Chapter 4

Applications of Derivatives

Section 4.1

Extreme Values of Functions on Closed Intervals

DEFINITIONS Let f be a function with domain D. Then f has an **absolute maximum** value on D at a point c if

$$f(x) \leq f(c) \qquad \text{for all } x \text{ in } D$$

and an **absolute minimum** value on D at c if

$$f(x) \geq f(c) \qquad \text{for all } x \text{ in } D.$$

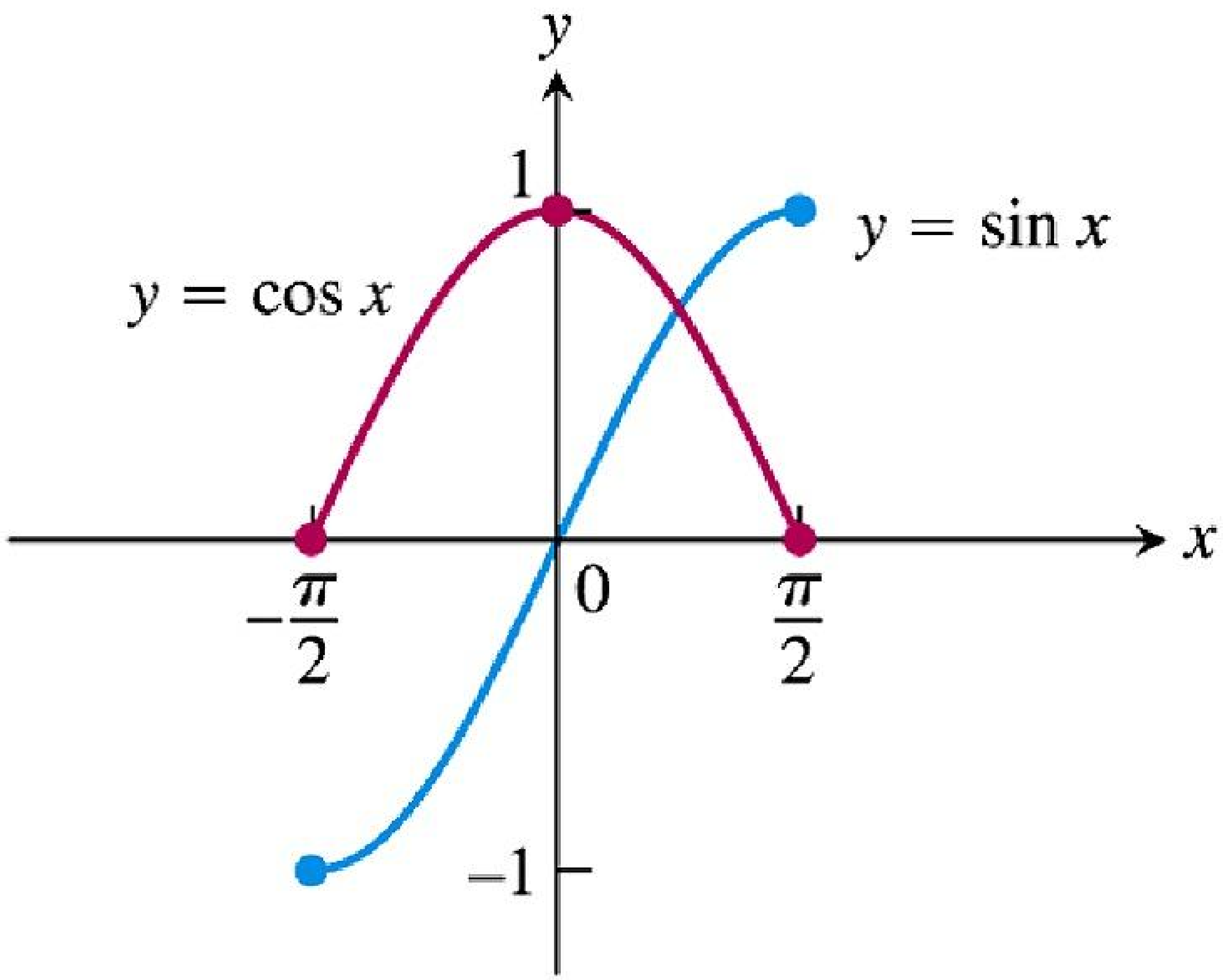


FIGURE 4.1 Absolute extrema for the sine and cosine functions on $[-\pi/2, \pi/2]$. These values can depend on the domain of a function.

Function rule	**Domain *D***	**Absolute extrema on *D***
(a) $y = x^2$	$(-\infty, \infty)$	No absolute maximum. Absolute minimum of 0 at $x = 0$.
(b) $y = x^2$	$[0, 2]$	Absolute maximum of 4 at $x = 2$. Absolute minimum of 0 at $x = 0$.
(c) $y = x^2$	$(0, 2]$	Absolute maximum of 4 at $x = 2$. No absolute minimum.
(d) $y = x^2$	$(0, 2)$	No absolute extrema.

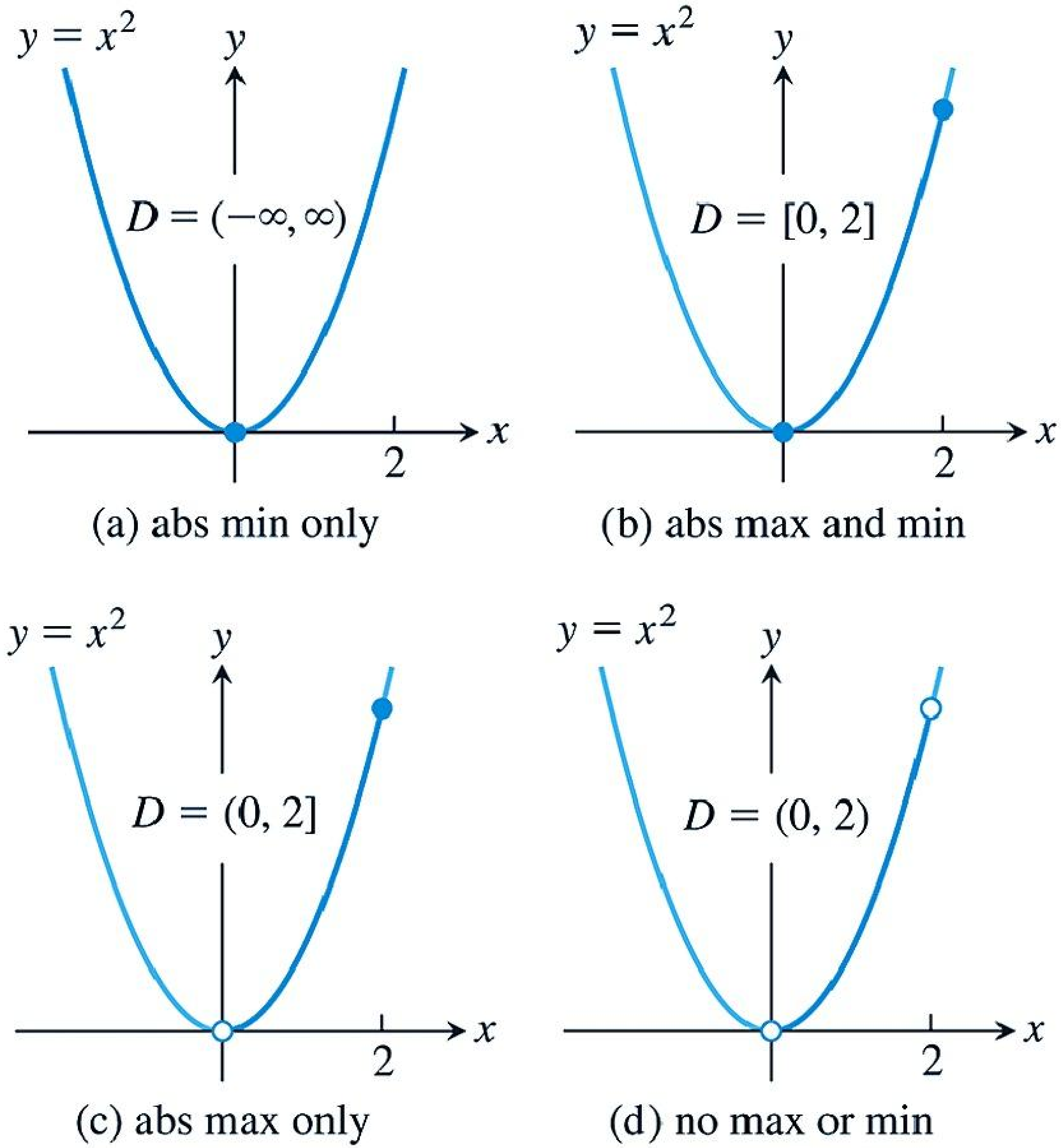


FIGURE 4.2 Graphs for Example 1.

THEOREM 1—The Extreme Value Theorem If f is continuous on a closed interval $[a, b]$, then f attains both an absolute maximum value M and an absolute minimum value m in $[a, b]$. That is, there are numbers x_1 and x_2 in $[a, b]$ with $f(x_1) = m$, $f(x_2) = M$, and $m \leq f(x) \leq M$ for every other x in $[a, b]$.

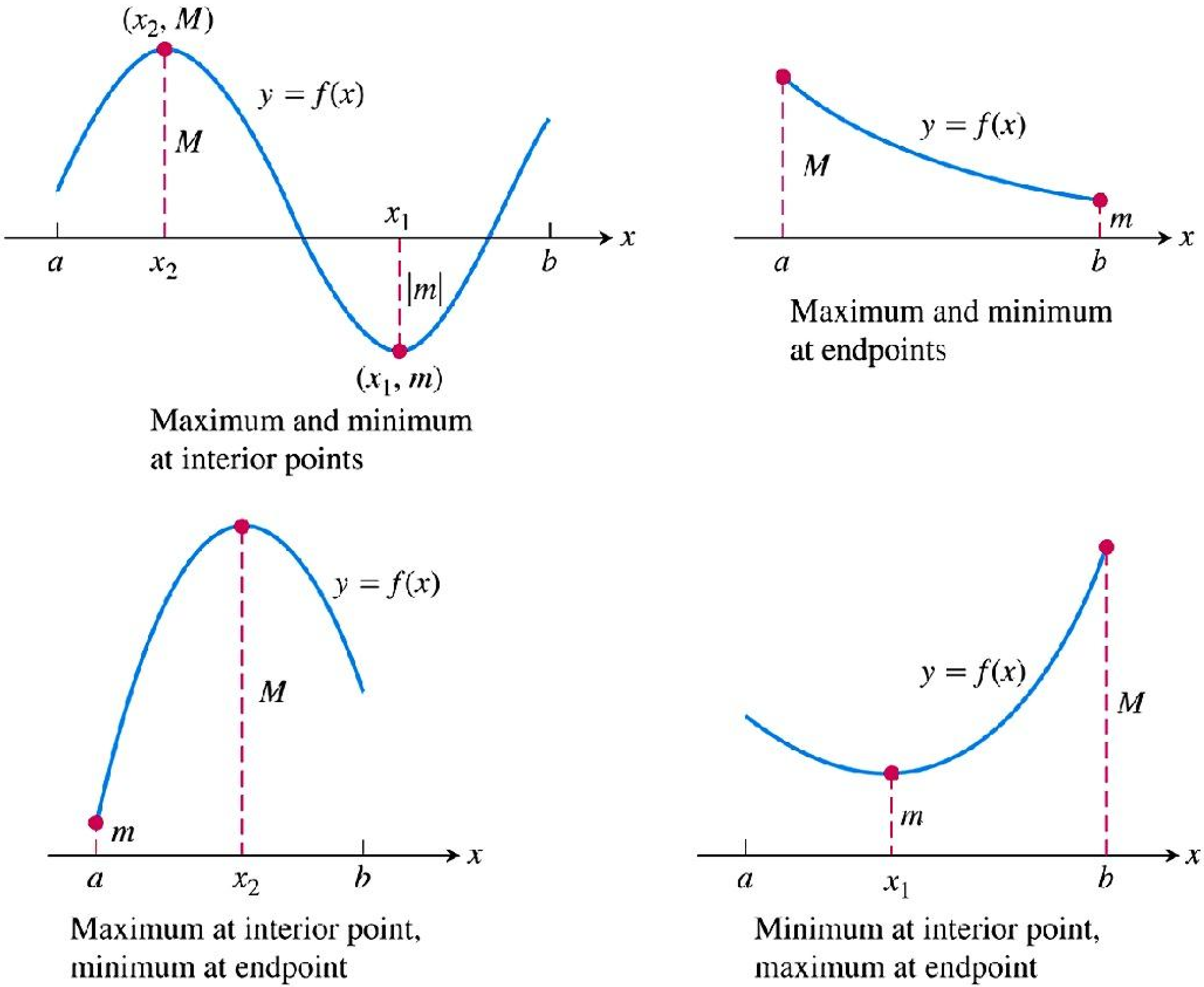


FIGURE 4.3 Some possibilities for a continuous function's maximum and minimum on a closed interval $[a, b]$.

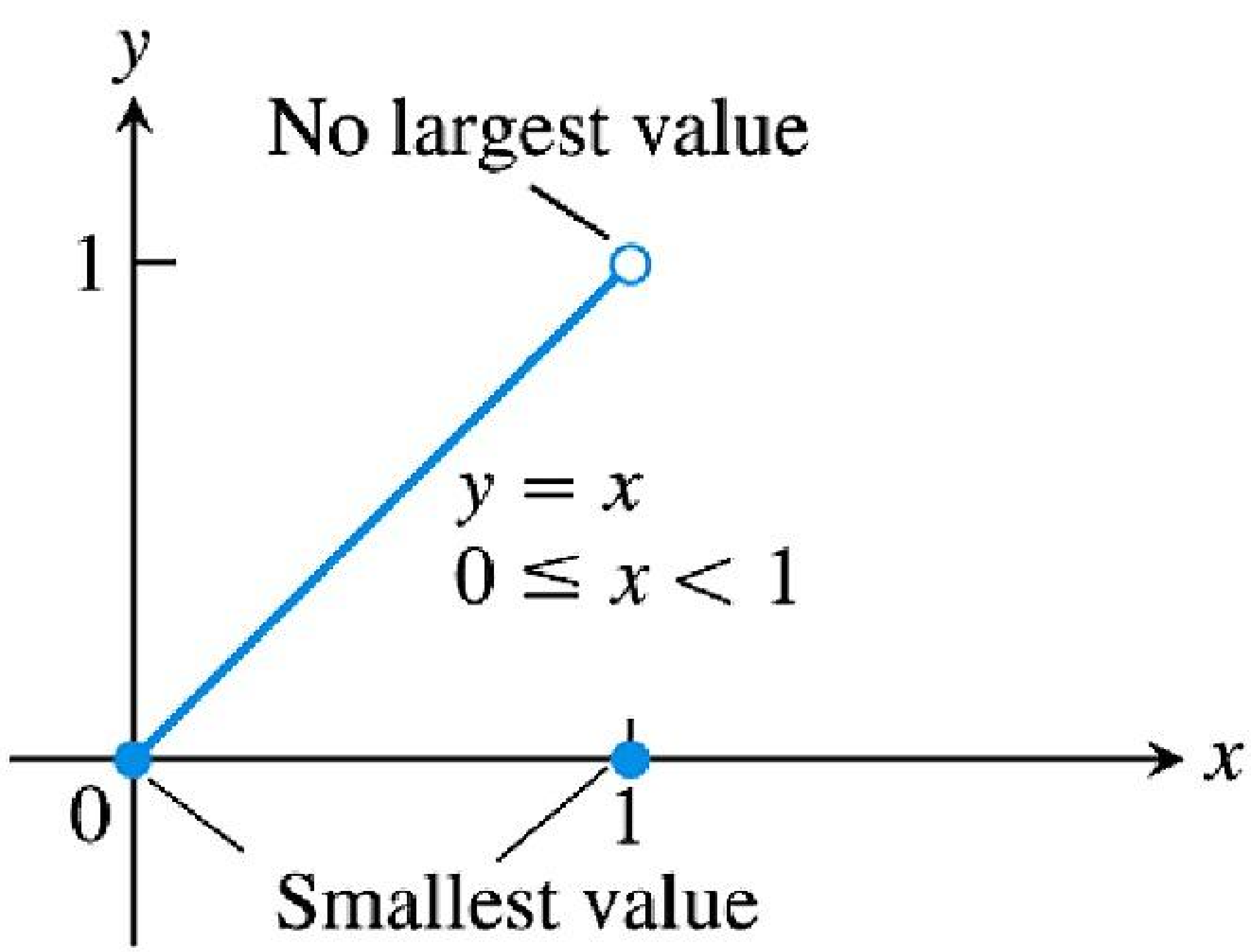


FIGURE 4.4 Even a single point of discontinuity can keep a function from having either a maximum or minimum value on a closed interval. The function

$$y = \begin{cases} x, & 0 \leq x < 1 \\ 0, & x = 1 \end{cases}$$

is continuous at every point of [0, 1] except $x = 1$, yet its graph over [0, 1] does not have a highest point.

DEFINITIONS A function f has a **local maximum** value at a point c within its domain D if $f(x) \leq f(c)$ for all $x \in D$ lying in some open interval containing c.

A function f has a **local minimum** value at a point c within its domain D if $f(x) \geq f(c)$ for all $x \in D$ lying in some open interval containing c.

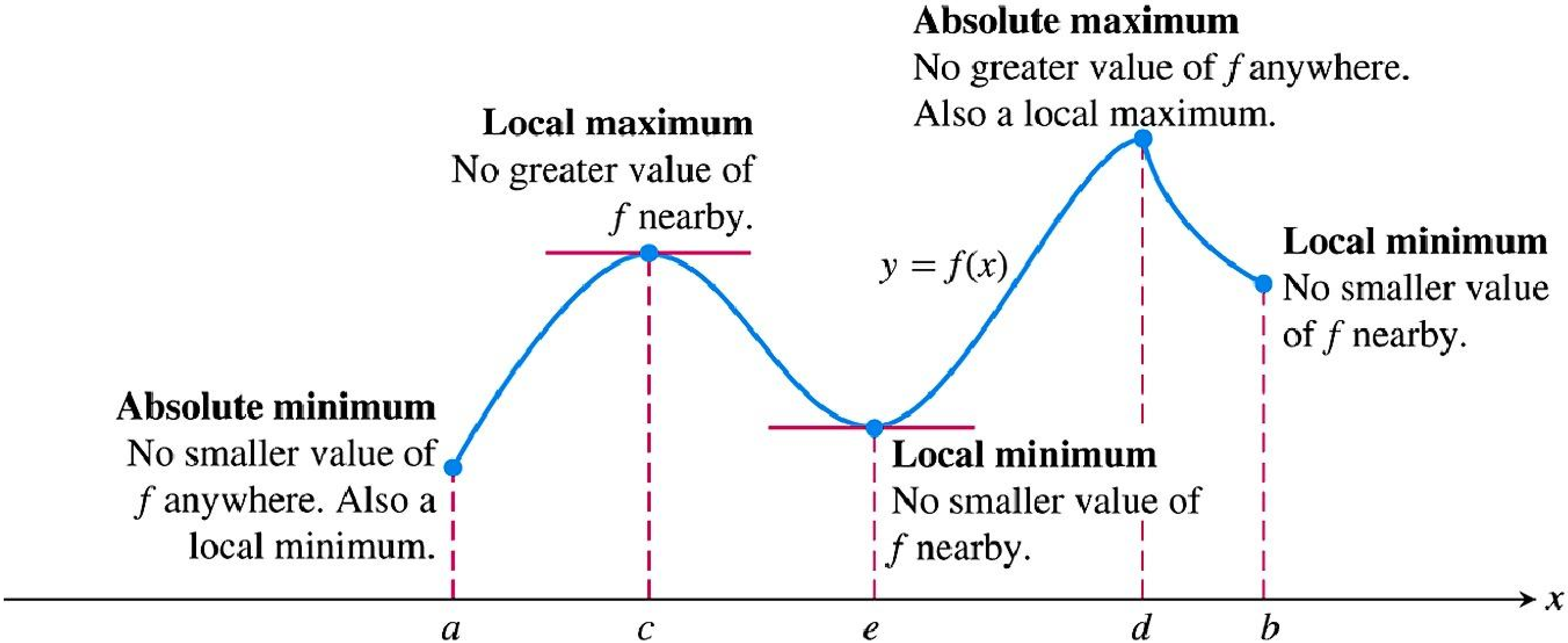


FIGURE 4.5 How to identify types of maxima and minima for a function with domain $a \leq x \leq b$.

THEOREM 2—The First Derivative Theorem for Local Extreme Values If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then

$$f'(c) = 0.$$

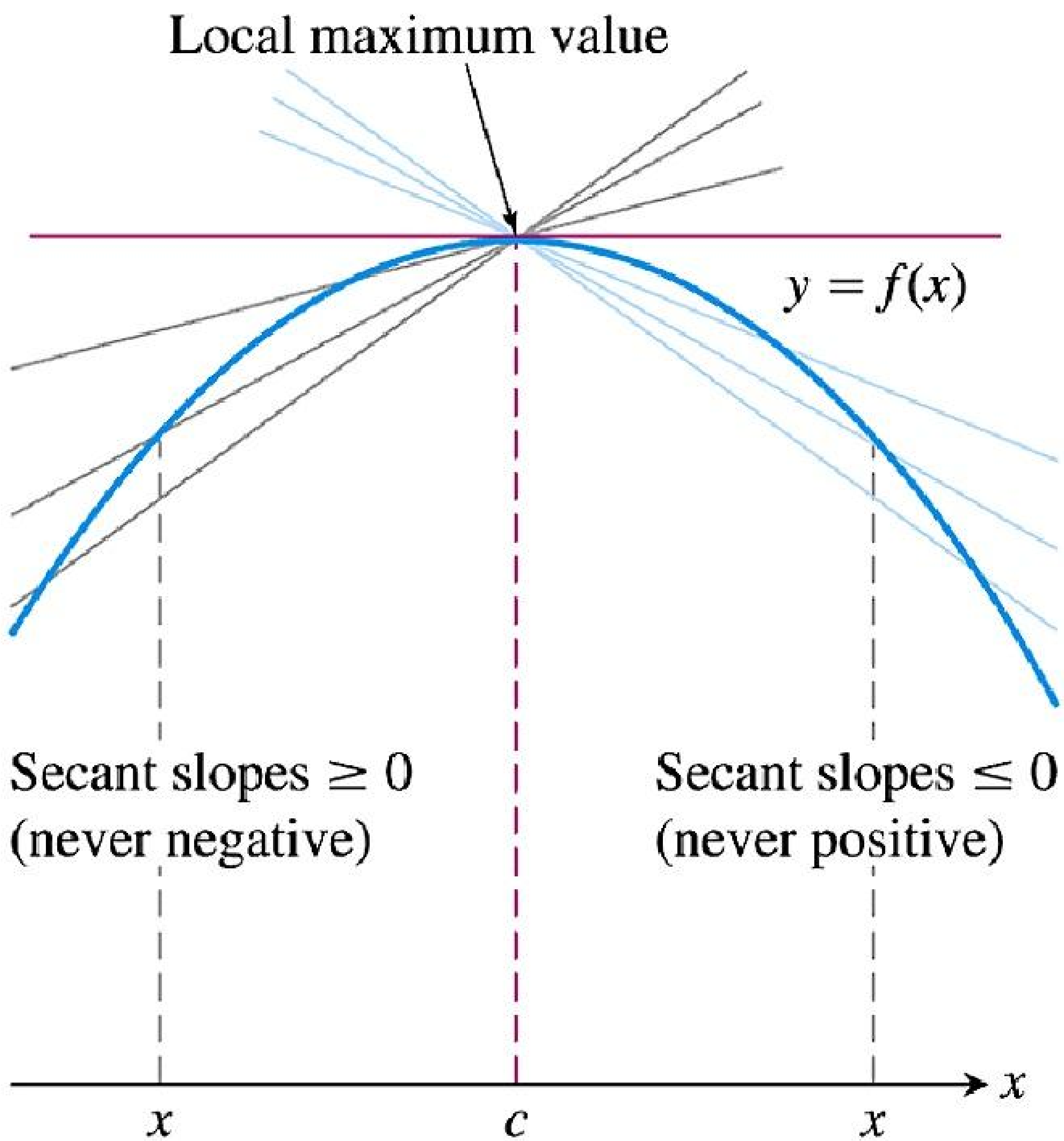


FIGURE 4.6 A curve with a local maximum value. The slope at c, simultaneously the limit of nonpositive numbers and nonnegative numbers, is zero.

DEFINITION An interior point of the domain of a function f where f' is zero or undefined is a **critical point** of f.

Finding the Absolute Extrema of a Continuous Function f on a Finite Closed Interval

1. Find all critical points of f on the interval.
2. Evaluate f at all critical points and endpoints.
3. Take the largest and smallest of these values.

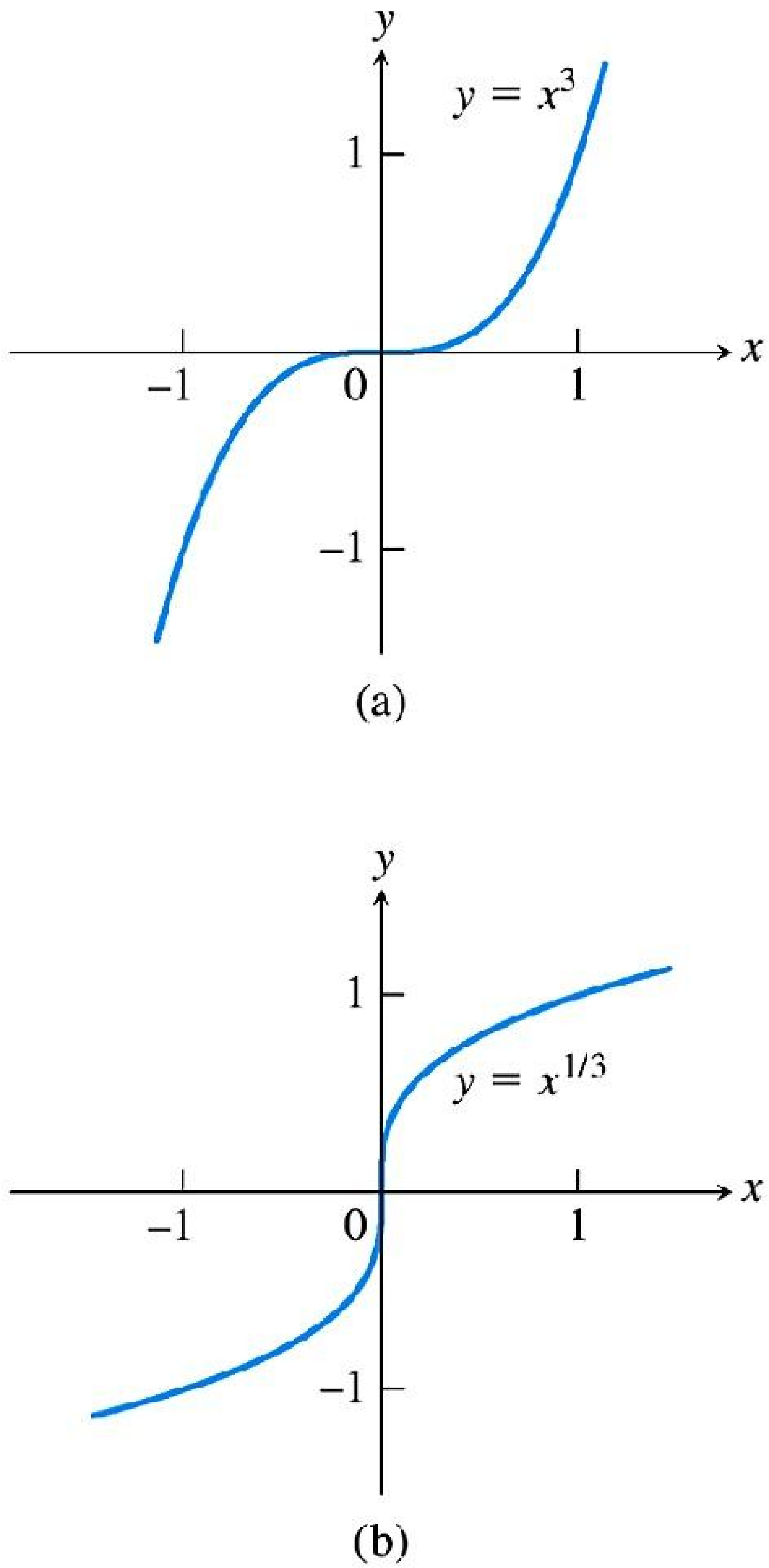


FIGURE 4.7 Critical points without extreme values. (a) $y' = 3x^2$ is 0 at $x = 0$, but $y = x^3$ has no extremum there. (b) $y' = (1/3)x^{-2/3}$ is undefined at $x = 0$, but $y = x^{1/3}$ has no extremum there.

