

Chapter 10

Parametric Equations and Polar Coordinates

Thomas' Calculus, 14e in SI Units

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Section 10.1

Parametrizations of Plane Curves

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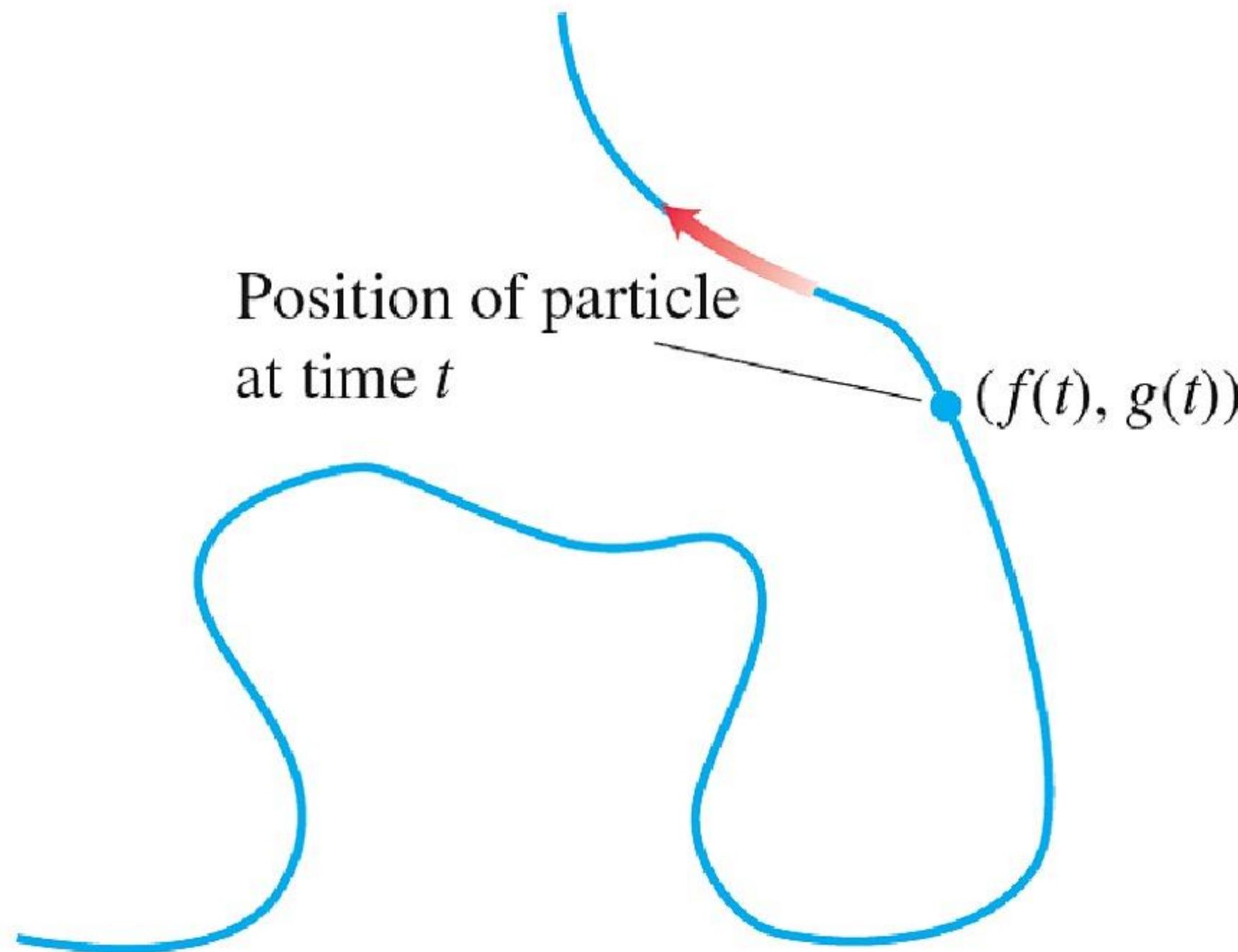


FIGURE 10.1 The curve or path traced by a particle moving in the xy -plane is not always the graph of a function or single equation.

DEFINITION

If x and y are given as functions

$$x = f(t), \quad y = g(t)$$

over an interval I of t -values, then the set of points $(x, y) = (f(t), g(t))$ defined by these equations is a **parametric curve**. The equations are **parametric equations** for the curve.

TABLE 10.1 Values of $x = \sin \pi t/2$
and $y = t$ for selected values of t .

t	x	y
0	0	0
1	1	1
2	0	2
3	-1	3
4	0	4
5	1	5
6	0	6

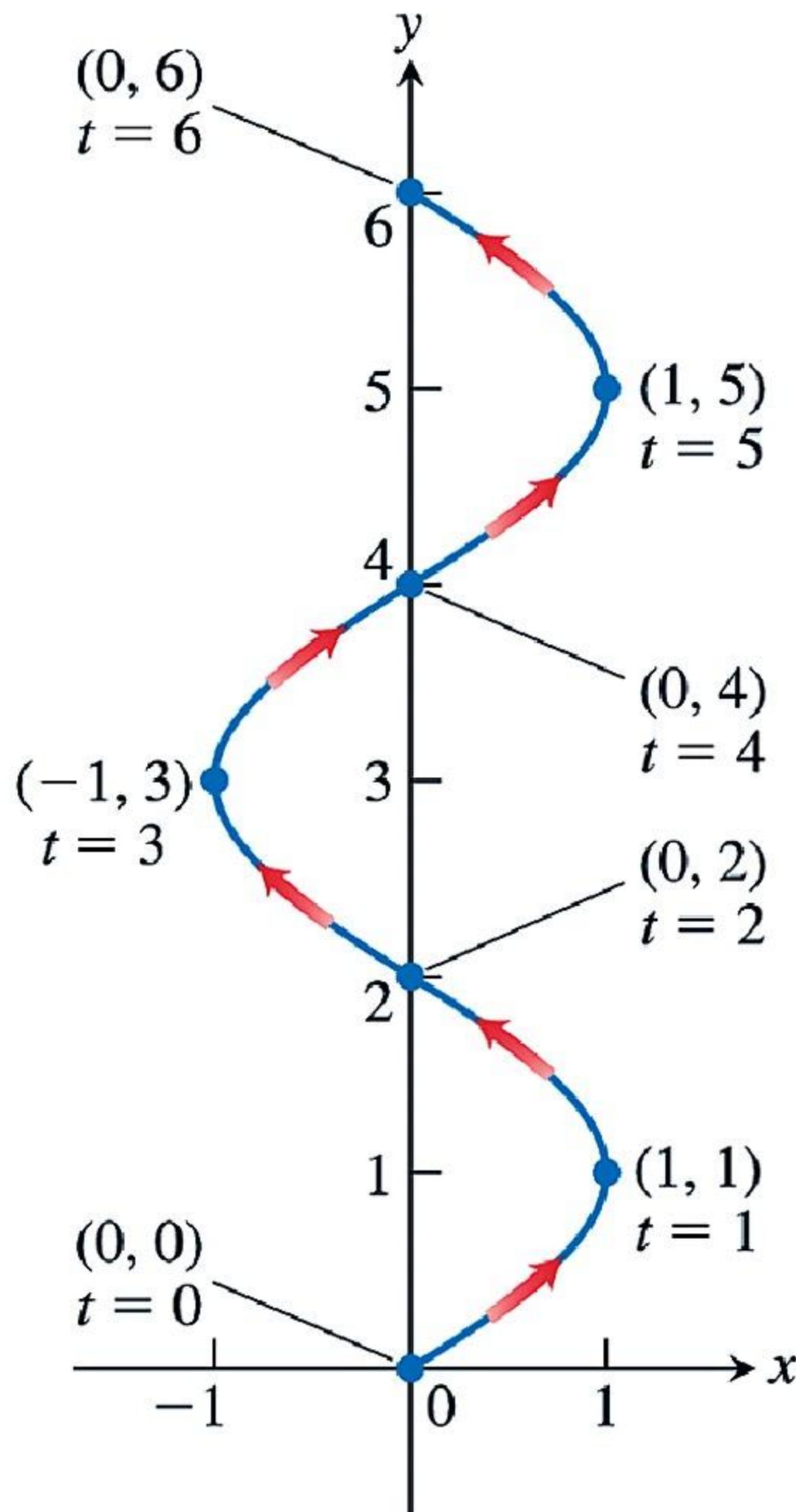


FIGURE 10.2 The curve given by the parametric equations $x = \sin \pi t/2$ and $y = t$ (Example 1).

TABLE 10.2 Values of $x = t^2$ and
 $y = t + 1$ for selected values of t .

t	x	y
-3	9	-2
-2	4	-1
-1	1	0
0	0	1
1	1	2
2	4	3
3	9	4

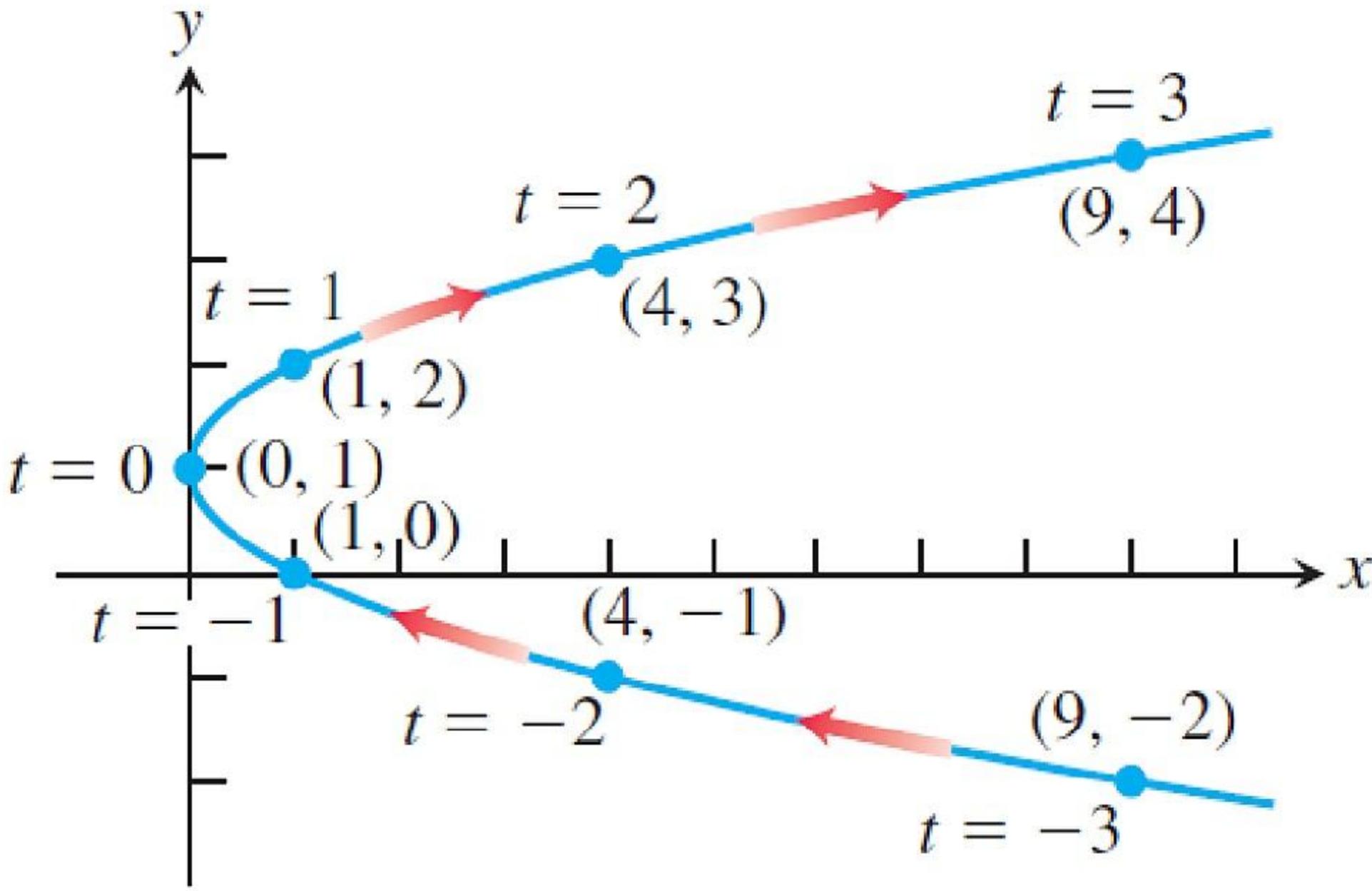


FIGURE 10.3 The curve given by the parametric equations $x = t^2$ and $y = t + 1$ (Example 2).

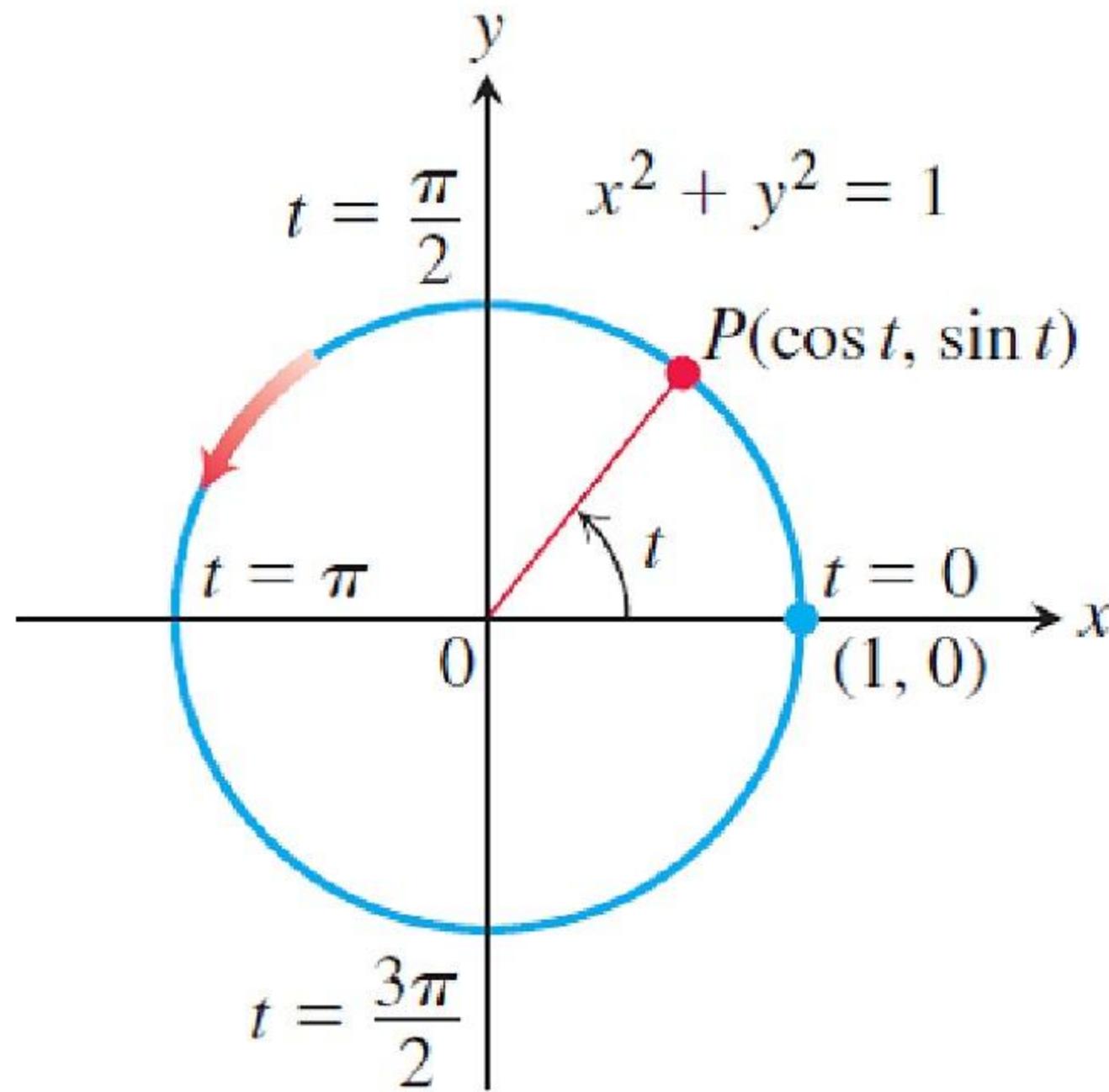


FIGURE 10.4 The equations $x = \cos t$ and $y = \sin t$ describe motion on the circle $x^2 + y^2 = 1$. The arrow shows the direction of increasing t (Example 3).

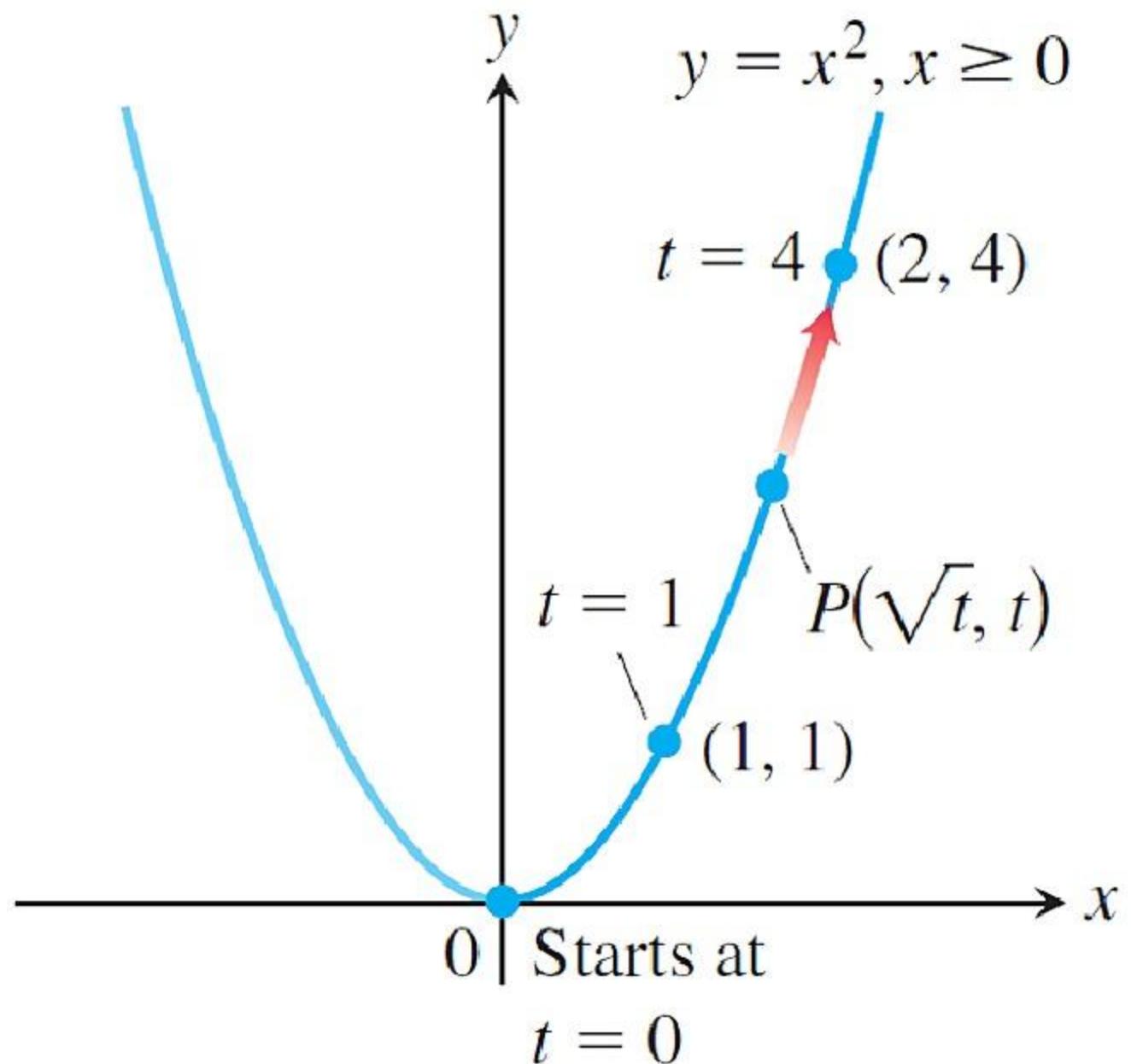


FIGURE 10.5 The equations $x = \sqrt{t}$ and $y = t$ and the interval $t \geq 0$ describe the path of a particle that traces the right-hand half of the parabola $y = x^2$ (Example 4).

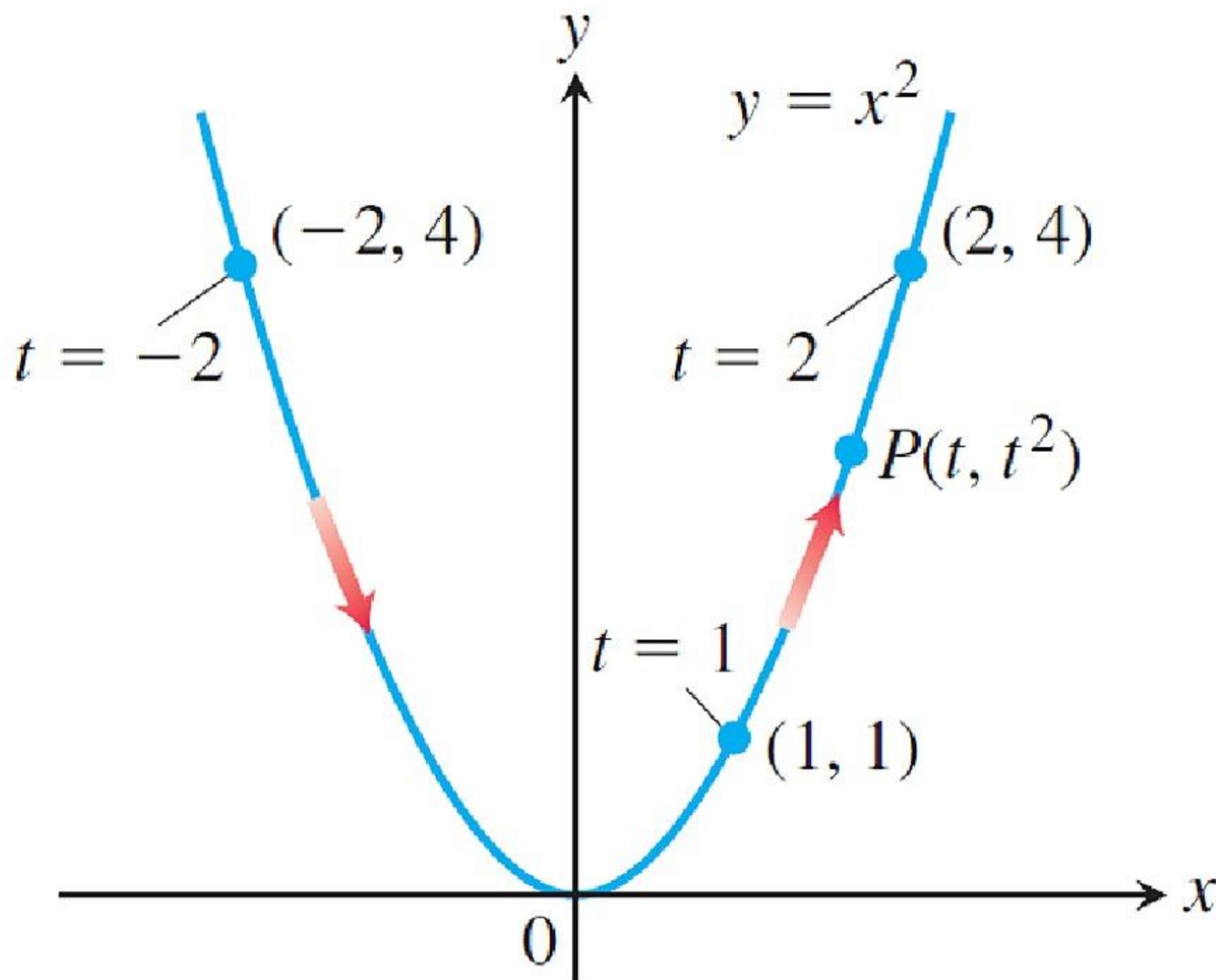


FIGURE 10.6 The path defined by $x = t, y = t^2, -\infty < t < \infty$ is the entire parabola $y = x^2$ (Example 5).

TABLE 10.3 Values of $x = t + (1/t)$
and $y = t - (1/t)$ for selected
values of t .

t	$1/t$	x	y
0.1	10.0	10.1	-9.9
0.2	5.0	5.2	-4.8
0.4	2.5	2.9	-2.1
1.0	1.0	2.0	0.0
2.0	0.5	2.5	1.5
5.0	0.2	5.2	4.8
10.0	0.1	10.1	9.9

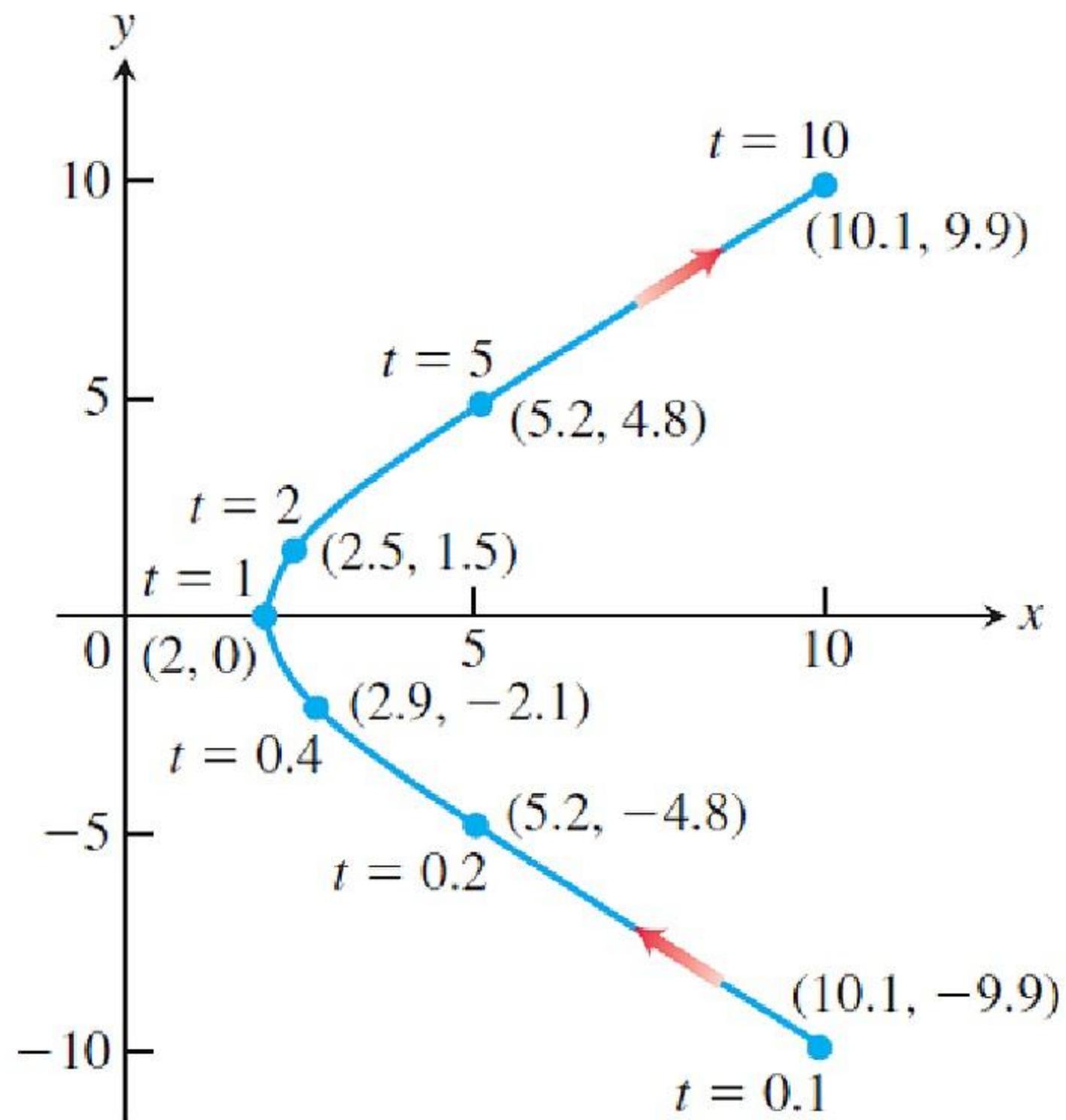


FIGURE 10.7 The curve for
 $x = t + (1/t)$, $y = t - (1/t)$, $t > 0$
in Example 7. (The part shown is for
 $0.1 \leq t \leq 10$.)

