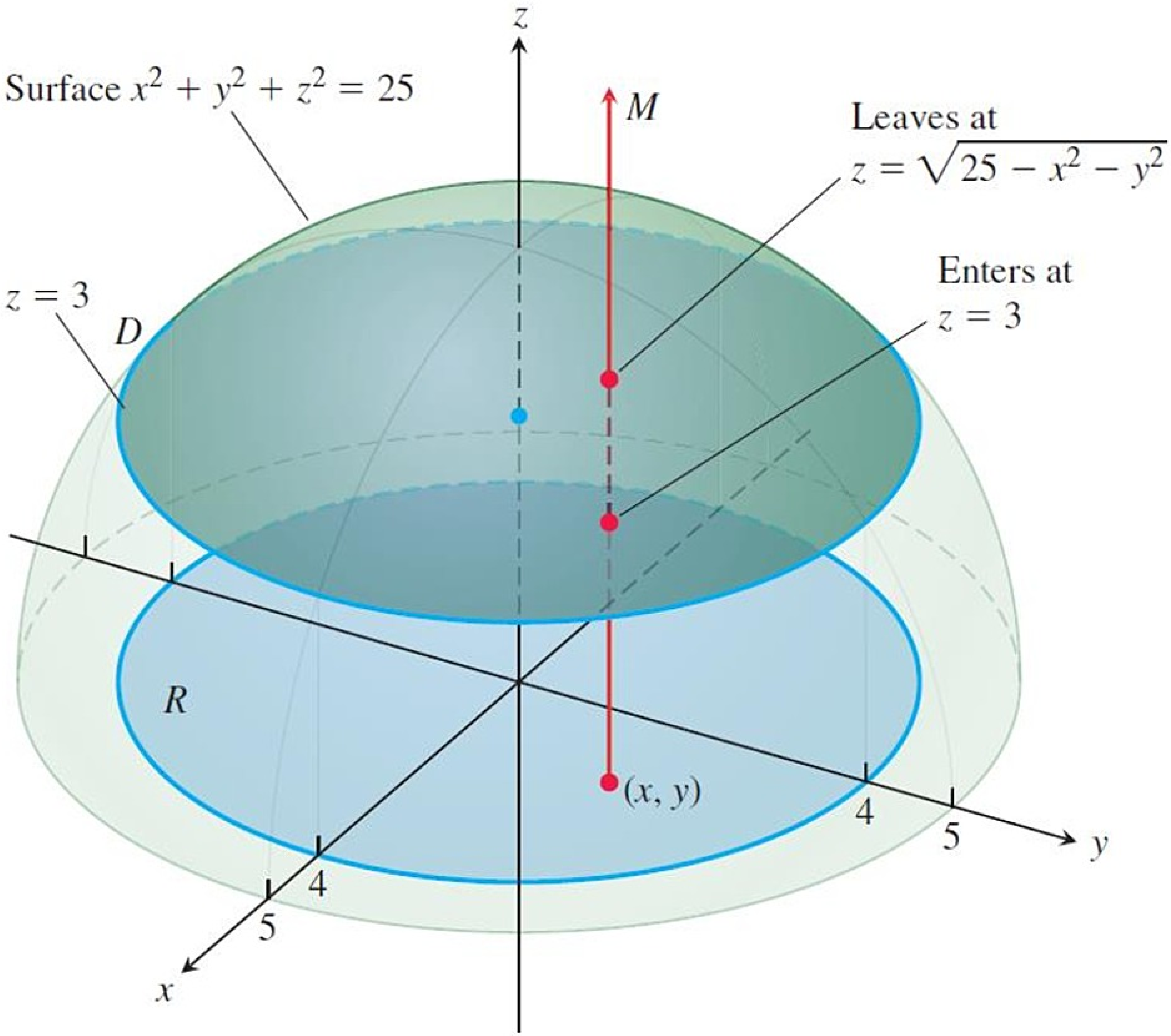


FIGURE 14.31 Finding the limits of integration for evaluating the triple integral of a function defined over the portion of the sphere of radius 5 that lies above the plane $z = 3$ (Example 1).

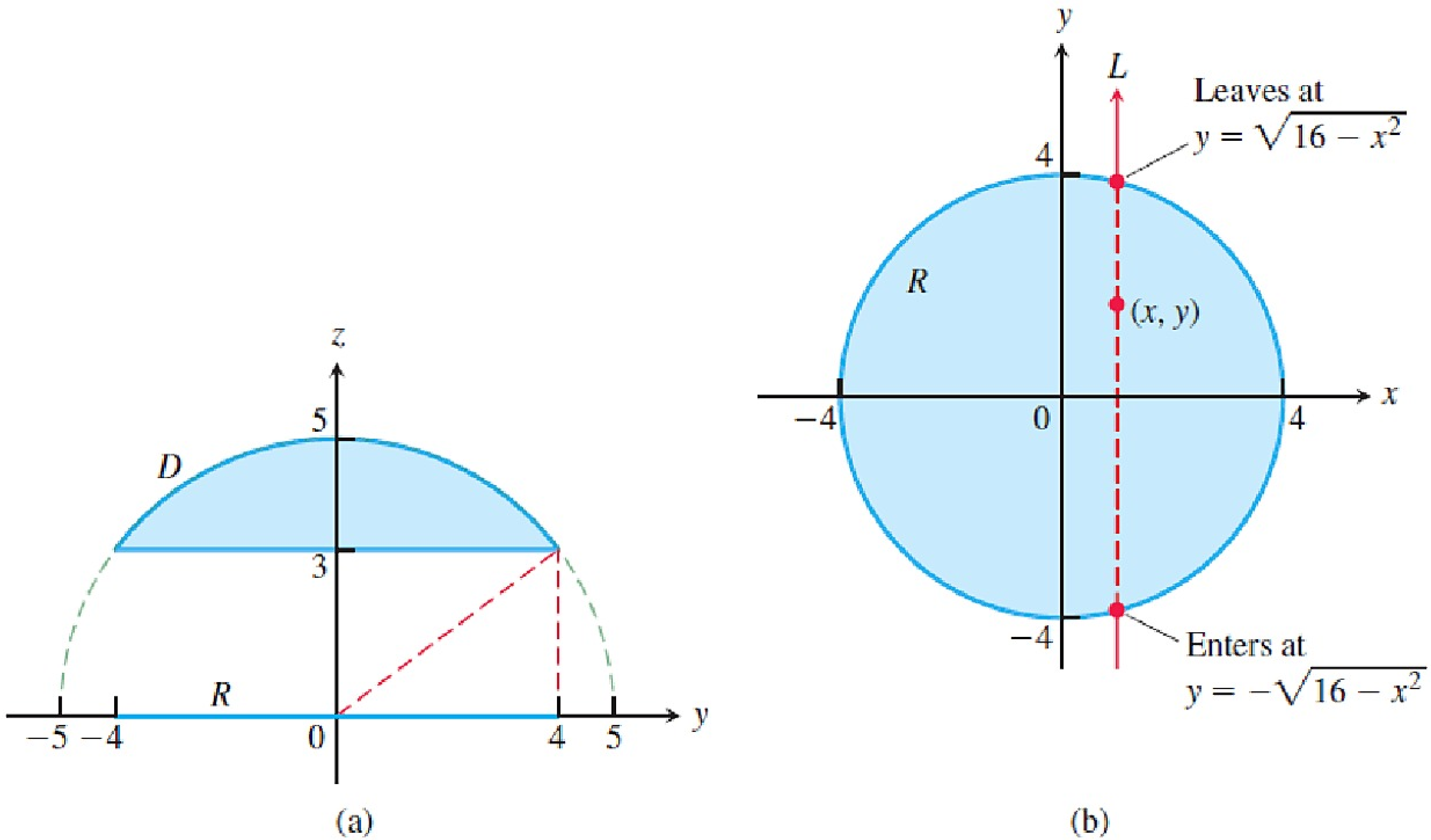
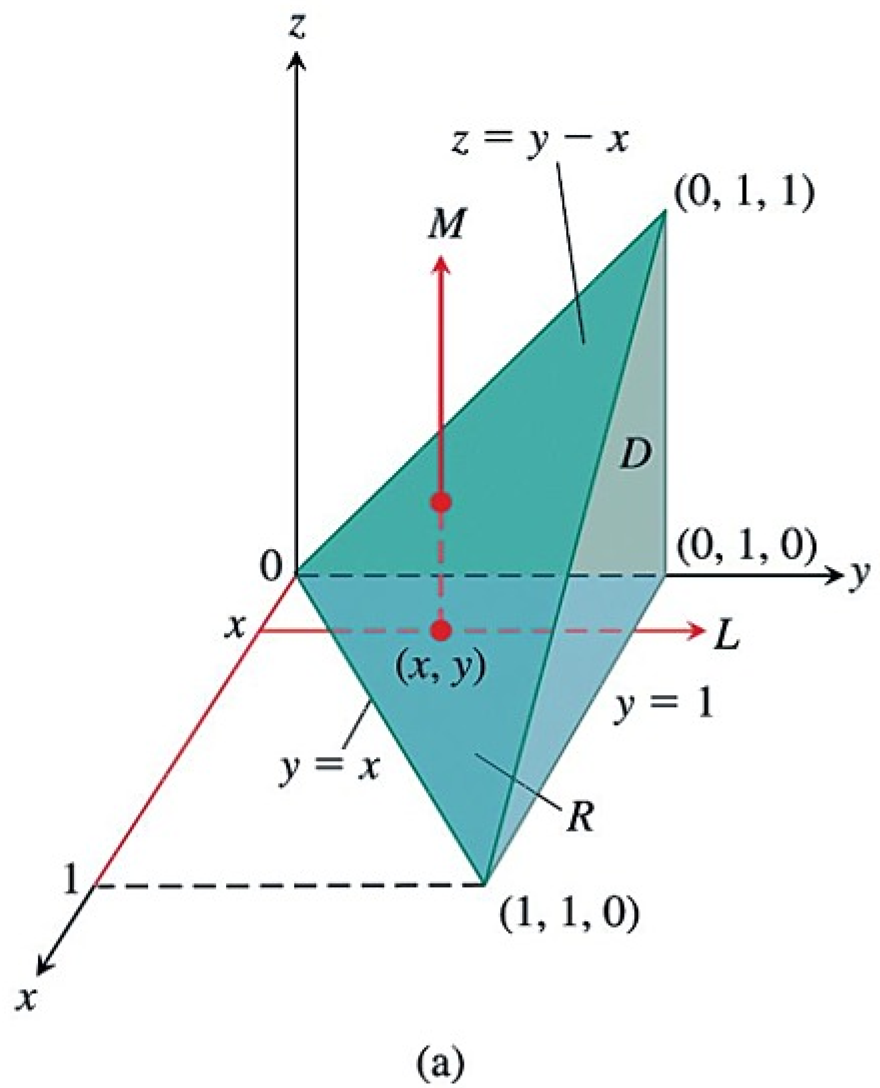


FIGURE 14.32 (a) Side view of the region from Example 1, looking down the x-axis. The dashed right triangle has a hypotenuse of length 5 and sides of lengths 3 and 4. In this side view, the shadow region R lies between -4 and 4 on the y-axis. (b) The "shadow region" R shown face-on in the xy-plane.

Pearson



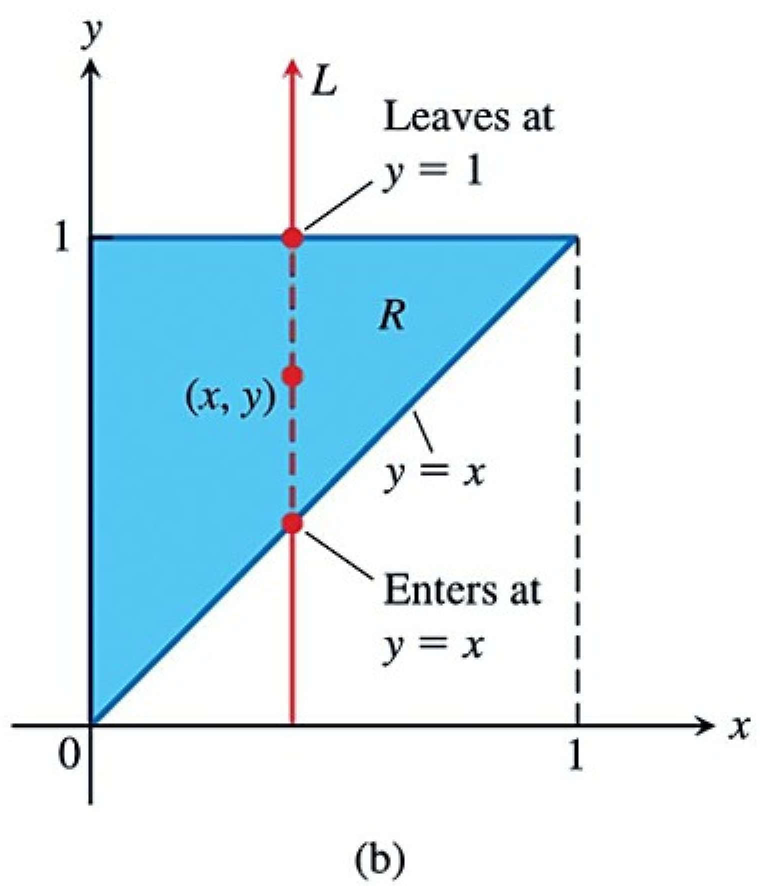


FIGURE 14.33 (a) The tetrahedron in Example 2 showing how the limits of integration are found for the order $dz\,dy\,dx$. (b) The "shadow region" R shown face-on in the xy-plane.

Pearson

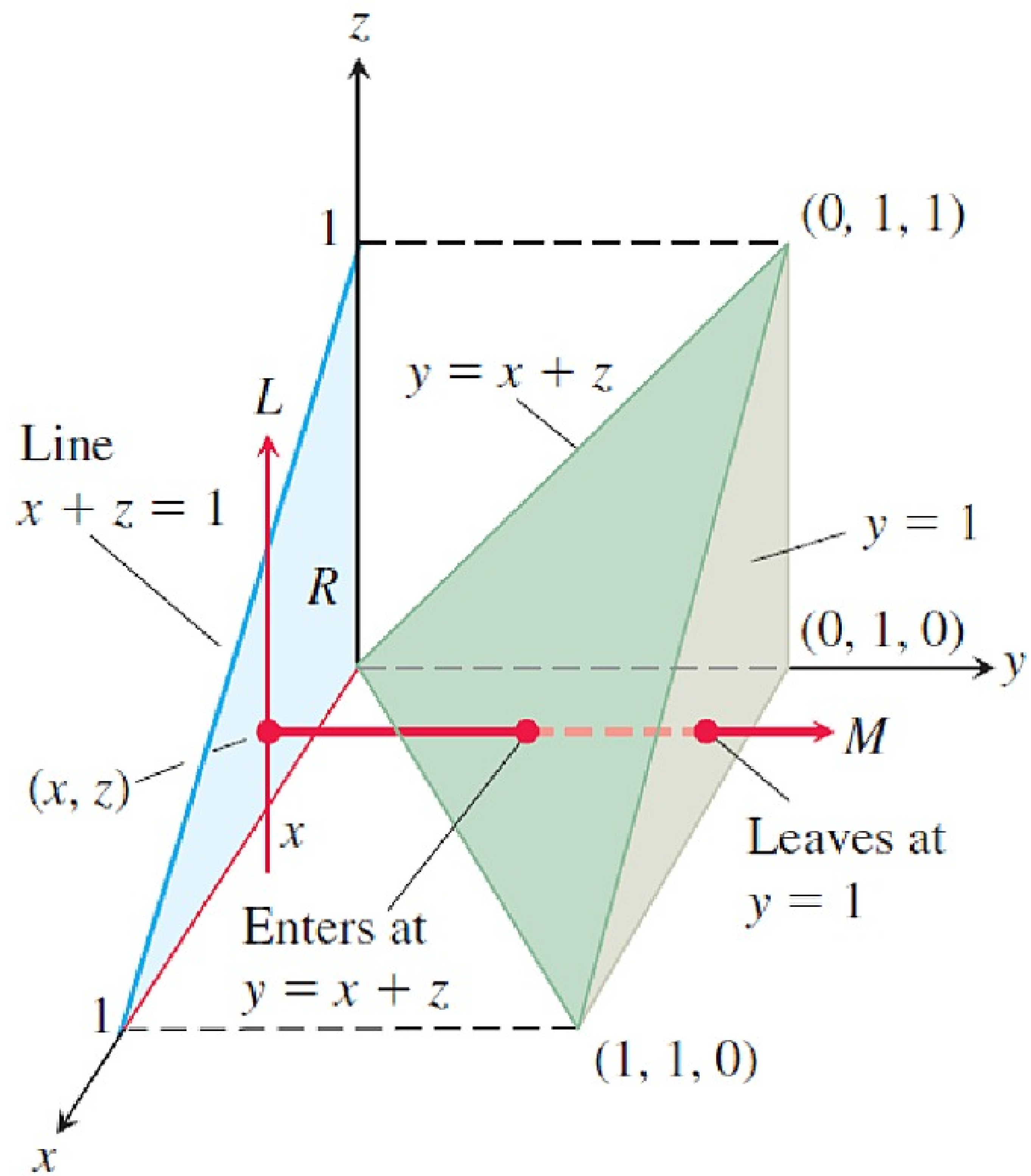


FIGURE 14.34 Finding the limits of integration for evaluating the triple integral of a function defined over the tetrahedron D (Example 3).