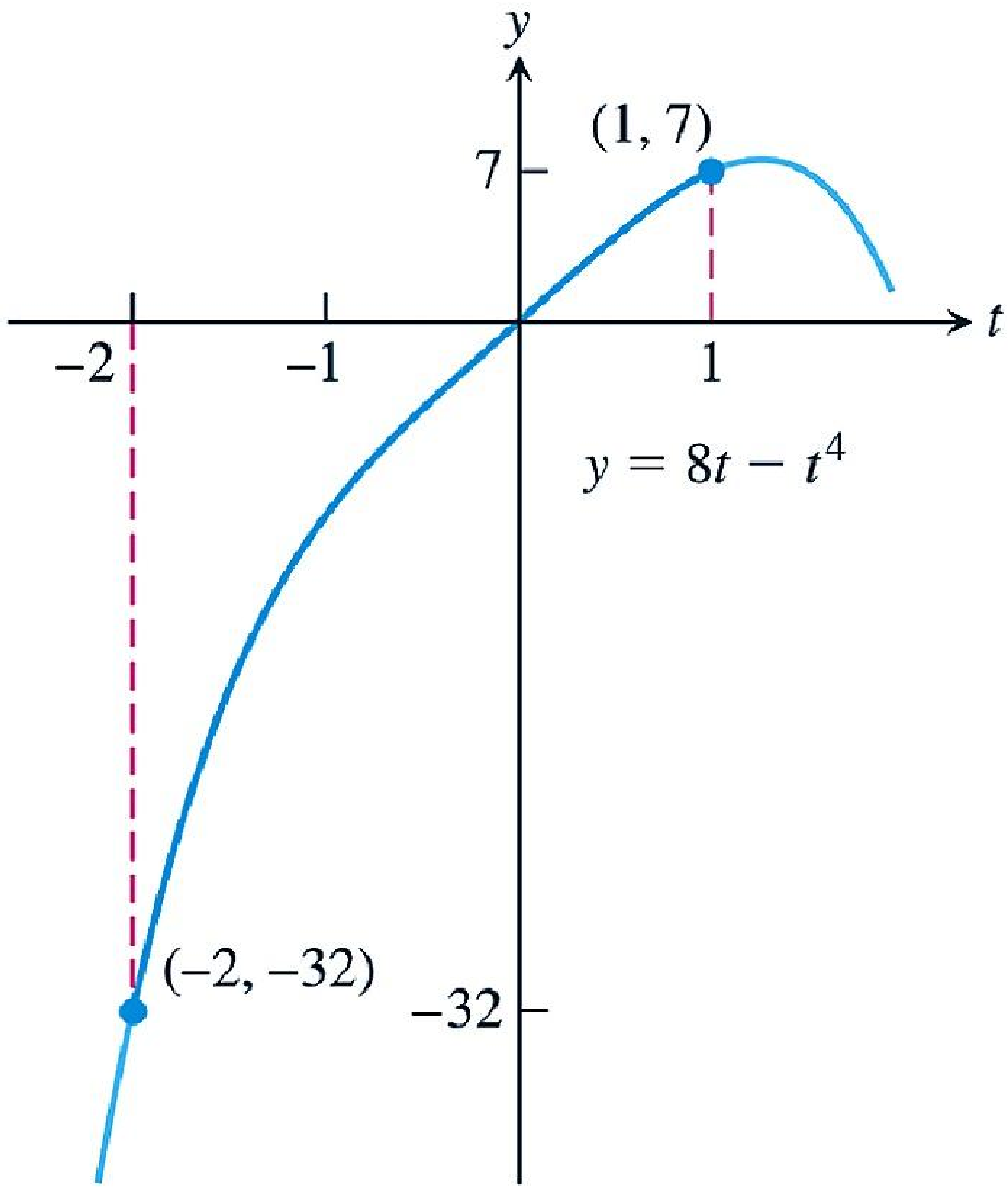


FIGURE 4.8 The extreme values of $g(t) = 8t - t^4$ on $[-2, 1]$ (Example 3).

Pearson

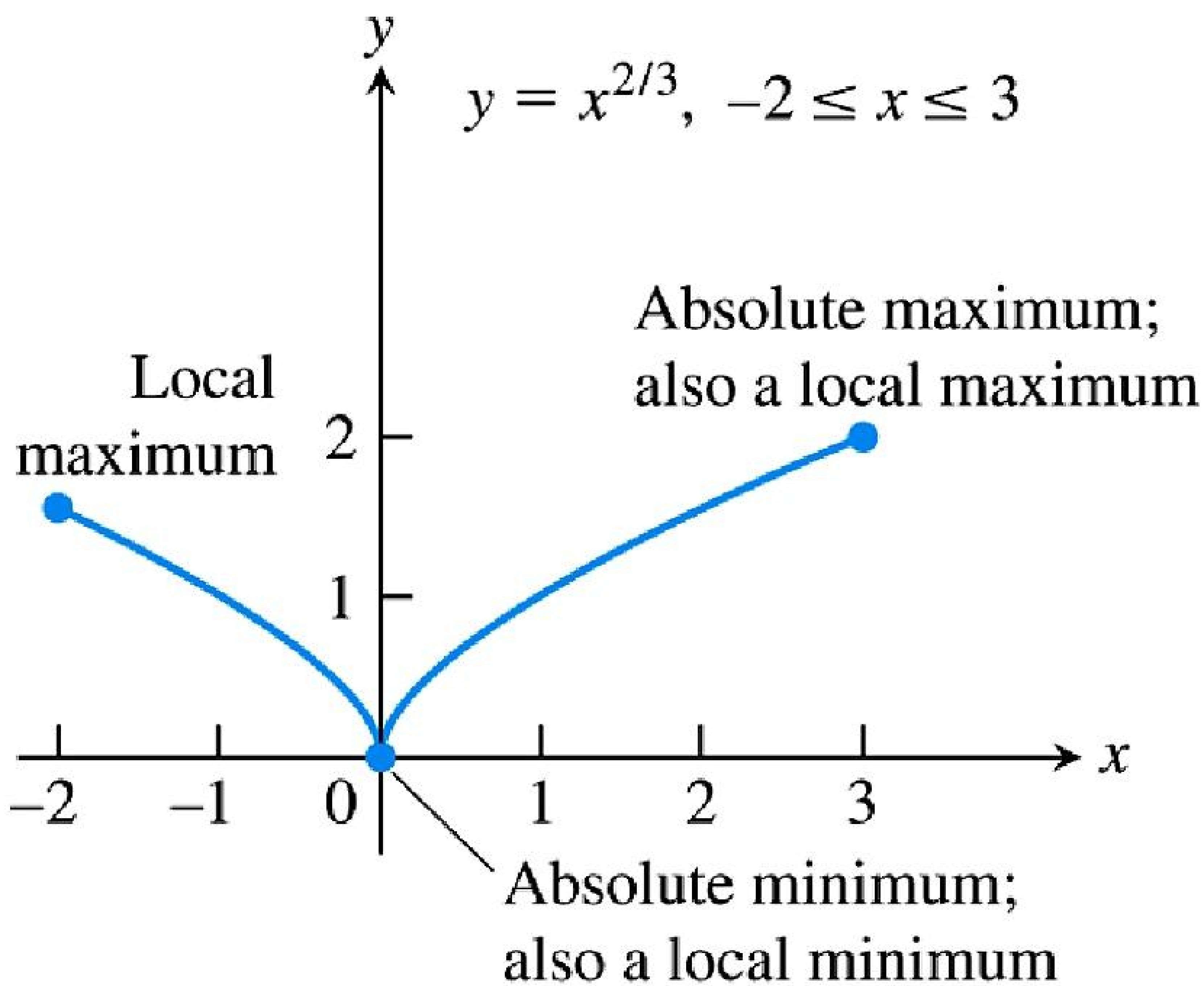
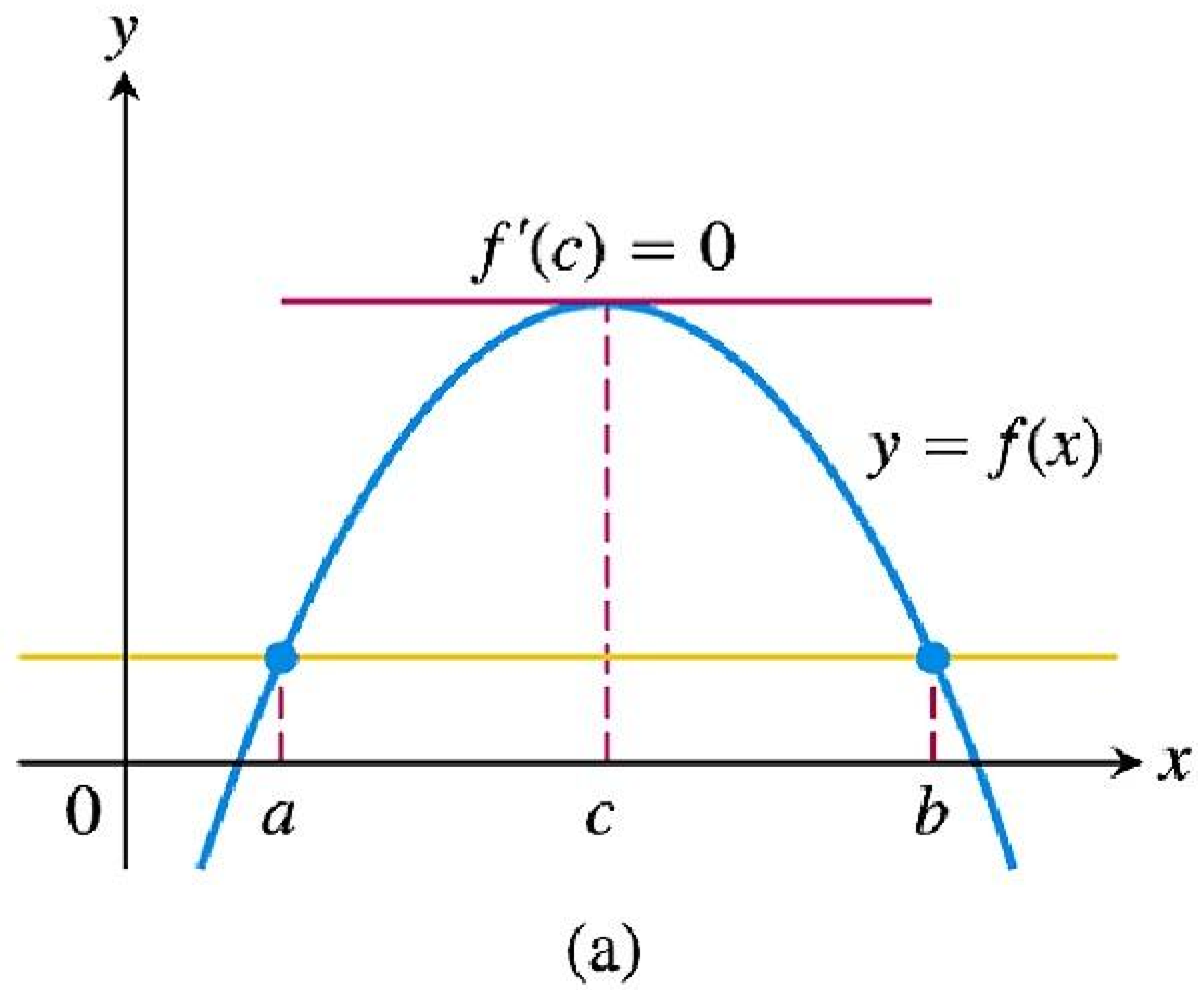


FIGURE 4.9 The extreme values of $f(x) = x^{2/3}$ on $[-2, 3]$ occur at $x = 0$ and $x = 3$ (Example 4).

Pearson

Section 4.2

The Mean Value Theorem



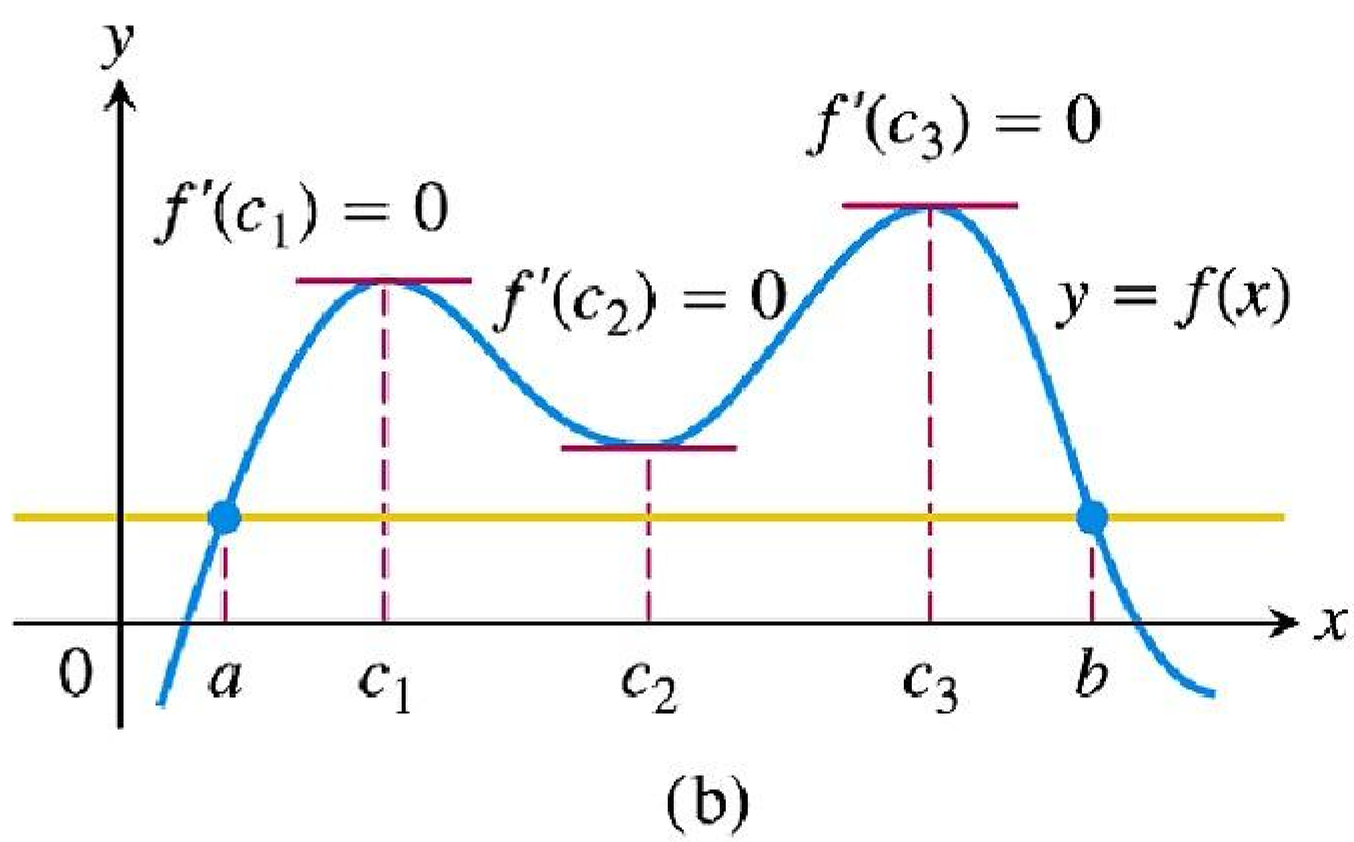
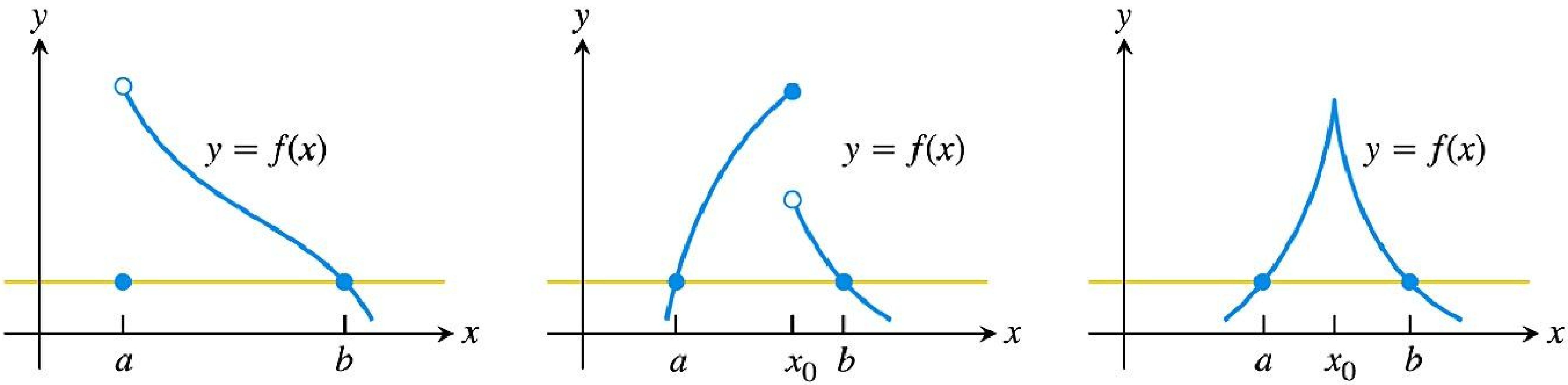


FIGURE 4.10 Rolle's Theorem says that a differentiable curve has at least one horizontal tangent between any two points where it crosses a horizontal line. It may have just one (a), or it may have more (b).

THEOREM 3—Rolle's Theorem
Suppose that $y = f(x)$ is continuous over the closed interval $[a, b]$ and differentiable at every point of its interior (a, b). If $f(a) = f(b)$, then there is at least one number c in (a, b) at which $f'(c) = 0$.



(a) Discontinuous at an endpoint of $[a, b]$

(b) Discontinuous at an interior point of $[a, b]$

(c) Continuous on $[a, b]$ but not differentiable at an interior point

FIGURE 4.11 There may be no horizontal tangent if the hypotheses of Rolle's Theorem do not hold.

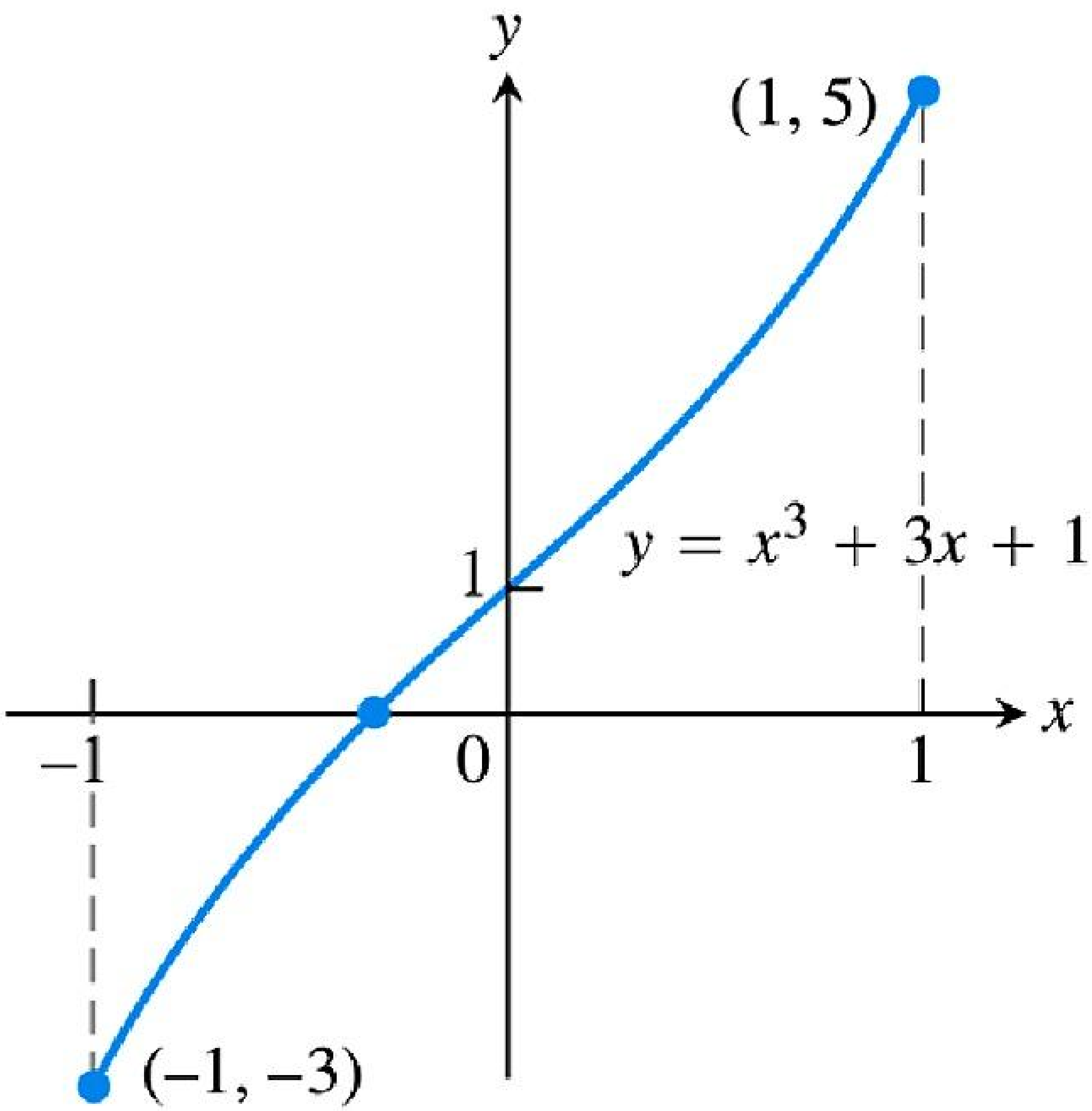


FIGURE 4.12 The only real zero of the polynomial $y = x^3 + 3x + 1$ is the one shown here where the curve crosses the x-axis between -1 and 0 (Example 1).

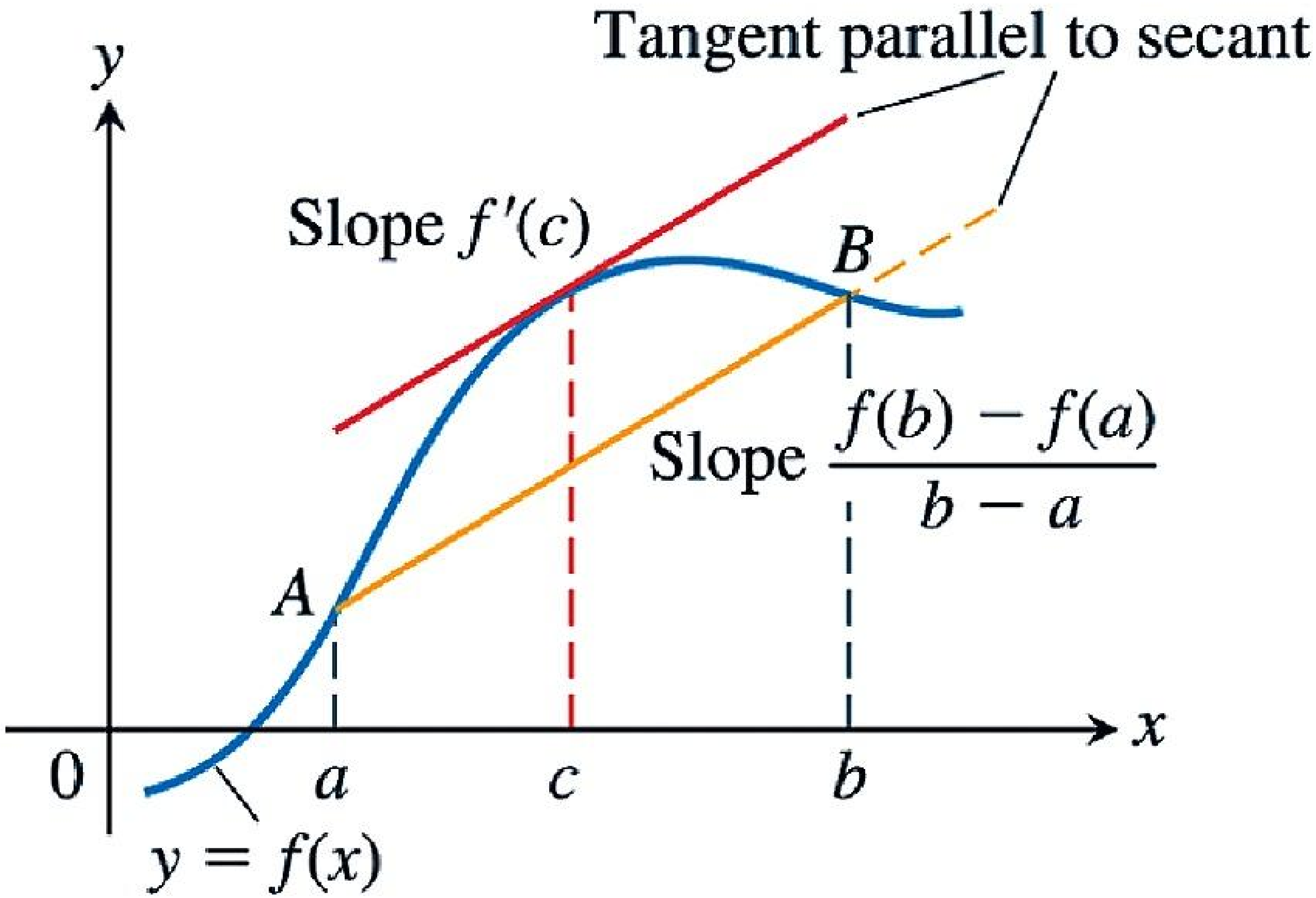


FIGURE 4.13 Geometrically, the Mean Value Theorem says that somewhere between a and b the curve has at least one tangent line parallel to the secant line that joins A and B.