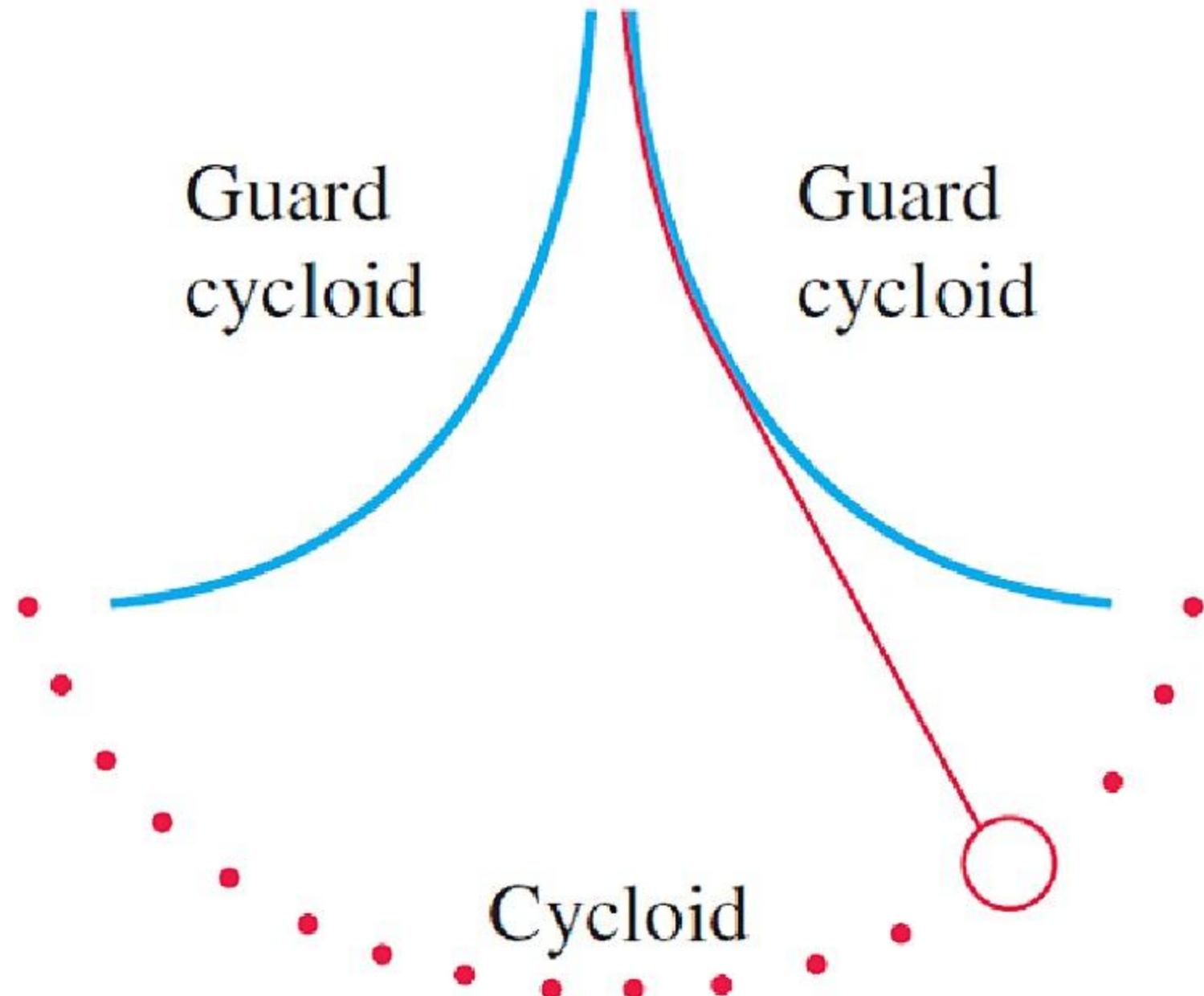


FIGURE 10.8 In Huygens' pendulum clock, the bob swings in a cycloid, so the frequency is independent of the amplitude.

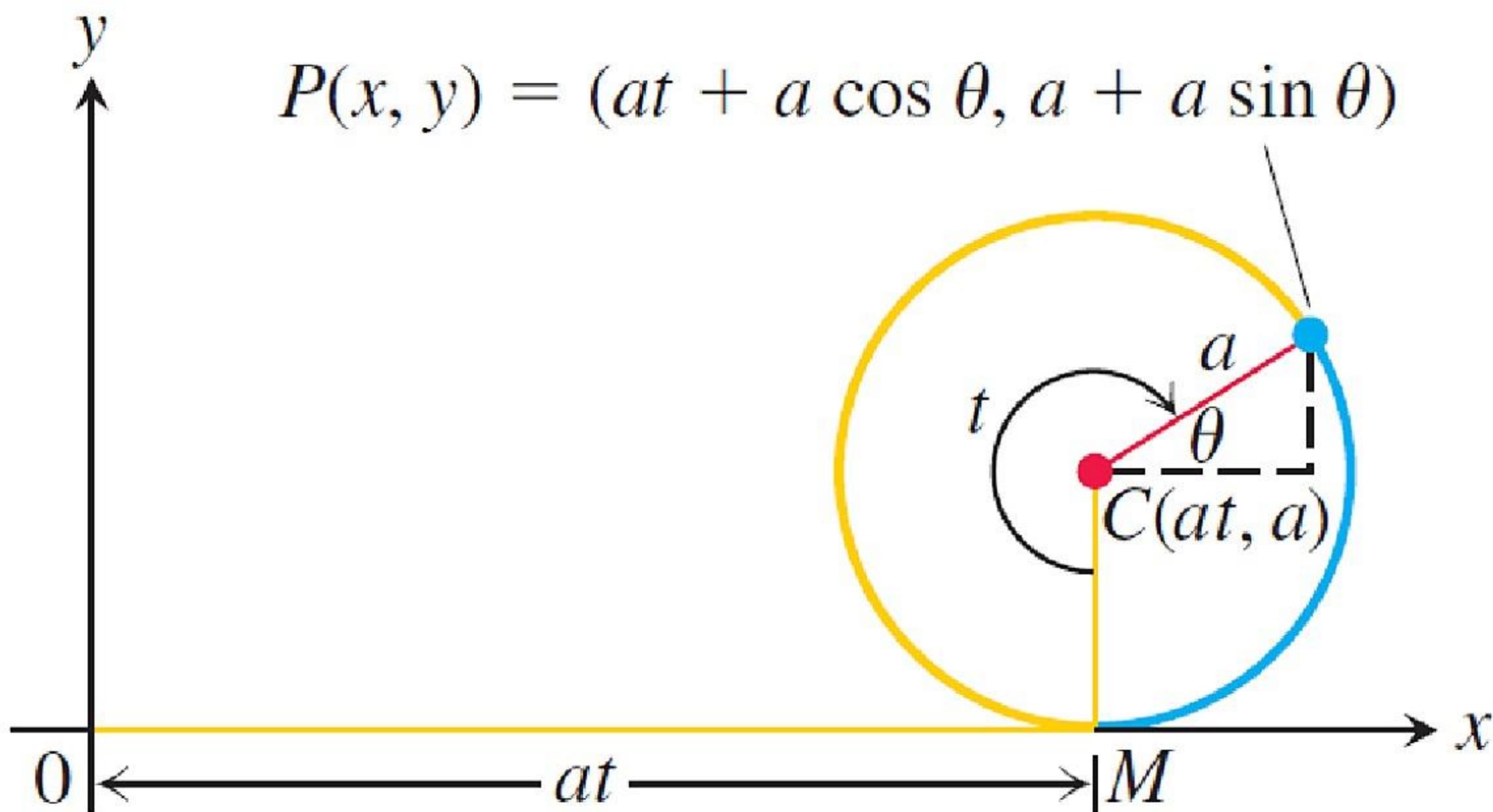


FIGURE 10.9 The position of $P(x, y)$ on the rolling wheel at angle t (Example 8).

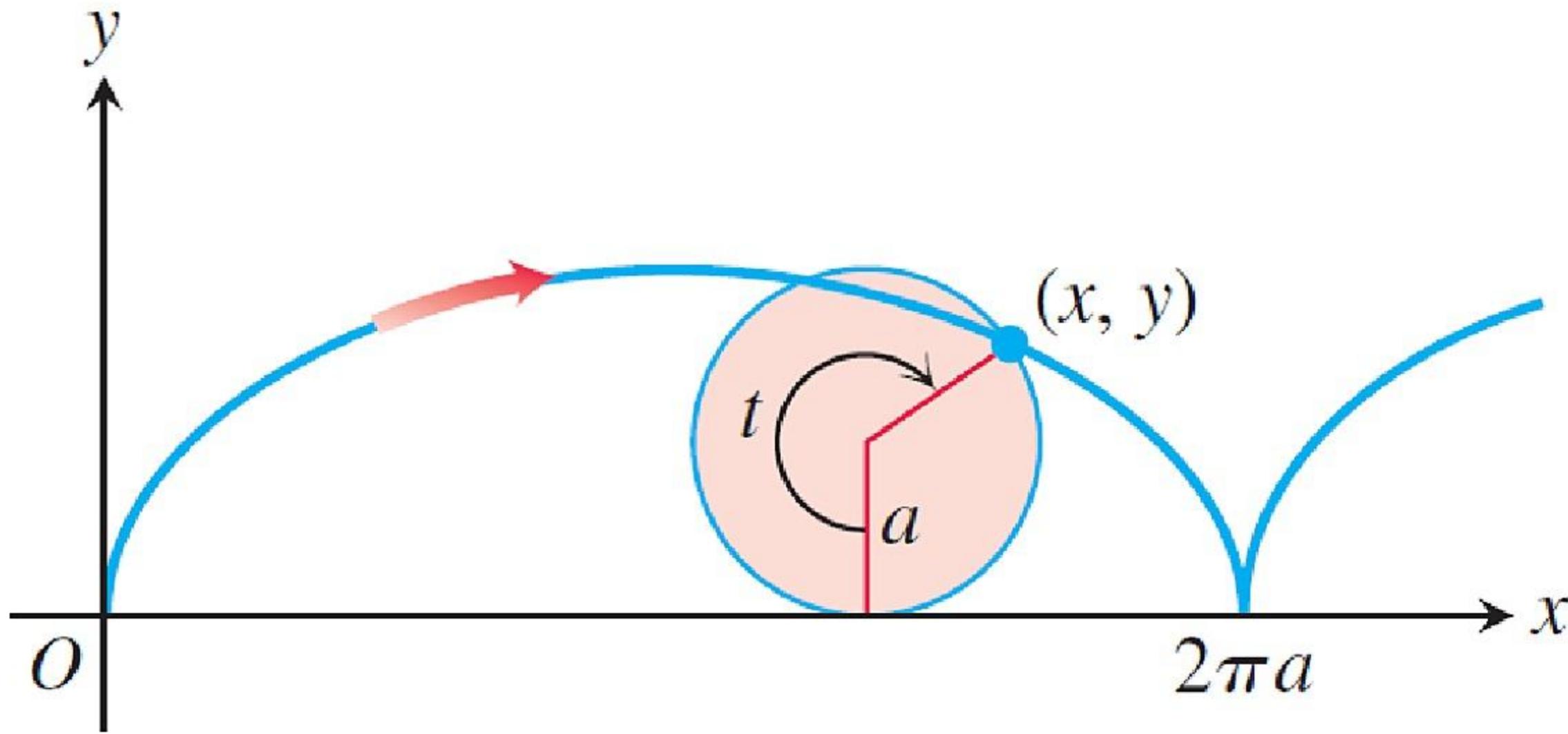


FIGURE 10.10 The cycloid curve

$x = a(t - \sin t)$, $y = a(1 - \cos t)$, for
 $t \geq 0$.

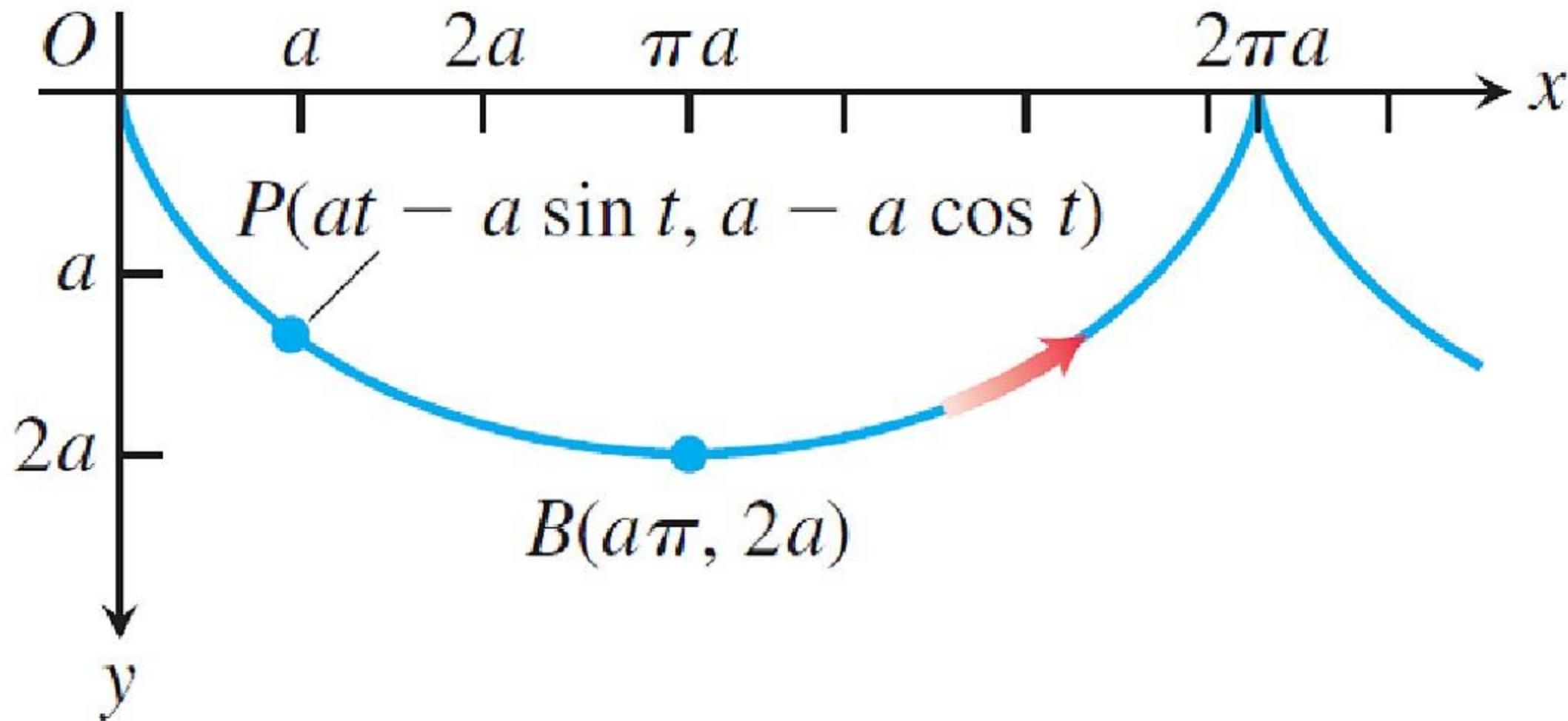


FIGURE 10.11 Turning Figure 10.10 upside down, the y -axis points downward, indicating the direction of the gravitational force. Equations (2) still describe the curve parametrically.

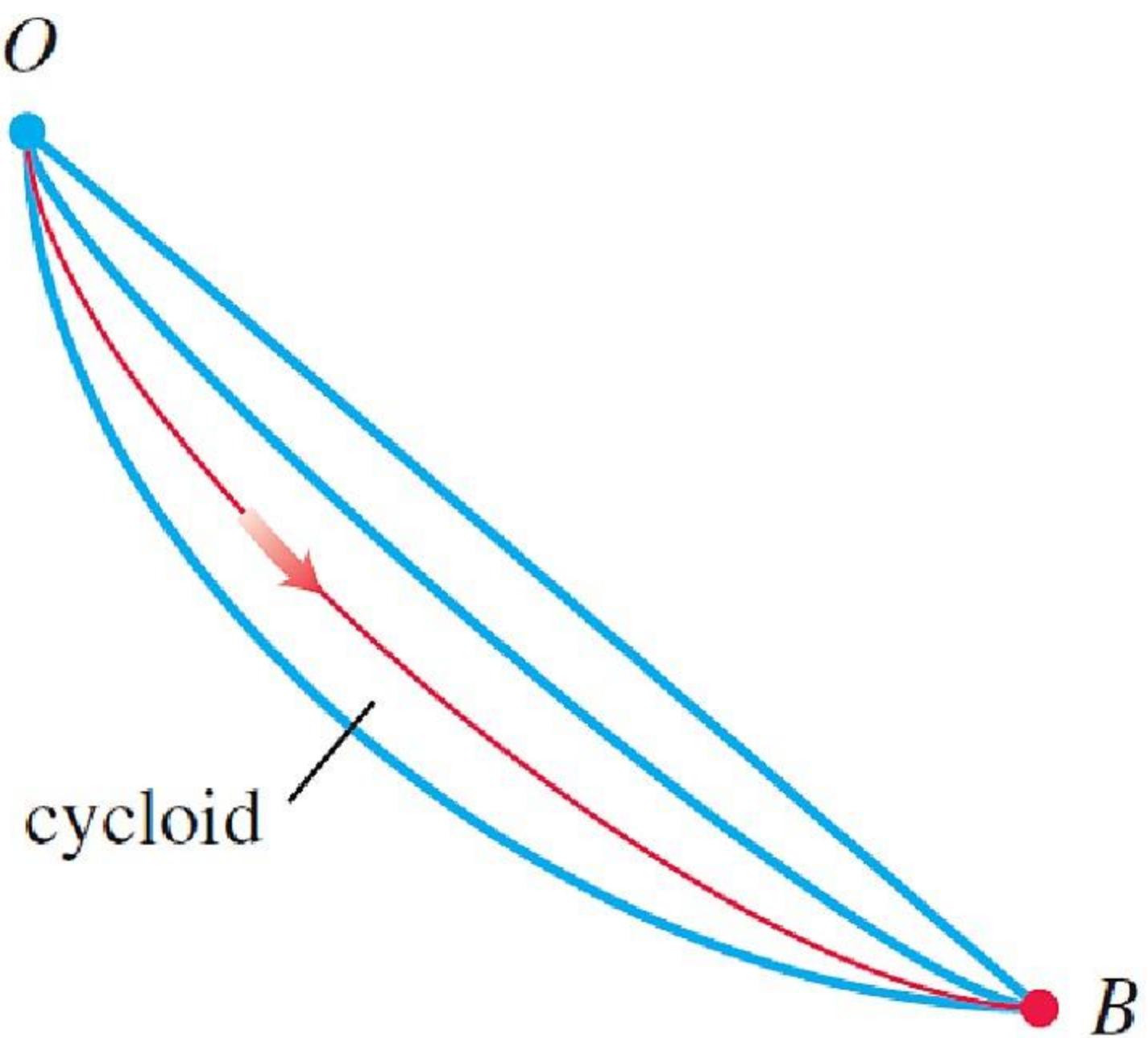


FIGURE 10.12 The cycloid is the unique curve which minimizes the time it takes for a frictionless bead to slide from point O to B .

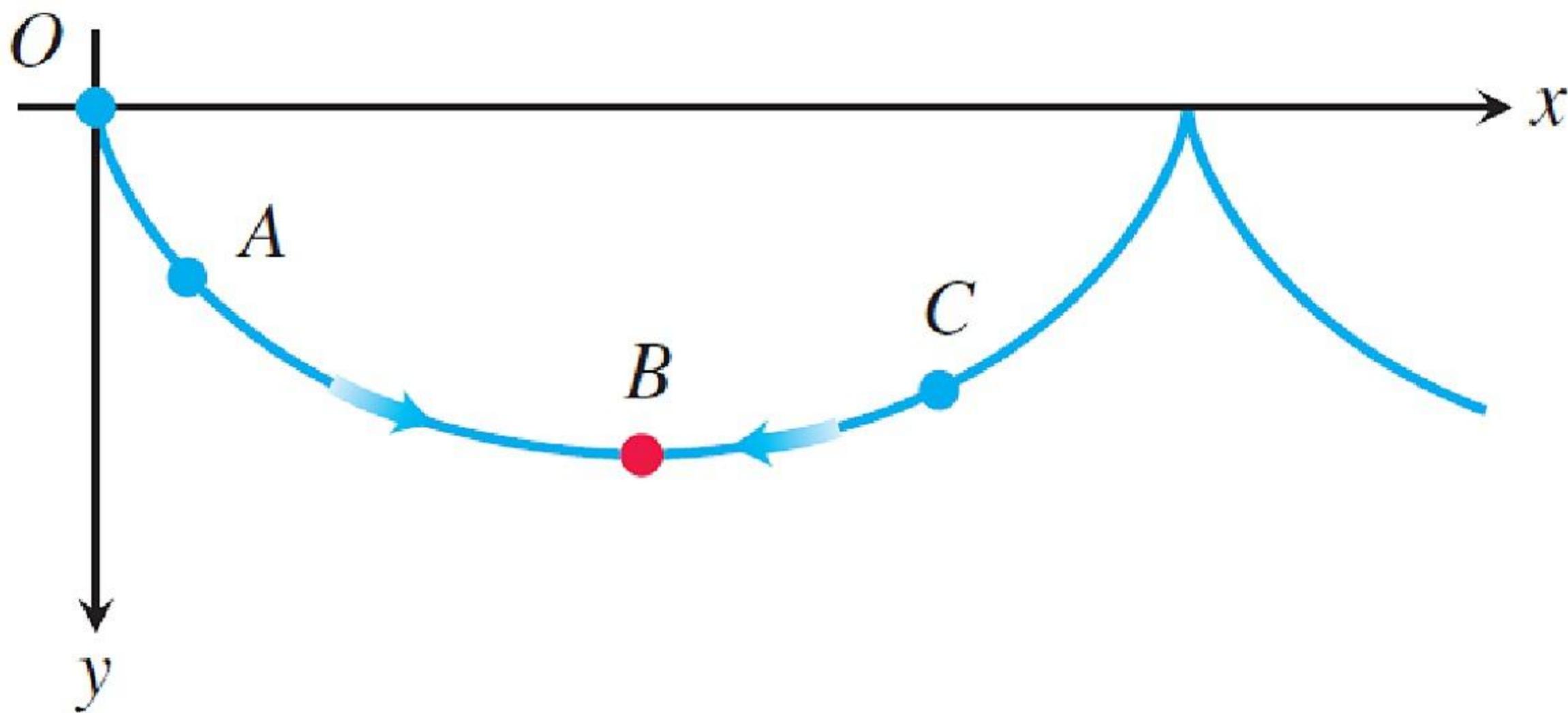


FIGURE 10.13 Beads released simultaneously on the upside-down cycloid at O , A , and C will reach B at the same time.

Section 10.2

Calculus with Parametric Curves

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Parametric Formula for dy/dx

If all three derivatives exist and $dx/dt \neq 0$,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}. \quad (1)$$

Parametric Formula for d^2y/dx^2

If the equations $x = f(t)$, $y = g(t)$ define y as a twice-differentiable function of x , then at any point where $dx/dt \neq 0$ and $y' = dy/dx$,

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}. \quad (2)$$

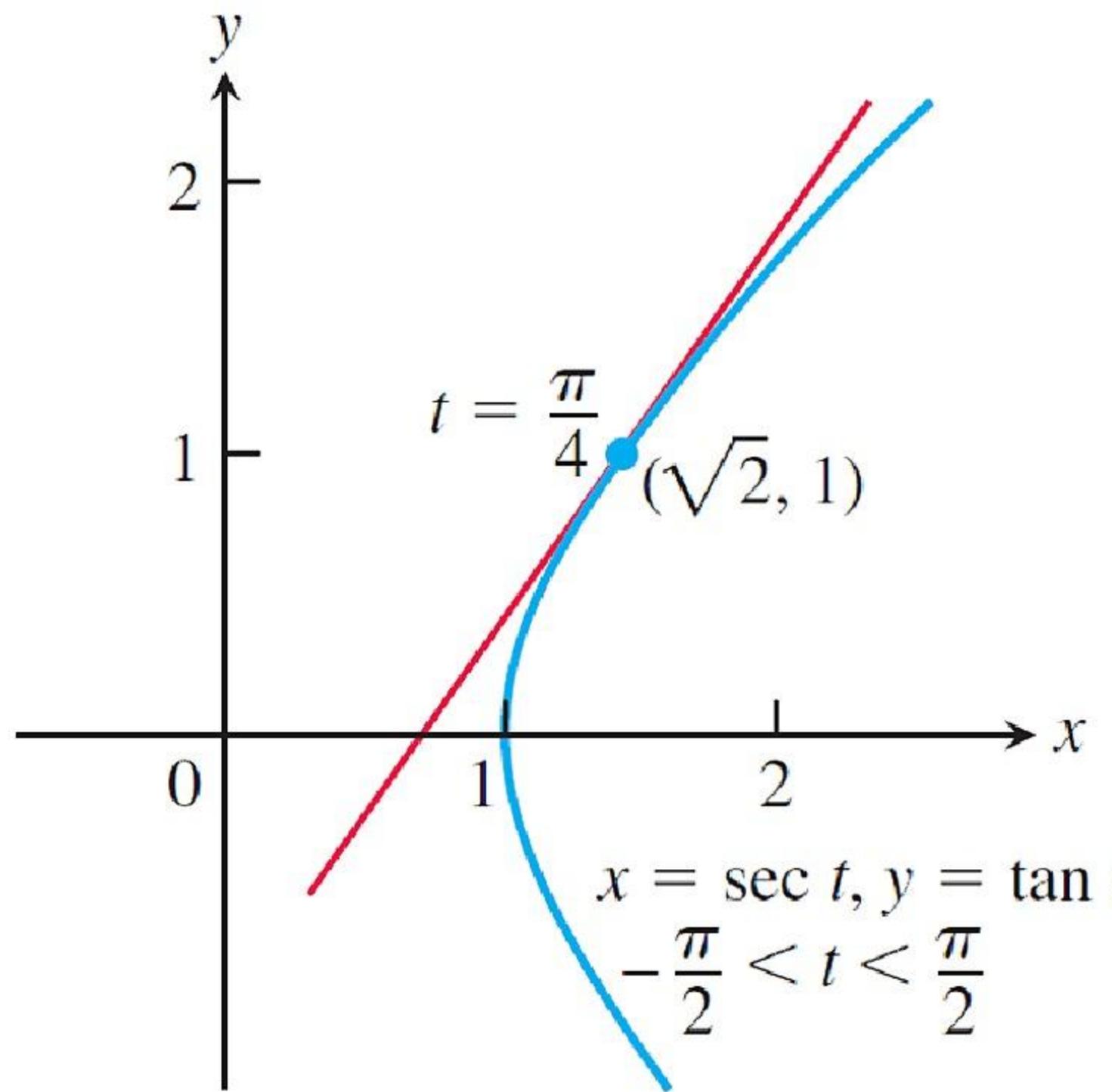


FIGURE 10.14 The curve in Example 1 is the right-hand branch of the hyperbola $x^2 - y^2 = 1$.

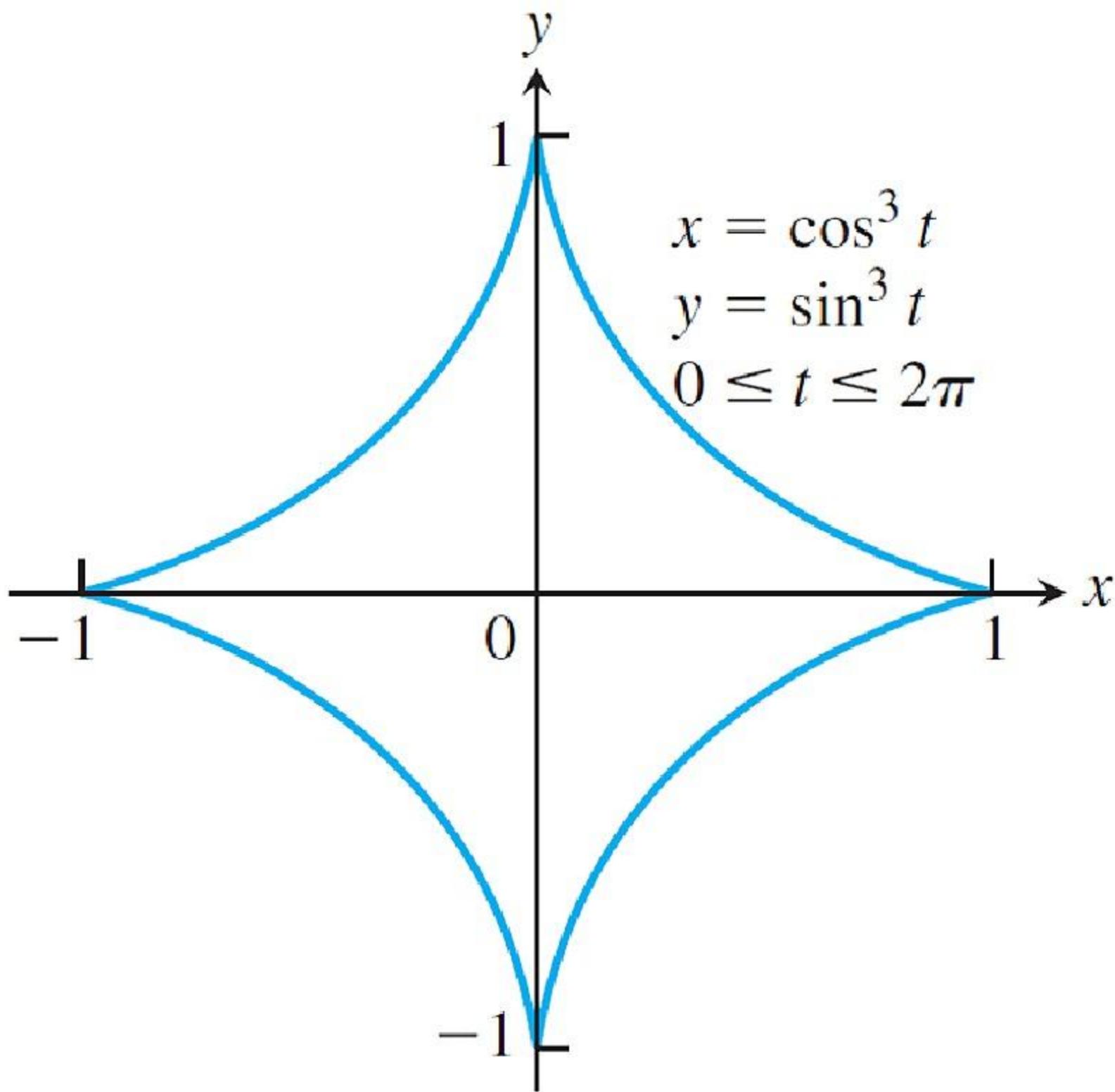


FIGURE 10.15 The astroid in Example 3.