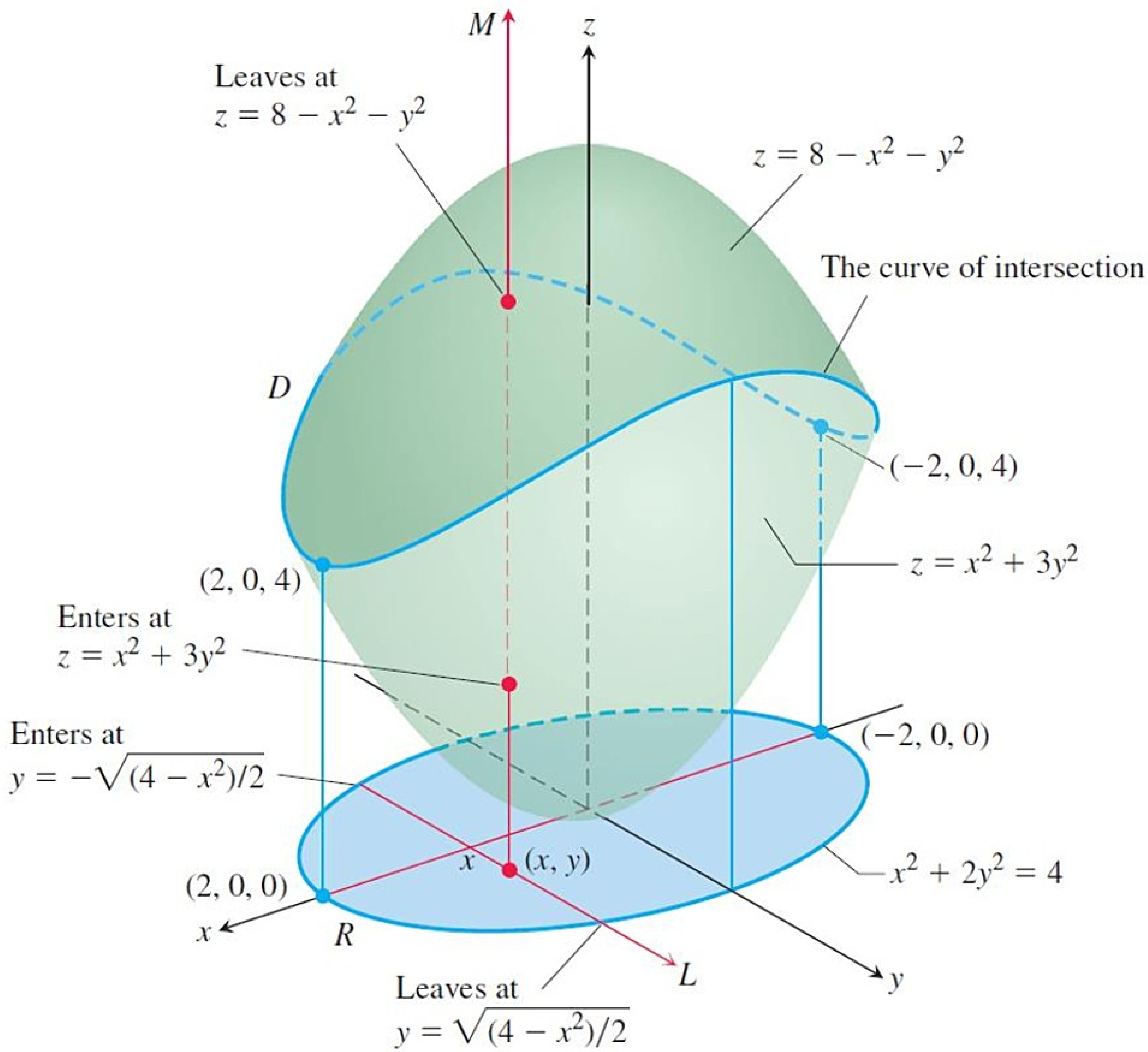


FIGURE 14.35 The volume of the region enclosed by two paraboloids, calculated in Example 4.

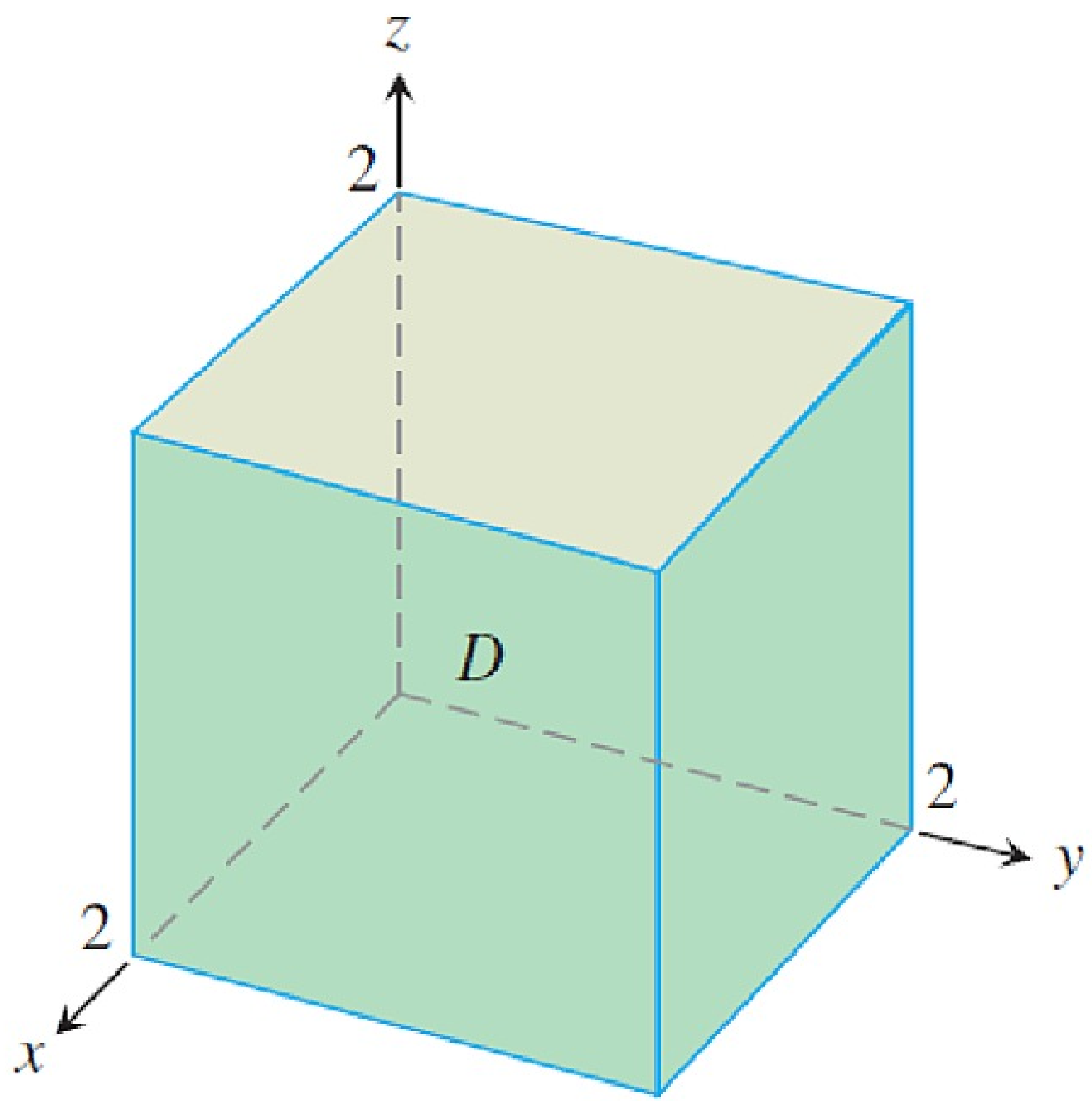


FIGURE 14.36 The region of integration in Example 5.

Section 14.6

Applications

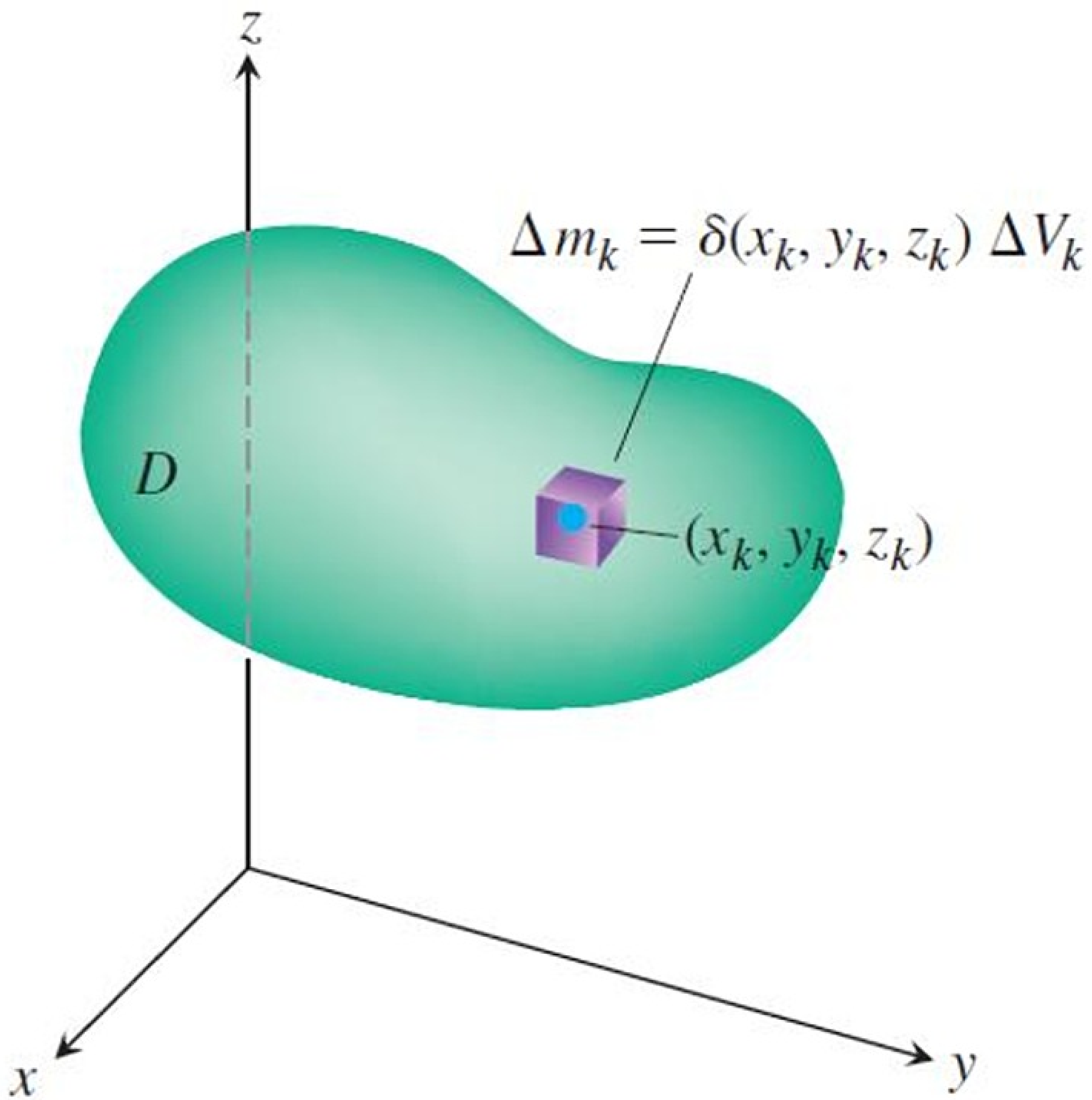


FIGURE 14.37 To define an object's mass, we first imagine it to be partitioned into a finite number of mass elements Δm_k.

TABLE 14.1 Mass and first moment formulas

THREE-DIMENSIONAL SOLID

Mass: $M = \iiint\limits_D \delta \, dV$ $\delta = \delta(x, y, z)$ is the density at (x, y, z).

First moments about the coordinate planes:

$$M_{yz} = \iiint\limits_D x \, \delta \, dV, \qquad M_{xz} = \iiint\limits_D y \, \delta \, dV, \qquad M_{xy} = \iiint\limits_D z \, \delta \, dV$$

Center of mass: $\bar{x} = \dfrac{M_{yz}}{M}, \quad \bar{y} = \dfrac{M_{xz}}{M}, \quad \bar{z} = \dfrac{M_{xy}}{M}$

TWO-DIMENSIONAL PLATE

Mass: $M = \iint\limits_R \delta \, dA$ $\delta = \delta(x, y)$ is the density at (x, y).

First moments: $M_y = \iint\limits_R x \, \delta \, dA, \quad M_x = \iint\limits_R y \, \delta \, dA$

Center of mass: $\bar{x} = \dfrac{M_y}{M}, \quad \bar{y} = \dfrac{M_x}{M}$

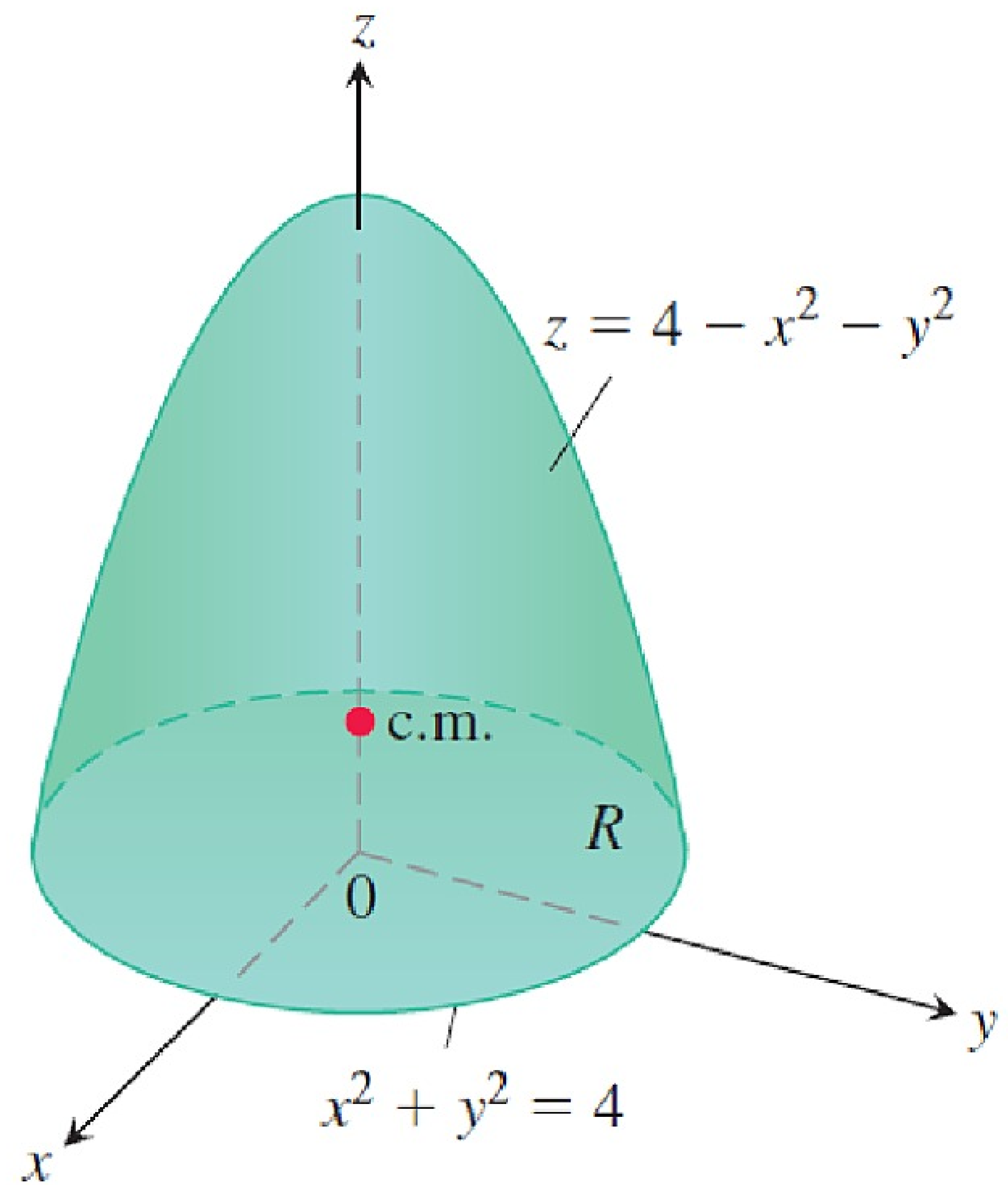


FIGURE 14.38 Finding the center of mass of a solid (Example 1).

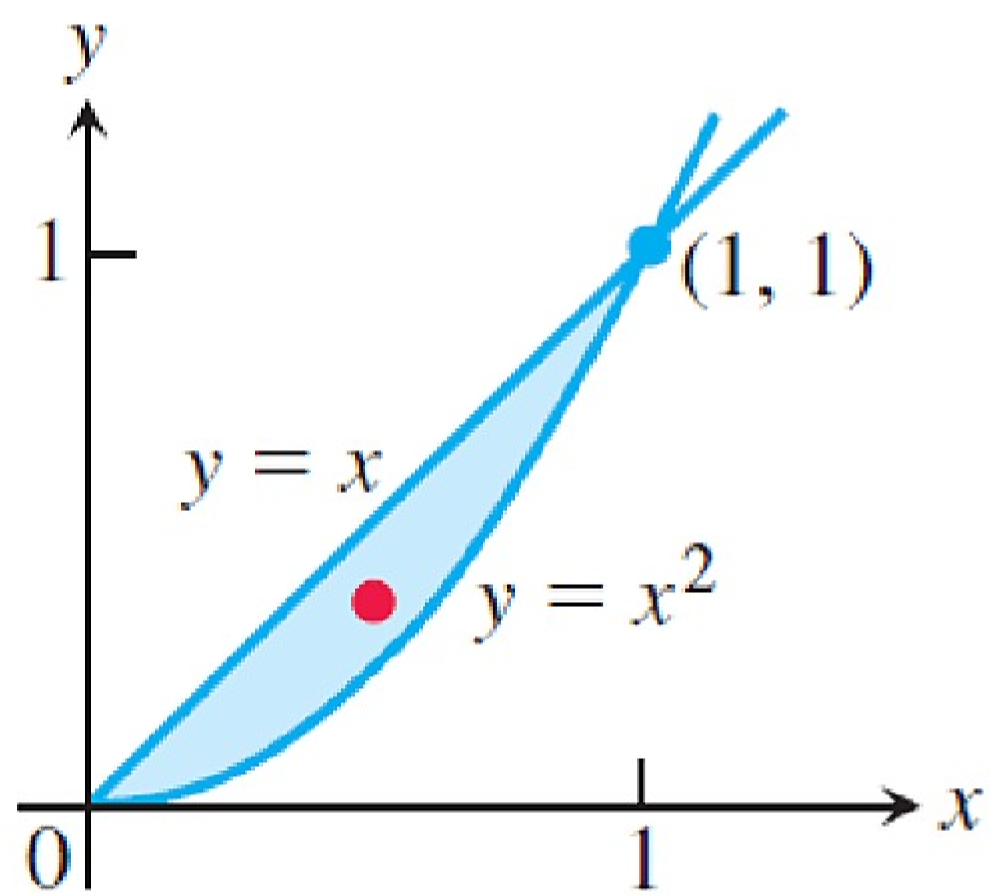


FIGURE 14.39 The centroid of this region is found in Example 2.

