

Chapter 11

Vectors and the Geometry of Space

Thomas' Calculus, 14e in SI Units

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Section 11.1

Three-Dimensional Coordinate Systems

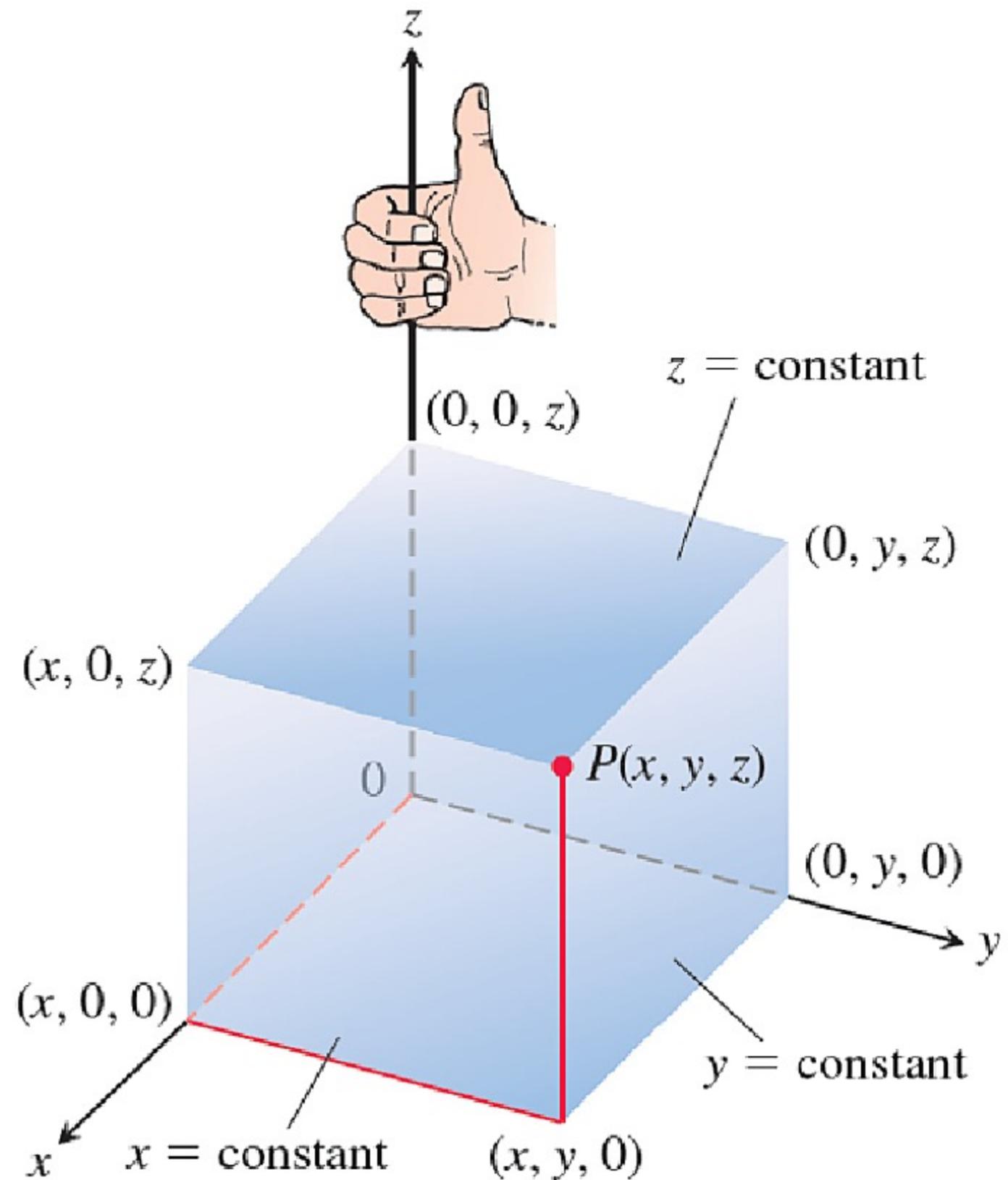


FIGURE 11.1 The Cartesian coordinate system is right-handed.

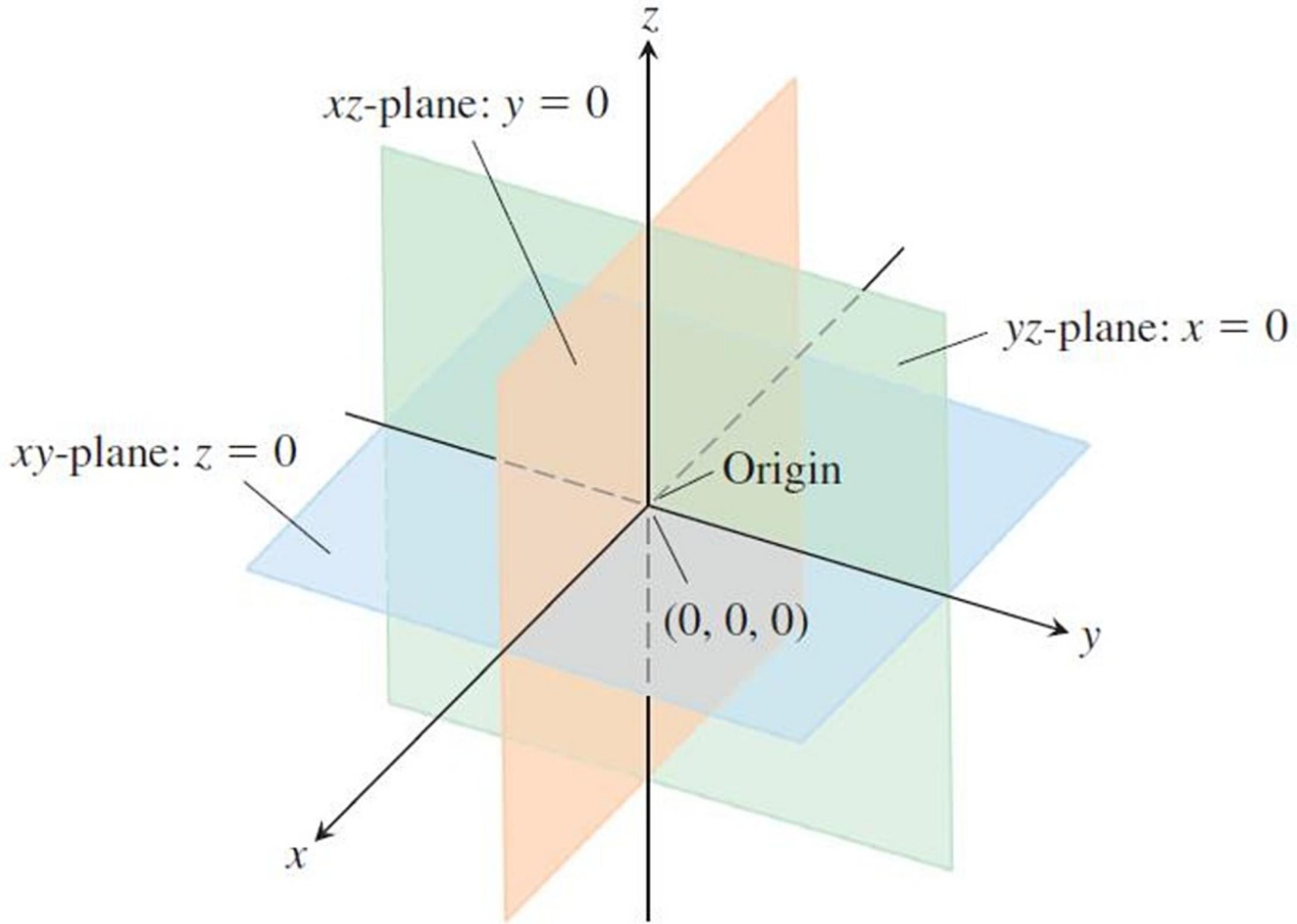


FIGURE 11.2 The planes $x = 0$, $y = 0$, and $z = 0$ divide space into eight octants.

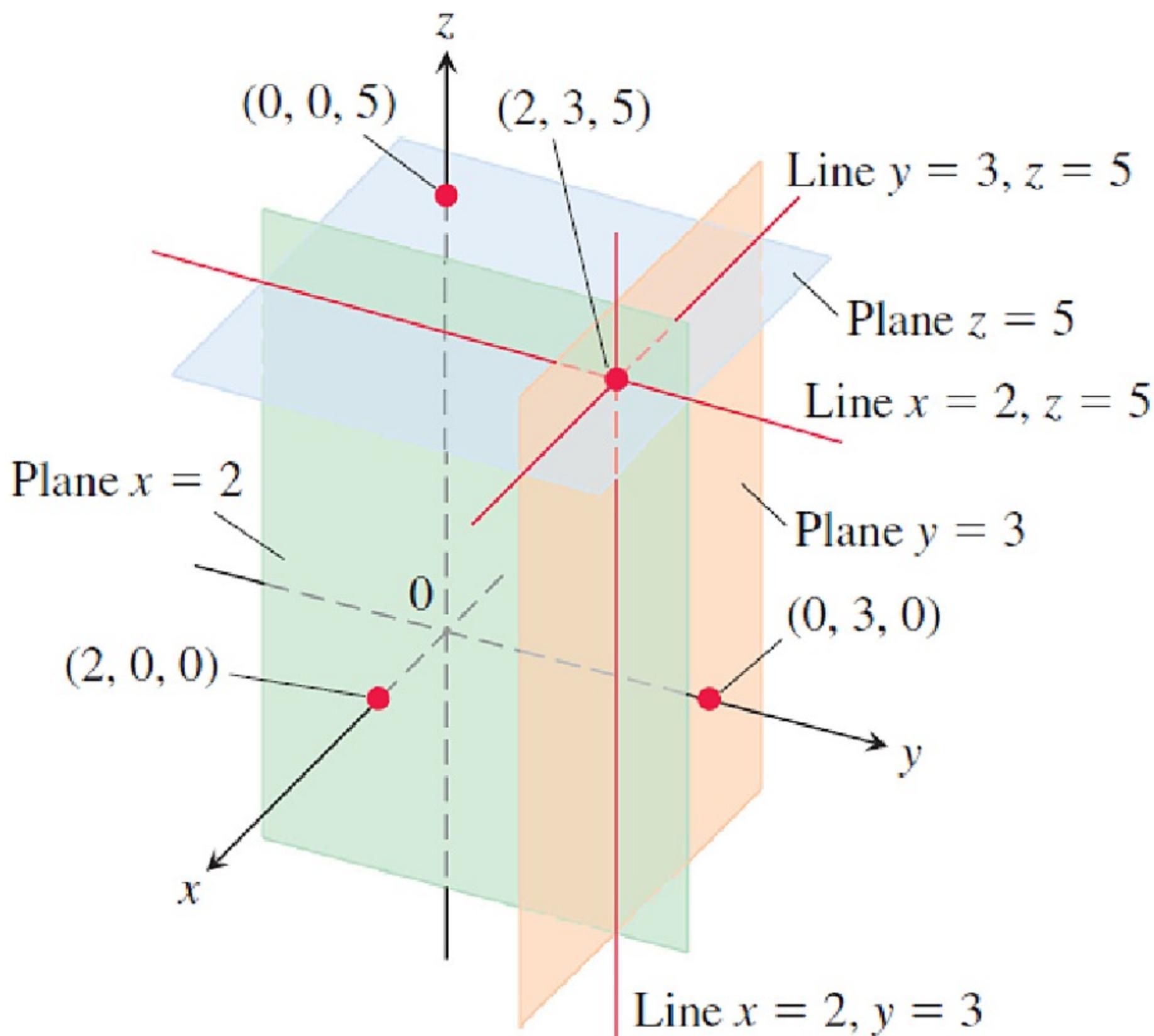


FIGURE 11.3 The planes $x = 2$, $y = 3$, and $z = 5$ determine three lines through the point $(2, 3, 5)$.

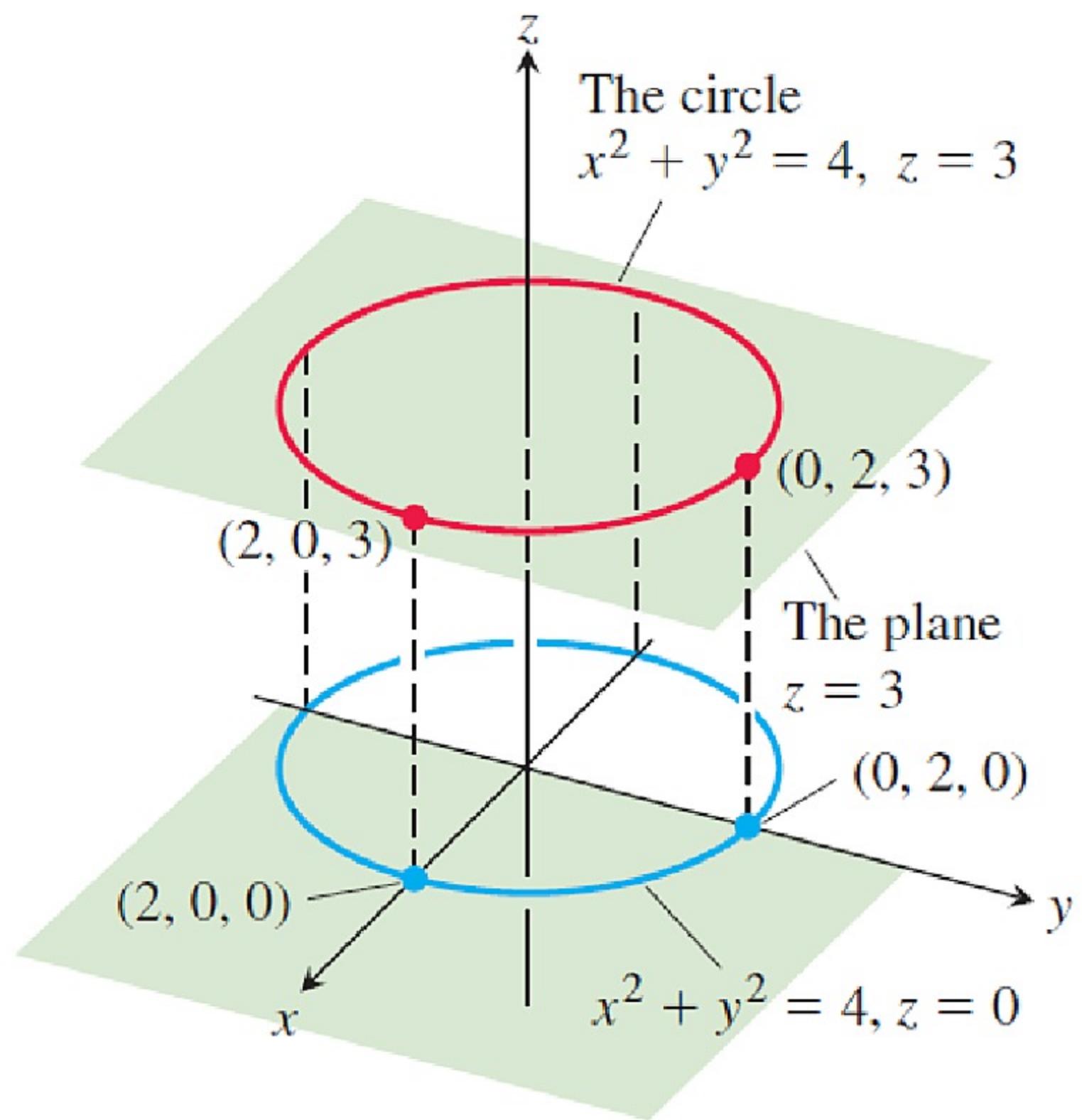


FIGURE 11.4 The circle $x^2 + y^2 = 4$ in the plane $z = 3$ (Example 2).

The Distance Between $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

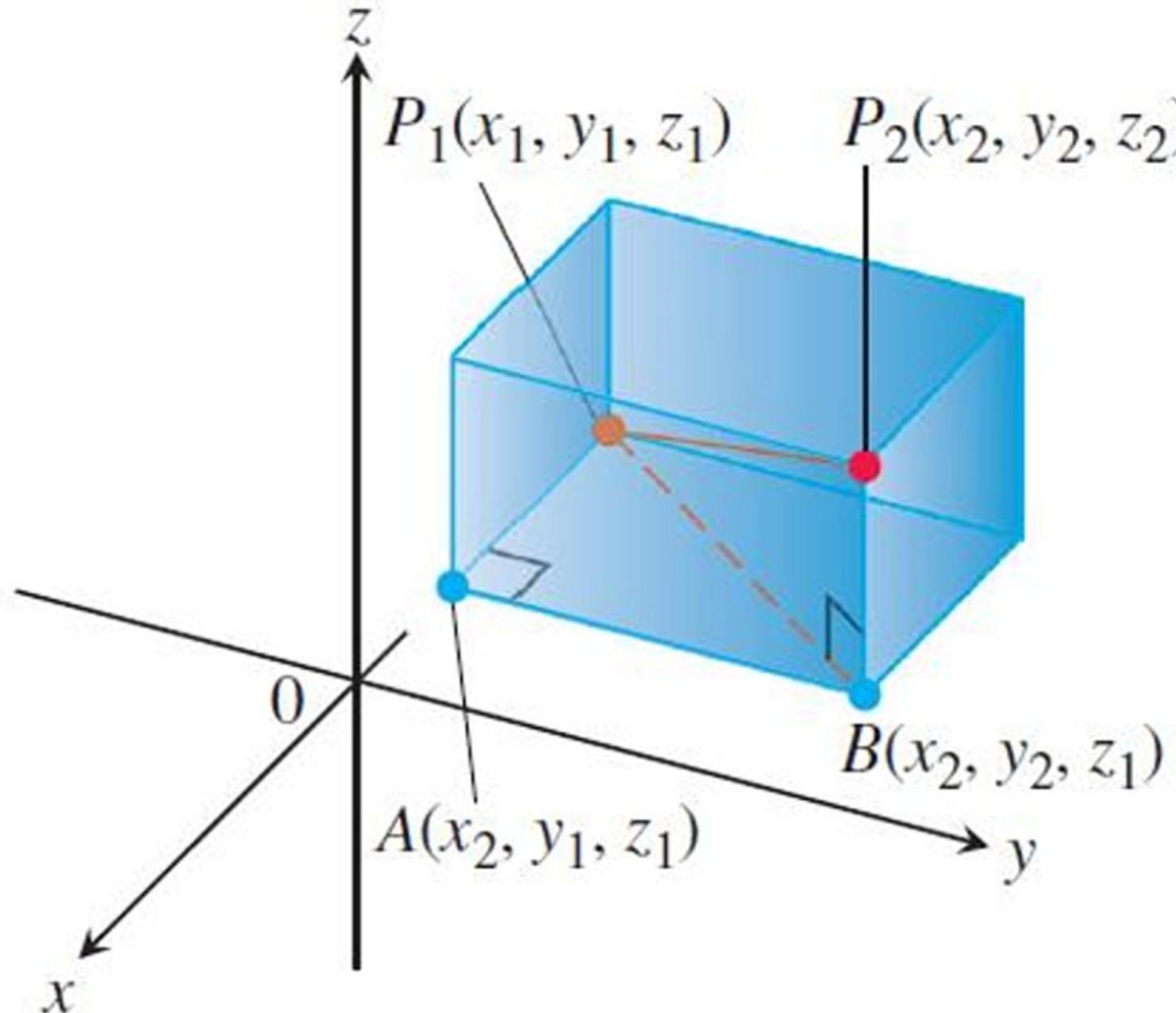


FIGURE 11.5 We find the distance between P_1 and P_2 by applying the Pythagorean theorem to the right triangles P_1AB and P_1BP_2 .

EXAMPLE 3

The distance between $P_1(2, 1, 5)$ and $P_2(-2, 3, 0)$ is

$$\begin{aligned}|P_1P_2| &= \sqrt{(-2 - 2)^2 + (3 - 1)^2 + (0 - 5)^2} \\&= \sqrt{16 + 4 + 25} \\&= \sqrt{45} \approx 6.708.\end{aligned}$$

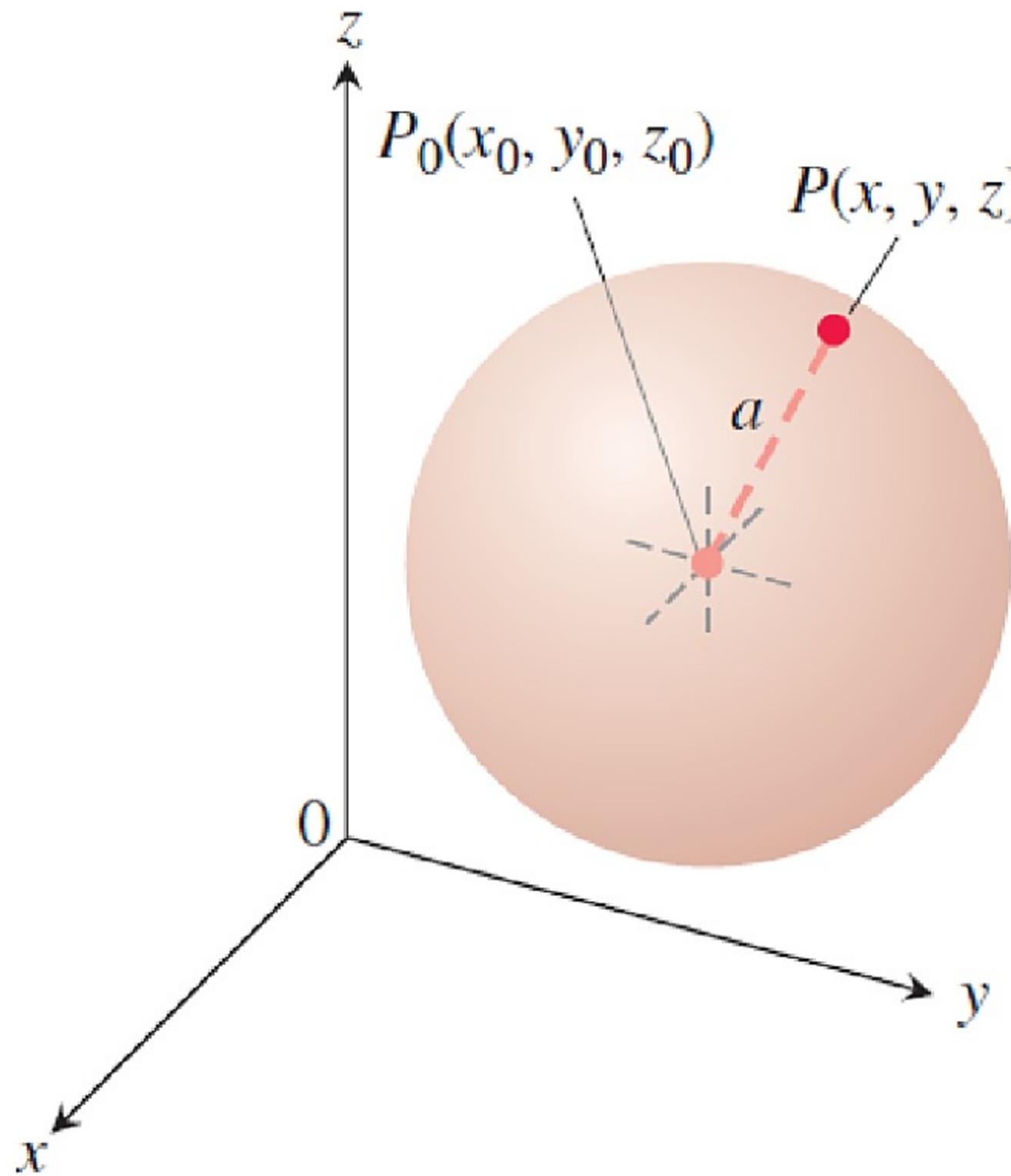


FIGURE 11.6 The sphere of radius a centered at the point (x_0, y_0, z_0) .

The Standard Equation for the Sphere of Radius a and Center (x_0, y_0, z_0)

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2$$

EXAMPLE 4 Find the center and radius of the sphere

$$x^2 + y^2 + z^2 + 3x - 4z + 1 = 0.$$

Solution We find the center and radius of a sphere the way we find the center and radius of a circle: Complete the squares on the x -, y -, and z -terms as necessary and write each quadratic as a squared linear expression. Then, from the equation in standard form, read off the center and radius. For the sphere here, we have

$$\begin{aligned}x^2 + y^2 + z^2 + 3x - 4z + 1 &= 0 \\(x^2 + 3x) + y^2 + (z^2 - 4z) &= -1 \\ \left(x^2 + 3x + \left(\frac{3}{2}\right)^2\right) + y^2 + \left(z^2 - 4z + \left(\frac{-4}{2}\right)^2\right) &= -1 + \left(\frac{3}{2}\right)^2 + \left(\frac{-4}{2}\right)^2 \\ \left(x + \frac{3}{2}\right)^2 + y^2 + (z - 2)^2 &= -1 + \frac{9}{4} + 4 = \frac{21}{4}.\end{aligned}$$

From this standard form, we read that $x_0 = -3/2$, $y_0 = 0$, $z_0 = 2$, and $a = \sqrt{21}/2$. The center is $(-3/2, 0, 2)$. The radius is $\sqrt{21}/2$.

Section 11.2

Vectors

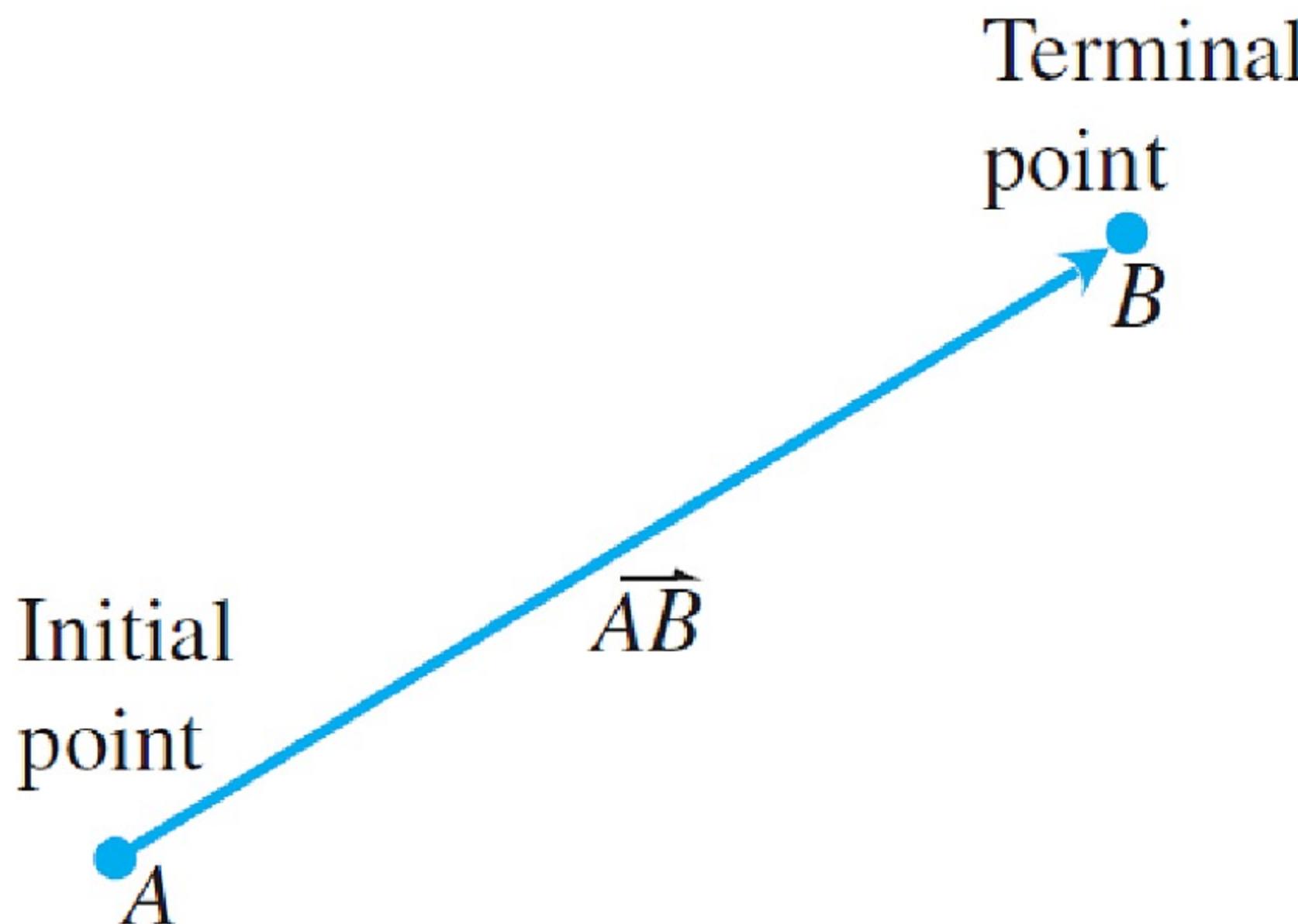
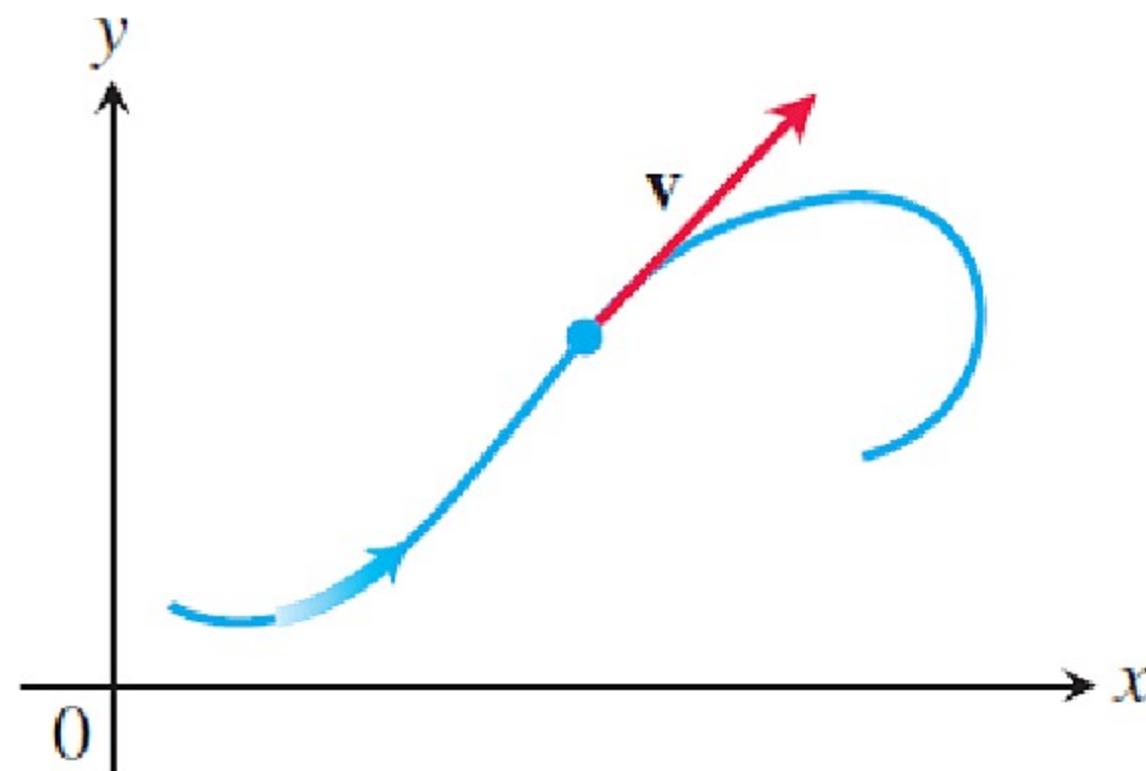
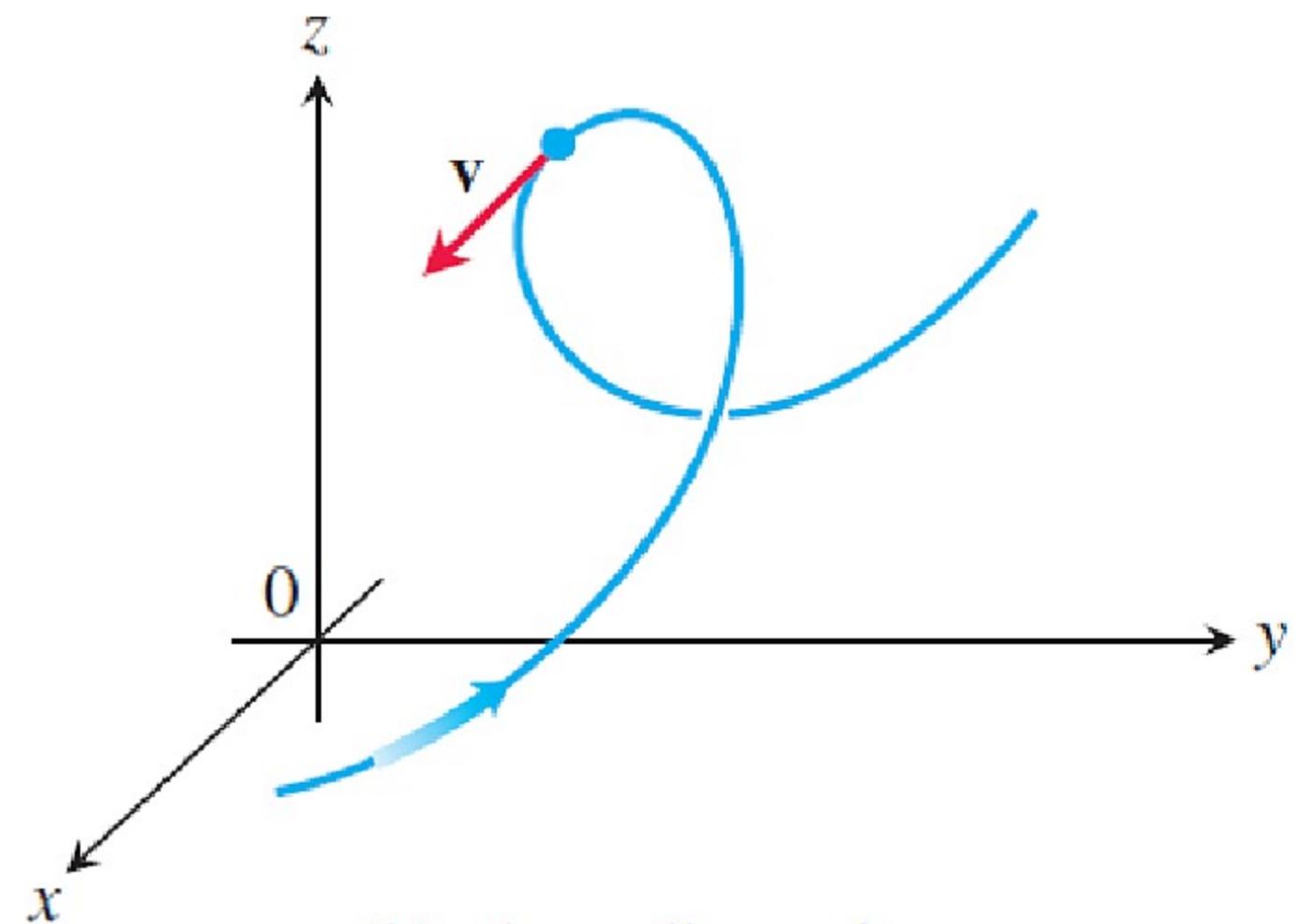


FIGURE 11.7 The directed line segment \vec{AB} is called a vector.



