

# Chapter 2

# Limits and Continuity

# Section 2.1

## Rates of Change and Tangents Lines to Curves

## Average Speed

When  $f(t)$  measures the distance traveled at time  $t$ ,

$$\text{Average speed over } [t_1, t_2] = \frac{\text{distance traveled}}{\text{elapsed time}} = \frac{f(t_2) - f(t_1)}{t_2 - t_1}$$

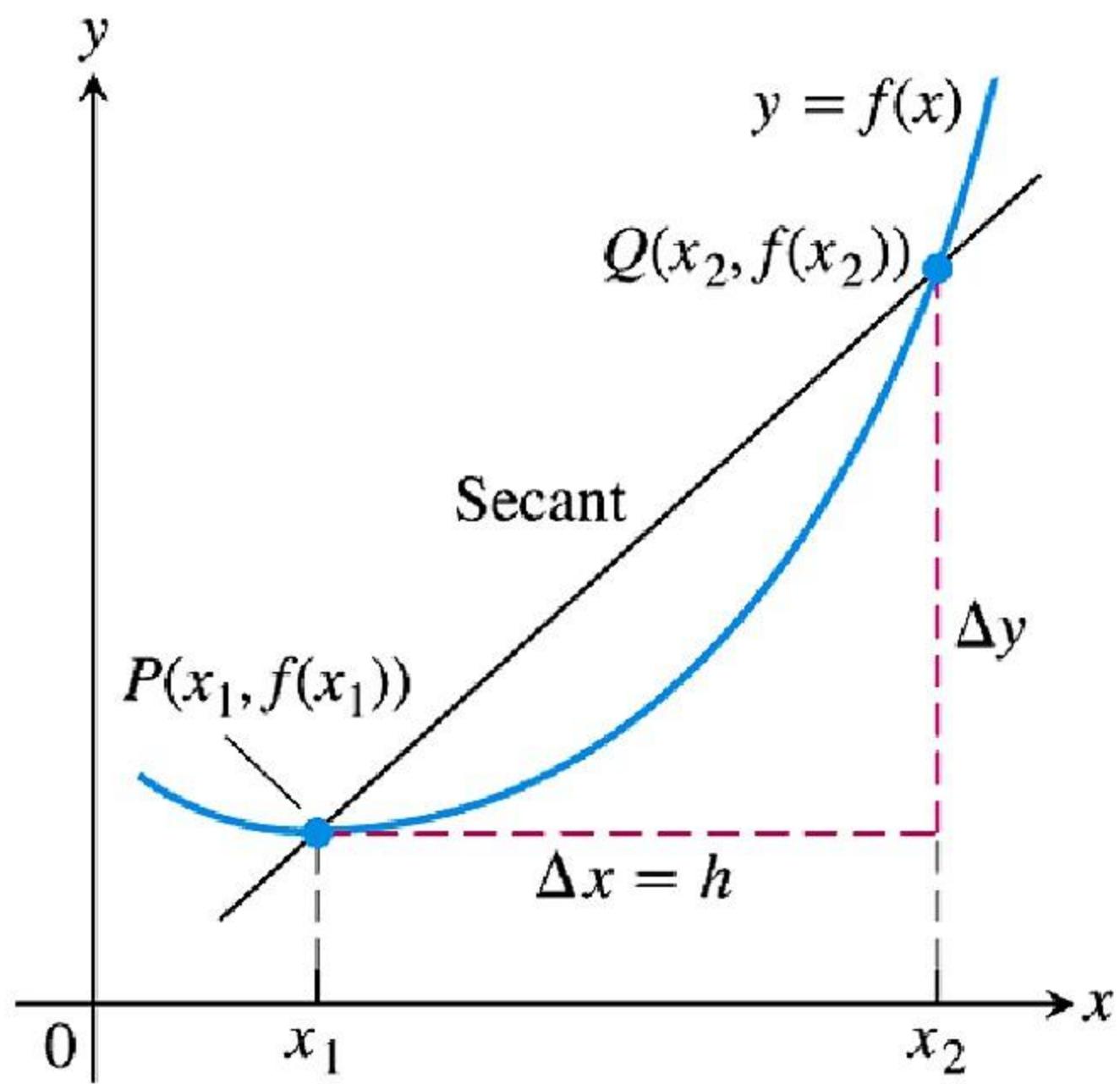
**TABLE 2.1** Average speeds over short time intervals  $[t_0, t_0 + h]$ 

$$\text{Average speed: } \frac{\Delta y}{\Delta t} = \frac{16(t_0 + h)^2 - 16t_0^2}{h}$$

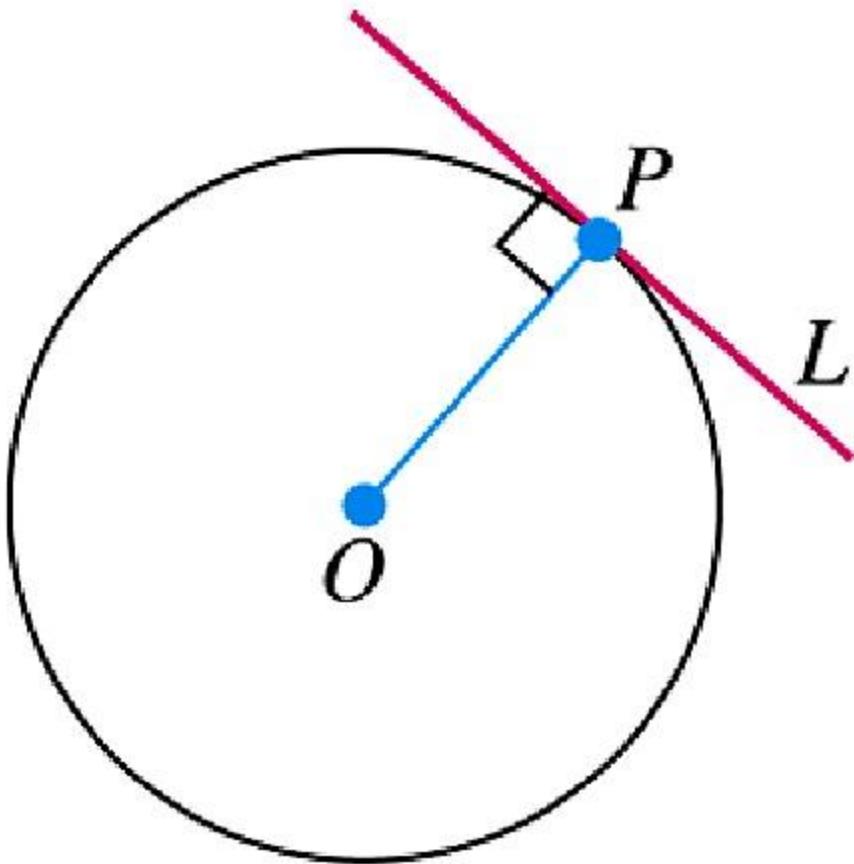
Length of time interval $h$	Average speed over interval of length $h$ starting at $t_0 = 1$	Average speed over interval of length $h$ starting at $t_0 = 2$
1	48	80
0.1	33.6	65.6
0.01	32.16	64.16
0.001	32.016	64.016
0.0001	32.0016	64.0016

**DEFINITION** The **average rate of change** of  $y = f(x)$  with respect to  $x$  over the interval  $[x_1, x_2]$  is

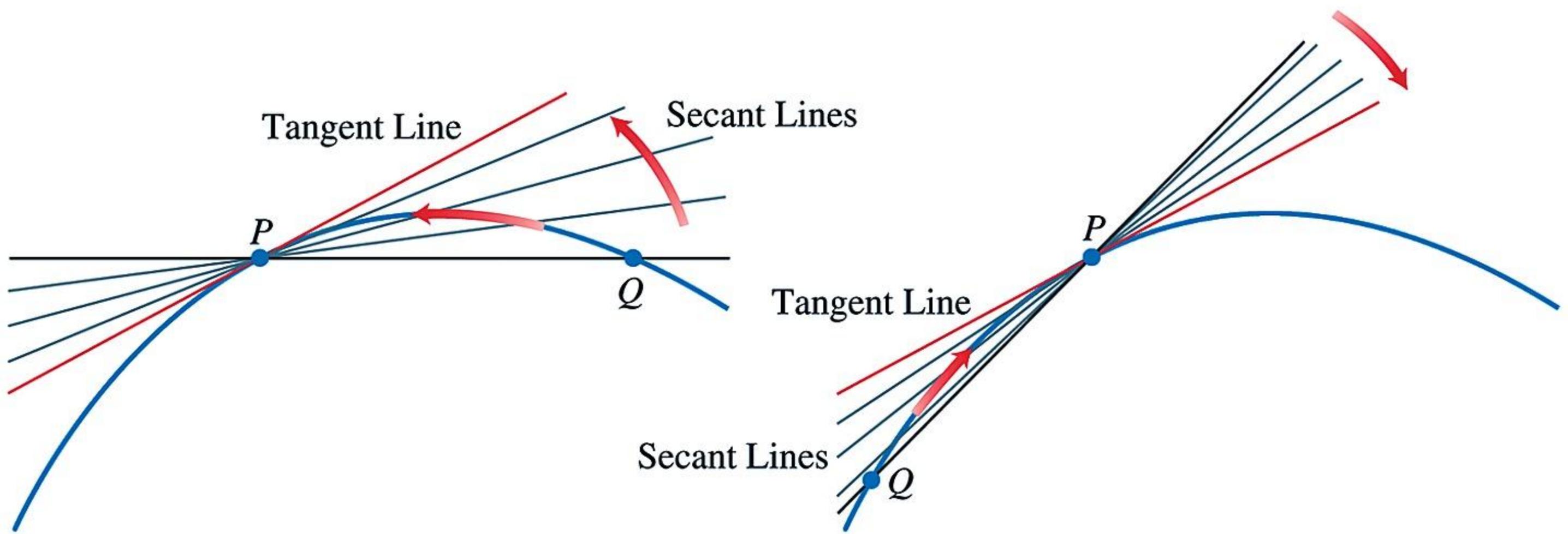
$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}, \quad h \neq 0.$$



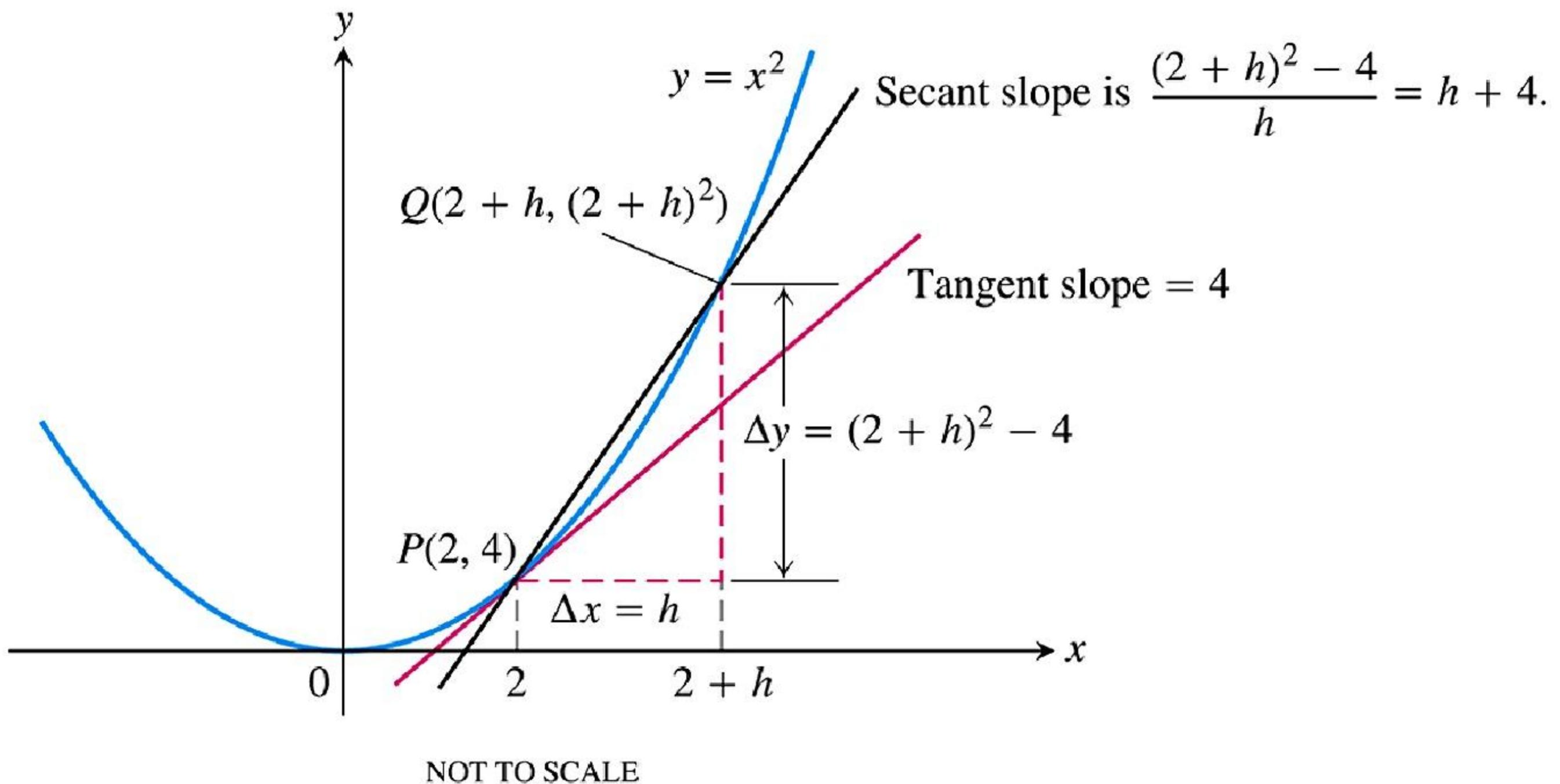
**FIGURE 2.1** A secant to the graph  $y = f(x)$ . Its slope is  $\Delta y/\Delta x$ , the average rate of change of  $f$  over the interval  $[x_1, x_2]$ .



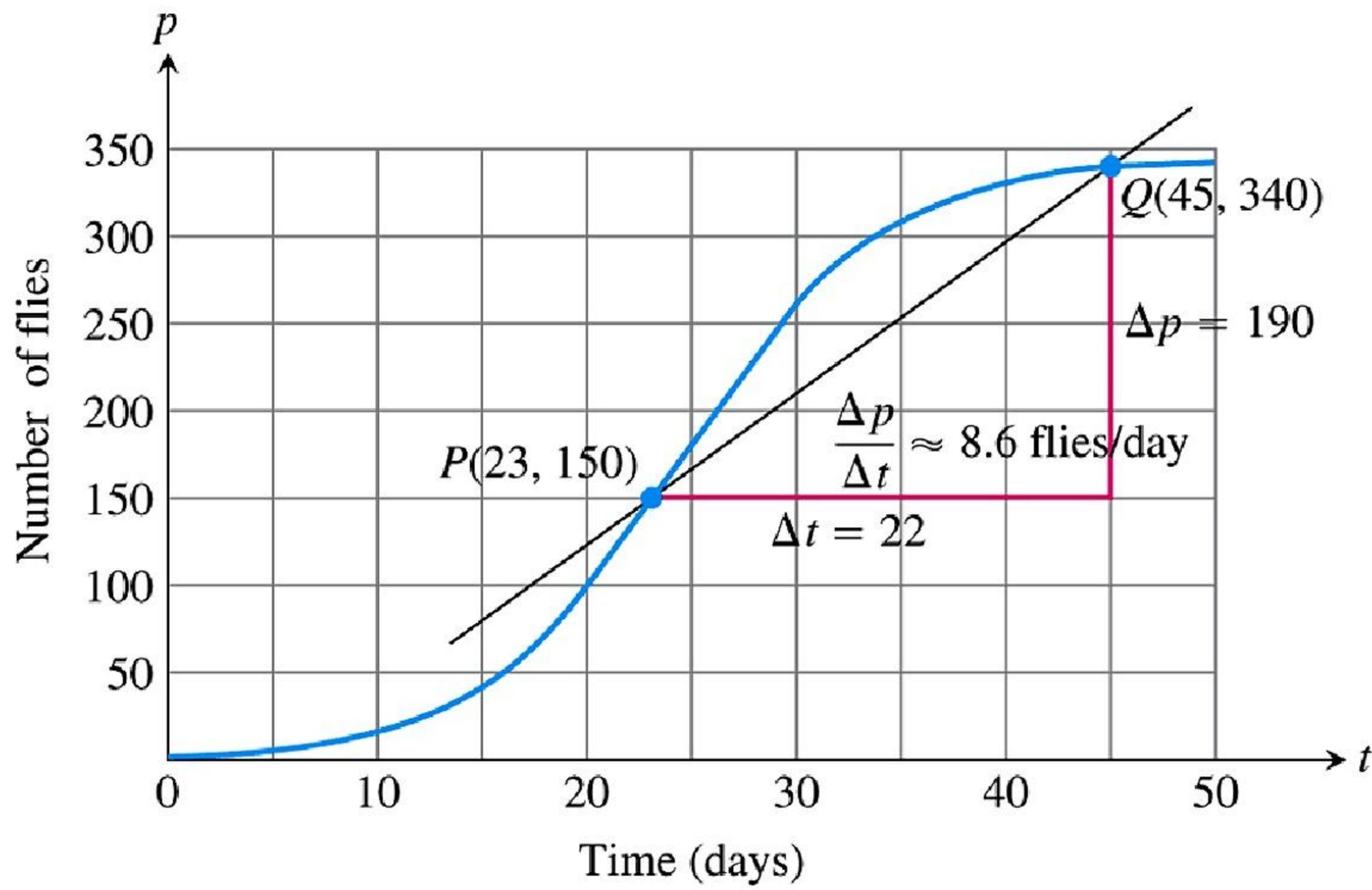
**FIGURE 2.2**  $L$  is tangent to the circle at  $P$  if it passes through  $P$  perpendicular to radius  $OP$ .



**FIGURE 2.3** The tangent line to the curve at  $P$  is the line through  $P$  whose slope is the limit of the secant line slopes as  $Q \rightarrow P$  from either side.



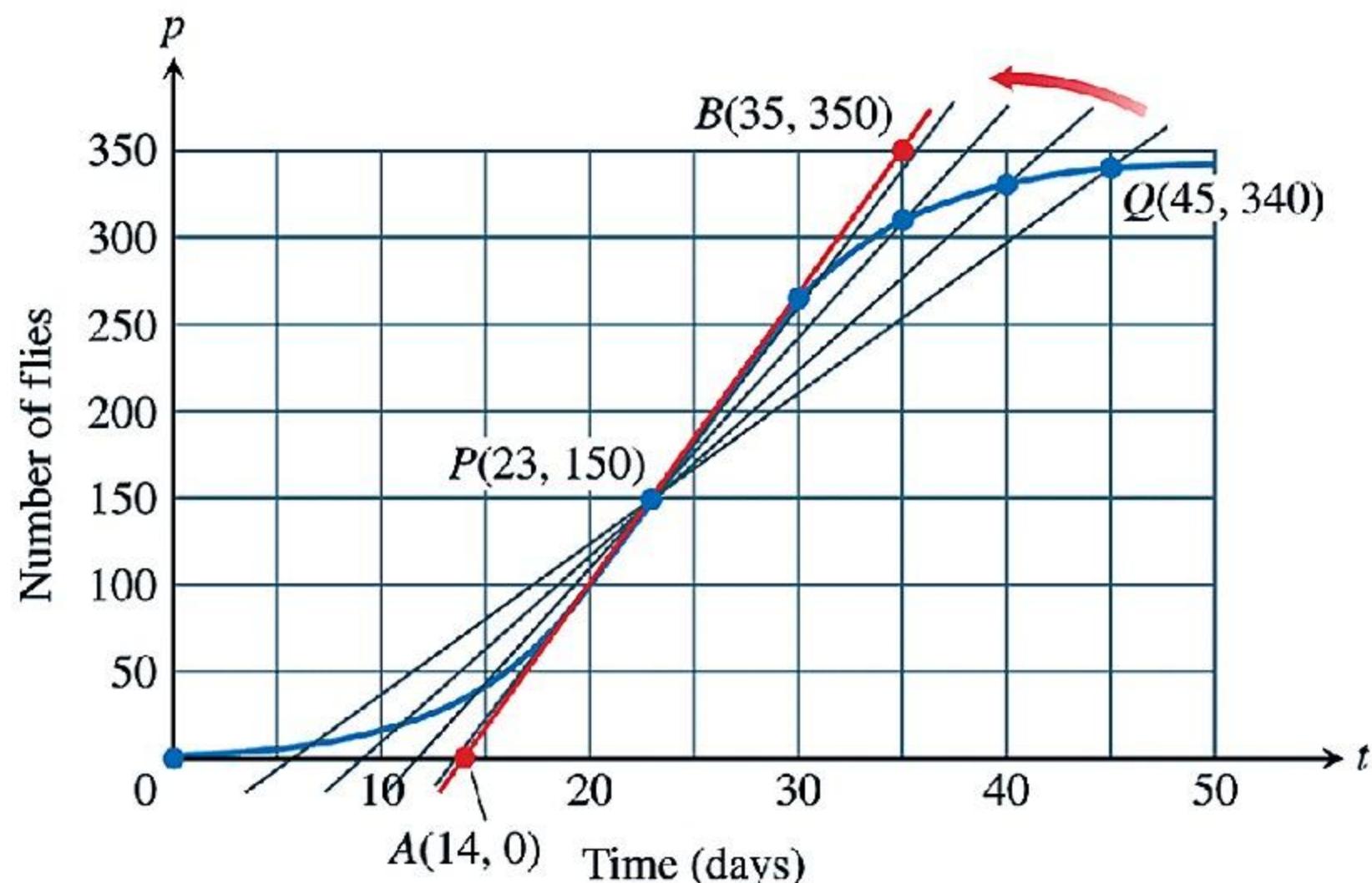
**FIGURE 2.4** Finding the slope of the parabola  $y = x^2$  at the point  $P(2, 4)$  as the limit of secant slopes (Example 3).



**FIGURE 2.5** Growth of a fruit fly population in a controlled experiment. The average rate of change over 22 days is the slope  $\Delta p/\Delta t$  of the secant line (Example 4).

**Slope of  $PQ$  =  $\Delta p / \Delta t$**   
**(flies / day)**

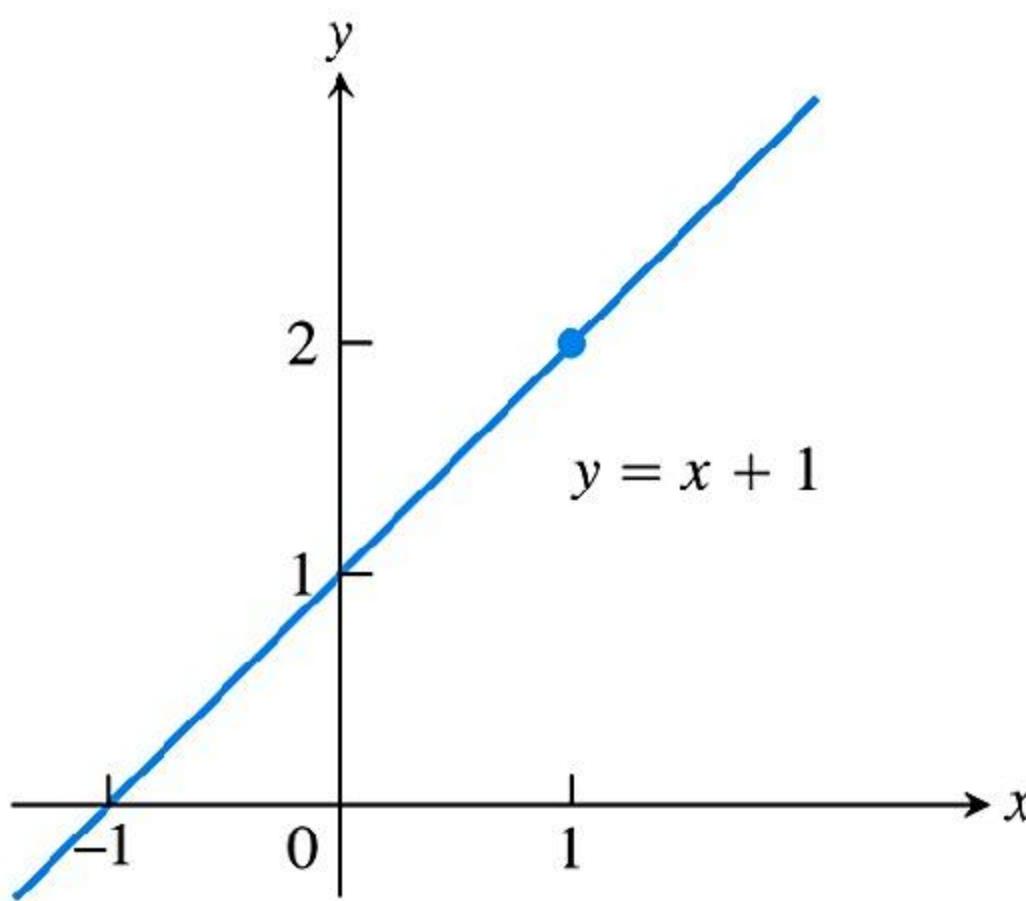
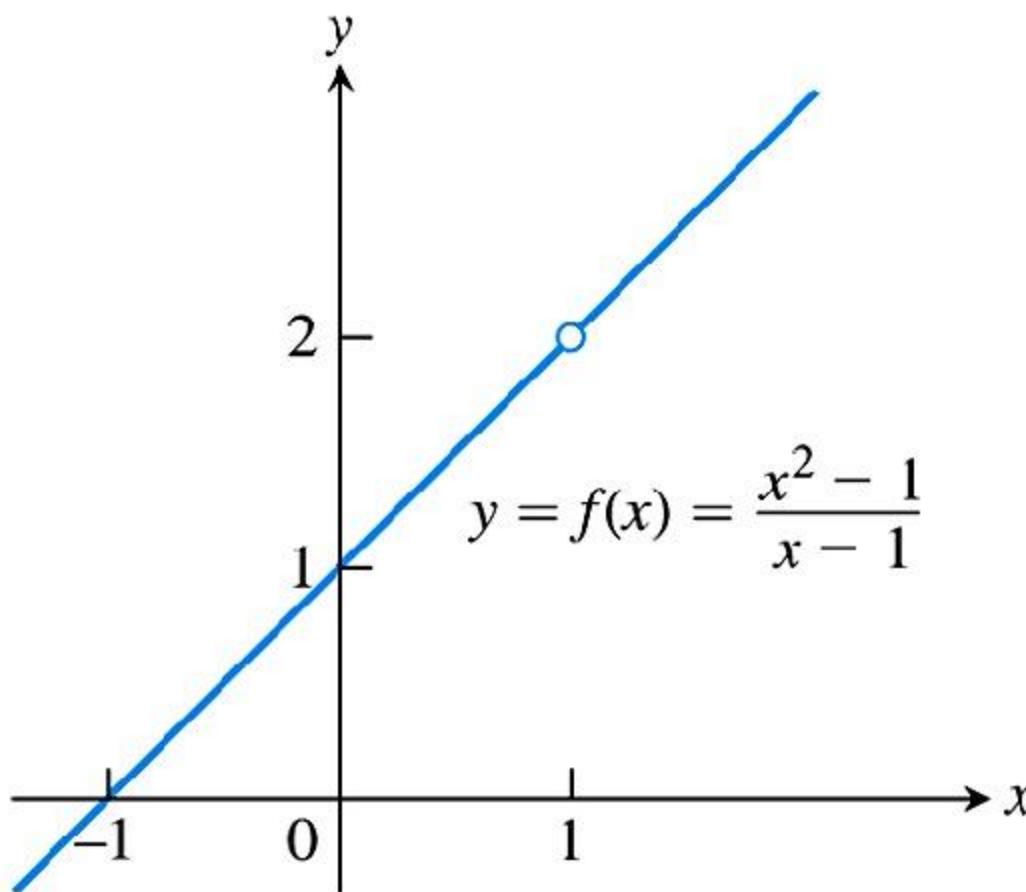
$Q$	
(45, 340)	$\frac{340 - 150}{45 - 23} \approx 8.6$
(40, 330)	$\frac{330 - 150}{40 - 23} \approx 10.6$
(35, 310)	$\frac{310 - 150}{35 - 23} \approx 13.3$
(30, 265)	$\frac{265 - 150}{30 - 23} \approx 16.4$



**FIGURE 2.6** The positions and slopes of four secant lines through the point  $P$  on the fruit fly graph (Example 5).

# Section 2.2

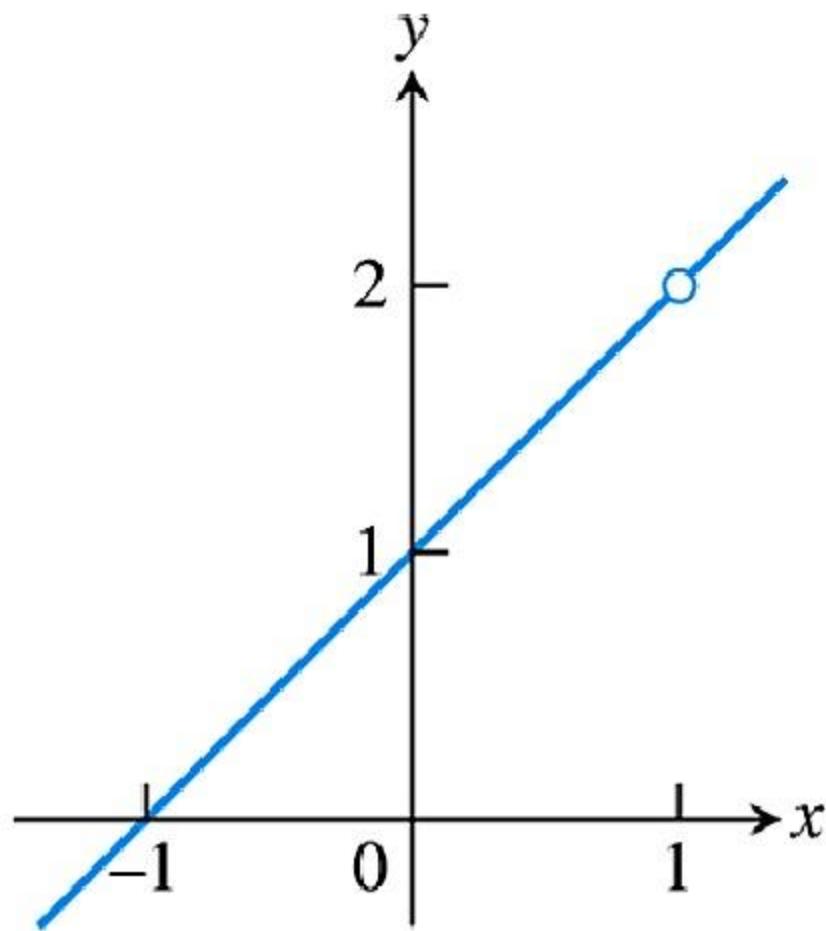
## Limit of a Function and Limit Laws



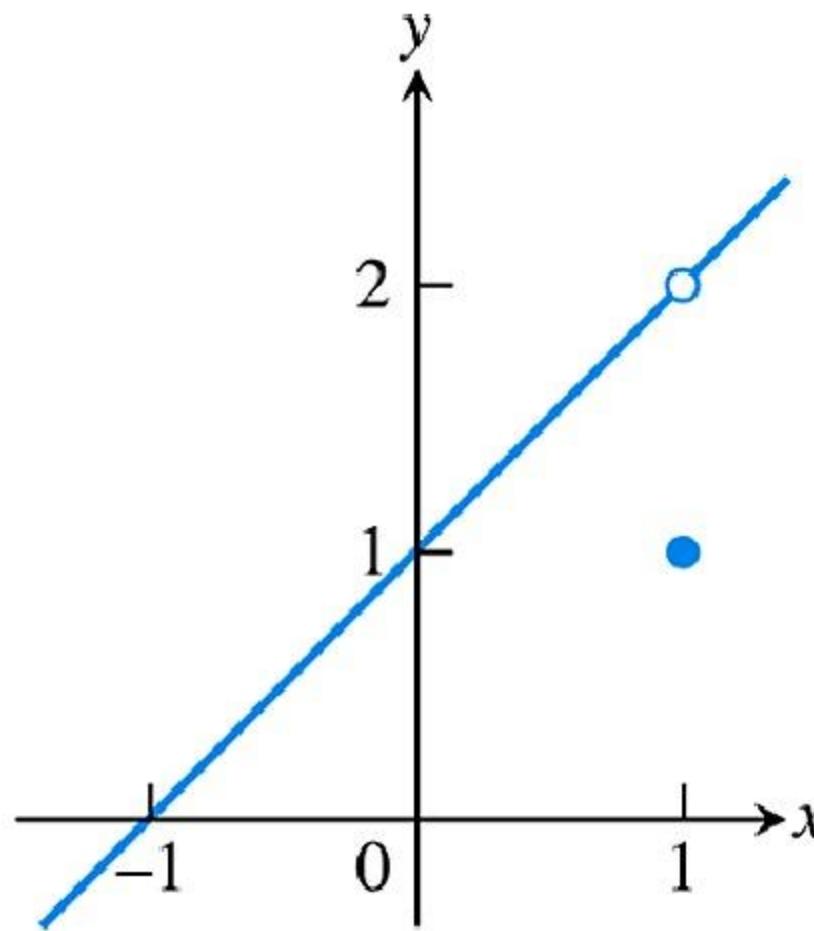
**FIGURE 2.7** The graph of  $f$  is identical with the line  $y = x + 1$  except at  $x = 1$ , where  $f$  is not defined (Example 1).

**TABLE 2.2** As  $x$  gets closer to 1,  
 $f(x)$  gets closer to 2.

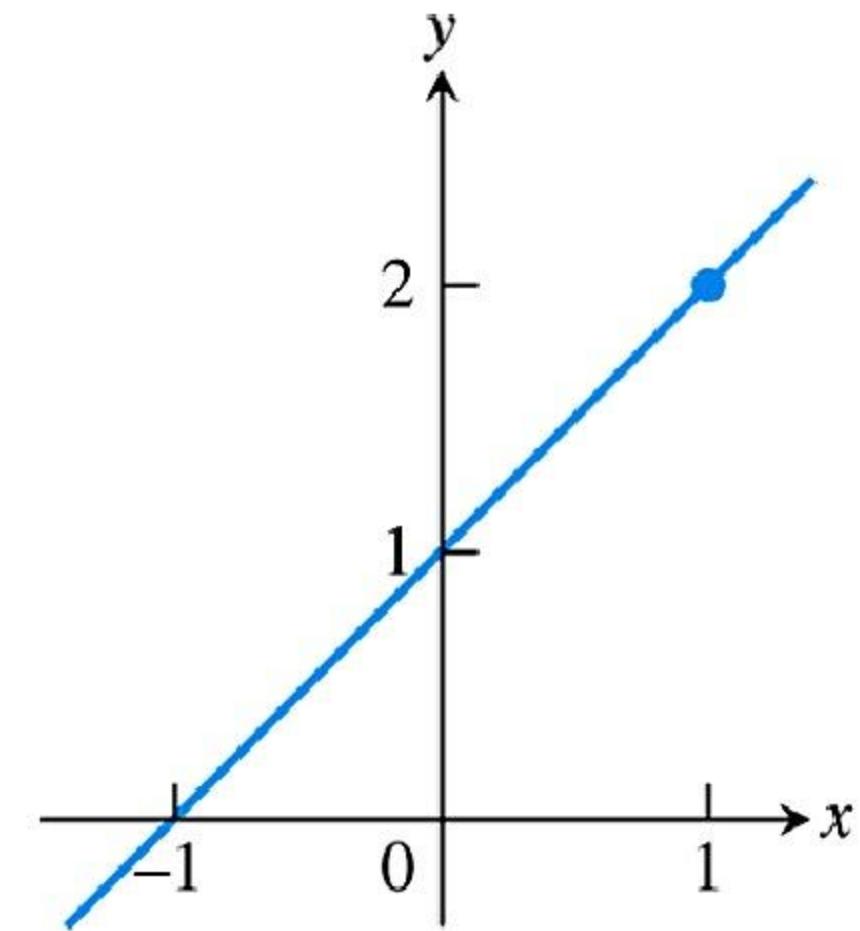
$x$	$f(x) = \frac{x^2 - 1}{x - 1}$
0.9	1.9
1.1	2.1
0.99	1.99
1.01	2.01
0.999	1.999
1.001	2.001
0.999999	1.999999
1.000001	2.000001



(a)  $f(x) = \frac{x^2 - 1}{x - 1}$

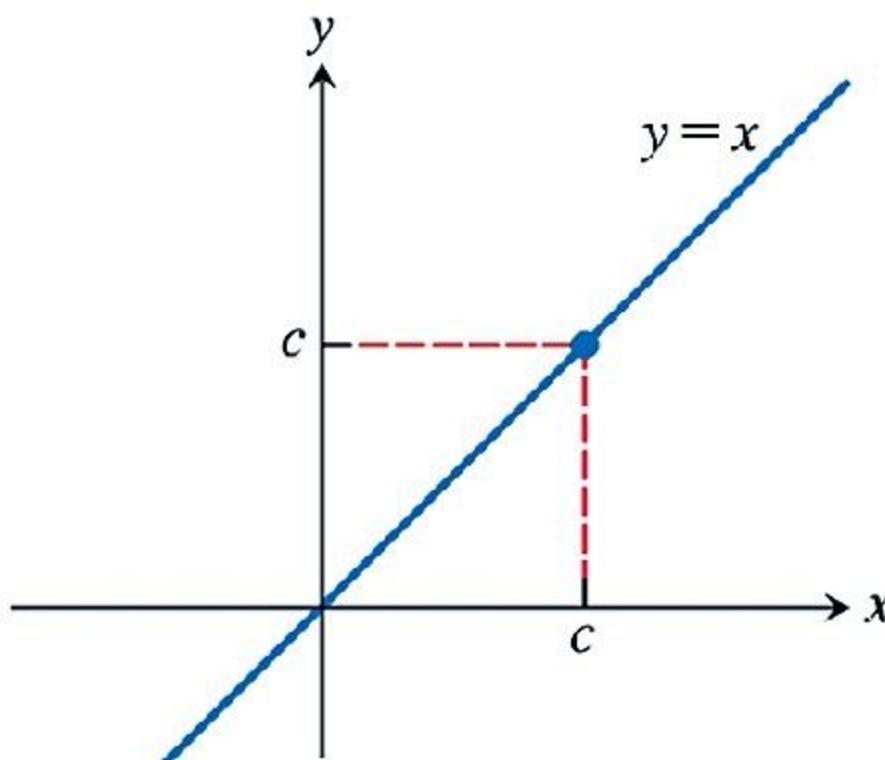


(b)  $g(x) = \begin{cases} \frac{x^2 - 1}{x - 1}, & x \neq 1 \\ 1, & x = 1 \end{cases}$

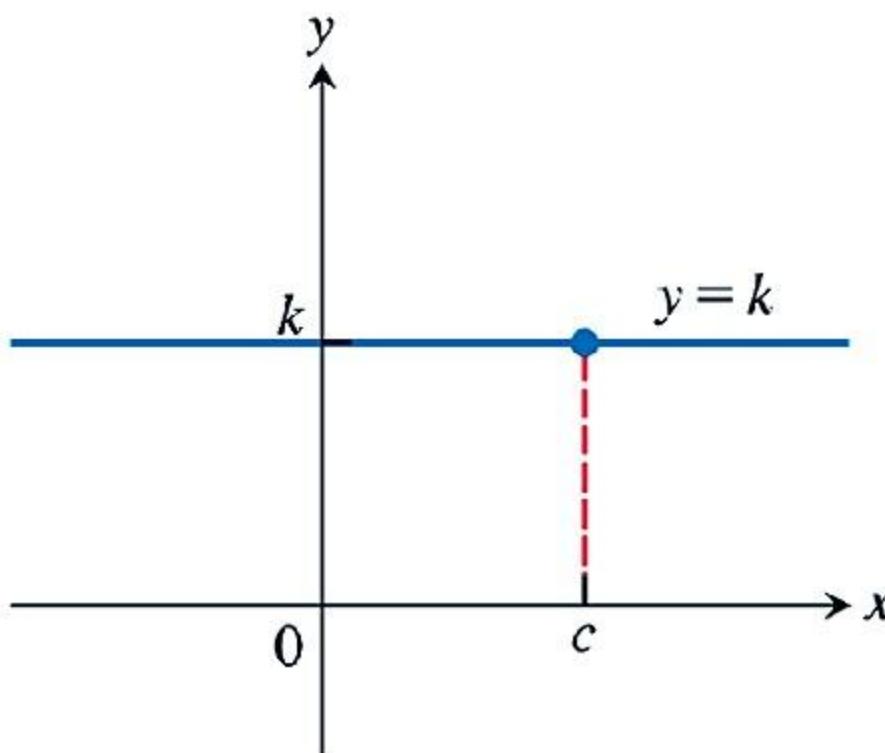


(c)  $h(x) = x + 1$

**FIGURE 2.8** The limits of  $f(x)$ ,  $g(x)$ , and  $h(x)$  all equal 2 as  $x$  approaches 1. However, only  $h(x)$  has the same function value as its limit at  $x = 1$  (Example 2).