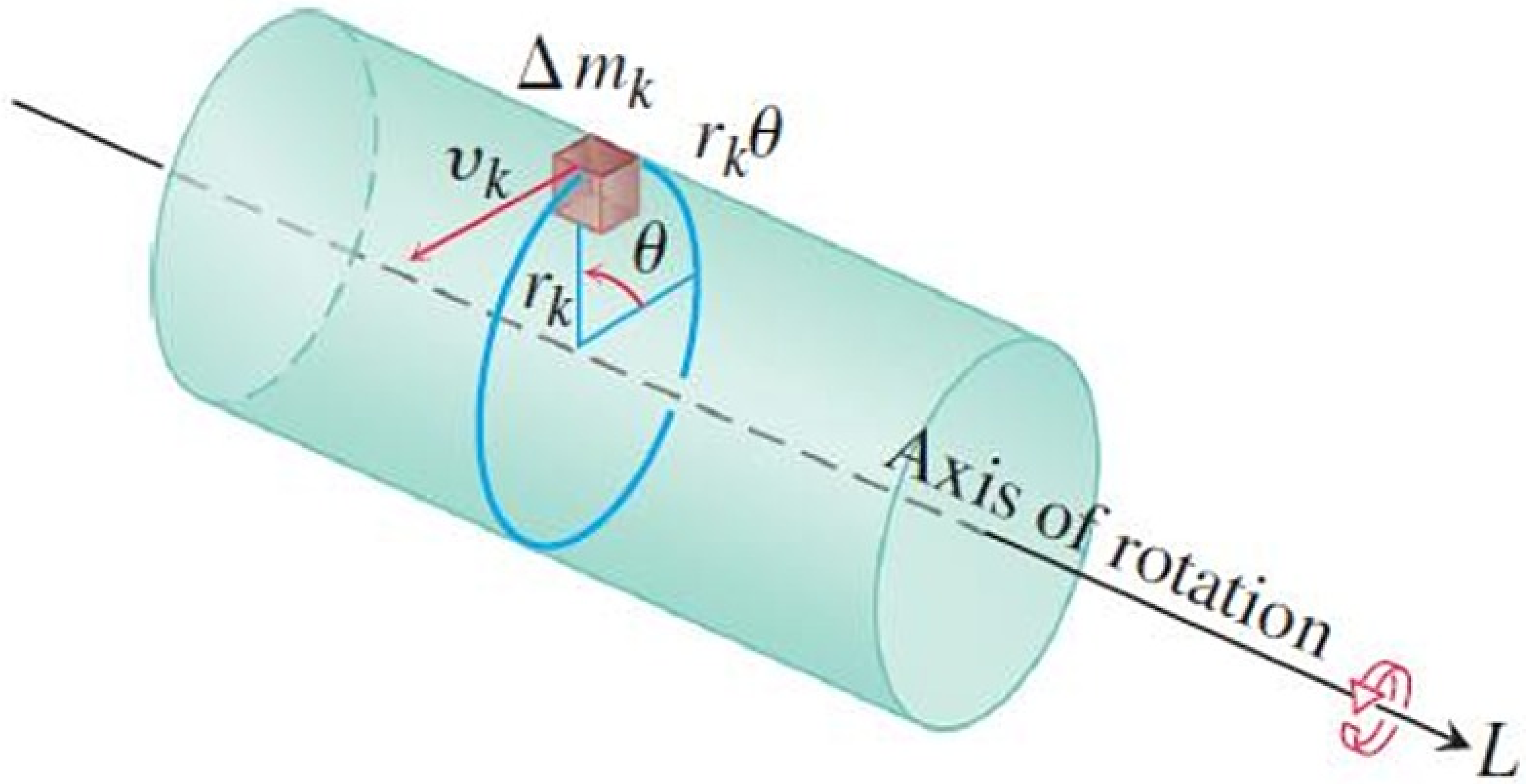


FIGURE 14.40 To find an integral for the amount of energy stored in a rotating shaft, we first imagine the shaft to be partitioned into small blocks. Each block has its own kinetic energy. We add the contributions of the individual blocks to find the kinetic energy of the shaft.

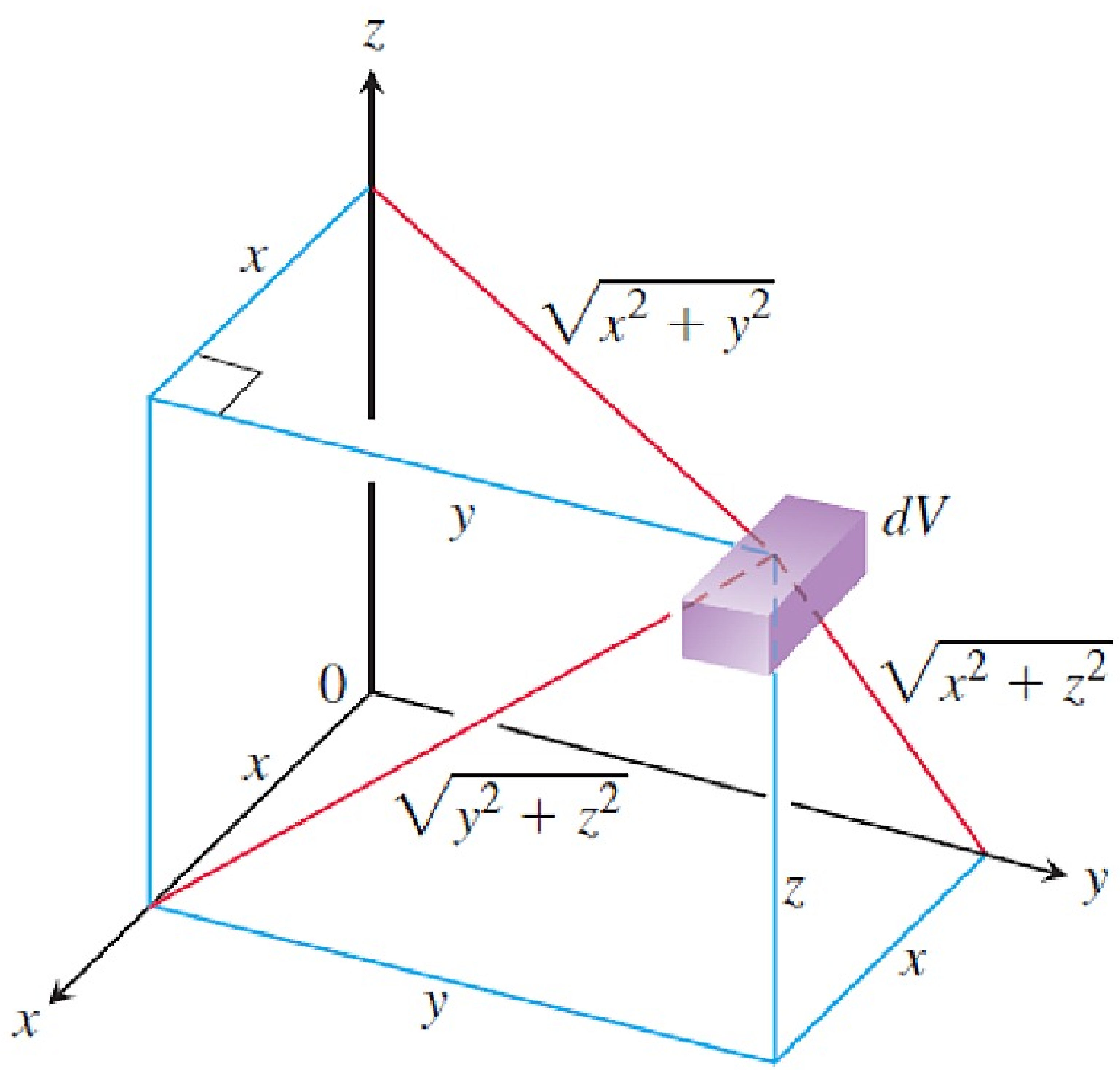


FIGURE 14.41 Distances from dV to the coordinate planes and axes.

TABLE 14.2 Moments of inertia (second moments) formulas

THREE-DIMENSIONAL SOLID		
About the x-axis:	$I_x = \iiint (y^2 + z^2)\,\delta\,dV$	$\delta = \delta(x, y, z)$
About the y-axis:	$I_y = \iiint (x^2 + z^2)\,\delta\,dV$	
About the z-axis:	$I_z = \iiint (x^2 + y^2)\,\delta\,dV$	
About a line L:	$I_L = \iiint r^2(x, y, z)\,\delta\,dV$	$r(x, y, z) =$ distance from the point (x, y, z) to line L
TWO-DIMENSIONAL PLATE		
About the x-axis:	$I_x = \iint y^2\,\delta\,dA$	$\delta = \delta(x, y)$
About the y-axis:	$I_y = \iint x^2\,\delta\,dA$	
About a line L:	$I_L = \iint r^2(x, y)\,\delta\,dA$	$r(x, y) =$ distance from (x, y) to L
About the origin (polar moment):	$I_0 = \iint (x^2 + y^2)\,\delta\,dA = I_x + I_y$	

EXAMPLE 3 Find I_x, I_y, I_z for the rectangular solid of constant density δ shown in Figure 14.42.

Solution The formula for I_x gives

$$I_x = \int_{-c/2}^{c/2} \int_{-b/2}^{b/2} \int_{-a/2}^{a/2} (y^2 + z^2)\, \delta\, dx\, dy\, dz.$$

We can avoid some of the work of integration by observing that $(y^2 + z^2)\delta$ is an even function of x, y, and z since δ is constant. The rectangular solid consists of eight symmetric pieces, one in each octant. We can evaluate the integral on one of these pieces and then multiply by 8 to get the total value.

$$I_x = 8\int_{0}^{c/2} \int_{0}^{b/2} \int_{0}^{a/2} (y^2 + z^2)\, \delta\, dx\, dy\, dz = 4a\delta \int_{0}^{c/2} \int_{0}^{b/2} (y^2 + z^2)\, dy\, dz$$

$$= 4a\delta \int_{0}^{c/2} \left[\frac{y^3}{3} + z^2 y\right]_{y=0}^{y=b/2} dz$$

$$= 4a\delta \int_{0}^{c/2} \left(\frac{b^3}{24} + \frac{z^2 b}{2}\right) dz$$

$$= 4a\delta \left(\frac{b^3 c}{48} + \frac{c^3 b}{48}\right) = \frac{abc\delta}{12}(b^2 + c^2) = \frac{M}{12}(b^2 + c^2).$$ $M = abc\delta$

Similarly,

$$I_y = \frac{M}{12}(a^2 + c^2) \quad \text{and} \quad I_z = \frac{M}{12}(a^2 + b^2).$$

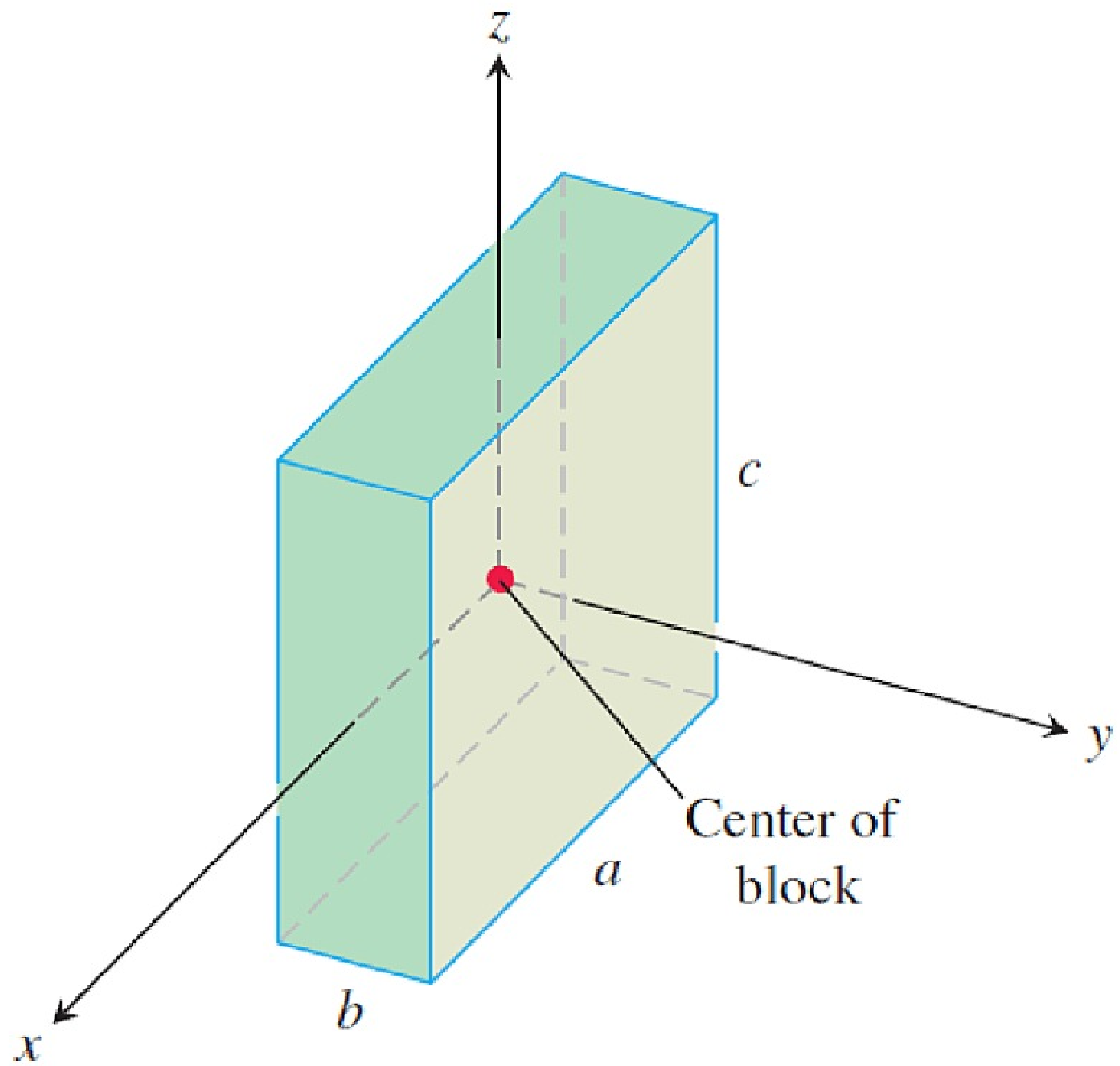


FIGURE 14.42 Finding I_x, I_y, and I_z for the block shown here. The origin lies at the center of the block (Example 3).

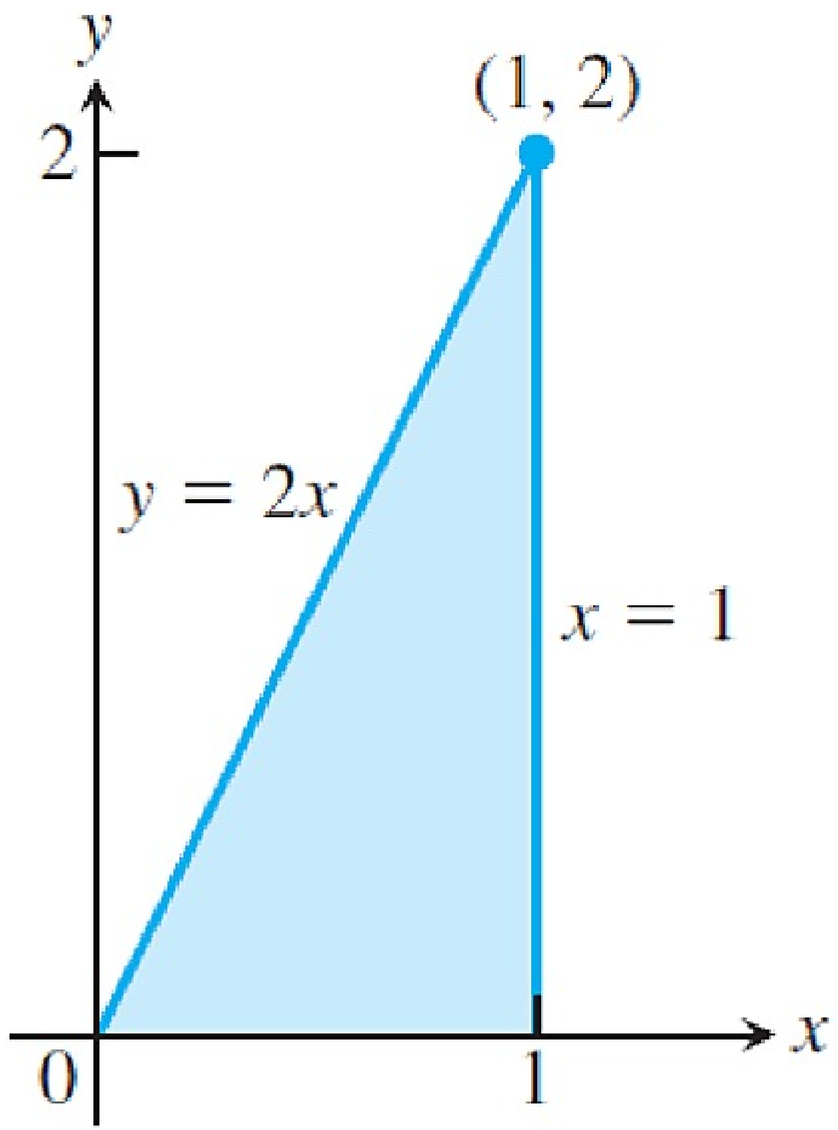


FIGURE 14.43 The triangular region covered by the plate in Example 4.