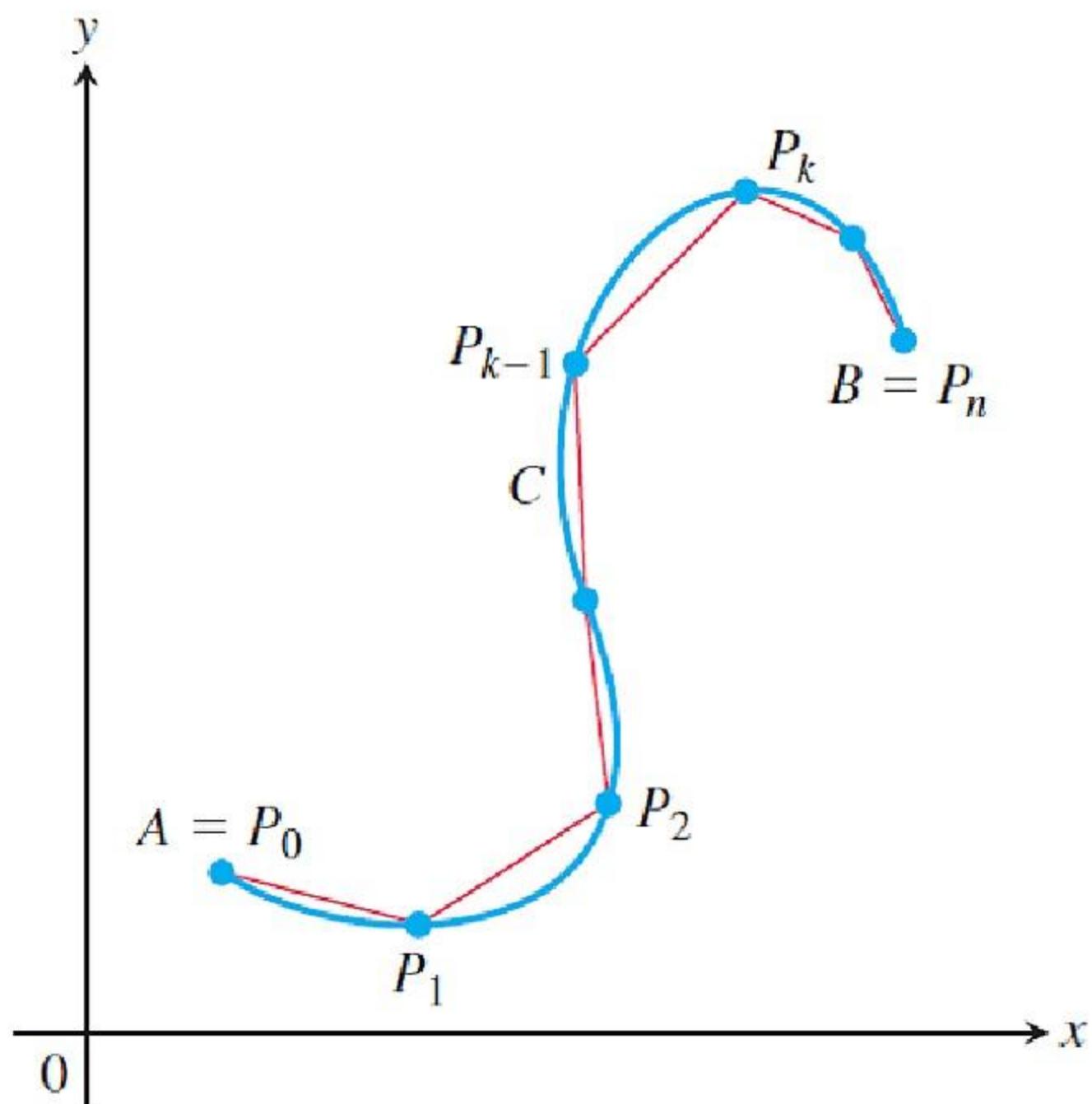


FIGURE 10.16 The length of the smooth curve C from A to B is approximated by the sum of the lengths of the polygonal path (straight-line segments) starting at $A = P_0$, then to P_1 , and so on, ending at $B = P_n$.

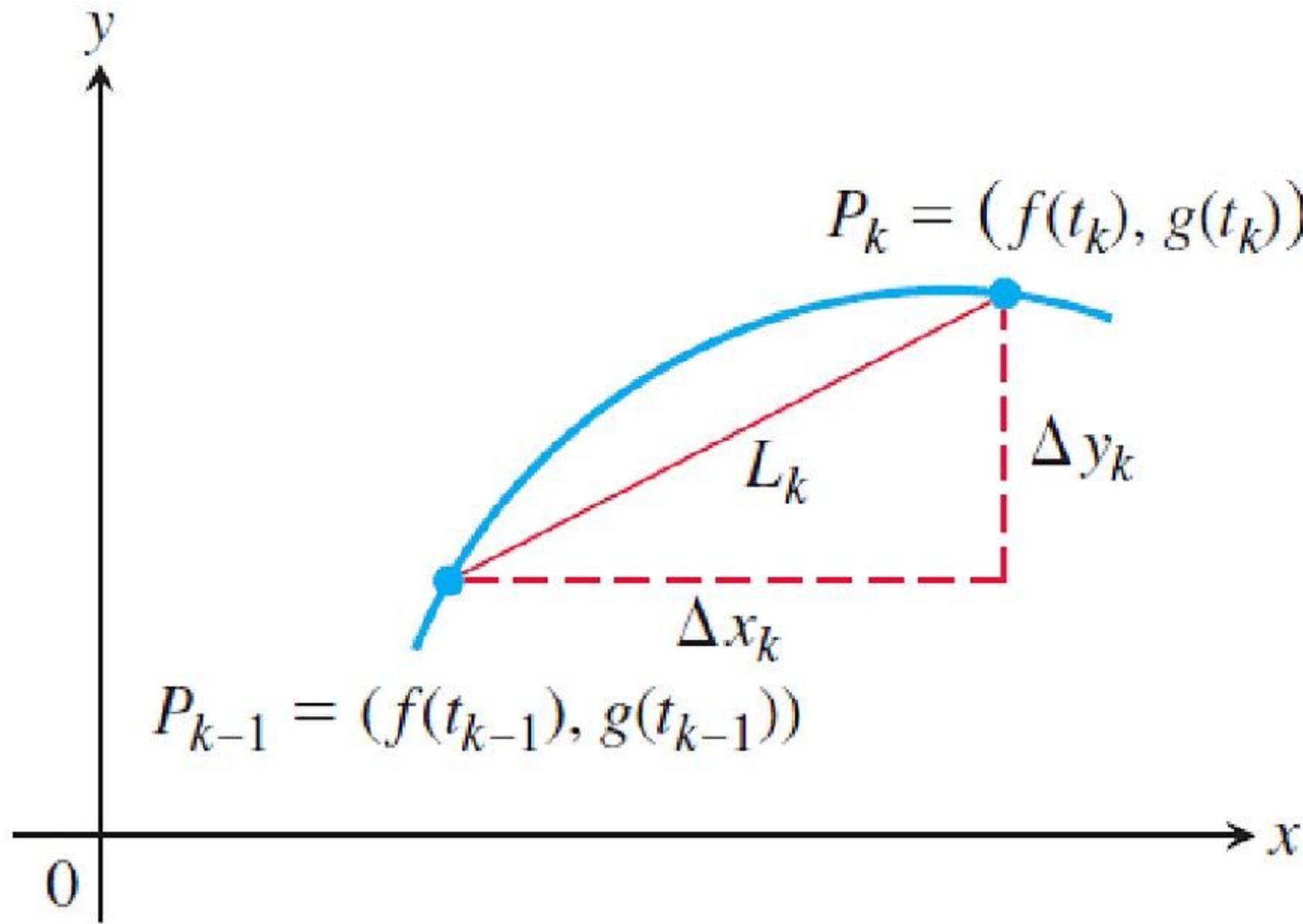


FIGURE 10.17 The arc $P_{k-1}P_k$ is approximated by the straight-line segment shown here, which has length $L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$.

DEFINITION If a curve C is defined parametrically by $x = f(t)$ and $y = g(t)$, $a \leq t \leq b$, where f' and g' are continuous and not simultaneously zero on $[a, b]$, and C is traversed exactly once as t increases from $t = a$ to $t = b$, then **the length of C** is the definite integral

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2} dt.$$

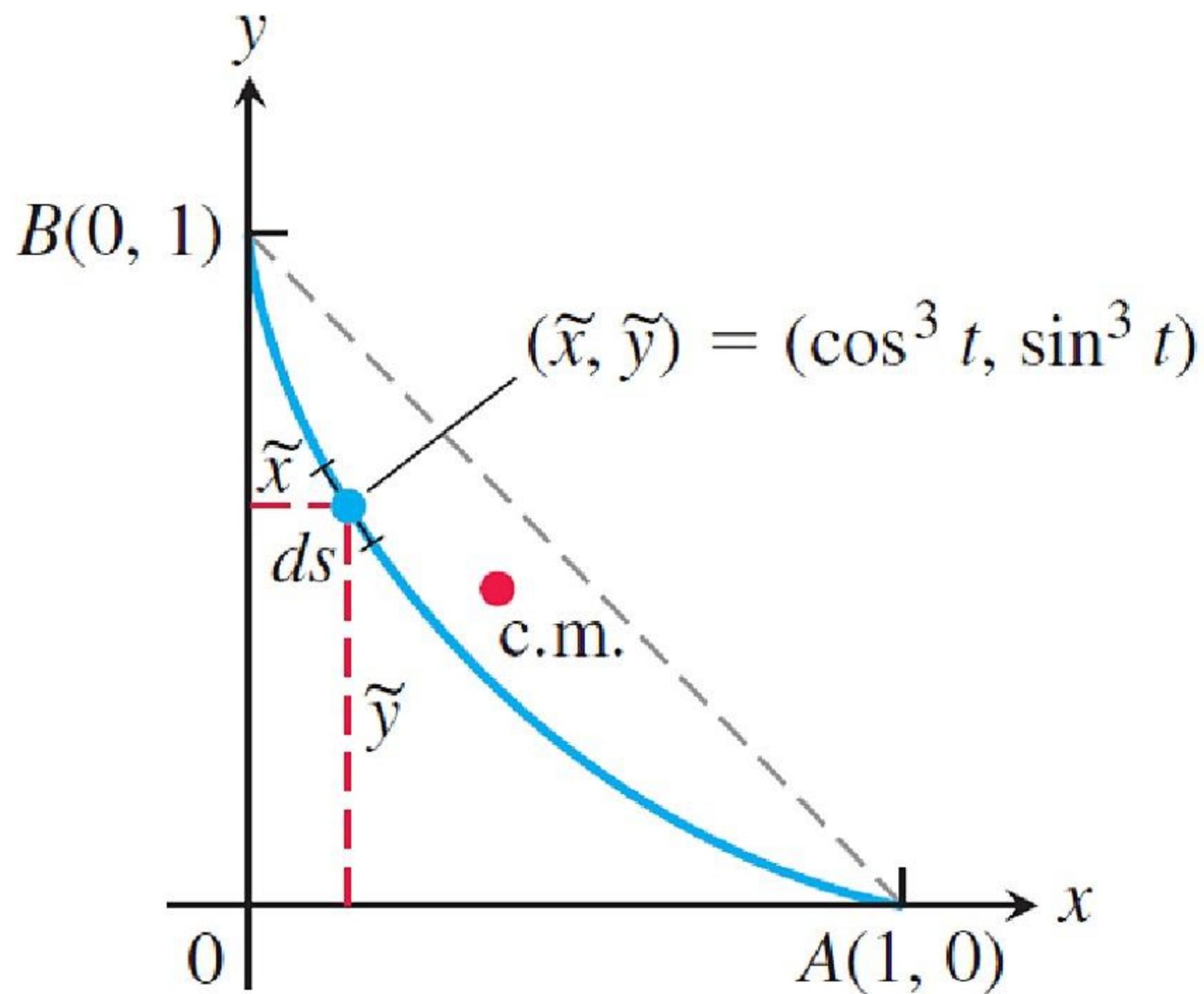


FIGURE 10.18 The centroid (c.m.) of the astroid arc in Example 7.

Area of Surface of Revolution for Parametrized Curves

If a smooth curve $x = f(t)$, $y = g(t)$, $a \leq t \leq b$, is traversed exactly once as t increases from a to b , then the areas of the surfaces generated by revolving the curve about the coordinate axes are as follows.

1. Revolution about the x -axis ($y \geq 0$):

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \quad (5)$$

2. Revolution about the y -axis ($x \geq 0$):

$$S = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \quad (6)$$

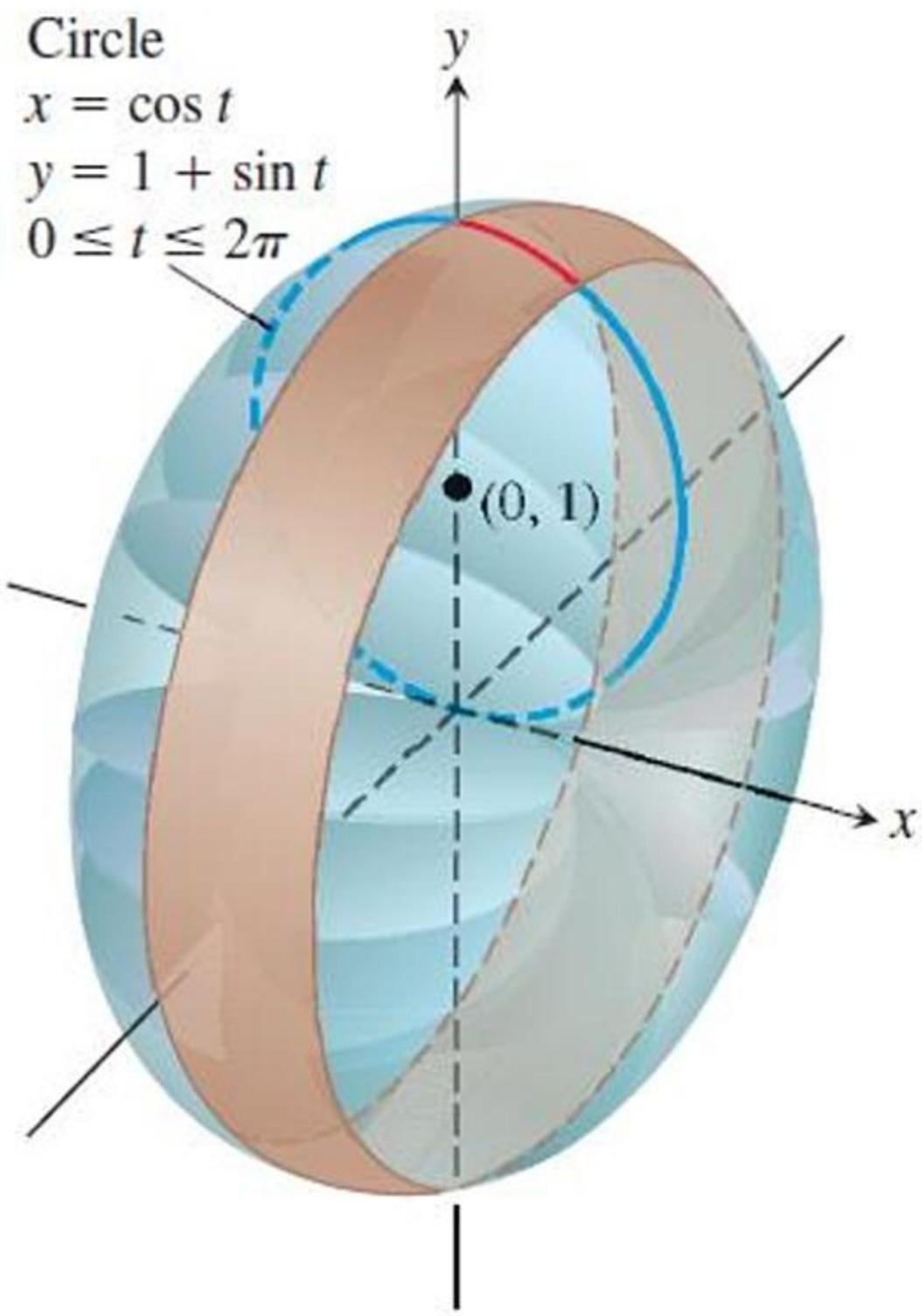


FIGURE 10.19 In Example 9 we calculate the area of the surface of revolution swept out by this parametrized curve.

10.3

Polar Coordinates

Polar Coordinates

$P(r, \theta)$

Directed distance
from O to P

Directed angle from
initial ray to OP

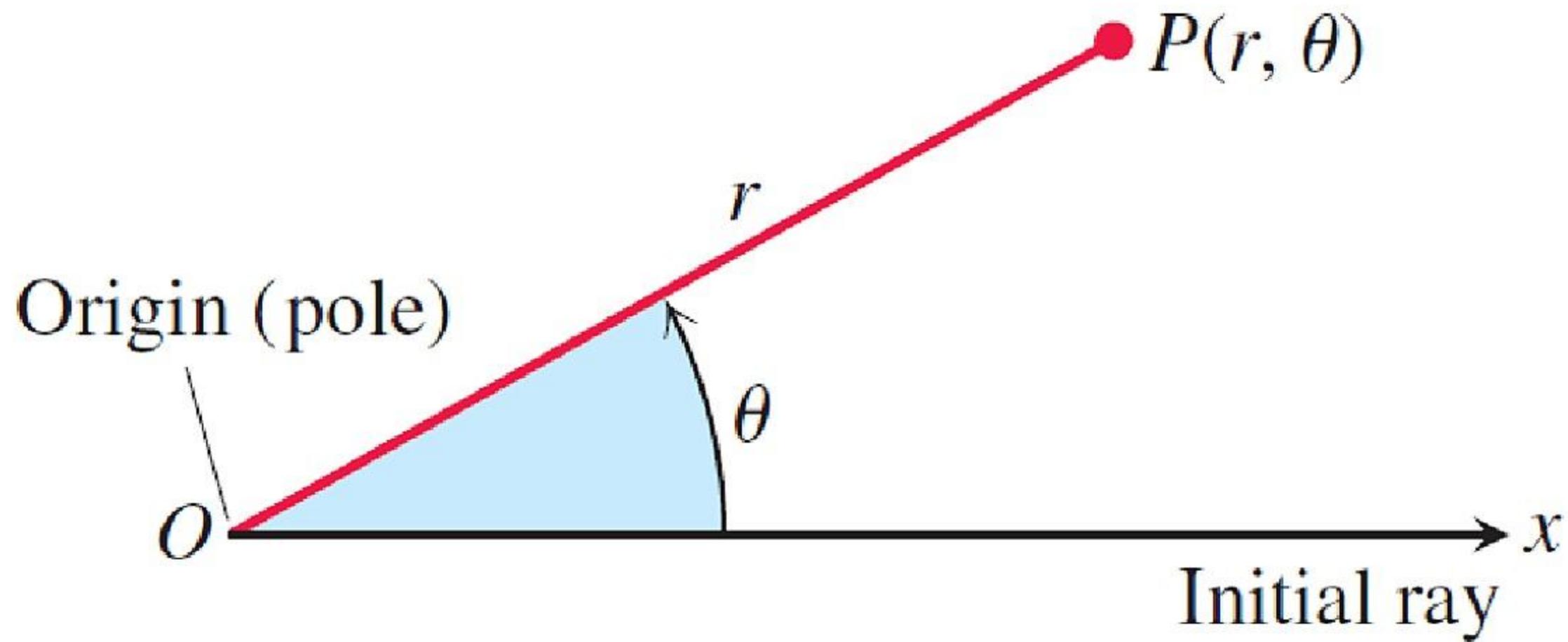


FIGURE 10.20 To define polar coordinates for the plane, we start with an origin, called the pole, and an initial ray.

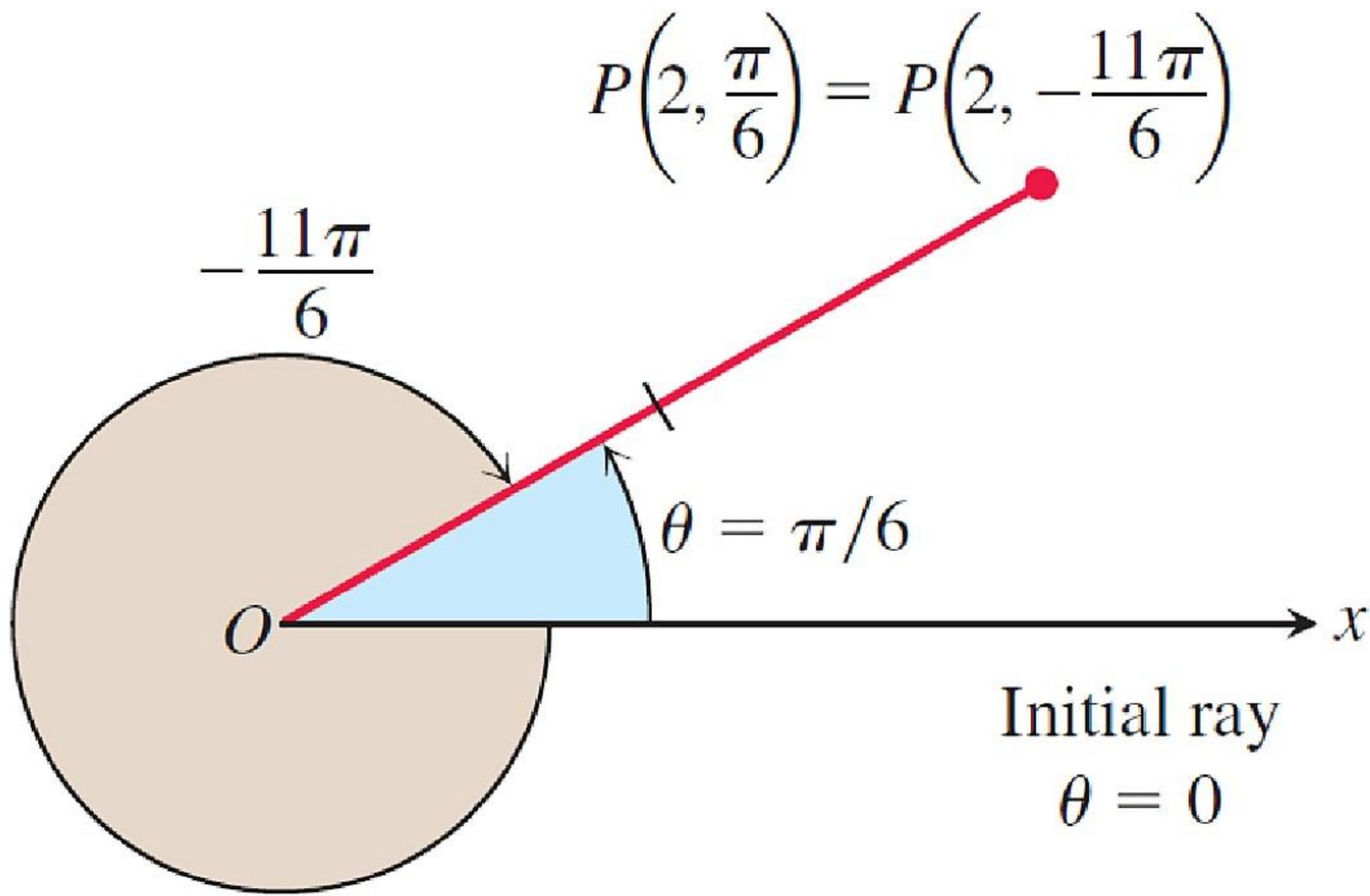


FIGURE 10.21 Polar coordinates are not unique.

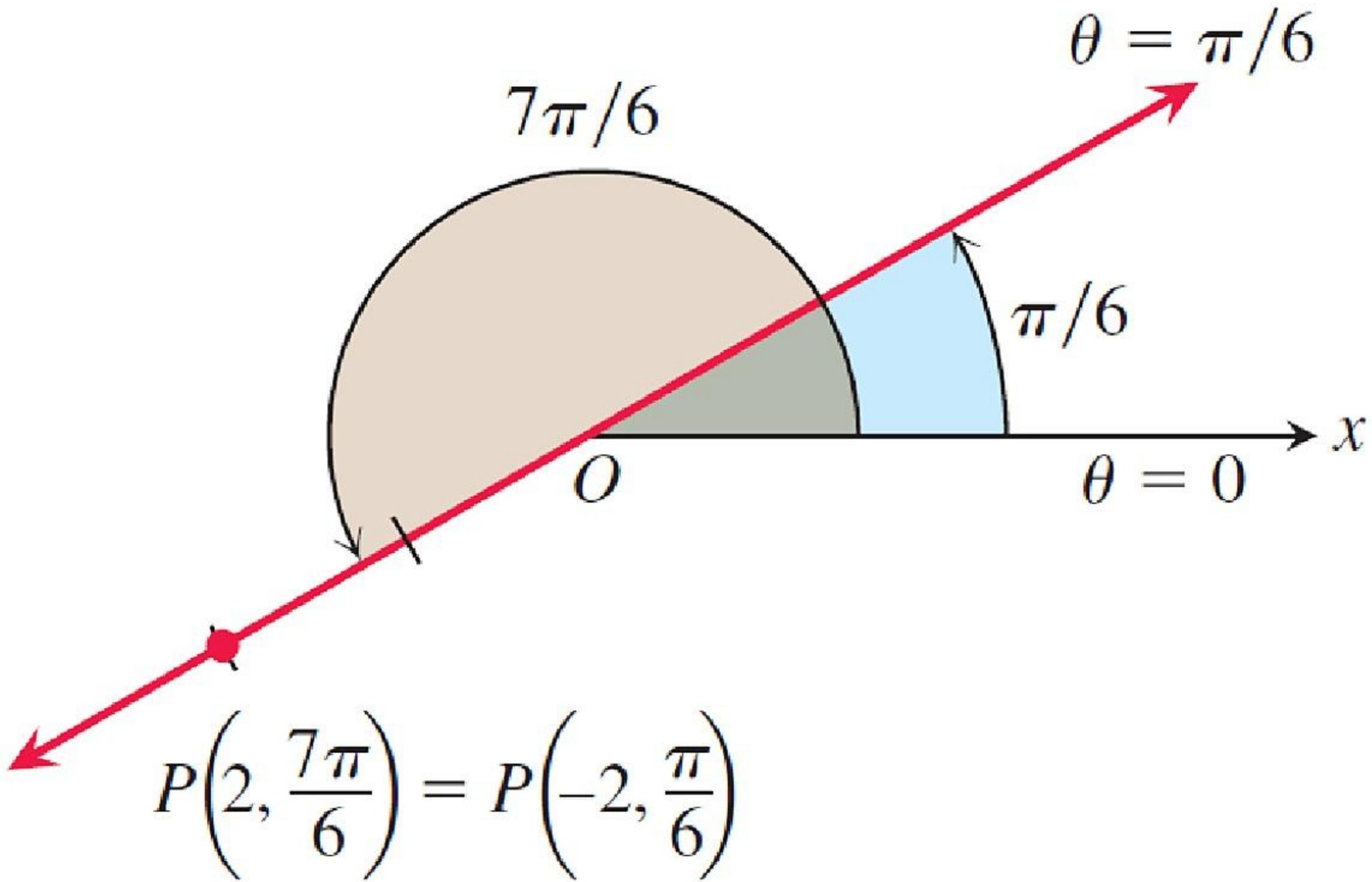


FIGURE 10.22 Polar coordinates can have negative r -values.

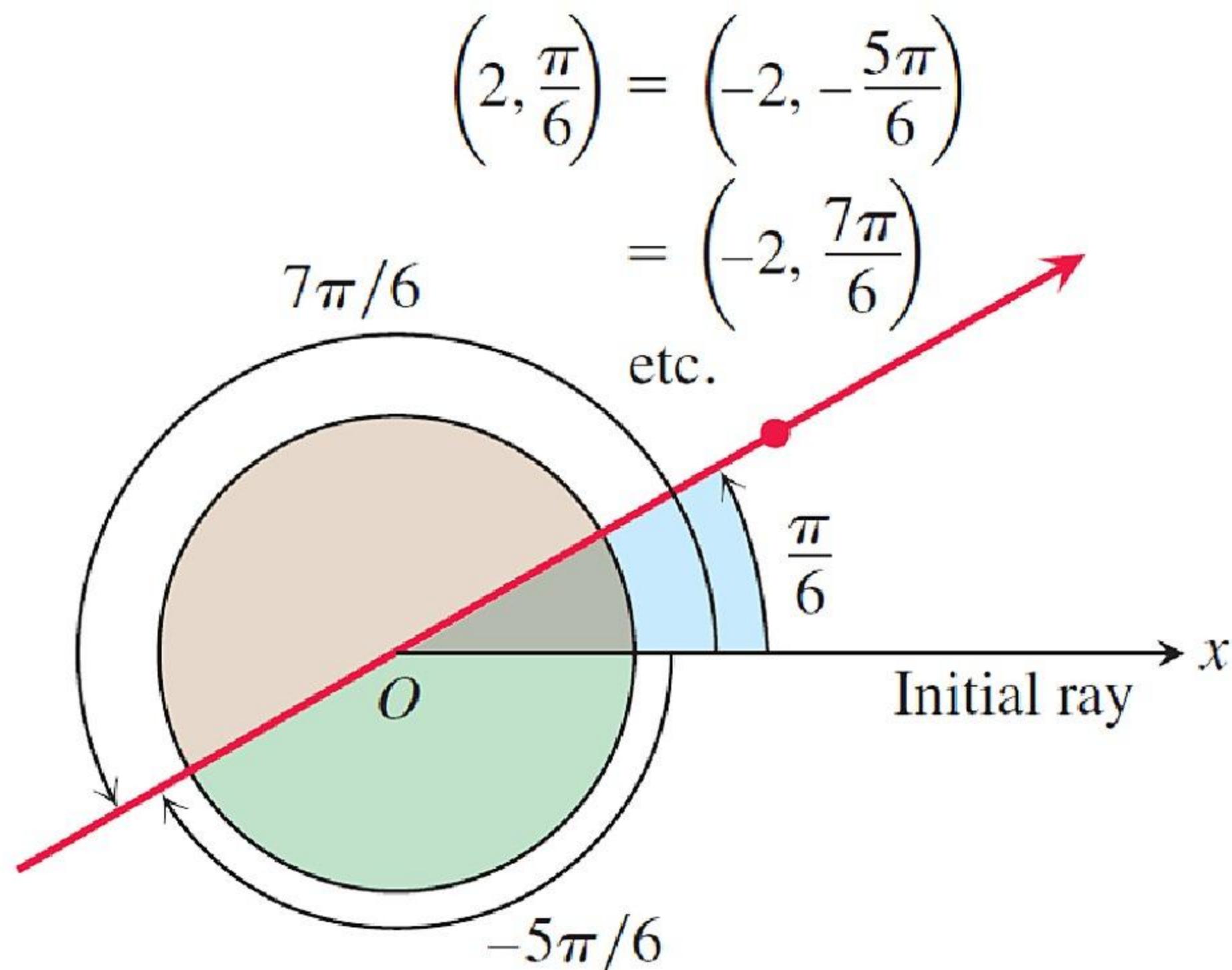


FIGURE 10.23 The point $P(2, \pi/6)$ has infinitely many polar coordinate pairs (Example 1).

