Chapter 15

Integrals and Vector Fields

Section 15.1

Line Integrals of Scalar Functions

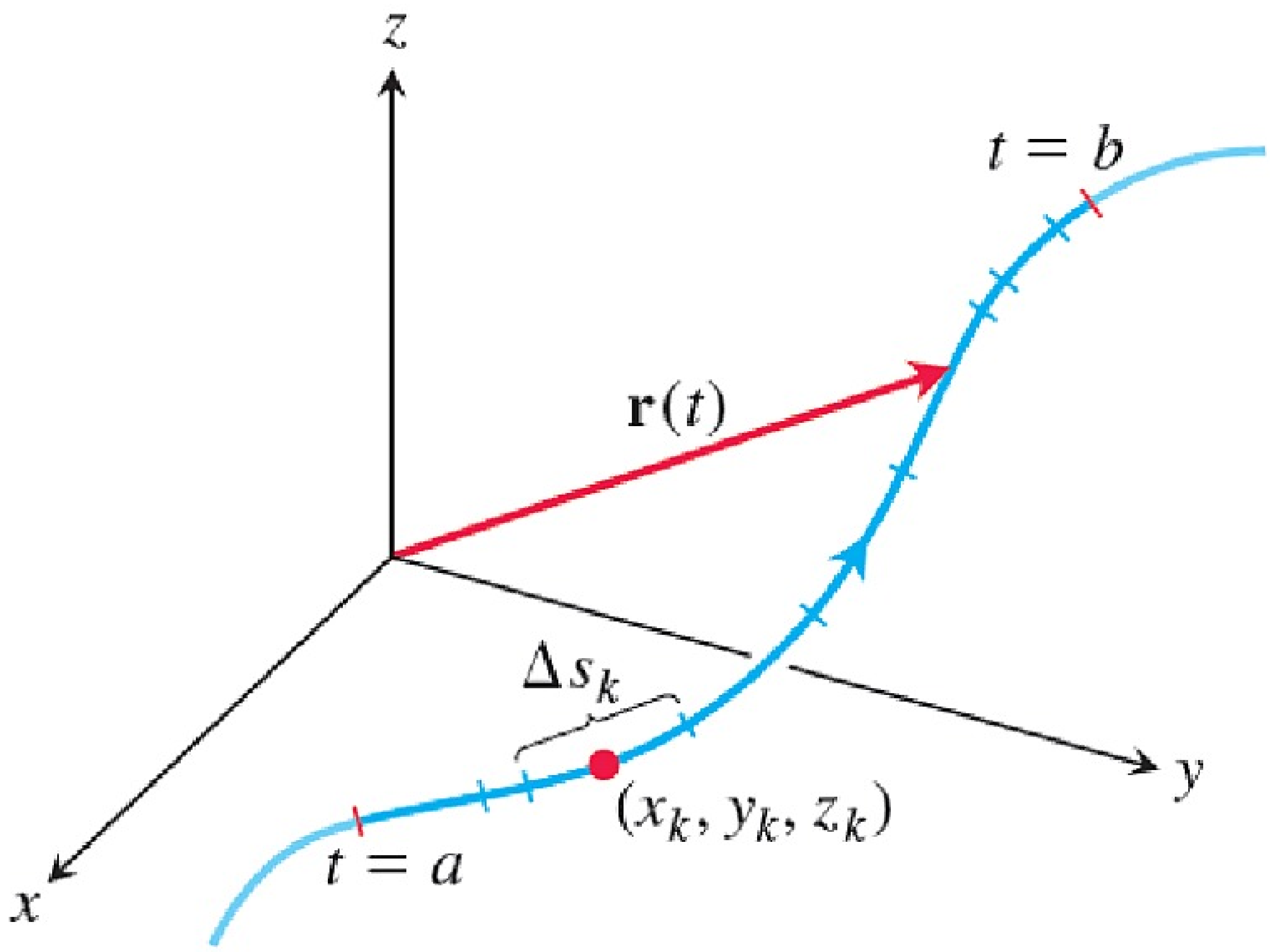


FIGURE 15.1 The curve $\mathbf{r}(t)$ partitioned into small arcs from $t = a$ to $t = b$. The length of a typical subarc is Δs_k.

DEFINITION If f is defined on a curve C given parametrically by $\mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}$, $a \le t \le b$, then the **line integral of f over C** is

$$\int_C f(x, y, z)\, ds = \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k, y_k, z_k)\, \Delta s_k\,, \tag{1}$$

provided this limit exists.

How to Evaluate a Line Integral

To integrate a continuous function $f(x, y, z)$ over a curve C:

1. Find a smooth parametrization of C,

$$\mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}, \qquad a \le t \le b.$$

2. Evaluate the integral as

$$\int_C f(x, y, z)\, ds = \int_a^b f(g(t), h(t), k(t))\,|\mathbf{v}(t)|\, dt.$$

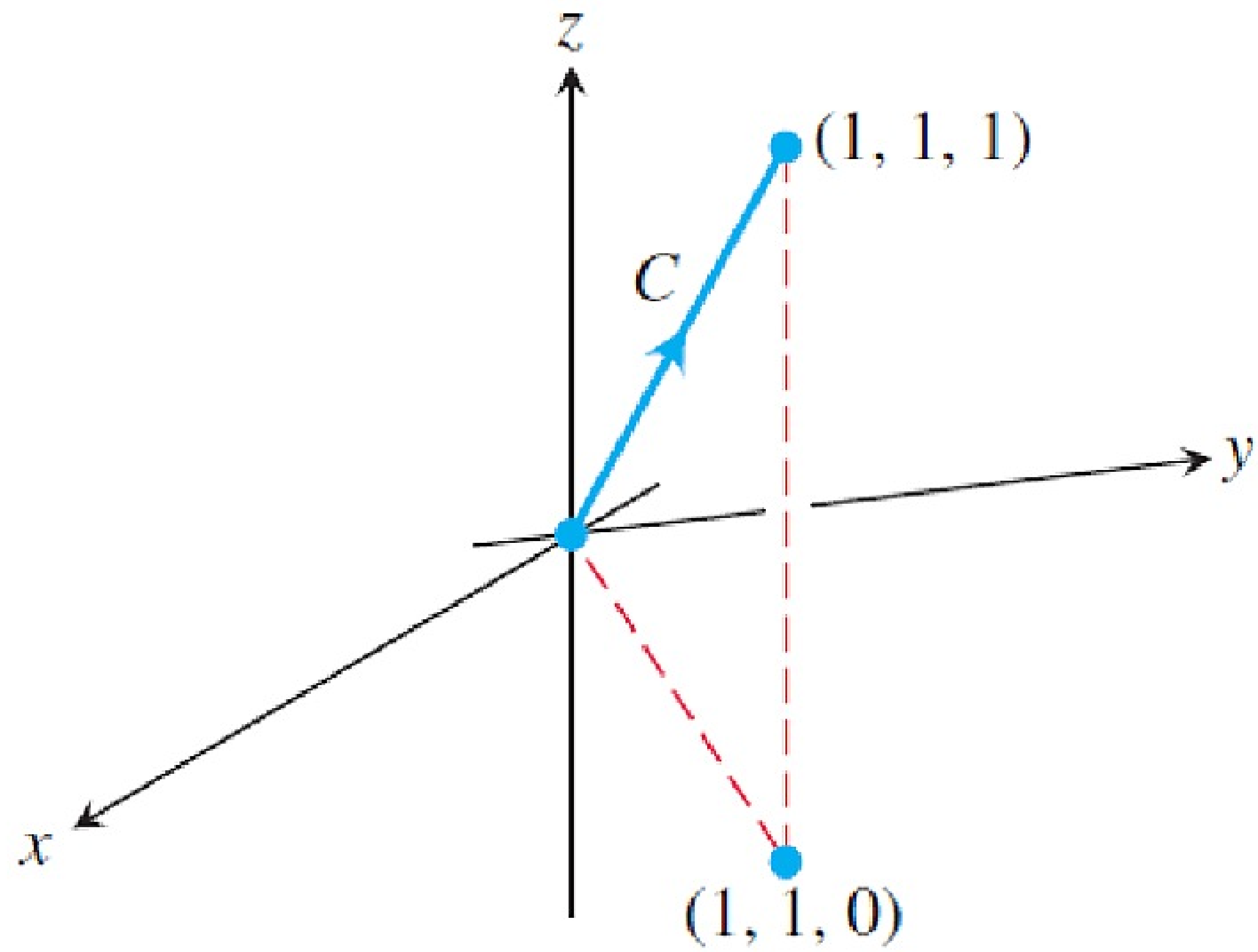


FIGURE 15.2 The integration path in Example 1.

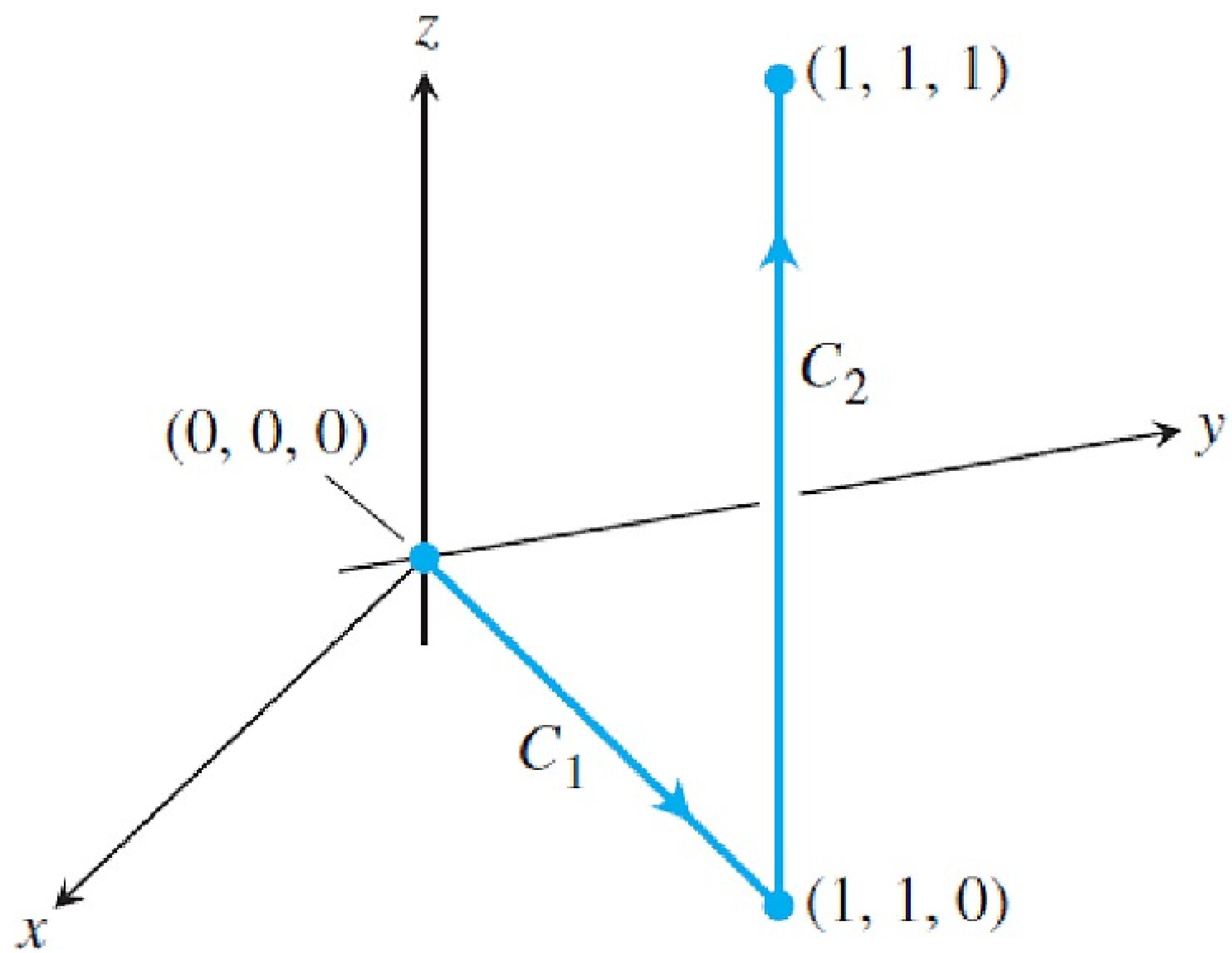


FIGURE 15.3 The path of integration in Example 2.

The value of the line integral along a path joining two points can change if you change the path between them.

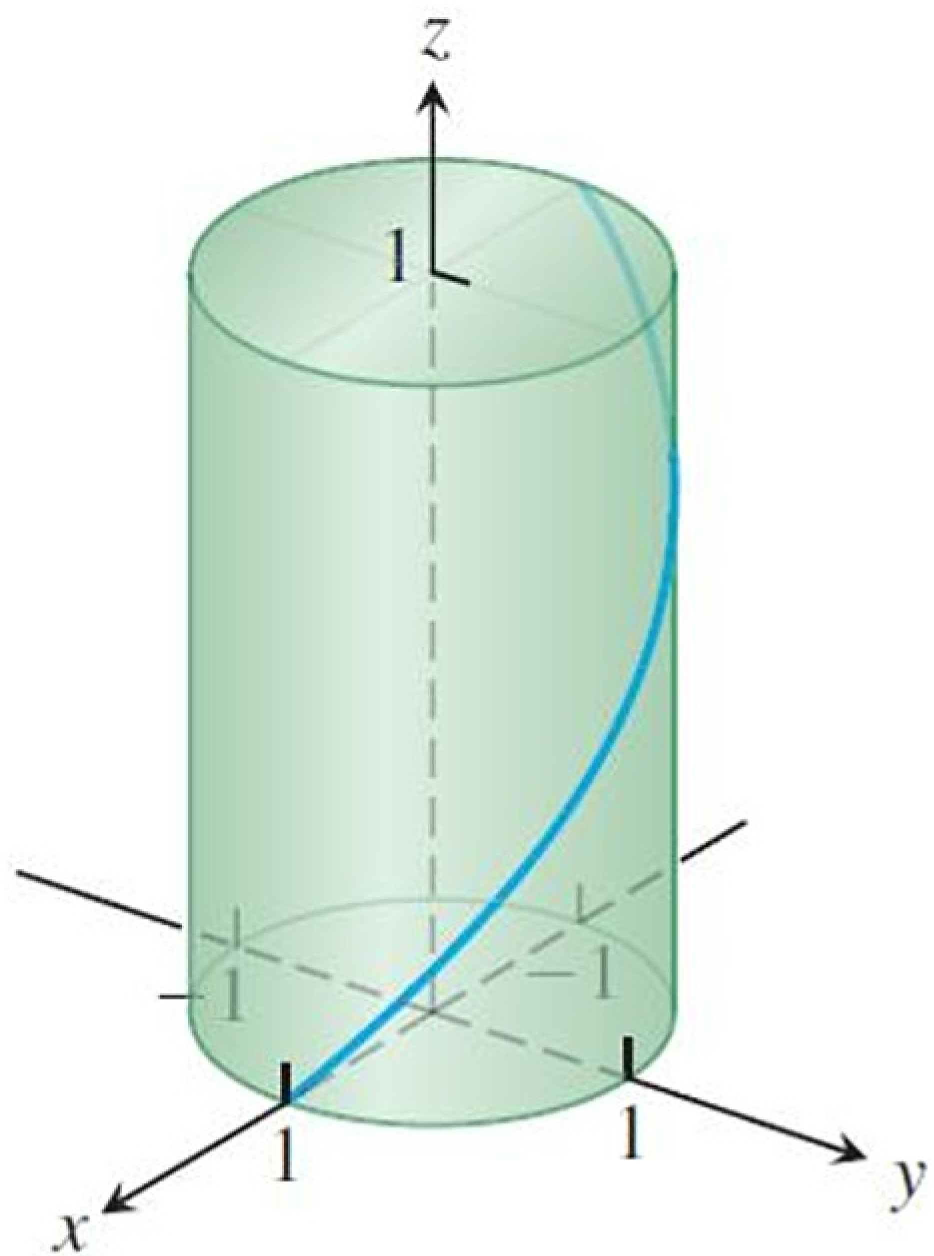


FIGURE 15.4 A line integral is taken over a curve such as this helix from Example 3.

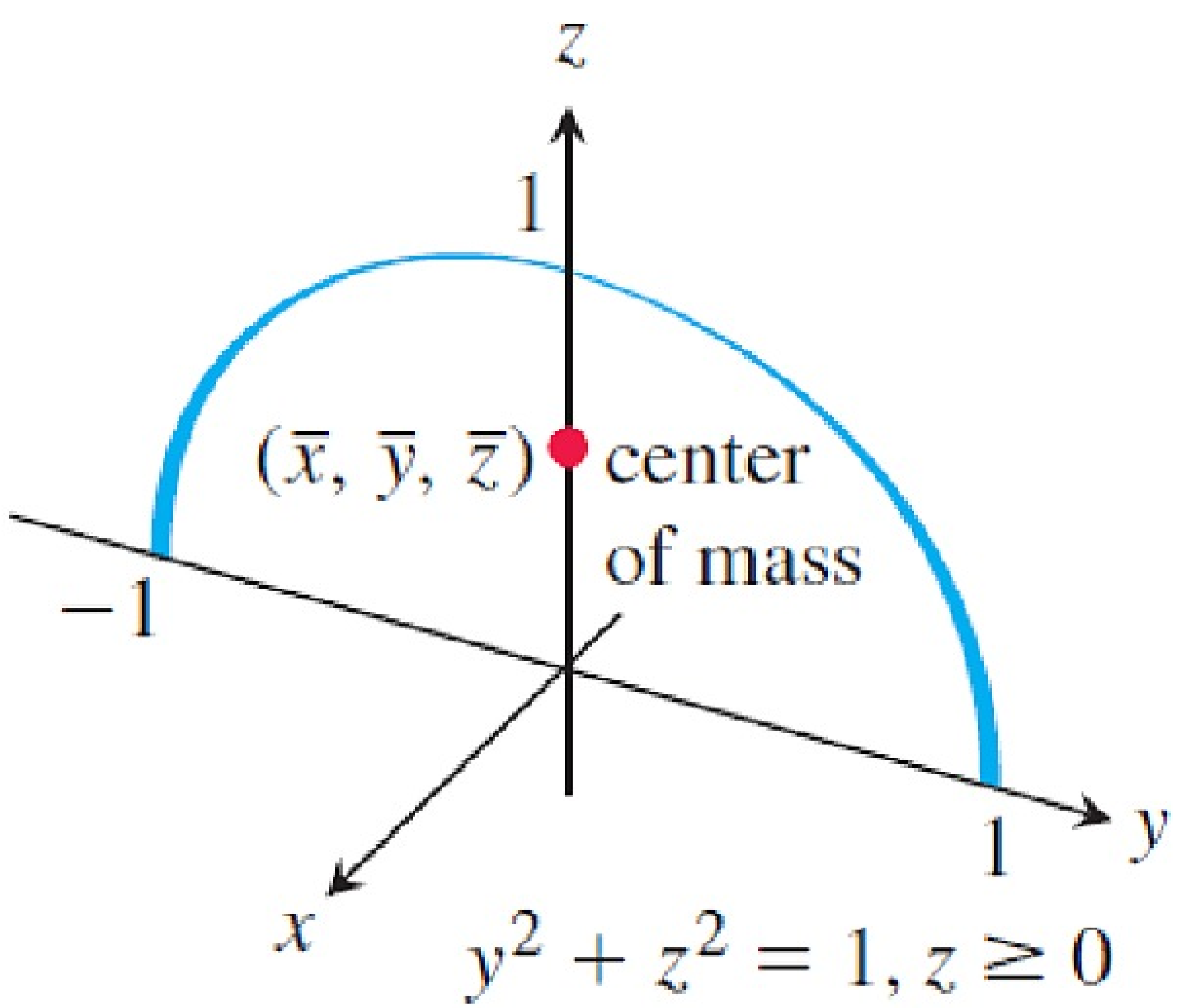


FIGURE 15.5 Example 4 shows how to find the center of mass of a circular arch of variable density.

TABLE 15.1 Mass and moment formulas for coil springs, wires, and thin rods lying along a smooth curve C in space

Mass: $M = \int_C \delta \, ds$ $\quad \delta = \delta(x, y, z)$ is the density at (x, y, z)

First moments about the coordinate planes:

$$M_{yz} = \int_C x\,\delta\, ds, \qquad M_{xz} = \int_C y\,\delta\, ds, \qquad M_{xy} = \int_C z\,\delta\, ds$$

Coordinates of the center of mass:

$$\bar{x} = M_{yz}/M, \qquad \bar{y} = M_{xz}/M, \qquad \bar{z} = M_{xy}/M$$

Moments of inertia about axes and other lines:

$$I_x = \int_C (y^2 + z^2)\,\delta\, ds, \qquad I_y = \int_C (x^2 + z^2)\,\delta\, ds, \qquad I_z = \int_C (x^2 + y^2)\,\delta\, ds,$$

$$I_L = \int_C r^2 \delta\, ds$$ $\quad r(x, y, z) =$ distance from the point (x, y, z) to line L

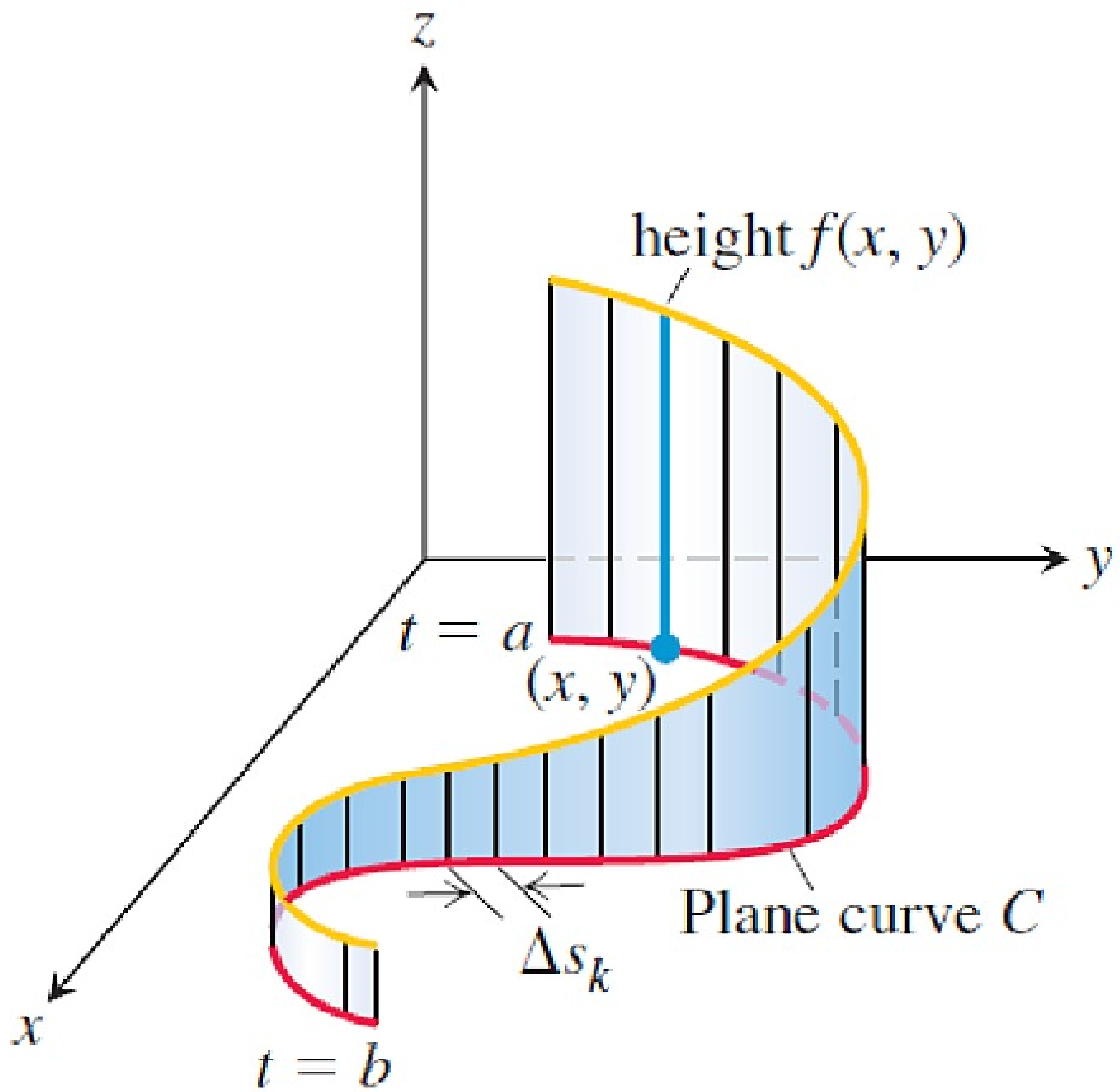


FIGURE 15.6 The line integral $\int_C f\, ds$ gives the area of the portion of the cylindrical surface or "wall" beneath $z = f(x, y) \geq 0$.

Section 15.2

Vector Fields and Line Integrals: Work, Circulation, and Flux

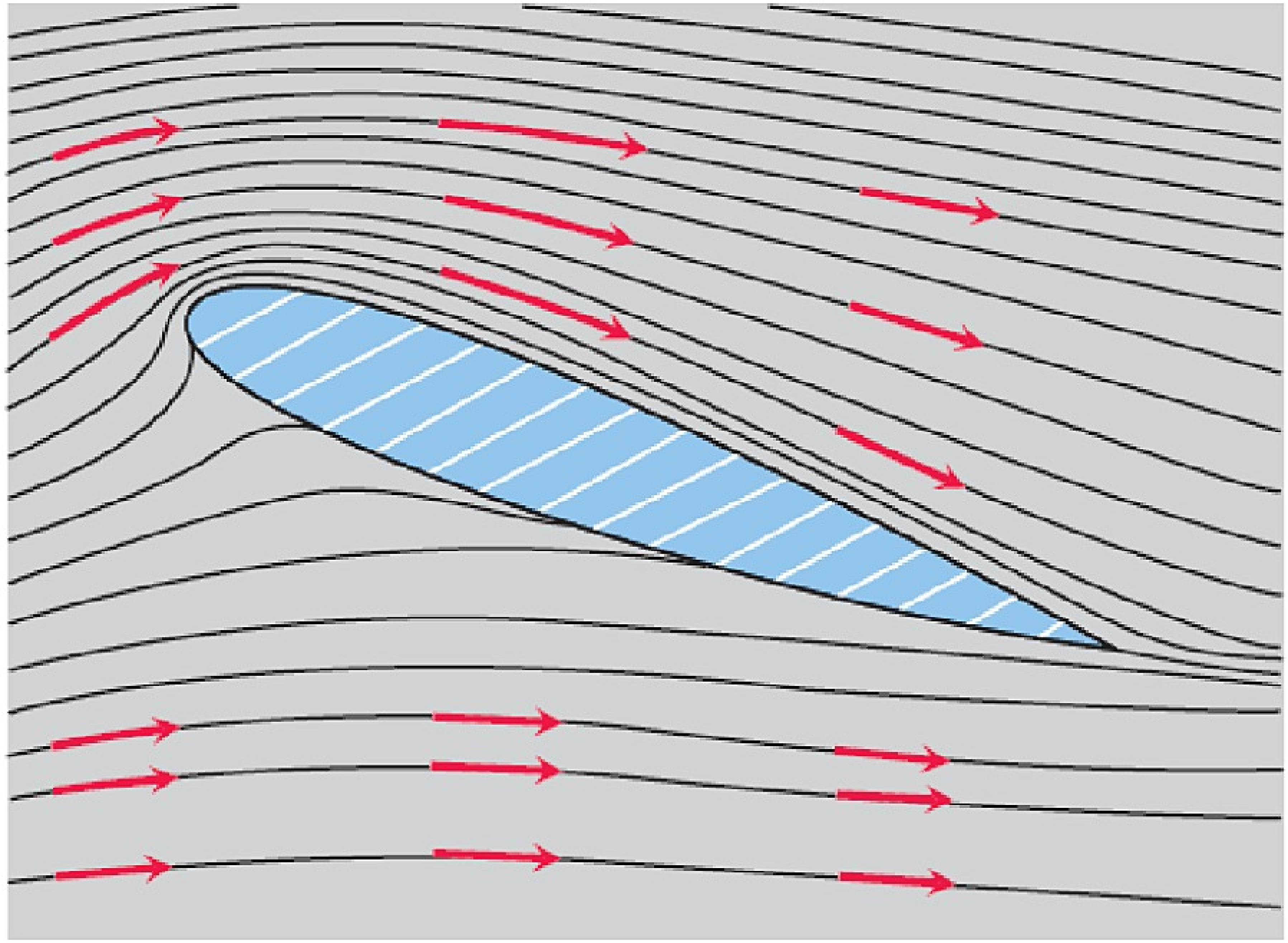


FIGURE 15.7 Velocity vectors of a flow around an airfoil in a wind tunnel.

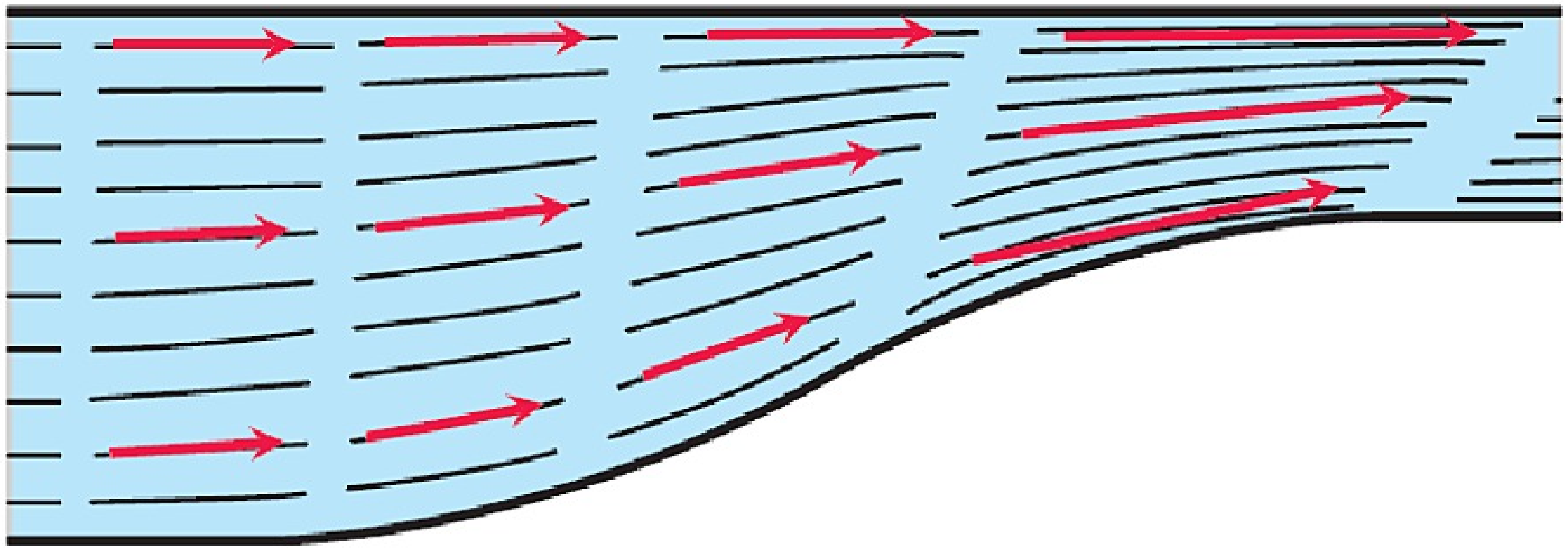


FIGURE 15.8 Streamlines in a contracting channel. The water speeds up as the channel narrows and the velocity vectors increase in length.

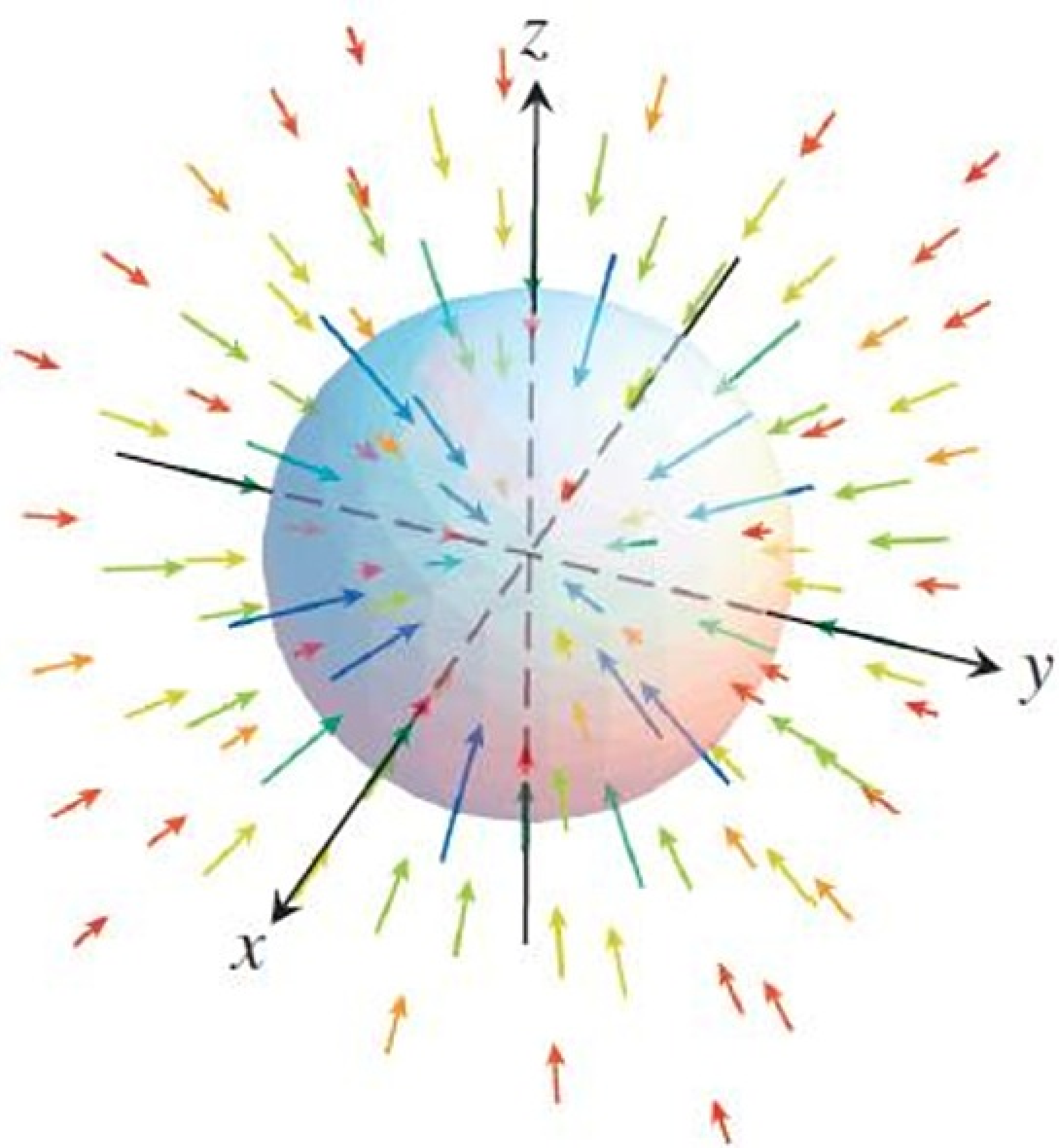


FIGURE 15.9 Vectors in a gravitational field point toward the center of mass that gives the source of the field.

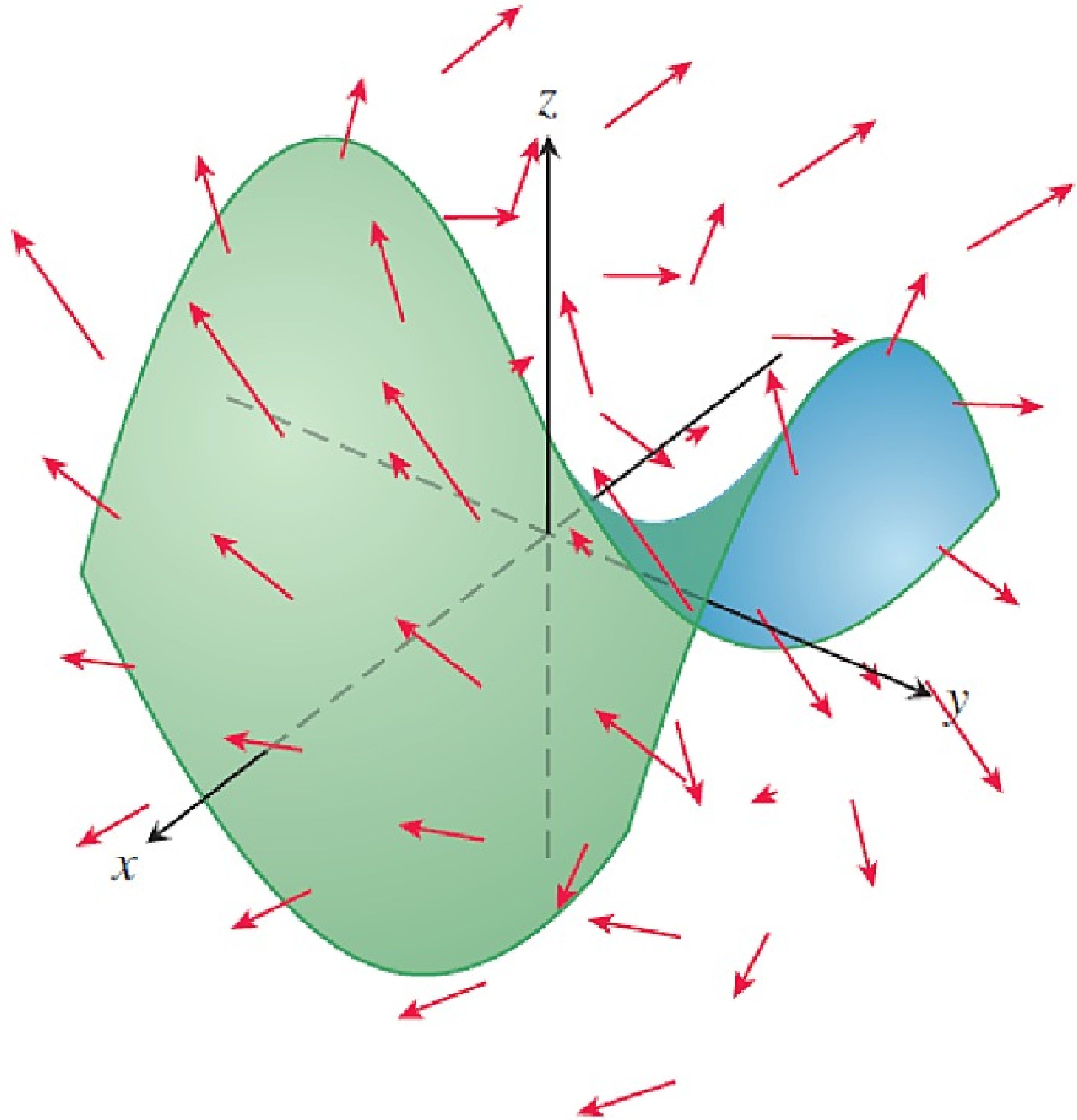


FIGURE 15.10 A surface might represent a filter, or a net, or a parachute, in a vector field representing water or wind flow velocity vectors. The arrows show the direction of fluid flow, and their lengths indicate speed.

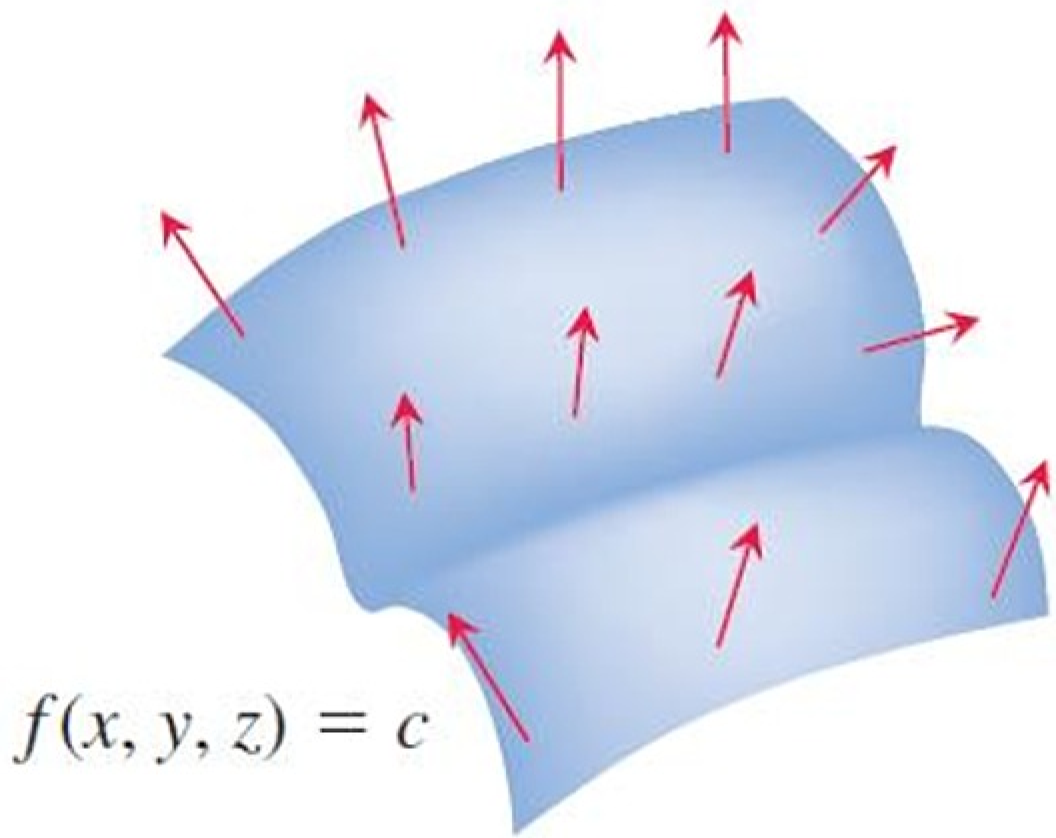


FIGURE 15.11 The field of gradient vectors ∇f on a level surface $f(x, y, z) = c$. The function f is constant on the surface, and each vector points in the direction where f is increasing fastest.