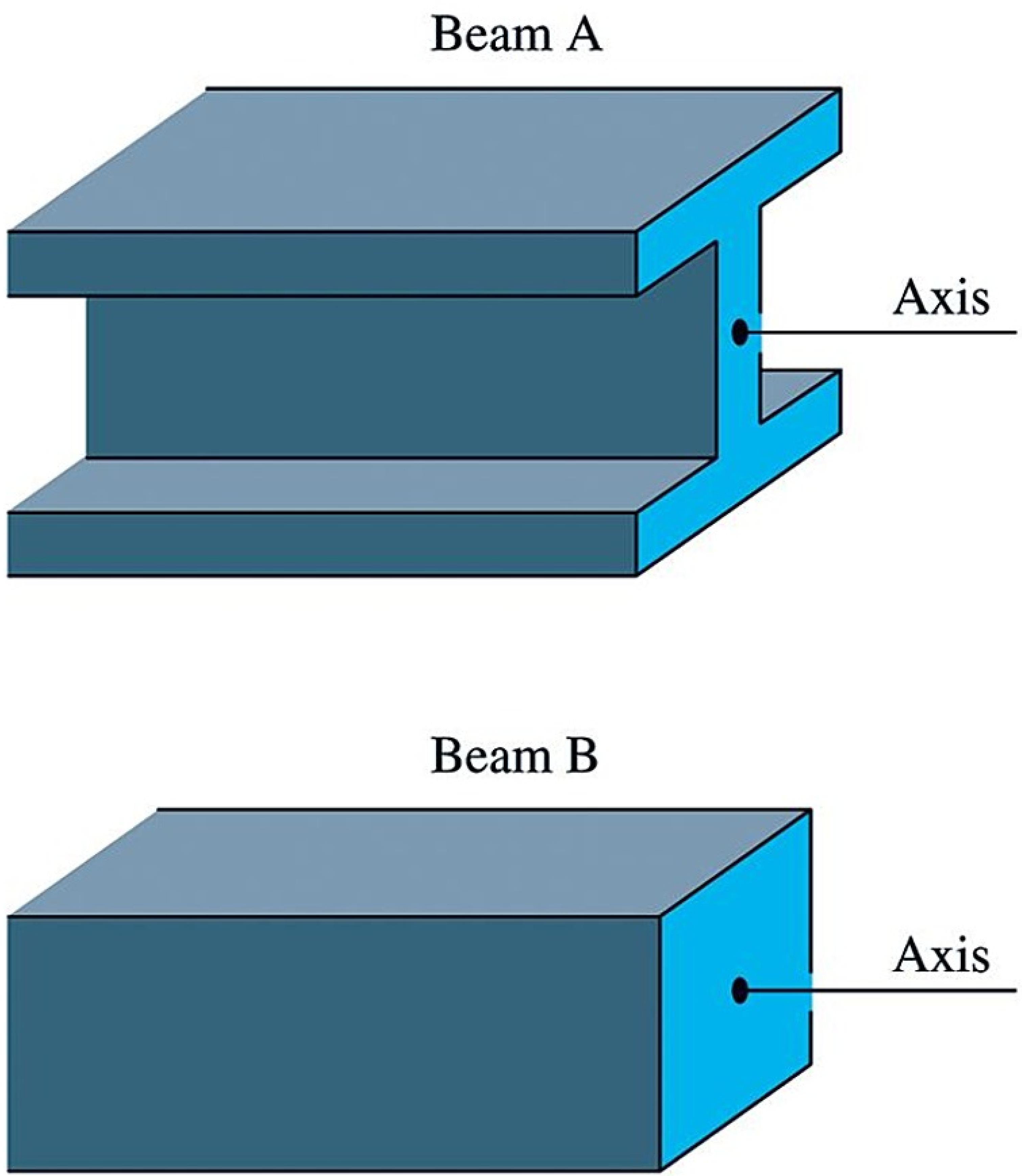


FIGURE 14.44 The greater the polar moment of inertia of the cross-section of a beam about the beam's longitudinal axis, the stiffer the beam. Beams A and B have the same cross-sectional area, but A is stiffer.

DEFINITION A **joint probability density function** f is a function that satisfies three conditions:

1. $f(x, y) \geq 0$
2. $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\, dx\, dy = 1$
3. $\displaystyle P\big((X, Y) \in R\big) = \iint_R f(x, y)\, dx\, dy.$

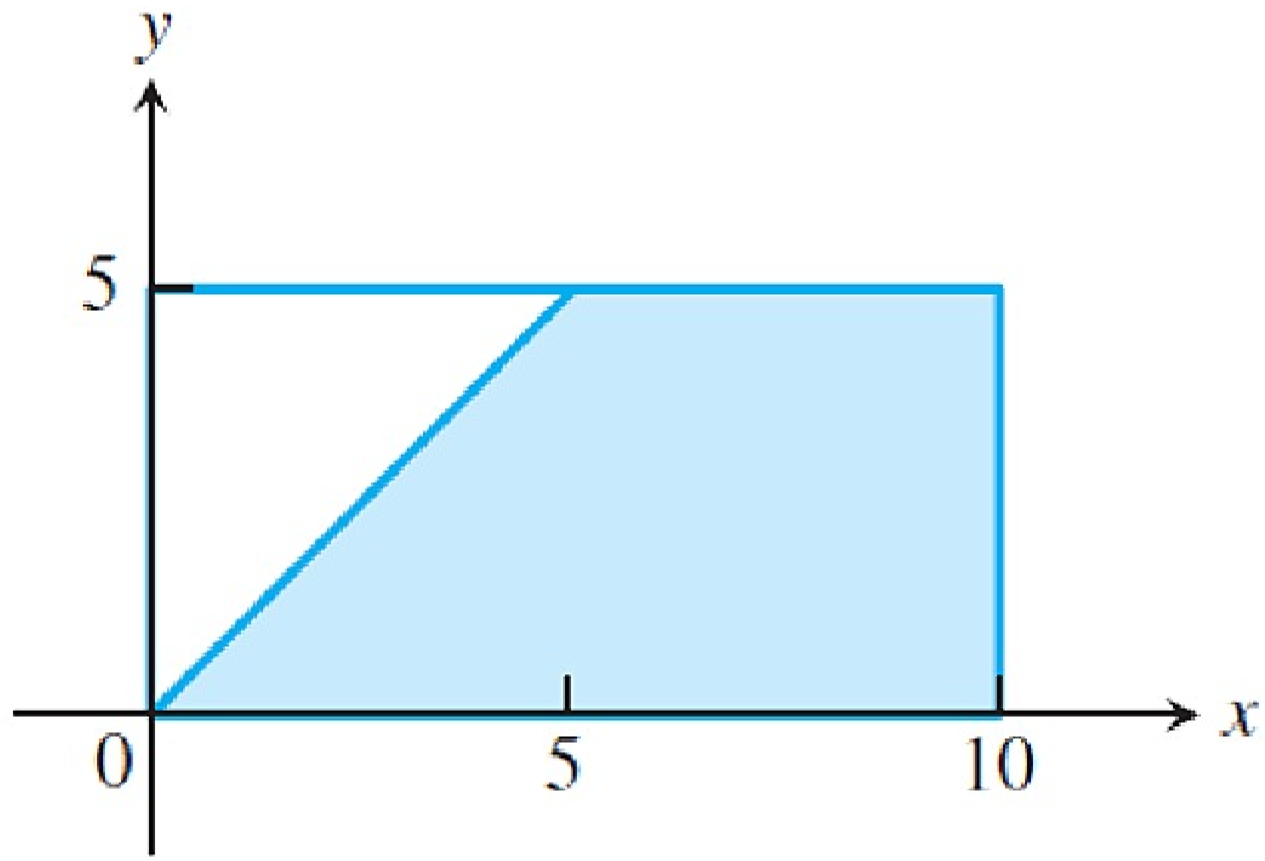


FIGURE 14.45 The pair of random variables X and Y take values anywhere in this rectangle with equal probability. In the shaded region we have $X > Y$.

Section 14.7

Triple Integrals in Cylindrical and Spherical Coordinates

DEFINITION **Cylindrical coordinates** represent a point P in space by ordered triples (r, θ, z) in which $r \geq 0$,

1. r and θ are polar coordinates for the vertical projection of P on the xy-plane
2. z is the rectangular vertical coordinate.

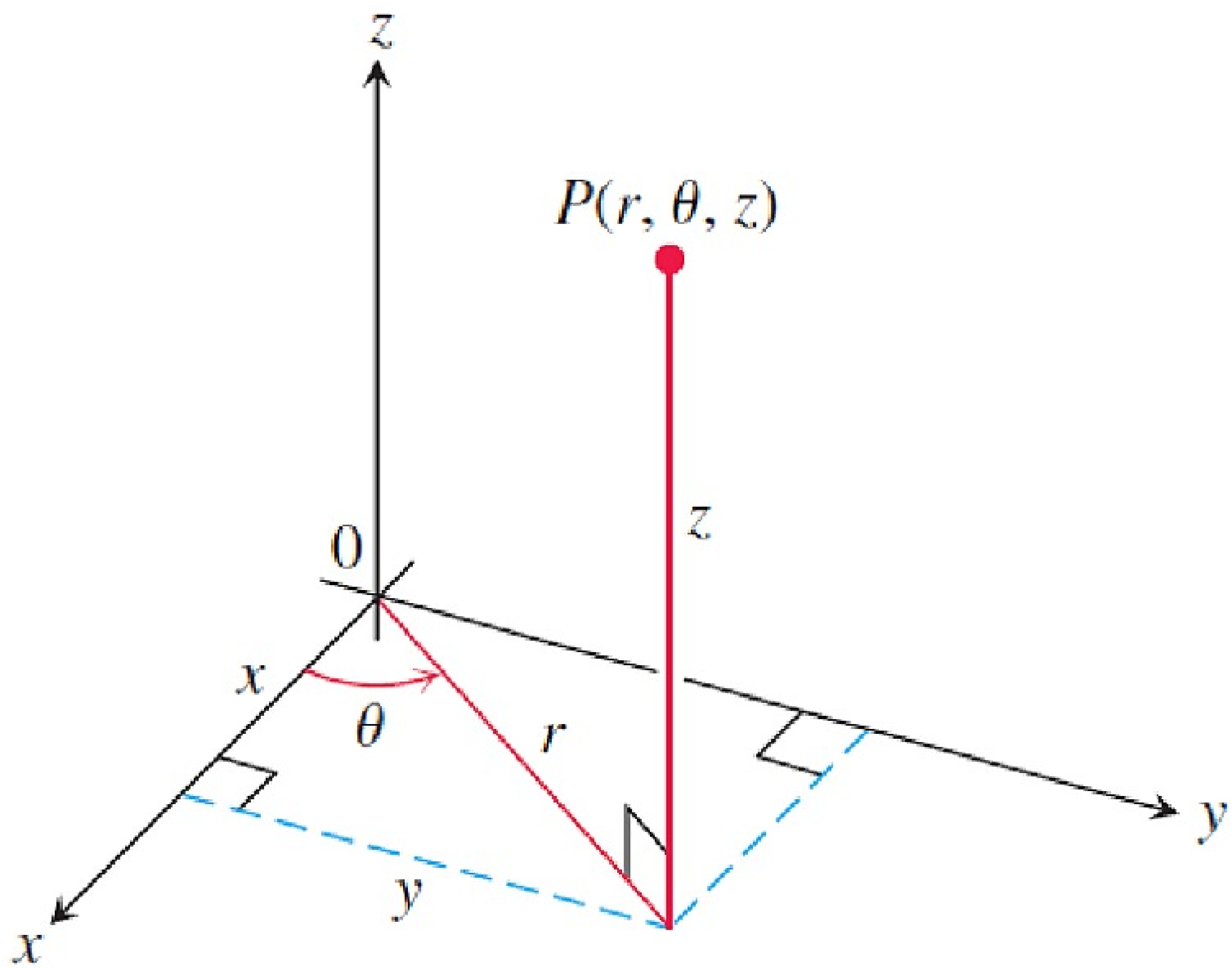


FIGURE 14.46 The cylindrical coordinates of a point in space are r, θ, and z.

Equations Relating Rectangular (x, y, z) and Cylindrical (r, θ, z) Coordinates

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z,$$

$$r^2 = x^2 + y^2, \qquad \tan\theta = y/x$$

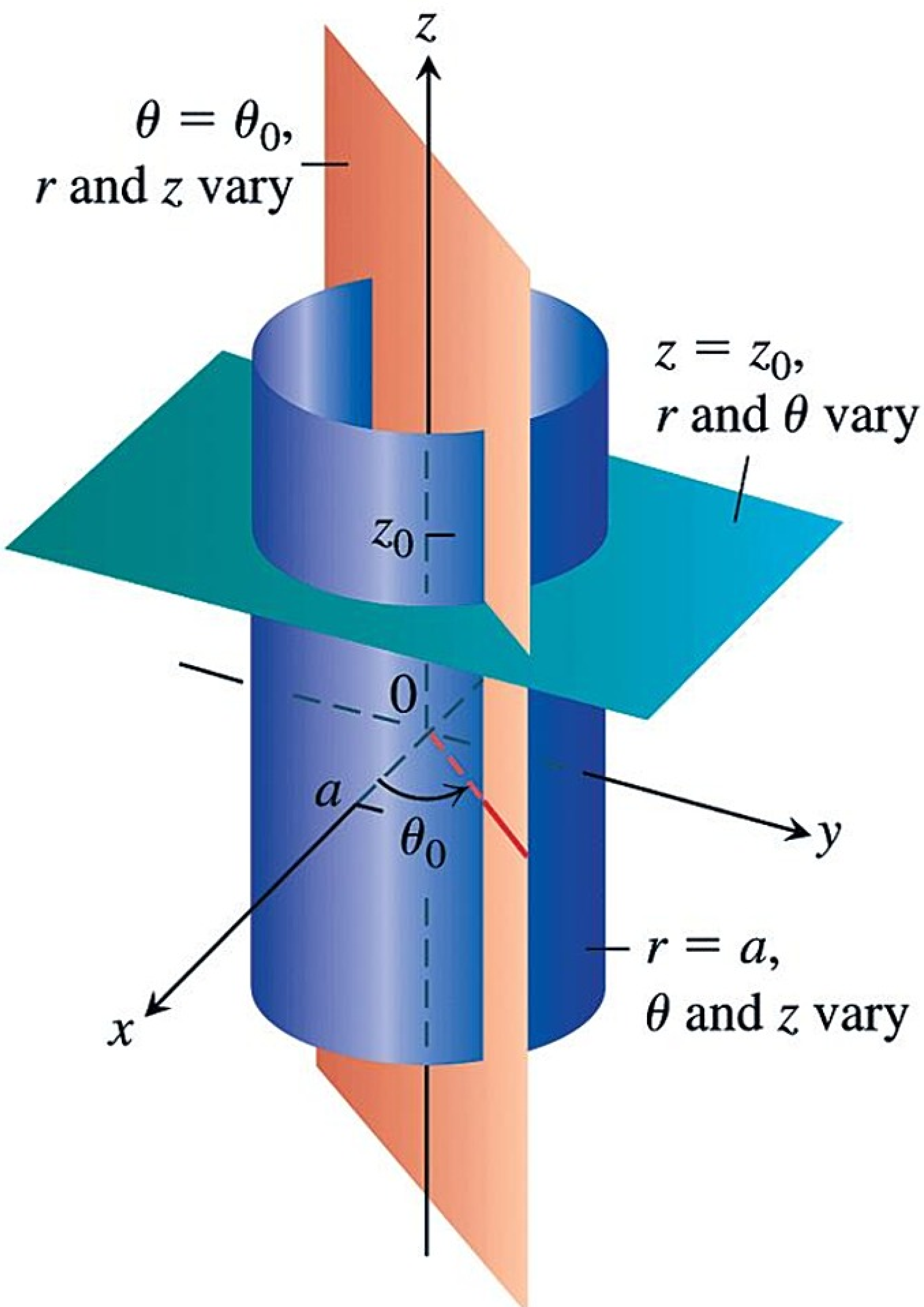


FIGURE 14.47 Constant-coordinate equations in cylindrical coordinates yield cylinders and planes.

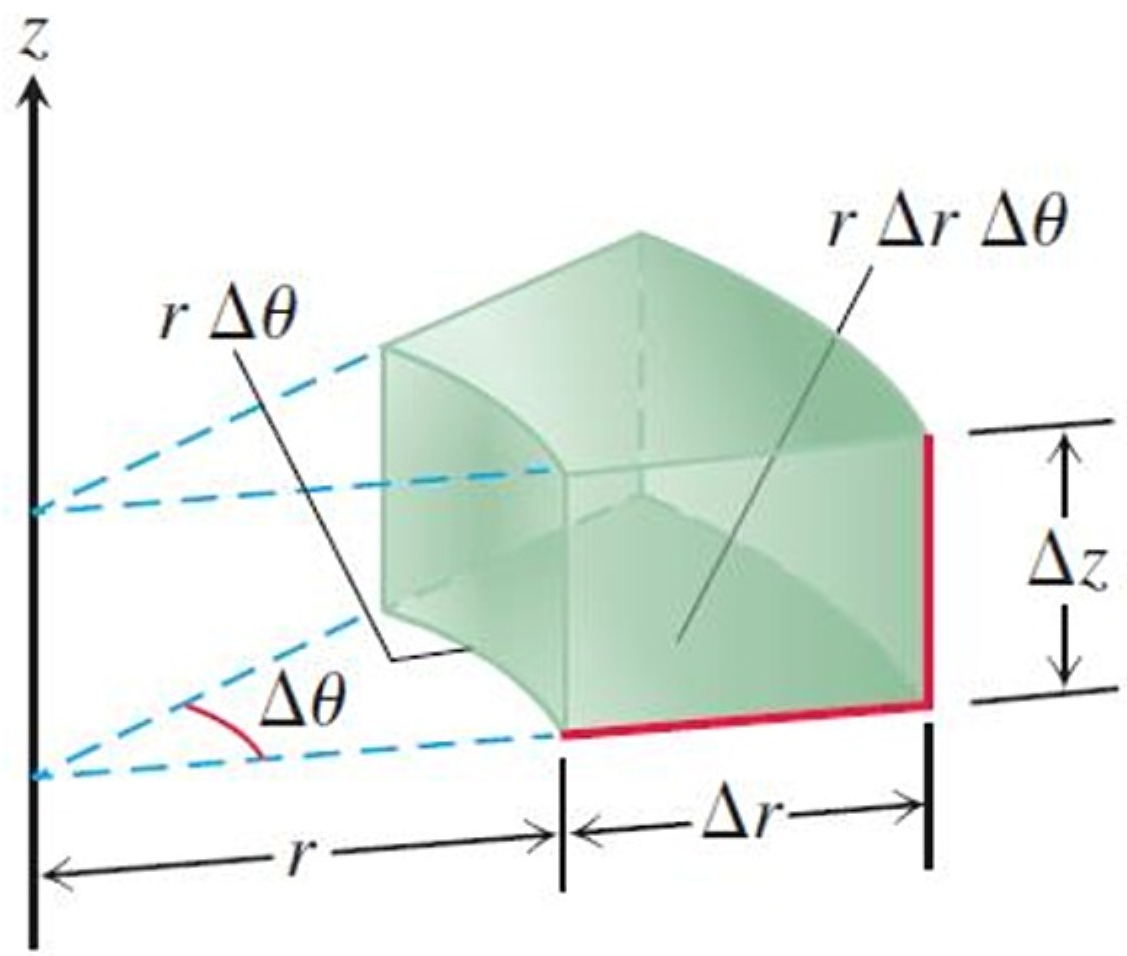


FIGURE 14.48 In cylindrical coordinates the volume of the wedge is approximated by the product $\Delta V = \Delta z\, r\, \Delta r\, \Delta\theta$.