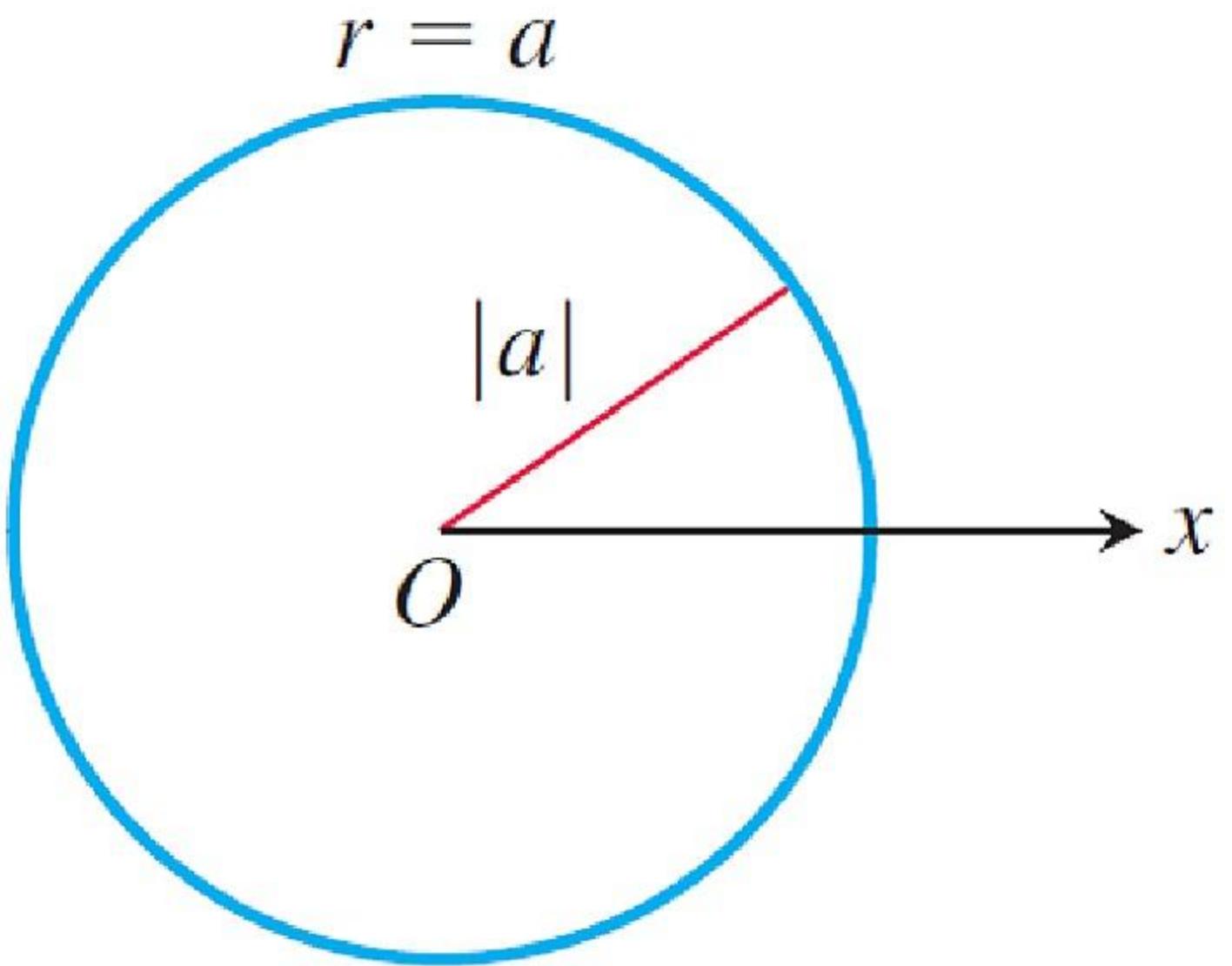
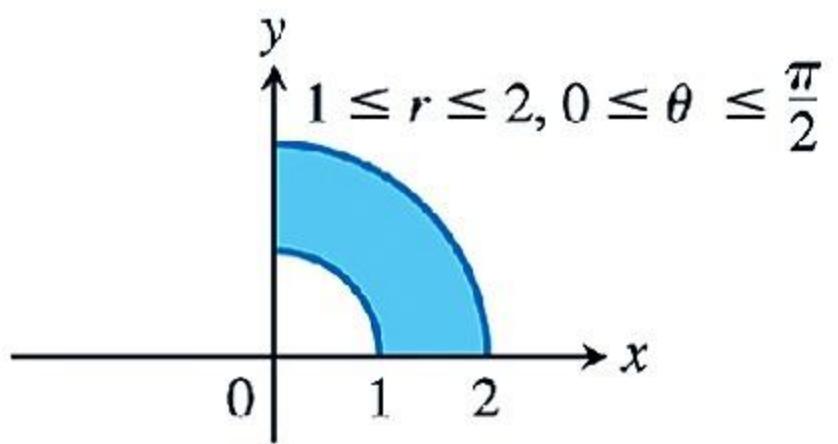
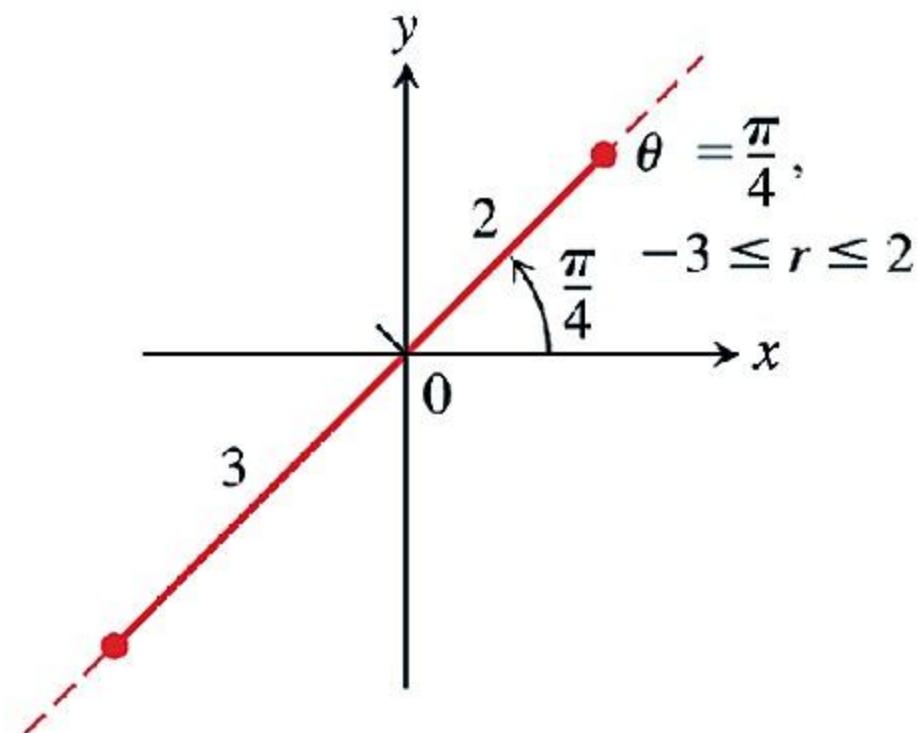


FIGURE 10.24 The polar equation for a circle is $r = a$.

(a)



(b)



(c)

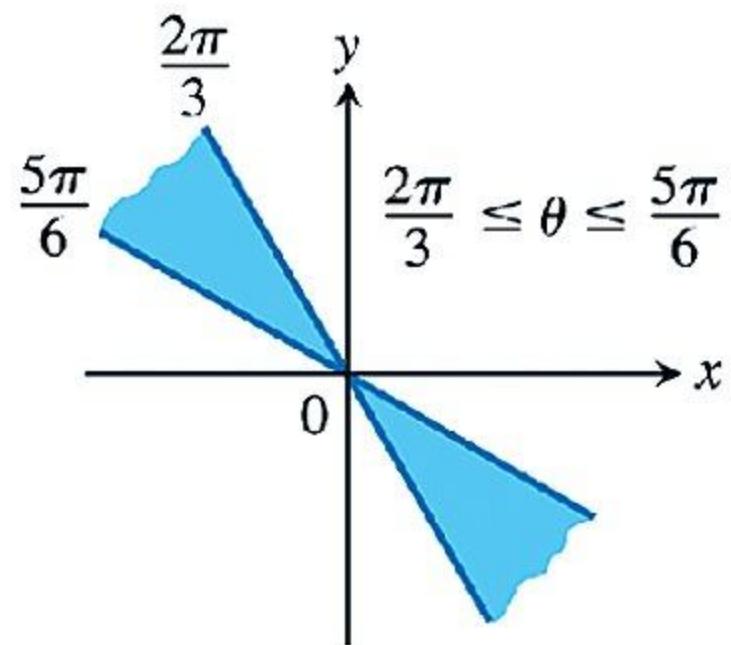


FIGURE 10.25 The graphs of typical inequalities in r and θ (Example 3).

Equations Relating Polar and Cartesian Coordinates

$$x = r \cos \theta, \quad y = r \sin \theta, \quad r^2 = x^2 + y^2, \quad \tan \theta = \frac{y}{x}$$

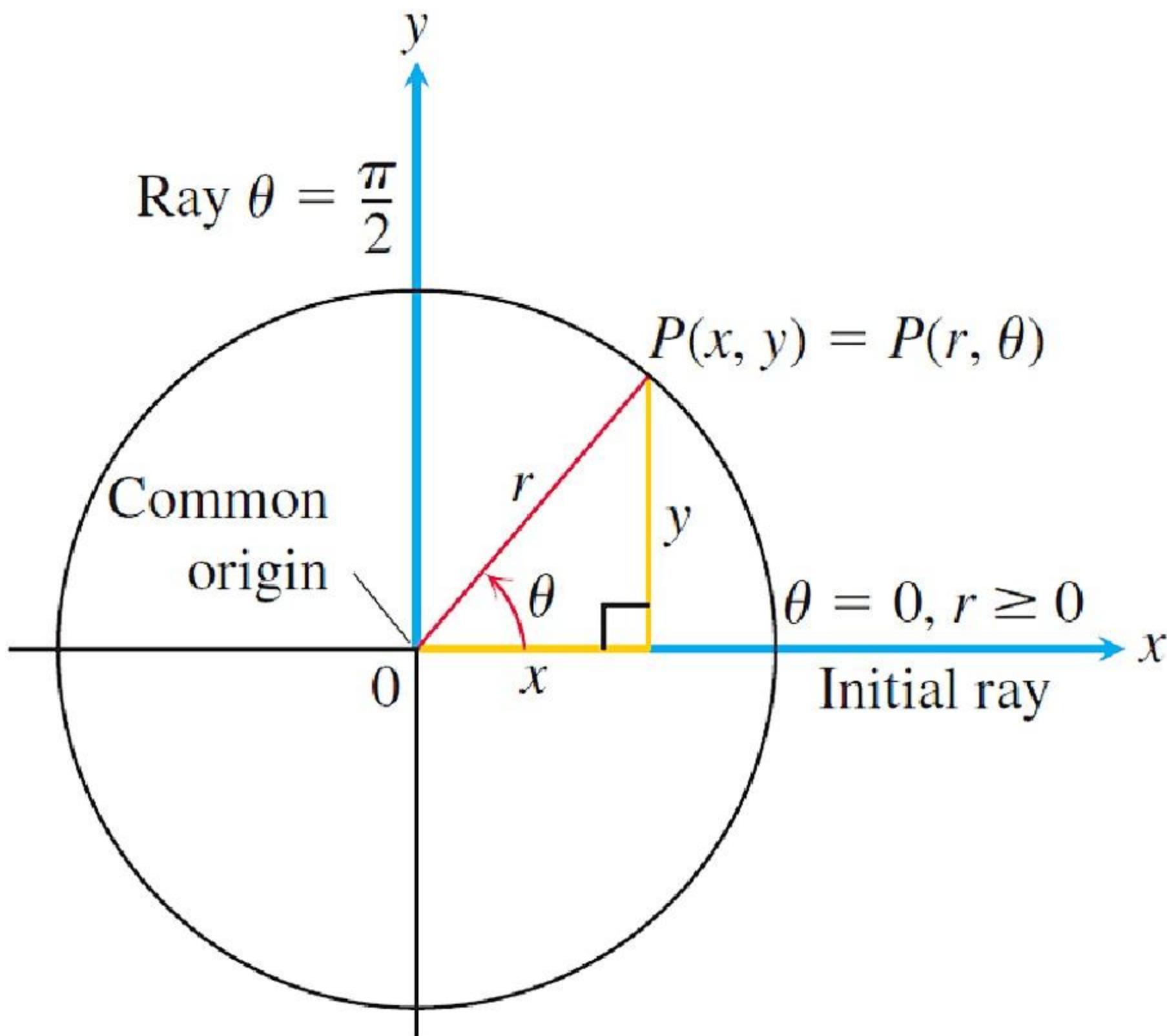


FIGURE 10.26 The usual way to relate polar and Cartesian coordinates.

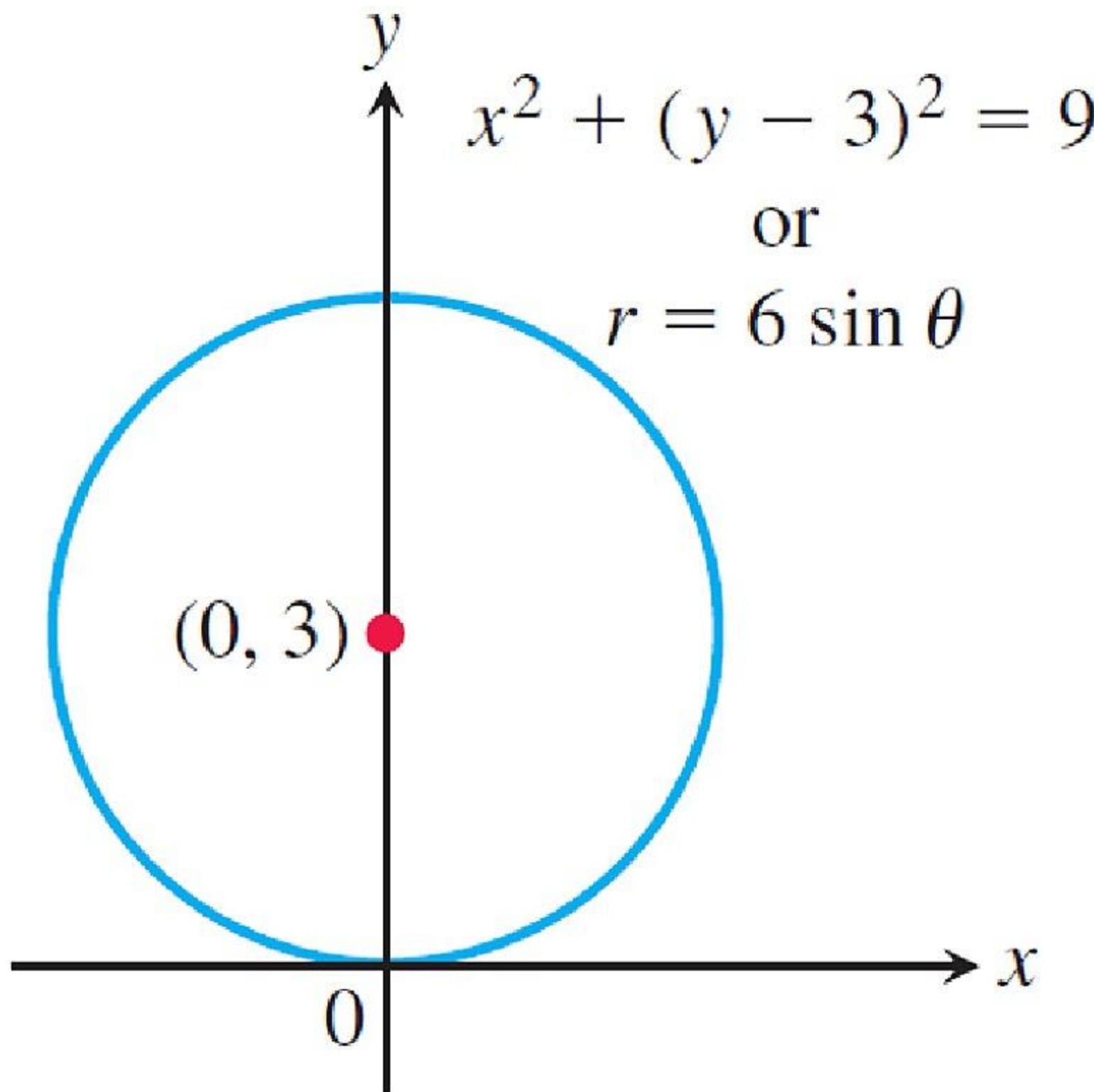


FIGURE 10.27 The circle in Example 5.

Section 10.4

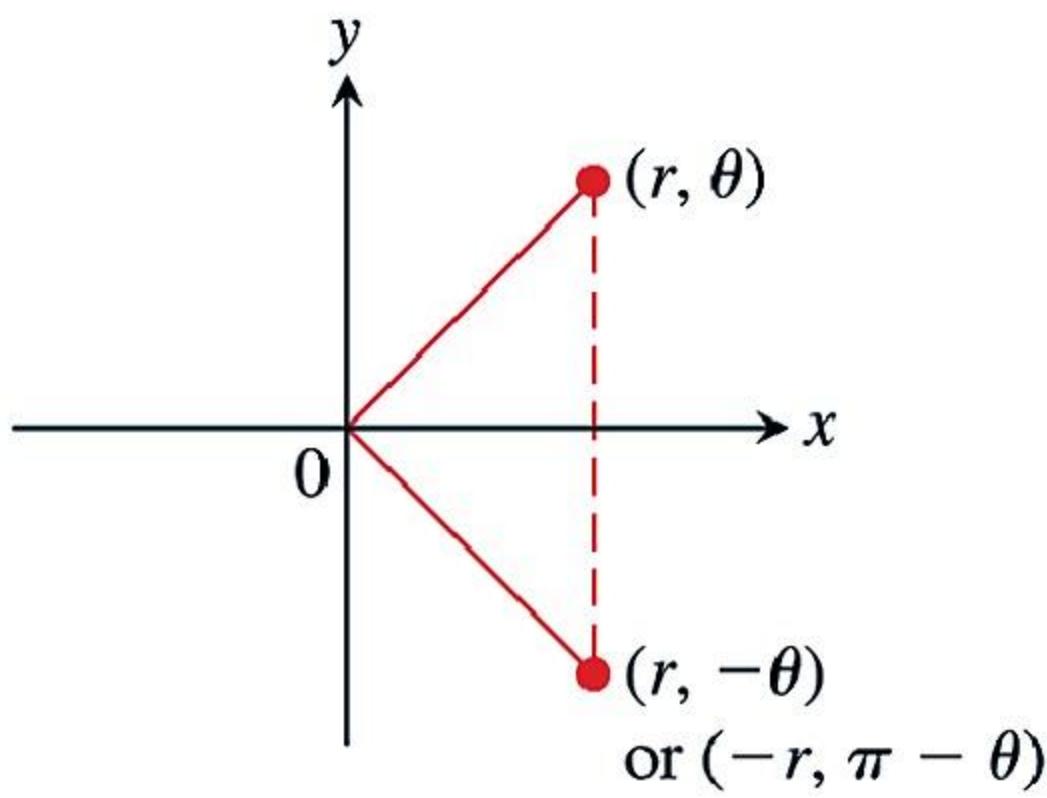
Graphing Polar Coordinate Equations

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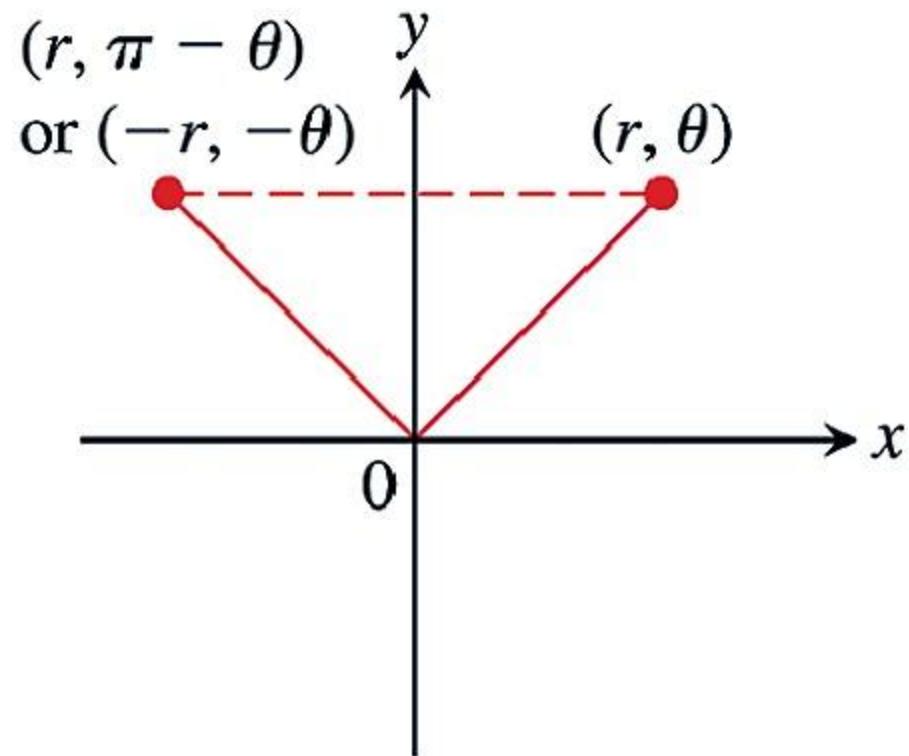
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Symmetry Tests for Polar Graphs in the Cartesian xy -Plane

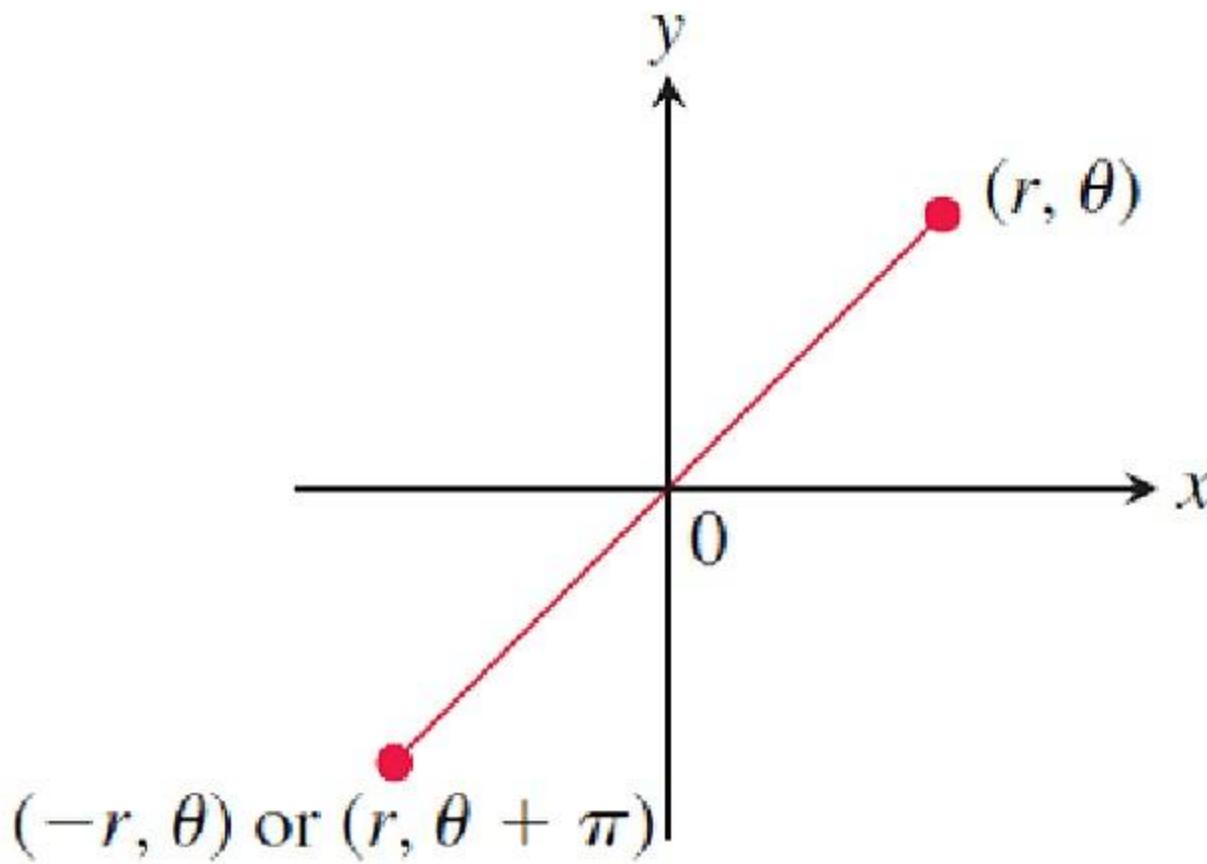
1. *Symmetry about the x -axis:* If the point (r, θ) lies on the graph, then the point $(r, -\theta)$ or $(-r, \pi - \theta)$ lies on the graph (Figure 10.28a).
2. *Symmetry about the y -axis:* If the point (r, θ) lies on the graph, then the point $(r, \pi - \theta)$ or $(-r, -\theta)$ lies on the graph (Figure 10.28b).
3. *Symmetry about the origin:* If the point (r, θ) lies on the graph, then the point $(-r, \theta)$ or $(r, \theta + \pi)$ lies on the graph (Figure 10.28c).



(a) About the x -axis



(b) About the y -axis



(c) About the origin

FIGURE 10.28 Three tests for symmetry in polar coordinates.

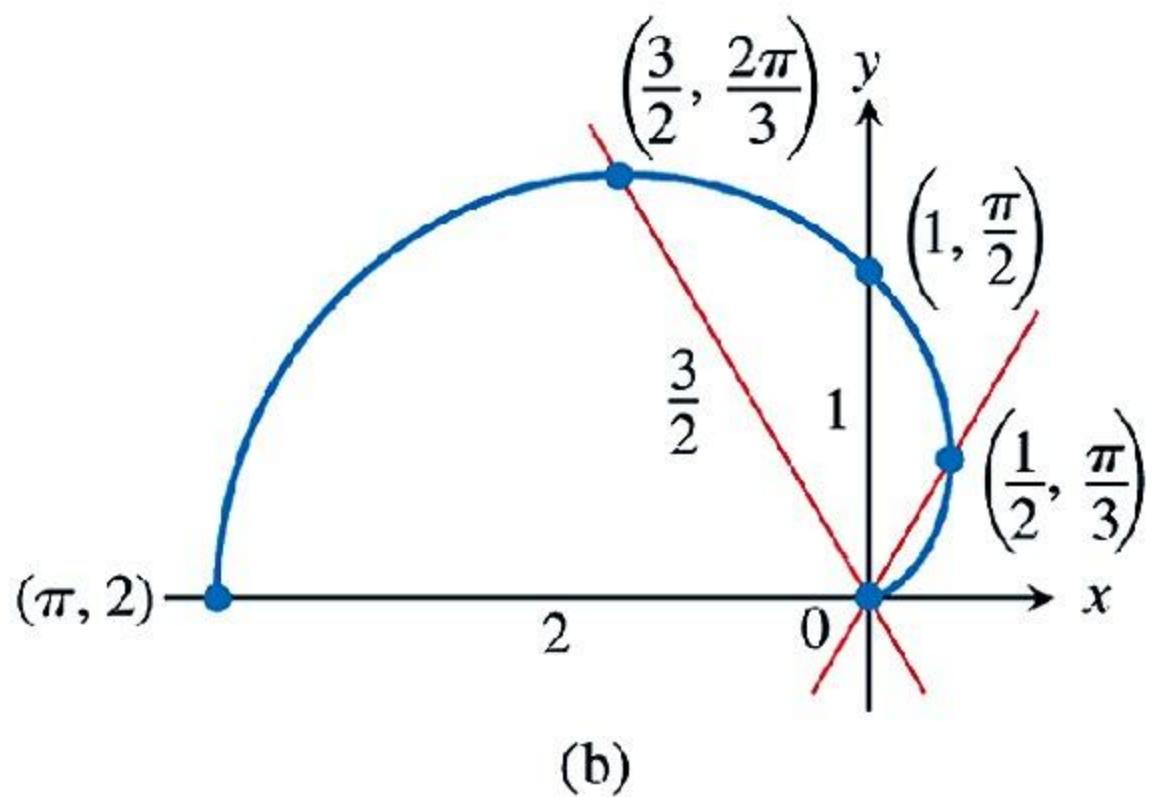
Slope of the Curve $r = f(\theta)$ in the Cartesian xy -Plane

$$\left. \frac{dy}{dx} \right|_{(r, \theta)} = \frac{f'(\theta) \sin \theta + f(\theta) \cos \theta}{f'(\theta) \cos \theta - f(\theta) \sin \theta} \quad (1)$$

provided $dx/d\theta \neq 0$ at (r, θ) .