THEOREM 4—The Mean Value Theorem

Suppose $y = f(x)$ is continuous over a closed interval $[a, b]$ and differentiable on the interval's interior (a, b). Then there is at least one point c in (a, b) at which

$$\frac{f(b) - f(a)}{b - a} = f'(c). \tag{1}$$

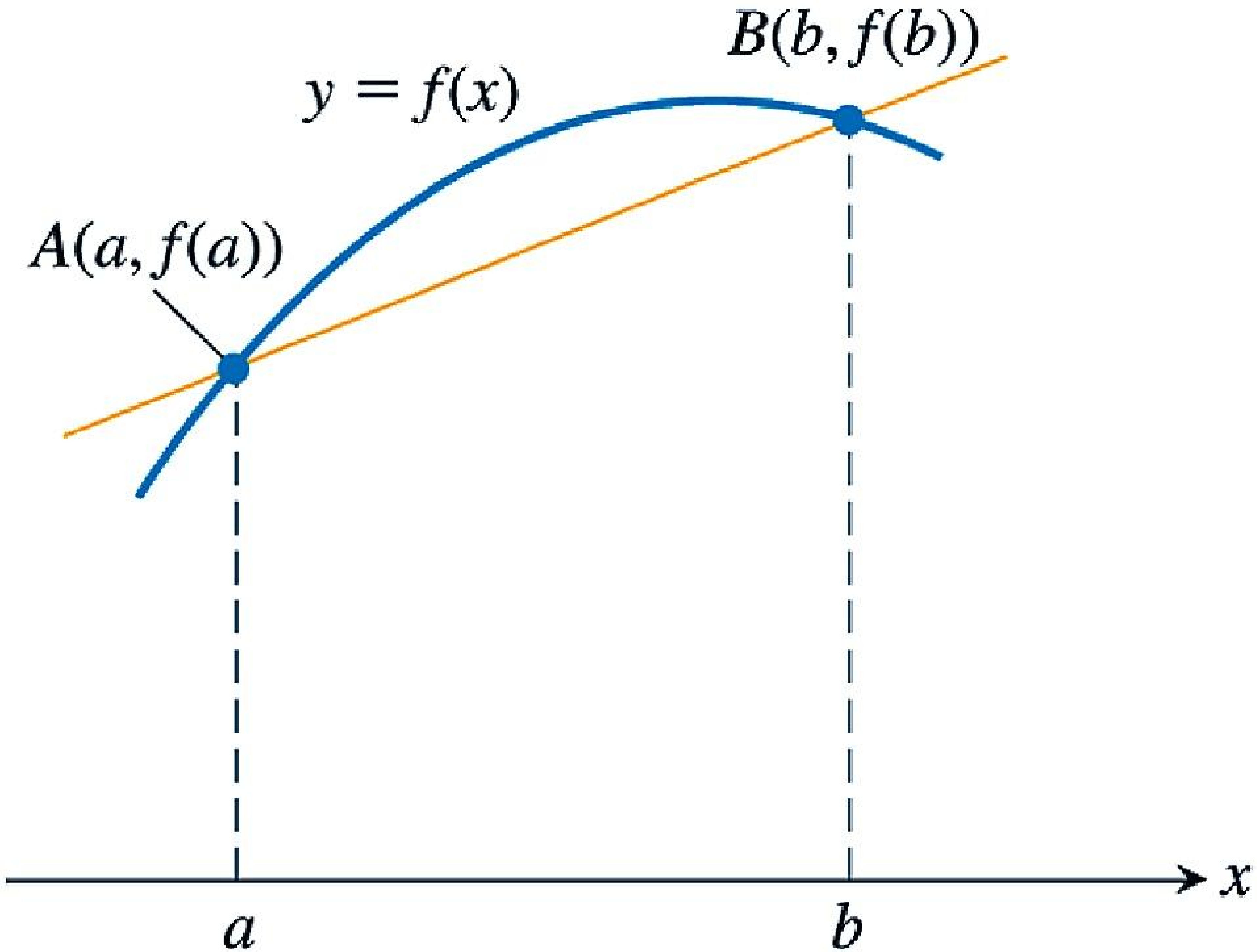


FIGURE 4.14 The graph of f and the secant AB over the interval $[a, b]$.

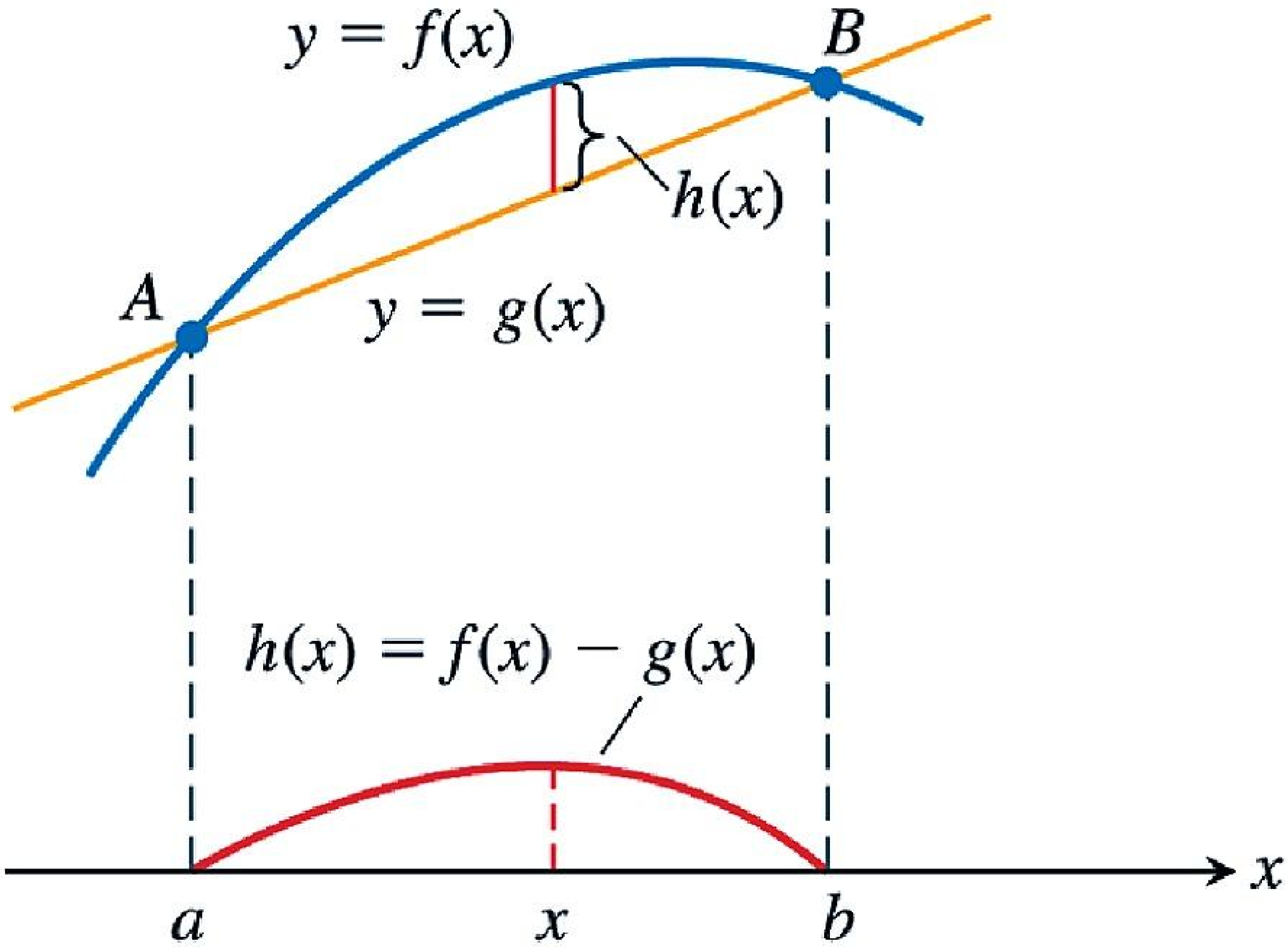


FIGURE 4.15 The secant AB is the graph of the function $g(x)$. The function $h(x) = f(x) - g(x)$ gives the vertical distance between the graphs of f and g at x.

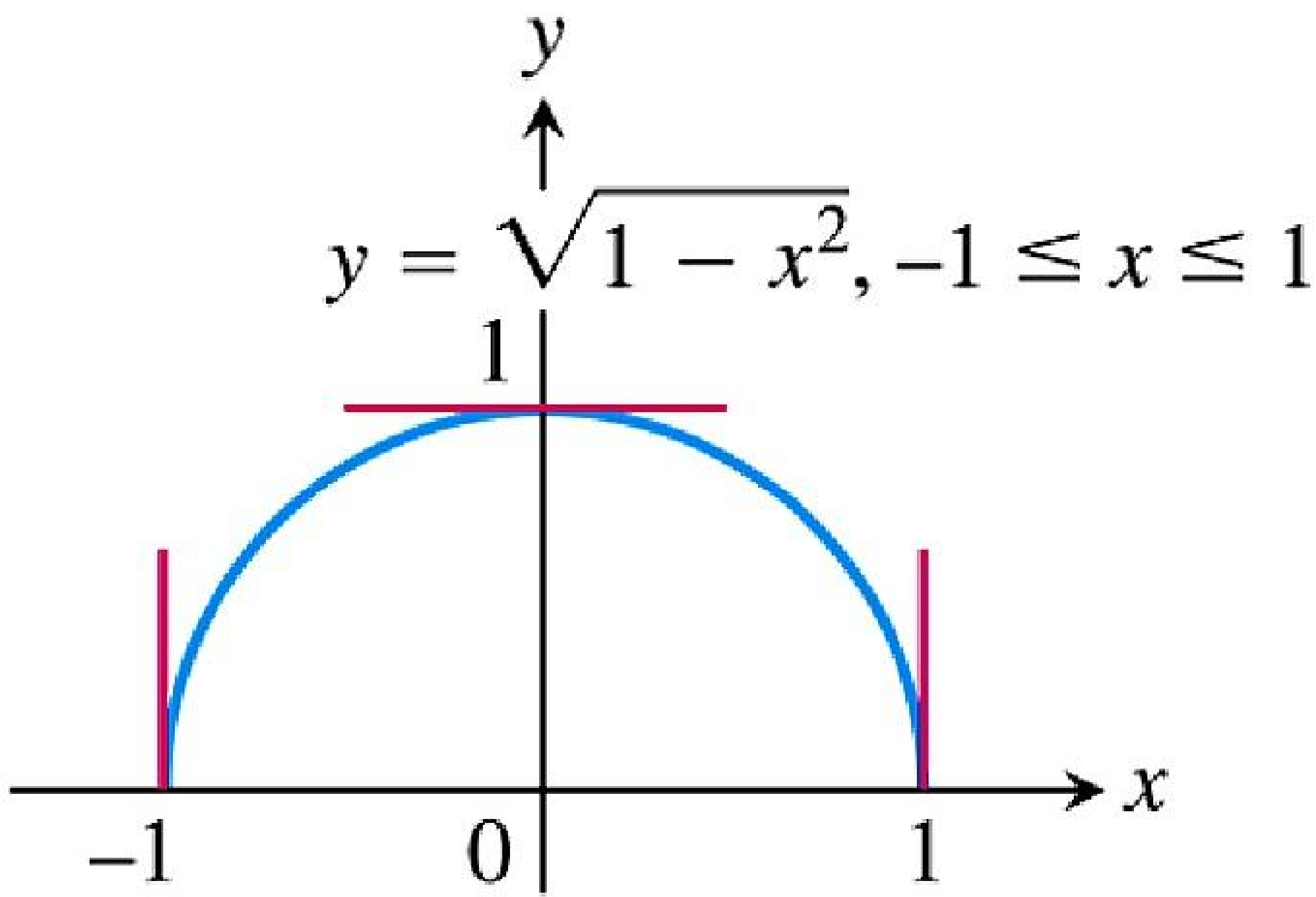


FIGURE 4.16 The function $f(x) = \sqrt{1 - x^2}$ satisfies the hypotheses (and conclusion) of the Mean Value Theorem on $[-1, 1]$ even though f is not differentiable at -1 and 1.

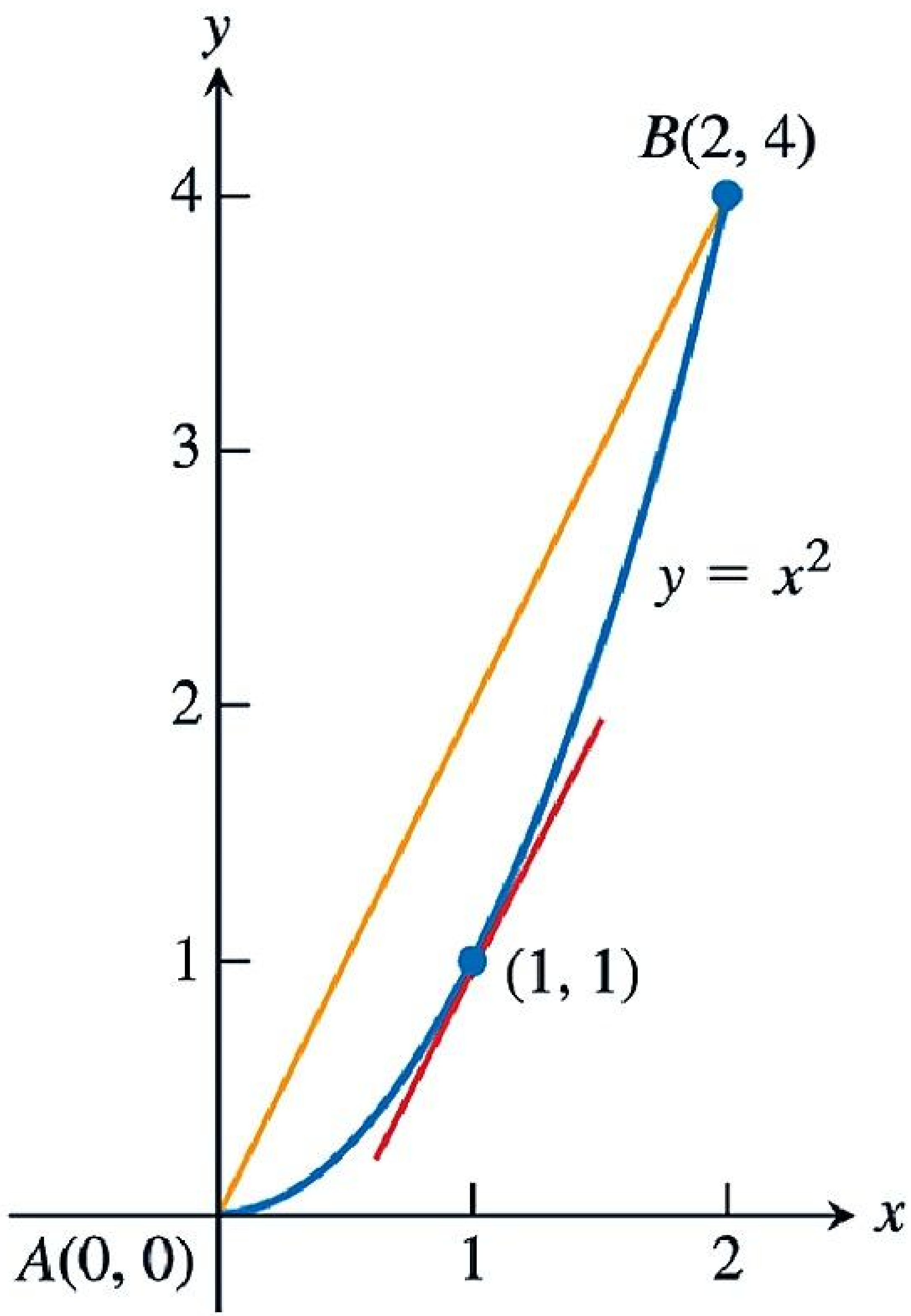


FIGURE 4.17 As we find in Example 2, $c = 1$ is where the tangent is parallel to the secant line.

COROLLARY 1 If $f'(x) = 0$ at each point x of an open interval (a, b), then $f(x) = C$ for all $x \in (a, b)$, where C is a constant.

COROLLARY 2 If $f'(x) = g'(x)$ at each point x in an open interval (a, b), then there exists a constant C such that $f(x) = g(x) + C$ for all $x \in (a, b)$. That is, $f - g$ is a constant function on (a, b).

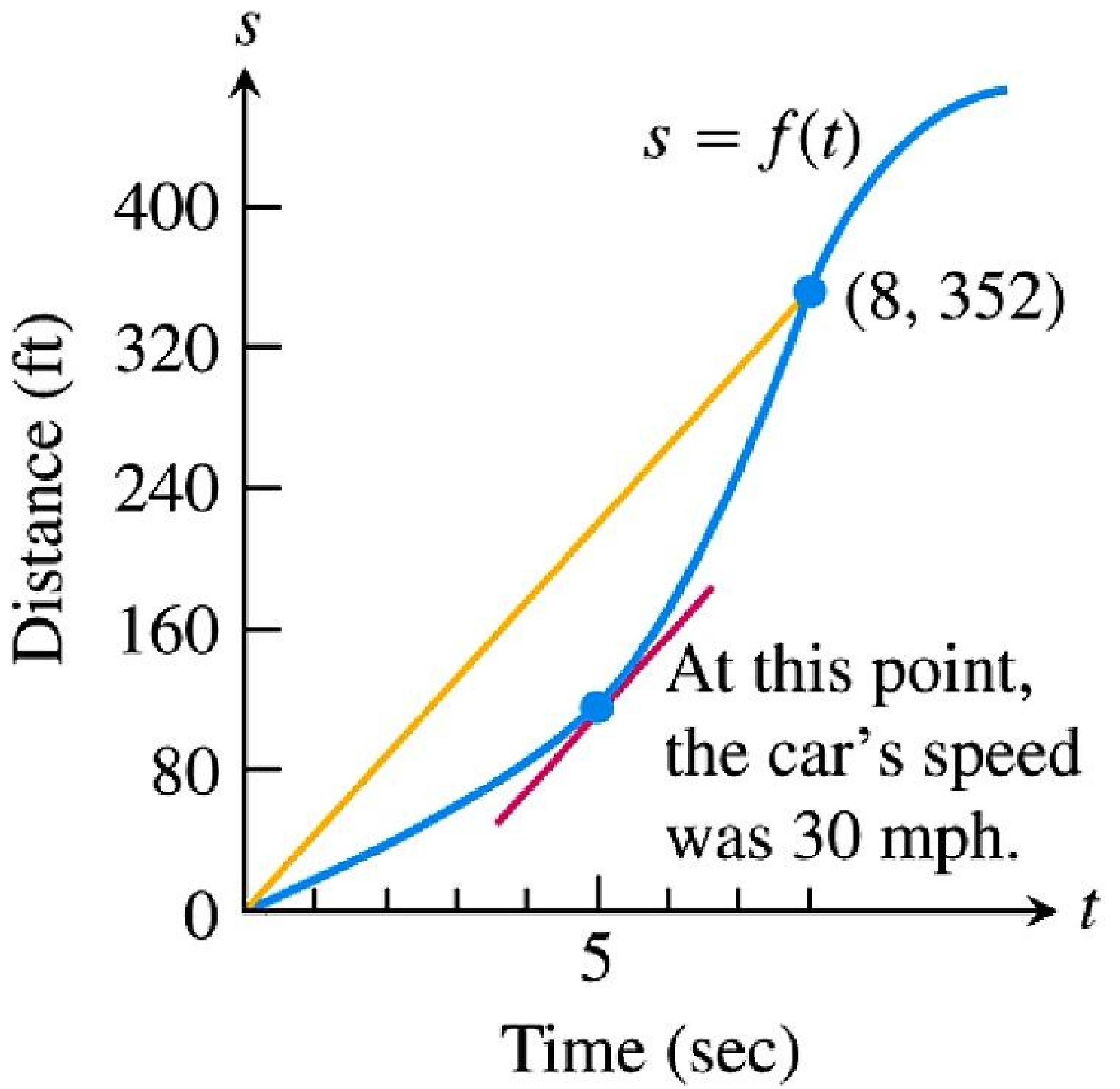


FIGURE 4.18 Distance versus elapsed time for the car in Example 3.

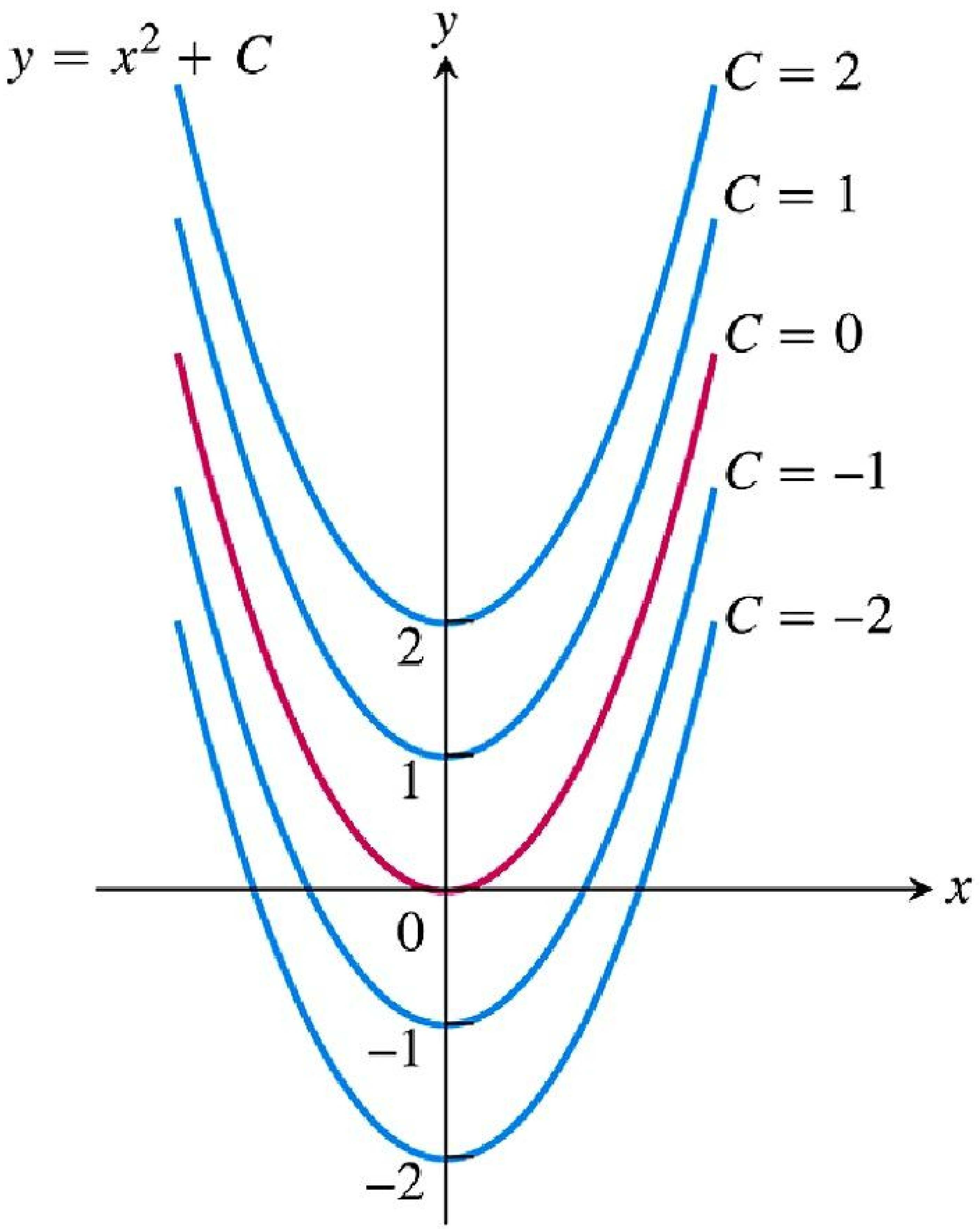


FIGURE 4.19 From a geometric point of view, Corollary 2 of the Mean Value Theorem says that the graphs of functions with identical derivatives on an interval can differ only by a vertical shift. The graphs of the functions with derivative $2x$ are the parabolas $y = x^2 + C$, shown here for several values of C.

Section 4.3

Monotonic Functions and The First Derivative Test

COROLLARY 3 Suppose that f is continuous on $[a, b]$ and differentiable on (a, b).

$$\text{If } f'(x) > 0 \text{ at each point } x \in (a, b), \text{ then } f \text{ is increasing on } [a, b].$$
$$\text{If } f'(x) < 0 \text{ at each point } x \in (a, b), \text{ then } f \text{ is decreasing on } [a, b].$$

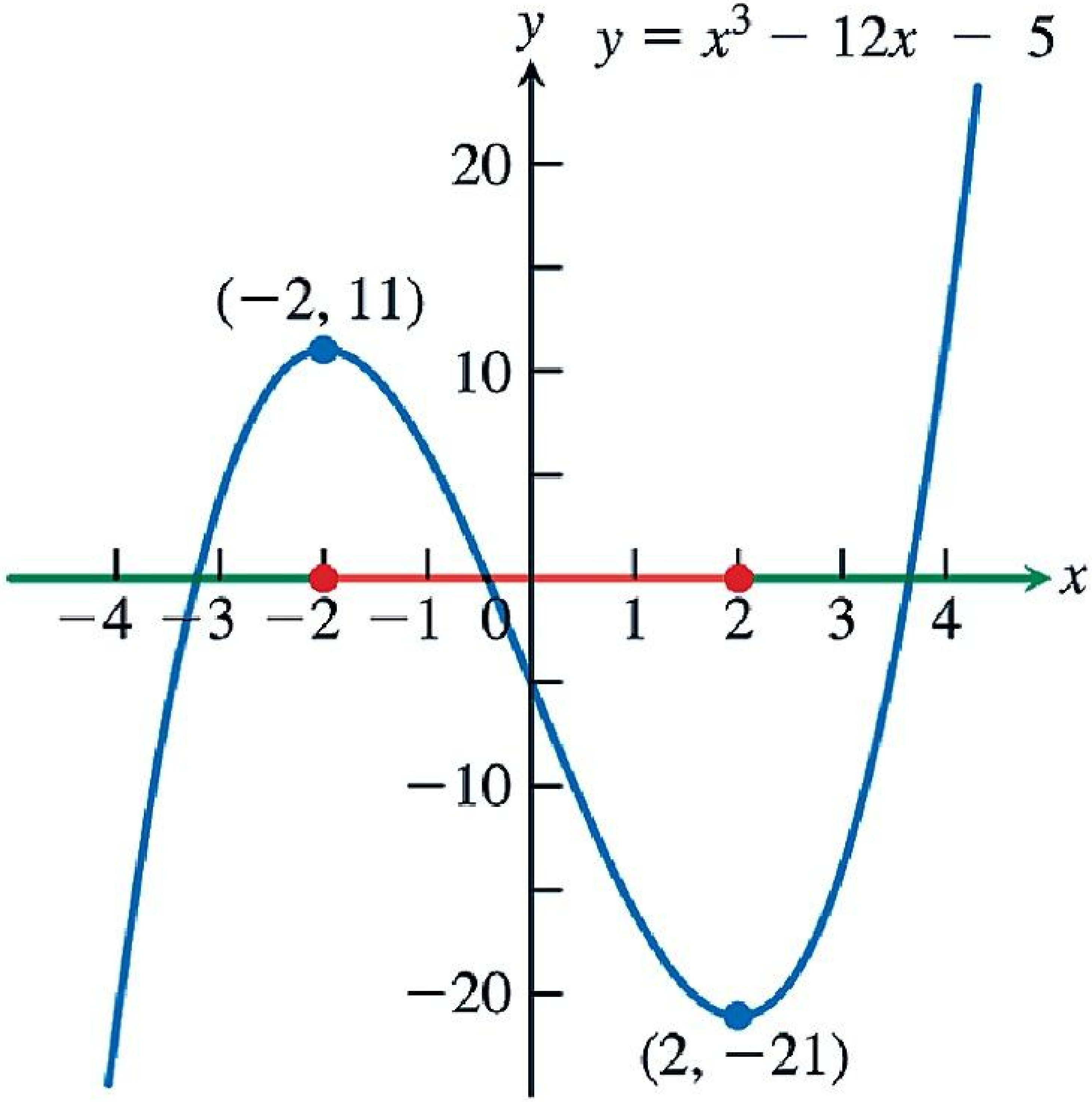
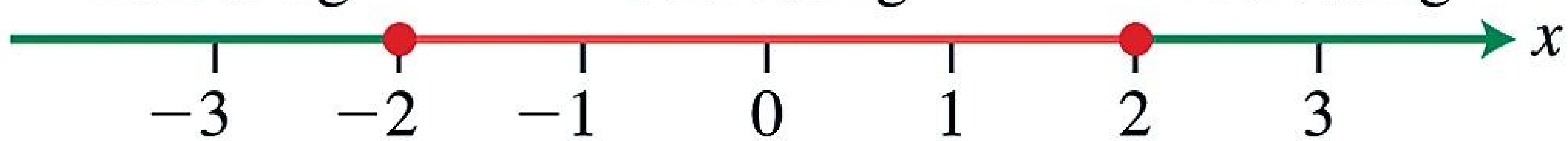


FIGURE 4.20 The function $f(x) = x^3 - 12x - 5$ is monotonic on three separate intervals (Example 1).