$$\lim_{n \to \infty} S_n = \iiint_D f \, dV = \iiint_D f \, dz \, r \, dr \, d\theta.$$

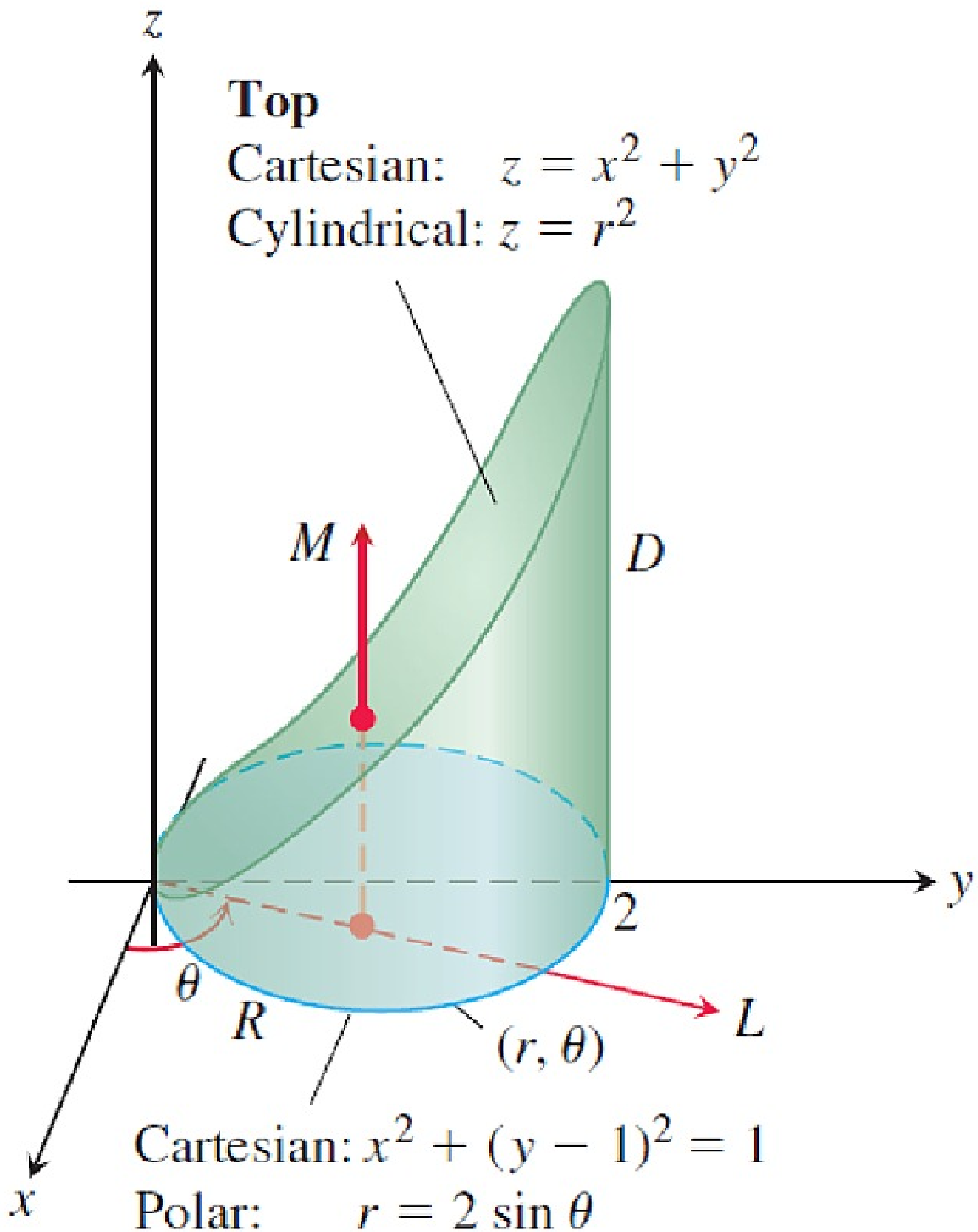


FIGURE 14.49 Finding the limits of integration for evaluating an integral in cylindrical coordinates (Example 1).

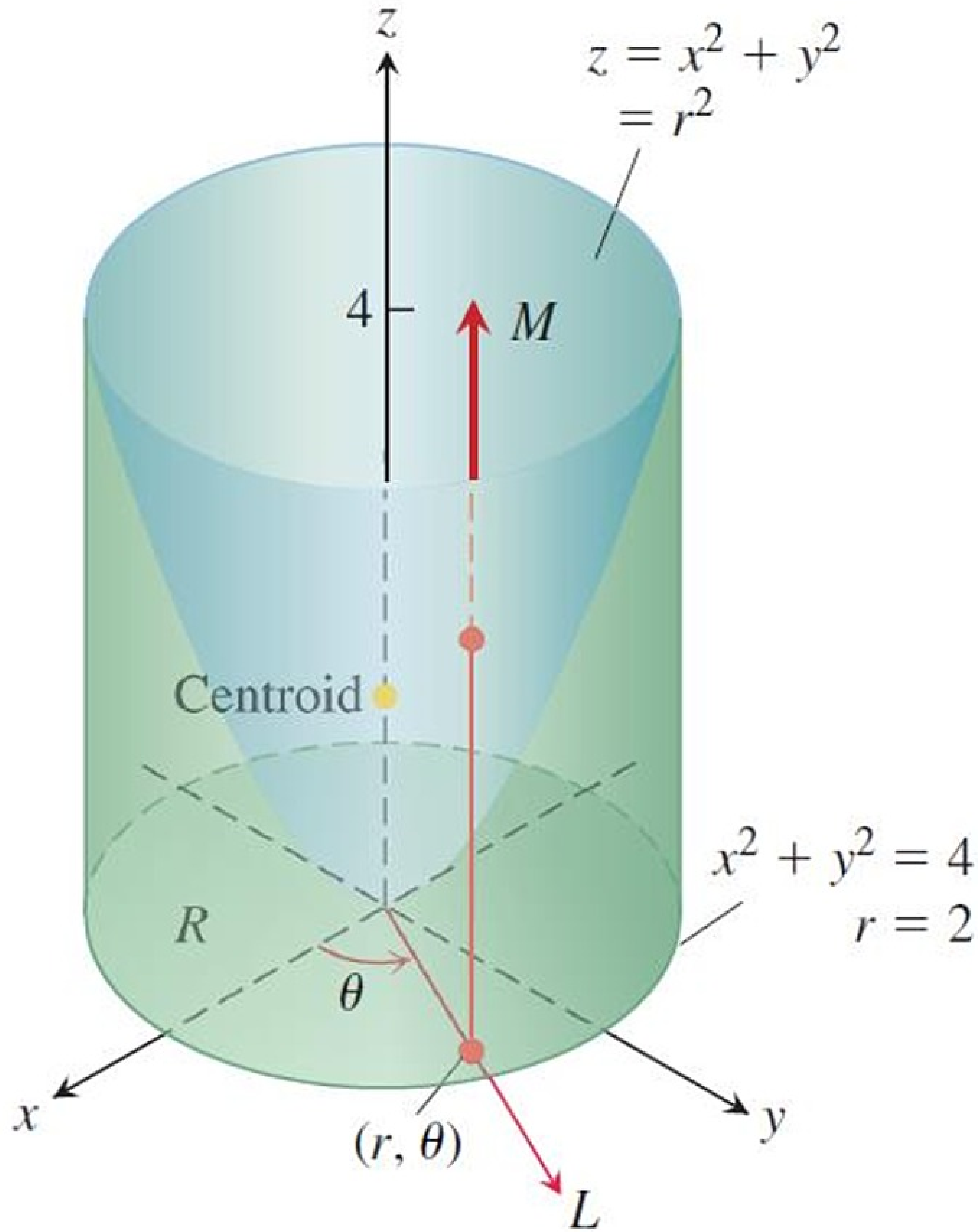


FIGURE 14.50 Example 2 shows how to find the centroid of this solid.

DEFINITION **Spherical coordinates** represent a point P in space by ordered triples (ρ, ϕ, θ) in which

1. ρ is the distance from P to the origin $(\rho \geq 0)$.
2. ϕ is the angle $\overrightarrow{OP}$ makes with the positive z-axis $(0 \leq \phi \leq \pi)$.
3. θ is the angle from cylindrical coordinates.

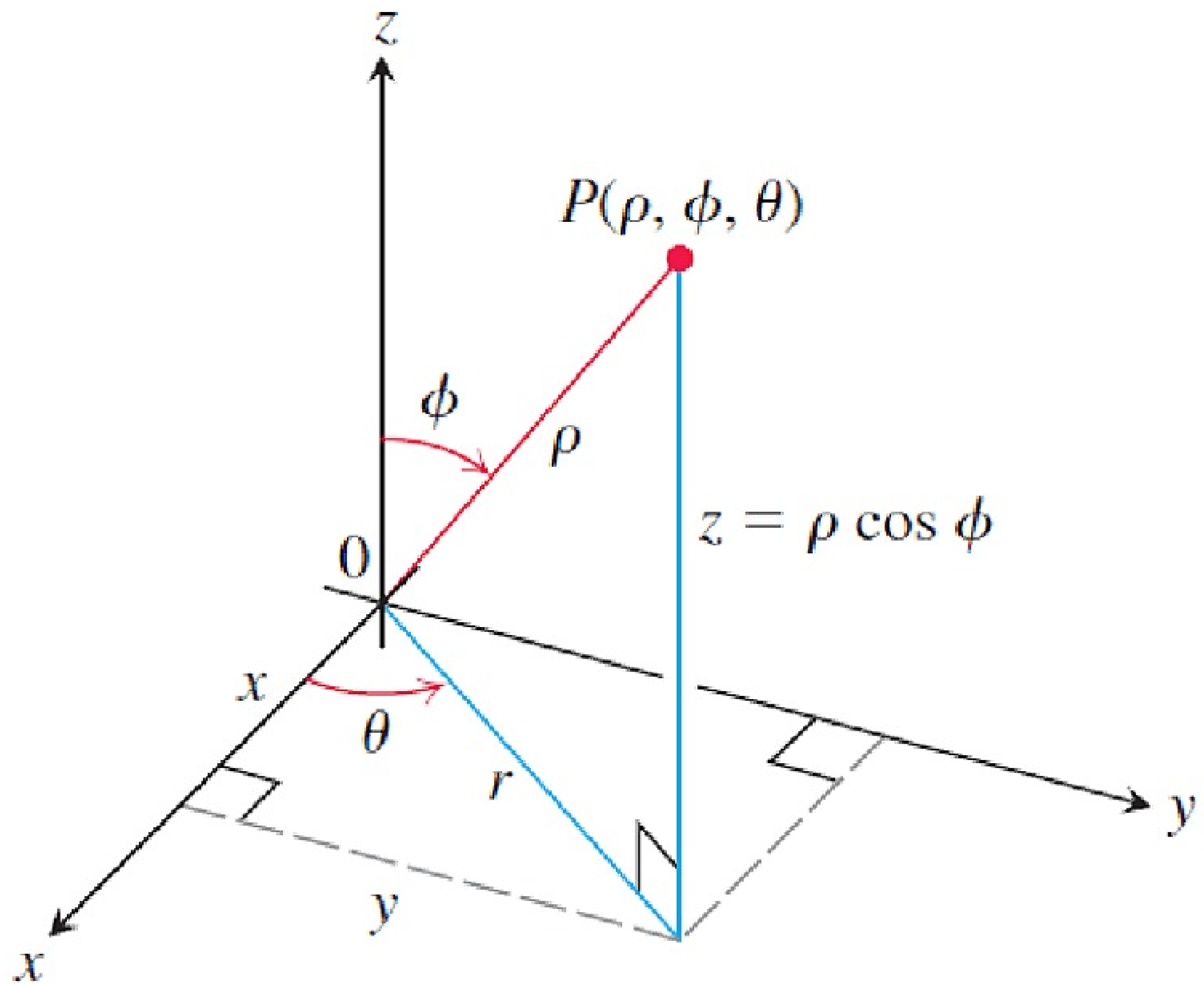


FIGURE 14.51 The spherical coordinates ρ, ϕ, and θ and their relation to x, y, z, and r.

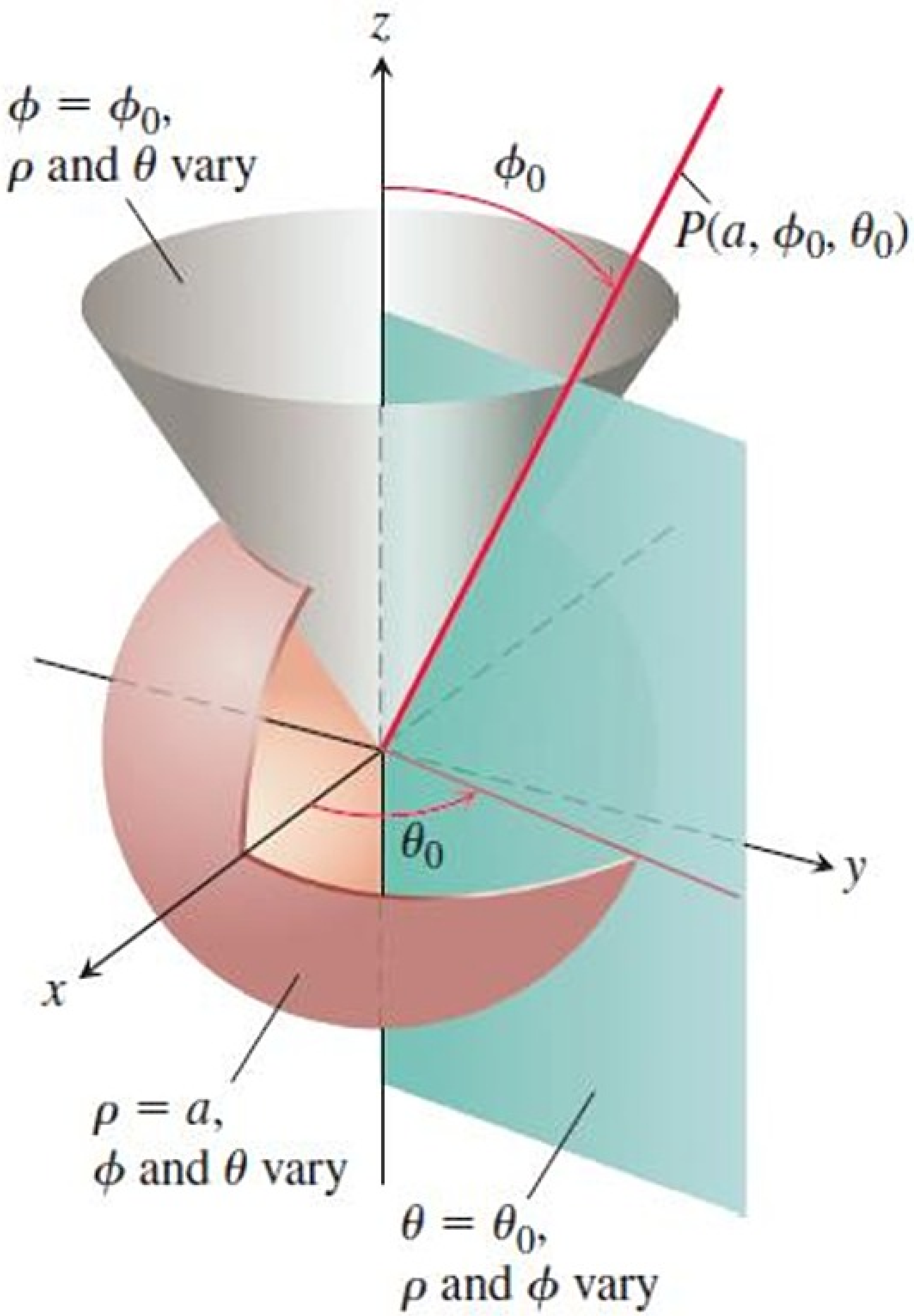


FIGURE 14.52 Constant-coordinate equations in spherical coordinates yield spheres, single cones, and half-planes.