(a) two dimensions



(b) three dimensions

FIGURE 11.8 The velocity vector of a particle moving along a path (a) in the plane (b) in space. The arrowhead on the path indicates the direction of motion of the particle.

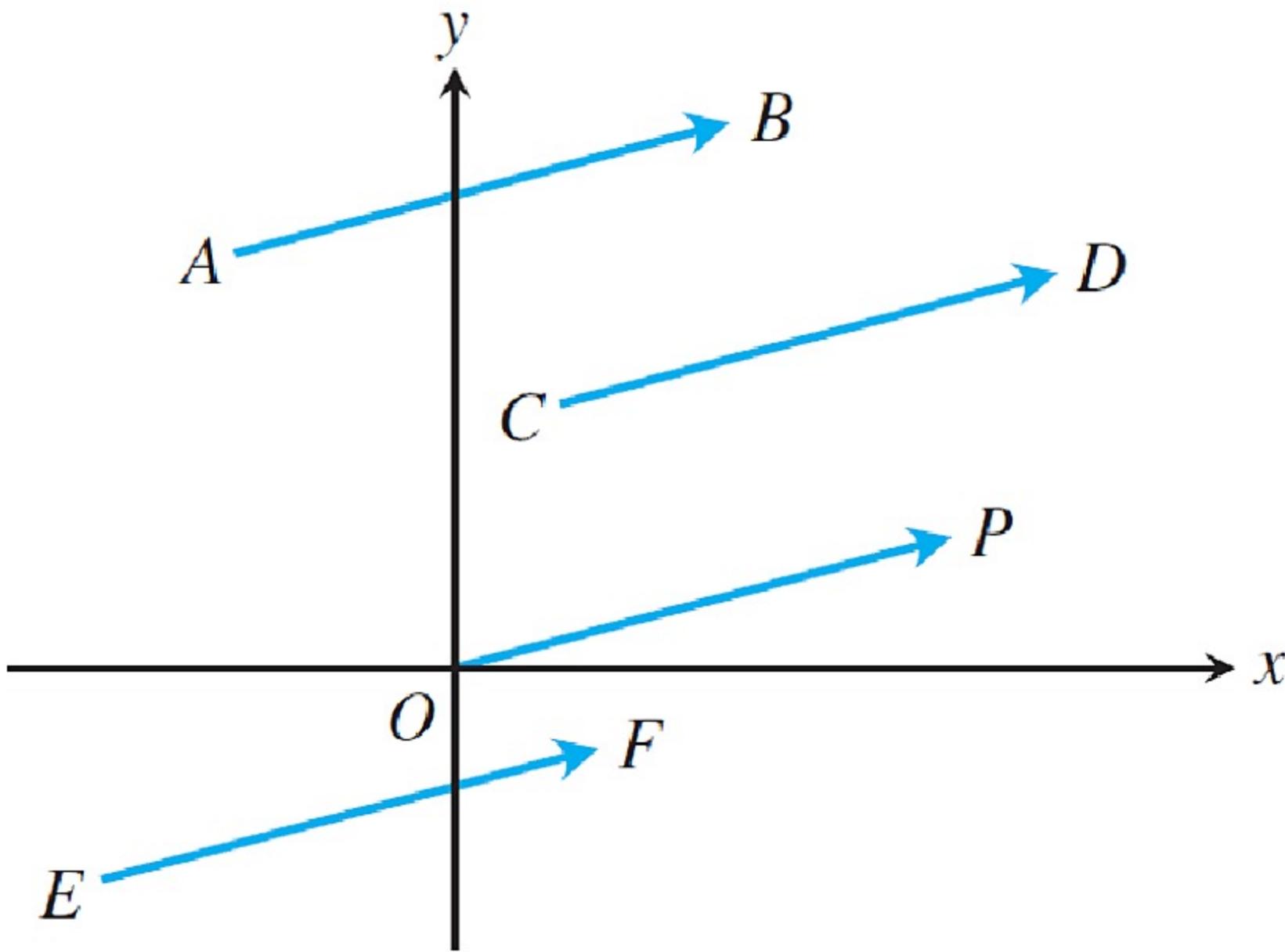


FIGURE 11.9 The four arrows in the plane (directed line segments) shown here have the same length and direction. They therefore represent the same vector, and we write $\overrightarrow{AB} = \overrightarrow{CD} = \overrightarrow{OP} = \overrightarrow{EF}$.

DEFINITIONS

The vector represented by the directed line segment \overrightarrow{AB} has **initial point A** and **terminal point B** and its **length** is denoted by $|\overrightarrow{AB}|$. Two vectors are **equal** if they have the same length and direction.

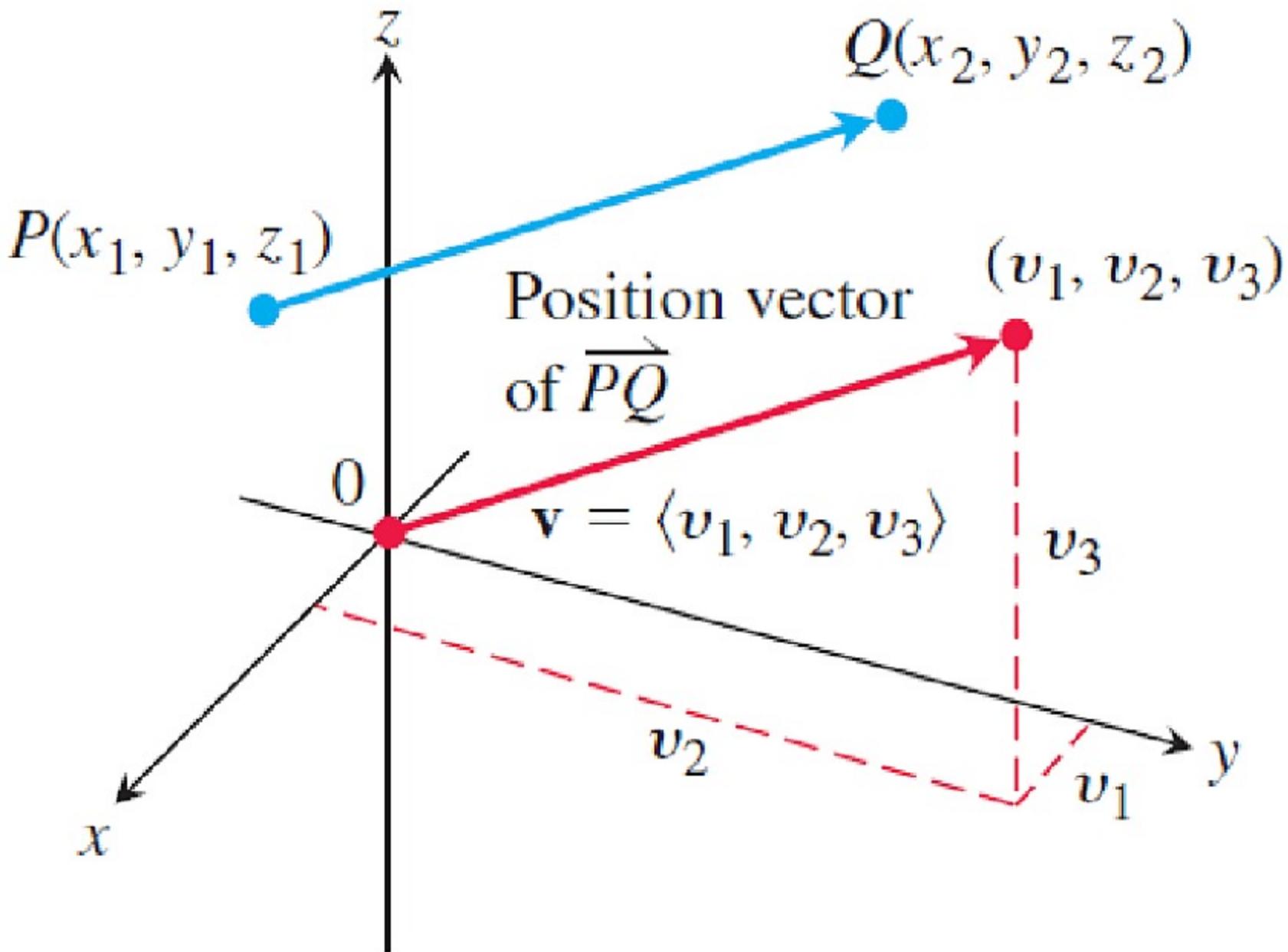


FIGURE 11.10 A vector \vec{PQ} in standard position has its initial point at the origin. The directed line segments \vec{PQ} and \mathbf{v} are parallel and have the same length.

DEFINITION

If \mathbf{v} is a **two-dimensional** vector in the plane equal to the vector with initial point at the origin and terminal point (v_1, v_2) , then the **component form** of \mathbf{v} is

$$\mathbf{v} = \langle v_1, v_2 \rangle.$$

If \mathbf{v} is a **three-dimensional** vector equal to the vector with initial point at the origin and terminal point (v_1, v_2, v_3) , then the **component form** of \mathbf{v} is

$$\mathbf{v} = \langle v_1, v_2, v_3 \rangle.$$

The **magnitude** or **length** of the vector $\mathbf{v} = \overrightarrow{PQ}$ is the nonnegative number

$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

(See Figure 12.10.)

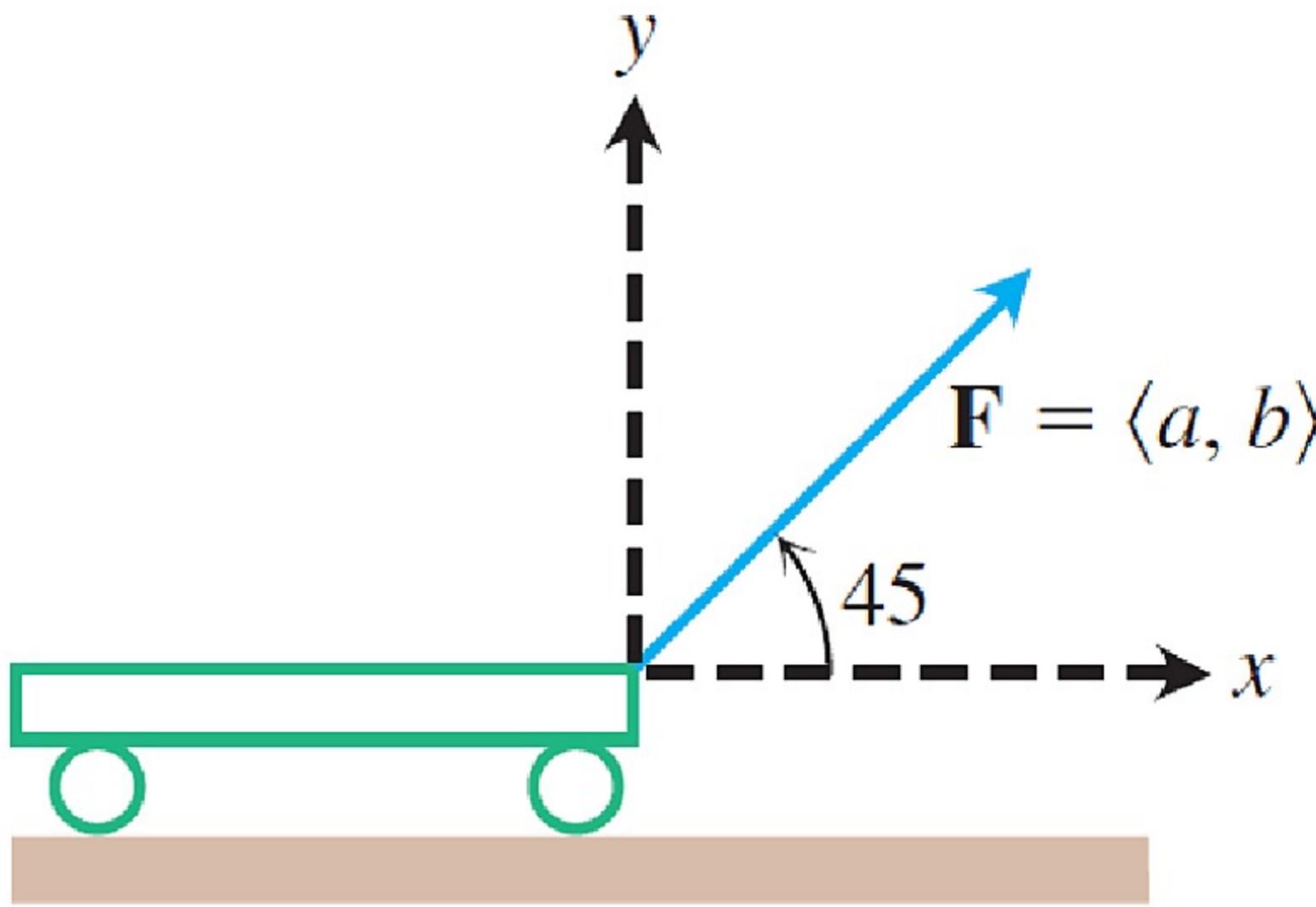


FIGURE 11.11 The force pulling the cart forward is represented by the vector \mathbf{F} whose horizontal component is the effective force (Example 2).

DEFINITIONS

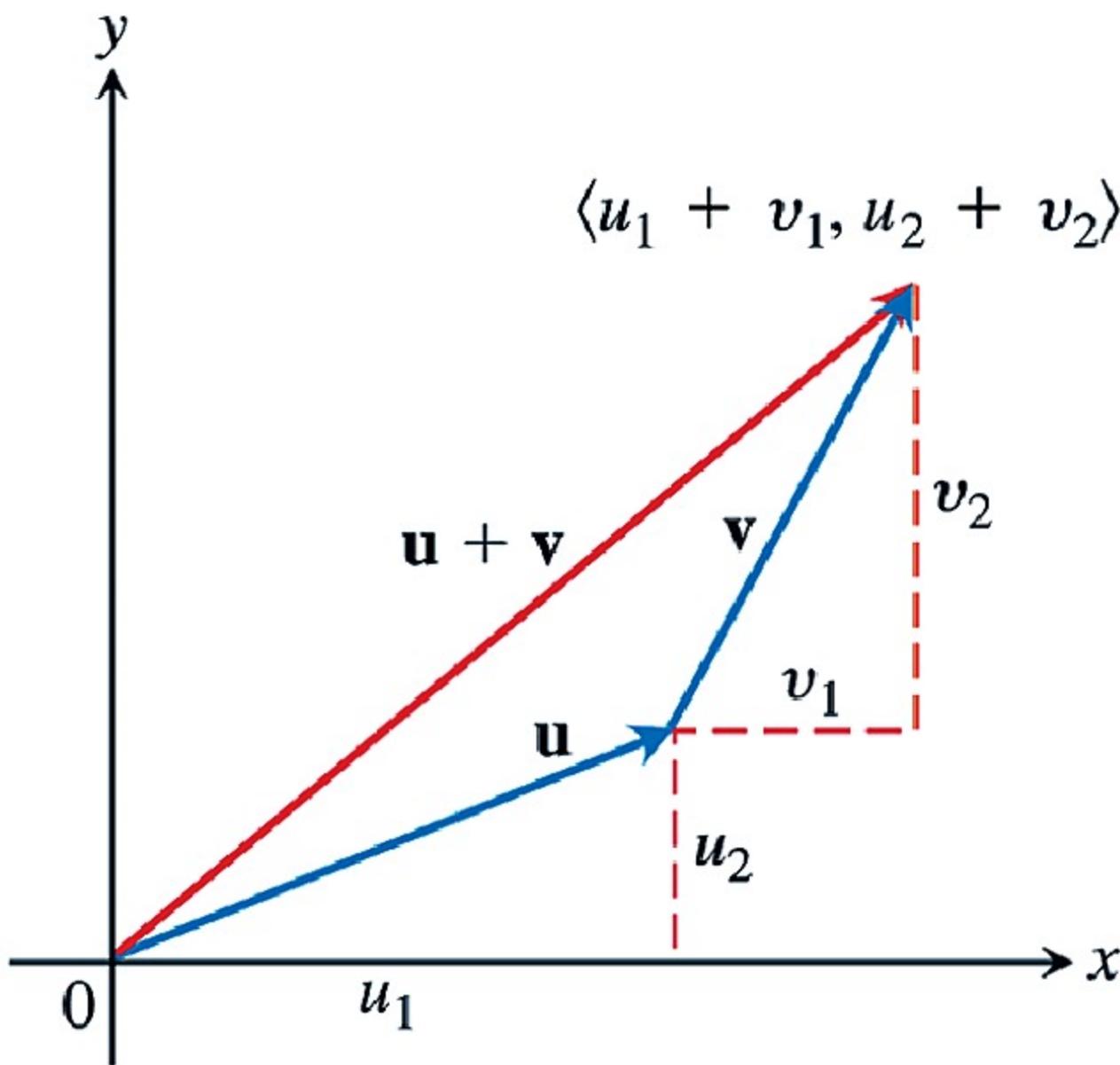
Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ be vectors with k a scalar.

Addition:

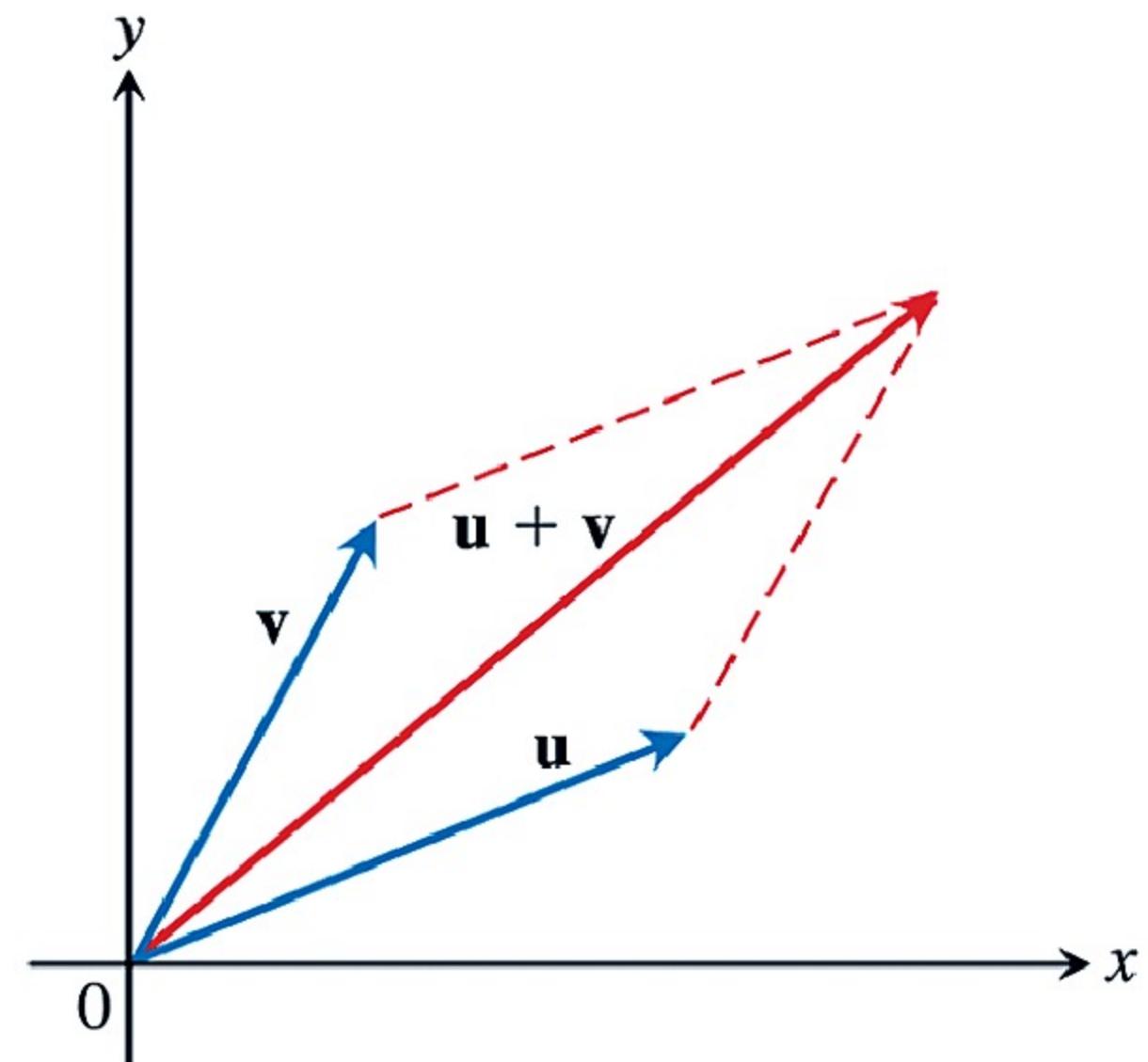
$$\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$$

Scalar multiplication:

$$k\mathbf{u} = \langle ku_1, ku_2, ku_3 \rangle$$



(a)



(b)

FIGURE 11.12 (a) Geometric interpretation of the vector sum. (b) The parallelogram law of vector addition in which both vectors are in standard position.

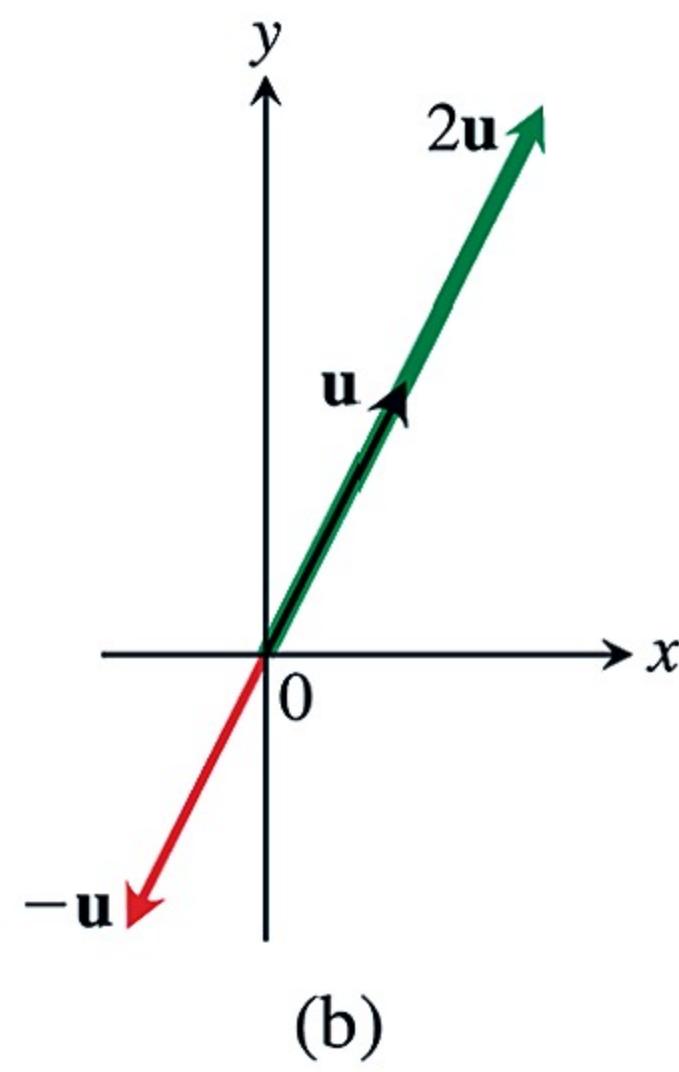
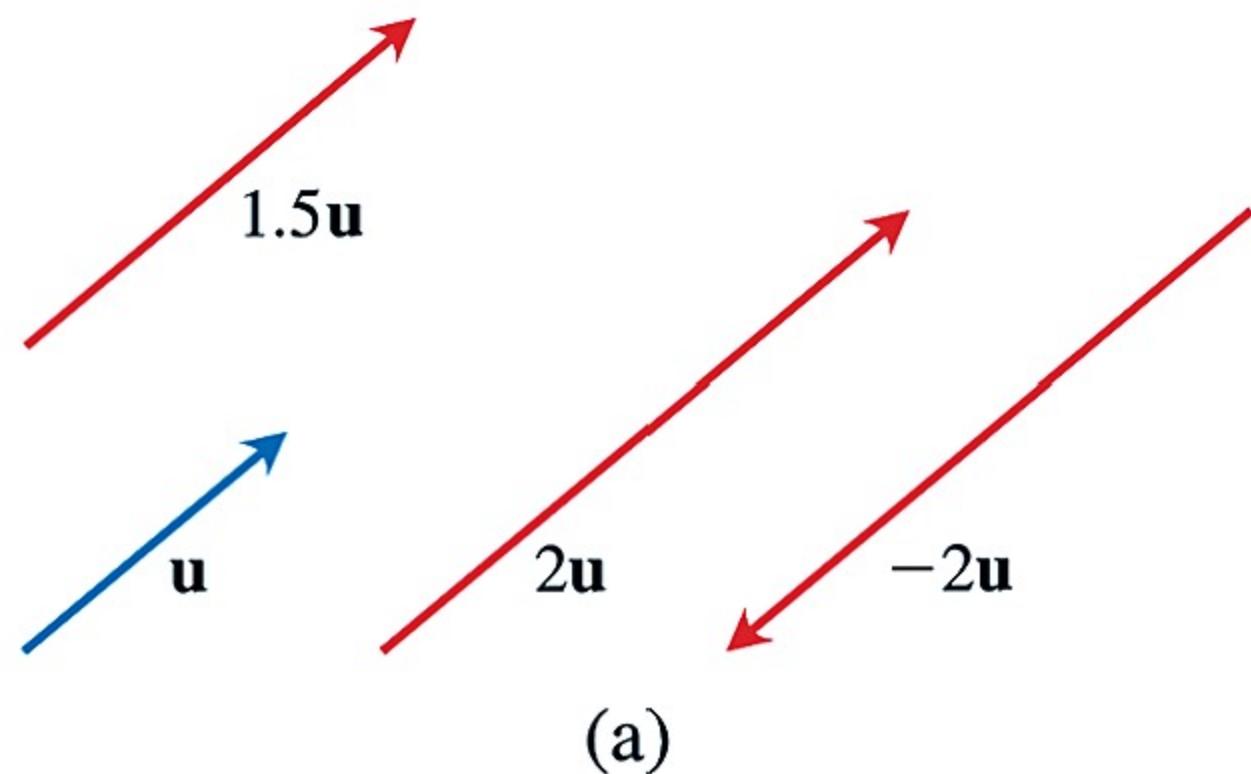
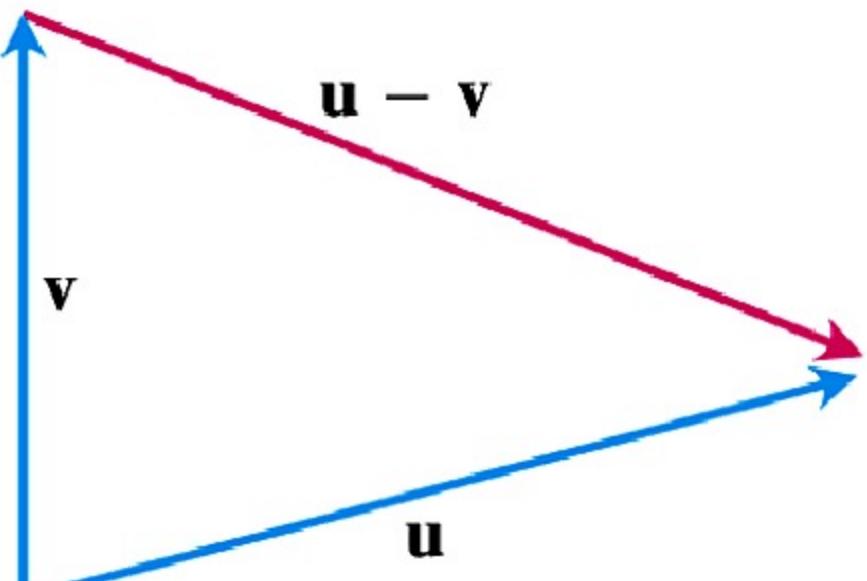
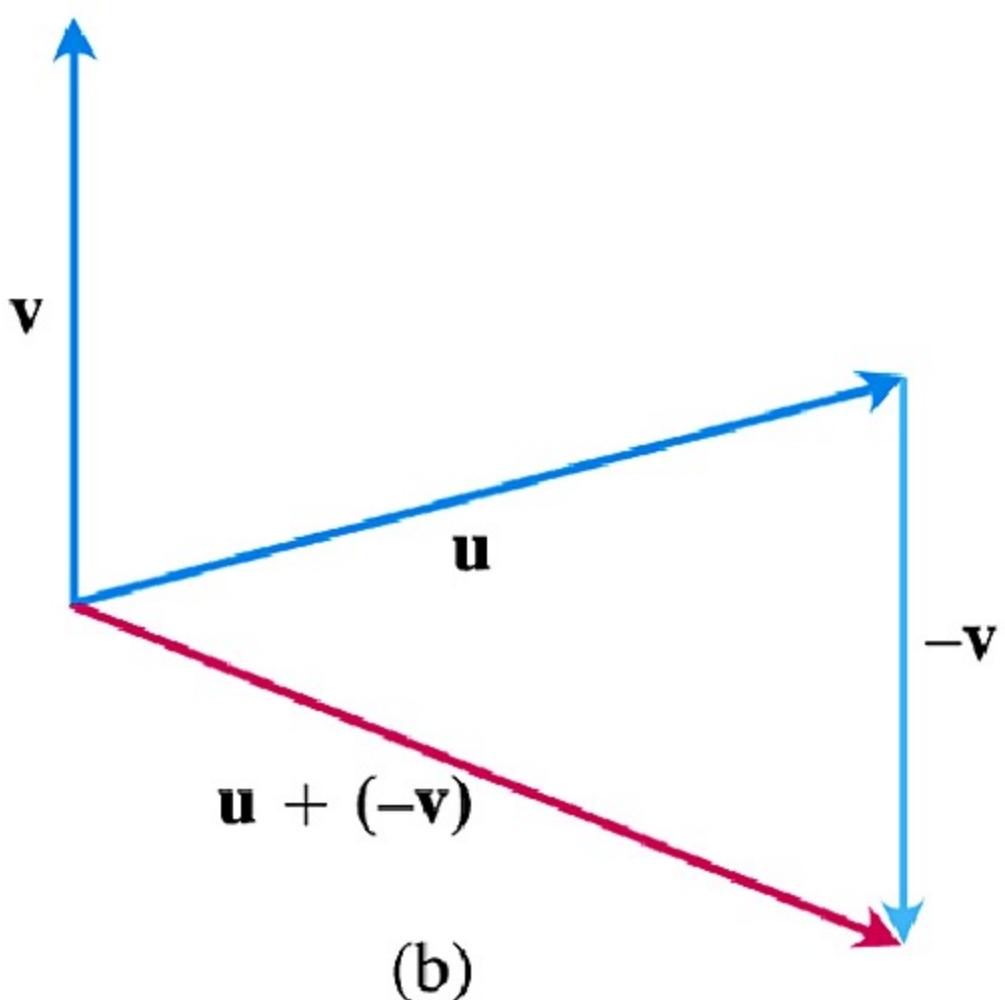


FIGURE 11.13 (a) Scalar multiples of \mathbf{u} . (b) Scalar multiples of a vector \mathbf{u} in standard position.