Interval	$-\infty < x < -2$	$-2 < x < 2$	$2 < x < \infty$
f' evaluated	$f'(-3) = 15$	$f'(0) = -12$	$f'(3) = 15$
Sign of f'	$+$	$-$	$+$
Behavior of f	increasing	decreasing	increasing



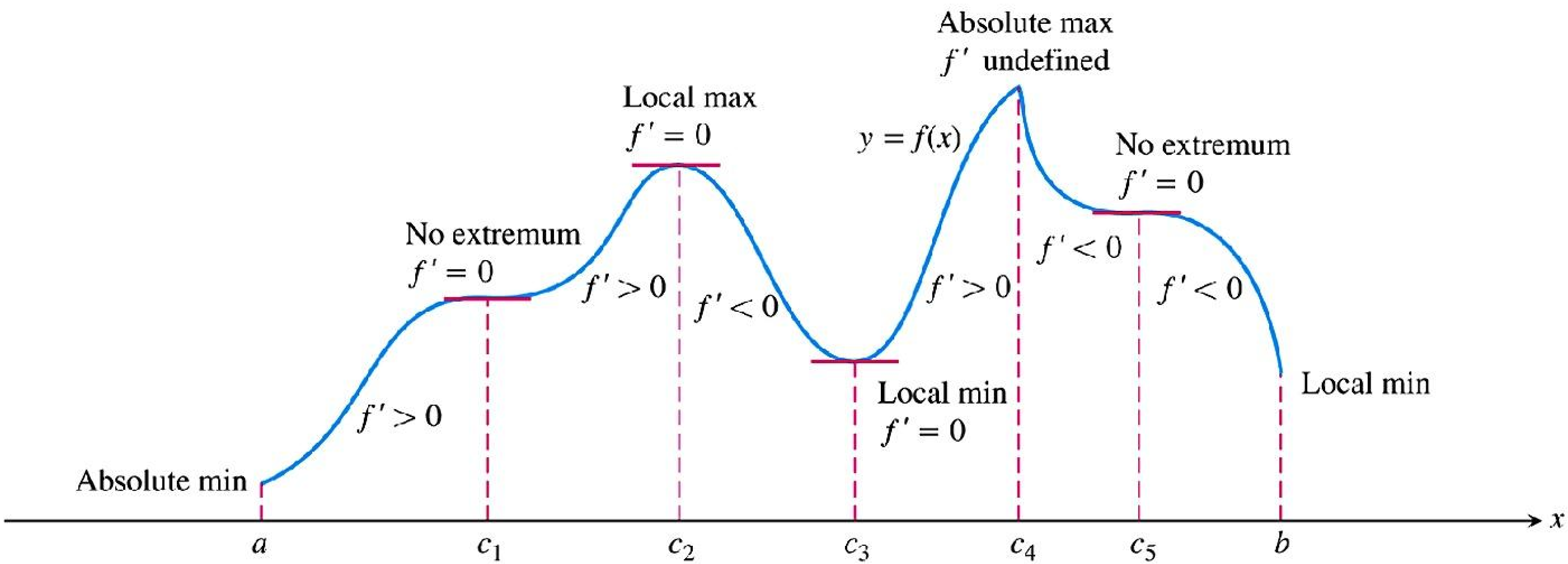


FIGURE 4.21 The critical points of a function locate where it is increasing and where it is decreasing. The first derivative changes sign at a critical point where a local extremum occurs.

First Derivative Test for Local Extrema

Suppose that c is a critical point of a continuous function f, and that f is differentiable at every point in some interval containing c except possibly at c itself. Moving across this interval from left to right,

1. if f' changes from negative to positive at c, then f has a local minimum at c;
2. if f' changes from positive to negative at c, then f has a local maximum at c;
3. if f' does not change sign at c (that is, f' is positive on both sides of c or negative on both sides), then f has no local extremum at c.

Interval	$x < 0$	$0 < x < 1$	$x > 1$
Sign of f'	$-$	$-$	$+$
Behavior of f	decreasing	decreasing	increasing

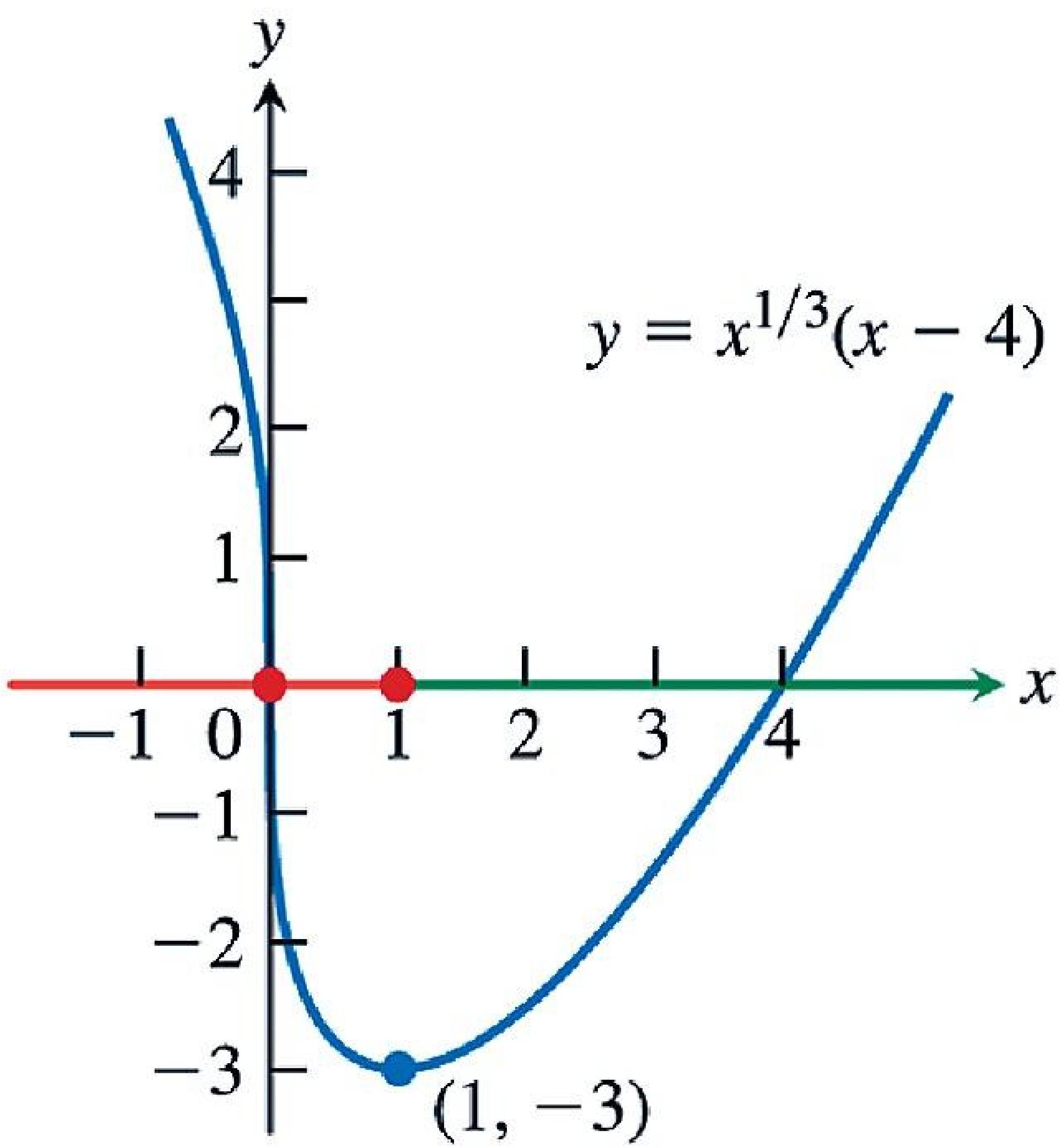


FIGURE 4.22 The function $f(x) = x^{1/3}(x - 4)$ decreases when $x < 1$ and increases when $x > 1$ (Example 2).

Interval	$\left(0, \frac{\pi}{6}\right)$	$\left(\frac{\pi}{6}, \frac{\pi}{2}\right)$	$\left(\frac{\pi}{2}, \frac{5\pi}{6}\right)$	$\left(\frac{5\pi}{6}, \frac{3\pi}{2}\right)$	$\left(\frac{3\pi}{2}, 2\pi\right)$
Sign of f'	$-$	$+$	$-$	$+$	$-$
Behavior of f	dec	inc	dec	increasing	decreasing

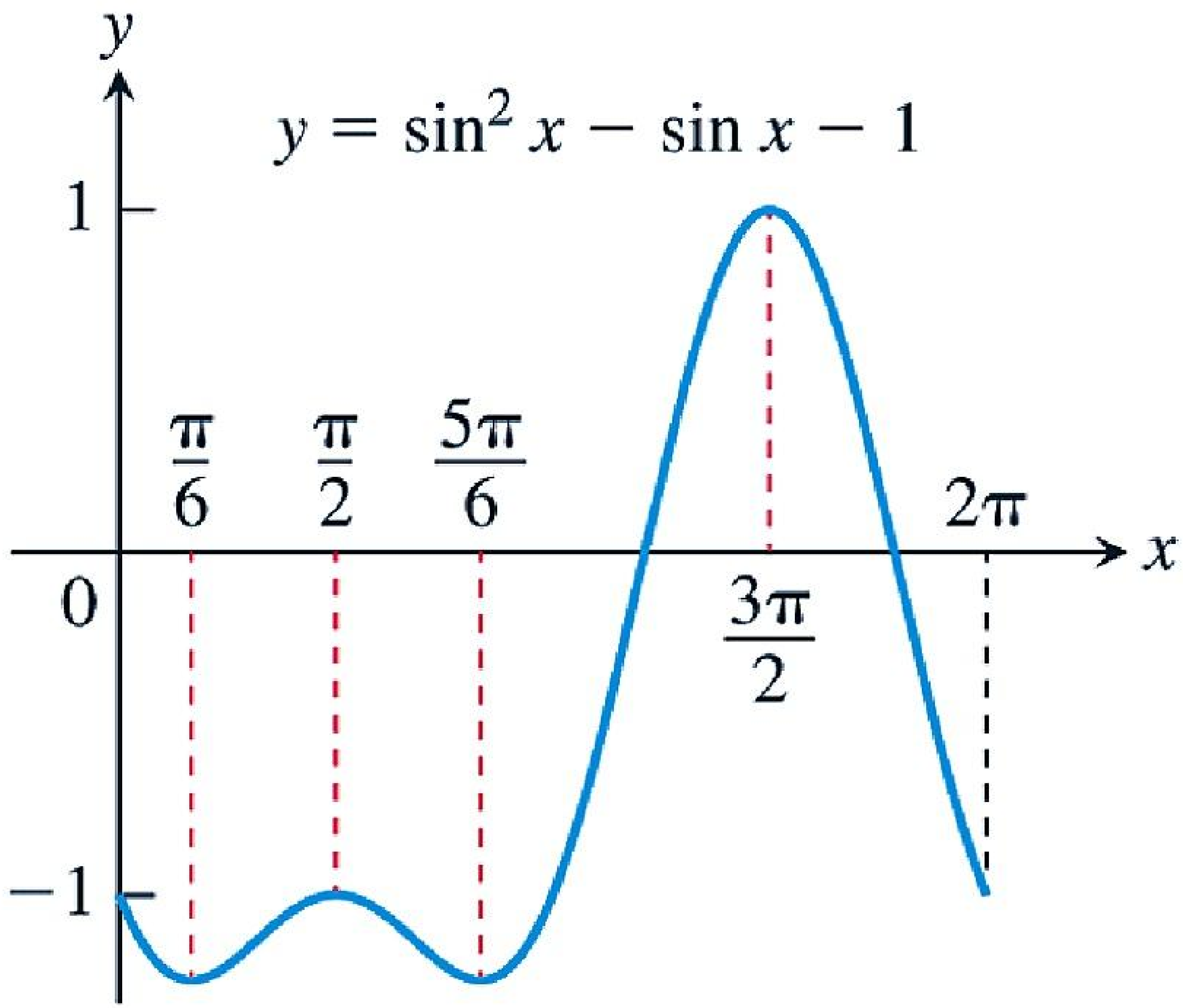


FIGURE 4.23 The graph of the function in Example 3.

Section 4.4

Concavity and Curve Sketching

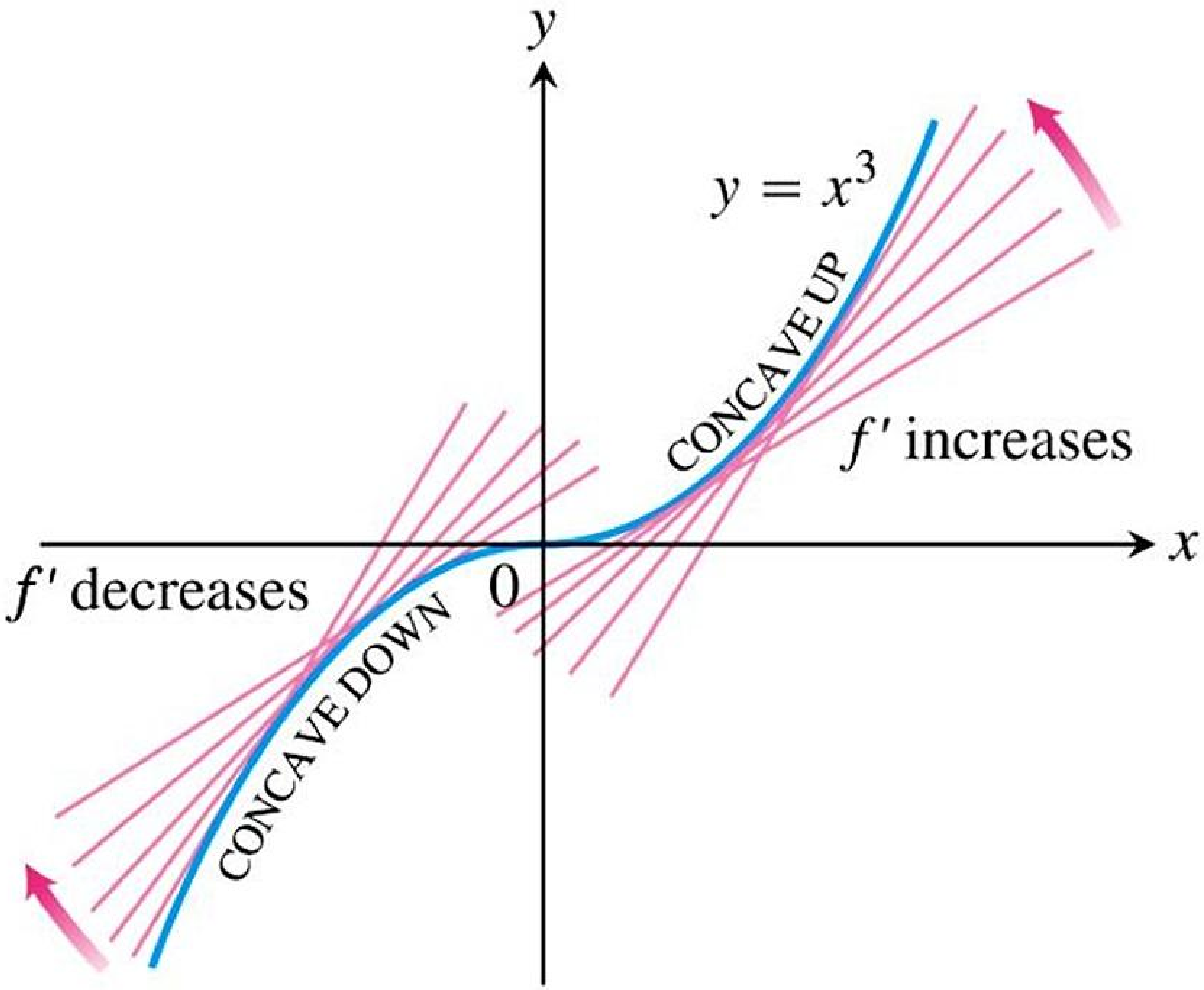


FIGURE 4.24 The graph of $f(x) = x^3$ is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$ (Example 1a).

DEFINITION The graph of a differentiable function $y = f(x)$ is

(a) **concave up** on an open interval I if f' is increasing on I;

(b) **concave down** on an open interval I if f' is decreasing on I.

The Second Derivative Test for Concavity

Let $y = f(x)$ be twice-differentiable on an interval I.

1. If $f'' > 0$ on I, the graph of f over I is concave up.
2. If $f'' < 0$ on I, the graph of f over I is concave down.

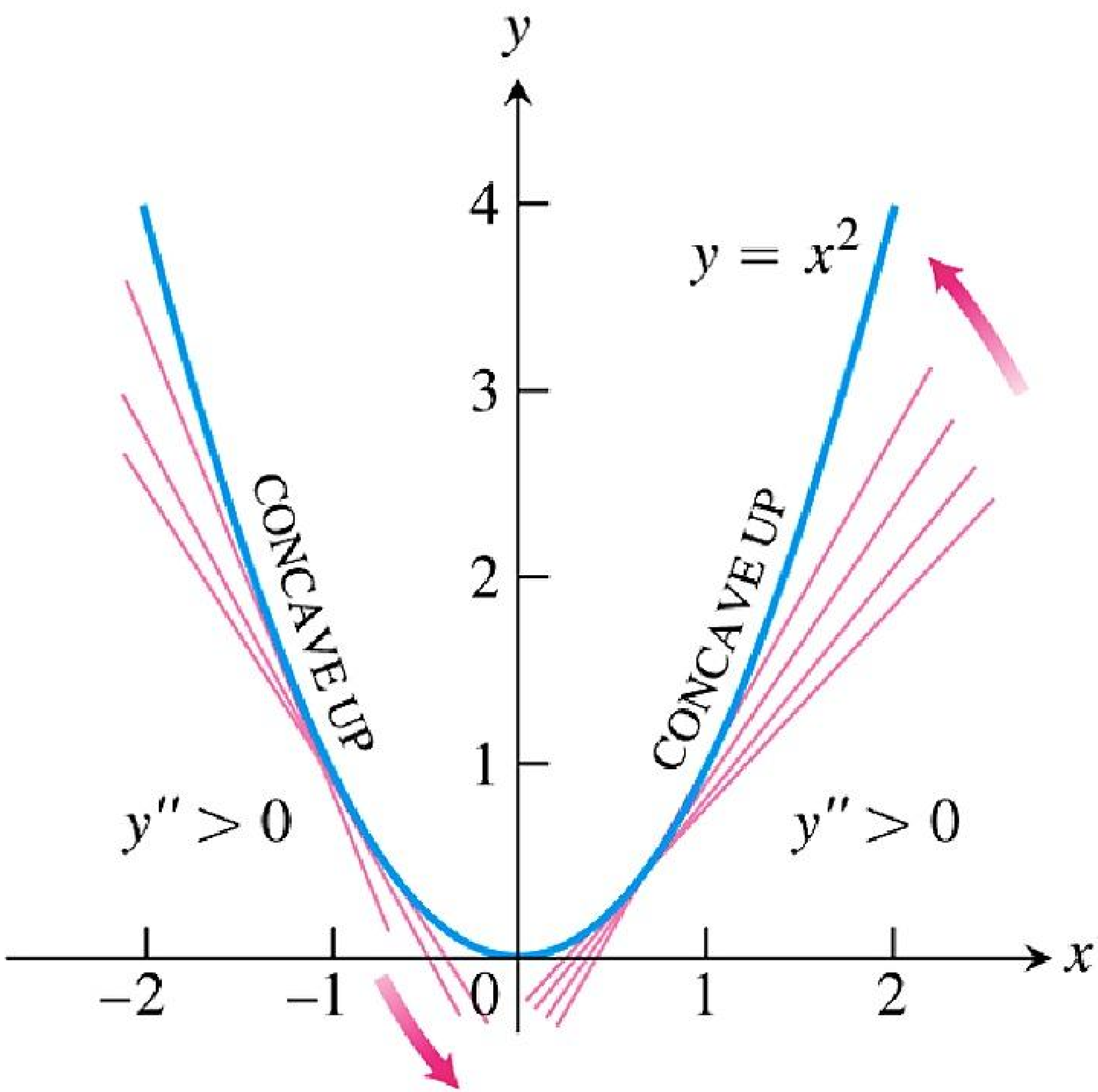


FIGURE 4.25 The graph of $f(x) = x^2$ is concave up on every interval (Example 1b).

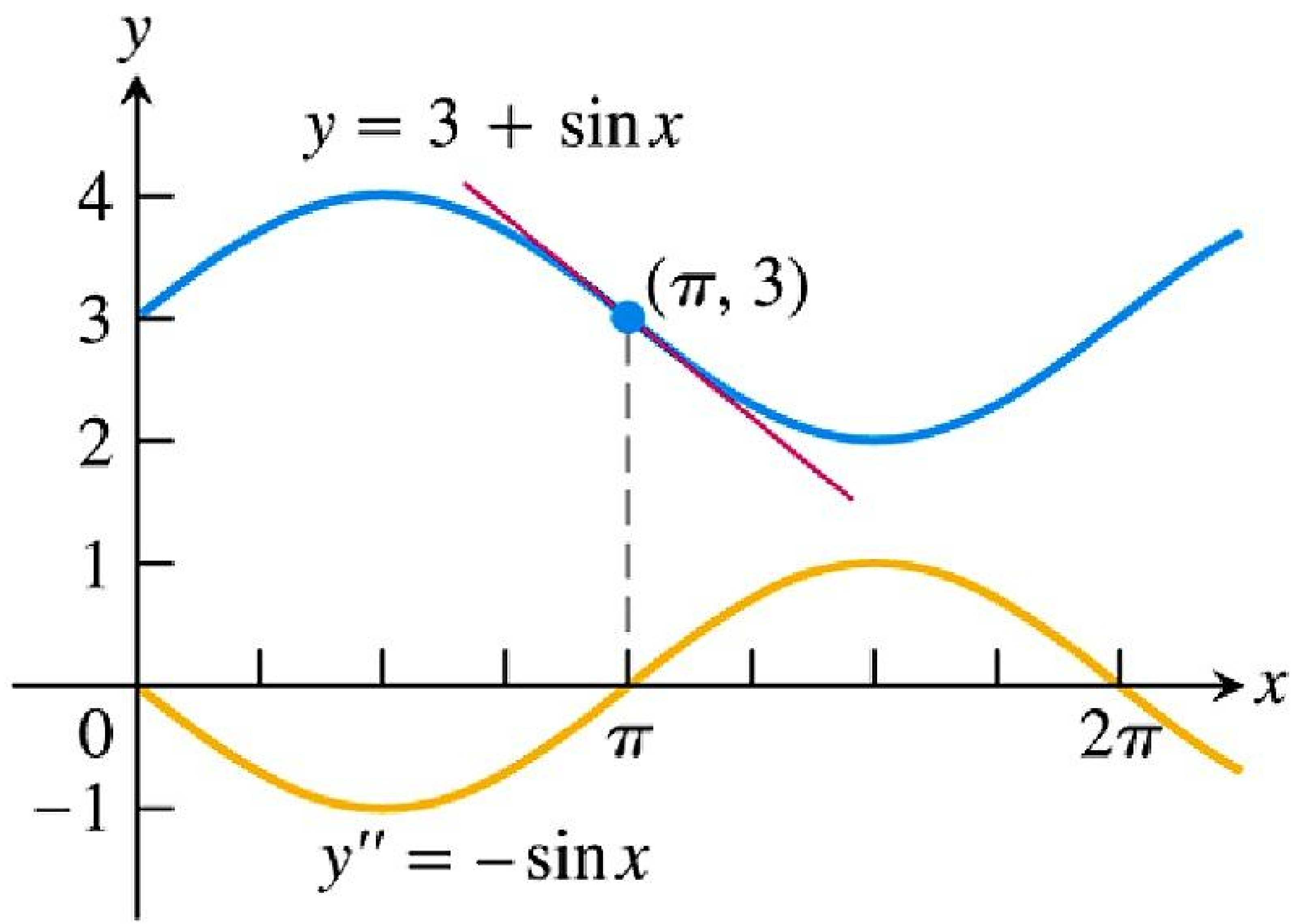


FIGURE 4.26 Using the sign of y'' to determine the concavity of y (Example 2).

DEFINITION A point $(c, f(c))$ where the graph of a function has a tangent line and where the concavity changes is a **point of inflection.**

At a point of inflection $(c, f(c))$, either $f''(c) = 0$ or $f''(c)$ fails to exist.

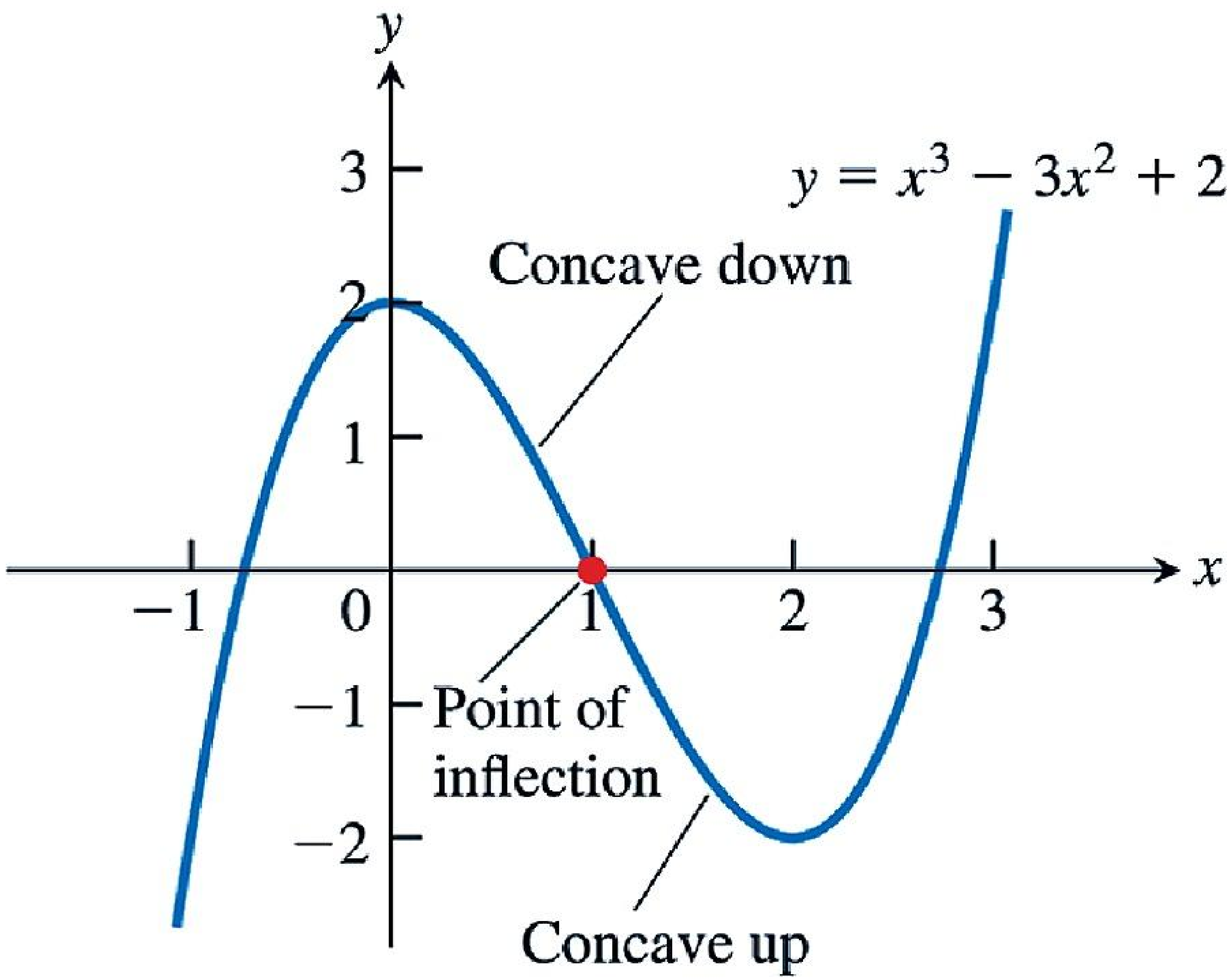


FIGURE 4.27 The concavity of the graph of f changes from concave down to concave up at the inflection point.