Equations Relating Spherical Coordinates to Cartesian and Cylindrical Coordinates

$$
\begin{aligned}
r &= \rho \sin \phi, \qquad & x &= r \cos \theta = \rho \sin \phi \cos \theta, \\
z &= \rho \cos \phi, \qquad & y &= r \sin \theta = \rho \sin \phi \sin \theta,
\end{aligned} \tag{1}
$$
$$
\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2}.
$$

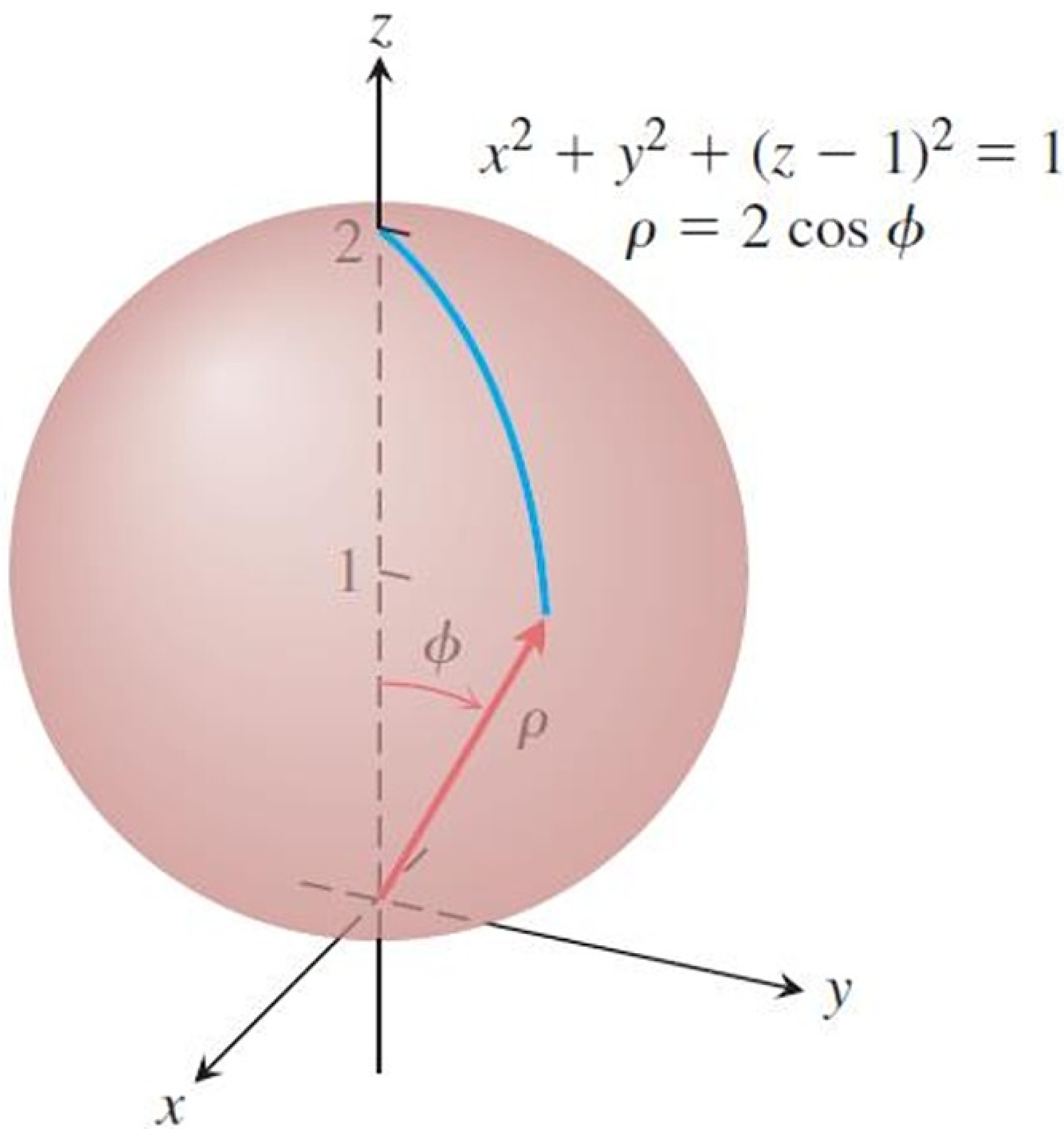


FIGURE 14.53 The sphere in Example 3.

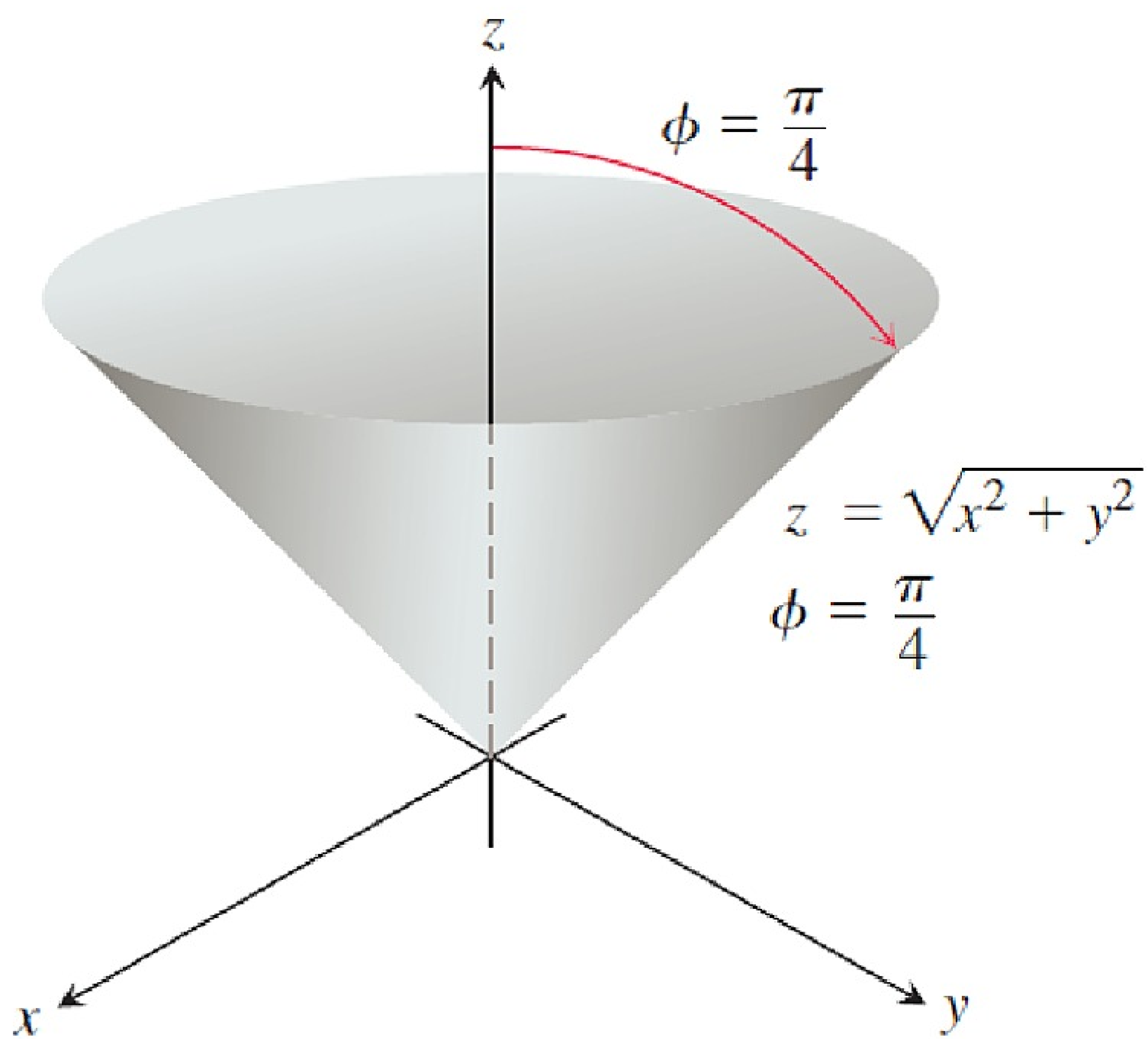


FIGURE 14.54 The cone in Example 4.

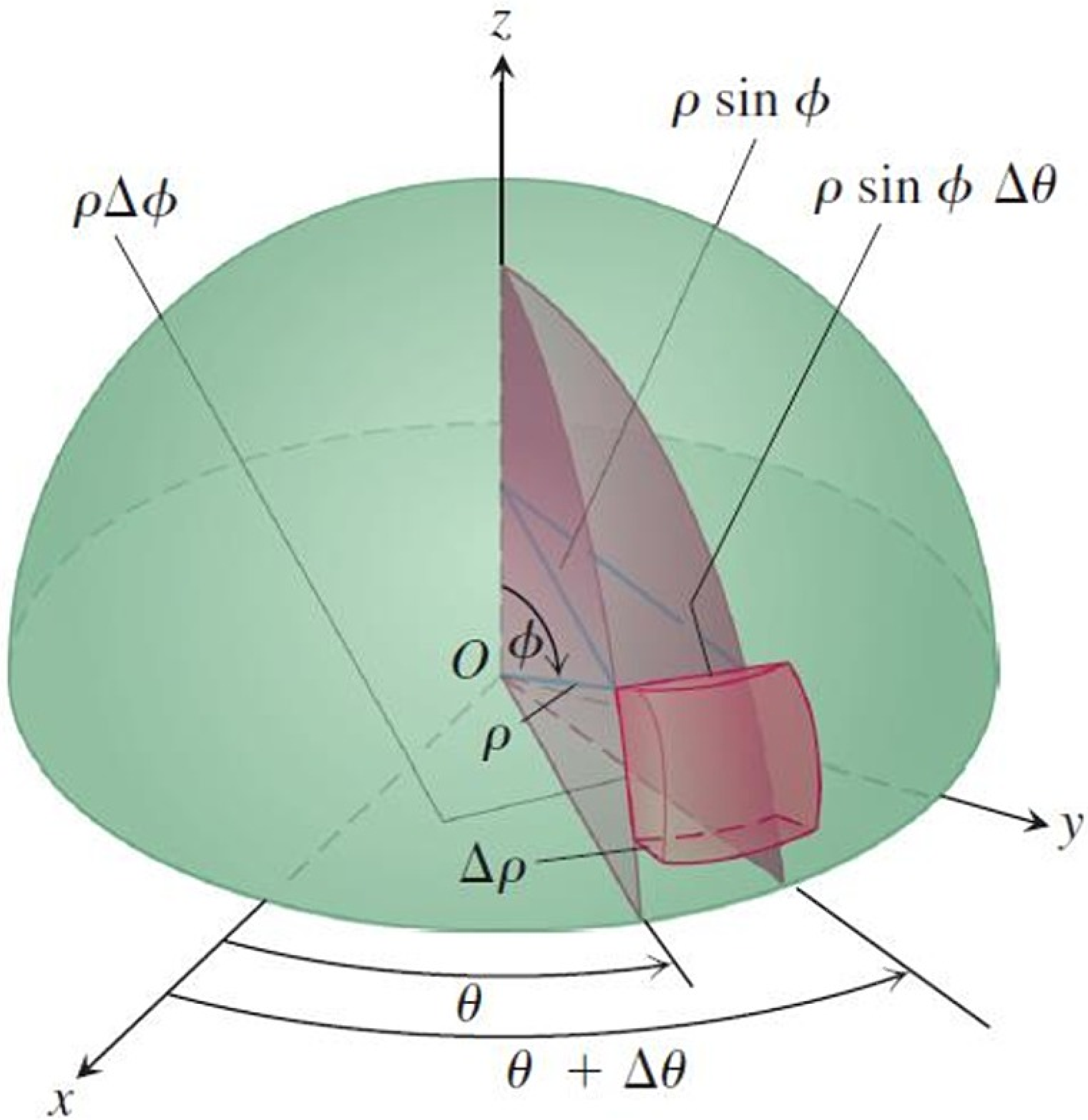


FIGURE 14.55 In spherical coordinates we use the volume of a spherical wedge, which closely approximates that of a cube.

$$\lim_{n\to\infty} S_n = \iiint_D f(\rho, \phi, \theta)\, dV = \iiint_D f(\rho, \phi, \theta)\, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$$

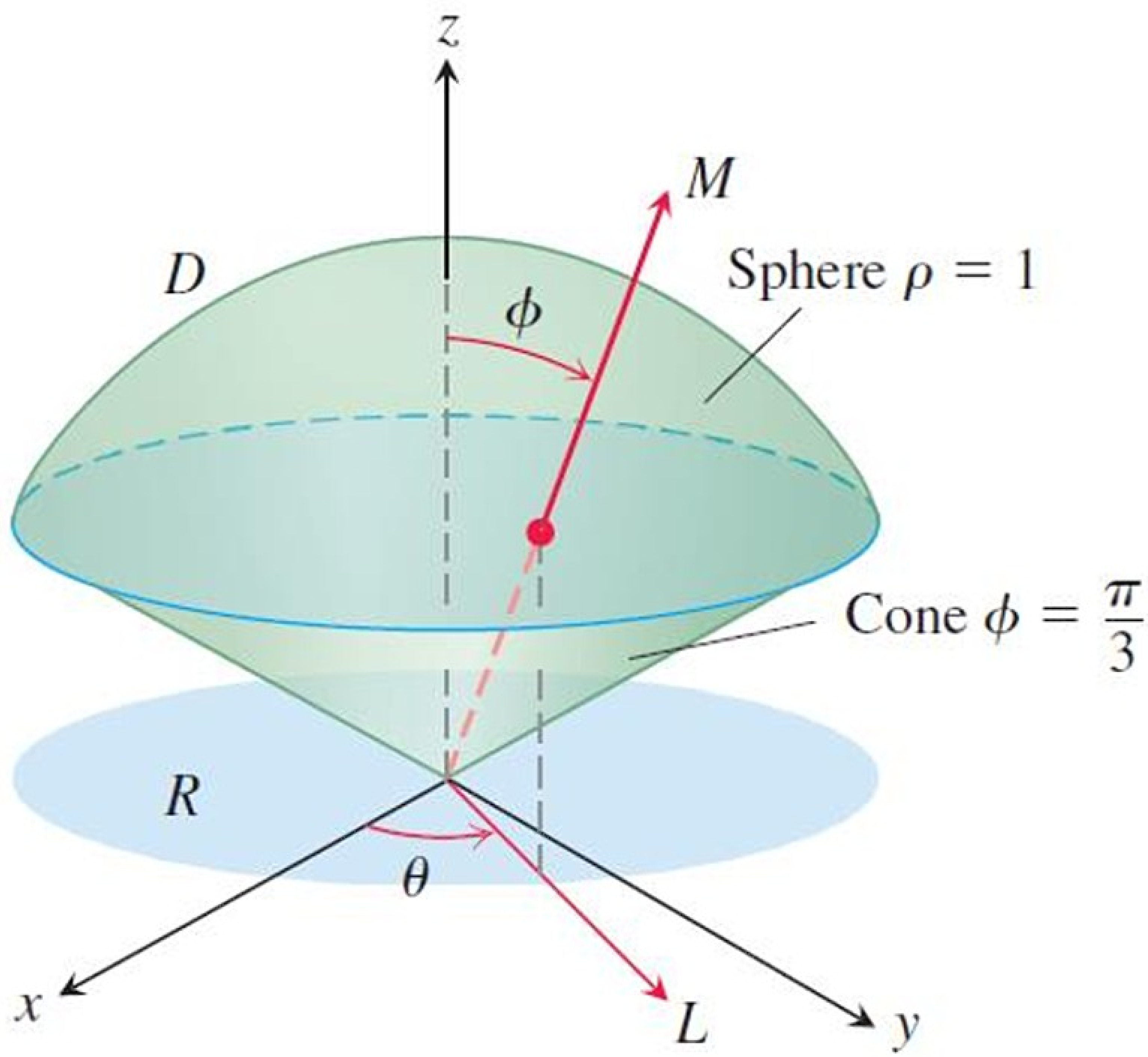


FIGURE 14.56 The ice cream cone in Example 5.

Coordinate Conversion Formulas

Cylindrical to Rectangular	Spherical to Rectangular	Spherical to Cylindrical
$x = r\cos\theta$	$x = \rho\sin\phi\cos\theta$	$r = \rho\sin\phi$
$y = r\sin\theta$	$y = \rho\sin\phi\sin\theta$	$z = \rho\cos\phi$
$z = z$	$z = \rho\cos\phi$	$\theta = \theta$

Corresponding formulas for dV in triple integrals:

$$
\begin{aligned}
dV &= dx\,dy\,dz \\
&= dz\,r\,dr\,d\theta \\
&= \rho^2\sin\phi\,d\rho\,d\phi\,d\theta
\end{aligned}
$$

Section 14.8

Substitutions in Multiple Integrals

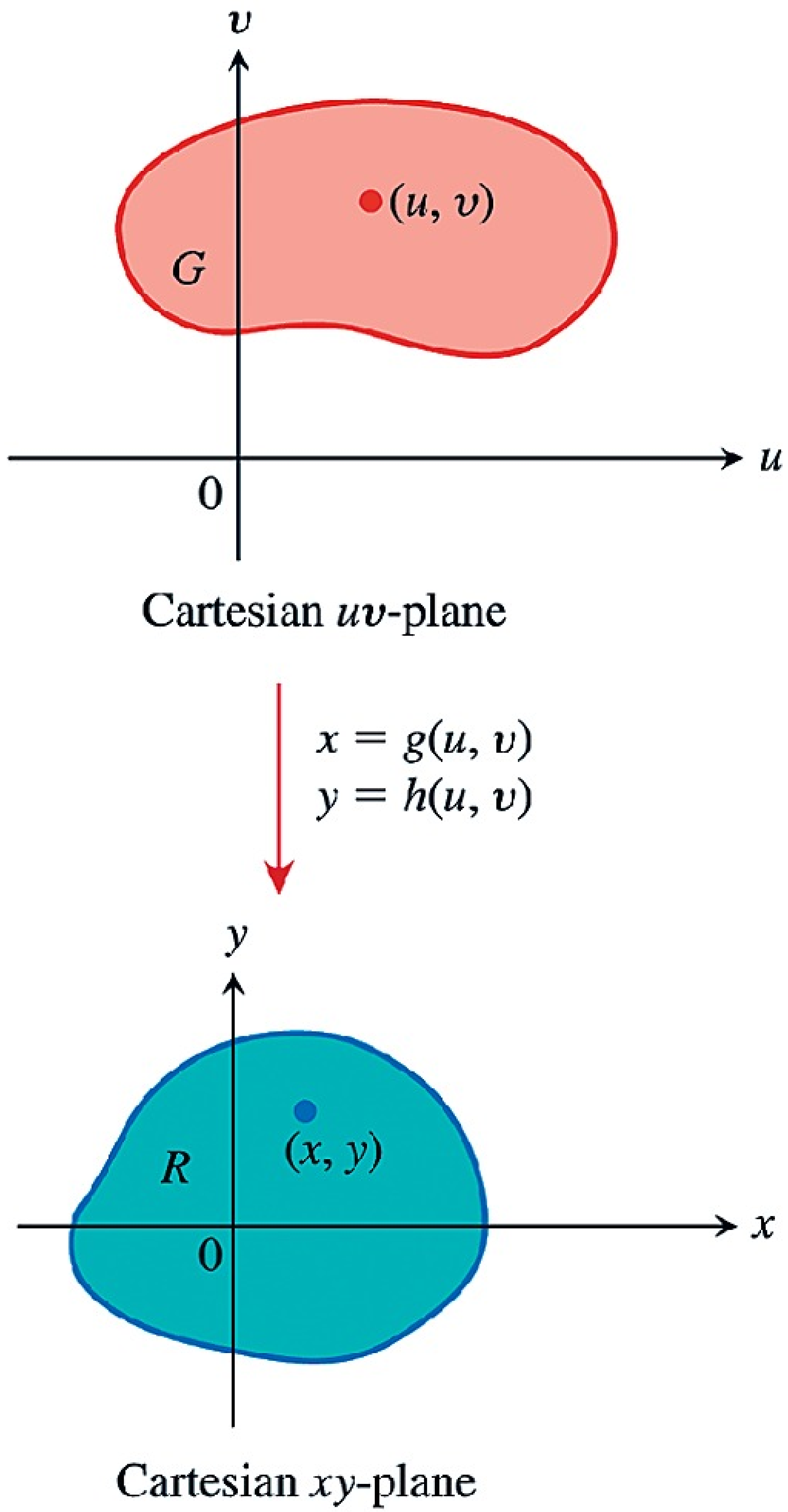


FIGURE 14.57 The equations $x = g(u, v)$ and $y = h(u, v)$ allow us to change an integral over a region R in the xy-plane into an integral over a region G in the uv-plane.