θ	$r = 1 - \cos \theta$
0	0
$\frac{\pi}{3}$	$\frac{1}{2}$
$\frac{\pi}{2}$	1
$\frac{2\pi}{3}$	$\frac{3}{2}$
π	2

(a)



(b)

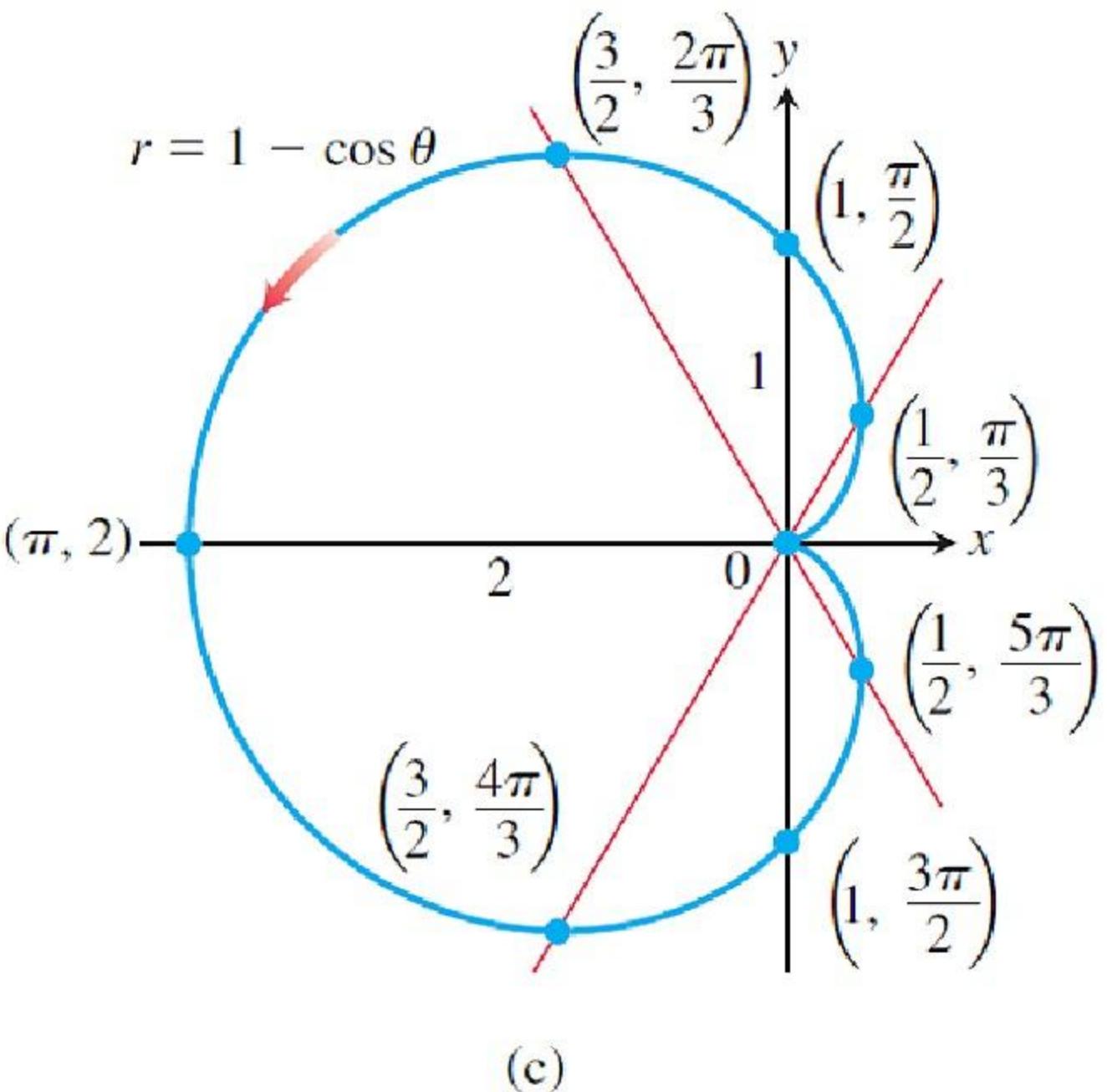
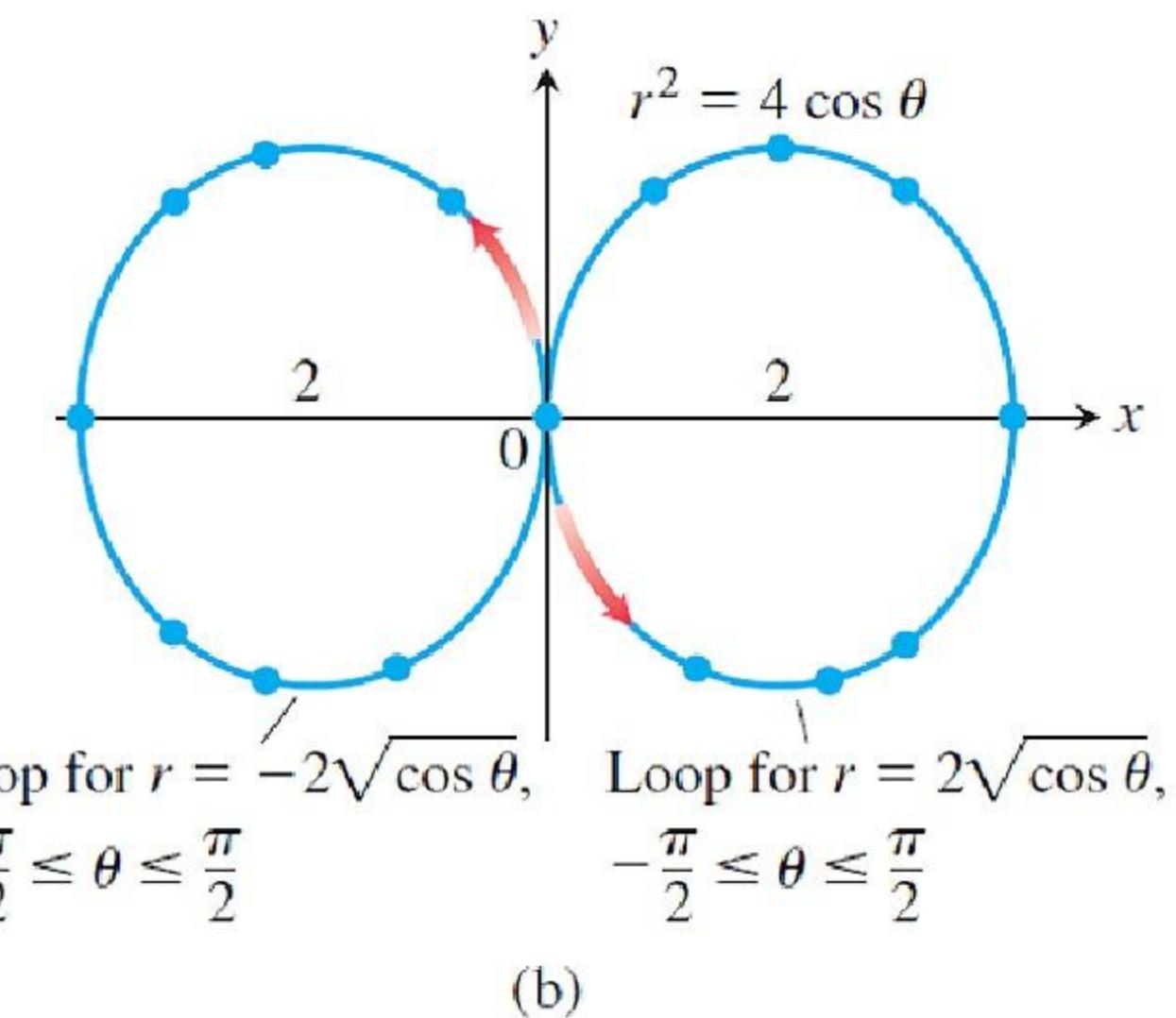


FIGURE 10.29 The steps in graphing the cardioid $r = 1 - \cos \theta$ (Example 1). The arrow shows the direction of increasing θ .

θ	$\cos \theta$	$r = \pm 2\sqrt{\cos \theta}$
0	1	± 2
$\pm \frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\approx \pm 1.9$
$\pm \frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\approx \pm 1.7$
$\pm \frac{\pi}{3}$	$\frac{1}{2}$	$\approx \pm 1.4$
$\pm \frac{\pi}{2}$	0	0

(a)



(b)

FIGURE 10.30 The graph of $r^2 = 4 \cos \theta$. The arrows show the direction of increasing θ . The values of r in the table are rounded (Example 2). ■

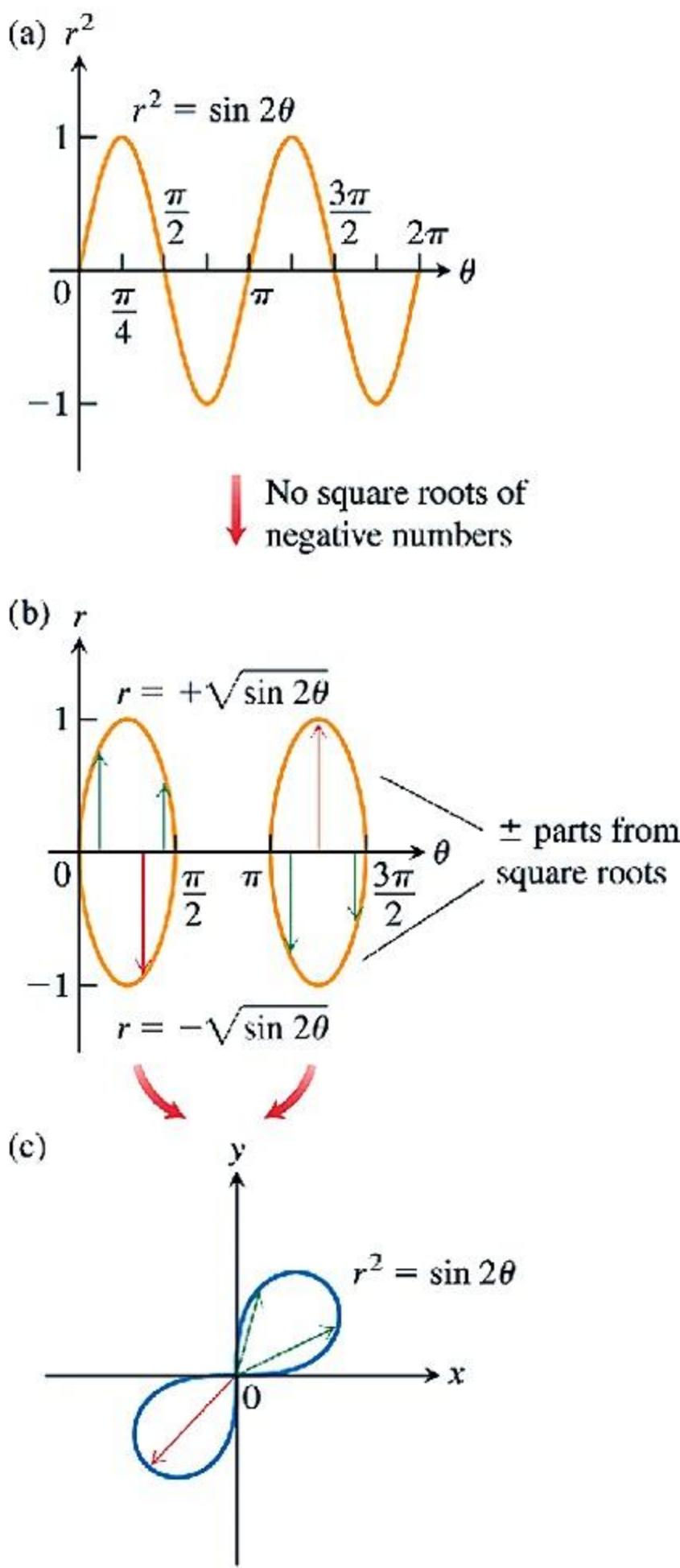


FIGURE 10.31 To plot $r = f(\theta)$ in the Cartesian $r\theta$ -plane in (b), we first plot $r^2 = \sin 2\theta$ in the $r^2\theta$ -plane in (a) and then ignore the values of θ for which $\sin 2\theta$ is negative. The radii from the sketch in (b) cover the polar graph of the lemniscate in (c) twice (Example 3).

Section 10.7

Areas and Lengths in Polar Coordinates

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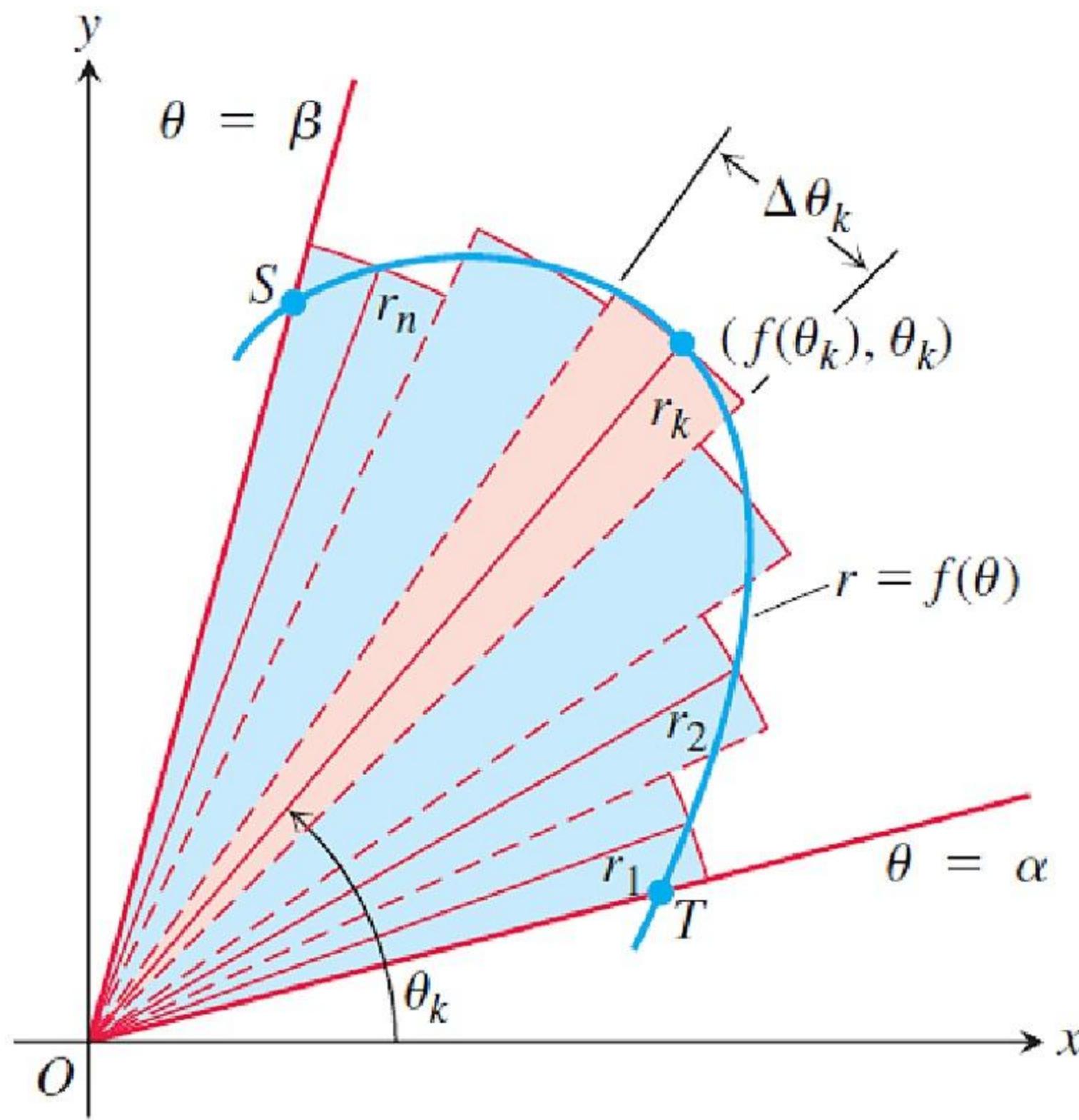


FIGURE 10.32 To derive a formula for the area of region OTS , we approximate the region with fan-shaped circular sectors.

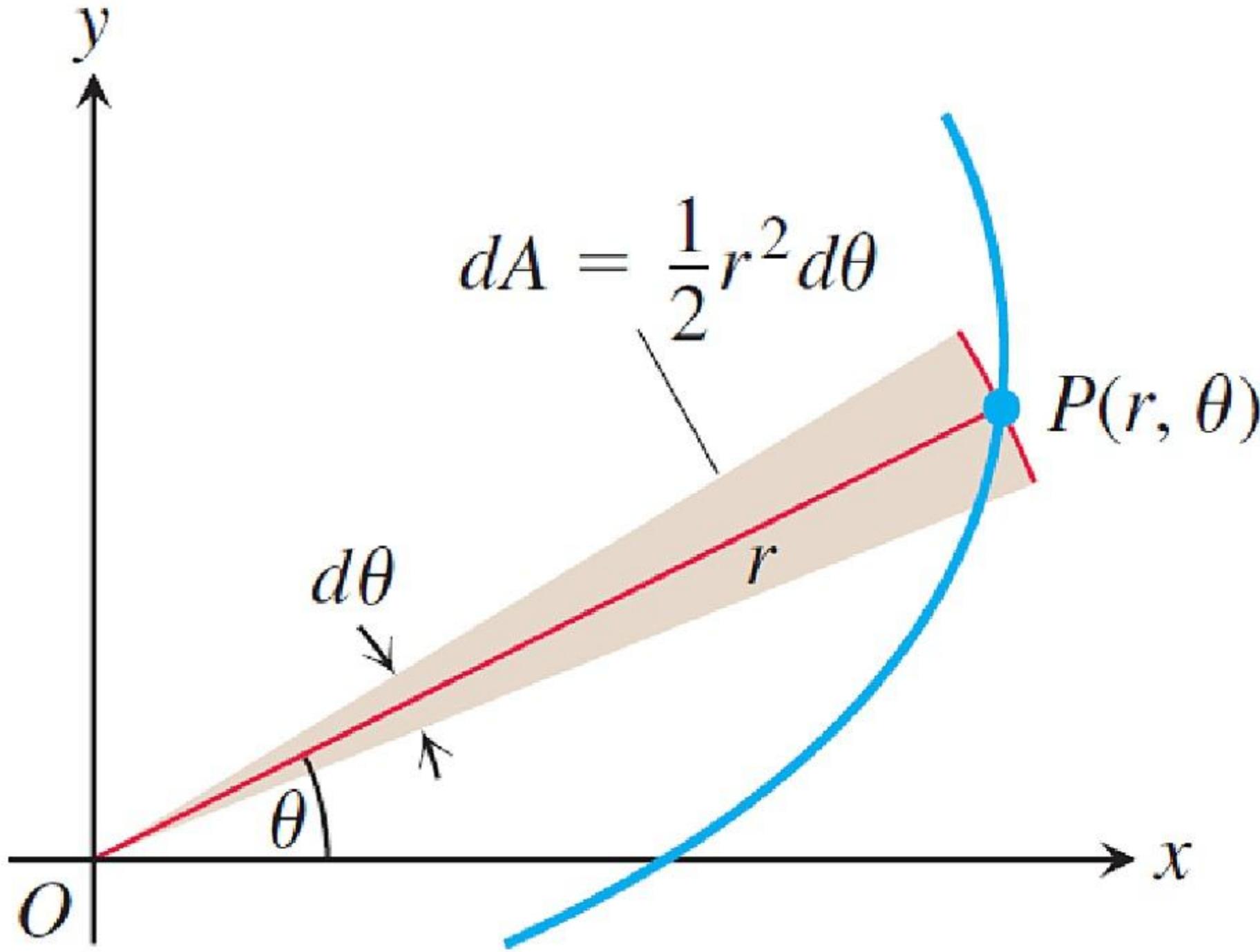


FIGURE 10.33 The area differential dA for the curve $r = f(\theta)$.

**Area of the Fan-Shaped Region Between the Origin and the Curve
 $r = f(\theta)$ when $\alpha \leq \theta \leq \beta$, $r \geq 0$, and $\beta - \alpha \leq 2\pi$.**

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta$$

This is the integral of the **area differential** (Figure 10.33)

$$dA = \frac{1}{2} r^2 d\theta = \frac{1}{2} (f(\theta))^2 d\theta.$$

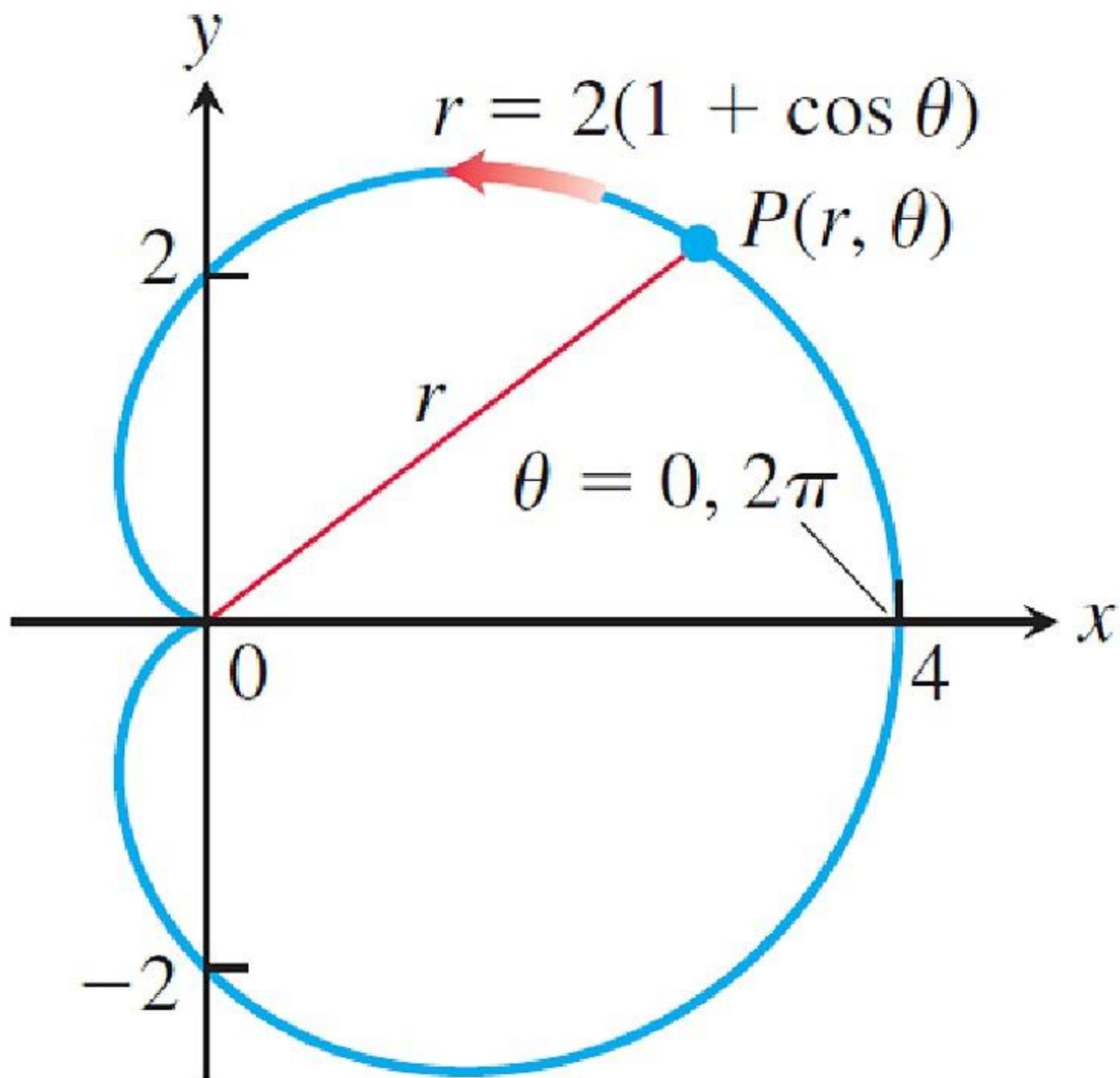


FIGURE 10.34 The cardioid in Example 1.

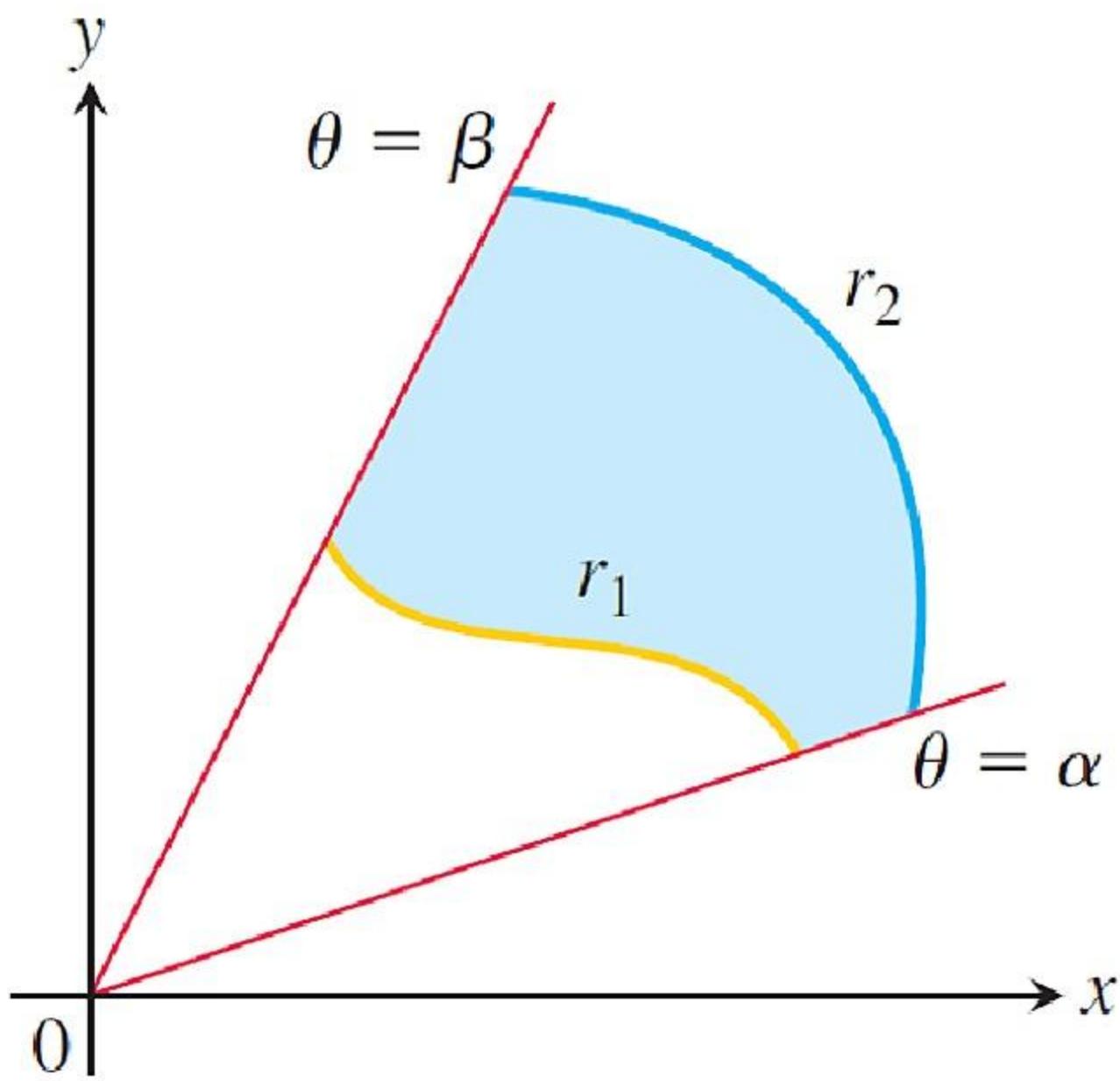


FIGURE 10.35 The area of the shaded region is calculated by subtracting the area of the region between r_1 and the origin from the area of the region between r_2 and the origin.

Area of the Region $0 \leq r_1(\theta) \leq r \leq r_2(\theta)$, $\alpha \leq \theta \leq \beta$, and $\beta - \alpha \leq 2\pi$.

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r_2^2 d\theta - \int_{\alpha}^{\beta} \frac{1}{2} r_1^2 d\theta = \int_{\alpha}^{\beta} \frac{1}{2} (r_2^2 - r_1^2) d\theta \quad (1)$$

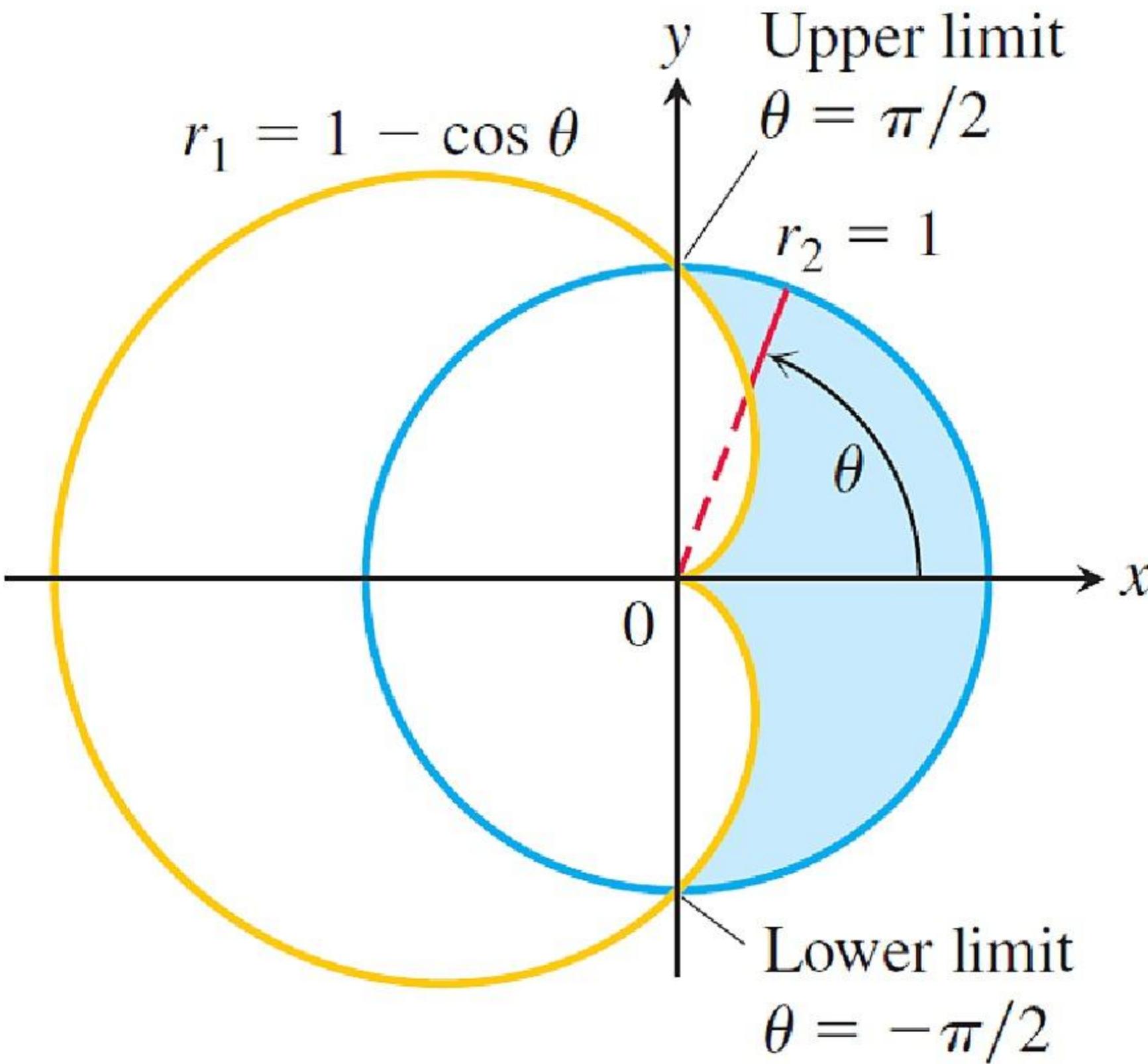


FIGURE 10.36 The region and limits of integration in Example 2.