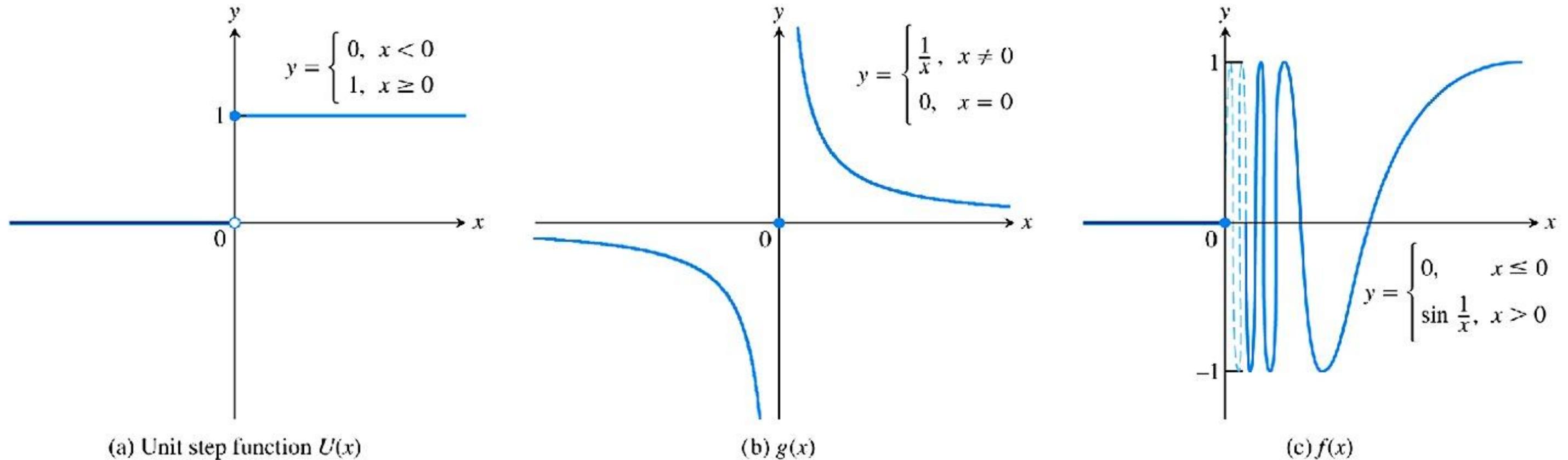


(a) Identity function



(b) Constant function

**FIGURE 2.9** The functions in Example 3 have limits at all points  $c$ .



**FIGURE 2.10** None of these functions has a limit as  $x$  approaches 0 (Example 4).

## THEOREM 1 – Limit Laws

If  $L, M, c$ , and  $k$  are real numbers and

$$\lim_{x \rightarrow c} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow c} g(x) = M, \quad \text{then}$$

1. *Sum Rule:*

$$\lim_{x \rightarrow c} (f(x) + g(x)) = L + M$$

2. *Difference Rule:*

$$\lim_{x \rightarrow c} (f(x) - g(x)) = L - M$$

3. *Constant Multiple Rule:*

$$\lim_{x \rightarrow c} (k \cdot f(x)) = k \cdot L$$

4. *Product Rule:*

$$\lim_{x \rightarrow c} (f(x) \cdot g(x)) = L \cdot M$$

5. *Quotient Rule:*

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0$$

6. *Power Rule:*

$$\lim_{x \rightarrow c} [f(x)]^n = L^n, \quad n \text{ a positive integer}$$

7. *Root Rule:*

$$\lim_{x \rightarrow c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}, \quad n \text{ a positive integer}$$

(If  $n$  is even, we assume that  $f(x) \geq 0$  for  $x$  in an interval containing  $c$ .)

## THEOREM 2—Limits of Polynomials

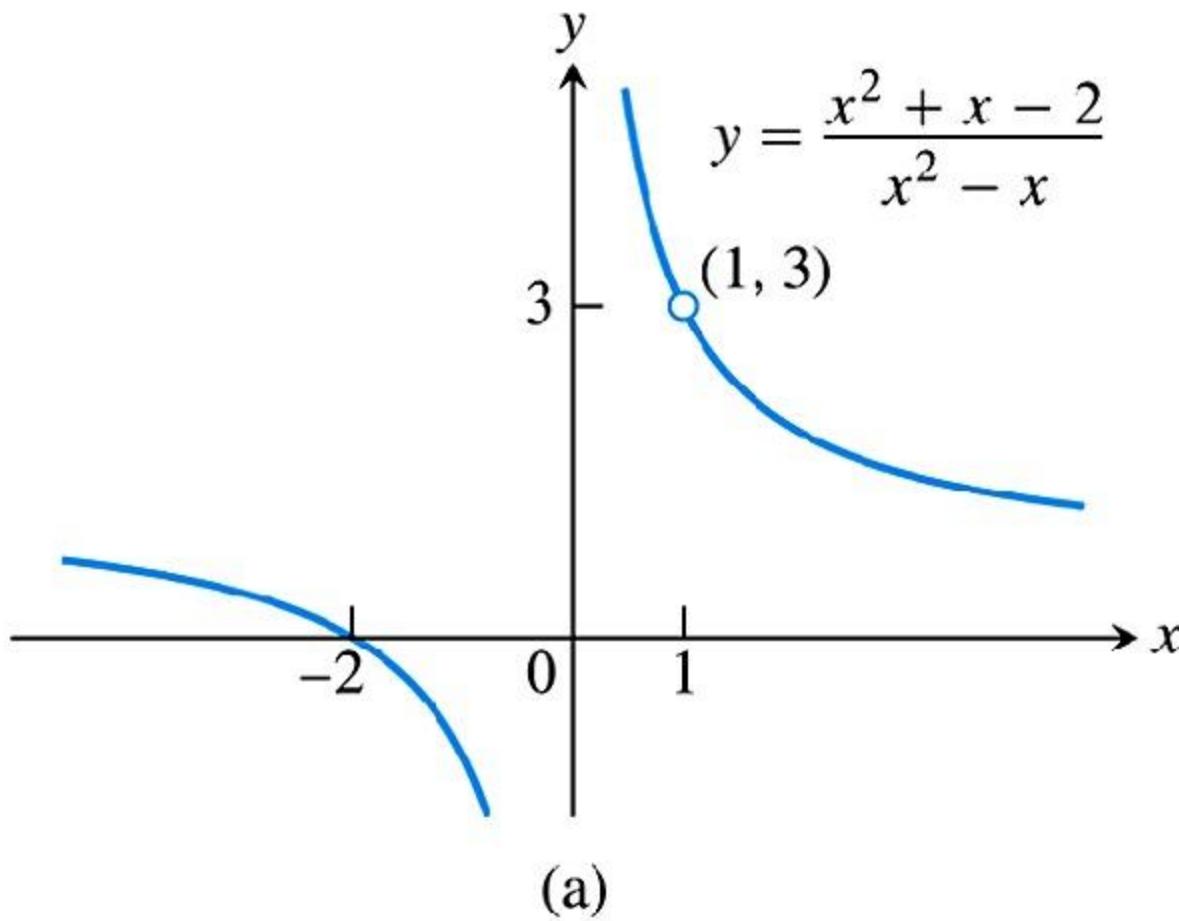
If  $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ , then

$$\lim_{x \rightarrow c} P(x) = P(c) = a_n c^n + a_{n-1} c^{n-1} + \cdots + a_0.$$

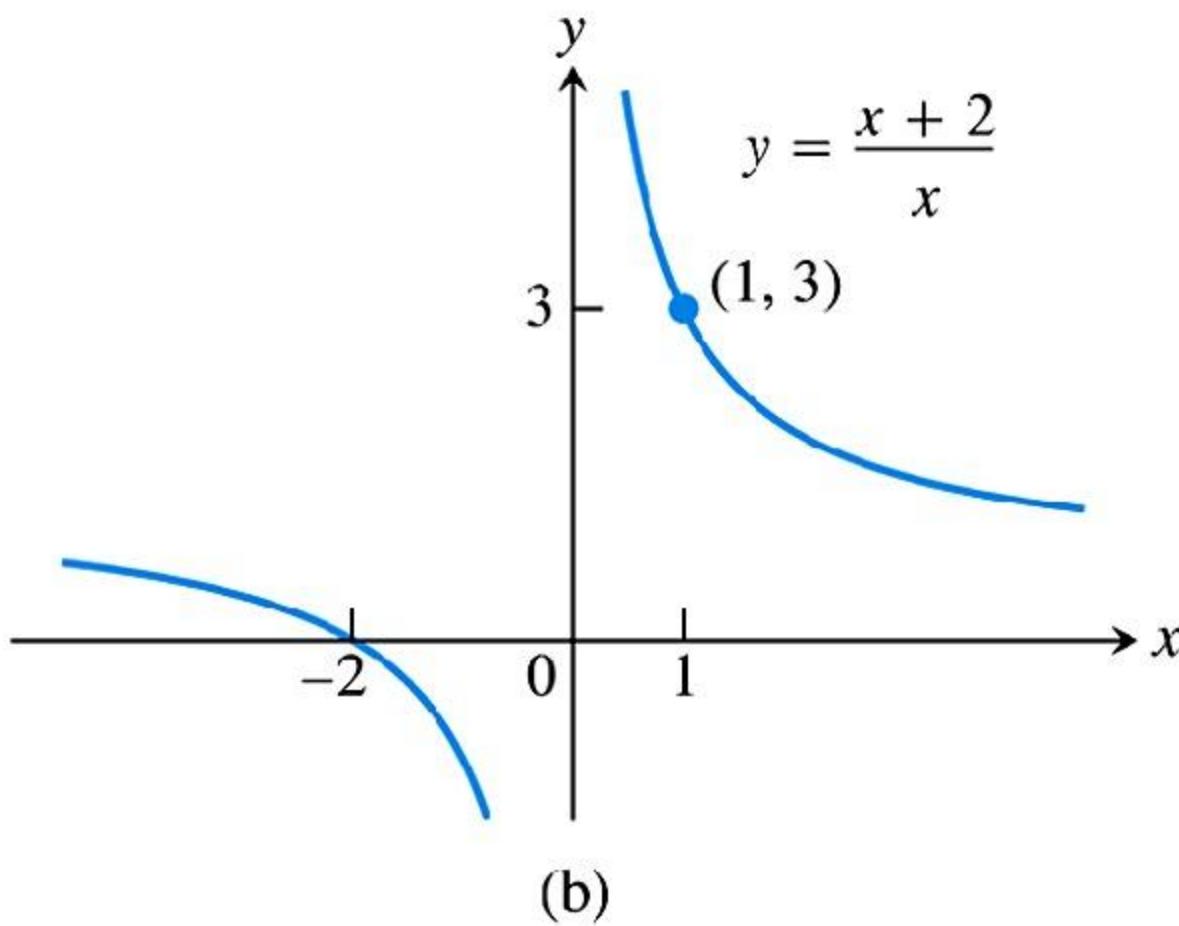
### THEOREM 3—Limits of Rational Functions

If  $P(x)$  and  $Q(x)$  are polynomials and  $Q(c) \neq 0$ , then

$$\lim_{x \rightarrow c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$



(a)



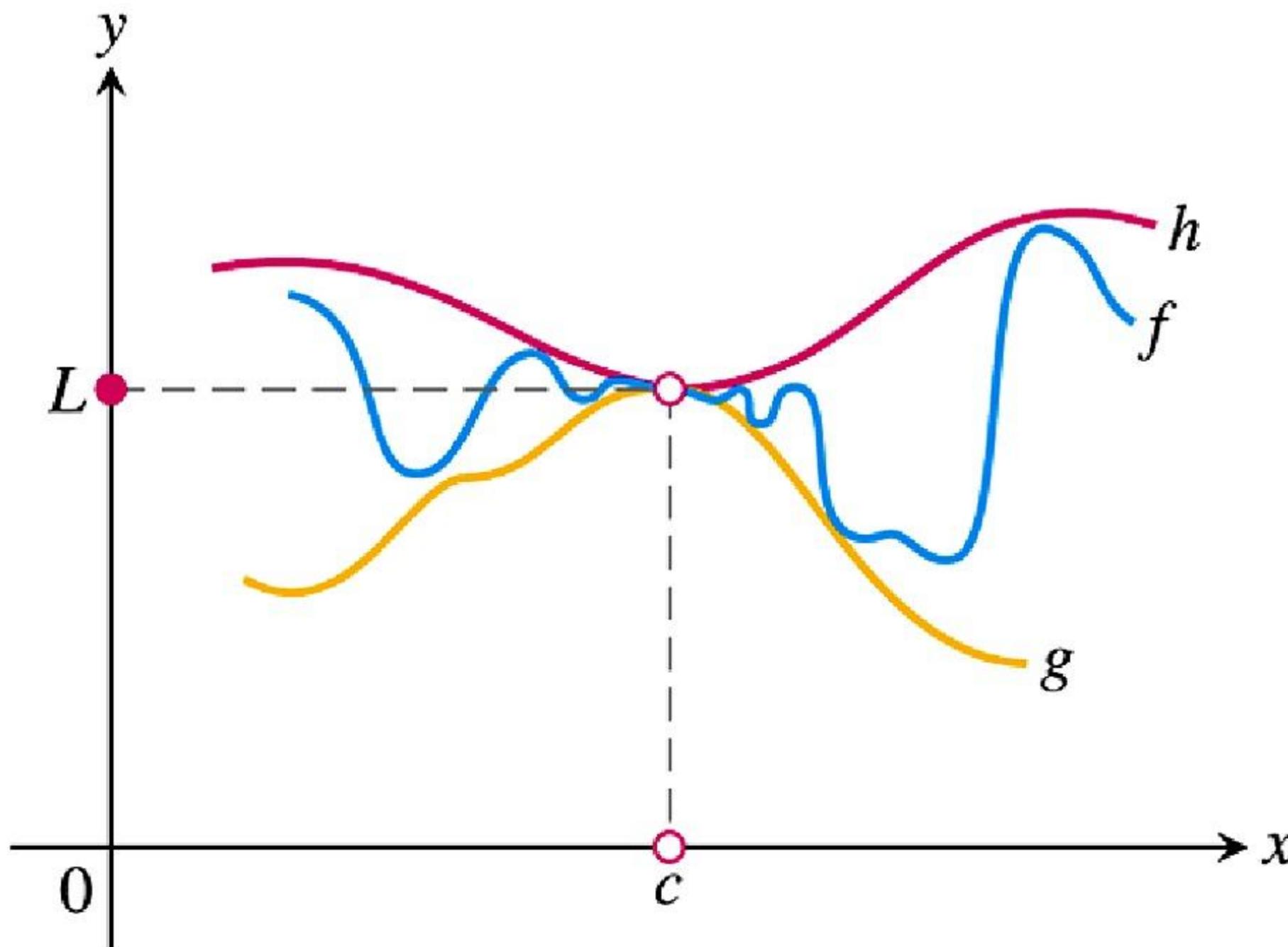
(b)

**FIGURE 2.11** The graph of  $f(x) = (x^2 + x - 2)/(x^2 - x)$  in part (a) is the same as the graph of  $g(x) = (x + 2)/x$  in part (b) except at  $x = 1$ , where  $f$  is undefined. The functions have the same limit as  $x \rightarrow 1$  (Example 7).

**TABLE 2.3** Computed values of  $f(x) = \frac{\sqrt{x^2 + 100} - 10}{x^2}$  near  $x = 0$

$x$	$f(x)$
$\pm 1$	0.049876
$\pm 0.5$	0.049969
$\pm 0.1$	0.049999
$\pm 0.01$	0.050000
$\pm 0.0005$	0.050000
$\pm 0.0001$	0.000000
$\pm 0.00001$	0.000000
$\pm 0.000001$	0.000000

} approaches 0.05?  
 } approaches 0?



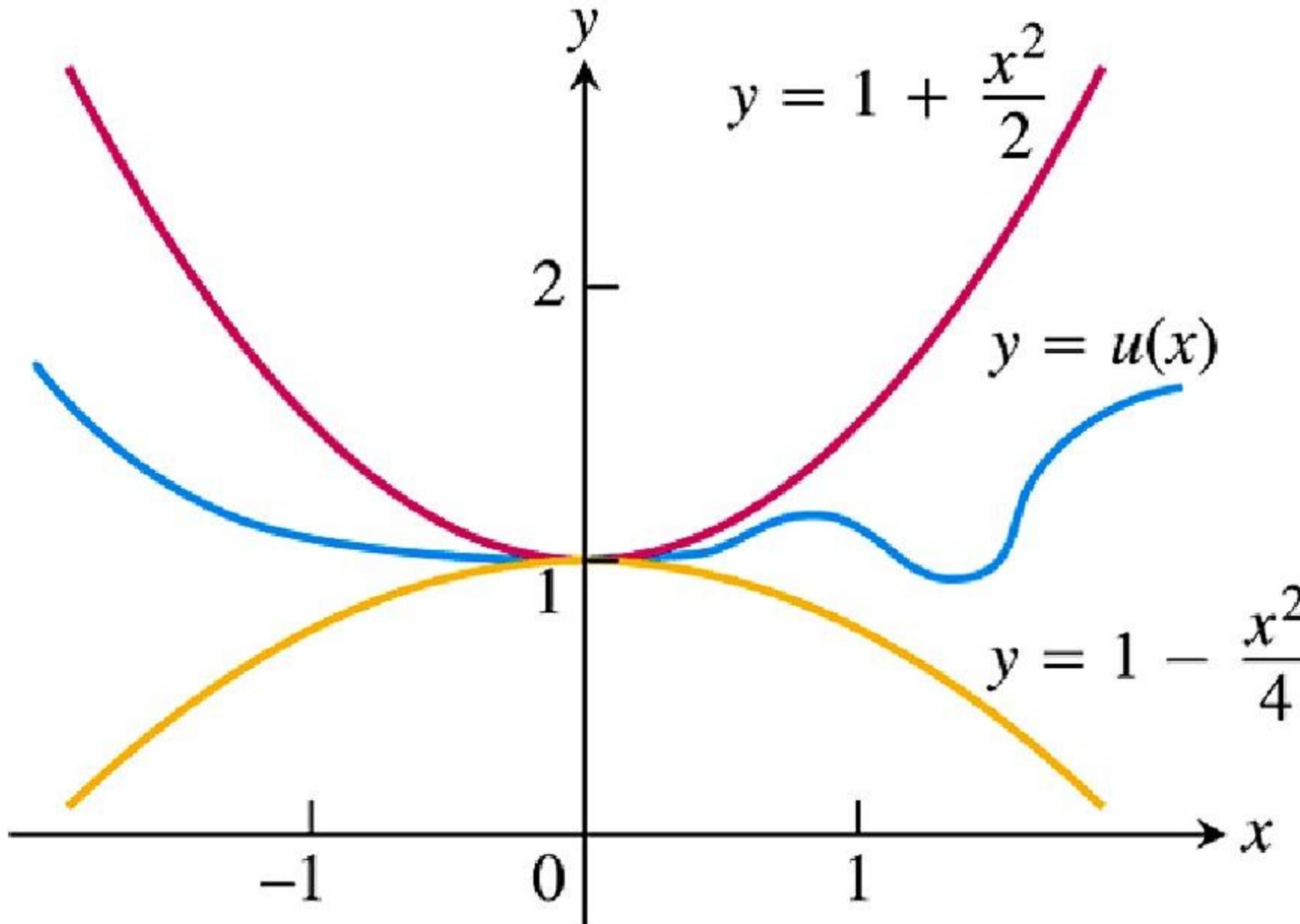
**FIGURE 2.12** The graph of  $f$  is sandwiched between the graphs of  $g$  and  $h$ .

## THEOREM 4—The Sandwich Theorem

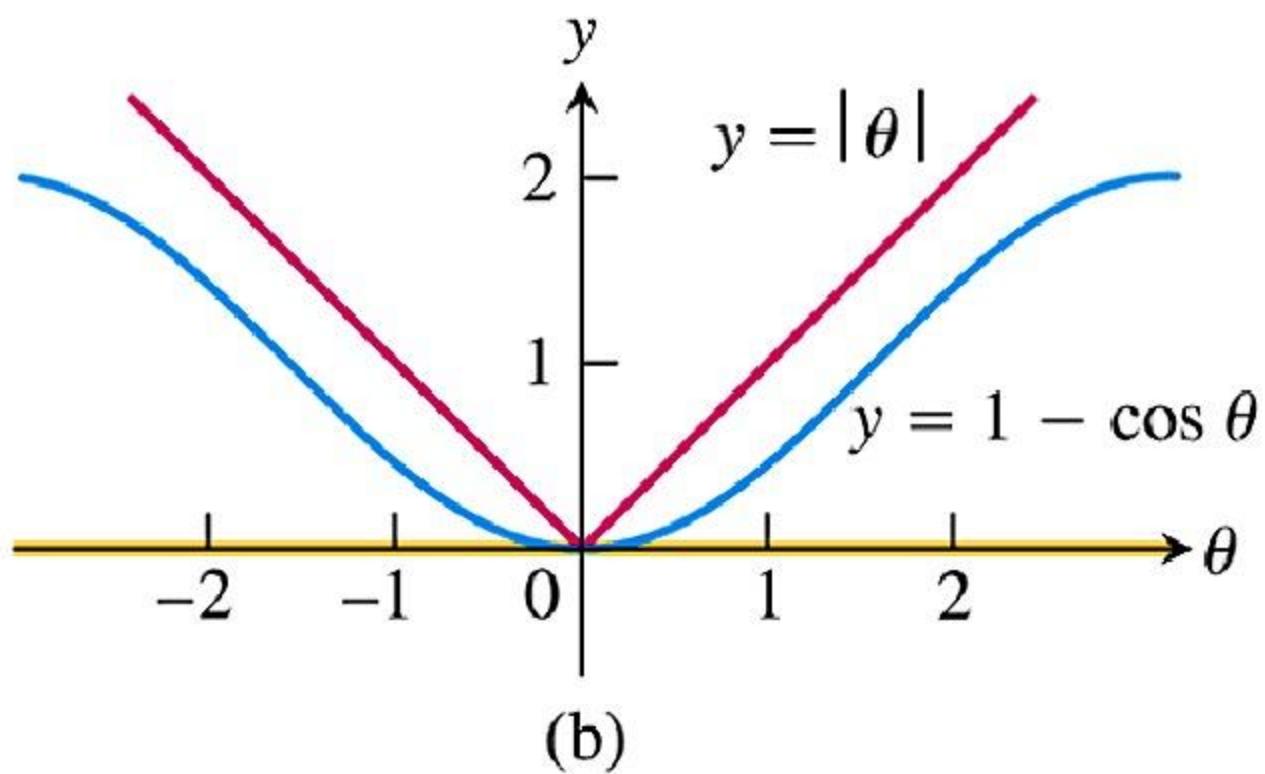
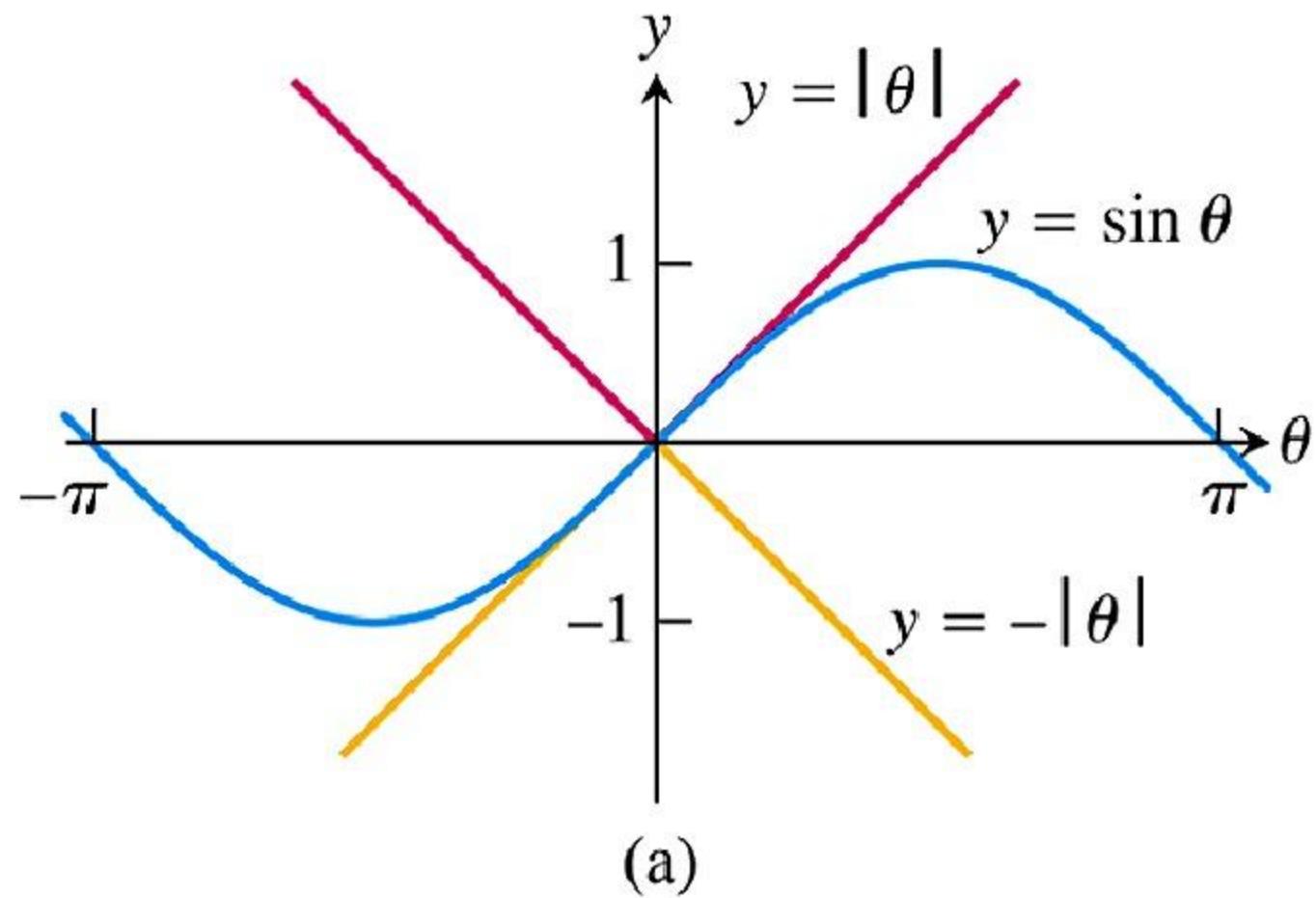
Suppose that  $g(x) \leq f(x) \leq h(x)$  for all  $x$  in some open interval containing  $c$ , except possibly at  $x = c$  itself. Suppose also that

$$\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} h(x) = L.$$

Then  $\lim_{x \rightarrow c} f(x) = L$ .



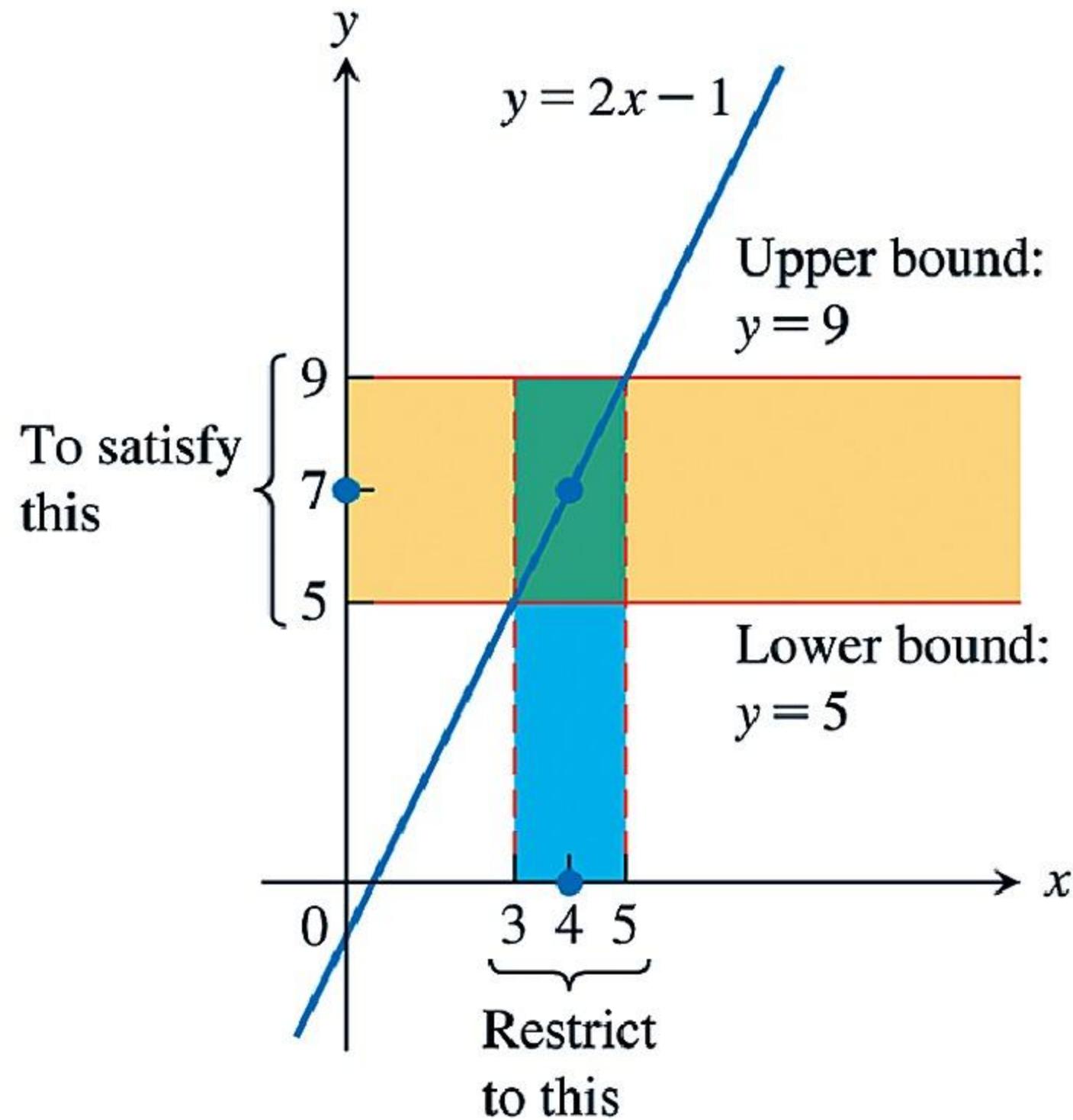
**FIGURE 2.13** Any function  $u(x)$  whose graph lies in the region between  $y = 1 + (x^2/2)$  and  $y = 1 - (x^2/4)$  has limit 1 as  $x \rightarrow 0$  (Example 10).



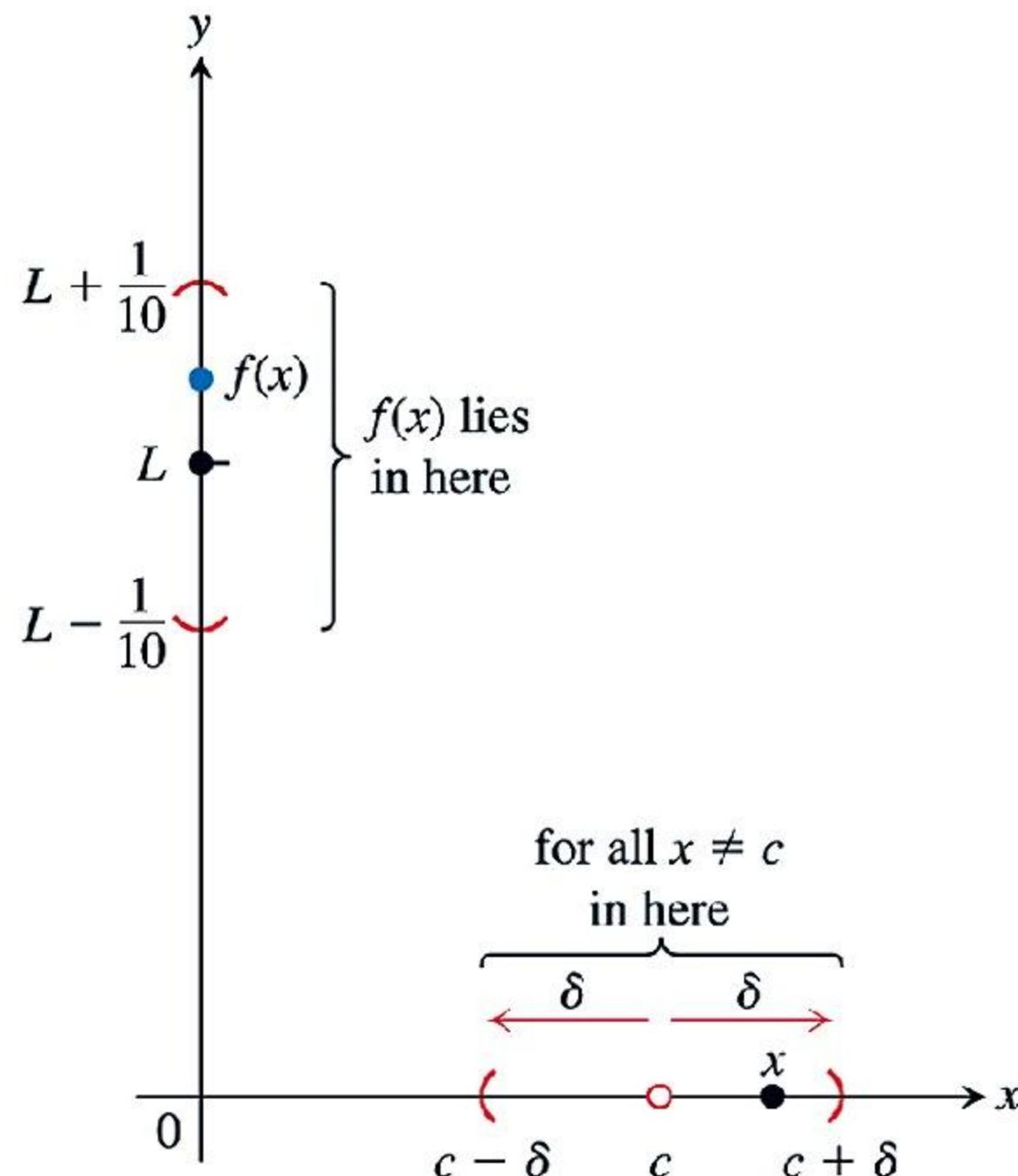
**FIGURE 2.14** The Sandwich Theorem confirms the limits in Example 11.