DEFINITION The **Jacobian determinant** or **Jacobian** of the coordinate transformation $x = g(u, v)$, $y = h(u, v)$ is

$$J(u, v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}. \qquad (2)$$

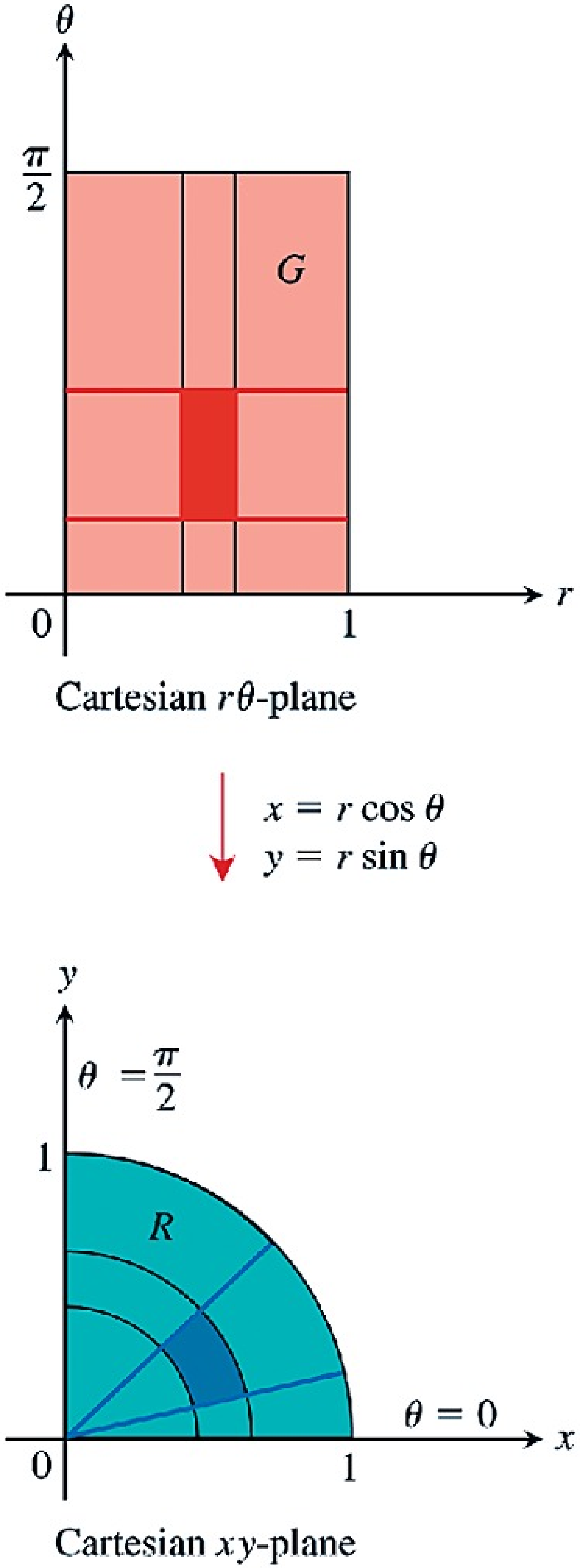


FIGURE 14.58 The equations $x = r\cos\theta$, $y = r\sin\theta$ transform G into R. The Jacobian factor r, calculated in Example 1, scales the differential rectangle $dr\,d\theta$ in G to match with the differential area element $dx\,dy$ in R.

THEOREM 3—Substitution for Double Integrals

Suppose that $f(x, y)$ is continuous over the region R. Let G be the preimage of R under the transformation $x = g(u, v)$, $y = h(u, v)$, which is assumed to be one-to-one on the interior of G. If the functions g and h have continuous first partial derivatives within the interior of G, then

$$\iint_R f(x, y)\, dx\, dy = \iint_G f(g(u, v), h(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv. \qquad (2)$$

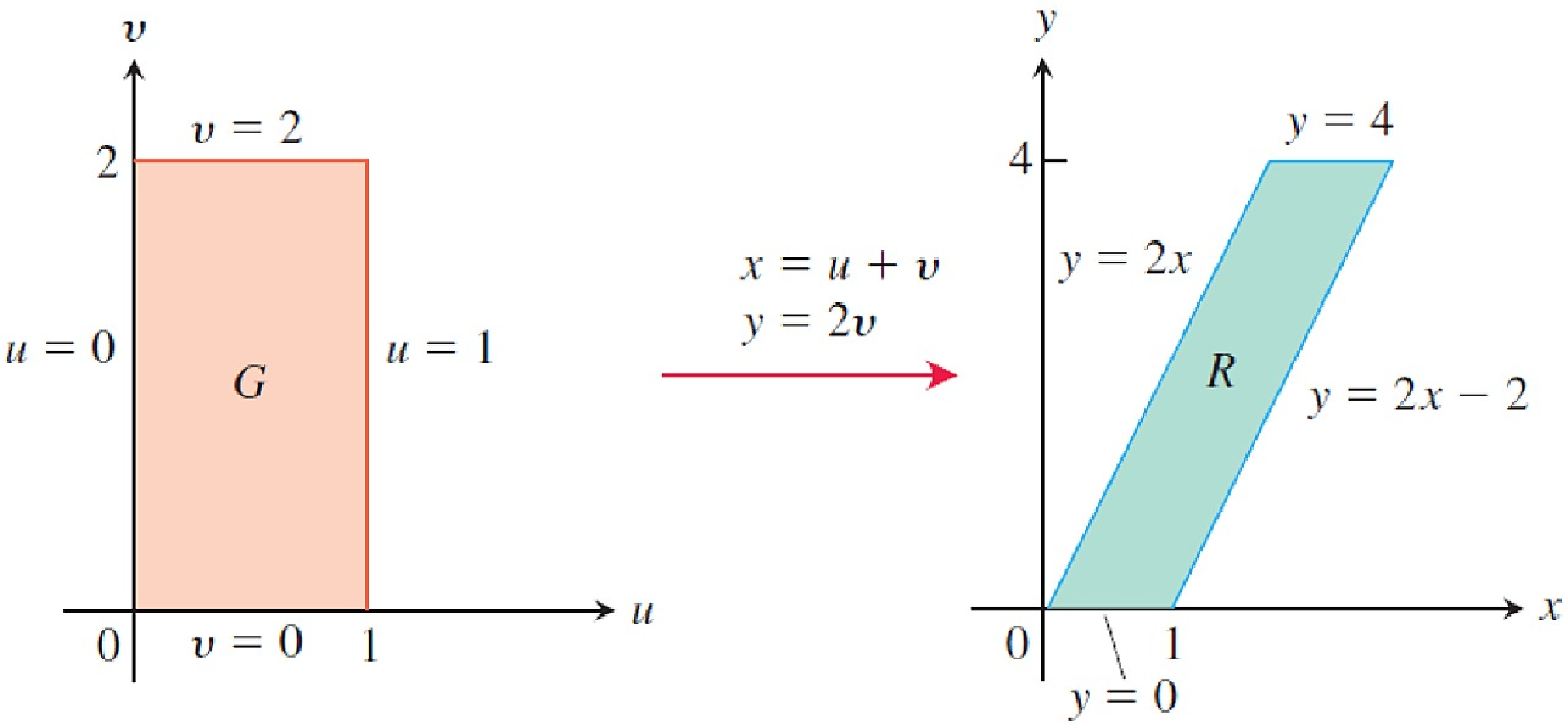


FIGURE 14.59 The equations $x = u + v$ and $y = 2v$ transform G into R. Reversing the transformation by the equations $u = (2x - y)/2$ and $v = y/2$ transforms R into G (Example 2).

xy*-equations for the boundary of *R	**Corresponding *uv*-equations for the boundary of *G***	**Simplified *uv*-equations**
$x = y/2$	$u + v = 2v/2 = v$	$u = 0$
$x = (y/2) + 1$	$u + v = (2v/2) + 1 = v + 1$	$u = 1$
$y = 0$	$2v = 0$	$v = 0$
$y = 4$	$2v = 4$	$v = 2$

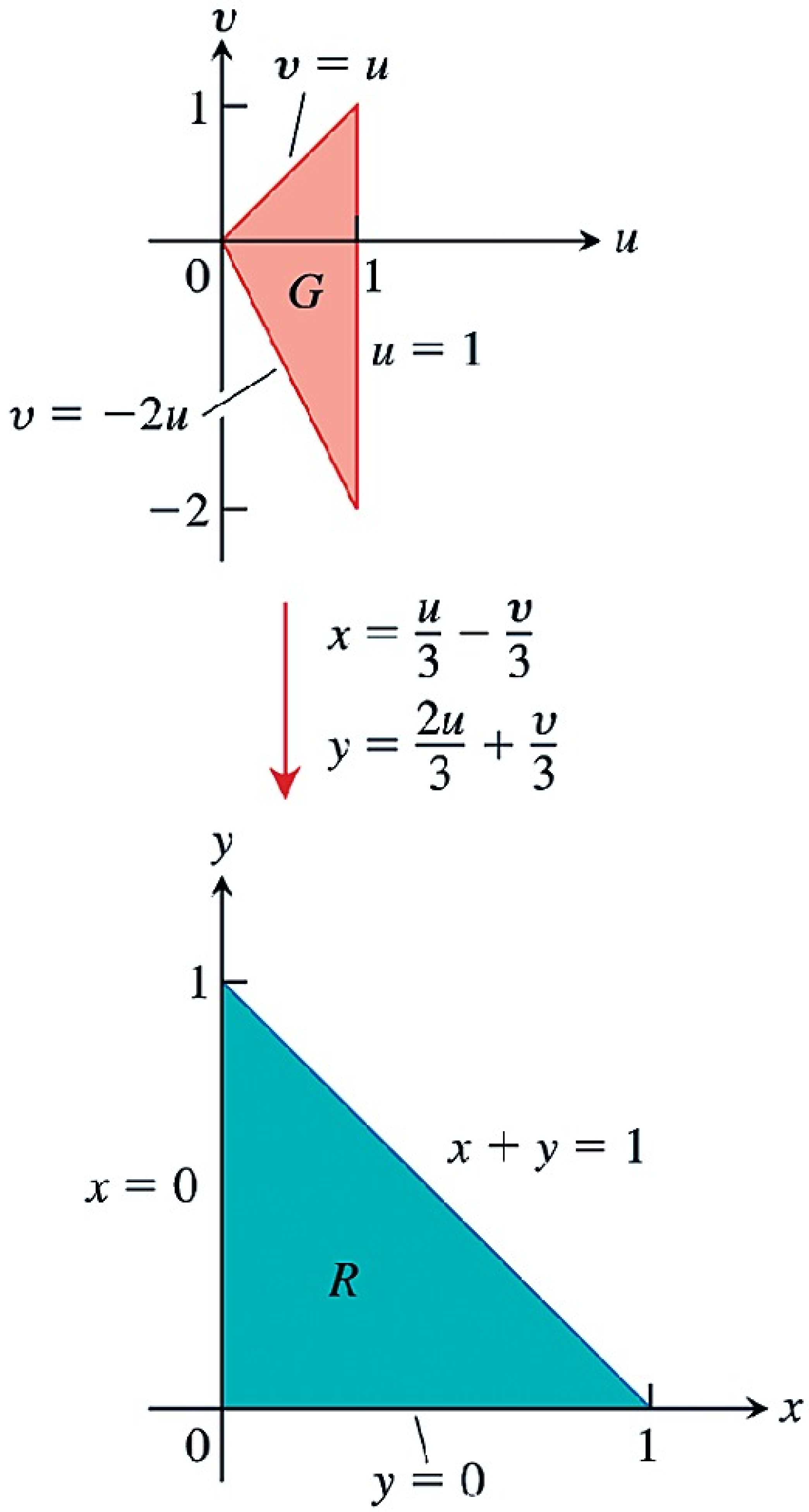


FIGURE 14.60 The equations $x = (u/3) - (v/3)$ and $y = (2u/3) + (v/3)$ transform G into R. Reversing the transformation by the equations $u = x + y$ and $v = y - 2x$ transforms R into G (Example 3).

xy*-equations for the boundary of *R	**Corresponding *uv*-equations for the boundary of *G***	**Simplified *uv*-equations**
$x + y = 1$	$\left(\frac{u}{3} - \frac{v}{3}\right) + \left(\frac{2u}{3} + \frac{v}{3}\right) = 1$	$u = 1$
$x = 0$	$\frac{u}{3} - \frac{v}{3} = 0$	$v = u$
$y = 0$	$\frac{2u}{3} + \frac{v}{3} = 0$	$v = -2u$

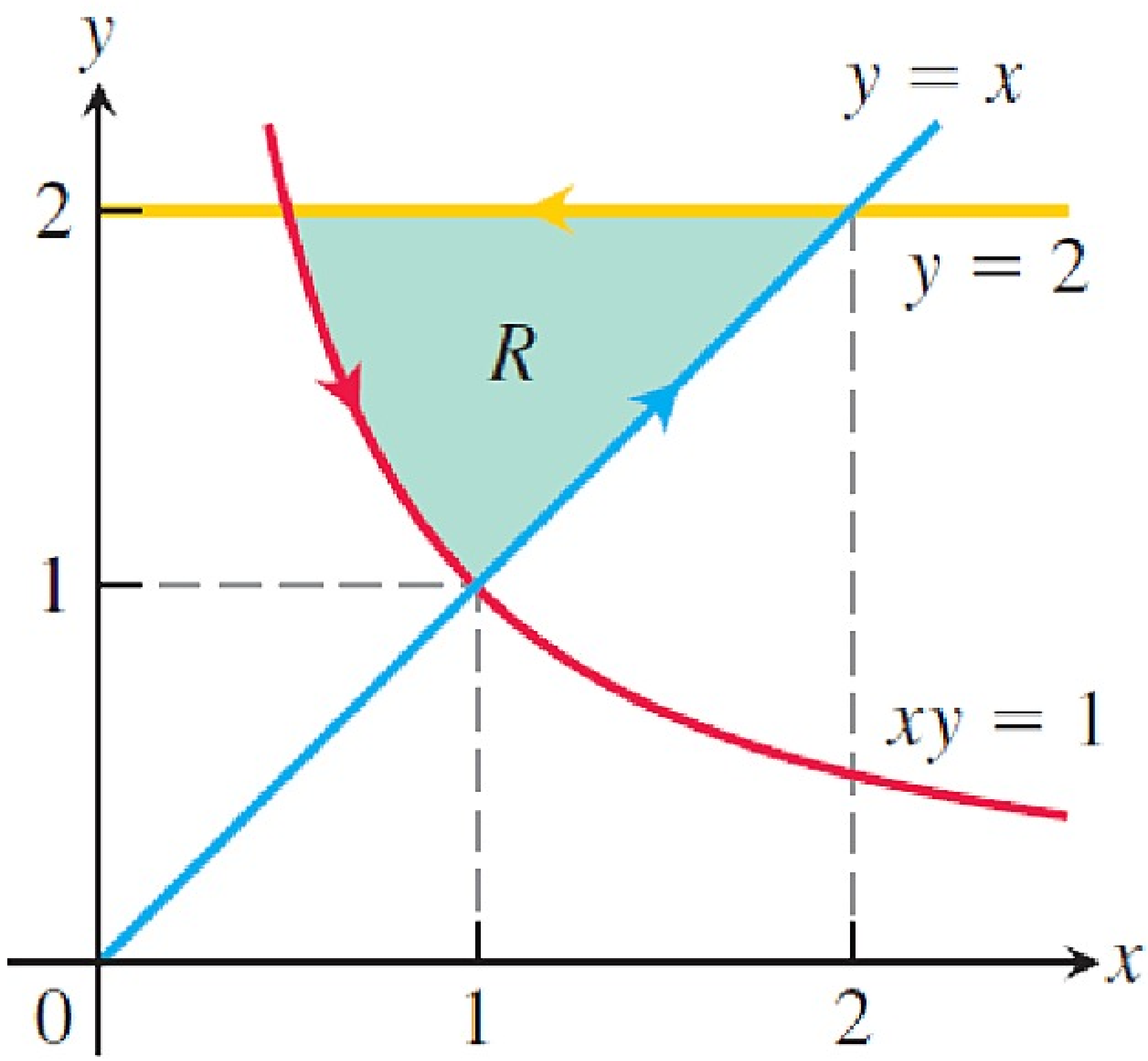


FIGURE 14.61 The region of integration R in Example 4.