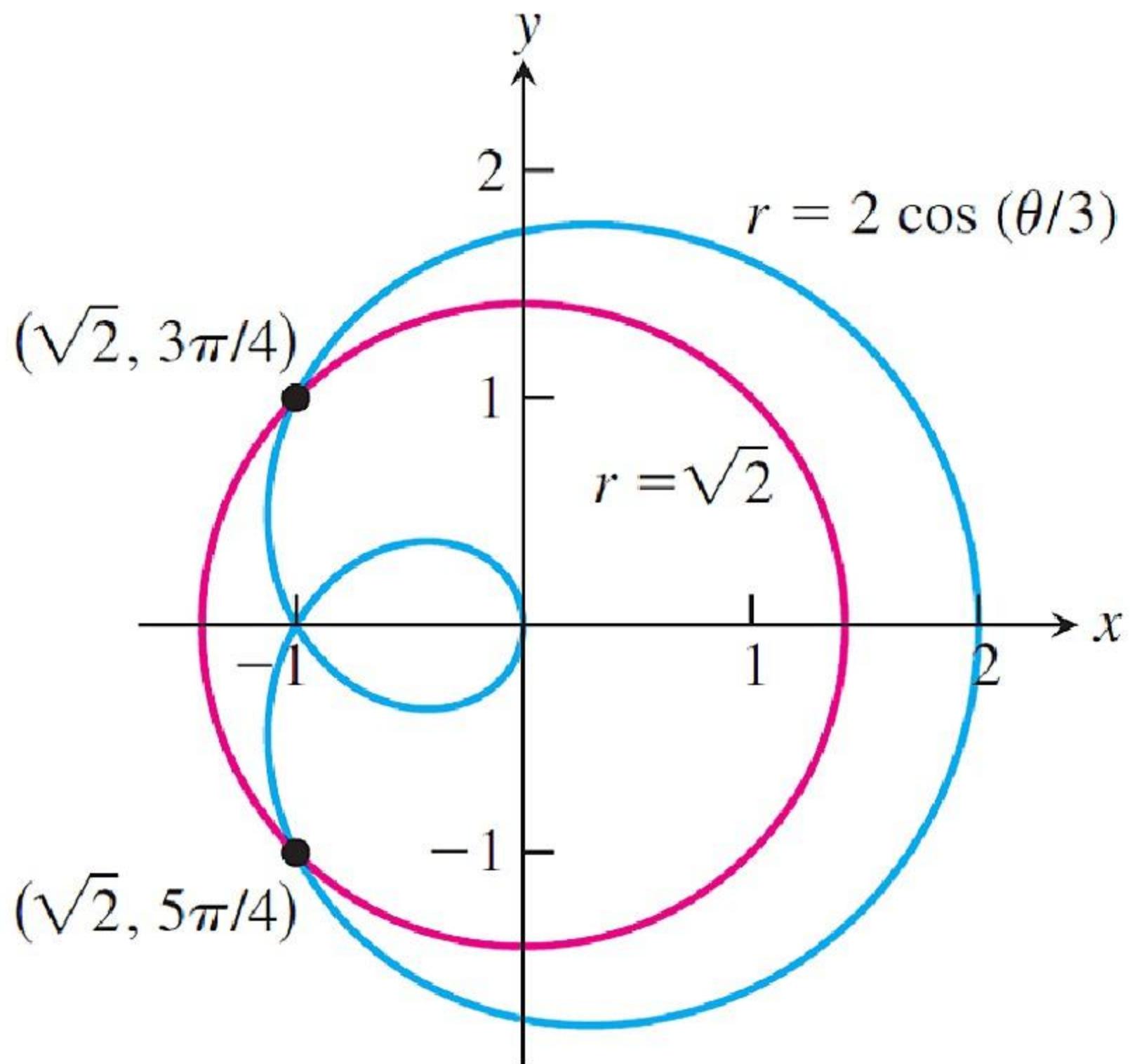


FIGURE 10.37 The curves $r = 2 \cos(\theta/3)$ and $r = \sqrt{2}$ intersect at two points (Example 3).

Length of a Polar Curve

If $r = f(\theta)$ has a continuous first derivative for $\alpha \leq \theta \leq \beta$ and if the point $P(r, \theta)$ traces the curve $r = f(\theta)$ exactly once as θ runs from α to β , then the length of the curve is

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta. \quad (3)$$

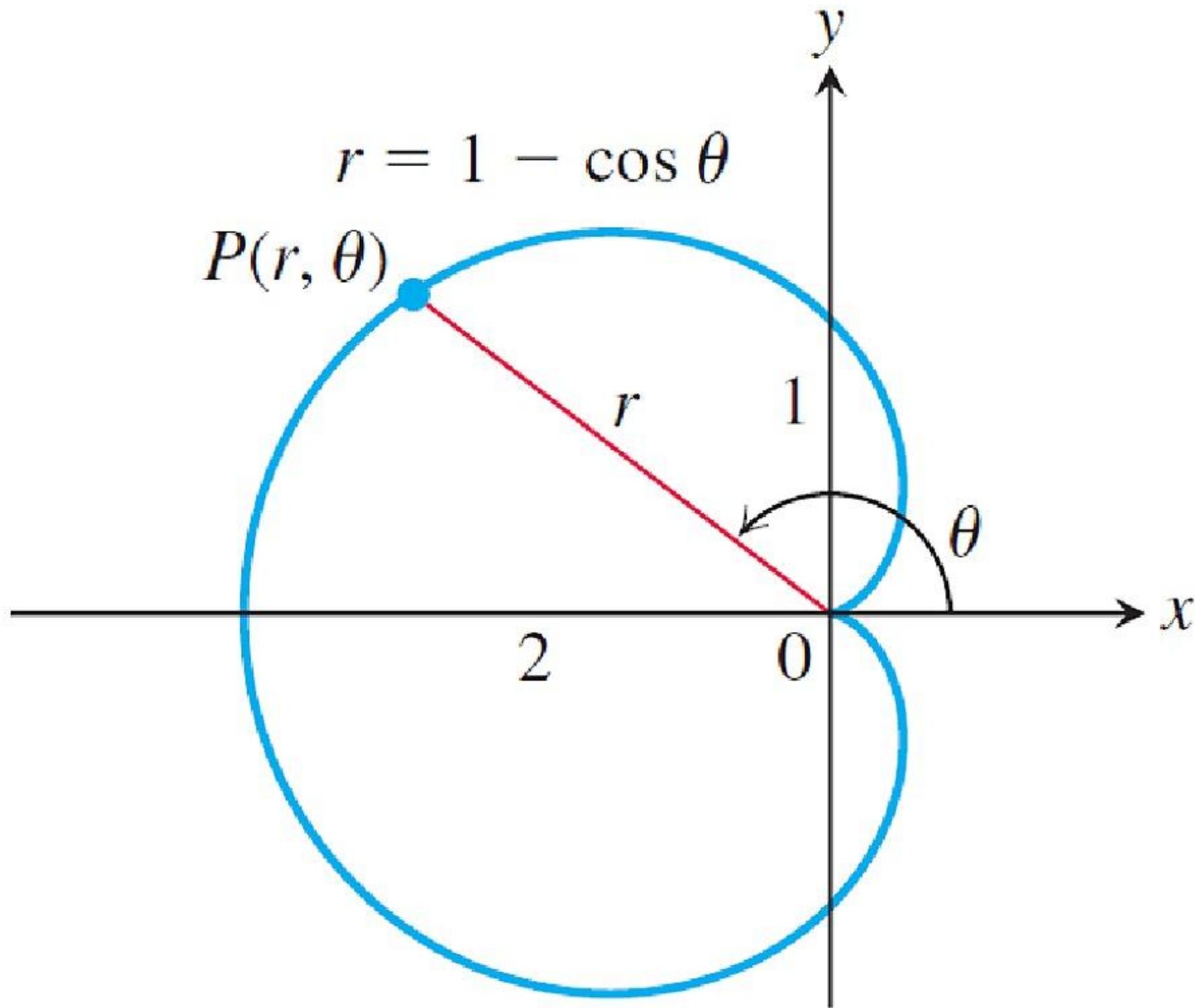
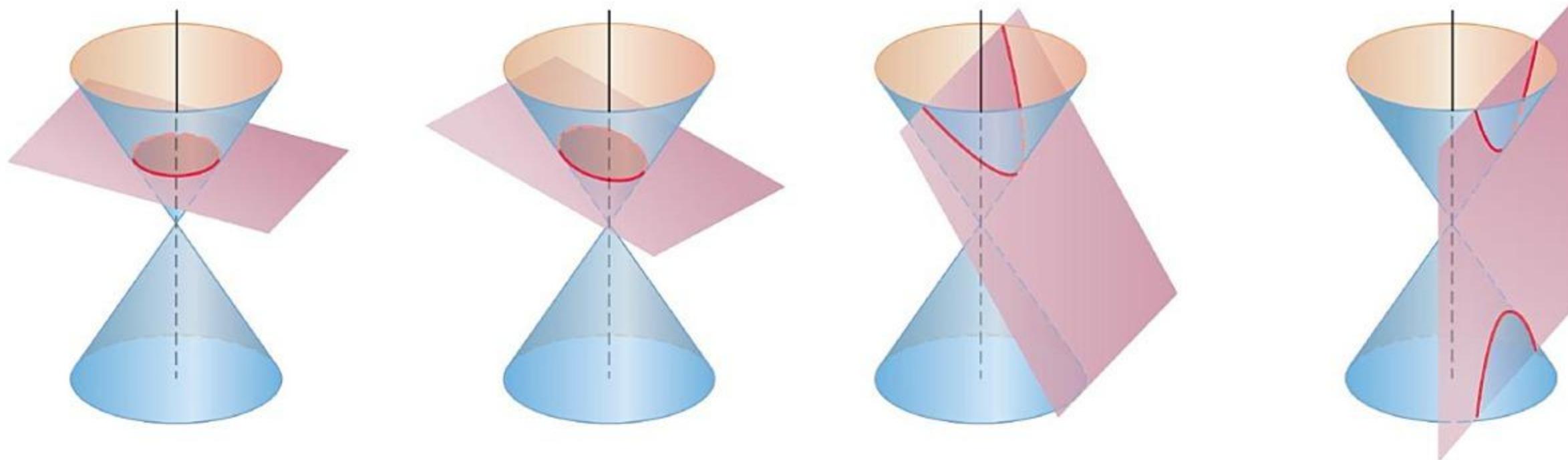


FIGURE 10.38 Calculating the length of a cardioid (Example 4).

Section 10.6

Conic Sections



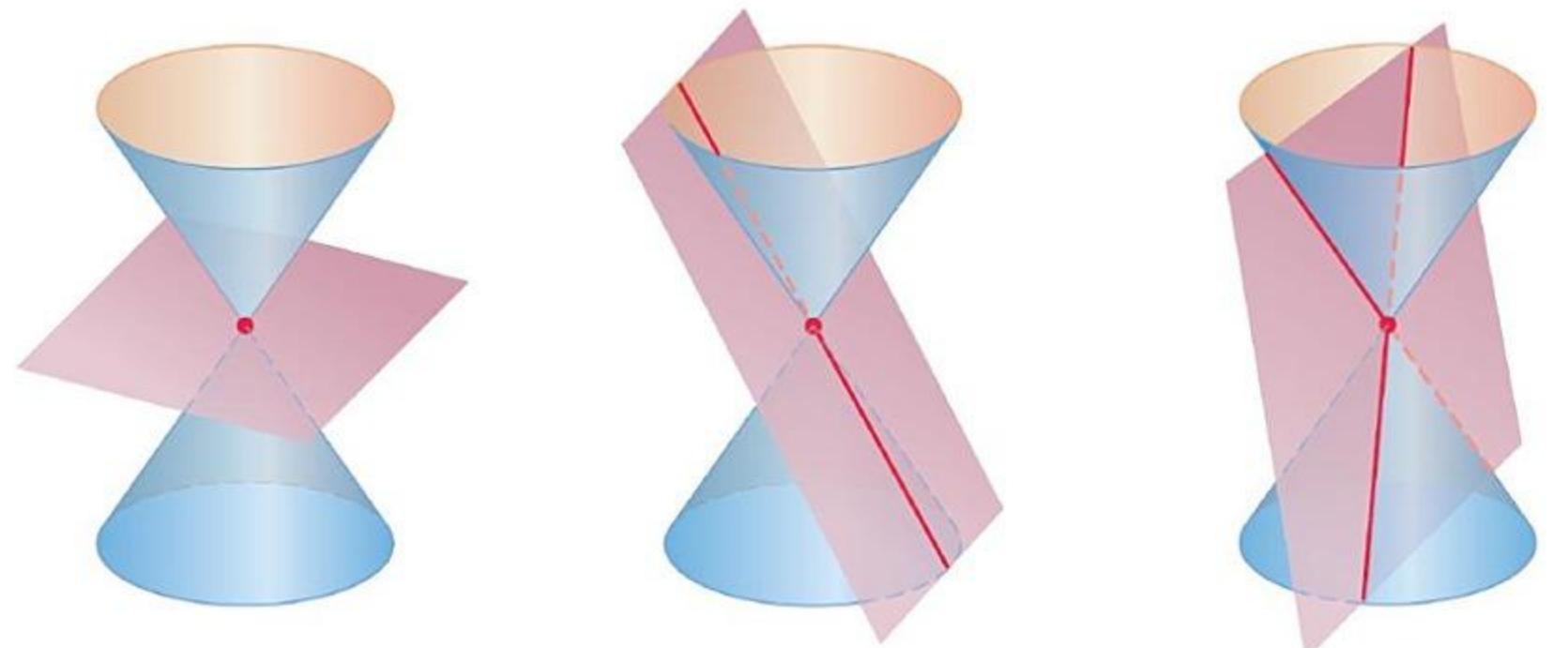
Circle: plane perpendicular
to cone axis

Ellipse: plane oblique
to cone axis

Parabola: plane parallel
to side of cone

Hyperbola: plane
parallel to cone axis

(a)



Point: plane through
cone vertex only

Single line: plane
tangent to cone

Pair of intersecting lines

(b)

FIGURE 10.39 The standard conic sections (a) are the curves in which a plane cuts a *double cone*. Hyperbolas come in two parts, called *branches*. The point and lines obtained by passing the plane through the cone's vertex (b) are *degenerate* conic sections.

DEFINITIONS A set that consists of all the points in a plane equidistant from a given fixed point and a given fixed line in the plane is a **parabola**. The fixed point is the **focus** of the parabola. The fixed line is the **directrix**.

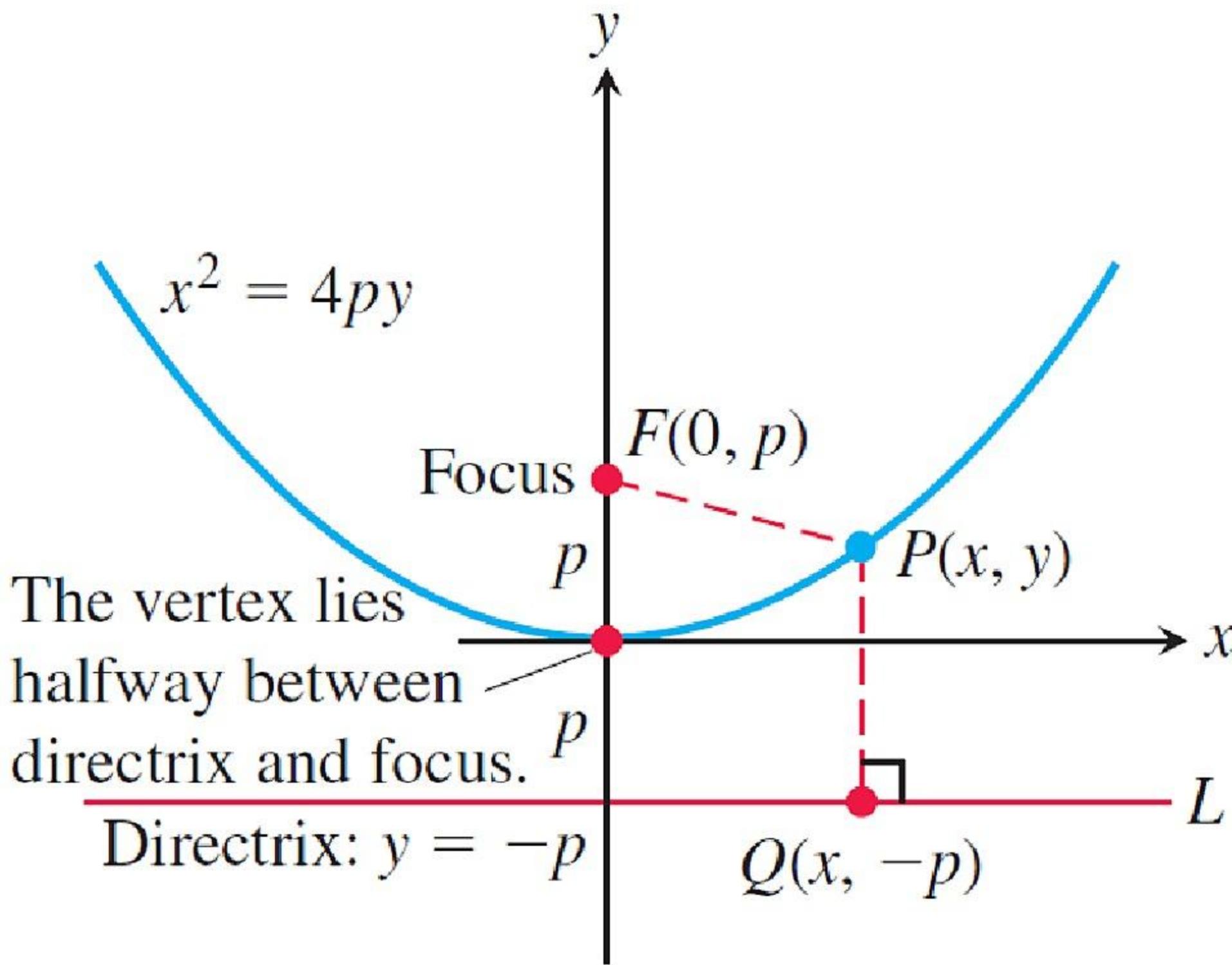
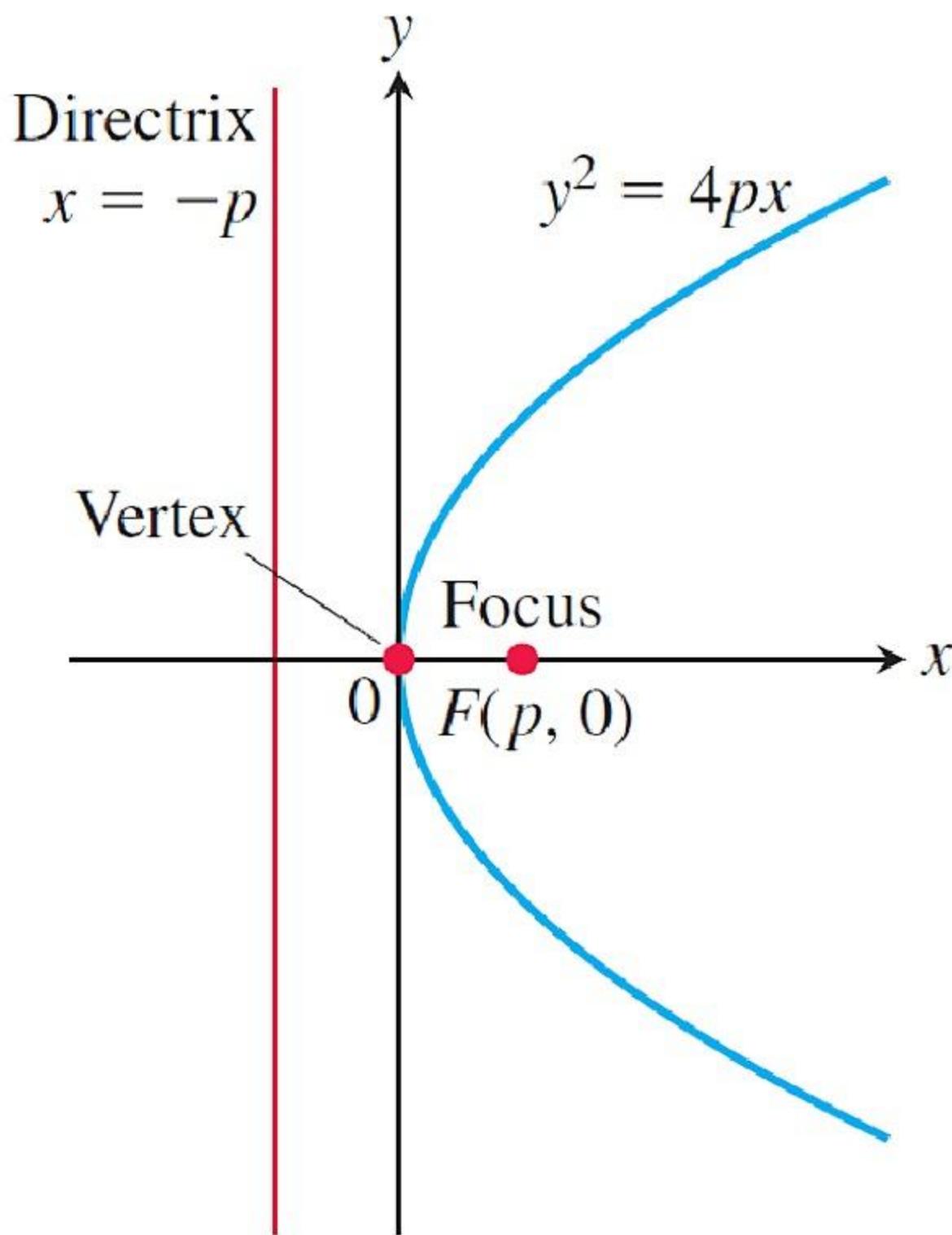
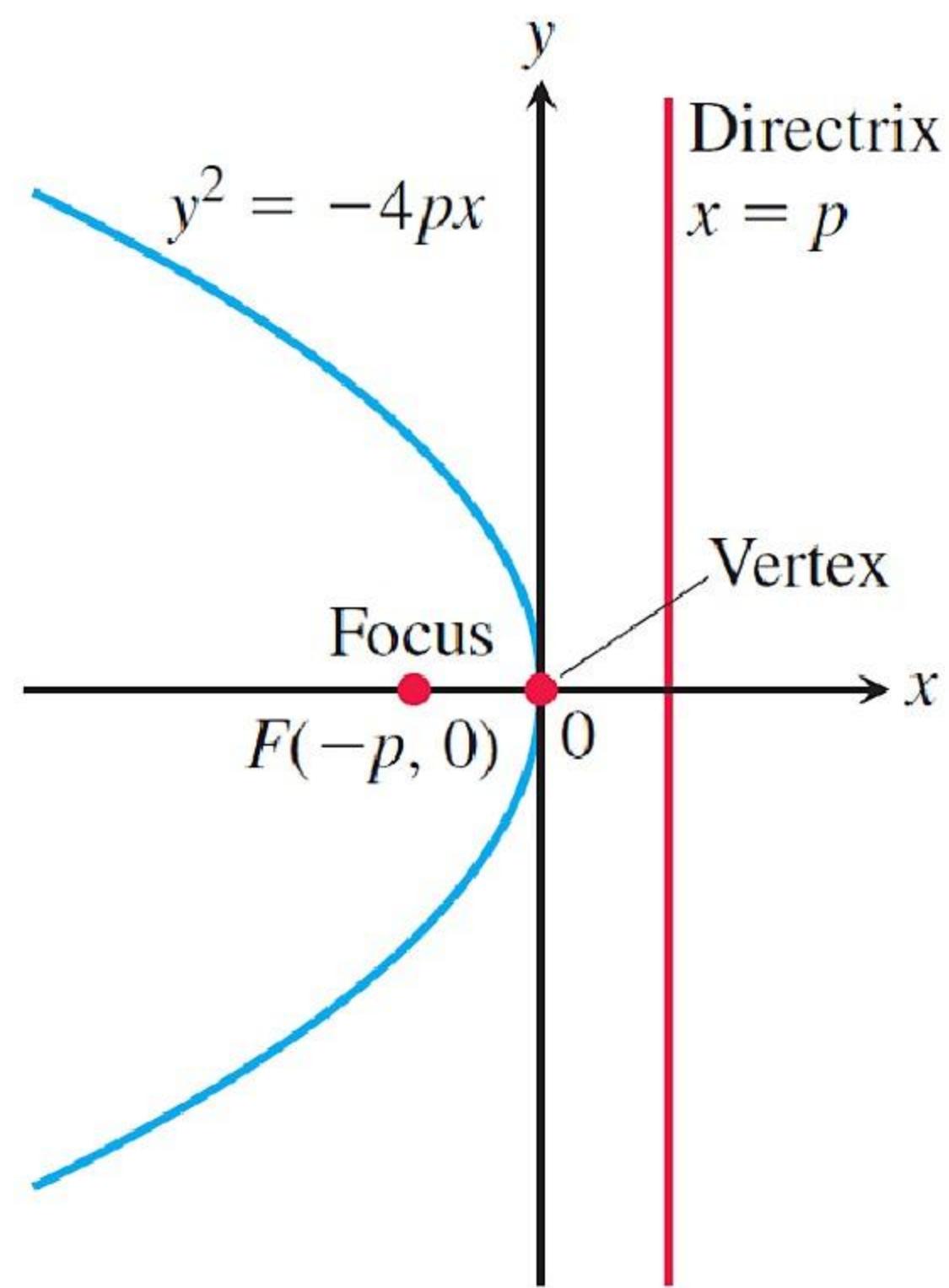


FIGURE 10.40 The standard form of the parabola $x^2 = 4py, p > 0$.



(a)



(b)

FIGURE 10.41 (a) The parabola $y^2 = 4px$. (b) The parabola $y^2 = -4px$.

DEFINITIONS An **ellipse** is the set of points in a plane whose distances from two fixed points in the plane have a constant sum. The two fixed points are the **foci** of the ellipse.

The line through the foci of an ellipse is the ellipse's **focal axis**. The point on the axis halfway between the foci is the **center**. The points where the focal axis and ellipse cross are the ellipse's **vertices** (Figure 10.42).

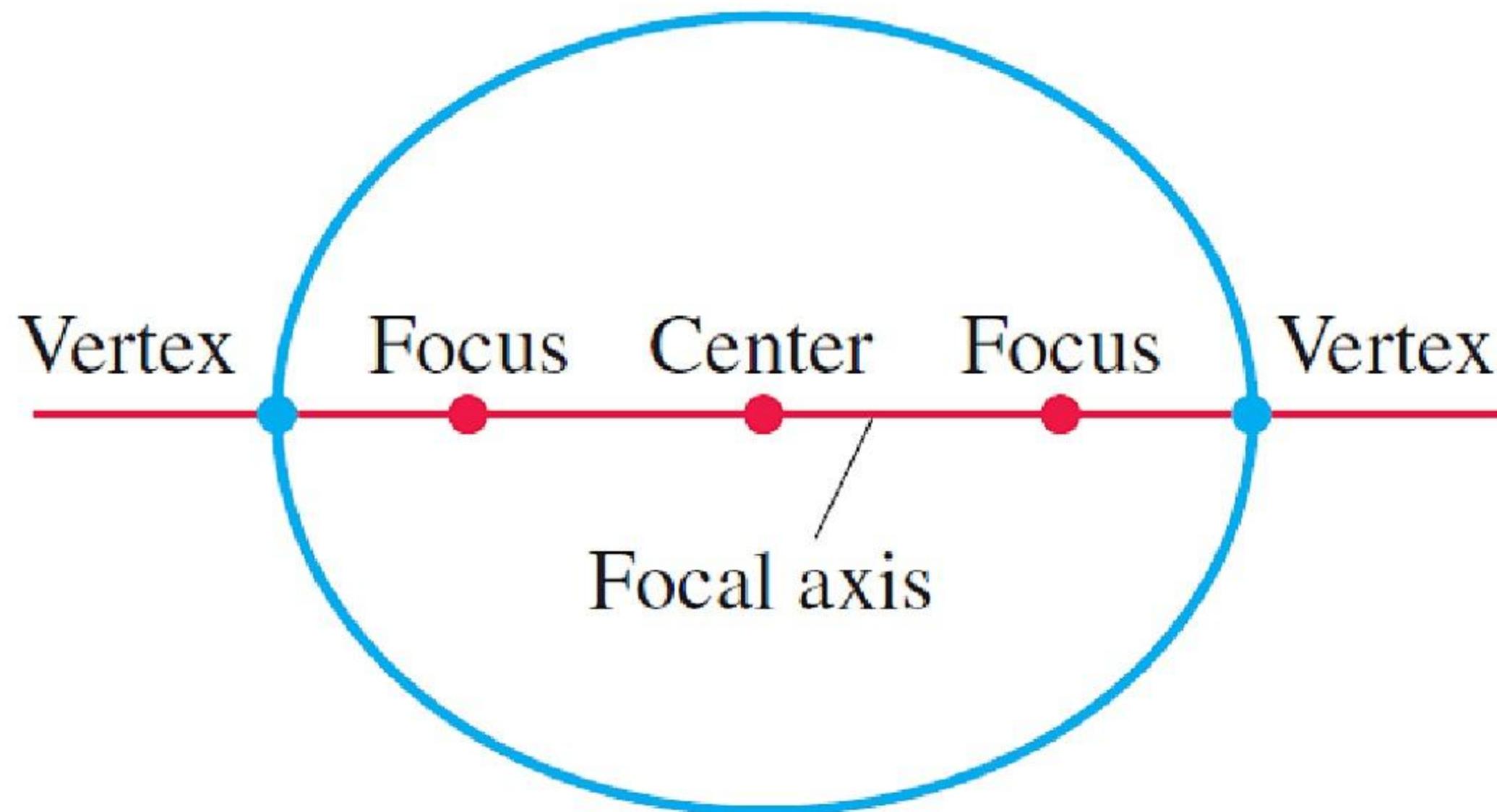


FIGURE 10.42 Points on the focal axis
of an ellipse.