# Section 2.3

## The Precise Definition of a Limit



**FIGURE 2.15** Keeping  $x$  within 1 unit of  $x = 4$  will keep  $y$  within 2 units of  $y = 7$  (Example 1).



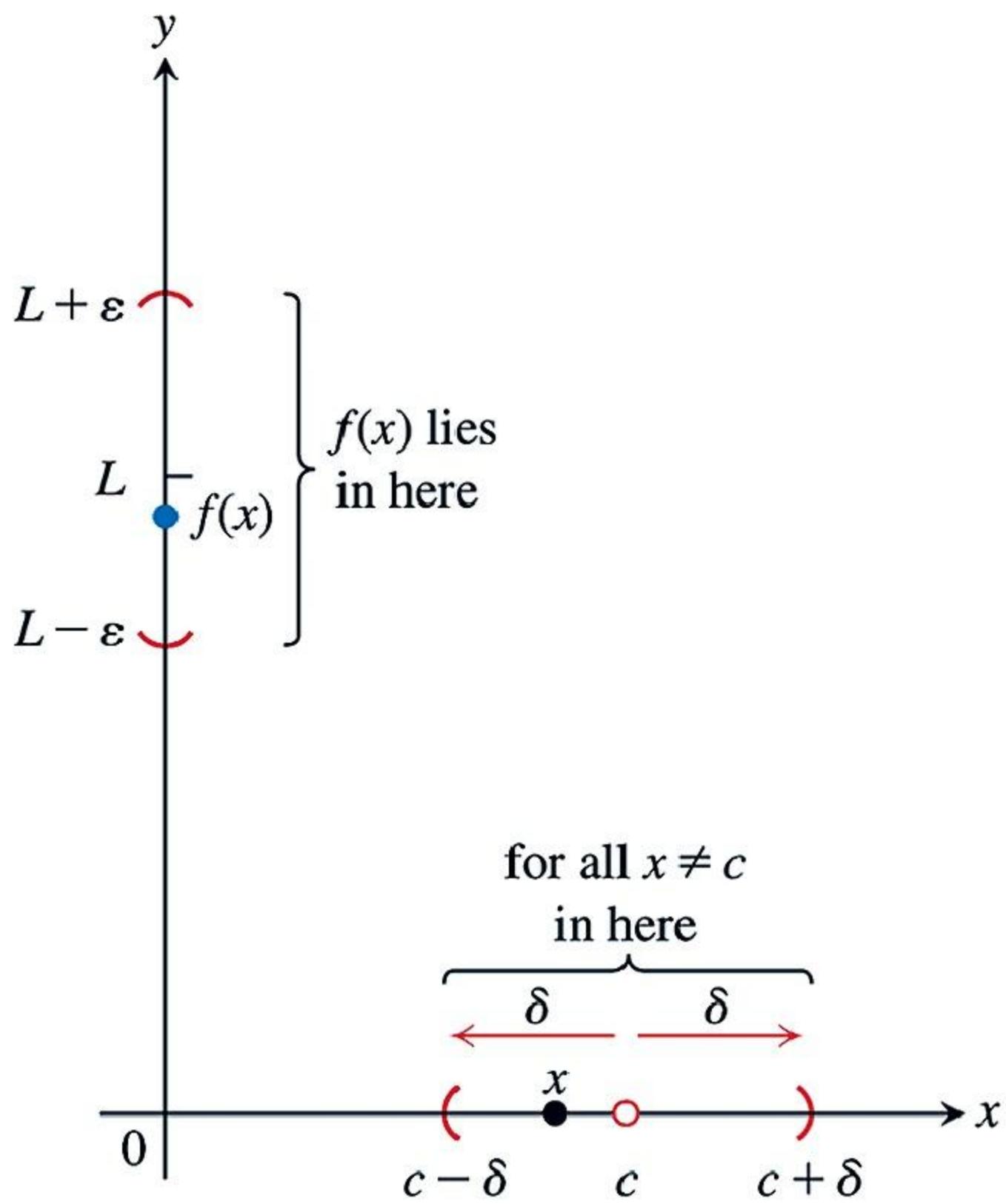
**FIGURE 2.16** How should we define  $\delta > 0$  so that keeping  $x$  within the interval  $(c - \delta, c + \delta)$  will keep  $f(x)$  within the interval  $\left(L - \frac{1}{10}, L + \frac{1}{10}\right)$ ?

**DEFINITION** Let  $f(x)$  be defined on an open interval about  $c$ , except possibly at  $c$  itself. We say that the **limit of  $f(x)$  as  $x$  approaches  $c$  is the number  $L$** , and write

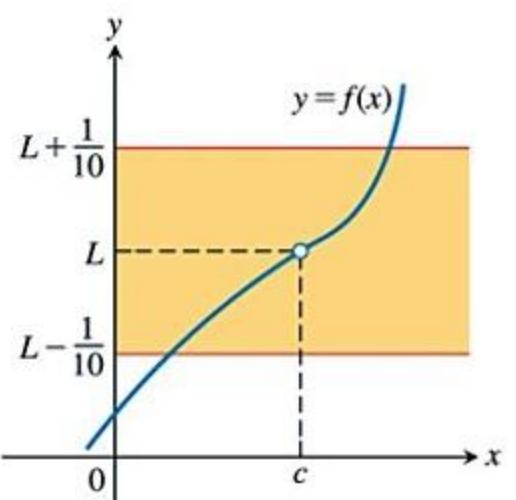
$$\lim_{x \rightarrow c} f(x) = L,$$

if, for every number  $\varepsilon > 0$ , there exists a corresponding number  $\delta > 0$  such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

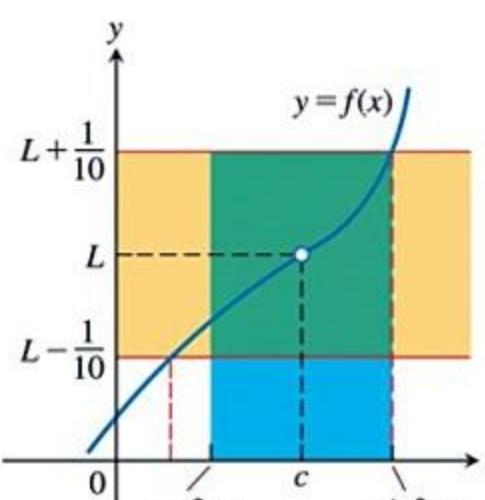


**FIGURE 2.17** The relation of  $\delta$  and  $\varepsilon$  in the definition of limit.



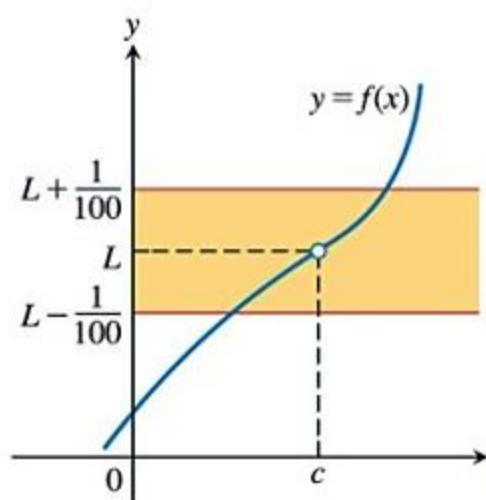
The challenge:

$$\text{Make } |f(x) - L| < \varepsilon = \frac{1}{10}$$



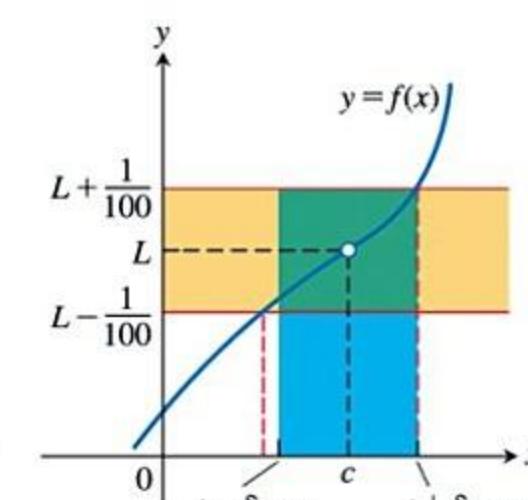
Response:

$$|x - c| < \delta_{1/10} \text{ (a number)}$$



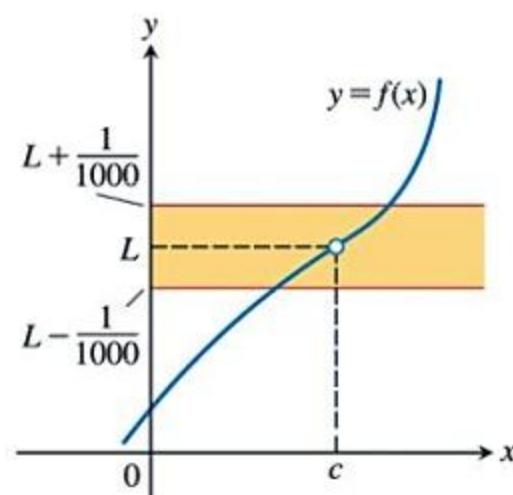
New challenge:

$$\text{Make } |f(x) - L| < \varepsilon = \frac{1}{100}$$



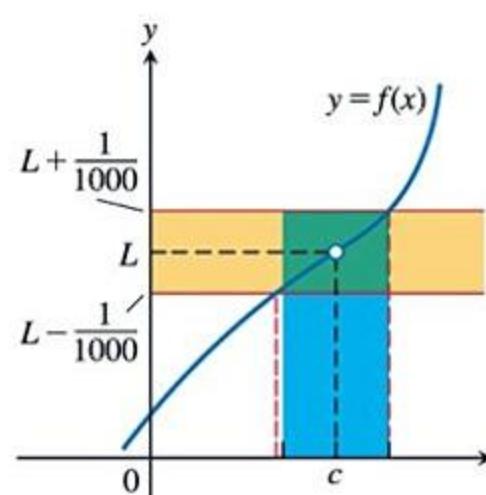
Response:

$$|x - c| < \delta_{1/100}$$



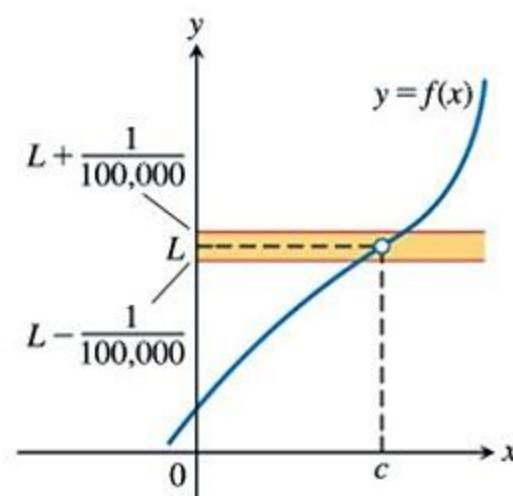
New challenge:

$$\varepsilon = \frac{1}{1000}$$



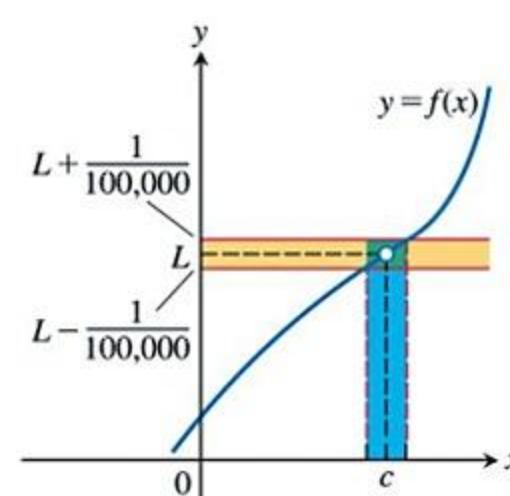
Response:

$$|x - c| < \delta_{1/1000}$$



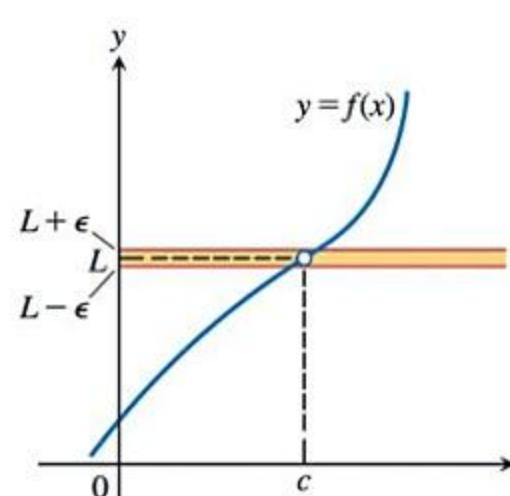
New challenge:

$$\varepsilon = \frac{1}{100,000}$$



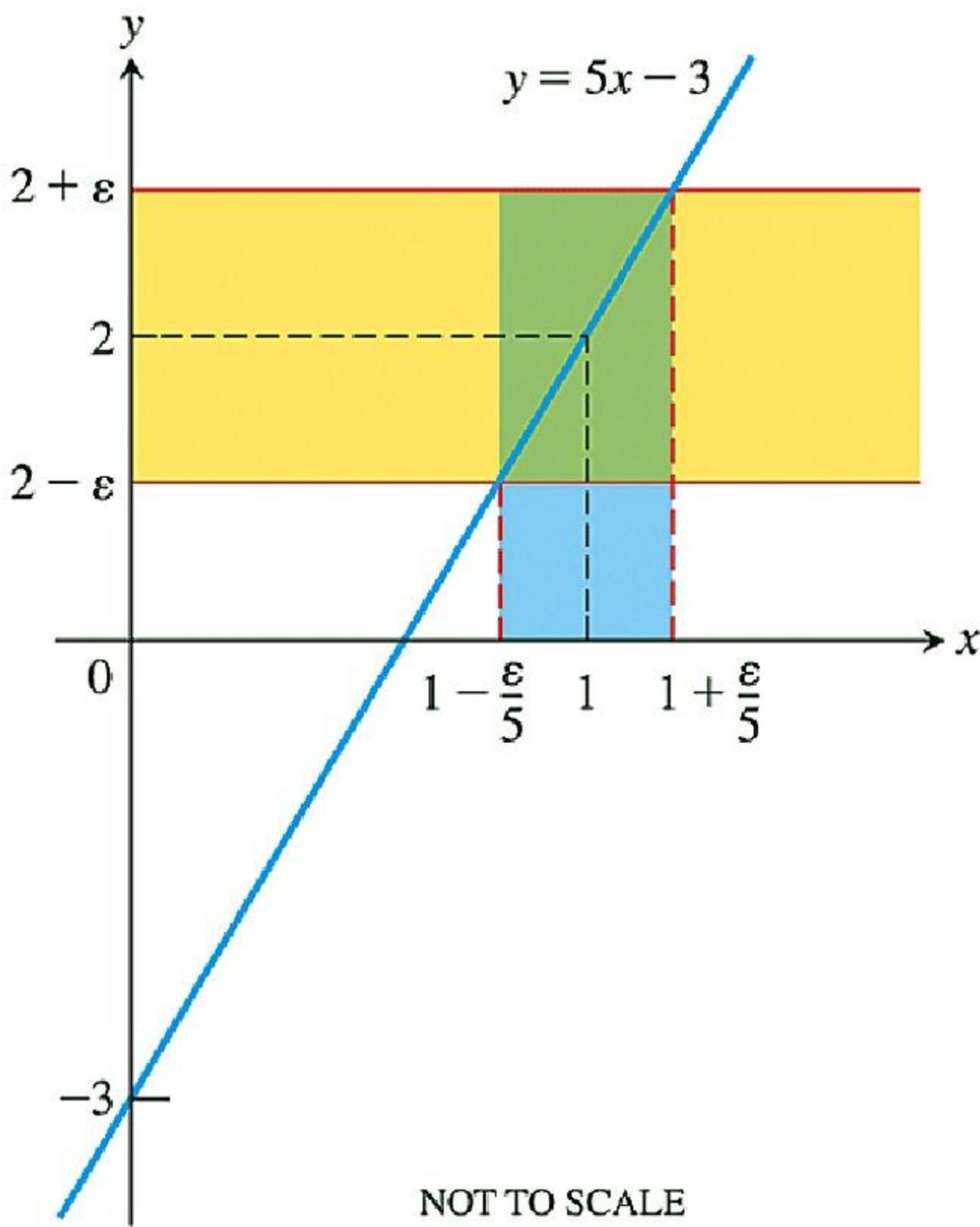
Response:

$$|x - c| < \delta_{1/100,000}$$

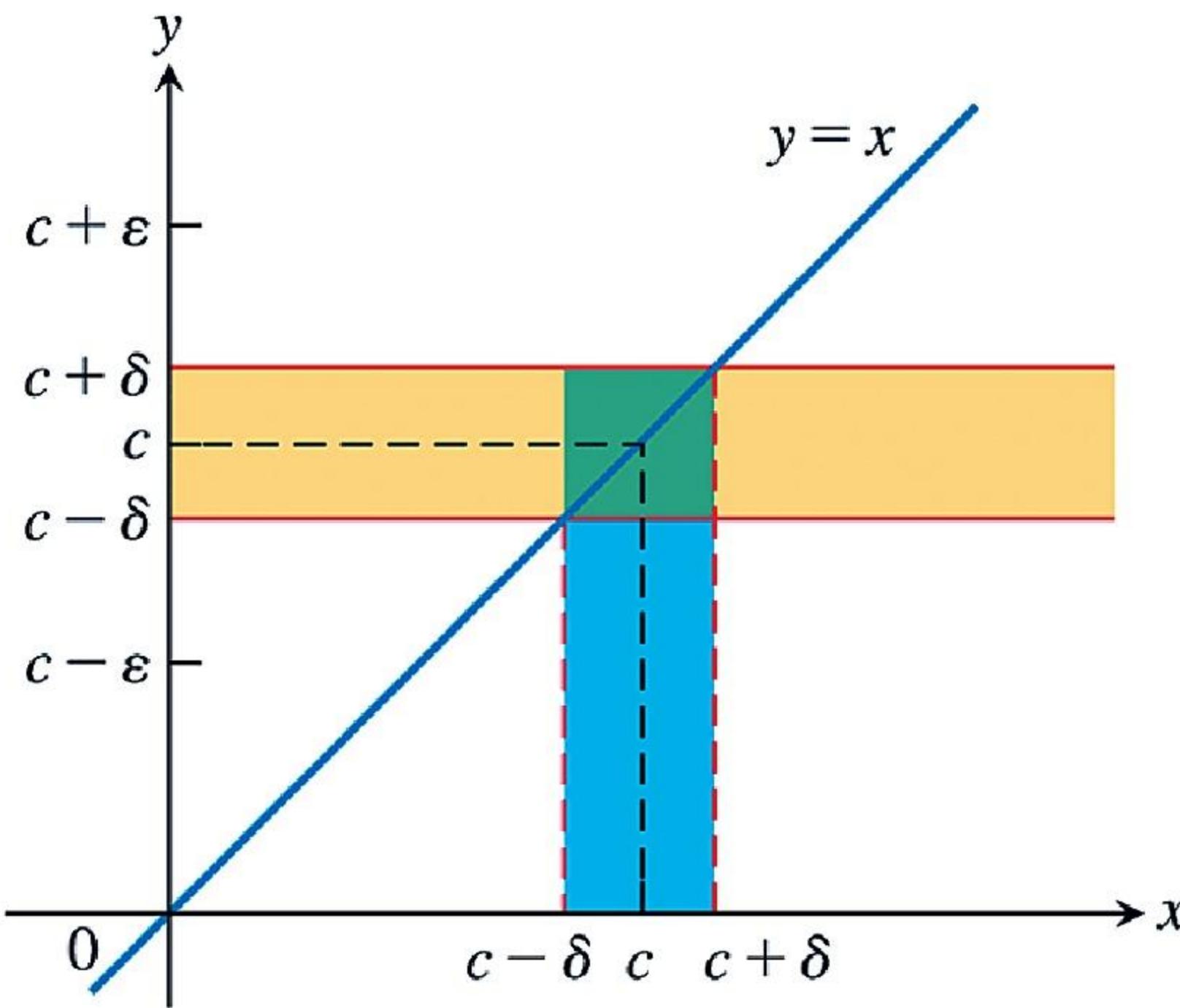


New challenge:

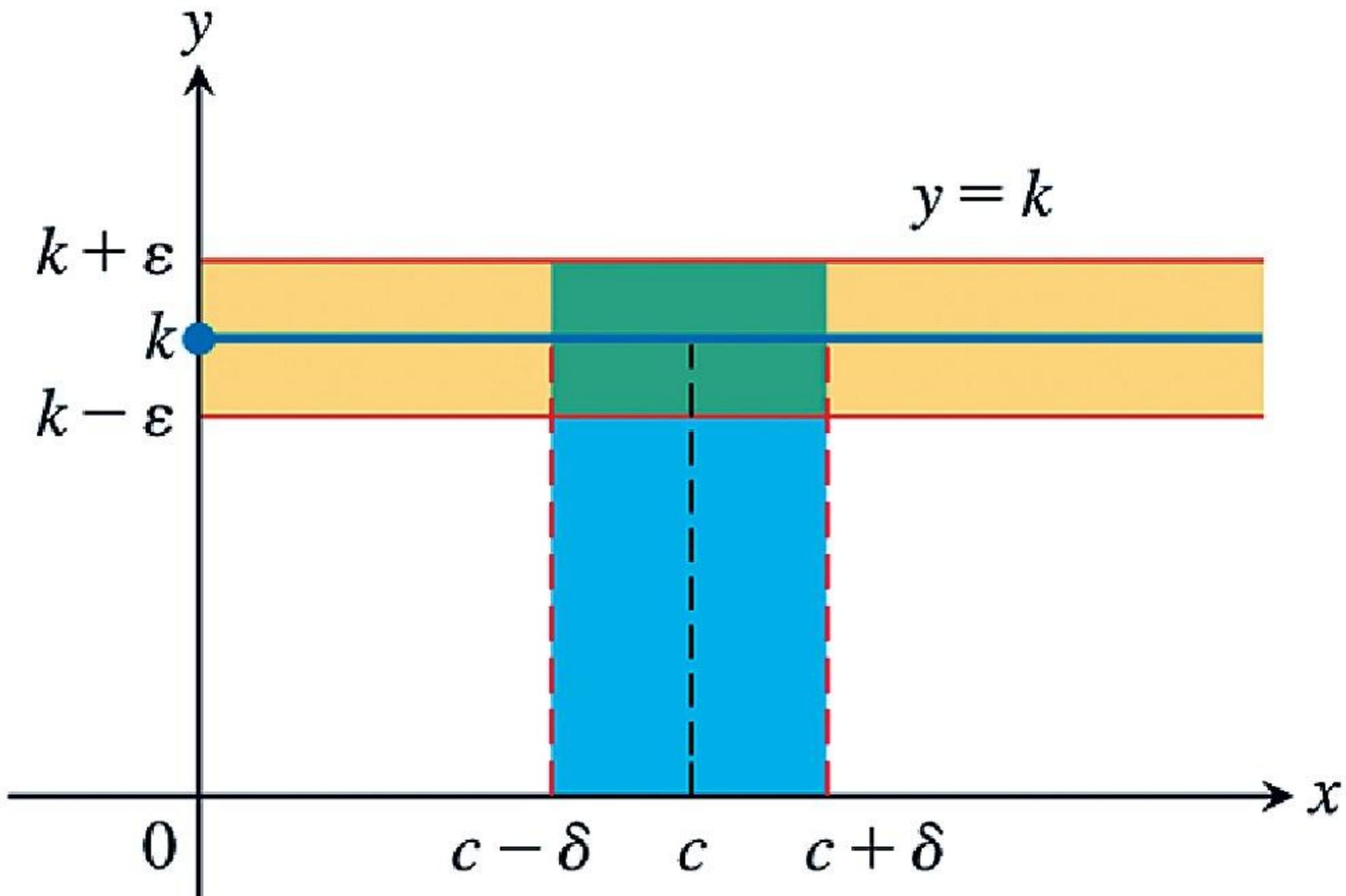
$$\varepsilon = \dots$$



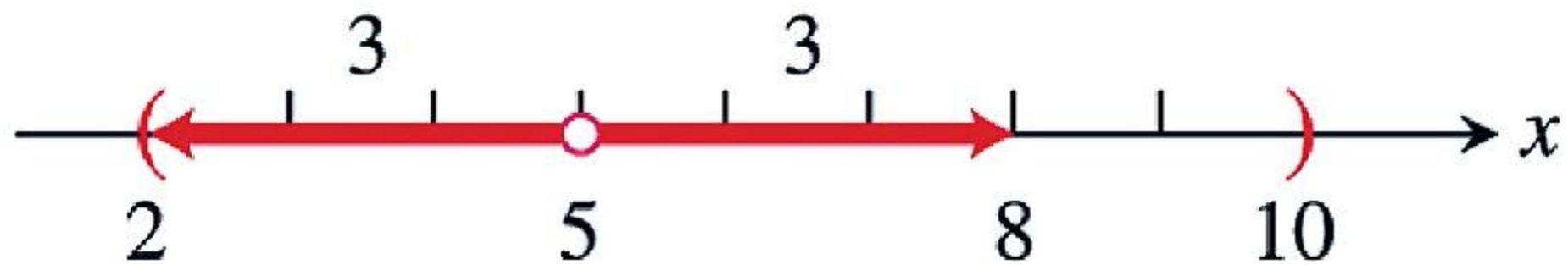
**FIGURE 2.18** If  $f(x) = 5x - 3$ , then  
 $0 < |x - 1| < \varepsilon/5$  guarantees that  
 $|f(x) - 2| < \varepsilon$  (Example 2).



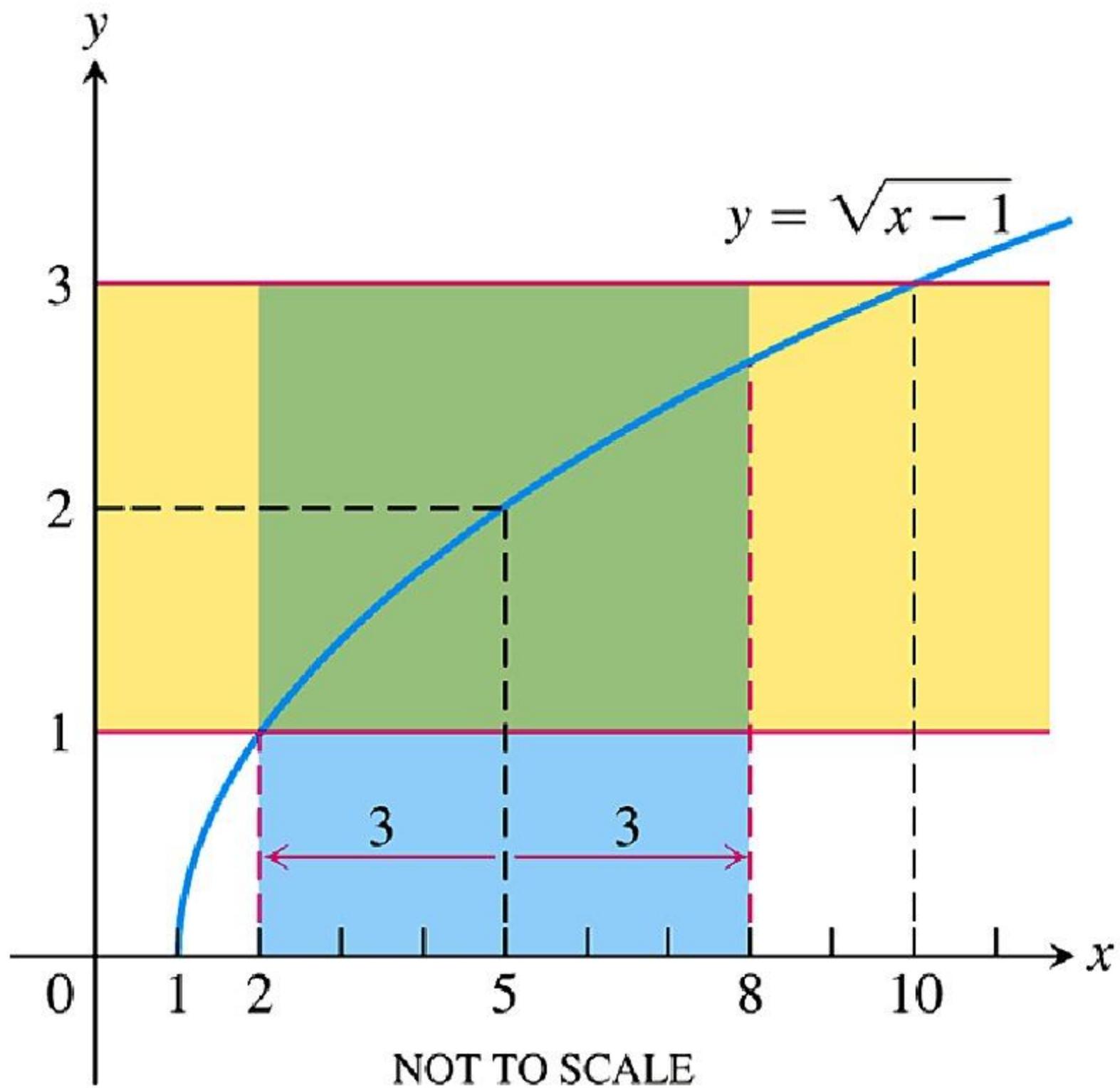
**FIGURE 2.19** For the function  $f(x) = x$ , we find that  $0 < |x - c| < \delta$  will guarantee  $|f(x) - c| < \varepsilon$  whenever  $\delta \leq \varepsilon$  (Example 3a).



**FIGURE 2.20** For the function  $f(x) = k$ , we find that  $|f(x) - k| < \varepsilon$  for any positive  $\delta$  (Example 3b).



**FIGURE 2.21** An open interval of radius 3 about  $x = 5$  will lie inside the open interval  $(2, 10)$ .



**FIGURE 2.22** The function and intervals  
in Example 4.

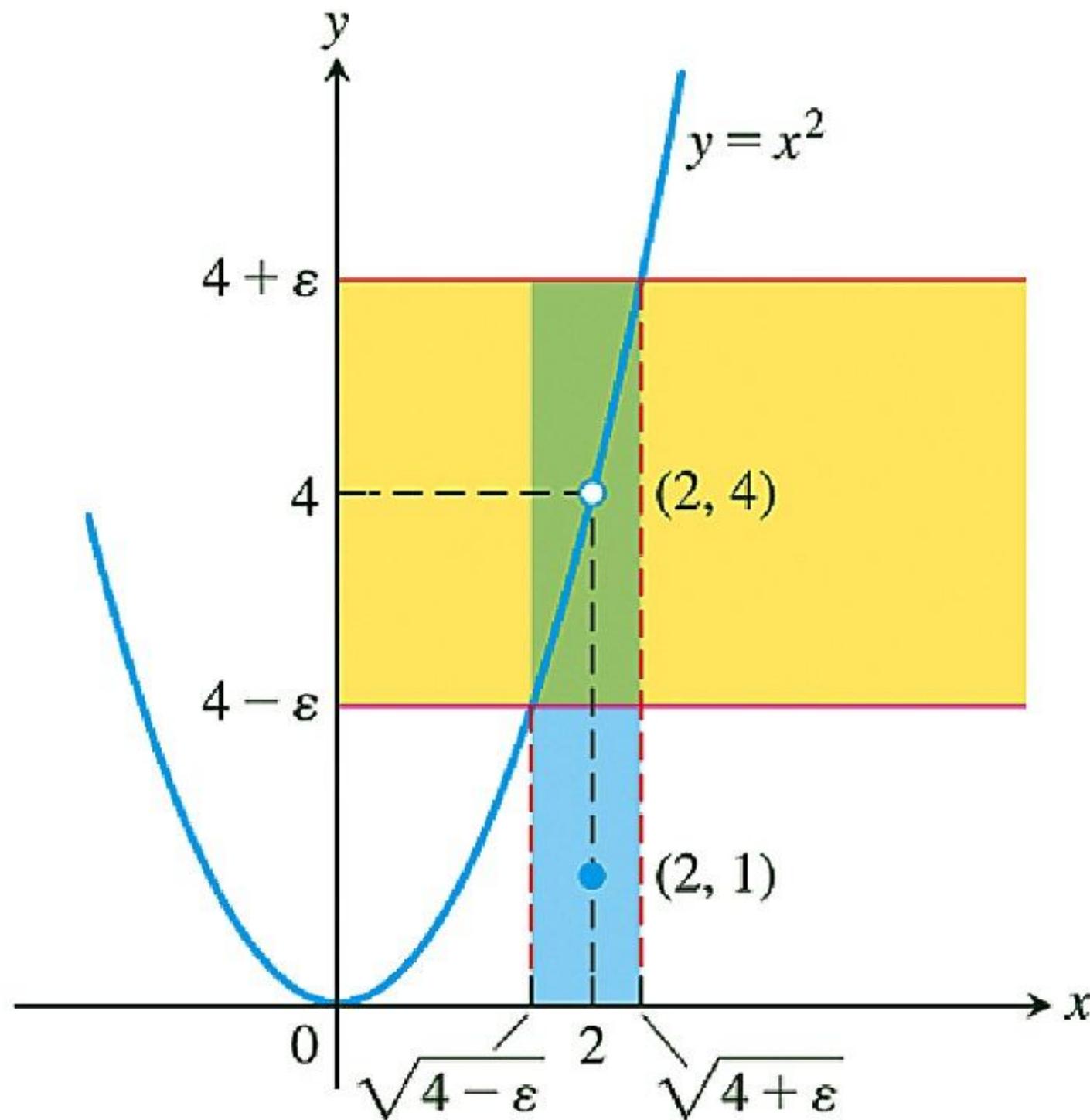
## How to Find Algebraically a $\delta$ for a Given $f, L, c$ , and $\varepsilon > 0$

The process of finding a  $\delta > 0$  such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta$$

can be accomplished in two steps.

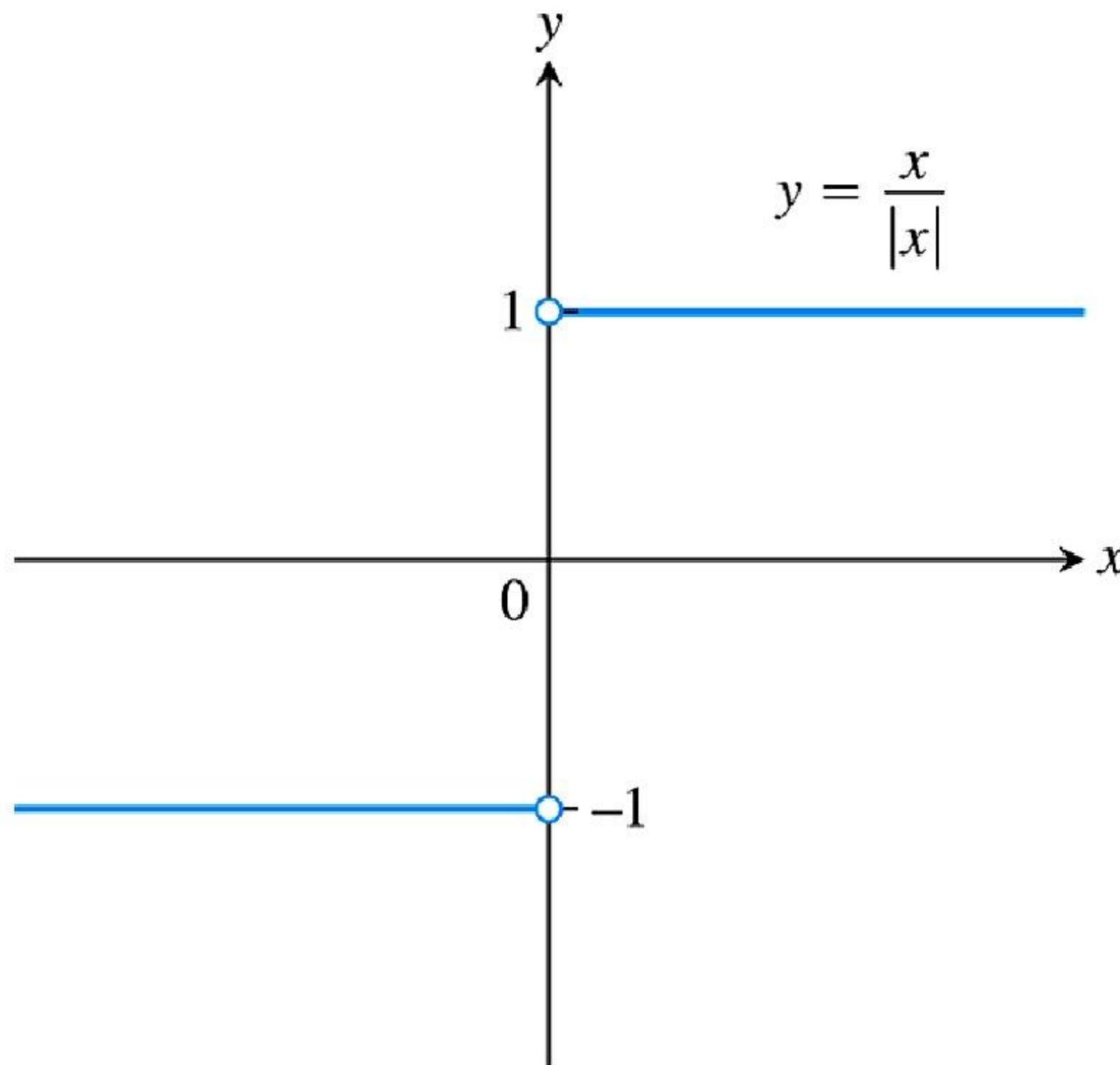
1. *Solve the inequality*  $|f(x) - L| < \varepsilon$  *to find an open interval*  $(a, b)$  *containing*  $c$  *on which the inequality holds for all*  $x \neq c$ . Note that we do not require the inequality to hold at  $x = c$ . It may hold there or it may not, but the value of  $f$  at  $x = c$  does not influence the existence of a limit.
2. *Find a value of*  $\delta > 0$  *that places the open interval*  $(c - \delta, c + \delta)$  *centered at*  $c$  *inside the interval*  $(a, b)$ . The inequality  $|f(x) - L| < \varepsilon$  *will hold for all*  $x \neq c$  *in this*  $\delta$ *-interval.*



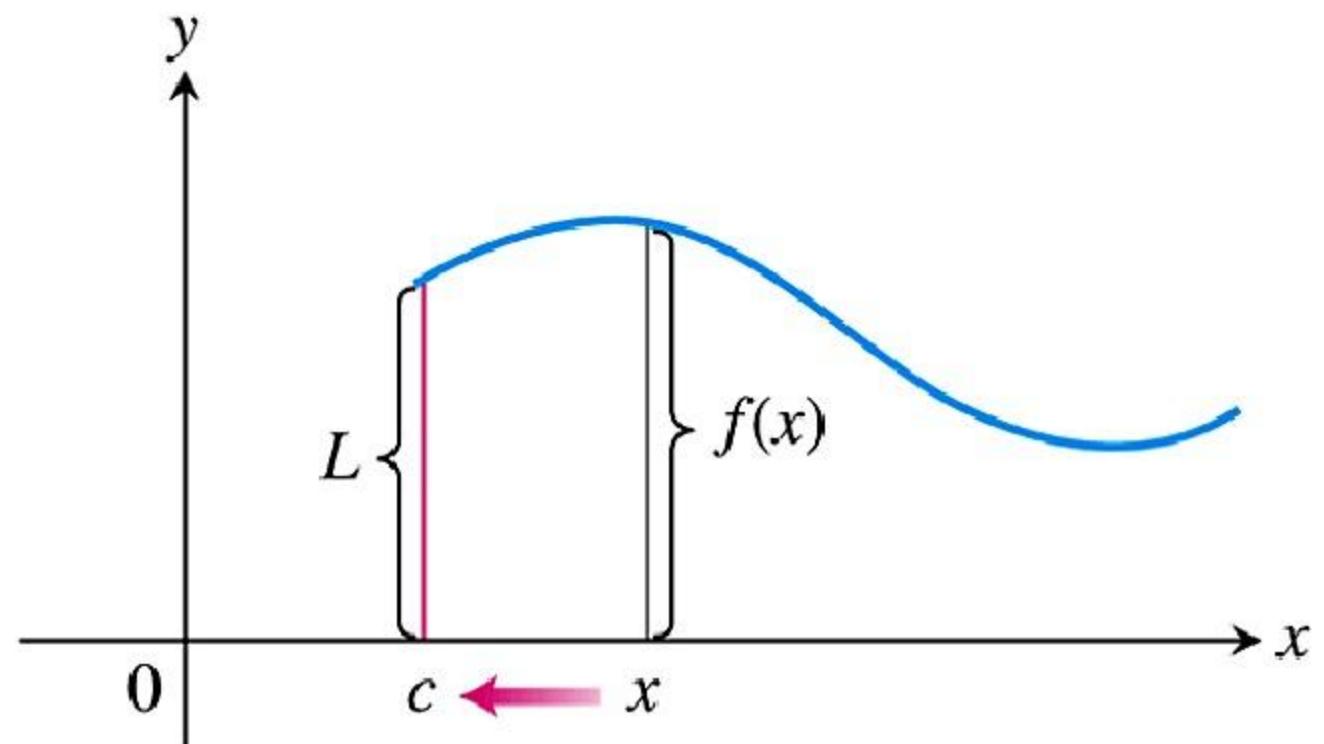
**FIGURE 2.23** An interval containing  $x = 2$  so that the function in Example 5 satisfies  $|f(x) - 4| < \varepsilon$ .