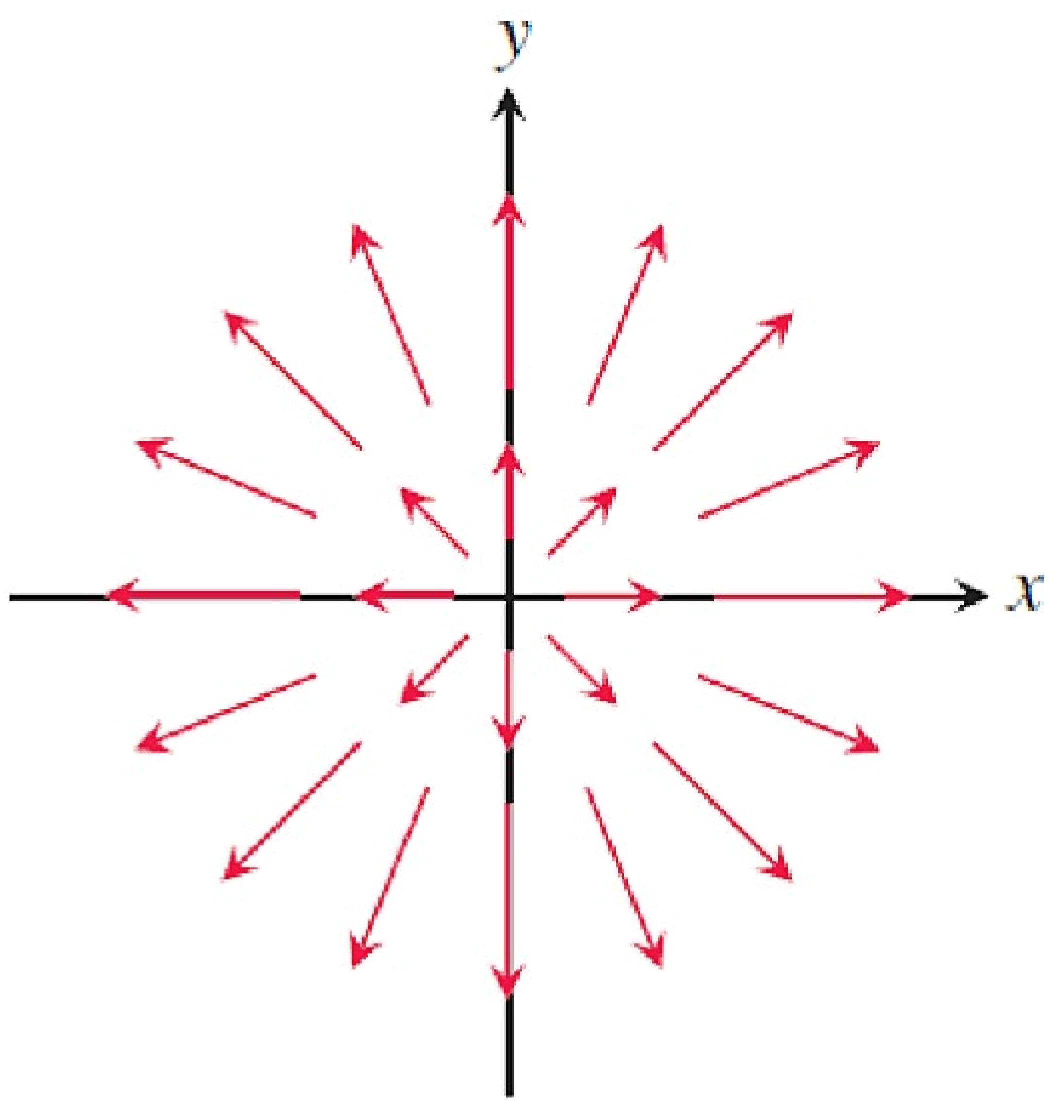


FIGURE 15.12 The radial field $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$ formed by the position vectors of points in the plane. Notice the convention that an arrow is drawn with its tail, not its head, at the point where $\mathbf{F}$ is evaluated.

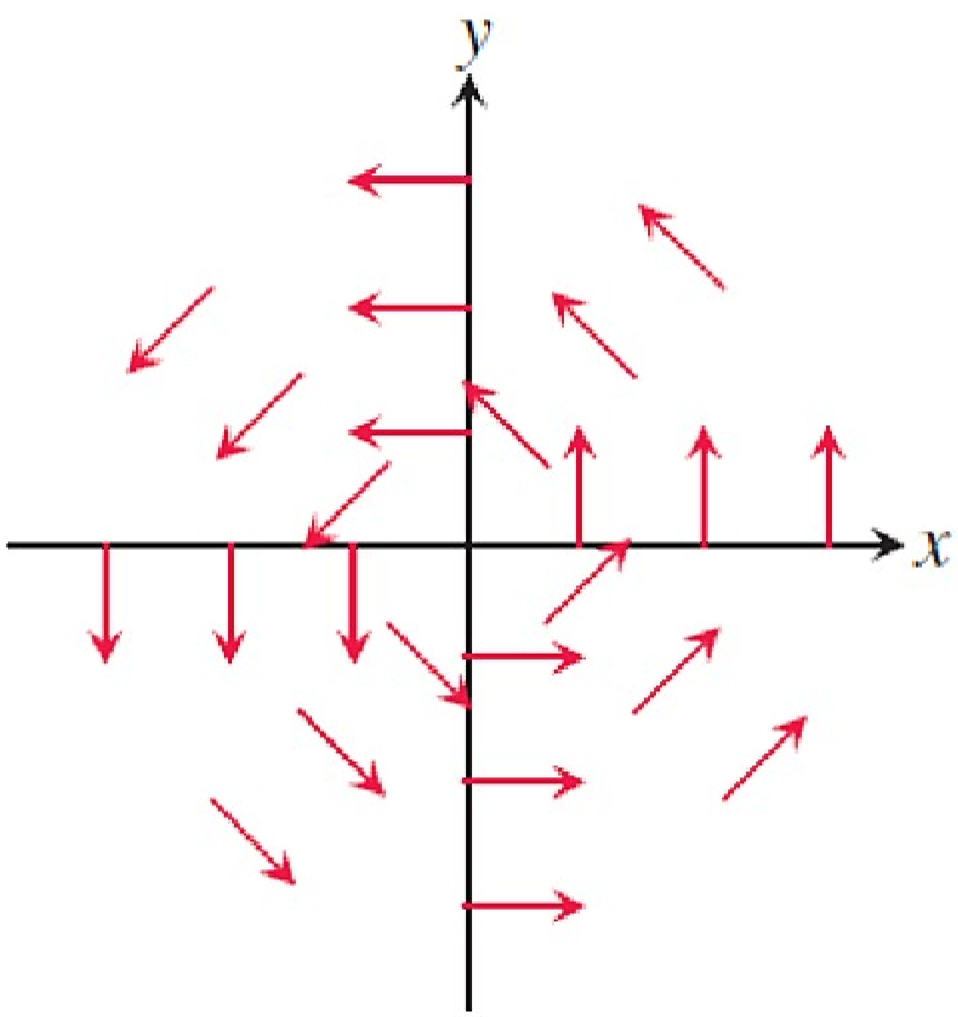


FIGURE 15.13 A "spin" field of rotating unit vectors

$$\mathbf{F} = (-y\mathbf{i} + x\mathbf{j})/(x^2 + y^2)^{1/2}$$

in the plane. The field is not defined at the origin.

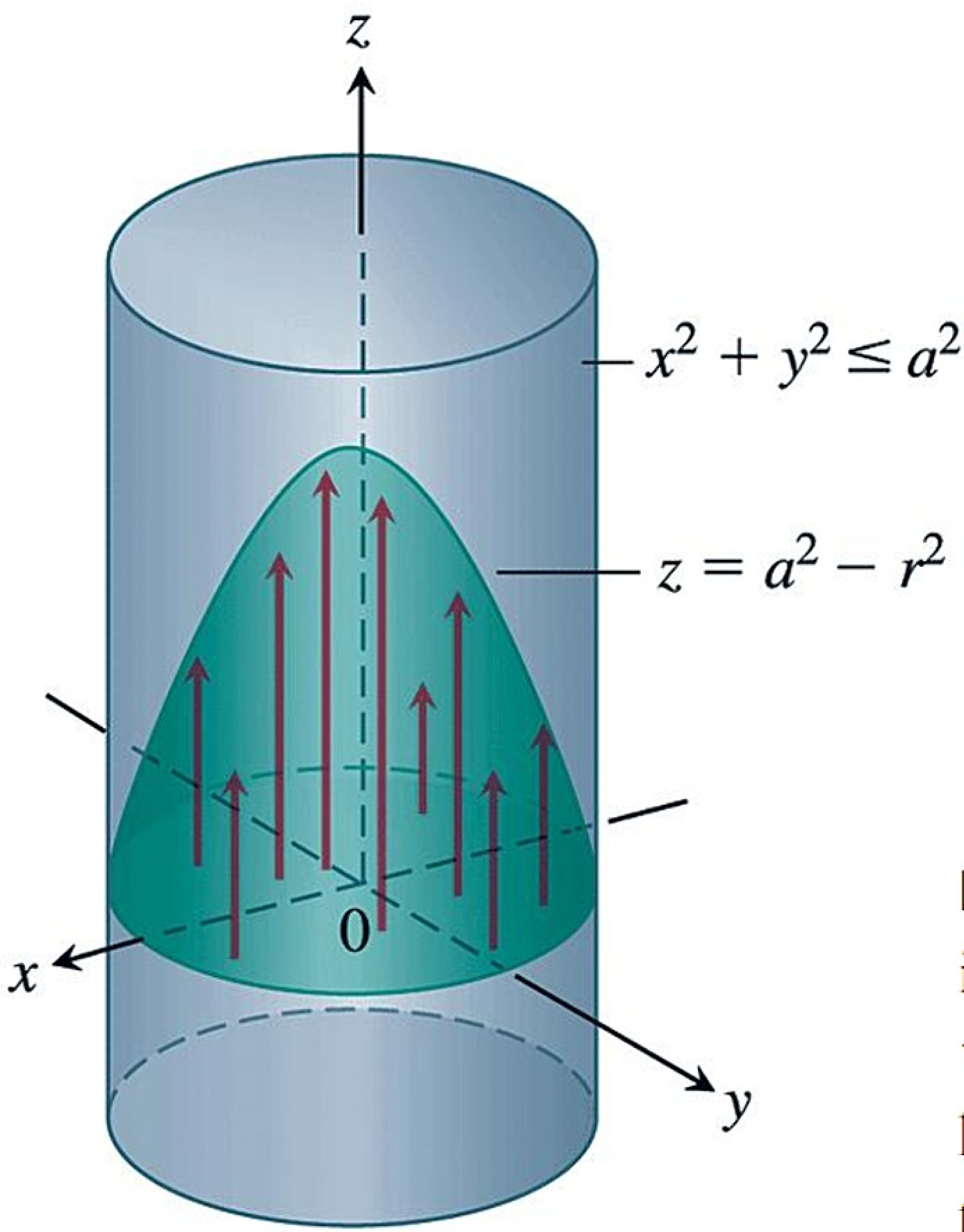


FIGURE 15.14 The flow of fluid in a long cylindrical pipe. The vectors $\mathbf{v} = (a^2 - r^2)\mathbf{k}$ inside the cylinder that have their bases in the xy-plane have their tips on the paraboloid $z = a^2 - r^2$.

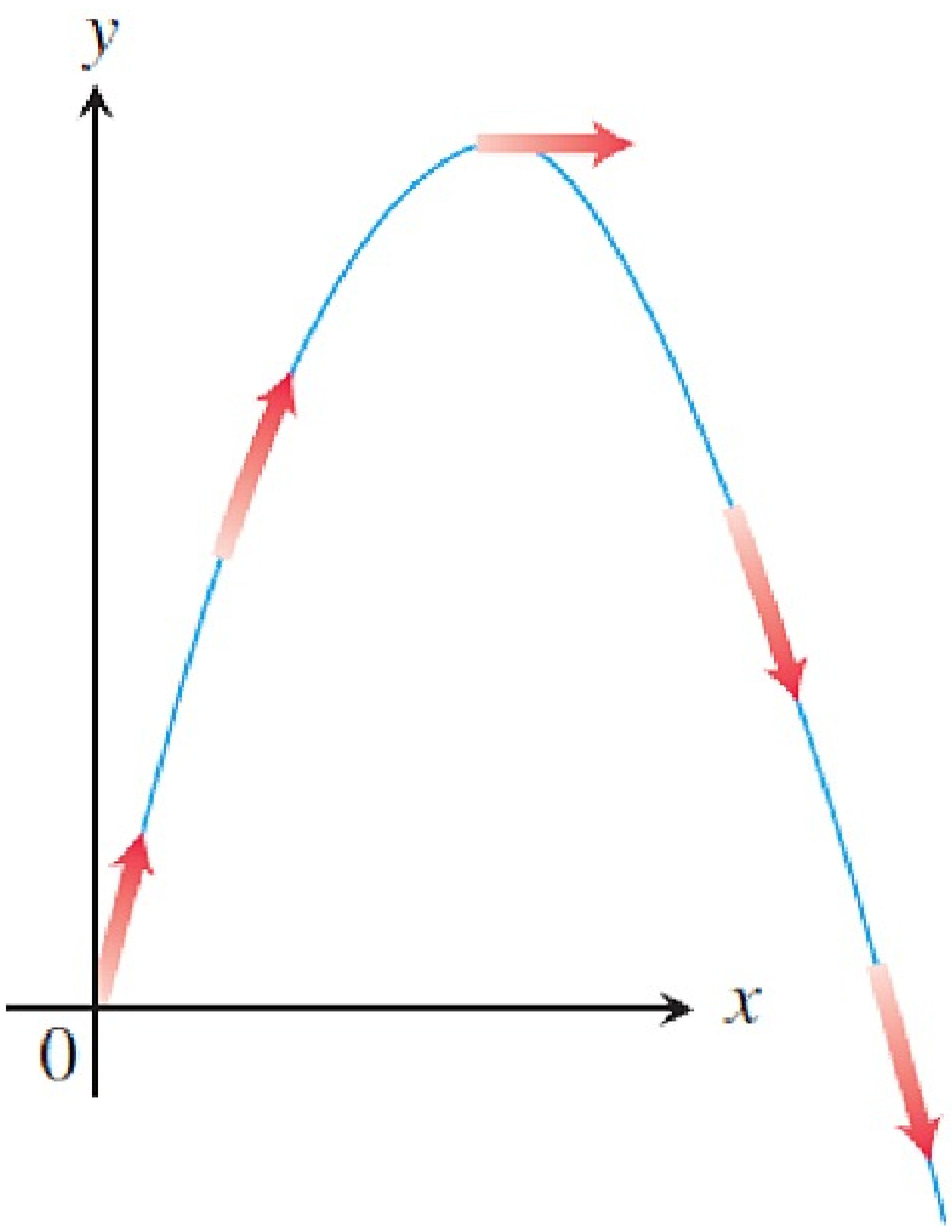


FIGURE 15.15 The velocity vectors $\mathbf{v}(t)$ of a projectile's motion make a vector field along the trajectory.

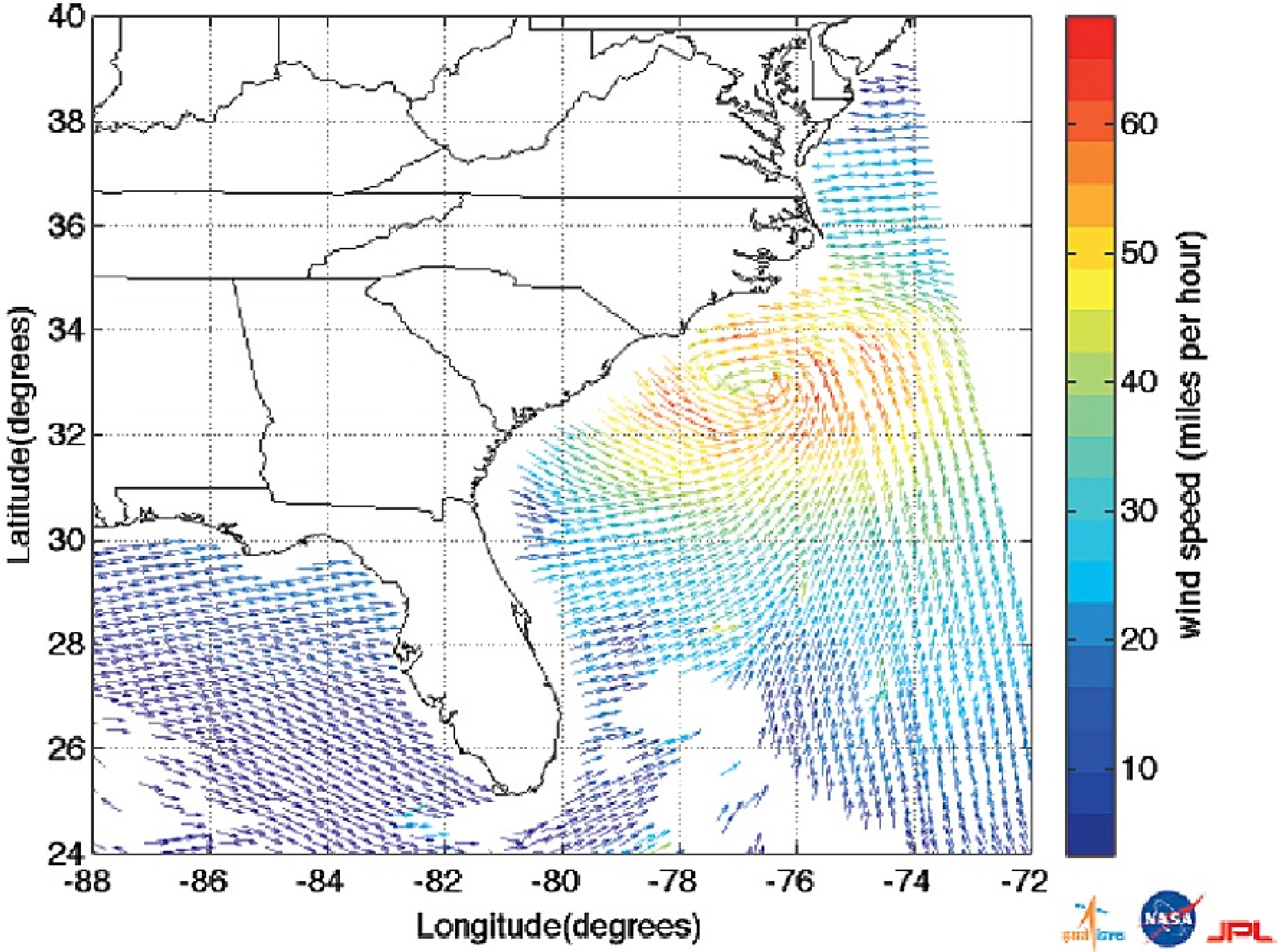


FIGURE 15.16 Data from NASA's QuikSCAT satellite were used to create this representation of windspeed and wind direction in Hurricane Irene approximately six hours before it made landfall in North Carolina on August 27, 2011. The arrows show wind direction, while speed is indicated by color (rather than length). The maximum wind speeds (over 130 km/hour) occurred over a region too small to be resolved in this illustration.

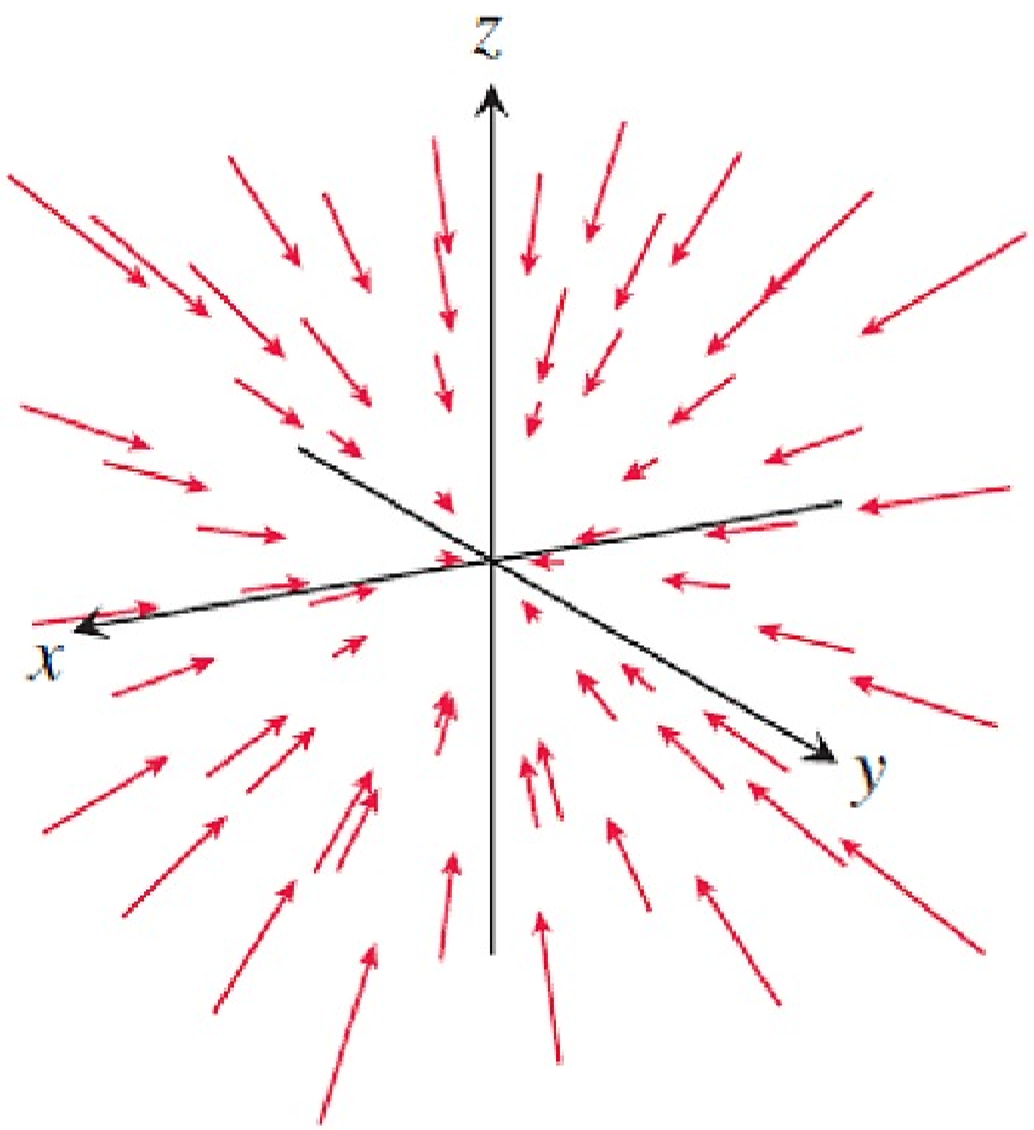


FIGURE 15.17 The vectors in a temperature gradient field point in the direction of greatest increase in temperature. In this case they are pointing toward the origin.

DEFINITION Let $\mathbf{F}$ be a vector field with continuous components defined along a smooth curve C parametrized by $\mathbf{r}(t)$, $a \le t \le b$. Then the **line integral of F along C** is

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_C \left(\mathbf{F} \cdot \frac{d\mathbf{r}}{ds} \right) ds = \int_C \mathbf{F} \cdot d\mathbf{r}.$$

Evaluating the Line Integral of $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ Along C: $\mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}$

1. Express the vector field $\mathbf{F}$ along the parametrized curve C as $\mathbf{F}(\mathbf{r}(t))$ by substituting the components $x = g(t)$, $y = h(t)$, $z = k(t)$ of $\mathbf{r}$ into the scalar components $M(x, y, z)$, $N(x, y, z)$, $P(x, y, z)$ of $\mathbf{F}$.
2. Find the derivative (velocity) vector $d\mathbf{r}/dt$.
3. Evaluate the line integral with respect to the parameter t, $a \leq t \leq b$, to obtain

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt}\, dt. \tag{2}$$

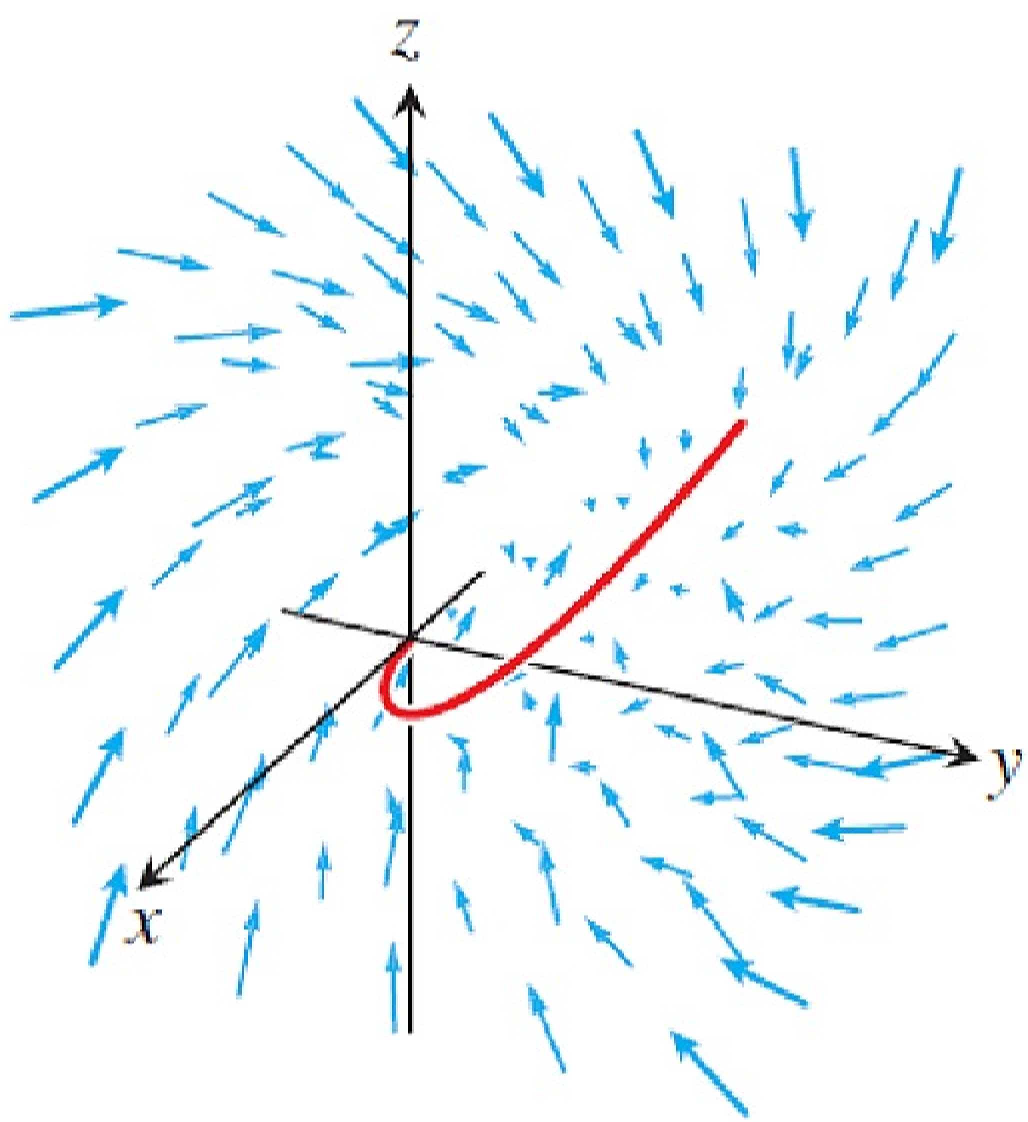


FIGURE 15.18 The curve (in red) winds through the vector field in Example 2. The line integral is determined by the vectors that lie along the curve.

$$\int_C M(x, y, z)\, dx = \int_a^b M(g(t), h(t), k(t))\, g'(t)\, dt \tag{3}$$

$$\int_C N(x, y, z)\, dy = \int_a^b N(g(t), h(t), k(t))\, h'(t)\, dt \tag{4}$$

$$\int_C P(x, y, z)\, dz = \int_a^b P(g(t), h(t), k(t))\, k'(t)\, dt \tag{5}$$

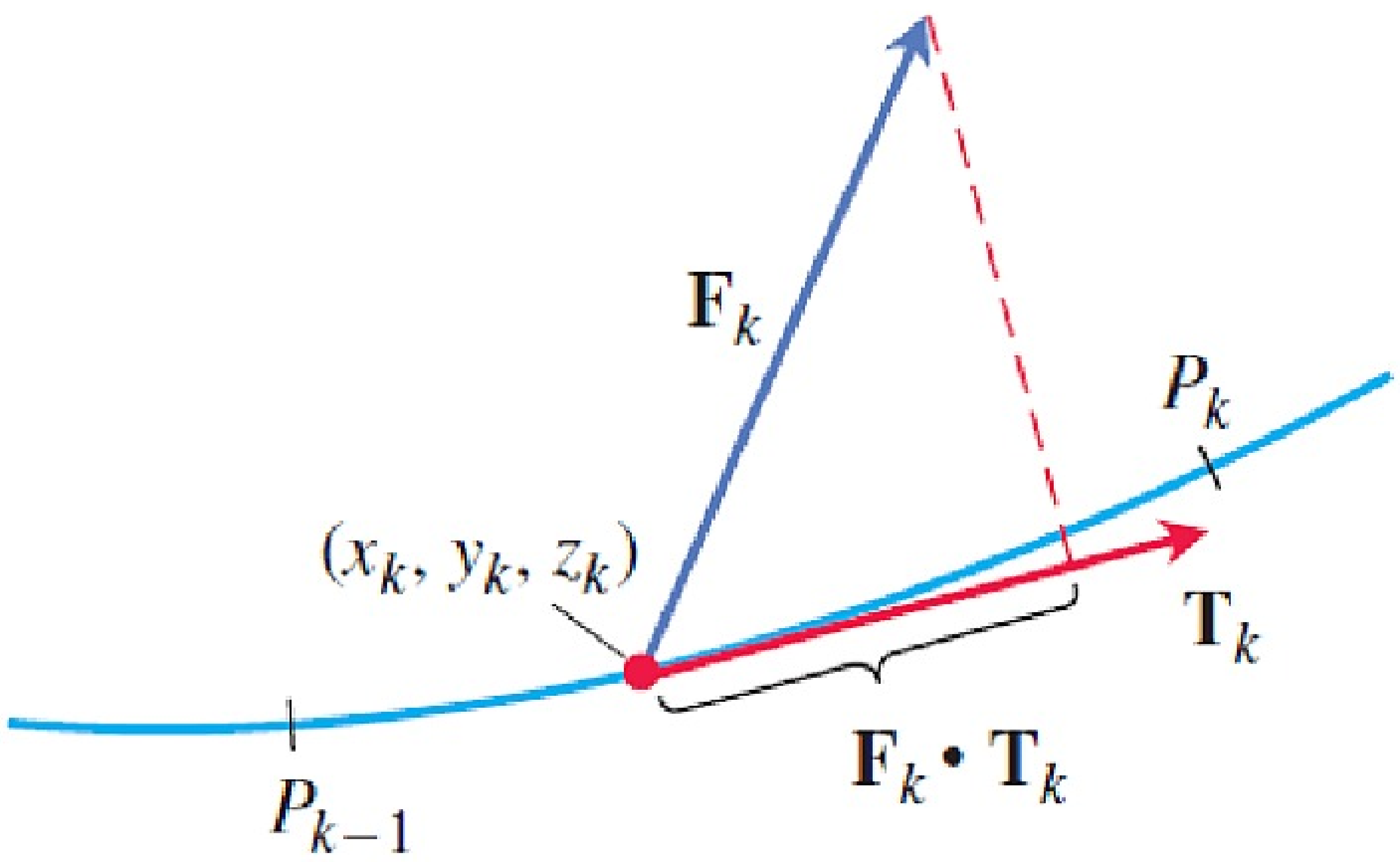


FIGURE 15.19 The work done along the subarc shown here is approximately $\mathbf{F}_k \cdot \mathbf{T}_k \, \Delta s_k$, where $\mathbf{F}_k = \mathbf{F}(x_k, y_k, z_k)$ and $\mathbf{T}_k = \mathbf{T}(x_k, y_k, z_k)$.

DEFINITION Let C be a smooth curve parametrized by $\mathbf{r}(t)$, $a \leq t \leq b$, and let $\mathbf{F}$ be a continuous force field over a region containing C. Then the **work** done in moving an object from the point $A = \mathbf{r}(a)$ to the point $B = \mathbf{r}(b)$ along C is

$$W = \int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt}\, dt. \qquad (6)$$

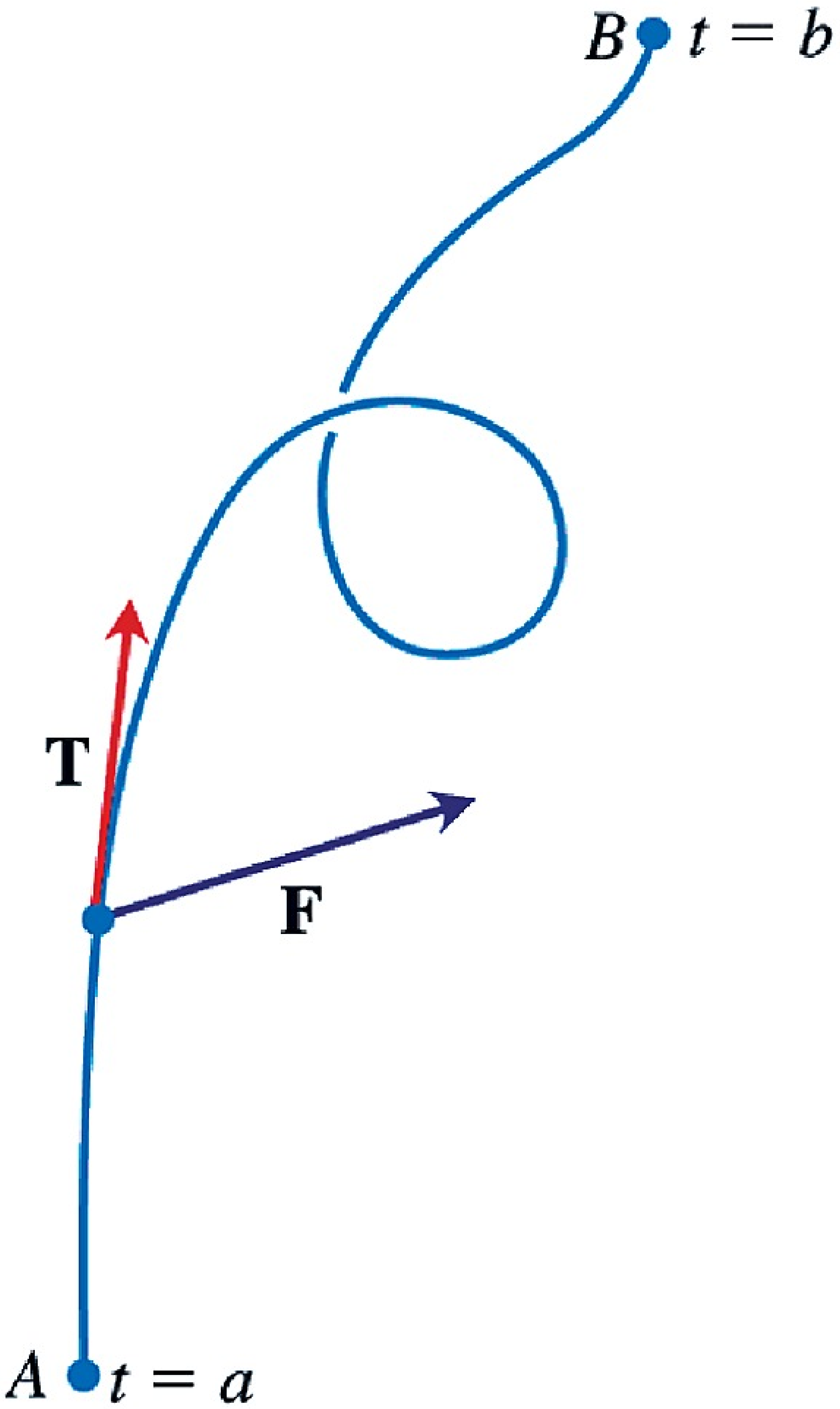


FIGURE 15.20 The work done by a force $\mathbf{F}$ is the line integral of the scalar component $\mathbf{F} \cdot \mathbf{T}$ over the smooth curve from A to B.

TABLE 15.2 Different ways to write the work integral for $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ over the curve $C: \mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k},\ a \le t \le b$

$W = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$	The definition
$= \int_C \mathbf{F} \cdot d\mathbf{r}$	Vector differential form
$= \int_a^b \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\, dt$	Parametric vector evaluation
$= \int_a^b \left(M g'(t) + N h'(t) + P k'(t)\right) dt$	Parametric scalar evaluation
$= \int_C M\, dx + N\, dy + P\, dz$	Scalar differential form

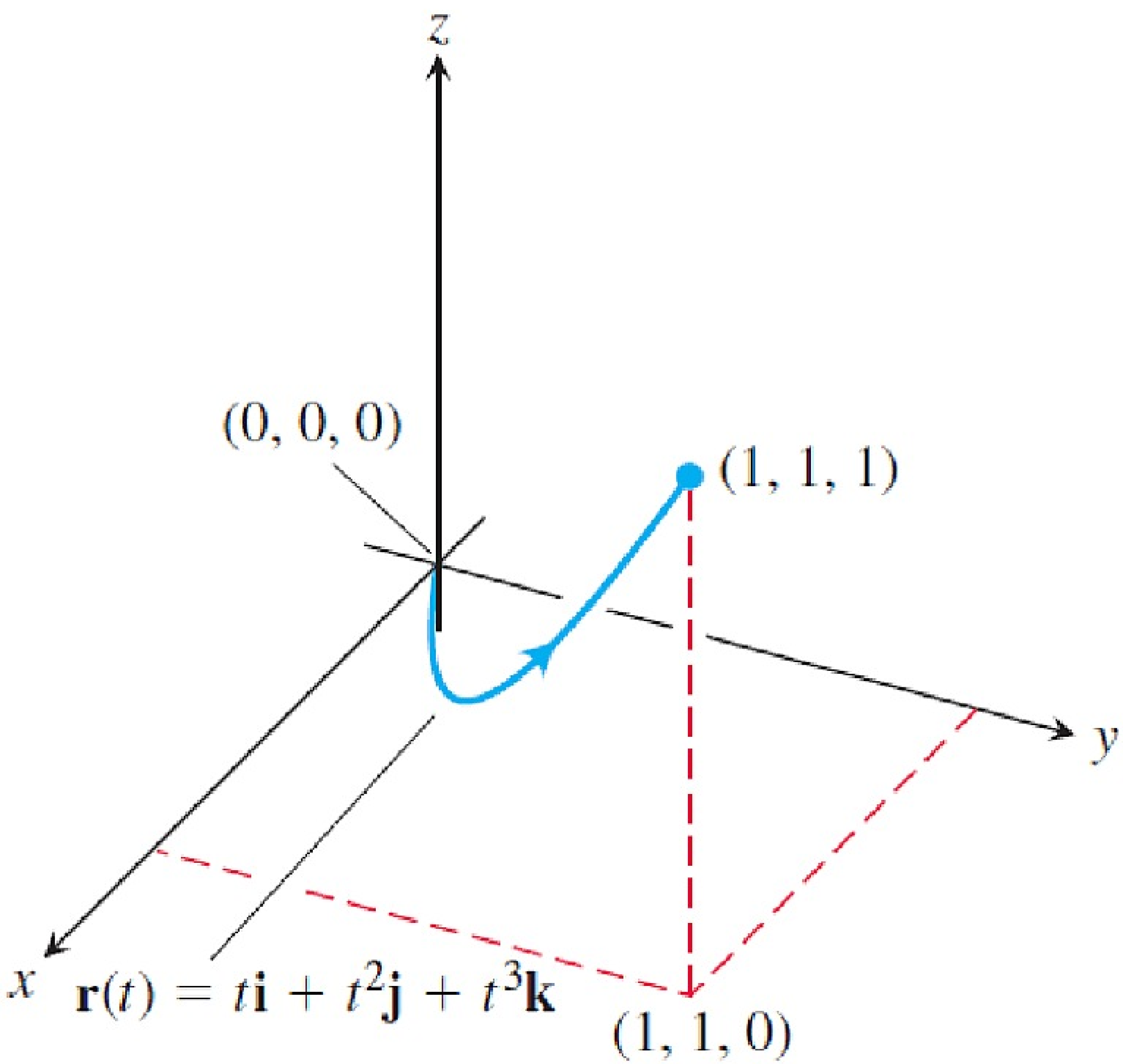


FIGURE 15.21 The curve in Example 4.

DEFINITION If $\mathbf{r}(t)$ parametrizes a smooth curve C in the domain of a continuous velocity field $\mathbf{F}$, the **flow** along the curve from $A = \mathbf{r}(a)$ to $B = \mathbf{r}(b)$ is

$$\text{Flow} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds. \tag{7}$$

The integral is called a **flow integral**. If the curve starts and ends at the same point, so that $A = B$, the flow is called the **circulation** around the curve.

