

Chapter 12

Vector-Valued Functions and Motion in Space

Thomas' Calculus, 14e in SI Units

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Section 12.1

Curves in Space and Their Tangents

Thomas' Calculus, 14e in SI Units

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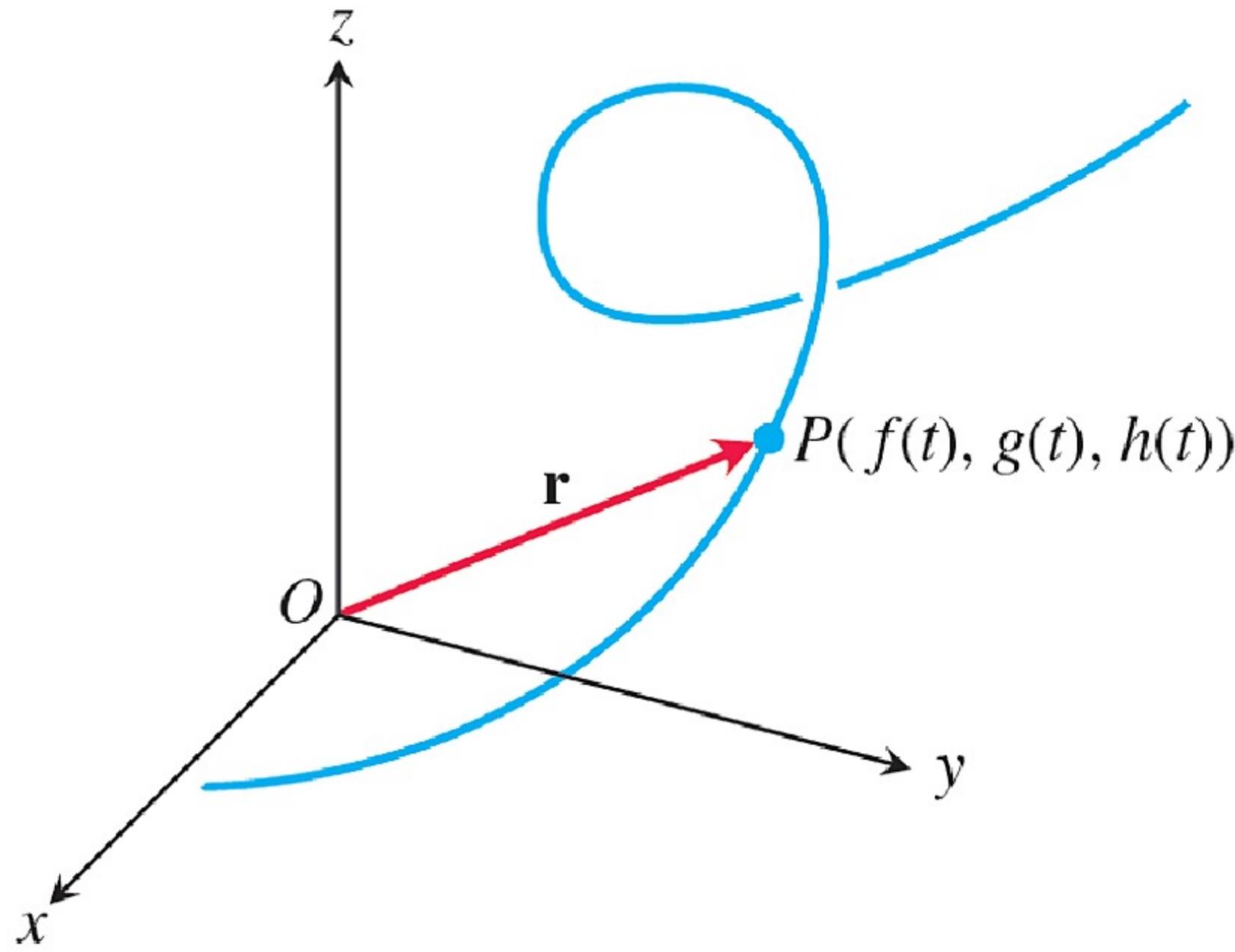
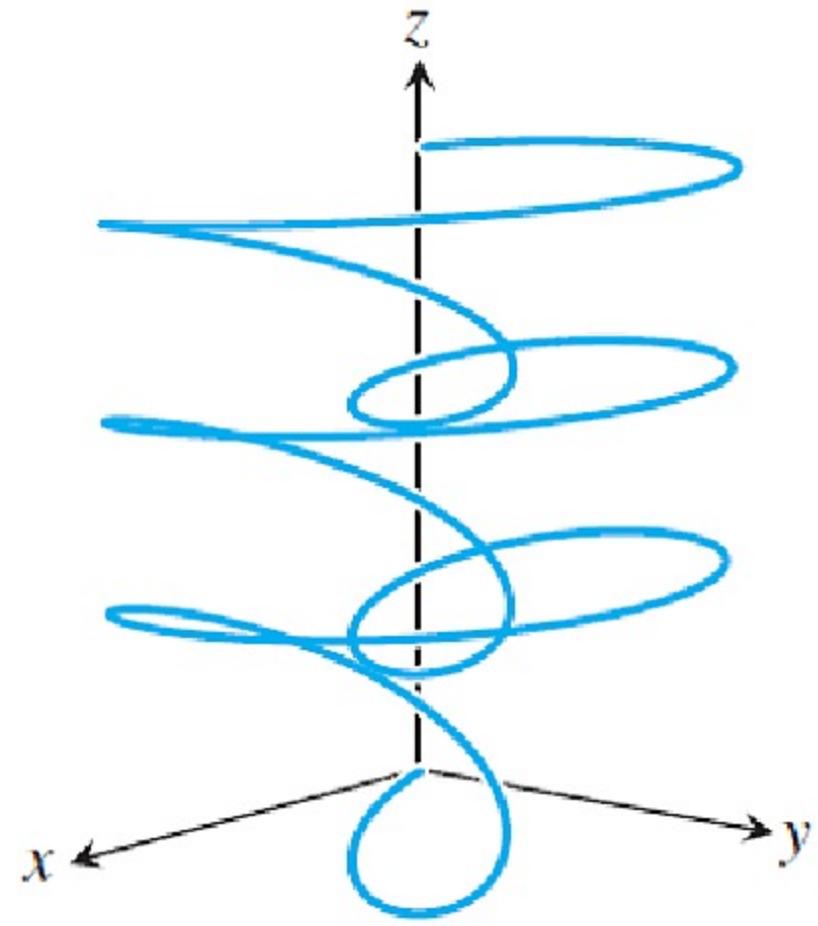


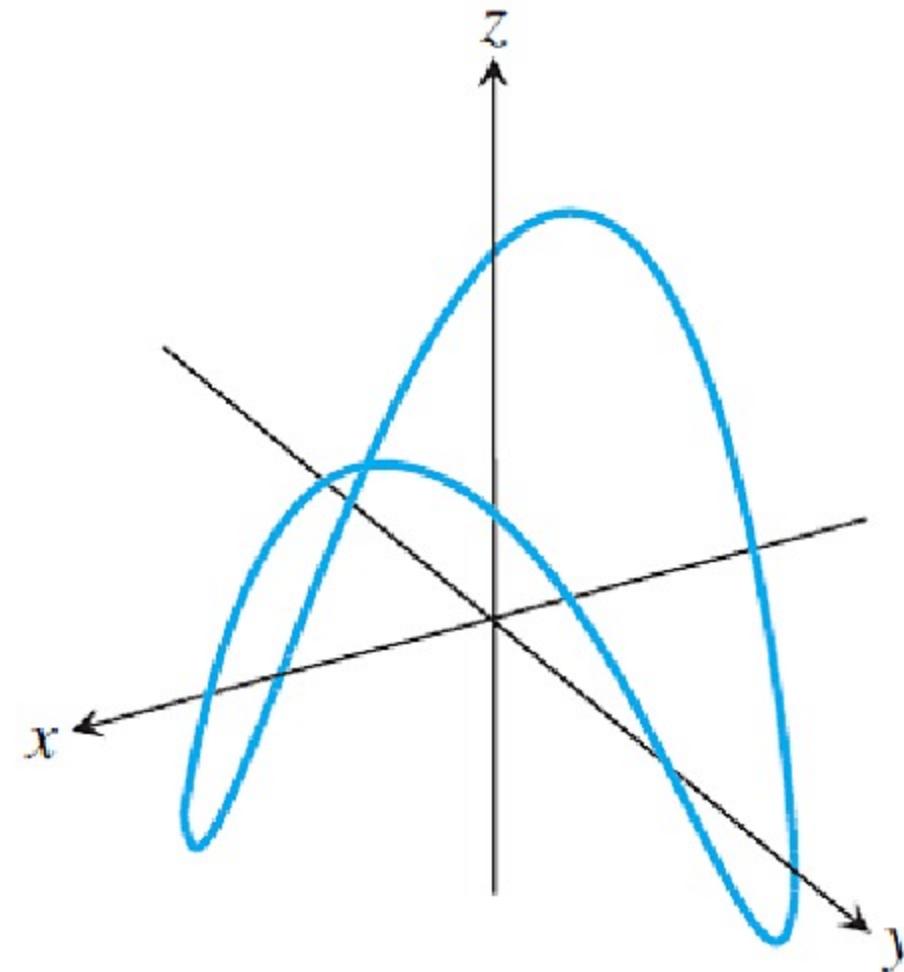
FIGURE 12.1 The position vector $\mathbf{r} = \overrightarrow{OP}$ of a particle moving through space is a function of time.

$$\mathbf{r}(t) = \overrightarrow{OP} = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} \quad (2)$$



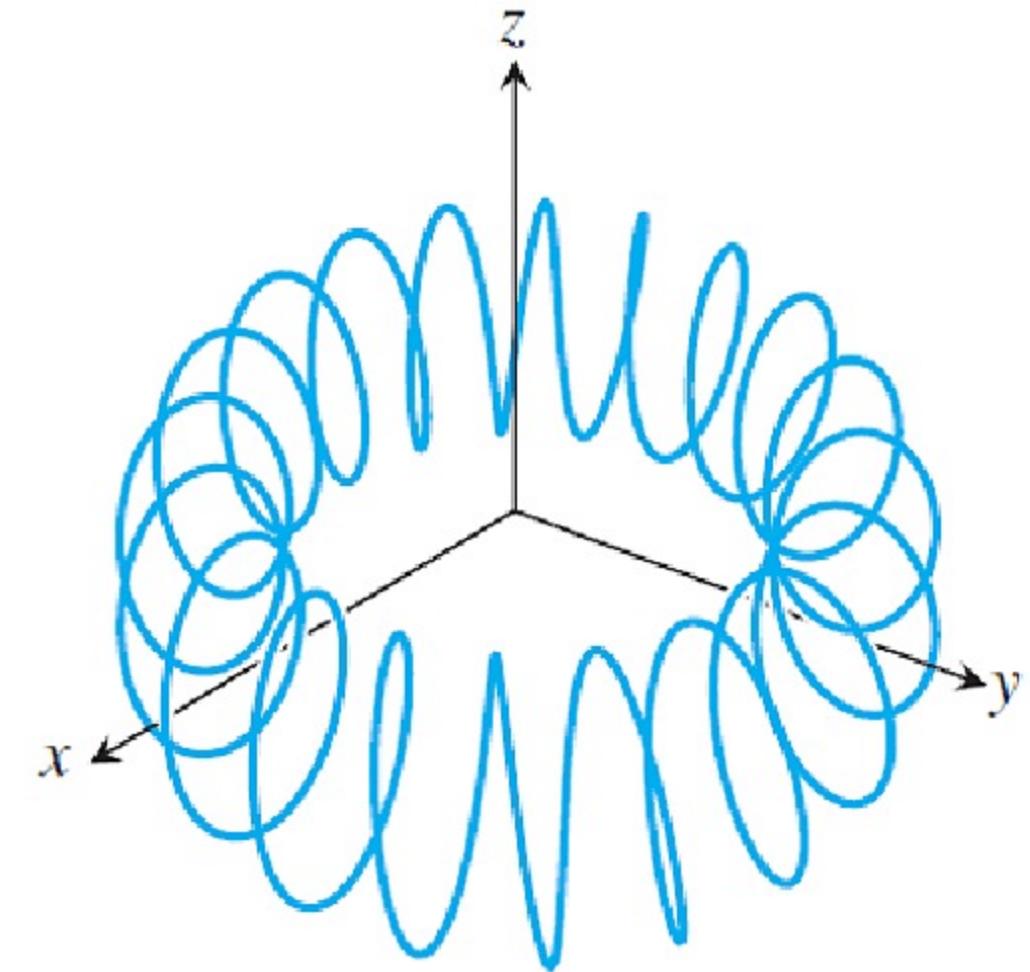
$$\mathbf{r}(t) = (\sin 3t)\cos t \mathbf{i} + (\sin 3t)\sin t \mathbf{j} + t \mathbf{k}$$

(a)



$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + (\sin 2t)\mathbf{k}$$

(b)



$$\mathbf{r}(t) = (4 + \sin 20t)\cos t \mathbf{i} + (4 + \sin 20t)\sin t \mathbf{j} + (\cos 20t)\mathbf{k}$$

(c)

FIGURE 12.2 Space curves are defined by the position vectors $\mathbf{r}(t)$.

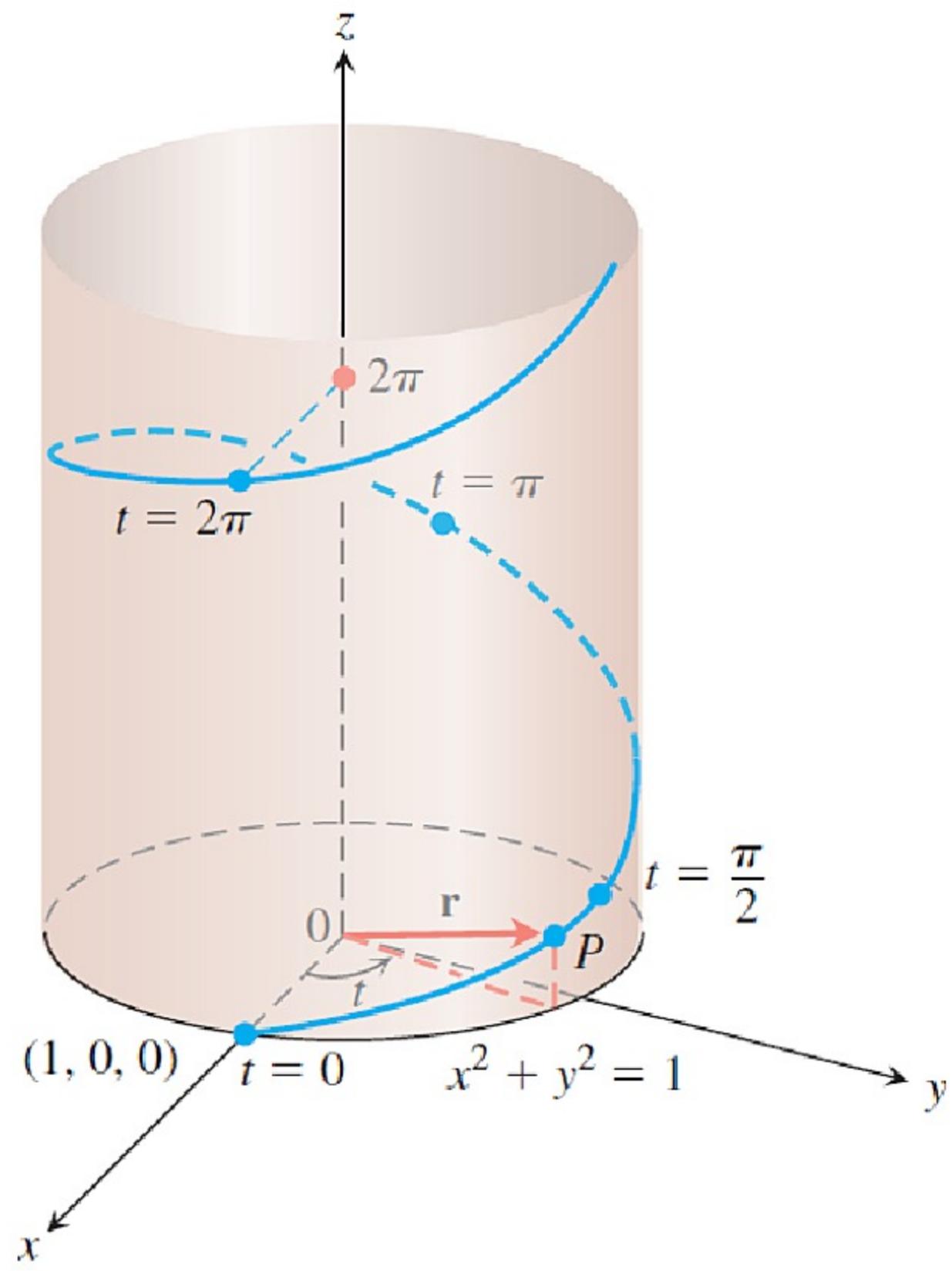
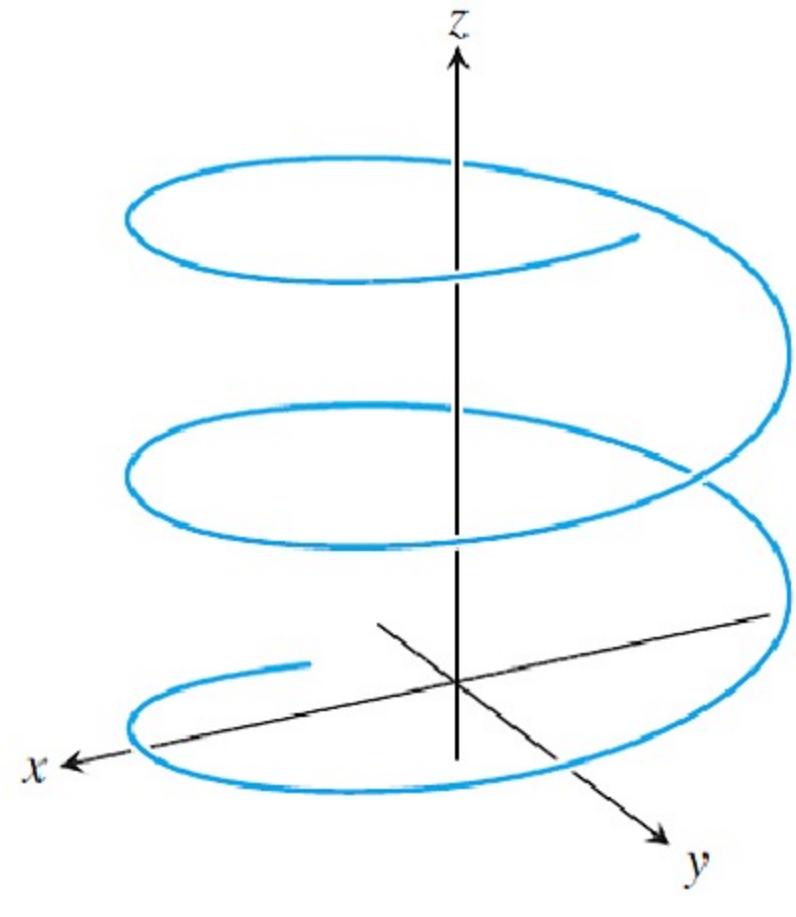
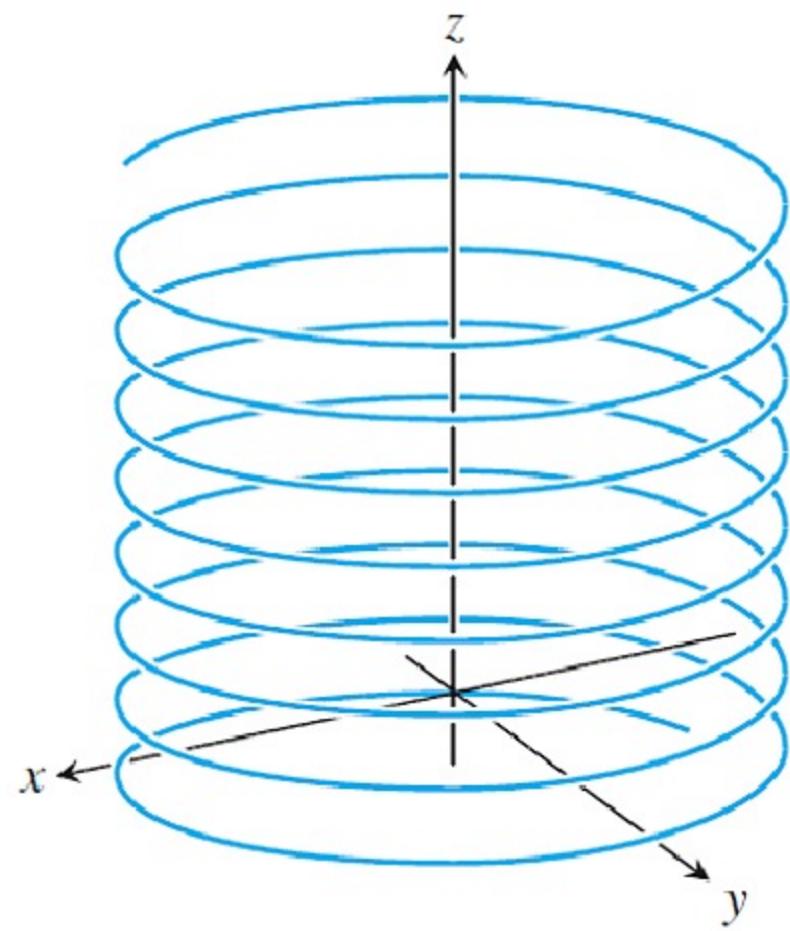


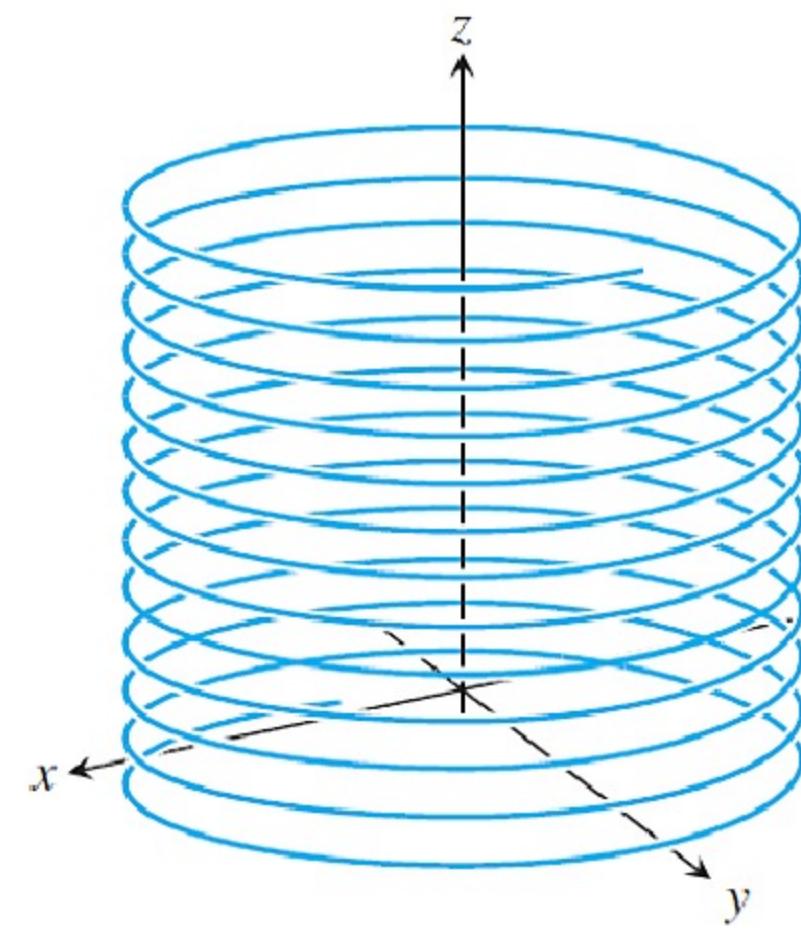
FIGURE 12.3 The upper half of the helix $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$ (Example 1).