# Section 2.4

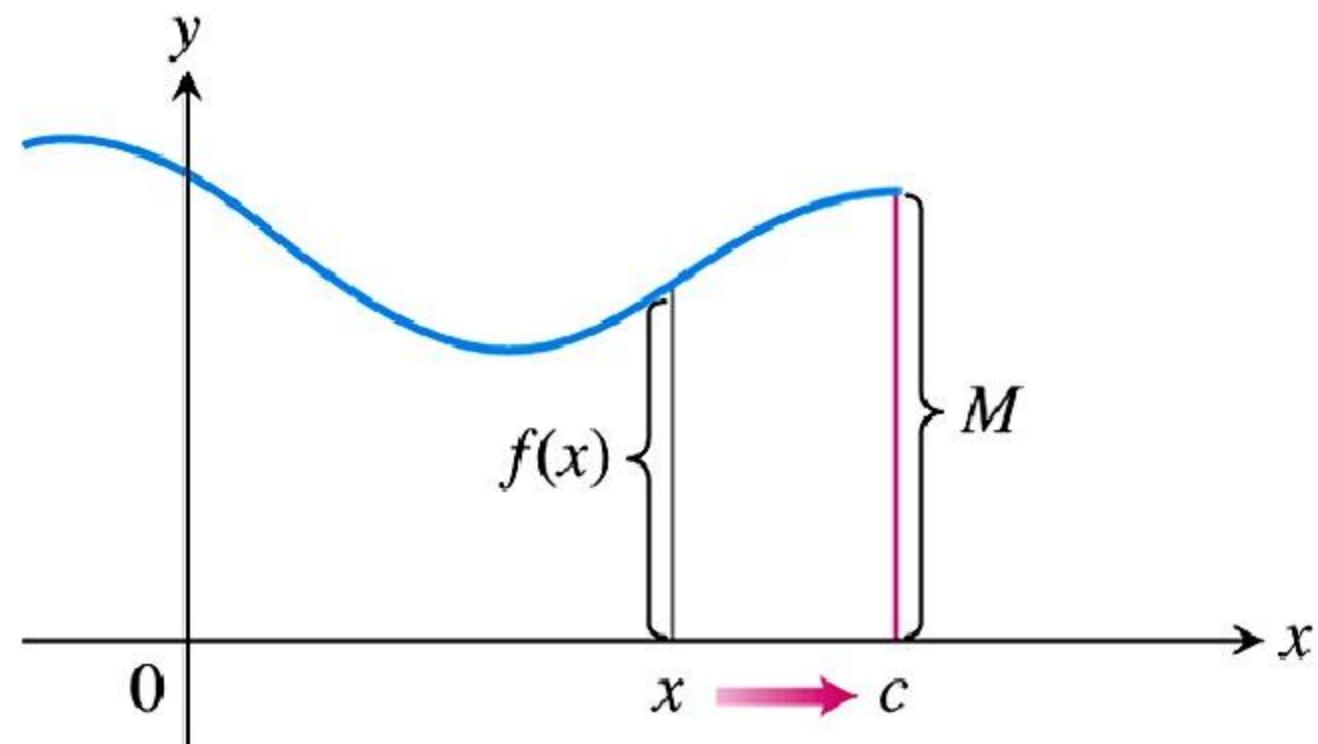
## One-Sided Limits



**FIGURE 2.24** Different right-hand and left-hand limits at the origin.

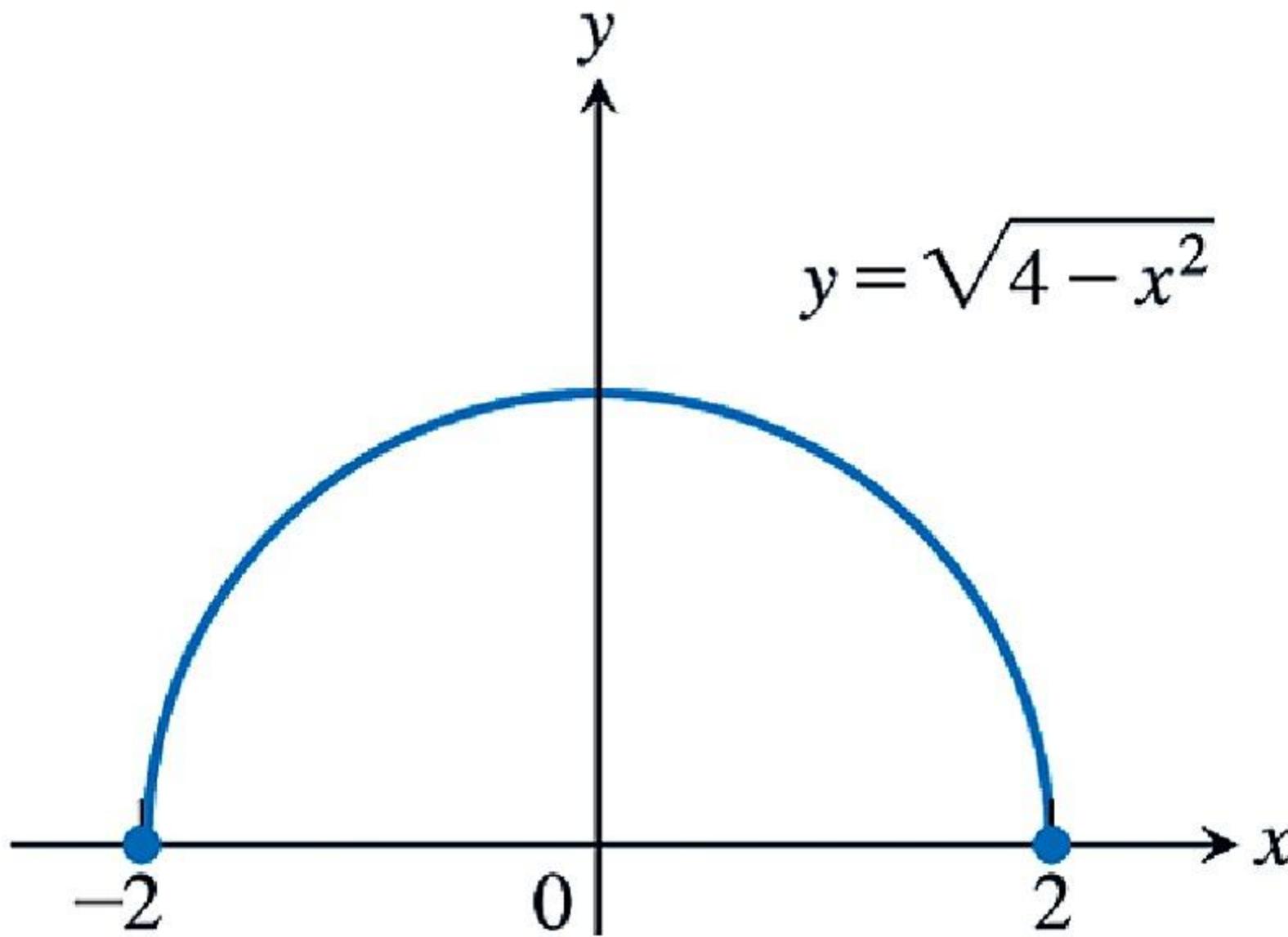


$$(a) \lim_{x \rightarrow c^+} f(x) = L$$



$$(b) \lim_{x \rightarrow c^-} f(x) = M$$

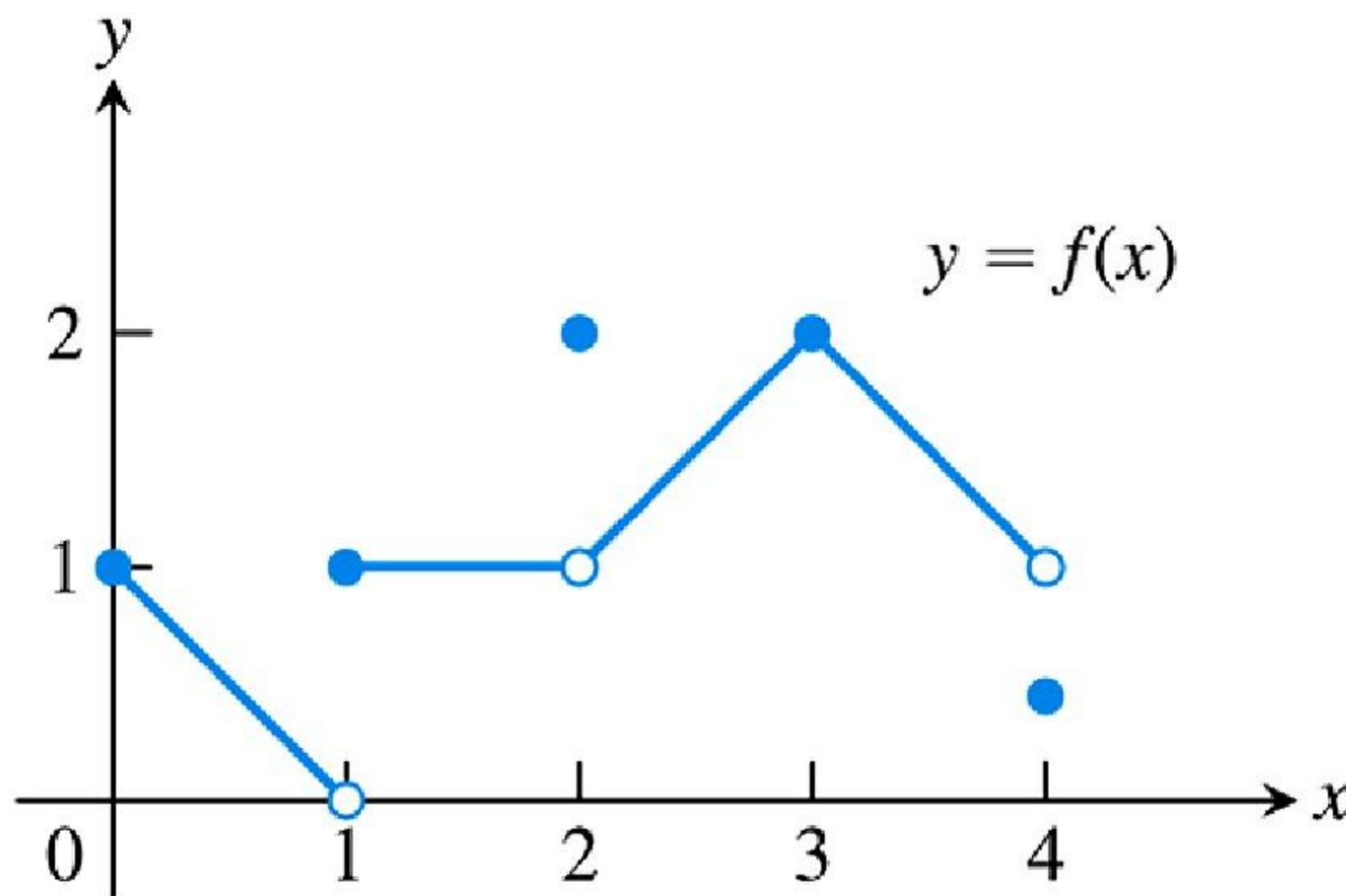
**FIGURE 2.25** (a) Right-hand limit as  $x$  approaches  $c$ . (b) Left-hand limit as  $x$  approaches  $c$ .



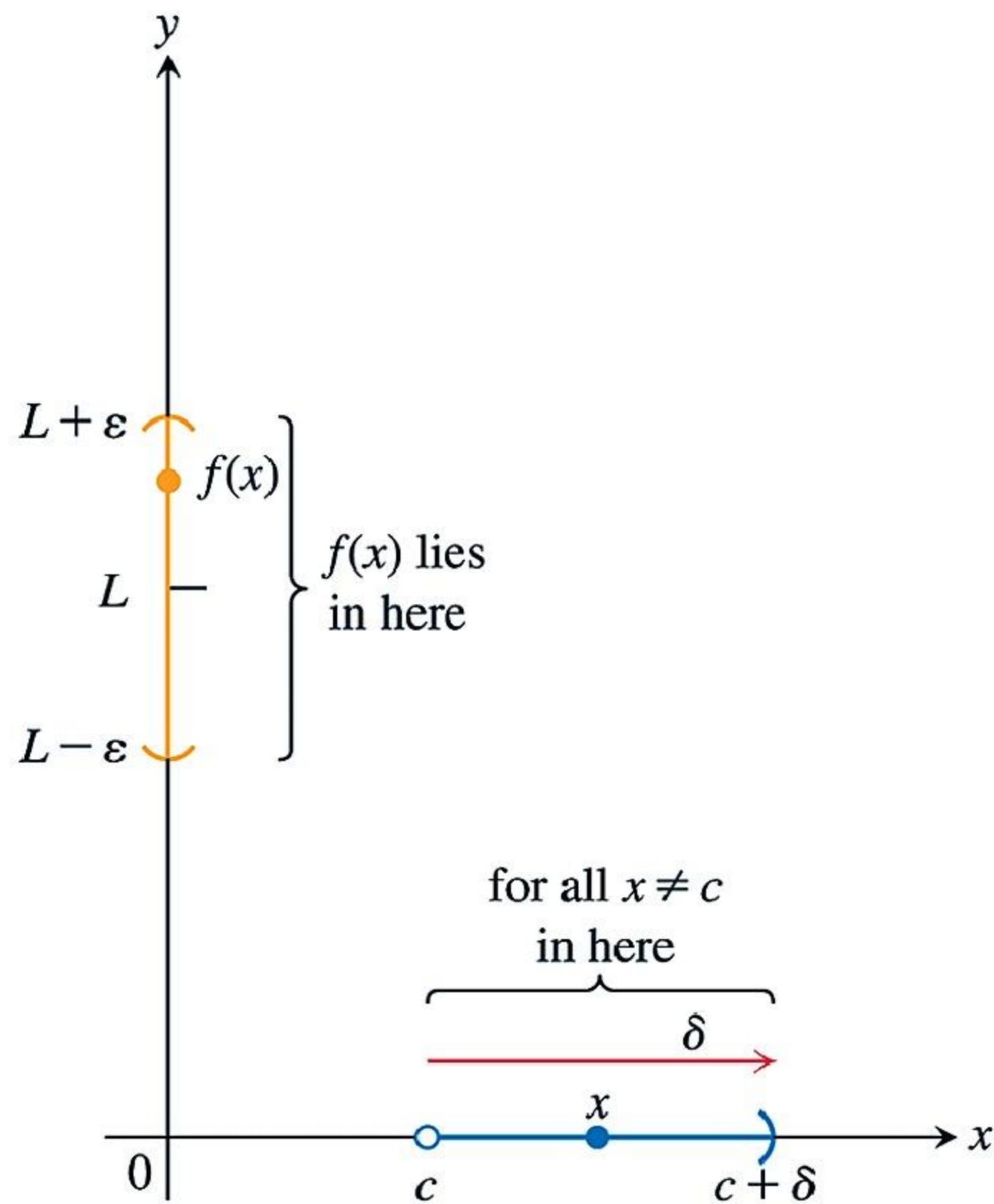
**FIGURE 2.26** The function  $f(x) = \sqrt{4 - x^2}$  has a right-hand limit 0 at  $x = -2$  and a left-hand limit 0 at  $x = 2$  (Example 1).

**THEOREM 6** Suppose that a function  $f$  is defined on an open interval containing  $c$ , except perhaps at  $c$  itself. Then  $f(x)$  has a limit as  $x$  approaches  $c$  if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x \rightarrow c} f(x) = L \iff \lim_{x \rightarrow c^-} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow c^+} f(x) = L.$$



**FIGURE 2.27** Graph of the function  
in Example 2.



**FIGURE 2.28** Intervals associated with the definition of right-hand limit.

**DEFINITIONS** (a) Assume the domain of  $f$  contains an interval  $(c, d)$  to the right of  $c$ . We say that  $f(x)$  has **right-hand limit  $L$  at  $c$** , and write

$$\lim_{x \rightarrow c^+} f(x) = L$$

if for every number  $\varepsilon > 0$  there exists a corresponding number  $\delta > 0$  such that

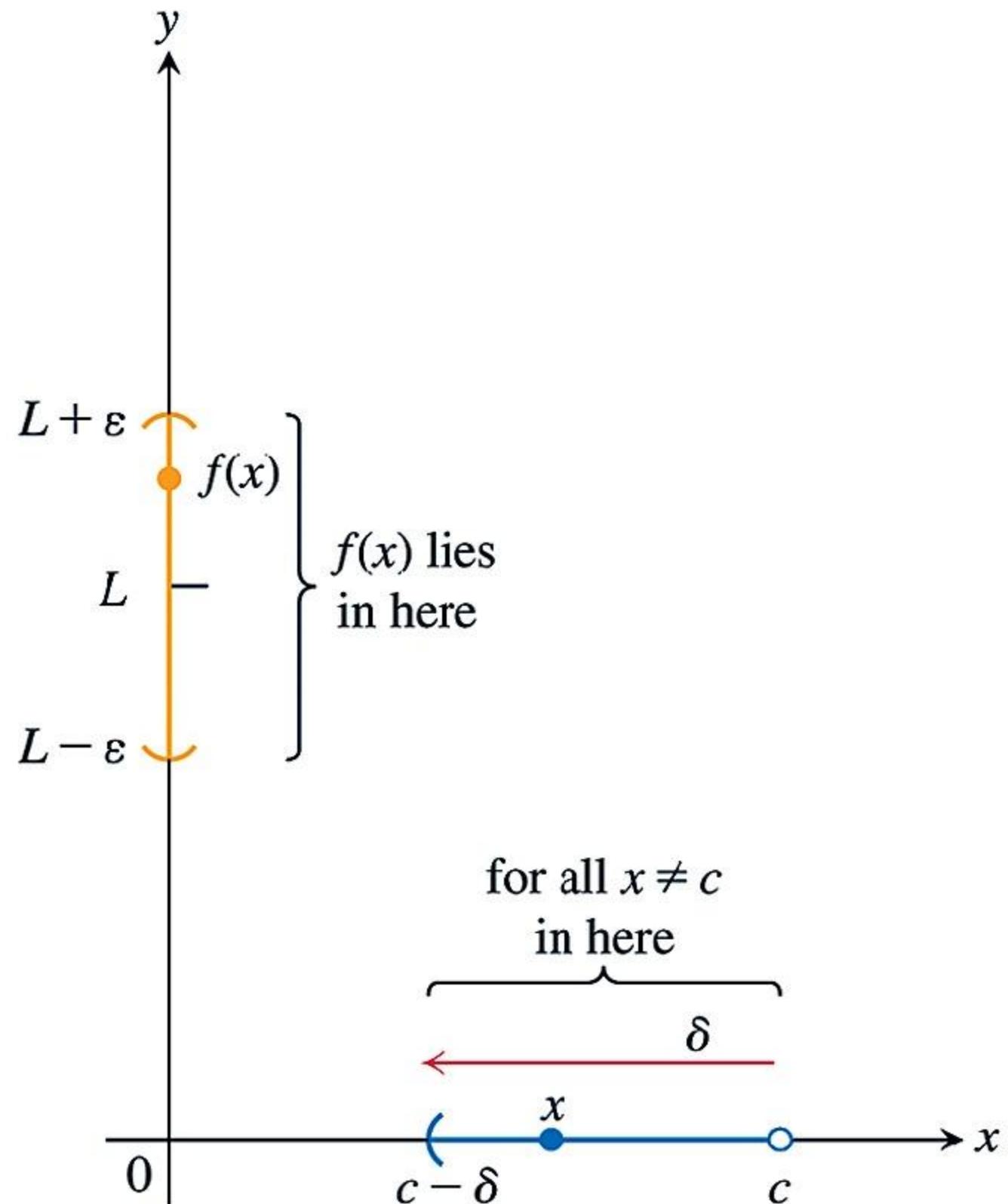
$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad c < x < c + \delta.$$

(b) Assume the domain of  $f$  contains an interval  $(b, c)$  to the left of  $c$ . We say that  $f$  has **left-hand limit  $L$  at  $c$** , and write

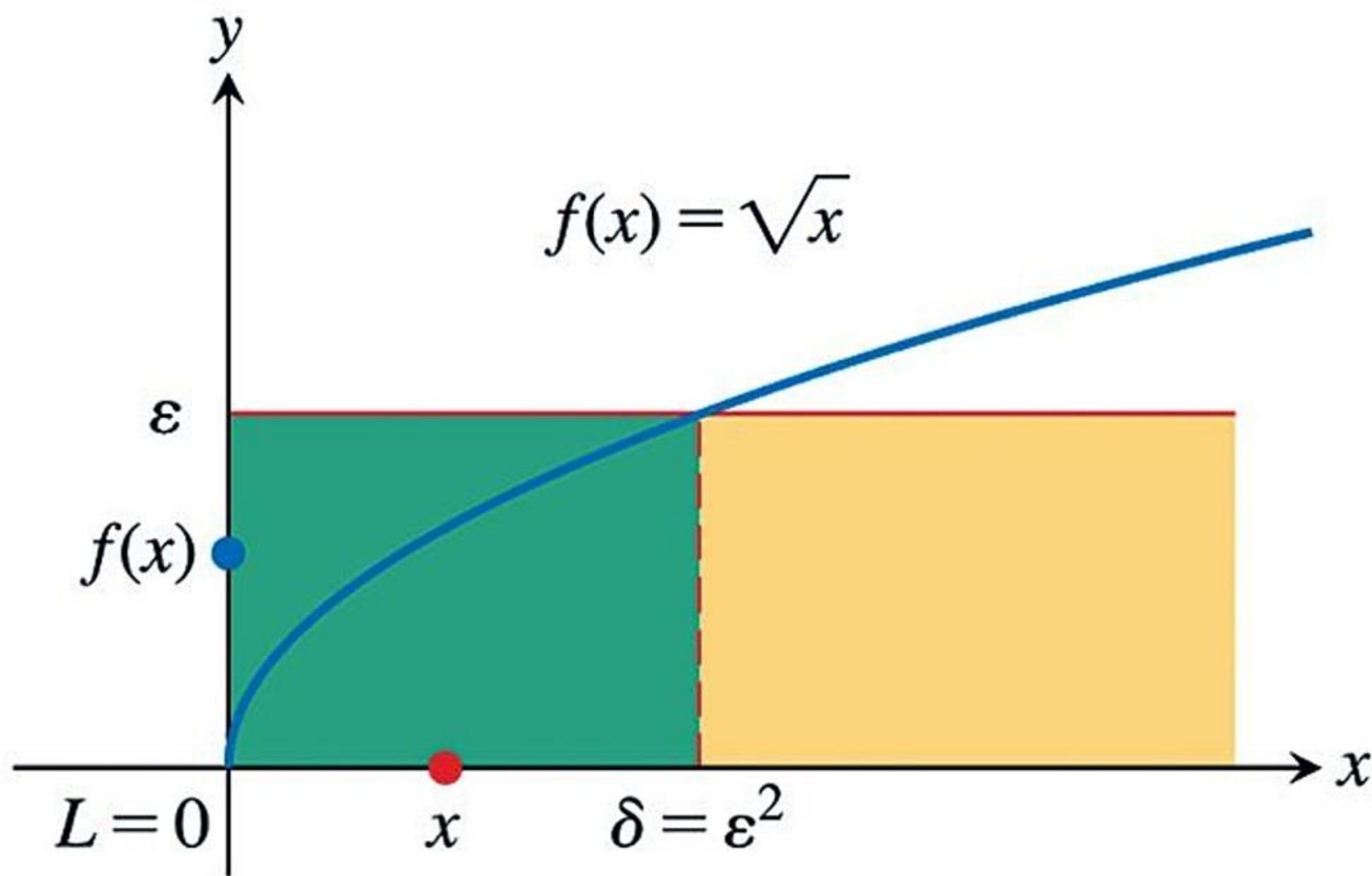
$$\lim_{x \rightarrow c^-} f(x) = L$$

if for every number  $\varepsilon > 0$  there exists a corresponding number  $\delta > 0$  such that

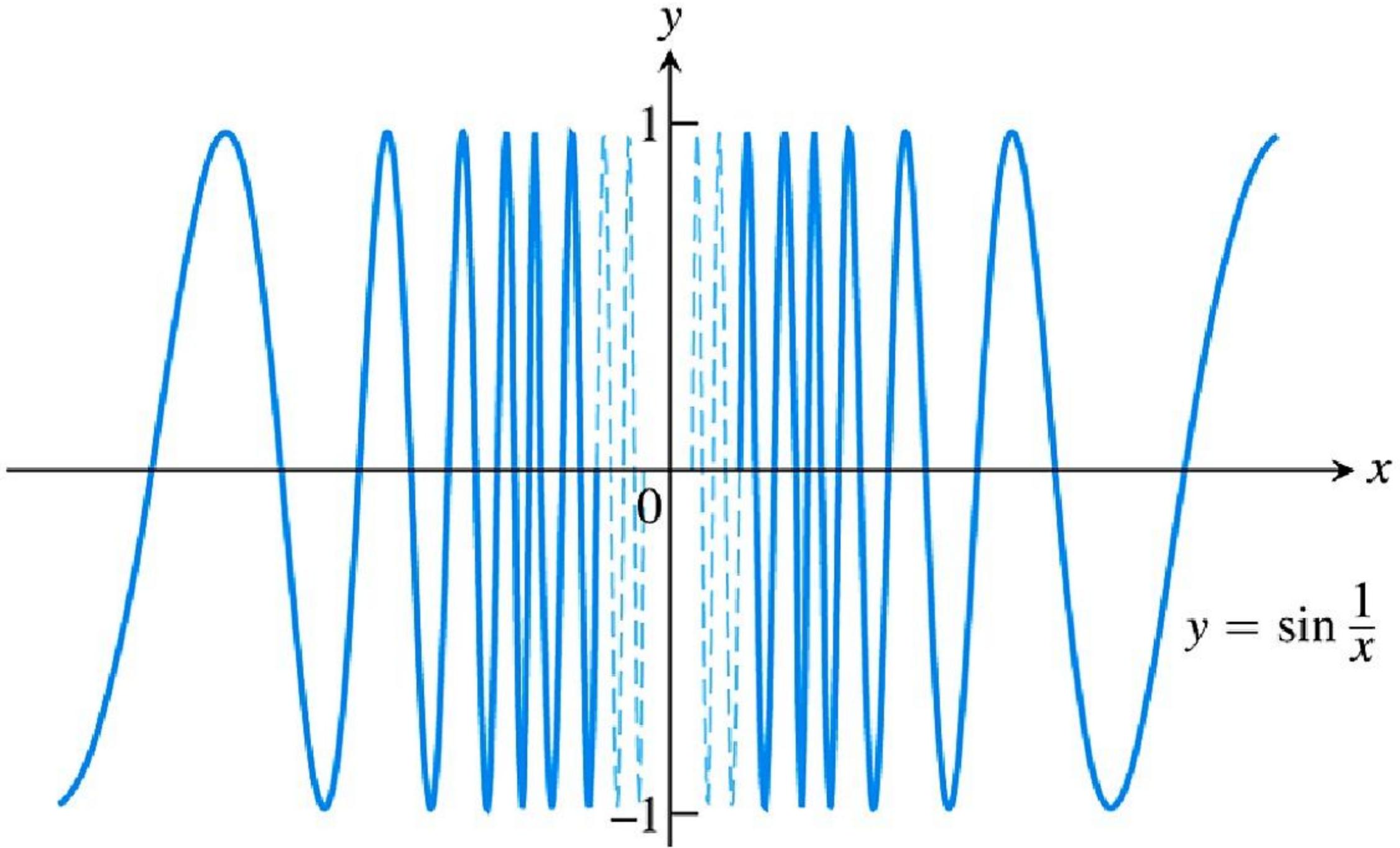
$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad c - \delta < x < c.$$



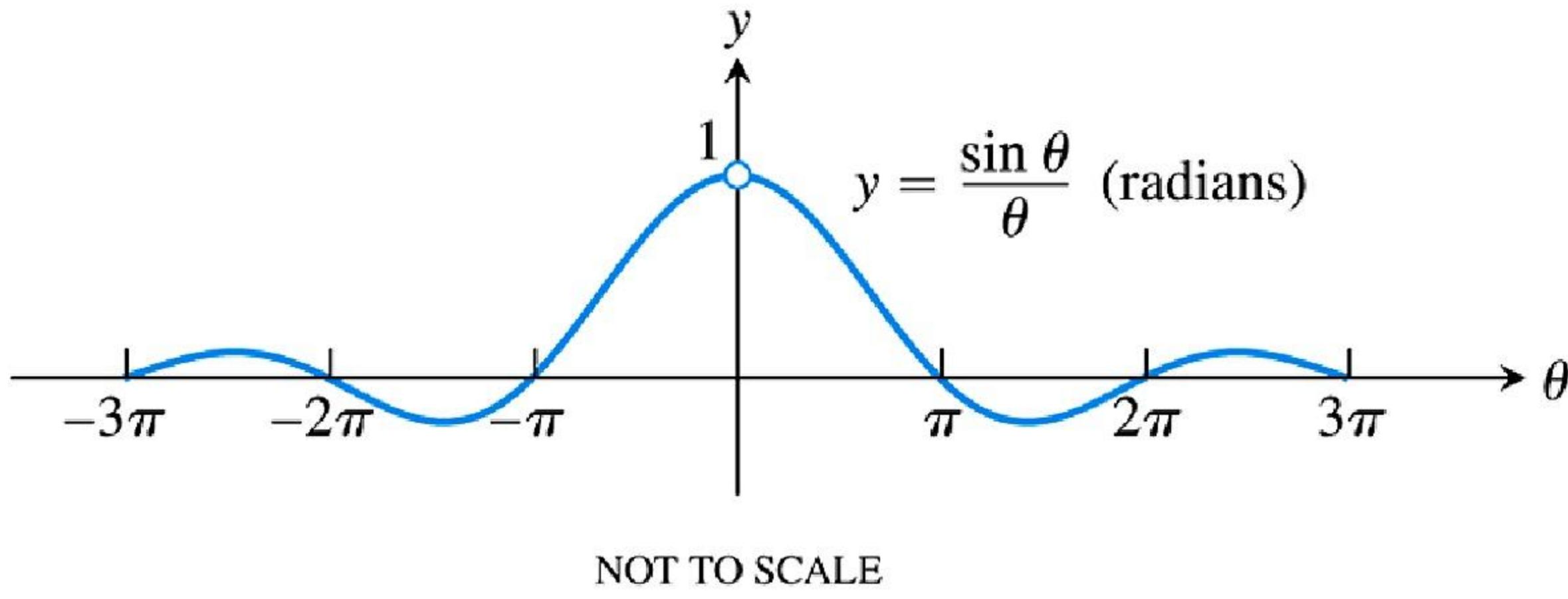
**FIGURE 2.29** Intervals associated with the definition of left-hand limit.



**FIGURE 2.30**  $\lim_{x \rightarrow 0^+} \sqrt{x} = 0$  in Example 3.



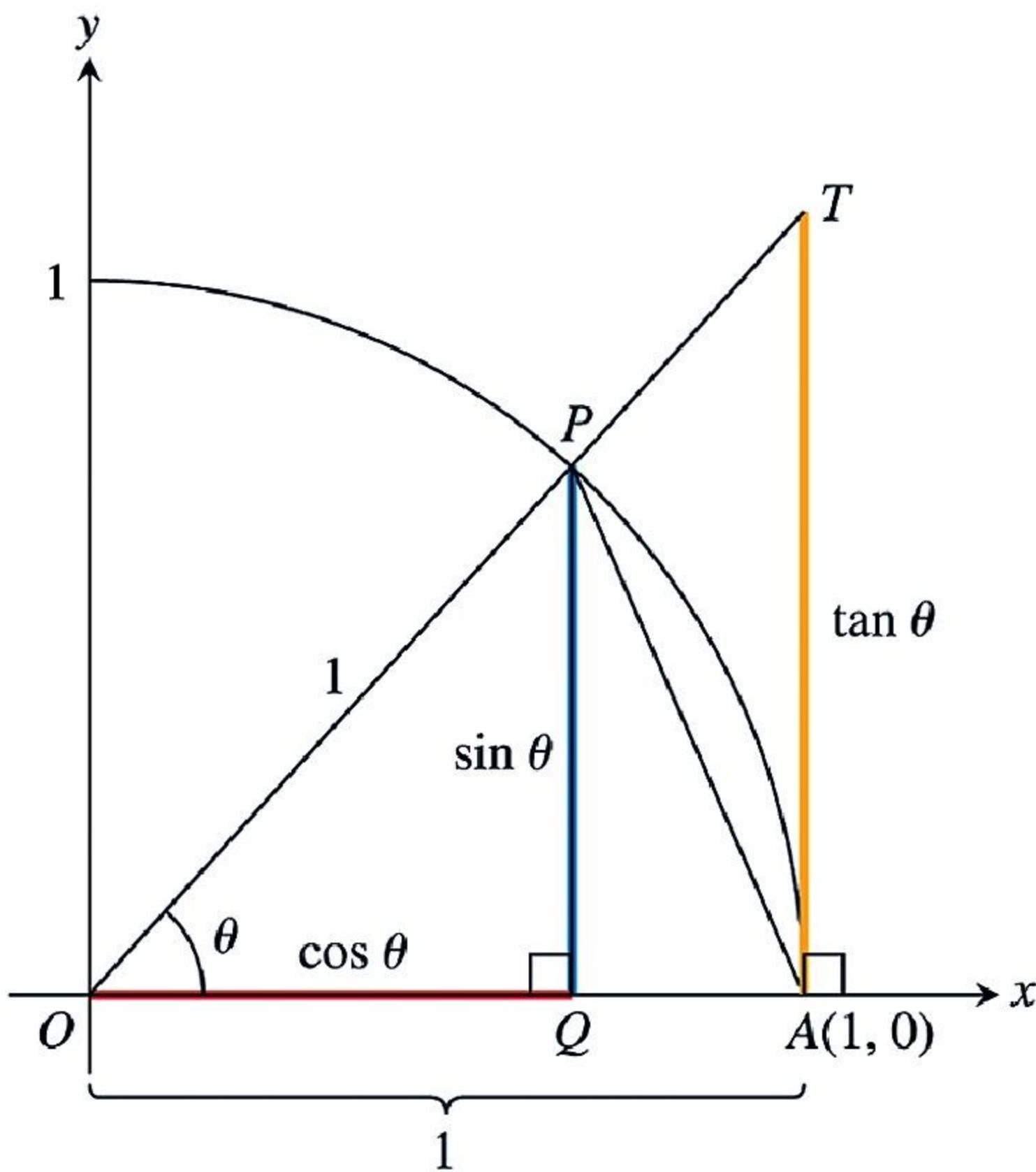
**FIGURE 2.31** The function  $y = \sin (1/x)$  has neither a right-hand nor a left-hand limit as  $x$  approaches zero (Example 4). The graph here omits values very near the  $y$ -axis.



**FIGURE 2.32** The graph of  $f(\theta) = (\sin \theta)/\theta$  suggests that the right- and left-hand limits as  $\theta$  approaches 0 are both 1.