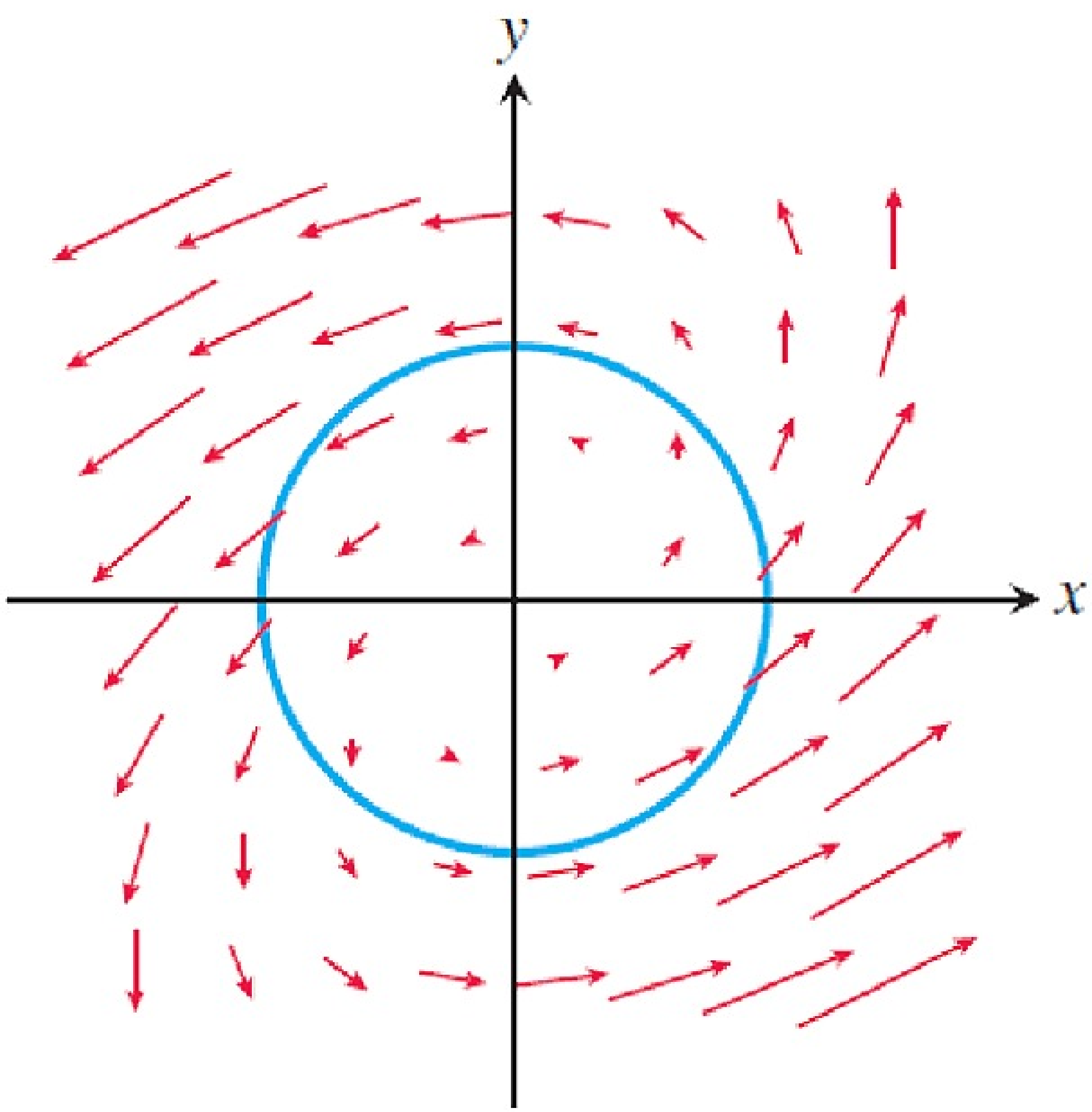


FIGURE 15.22 The vector field **F** and curve **r**(t) in Example 7.

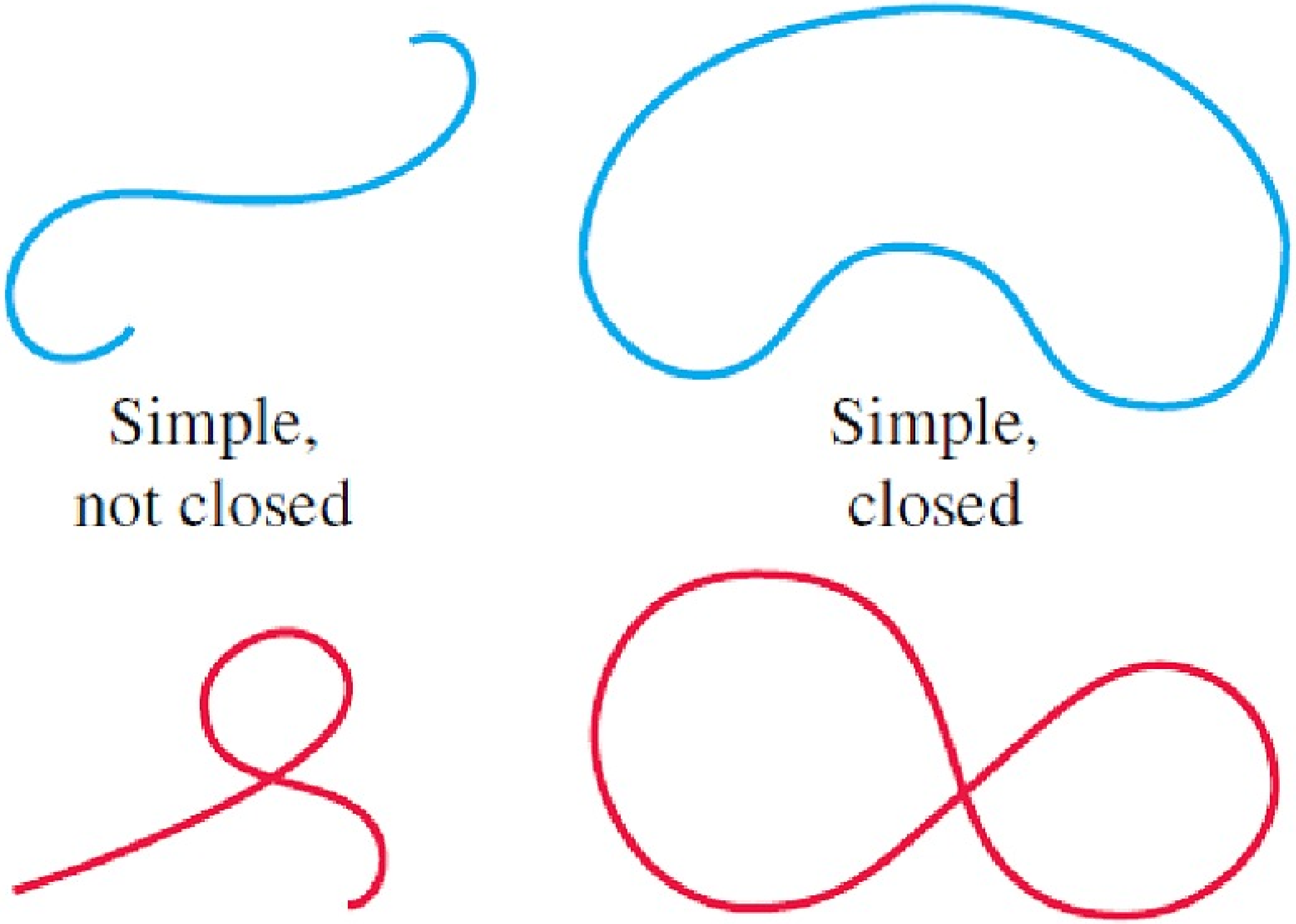


FIGURE 15.23 Distinguishing curves that are simple or closed. Closed curves are also called loops.

DEFINITION If C is a smooth simple closed curve in the domain of a continuous vector field $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ in the plane, and if $\mathbf{n}$ is the outward-pointing unit normal vector on C, the **flux** of $\mathbf{F}$ across C is

$$\text{Flux of } \mathbf{F} \text{ across C} = \int_{C} \mathbf{F} \cdot \mathbf{n}\, ds. \tag{8}$$

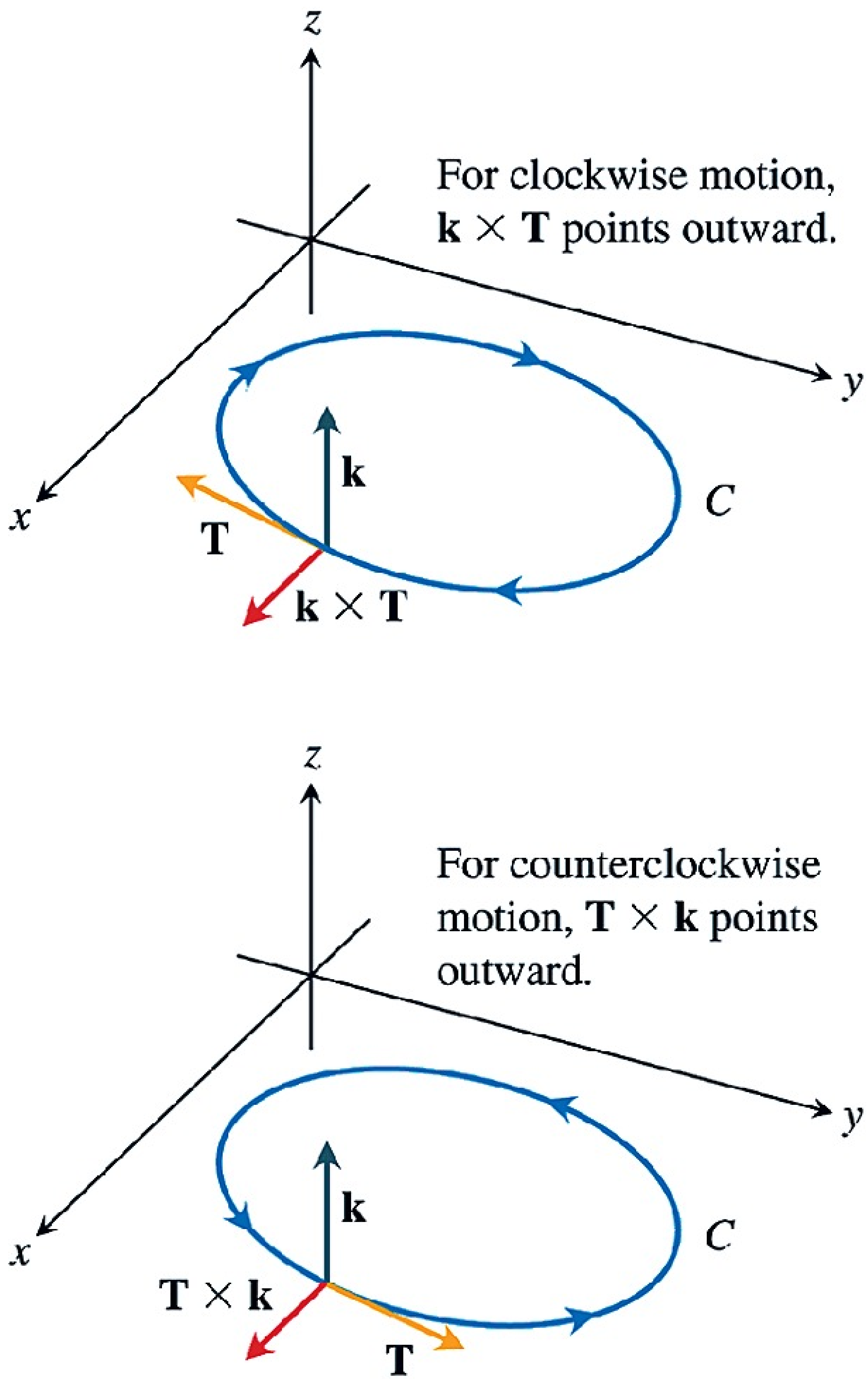


FIGURE 15.24 To find an outward unit normal vector for a smooth simple curve C in the xy-plane that is traversed counterclockwise as t increases, we take $\mathbf{n} = \mathbf{T} \times \mathbf{k}$. For clockwise motion, we take $\mathbf{n} = \mathbf{k} \times \mathbf{T}$.

Calculating Flux Across a Smooth Closed Plane Curve

$$\text{Flux of } \mathbf{F} = M\mathbf{i} + N\mathbf{j} \text{ across } C = \oint_C M\,dy - N\,dx \tag{9}$$

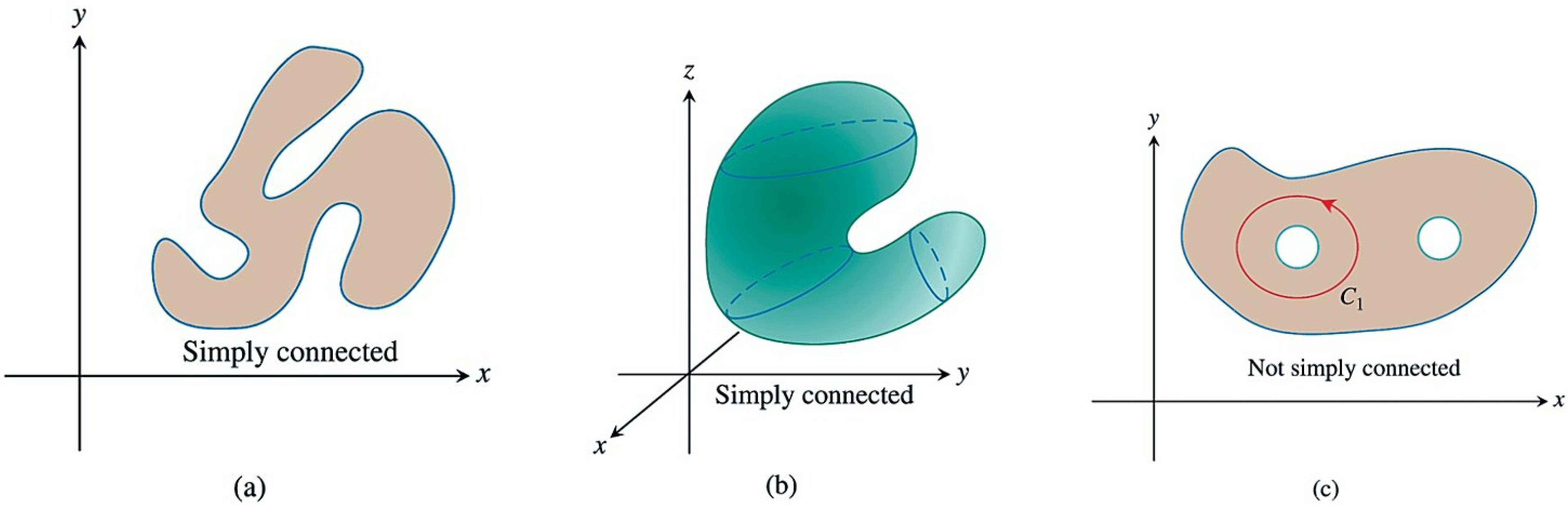
The integral can be evaluated from any smooth parametrization $x = g(t)$, $y = h(t)$, $a \leq t \leq b$, that traces C counterclockwise exactly once.

Section 15.3

Path Independence, Conservative Fields, and Potential Functions

DEFINITIONS Let $\mathbf{F}$ be a vector field defined on an open region D in space, and suppose that for any two points A and B in D the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ along a path C from A to B in D is the same over all paths from A to B. Then the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is **path independent in *D*** and the field $\mathbf{F}$ is **conservative on *D***.

DEFINITION If $\mathbf{F}$ is a vector field defined on D and $\mathbf{F} = \nabla f$ for some scalar function f on D, then f is called a **potential function for F**.



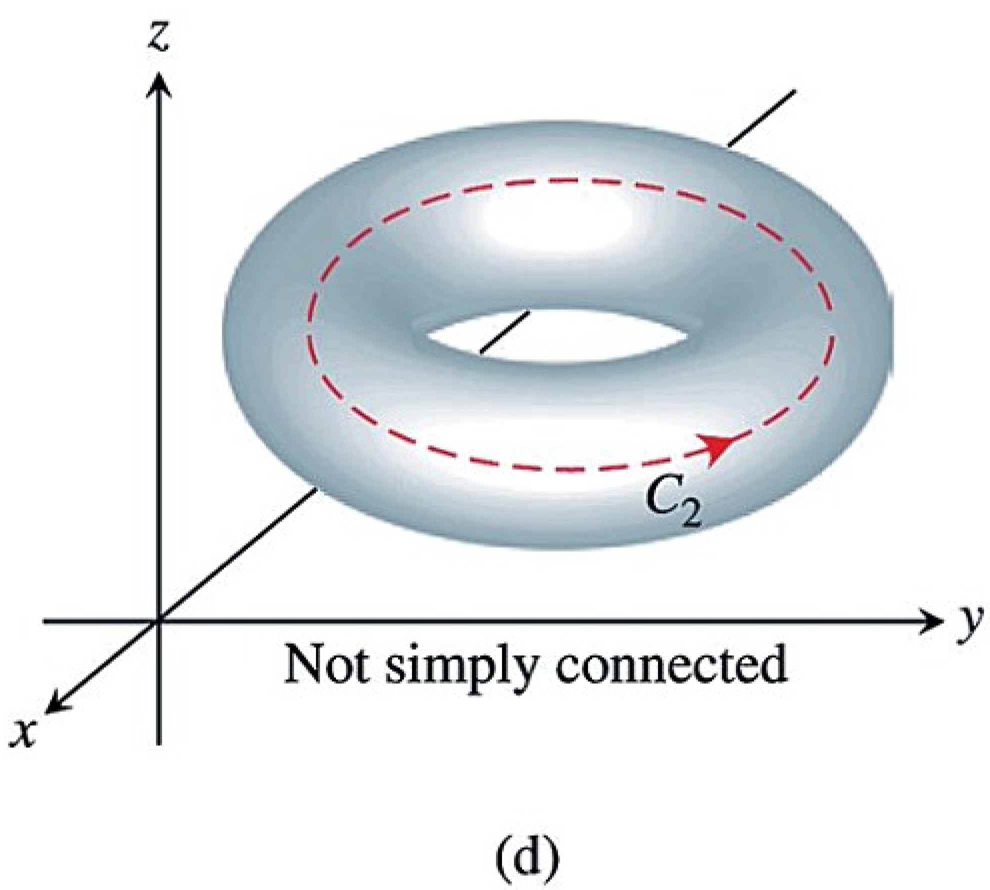


FIGURE 15.25 Four connected regions. In (a) and (b), the regions are simply connected. In (c) and (d), the regions are not simply connected because the curves C_1 and C_2 cannot be contracted to a point inside the regions containing them.

THEOREM 1—Fundamental Theorem of Line Integrals Let C be a smooth curve joining the point A to the point B in the plane or in space and parametrized by $\mathbf{r}(t)$. Let f be a differentiable function with a continuous gradient vector $\mathbf{F} = \nabla f$ on a domain D containing C. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A).$$

THEOREM 2—Conservative Fields are Gradient Fields Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ be a vector field whose components are continuous throughout an open connected region D in space. Then $\mathbf{F}$ is conservative if and only if $\mathbf{F}$ is a gradient field ∇f for a differentiable function f.

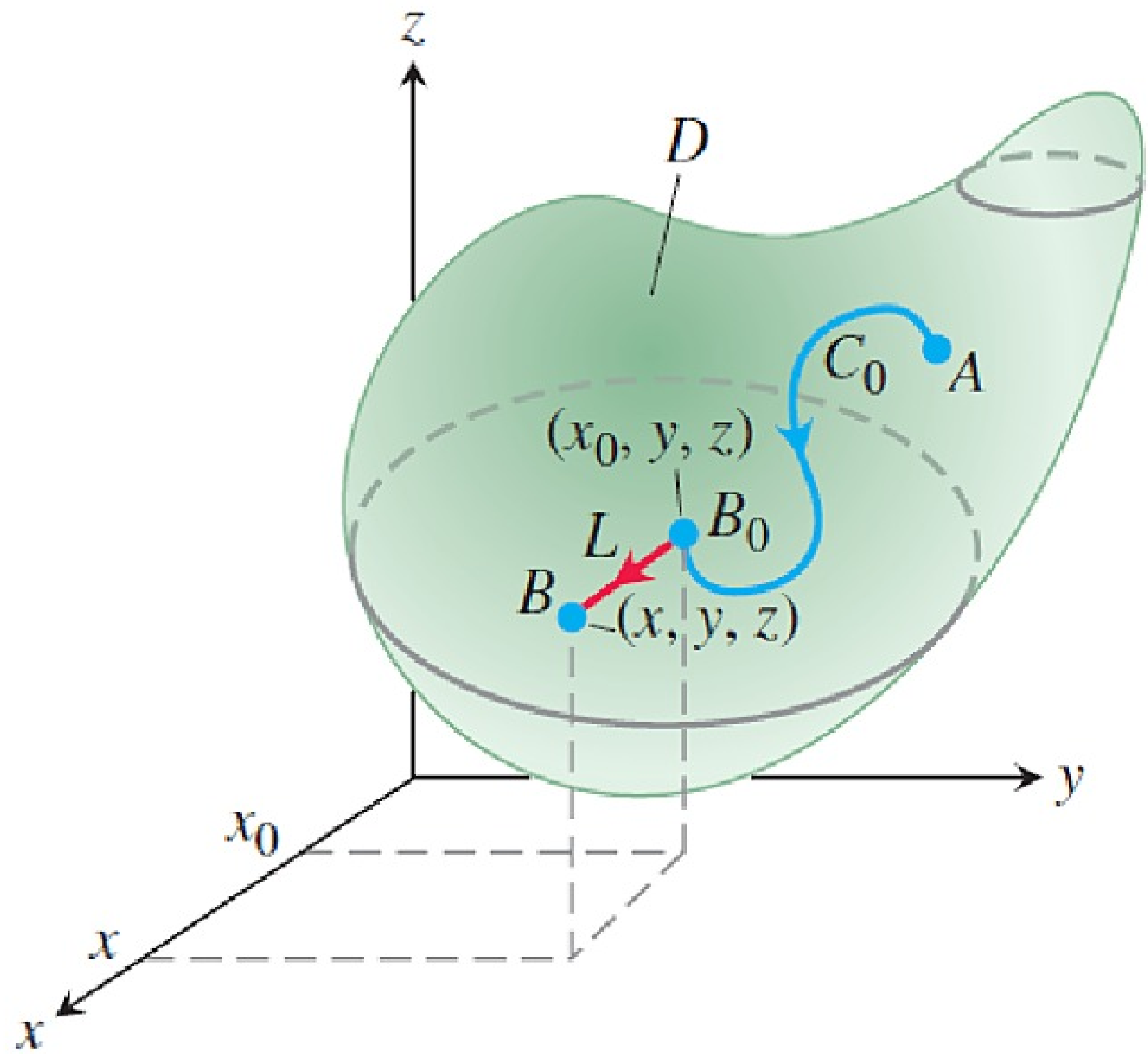


FIGURE 15.26 The function $f(x, y, z)$ in the proof of Theorem 2 is computed by a line integral $\int_{C_0} \mathbf{F} \cdot d\mathbf{r} = f(B_0)$ from A to B_0, plus a line integral $\int_L \mathbf{F} \cdot d\mathbf{r}$ along a line segment L parallel to the x-axis and joining B_0 to B located at (x, y, z). The value of f at A is $f(A) = 0$.

THEOREM 3—Loop Property of Conservative Fields The following statements are equivalent.

1. $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ around every loop (that is, closed curve C) in D.
2. The field $\mathbf{F}$ is conservative on D.

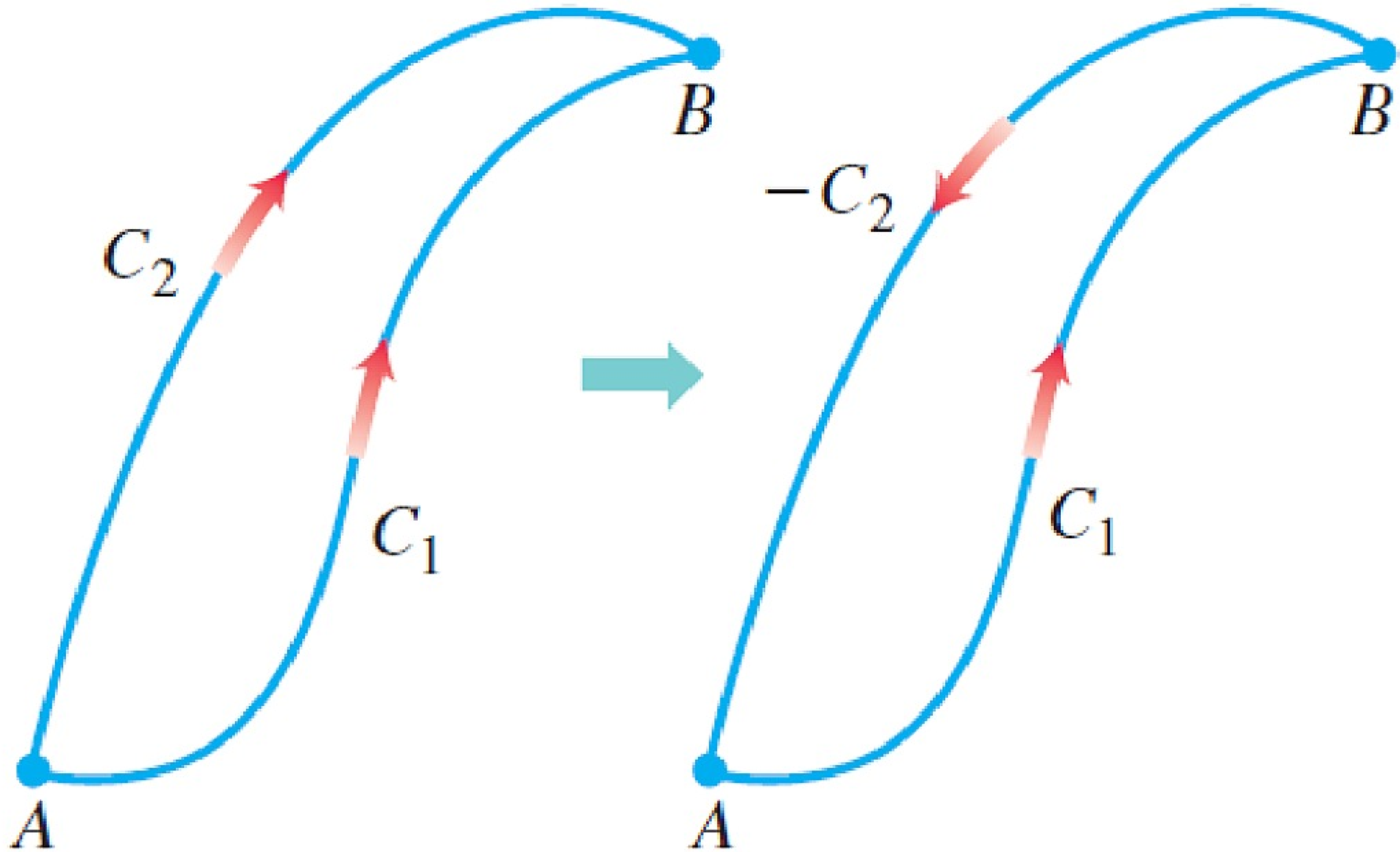


FIGURE 15.27 If we have two paths from A to B, one of them can be reversed to make a loop.

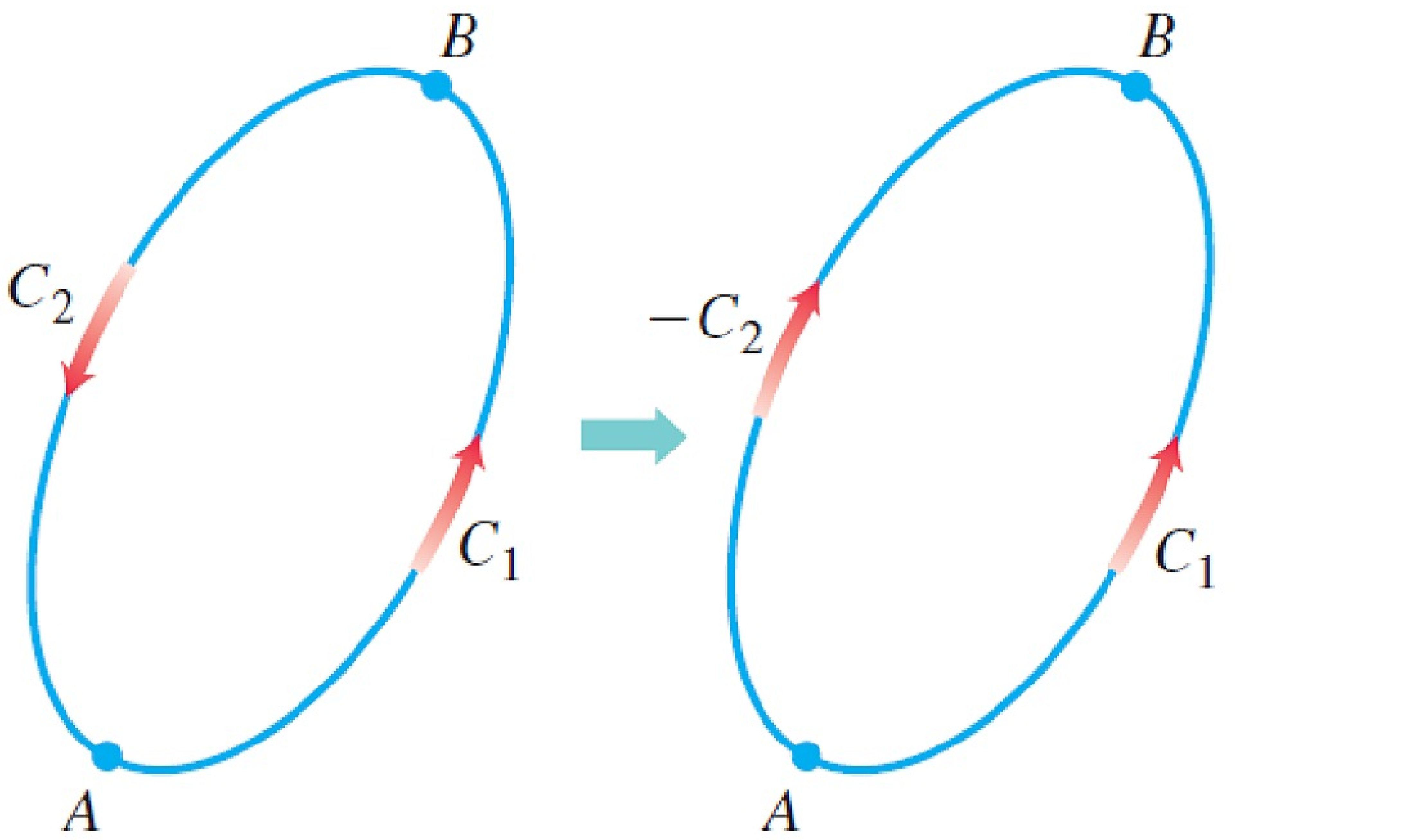


FIGURE 15.28 If A and B lie on a loop, we can reverse part of the loop to make two paths from A to B.

Component Test for Conservative Fields

Let $\mathbf{F} = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$ be a field on an open simply connected domain whose component functions have continuous first partial derivatives. Then, $\mathbf{F}$ is conservative if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \qquad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \qquad \text{and} \qquad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}. \tag{2}$$

DEFINITIONS Any expression $M(x, y, z)\,dx + N(x, y, z)\,dy + P(x, y, z)\,dz$ is a **differential form**. A differential form is **exact** on a domain D in space if

$$M\,dx + N\,dy + P\,dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = df$$

for some scalar function f throughout D.

Component Test for Exactness of $M\,dx + N\,dy + P\,dz$

The differential form $M\,dx + N\,dy + P\,dz$ is exact on an open simply connected domain if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \qquad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \qquad \text{and} \qquad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$

This is equivalent to saying that the field $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is conservative.

Section 15.4

Green’s Theorem in the Plane

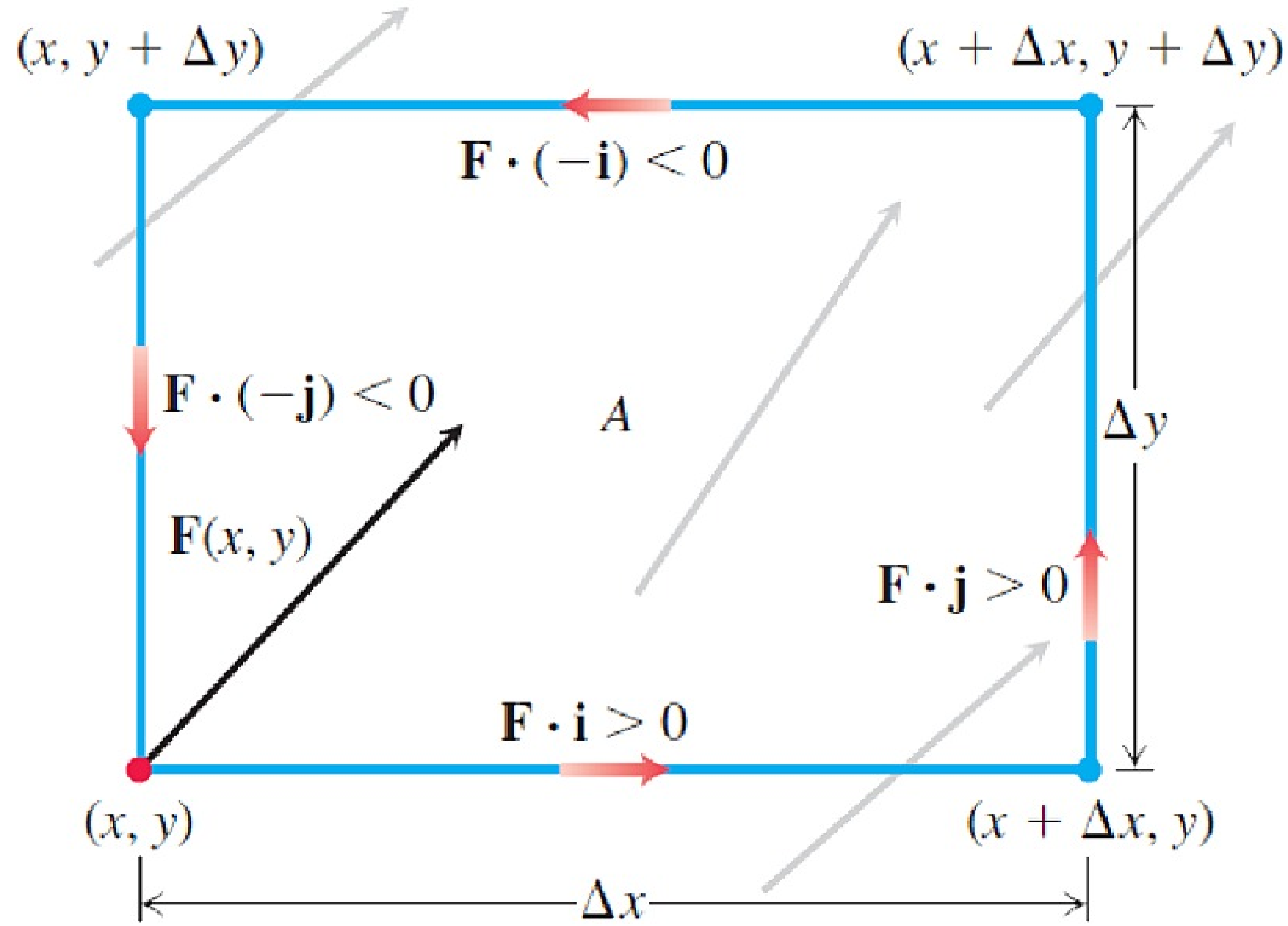
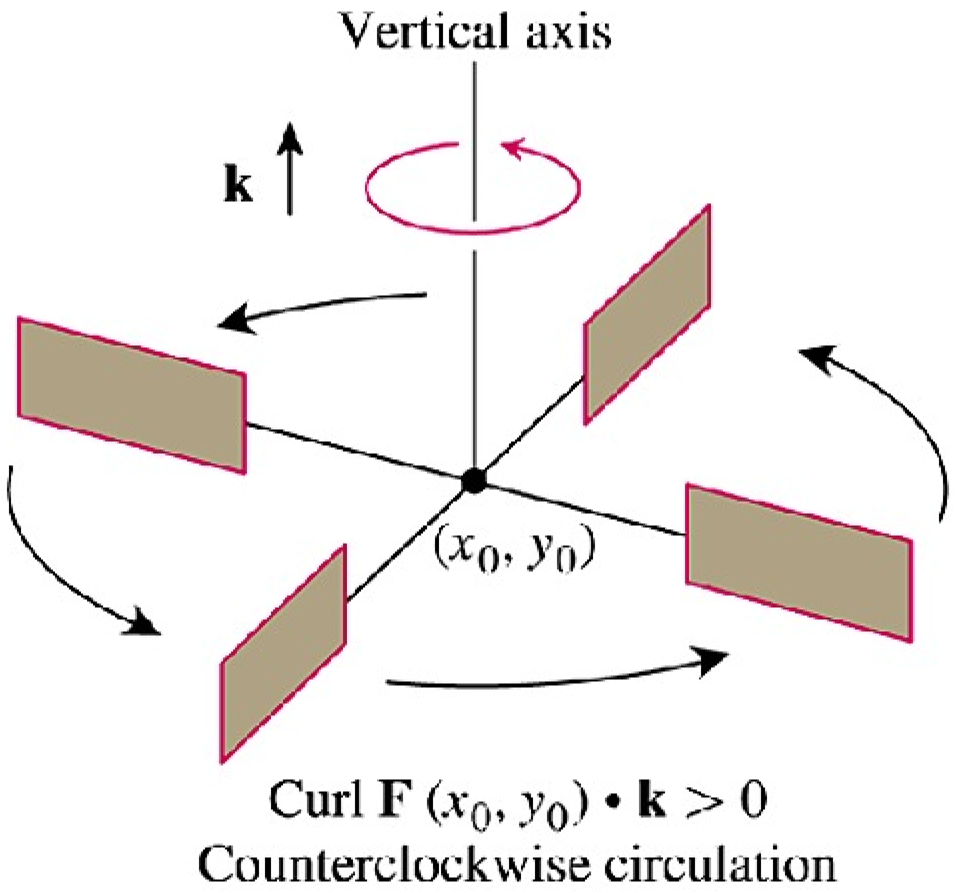


FIGURE 15.29 The rate at which a fluid flows along the bottom edge of a rectangular region A in the direction $\mathbf{i}$ is approximately $\mathbf{F}(x, y) \cdot \mathbf{i}\, \Delta x$, which is positive for the vector field $\mathbf{F}$ shown here. To approximate the rate of circulation at the point (x, y), we calculate the (approximate) flow rates along each edge in the directions of the red arrows, sum these rates, and then divide the sum by the area of A. Taking the limit as $\Delta x \to 0$ and $\Delta y \to 0$ gives the rate of the circulation per unit area.



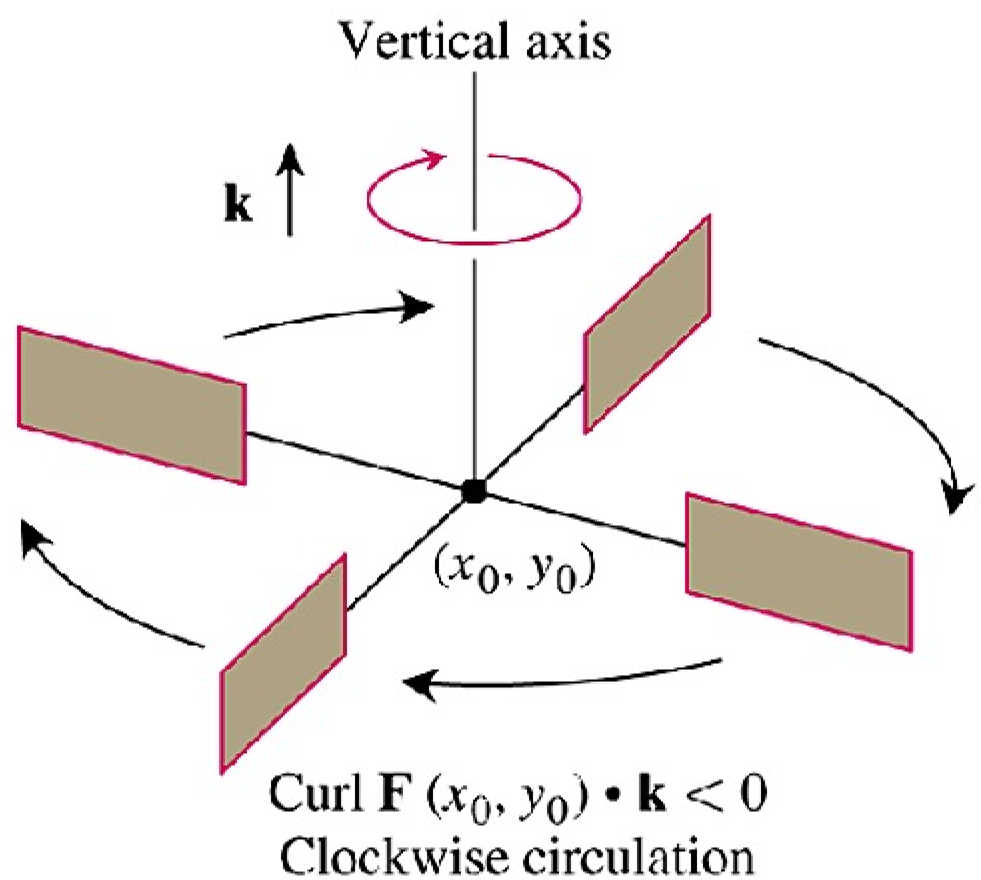


FIGURE 15.30 In the flow of an incompressible fluid over a plane region, the **k**-component of the curl measures the rate of the fluid's rotation at a point. The **k**-component of the curl is positive at points where the rotation is counterclockwise and negative where the rotation is clockwise.

DEFINITION The **circulation density** of a vector field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ at the point (x, y) is the scalar expression

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}. \tag{1}$$

This expression is also called **the k-component of the curl**, denoted by $(\text{curl } \mathbf{F}) \cdot \mathbf{k}$.

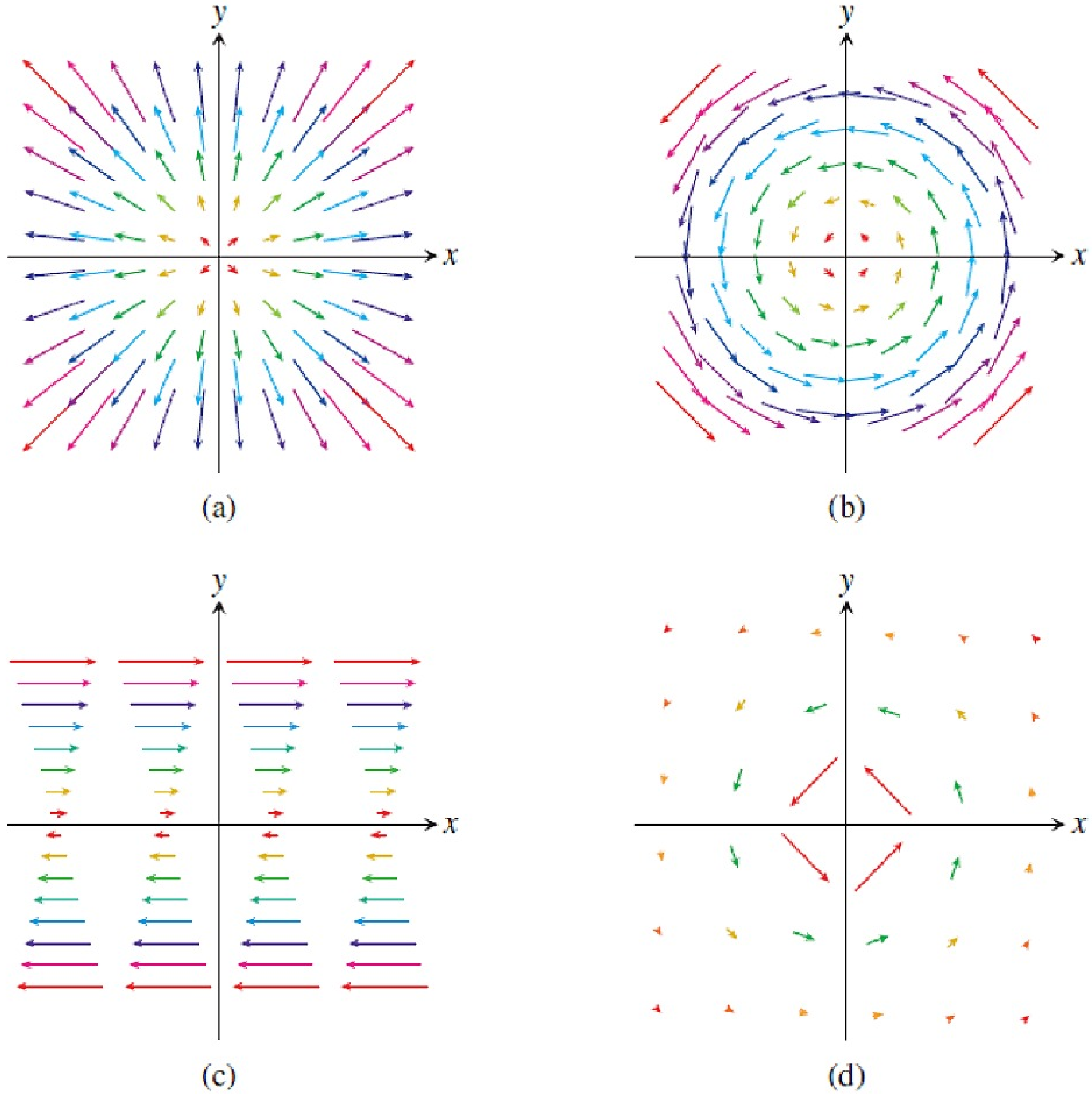


FIGURE 15.31 Velocity fields of a gas flowing in the plane (Example 1).