$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$$



$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + 0.3t\mathbf{k}$$



$$\mathbf{r}(t) = (\cos 5t)\mathbf{i} + (\sin 5t)\mathbf{j} + t\mathbf{k}$$

FIGURE 12.4 Helices spiral upward around a cylinder, like coiled springs.

DEFINITION Let $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ be a vector function with domain D , and \mathbf{L} a vector. We say that \mathbf{r} has **limit \mathbf{L}** as t approaches t_0 and write

$$\lim_{t \rightarrow t_0} \mathbf{r}(t) = \mathbf{L}$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all $t \in D$

$$|\mathbf{r}(t) - \mathbf{L}| < \epsilon \quad \text{whenever} \quad 0 < |t - t_0| < \delta.$$

DEFINITION A vector function $\mathbf{r}(t)$ is **continuous at a point** $t = t_0$ in its domain if $\lim_{t \rightarrow t_0} \mathbf{r}(t) = \mathbf{r}(t_0)$. The function is **continuous** if it is continuous at every point in its domain.

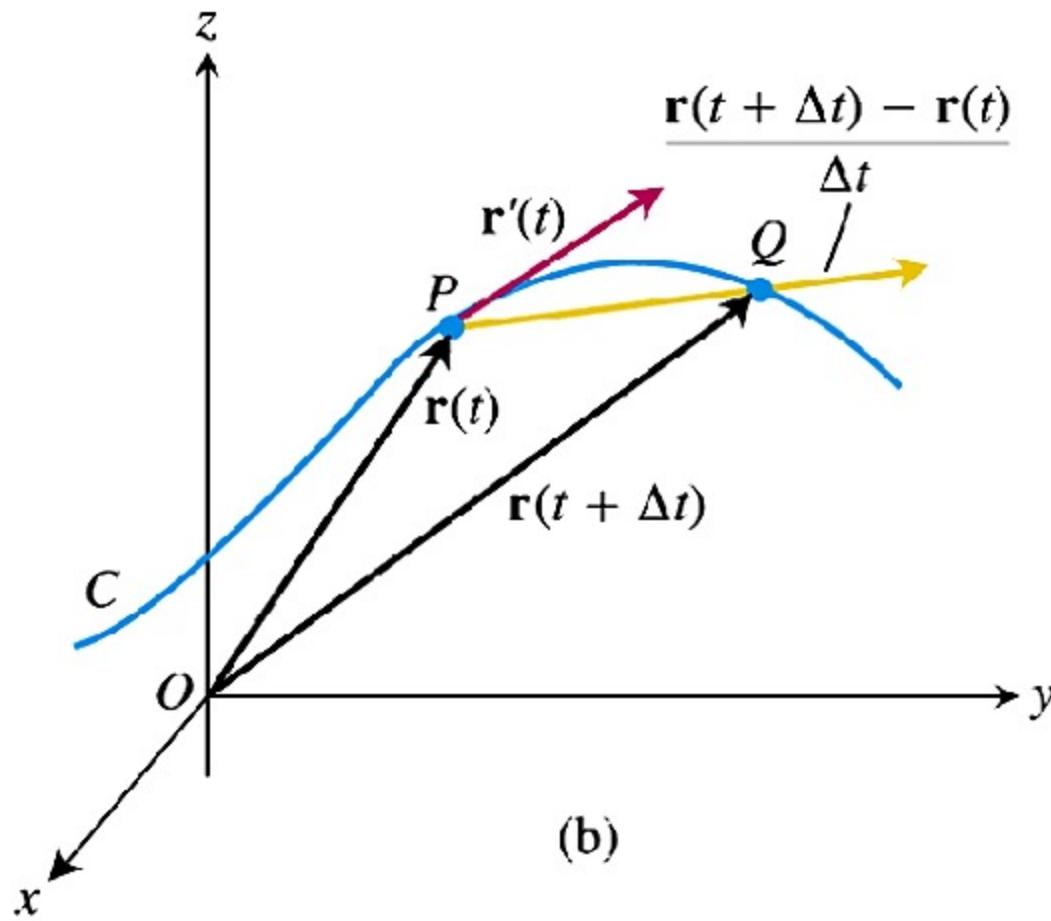
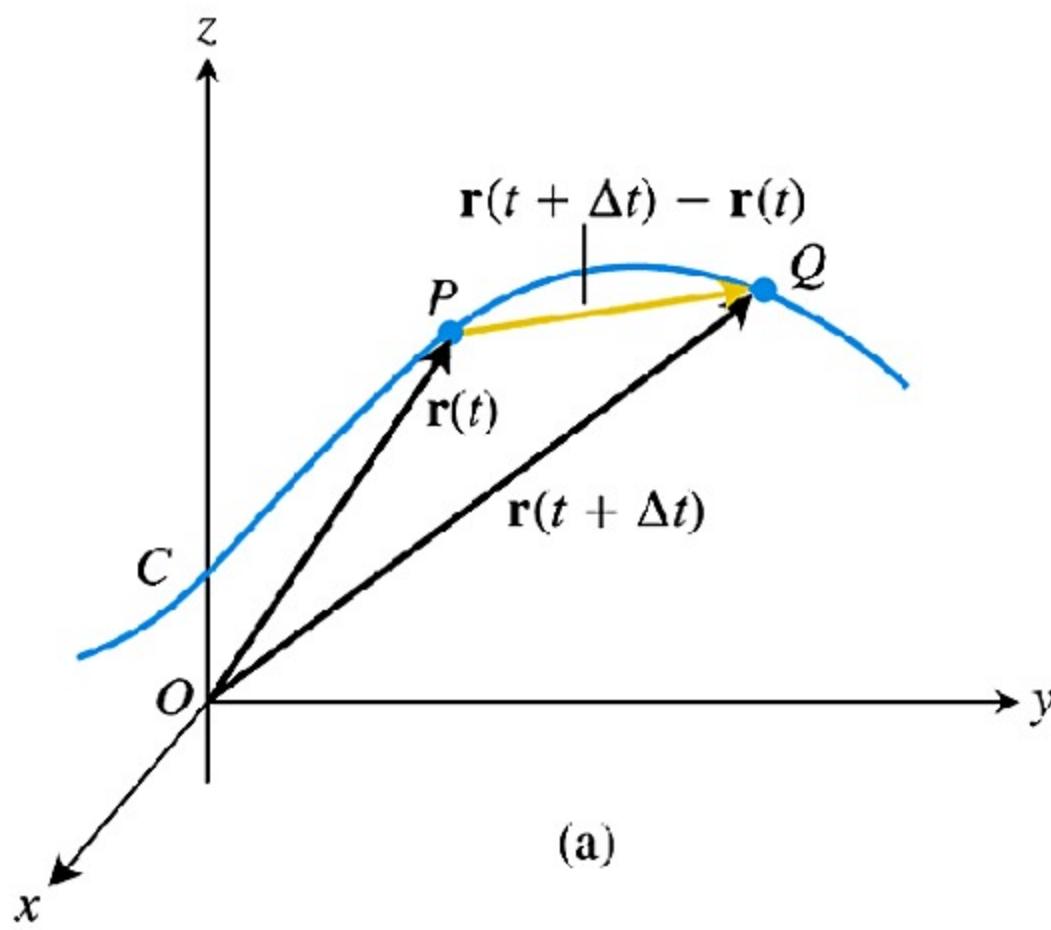


FIGURE 12.5 As $\Delta t \rightarrow 0$, the point Q approaches the point P along the curve C . In the limit, the vector $\overrightarrow{PQ} / \Delta t$ becomes the tangent vector $\mathbf{r}'(t)$.

DEFINITION The vector function $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ has a **derivative (is differentiable)** at t if f , g , and h have derivatives at t . The derivative is the vector function

$$\mathbf{r}'(t) = \frac{d\mathbf{r}}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} = \frac{df}{dt}\mathbf{i} + \frac{dg}{dt}\mathbf{j} + \frac{dh}{dt}\mathbf{k}.$$

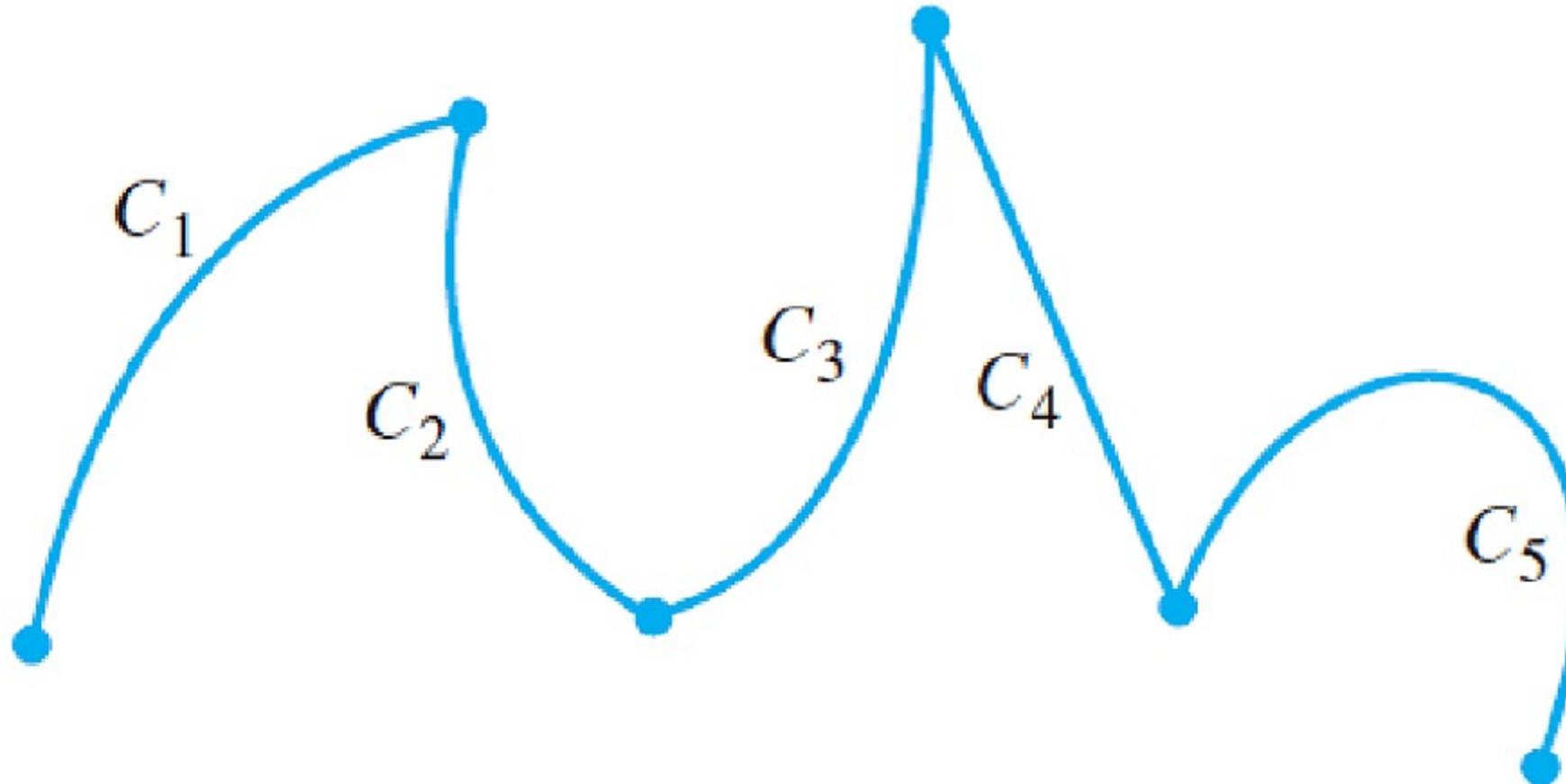


FIGURE 12.6 A piecewise smooth curve made up of five smooth curves connected end to end in a continuous fashion. The curve here is not smooth at the points joining the five smooth curves.

DEFINITIONS If \mathbf{r} is the position vector of a particle moving along a smooth curve in space, then

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt}$$

is the particle's **velocity vector**, tangent to the curve. At any time t , the direction of \mathbf{v} is the **direction of motion**, the magnitude of \mathbf{v} is the particle's **speed**, and the derivative $\mathbf{a} = d\mathbf{v}/dt$, when it exists, is the particle's **acceleration vector**. In summary,

1. Velocity is the derivative of position: $\mathbf{v} = \frac{d\mathbf{r}}{dt}$.
2. Speed is the magnitude of velocity: $\text{Speed} = |\mathbf{v}|$.
3. Acceleration is the derivative of velocity: $\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2}$.
4. The unit vector $\mathbf{v}/|\mathbf{v}|$ is the direction of motion at time t .

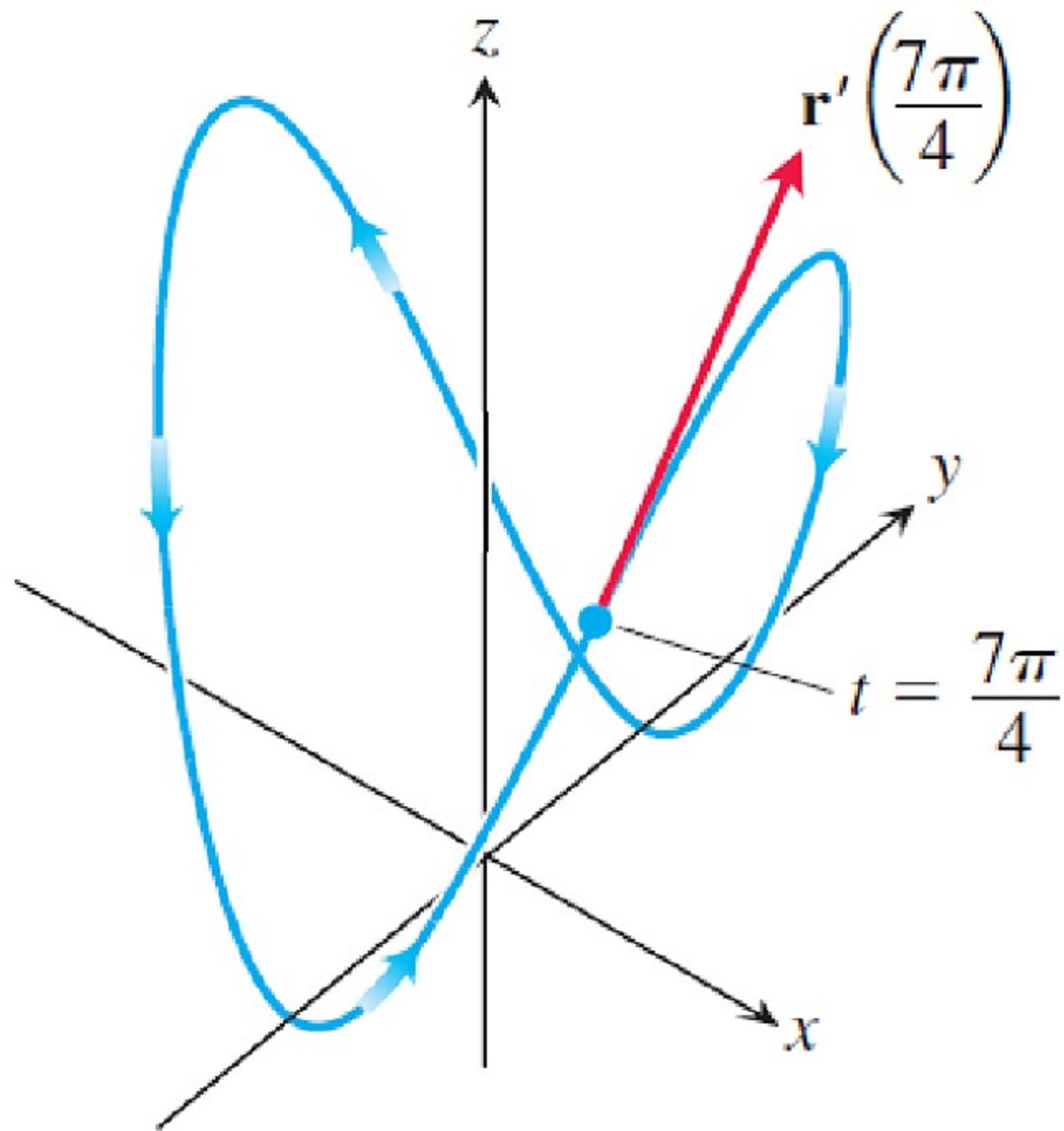


FIGURE 12.7 The curve and the velocity vector when $t = 7\pi/4$ for the motion given in Example 4.

Differentiation Rules for Vector Functions

Let \mathbf{u} and \mathbf{v} be differentiable vector functions of t , \mathbf{C} a constant vector, c any scalar, and f any differentiable scalar function.