(a)



(b)

FIGURE 11.14 (a) The vector $u - v$, when added to v , gives u .
(b) $u - v = u + (-v)$.

EXAMPLE 3 Let $\mathbf{u} = \langle -1, 3, 1 \rangle$ and $\mathbf{v} = \langle 4, 7, 0 \rangle$. Find the components of

- (a) $2\mathbf{u} + 3\mathbf{v}$ (b) $\mathbf{u} - \mathbf{v}$ (c) $\left| \frac{1}{2}\mathbf{u} \right|$.

Solution

(a) $2\mathbf{u} + 3\mathbf{v} = 2\langle -1, 3, 1 \rangle + 3\langle 4, 7, 0 \rangle = \langle -2, 6, 2 \rangle + \langle 12, 21, 0 \rangle = \langle 10, 27, 2 \rangle$

(b) $\mathbf{u} - \mathbf{v} = \langle -1, 3, 1 \rangle - \langle 4, 7, 0 \rangle = \langle -1 - 4, 3 - 7, 1 - 0 \rangle = \langle -5, -4, 1 \rangle$

(c) $\left| \frac{1}{2}\mathbf{u} \right| = \left| \left\langle -\frac{1}{2}, \frac{3}{2}, \frac{1}{2} \right\rangle \right| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{1}{2}\sqrt{11}.$

Properties of Vector Operations

Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be vectors and a, b be scalars.

- | | |
|--|--|
| 1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ | 2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ |
| 3. $\mathbf{u} + \mathbf{0} = \mathbf{u}$ | 4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ |
| 5. $0\mathbf{u} = \mathbf{0}$ | 6. $1\mathbf{u} = \mathbf{u}$ |
| 7. $a(b\mathbf{u}) = (ab)\mathbf{u}$ | 8. $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$ |
| 9. $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$ | |

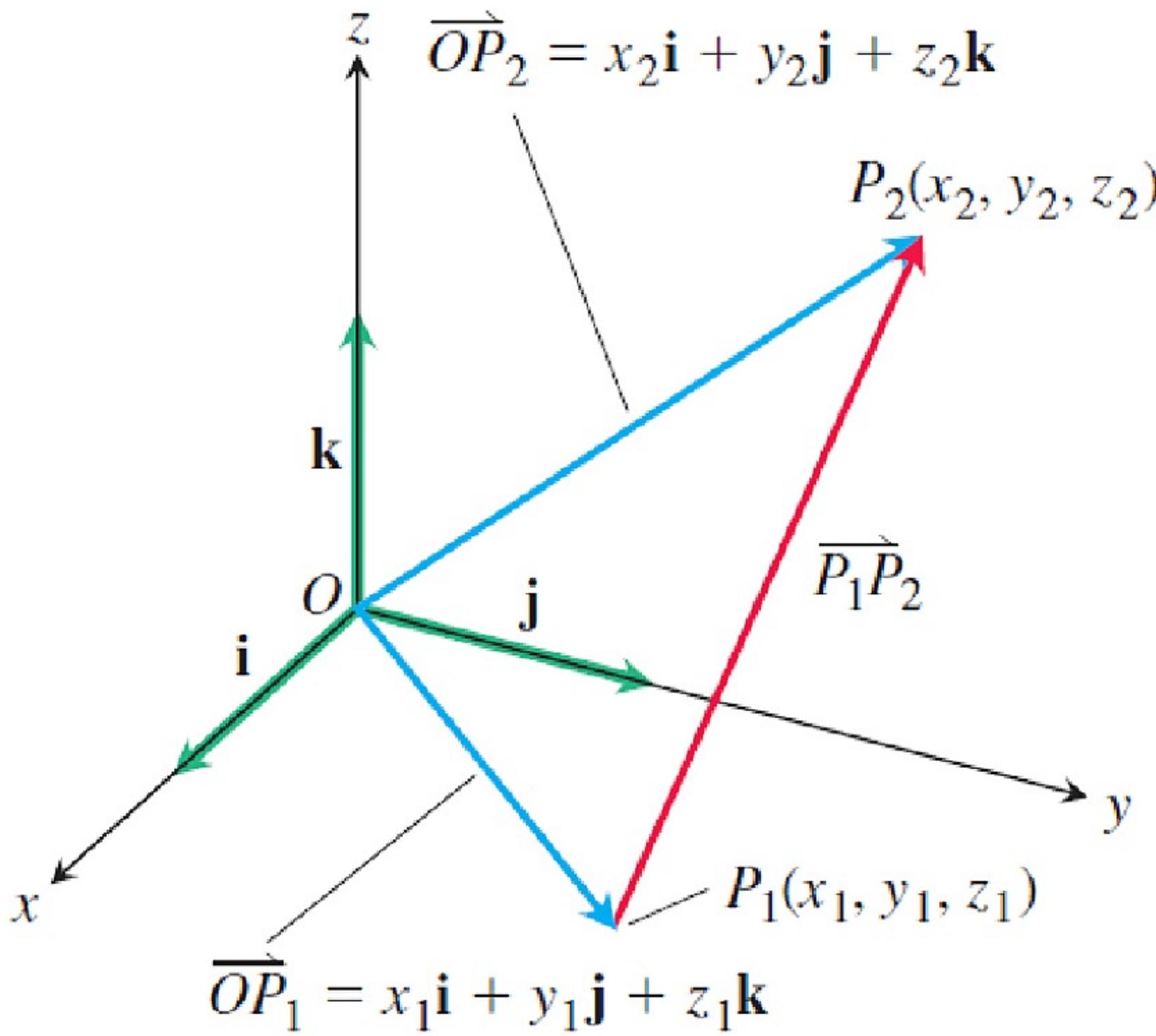


FIGURE 11.15 The vector from P_1 to P_2 is $\overrightarrow{P_1P_2} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}$.

EXAMPLE 5 If $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$ is a velocity vector, express \mathbf{v} as a product of its speed times its direction of motion.

Solution Speed is the magnitude (length) of \mathbf{v} :

$$|\mathbf{v}| = \sqrt{(3)^2 + (-4)^2} = \sqrt{9 + 16} = 5.$$

The unit vector $\mathbf{v}/|\mathbf{v}|$ is the direction of \mathbf{v} :

$$\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{3\mathbf{i} - 4\mathbf{j}}{5} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}.$$

So

$$\mathbf{v} = 3\mathbf{i} - 4\mathbf{j} = 5 \left(\underbrace{\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}}_{\text{Length } \text{Direction of motion } (\text{speed})} \right).$$

If $\mathbf{v} \neq \mathbf{0}$, then

1. $\frac{\mathbf{v}}{|\mathbf{v}|}$ is a unit vector in the direction of \mathbf{v} ;
2. the equation $\mathbf{v} = |\mathbf{v}| \frac{\mathbf{v}}{|\mathbf{v}|}$ expresses \mathbf{v} as its length times its direction.

The **midpoint** M of the line segment joining points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is the point

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right).$$

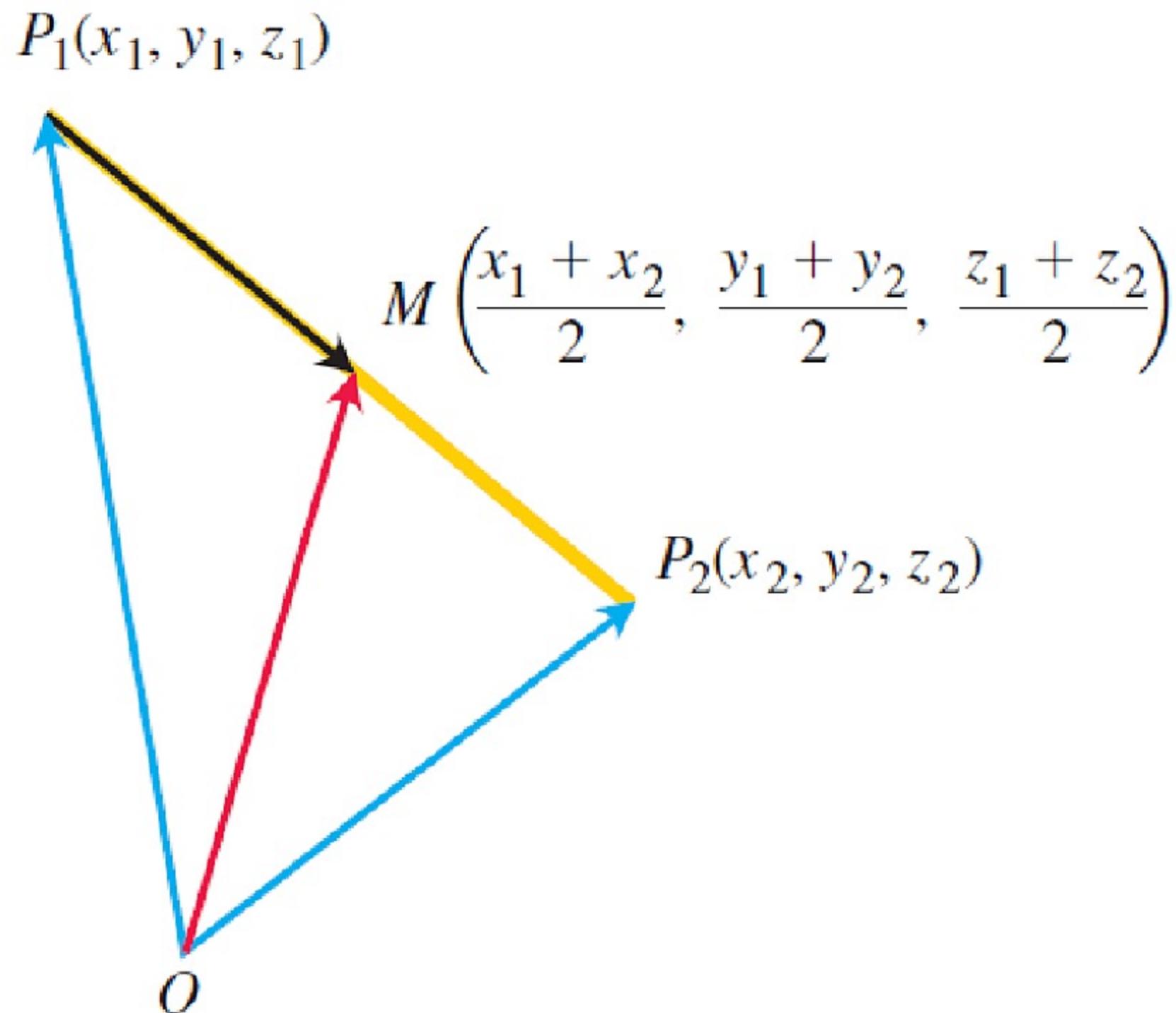
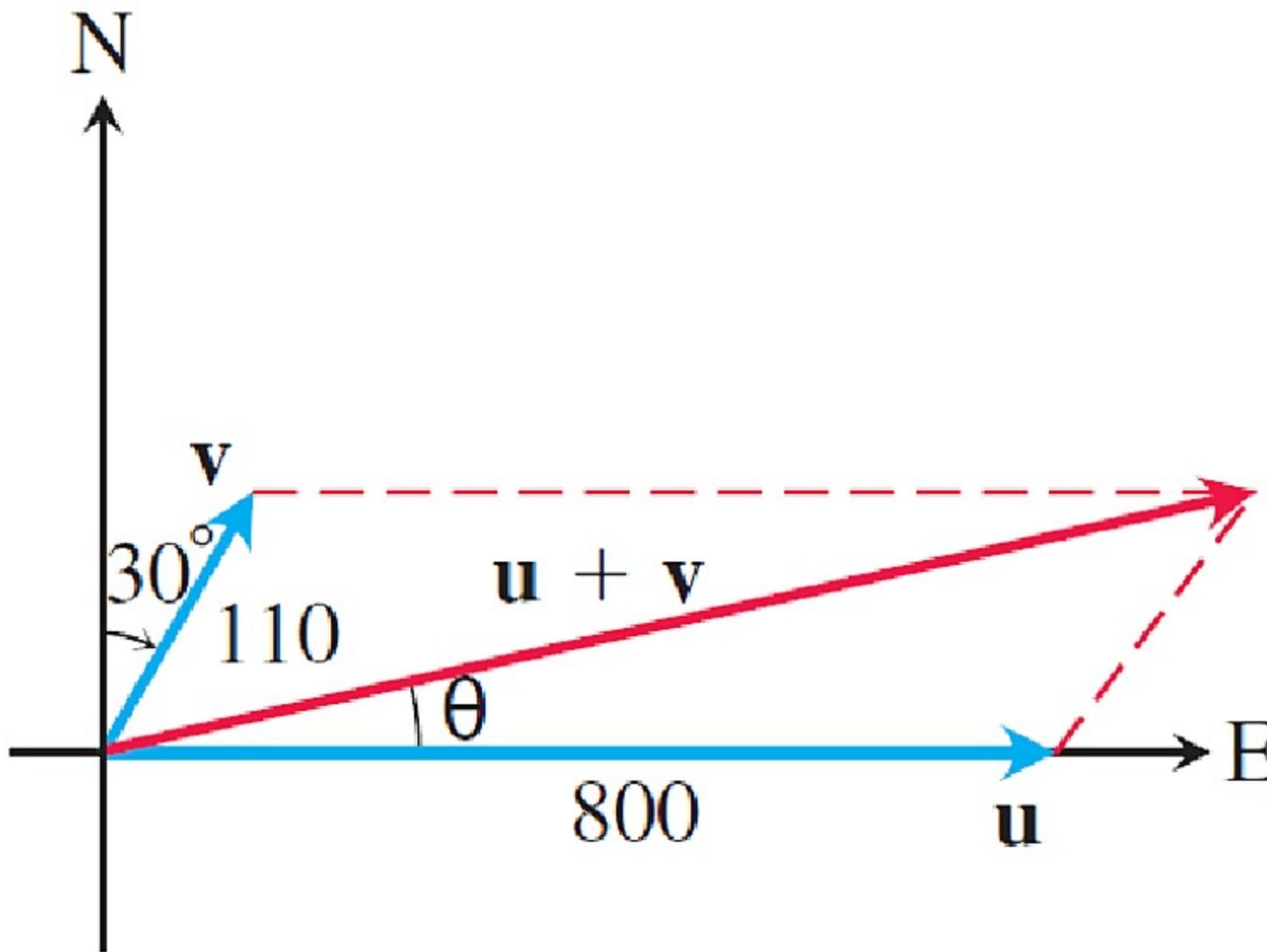
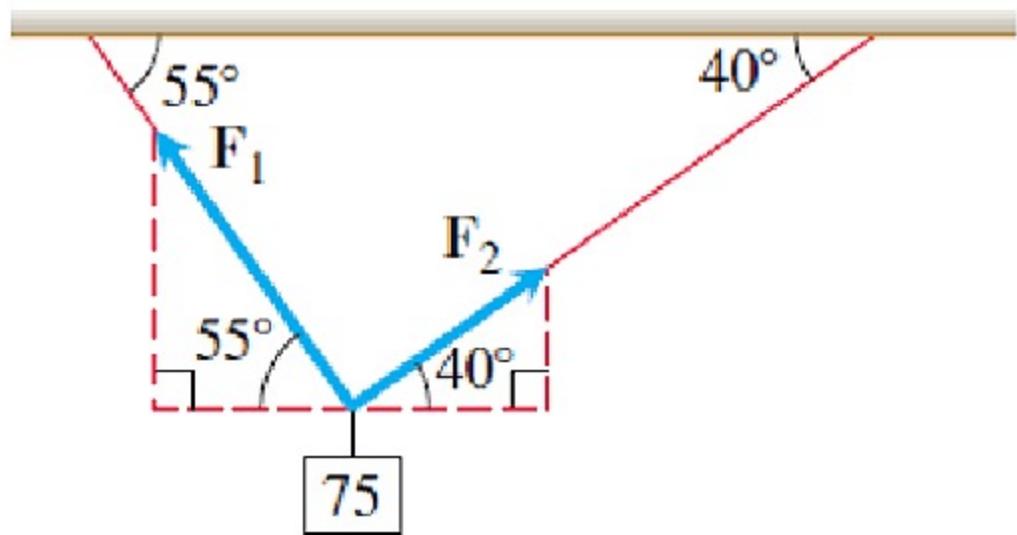


FIGURE 11.16 The coordinates of the midpoint are the averages of the coordinates of P_1 and P_2 .

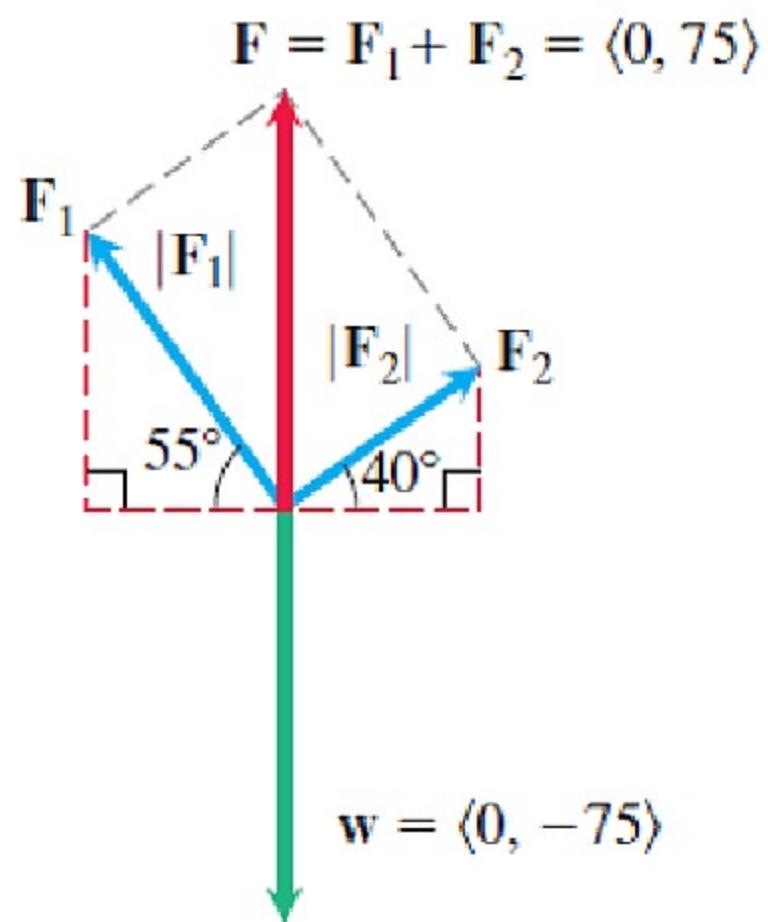


NOT TO SCALE

FIGURE 11.17 Vectors representing the velocities of the airplane \mathbf{u} and tailwind \mathbf{v} in Example 8.



(a)

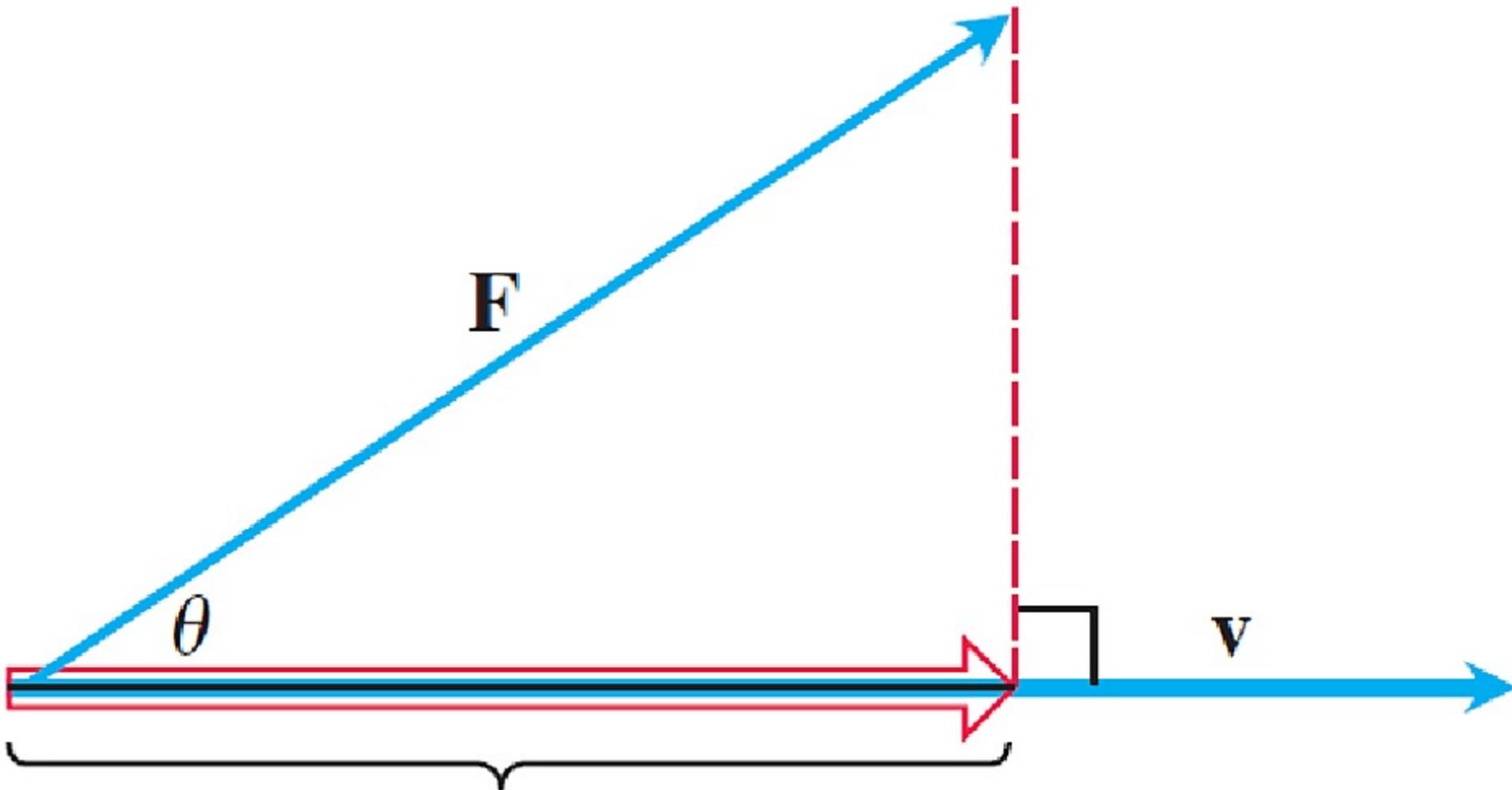


(b)

FIGURE 11.18 The suspended weight in Example 9.

Section 11.3

The Dot Product



$$\text{Length} = |\mathbf{F}| \cos \theta$$

FIGURE 11.19 The magnitude of the force \mathbf{F} in the direction of vector \mathbf{v} is the length $|\mathbf{F}| \cos \theta$ of the projection of \mathbf{F} onto \mathbf{v} .

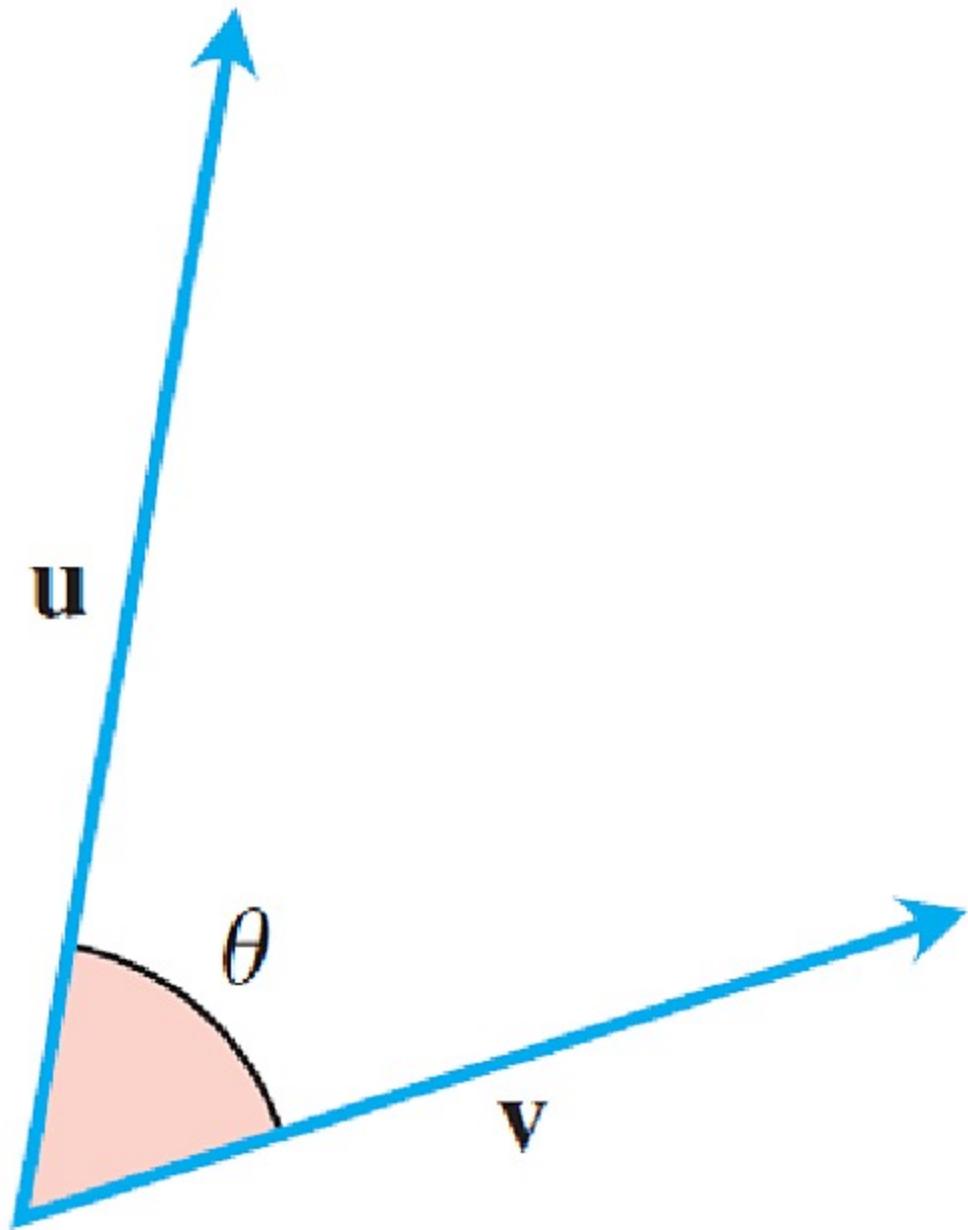


FIGURE 11.20 The angle between \mathbf{u} and \mathbf{v} given by Theorem 1 lies in the interval $[0, \pi]$.