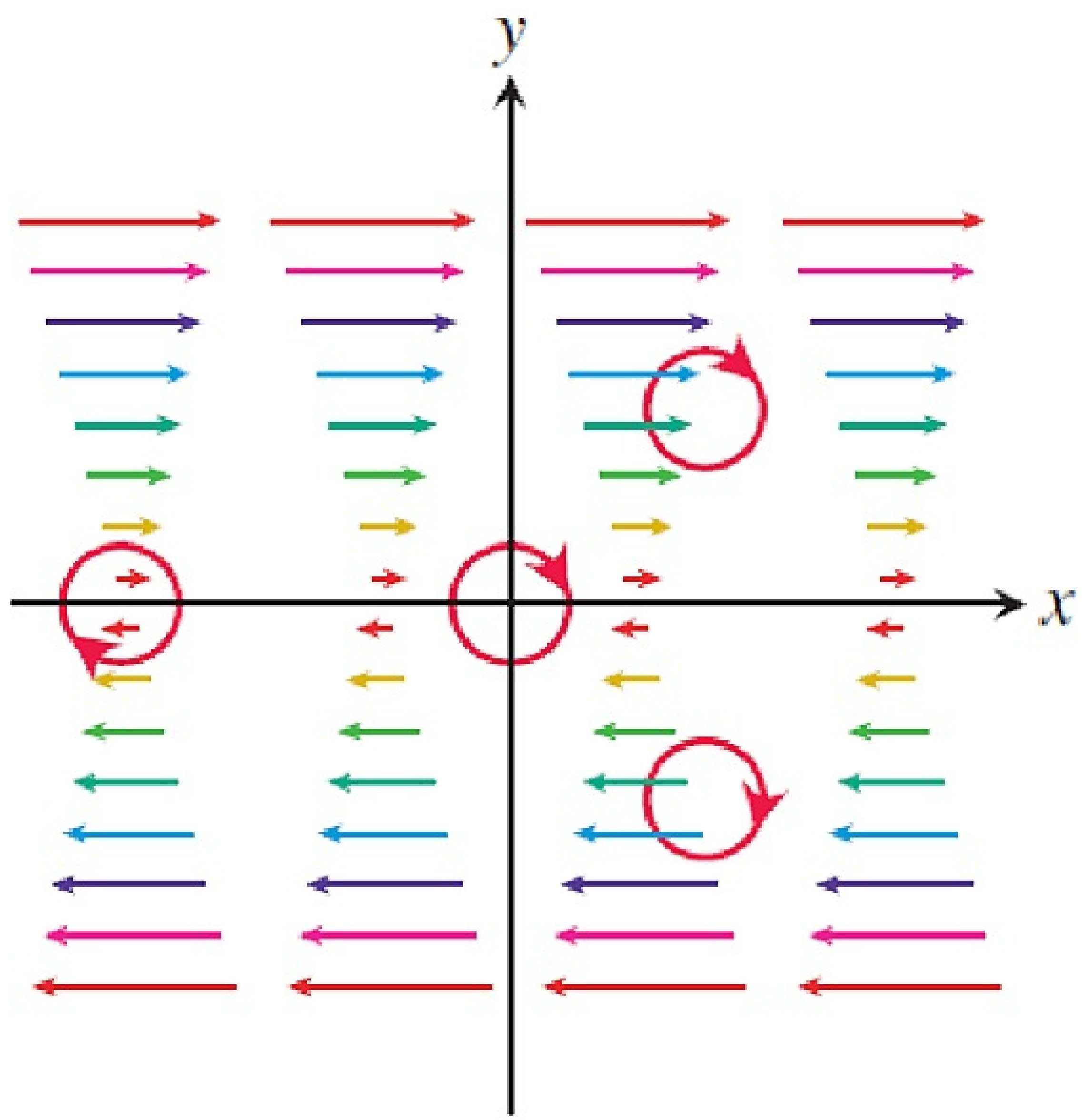


FIGURE 15.32 A shearing flow pushes the fluid clockwise around each point (Example 1c).

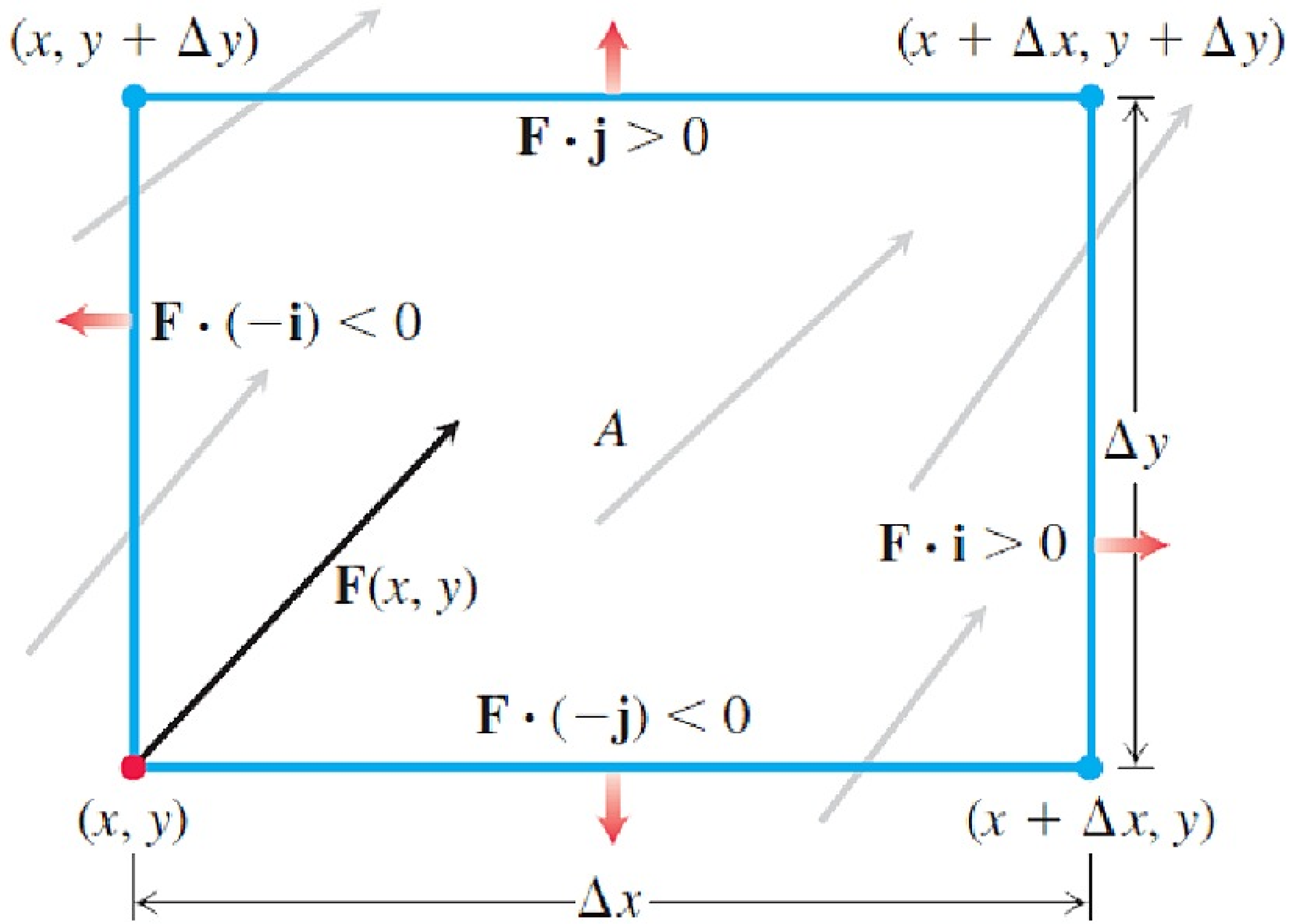


FIGURE 15.33 The rate at which the fluid leaves the rectangular region A across the bottom edge in the direction of the outward normal $-\mathbf{j}$ is approximately $\mathbf{F}(x, y) \cdot (-\mathbf{j})\, \Delta x$, which is negative for the vector field $\mathbf{F}$ shown here. To approximate the flow rate at the point (x, y), we calculate the (approximate) flow rates across each edge in the directions of the red arrows, sum these rates, and then divide the sum by the area of A. Taking the limit as $\Delta x \to 0$ and $\Delta y \to 0$ gives the flow rate per unit area.

DEFINITION The **divergence (flux density)** of a vector field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ at the point (x, y) is

$$\text{div}\,\mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}. \tag{2}$$

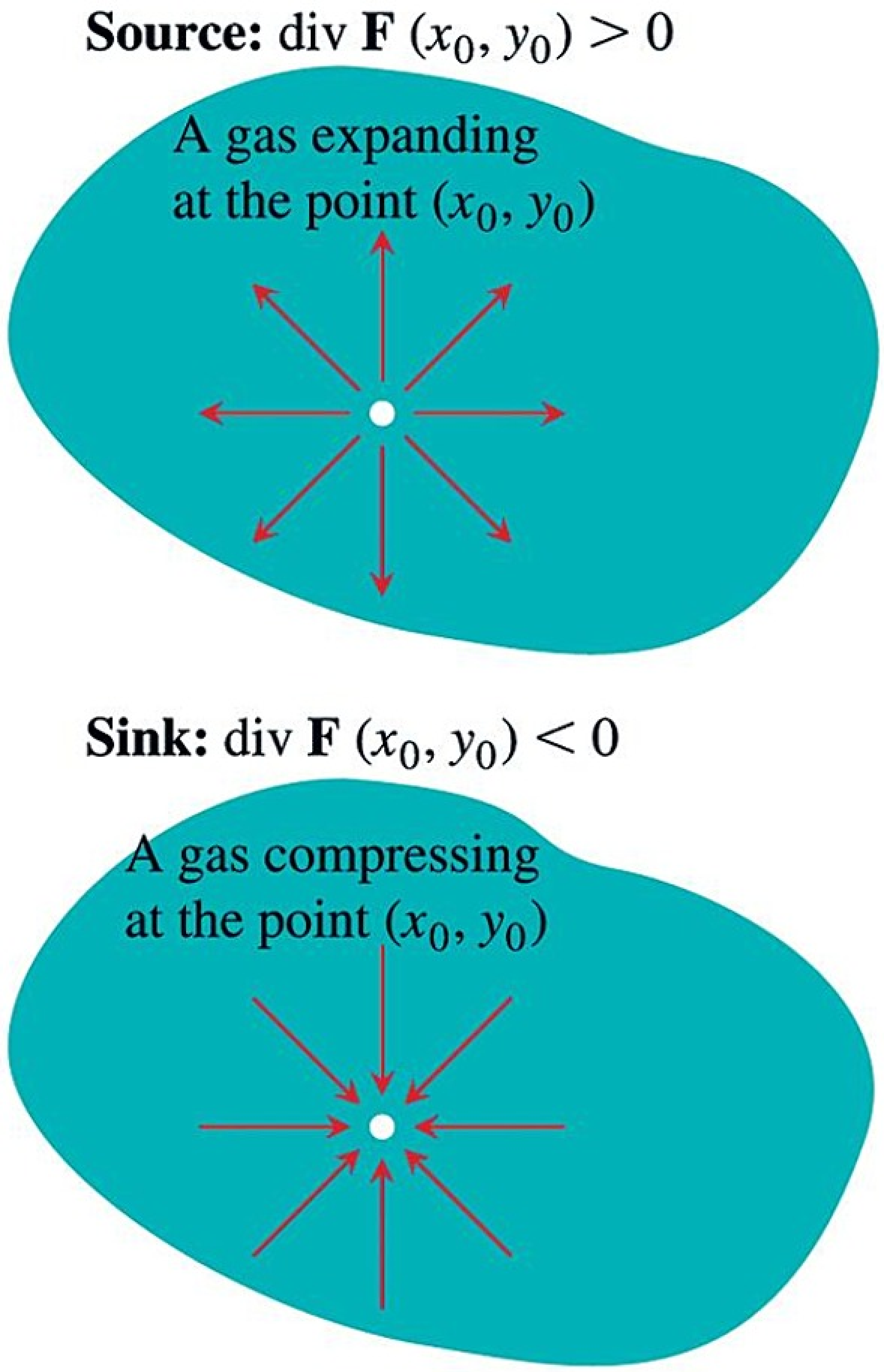


FIGURE 15.34 If a gas is expanding at a point (x_0, y_0), the lines of flow have positive divergence; if the gas is compressing, the divergence is negative.

THEOREM 4—Green's Theorem (Circulation-Curl or Tangential Form)
Let C be a piecewise smooth, simple closed curve enclosing a region R in the plane. Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ be a vector field with M and N having continuous first partial derivatives in an open region containing R. Then the counterclockwise circulation of $\mathbf{F}$ around C equals the double integral of (curl $\mathbf{F}$) $\cdot$ $\mathbf{k}$ over R.

$$\underbrace{\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \oint_C M\, dx + N\, dy}_{\text{Counterclockwise circulation}} = \underbrace{\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy}_{\text{Curl integral}} \tag{3}$$

THEOREM 5—Green's Theorem (Flux-Divergence or Normal Form)
Let C be a piecewise smooth, simple closed curve enclosing a region R in the plane. Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ be a vector field with M and N having continuous first partial derivatives in an open region containing R. Then the outward flux of $\mathbf{F}$ across C equals the double integral of div $\mathbf{F}$ over the region R enclosed by C.

$$\underbrace{\oint_C \mathbf{F}\cdot\mathbf{n}\,ds}_{\text{Outward flux}} = \oint_C M\,dy - N\,dx = \underbrace{\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy}_{\text{Divergence integral}} \qquad (4)$$

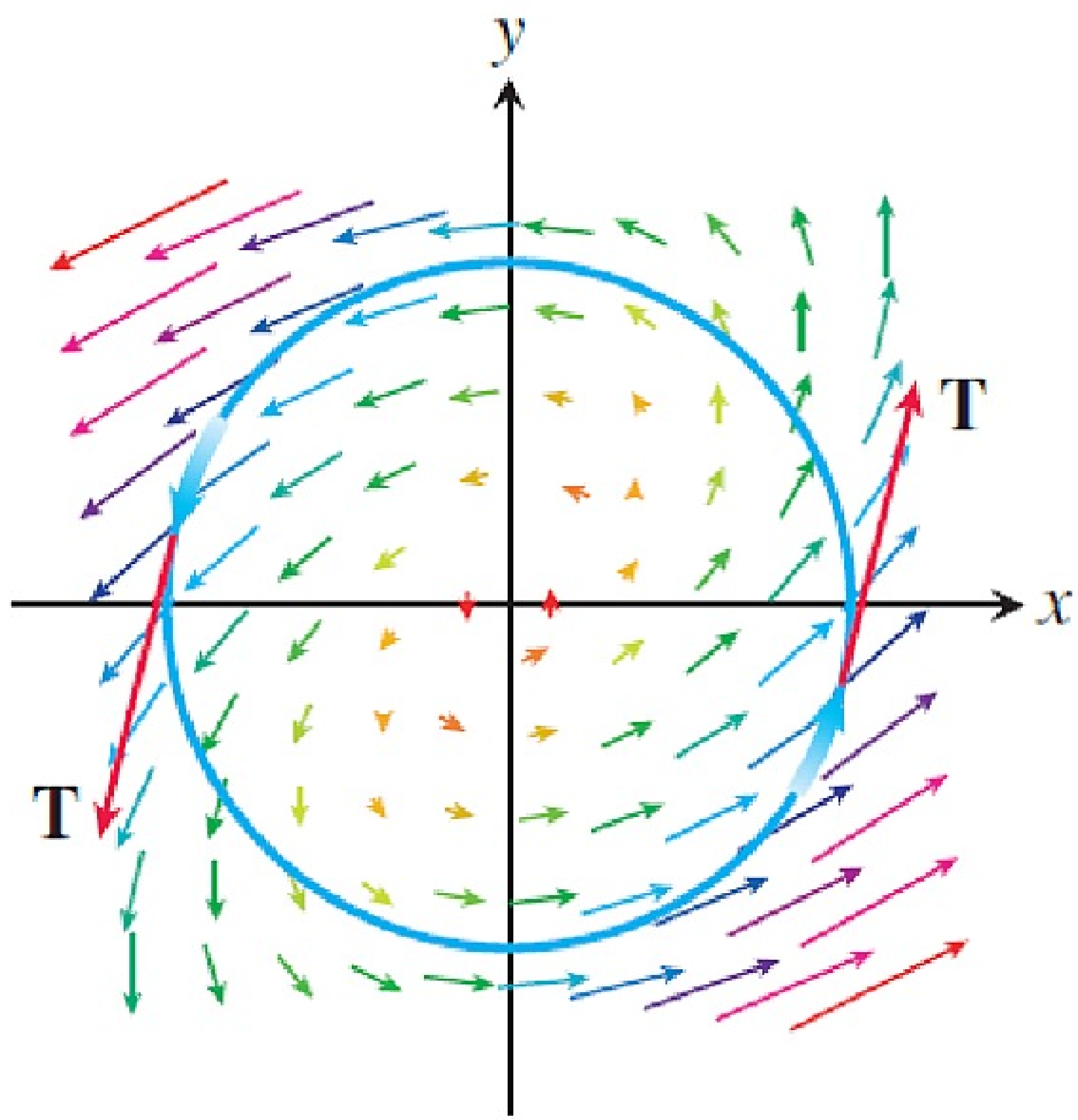


FIGURE 15.35 The vector field in Example 3 has a counterclockwise circulation of 2π around the unit circle.

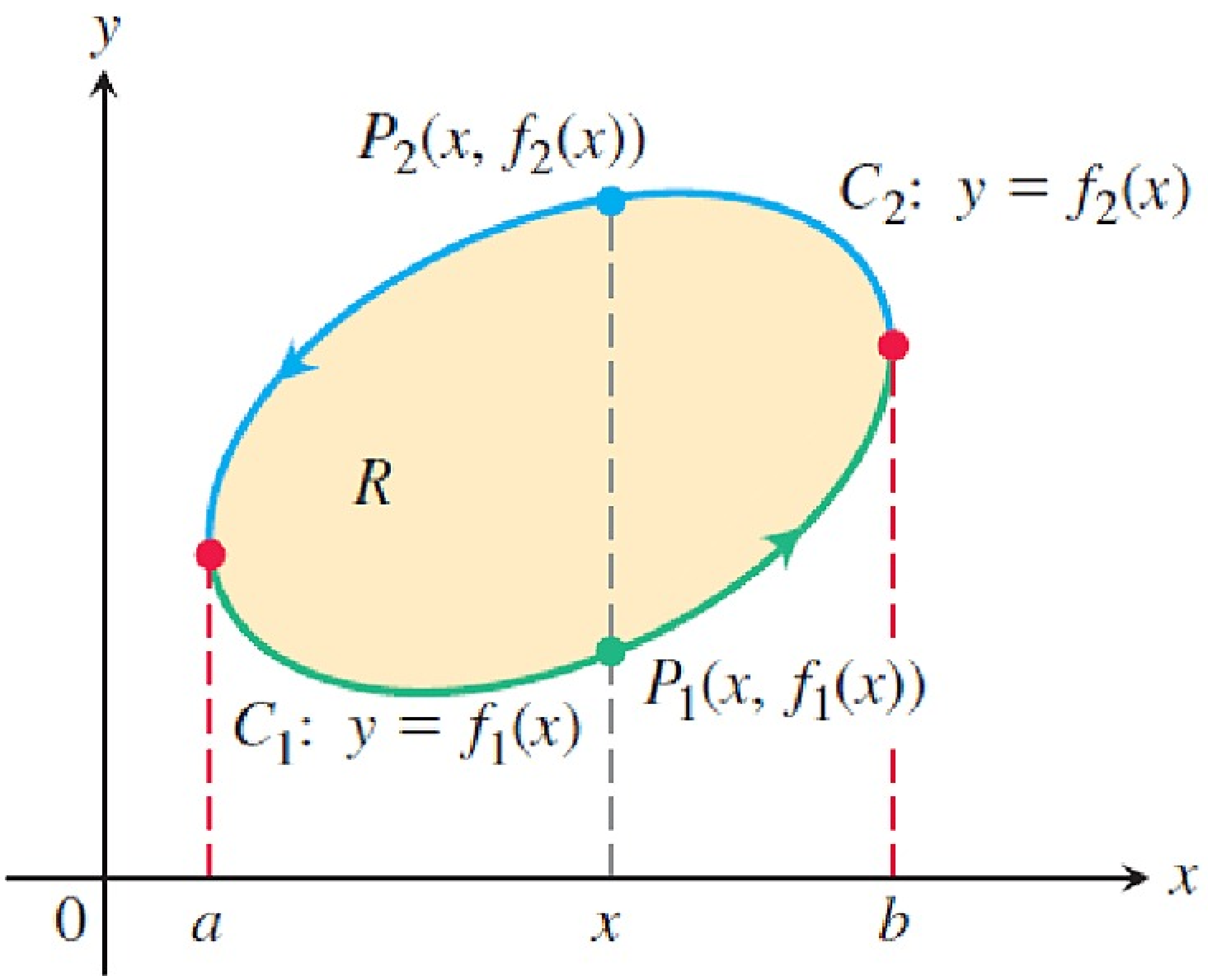


FIGURE 15.36 The boundary curve C is made up of C_1, the graph of $y = f_1(x)$, and C_2, the graph of $y = f_2(x)$.

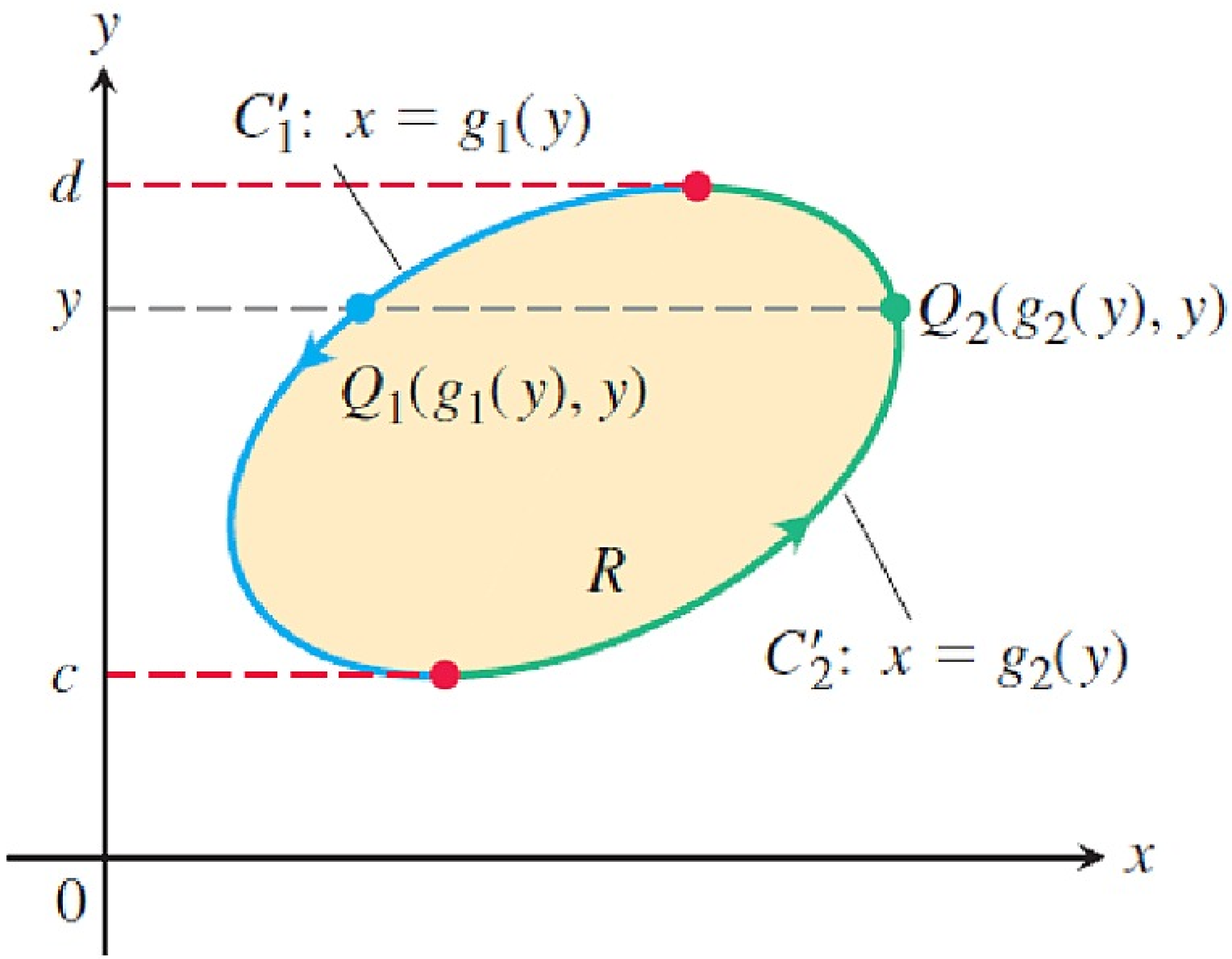


FIGURE 15.37 The boundary curve C is made up of C_1', the graph of $x = g_1(y)$, and C_2', the graph of $x = g_2(y)$.

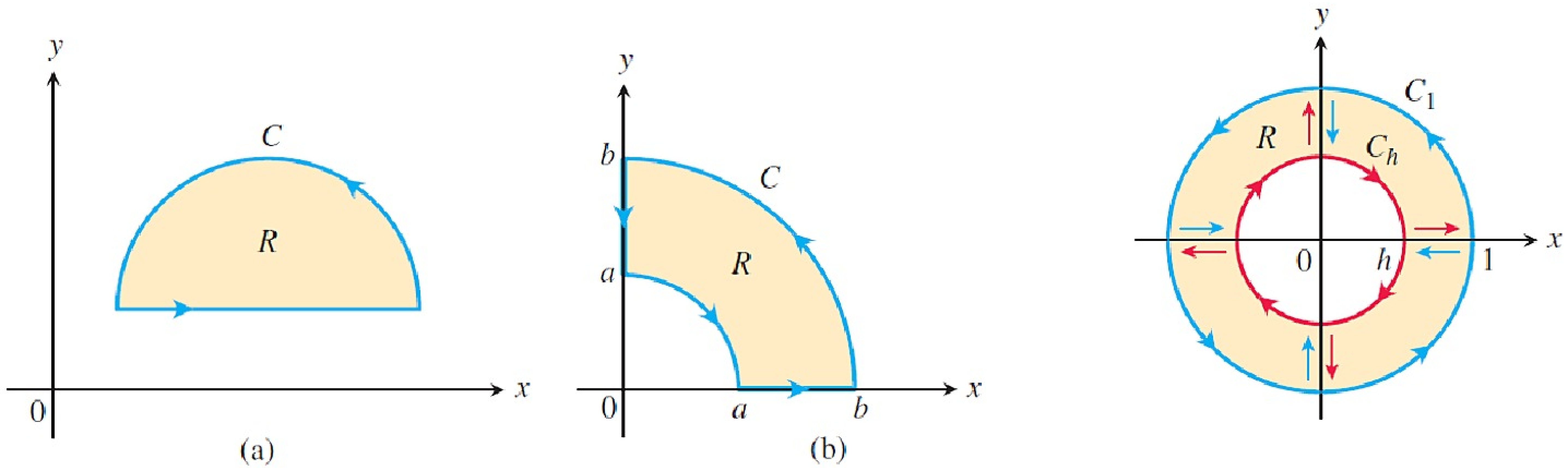


FIGURE 15.38 Other regions to which Green's Theorem applies. In (c) the axes convert the region into four simply connected regions, and we sum the line integrals along the oriented boundaries.

Section 15.5

Surfaces and Area

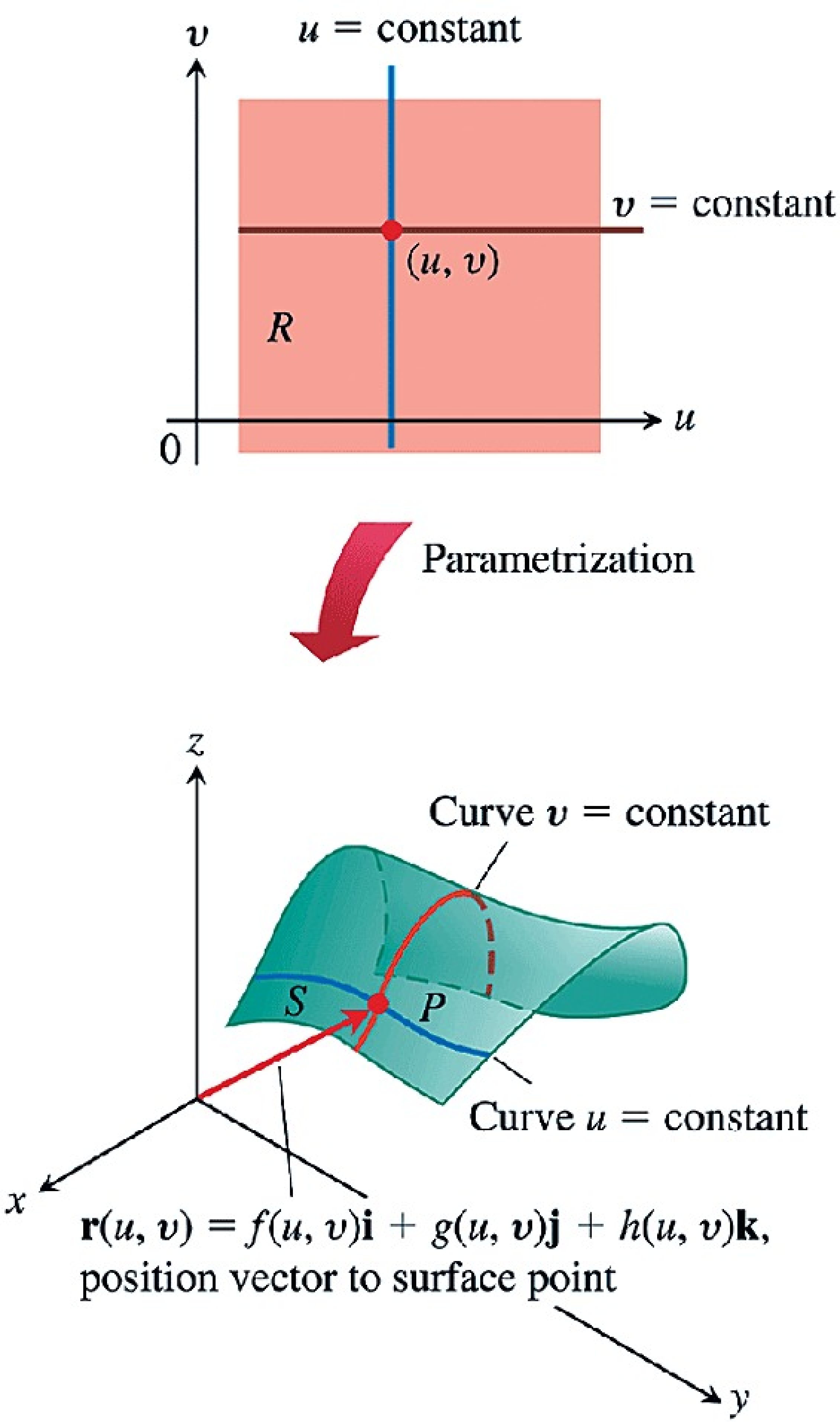


FIGURE 15.39 A parametrized surface S expressed as a vector function of two variables defined on a region R.

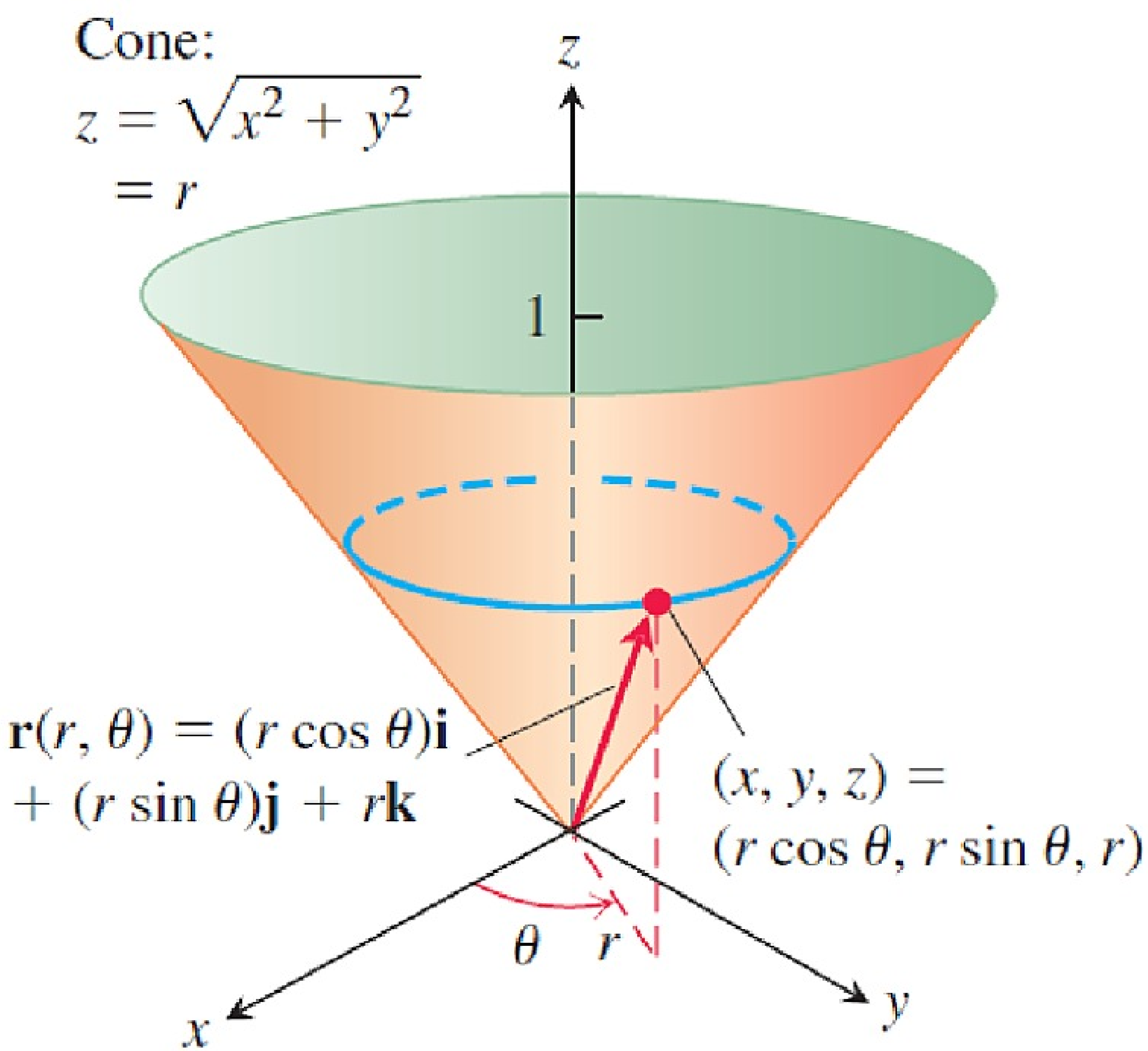


FIGURE 15.40 The cone in Example 1 can be parametrized using cylindrical coordinates.

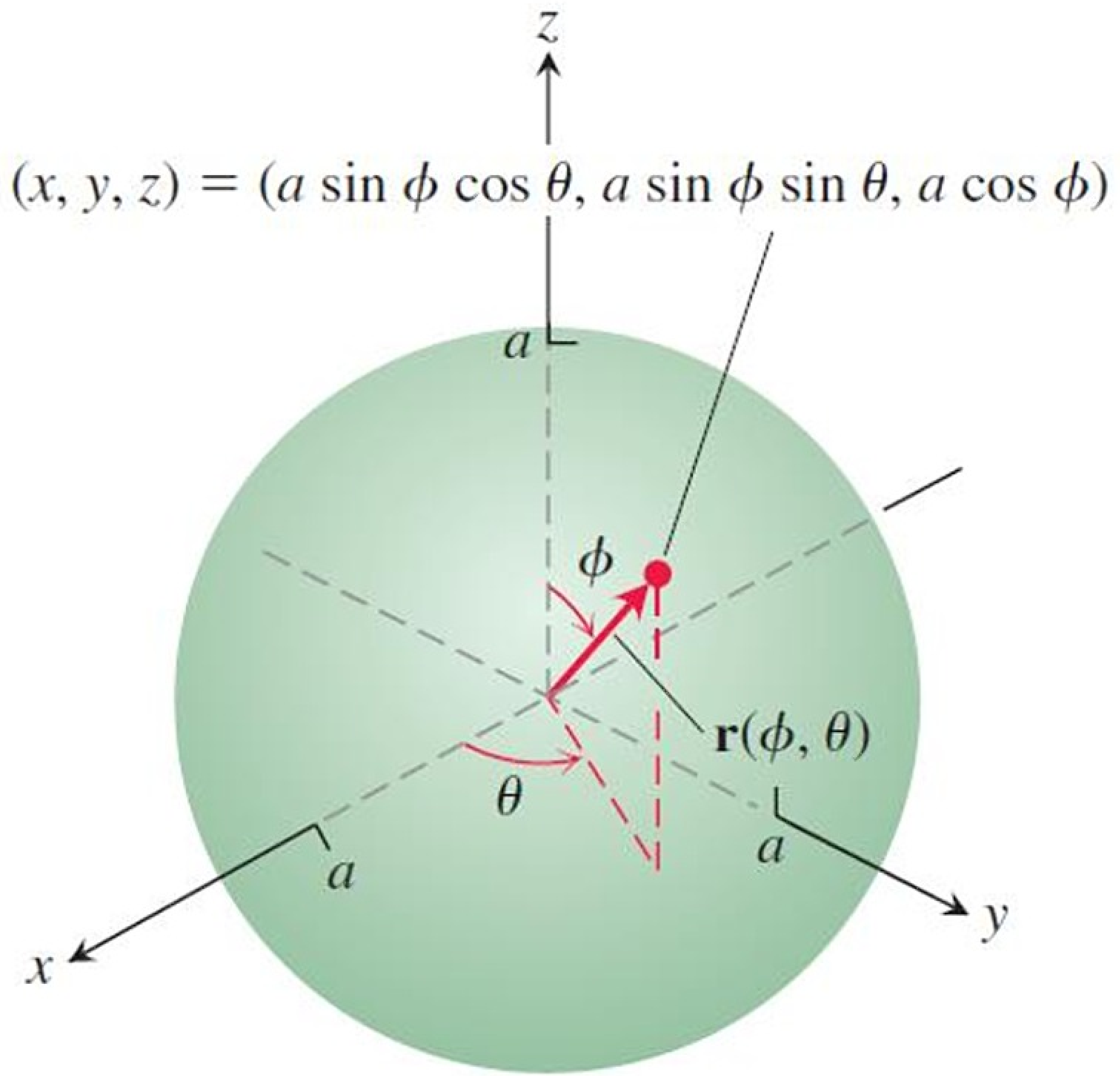


FIGURE 15.41 The sphere in Example 2 can be parametrized using spherical coordinates.