1. *Constant Function Rule:* $\frac{d}{dt} \mathbf{C} = \mathbf{0}$

2. *Scalar Multiple Rules:* $\frac{d}{dt} [c\mathbf{u}(t)] = c\mathbf{u}'(t)$

$$\frac{d}{dt} [f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$$

3. *Sum Rule:* $\frac{d}{dt} [\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$

4. *Difference Rule:* $\frac{d}{dt} [\mathbf{u}(t) - \mathbf{v}(t)] = \mathbf{u}'(t) - \mathbf{v}'(t)$

5. *Dot Product Rule:* $\frac{d}{dt} [\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$

6. *Cross Product Rule:* $\frac{d}{dt} [\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$

7. *Chain Rule:* $\frac{d}{dt} [\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$

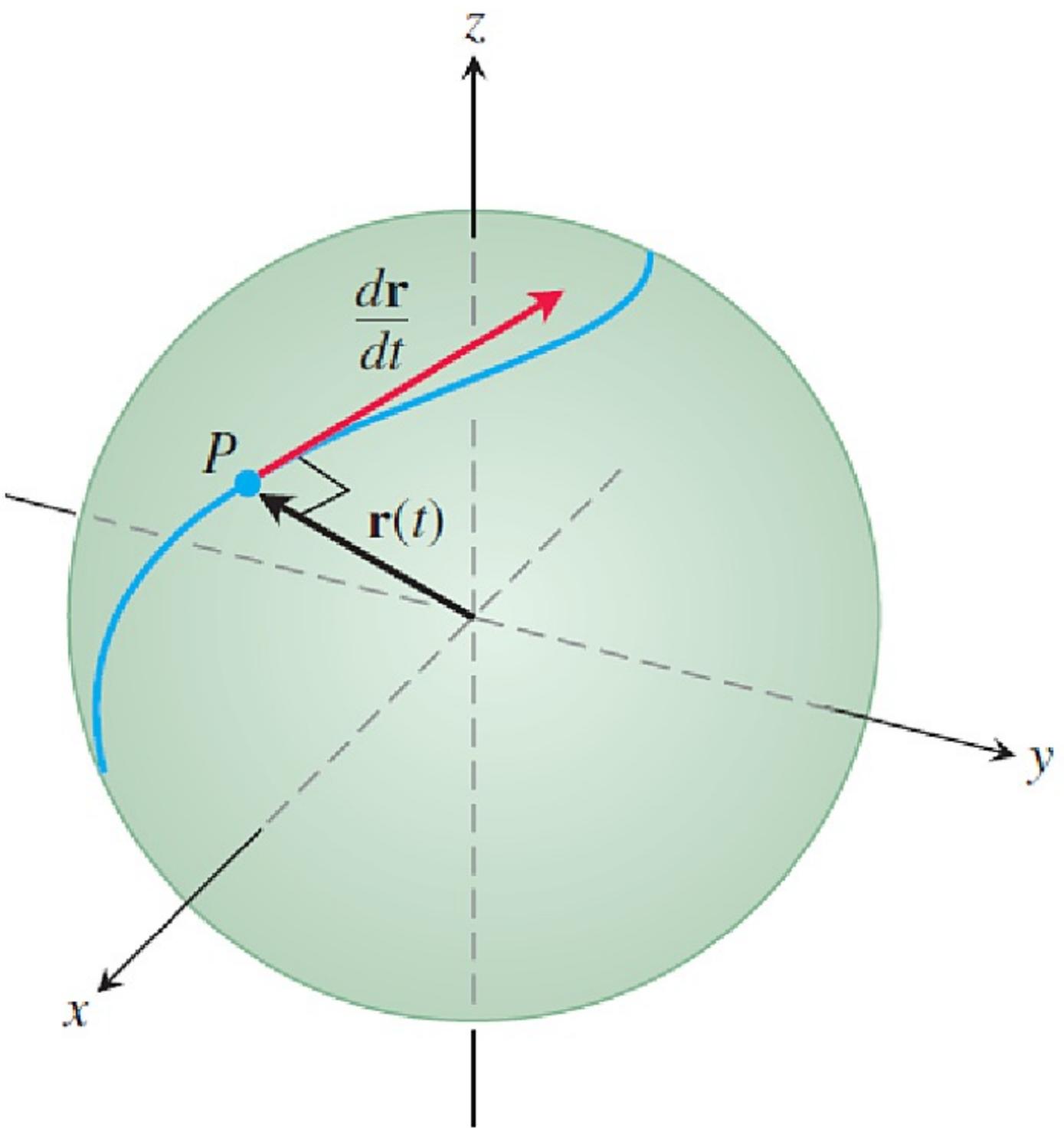


FIGURE 12.8 If a particle moves on a sphere in such a way that its position \mathbf{r} is a differentiable function of time, then $\mathbf{r} \cdot (d\mathbf{r}/dt) = 0$.

If \mathbf{r} is a differentiable vector function of t of constant length, then

$$\mathbf{r} \cdot \frac{d\mathbf{r}}{dt} = 0. \quad (4)$$

12.2

Integrals of Vector Functions; Projectile Motion

DEFINITION The **indefinite integral** of \mathbf{r} with respect to t is the set of all antiderivatives of \mathbf{r} , denoted by $\int \mathbf{r}(t) dt$. If \mathbf{R} is any antiderivative of \mathbf{r} , then

$$\int \mathbf{r}(t) dt = \mathbf{R}(t) + \mathbf{C}.$$

DEFINITION If the components of $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ are integrable over $[a, b]$, then so is \mathbf{r} , and the **definite integral** of \mathbf{r} from a to b is

$$\int_a^b \mathbf{r}(t) dt = \left(\int_a^b f(t) dt \right) \mathbf{i} + \left(\int_a^b g(t) dt \right) \mathbf{j} + \left(\int_a^b h(t) dt \right) \mathbf{k}.$$

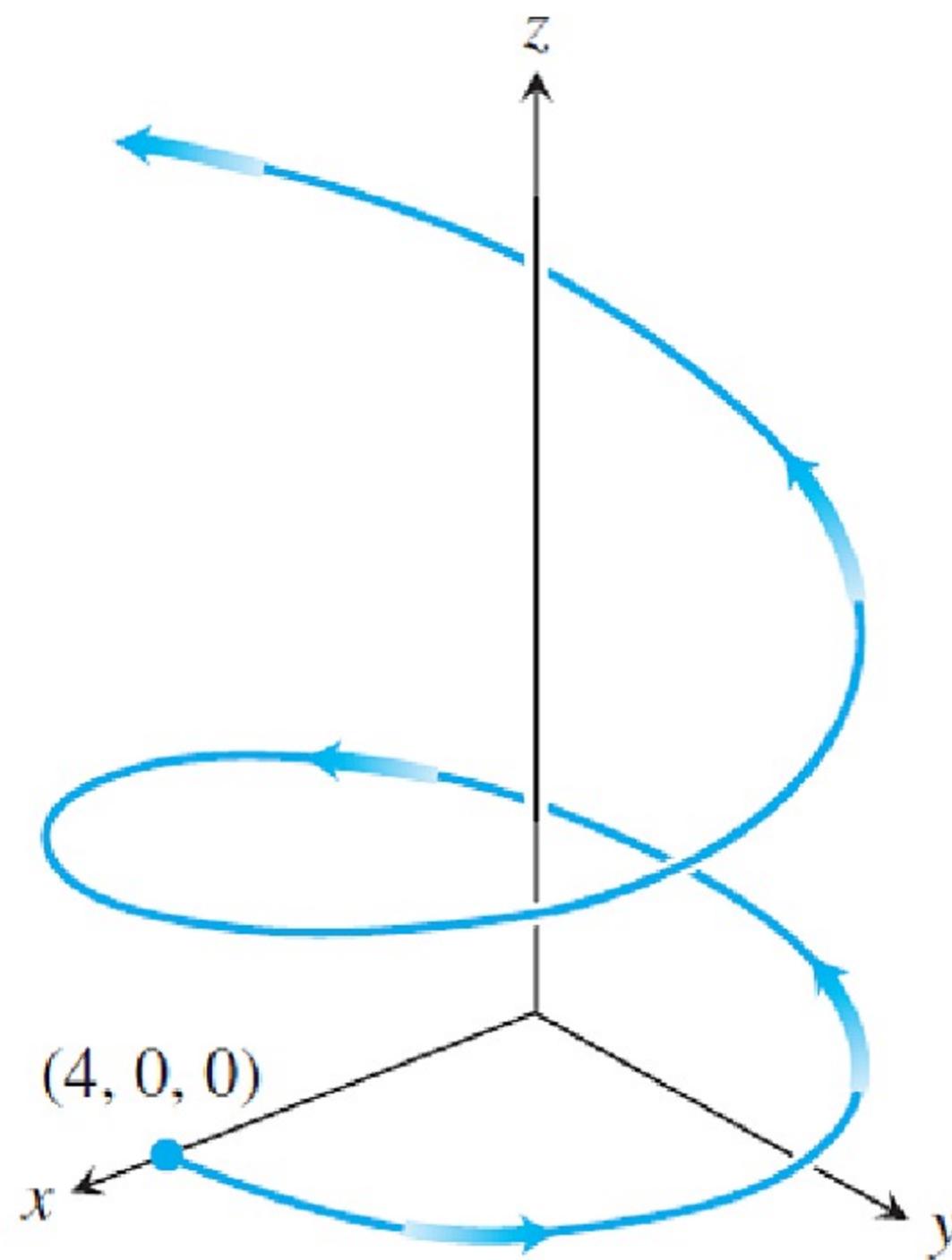
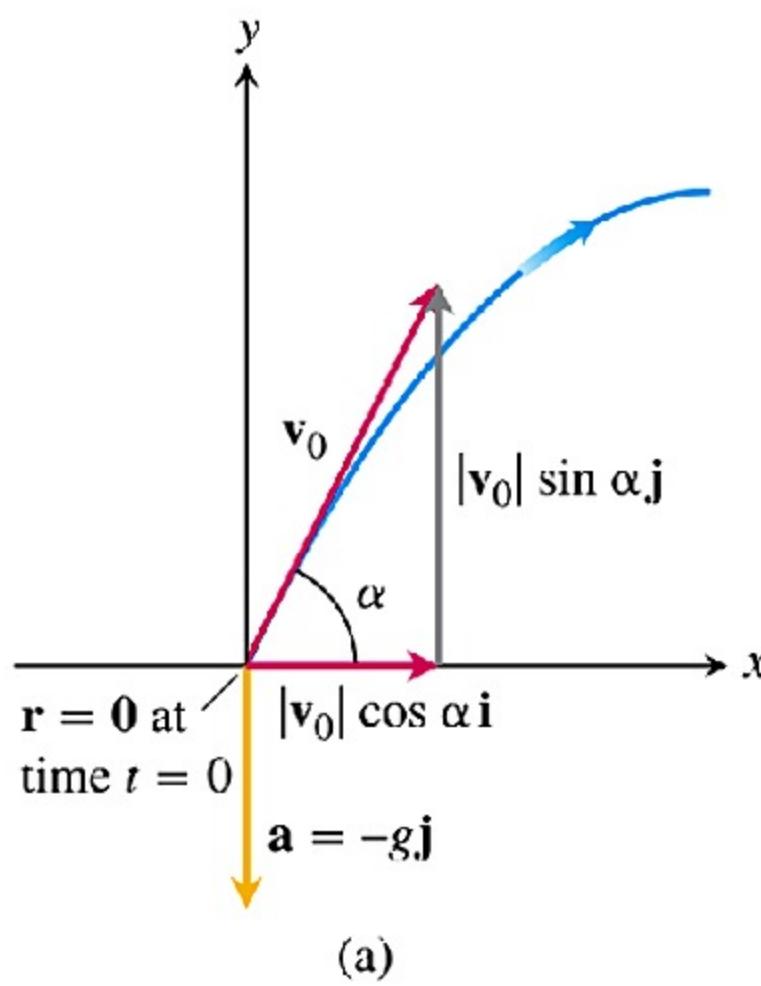


FIGURE 12.9 The path of the hang glider in Example 3. Although the path spirals around the z -axis, it is not a helix.



$$\mathbf{v}_0 = (|v_0| \cos \alpha) \mathbf{i} + (|v_0| \sin \alpha) \mathbf{j}$$

(a)

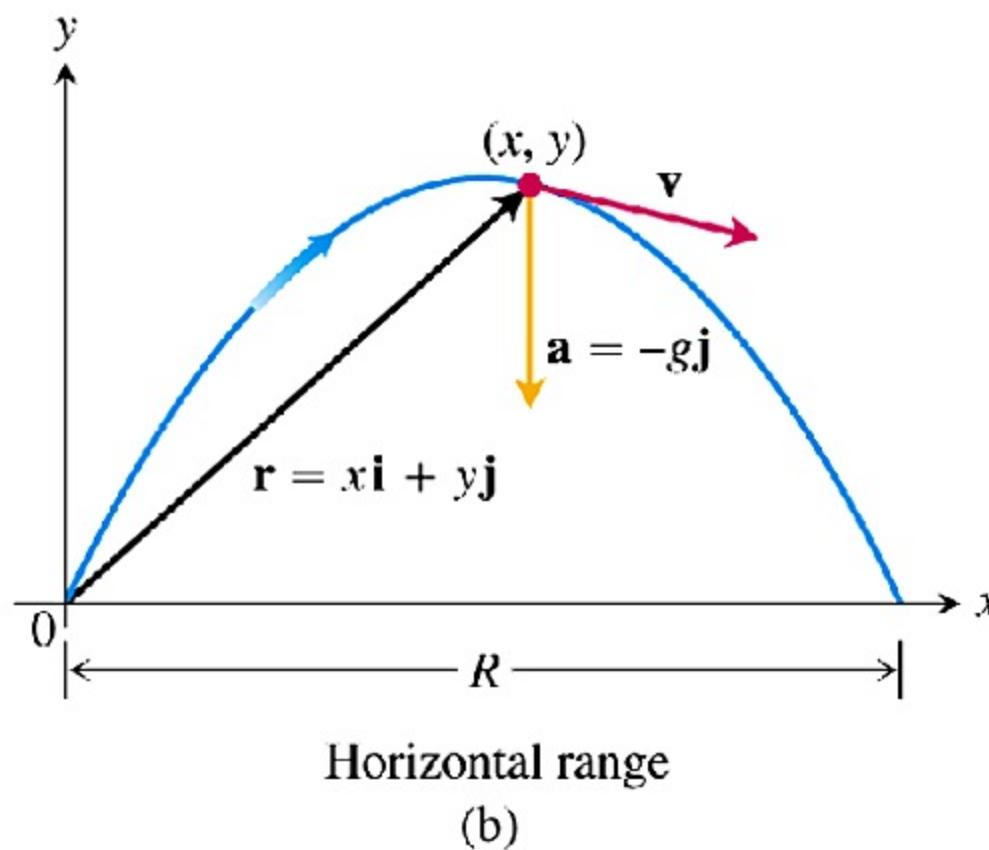


FIGURE 12.10 (a) Position, velocity, acceleration, and launch angle at $t = 0$. (b) Position, velocity, and acceleration at a later time t .

Ideal Projectile Motion Equation

$$\mathbf{r} = (v_0 \cos \alpha)t\mathbf{i} + \left((v_0 \sin \alpha)t - \frac{1}{2}gt^2 \right)\mathbf{j}. \quad (5)$$

Height, Flight Time, and Range for Ideal Projectile Motion

For ideal projectile motion when an object is launched from the origin over a horizontal surface with initial speed v_0 and launch angle α :

Maximum height: $y_{\max} = \frac{(v_0 \sin \alpha)^2}{2g}$

Flight time: $t = \frac{2v_0 \sin \alpha}{g}$

Range: $R = \frac{v_0^2}{g} \sin 2\alpha.$

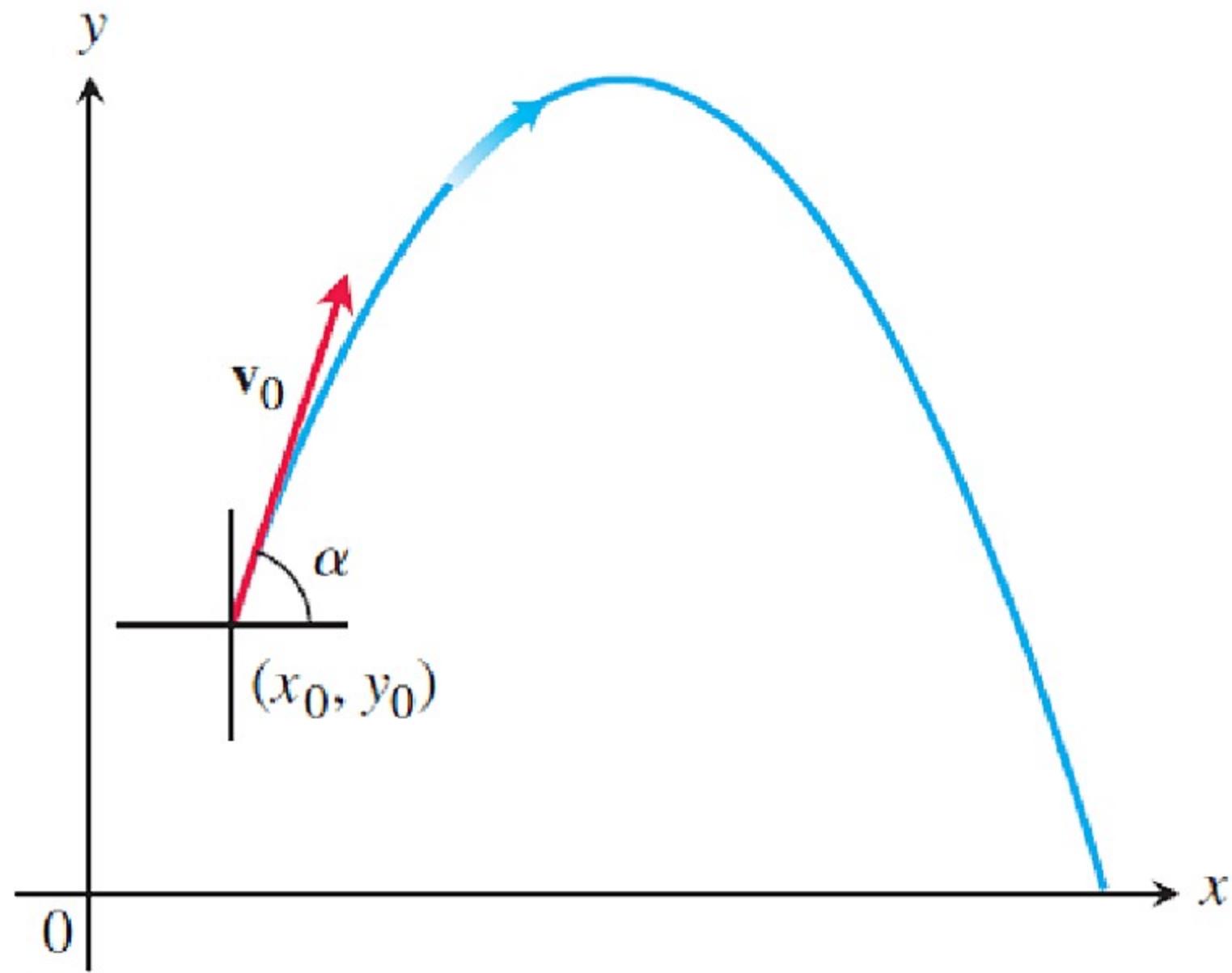


FIGURE 12.11 The path of a projectile fired from (x_0, y_0) with an initial velocity \mathbf{v}_0 at an angle of α degrees with the horizontal.

Section 12.3

Arc Length in Space

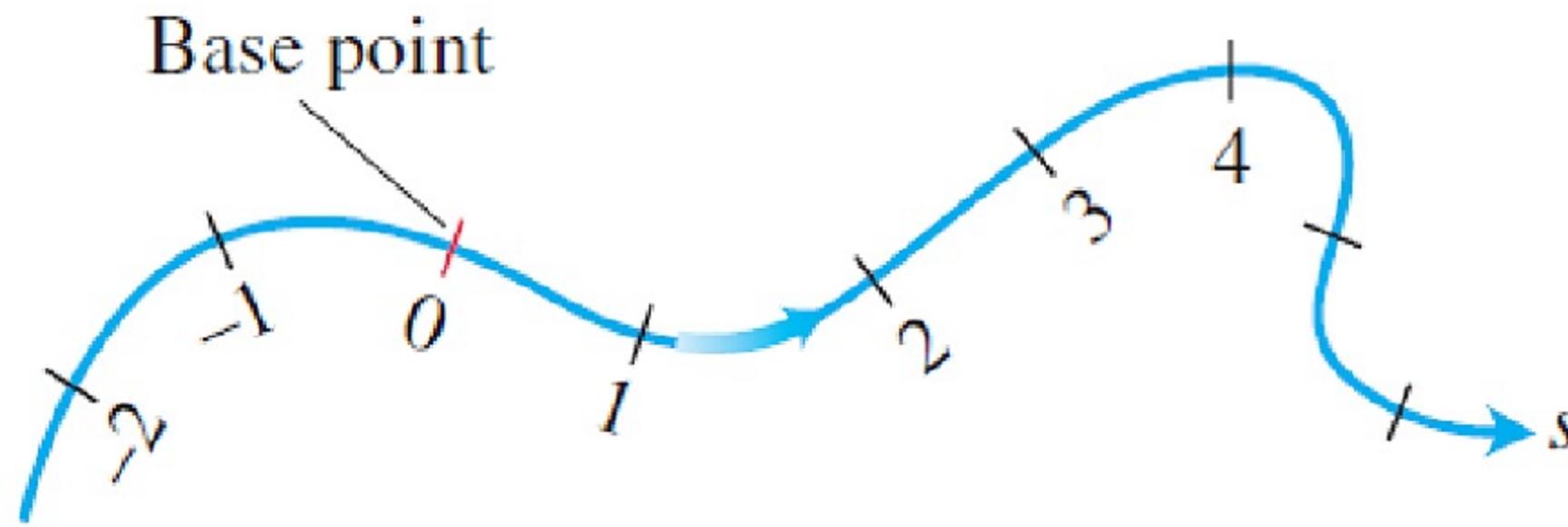


FIGURE 12.12 Smooth curves can be scaled like number lines, the coordinate of each point being its directed distance along the curve from a preselected base point.