THEOREM 1—Angle Between Two Vectors The angle θ between two nonzero vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is given by

$$\theta = \cos^{-1} \left(\frac{u_1 v_1 + u_2 v_2 + u_3 v_3}{|\mathbf{u}| |\mathbf{v}|} \right).$$

DEFINITION The **dot product** $\mathbf{u} \cdot \mathbf{v}$ (“ \mathbf{u} dot \mathbf{v} ”) of vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is the scalar

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

EXAMPLE 1

$$\begin{aligned}\text{(a)} \quad \langle 1, -2, -1 \rangle \cdot \langle -6, 2, -3 \rangle &= (1)(-6) + (-2)(2) + (-1)(-3) \\ &= -6 - 4 + 3 = -7\end{aligned}$$

$$\text{(b)} \quad \left(\frac{1}{2} \mathbf{i} + 3\mathbf{j} + \mathbf{k} \right) \cdot (4\mathbf{i} - \mathbf{j} + 2\mathbf{k}) = \left(\frac{1}{2} \right)(4) + (3)(-1) + (1)(2) = 1$$

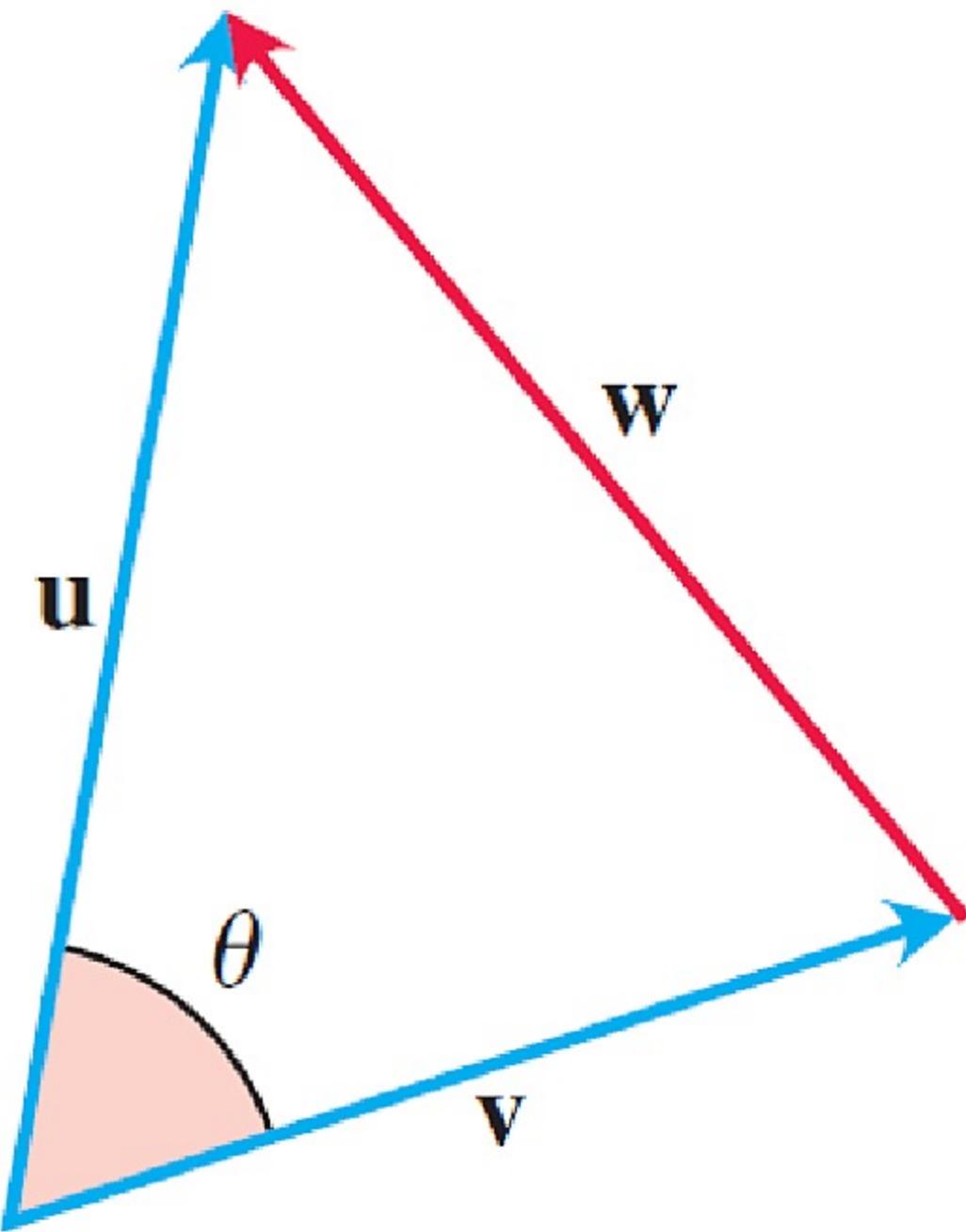


FIGURE 11.21 The parallelogram law of addition of vectors gives $\mathbf{w} = \mathbf{u} - \mathbf{v}$.

Dot Product and Angles

The angle between two nonzero vectors \mathbf{u} and \mathbf{v} is $\theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\right)$.

The dot product of two vectors \mathbf{u} and \mathbf{v} is given by $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos \theta$.

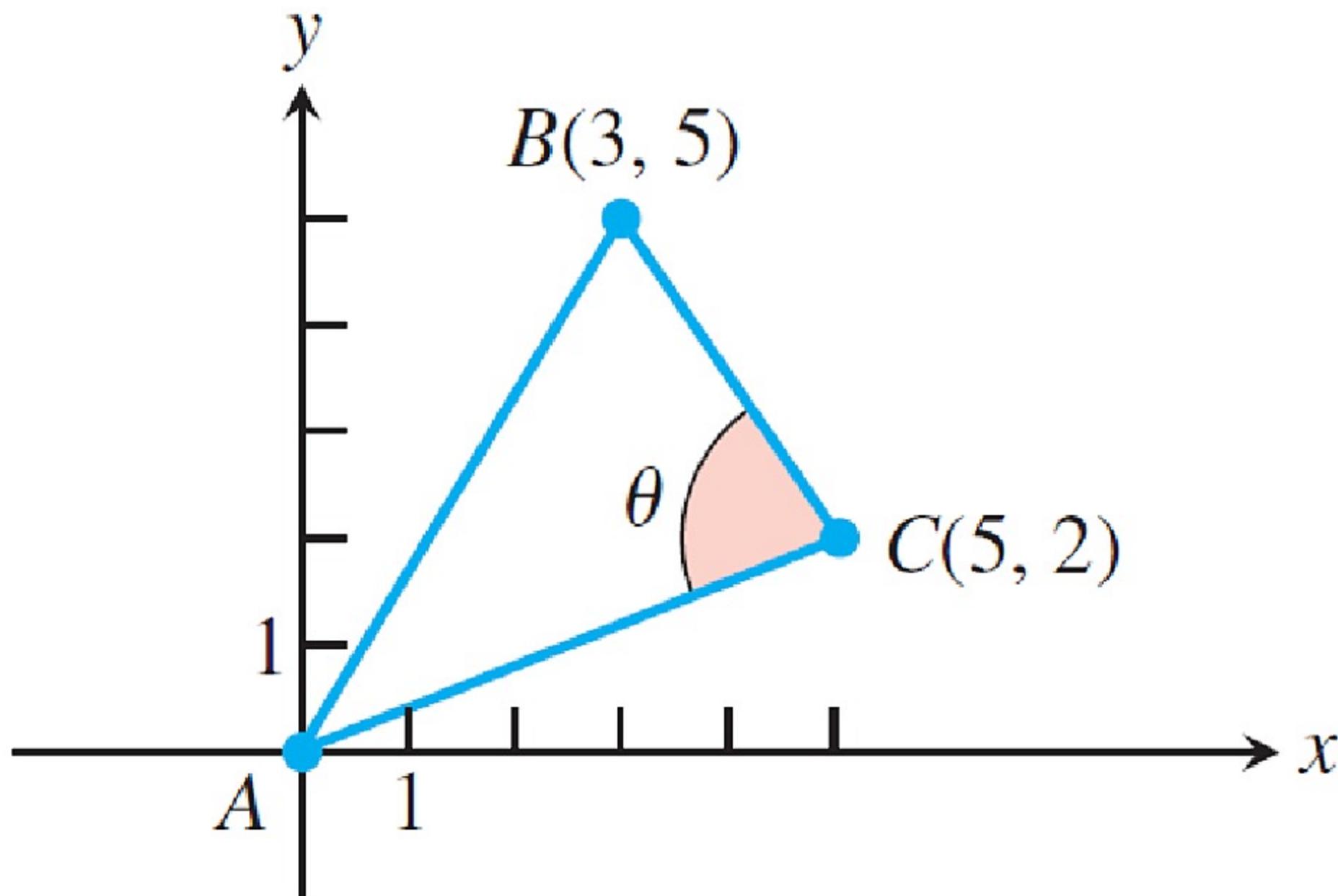


FIGURE 11.22 The triangle in Example 3.

DEFINITION Vectors \mathbf{u} and \mathbf{v} are orthogonal if $\mathbf{u} \cdot \mathbf{v} = 0$.

Properties of the Dot Product

If \mathbf{u} , \mathbf{v} , and \mathbf{w} are any vectors and c is a scalar, then

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $(c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v})$
3. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
4. $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$
5. $\mathbf{0} \cdot \mathbf{u} = 0.$

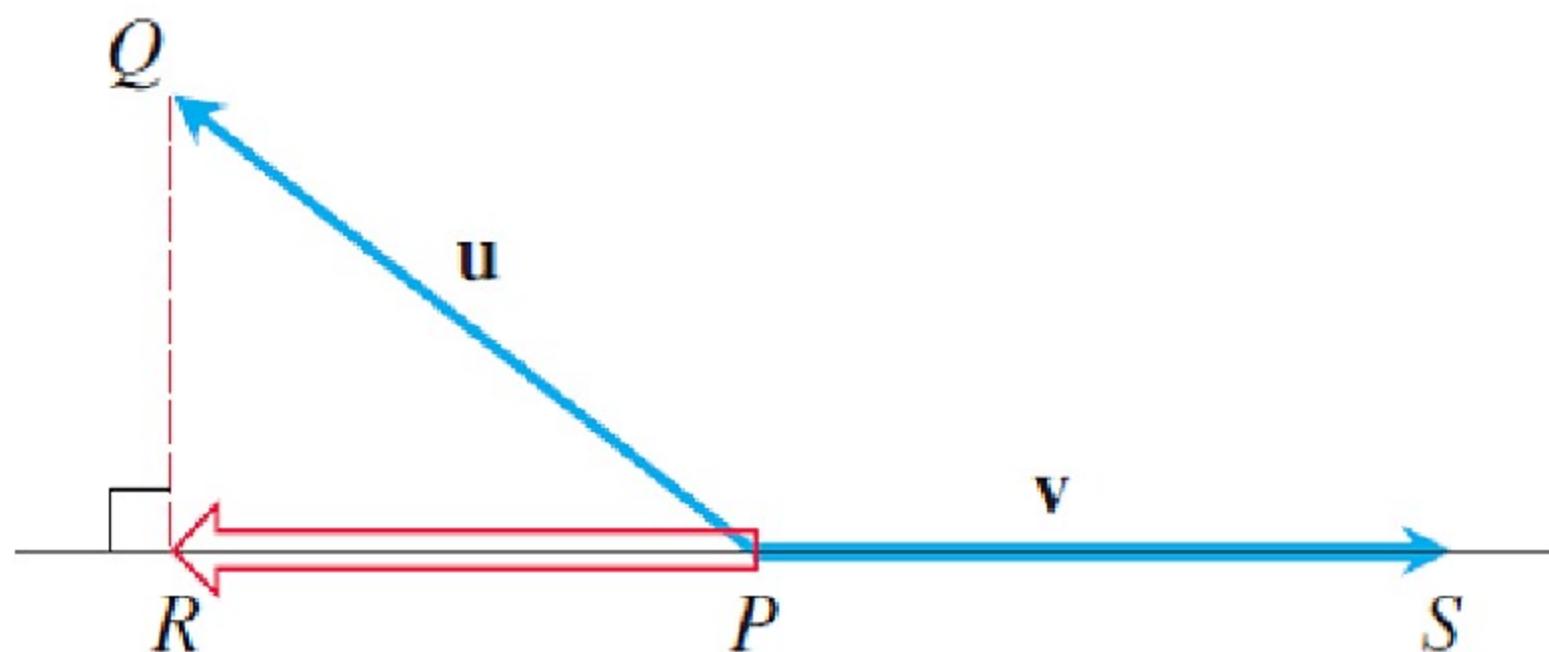
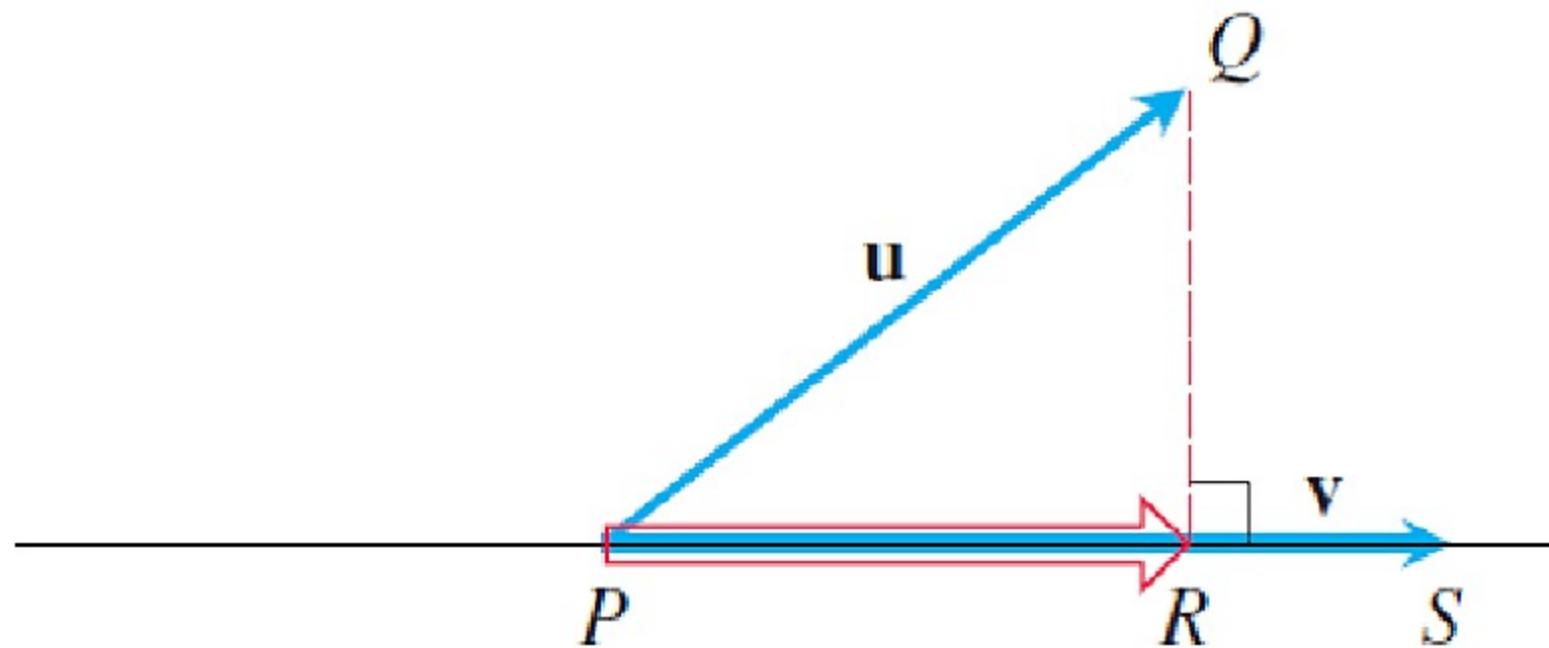


FIGURE 11.23 The vector projection of \mathbf{u} onto \mathbf{v} .

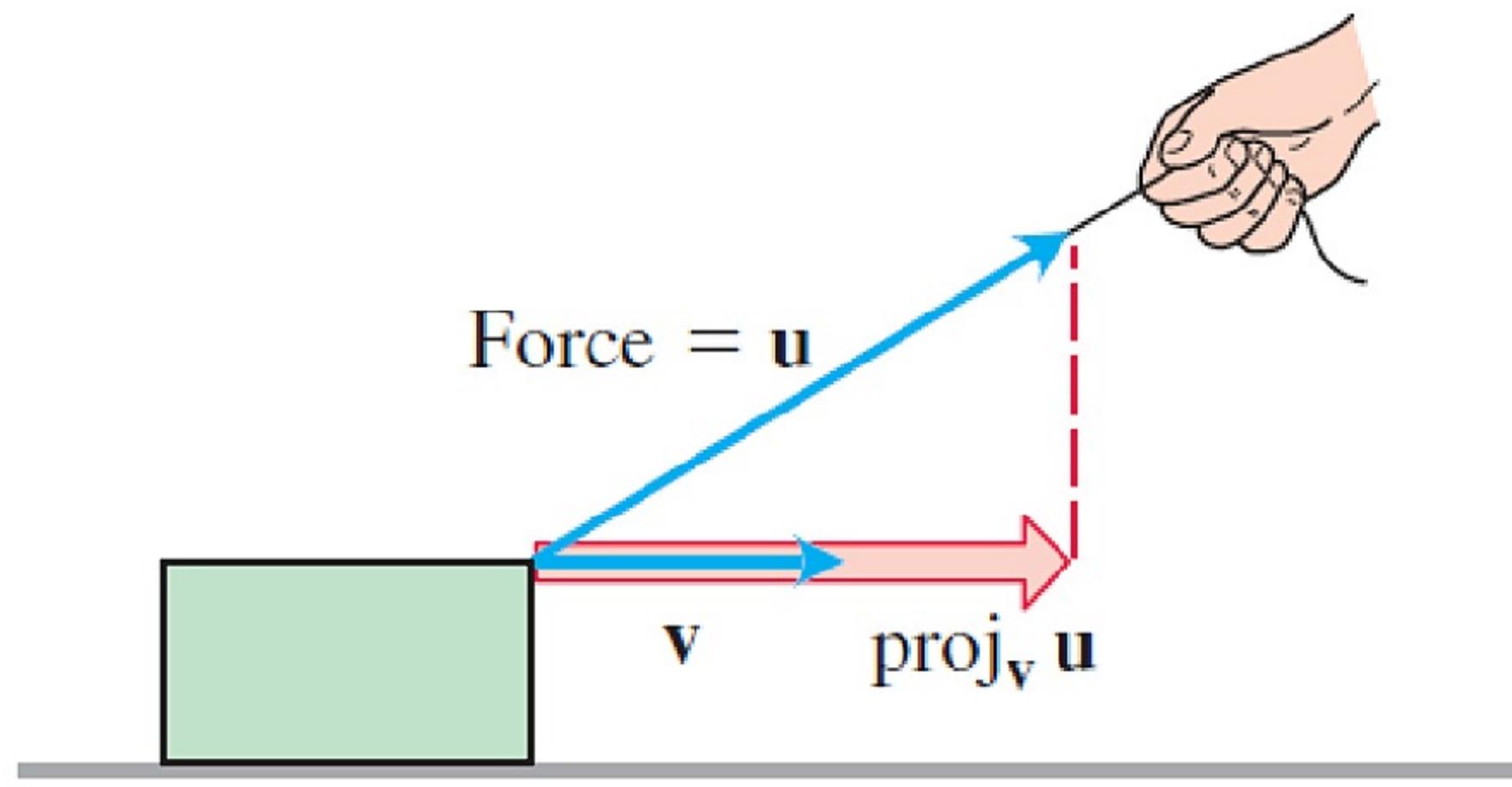
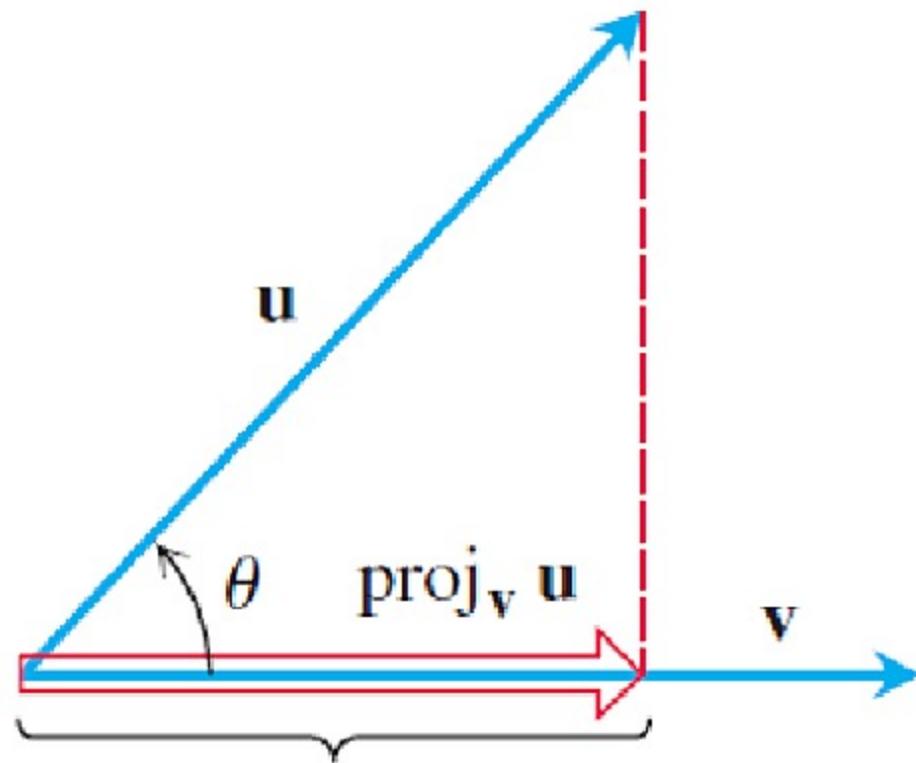
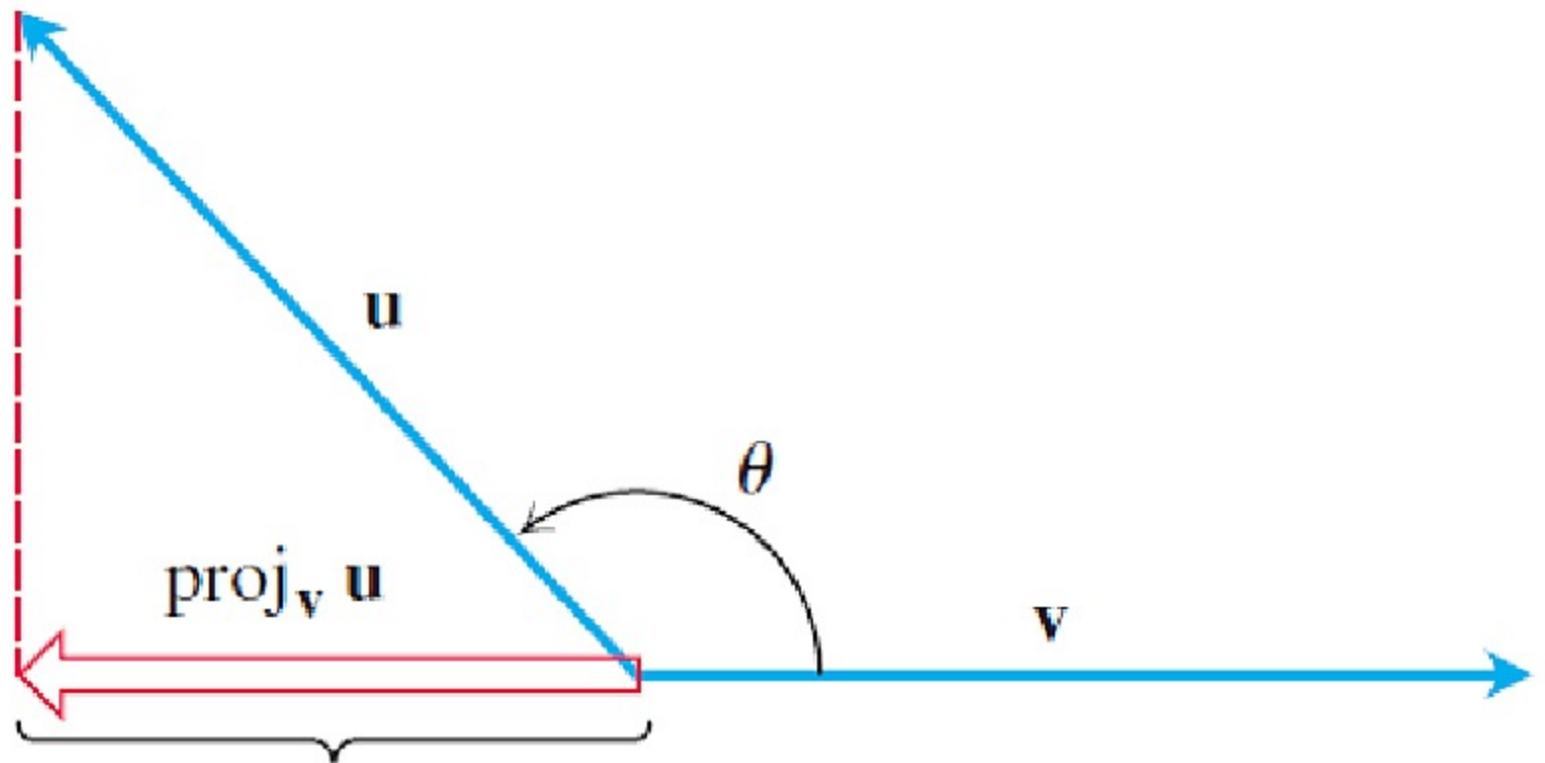


FIGURE 11.24 If we pull on the box with force \mathbf{u} , the effective force moving the box forward in the direction \mathbf{v} is the projection of \mathbf{u} onto \mathbf{v} .



Length = $|\mathbf{u}| \cos \theta$

(a)



Length = $-|\mathbf{u}| \cos \theta$

(b)

FIGURE 11.25 The length of $\text{proj}_v \mathbf{u}$ is (a) $|\mathbf{u}| \cos \theta$ if $\cos \theta \geq 0$ and (b) $-|\mathbf{u}| \cos \theta$ if $\cos \theta < 0$.

The vector projection of \mathbf{u} onto \mathbf{v} is the vector

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \right) \mathbf{v} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|} \right) \frac{\mathbf{v}}{|\mathbf{v}|}. \quad (1)$$

The scalar component of \mathbf{u} in the direction of \mathbf{v} is the scalar

$$|\mathbf{u}| \cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|} = \mathbf{u} \cdot \frac{\mathbf{v}}{|\mathbf{v}|}. \quad (2)$$

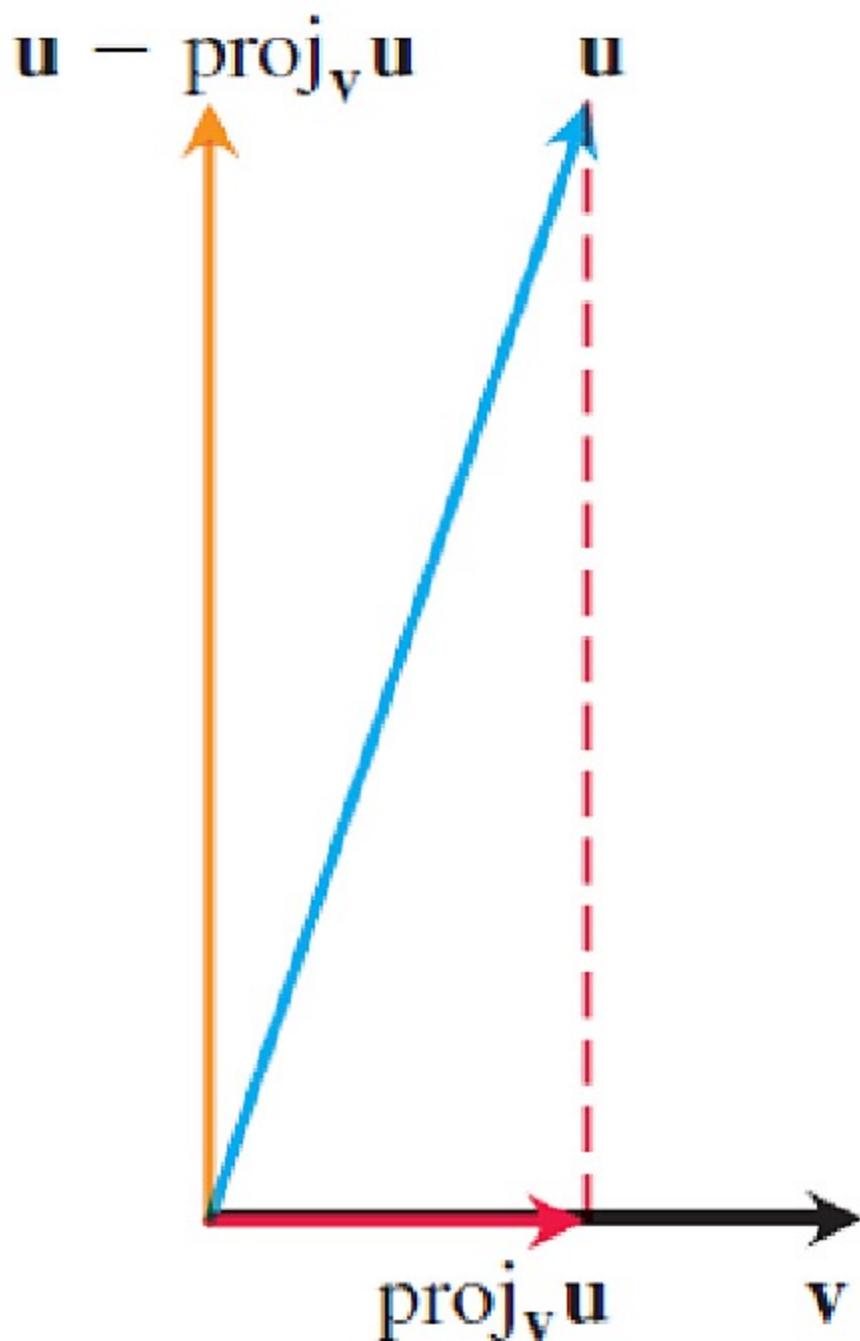


FIGURE 11.26 The vector \mathbf{u} is the sum of two perpendicular vectors: a vector $\text{proj}_{\mathbf{v}} \mathbf{u}$, parallel to \mathbf{v} , and a vector $\mathbf{u} - \text{proj}_{\mathbf{v}} \mathbf{u}$, perpendicular to \mathbf{v} .