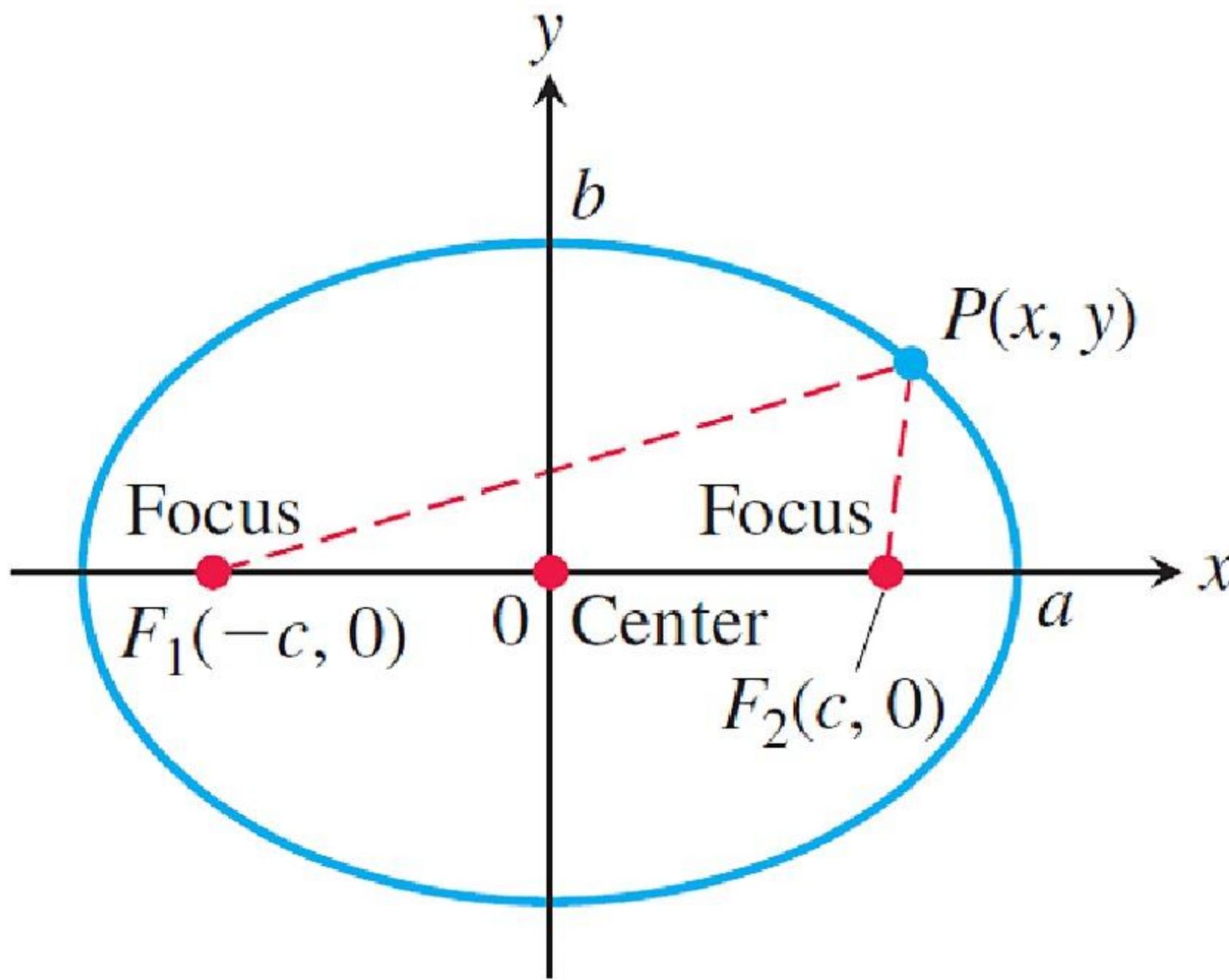


FIGURE 10.43 The ellipse defined by the equation $PF_1 + PF_2 = 2a$ is the graph of the equation $(x^2/a^2) + (y^2/b^2) = 1$, where $b^2 = a^2 - c^2$.

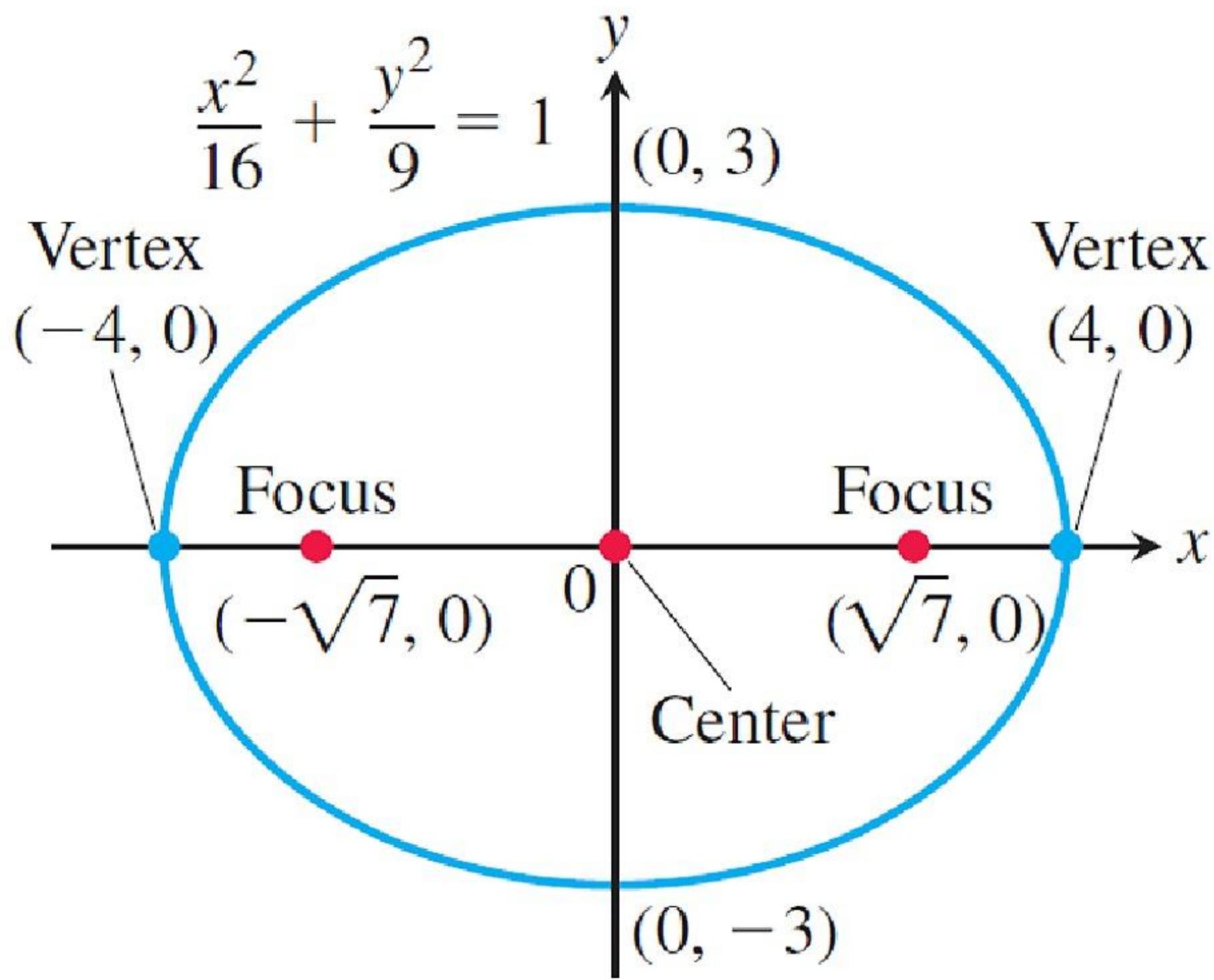


FIGURE 10.44 An ellipse with its major axis horizontal (Example 2).

Standard-Form Equations for Ellipses Centered at the Origin

Foci on the x-axis: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (a > b)$

Center-to-focus distance: $c = \sqrt{a^2 - b^2}$

Foci: $(\pm c, 0)$

Vertices: $(\pm a, 0)$

Foci on the y-axis: $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \quad (a > b)$

Center-to-focus distance: $c = \sqrt{a^2 - b^2}$

Foci: $(0, \pm c)$

Vertices: $(0, \pm a)$

In each case, a is the semimajor axis and b is the semiminor axis.

DEFINITIONS A **hyperbola** is the set of points in a plane whose distances from two fixed points in the plane have a constant difference. The two fixed points are the **foci** of the hyperbola.

The line through the foci of a hyperbola is the **focal axis**. The point on the axis halfway between the foci is the hyperbola's **center**. The points where the focal axis and hyperbola cross are the **vertices** (Figure 10.45).

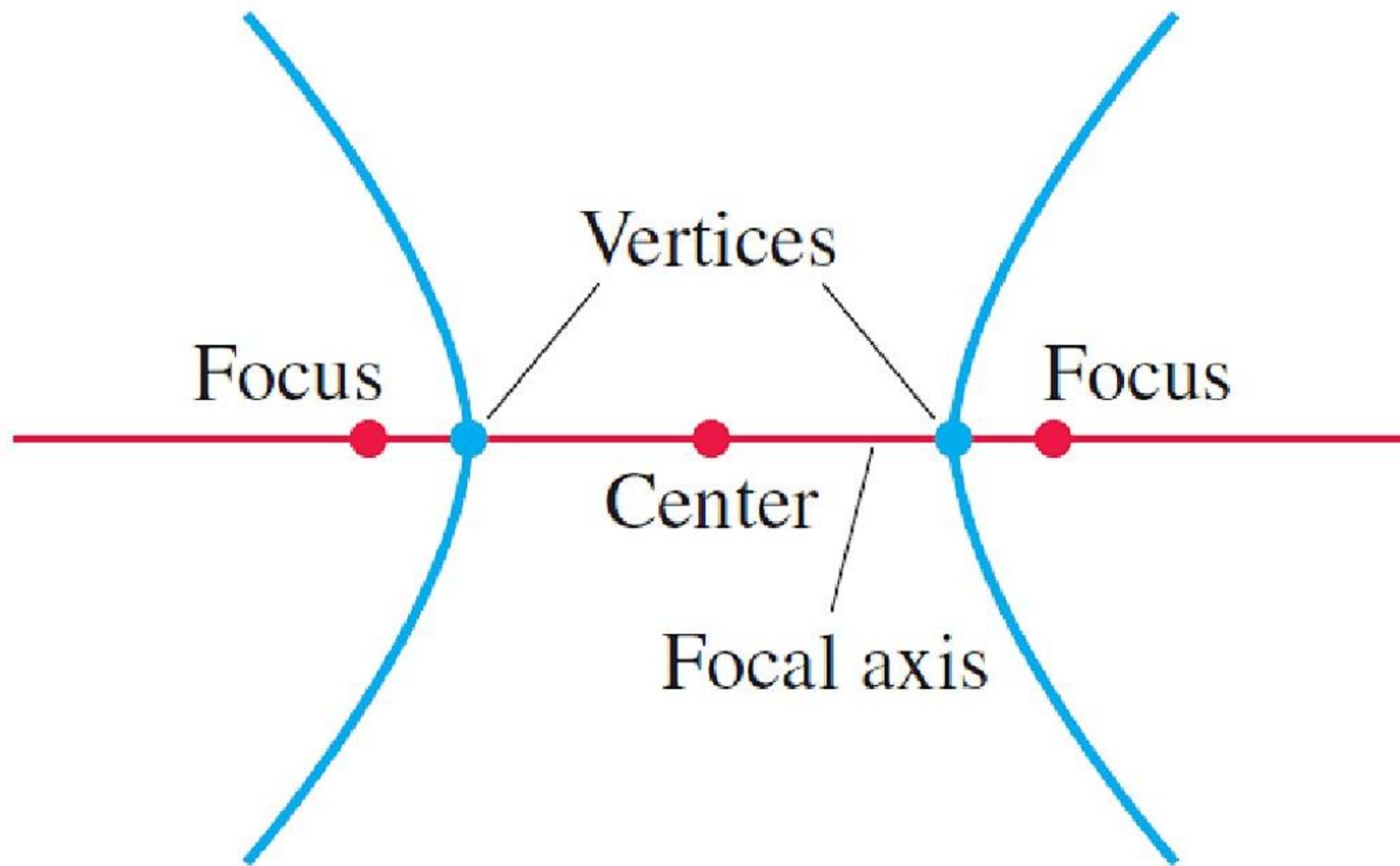


FIGURE 10.45 Points on the focal axis
of a hyperbola.

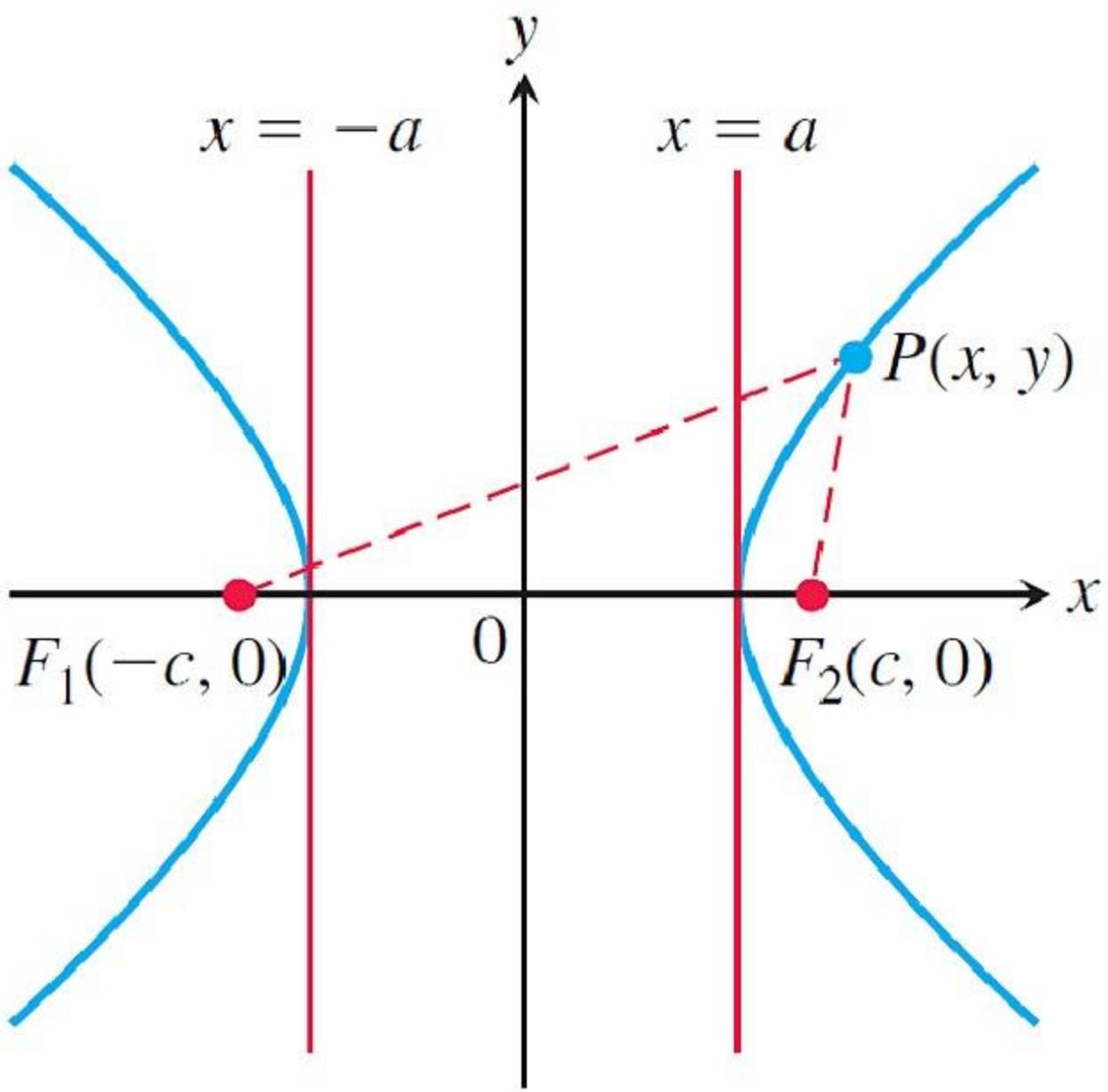


FIGURE 10.46 Hyperbolas have two branches. For points on the right-hand branch of the hyperbola shown here, $PF_1 - PF_2 = 2a$. For points on the left-hand branch, $PF_2 - PF_1 = 2a$. We then let $b = \sqrt{c^2 - a^2}$.

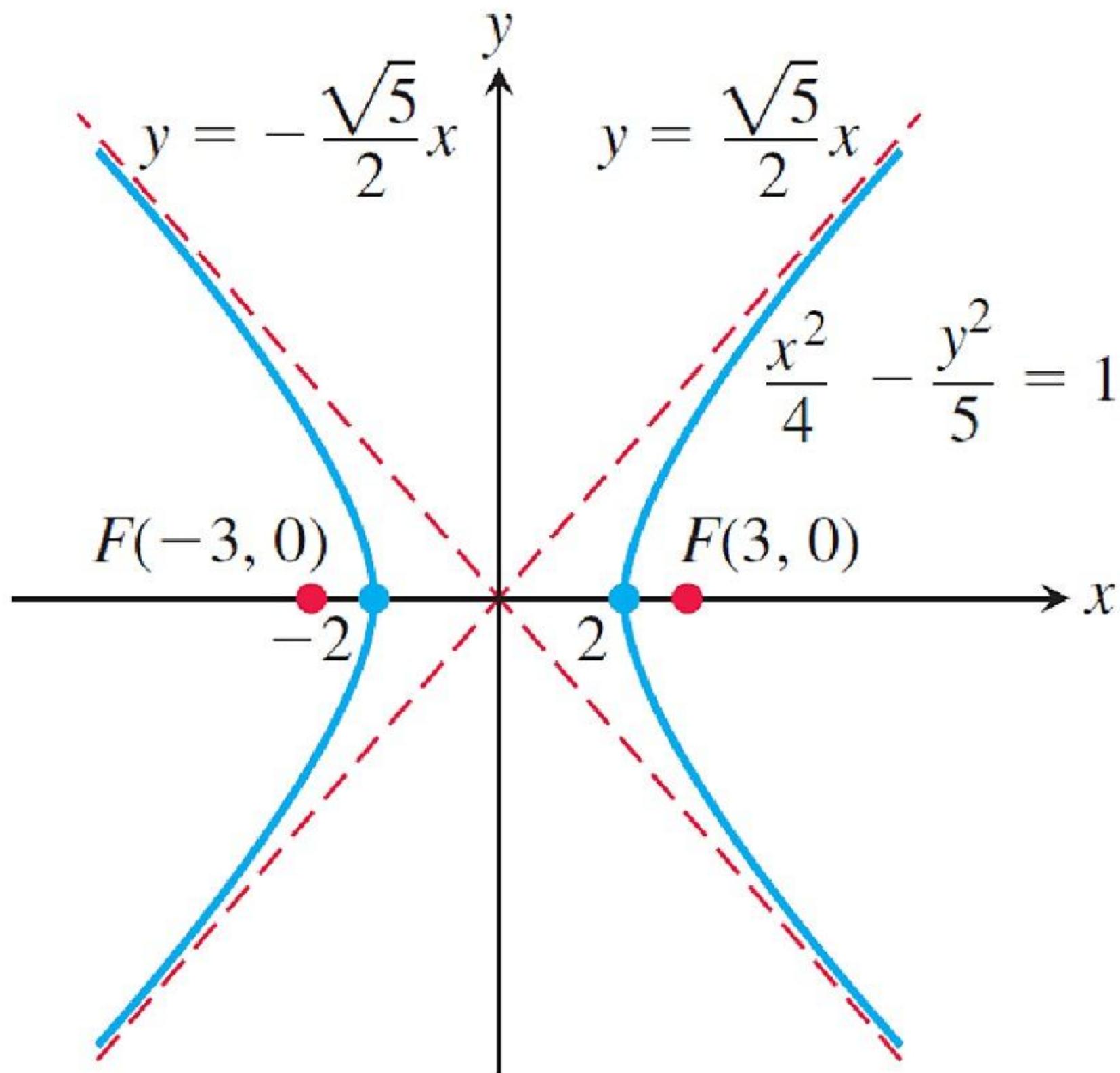


FIGURE 10.47 The hyperbola and its asymptotes in Example 3.

Standard-Form Equations for Hyperbolas Centered at the Origin

$$\text{Foci on the } x\text{-axis: } \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\text{Center-to-focus distance: } c = \sqrt{a^2 + b^2}$$

$$\text{Foci: } (\pm c, 0)$$

$$\text{Vertices: } (\pm a, 0)$$

$$\text{Asymptotes: } \frac{x^2}{a^2} - \frac{y^2}{b^2} = 0 \quad \text{or} \quad y = \pm \frac{b}{a}x$$

$$\text{Foci on the } y\text{-axis: } \frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

$$\text{Center-to-focus distance: } c = \sqrt{a^2 + b^2}$$

$$\text{Foci: } (0, \pm c)$$

$$\text{Vertices: } (0, \pm a)$$

$$\text{Asymptotes: } \frac{y^2}{a^2} - \frac{x^2}{b^2} = 0 \quad \text{or} \quad y = \pm \frac{a}{b}x$$

Notice the difference in the asymptote equations (b/a in the first, a/b in the second).

Section 10.7

Conics in Polar Coordinates

Thomas' Calculus, 14e in SI Units

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DEFINITION

The **eccentricity** of the ellipse $(x^2/a^2) + (y^2/b^2) = 1$ ($a > b$) is

$$e = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a}.$$

The **eccentricity** of the hyperbola $(x^2/a^2) - (y^2/b^2) = 1$ is

$$e = \frac{c}{a} = \frac{\sqrt{a^2 + b^2}}{a}.$$

The **eccentricity** of a parabola is $e = 1$.

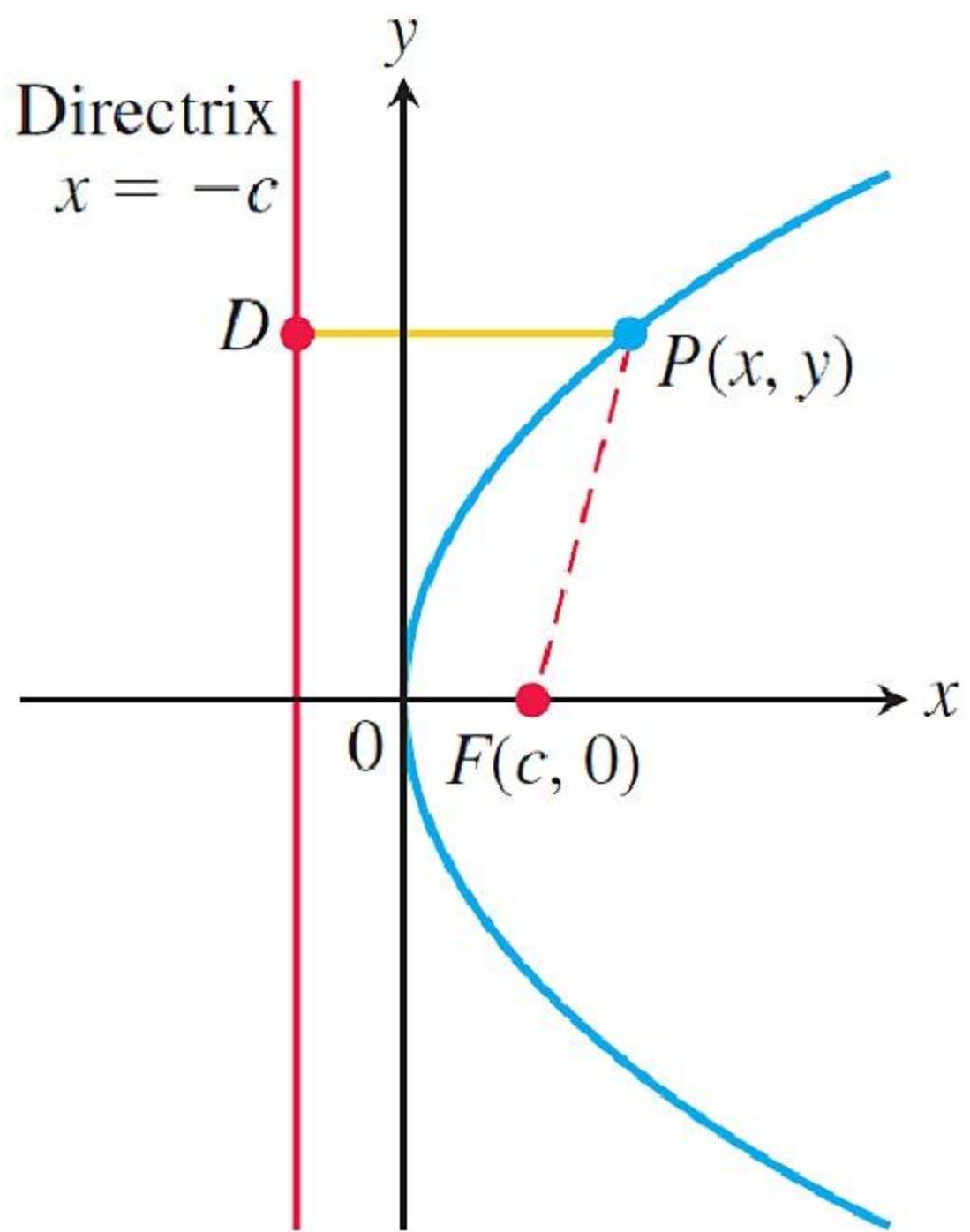


FIGURE 10.48 The distance from the focus F to any point P on a parabola equals the distance from P to the nearest point D on the directrix, so $PF = PD$.

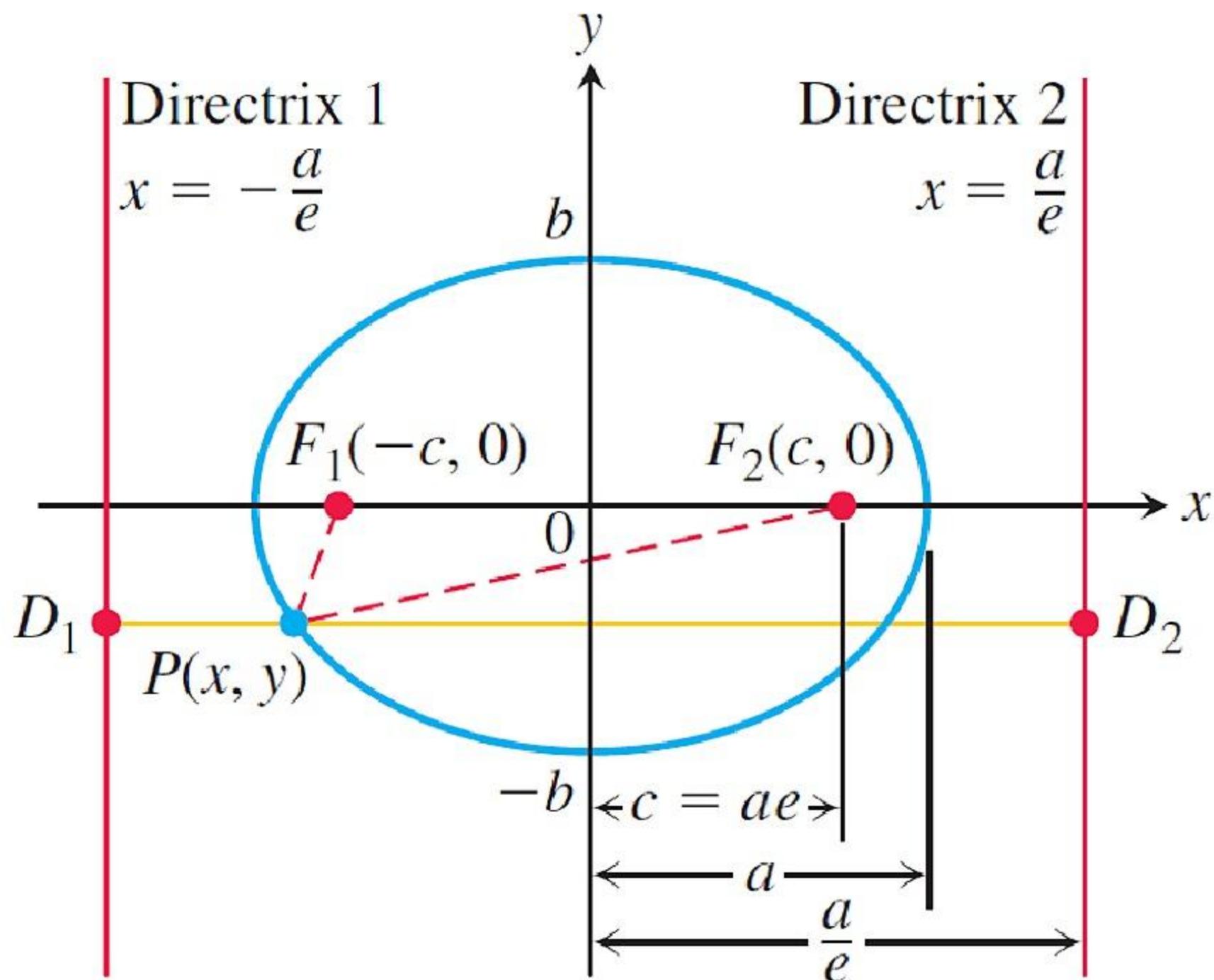


FIGURE 10.49 The foci and directrices of the ellipse $(x^2/a^2) + (y^2/b^2) = 1$. Directrix 1 corresponds to focus F_1 and directrix 2 to focus F_2 .