DEFINITION The **length** of a smooth curve $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, $a \leq t \leq b$, that is traced exactly once as t increases from $t = a$ to $t = b$, is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt. \quad (1)$$

Arc Length Formula

$$L = \int_a^b |\mathbf{v}| dt \quad (2)$$

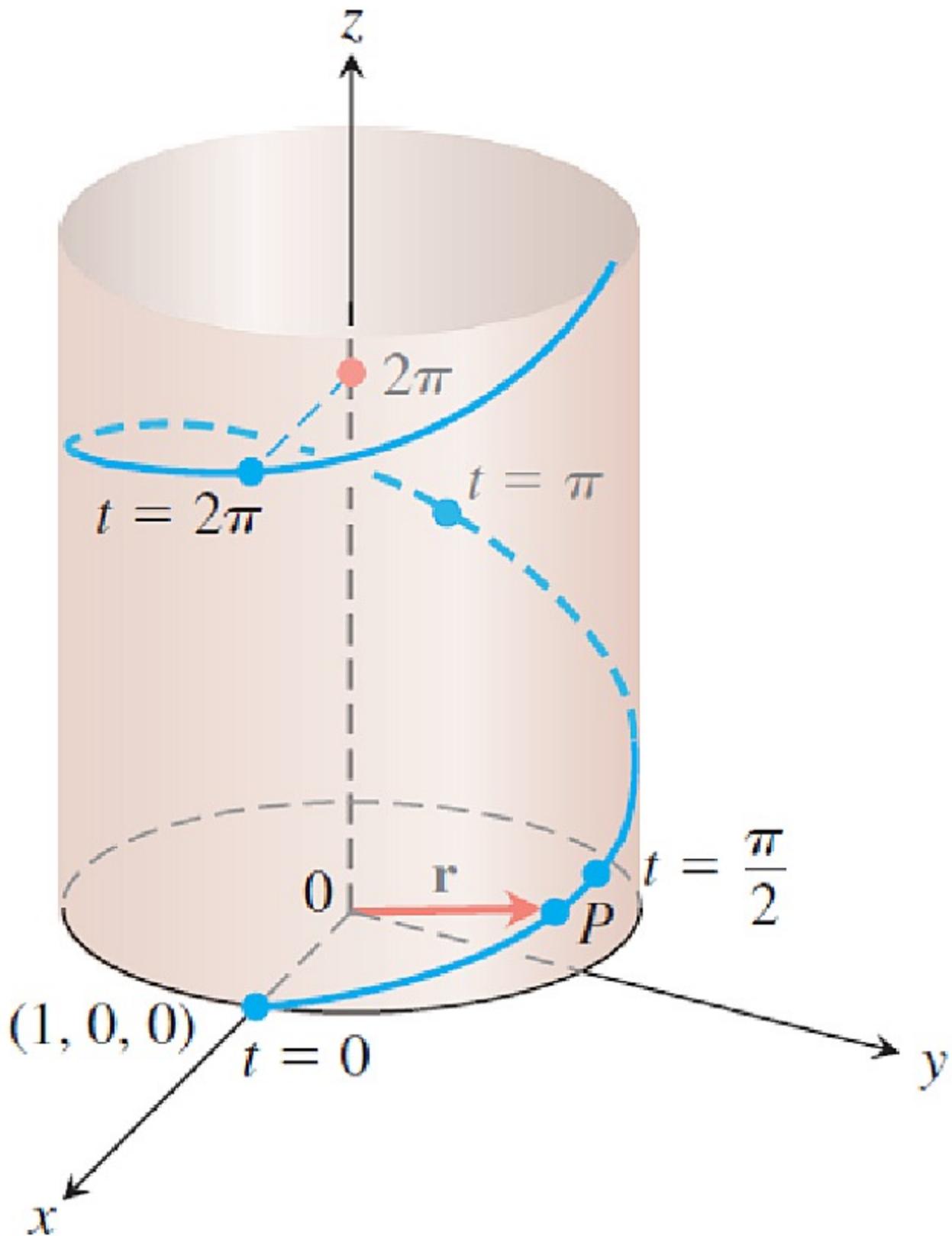


FIGURE 12.13 The helix in Example 1,
 $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + tk\mathbf{k}$.

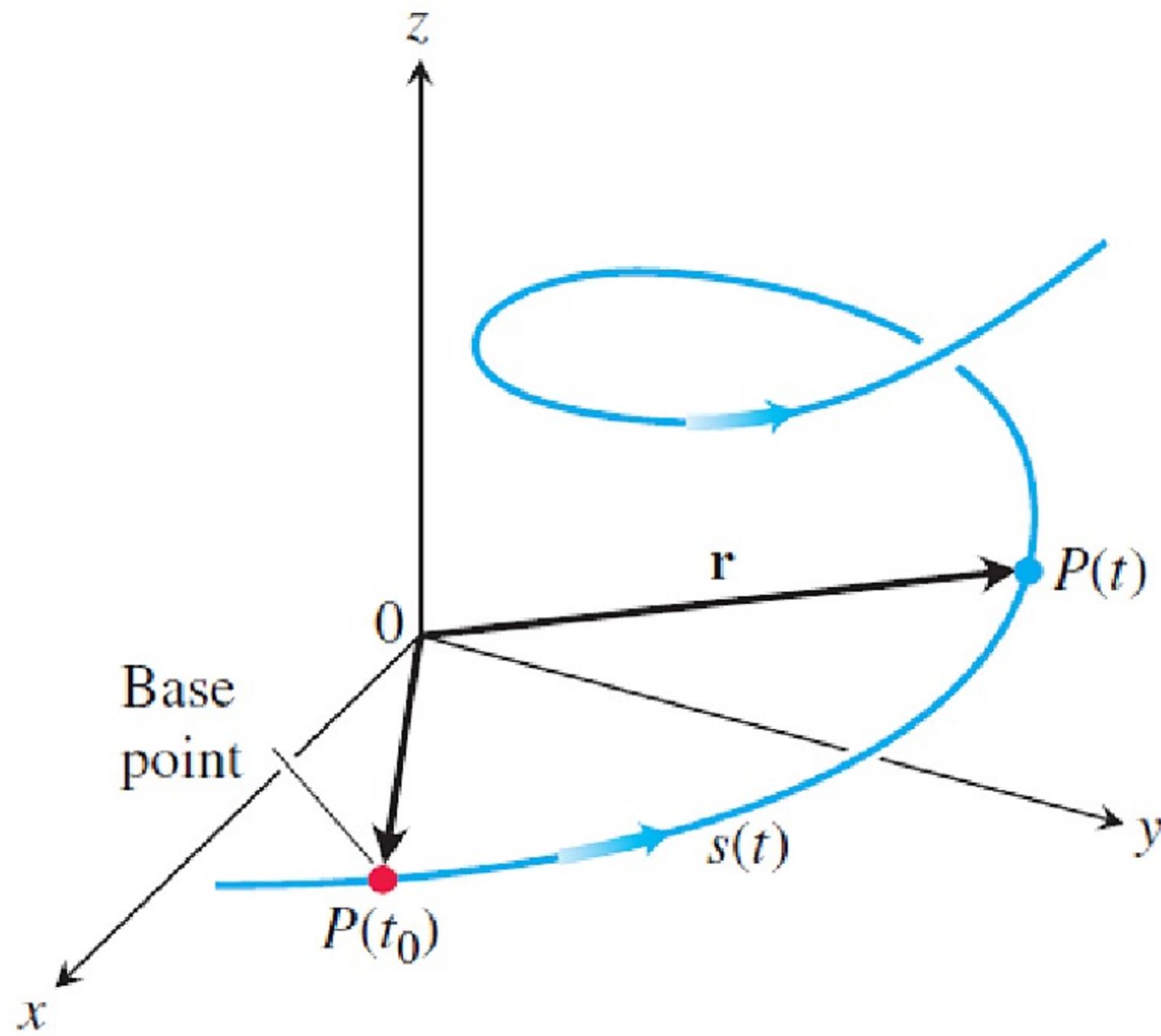


FIGURE 12.14 The directed distance along the curve from $P(t_0)$ to any point $P(t)$ is $s(t) = \int_{t_0}^t |\mathbf{v}(\tau)| d\tau$.

$$P(t) \text{ is } s(t) = \int_{t_0}^t |\mathbf{v}(\tau)| d\tau.$$

Arc Length Parameter with Base Point $P(t_0)$

$$s(t) = \int_{t_0}^t \sqrt{[x'(\tau)]^2 + [y'(\tau)]^2 + [z'(\tau)]^2} d\tau = \int_{t_0}^t |\mathbf{v}(\tau)| d\tau \quad (3)$$

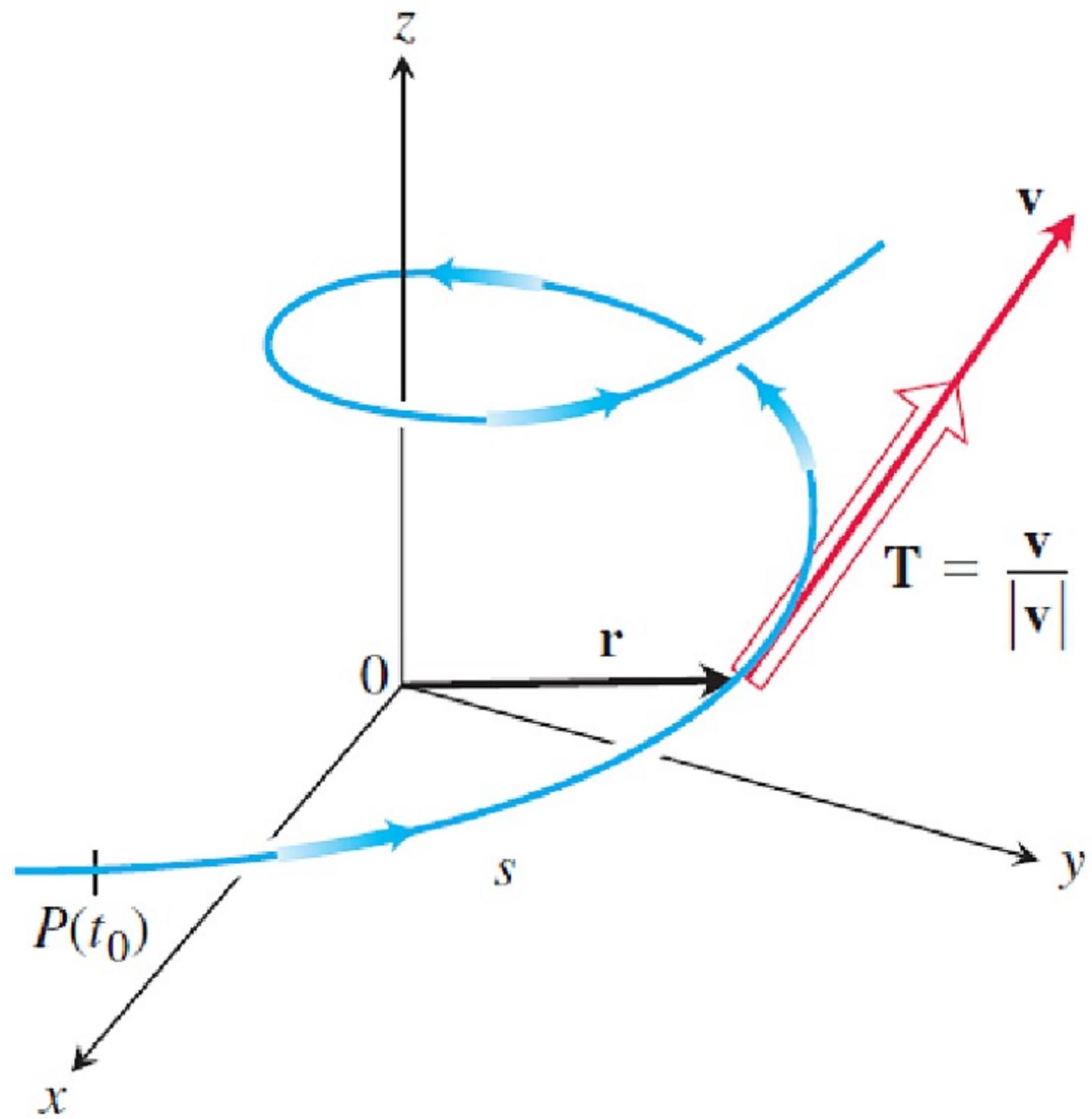


FIGURE 12.15 We find the unit tangent vector \mathbf{T} by dividing \mathbf{v} by its length $|\mathbf{v}|$.

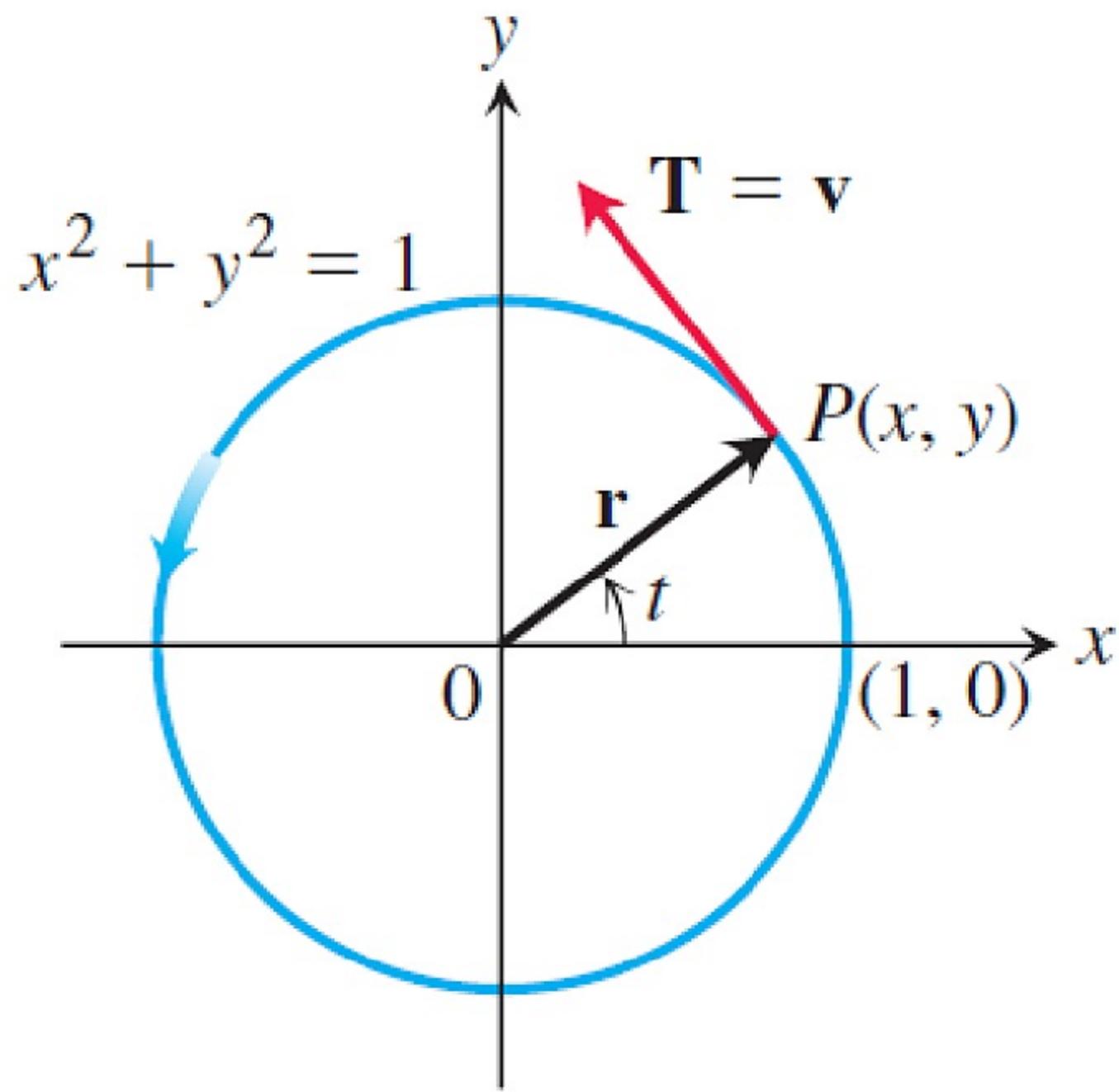


FIGURE 12.16 Counterclockwise motion around the unit circle.

Section 12.4

Curvature and Normal Vectors of a Curve

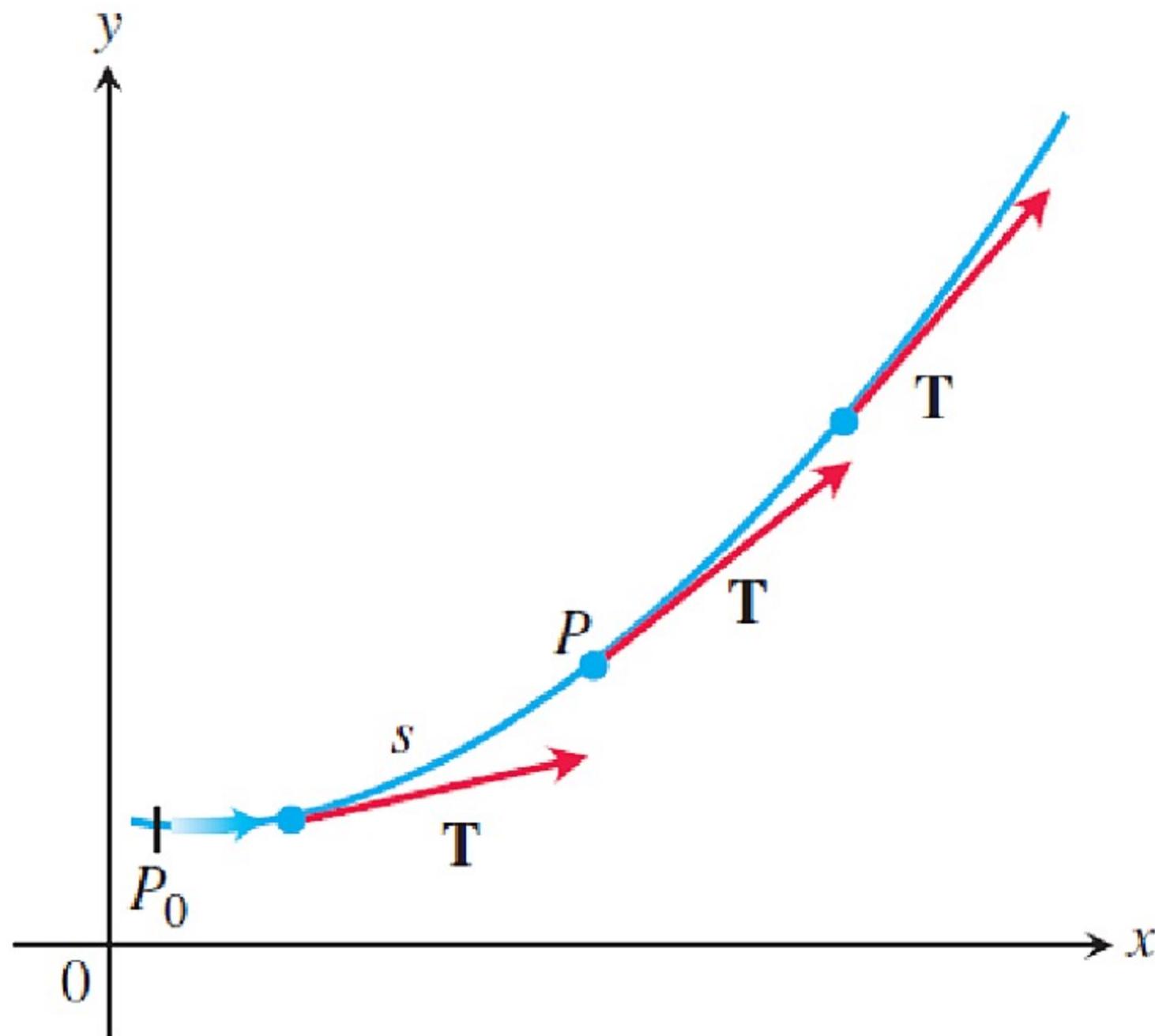


FIGURE 12.17 As P moves along the curve in the direction of increasing arc length, the unit tangent vector turns. The value of $|d\mathbf{T}/ds|$ at P is called the *curvature* of the curve at P .

DEFINITION If \mathbf{T} is the unit vector of a smooth curve, the **curvature** function of the curve is

$$\kappa = \left| \frac{d\mathbf{T}}{ds} \right|.$$

Formula for Calculating Curvature

If $\mathbf{r}(t)$ is a smooth curve, then the curvature is the scalar function

$$\kappa = \frac{1}{|\mathbf{v}|} \left| \frac{d\mathbf{T}}{dt} \right|, \quad (1)$$

where $\mathbf{T} = \mathbf{v}/|\mathbf{v}|$ is the unit tangent vector.