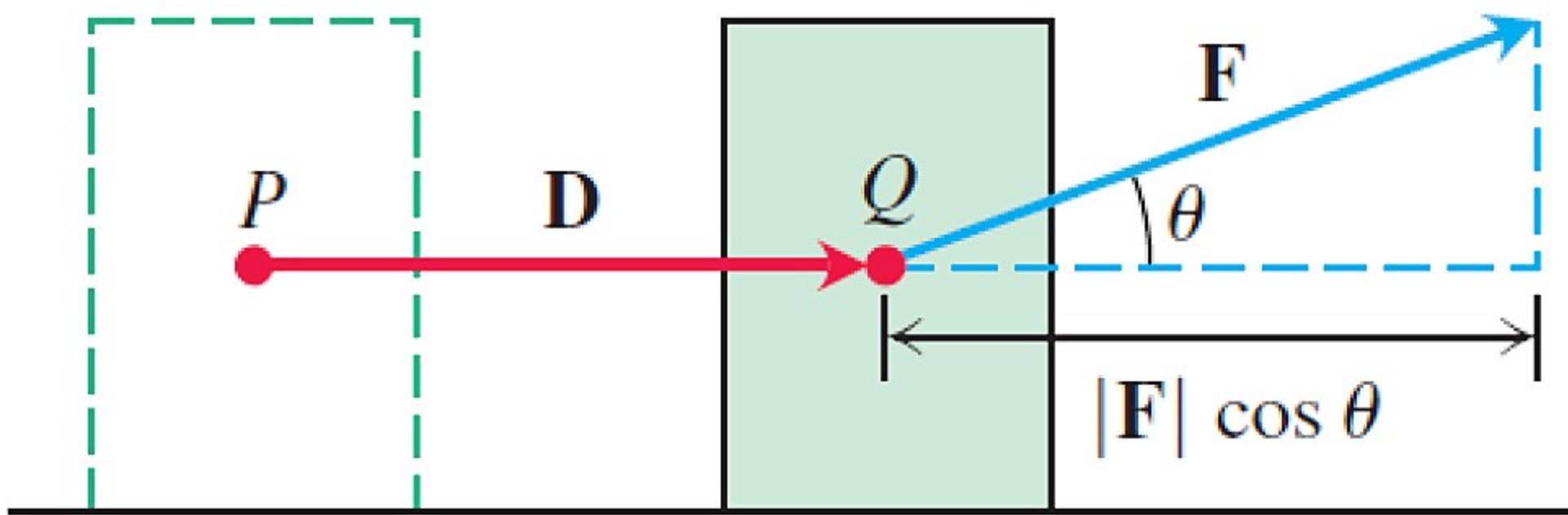


FIGURE 11.27 The work done by a constant force \mathbf{F} during a displacement \mathbf{D} is $(|\mathbf{F}| \cos \theta)|\mathbf{D}|$, which is the dot product $\mathbf{F} \cdot \mathbf{D}$.

DEFINITION

The **work** done by a constant force \mathbf{F} acting through a displacement $\mathbf{D} = \overrightarrow{PQ}$ is

$$W = \mathbf{F} \cdot \mathbf{D}.$$

Section 11.4

The Cross Product

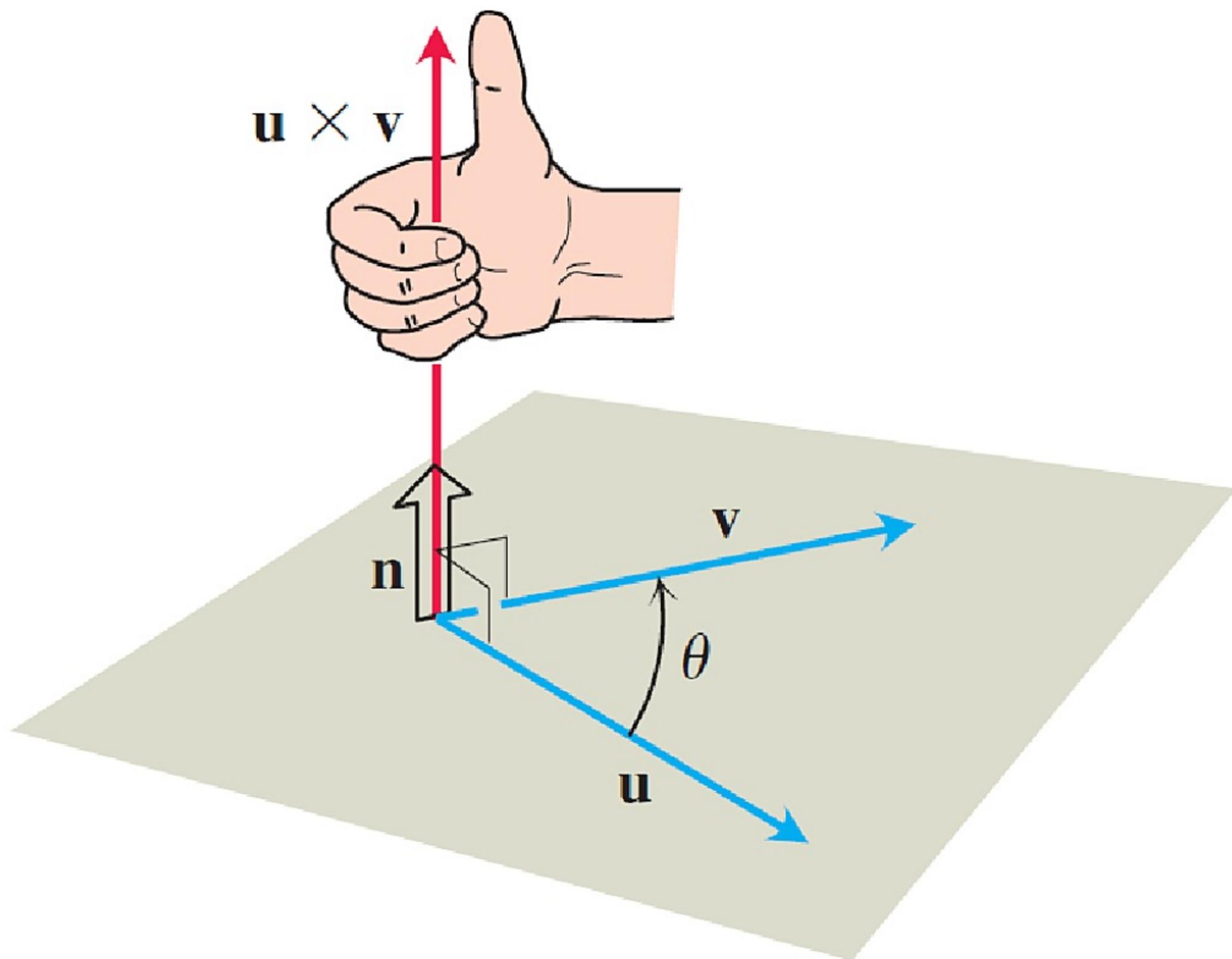


FIGURE 11.28 The construction of $\mathbf{u} \times \mathbf{v}$.

DEFINITION The **cross product** $\mathbf{u} \times \mathbf{v}$ (“ \mathbf{u} cross \mathbf{v} ”) is the vector

$$\mathbf{u} \times \mathbf{v} = (\|\mathbf{u}\| \|\mathbf{v}\| \sin \theta) \mathbf{n}.$$

Parallel Vectors

Nonzero vectors \mathbf{u} and \mathbf{v} are parallel if and only if $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.

Properties of the Cross Product

If \mathbf{u} , \mathbf{v} , and \mathbf{w} are any vectors and r, s are scalars, then

1. $(r\mathbf{u}) \times (s\mathbf{v}) = (rs)(\mathbf{u} \times \mathbf{v})$
2. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$
3. $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v})$
4. $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = \mathbf{v} \times \mathbf{u} + \mathbf{w} \times \mathbf{u}$
5. $\mathbf{0} \times \mathbf{u} = \mathbf{0}$
6. $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$

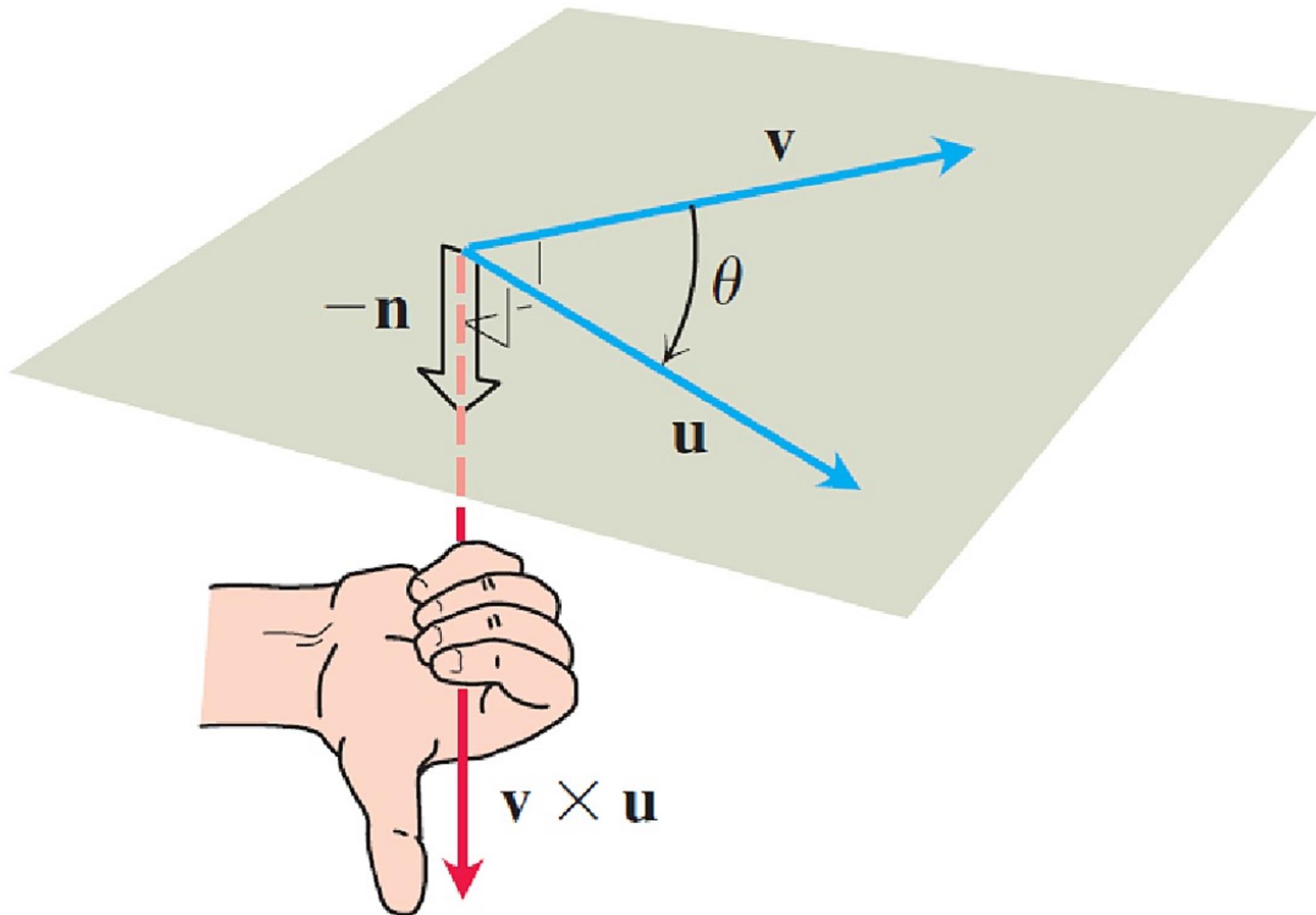


FIGURE 11.29 The construction of $\mathbf{v} \times \mathbf{u}$.

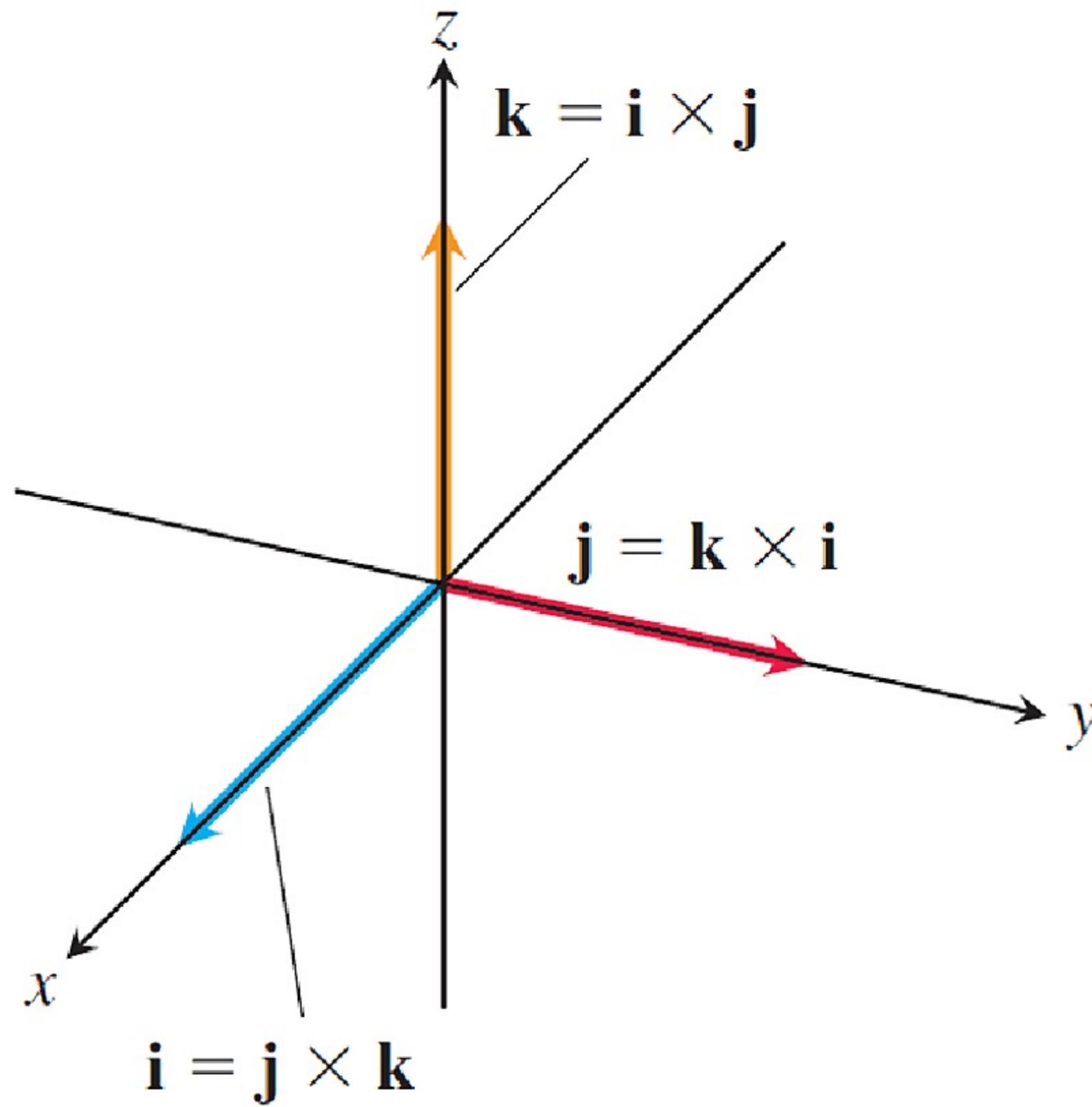


FIGURE 11.30 The pairwise cross products of \mathbf{i} , \mathbf{j} , and \mathbf{k} .

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}| |\mathbf{v}| |\sin \theta| |\mathbf{n}| = |\mathbf{u}| |\mathbf{v}| \sin \theta.$$

$$\begin{aligned}\text{Area} &= \text{base} \cdot \text{height} \\ &= |\mathbf{u}| \cdot |\mathbf{v}| |\sin \theta| \\ &= |\mathbf{u} \times \mathbf{v}|\end{aligned}$$

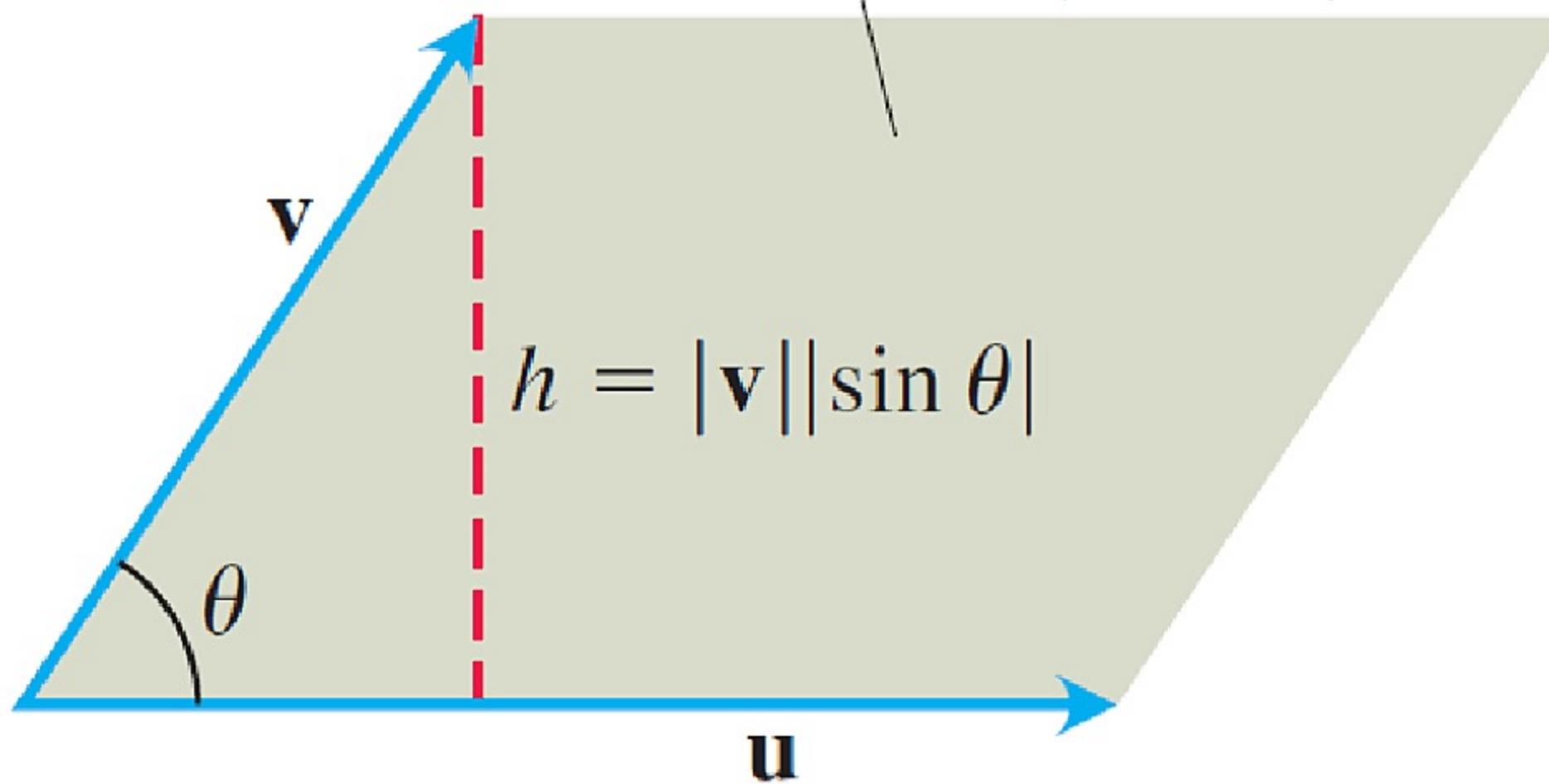


FIGURE 11.31 The parallelogram determined by \mathbf{u} and \mathbf{v} .

Calculating the Cross Product as a Determinant

If $\mathbf{u} = u_1 \mathbf{i} + u_2 \mathbf{j} + u_3 \mathbf{k}$ and $\mathbf{v} = v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}$, then

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

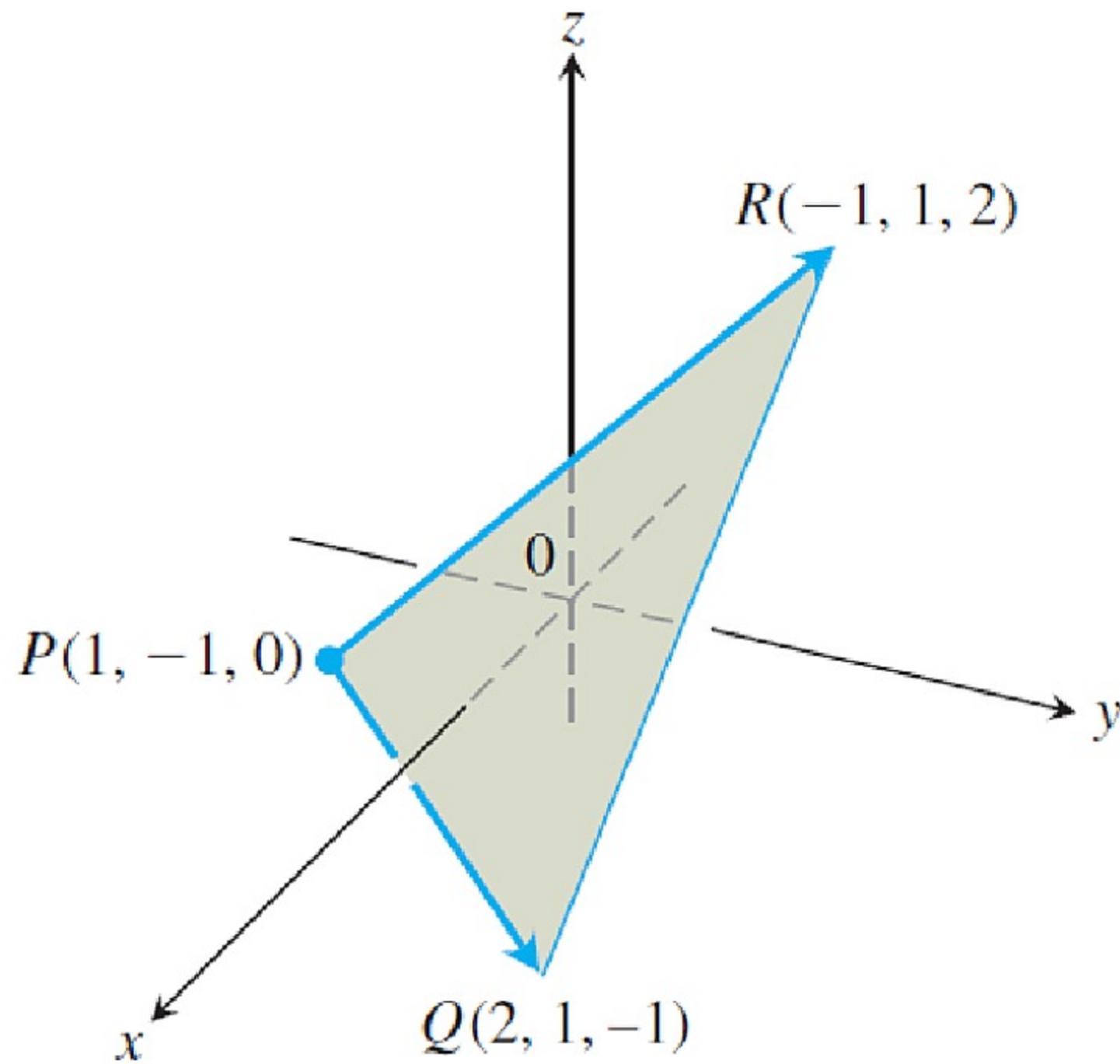
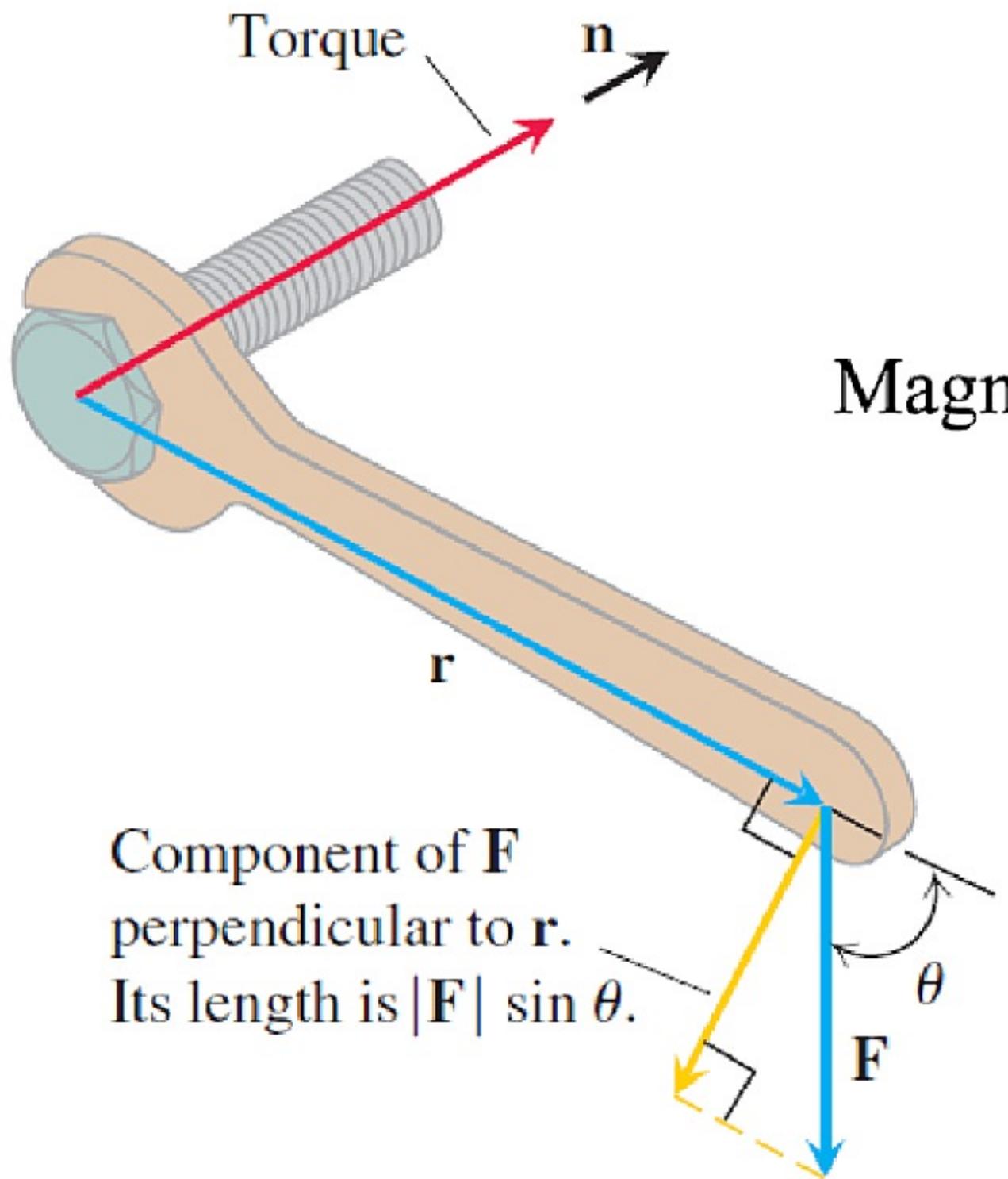


FIGURE 11.32 The vector $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to the plane of triangle PQR (Example 2). The area of triangle PQR is half of $|\overrightarrow{PQ} \times \overrightarrow{PR}|$ (Example 3).



Magnitude of torque vector = $|r| |F| \sin \theta$,

FIGURE 11.33 The torque vector describes the tendency of the force F to drive the bolt forward.

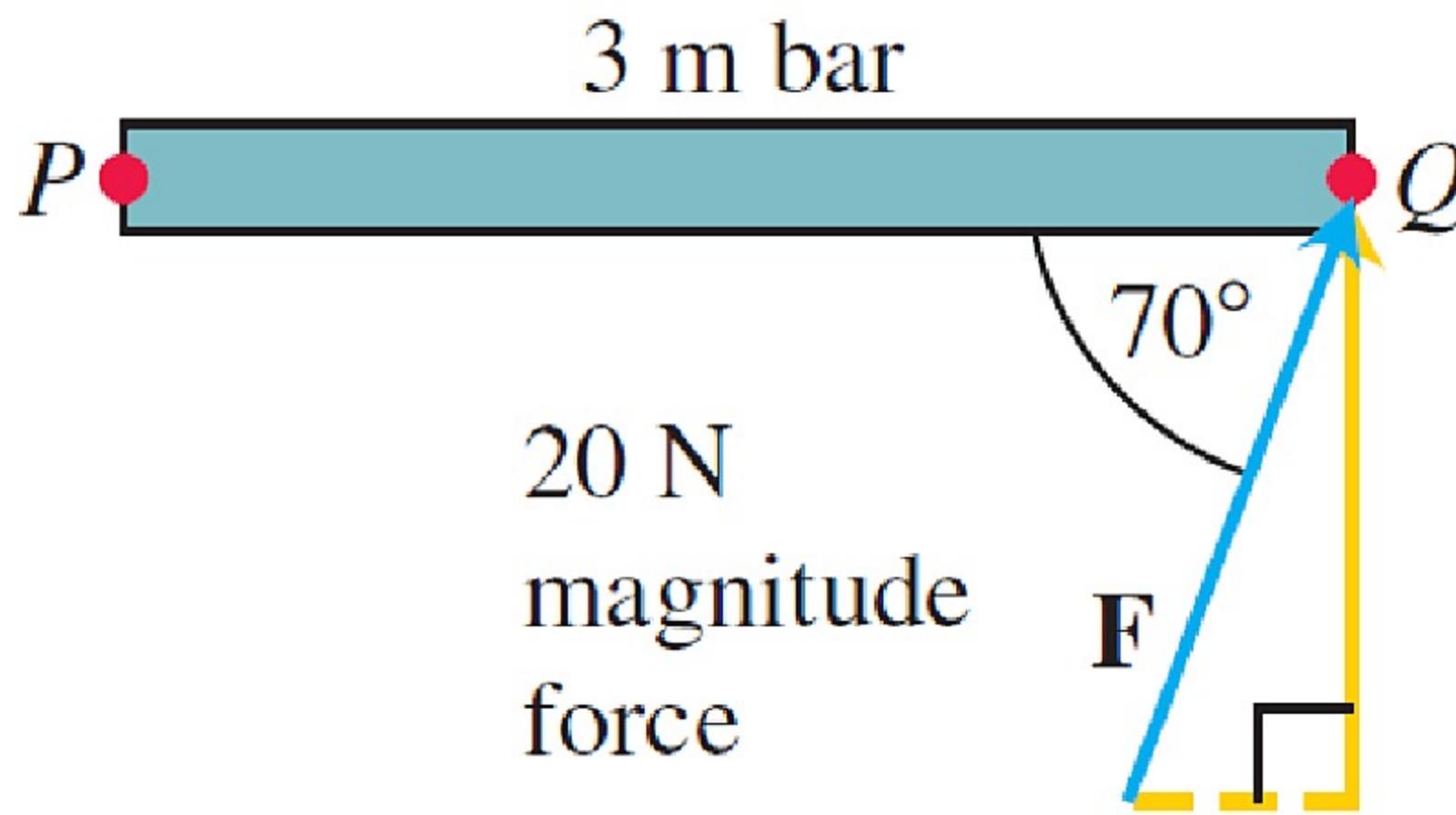
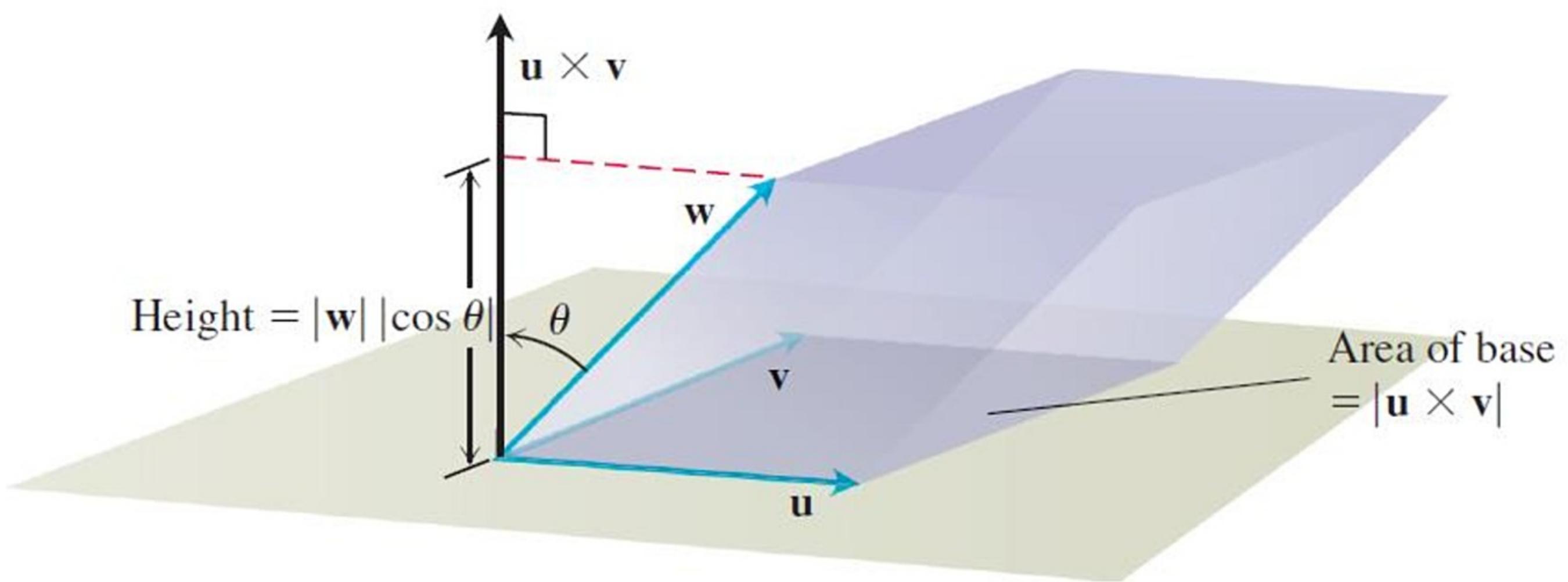


FIGURE 11.34 The magnitude of the torque exerted by \mathbf{F} at P is about $56.4 \text{ N}\cdot\text{m}$ (Example 5). The bar rotates counterclockwise around P .



$$\begin{aligned}
 \text{Volume} &= \text{area of base} \cdot \text{height} \\
 &= |\mathbf{u} \times \mathbf{v}| |\mathbf{w}| |\cos \theta| \\
 &= |(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}|
 \end{aligned}$$

FIGURE 11.35 The number $|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}|$ is the volume of a parallelepiped.