## **THEOREM 7—Limit of the Ratio $\sin \theta/\theta$ as $\theta \rightarrow 0$**

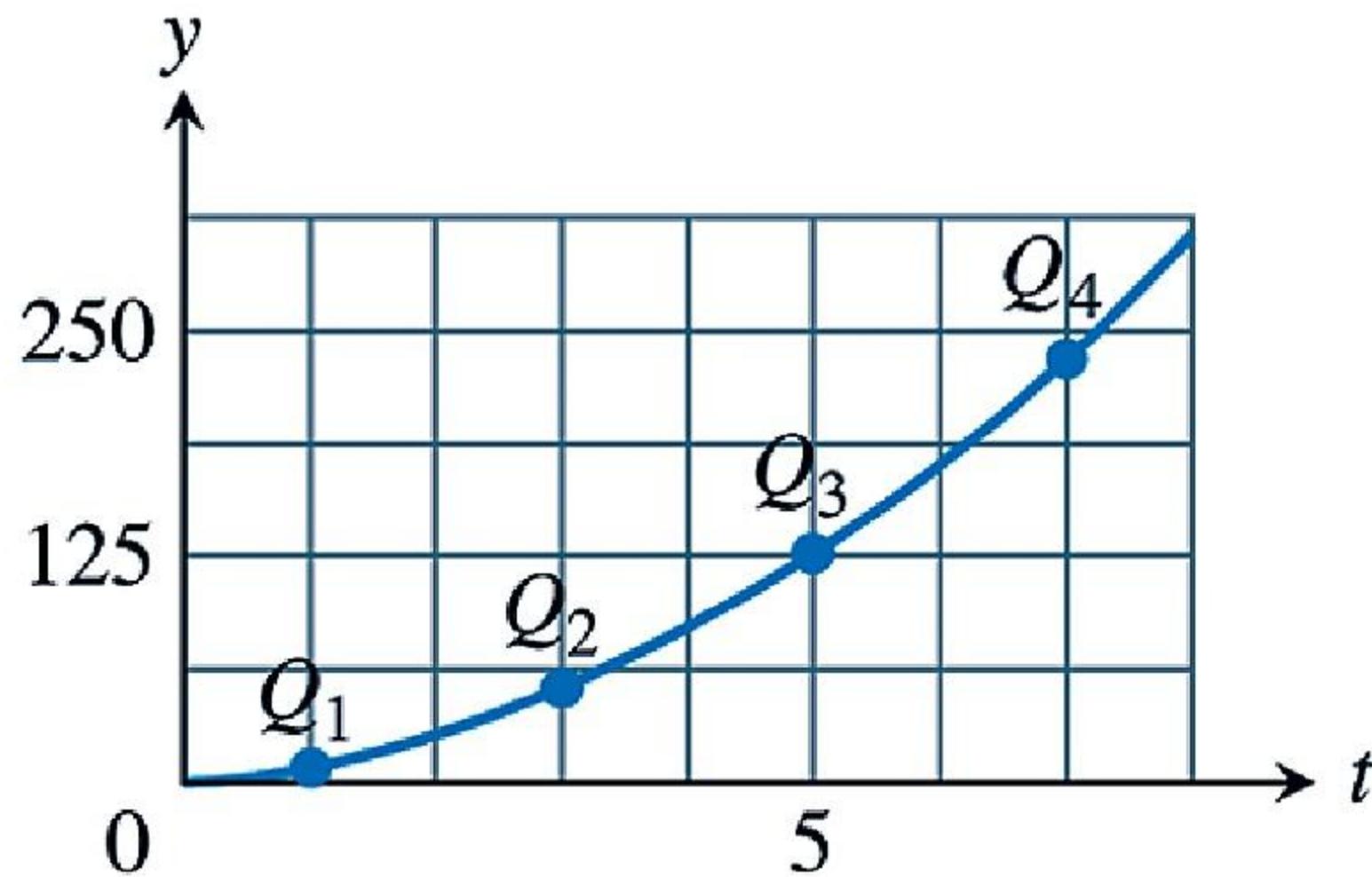
$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 \quad (\theta \text{ in radians}) \quad (1)$$



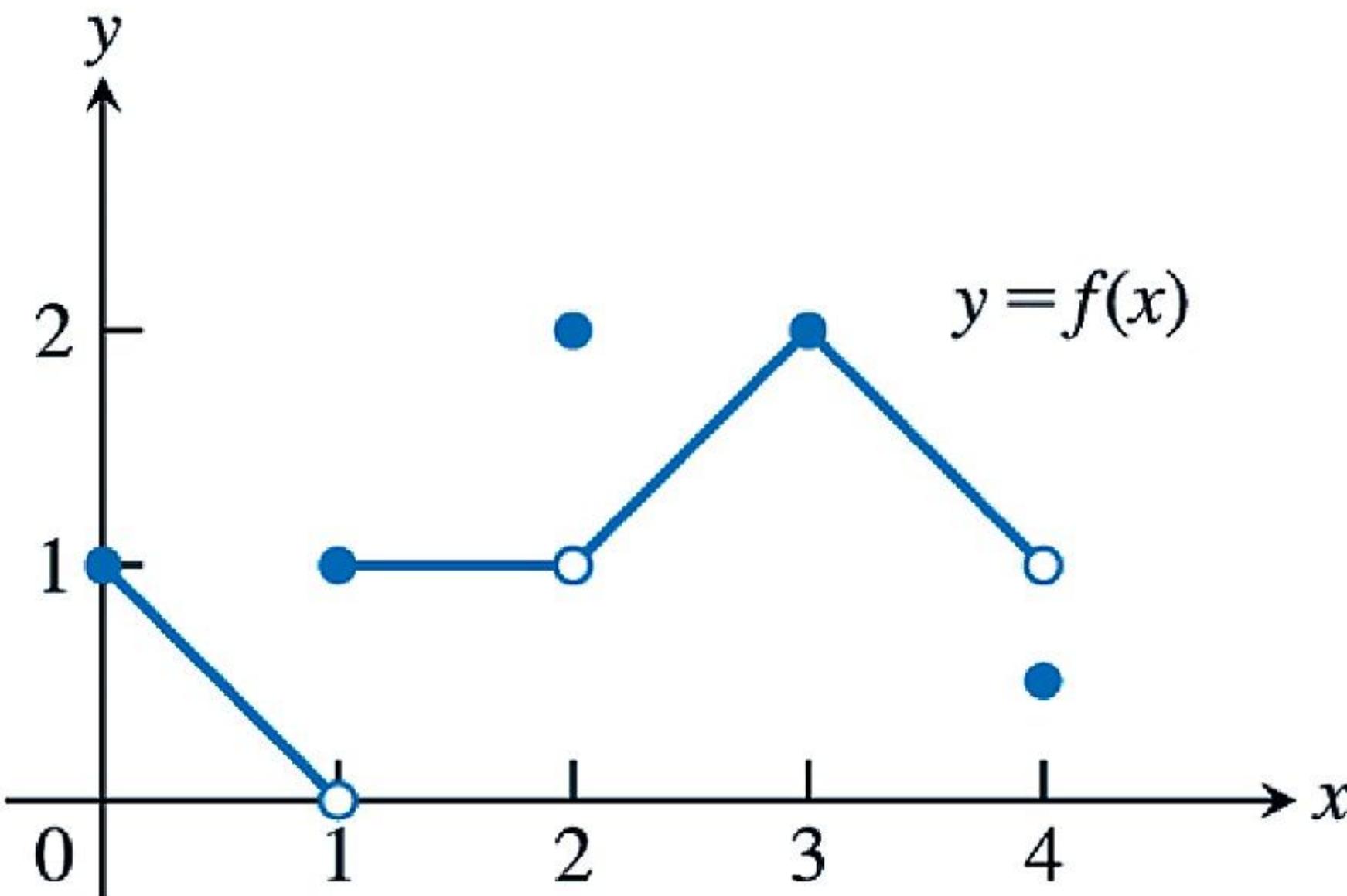
**FIGURE 2.33** The ratio  $TA/OA = \tan \theta$ , and  $OA = 1$ , so  $TA = \tan \theta$ .

# 2.5

## Continuity



**FIGURE 2.34** Connecting plotted points.



**FIGURE 2.35** The function is not continuous at  $x = 1$ ,  $x = 2$ , and  $x = 4$  (Example 1).

**DEFINITIONS** Let  $c$  be a real number that is either an interior point or an endpoint of an interval in the domain of  $f$ .

The function  $f$  is **continuous at  $c$**  if

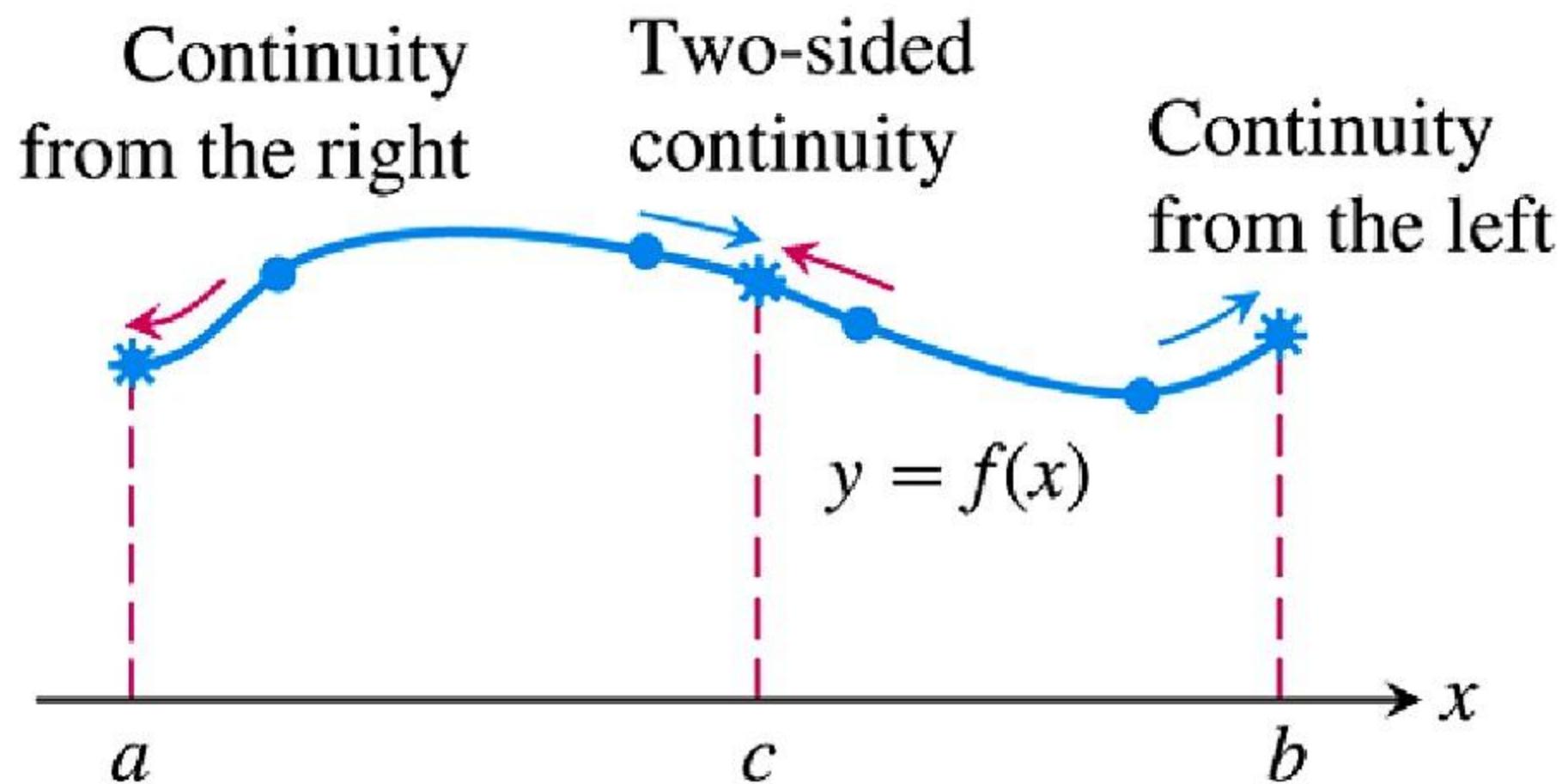
$$\lim_{x \rightarrow c} f(x) = f(c).$$

The function  $f$  is **right-continuous at  $c$  (or continuous from the right)** if

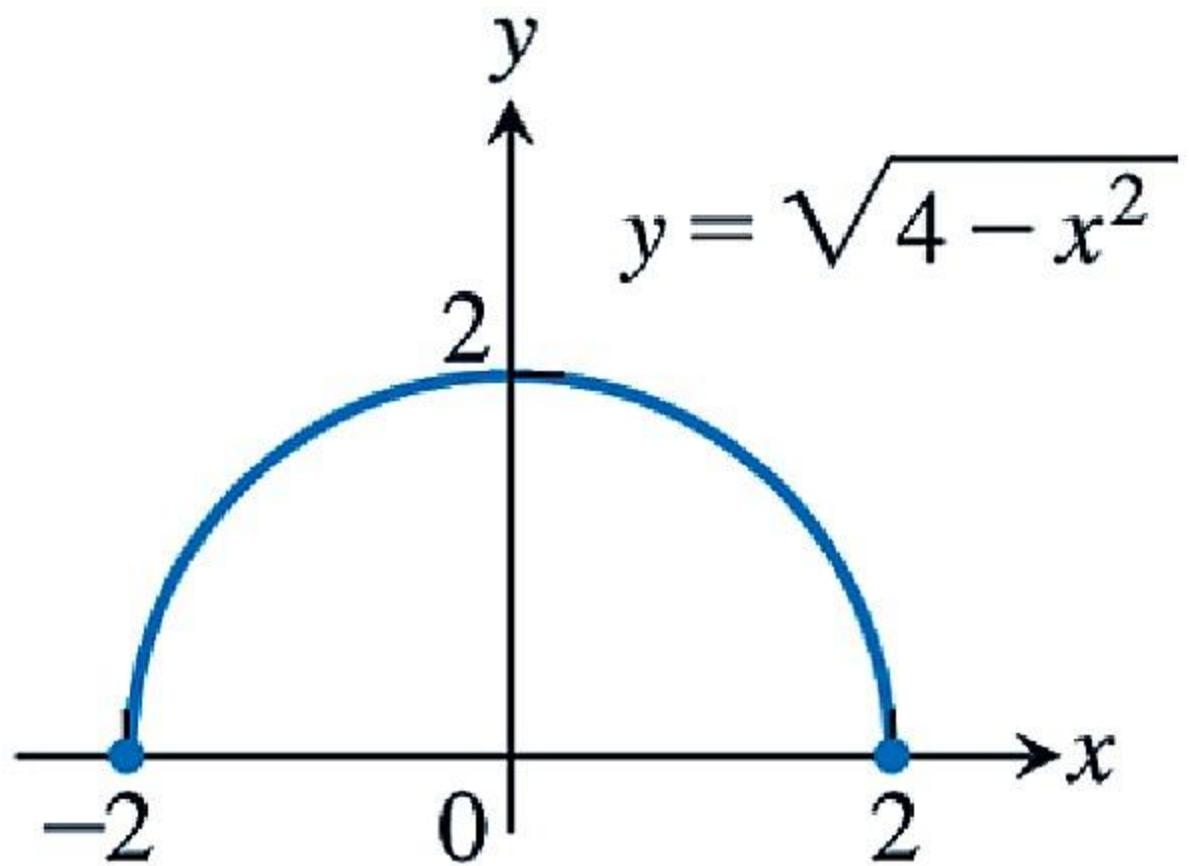
$$\lim_{x \rightarrow c^+} f(x) = f(c).$$

The function  $f$  is **left-continuous at  $c$  (or continuous from the left)** if

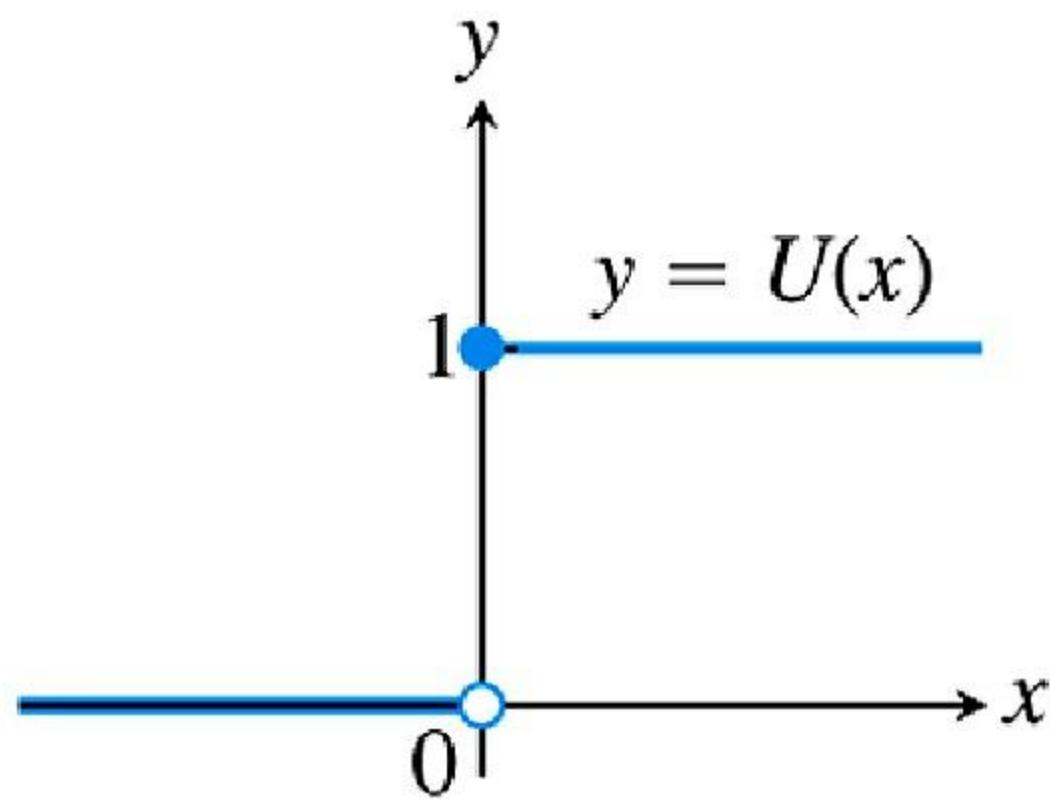
$$\lim_{x \rightarrow c^-} f(x) = f(c).$$



**FIGURE 2.36** Continuity at points  $a$ ,  $b$ , and  $c$ .



**FIGURE 2.37** A function that is continuous over its domain (Example 2).

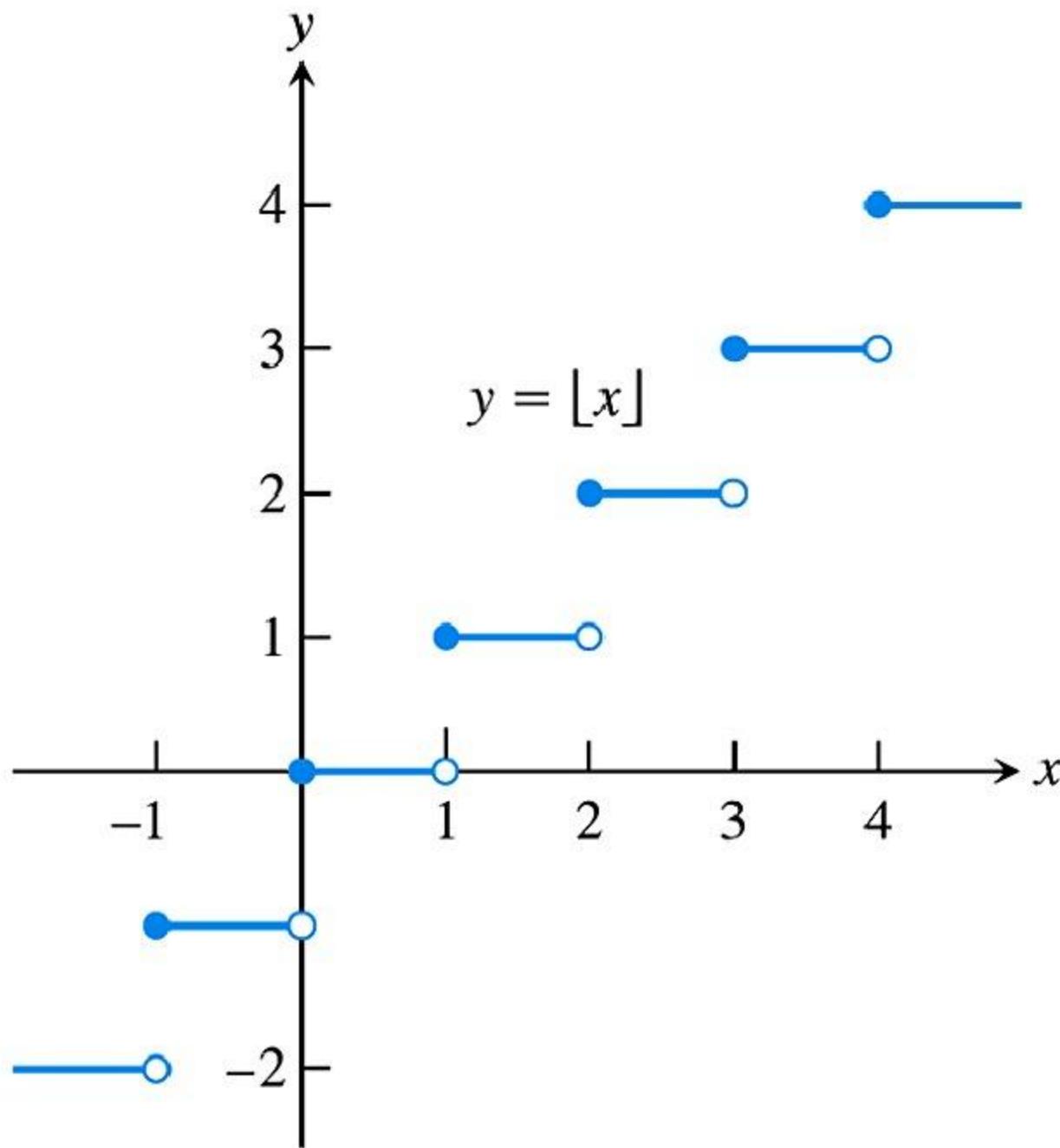


**FIGURE 2.38** A function  
that has a jump discontinuity  
at the origin (Example 3).

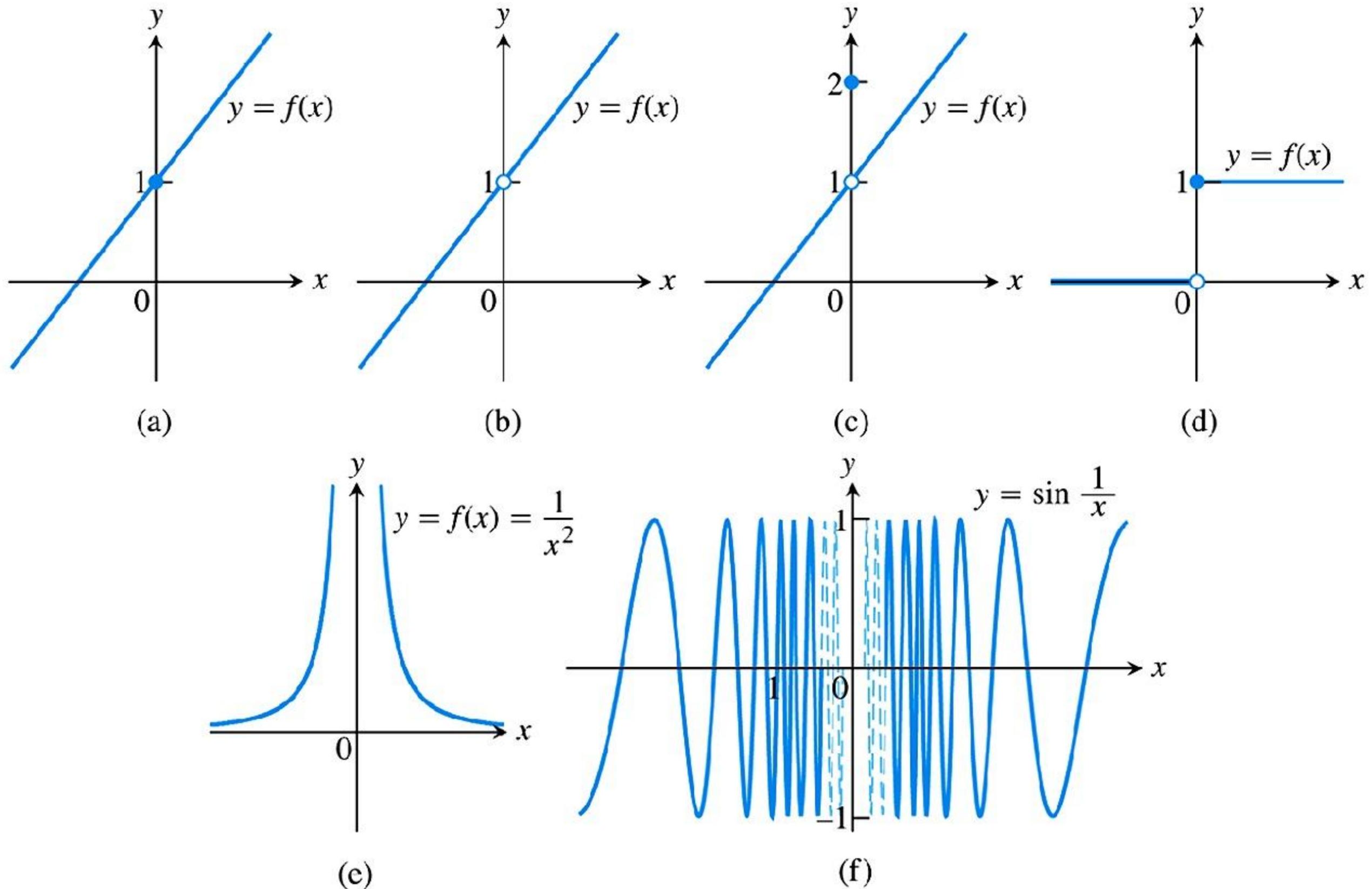
## Continuity Test

A function  $f(x)$  is continuous at a point  $x = c$  if and only if it meets the following three conditions.

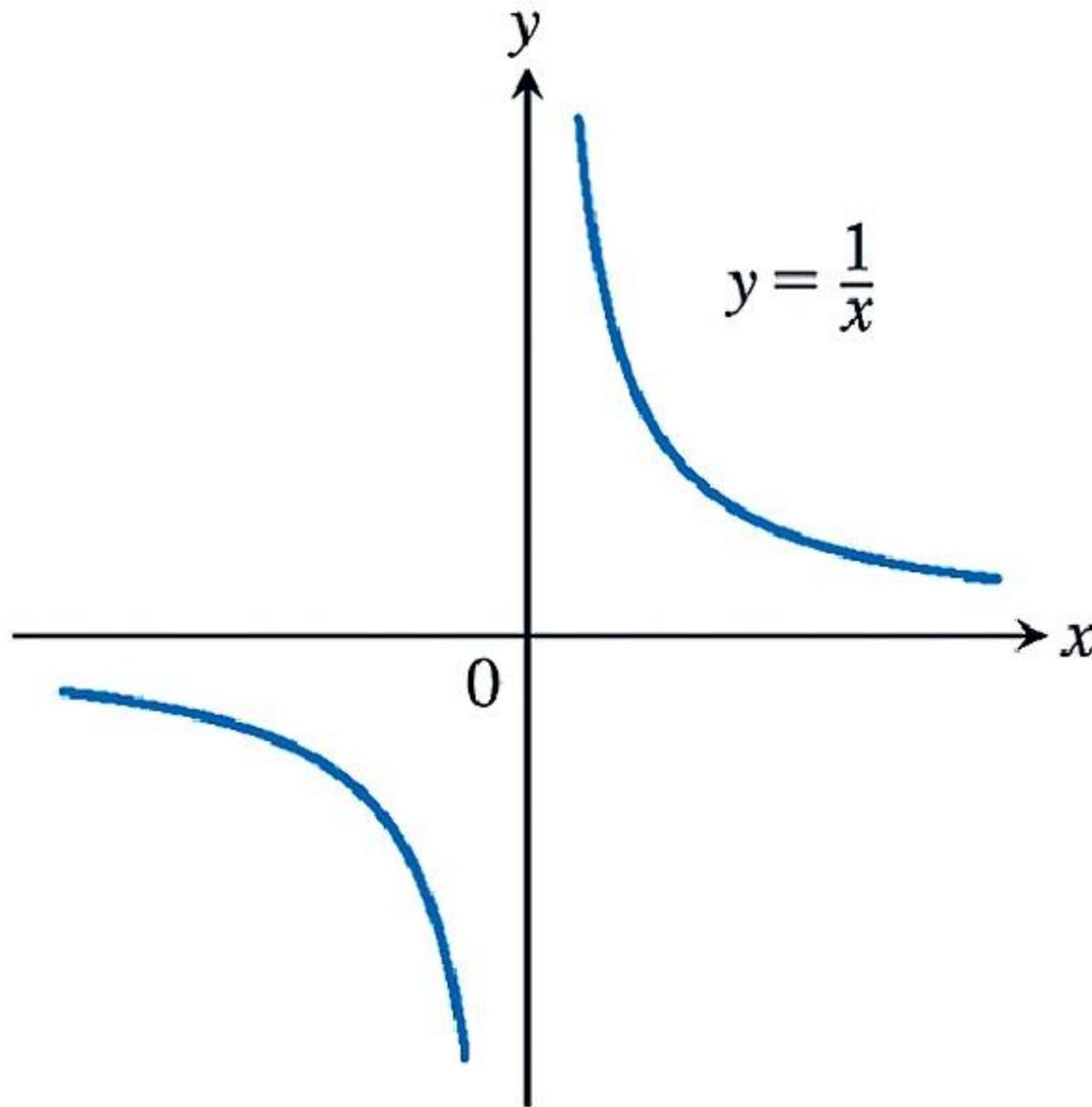
1.  $f(c)$  exists ( $c$  lies in the domain of  $f$ ).
2.  $\lim_{x \rightarrow c} f(x)$  exists ( $f$  has a limit as  $x \rightarrow c$ ).
3.  $\lim_{x \rightarrow c} f(x) = f(c)$  (the limit equals the function value).



**FIGURE 2.39** The greatest integer function is continuous at every noninteger point. It is right-continuous, but not left-continuous, at every integer point (Example 4).



**FIGURE 2.40** The function in (a) is continuous at  $x = 0$ ; the functions in (b) through (f) are not.



**FIGURE 2.41** The function  $f(x) = 1/x$  is continuous over its natural domain. It is not defined at the origin, so it is not continuous on any interval containing  $x = 0$  (Example 5).

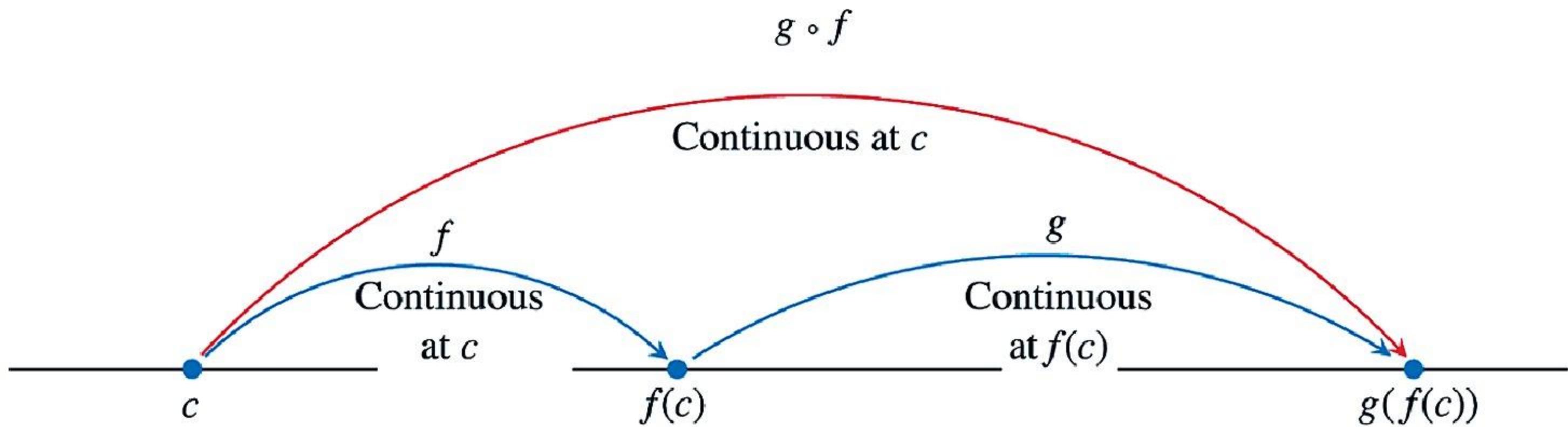
## THEOREM 8—Properties of Continuous Functions

If the functions  $f$  and  $g$  are continuous at  $x = c$ , then the following algebraic combinations are continuous at  $x = c$ .

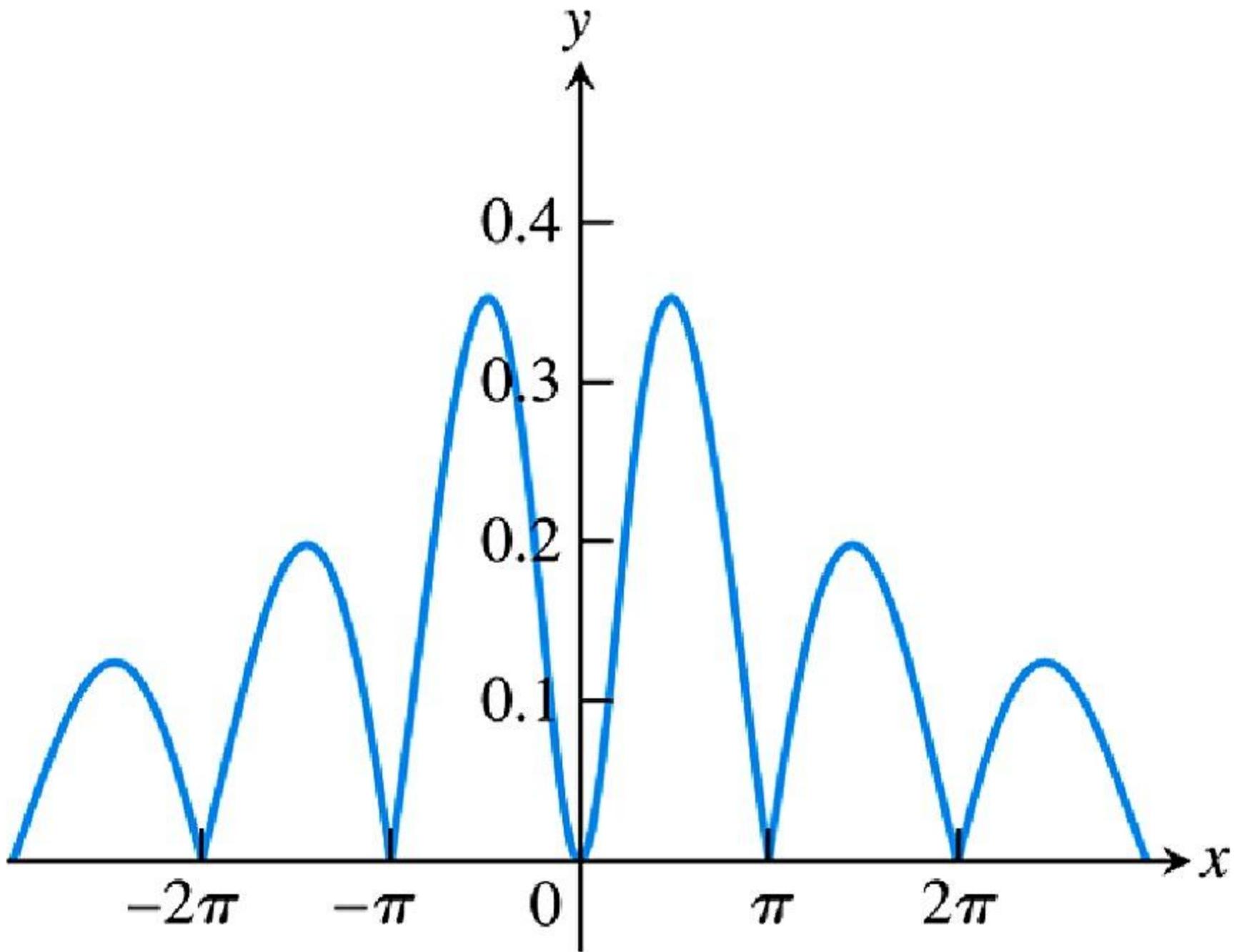
1. *Sums:*  $f + g$
2. *Differences:*  $f - g$
3. *Constant multiples:*  $k \cdot f$ , for any number  $k$
4. *Products:*  $f \cdot g$
5. *Quotients:*  $f/g$ , provided  $g(c) \neq 0$
6. *Powers:*  $f^n$ ,  $n$  a positive integer
7. *Roots:*  $\sqrt[n]{f}$ , provided it is defined on an interval containing  $c$ , where  $n$  is a positive integer

## **THEOREM 9—Compositions of Continuous Functions**

If  $f$  is continuous at  $c$  and  $g$  is continuous at  $f(c)$ , then the composition  $g \circ f$  is continuous at  $c$ .