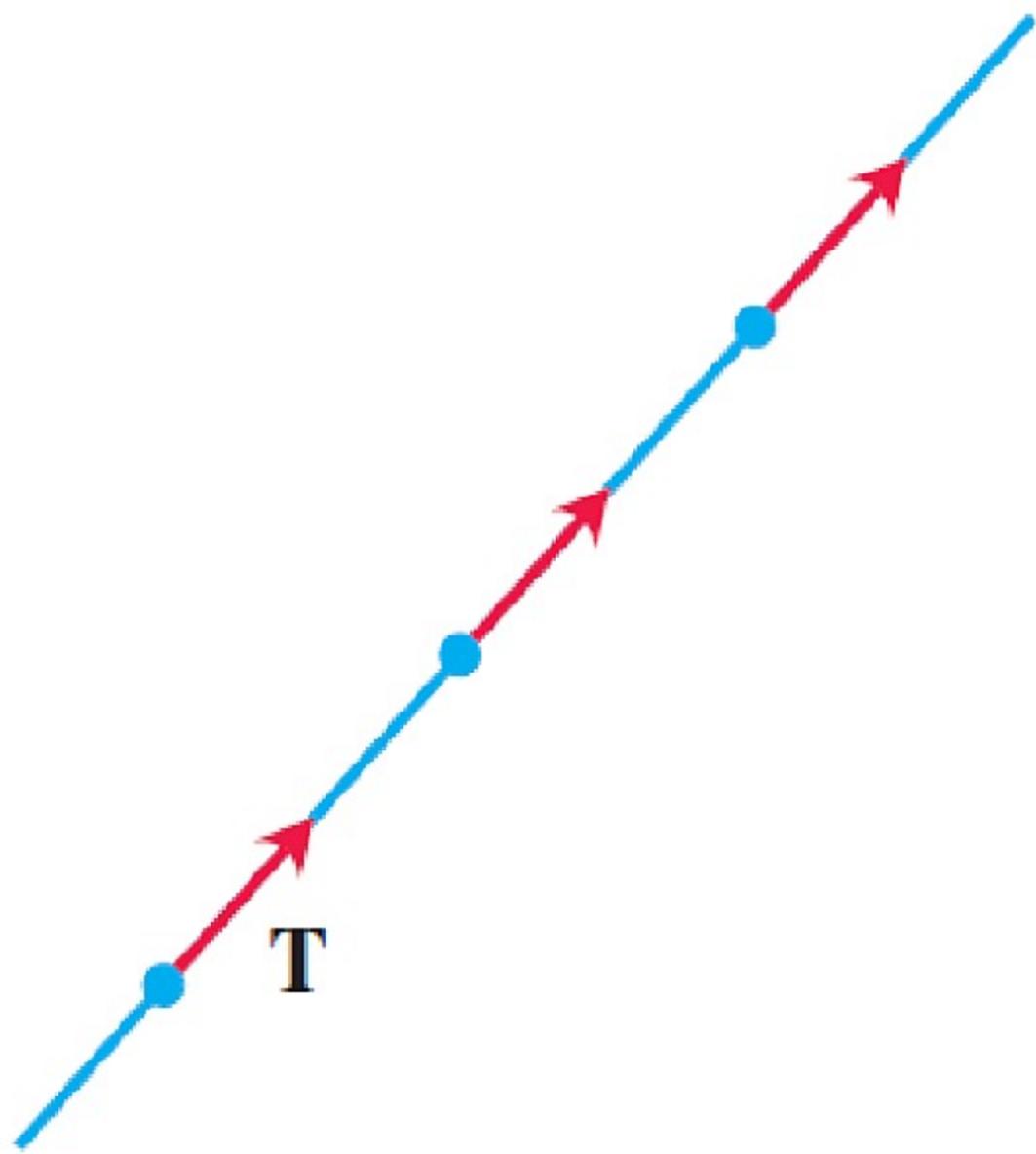


FIGURE 12.18 Along a straight line, \mathbf{T} always points in the same direction. The curvature, $|d\mathbf{T}/ds|$, is zero (Example 1).

DEFINITION At a point where $\kappa \neq 0$, the **principal unit normal vector** for a smooth curve in the plane is

$$\mathbf{N} = \frac{1}{\kappa} \frac{d\mathbf{T}}{ds}.$$

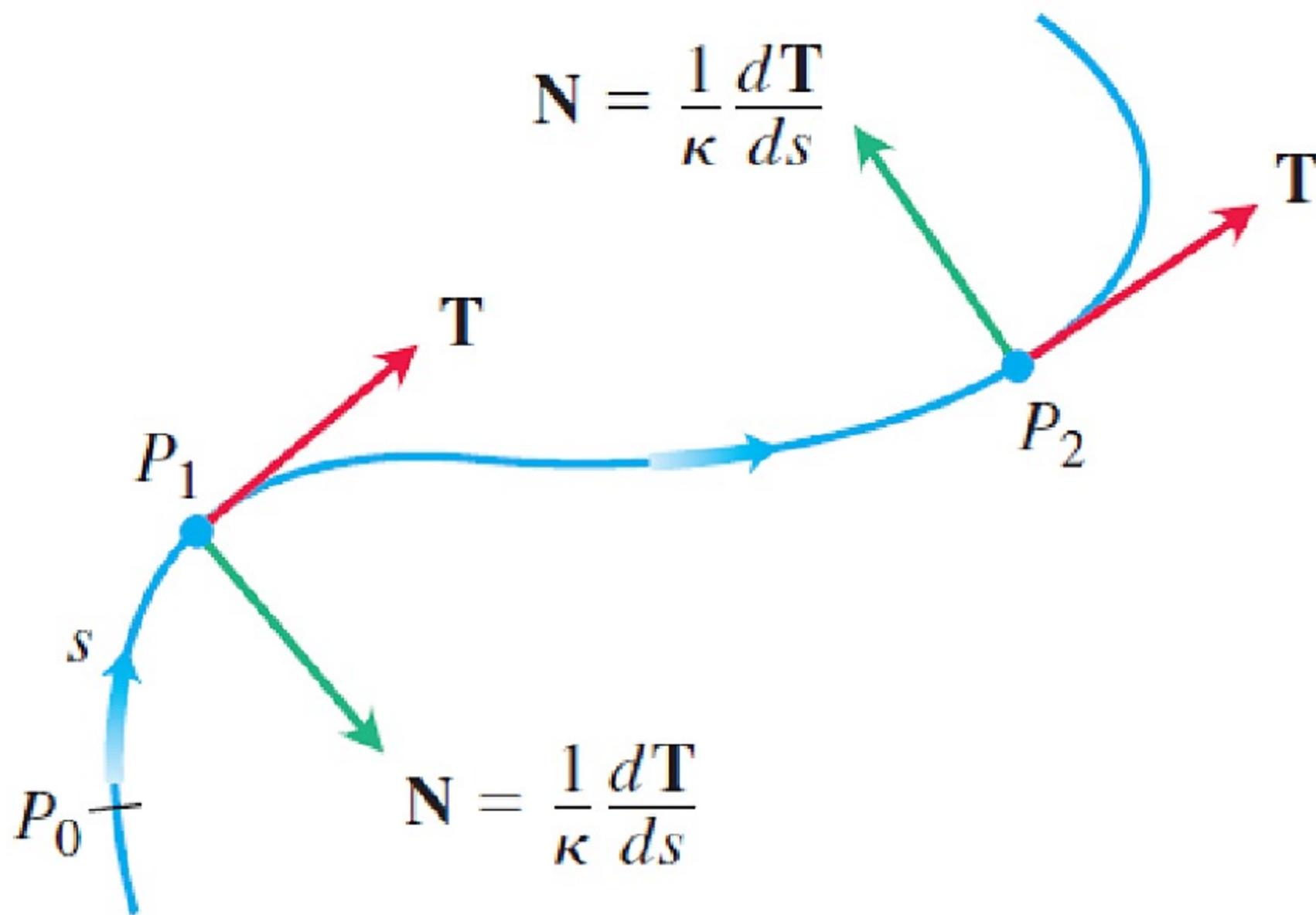


FIGURE 12.19 The vector $d\mathbf{T}/ds$, normal to the curve, always points in the direction in which \mathbf{T} is turning. The unit normal vector \mathbf{N} is the direction of $d\mathbf{T}/ds$.

Formula for Calculating \mathbf{N}

If $\mathbf{r}(t)$ is a smooth curve, then the principal unit normal is

$$\mathbf{N} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}, \quad (2)$$

where $\mathbf{T} = \mathbf{v}/|\mathbf{v}|$ is the unit tangent vector.

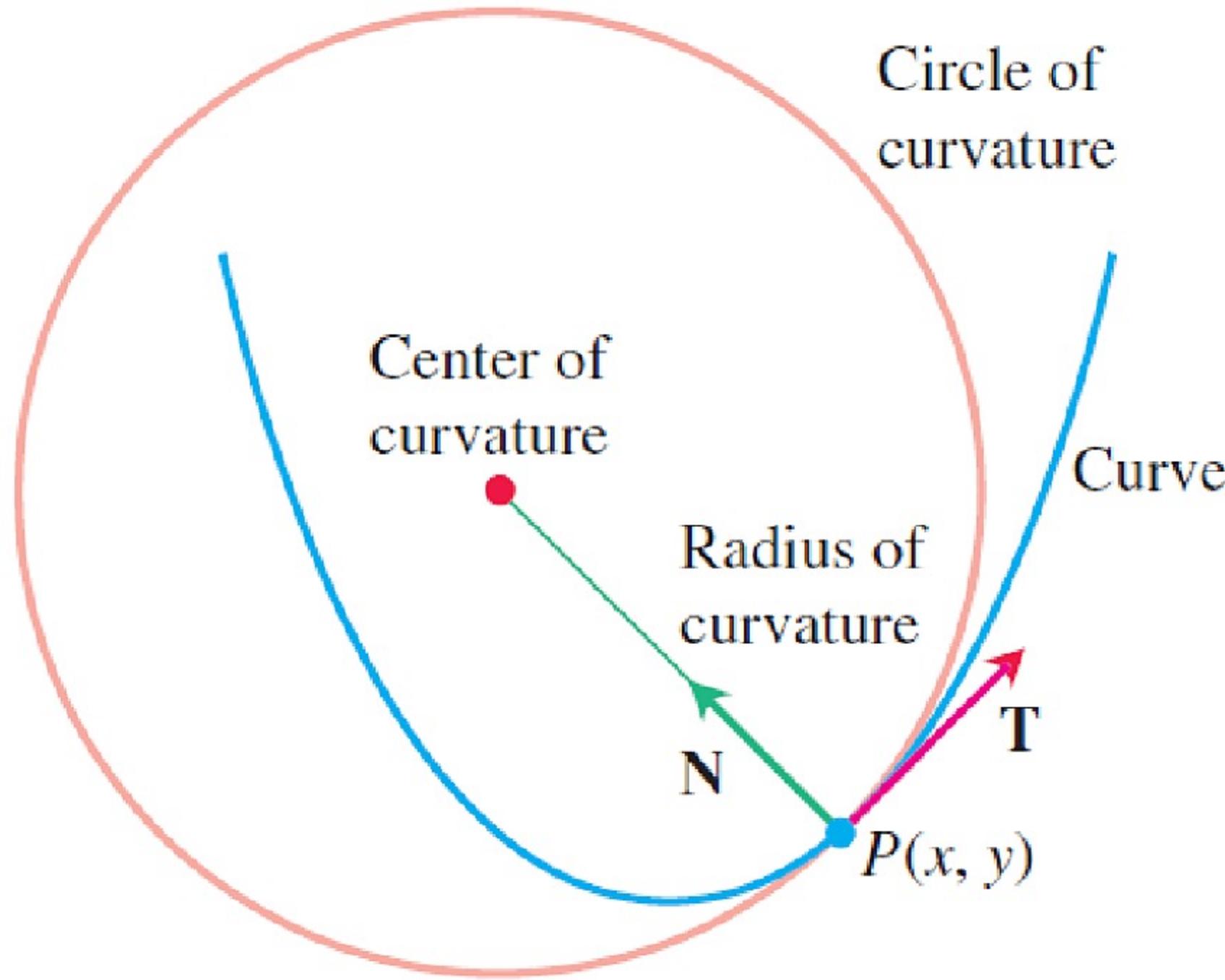


FIGURE 12.20 The center of the osculating circle at $P(x, y)$ lies toward the inner side of the curve.

Circle of Curvature for Plane Curves

The **circle of curvature** or **osculating circle** at a point P on a plane curve where $\kappa \neq 0$ is the circle in the plane of the curve that

1. is tangent to the curve at P (has the same tangent line the curve has)
2. has the same curvature the curve has at P
3. has center that lies toward the concave or inner side of the curve (as in Figure 12.20).

The **radius of curvature** of the curve at P is the radius of the circle of curvature, which, according to Example 2, is

$$\text{Radius of curvature} = \rho = \frac{1}{\kappa}.$$

To find ρ , we find κ and take the reciprocal. The **center of curvature** of the curve at P is the center of the circle of curvature.

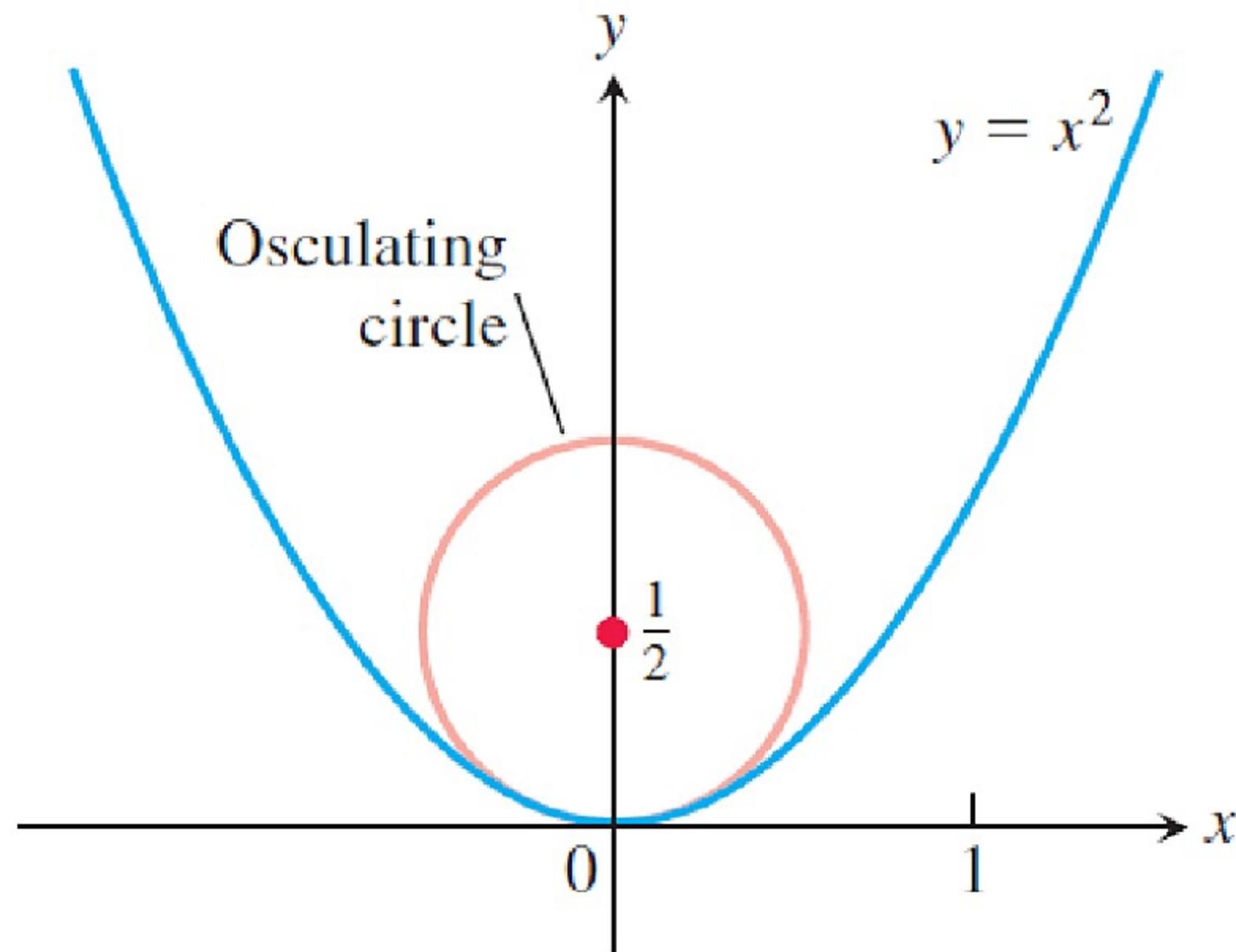


FIGURE 12.21 The osculating circle for the parabola $y = x^2$ at the origin (Example 4).

Curvature and Normal Vectors for Space Curves

If a smooth curve in space is specified by the position vector $\mathbf{r}(t)$ as a function of some parameter t , and if s is the arc length parameter of the curve, then the unit tangent vector \mathbf{T} is $d\mathbf{r}/ds = \mathbf{v}/|\mathbf{v}|$. The **curvature** in space is then defined to be

$$\kappa = \left| \frac{d\mathbf{T}}{ds} \right| = \frac{1}{|\mathbf{v}|} \left| \frac{d\mathbf{T}}{dt} \right| \quad (3)$$

just as for plane curves. The vector $d\mathbf{T}/ds$ is orthogonal to \mathbf{T} , and we define the **principal unit normal** to be

$$\mathbf{N} = \frac{1}{\kappa} \frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}. \quad (4)$$

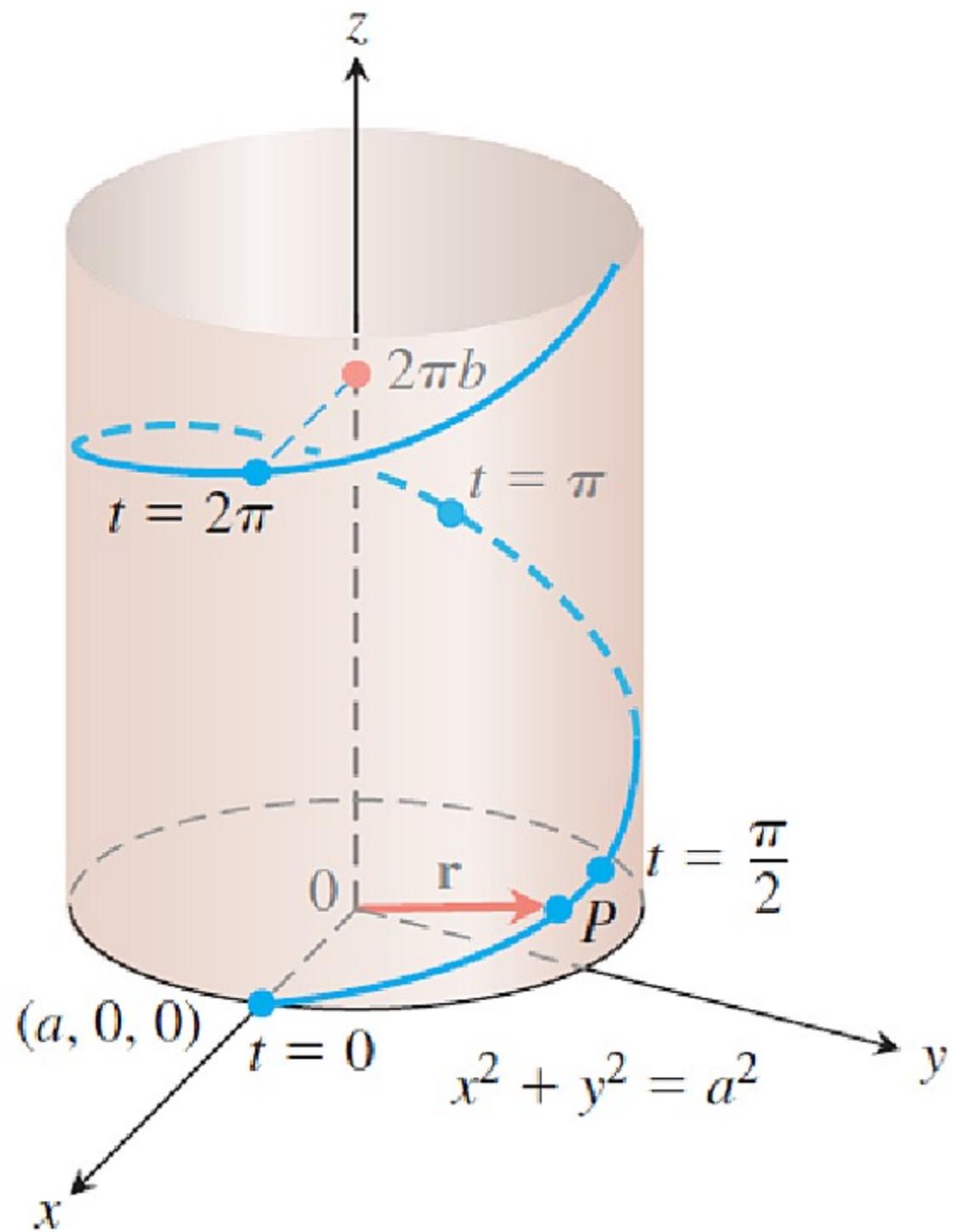


FIGURE 12.22 The helix

$$\mathbf{r}(t) = (a \cos t)\mathbf{i} + (a \sin t)\mathbf{j} + bt\mathbf{k},$$

drawn with a and b positive and $t \geq 0$ (Example 5).