Calculating the Triple Scalar Product as a Determinant

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Section 11.5

Lines and Planes in Space

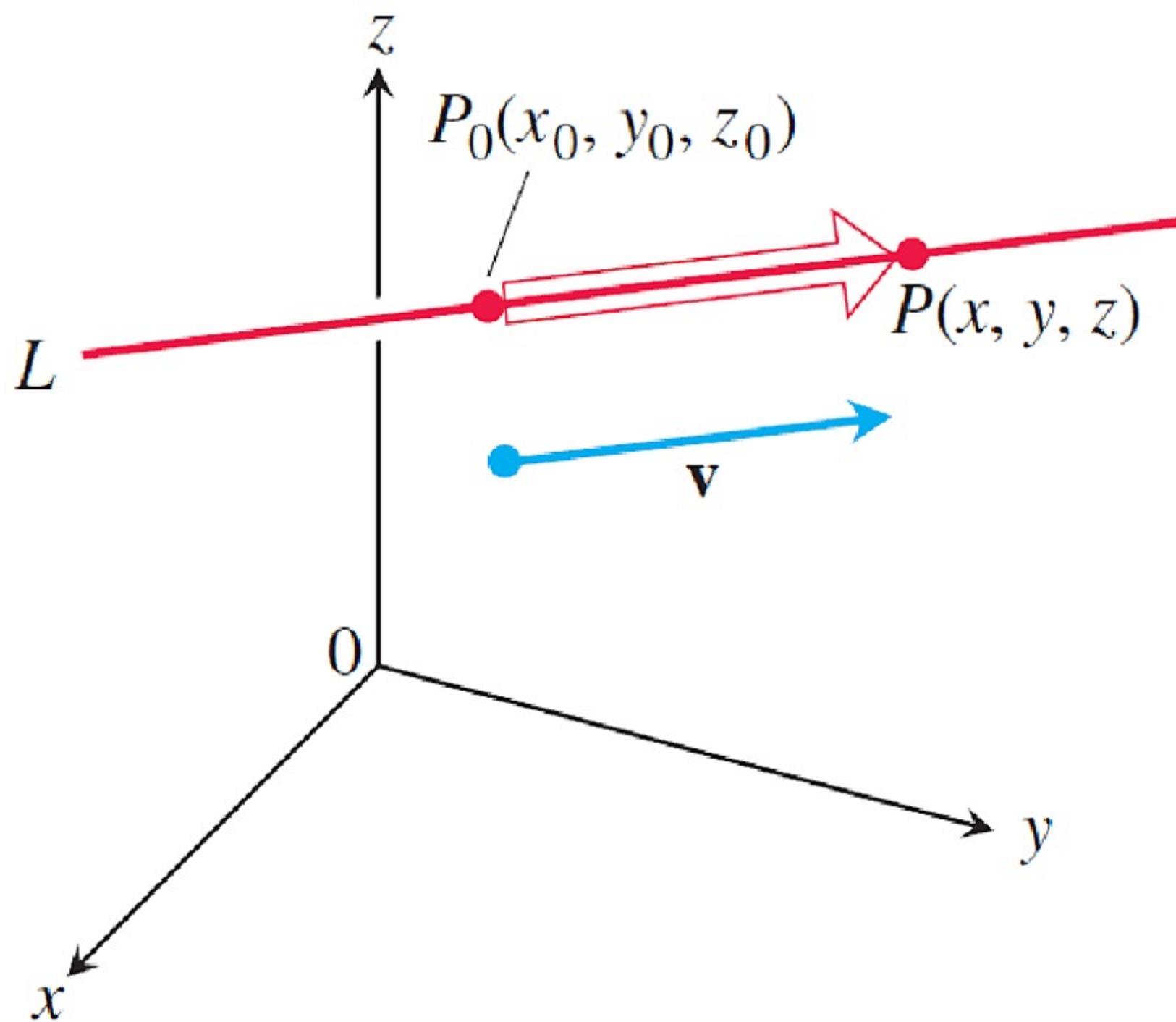


FIGURE 11.36 A point P lies on L through P_0 parallel to \mathbf{v} if and only if $\overrightarrow{P_0P}$ is a scalar multiple of \mathbf{v} .

Vector Equation for a Line

A vector equation for the line L through $P_0(x_0, y_0, z_0)$ parallel to \mathbf{v} is

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}, \quad -\infty < t < \infty, \quad (2)$$

where \mathbf{r} is the position vector of a point $P(x, y, z)$ on L and \mathbf{r}_0 is the position vector of $P_0(x_0, y_0, z_0)$.

Parametric Equations for a Line

The standard parametrization of the line through $P_0(x_0, y_0, z_0)$ parallel to $\mathbf{v} = v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}$ is

$$x = x_0 + tv_1, \quad y = y_0 + tv_2, \quad z = z_0 + tv_3, \quad -\infty < t < \infty \quad (3)$$

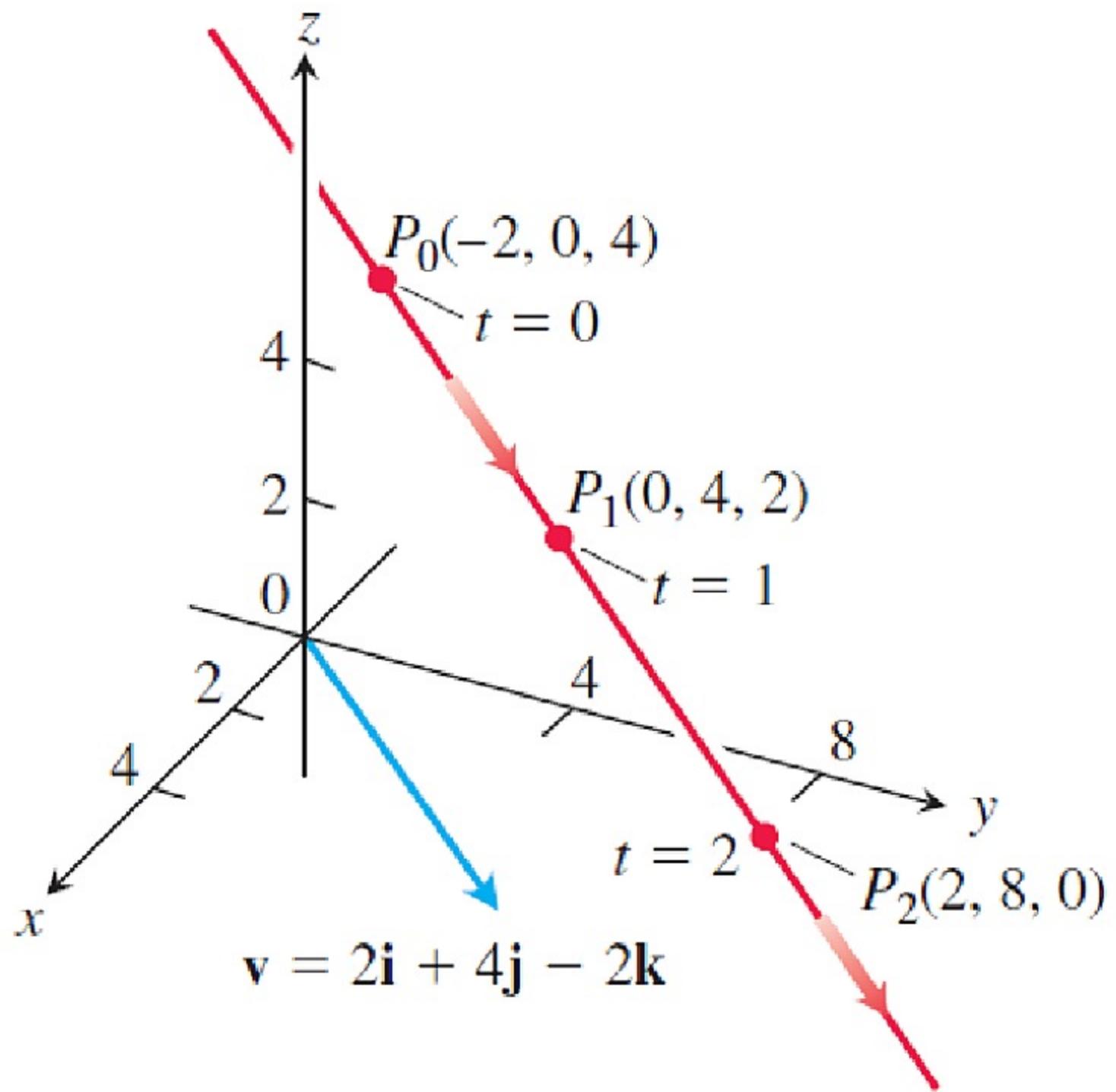


FIGURE 11.37 Selected points and parameter values on the line in Example 1. The arrows show the direction of increasing t .

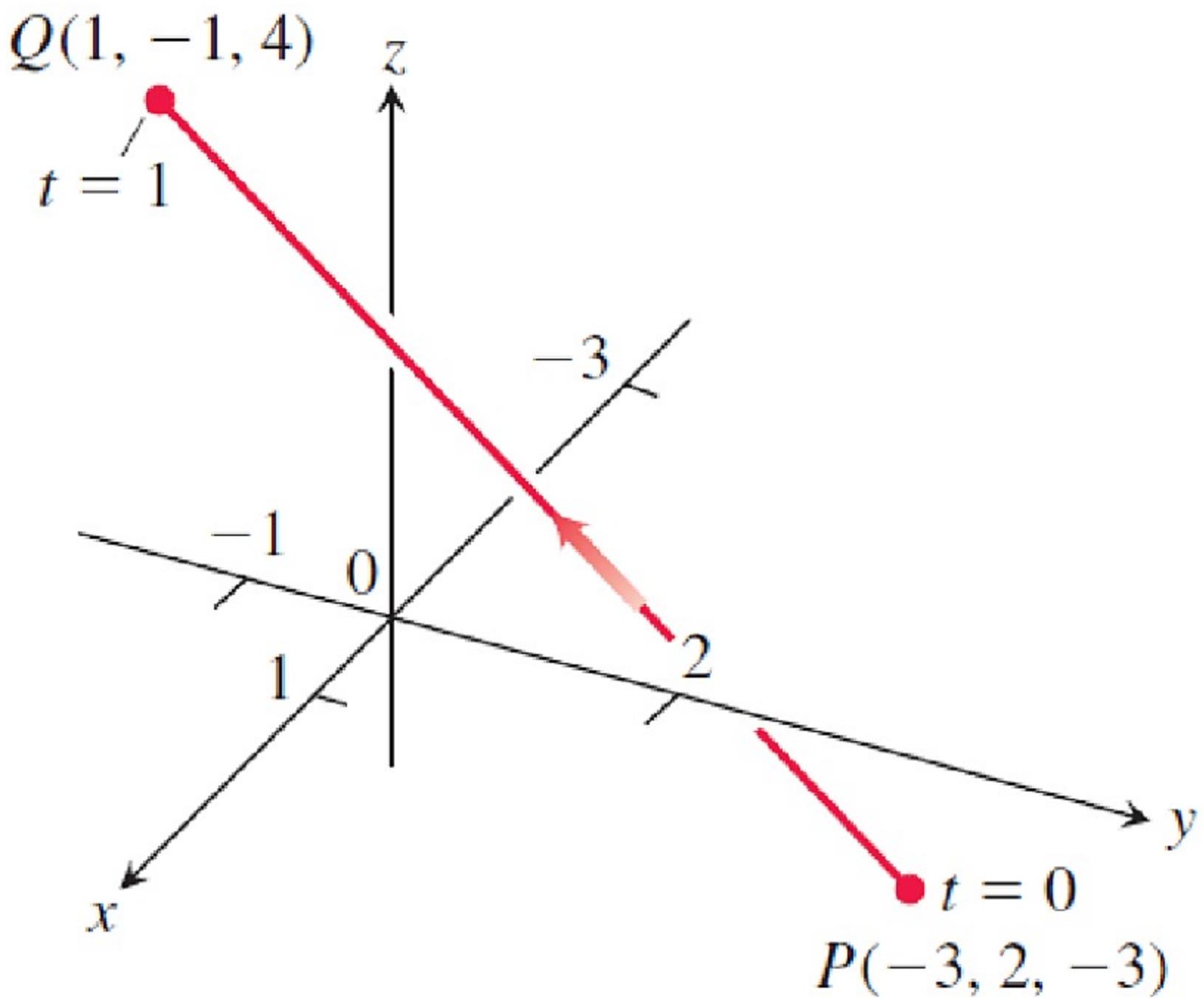


FIGURE 11.38 Example 3 derives a parametrization of line segment PQ . The arrow shows the direction of increasing t .

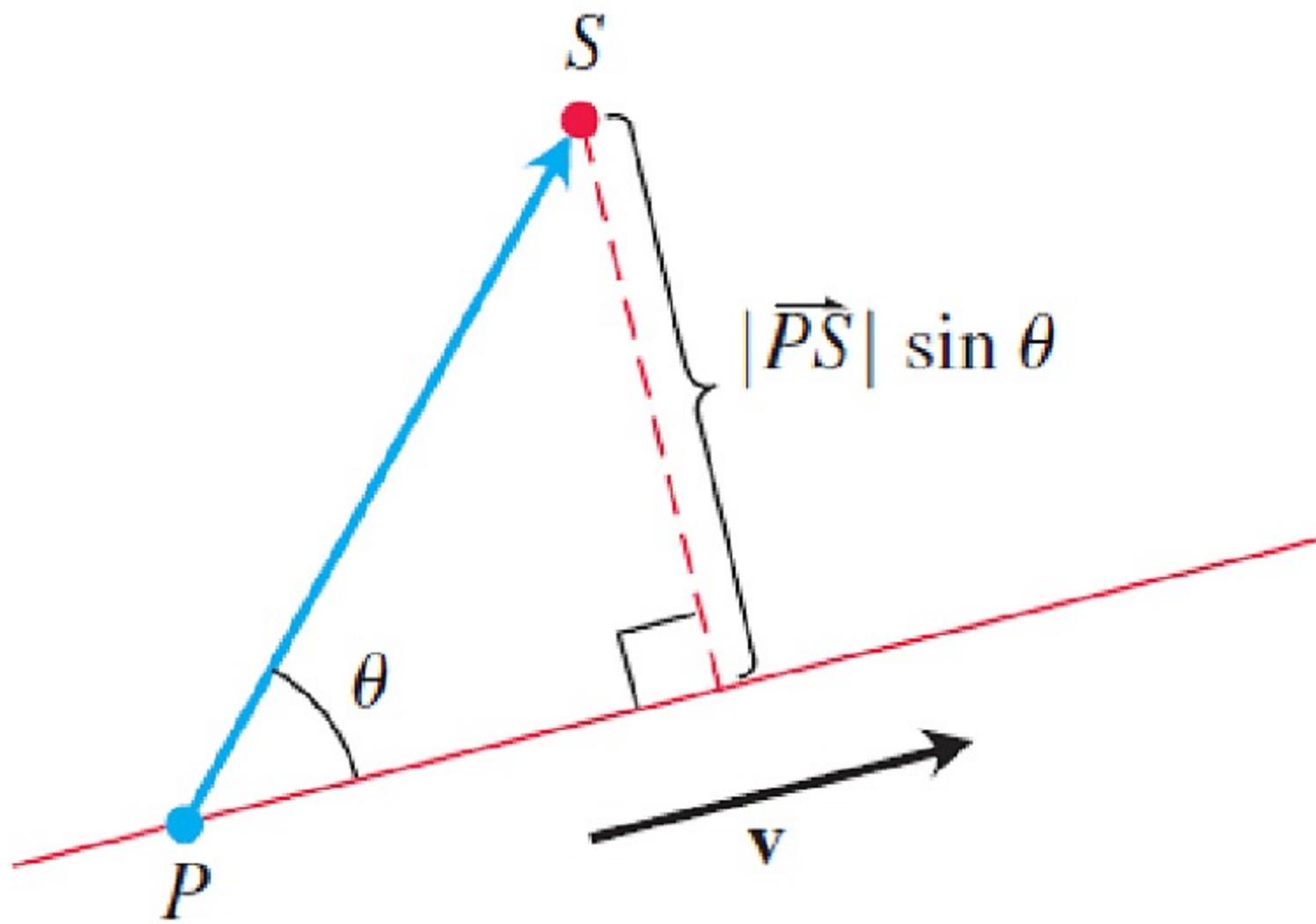


FIGURE 11.39 The distance from S to the line through P parallel to \mathbf{v} is $|\overrightarrow{PS}| \sin \theta$, where θ is the angle between \overrightarrow{PS} and \mathbf{v} .

Distance from a Point S to a Line Through P Parallel to \mathbf{v}

$$d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|} \quad (5)$$

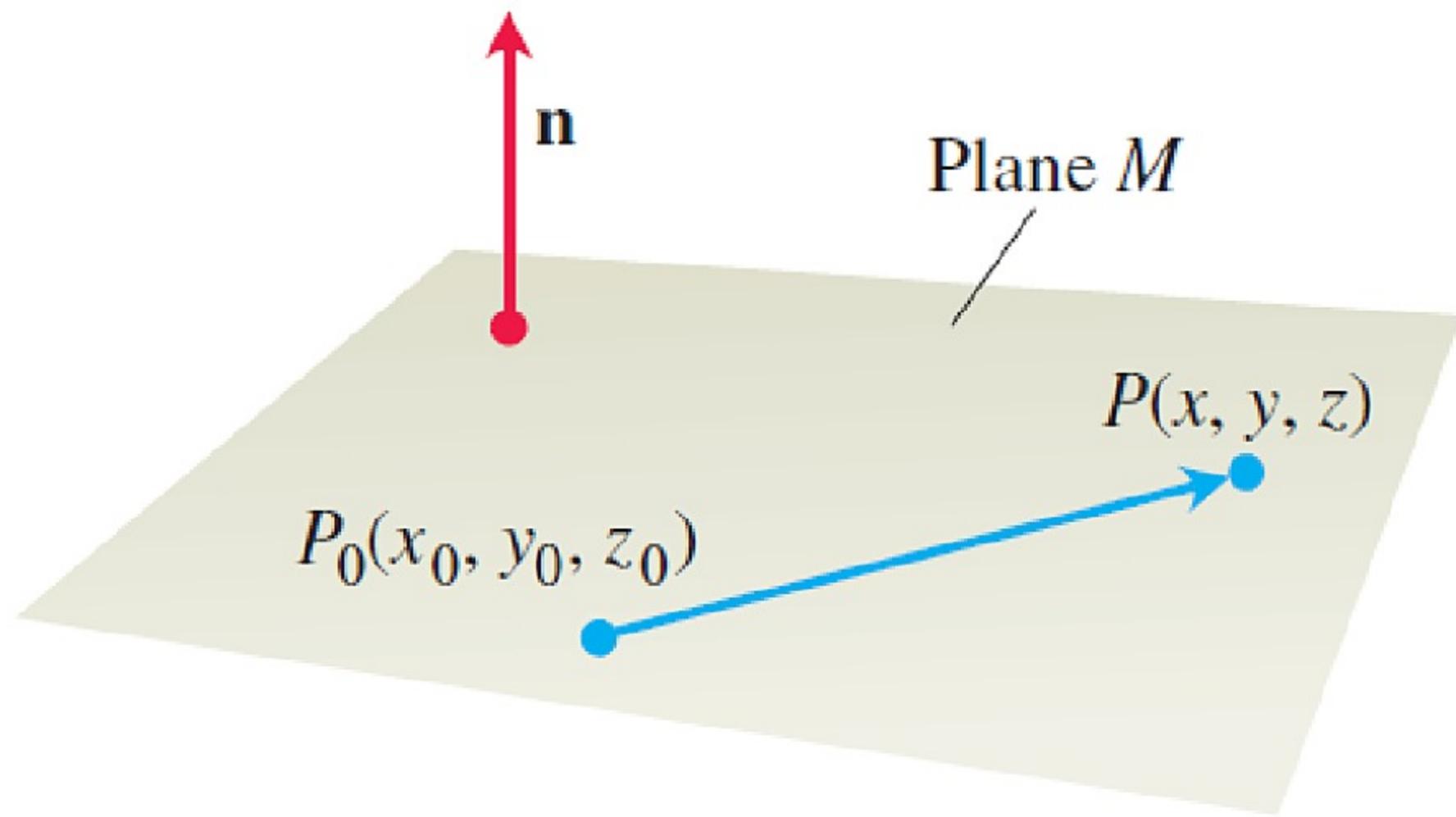


FIGURE 11.40 The standard equation for a plane in space is defined in terms of a vector normal to the plane: A point P lies in the plane through P_0 normal to \mathbf{n} if and only if $\mathbf{n} \cdot \overrightarrow{P_0P} = 0$.

Equation for a Plane

The plane through $P_0(x_0, y_0, z_0)$ normal to $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ has

Vector equation:

$$\mathbf{n} \cdot \overrightarrow{P_0P} = 0$$

Component equation:

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

Component equation simplified: $Ax + By + Cz = D,$ where

$$D = Ax_0 + By_0 + Cz_0$$

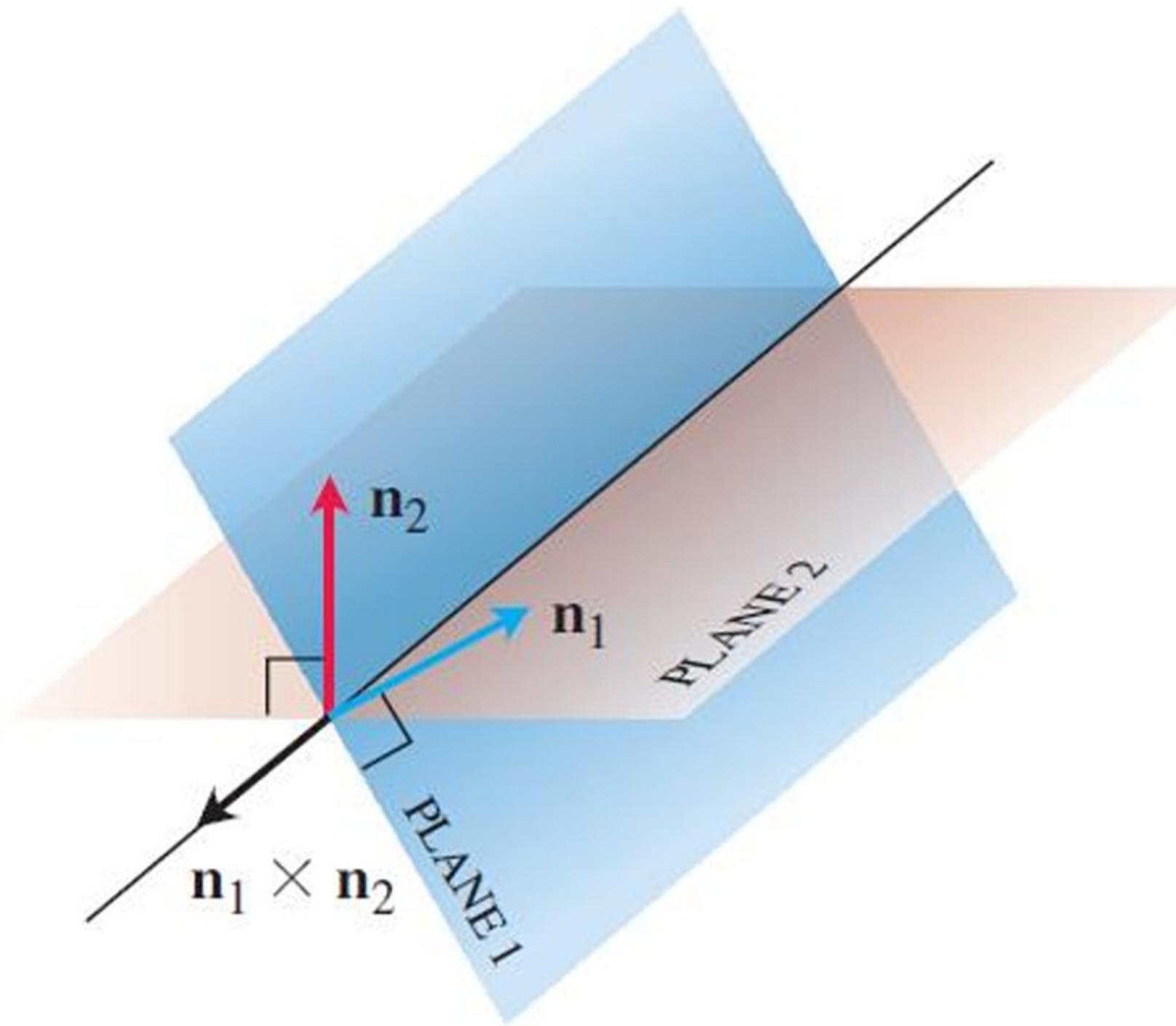


FIGURE 11.41 How the line of intersection of two planes is related to the planes' normal vectors (Example 8).

Distance from a Point S to a Plane with Normal \mathbf{n} at Point P

$$d = \left| \overrightarrow{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|} \right| \quad (6)$$

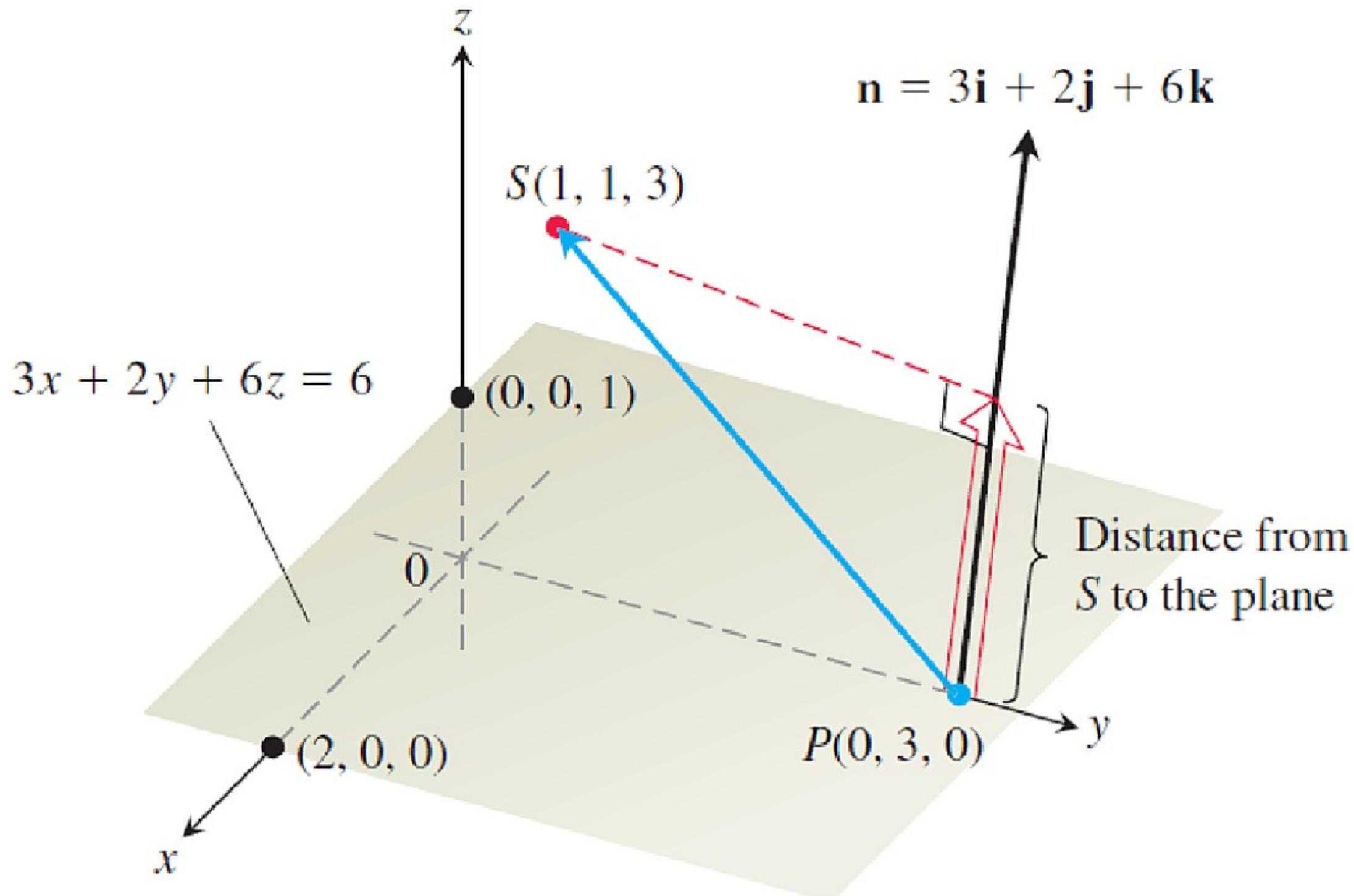


FIGURE 11.42 The distance from S to the plane is the length of the vector projection of \overrightarrow{PS} onto \mathbf{n} (Example 11).