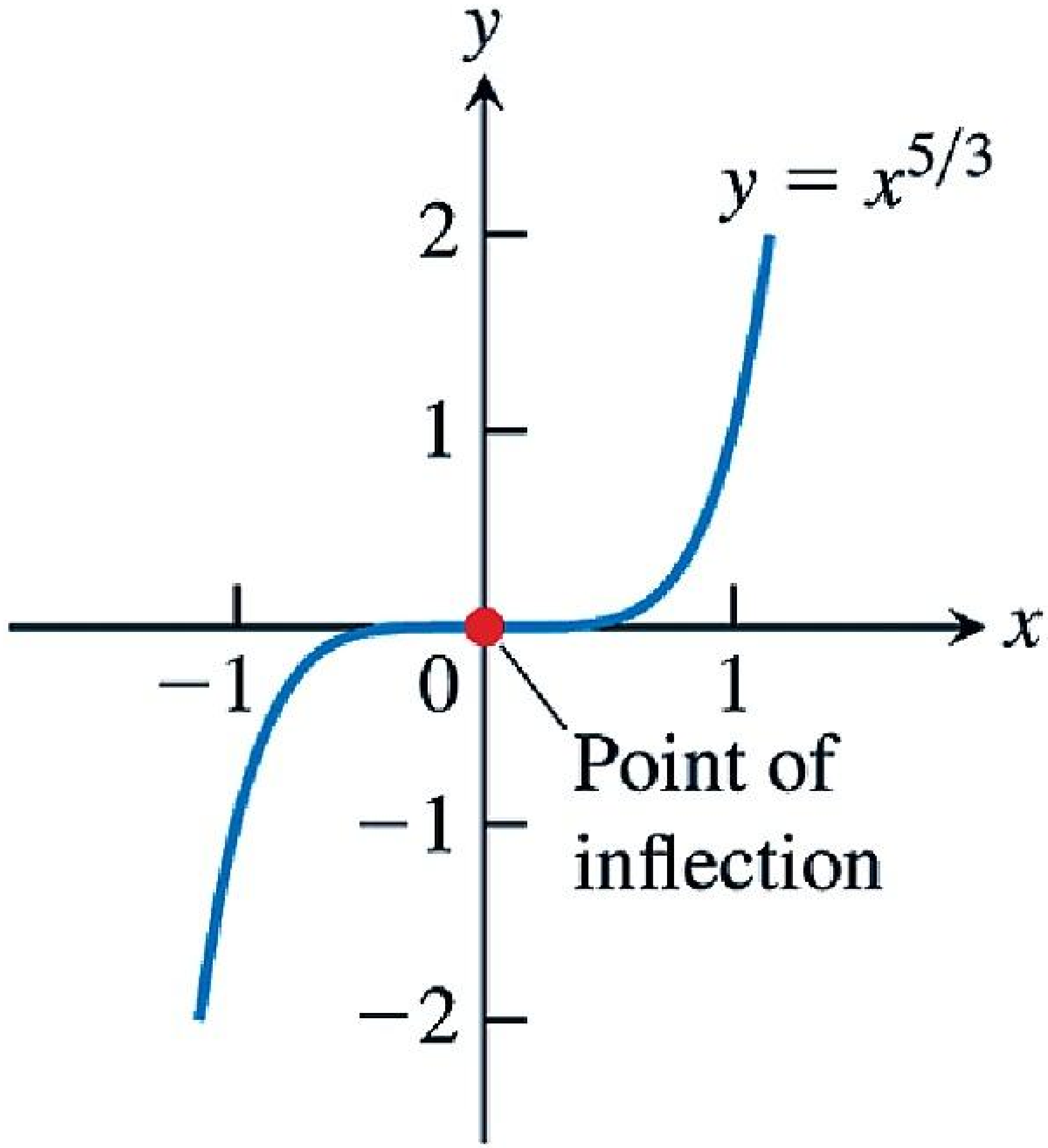


FIGURE 4.28 The graph of $f(x) = x^{5/3}$ has a horizontal tangent at the origin where the concavity changes, although f'' does not exist at $x = 0$ (Example 4).

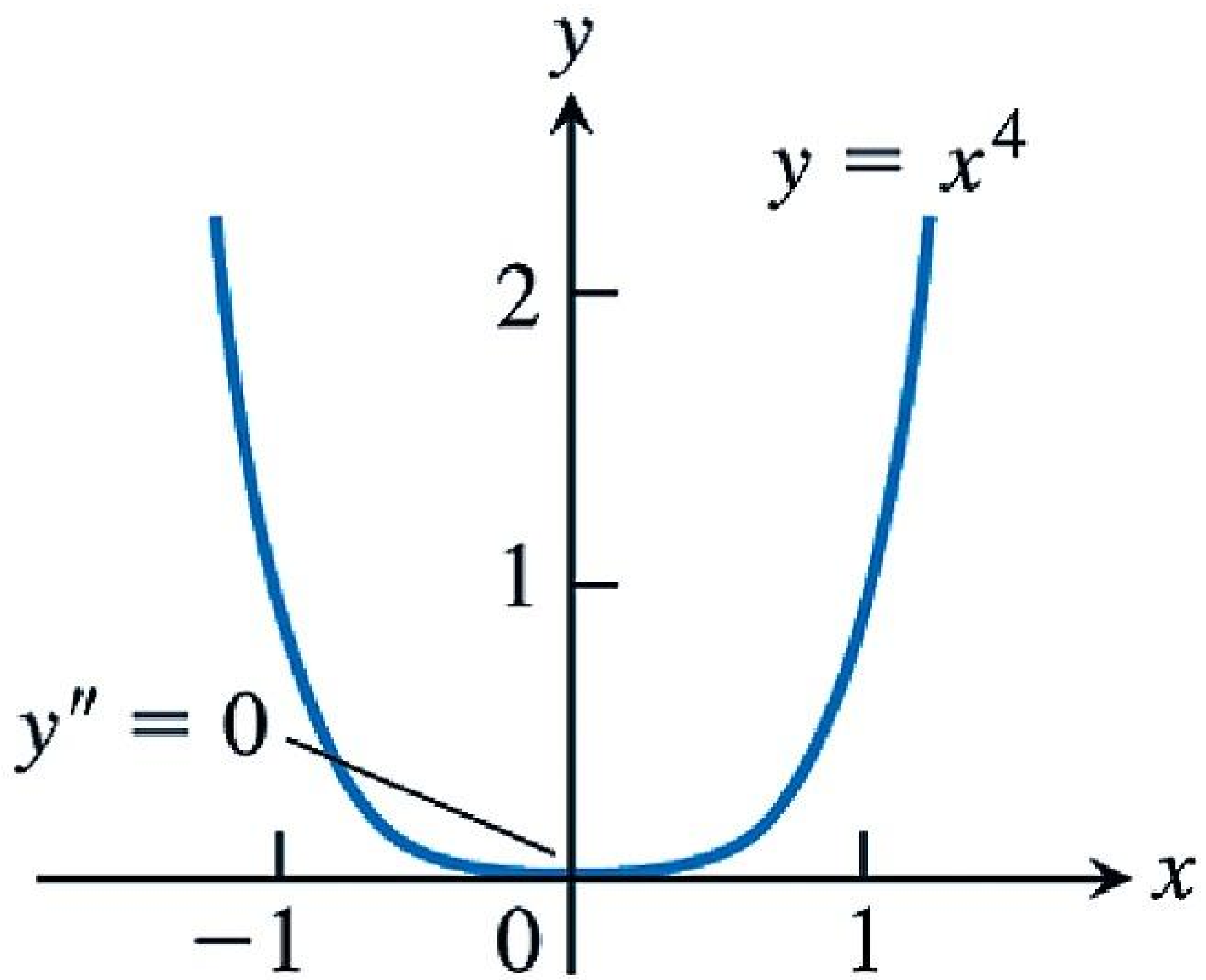


FIGURE 4.29 The graph of $y = x^4$ has no inflection point at the origin, even though $y'' = 0$ there (Example 5).

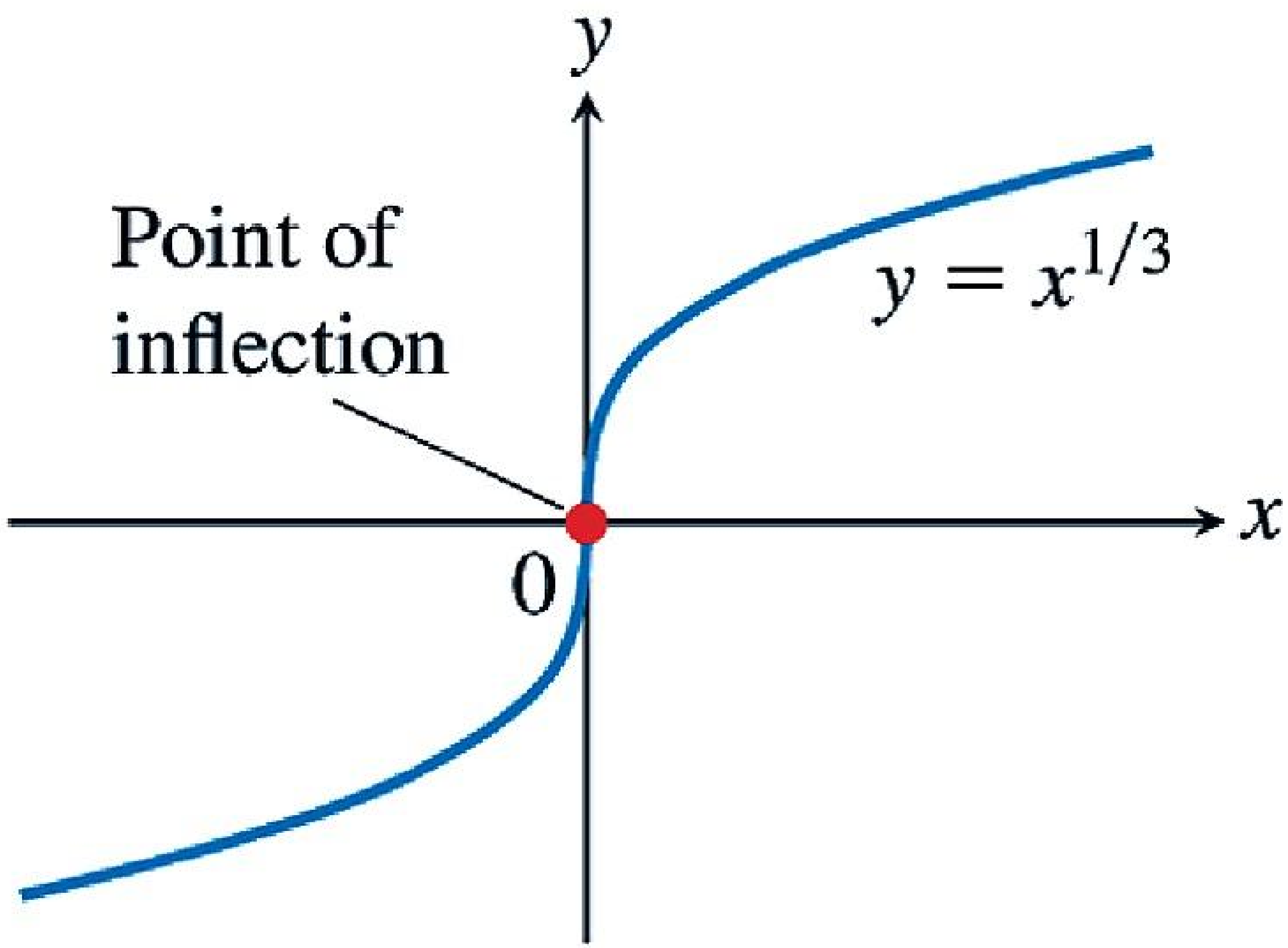


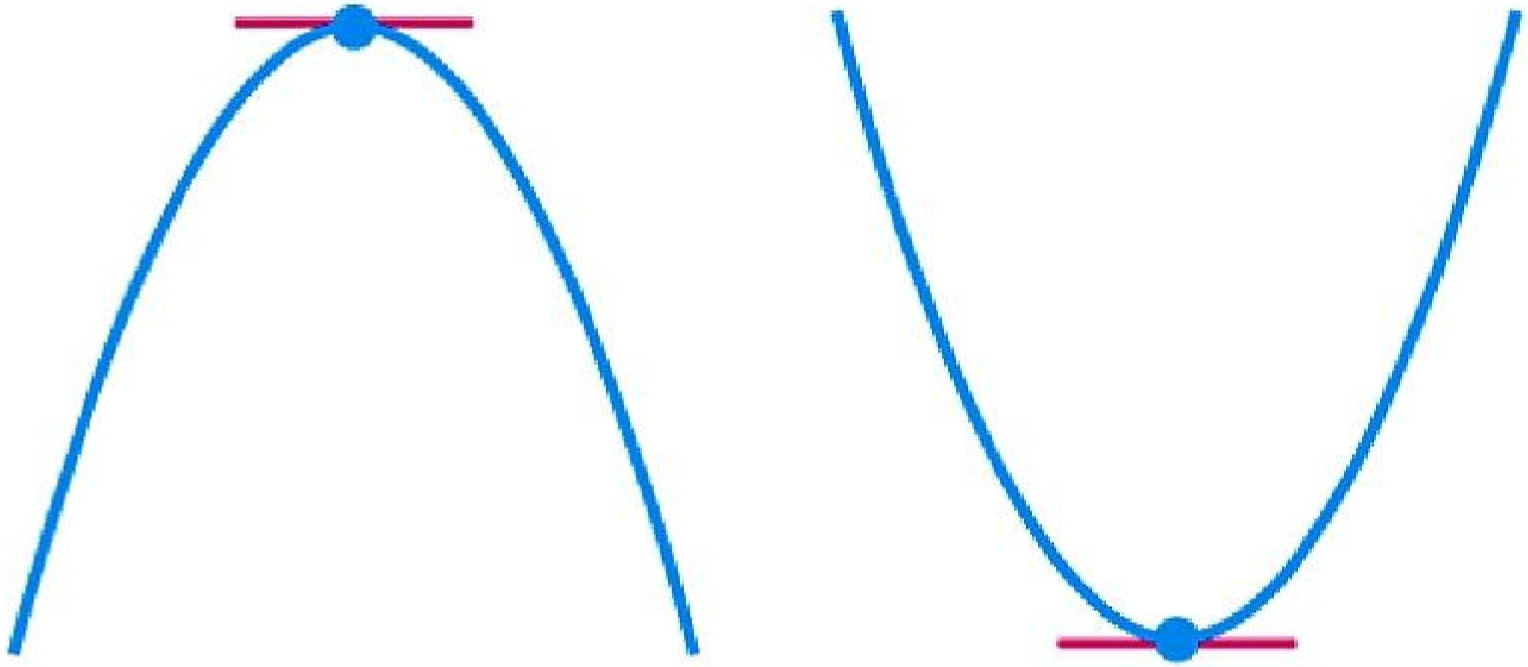
FIGURE 4.30 A point of inflection where y' and y'' fail to exist (Example 6).

Interval	$0 < t < 1$	$1 < t < 11/3$	$11/3 < t$
Sign of $v = s'$	$+$	$-$	$+$
Behavior of s	increasing	decreasing	increasing
Particle motion	right	left	right

Interval	$0 < t < 7/3$	$7/3 < t$
Sign of $a = s''$	$-$	$+$
Graph of s	concave down	concave up

THEOREM 5—Second Derivative Test for Local Extrema Suppose f'' is continuous on an open interval that contains $x = c$.

1. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.
2. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.
3. If $f'(c) = 0$ and $f''(c) = 0$, then the test fails. The function f may have a local maximum, a local minimum, or neither.



$f' = 0, f'' < 0$
$\Rightarrow$ local max

$f' = 0, f'' > 0$
$\Rightarrow$ local min

Interval	$x < 0$	$0 < x < 3$	$3 < x$
Sign of f'	$-$	$-$	$+$
Behavior of f	decreasing	decreasing	increasing

Interval	$x < 0$	$0 < x < 2$	$2 < x$
Sign of f''	$+$	$-$	$+$
Behavior of f	concave up	concave down	concave up

$\boldsymbol{x < 0}$	$\boldsymbol{0 < x < 2}$	$\boldsymbol{2 < x < 3}$	$\boldsymbol{3 < x}$
decreasing concave up	decreasing concave down	decreasing concave up	increasing concave up

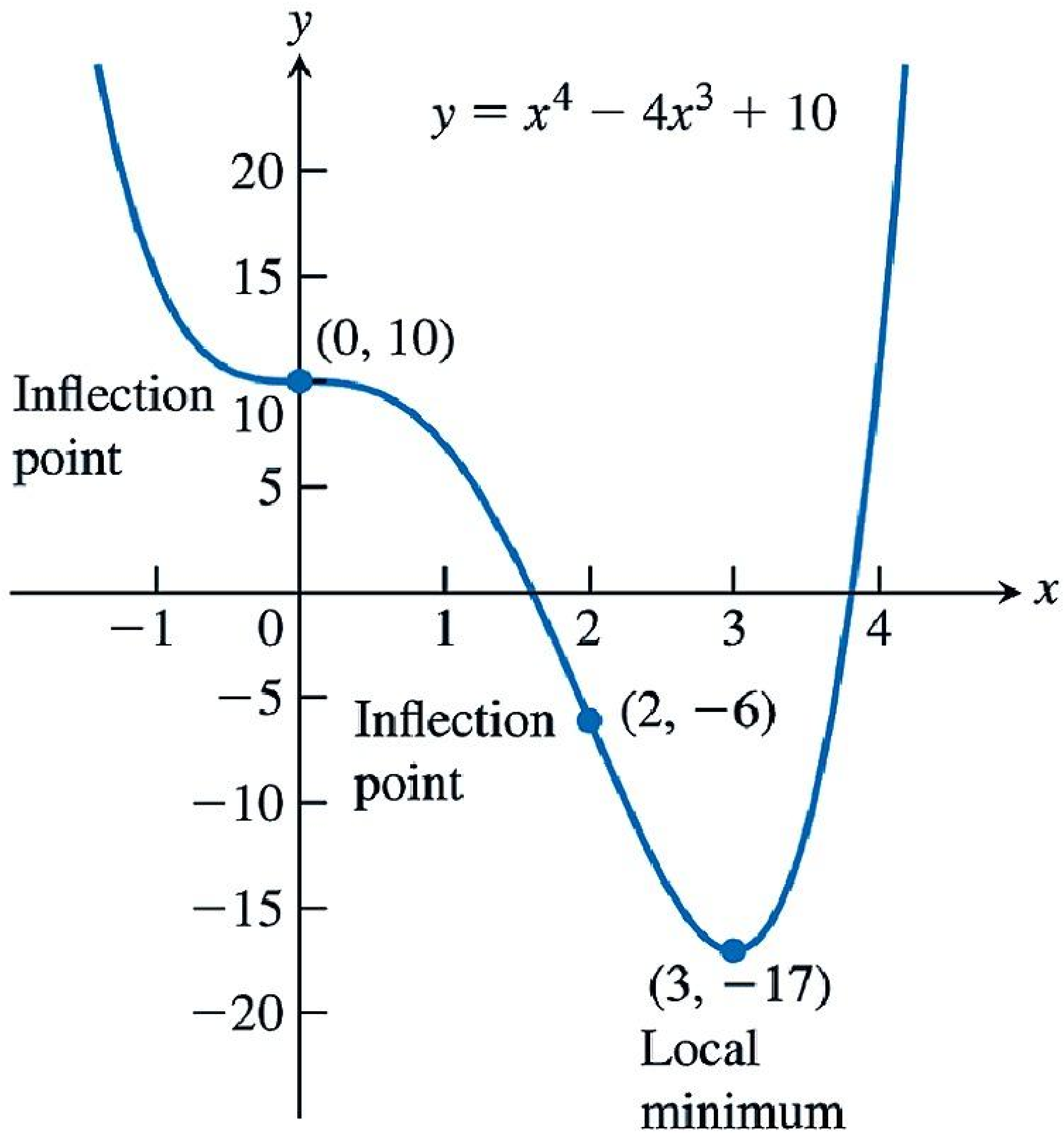


FIGURE 4.31 The graph of $f(x) = x^4 - 4x^3 + 10$ (Example 8).

Procedure for Graphing $y = f(x)$

1. Identify the domain of f and any symmetries the curve may have.
2. Find the derivatives y' and y''.
3. Find the critical points of f, if any, and identify the function's behavior at each one.
4. Find where the curve is increasing and where it is decreasing.
5. Find the points of inflection, if any occur, and determine the concavity of the curve.
6. Identify any asymptotes that may exist.
7. Plot key points, such as the intercepts and the points found in Steps 3–5, and sketch the curve together with any asymptotes that exist.

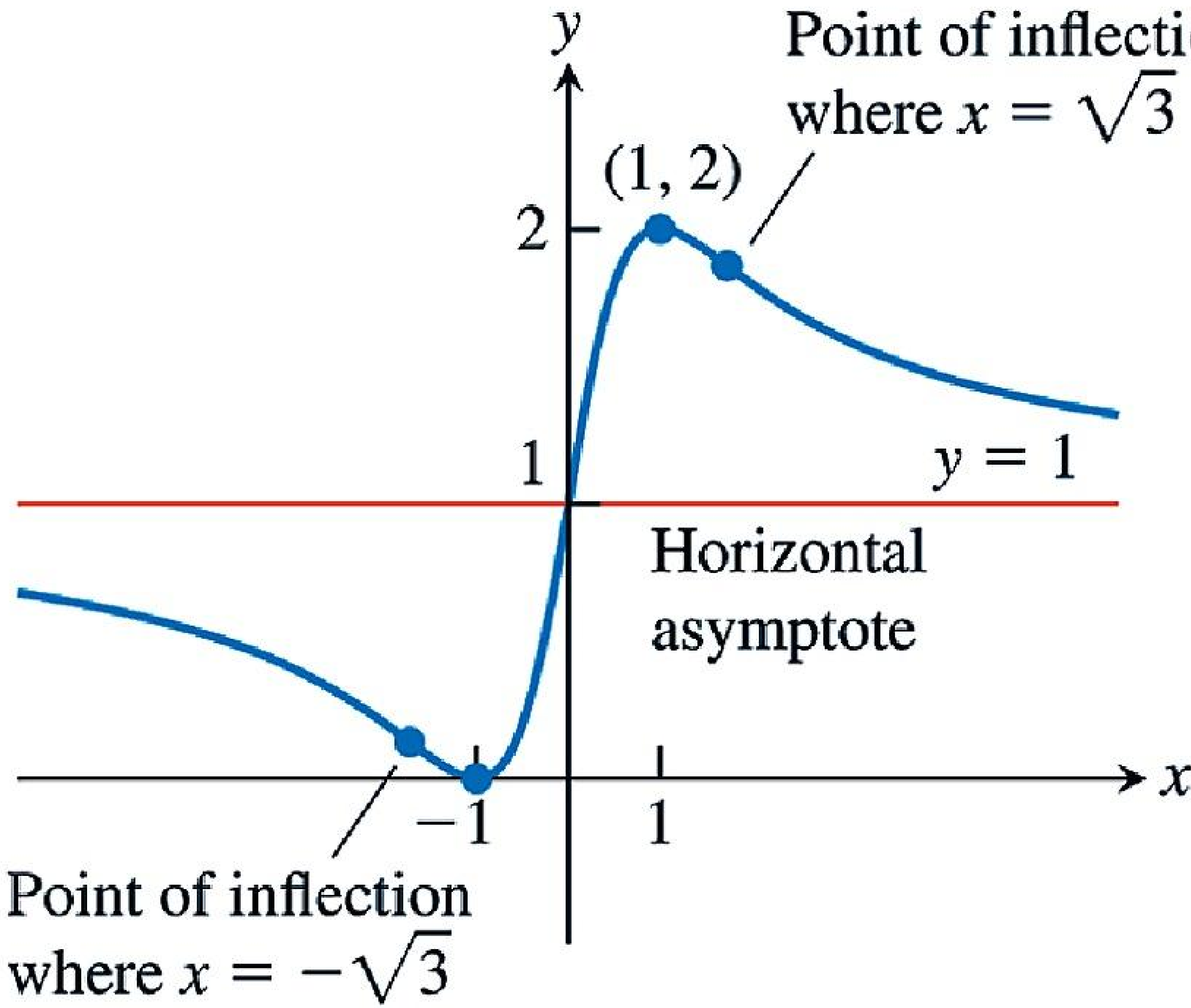


FIGURE 4.32 The graph of $y = \dfrac{(x+1)^2}{1+x^2}$

(Example 9).

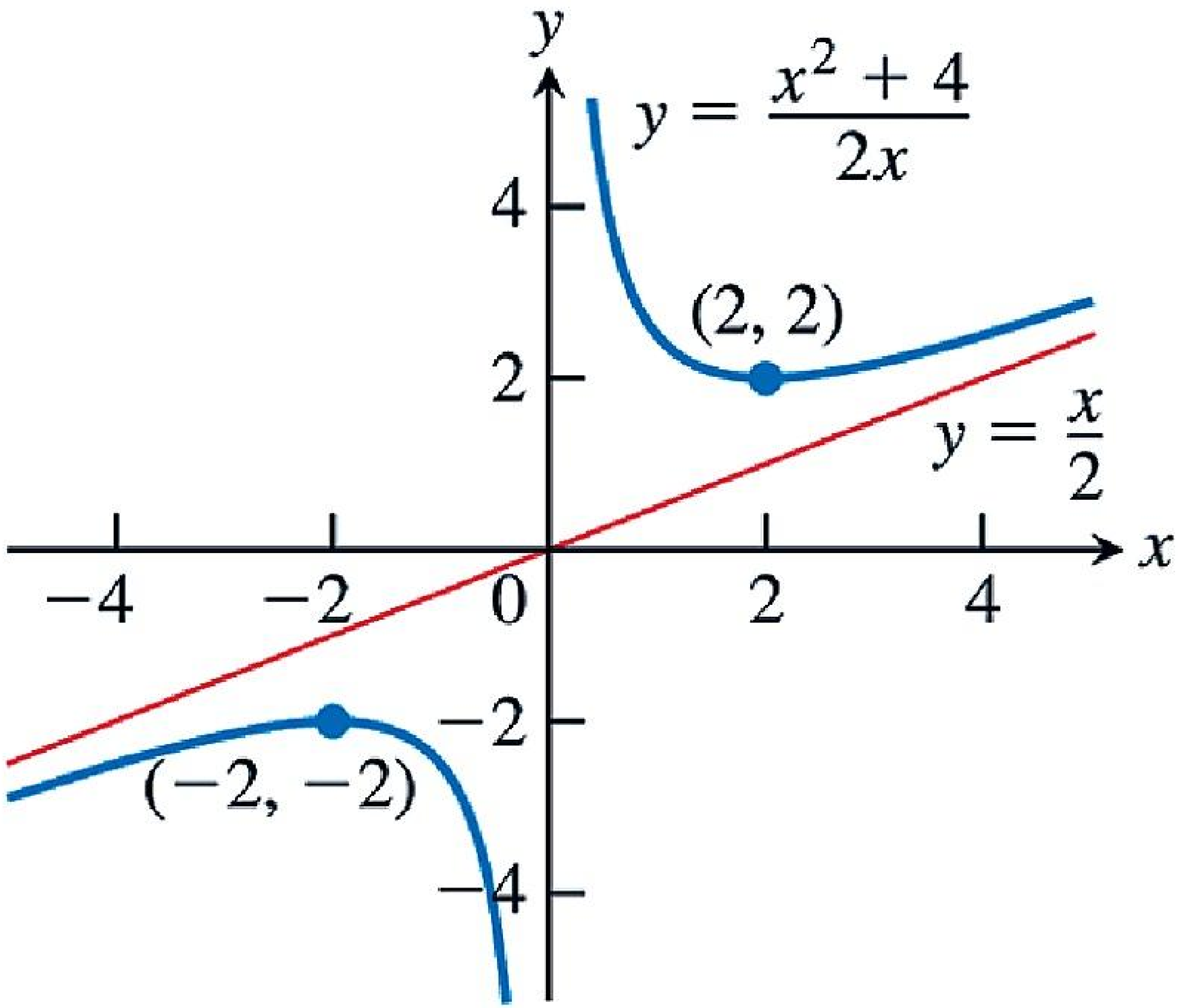


FIGURE 4.33 The graph of $y = \dfrac{x^2 + 4}{2x}$

(Example 10).