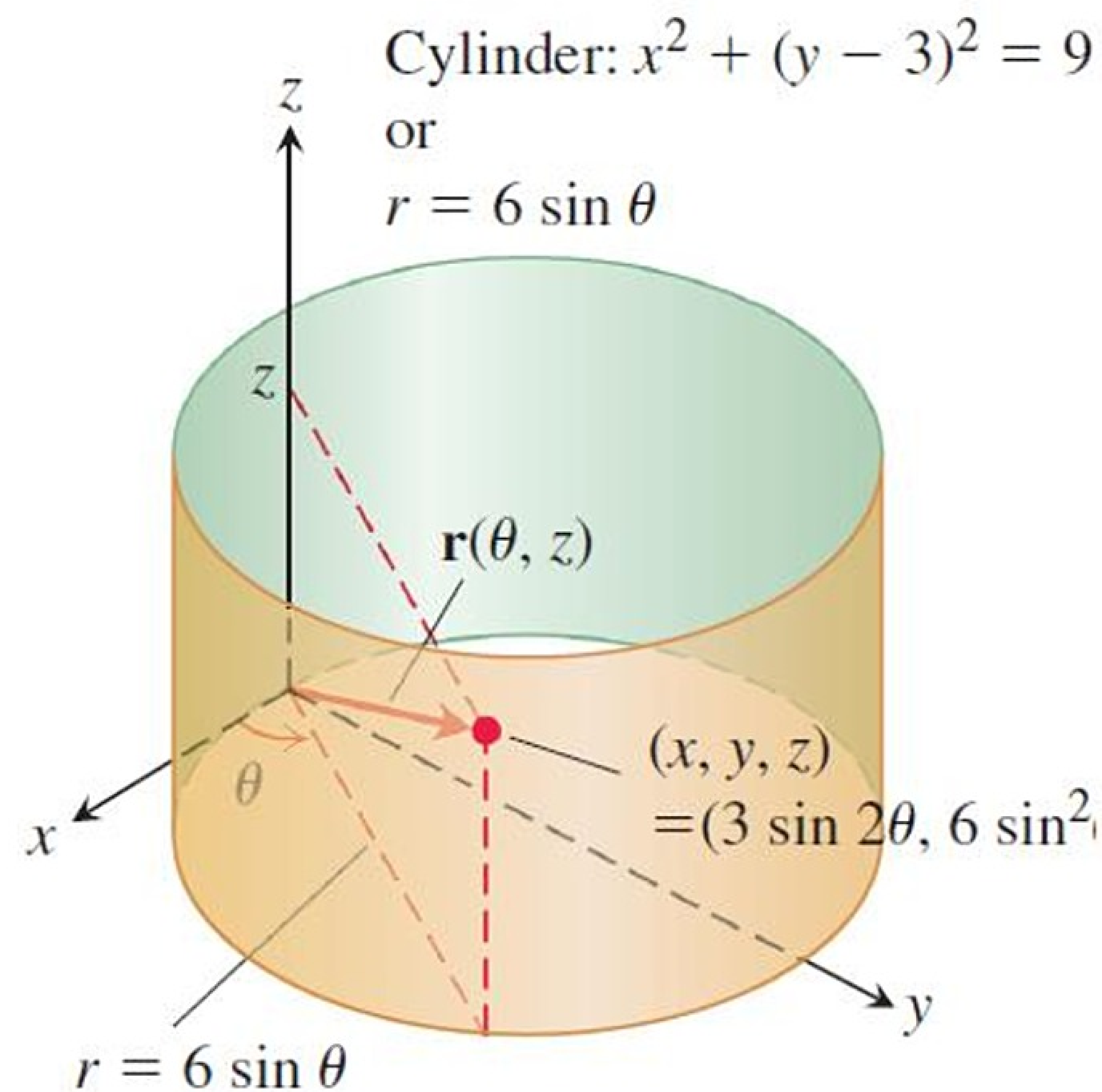


FIGURE 15.42 The cylinder in Example 3 can be parametrized using cylindrical coordinates.

DEFINITION A parametrized surface $\mathbf{r}(u, v) = f(u, v)\mathbf{i} + g(u, v)\mathbf{j} + h(u, v)\mathbf{k}$ is **smooth** if $\mathbf{r}_u$ and $\mathbf{r}_v$ are continuous and $\mathbf{r}_u \times \mathbf{r}_v$ is never zero on the interior of the parameter domain.

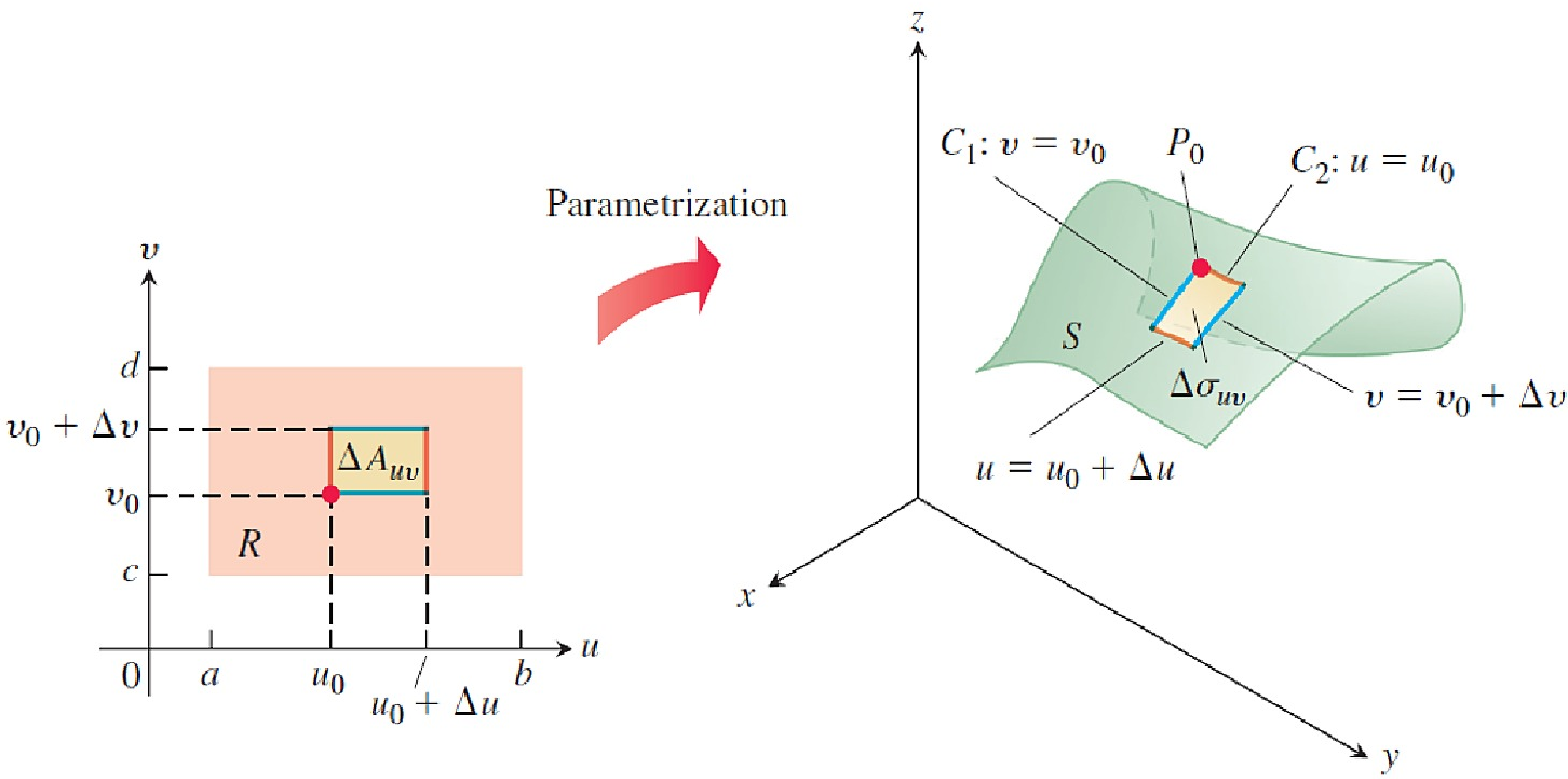


FIGURE 15.43 A rectangular area element ΔA_{uv} in the uv-plane maps onto a curved patch element $\Delta\sigma_{uv}$ on S.

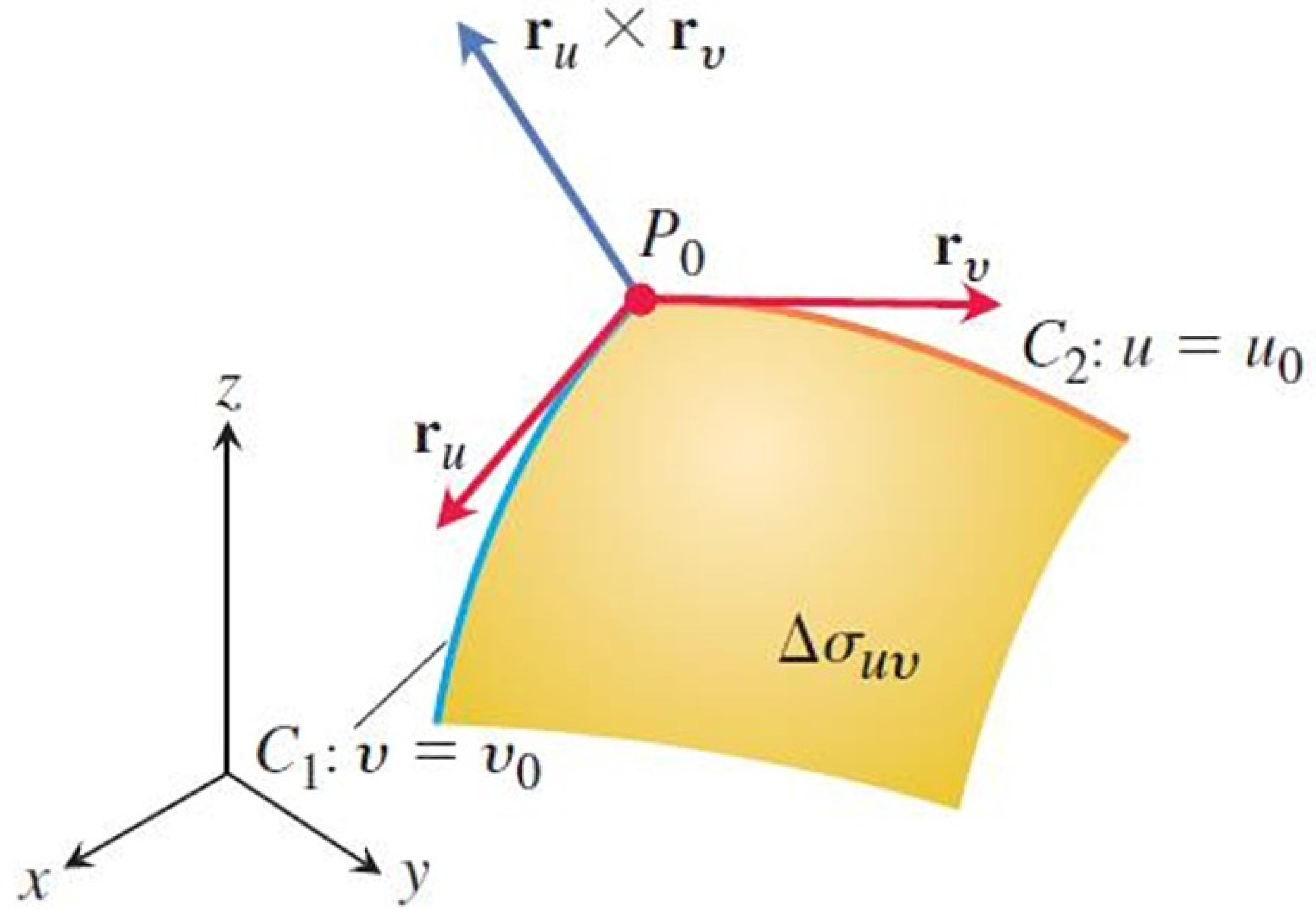


FIGURE 15.44 A magnified view of a surface patch element $\Delta\sigma_{uv}$.

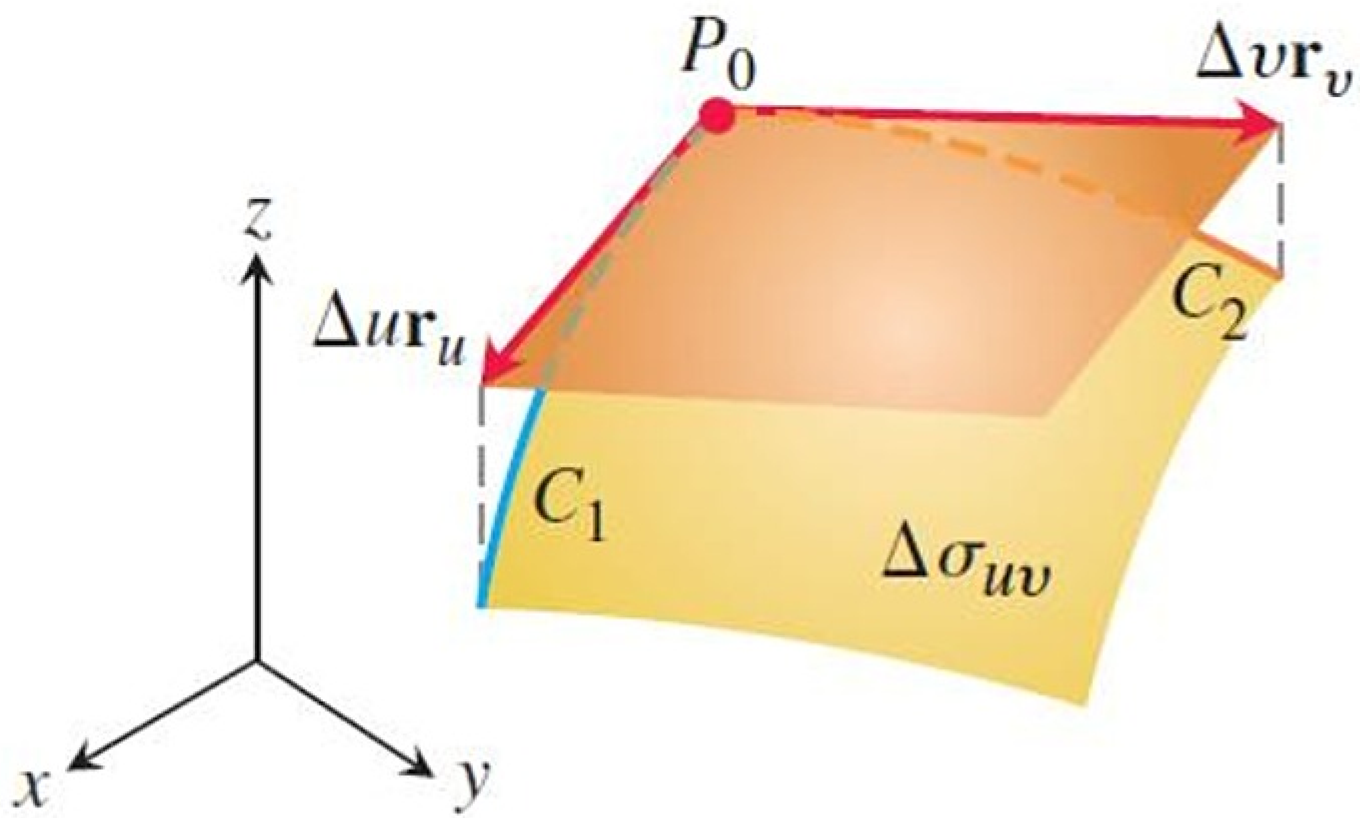


FIGURE 15.45 The area of the parallelogram determined by the vectors $\Delta u\,\mathbf{r}_u$ and $\Delta v\,\mathbf{r}_v$ approximates the area of the surface patch element $\Delta\sigma_{uv}$.

DEFINITION The **area** of the smooth surface

$$\mathbf{r}(u, v) = f(u, v)\mathbf{i} + g(u, v)\mathbf{j} + h(u, v)\mathbf{k}, \qquad a \le u \le b, \quad c \le v \le d$$

is

$$A = \iint_R |\mathbf{r}_u \times \mathbf{r}_v|\, dA = \int_c^d \int_a^b |\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv. \tag{4}$$

Surface Area Differential for a Parametrized Surface

$$d\sigma = |\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv \qquad \iint_S d\sigma \qquad (5)$$

Surface area differential, also called surface area element

Differential formula for surface area

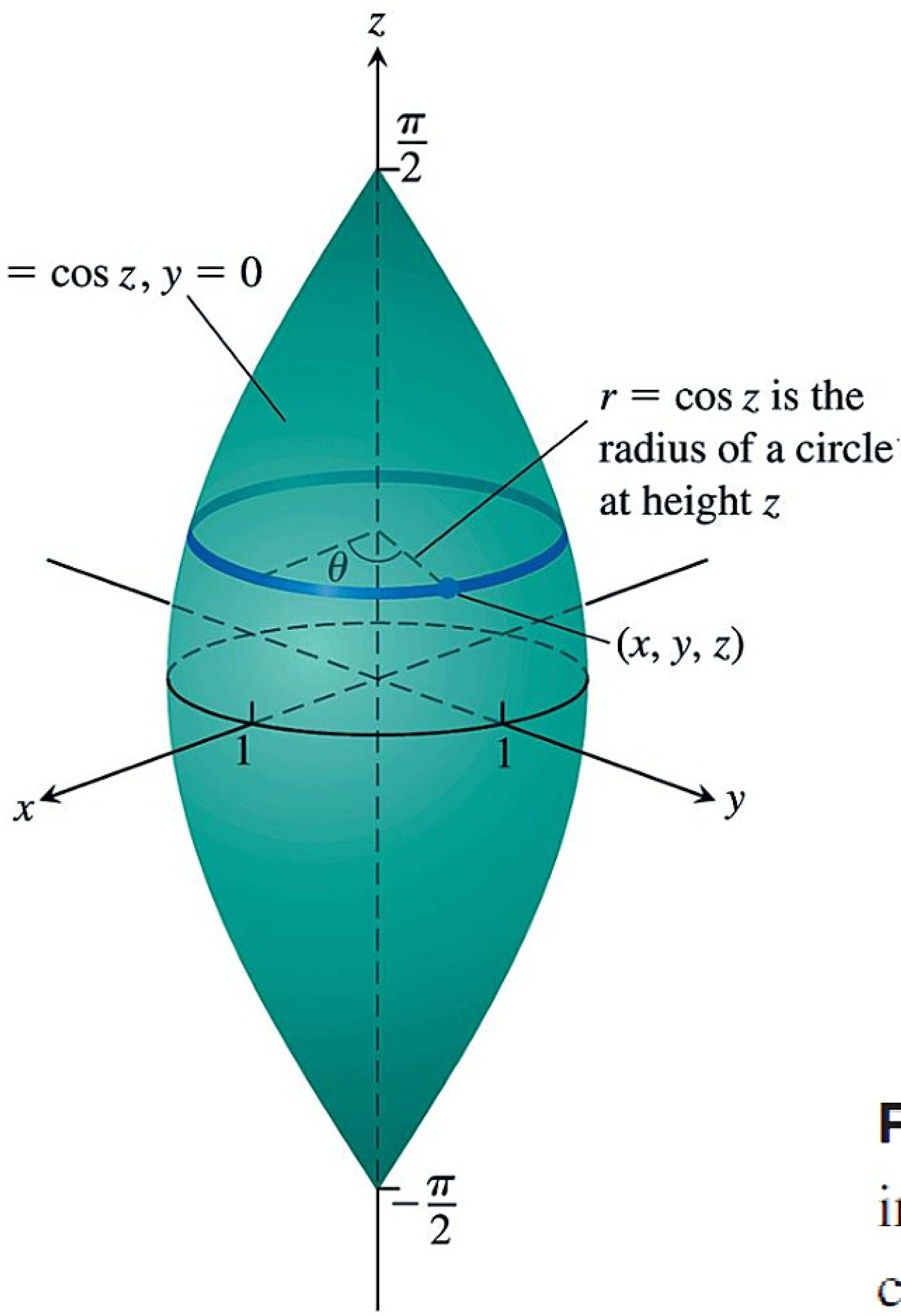


FIGURE 15.46 The "football" surface in Example 6 obtained by rotating the curve $x = \cos z$ about the z-axis.

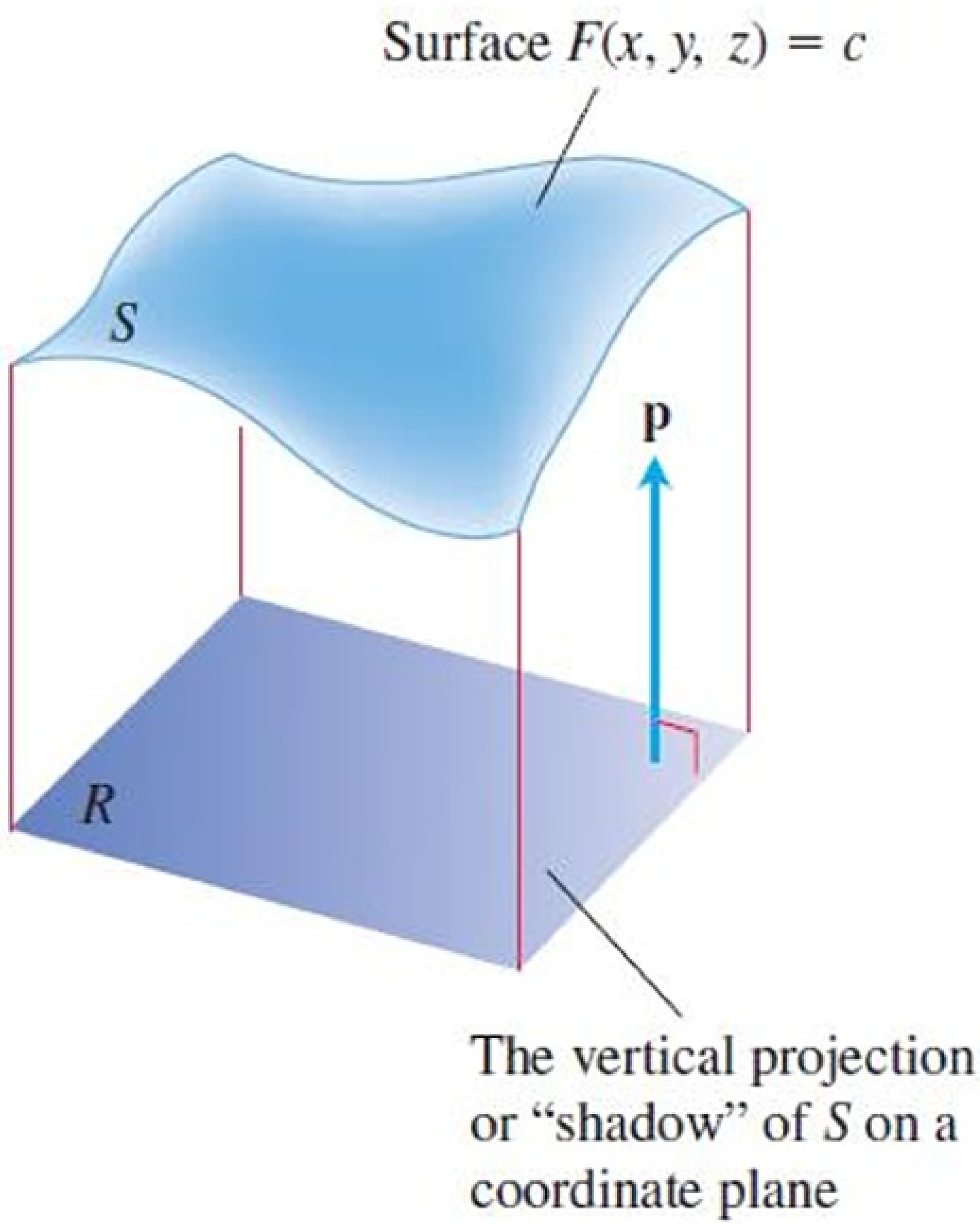


FIGURE 15.47 As we soon see, the area of a surface S in space can be calculated by evaluating a related double integral over the vertical projection or "shadow" of S on a coordinate plane. The unit vector $\mathbf{p}$ is normal to the plane.

Formula for the Surface Area of an Implicit Surface

The area of the surface $F(x, y, z) = c$ over a closed and bounded plane region R is

$$\text{Surface area} = \iint_R \frac{|\nabla F|}{|\nabla F \cdot \mathbf{p}|}\, dA, \tag{7}$$

where $\mathbf{p} = \mathbf{i}, \mathbf{j}$, or $\mathbf{k}$ is normal to R and $\nabla F \cdot \mathbf{p} \neq 0$.

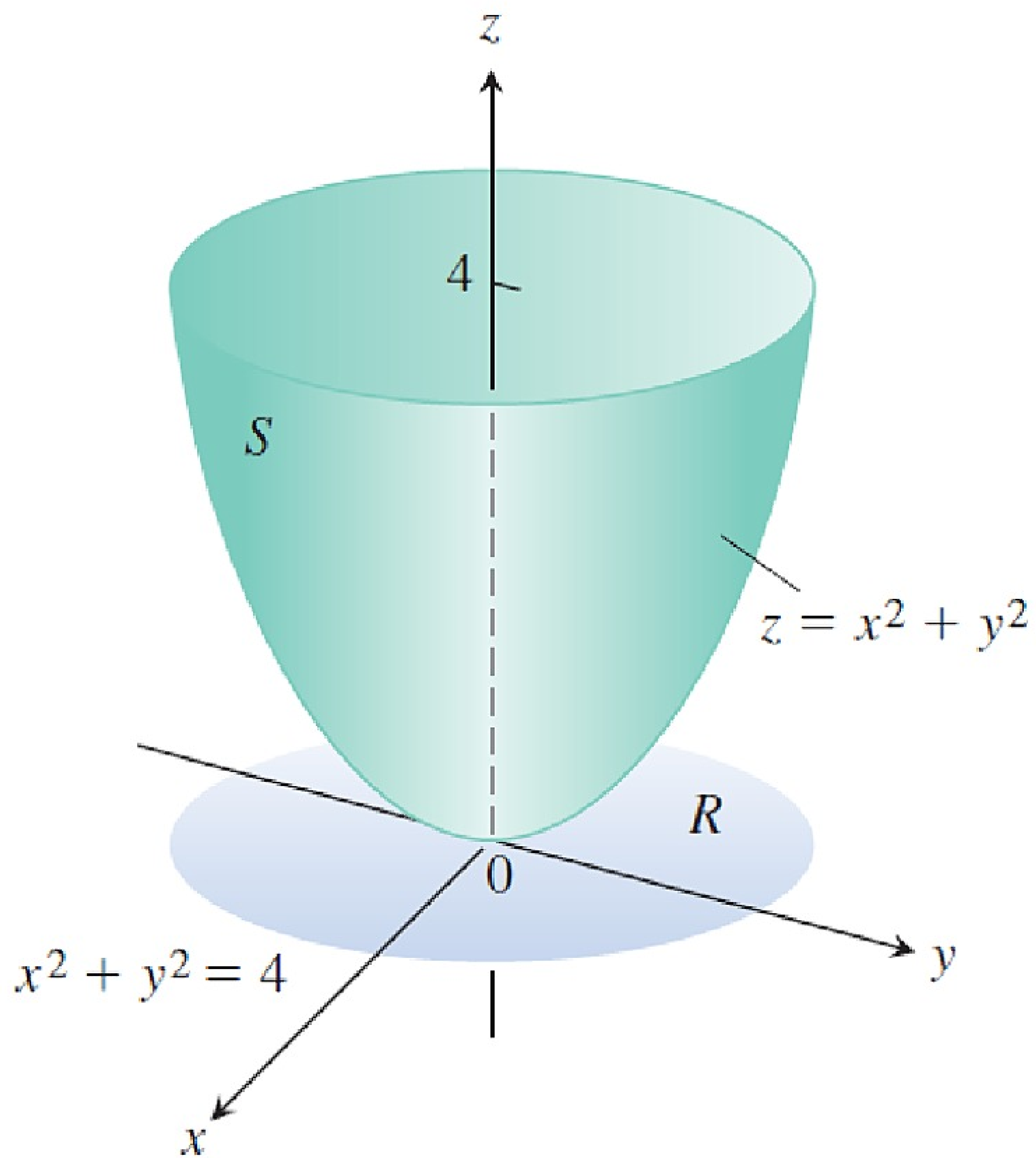


FIGURE 15.48 The area of this parabolic surface is calculated in Example 7.

Formula for the Surface Area of a Graph $z = f(x, y)$

For a graph $z = f(x, y)$ over a region R in the xy-plane, the surface area formula is

$$A = \iint_R \sqrt{f_x^2 + f_y^2 + 1}\, dx\, dy. \tag{8}$$

Section 15.6

Surface Integrals

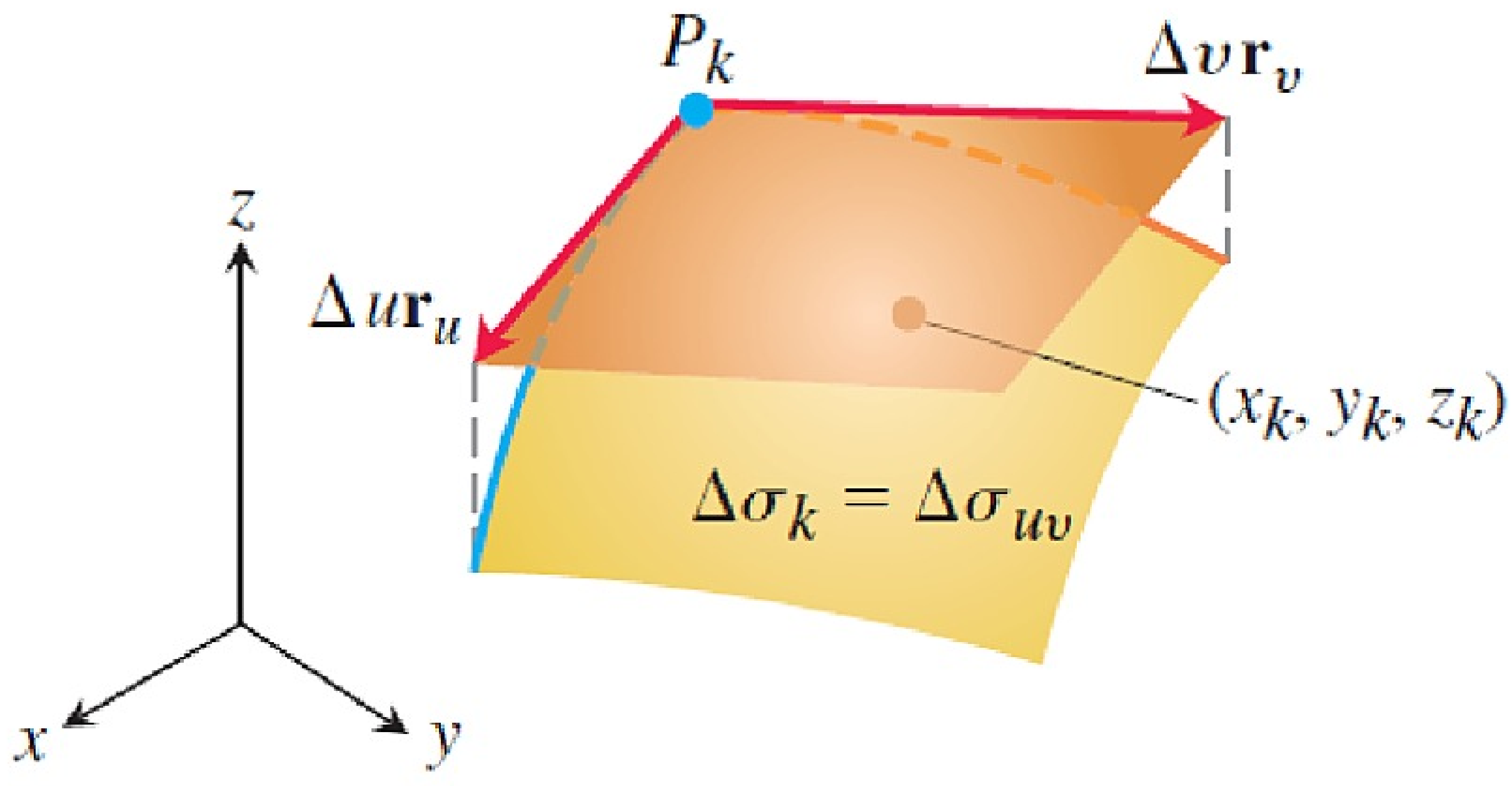


FIGURE 15.49 The area of the patch $\Delta\sigma_k$ is approximated by the area of the tangent parallelogram determined by the vectors $\Delta u\,\mathbf{r}_u$ and $\Delta v\,\mathbf{r}_v$. The point (x_k, y_k, z_k) lies on the surface patch, beneath the parallelogram shown here.

$$\iint_S G(x, y, z)\, d\sigma = \lim_{n\to\infty} \sum_{k=1}^{n} G(x_k, y_k, z_k)\, \Delta\sigma_k. \qquad (1)$$

Formulas for a Surface Integral

1. For a smooth surface S defined **parametrically** as $\mathbf{r}(u, v) = f(u, v)\mathbf{i} + g(u, v)\mathbf{j} + h(u, v)\mathbf{k}$, $(u, v) \in R$, and a continuous function $G(x, y, z)$ defined on S, the surface integral of G over S is given by the double integral over R,

$$\iint_S G(x, y, z)\, d\sigma = \iint_R G(f(u, v), g(u, v), h(u, v))\, |\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv. \qquad (2)$$

2. For a surface S given **implicitly** by $F(x, y, z) = c$, where F is a continuously differentiable function, with S lying above its closed and bounded shadow region R in the coordinate plane beneath it, the surface integral of the continuous function G over S is given by the double integral over R,

$$\iint_S G(x, y, z)\, d\sigma = \iint_R G(x, y, z) \frac{|\nabla F|}{|\nabla F \cdot \mathbf{p}|}\, dA, \qquad (3)$$

where $\mathbf{p}$ is a unit vector normal to R and $\nabla F \cdot \mathbf{p} \neq 0$.

3. For a surface S given **explicitly** as the graph of $z = f(x, y)$, where f is a continuously differentiable function over a region R in the xy-plane, the surface integral of the continuous function G over S is given by the double integral over R,

$$\iint_S G(x, y, z)\, d\sigma = \iint_R G(x, y, f(x, y)) \sqrt{f_x^{\,2} + f_y^{\,2} + 1}\, dx\, dy. \qquad (4)$$

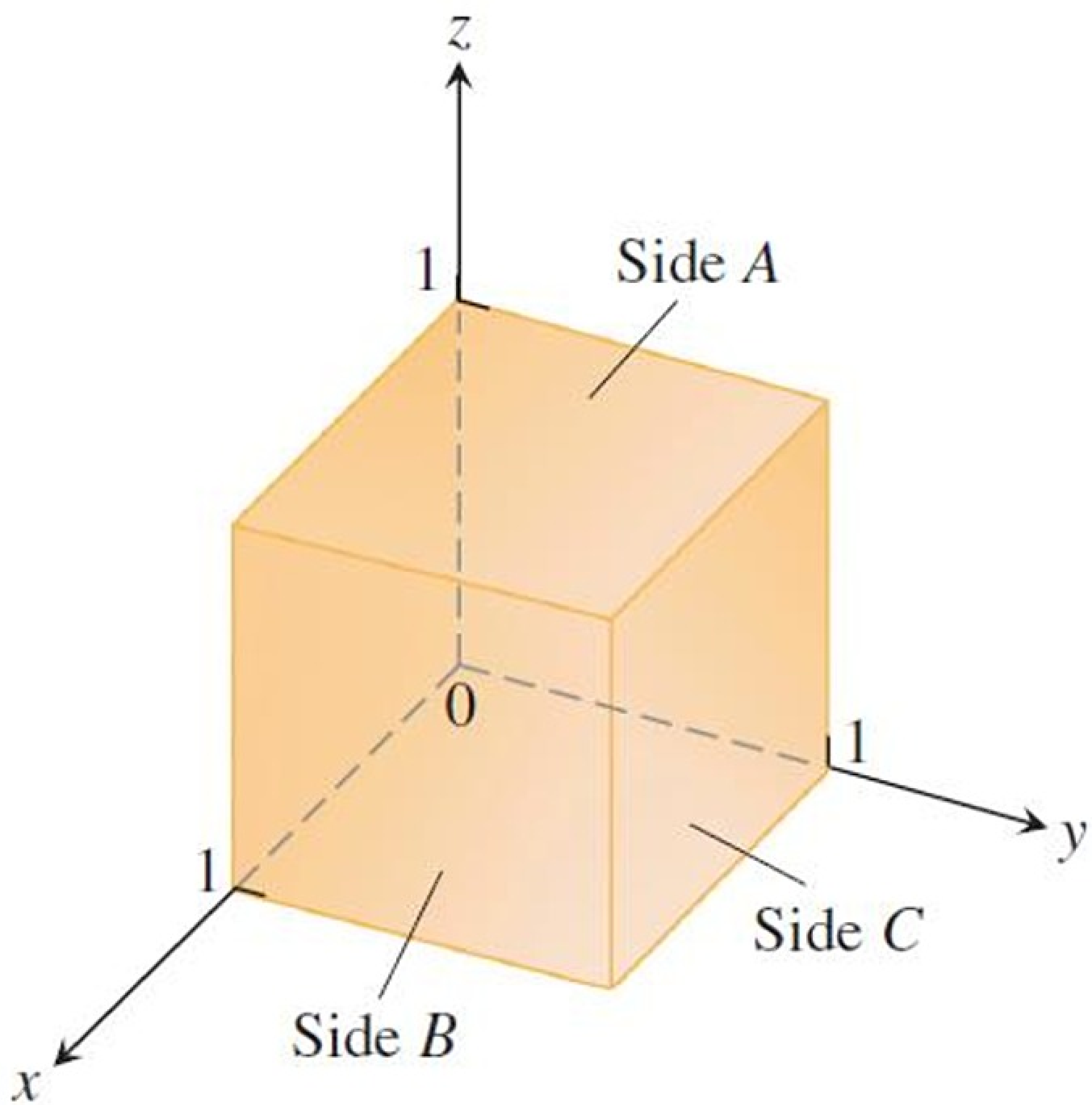


FIGURE 15.50 The cube in Example 2.

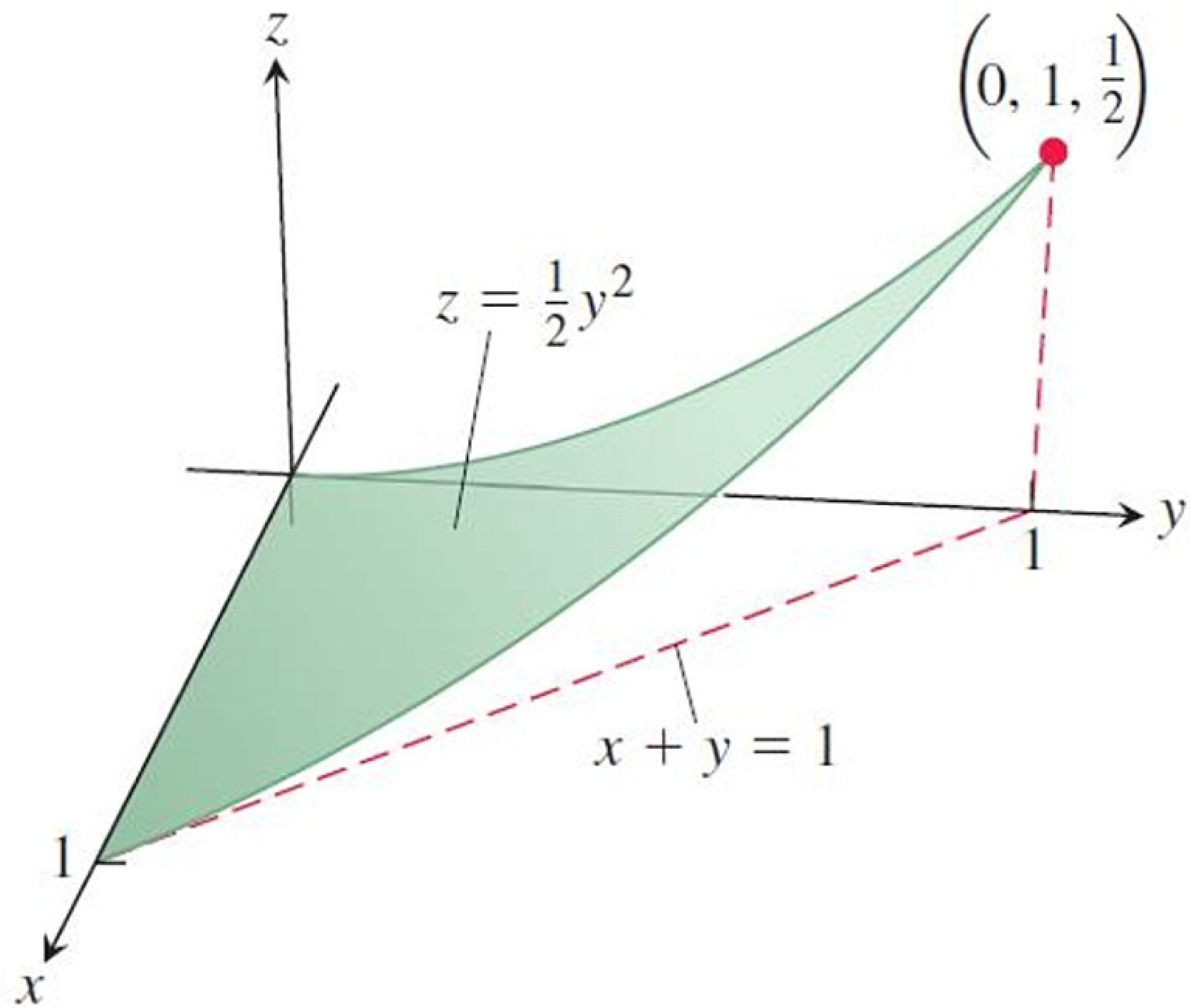


FIGURE 15.51 The surface S in Example 4.

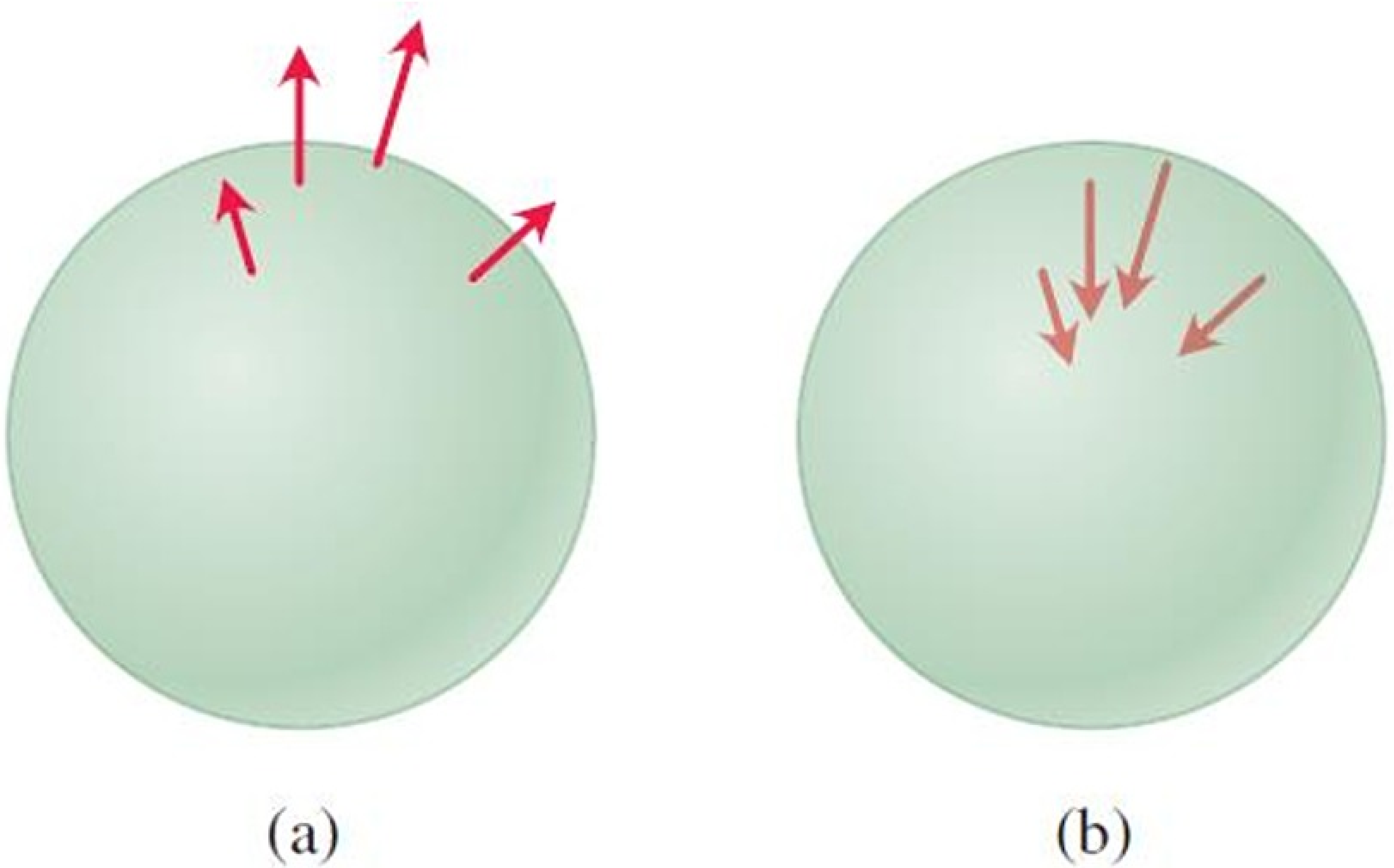


FIGURE 15.52 An outward-pointing vector field (a) and an inward-pointing vector field (b) give the two possible orientations of a sphere.