**FIGURE 2.42** Compositions of continuous functions are continuous.



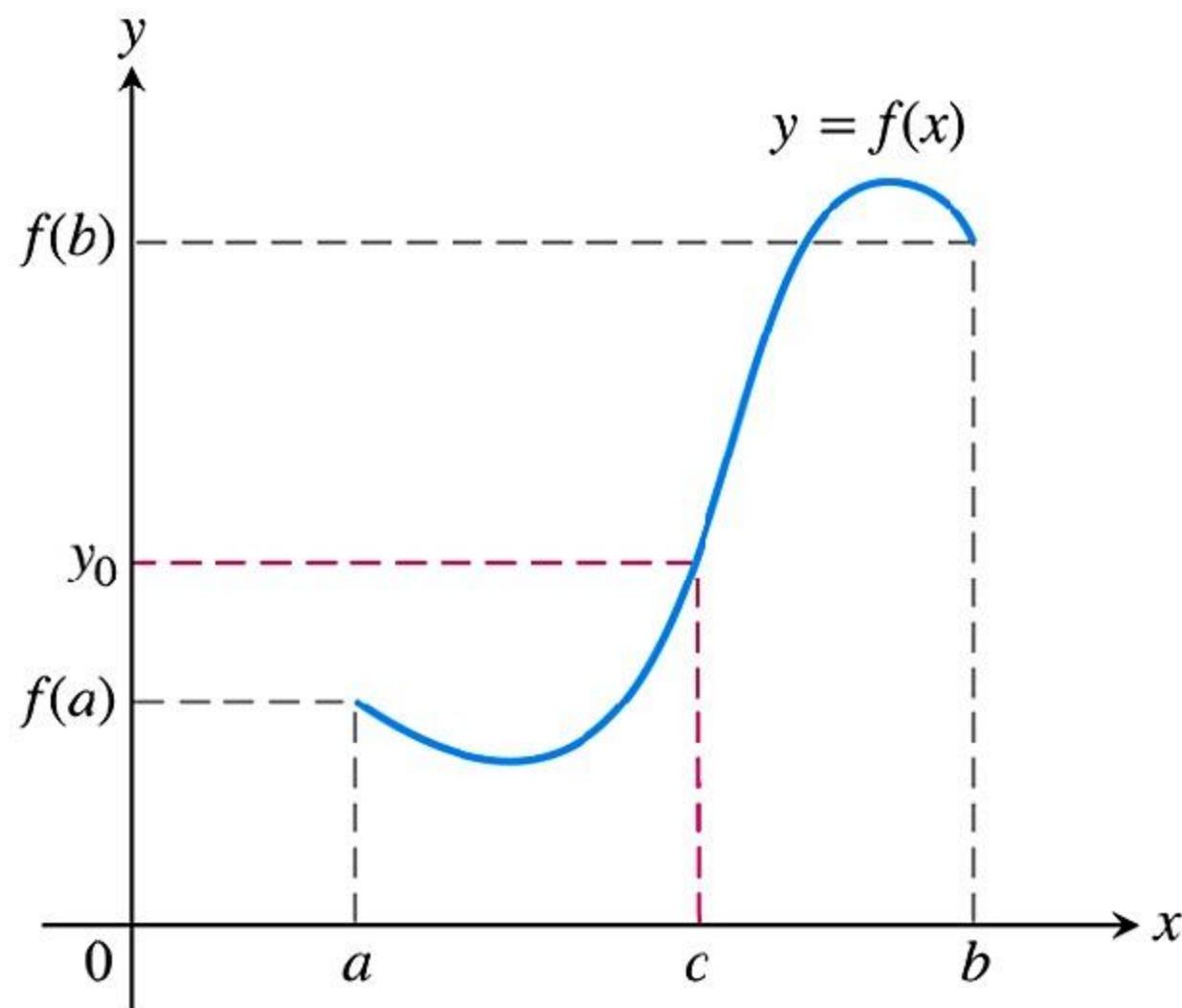
**FIGURE 2.43** The graph suggests that  $y = |(x \sin x)/(x^2 + 2)|$  is continuous (Example 8d).

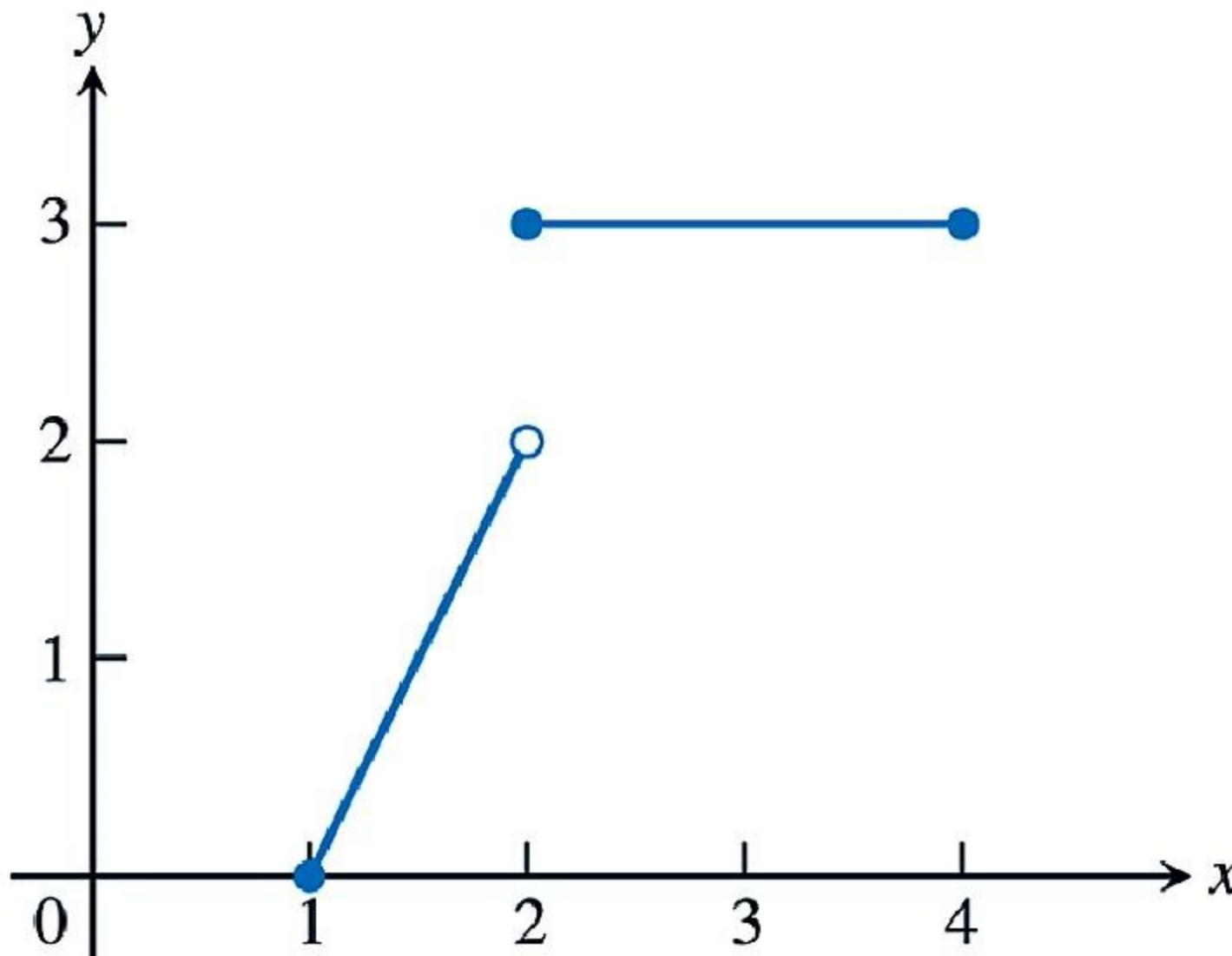
## **THEOREM 10—Limits of Continuous Functions**

If  $\lim_{x \rightarrow c} f(x) = b$  and  $g$  is continuous at the point  $b$ , then

$$\lim_{x \rightarrow c} g(f(x)) = g(b).$$

**THEOREM 11—The Intermediate Value Theorem for Continuous Functions** If  $f$  is a continuous function on a closed interval  $[a, b]$ , and if  $y_0$  is any value between  $f(a)$  and  $f(b)$ , then  $y_0 = f(c)$  for some  $c$  in  $[a, b]$ .



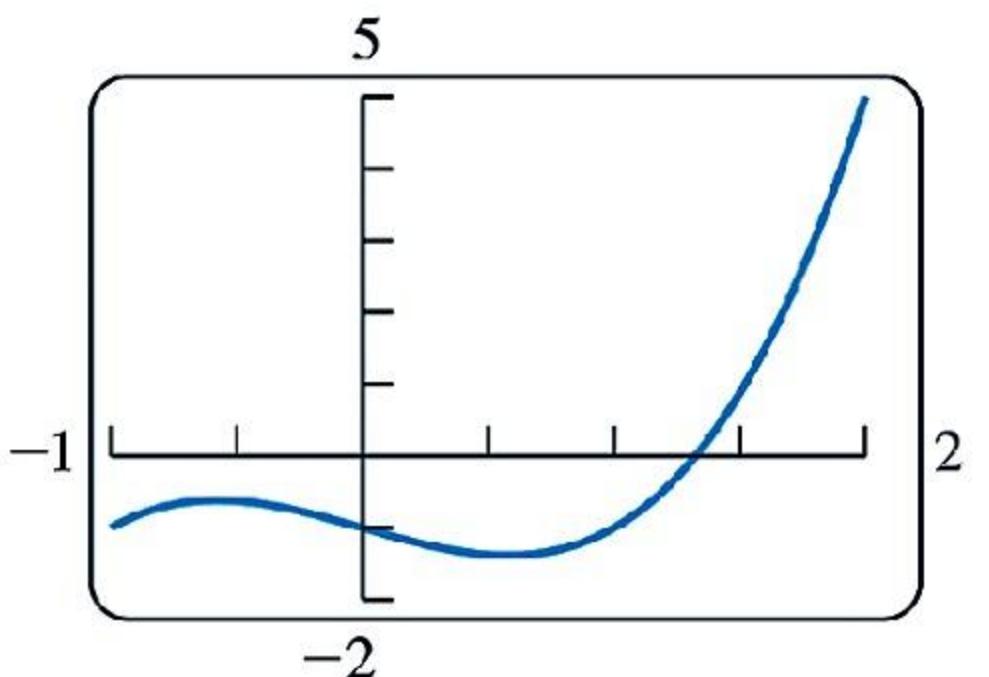


**FIGURE 2.44** The function

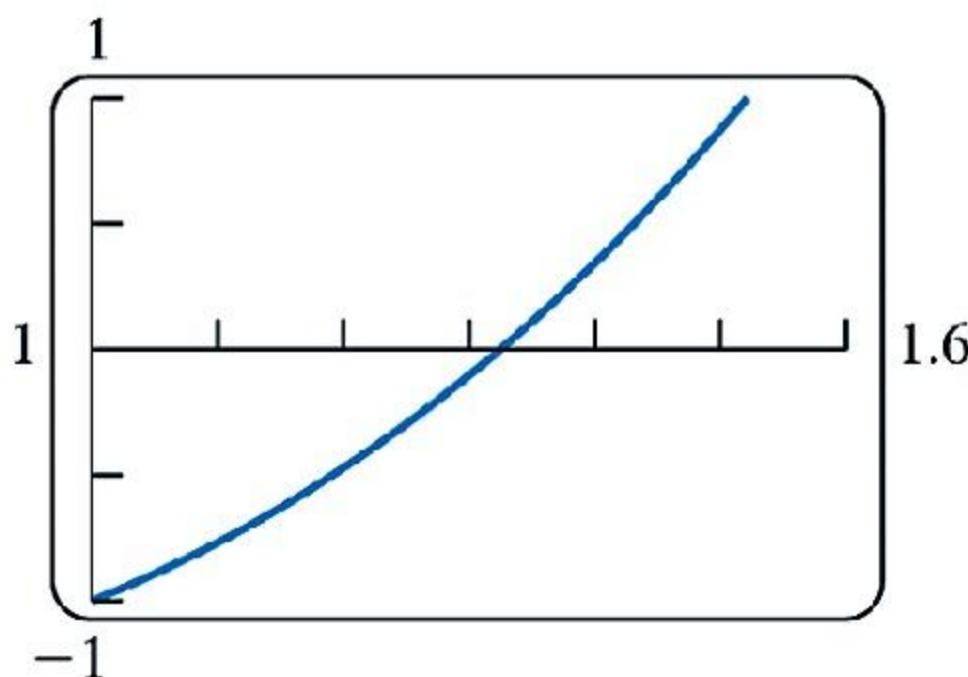
$$f(x) = \begin{cases} 2x - 2, & 1 \leq x < 2 \\ 3, & 2 \leq x \leq 4 \end{cases}$$

does not take on all values between

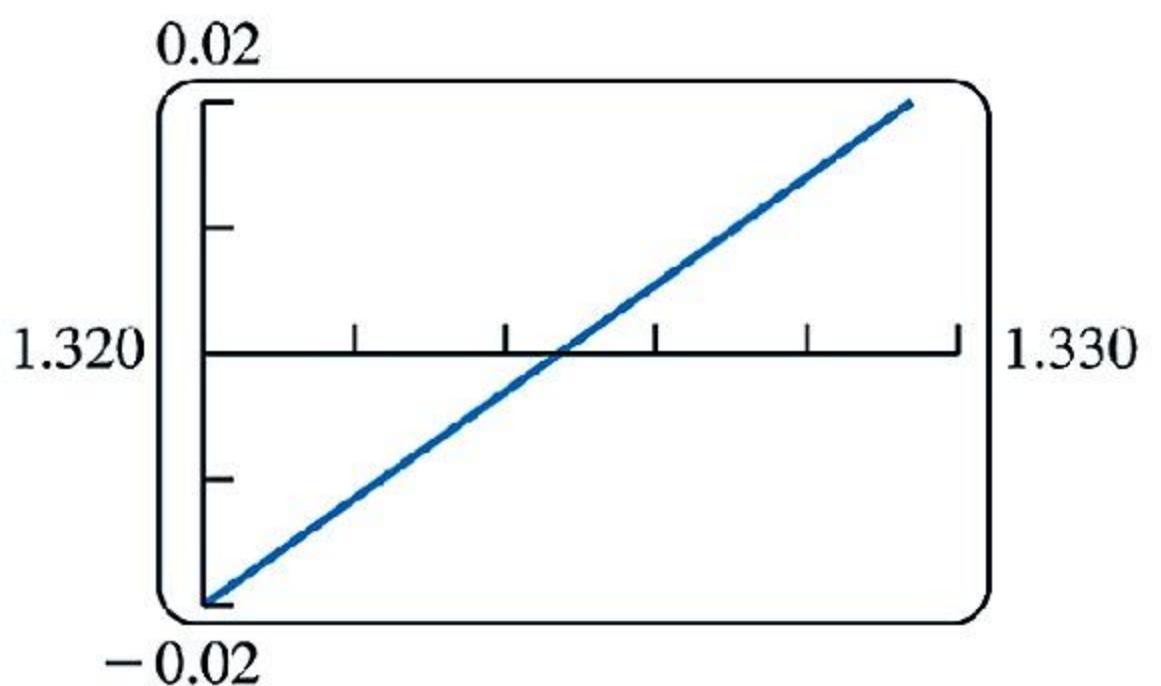
$f(1) = 0$  and  $f(4) = 3$ ; it misses all the values between 2 and 3.



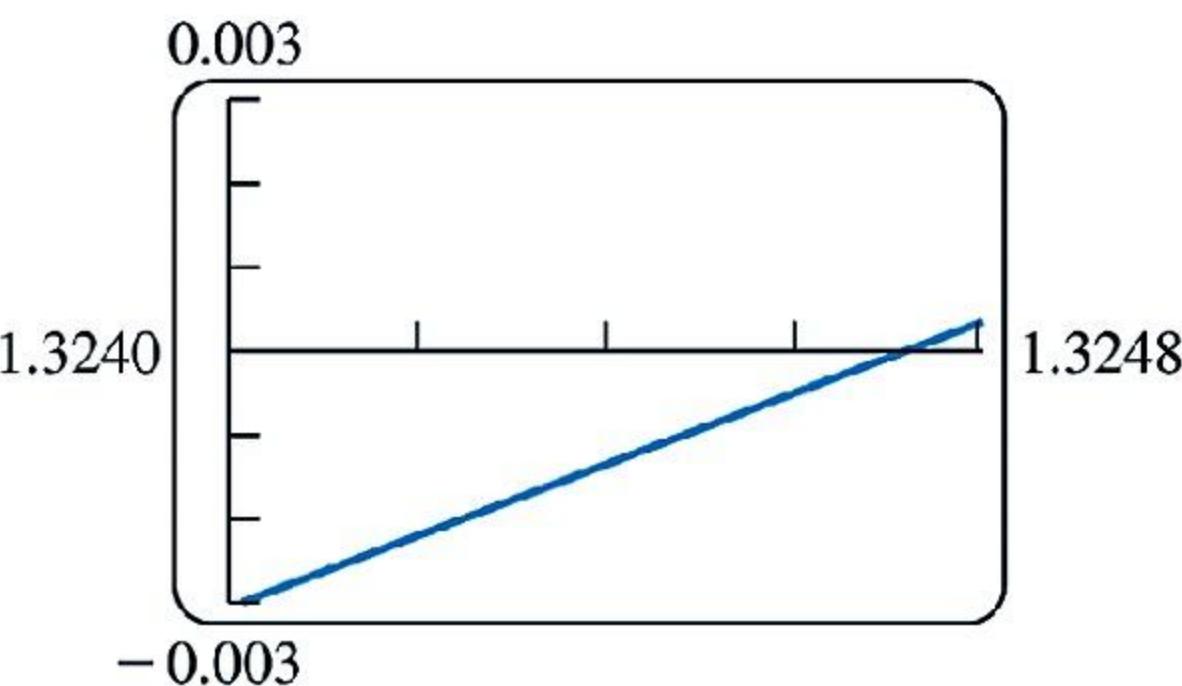
(a)



(b)

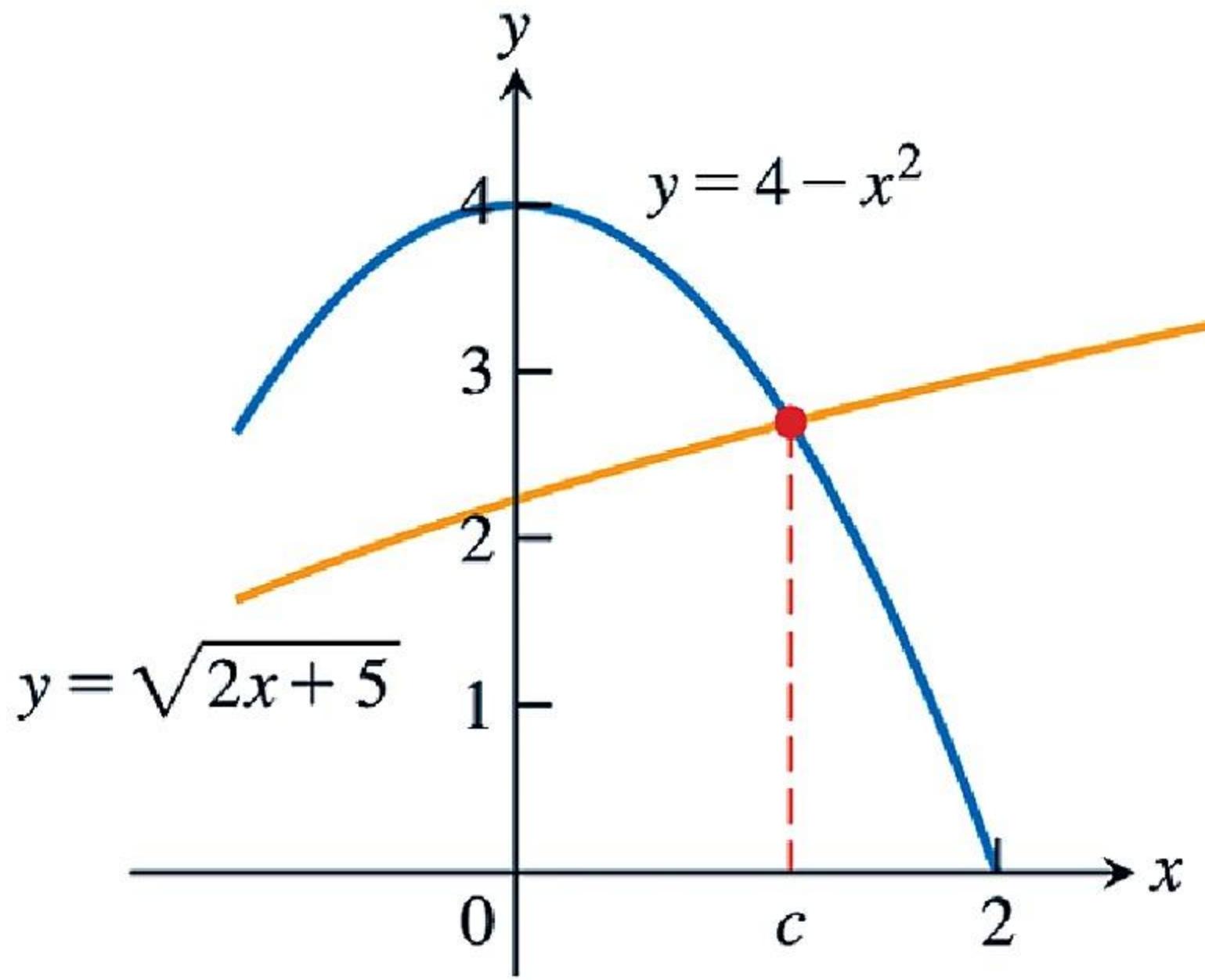


(c)

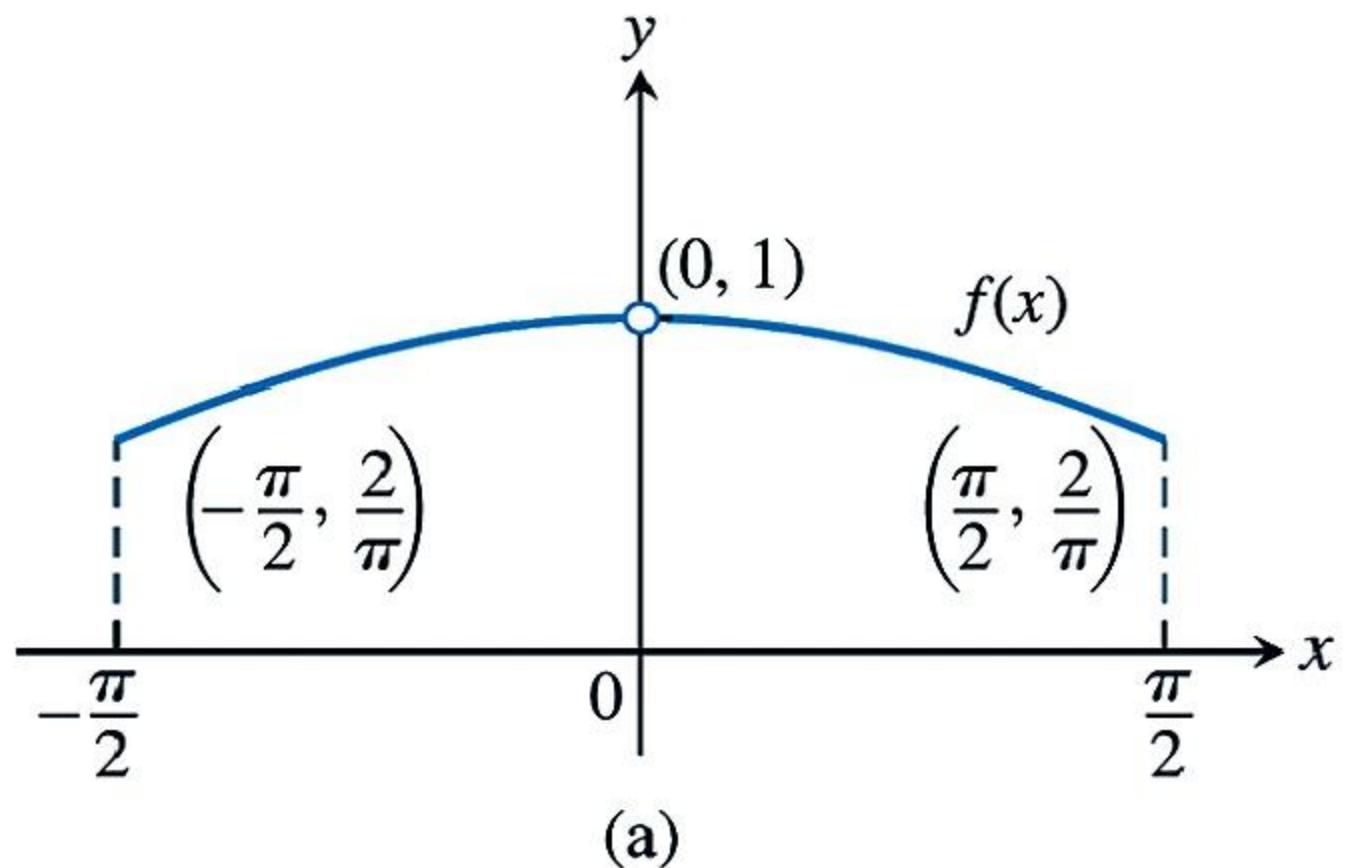


(d)

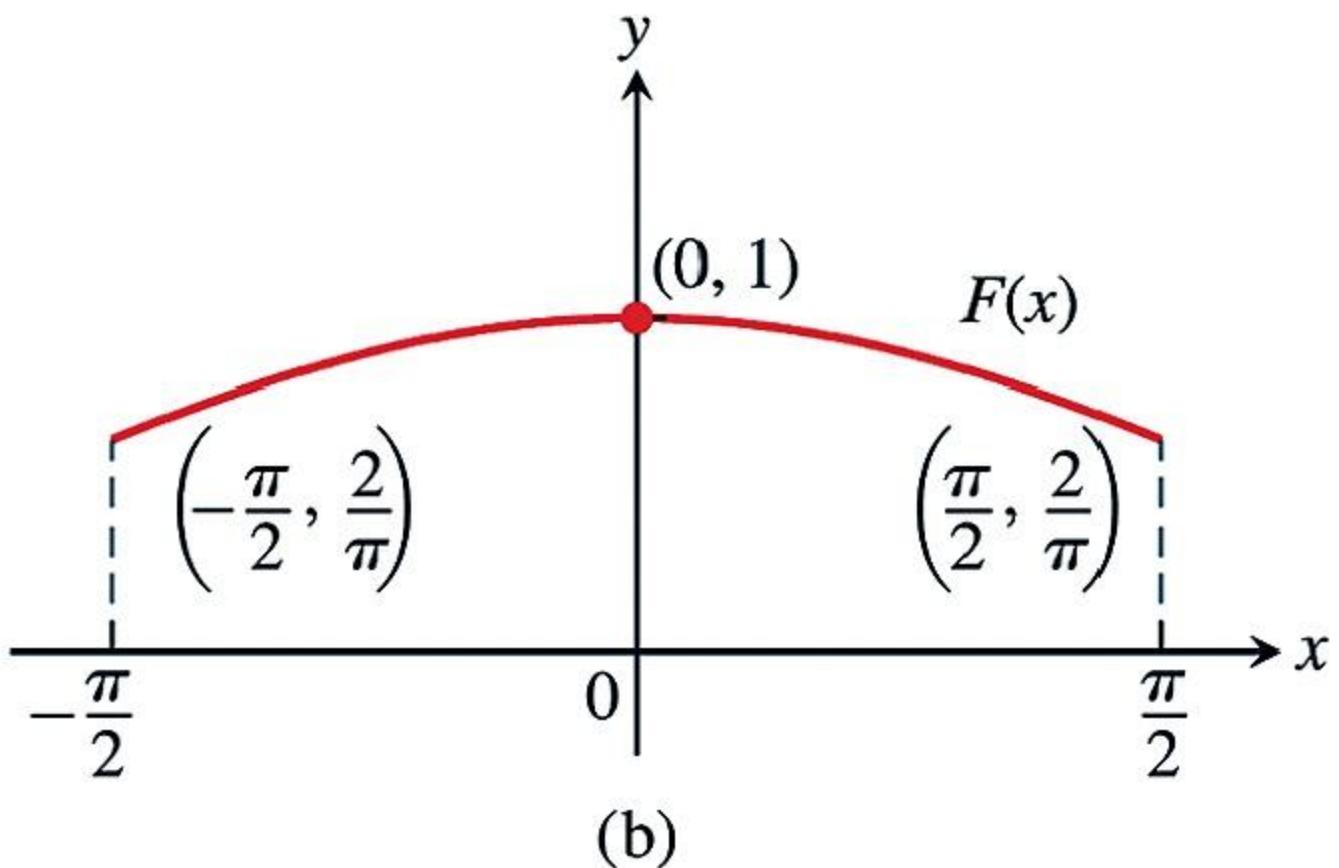
**FIGURE 2.45** Zooming in on a zero of the function  $f(x) = x^3 - x - 1$ . The zero is near  $x = 1.3247$  (Example 10).



**FIGURE 2.46** The curves  
 $y = \sqrt{2x + 5}$  and  $y = 4 - x^2$   
have the same value at  $x = c$  where  
 $\sqrt{2x + 5} + x^2 - 4 = 0$  (Example 11).

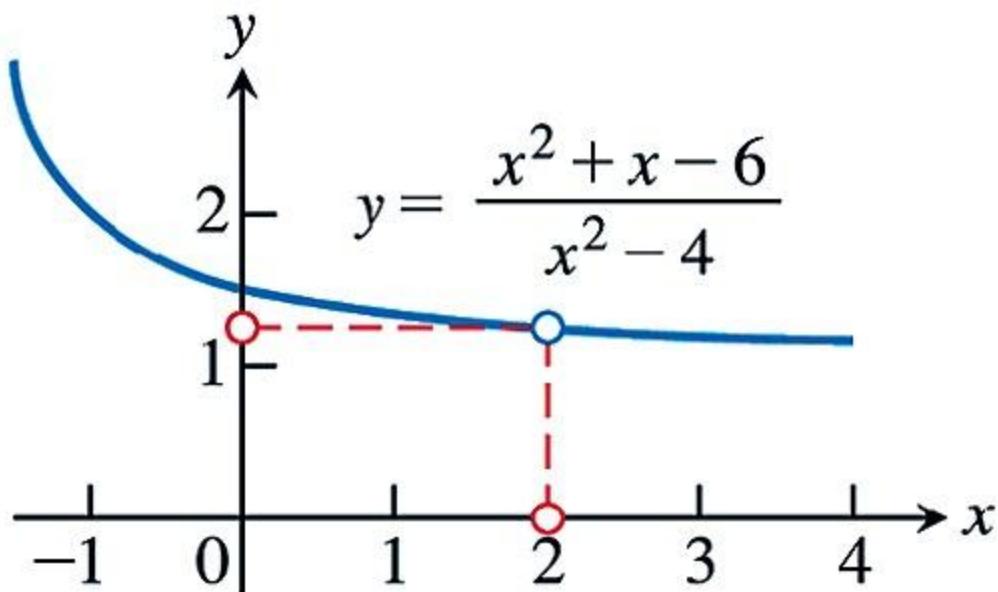


(a)

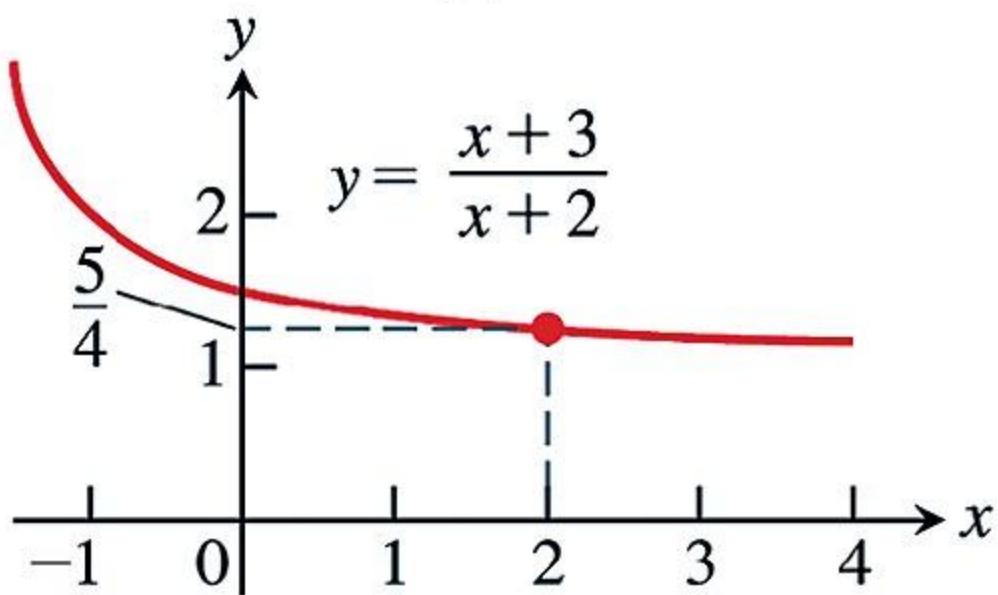


(b)

**FIGURE 2.47** (a) The graph of  $f(x) = (\sin x)/x$  for  $-\pi/2 \leq x \leq \pi/2$  does not include the point  $(0, 1)$  because the function is not defined at  $x = 0$ . (b) We can extend the domain to include  $x = 0$  by defining the new function  $F(x)$  with  $F(0) = 1$  and  $F(x) = f(x)$  everywhere else. Note that  $F(0) = \lim_{x \rightarrow 0} f(x)$  and  $F(x)$  is a continuous function at  $x = 0$ .