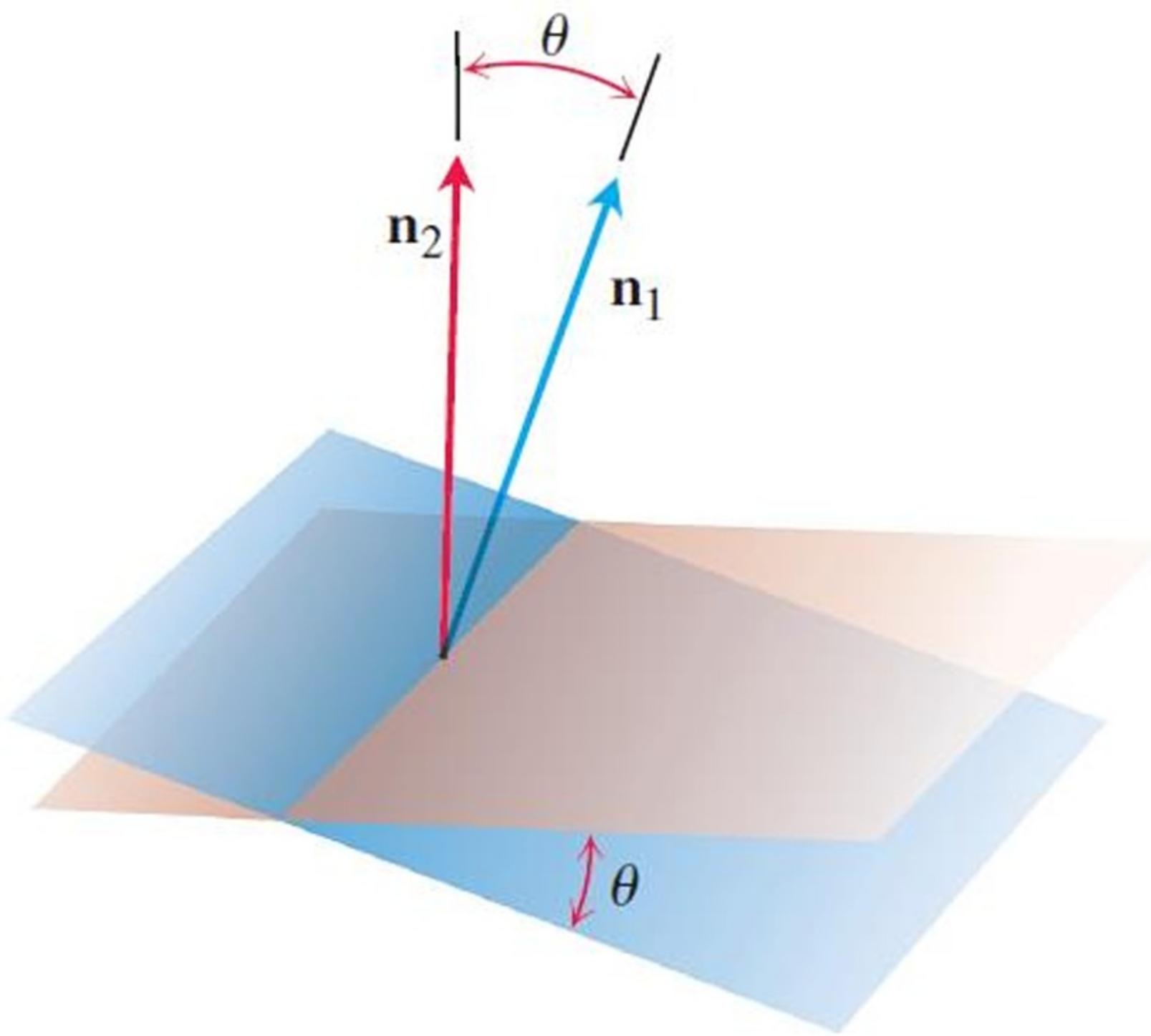


FIGURE 11.43 The angle between two planes is obtained from the angle between their normals.

Section 11.6

Cylinders and Quadric Surfaces

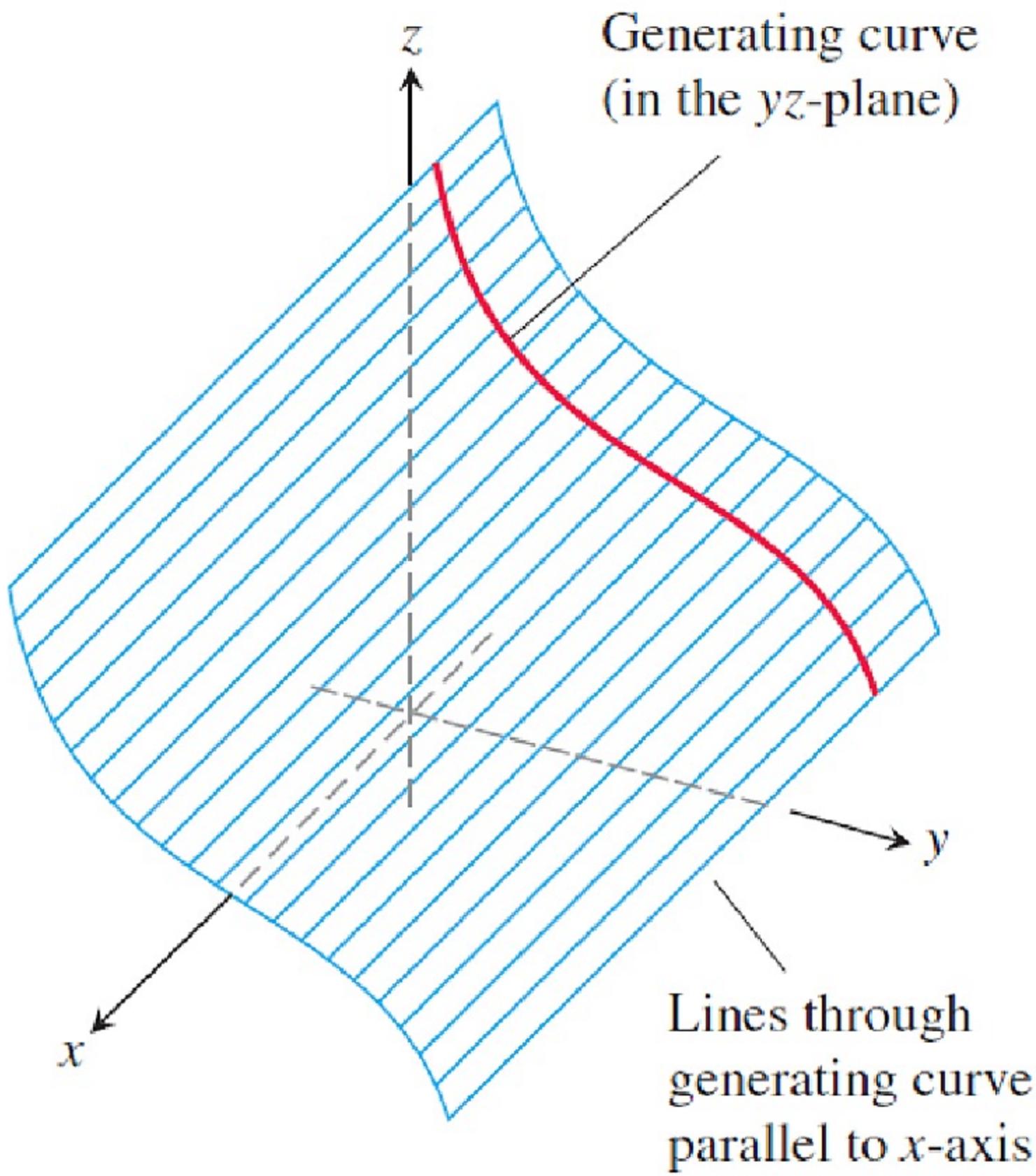


FIGURE 11.44 A cylinder and generating curve.

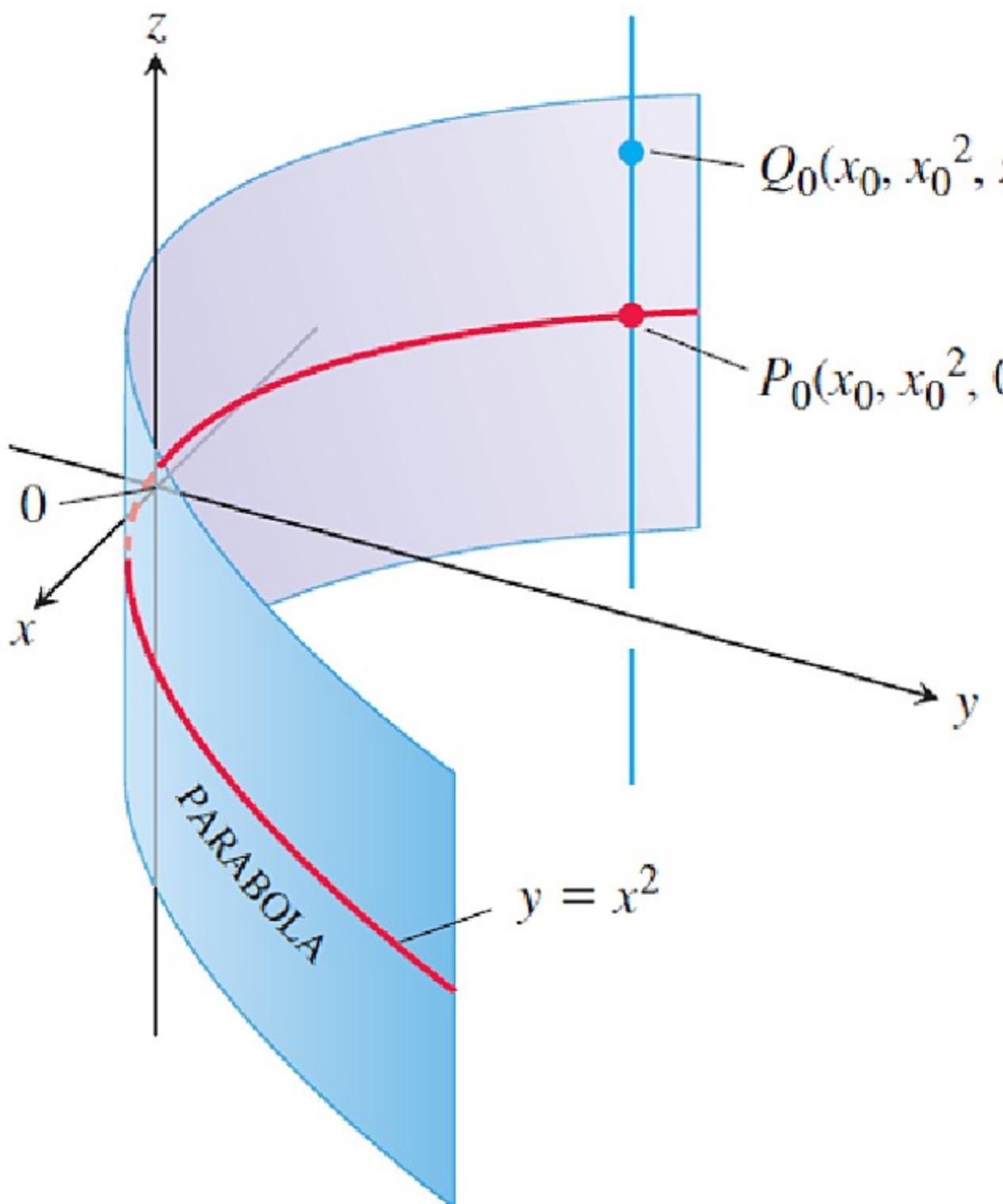


FIGURE 11.45 Every point of the cylinder in Example 1 has coordinates of the form (x_0, x_0^2, z) . We call it “the cylinder $y = x^2$.”

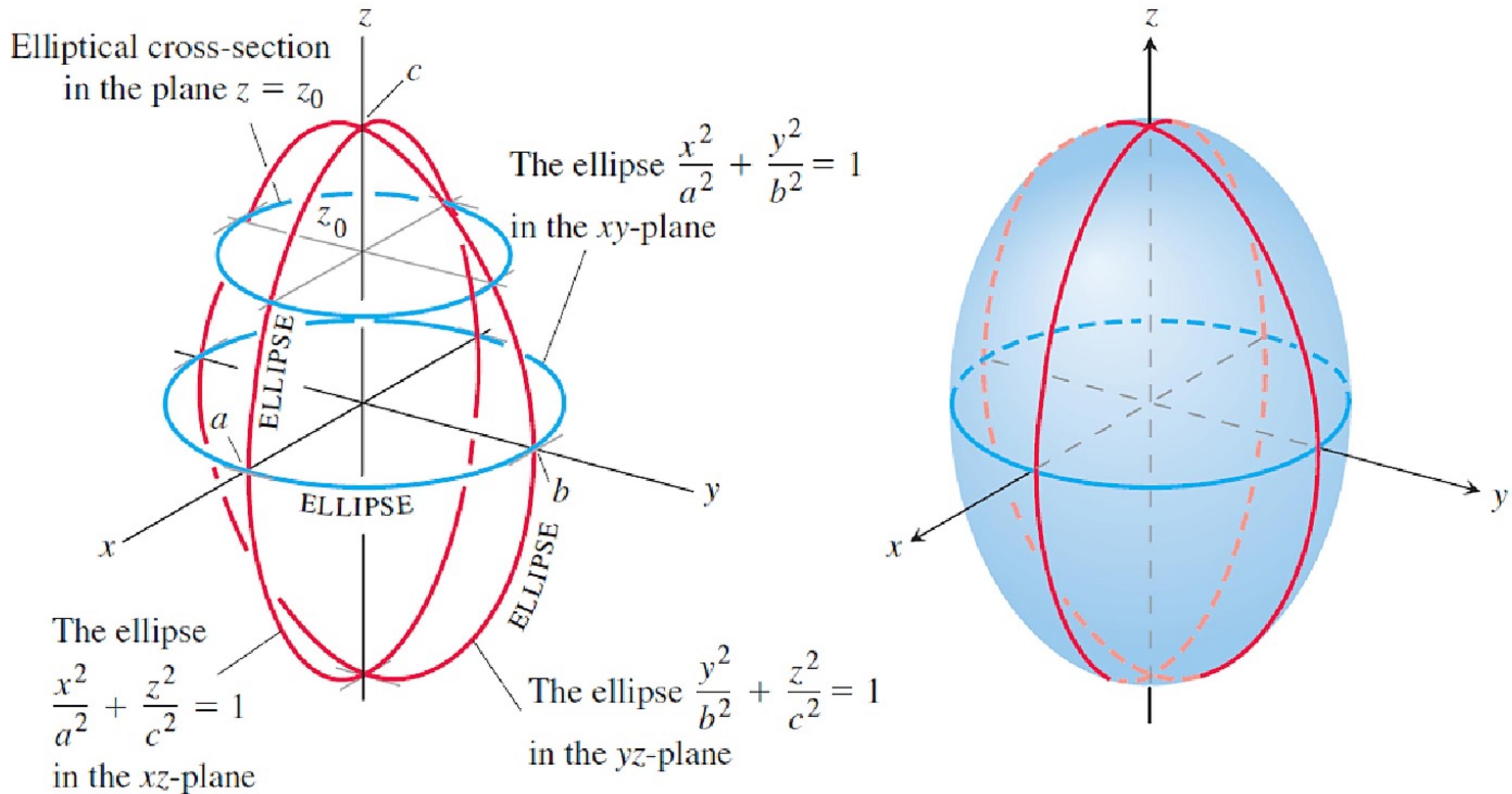
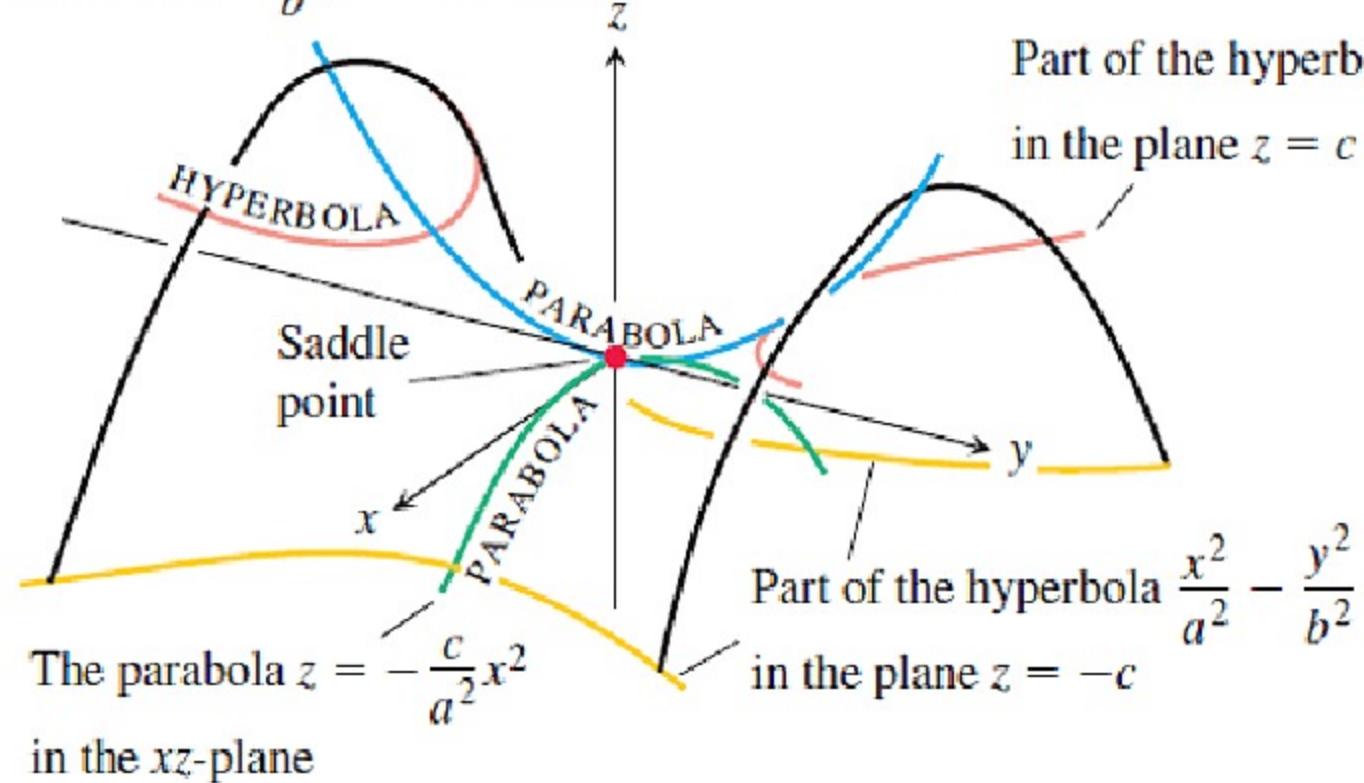


FIGURE 11.46 The ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

in Example 2 has elliptical cross-sections in each of the three coordinate planes.

The parabola $z = \frac{c}{b^2}y^2$ in the yz -plane



Part of the hyperbola $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$ in the plane $z = c$

Part of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ in the plane $z = -c$

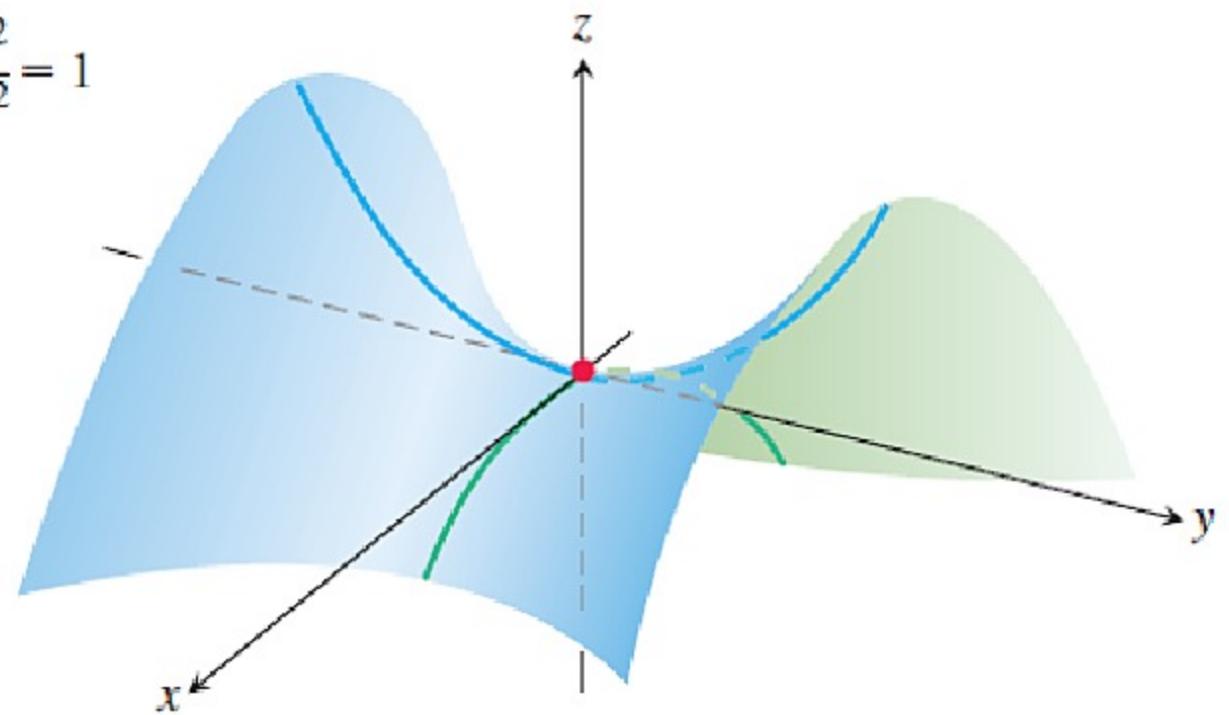


FIGURE 11.47 The hyperbolic paraboloid $(y^2/b^2) - (x^2/a^2) = z/c$, $c > 0$. The cross-sections in planes perpendicular to the z -axis above and below the xy -plane are hyperbolas. The cross-sections in planes perpendicular to the other axes are parabolas.

Table 11.1 Graphs of Quadric Surfaces

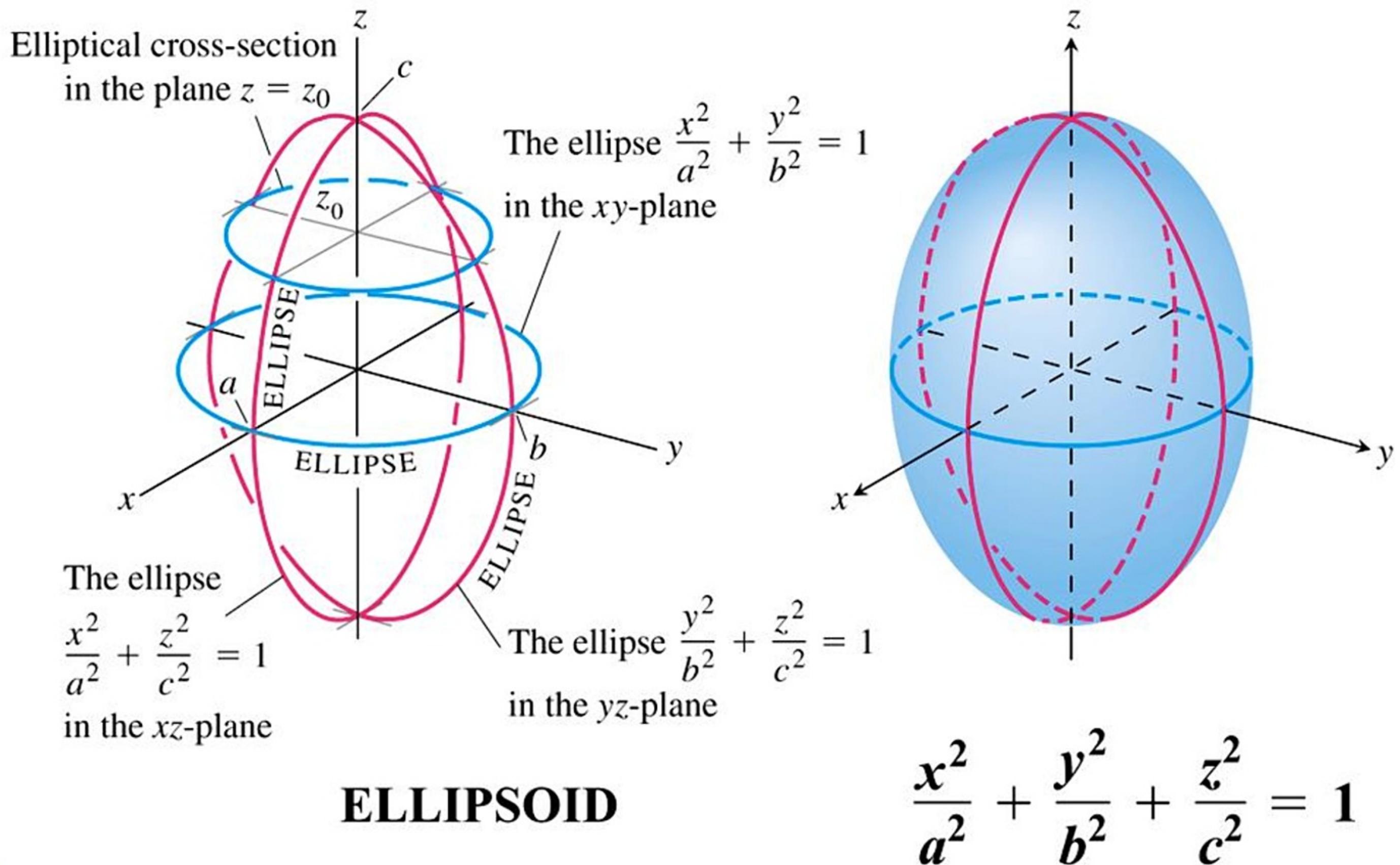


Table 11.1 Graphs of Quadric Surfaces (cont)