Section 12.5

Tangential and Normal
Components of
Acceleration

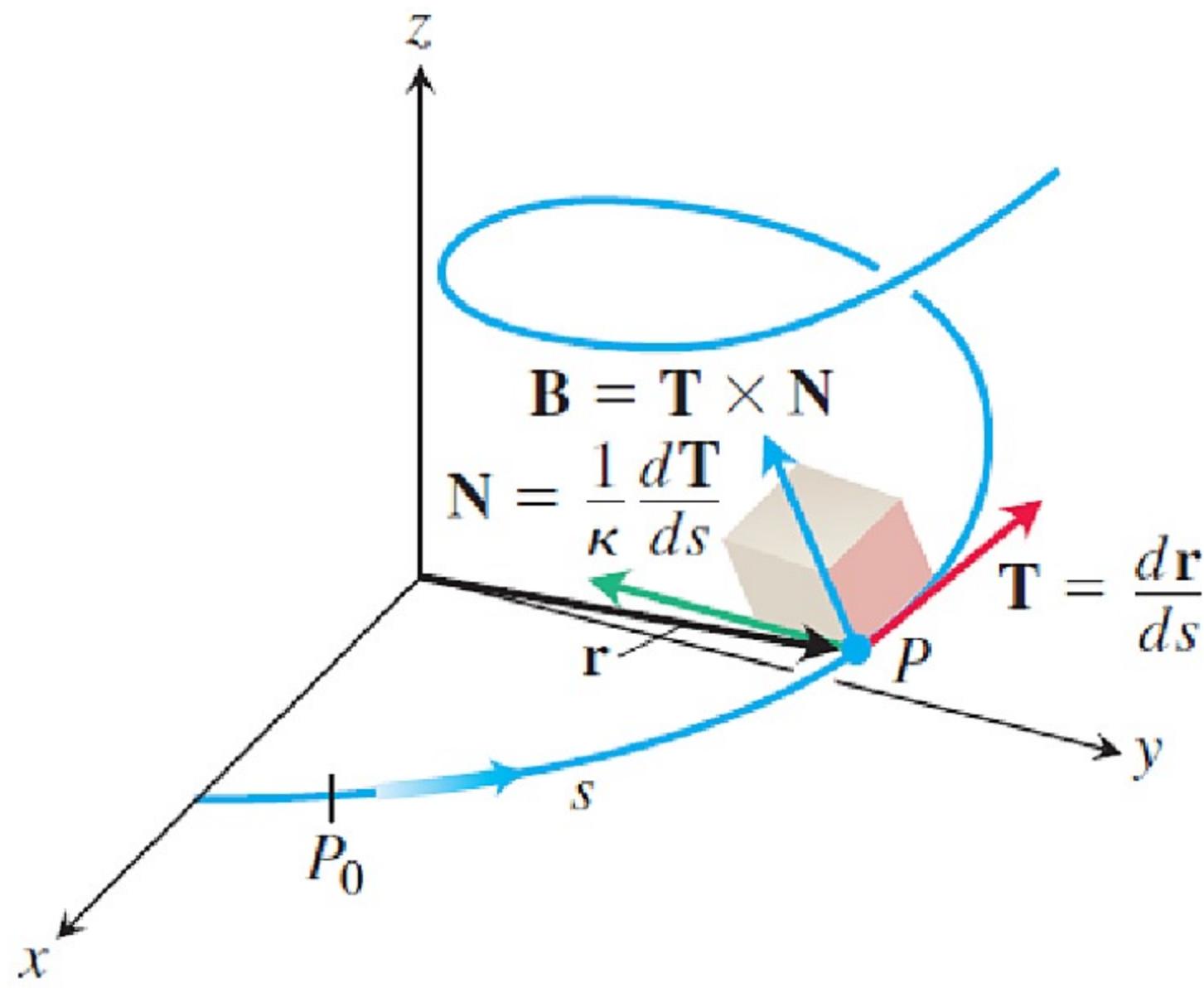


FIGURE 12.23 The **TNB** frame of mutually orthogonal unit vectors traveling along a curve in space.

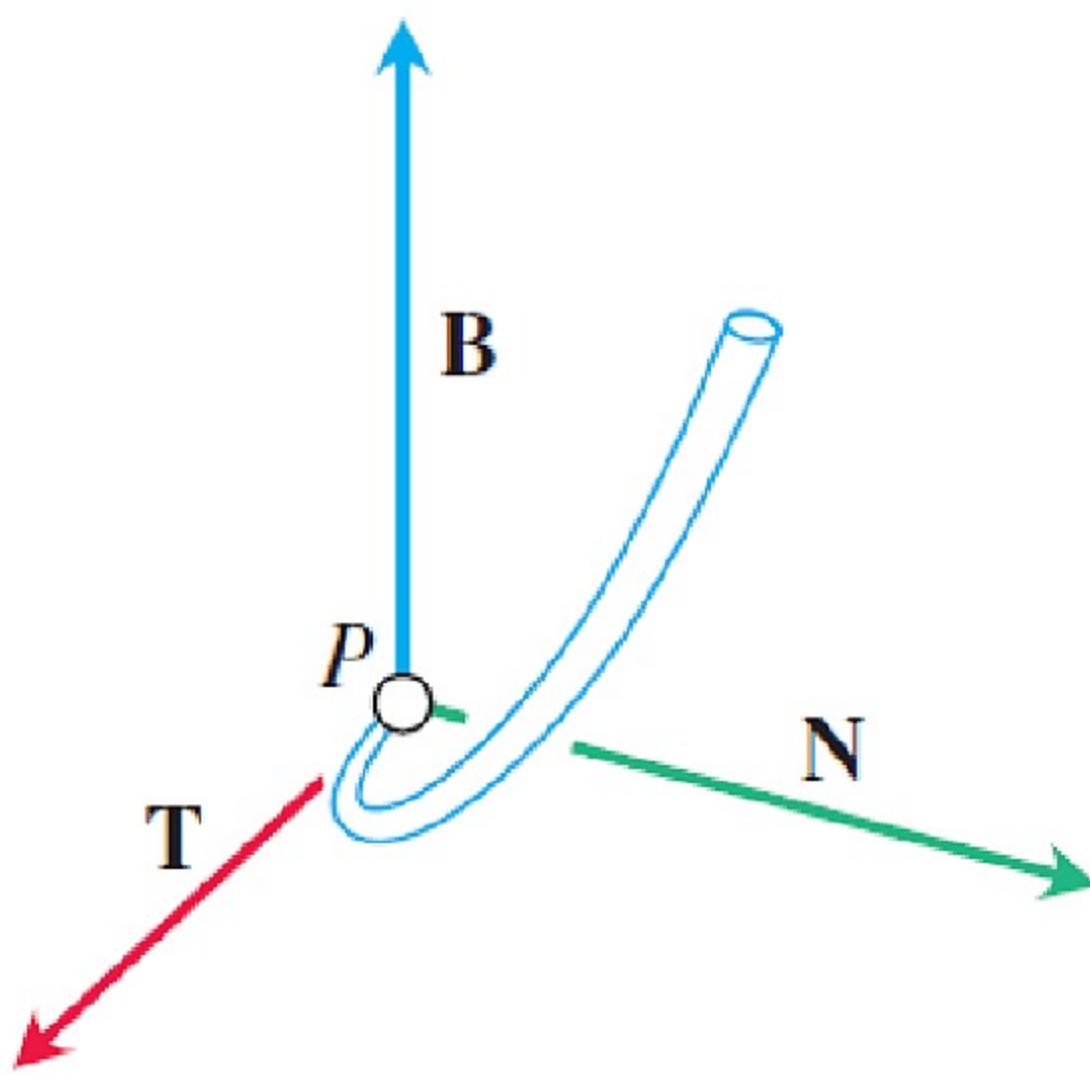


FIGURE 12.24 The vectors \mathbf{T} , \mathbf{N} , and \mathbf{B} (in that order) make a right-handed frame of mutually orthogonal unit vectors in space.

DEFINITION

If the acceleration vector is written as

$$\mathbf{a} = a_T \mathbf{T} + a_N \mathbf{N}, \quad (1)$$

then

$$a_T = \frac{d^2 s}{dt^2} = \frac{d}{dt} |\mathbf{v}| \quad \text{and} \quad a_N = \kappa \left(\frac{ds}{dt} \right)^2 = \kappa |\mathbf{v}|^2 \quad (2)$$

are the **tangential** and **normal** scalar components of acceleration.

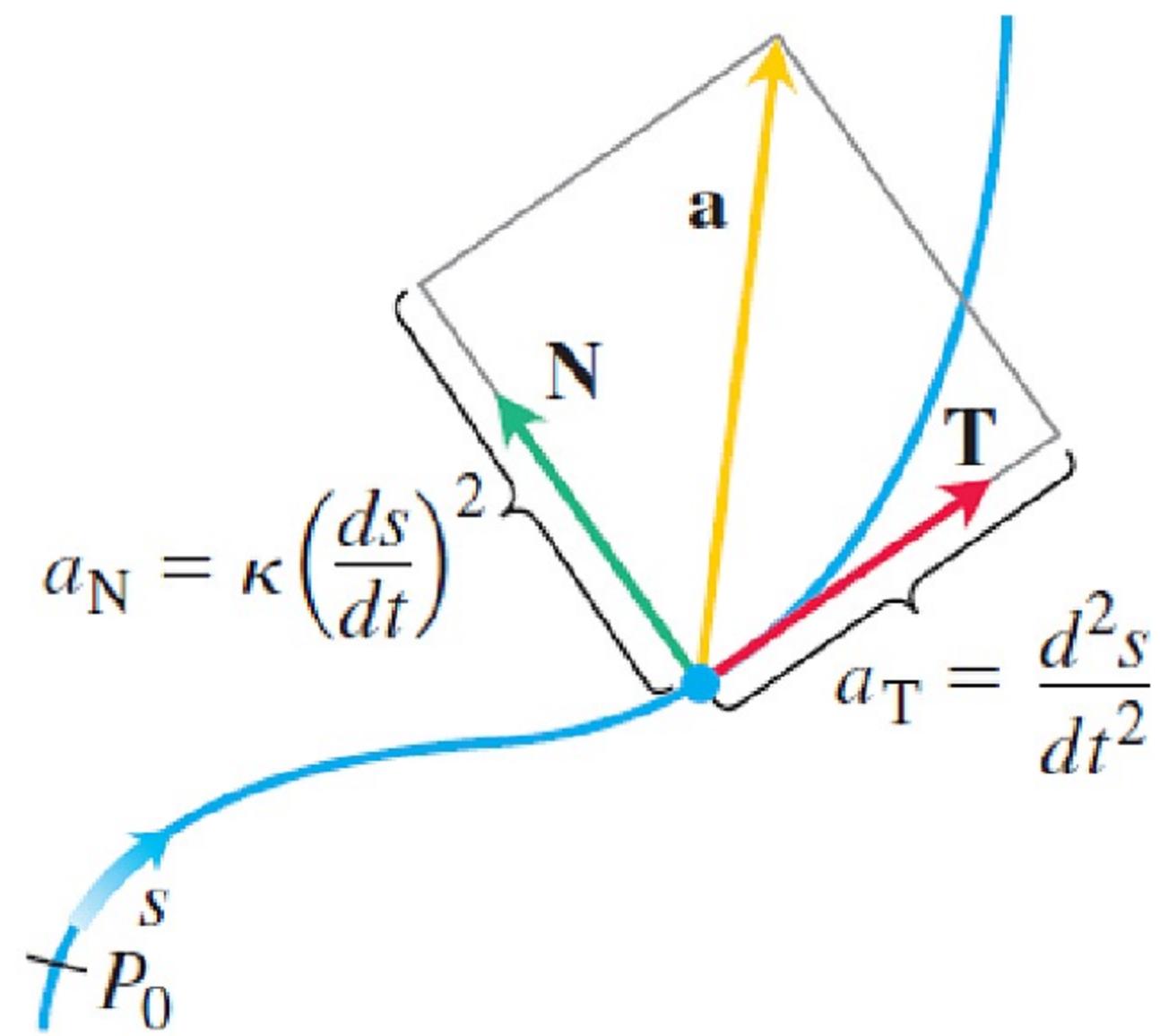


FIGURE 12.25 The tangential and normal components of acceleration. The acceleration **a** always lies in the plane of **T** and **N** and is orthogonal to **B**.

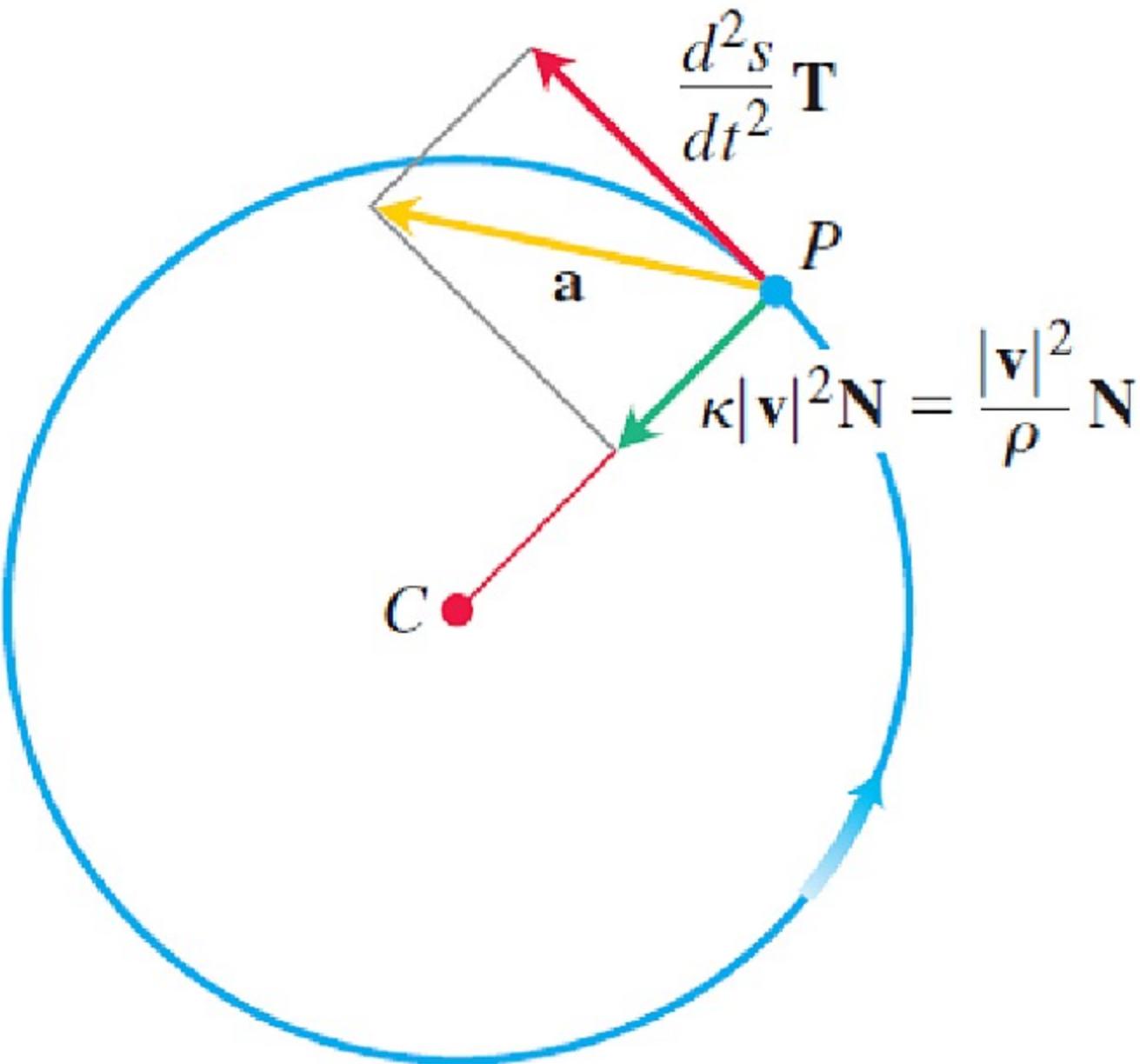


FIGURE 12.26 The tangential and normal components of the acceleration of an object that is speeding up as it moves counterclockwise around a circle of radius ρ .

Formula for Calculating the Normal Component of Acceleration

$$a_N = \sqrt{|\mathbf{a}|^2 - a_T^2} \quad (3)$$

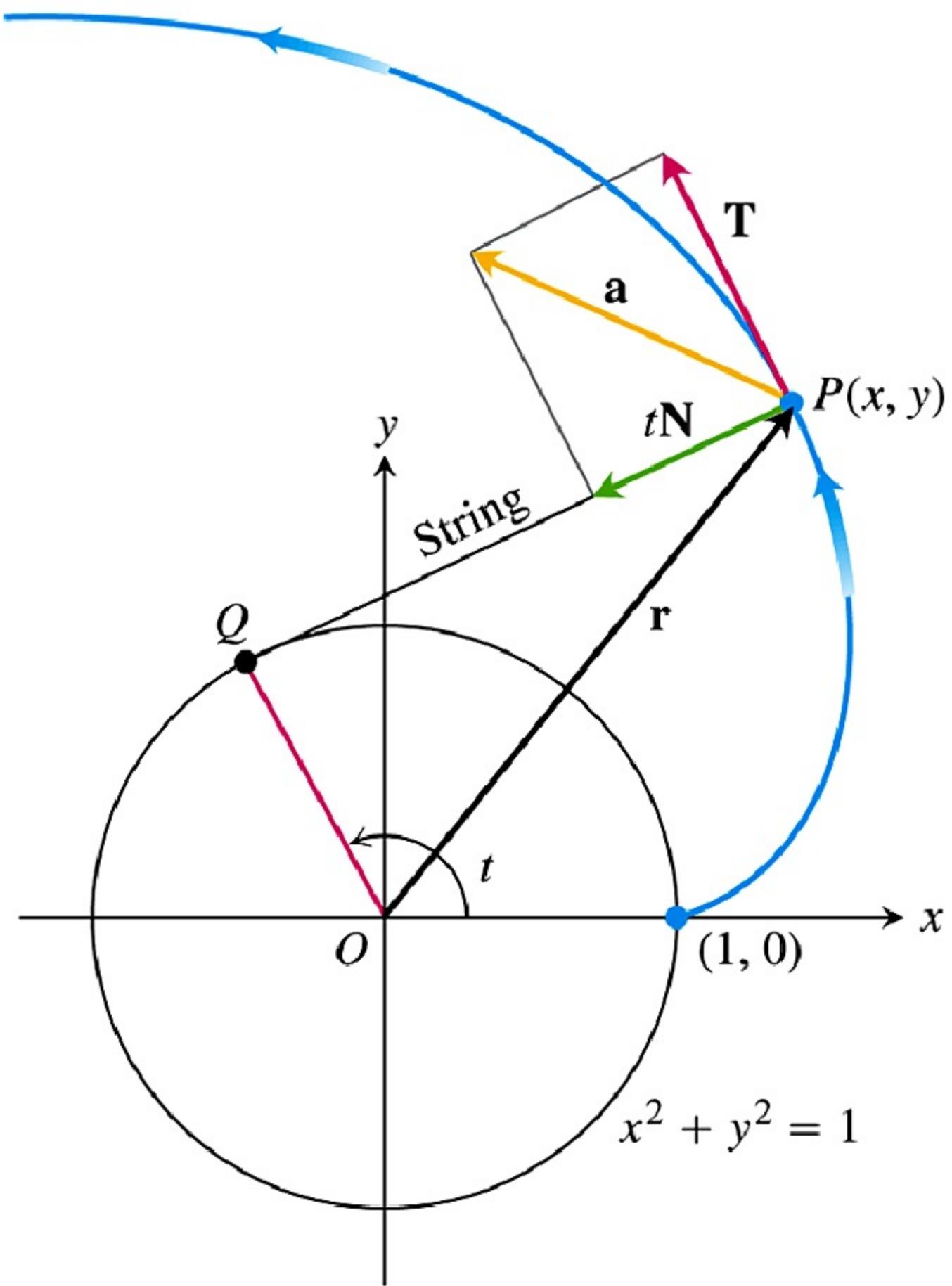


FIGURE 12.27 The tangential and normal components of the acceleration of the motion $\mathbf{r}(t) = (\cos t + t \sin t)\mathbf{i} + (\sin t - t \cos t)\mathbf{j}$, for $t > 0$. If a string wound around a fixed circle is unwound while held taut in the plane of the circle, its end P traces an involute of the circle (Example 1).