In both the ellipse and the hyperbola, the eccentricity is the ratio of the distance between the foci to the distance between the vertices (because $c/a = 2c/2a$).

$$\text{Eccentricity} = \frac{\text{distance between foci}}{\text{distance between vertices}}$$

The “focus–directrix” equation $PF = e \cdot PD$ unites the parabola, ellipse, and hyperbola in the following way. Suppose that the distance PF of a point P from a fixed point F (the focus) is a constant multiple of its distance from a fixed line (the directrix). That is, suppose

$$PF = e \cdot PD, \quad (4)$$

where e is the constant of proportionality. Then the path traced by P is

- (a) a *parabola* if $e = 1$,
- (b) an *ellipse* of eccentricity e if $e < 1$, and
- (c) a *hyperbola* of eccentricity e if $e > 1$.

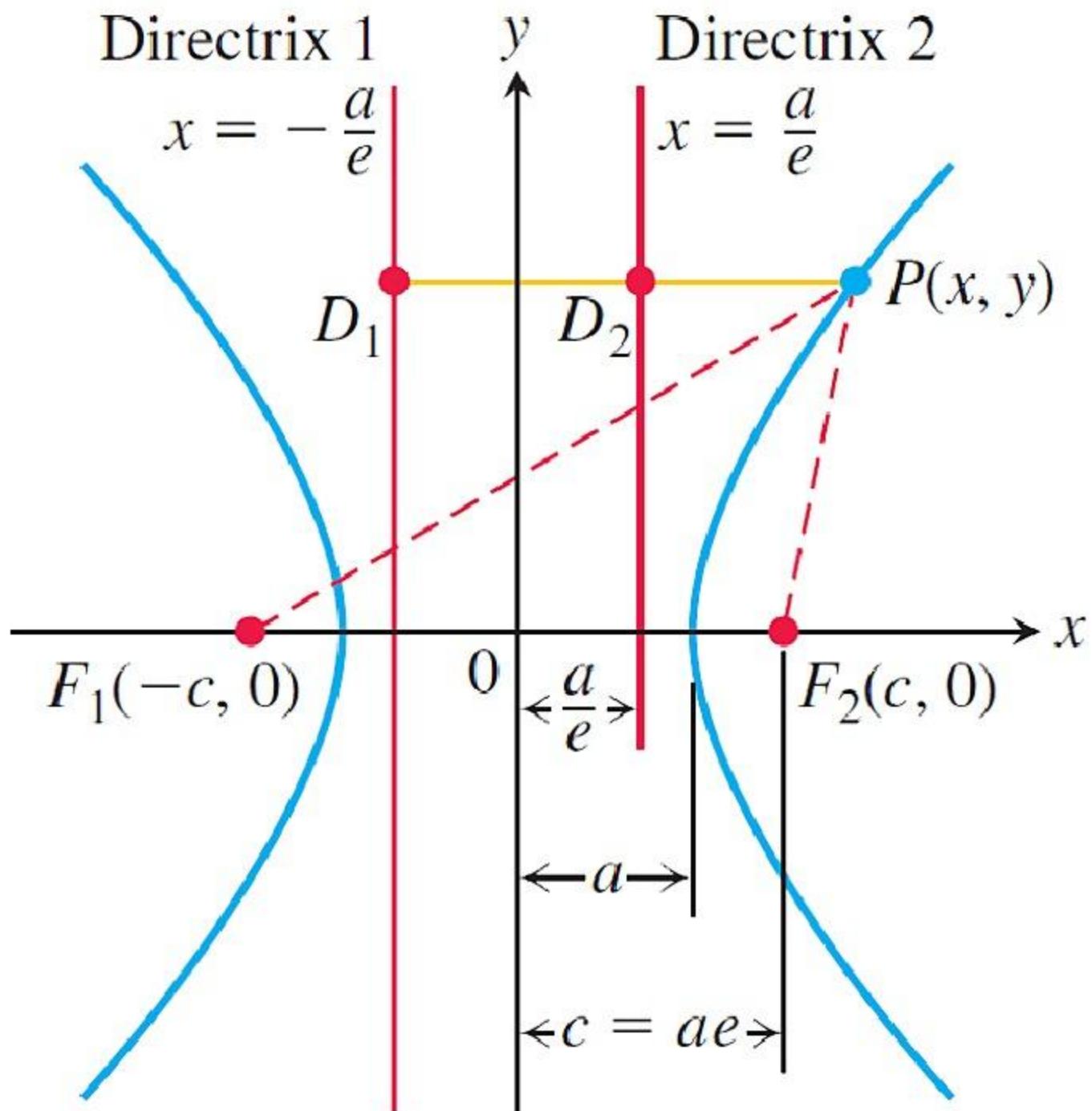


FIGURE 10.50 The foci and directrices of the hyperbola $(x^2/a^2) - (y^2/b^2) = 1$. No matter where P lies on the hyperbola, $PF_1 = e \cdot PD_1$ and $PF_2 = e \cdot PD_2$.

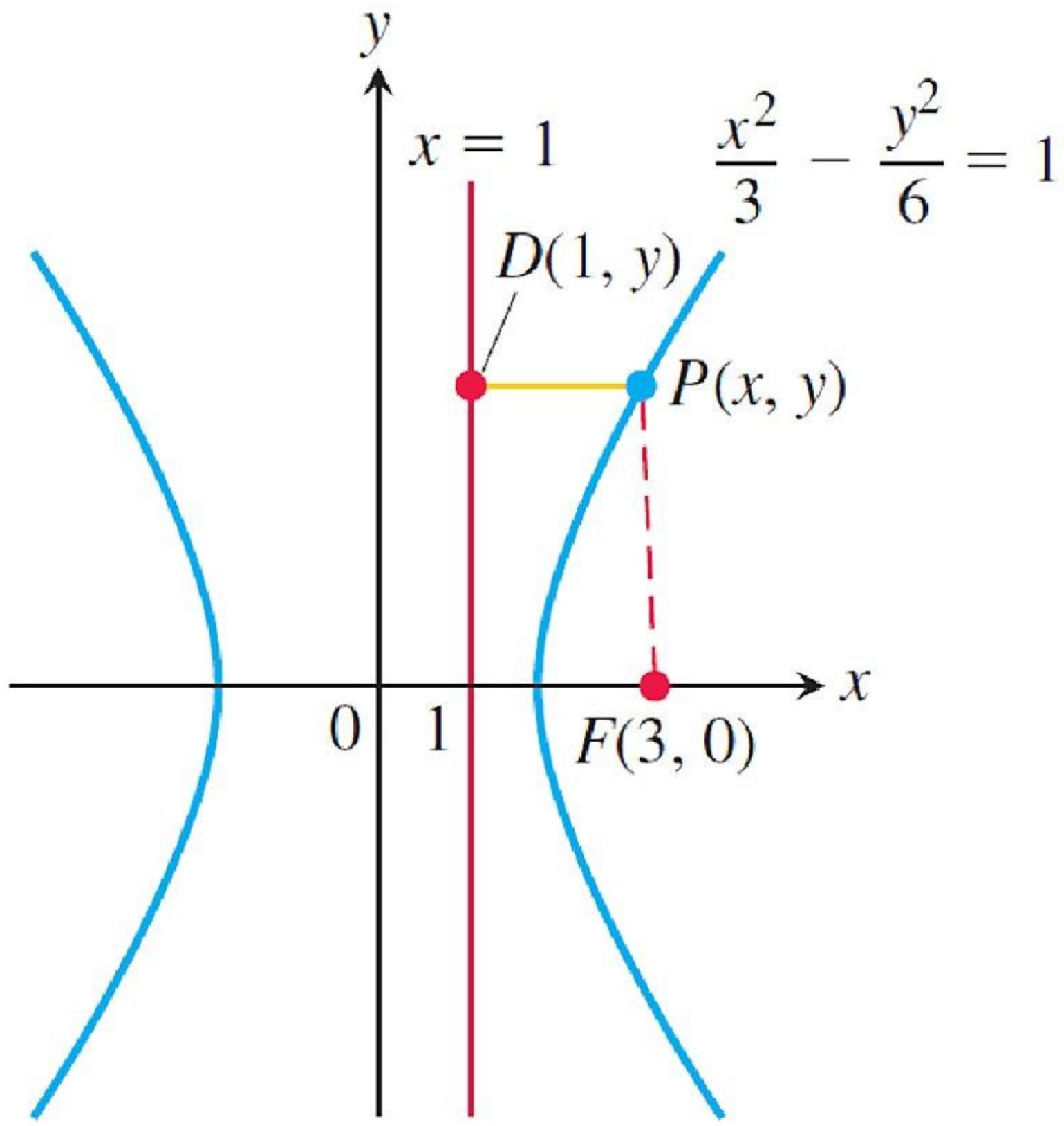


FIGURE 10.51 The hyperbola and directrix in Example 1.

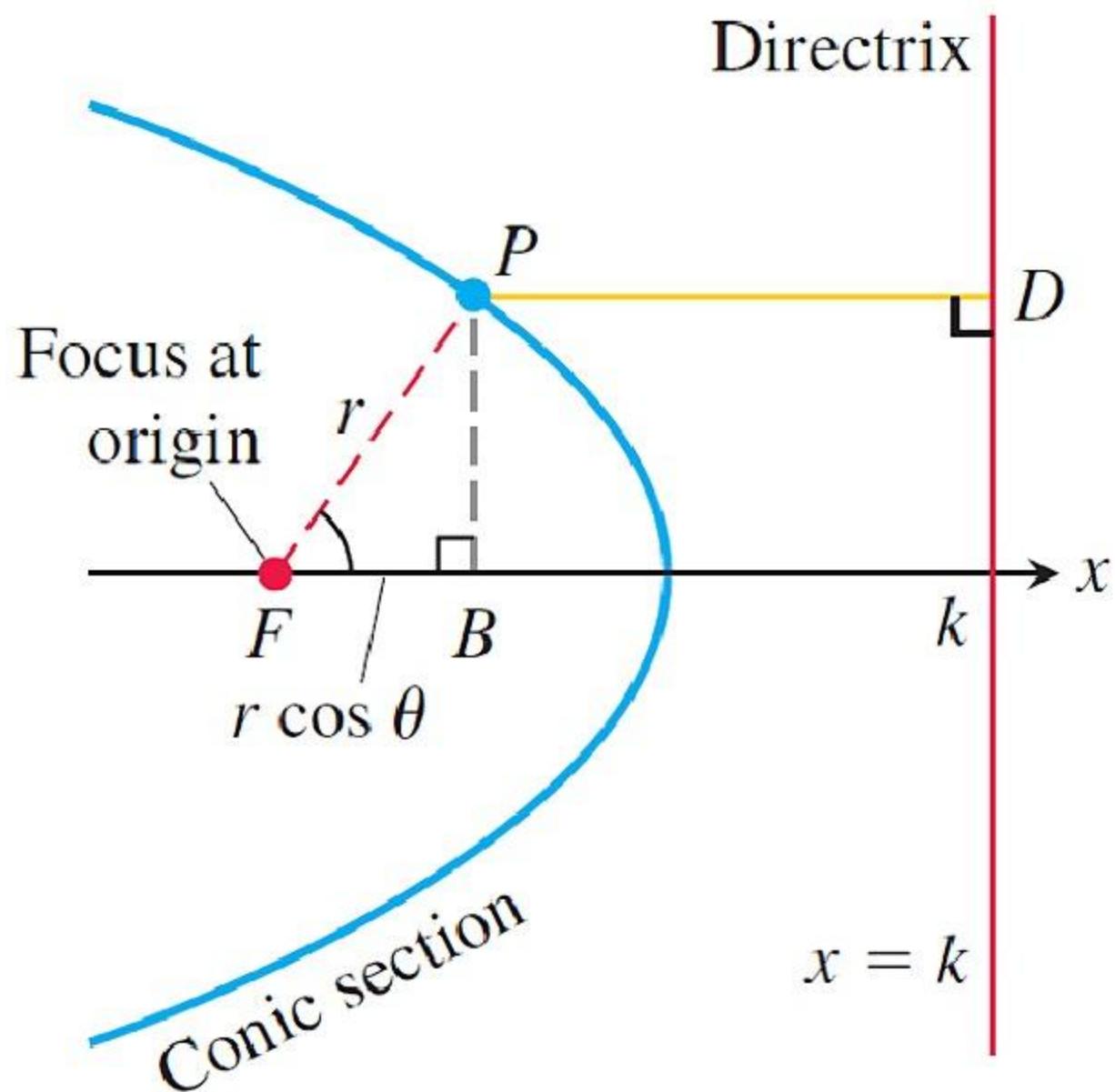


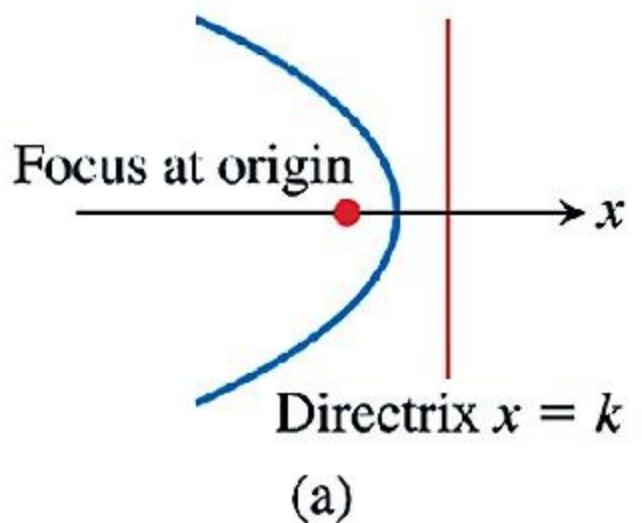
FIGURE 10.52 If a conic section is put in the position with its focus placed at the origin and a directrix perpendicular to the initial ray and right of the origin, we can find its polar equation from the conic's focus–directrix equation.

Polar Equation for a Conic with Eccentricity e

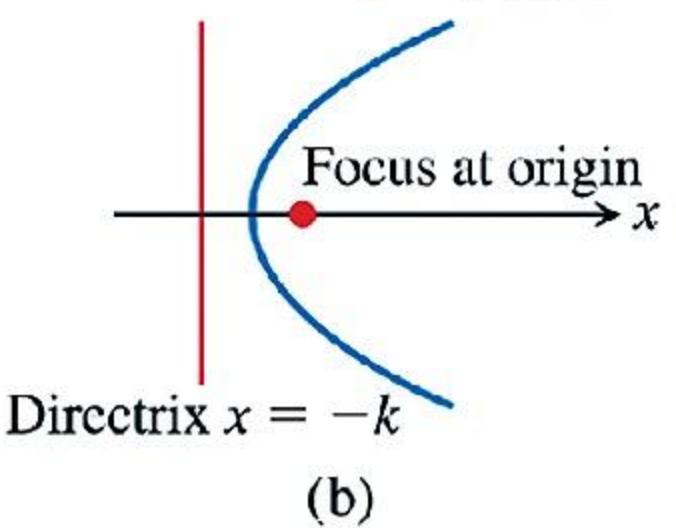
$$r = \frac{ke}{1 + e \cos \theta}, \quad (5)$$

where $x = k > 0$ is the vertical directrix.

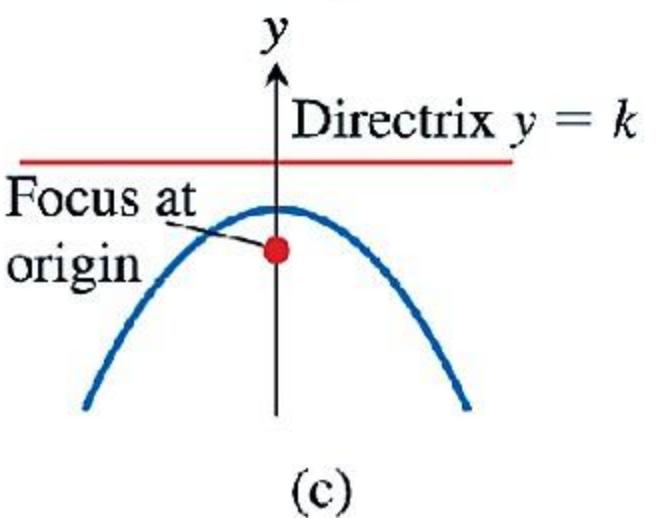
$$r = \frac{ke}{1 + e \cos \theta}$$



$$r = \frac{ke}{1 - e \cos \theta}$$



$$r = \frac{ke}{1 + e \sin \theta}$$



$$r = \frac{ke}{1 - e \sin \theta}$$

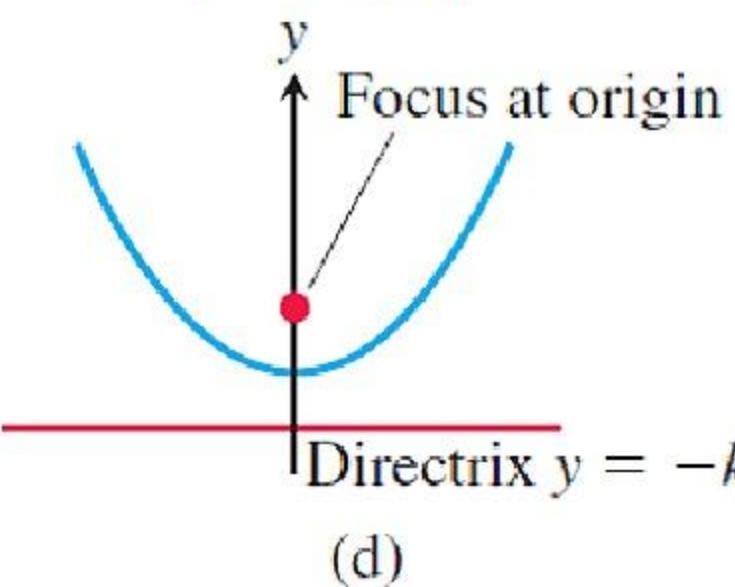


FIGURE 10.53 Equations for conic sections with eccentricity $e > 0$ but different locations of the directrix. The graphs here show a parabola, so $e = 1$.

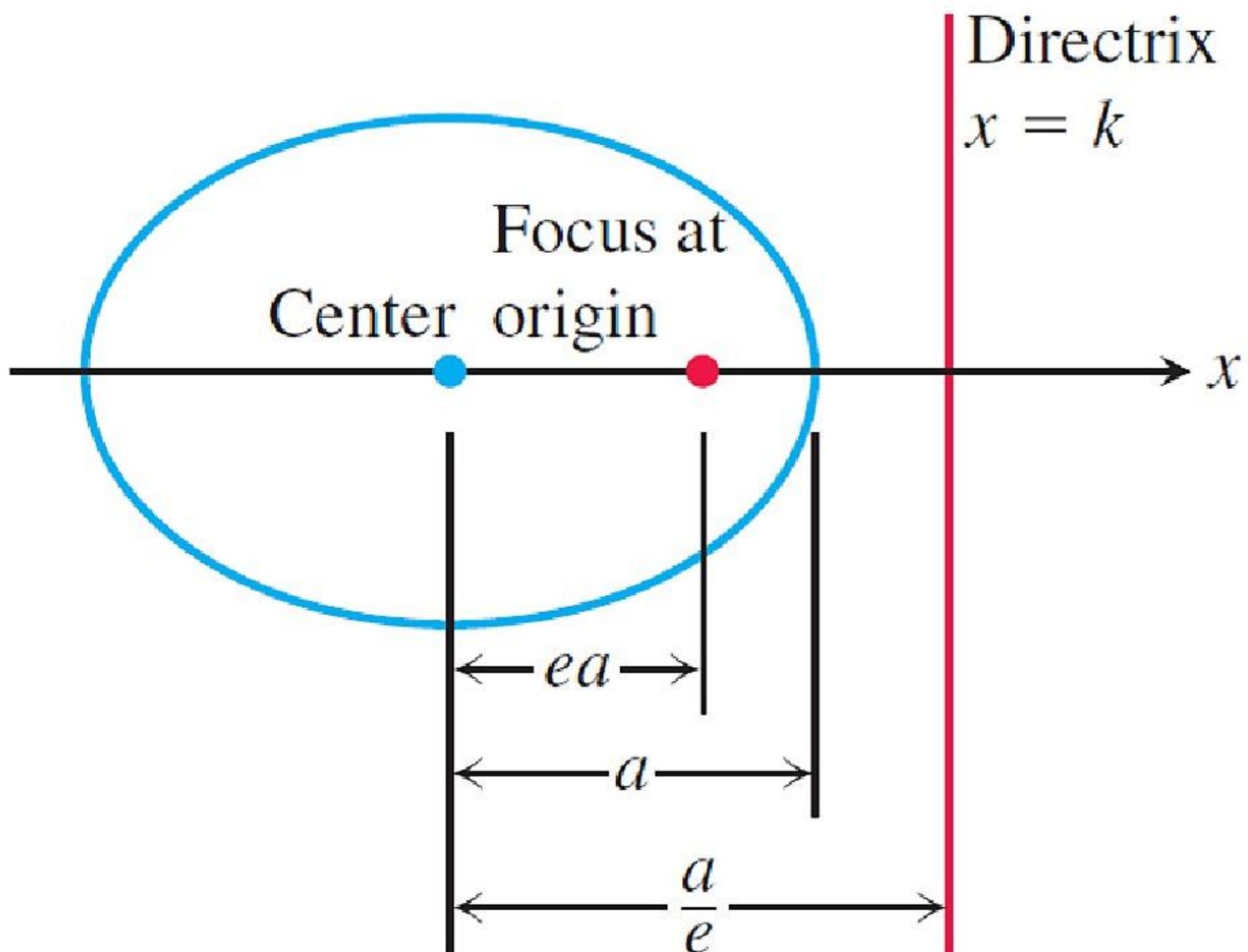


FIGURE 10.54 In an ellipse with semi-major axis a , the focus–directrix distance is $k = (a/e) - ea$, so $ke = a(1 - e^2)$.

Polar Equation for the Ellipse with Eccentricity e and Semimajor Axis a

$$r = \frac{a(1 - e^2)}{1 + e \cos \theta} \quad (6)$$

The Standard Polar Equation for Lines

If the point $P_0(r_0, \theta_0)$ is the foot of the perpendicular from the origin to the line L , and $r_0 \geq 0$, then an equation for L is

$$r \cos(\theta - \theta_0) = r_0. \quad (7)$$

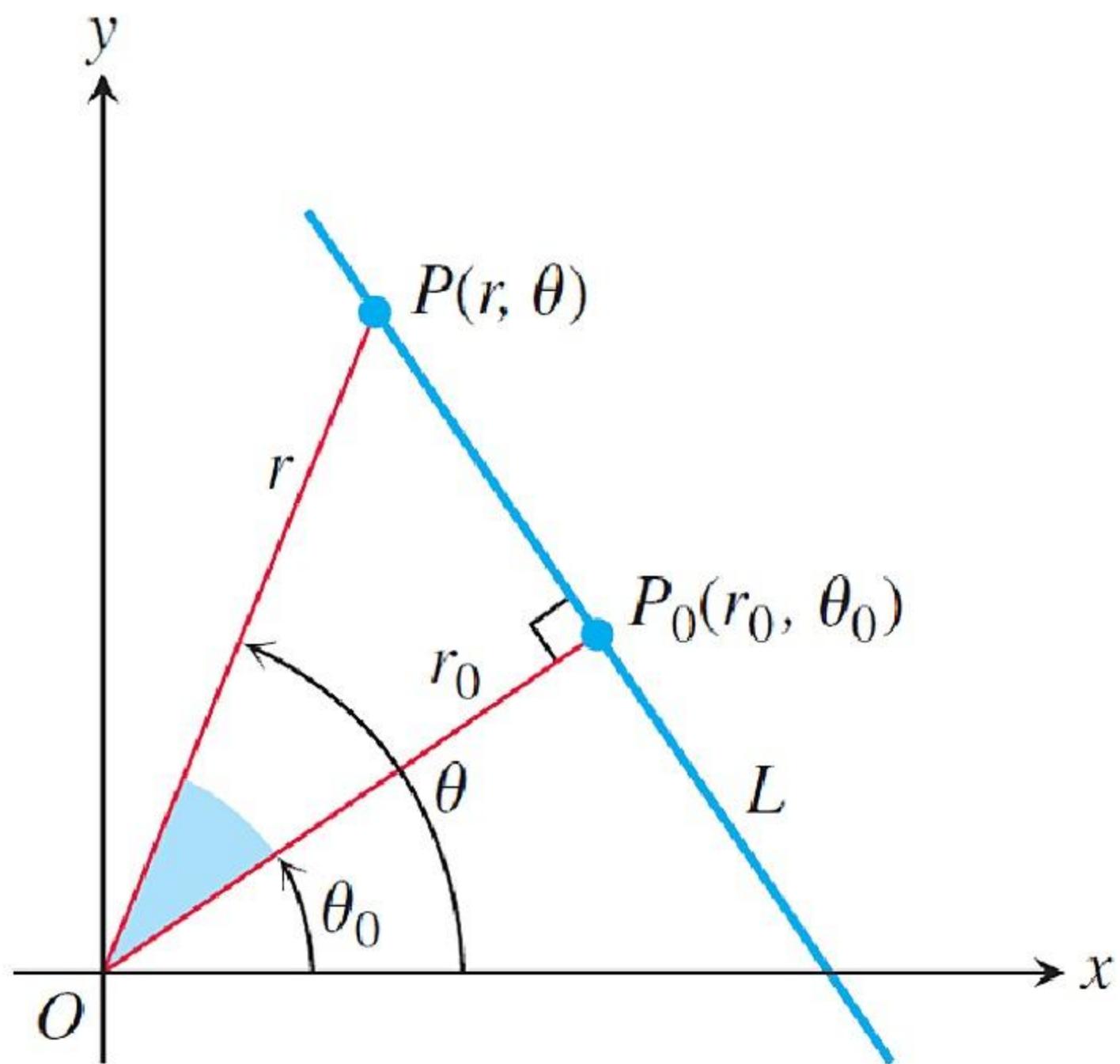


FIGURE 10.55 We can obtain a polar equation for line L by reading the relation $r_0 = r \cos(\theta - \theta_0)$ from the right triangle OP_0P .

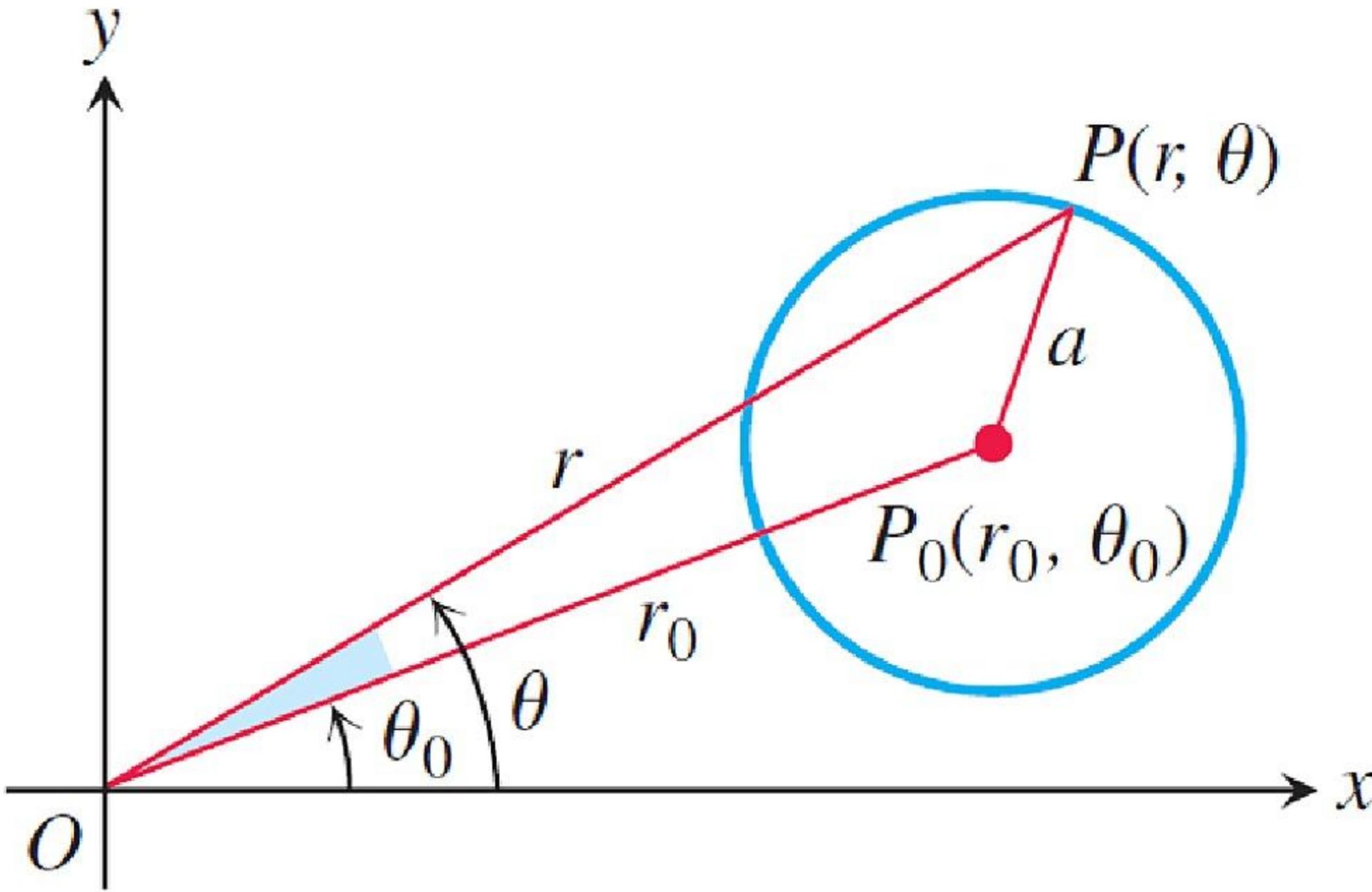


FIGURE 10.56 We can get a polar equation for this circle by applying the Law of Cosines to triangle OP_0P .