(a)

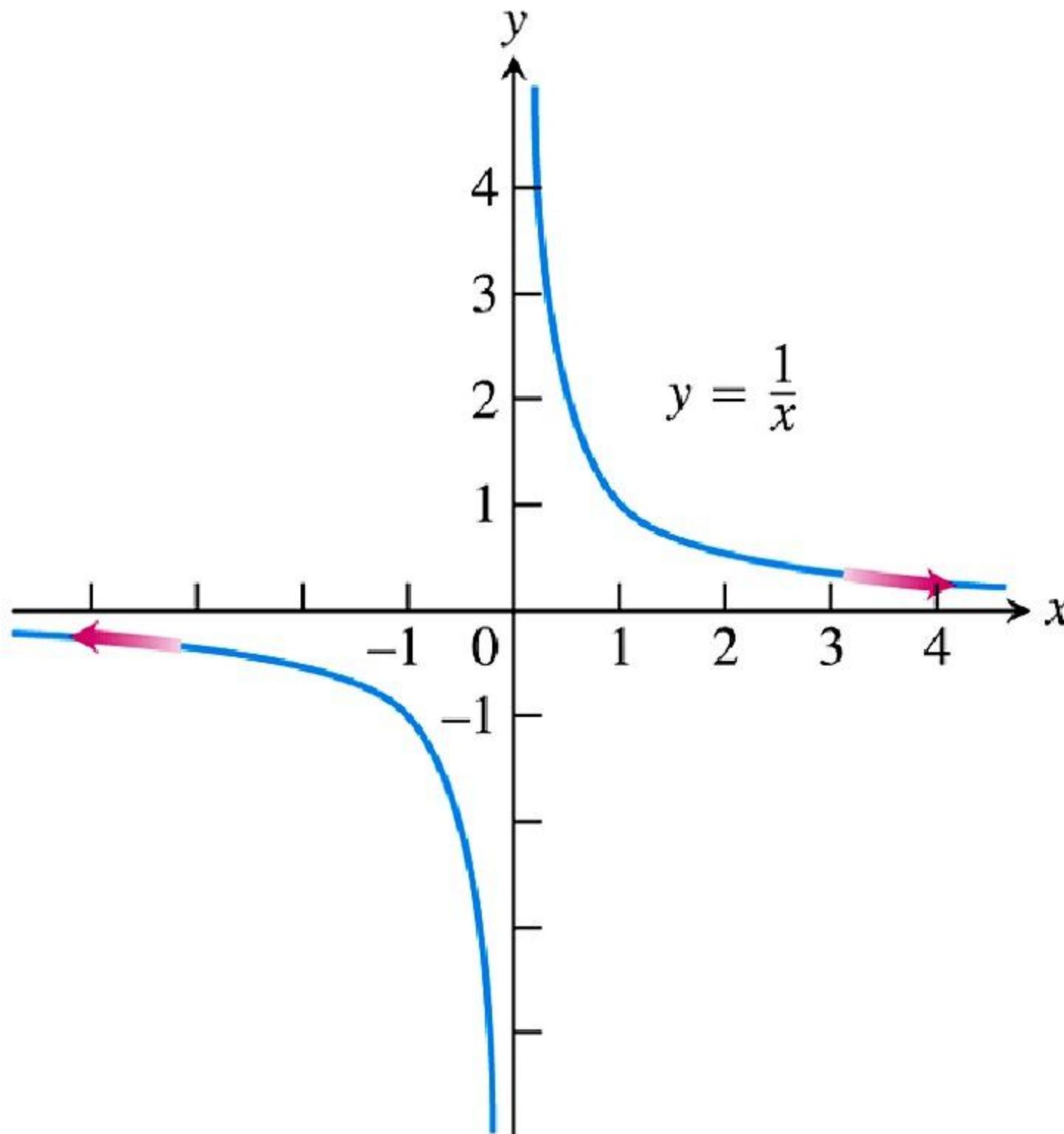


(b)

**FIGURE 2.48** (a) The graph of  $f(x)$  and (b) the graph of its continuous extension  $F(x)$  (Example 12).

# Section 2.6

## Limits Involving Infinity; Asymptotes of Graphs



**FIGURE 2.49** The graph of  $y = 1/x$  approaches 0 as  $x \rightarrow \infty$  or  $x \rightarrow -\infty$ .

## DEFINITIONS

1. We say that  $f(x)$  has the **limit  $L$  as  $x$  approaches infinity** and write

$$\lim_{x \rightarrow \infty} f(x) = L$$

if, for every number  $\varepsilon > 0$ , there exists a corresponding number  $M$  such that for all  $x$  in the domain of  $f$

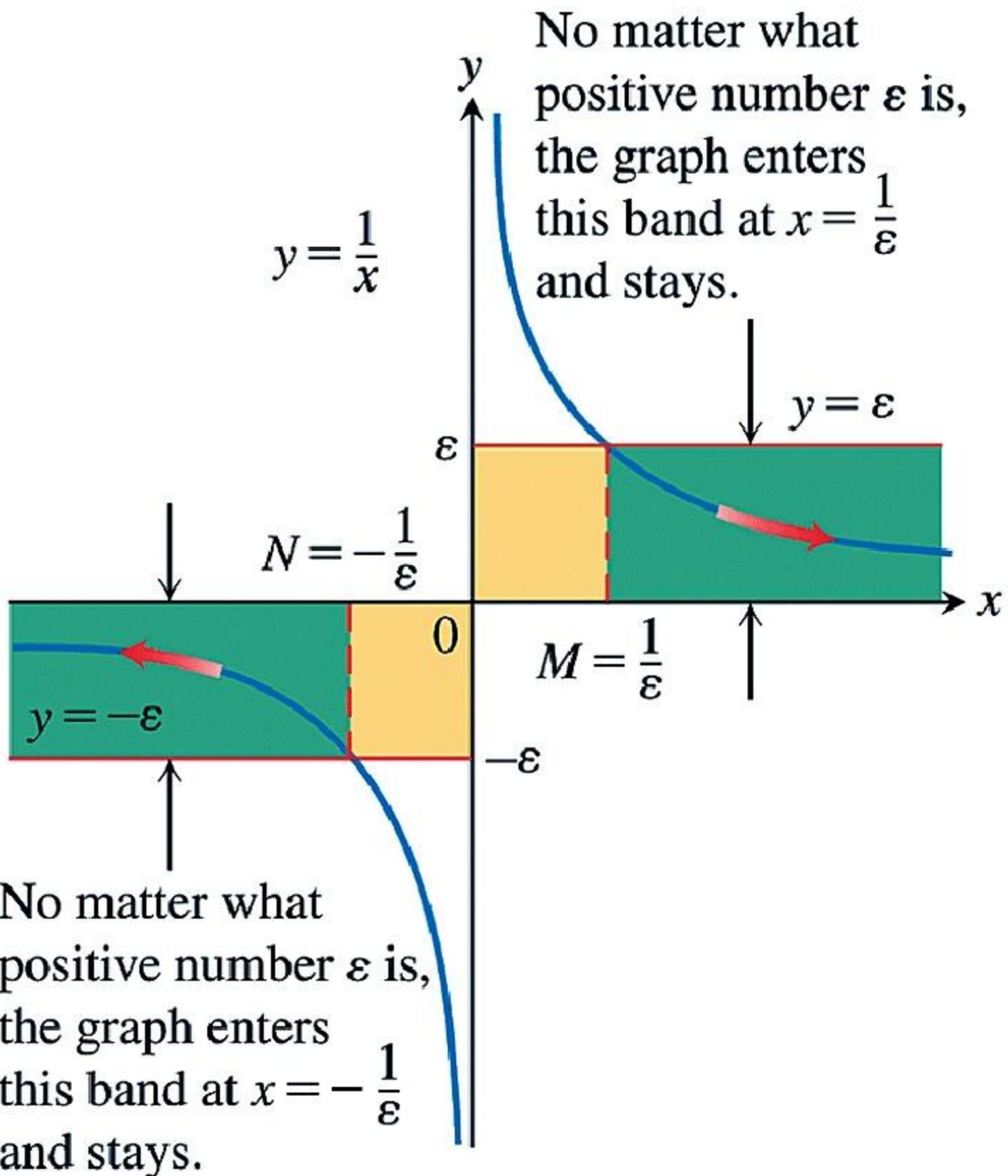
$$|f(x) - L| < \varepsilon \quad \text{whenever } x > M.$$

2. We say that  $f(x)$  has the **limit  $L$  as  $x$  approaches negative infinity** and write

$$\lim_{x \rightarrow -\infty} f(x) = L$$

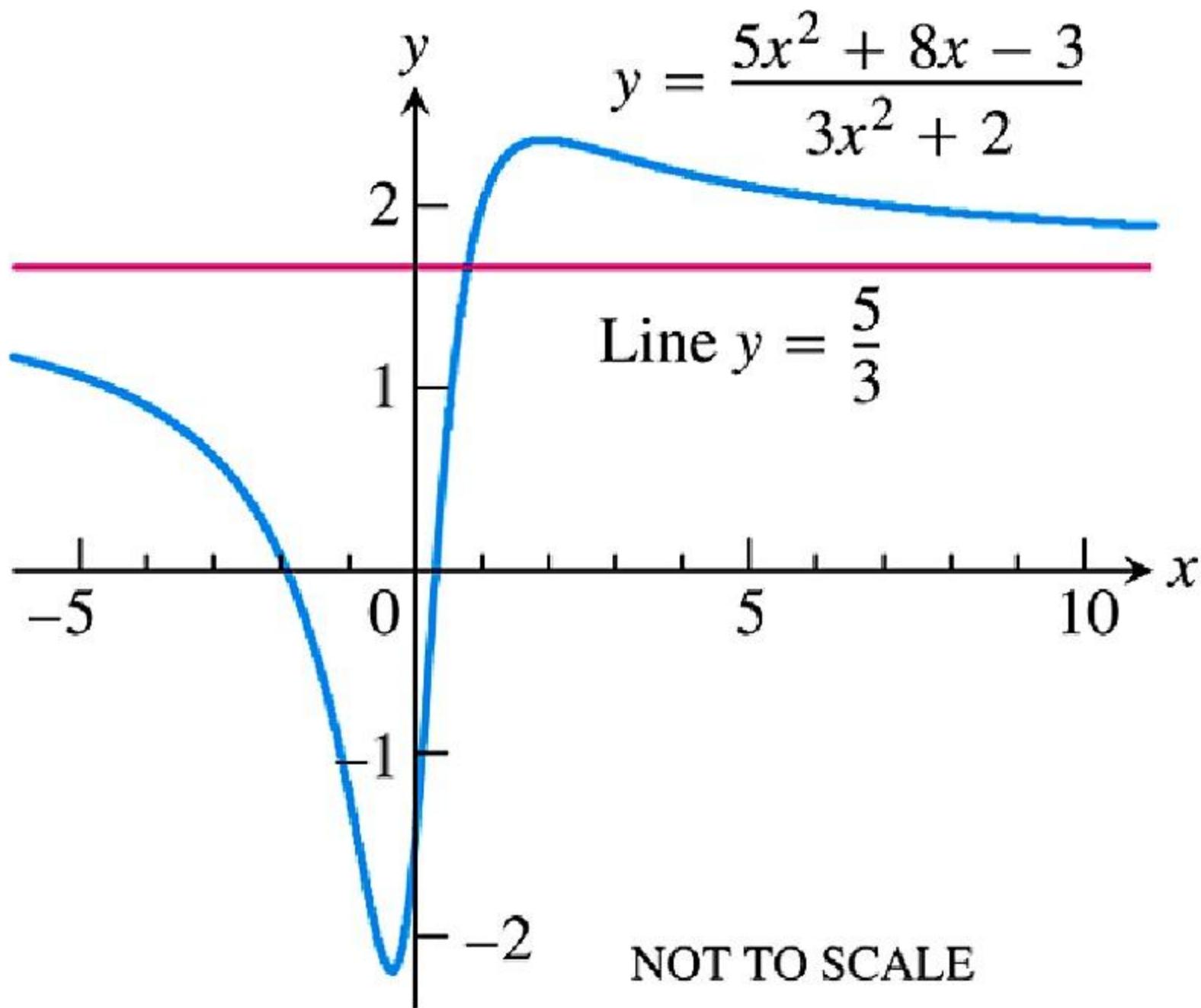
if, for every number  $\varepsilon > 0$ , there exists a corresponding number  $N$  such that for all  $x$  in the domain of  $f$

$$|f(x) - L| < \varepsilon \quad \text{whenever } x < N.$$

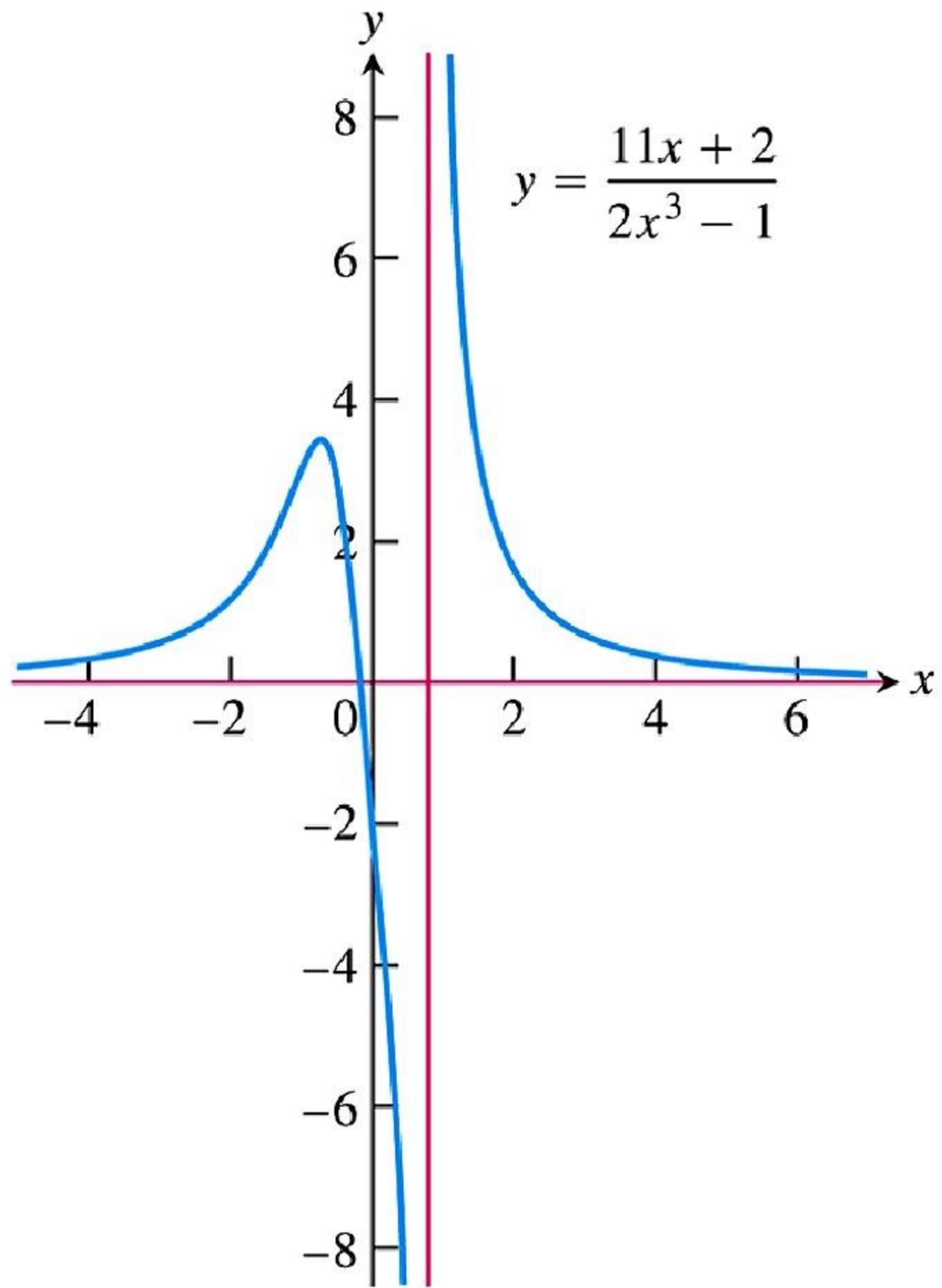


**FIGURE 2.50** The geometry behind the argument in Example 1.

**THEOREM 12** All the limit laws in Theorem 1 are true when we replace  $\lim_{x \rightarrow c}$  by  $\lim_{x \rightarrow \infty}$  or  $\lim_{x \rightarrow -\infty}$ . That is, the variable  $x$  may approach a finite number  $c$  or  $\pm\infty$ .



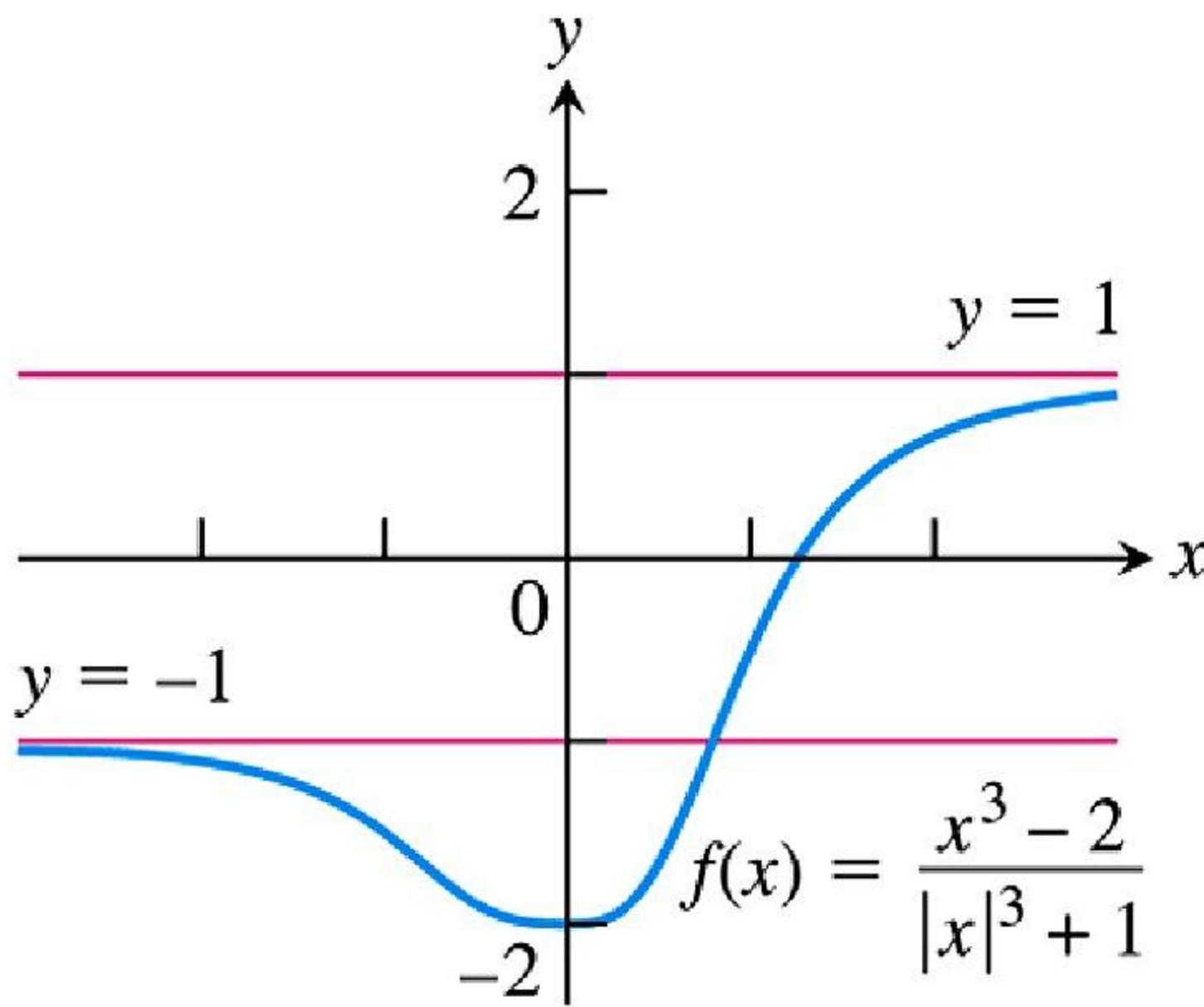
**FIGURE 2.51** The graph of the function in Example 3a. The graph approaches the line  $y = 5/3$  as  $|x|$  increases.



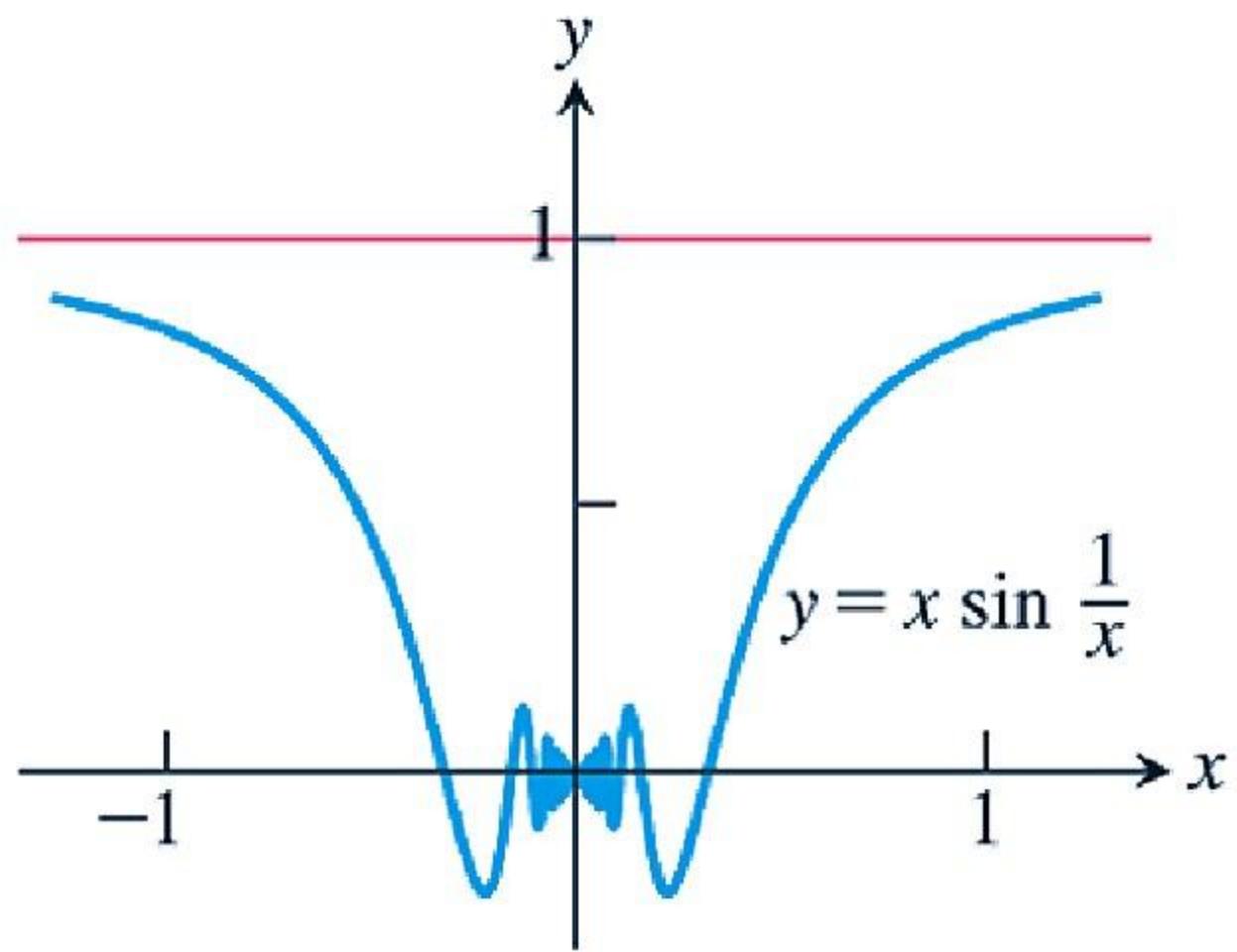
**FIGURE 2.52** The graph of the function in Example 3b. The graph approaches the  $x$ -axis as  $|x|$  increases.

**DEFINITION** A line  $y = b$  is a **horizontal asymptote** of the graph of a function  $y = f(x)$  if either

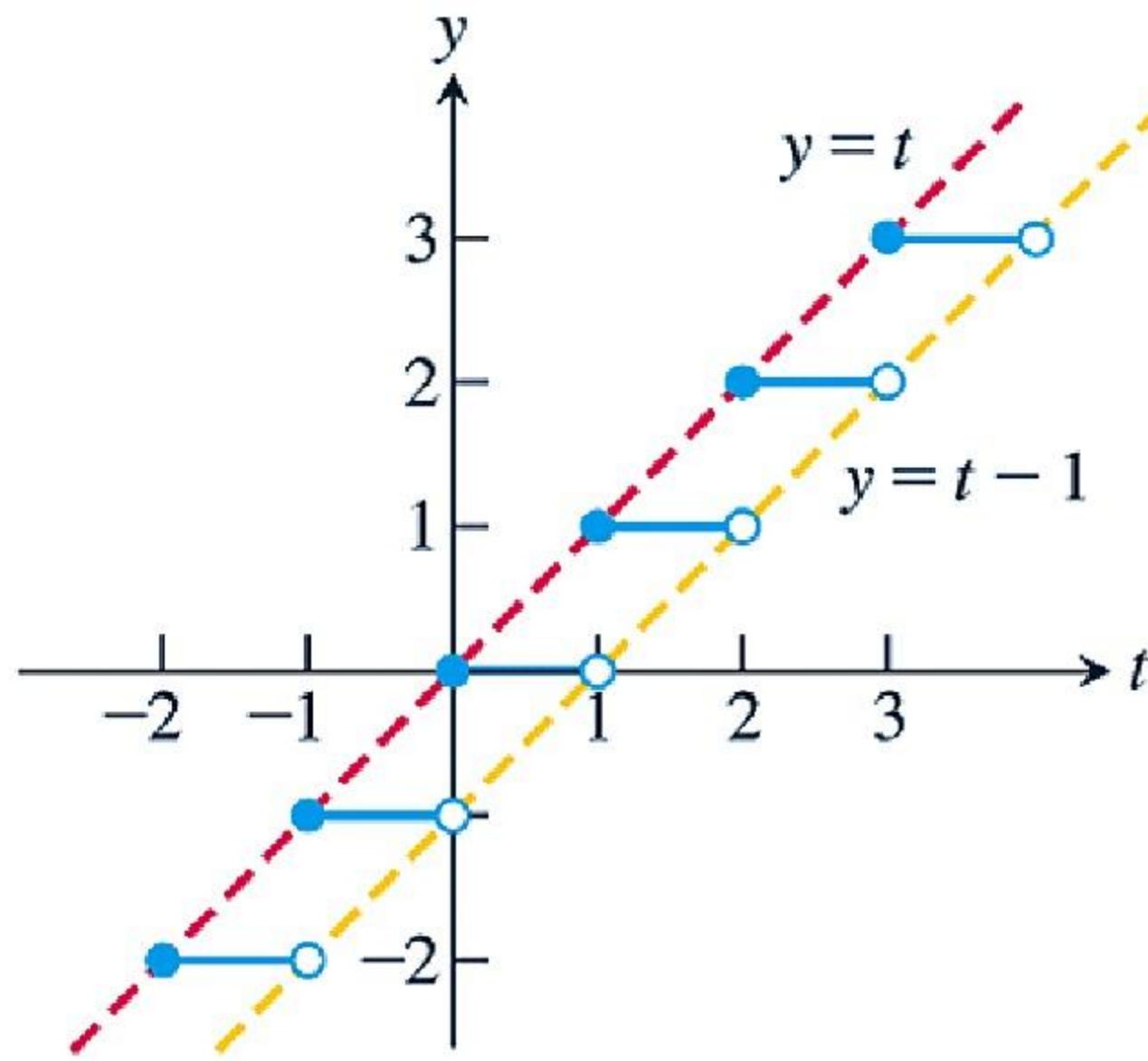
$$\lim_{x \rightarrow \infty} f(x) = b \quad \text{or} \quad \lim_{x \rightarrow -\infty} f(x) = b.$$



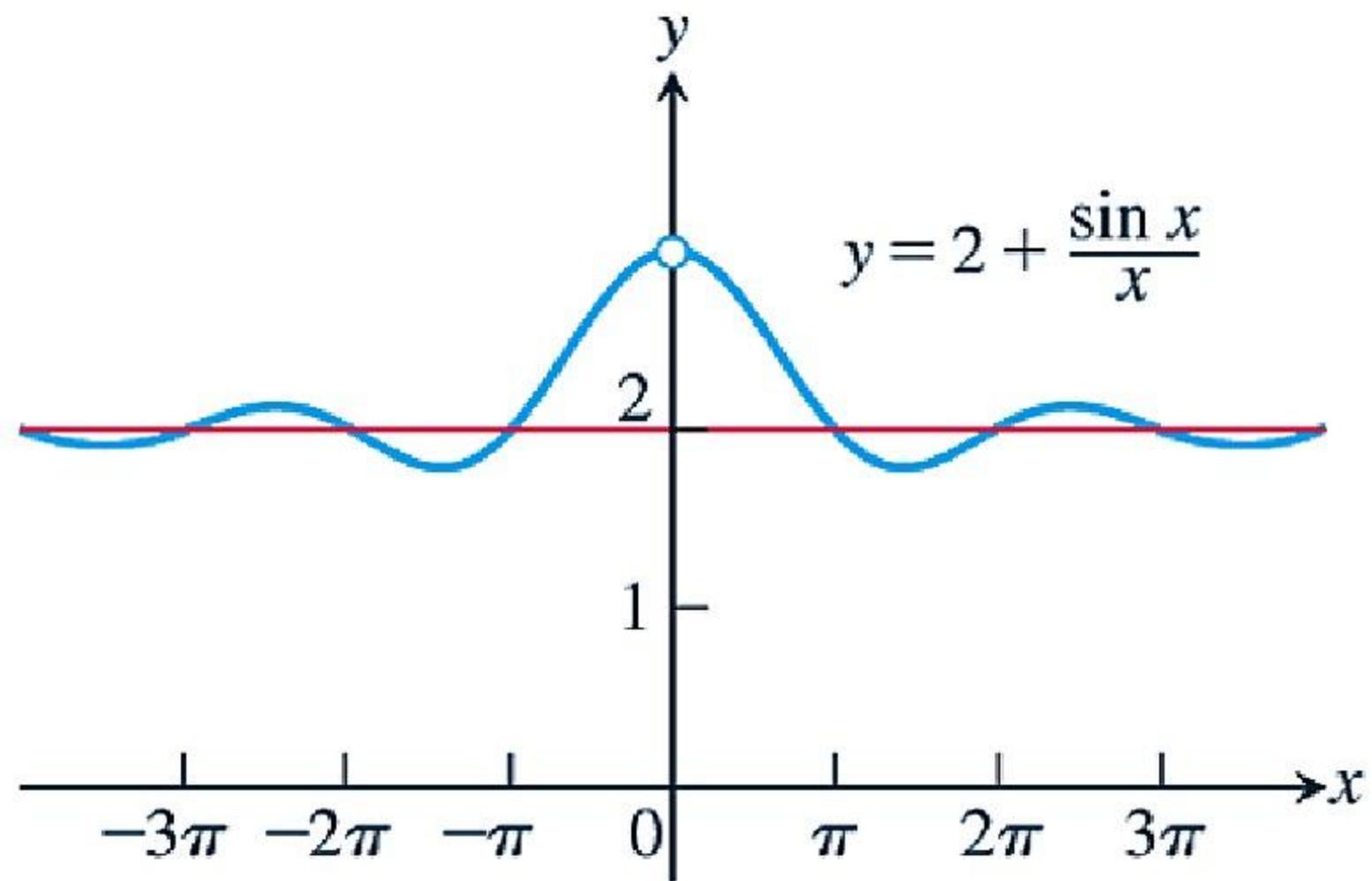
**FIGURE 2.53** The graph of the function in Example 4 has two horizontal asymptotes.



**FIGURE 2.54** The line  $y = 1$  is a horizontal asymptote of the function graphed here (Example 5b).

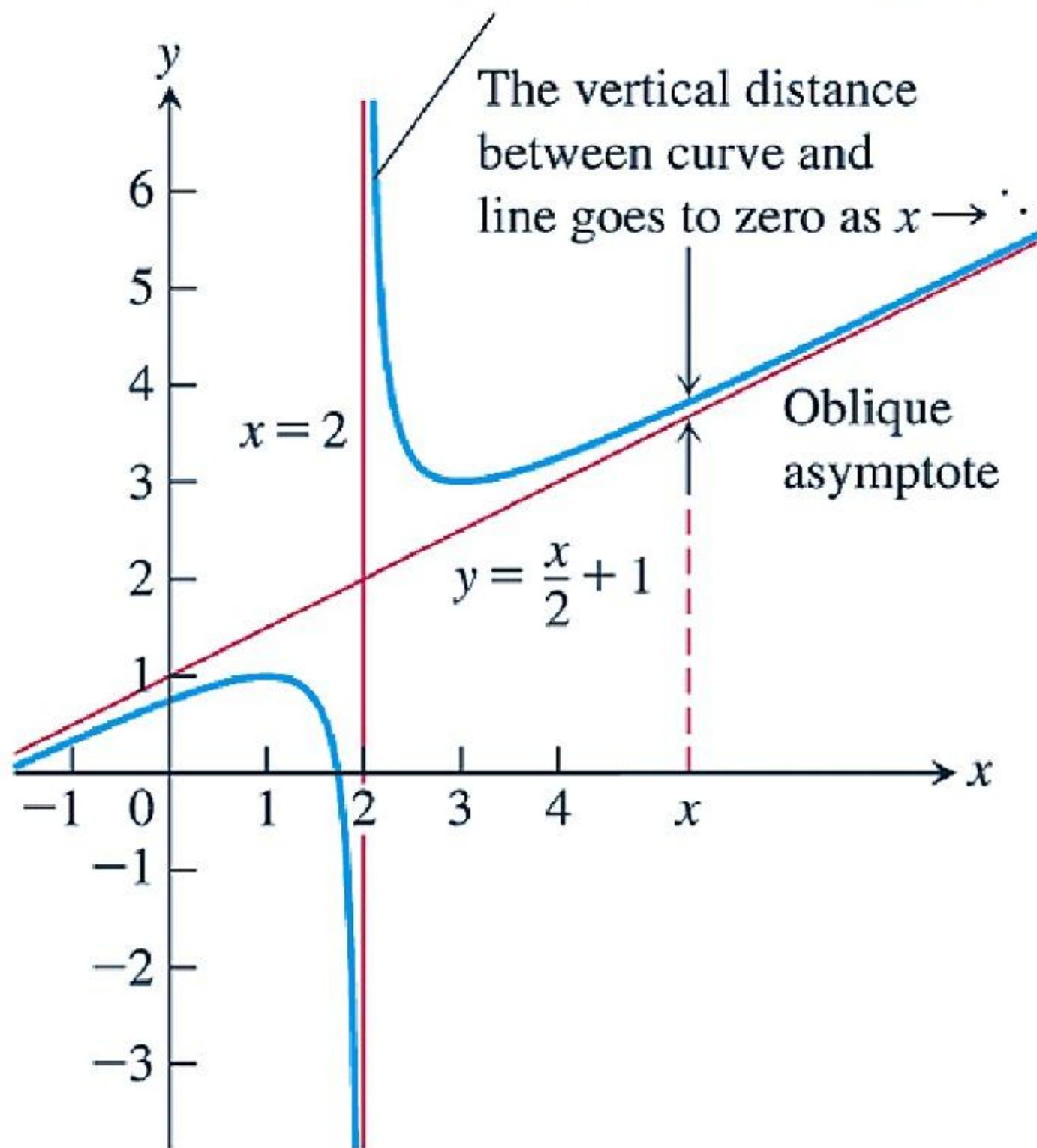


**FIGURE 2.55** The graph of the greatest integer function  $y = \lfloor t \rfloor$  is sandwiched between  $y = t - 1$  and  $y = t$ .