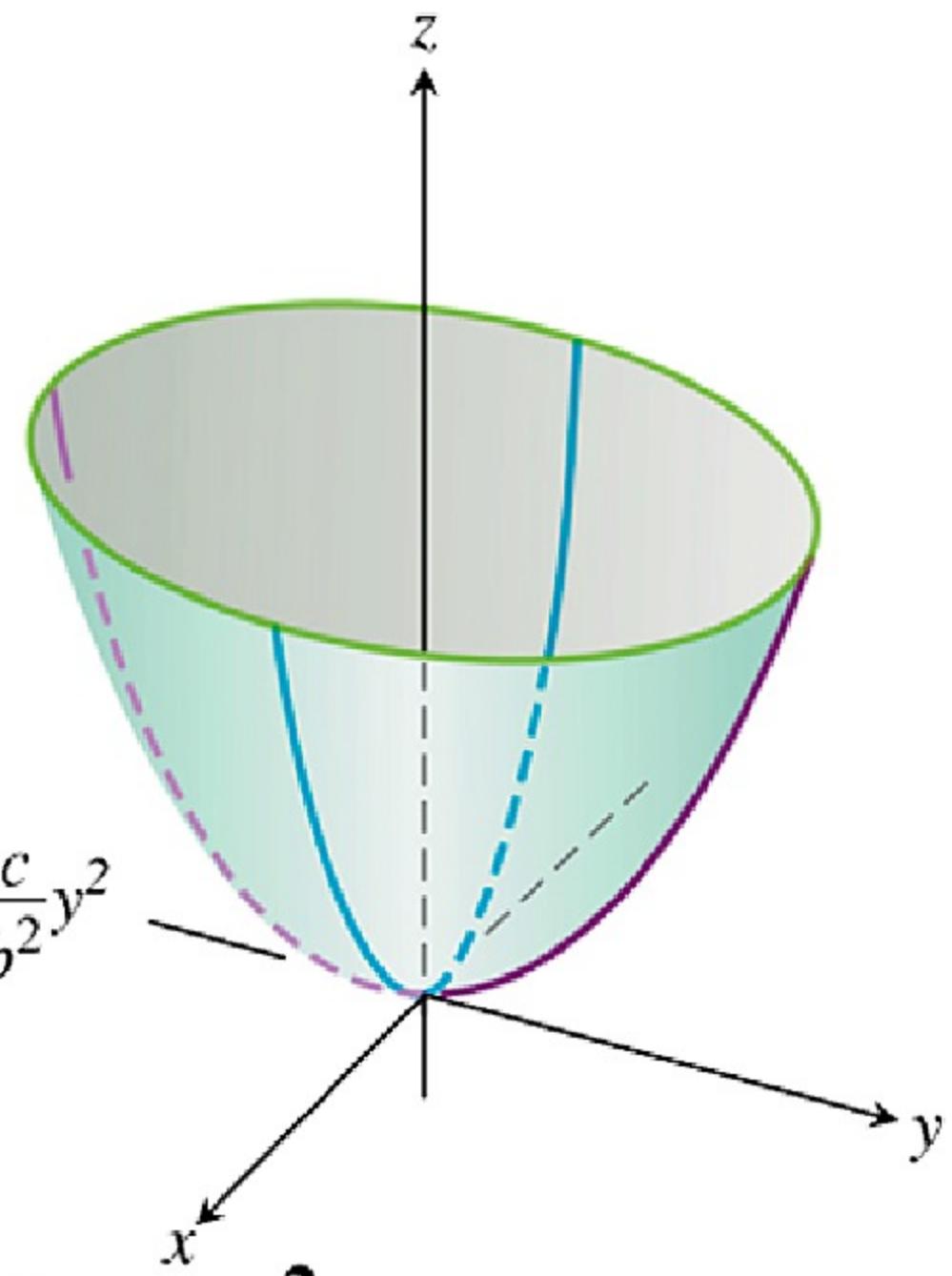
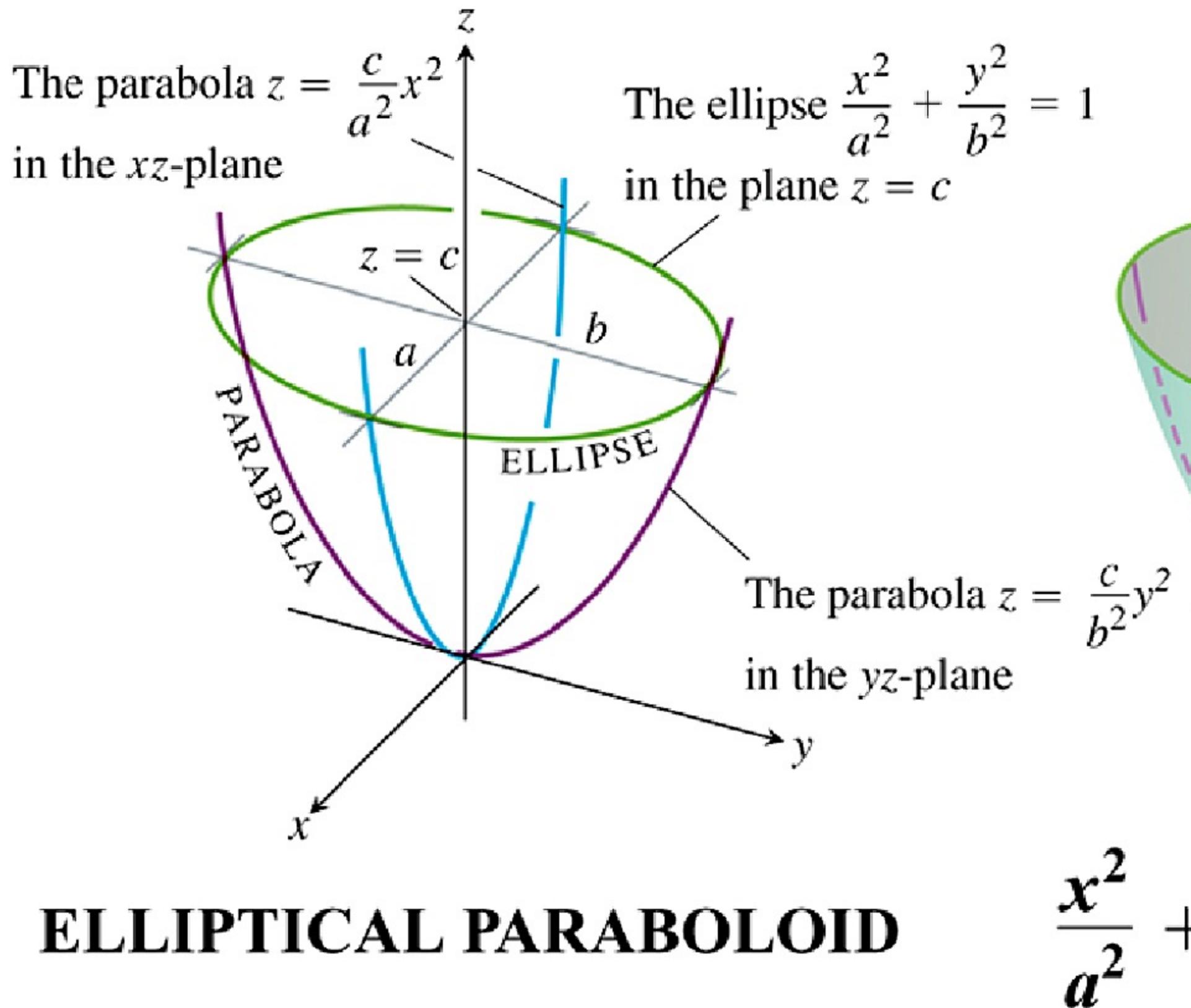
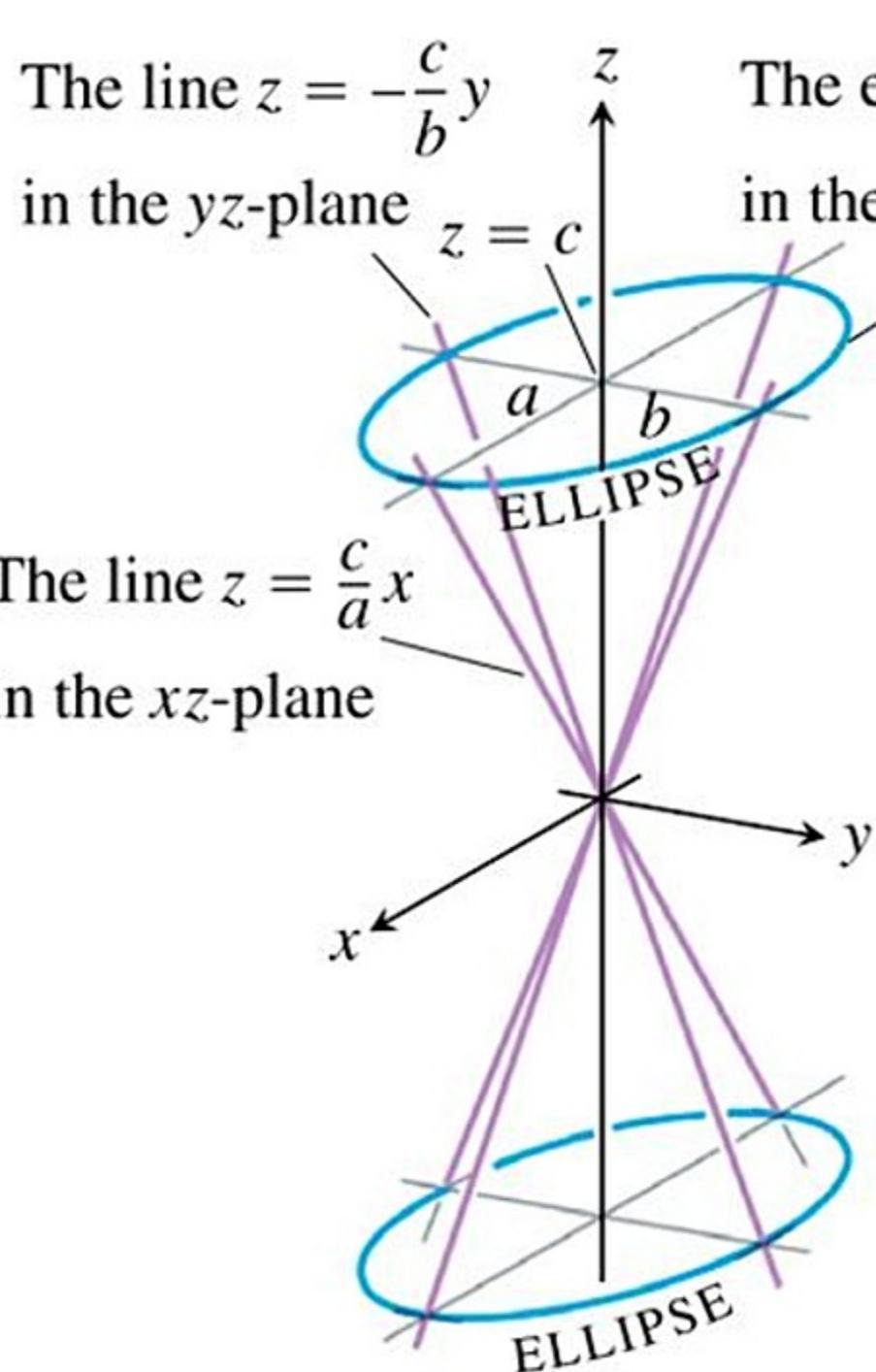
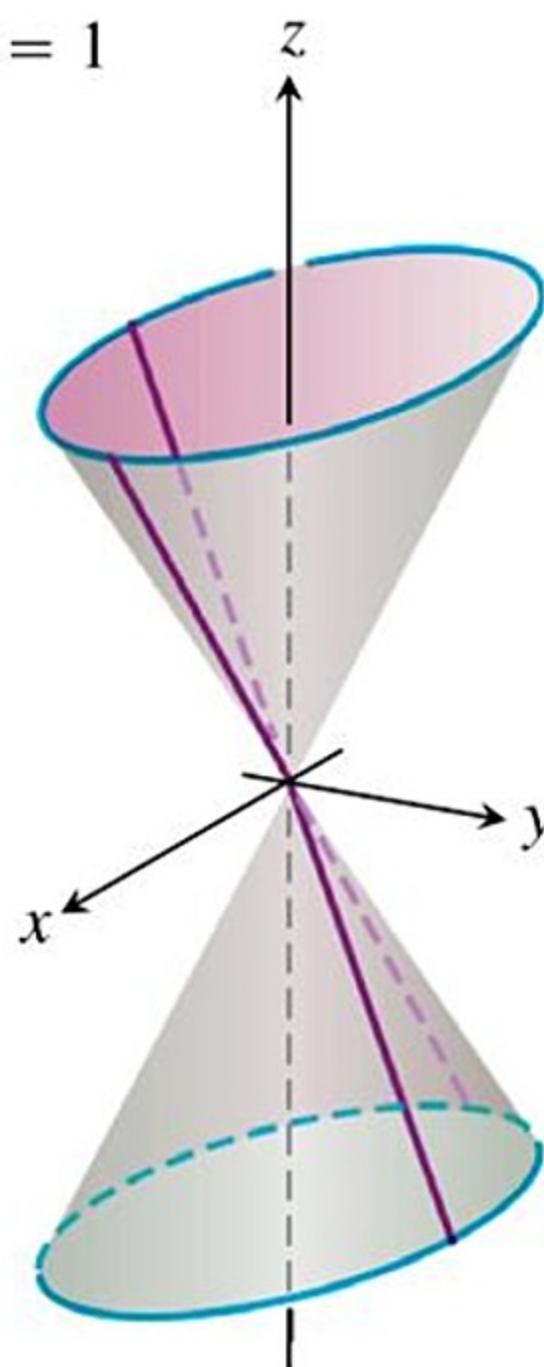


Table 11.1 Graphs of Quadric Surfaces (cont)



The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in the plane $z = c$



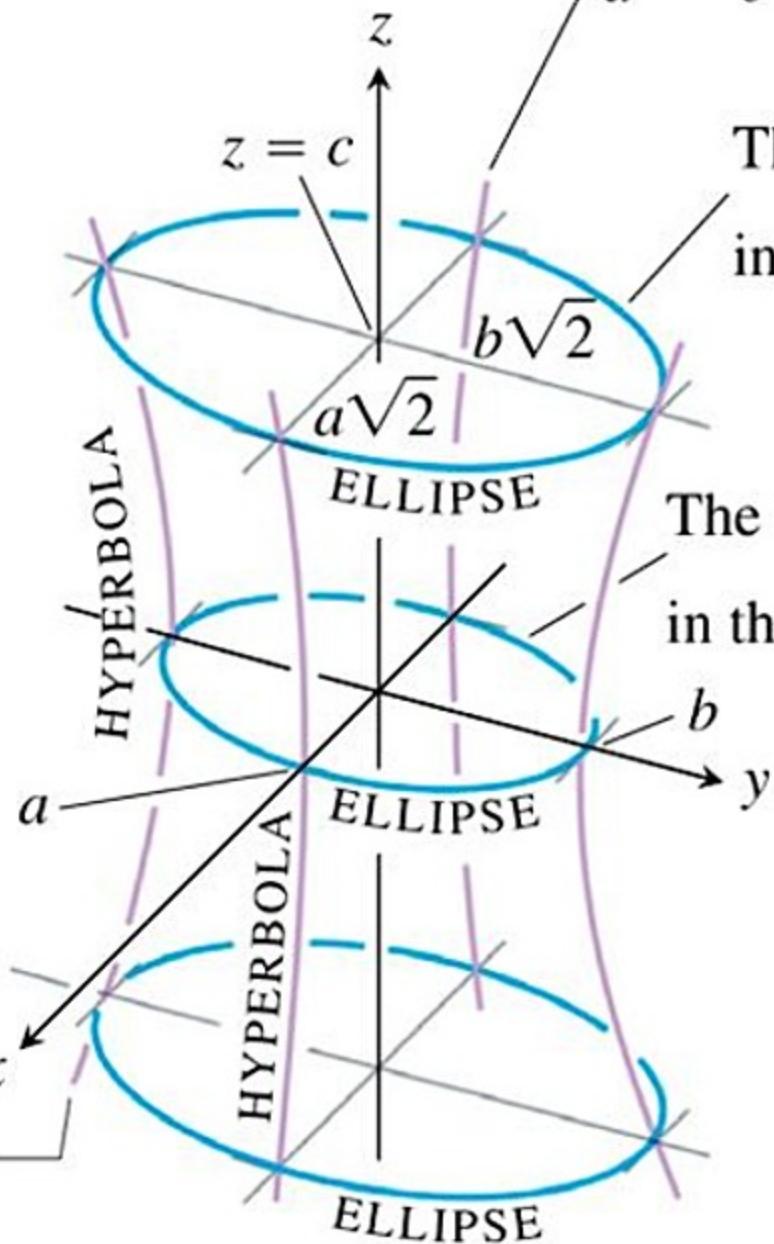
ELLIPTICAL CONE

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

Table 11.1 Graphs of Quadric Surfaces (cont)

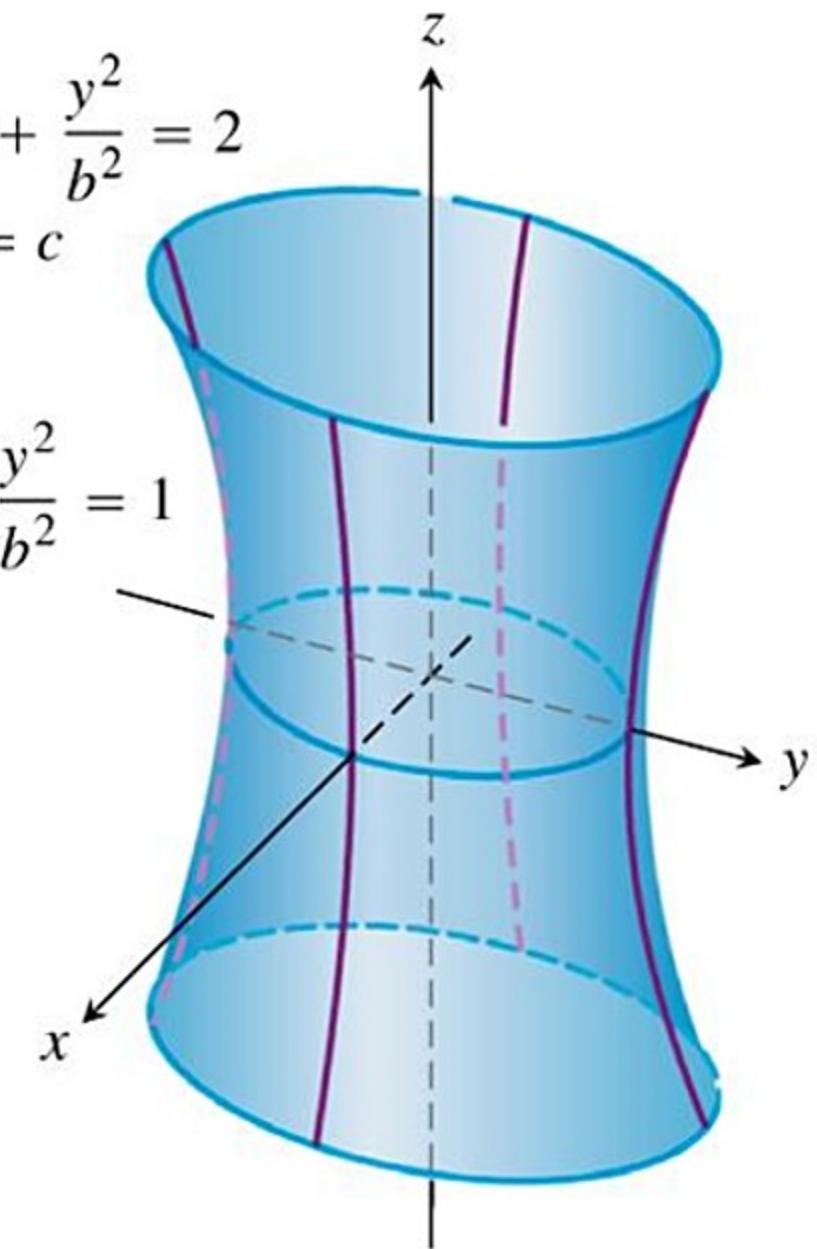
Part of the hyperbola $\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$
in the yz -plane

Part of the hyperbola $\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1$ in the xz -plane



The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
in the plane $z = c$

The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
in the xy -plane



HYPEROLOID OF ONE SHEET

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

Table 11.1 Graphs of Quadric Surfaces (cont)

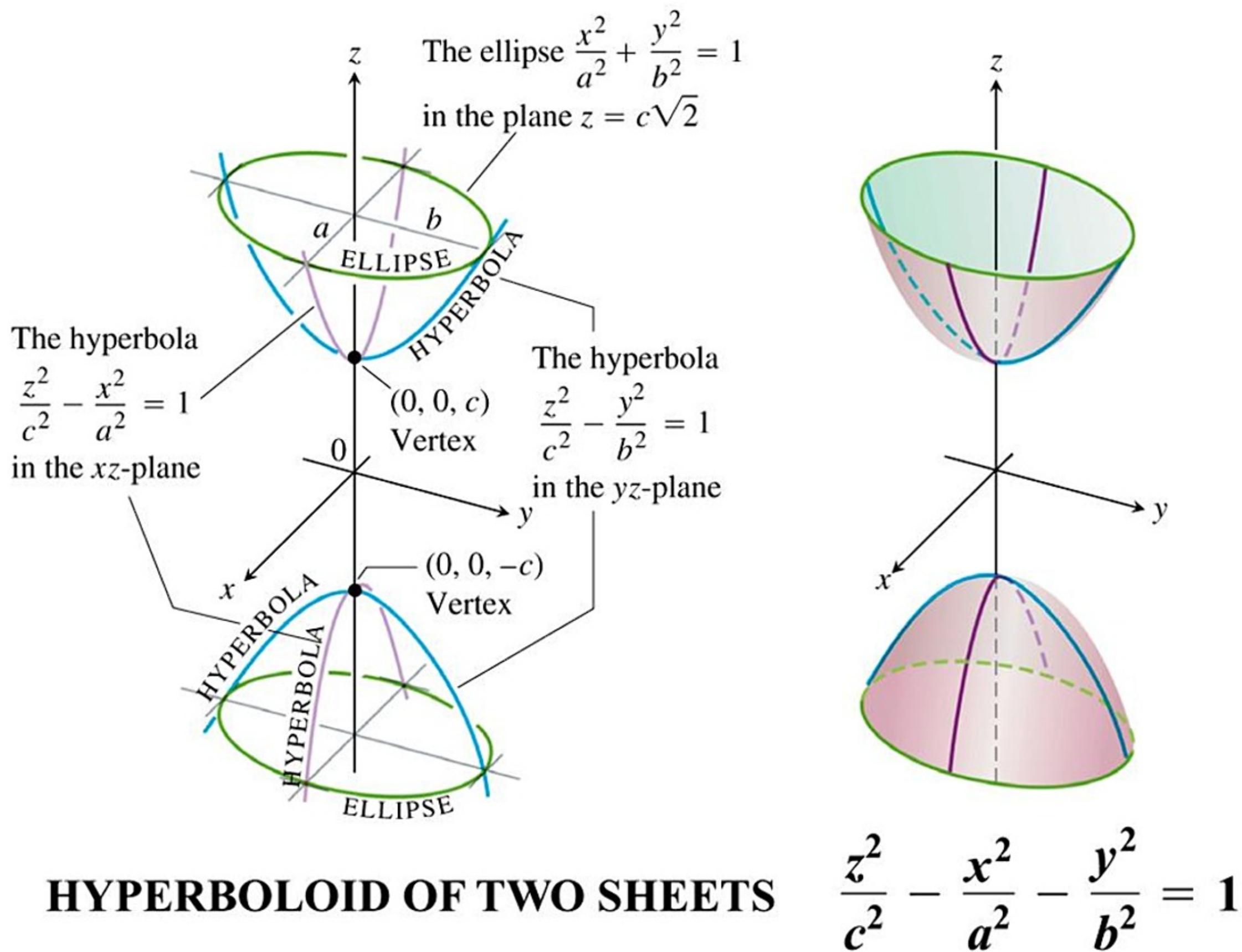
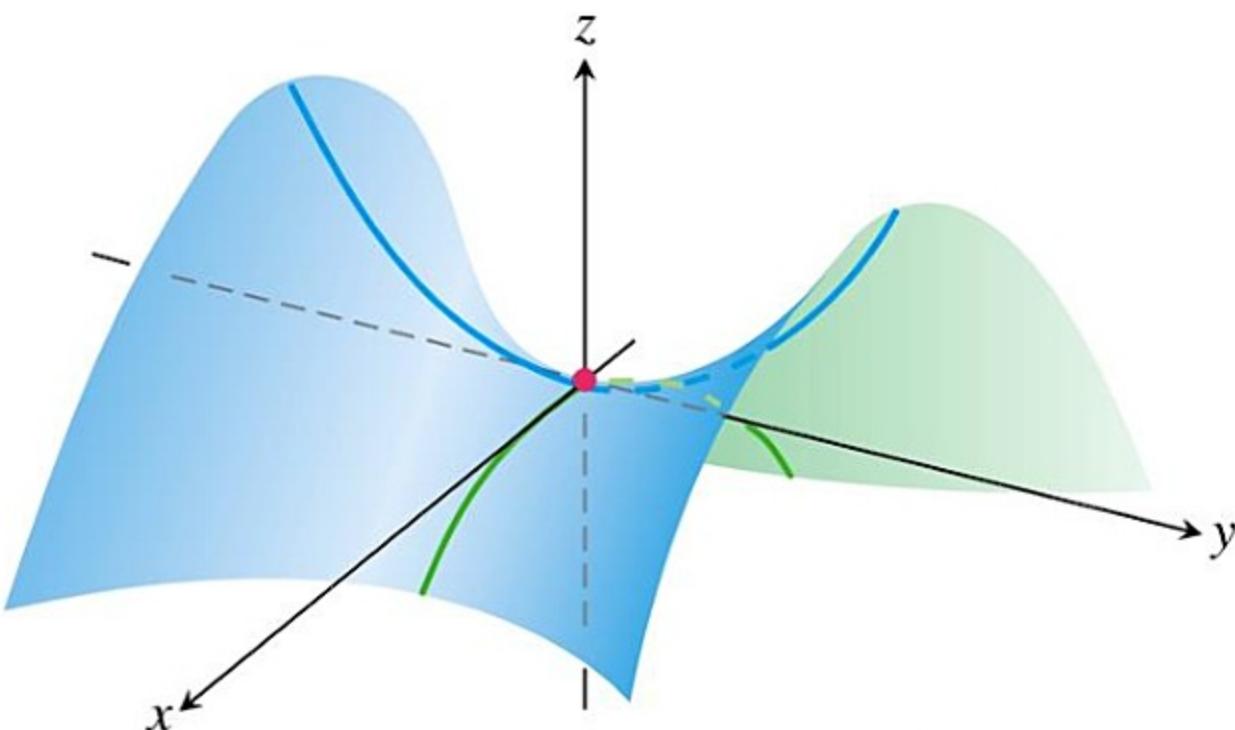
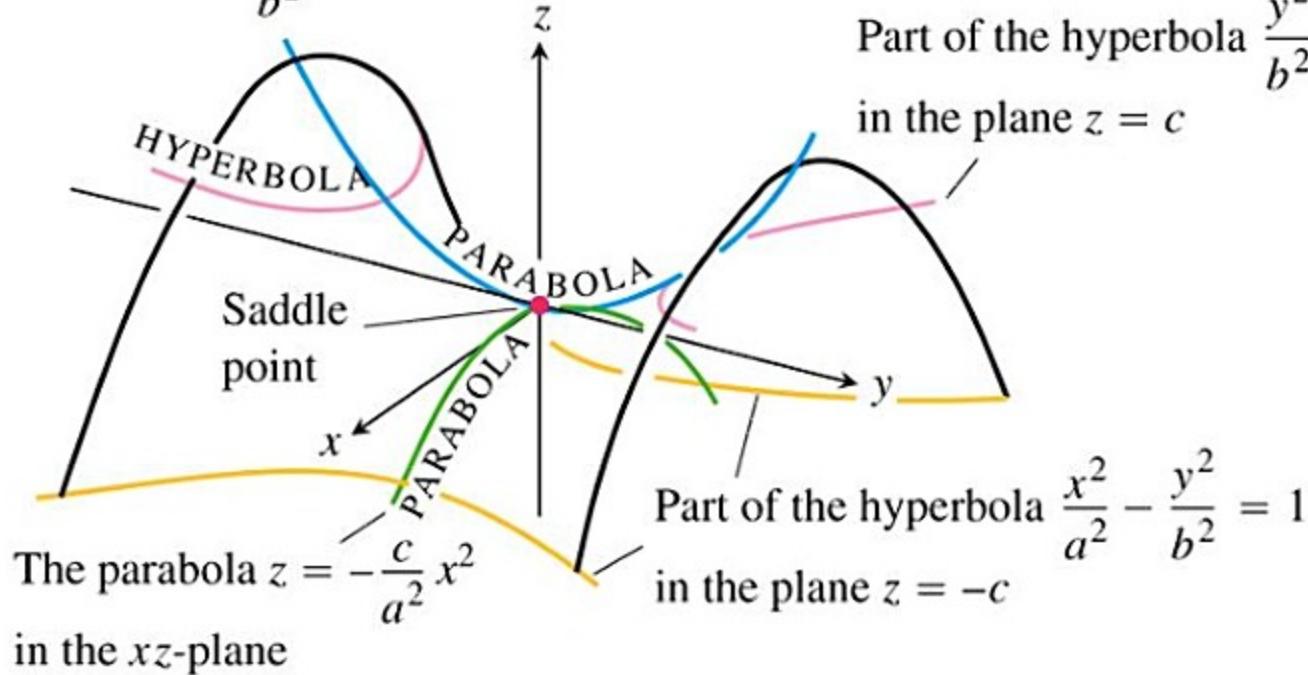


Table 11.1 Graphs of Quadric Surfaces (cont)

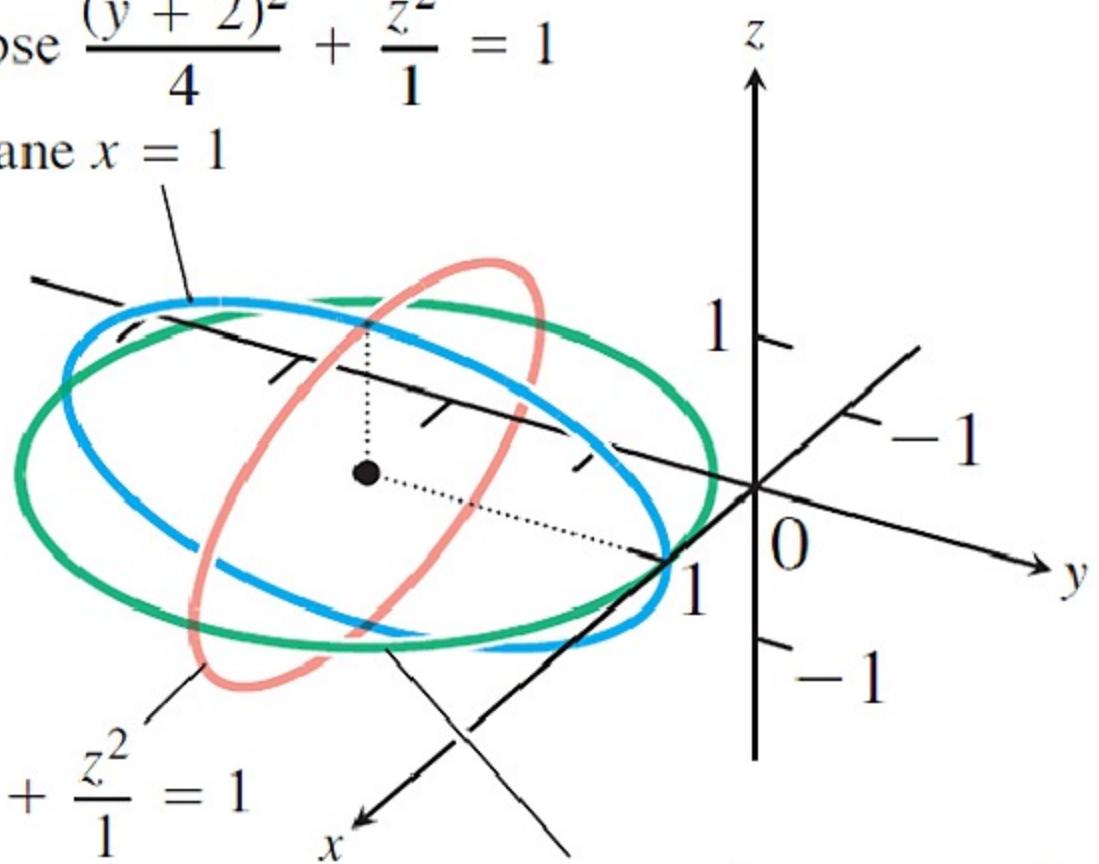
The parabola $z = \frac{c}{b^2} y^2$ in the yz -plane



HYPERBOLIC PARABOLOID $\frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z}{c}, \quad c > 0$

The ellipse $\frac{(y+2)^2}{4} + \frac{z^2}{1} = 1$

in the plane $x = 1$



The ellipse $\frac{(x-1)^2}{4} + \frac{z^2}{1} = 1$

in the plane $y = -2$

The ellipse $\frac{(x-1)^2}{4} + \frac{(y+2)^2}{4} = 1$

in the plane $z = 0$

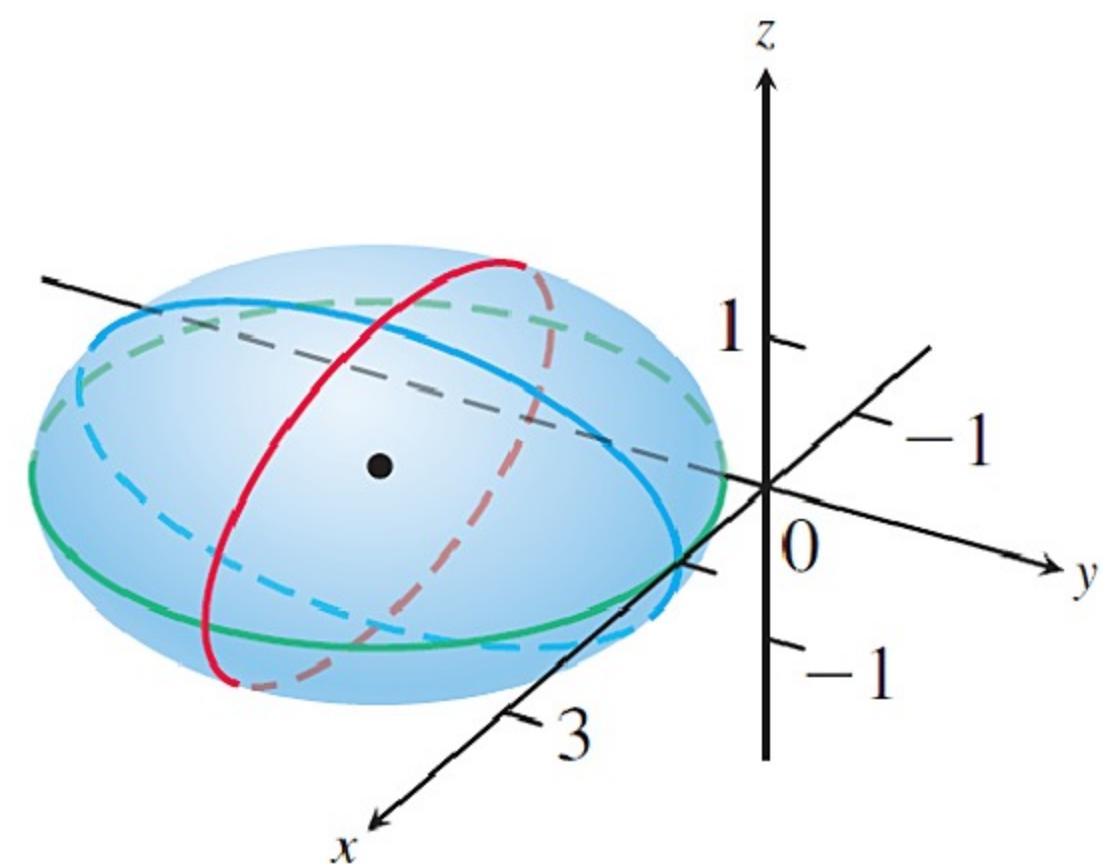


FIGURE 11.48 An ellipsoid centered at the point $(1, -2, 0)$.