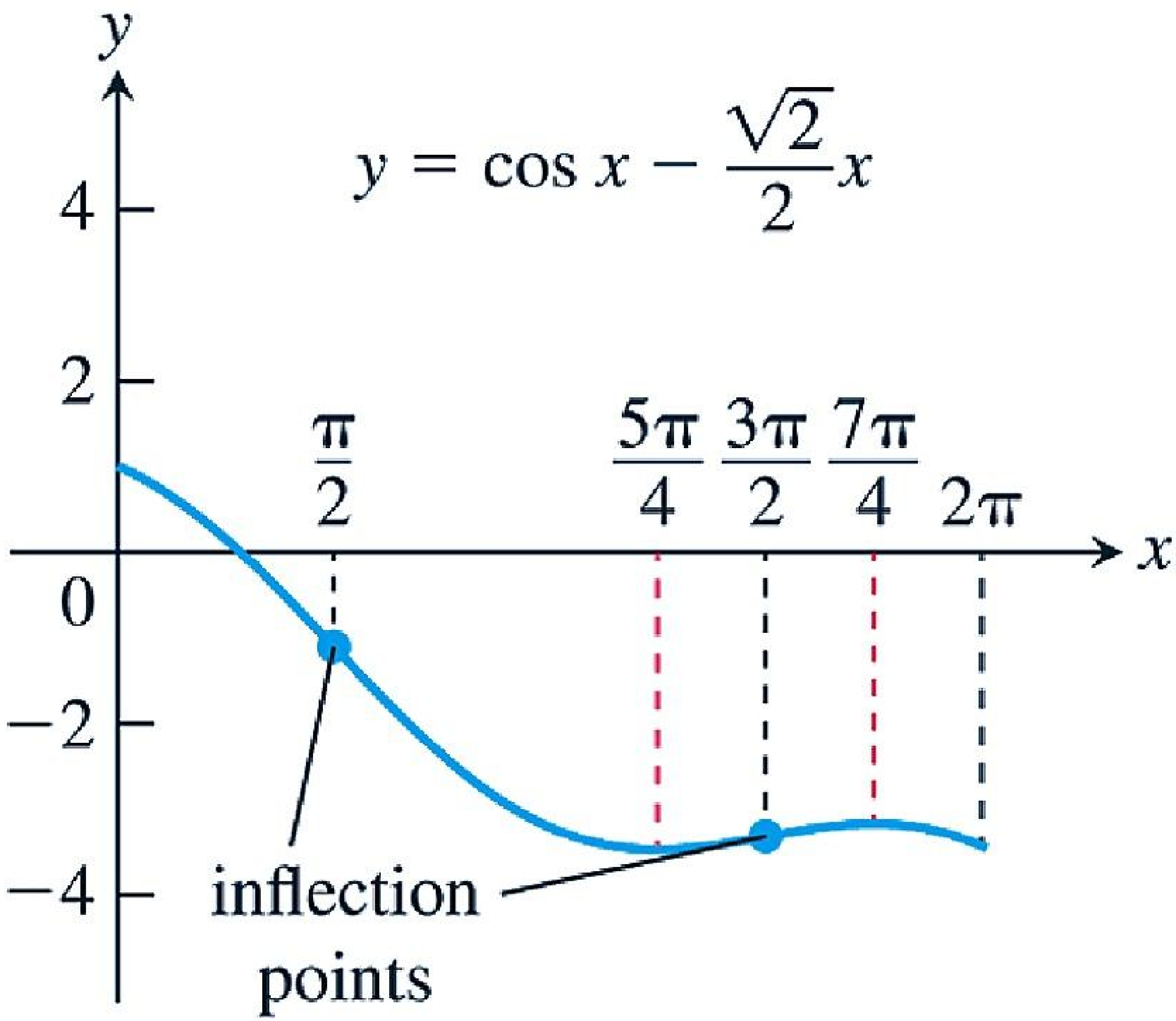
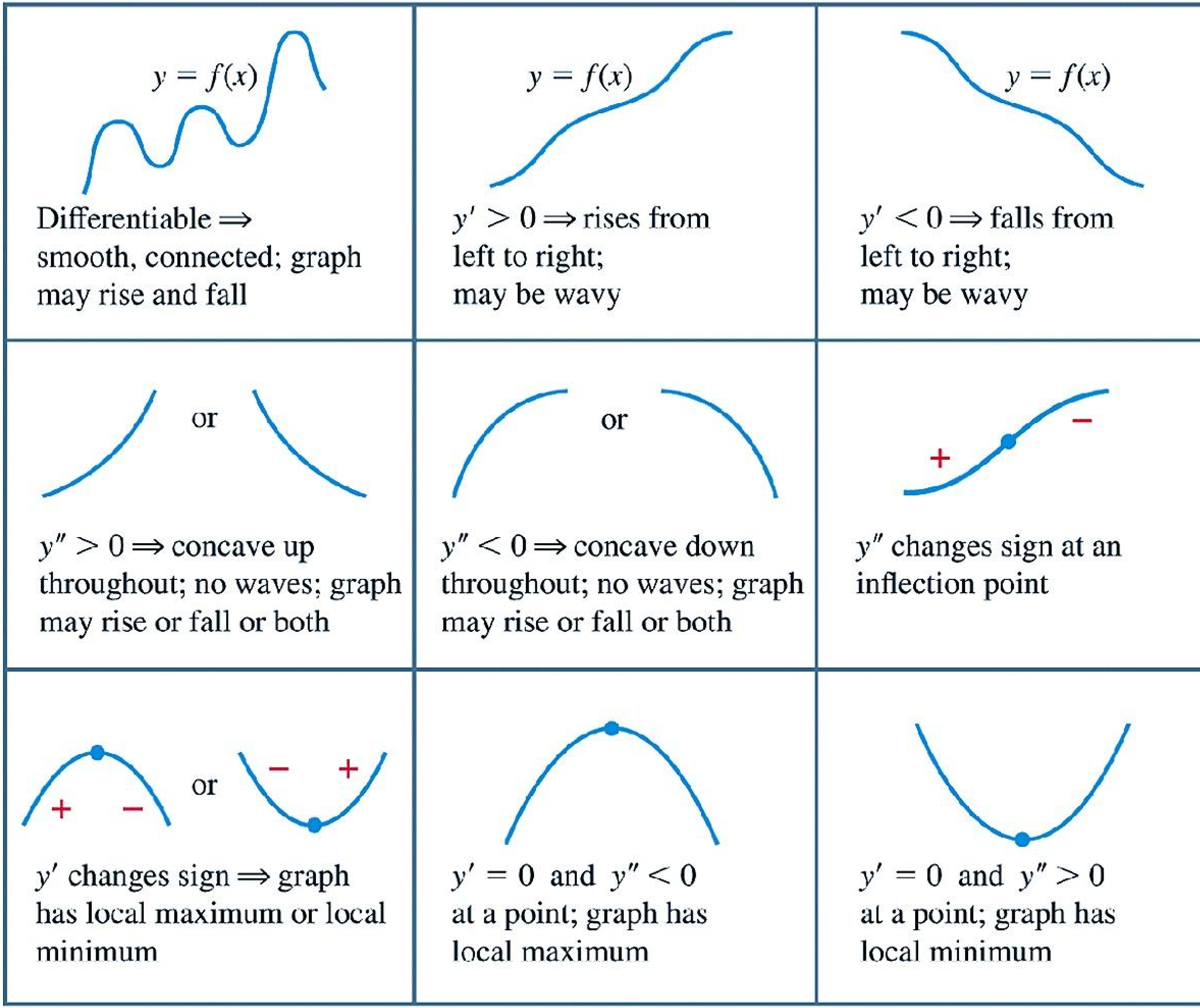


FIGURE 4.34 The graph of the function in Example 11.



y = f(x)
Differentiable ⇒ smooth, connected; graph may rise and fall
y = f(x)
y′ > 0 ⇒ rises from left to right; may be wavy
y = f(x)
y′ < 0 ⇒ falls from left to right; may be wavy
or
y″ > 0 ⇒ concave up throughout; no waves; graph may rise or fall or both
or
y″ < 0 ⇒ concave down throughout; no waves; graph may rise or fall or both
+
−
y″ changes sign at an inflection point
+
−
or
−
+
y′ changes sign ⇒ graph has local maximum or local minimum
y′ = 0 and y″ < 0 at a point; graph has local maximum
y′ = 0 and y″ > 0 at a point; graph has local minimum

Section 4.5

Applied Optimization

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Solving Applied Optimization Problems

1. *Read the problem.* Read the problem until you understand it. What is given? What is the unknown quantity to be optimized?
2. *Draw a picture.* Label any part that may be important to the problem.
3. *Introduce variables.* List every relation in the picture and in the problem as an equation or algebraic expression, and identify the unknown variable.
4. *Write an equation for the unknown quantity.* If you can, express the unknown as a function of a single variable or in two equations in two unknowns. This may require considerable manipulation.
5. *Test the critical points and endpoints in the domain of the unknown.* Use what you know about the shape of the function's graph. Use the first and second derivatives to identify and classify the function's critical points.

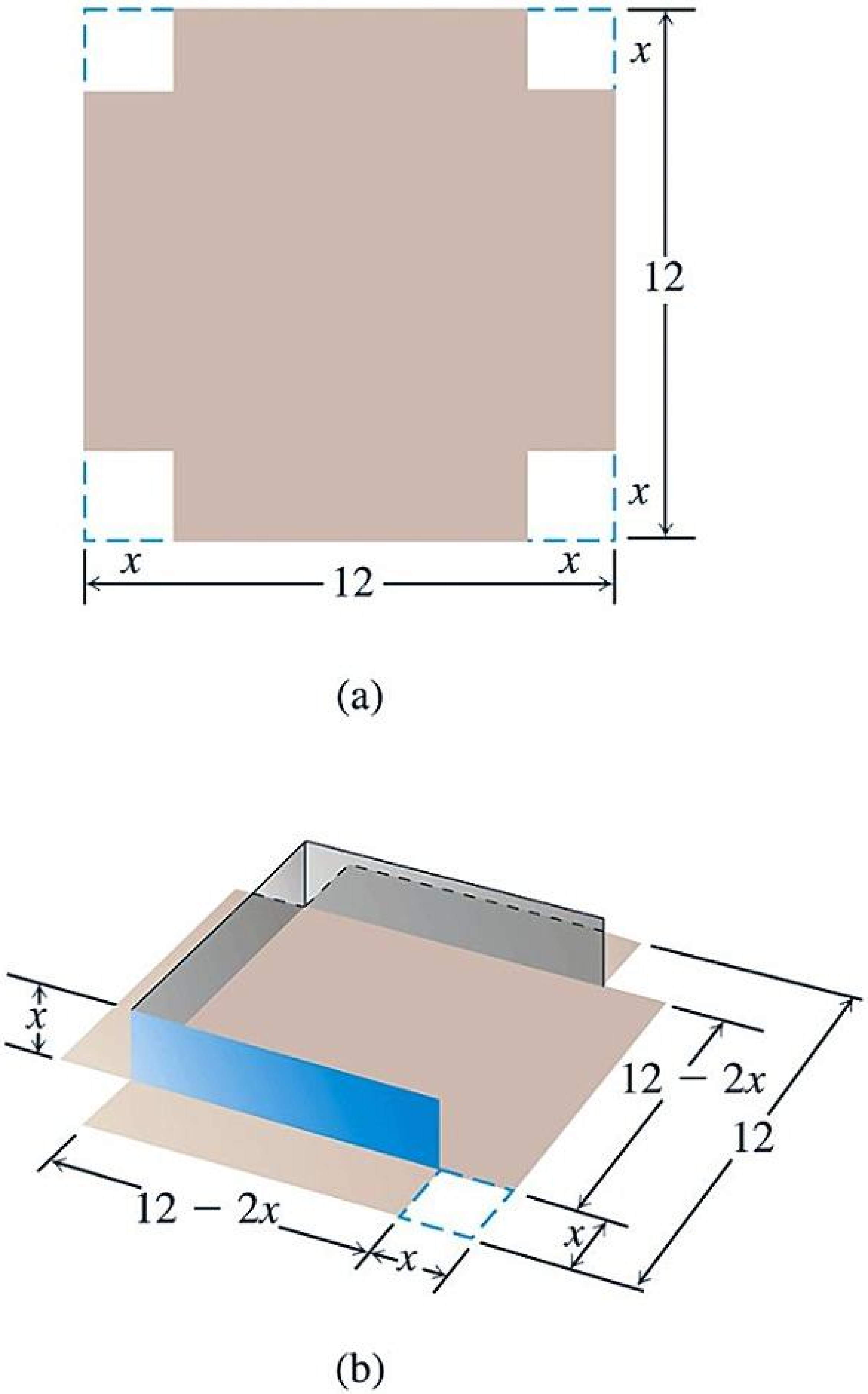

FIGURE 4.36 An open box made by cutting the corners from a square sheet of tin. What size corners maximize the box's volume (Example 1)?

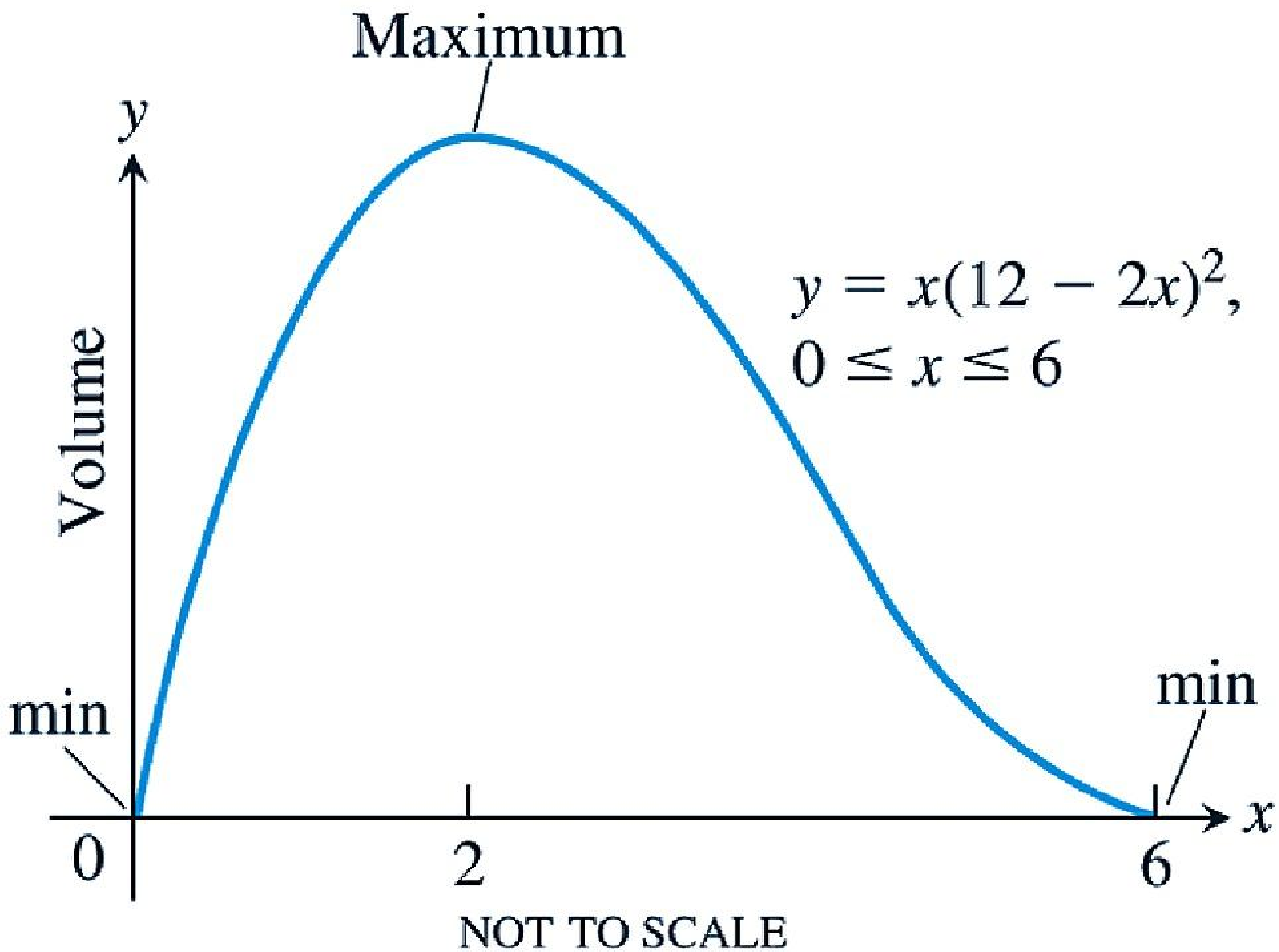

FIGURE 4.37 The volume of the box in Figure 4.36 graphed as a function of x.

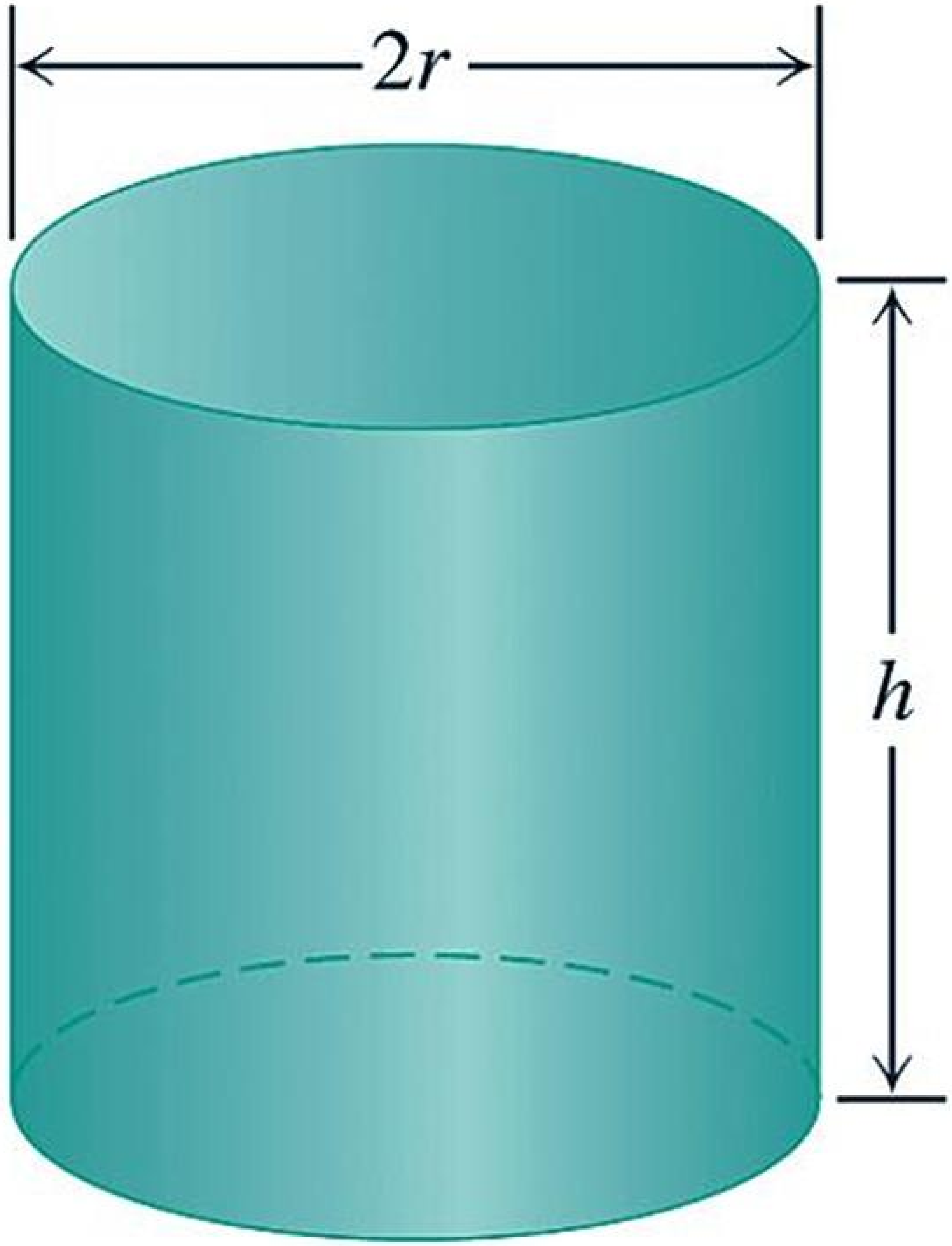

FIGURE 4.38 This one-liter can uses the least material when $h = 2r$ (Example 2).

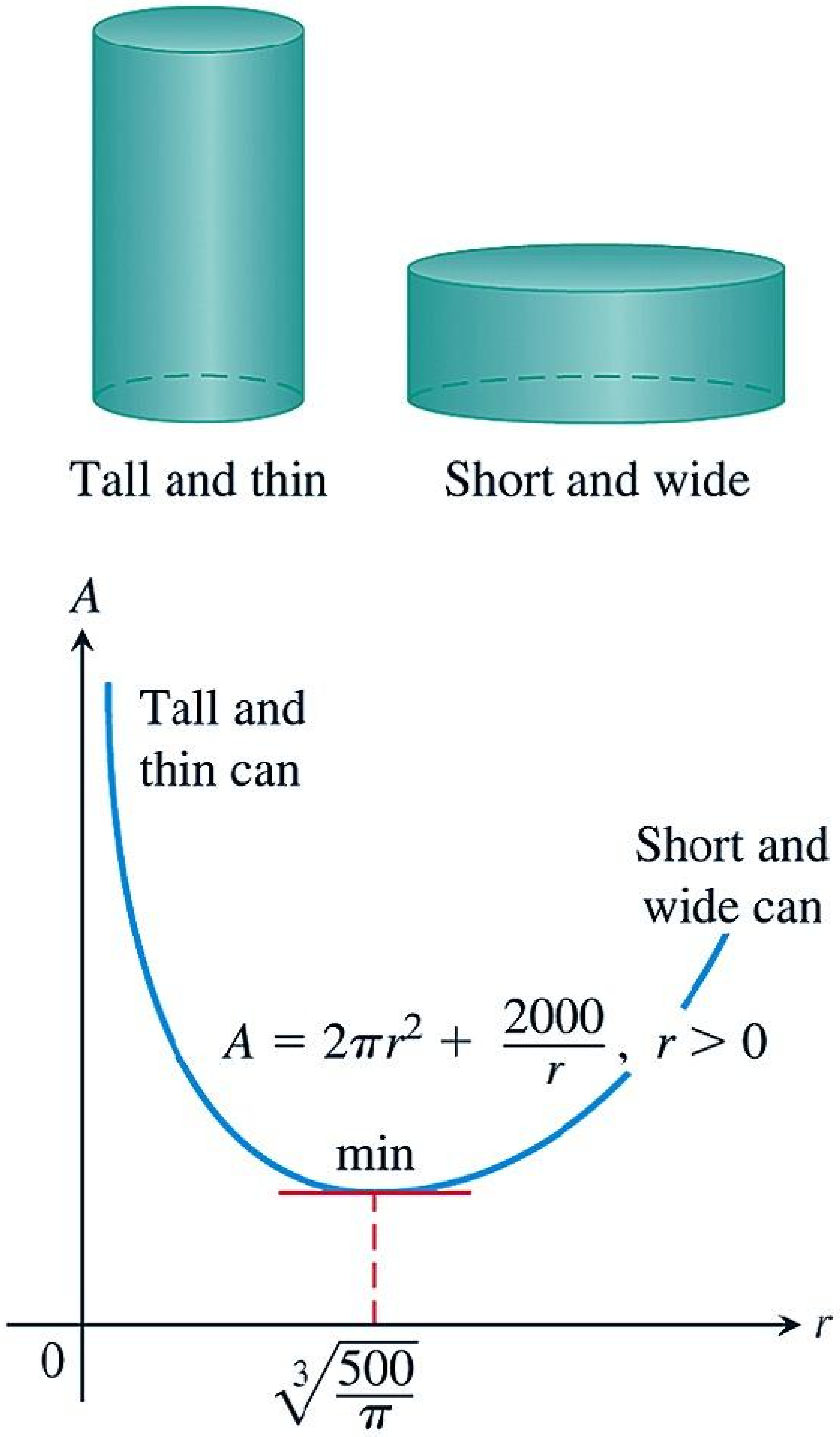

FIGURE 4.39 The graph of $A = 2\pi r^2 + 2000/r$ is concave up.