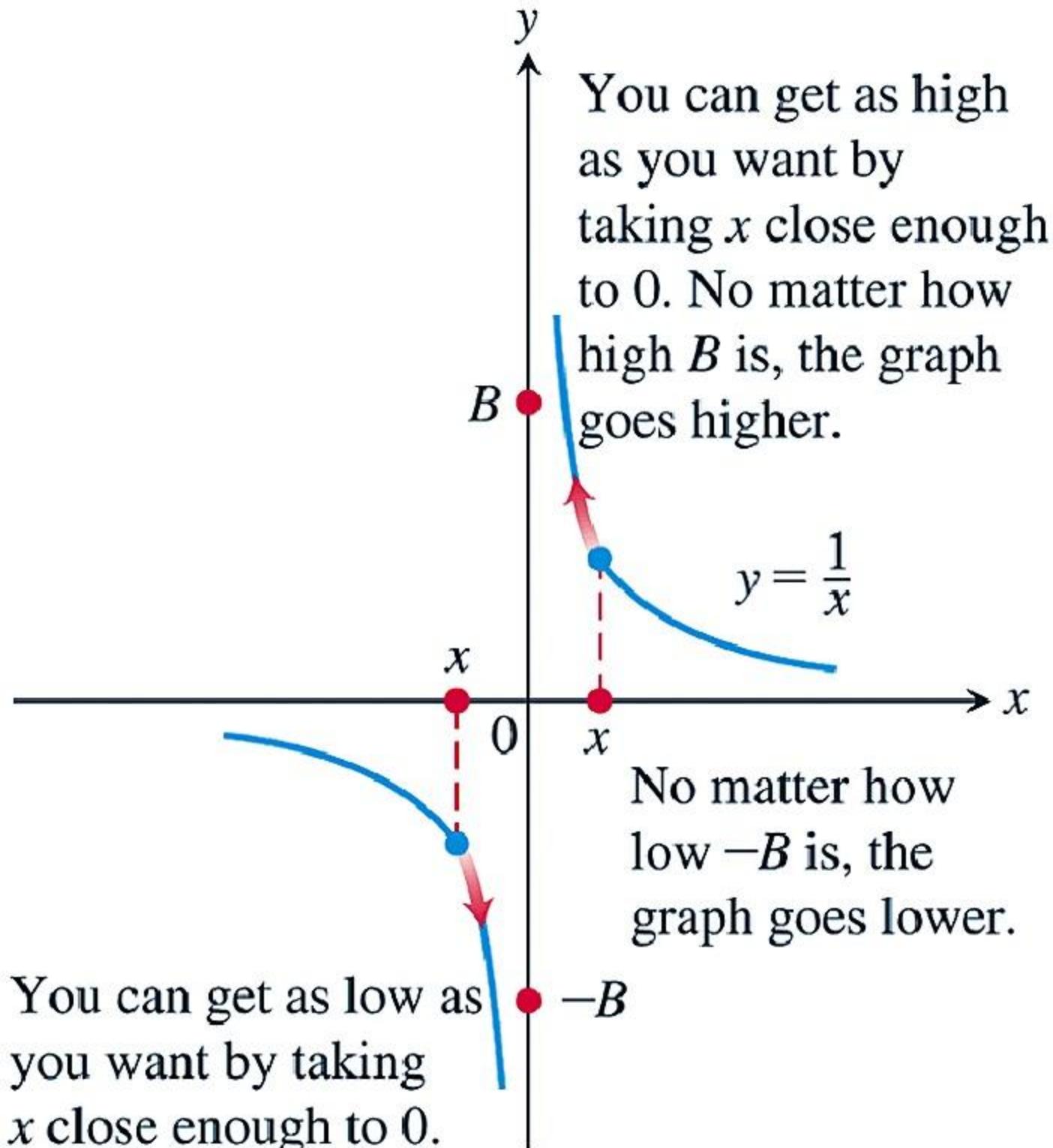


**FIGURE 2.56** A curve may cross one of its asymptotes infinitely often (Example 7).

$$y = \frac{x^2 - 3}{2x - 4} = \frac{x}{2} + 1 + \frac{1}{2x - 4}$$

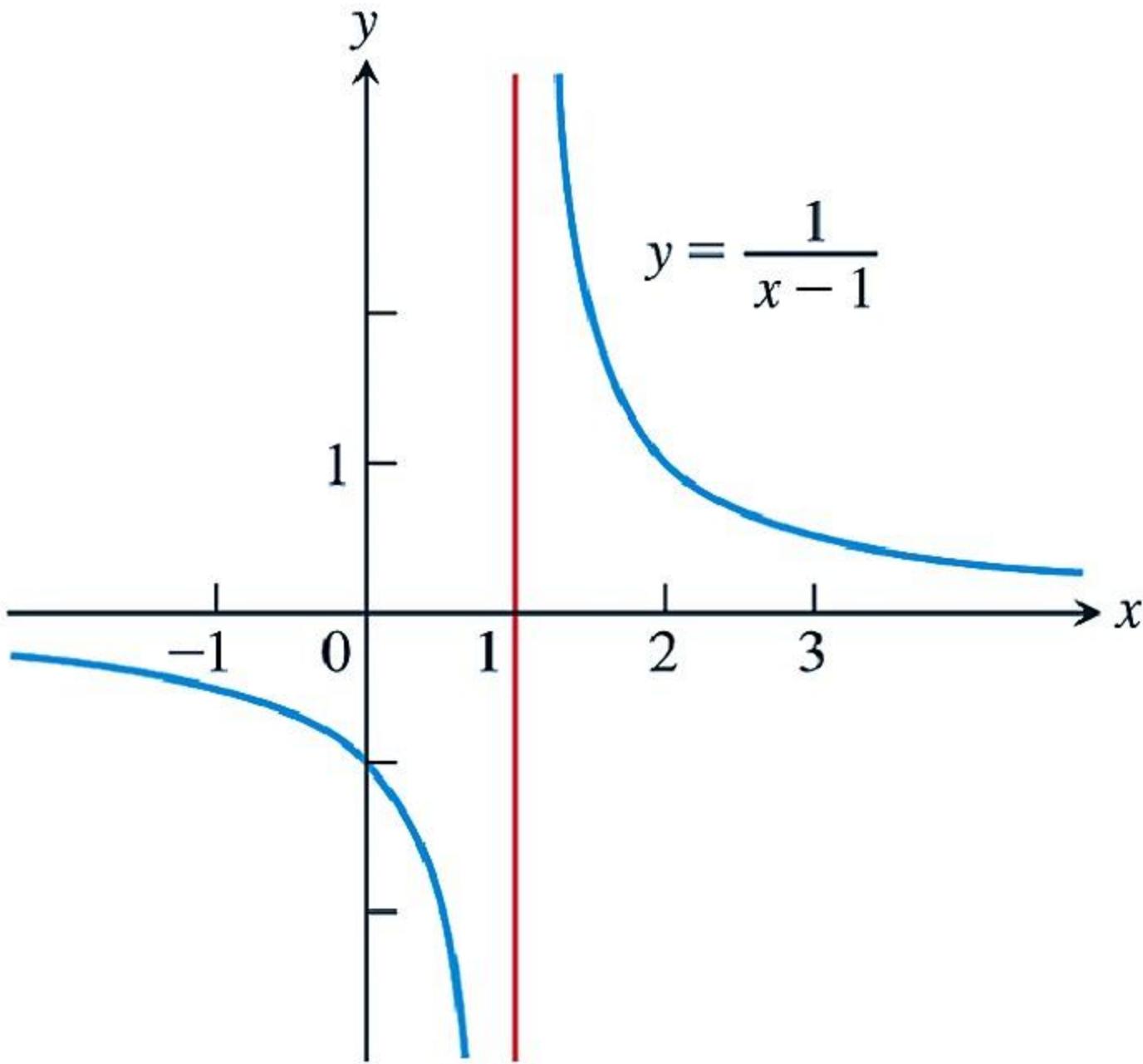


**FIGURE 2.57** The graph of the function in Example 9 has an oblique asymptote.

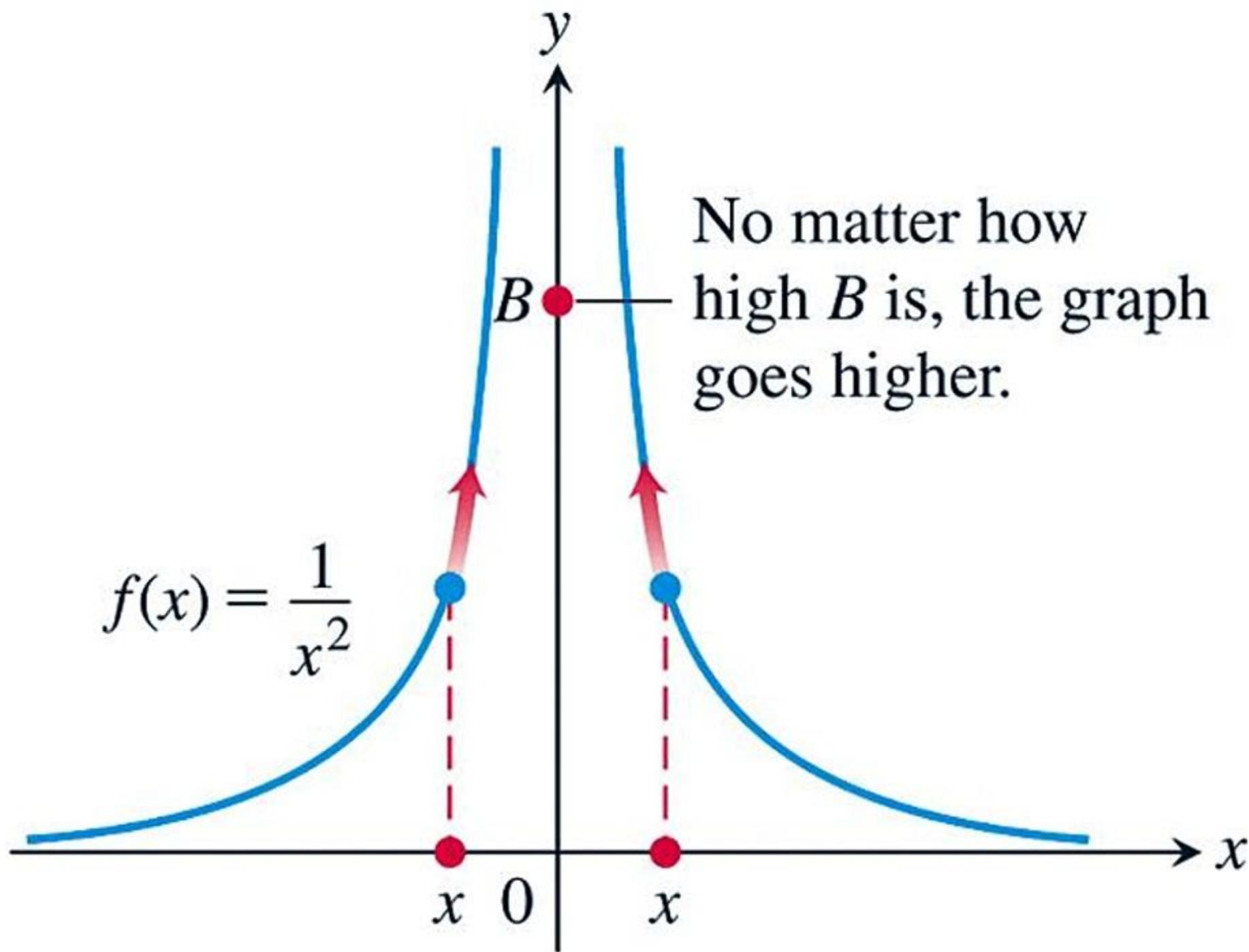


**FIGURE 2.58** One-sided infinite limits:

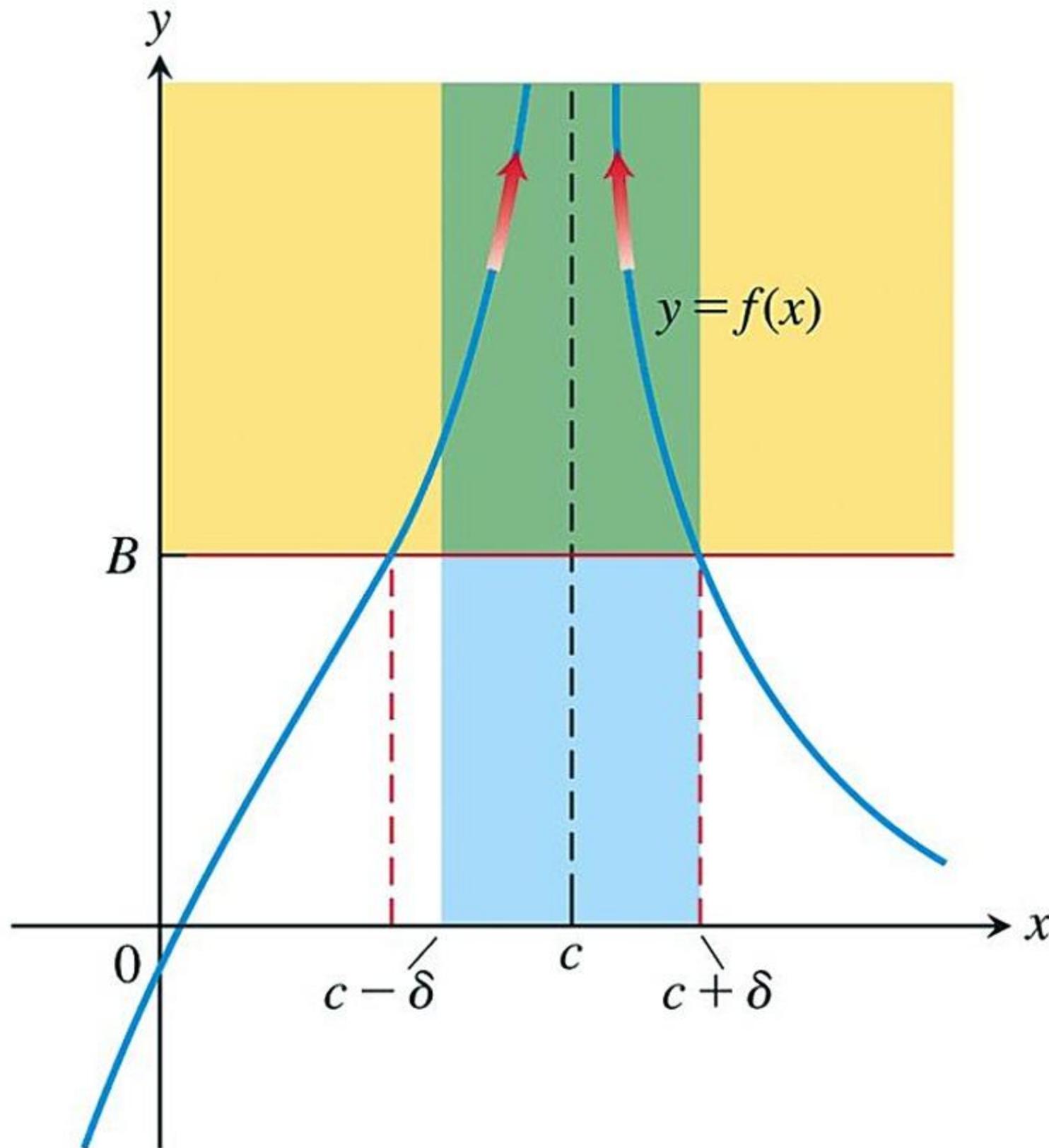
$$\lim_{x \rightarrow 0^+} \frac{1}{x} = \infty \quad \text{and} \quad \lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty.$$



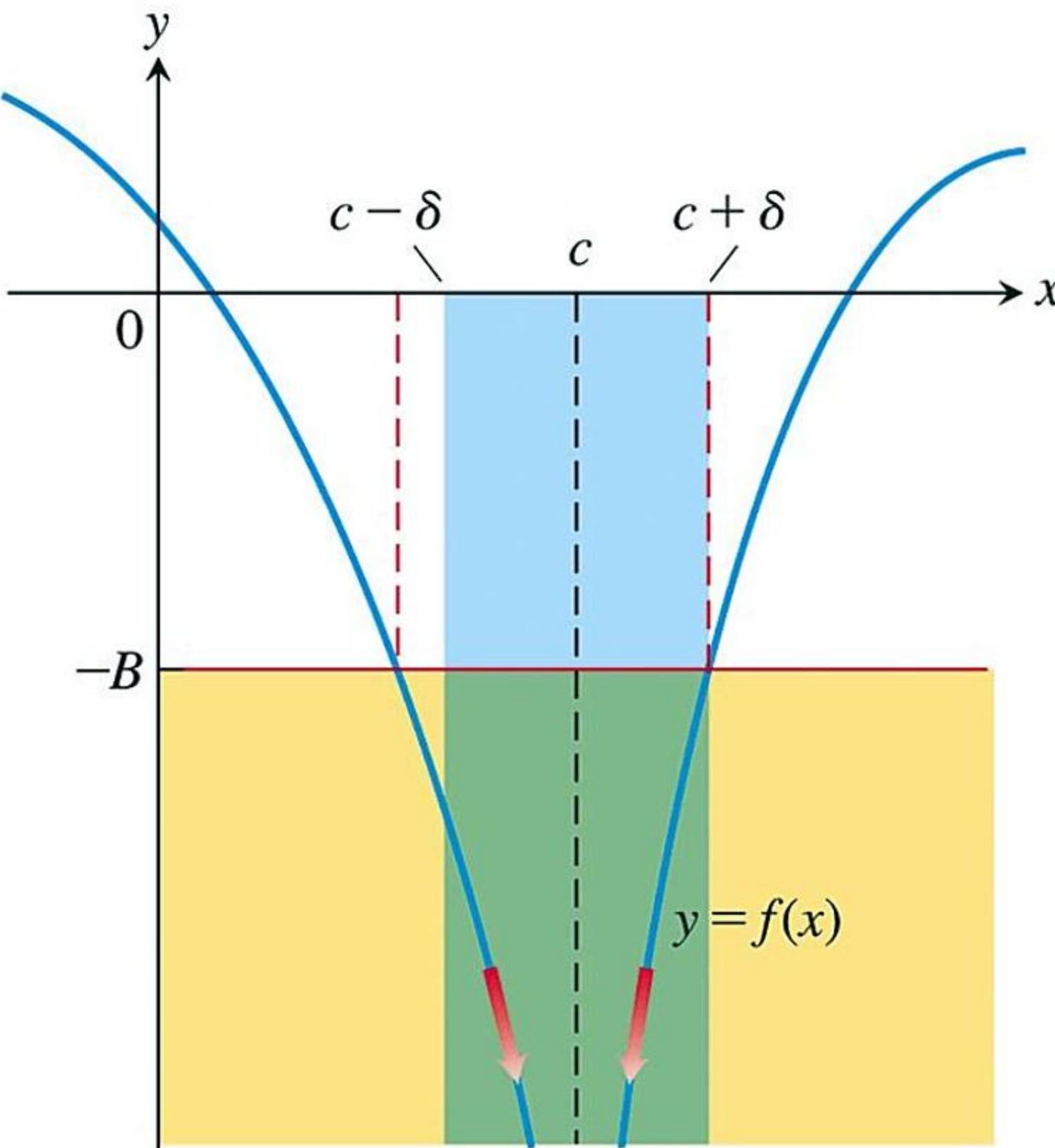
**FIGURE 2.59** Near  $x = 1$ , the function  $y = 1/(x - 1)$  behaves the way the function  $y = 1/x$  behaves near  $x = 0$ . Its graph is the graph of  $y = 1/x$  shifted 1 unit to the right (Example 10).



**FIGURE 2.60** The graph of  $f(x)$  in Example 11 approaches infinity as  $x \rightarrow 0$ .



**FIGURE 2.61** For  $c - \delta < x < c + \delta$ , the graph of  $f(x)$  lies above the line  $y = B$ .



**FIGURE 2.62** For  $c - \delta < x < c + \delta$ ,  
the graph of  $f(x)$  lies below the line  $y = -B$ .

## DEFINITIONS

1. We say that  **$f(x)$  approaches infinity as  $x$  approaches  $c$** , and write

$$\lim_{x \rightarrow c} f(x) = \infty,$$

if for every positive real number  $B$  there exists a corresponding  $\delta > 0$  such that

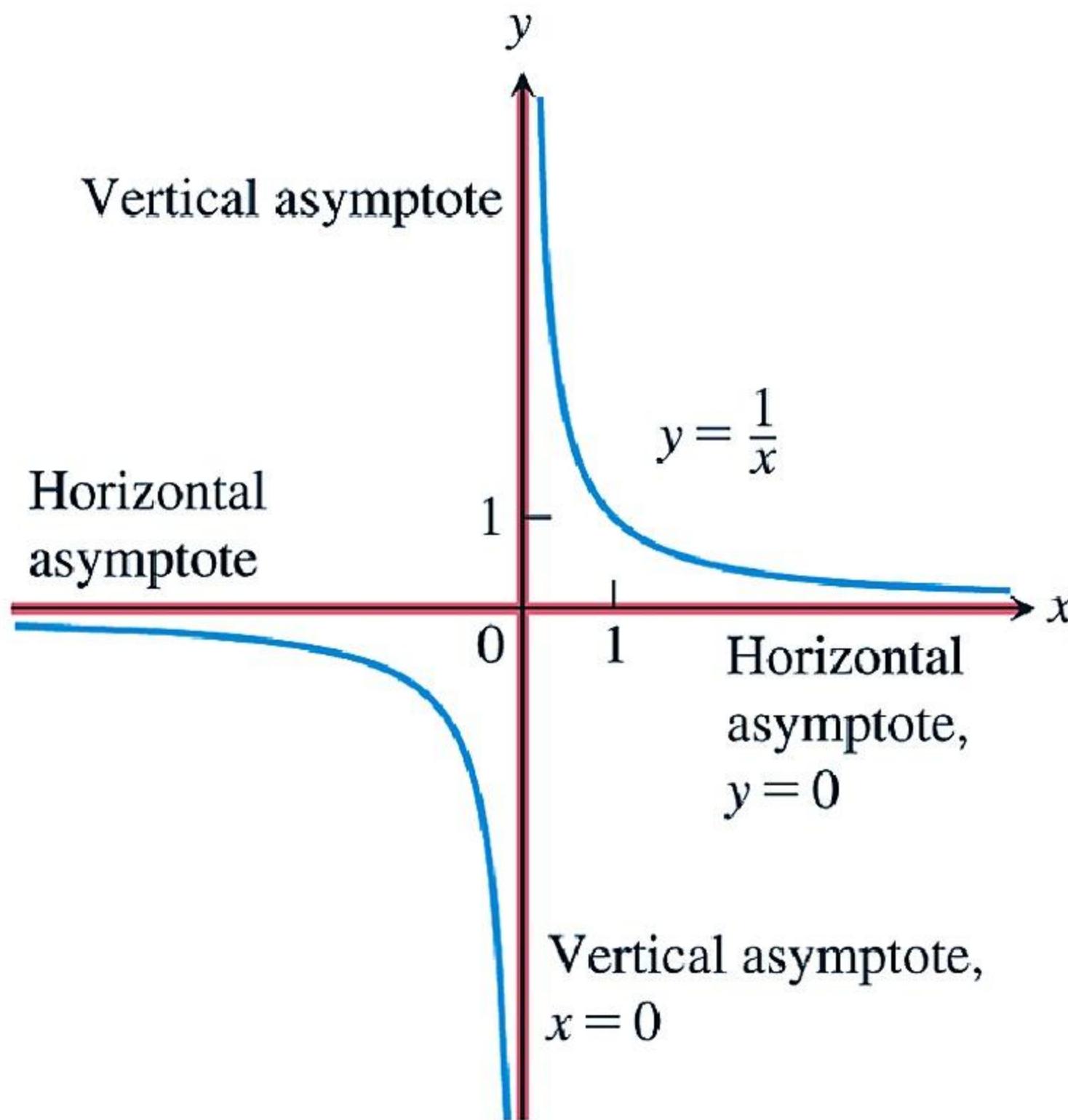
$$f(x) > B \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

2. We say that  **$f(x)$  approaches negative infinity as  $x$  approaches  $c$** , and write

$$\lim_{x \rightarrow c} f(x) = -\infty,$$

if for every negative real number  $-B$  there exists a corresponding  $\delta > 0$  such that

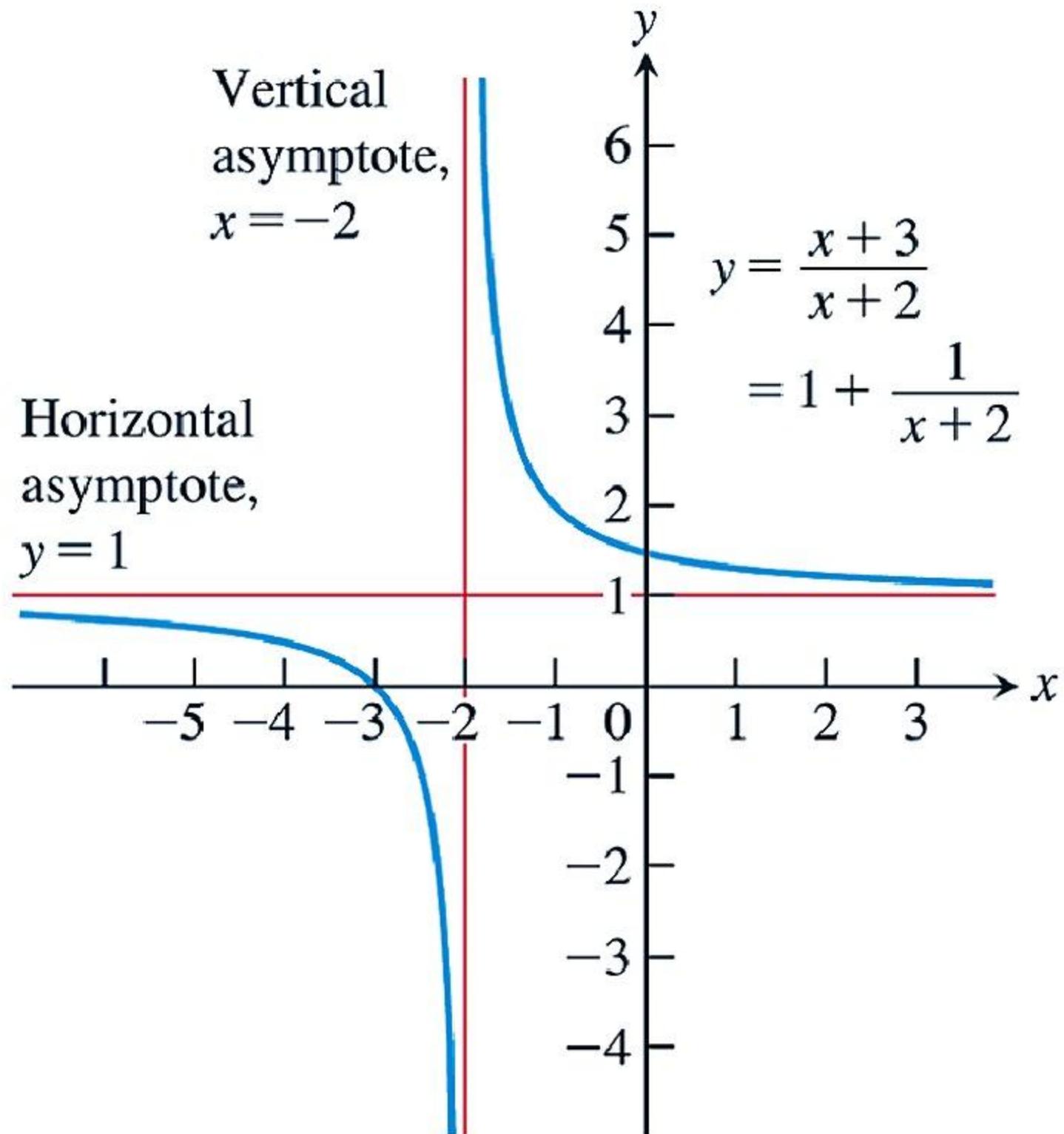
$$f(x) < -B \quad \text{whenever} \quad 0 < |x - c| < \delta.$$



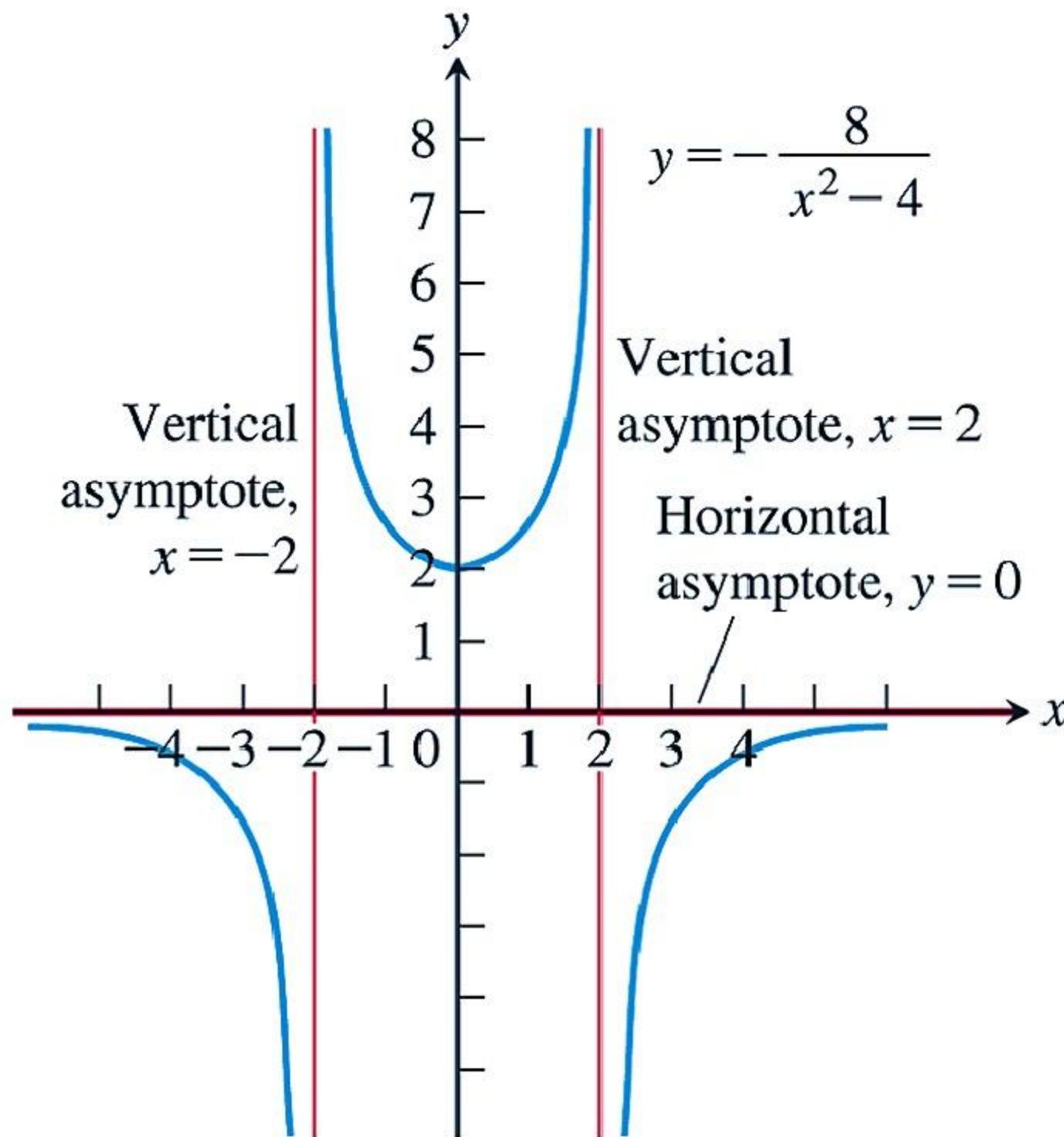
**FIGURE 2.63** The coordinate axes are asymptotes of both branches of the hyperbola  $y = 1/x$ .

**DEFINITION** A line  $x = a$  is a **vertical asymptote** of the graph of a function  $y = f(x)$  if either

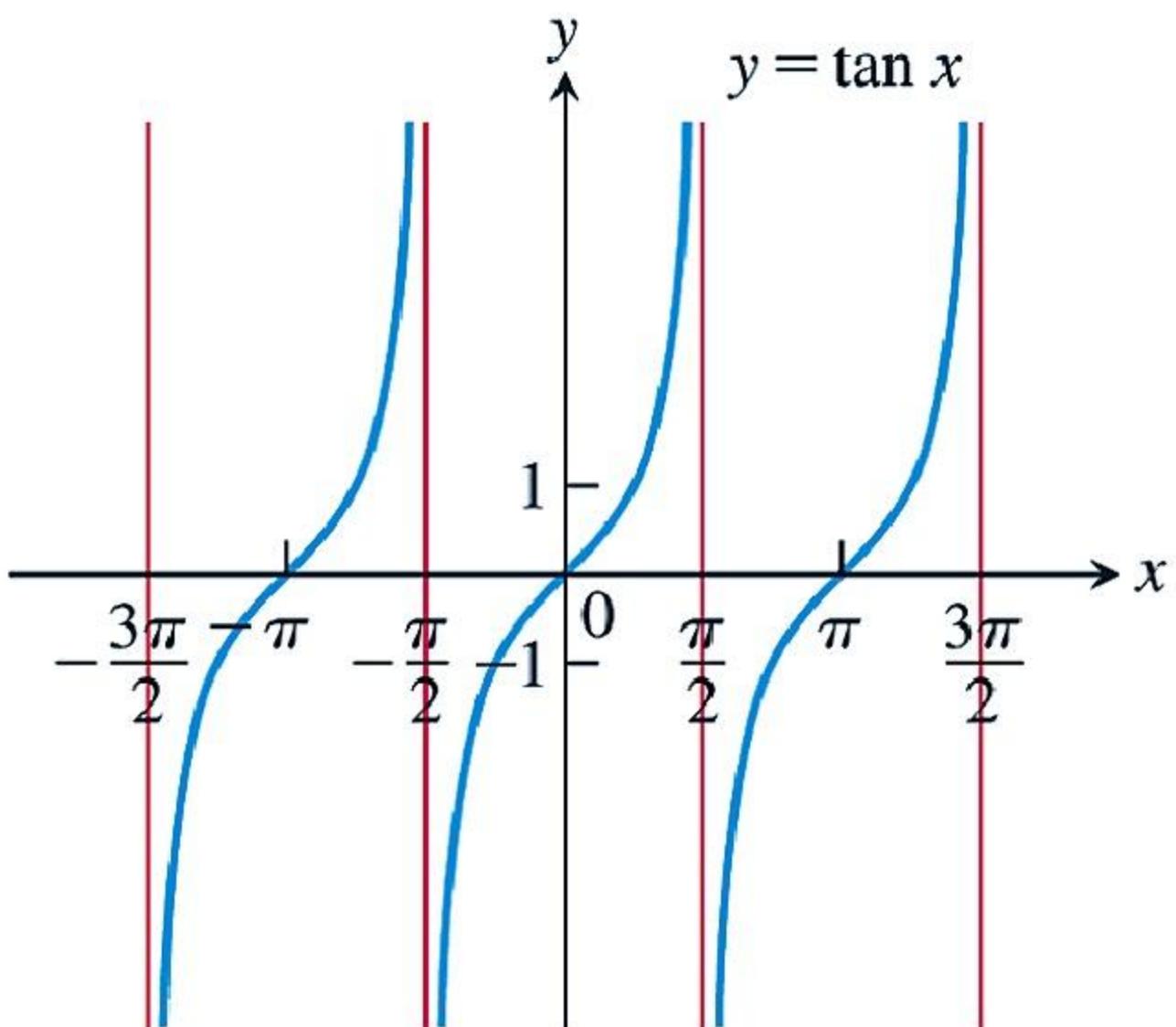
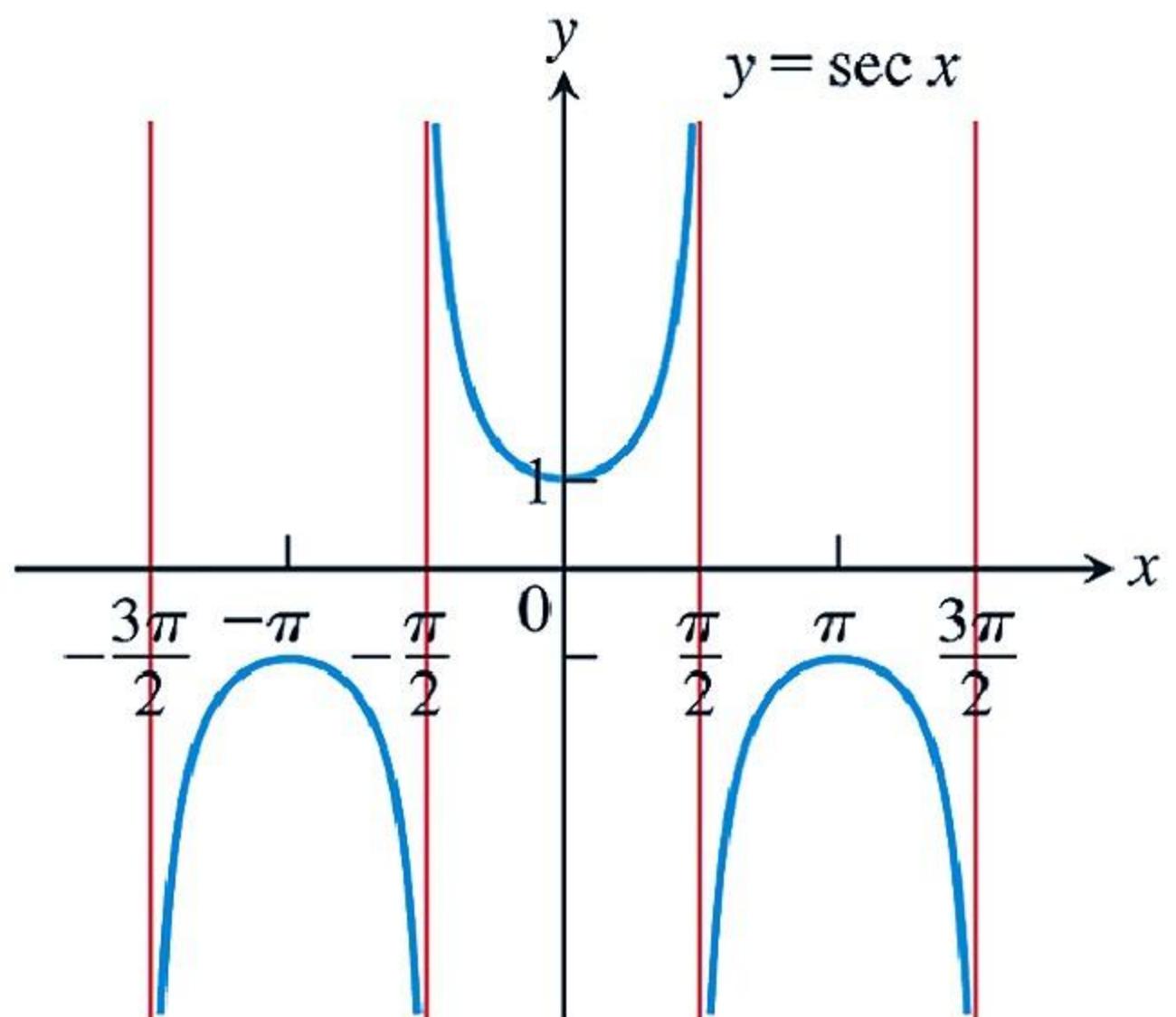
$$\lim_{x \rightarrow a^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \rightarrow a^-} f(x) = \pm\infty.$$



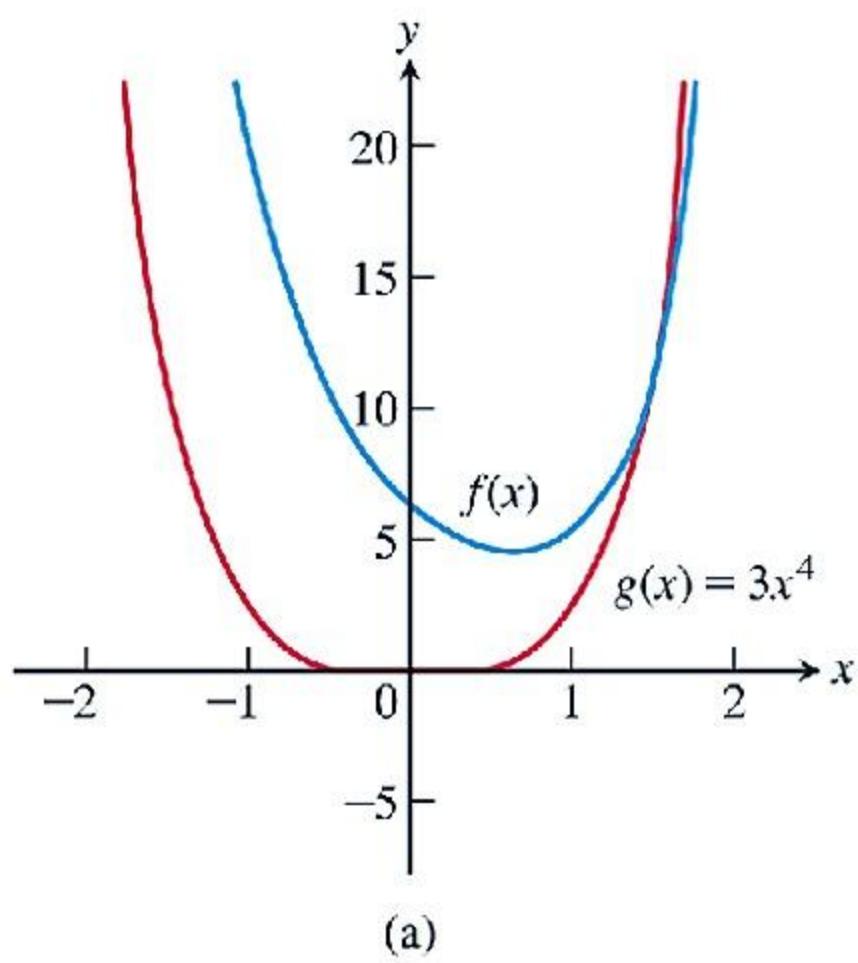
**FIGURE 2.64** The lines  $y = 1$  and  $x = -2$  are asymptotes of the curve in Example 15.



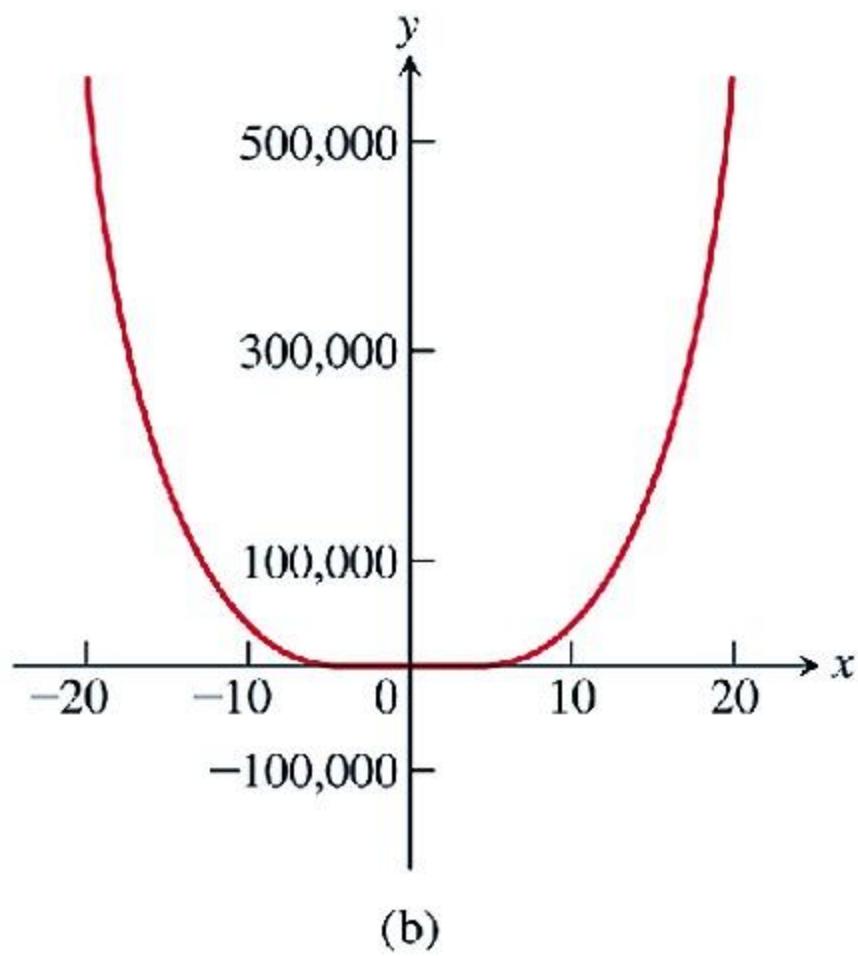
**FIGURE 2.65** Graph of the function in Example 16. Notice that the curve approaches the  $x$ -axis from only one side. Asymptotes do not have to be two-sided.



**FIGURE 2.66** The graphs of  $\sec x$  and  $\tan x$  have infinitely many vertical asymptotes (Example 17).



(a)



(b)

**FIGURE 2.67** The graphs of  $f$  and  $g$  are (a) distinct for  $|x|$  small, and (b) nearly identical for  $|x|$  large (Example 18).