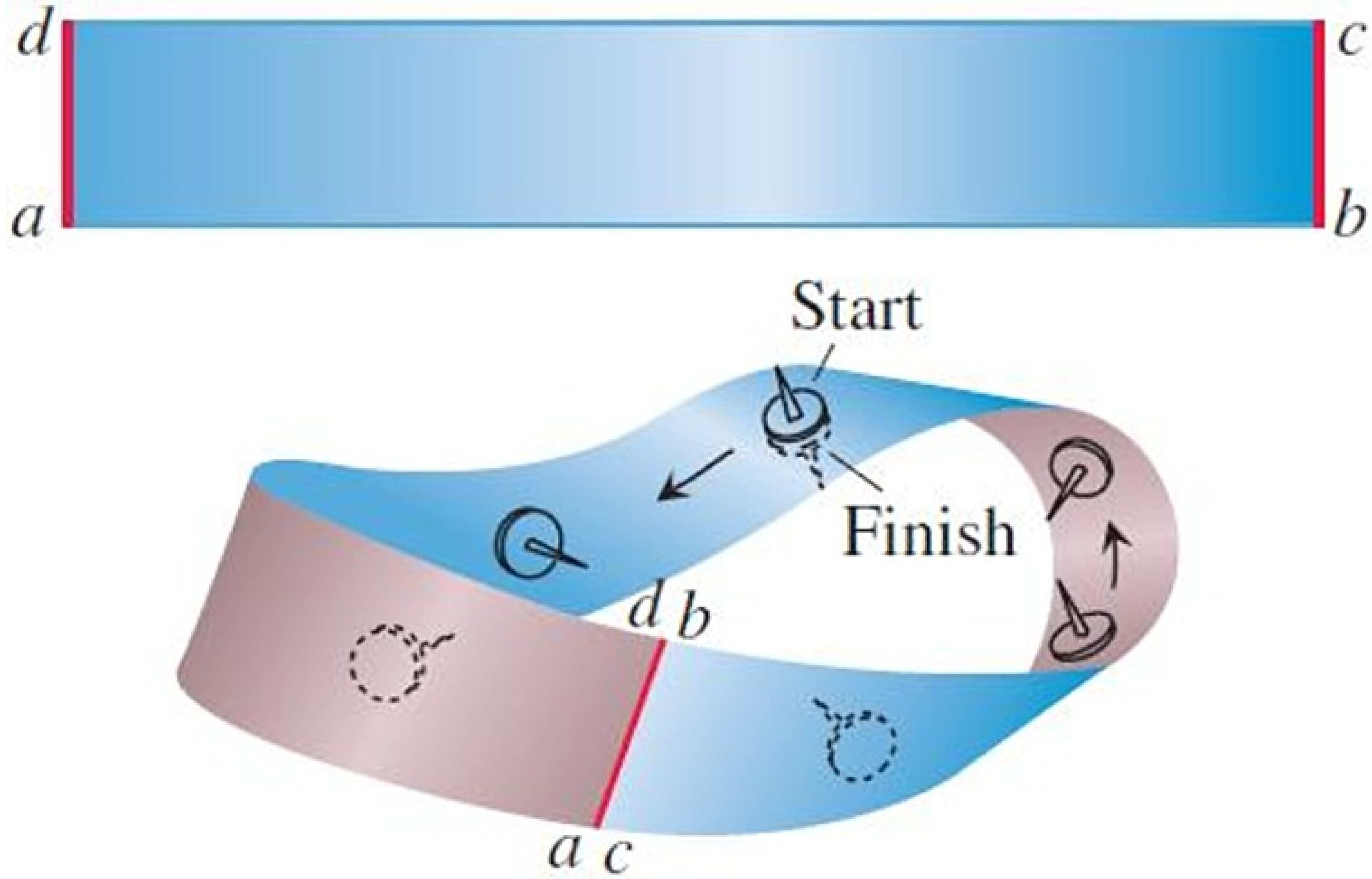


FIGURE 15.53 To make a Möbius band, take a rectangular strip of paper *abcd*, give the end *bc* a single twist, and paste the ends of the strip together to match *a* with *c* and *b* with *d*. The Möbius band is a nonorientable or one-sided surface.

DEFINITION Let $\mathbf{F}$ be a vector field in three-dimensional space with continuous components defined over a smooth surface S having a chosen field of normal unit vectors $\mathbf{n}$ orienting S. Then the **surface integral of F over *S*** is

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma. \tag{5}$$

This integral is also called the **flux** of the vector field $\mathbf{F}$ across S.

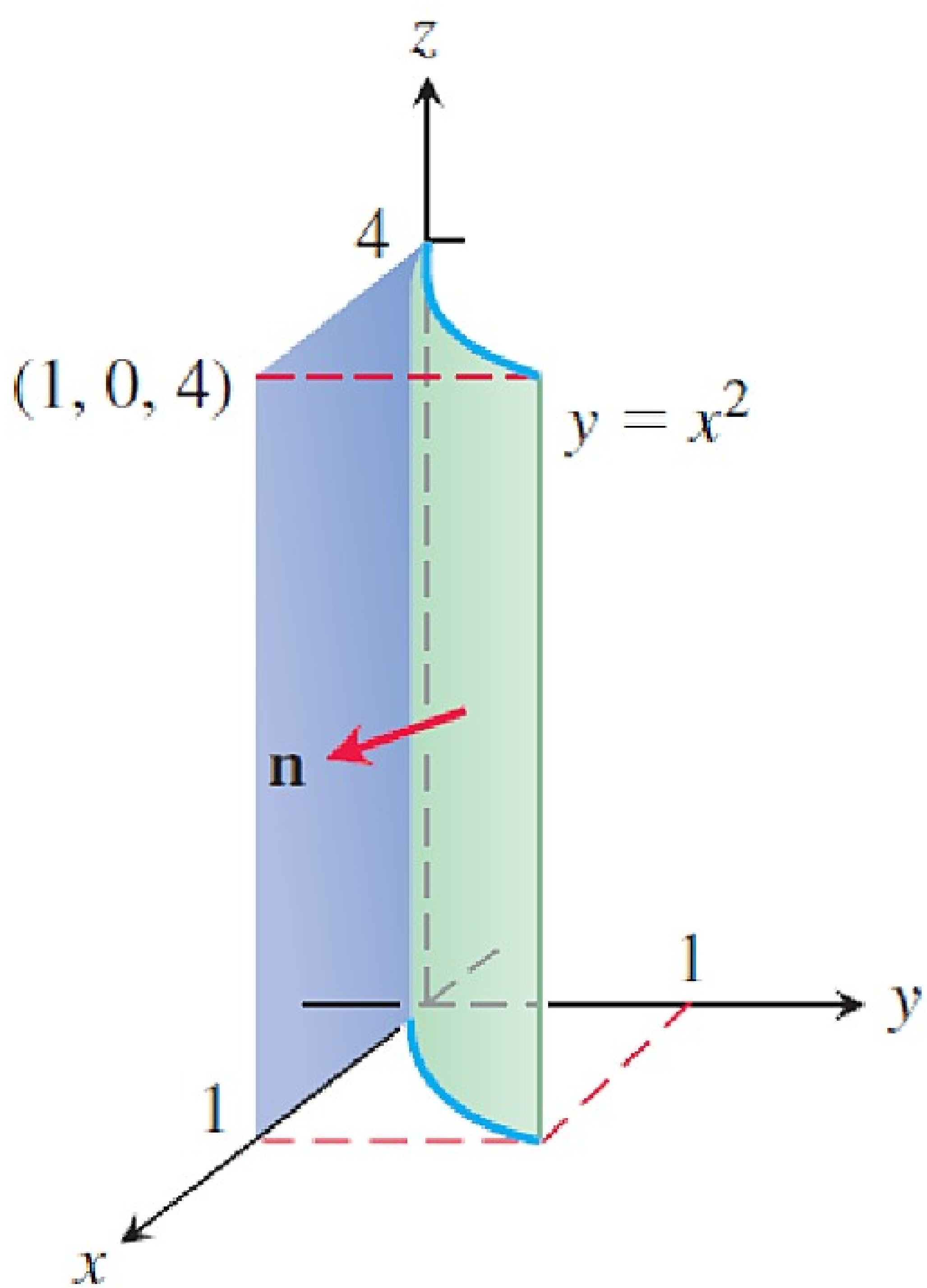


FIGURE 15.54 Finding the flux through the surface of a parabolic cylinder (Example 5).

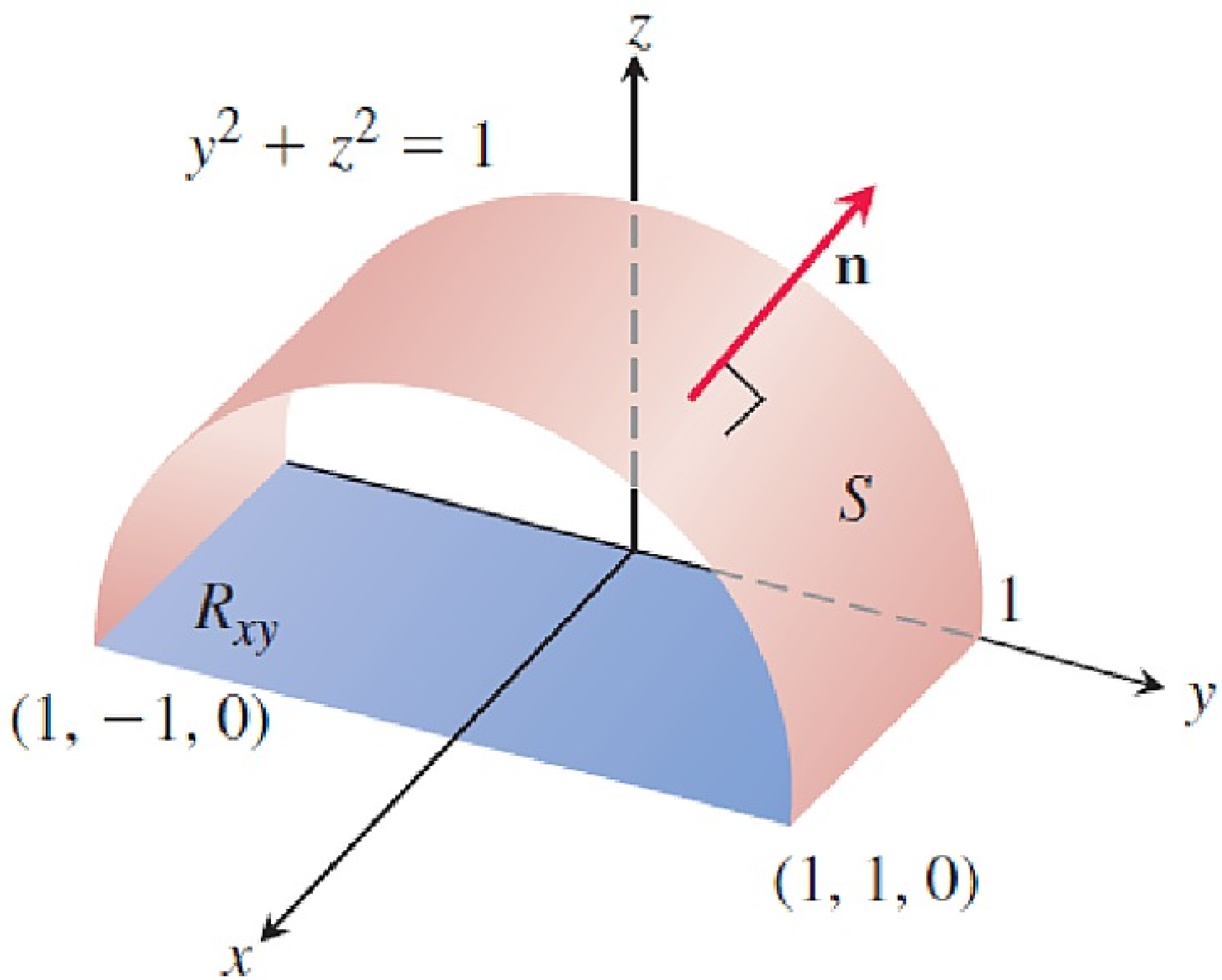


FIGURE 15.55 Calculating the flux of a vector field outward through the surface S. The area of the shadow region R_{xy} is 2 (Example 6).

TABLE 15.3 Mass and moment formulas for very thin shells

Mass: $M = \iint_S \delta \, d\sigma$ $\quad \delta = \delta(x, y, z) =$ density at (x, y, z) is mass per unit area

First moments about the coordinate planes:

$$M_{yz} = \iint_S x\,\delta\,d\sigma, \qquad M_{xz} = \iint_S y\,\delta\,d\sigma, \qquad M_{xy} = \iint_S z\,\delta\,d\sigma$$

Coordinates of center of mass:

$$\bar{x} = M_{yz}/M, \qquad \bar{y} = M_{xz}/M, \qquad \bar{z} = M_{xy}/M$$

Moments of inertia about coordinate axes:

$$I_x = \iint_S (y^2 + z^2)\,\delta\,d\sigma, \quad I_y = \iint_S (x^2 + z^2)\,\delta\,d\sigma, \quad I_z = \iint_S (x^2 + y^2)\,\delta\,d\sigma,$$

$I_L = \iint_S r^2 \delta \, d\sigma$ $\quad r(x, y, z) =$ distance from point (x, y, z) to line L

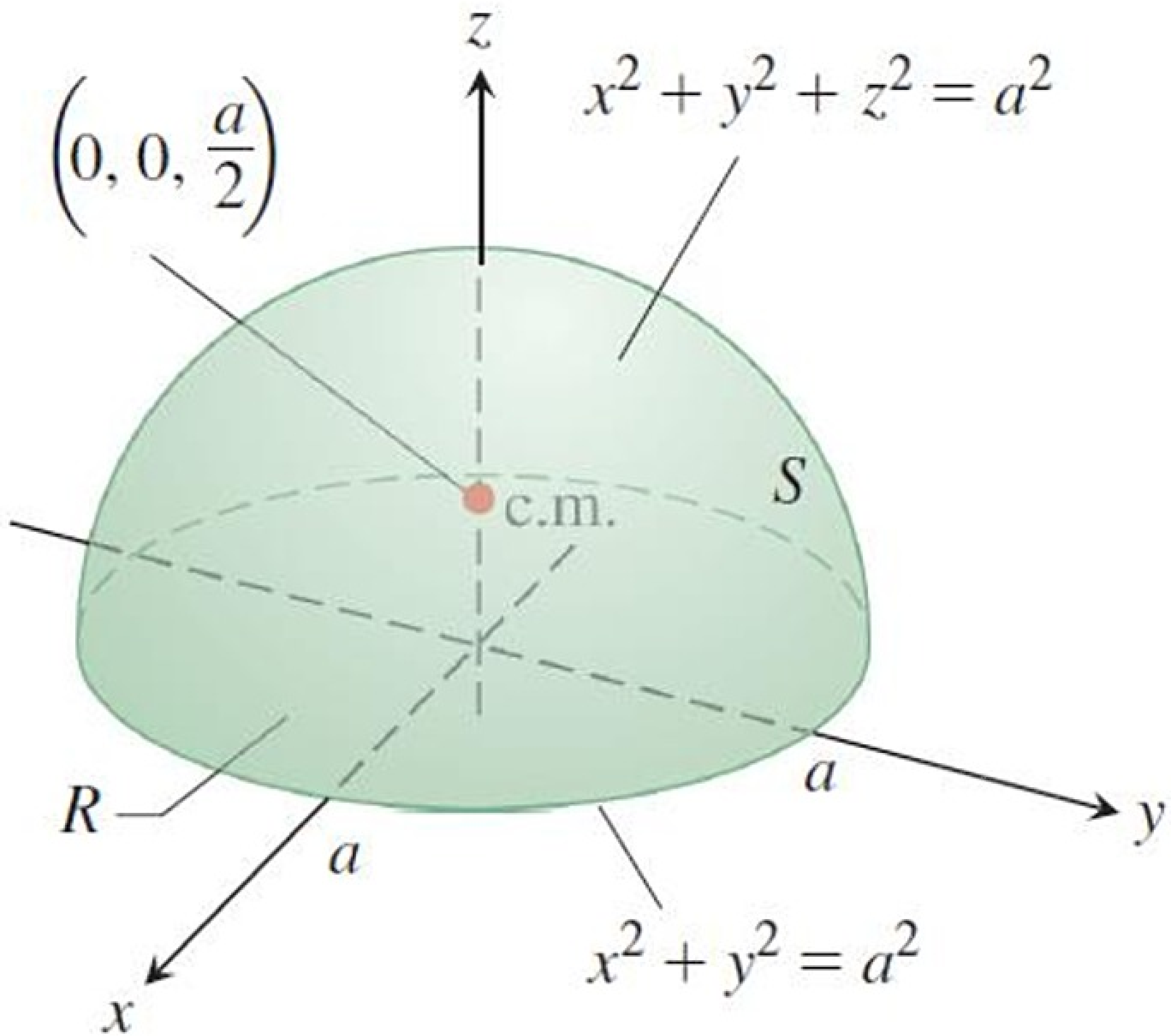


FIGURE 15.56 The center of mass of a thin hemispherical shell of constant density lies on the axis of symmetry halfway from the base to the top (Example 7).

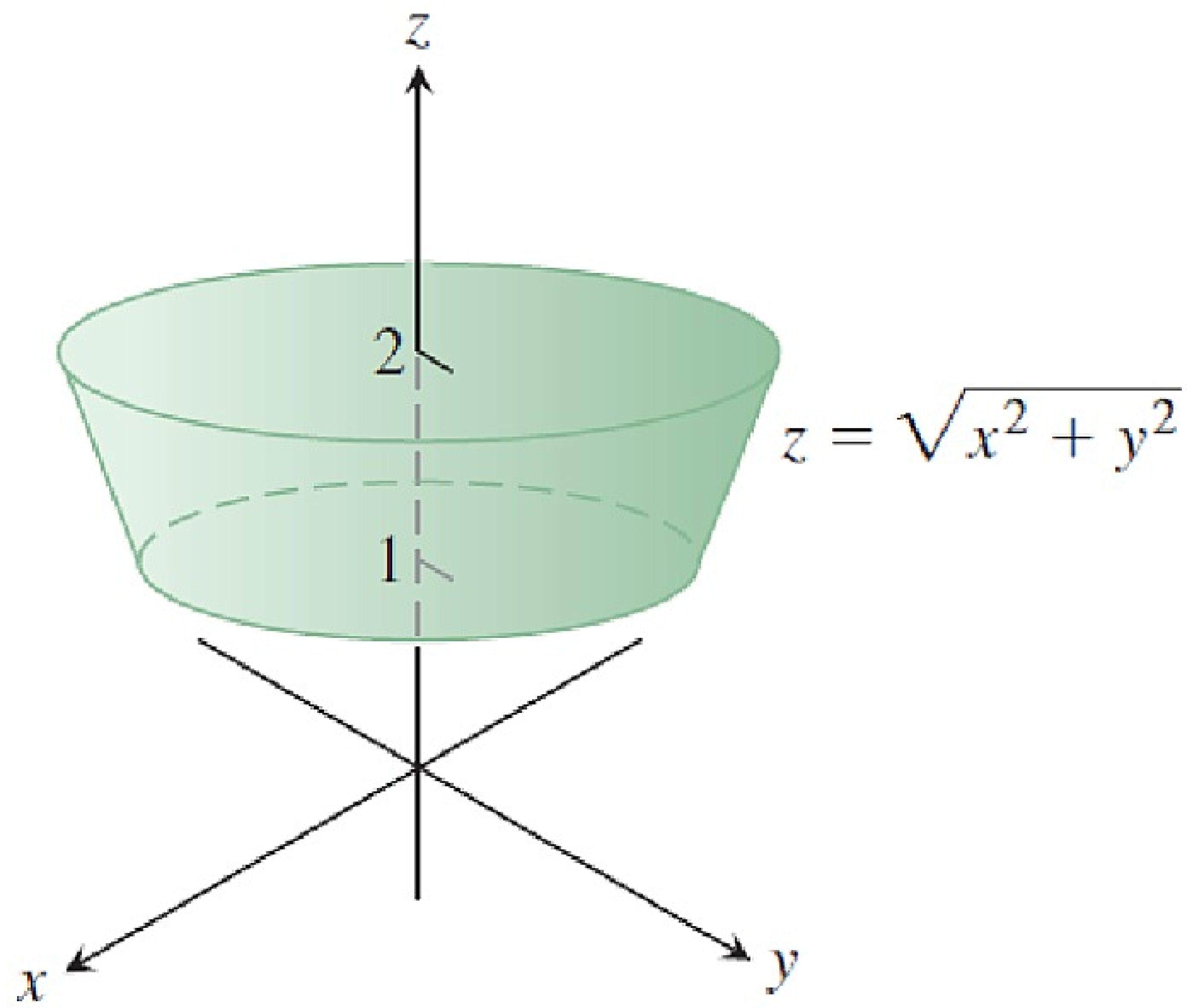


FIGURE 15.57 The cone frustum formed when the cone $z = \sqrt{x^2 + y^2}$ is cut by the planes $z = 1$ and $z = 2$ (Example 8).

Section 15.7

Stokes' Theorem

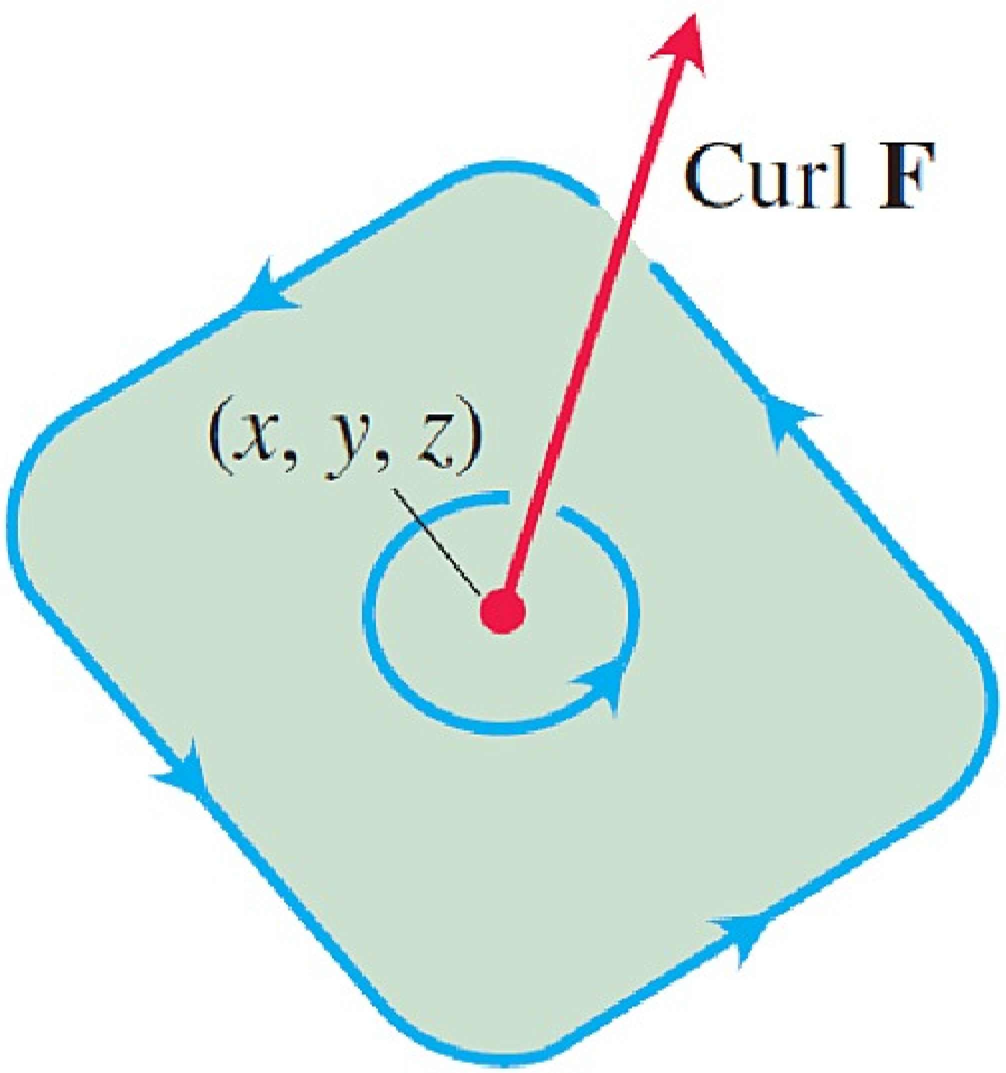


FIGURE 15.58 The circulation vector at a point (x, y, z) in a plane in a three-dimensional fluid flow. Notice its right-hand relation to the rotating particles in the fluid.

$$\boxed{\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} \qquad (3)}$$

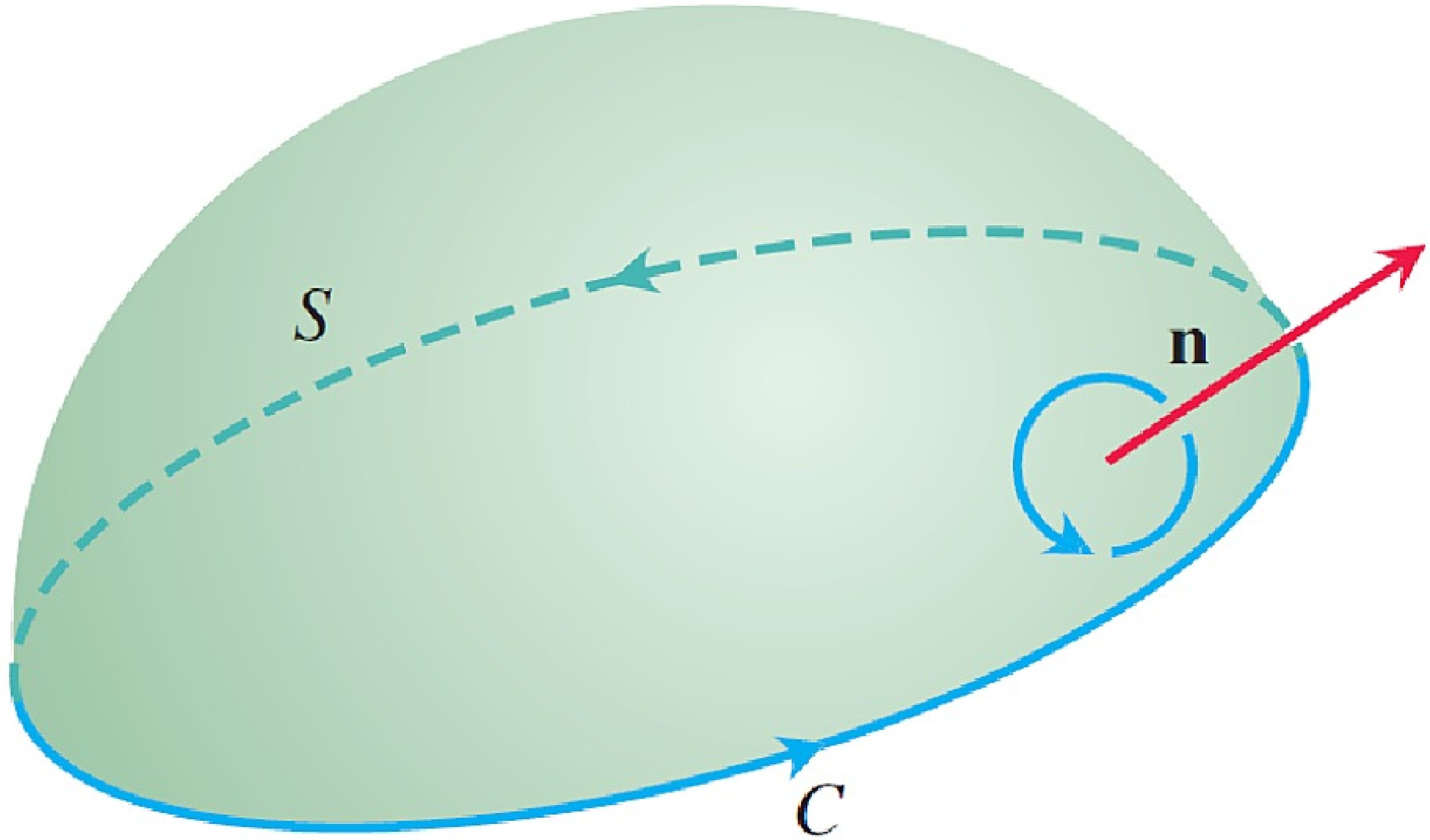


FIGURE 15.59 The orientation of the bounding curve C gives it a right-handed relation to the normal field $\mathbf{n}$. If the thumb of a right hand points along $\mathbf{n}$, the fingers curl in the direction of C.

THEOREM 6—Stokes' Theorem

Let S be a piecewise smooth oriented surface having a piecewise smooth boundary curve C. Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ be a vector field whose components have continuous first partial derivatives on an open region containing S. Then the circulation of $\mathbf{F}$ around C in the direction counterclockwise with respect to the surface's unit normal vector $\mathbf{n}$ equals the integral of the curl vector field $\nabla \times \mathbf{F}$ over S:

$$\underbrace{\oint_C \mathbf{F} \cdot d\mathbf{r}}_{\text{Counterclockwise circulation}} = \underbrace{\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, d\sigma}_{\text{Curl integral}} \qquad (4)$$

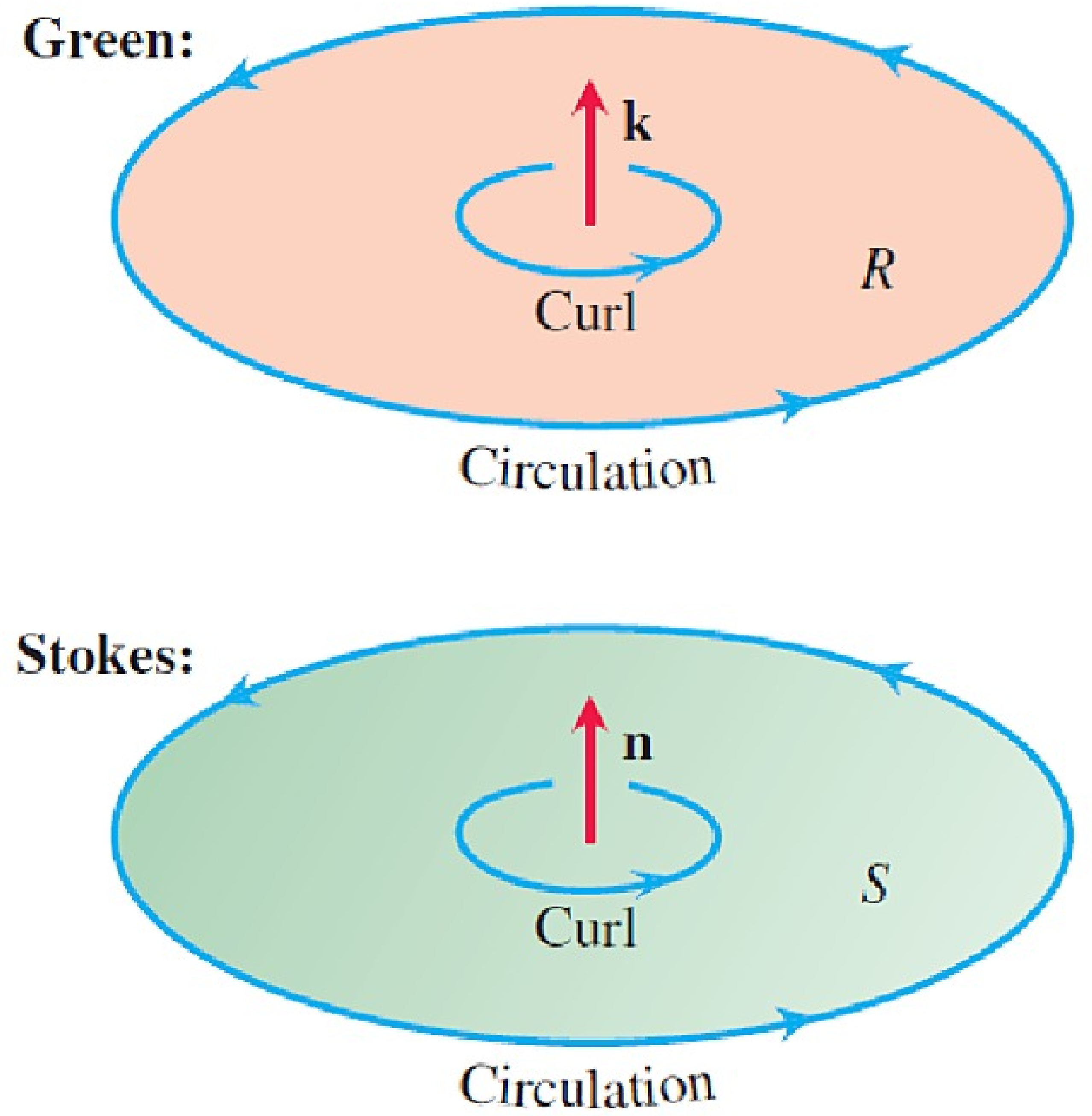


FIGURE 15.60 When applied to curves and surfaces in the plane, Stokes' Theorem gives the circulation-curl version of Green's Theorem. But Stokes' Theorem also applies more generally, to curves and surfaces not lying in the plane.

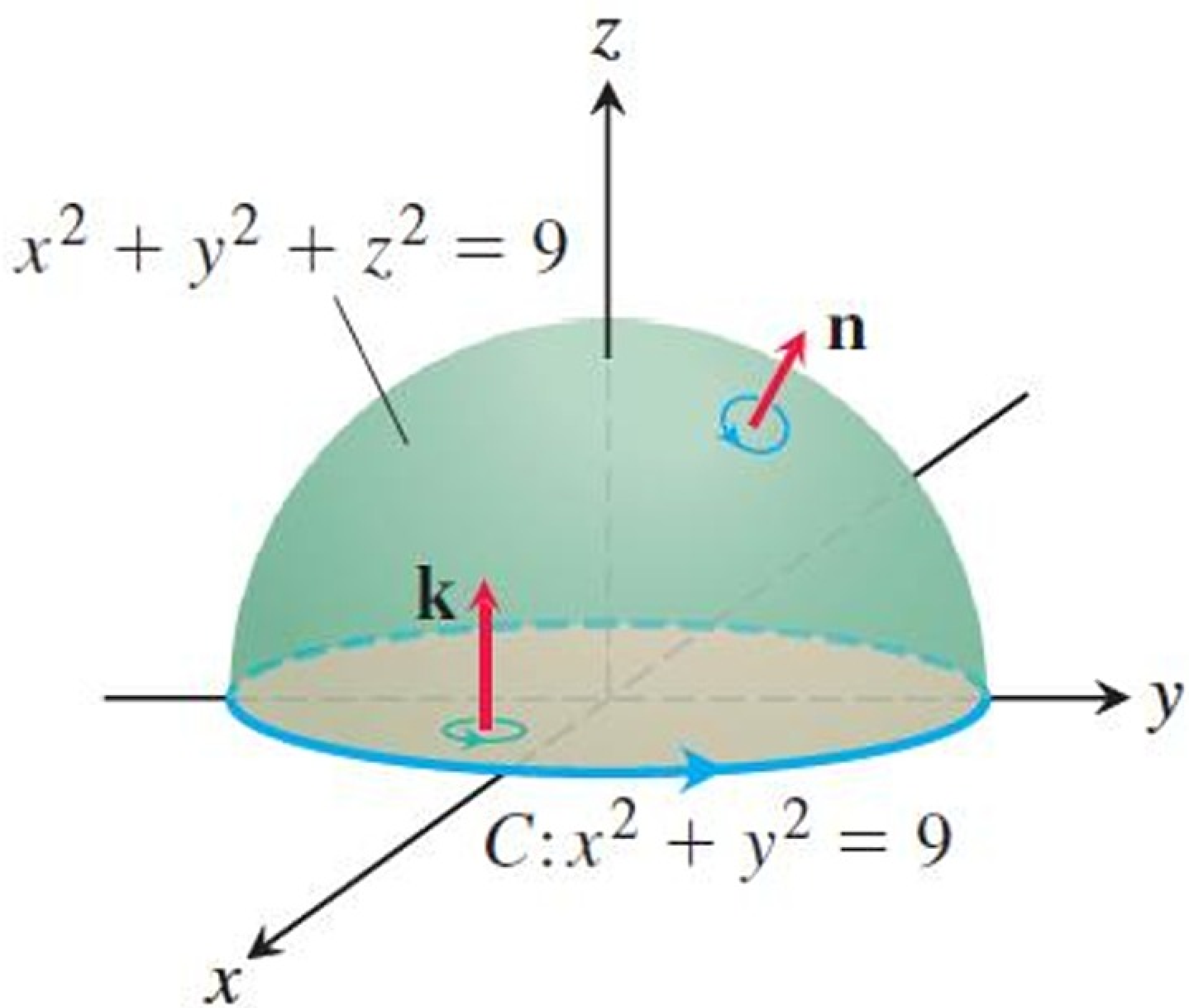


FIGURE 15.61 A hemisphere and a disk, each with boundary C (Examples 2 and 3).

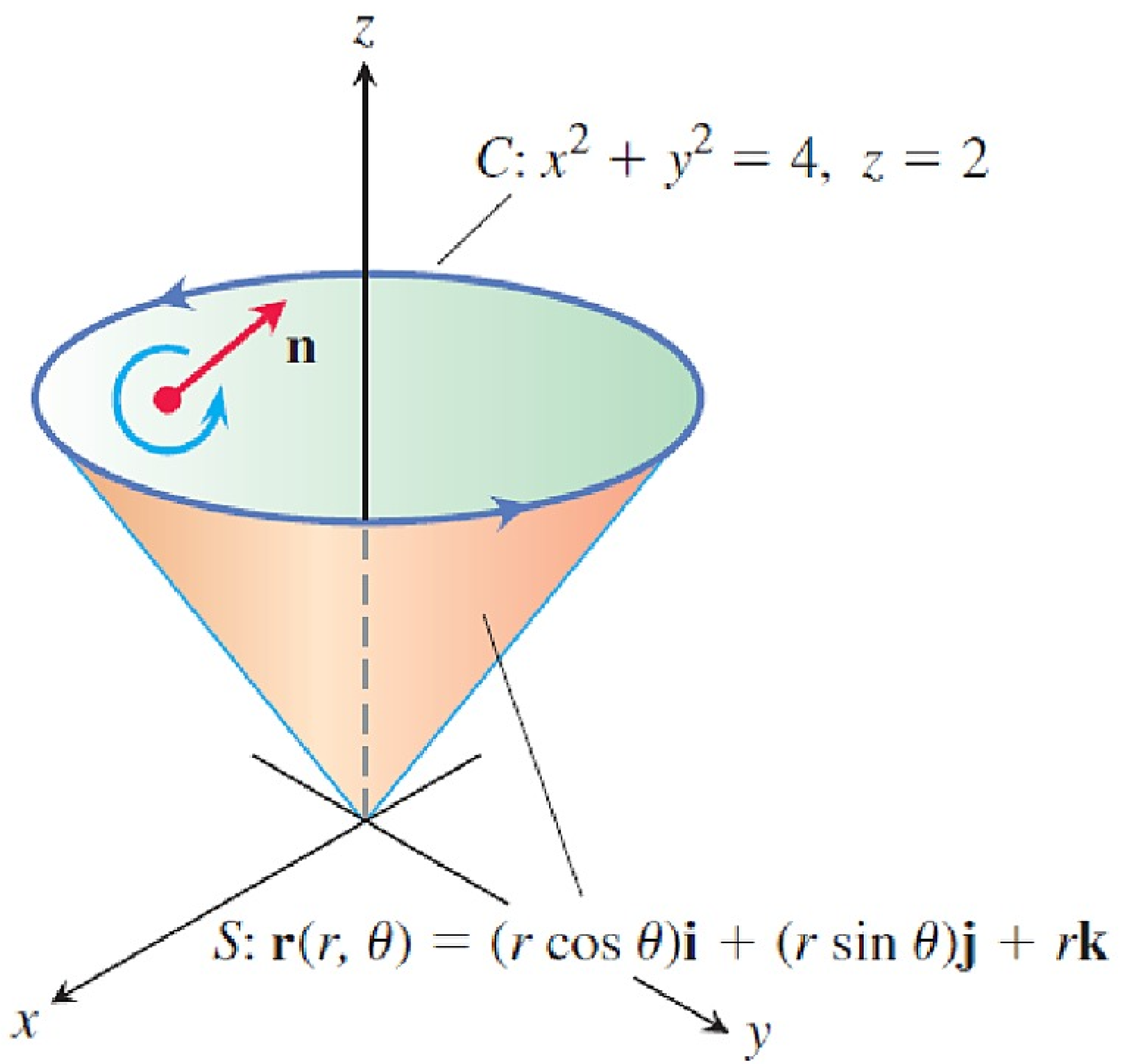
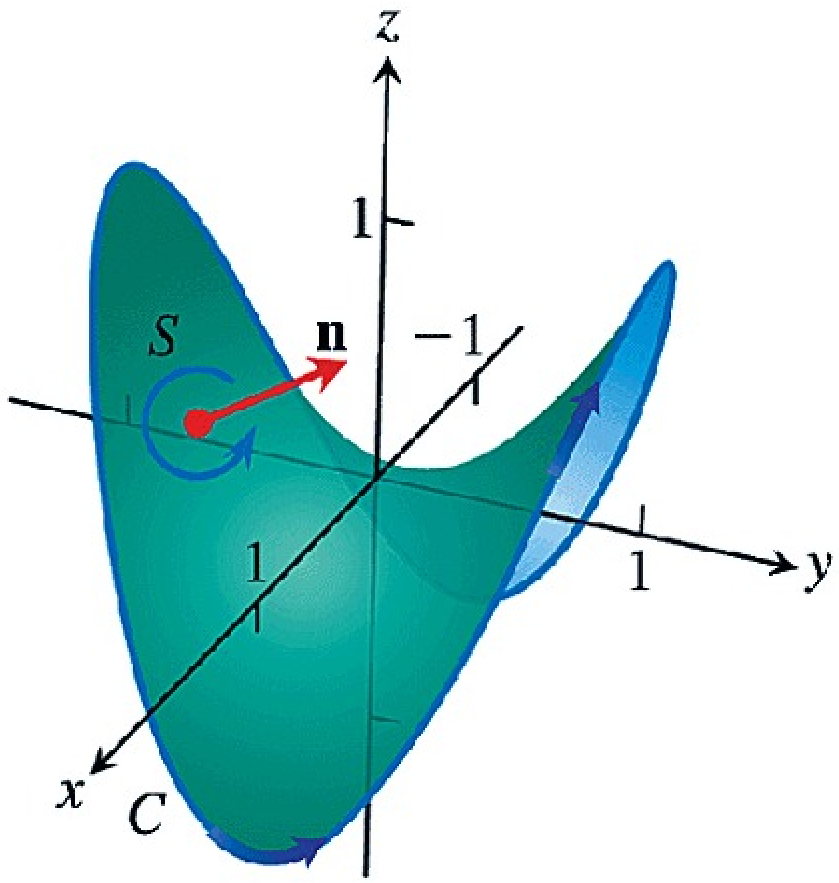


FIGURE 15.62 The curve C and cone S in Example 4.