DEFINITION Let $\mathbf{B} = \mathbf{T} \times \mathbf{N}$. The **torsion** function of a smooth curve is

$$\tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N}. \quad (4)$$

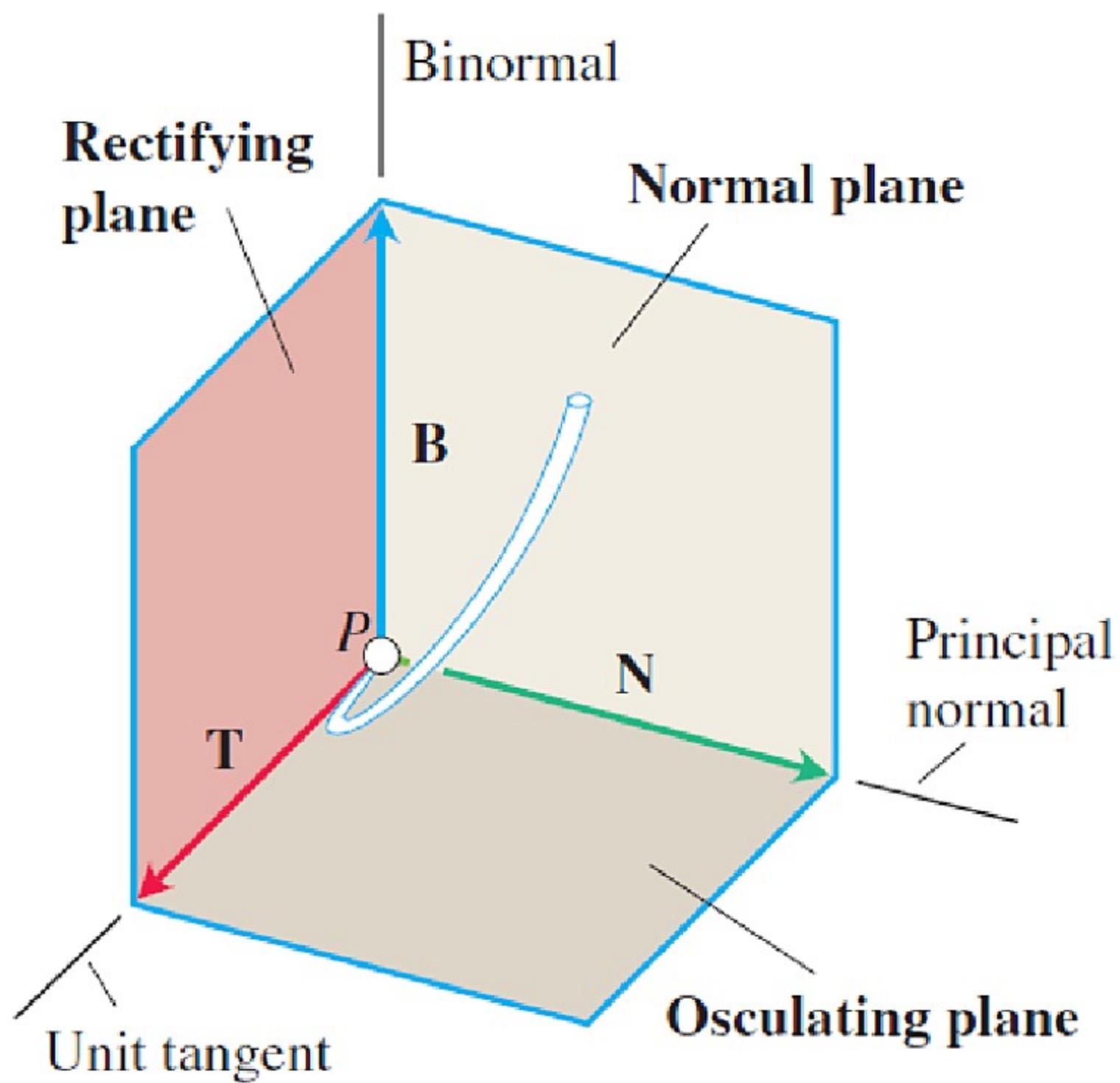


FIGURE 12.28 The names of the three planes determined by T , N , and B .

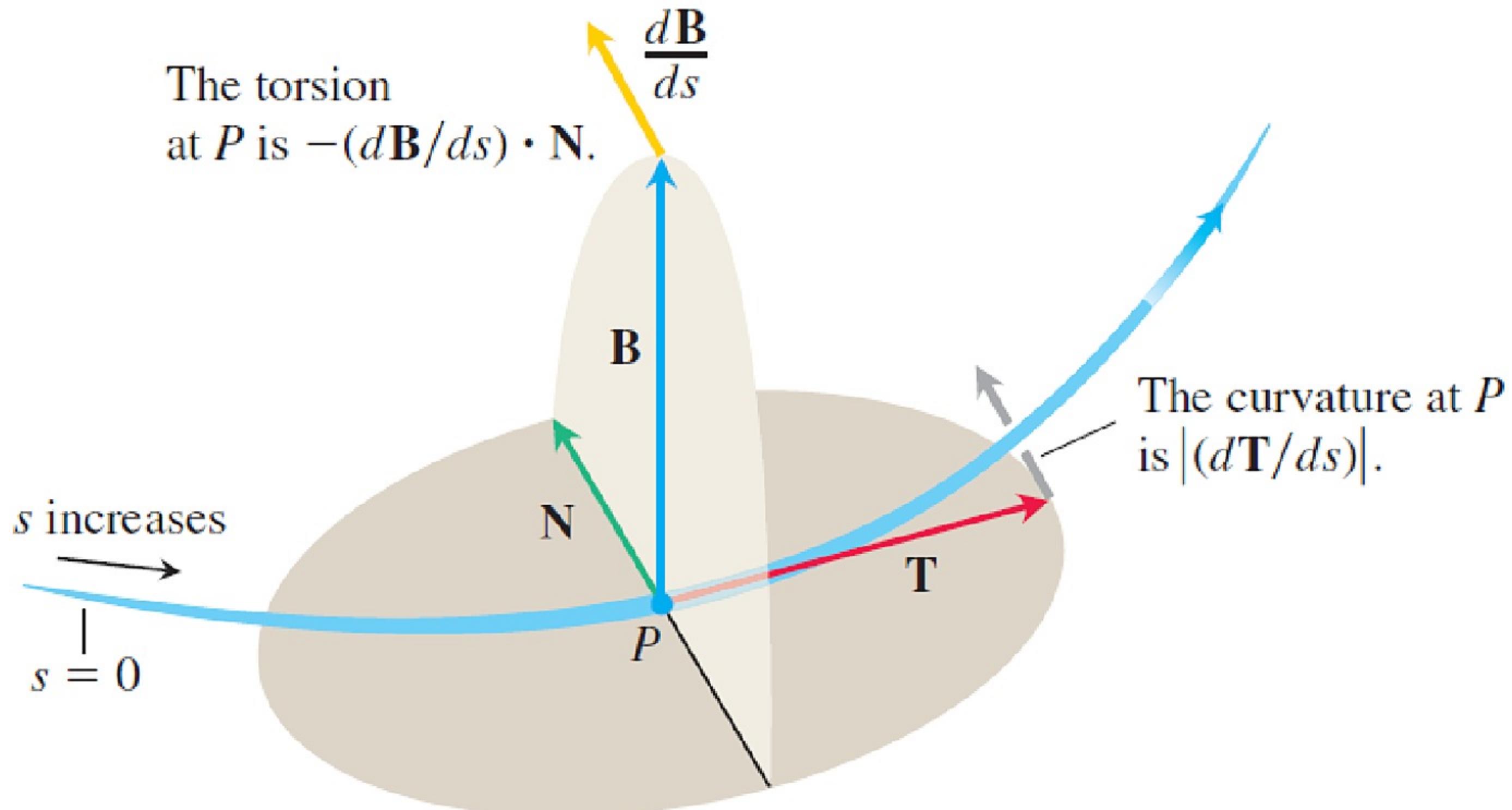


FIGURE 12.29 Every moving body travels with a **TNB** frame that characterizes the geometry of its path of motion.

Vector Formula for Curvature

$$\kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} \quad (5)$$

Formula for Torsion

$$\tau = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \ddot{\dot{x}} & \ddot{\dot{y}} & \ddot{\dot{z}} \end{vmatrix}}{|\mathbf{v} \times \mathbf{a}|^2} \quad (\text{if } \mathbf{v} \times \mathbf{a} \neq \mathbf{0}) \quad (6)$$

Computation Formulas for Curves in Space

Unit tangent vector:

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$$

Principal unit normal vector:

$$\mathbf{N} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$$

Binormal vector:

$$\mathbf{B} = \mathbf{T} \times \mathbf{N}$$

Curvature:

$$\kappa = \left| \frac{d\mathbf{T}}{ds} \right| = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}$$

Torsion:

$$\tau = - \frac{d\mathbf{B}}{ds} \cdot \mathbf{N} = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \end{vmatrix}}{|\mathbf{v} \times \mathbf{a}|^2}$$

Tangential and normal scalar components of acceleration:

$$\mathbf{a} = a_T \mathbf{T} + a_N \mathbf{N}$$

$$a_T = \frac{d}{dt} |\mathbf{v}|$$

$$a_N = \kappa |\mathbf{v}|^2 = \sqrt{|\mathbf{a}|^2 - a_T^2}$$

Section 12.6

Velocity and Acceleration in Polar Coordinates

Thomas' Calculus, 14e in SI Units

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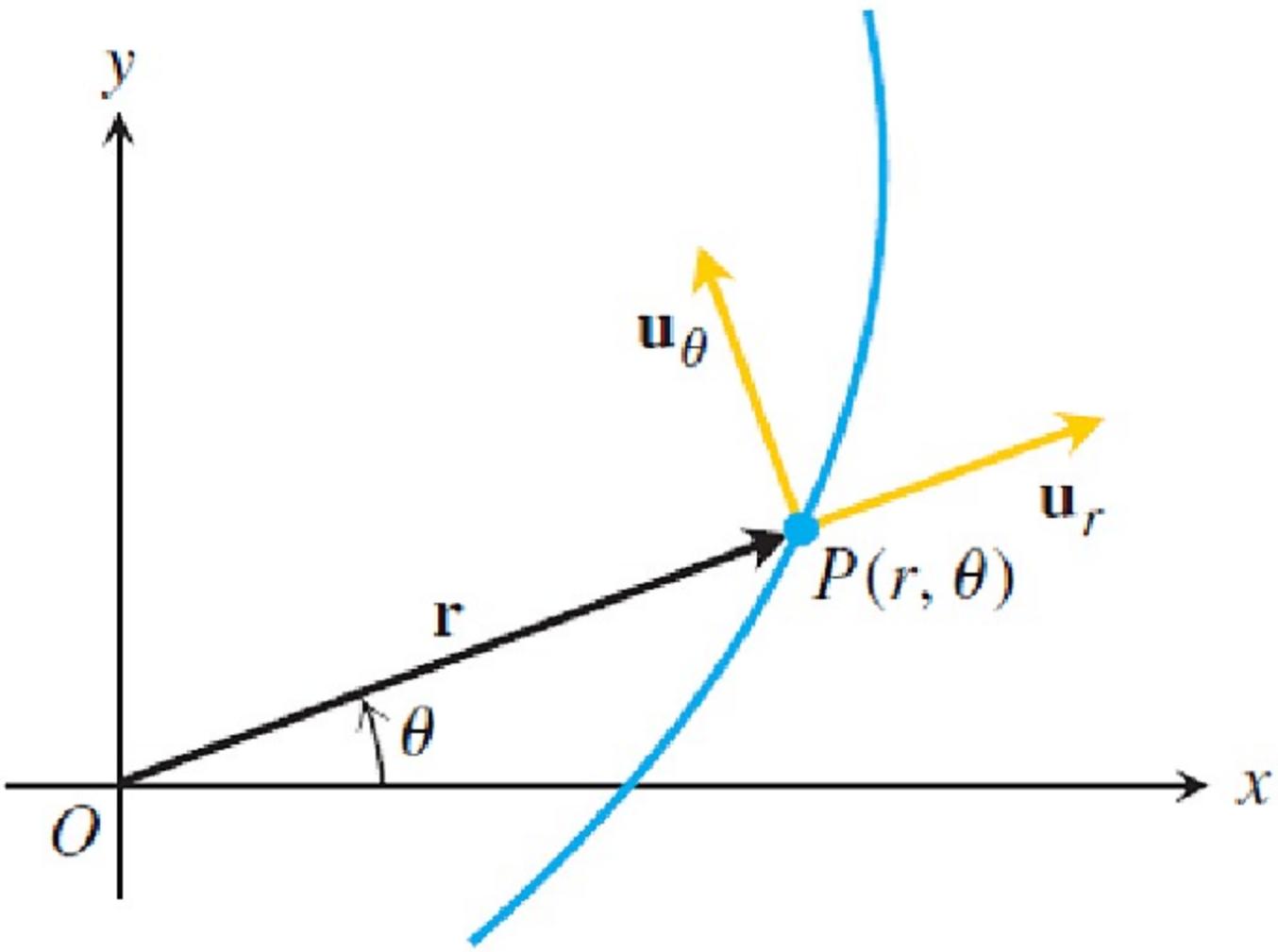


FIGURE 12.30 The length of \mathbf{r} is the positive polar coordinate r of the point P . Thus, \mathbf{u}_r , which is $\mathbf{r}/|\mathbf{r}|$, is also \mathbf{r}/r . Equations (1) express \mathbf{u}_r and \mathbf{u}_θ in terms of \mathbf{i} and \mathbf{j} .

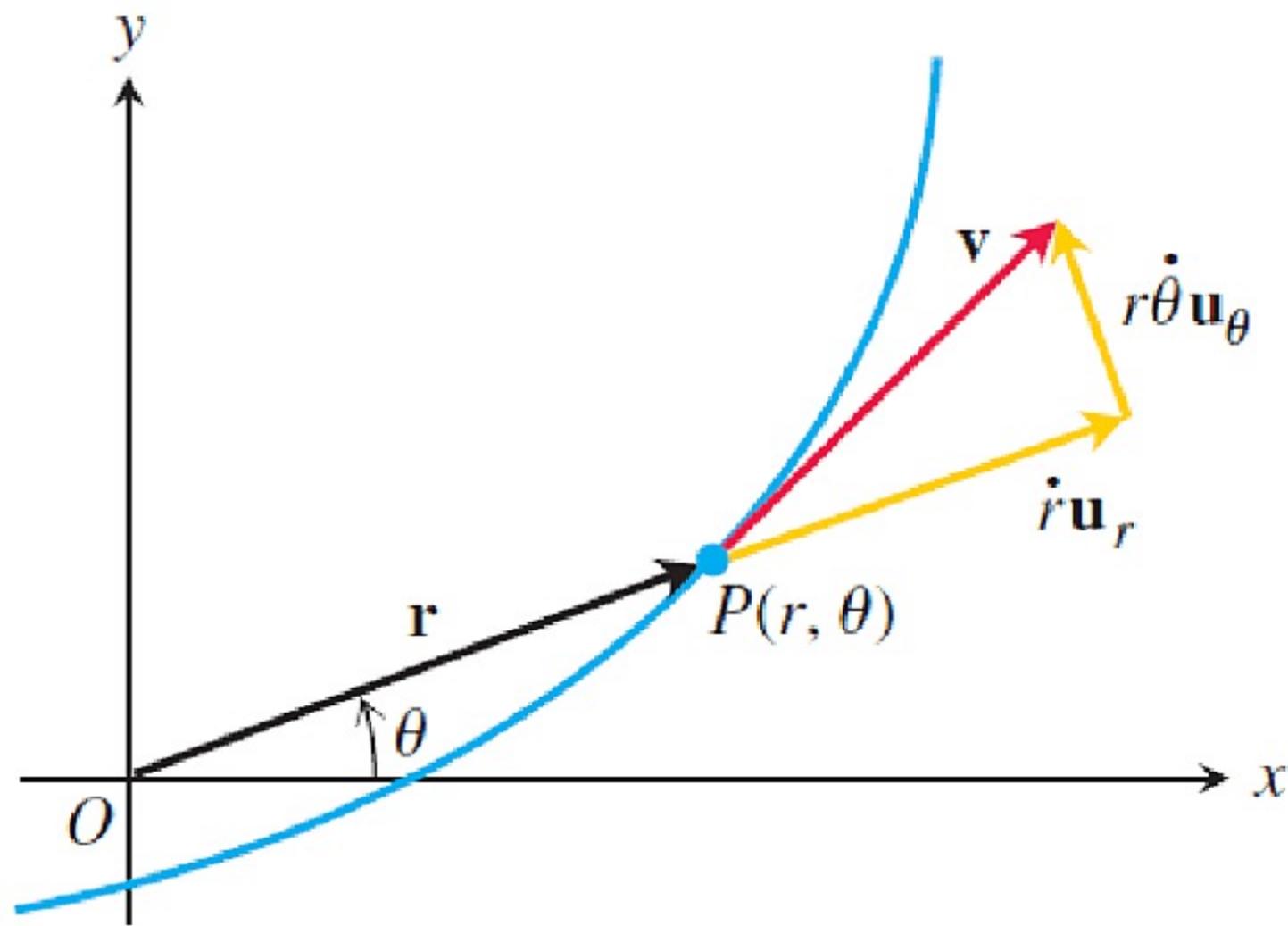


FIGURE 12.31 In polar coordinates, the velocity vector is

$$\mathbf{v} = \dot{r} \mathbf{u}_r + r \dot{\theta} \mathbf{u}_\theta.$$

Position:

$$\mathbf{r} = r\mathbf{u}_r + z\mathbf{k}$$

Velocity:

$$\mathbf{v} = \dot{r}\mathbf{u}_r + r\dot{\theta}\mathbf{u}_\theta + \dot{z}\mathbf{k} \quad (3)$$

Acceleration:

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\mathbf{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{u}_\theta + \ddot{z}\mathbf{k}$$

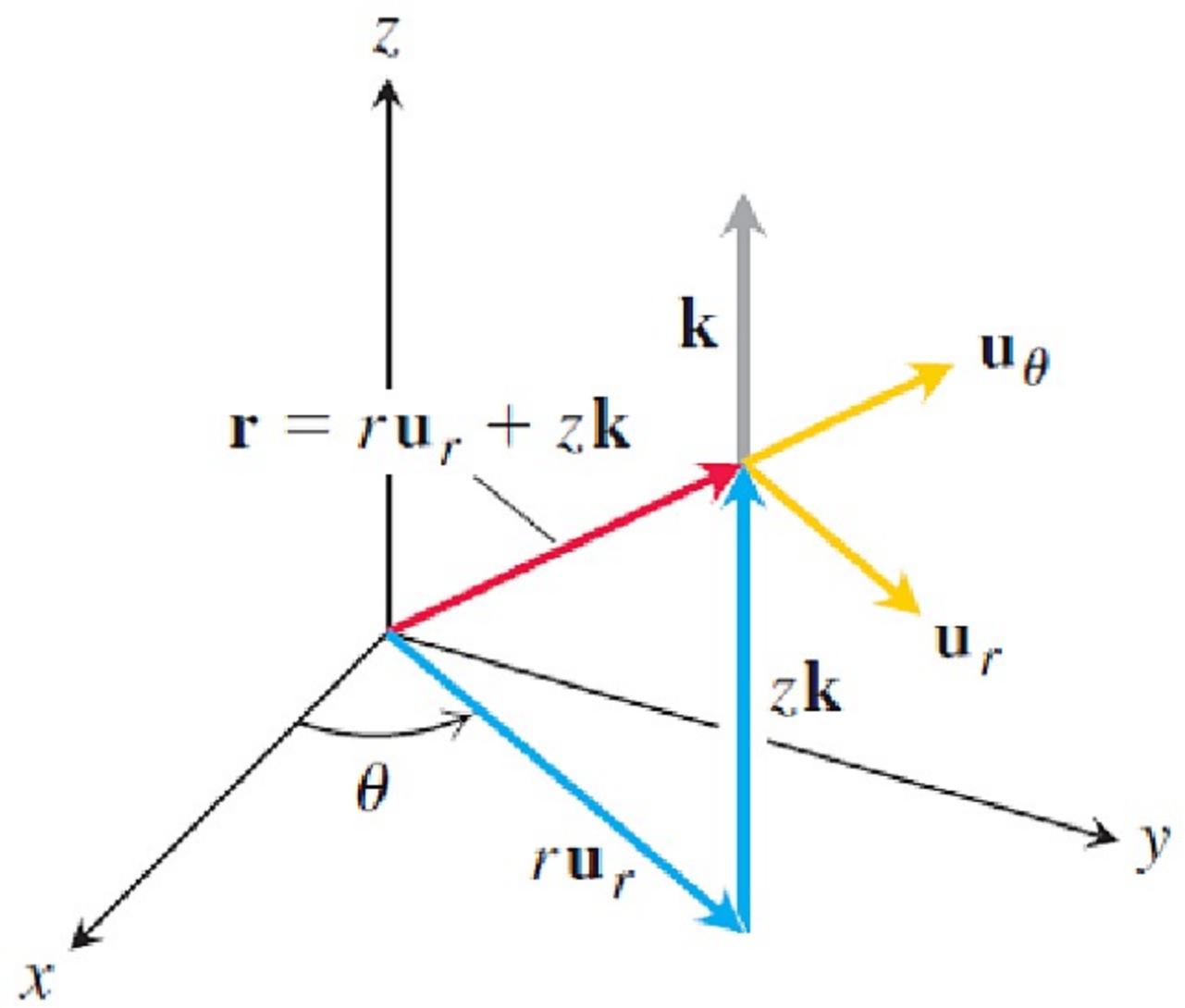


FIGURE 12.32 Position vector and basic unit vectors in cylindrical coordinates. Notice that $|\mathbf{r}| \neq r$ if $z \neq 0$ since $|\mathbf{r}| = \sqrt{r^2 + z^2}$.