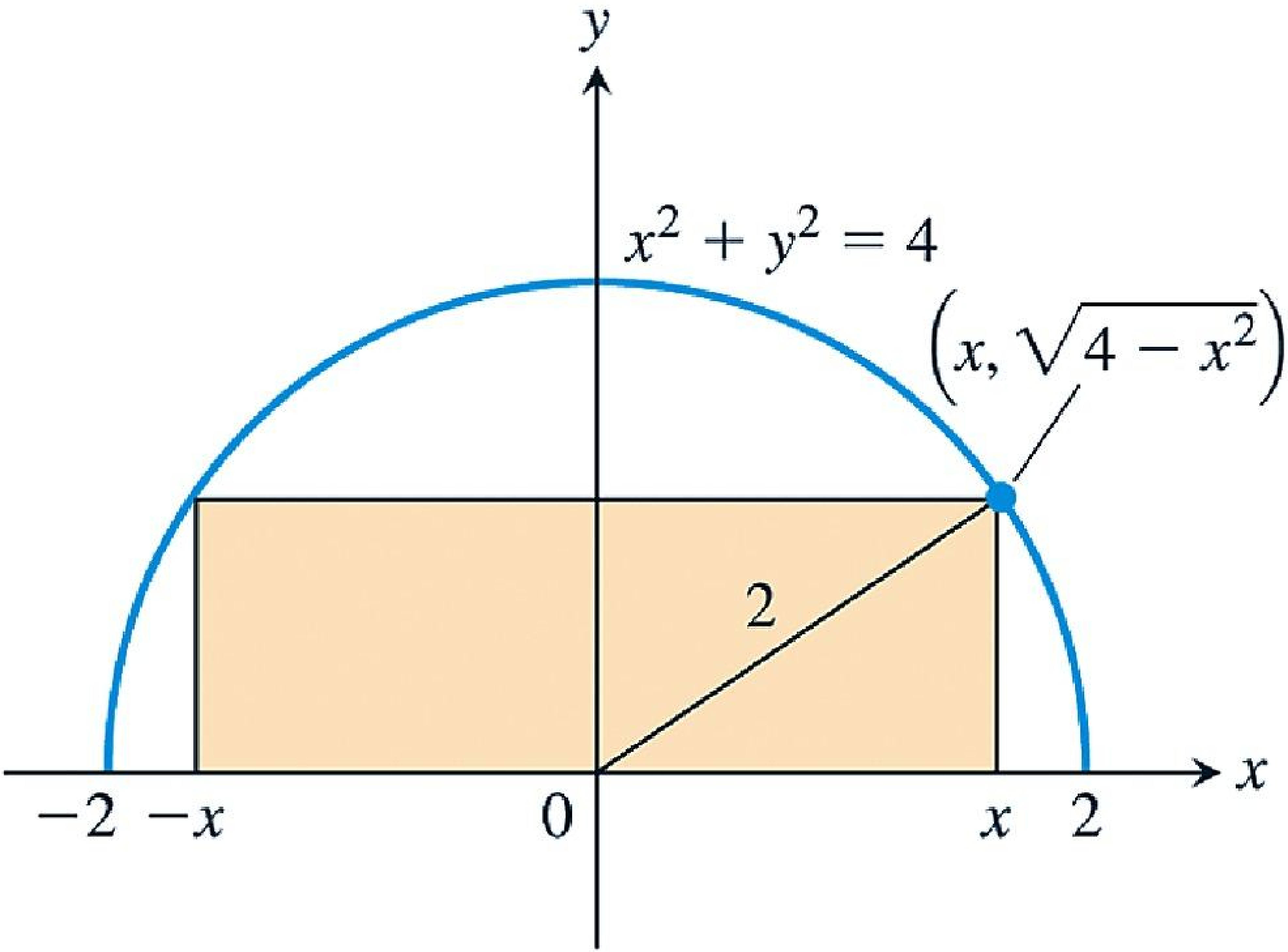

FIGURE 4.40 The rectangle inscribed in the semicircle in Example 3.

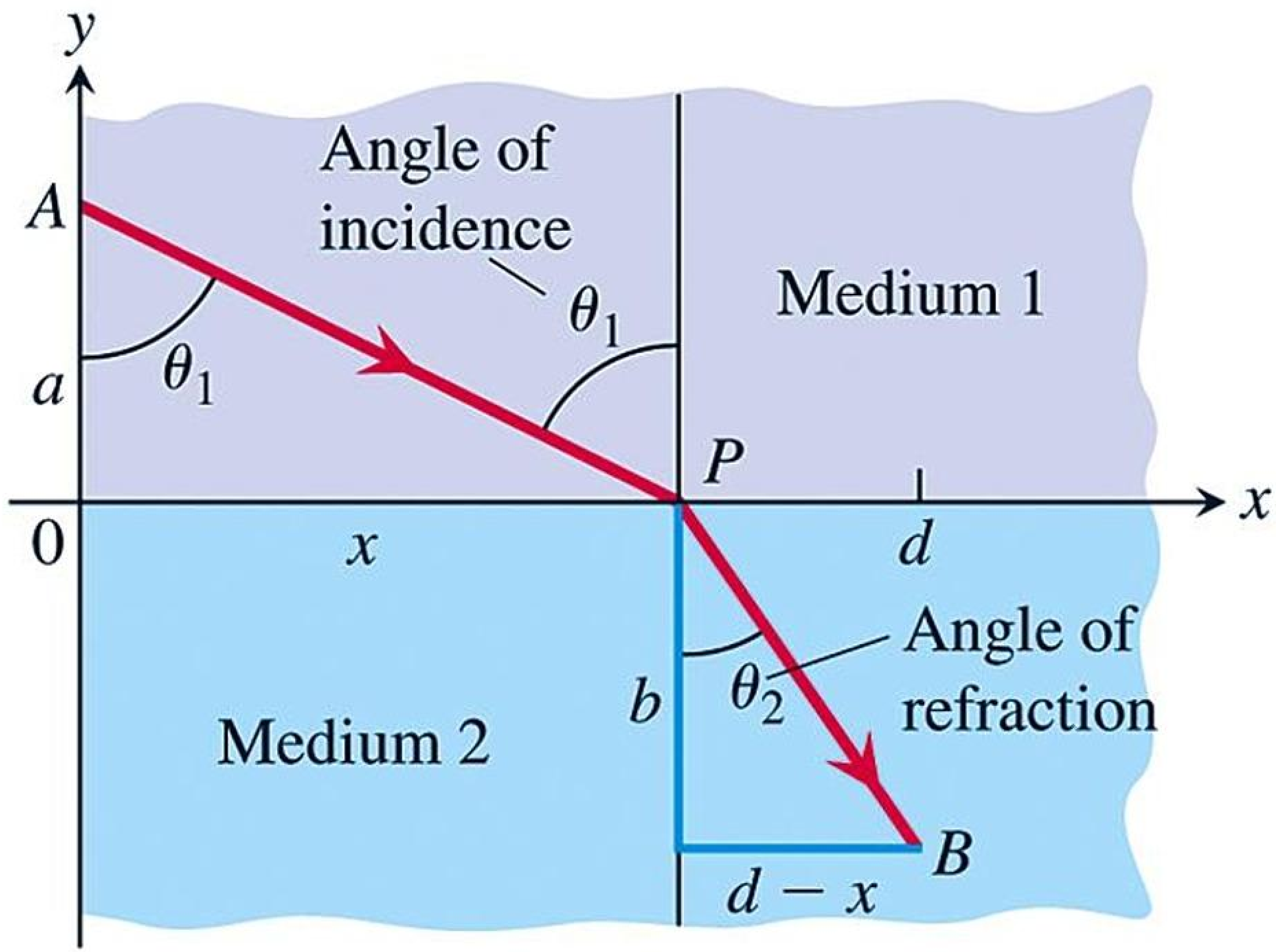

FIGURE 4.41 A light ray refracted (deflected from its path) as it passes from one medium to a denser medium (Example 4).

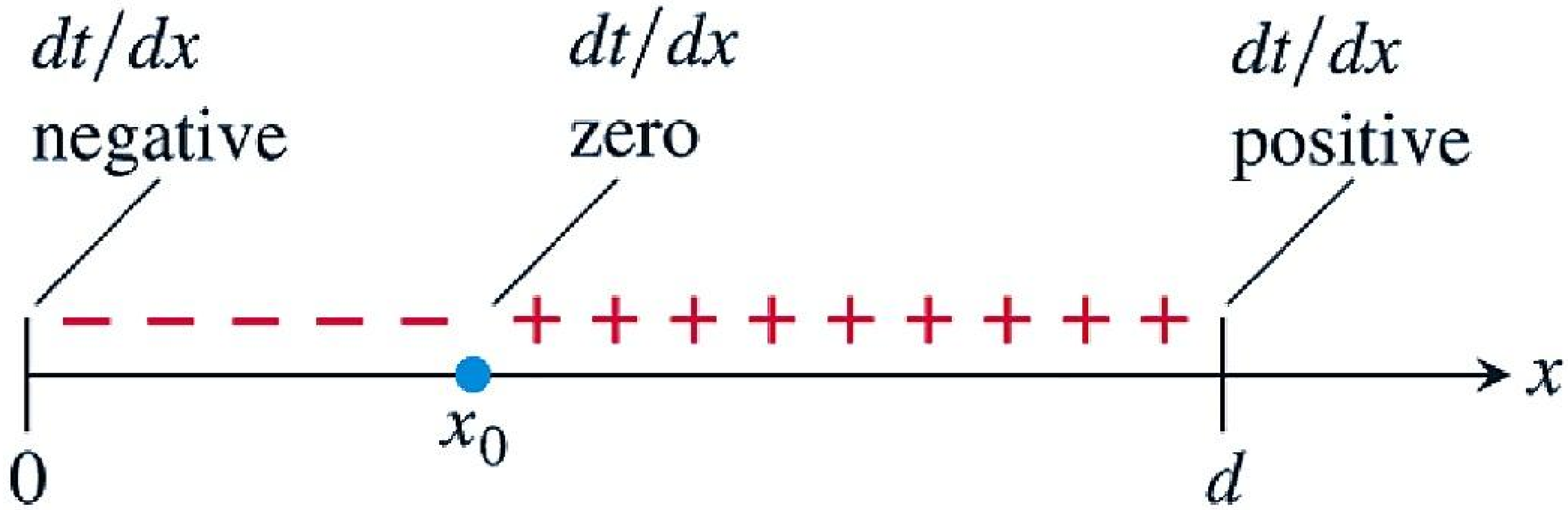

FIGURE 4.42 The sign pattern of dt/dx in Example 4.

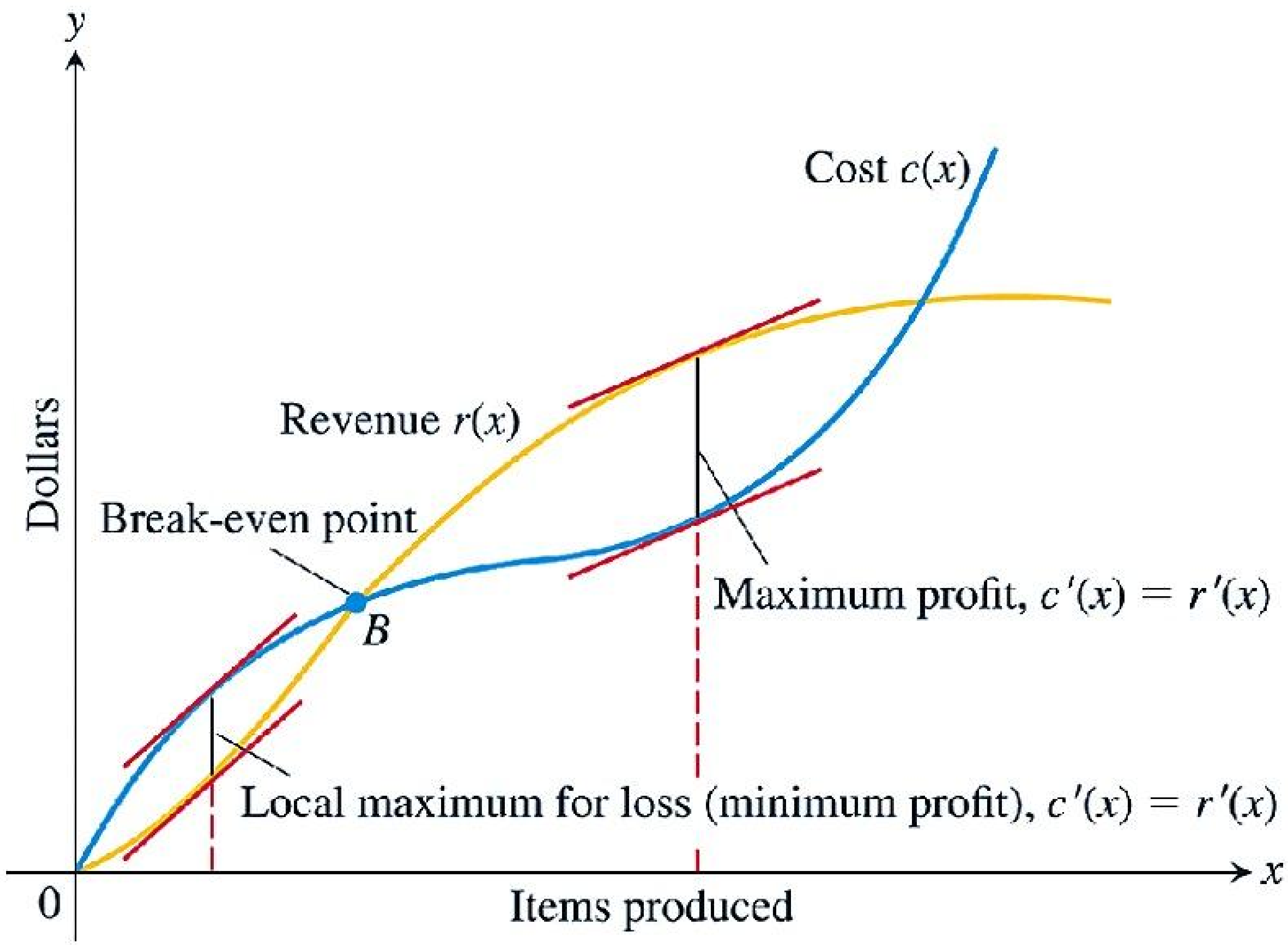

FIGURE 4.43 The graph of a typical cost function starts concave down and later turns concave up. It crosses the revenue curve at the break-even point B. To the left of B, the company operates at a loss. To the right, the company operates at a profit, with the maximum profit occurring where $c'(x) = r'(x)$. Farther to the right, cost exceeds revenue (perhaps because of a combination of rising labor and material costs and market saturation) and production levels become unprofitable again.

At a production level yielding maximum profit, marginal revenue equals marginal cost (Figure 4.43).

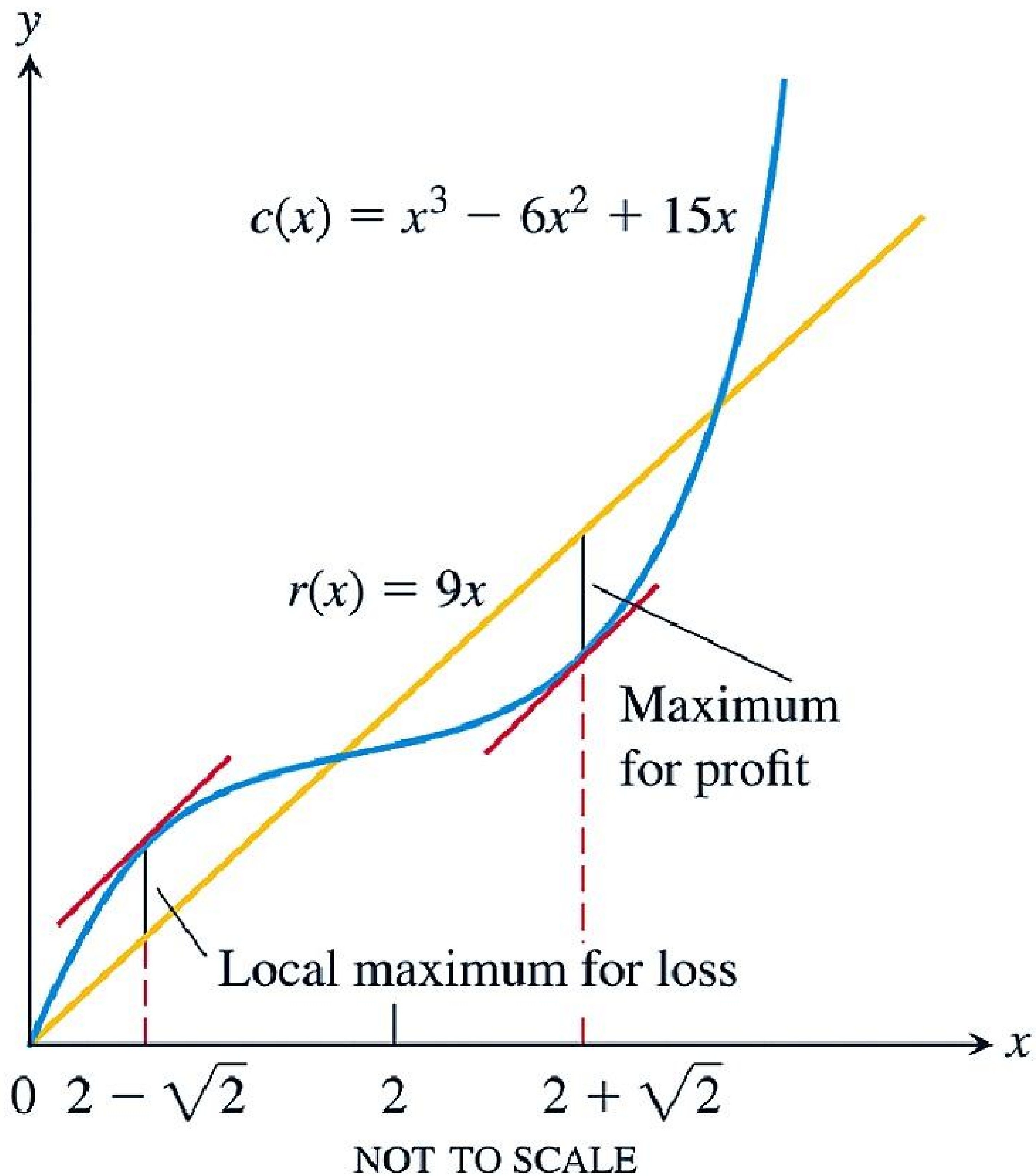

FIGURE 4.44 The cost and revenue curves for Example 5.

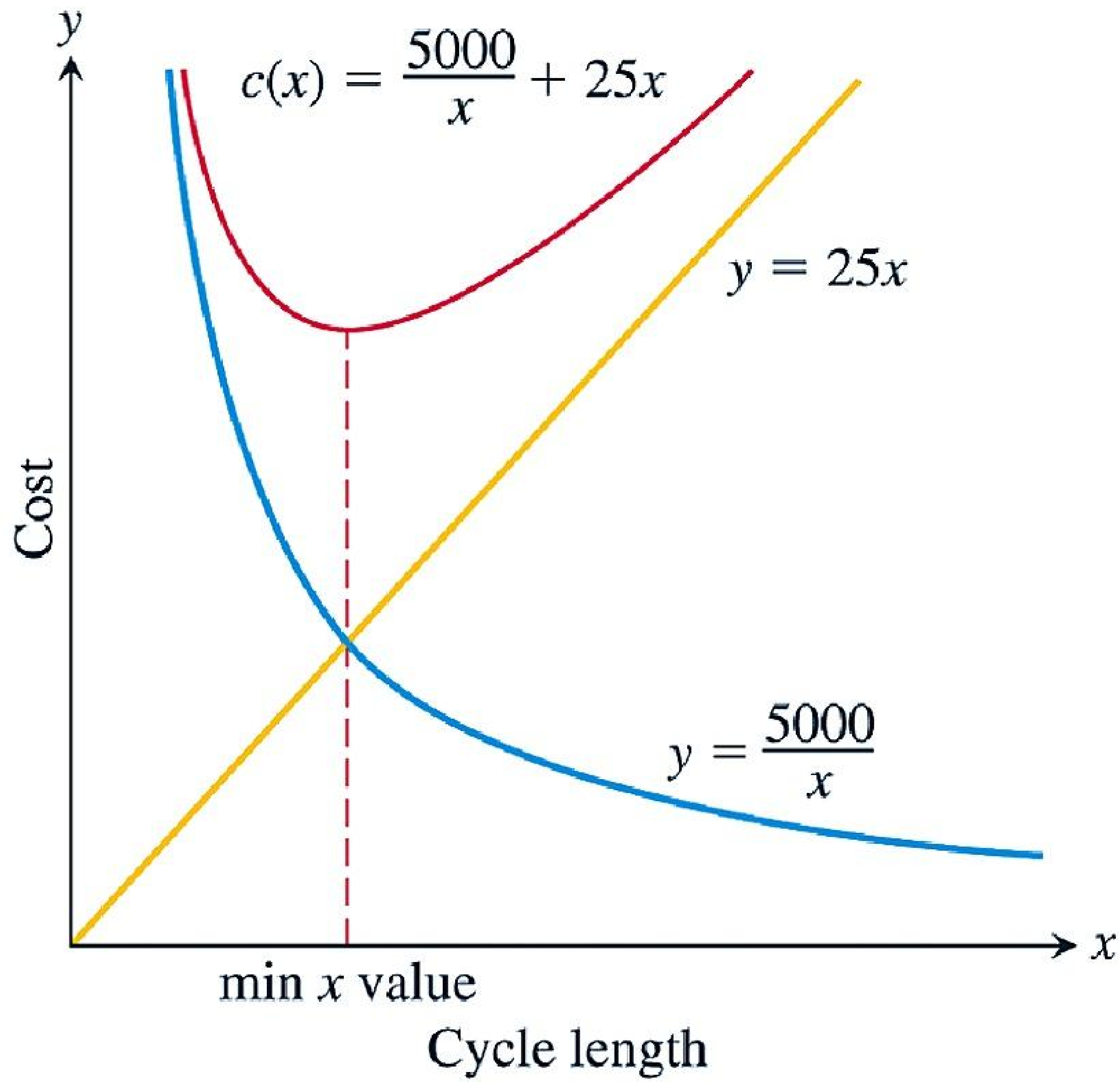

FIGURE 4.45 The average daily cost $c(x)$ is the sum of a hyperbola and a linear function (Example 6).

Section 4.6

Newton's Method

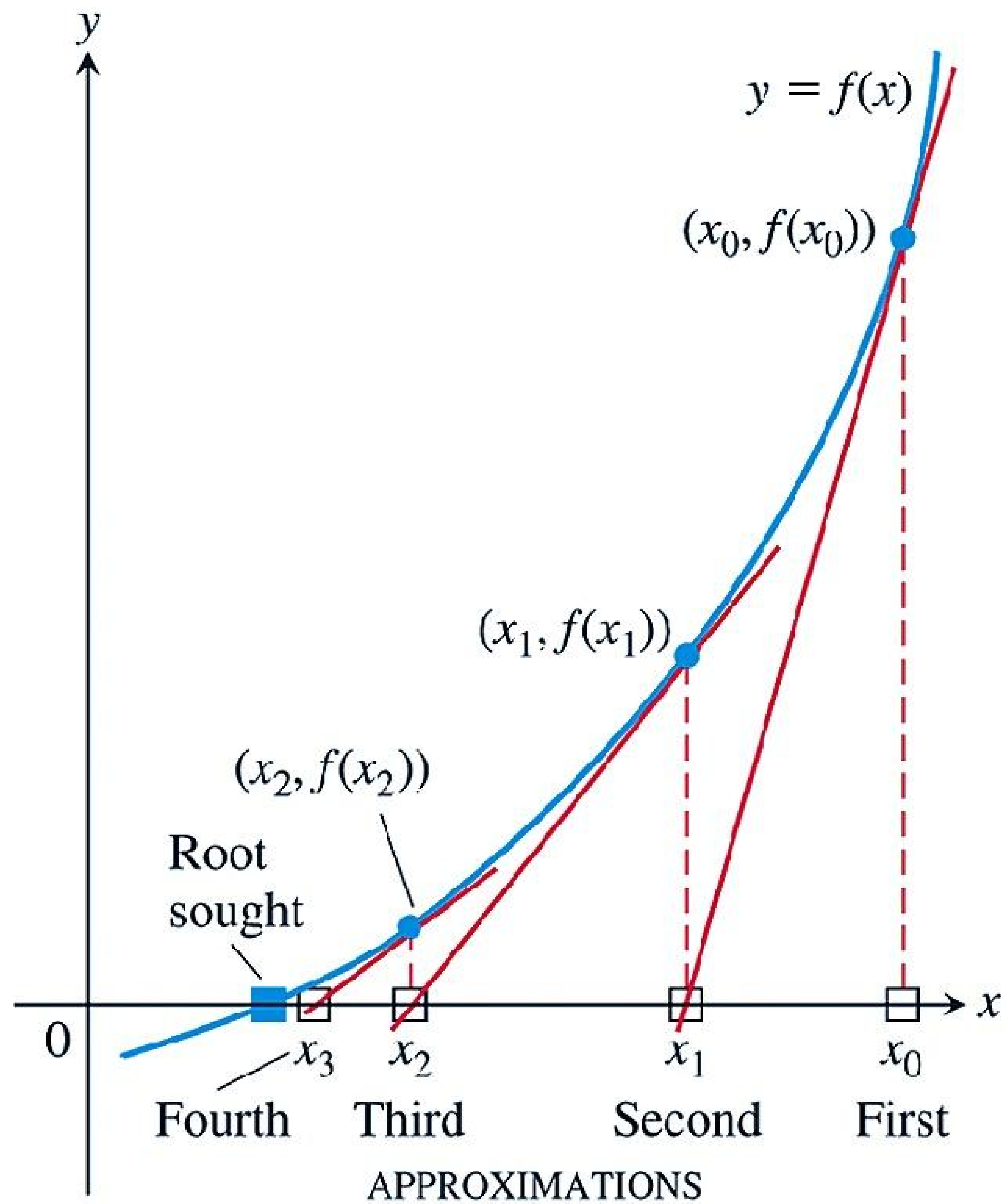

FIGURE 4.46 Newton's method starts with an initial guess x_0 and (under favorable circumstances) improves the guess one step at a time.

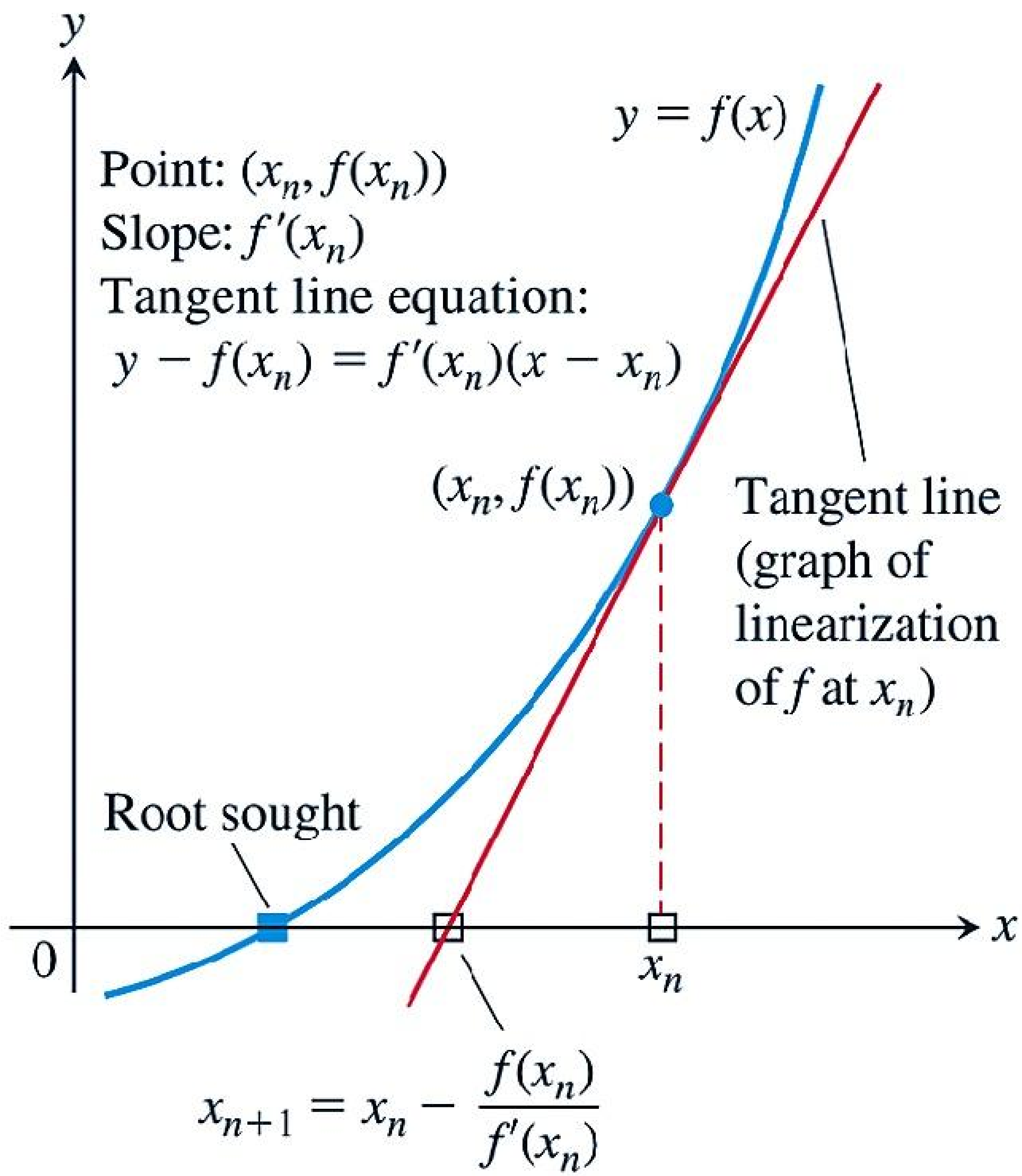

FIGURE 4.47 The geometry of the successive steps of Newton's method. From x_n we go up to the curve and follow the tangent line down to find x_{n+1}.