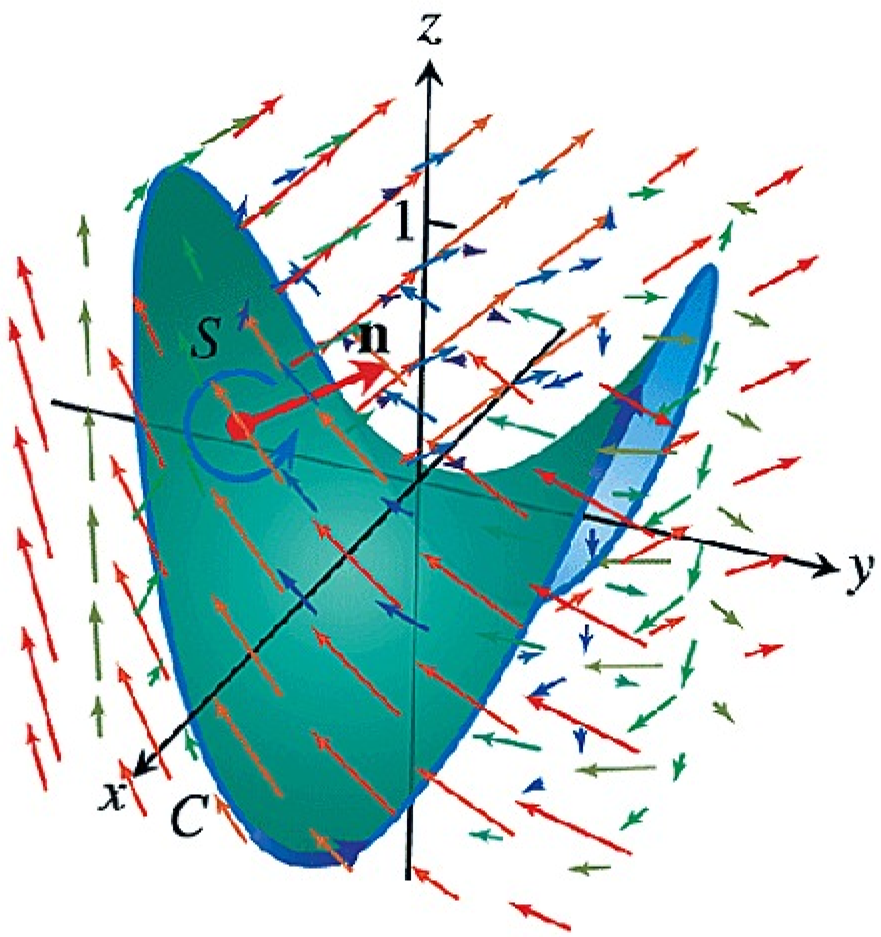


FIGURE 15.63 The surface and vector field for Example 6.

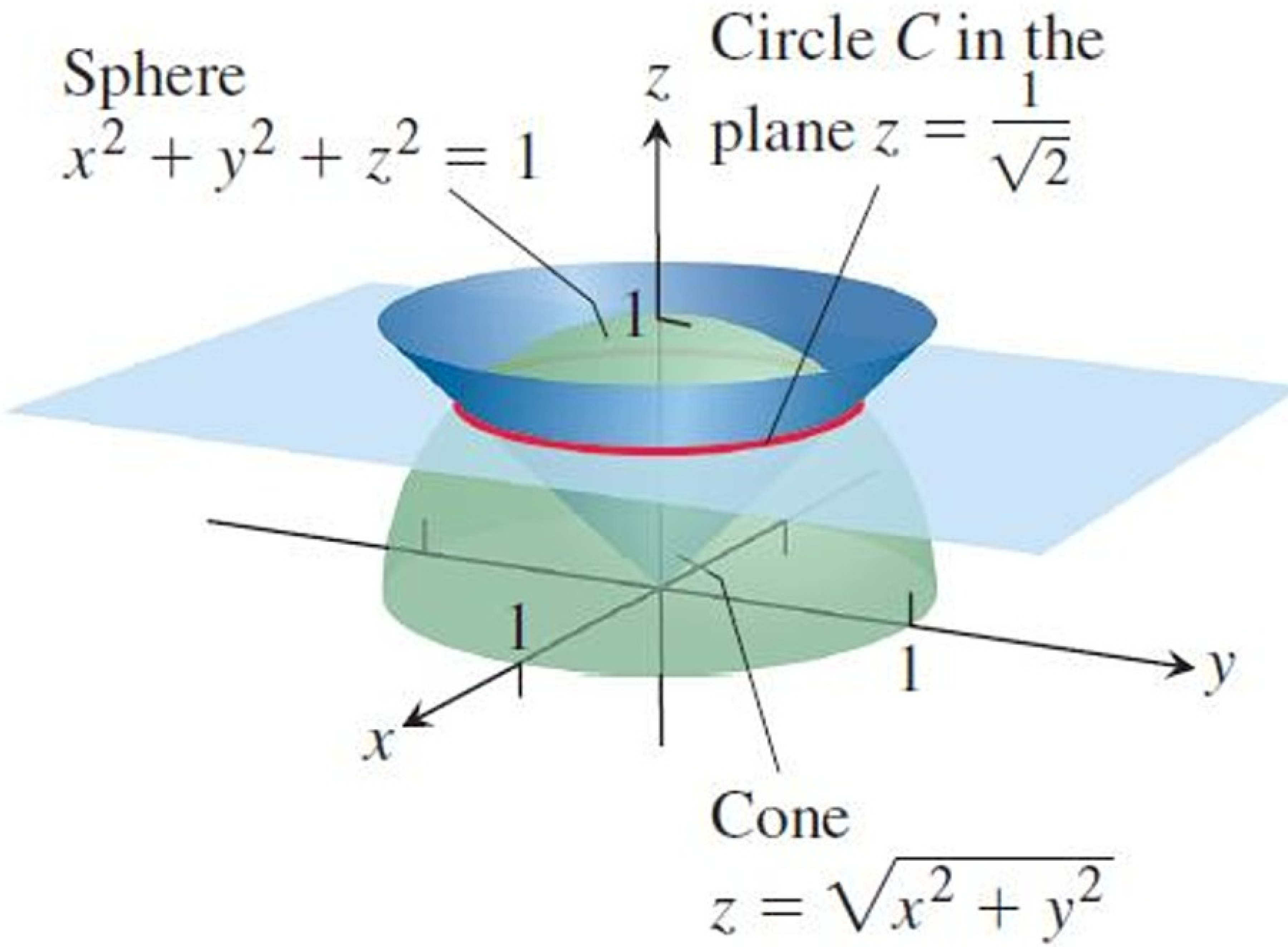


FIGURE 15.64 Circulation curve C in Example 7.

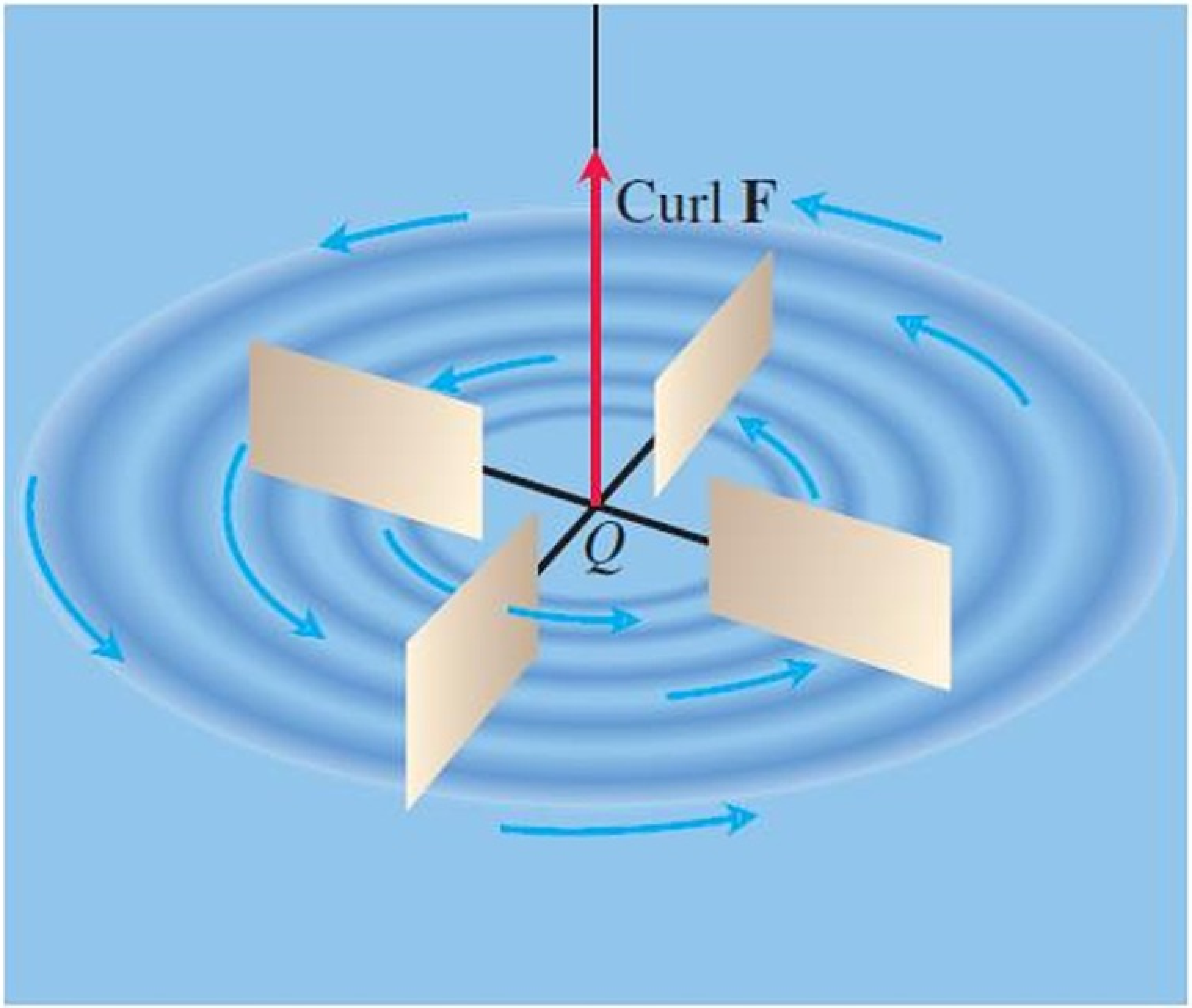


FIGURE 15.65 A small paddle wheel in a fluid spins fastest at point Q when its axle points in the direction of curl **F**.

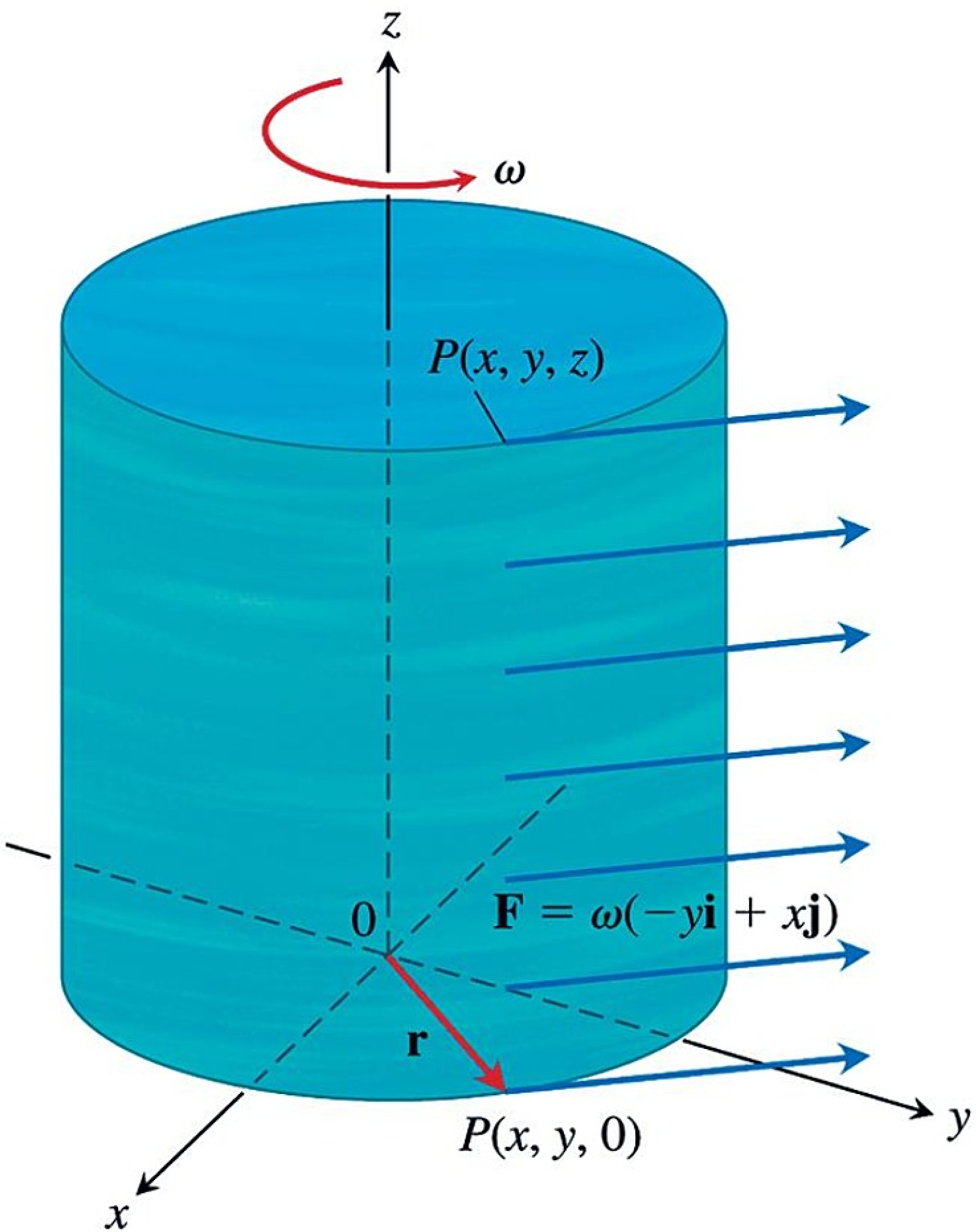


FIGURE 15.66 A steady rotational flow parallel to the xy-plane, with constant angular velocity $\boldsymbol{\omega}$ in the positive (counterclockwise) direction (Example 8).

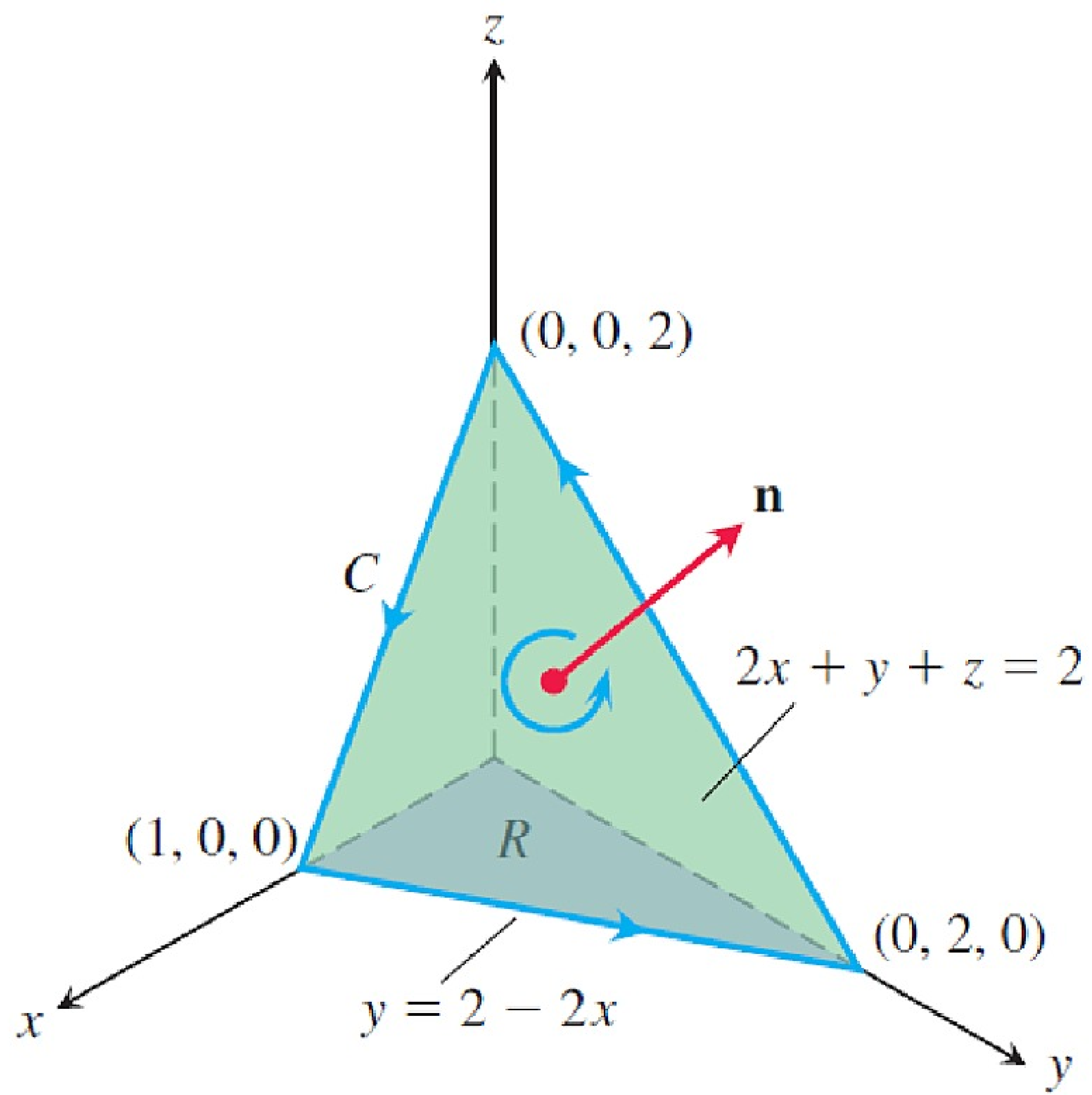


FIGURE 15.67 The planar surface in Example 9.

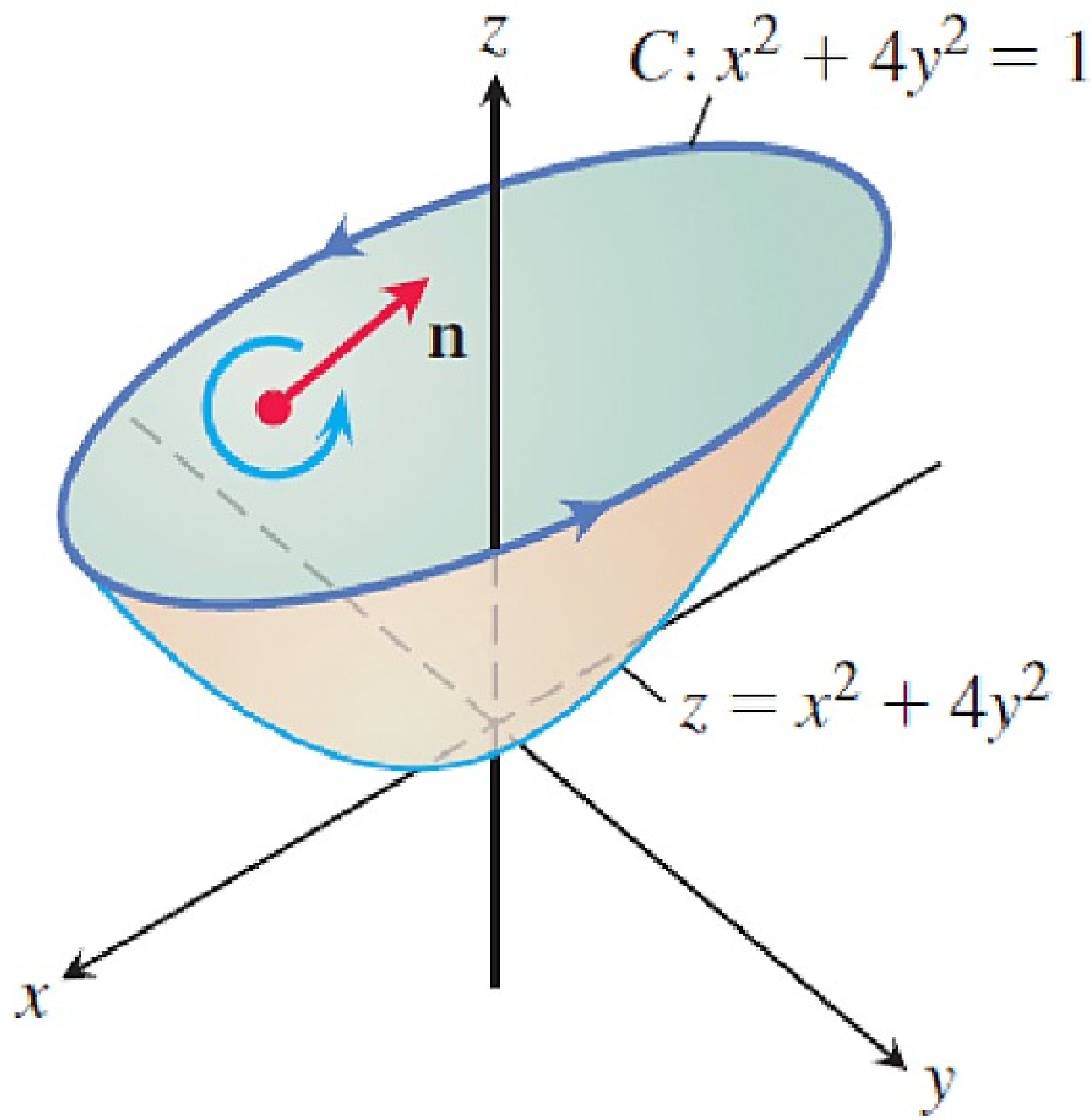


FIGURE 15.68 The portion of the ellipitical paraboloid in Example 10, showing its curve of intersection C with the plane $z = 1$ and its inner normal orientation by $\mathbf{n}$.

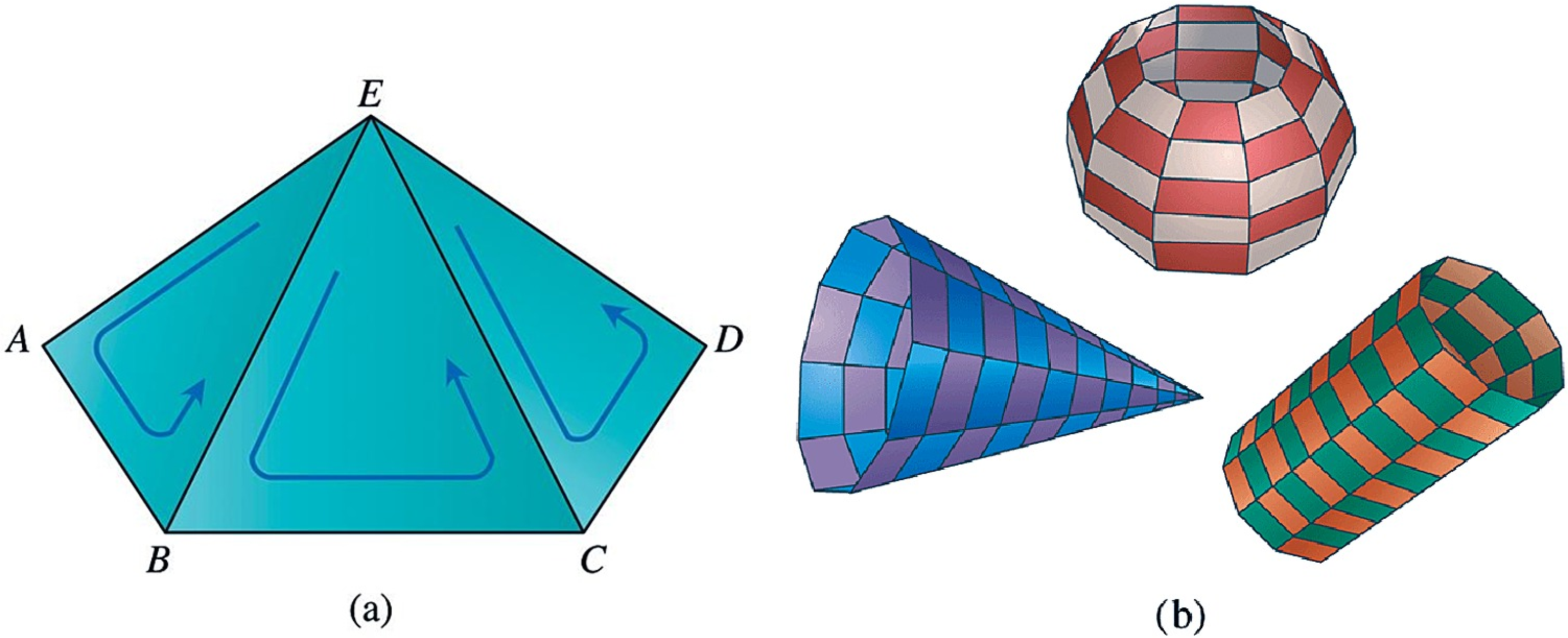


FIGURE 15.69 (a) Part of a polyhedral surface. (b) Other polyhedral surfaces.

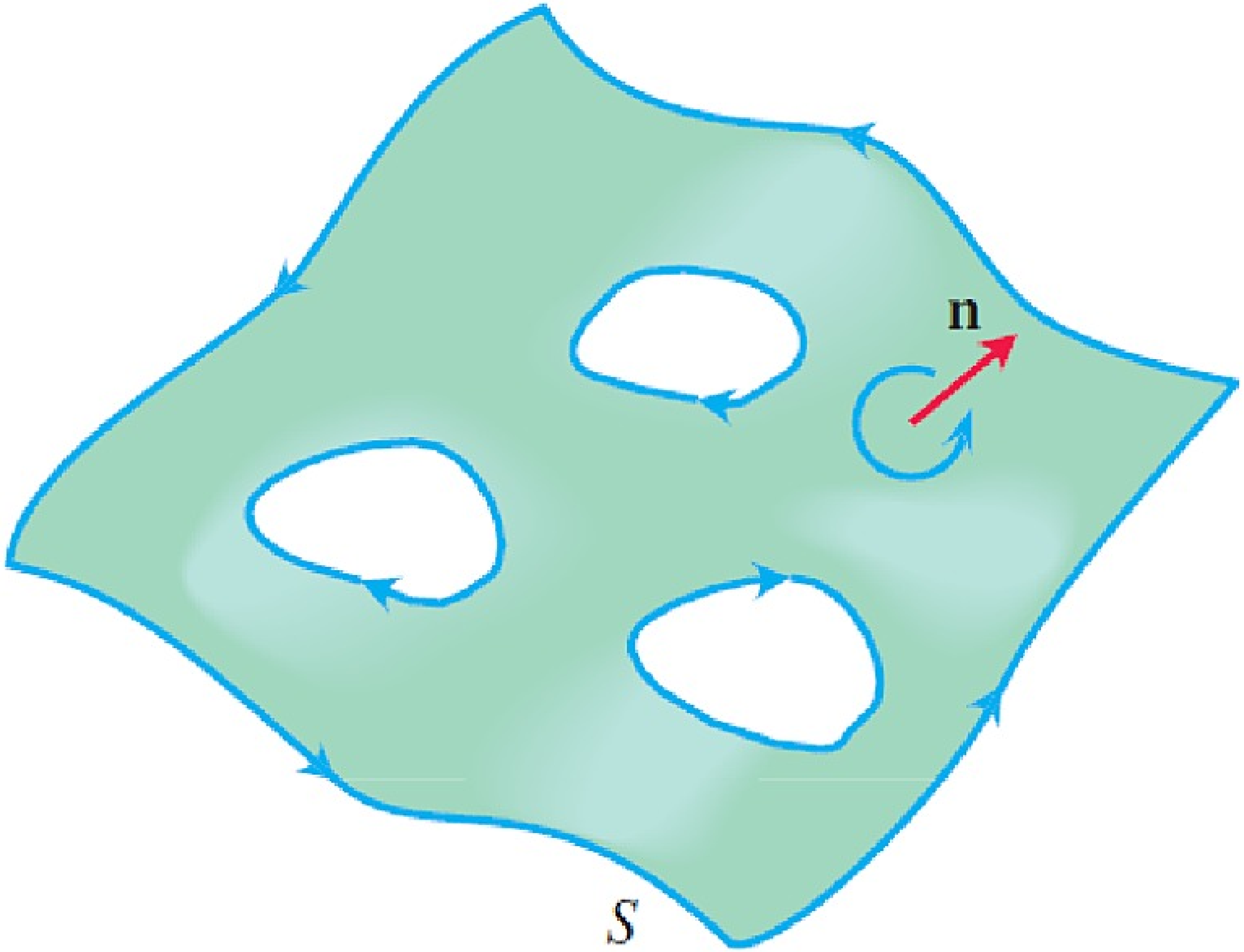


FIGURE 15.70 Stokes' Theorem also holds for oriented surfaces with holes. Consistent with the orientation of S, the outer curve is traversed counterclockwise around $\mathbf{n}$ and the inner curves surrounding the holes are traversed clockwise.

$$\text{curl grad } f = \mathbf{0} \qquad \text{or} \qquad \nabla \times \nabla f = \mathbf{0} \tag{8}$$

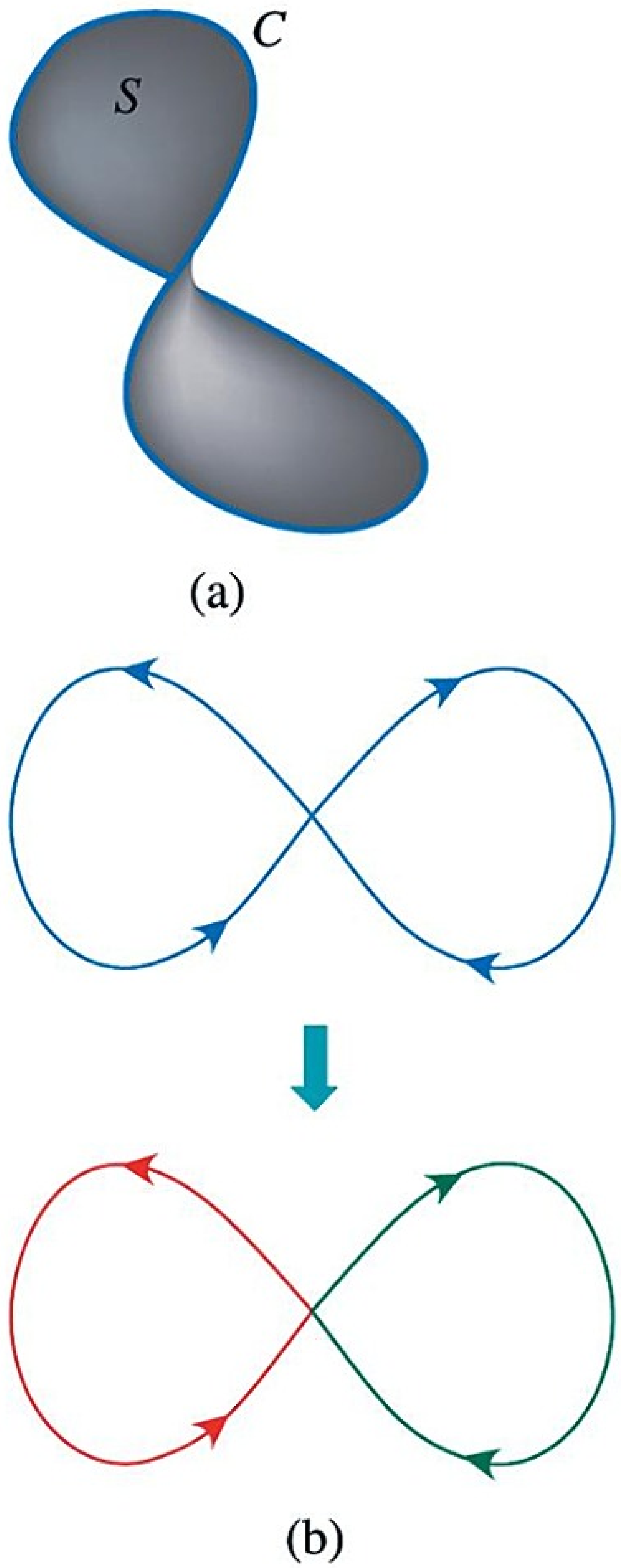


FIGURE 15.71 (a) In a simply connected open region in space, a simple closed curve C is the boundary of a smooth surface S. (b) Smooth curves that cross themselves can be divided into loops to which Stokes' Theorem applies.

THEOREM 7—Curl $\mathbf{F} = \mathbf{0}$ Related to the Closed-Loop Property If $\nabla \times \mathbf{F} = \mathbf{0}$ at every point of a simply connected open region D in space, then on any piecewise-smooth closed path C in D,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0.$$

The following diagram summarizes the results for conservative fields defined on connected, simply connected open regions. For such regions, the four statements are equivalent to each other.

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\mathbf{F} \text{ conservative on } D \arrow[r, Leftrightarrow, "\substack{\text{Theorem 2,}\\ \text{Section 15.3}}"] \arrow[d, Leftrightarrow, "\substack{\text{Theorem 3,}\\ \text{Section 15.3}}"'] & \mathbf{F} = \nabla f \text{ on } D \arrow[d, Rightarrow, "\substack{\text{Vector identity (Eq. 8)}\\ \text{(continuous second}\\ \text{partial derivatives)}}"] \\
\substack{\displaystyle\oint_C \mathbf{F} \cdot d\mathbf{r} = 0 \\ \text{over any closed} \\ \text{path in } D} & \nabla \times \mathbf{F} = \mathbf{0} \text{ throughout } D \arrow[l, Rightarrow, "\substack{\text{Theorem 7}\\ \text{Domain's simple}\\ \text{connectivity and}\\ \text{Stokes' Theorem}}"]
\end{tikzcd}
$$

Section 15.8

The Divergence Theorem and a Unified Theory

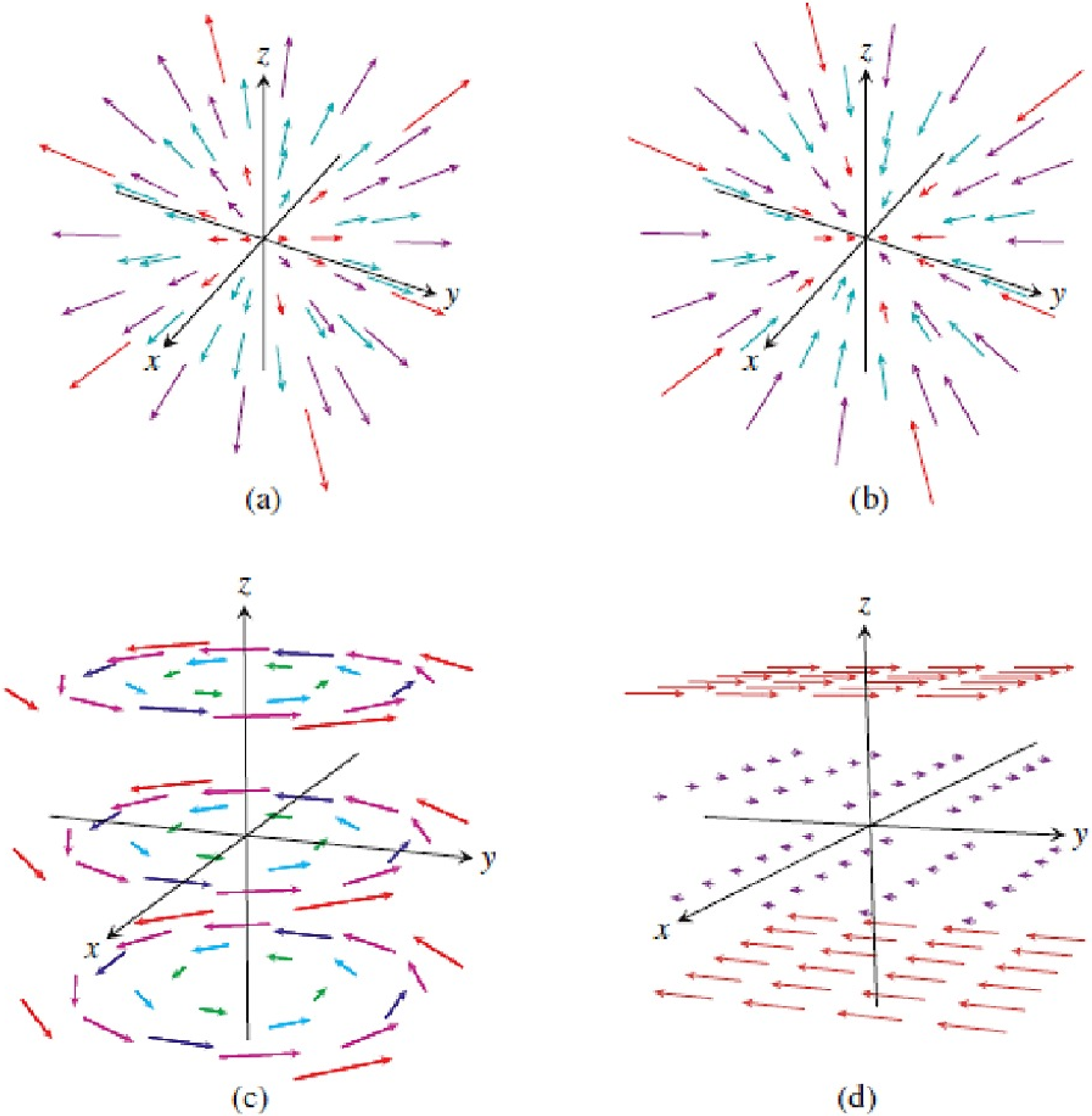


FIGURE 15.72 Velocity fields of a gas flowing in space (Example 1).

THEOREM 8—Divergence Theorem Let $\mathbf{F}$ be a vector field whose components have continuous first partial derivatives, and let S be a piecewise smooth oriented closed surface. The flux of $\mathbf{F}$ across S in the direction of the surface's outward unit normal field $\mathbf{n}$ equals the integral of $\nabla \cdot \mathbf{F}$ over the region D enclosed by the surface:

$$\underbrace{\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma}_{\text{Outward flux}} = \underbrace{\iiint_D \nabla \cdot \mathbf{F}\, dV}_{\text{Divergence integral}}. \tag{2}$$

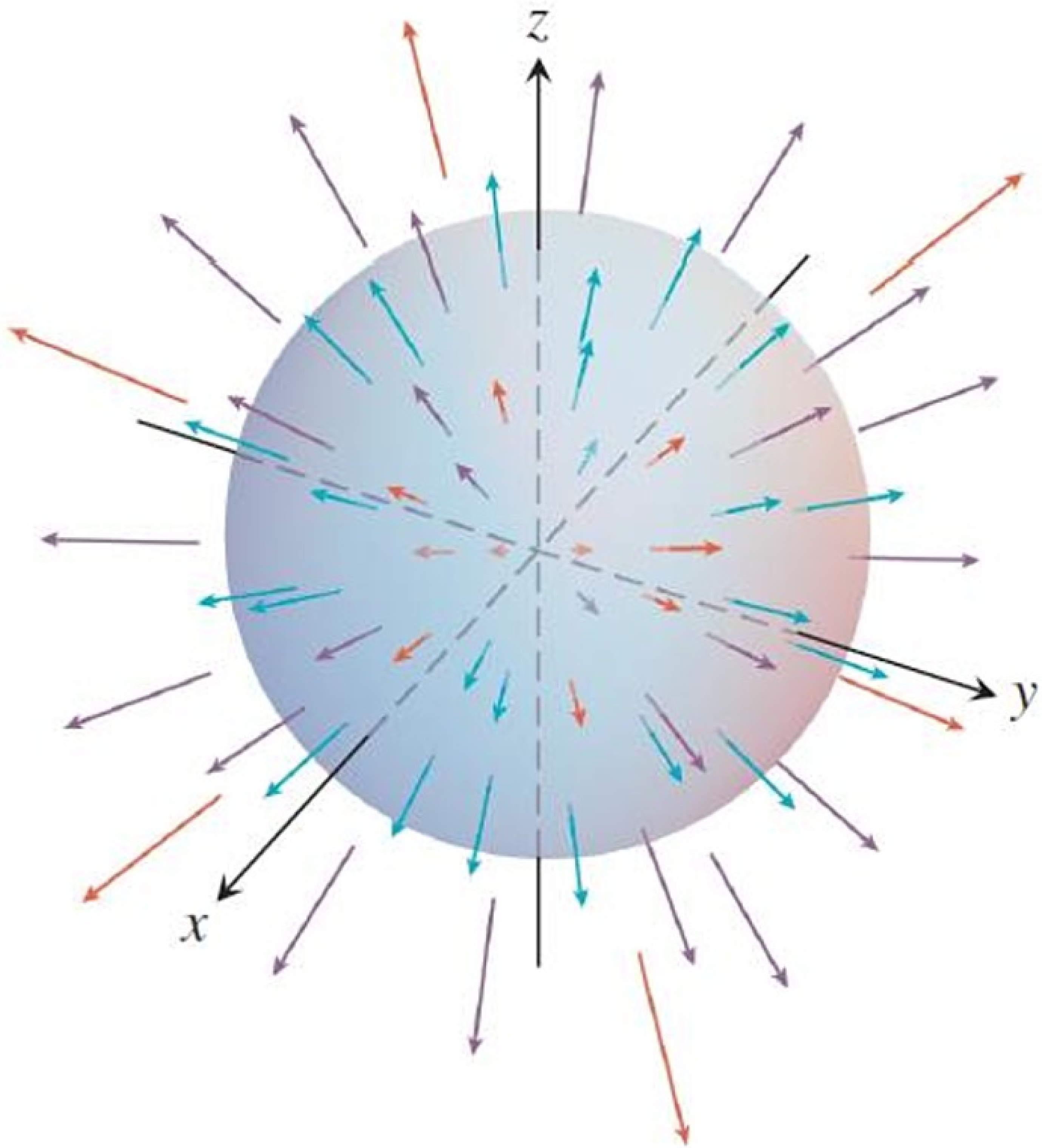


FIGURE 15.73 A uniformly expanding vector field and a sphere (Example 2).