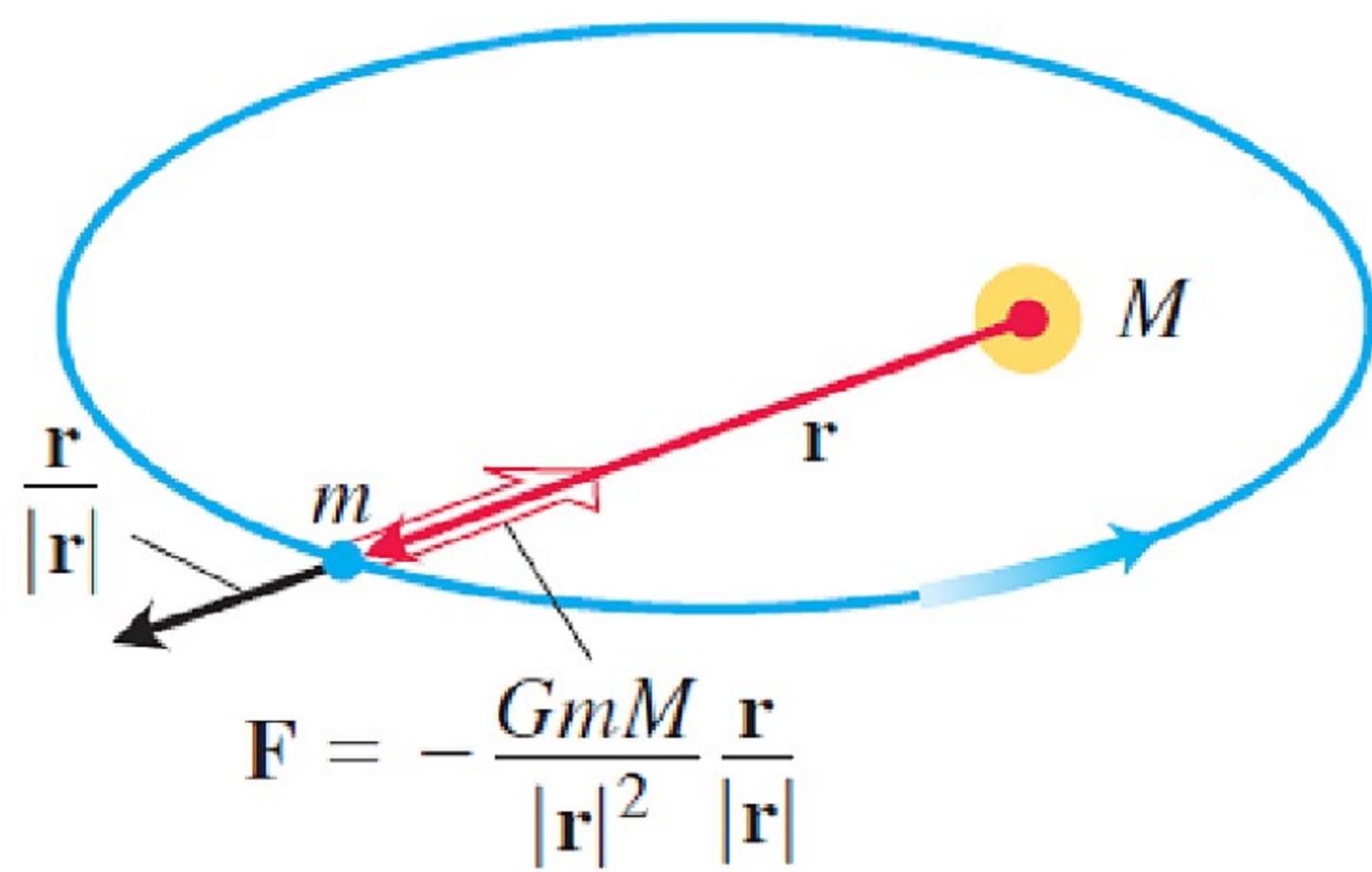


FIGURE 12.33 The force of gravity is directed along the line joining the centers of mass.

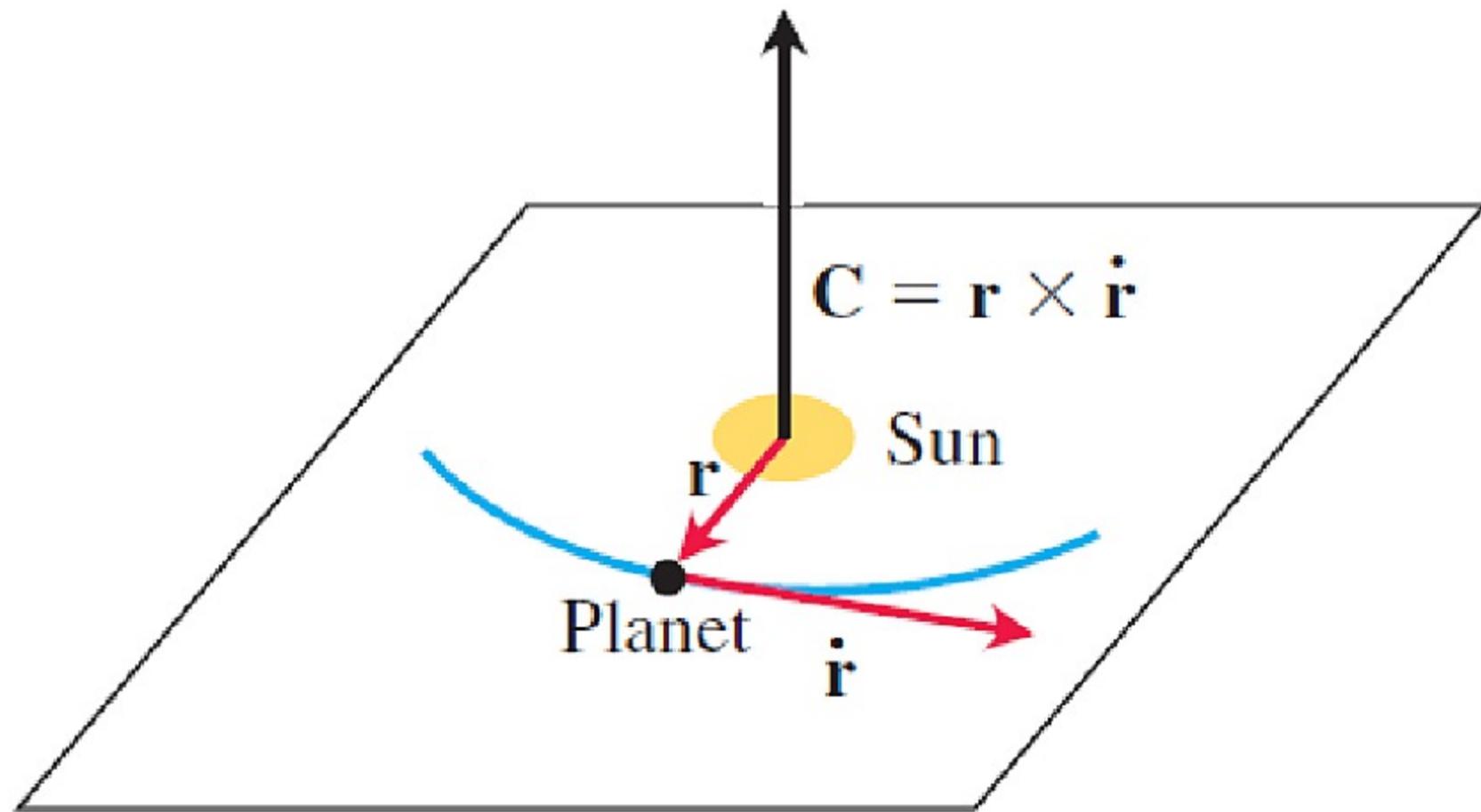


FIGURE 12.34 A planet that obeys Newton's laws of gravitation and motion travels in the plane through the sun's center of mass perpendicular to $\mathbf{C} = \mathbf{r} \times \dot{\mathbf{r}}$.

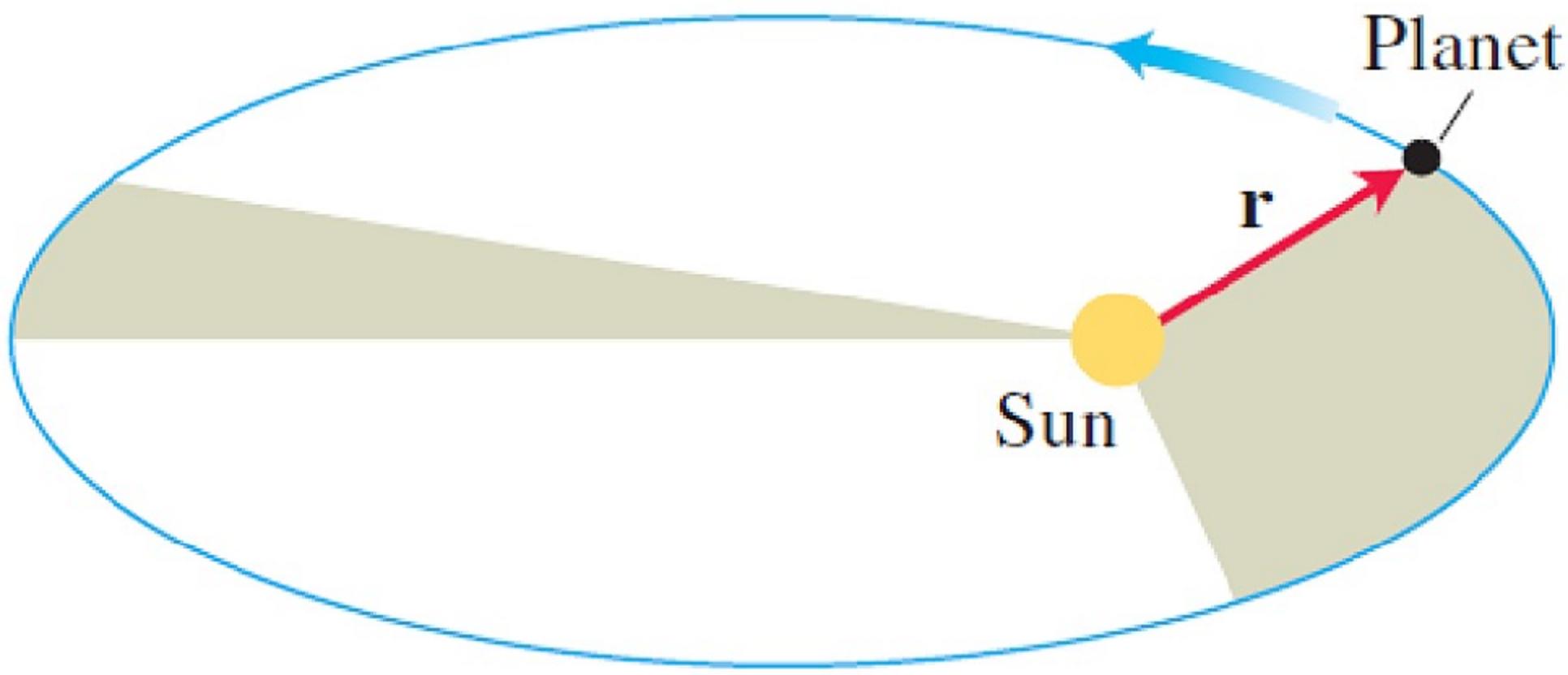


FIGURE 12.35 The line joining a planet to its sun sweeps over equal areas in equal times.

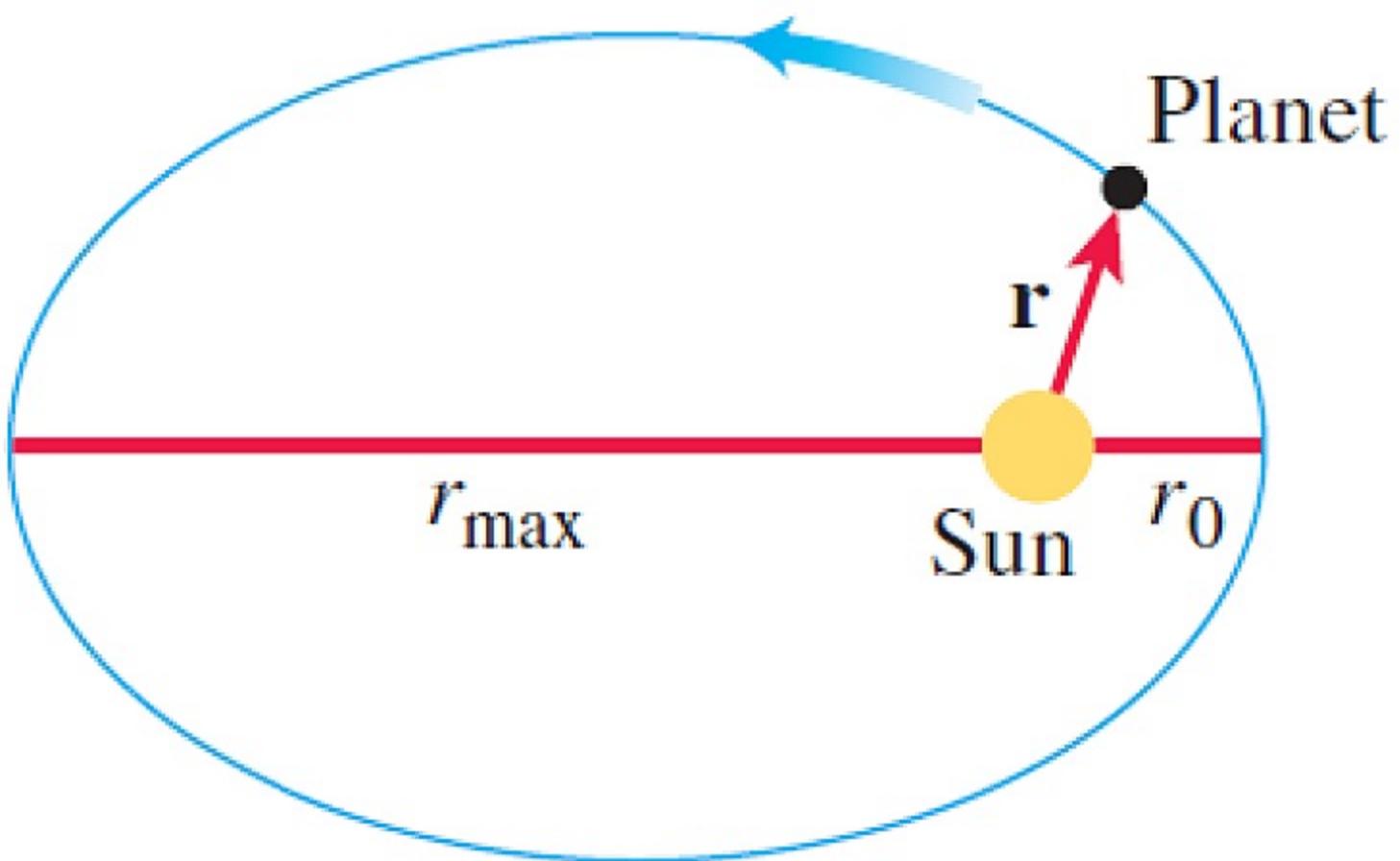


FIGURE 12.36 The length of the major axis of the ellipse is $2a = r_0 + r_{\max}$.

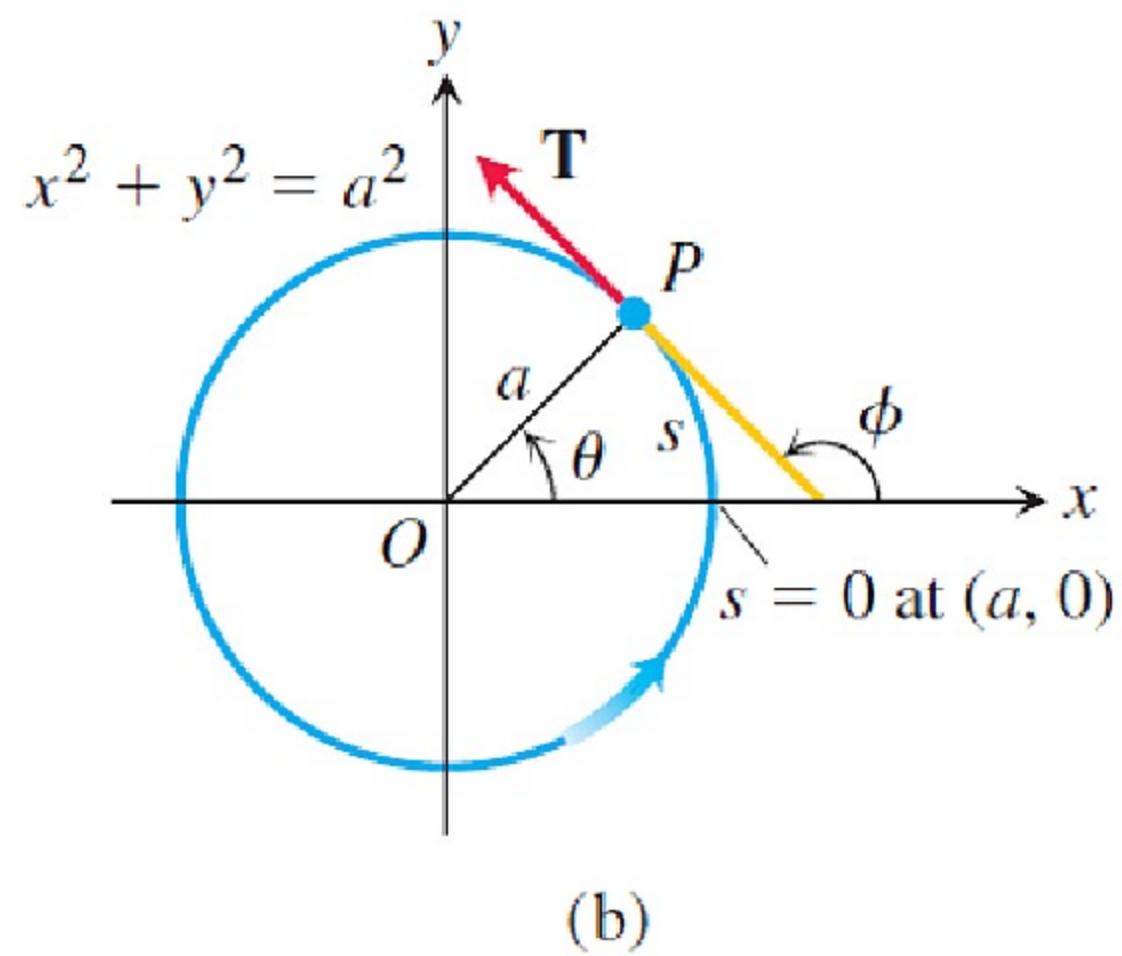
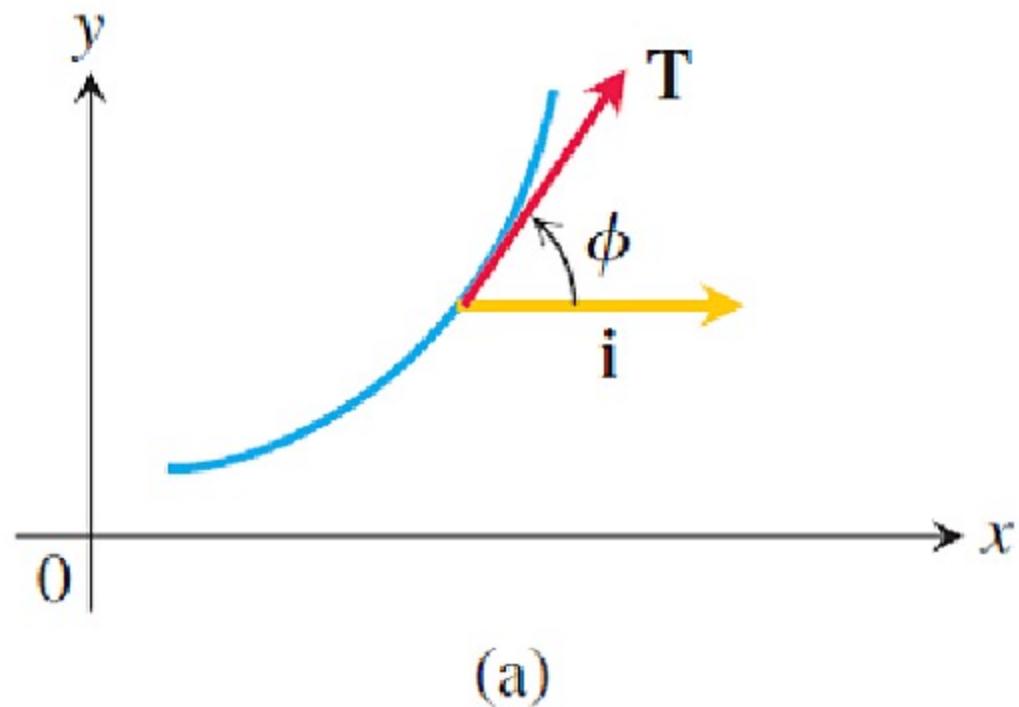


FIGURE 12.37 Figures for Exercise 31.