Newton's Method

1. Guess a first approximation to a solution of the equation $f(x) = 0$. A graph of $y = f(x)$ may help.
2. Use the first approximation to get a second, the second to get a third, and so on, using the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad \text{if } f'(x_n) \neq 0. \tag{1}$$

	Error	Number of correct digits
$x_0 = 1$	-0.41421	1
$x_1 = 1.5$	0.08579	1
$x_2 = 1.41667$	0.00246	3
$x_3 = 1.41422$	0.00001	5

Pearson

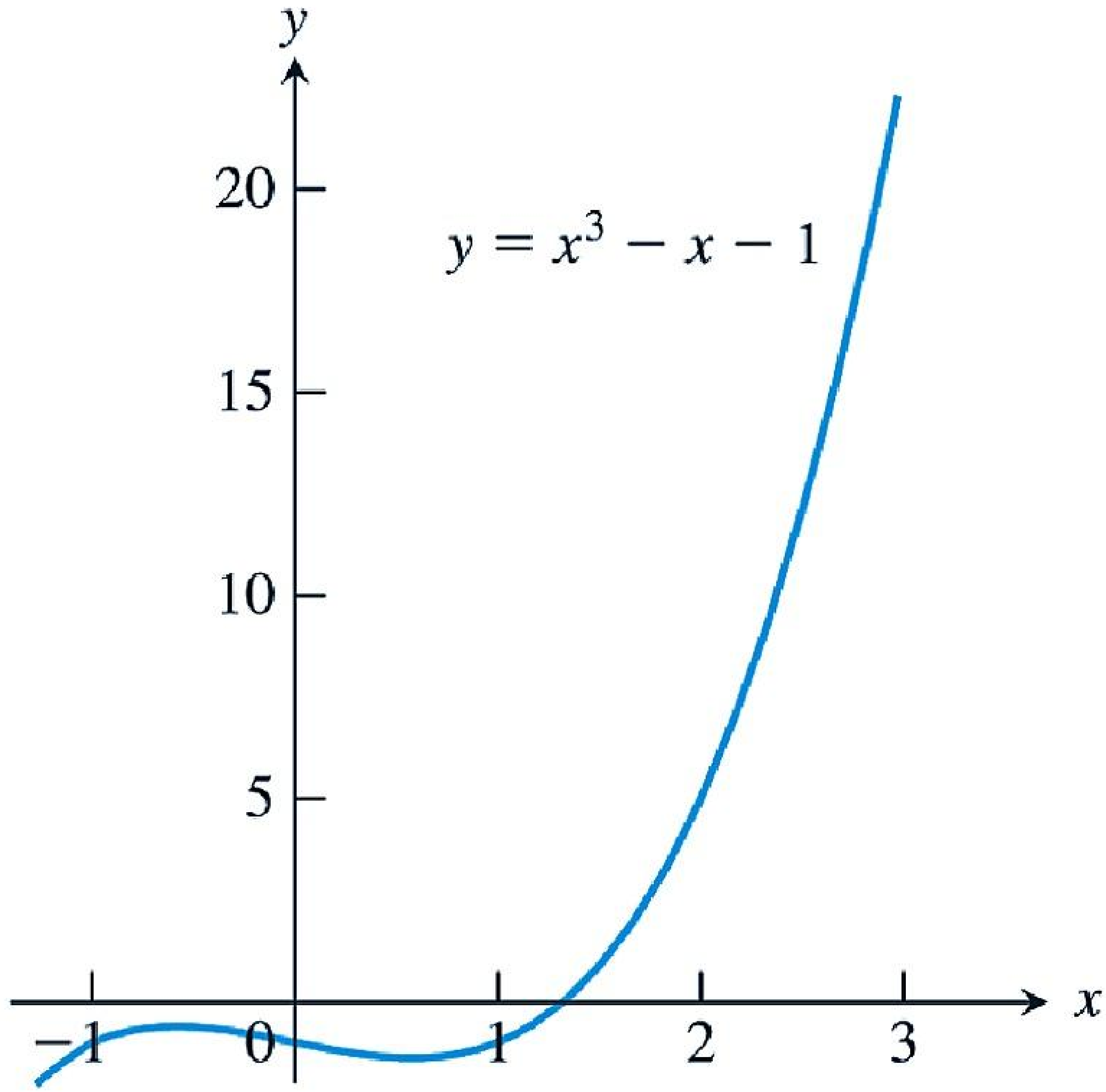

FIGURE 4.48 The graph of $f(x) = x^3 - x - 1$ crosses the x-axis once; this is the root we want to find (Example 2).

TABLE 4.1 The result of applying Newton's method to $f(x) = x^3 - x - 1$ with $x_0 = 1$

n	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$
0	1	−1	2	1.5
1	1.5	0.875	5.75	1.3478 26087
2	1.3478 26087	0.1006 82173	4.4499 05482	1.3252 00399
3	1.3252 00399	0.0020 58362	4.2684 68292	1.3247 18174
4	1.3247 18174	0.0000 00924	4.2646 34722	1.3247 17957
5	1.3247 17957	−1.8672E-13	4.2646 32999	1.3247 17957

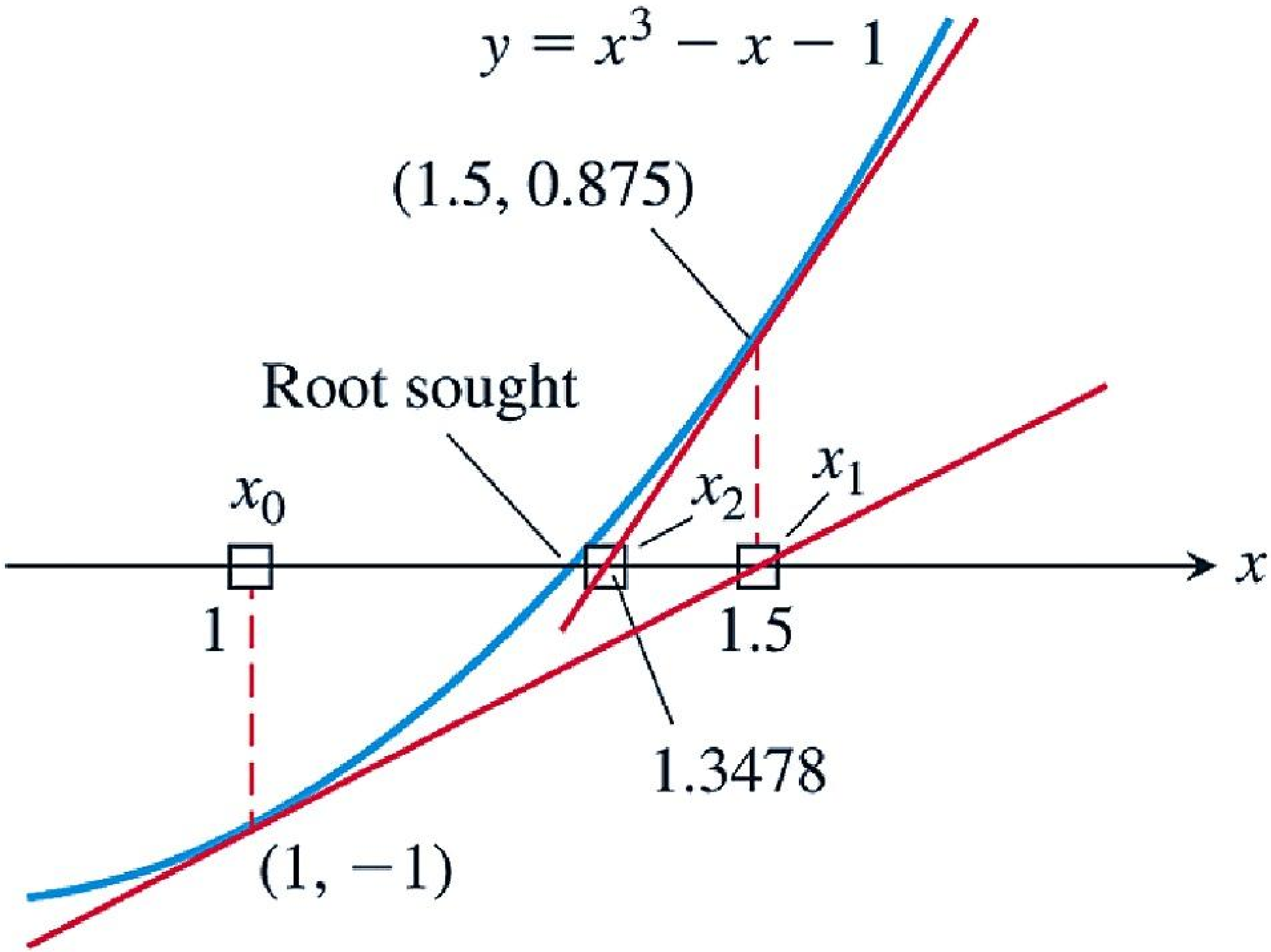

FIGURE 4.49 The first three x-values in Table 4.1 (four decimal places).

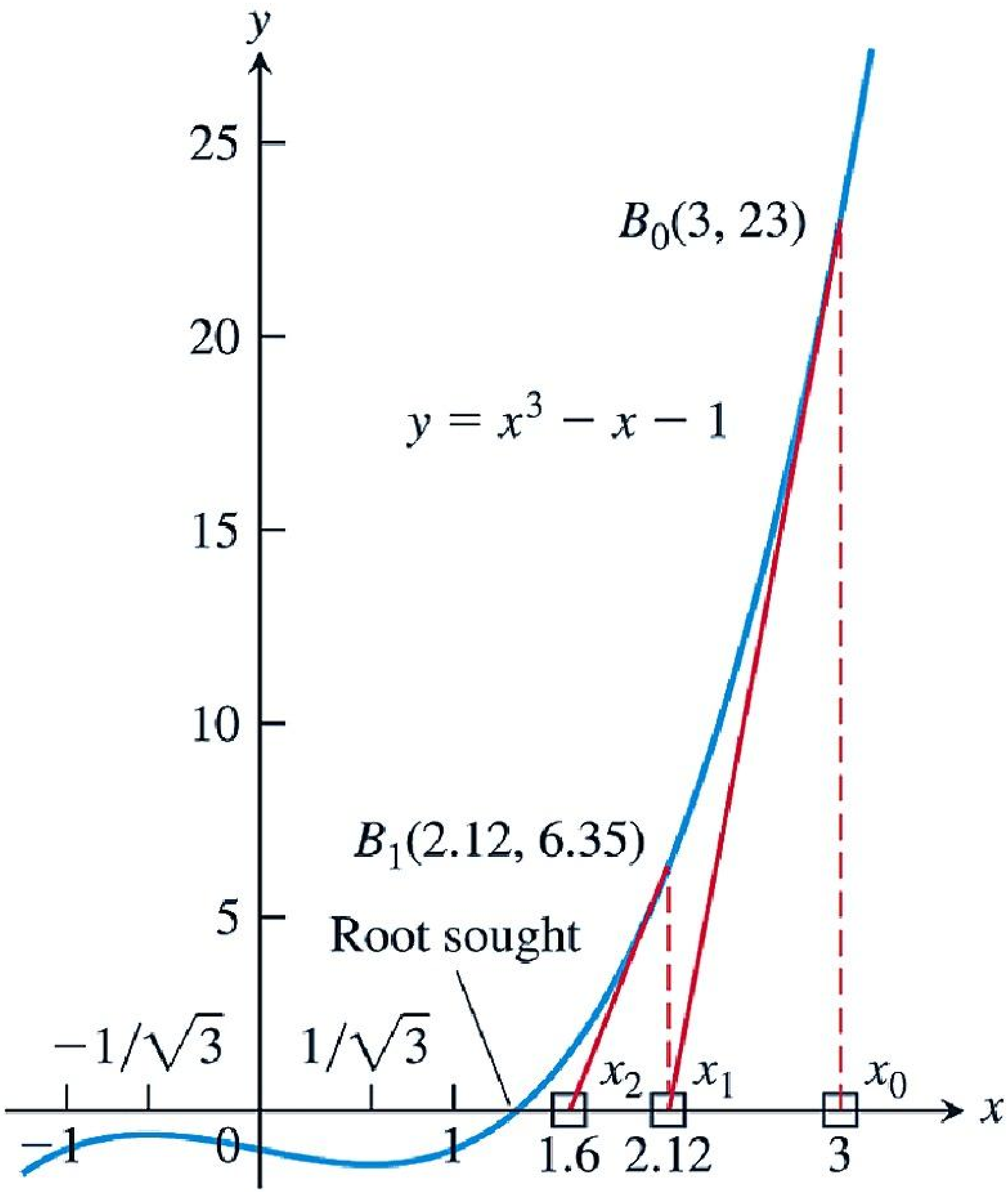

FIGURE 4.50 Any starting value x_0 to the right of $x = 1/\sqrt{3}$ will lead to the root in Example 2.

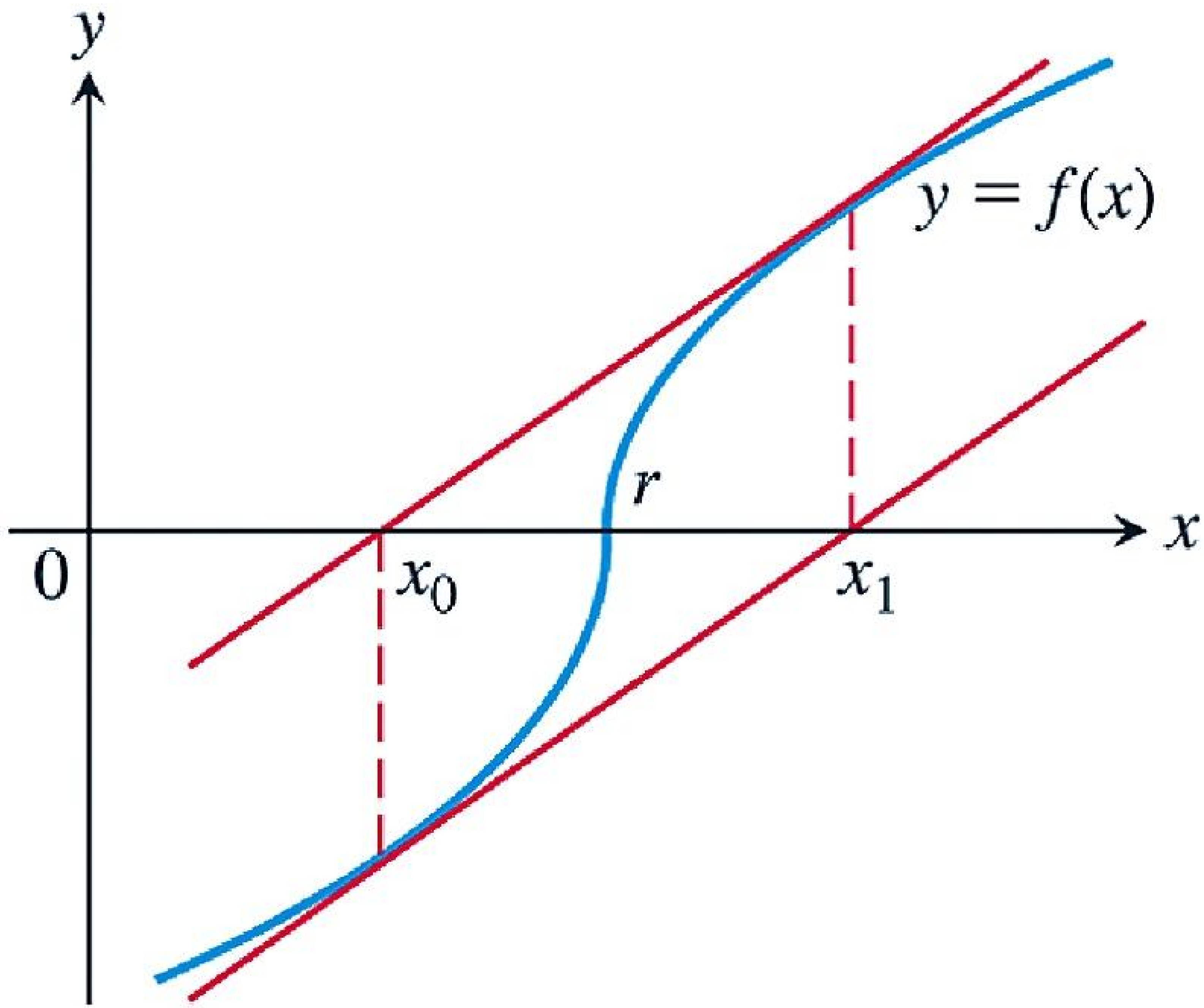

FIGURE 4.51 Newton's method fails to converge. You go from x_0 to x_1 and back to x_0, never getting any closer to r.

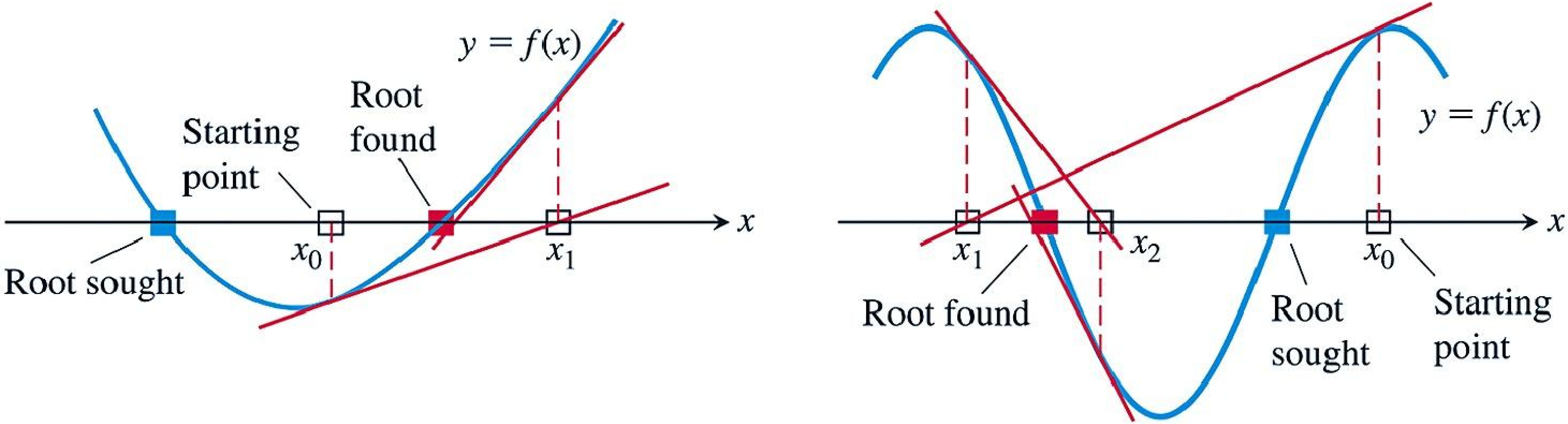

FIGURE 4.52 If you start too far away, Newton's method may miss the root you want.

Section 4.7

Antiderivatives

DEFINITION A function F is an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

THEOREM 8 If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + C$$

where C is an arbitrary constant.

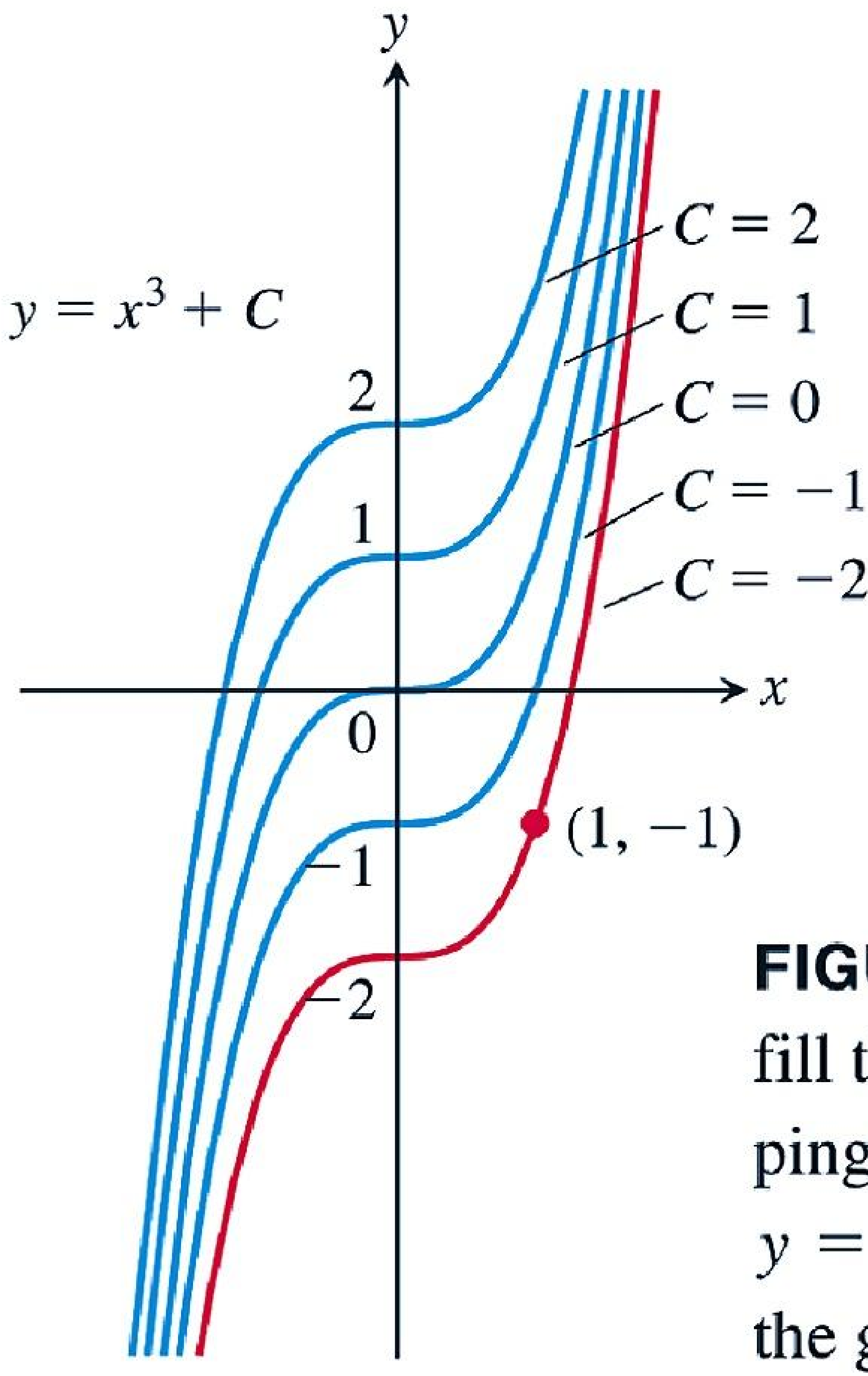

FIGURE 4.53 The curves $y = x^3 + C$ fill the coordinate plane without overlapping. In Example 2, we identify the curve $y = x^3 - 2$ as the one that passes through the given point $(1, -1)$.

TABLE 4.2 Antiderivative formulas, k a nonzero constant

Function	General antiderivative
1. x^n	$\frac{1}{n+1}x^{n+1} + C, \quad n \neq -1$
2. $\sin kx$	$-\frac{1}{k}\cos kx + C$
3. $\cos kx$	$\frac{1}{k}\sin kx + C$
4. $\sec^2 kx$	$\frac{1}{k}\tan kx + C$
5. $\csc^2 kx$	$-\frac{1}{k}\cot kx + C$
6. $\sec kx \tan kx$	$\frac{1}{k}\sec kx + C$
7. $\csc kx \cot kx$	$-\frac{1}{k}\csc kx + C$

TABLE 4.3 Antiderivative linearity rules

	Function	General antiderivative
1. *Constant Multiple Rule*:	$kf(x)$	$kF(x) + C, \quad k$ a constant
2. *Sum or Difference Rule*:	$f(x) \pm g(x)$	$F(x) \pm G(x) + C$

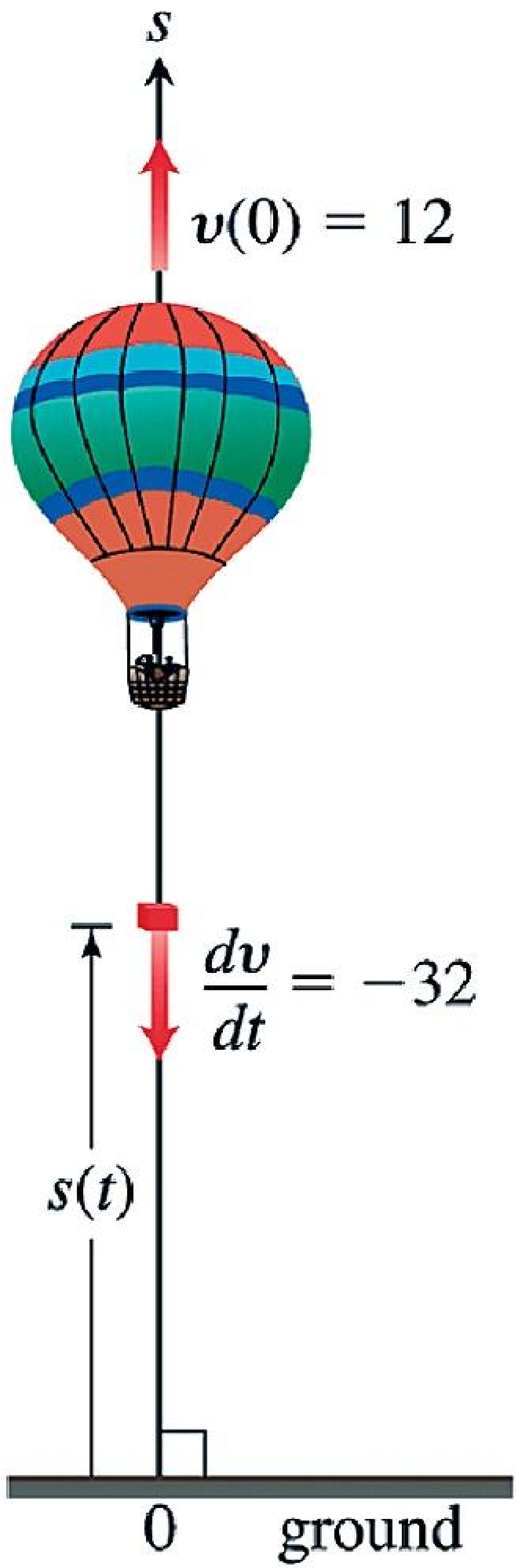

FIGURE 4.54 A package dropped from a rising hot-air balloon (Example 5).

DEFINITION The collection of all antiderivatives of f is called the **indefinite integral** of f with respect to x, and is denoted by

$$\int f(x)\,dx.$$

The symbol $\int$ is an **integral sign**. The function f is the **integrand** of the integral, and x is the **variable of integration**.

Pearson