COROLLARY The outward flux across a piecewise smooth oriented closed surface S is zero for any vector field $\mathbf{F}$ having zero divergence at every point of the region enclosed by the surface.

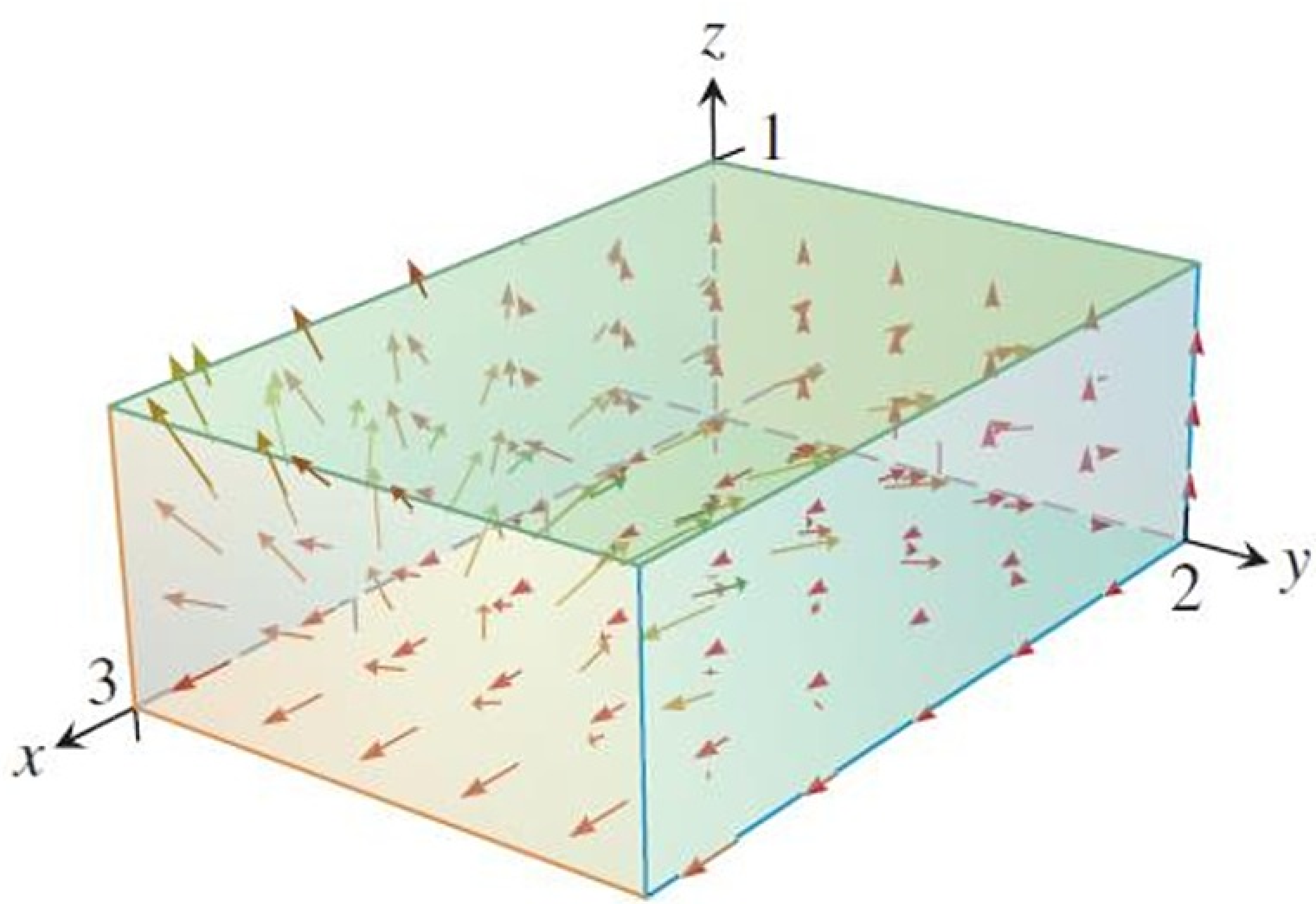


FIGURE 15.74 The integral of div **F** over this region equals the total flux across the six sides (Example 4).

Side	Unit normal $\mathbf{n}$	$\mathbf{F} \cdot \mathbf{n}$	Flux across side
$x = 0$	$-\mathbf{i}$	$-x^2 = 0$	0
$x = 3$	$\mathbf{i}$	$x^2 = 9$	18
$y = 0$	$-\mathbf{j}$	$-4xyz = 0$	0
$y = 2$	$\mathbf{j}$	$4xyz = 8xz$	18
$z = 0$	$-\mathbf{k}$	$-ze^x = 0$	0
$z = 1$	$\mathbf{k}$	$ze^x = e^x$	$2e^3 - 2$

THEOREM 9 If $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is a vector field with continuous second partial derivatives, then

$$\operatorname{div}(\operatorname{curl}\mathbf{F}) = \nabla \cdot (\nabla \times \mathbf{F}) = 0.$$

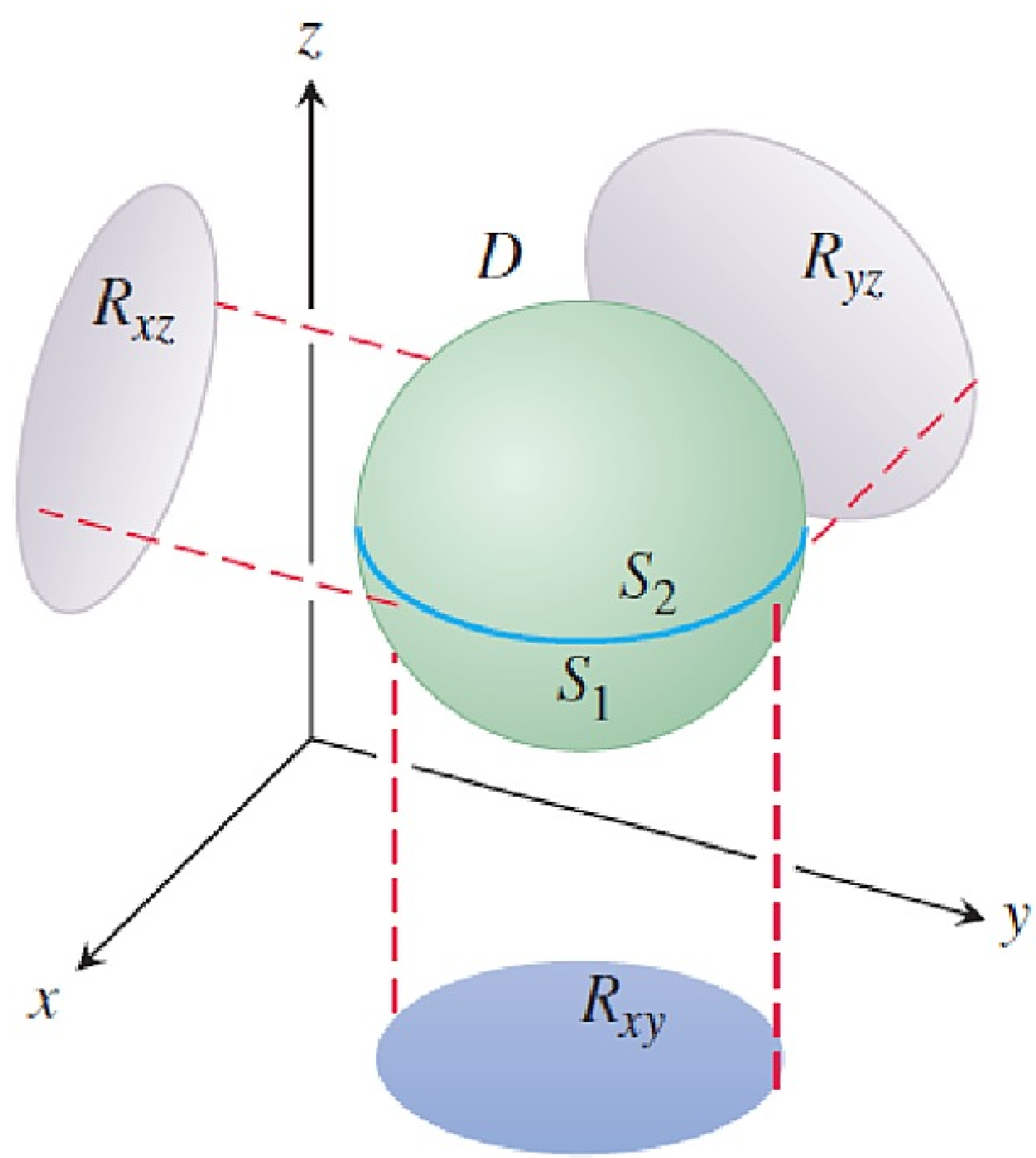


FIGURE 15.75 We prove the Divergence Theorem for the kind of three-dimensional region shown here.

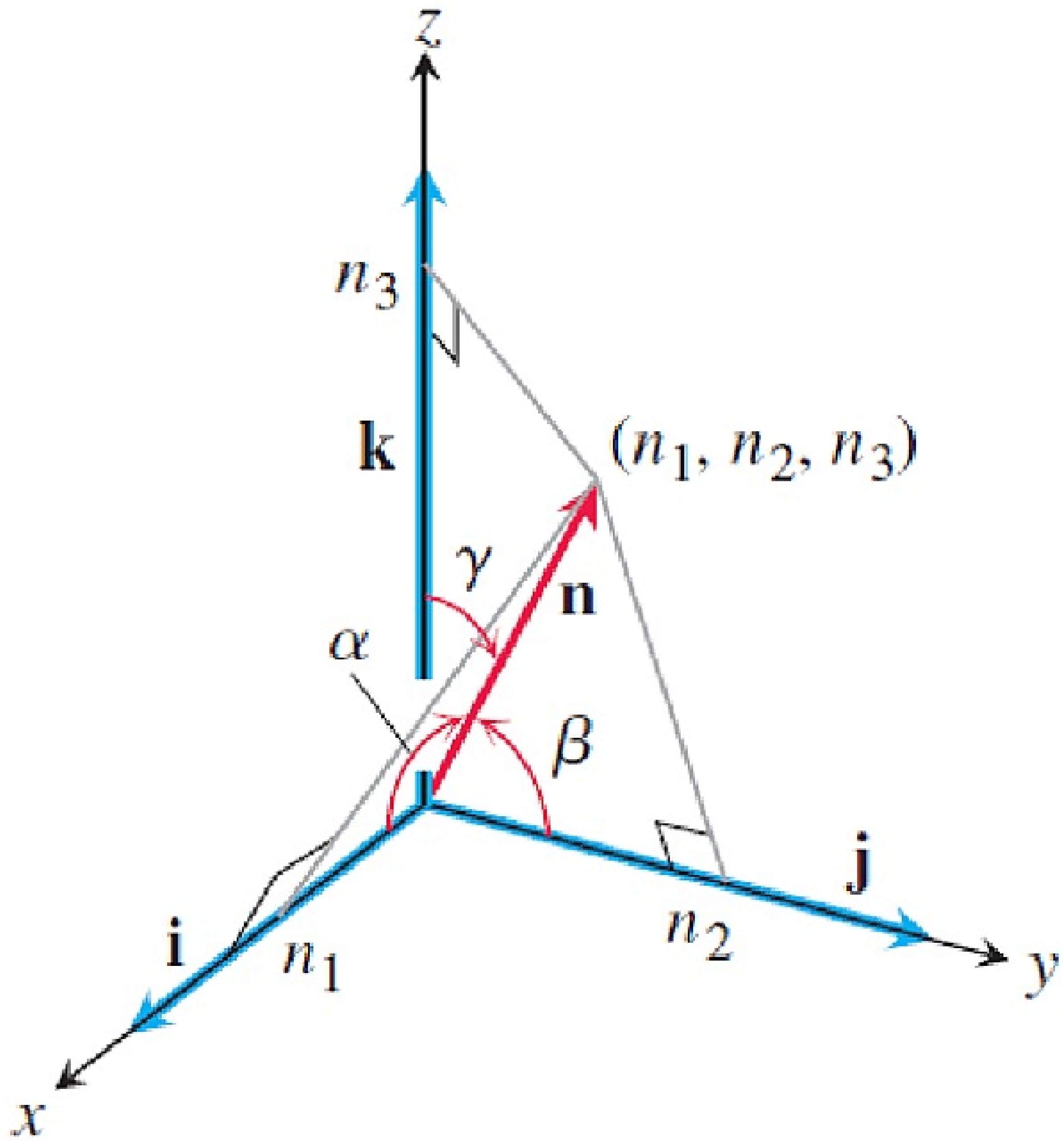


FIGURE 15.76 The components of **n** are the cosines of the angles α, β, and γ that it makes with **i**, **j**, and **k**.

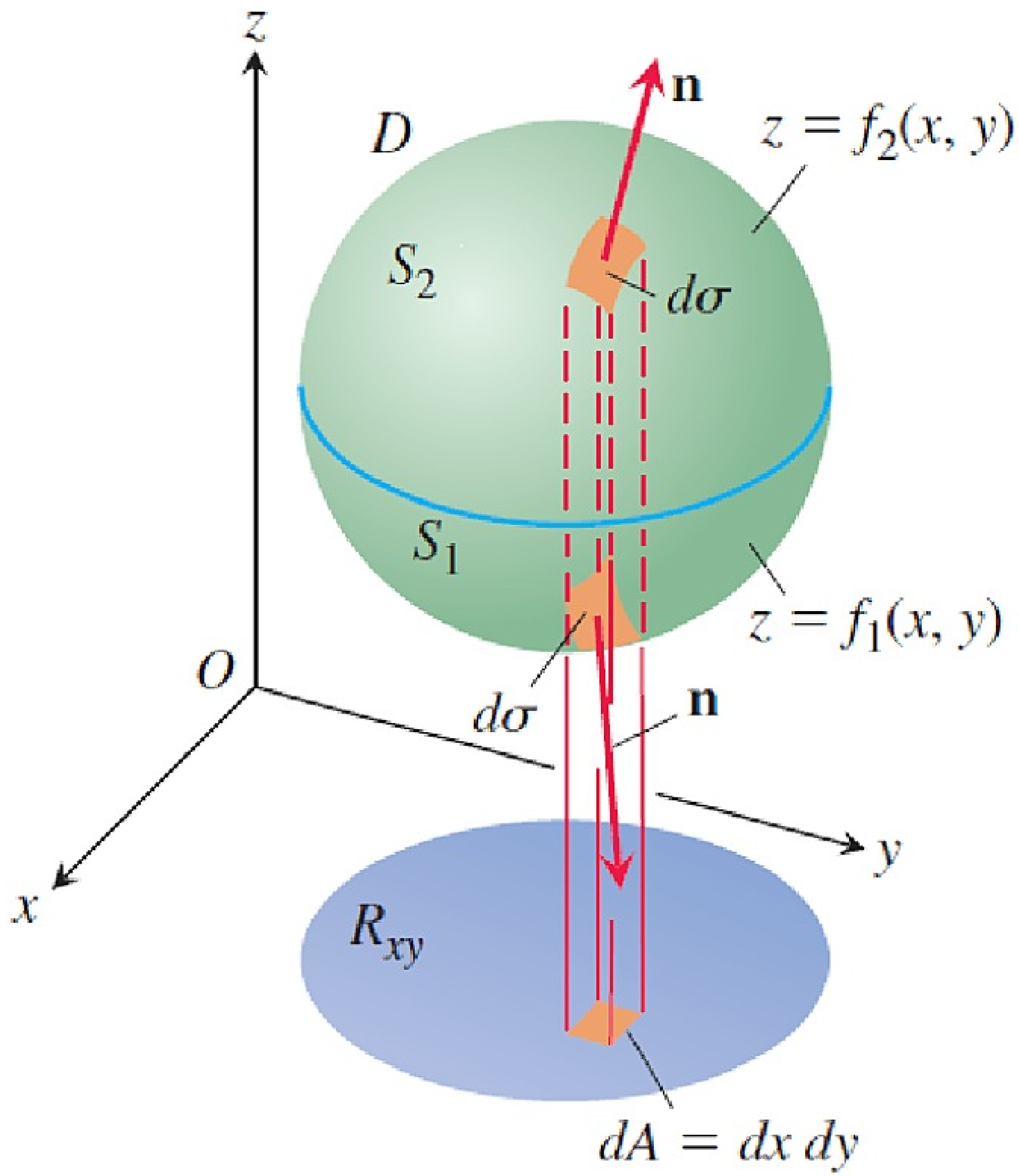


FIGURE 15.77 The region D enclosed by the surfaces S_1 and S_2 projects vertically onto R_{xy} in the xy-plane.

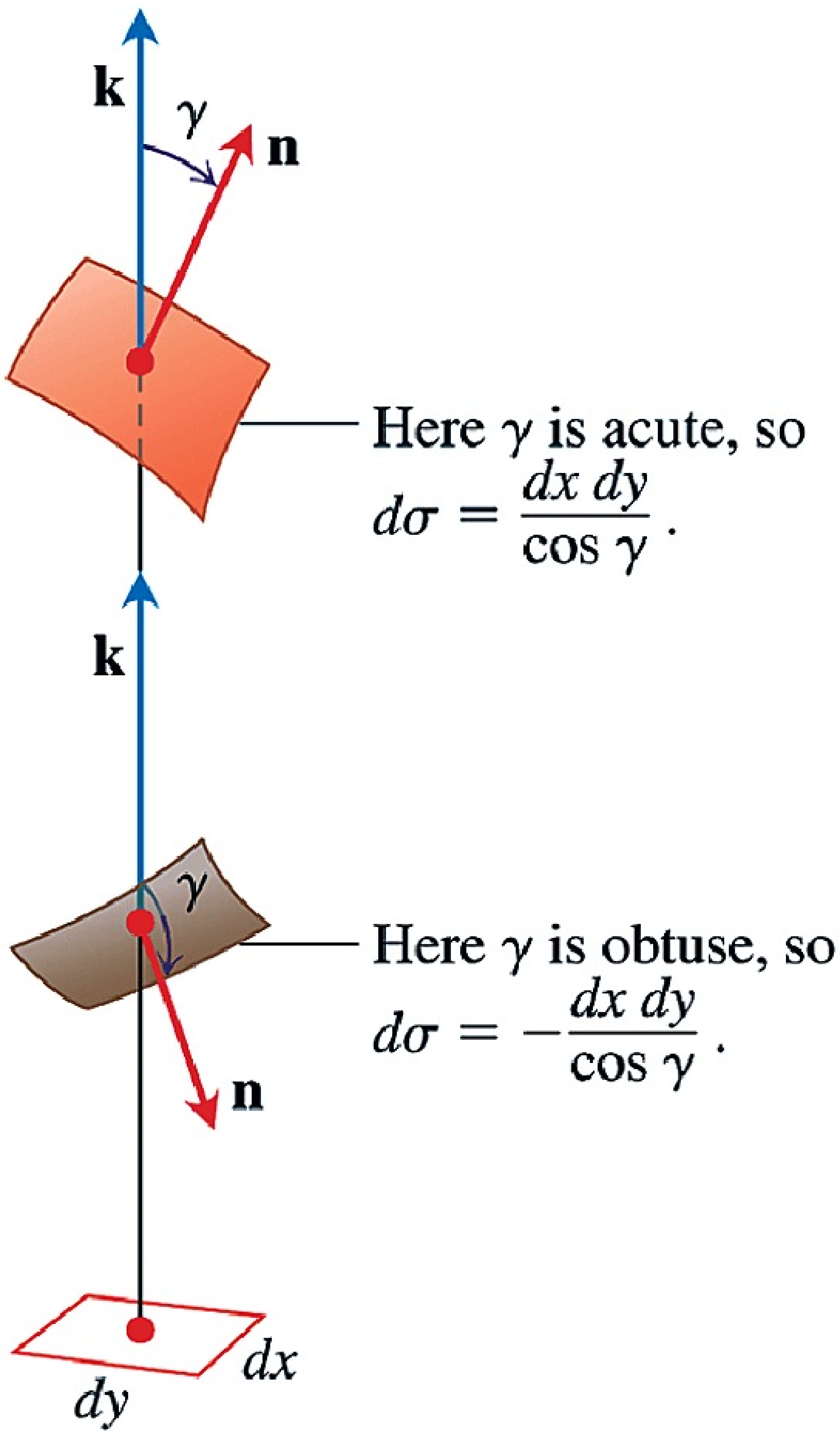


FIGURE 15.78 An enlarged view of the area patches in Figure 15.77. The relations $d\sigma = \pm\, dx\,dy/\cos\gamma$ come from Eq. (7) in Section 15.5 with $F = \mathbf{F}\cdot\mathbf{n}$.

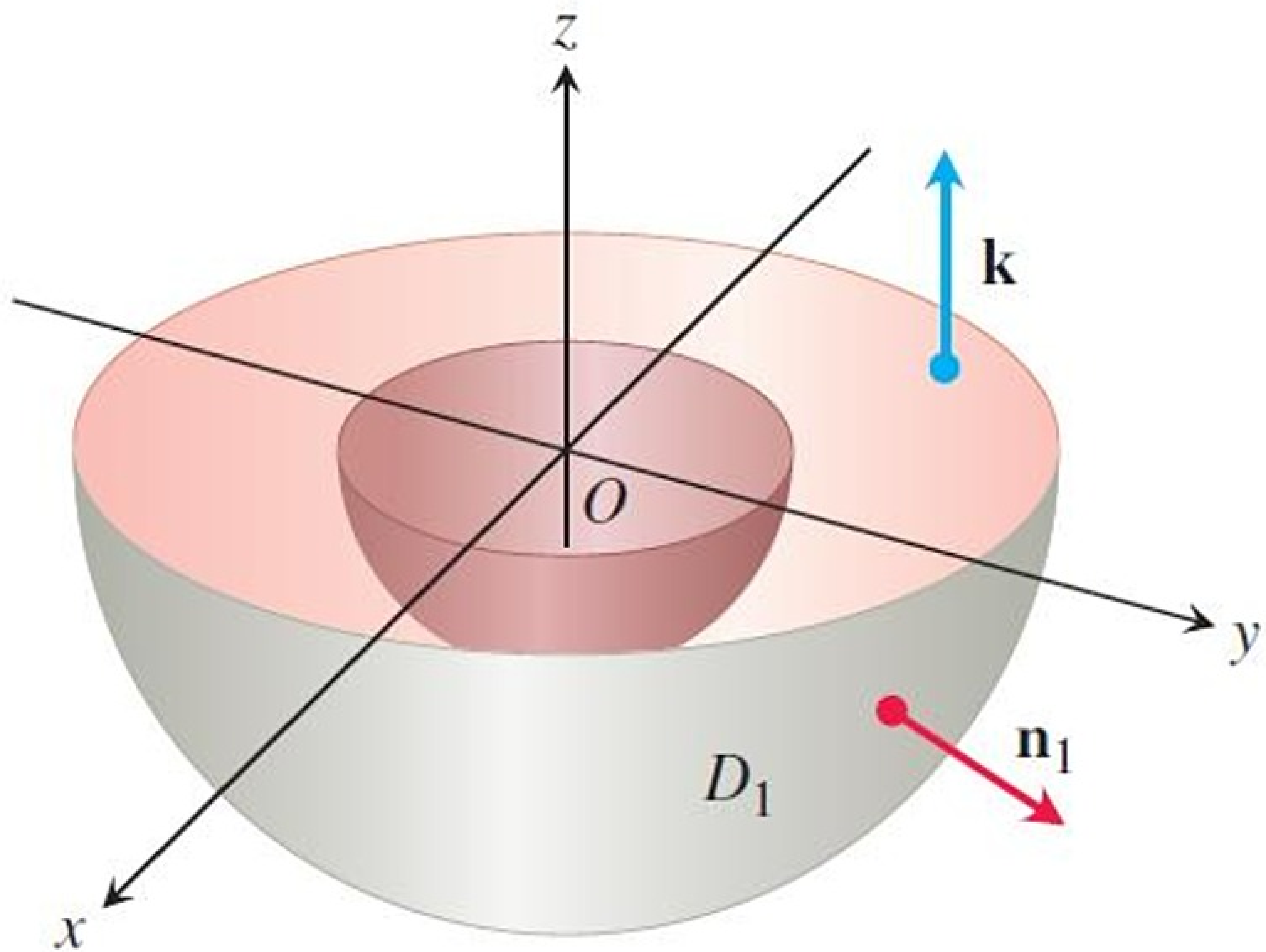


FIGURE 15.79 The lower half of the solid region between two concentric spheres.

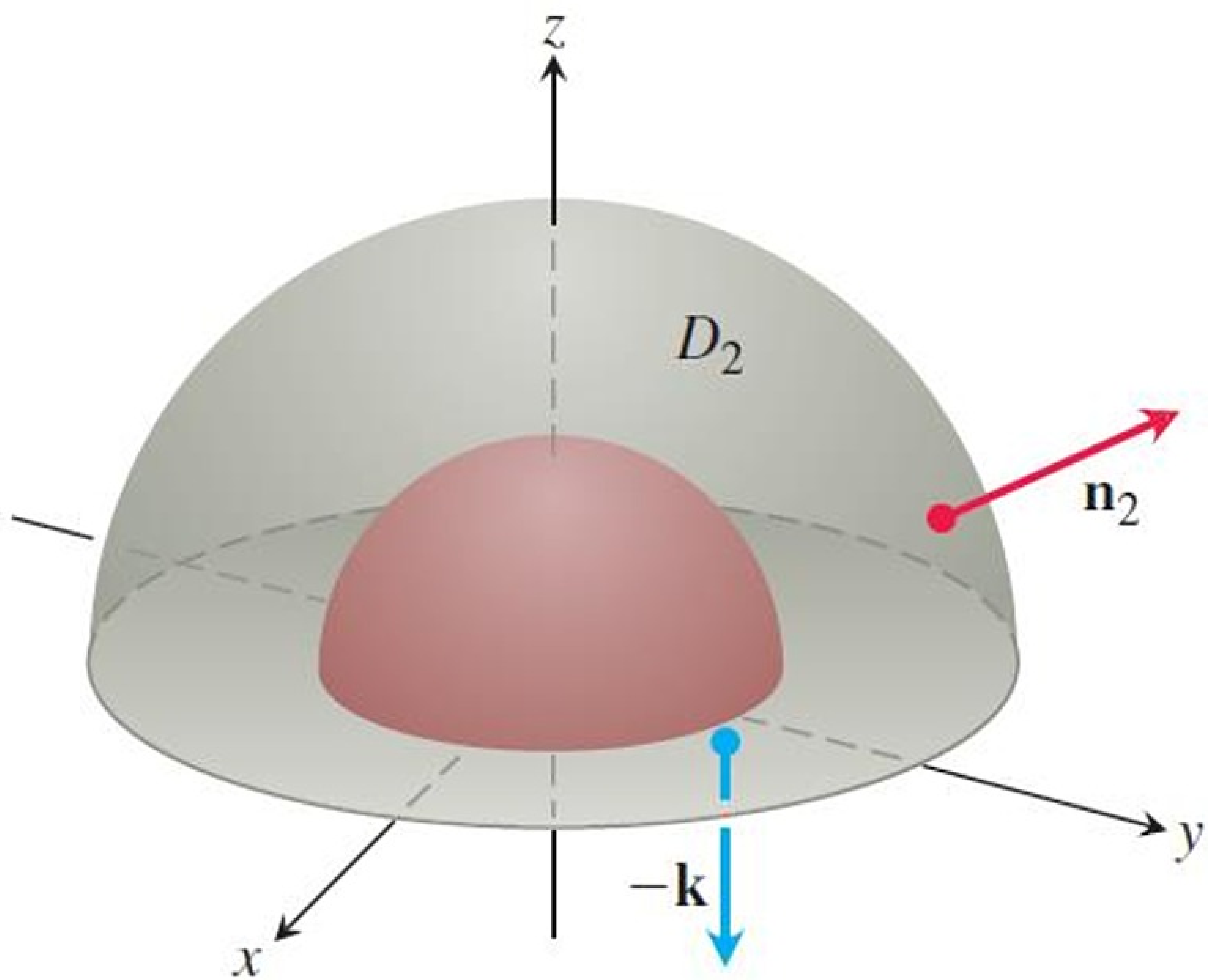


FIGURE 15.80 The upper half of the solid region between two concentric spheres.

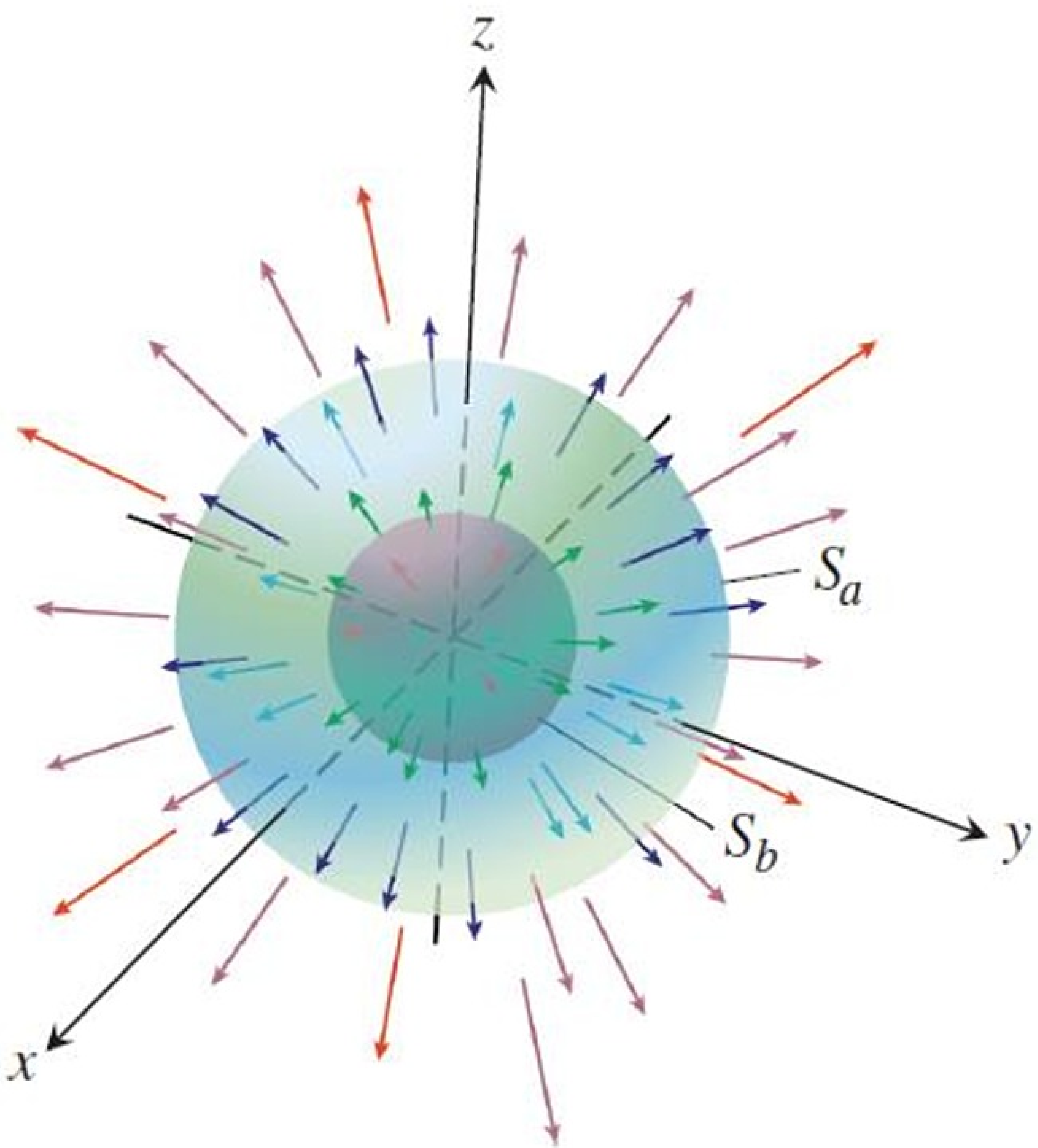


FIGURE 15.81 Two concentric spheres in an expanding vector field. The outer sphere S_a surrounds the inner sphere S_b.

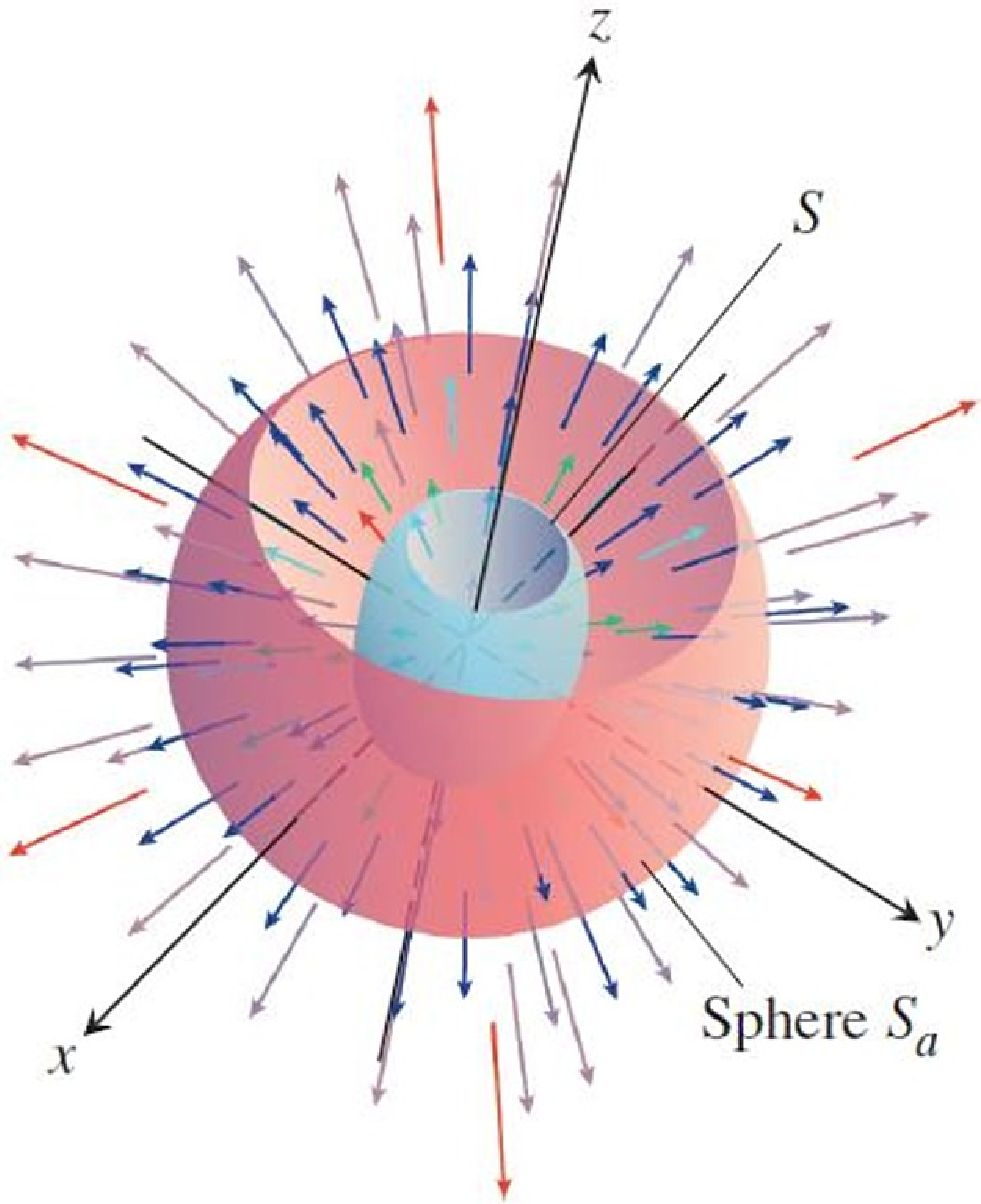


FIGURE 15.82 A sphere S_a surrounding another surface S. The tops of the surfaces are removed for visualization.

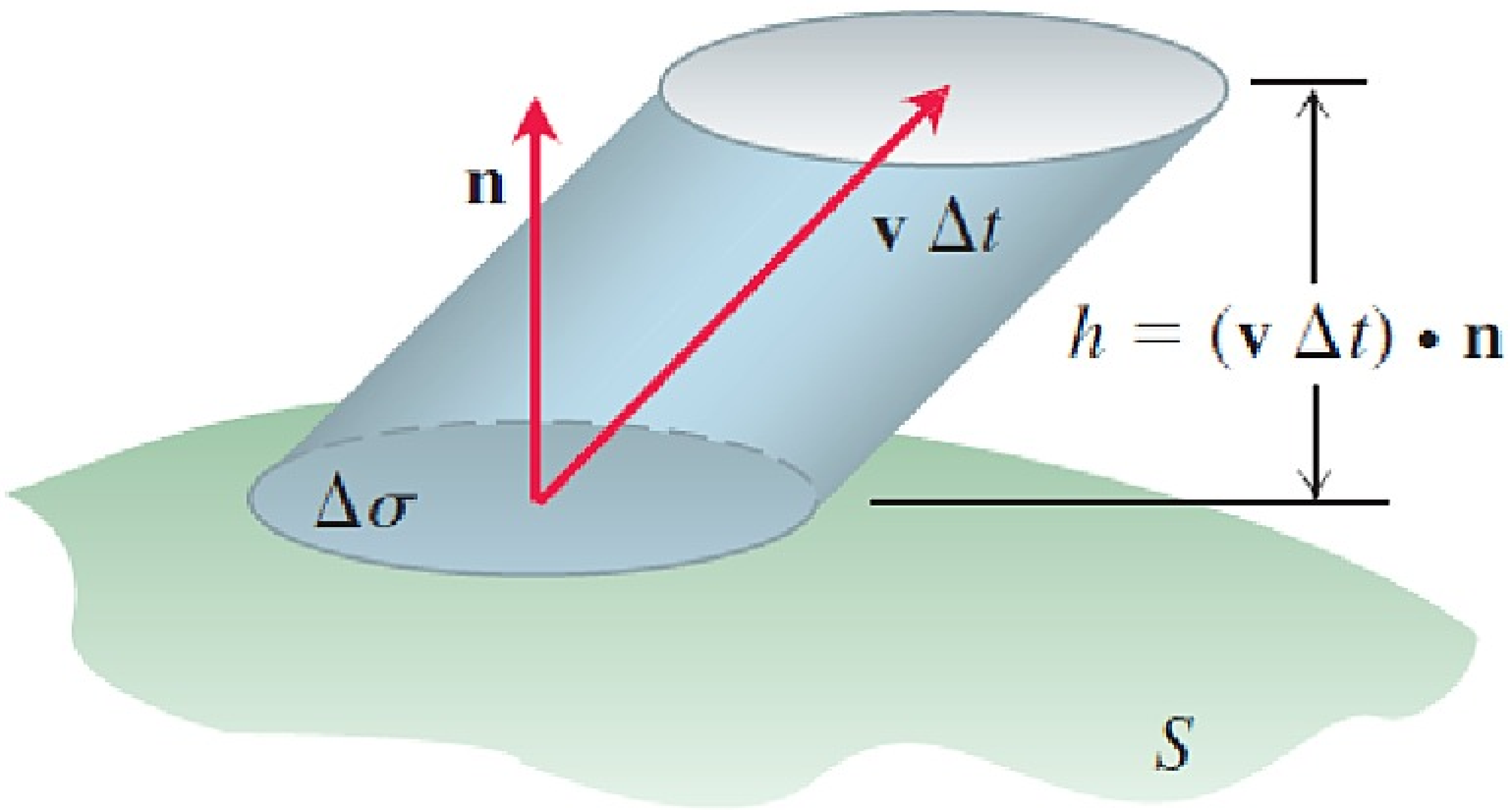


FIGURE 15.83 The fluid that flows upward through the patch $\Delta\sigma$ in a short time Δt fills a "cylinder" whose volume is approximately base $\times$ height $= \mathbf{v} \cdot \mathbf{n}\, \Delta\sigma\, \Delta t$.

Green's Theorem and Its Generalization to Three Dimensions

Normal form of Green's Theorem: $$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_R \nabla \cdot \mathbf{F}\, dA$$

Divergence Theorem: $$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_D \nabla \cdot \mathbf{F}\, dV$$

Tangential form of Green's Theorem: $$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \nabla \times \mathbf{F} \cdot \mathbf{k}\, dA$$

Stokes' Theorem: $$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma$$

FIGURE 15.84 The outward unit normals at the boundary of $[a, b]$ in one-dimensional space.

A Unifying Fundamental Theorem of Vector Integral Calculus

The integral of a differential operator acting on a field over a region equals the sum of the field components appropriate to the operator over the boundary of the region.