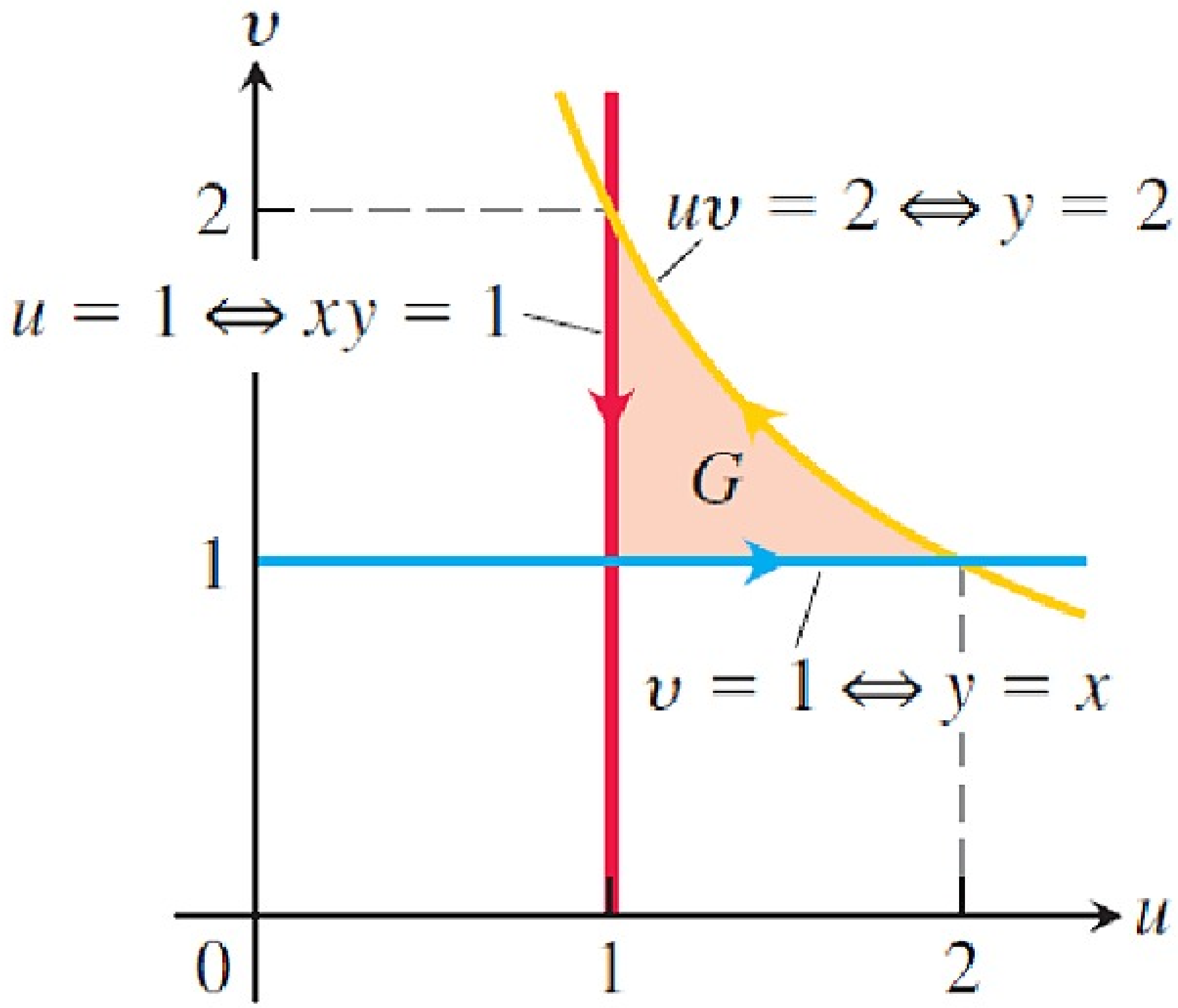


FIGURE 14.62 The boundaries of the region G correspond to those of region R in Figure 14.61. Notice as we move counterclockwise around the region R, we also move counterclockwise around the region G. The inverse transformation equations $u = \sqrt{xy}$, $v = \sqrt{y/x}$ produce the region G from the region R.

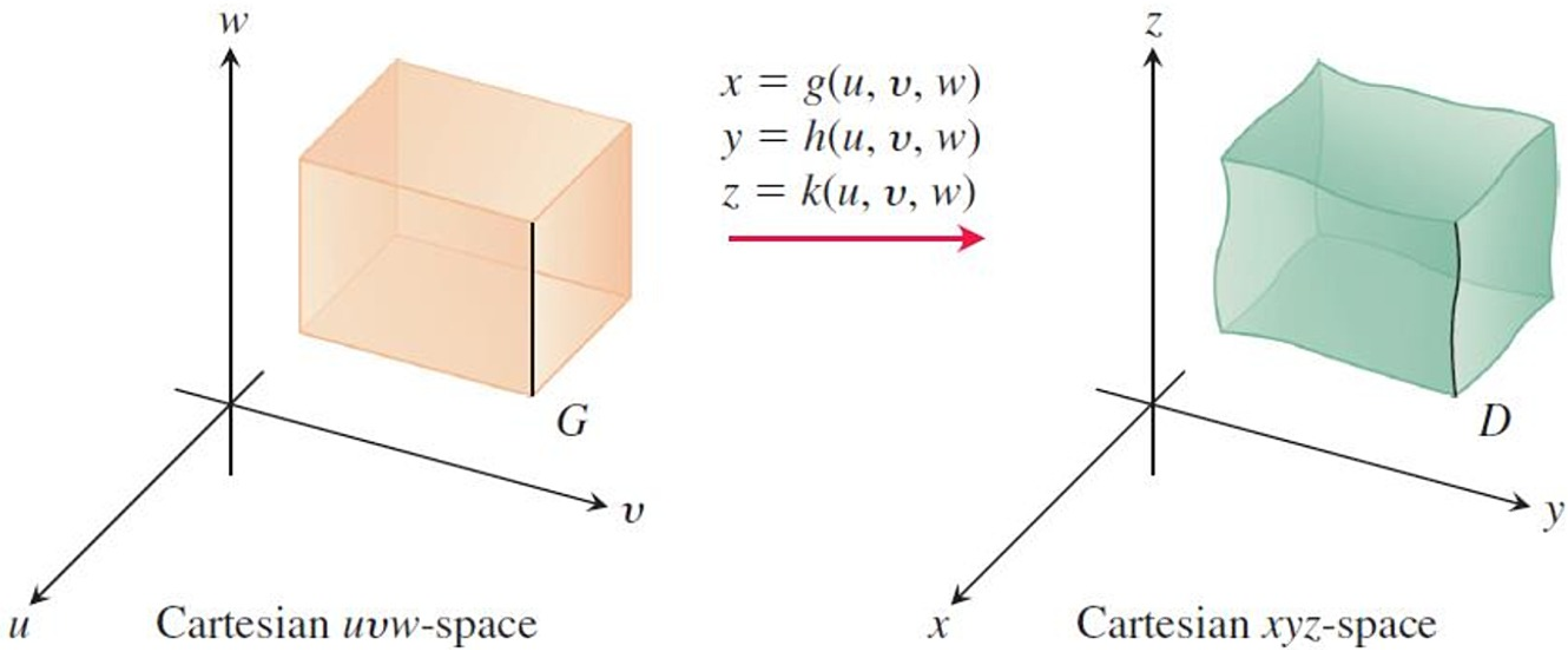


FIGURE 14.63 The equations $x = g(u, v, w)$, $y = h(u, v, w)$, and $z = k(u, v, w)$ allow us to change an integral over a region D in Cartesian xyz-space into an integral over a region G in Cartesian uvw-space using Equation (7).

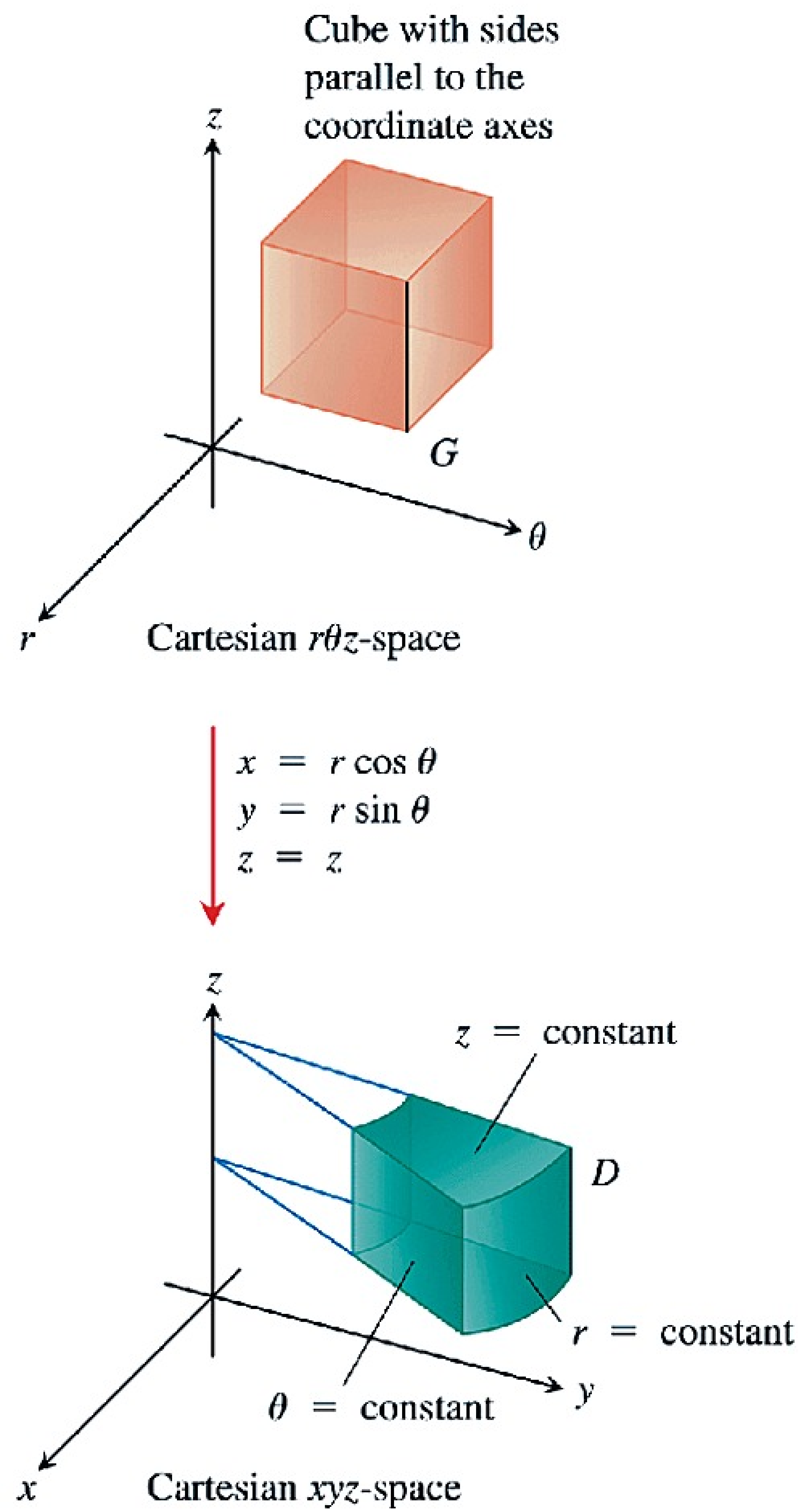


FIGURE 14.64 The equations $x = r\cos\theta$, $y = r\sin\theta$, and $z = z$ transform the cube G into a cylindrical wedge D.

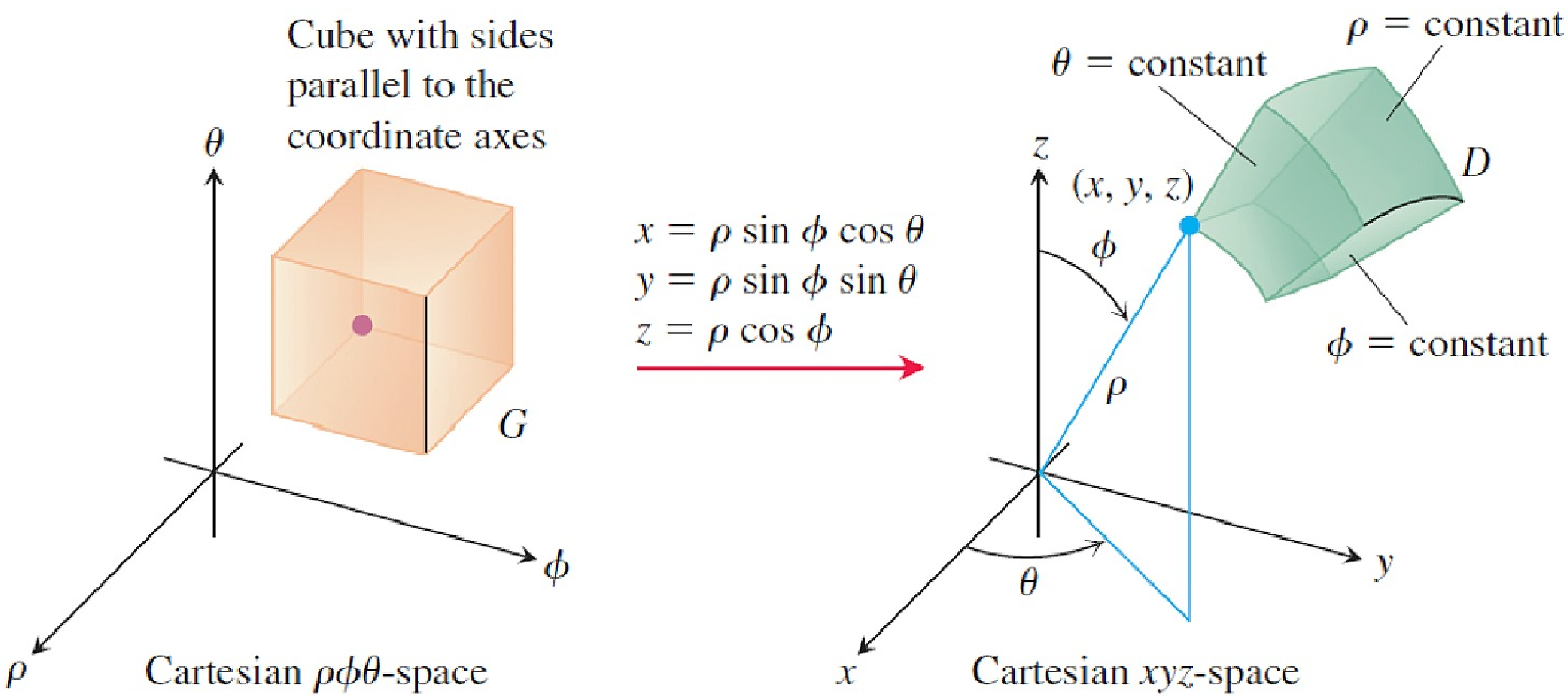


FIGURE 14.65 The equations $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$, and $z = \rho \cos\phi$ transform the cube G into the spherical wedge D.

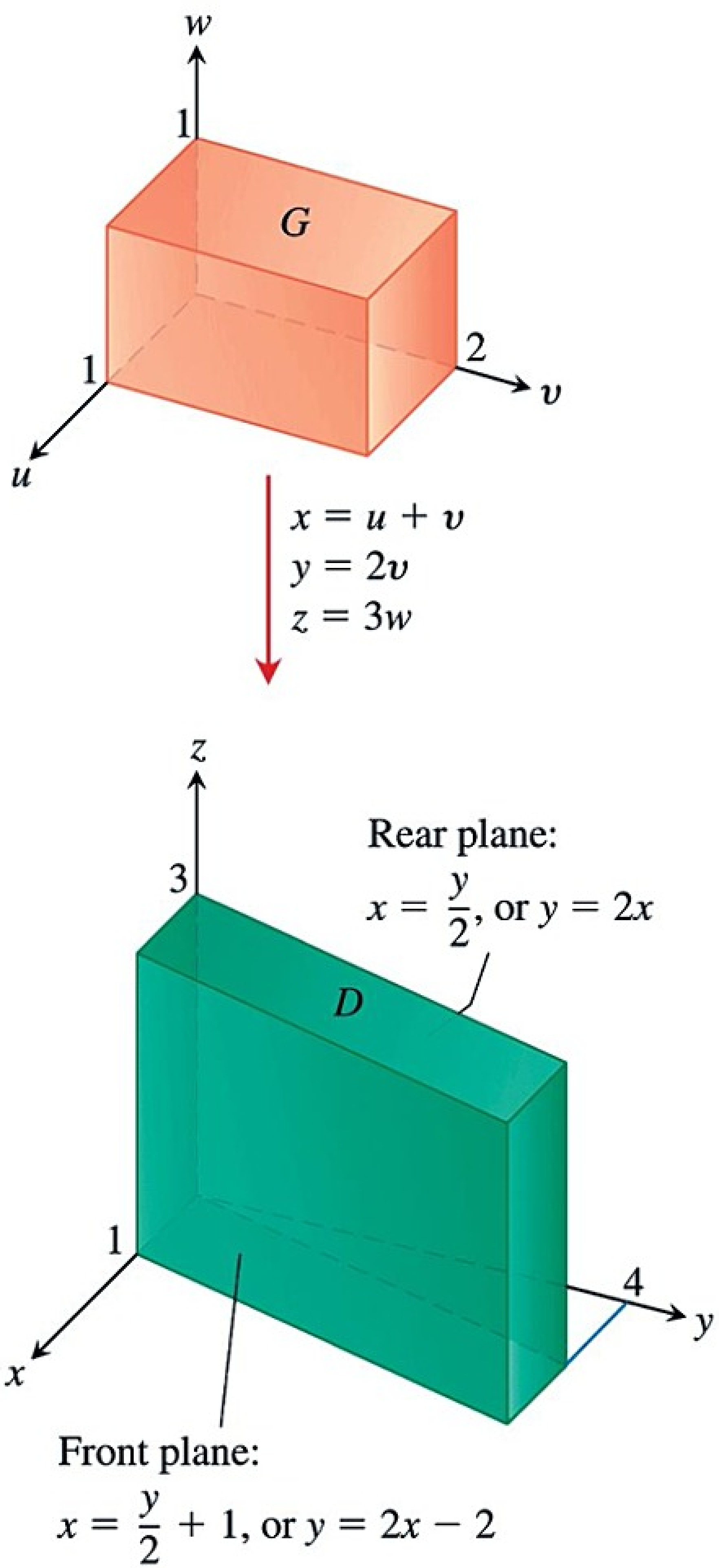


FIGURE 14.66 The equations $x = u + v$, $y = 2v$, and $z = 3w$ transform G into D. Reversing the transformation by the equations $u = (2x - y)/2$, $v = y/2$, and $w = z/3$ transforms D into G (Example 5).

xyz-equations for the boundary of *D*	Corresponding *uvw*-equations for the boundary of *G*	Simplified *uvw*-equations
$x = y/2$	$u + v = 2v/2 = v$	$u = 0$
$x = (y/2) + 1$	$u + v = (2v/2) + 1 = v + 1$	$u = 1$
$y = 0$	$2v = 0$	$v = 0$
$y = 4$	$2v = 4$	$v = 2$
$z = 0$	$3w = 0$	$w = 0$
$z = 3$	$3w = 3$	$w = 1$