4.4 LINEAR TRANSFORMATIONS AND POLYNOMIALS

In this section we shall apply our new knowledge of linear transformations to polynomials. This is the beginning of a general strategy of using our ideas about R^n to solve problems that are in different, yet somehow analogous, settings.

Polynomials and Vectors

Suppose that we have a polynomial function, say

$$p(x) = ax^2 + bx + c$$

where x is a real-valued variable. To form the related function $2p(x)$ we multiply each of its coefficients by 2:

$$2p(x) = 2ax^2 + 2bx + 2c$$

That is, if the coefficients of the polynomial $p(x)$ are a, b, c in descending order of the power of x with which they are associated, then $2p(x)$ is also a polynomial, and its coefficients are $2a$, $2b$, $2c$ in the same order.

Similarly, if $q(x) = dx^2 + ex + f$ is another polynomial function, then $p(x) + q(x)$ is also a polynomial, and its coefficients are $a + d$, $b + e$, $c + f$. We add polynomials by adding corresponding coefficients.

This suggests that associating a polynomial with the vector consisting of its coefficients may be useful.

EXAMPLE 1 Correspondence between Polynomials and Vectors

Consider the quadratic function $p(x) = ax^2 + bx + c$. Define the vector

$$\mathbf{z} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

consisting of the coefficients of this polynomial in descending order of the corresponding power of x. Then multiplication of $p(x)$ by a scalar s gives $sp(x) = sax^2 + sbx + sc$, and this corresponds exactly to the scalar multiple

$$s\mathbf{z} = \begin{bmatrix} sa \\ sb \\ sc \end{bmatrix}$$

of $\mathbf{z}$. Similarly, $p(x) + p(x)$ is $2ax^2 + 2bx + 2c$, and this corresponds exactly to the vector sum $\mathbf{z} + \mathbf{z}$:

$$\mathbf{z} + \mathbf{z} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} + \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$
$$= \begin{bmatrix} 2a \\ 2b \\ 2c \end{bmatrix}$$

◆

In general, given a polynomial $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ we associate with it the vector

$$\mathbf{z}=\begin{bmatrix} a_n \\ a_{n-1} \\ \vdots \\ a_1 \\ a_0 \end{bmatrix}$$

in R^{n+1} (Figure 4.4.1). It is then possible to view operations like $p(x) \longrightarrow 2p(x)$ as being equivalent to a linear transformation on R^{n+1}, namely $T(\mathbf{z})=2\mathbf{z}$. We can perform the desired operations in R^{n+1} rather than on the polynomials themselves.

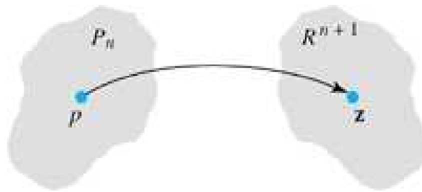

Figure 4.4.1
The vector *z* is associated with the polynomial *p*.

EXAMPLE 2 Addition of Polynomials by Adding Vectors

Let $p(x)=4x^3-2x+1$ and $q(x)=3x^3-3x^2+x$. Then to compute $r(x)=4p(x)-2q(x)$, we could define

$$\mathbf{u}=\begin{bmatrix} 4 \\ 0 \\ -2 \\ 1 \end{bmatrix}, \quad \mathbf{v}=\begin{bmatrix} 3 \\ -3 \\ 1 \\ 0 \end{bmatrix}$$

and perform the corresponding operation on these vectors:

$$\begin{aligned} 4\mathbf{u}-2\mathbf{v} &= 4\begin{bmatrix} 4 \\ 0 \\ -2 \\ 1 \end{bmatrix}-2\begin{bmatrix} 3 \\ -3 \\ 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 10 \\ 6 \\ -10 \\ 4 \end{bmatrix} \end{aligned}$$

Hence $r(x)=10x^3+6x^2-10x+4$.

◆

This association between polynomials of degree *n* and vectors in R^{n+1} would be useful for someone writing a computer program to perform polynomial computations, as in a computer algebra system. The coefficients of polynomial functions could be stored as vectors, and computations could be performed on these vectors.

For convenience, we define P_n to be the set of all polynomials of degree at most *n* (including the zero polynomial, all the coefficients of which are zero). This is also called the *space* of polynomials of degree at most *n*. The use of the word *space* indicates that this set has some sort of structure to it. The structure of P_n will be explored in Chapter 8.

EXAMPLE 3 Differentiation of Polynomials

Calculus Required

Differentiation takes polynomials of degree *n* to polynomials of degree $n-1$, so the corresponding transformation on vectors must take vectors in R^{n+1} to vectors in R^n. Hence, if differentiation corresponds to a linear transformation, it must be represented by a $n \times (n+1)$ matrix. For example, if *p* is an element of P_2—that is,

$$p(x) = ax^2 + bx + c$$

for some real numbers *a*, *b*, and *c*—then

$$\frac{d}{dx}p(x) = 2ax + b$$

Evidently, if $p(x)$ in P_2 corresponds to the vector (a, b, c) in R^3, then its derivative is in P_1 and corresponds to the vector $(2a, b)$ in R^2. Note that

$$\begin{bmatrix} 2a \\ b \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

The operation differentiation, $D\colon P_2 \longrightarrow P_1$, corresponds to a linear transformation $T_A\colon R^3 \longrightarrow R^2$, where

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Some transformations from P_n to P_m do not correspond to linear transformations from R^{n+1} to R^{m+1}. For example, if we consider the transformation of $ax^2 + bx + c$ in P_2 to $|a|$ in P_0, the space of all constants (viewed as polynomials of degree zero, plus the zero polynomial), then we find that there is no matrix that maps (a, b, c) in R^3 to $|a|$ in *R*. Other transformations may correspond to transformations that are not *quite* linear, in the following sense.

DEFINITION

An ***affine transformation*** from R^n to R^m is a mapping of the form $S(\mathbf{u}) = T(\mathbf{u}) + \mathbf{f}$, where *T* is a linear transformation from R^n to R^m and ***f*** is a (constant) vector in R^m.

The affine transformation *S* is a linear transformation if ***f*** is the zero vector. Otherwise, it isn't linear, because it doesn't satisfy Theorem 4.3.2. This may seem surprising because the form of *S* looks like a natural generalization of an equation describing a line, but linear transformations satisfy the *Principle of Superposition*

$$T(c_1\mathbf{u} + c_2\mathbf{v}) = c_1 T(\mathbf{u}) + c_2 T(\mathbf{v})$$

for any scalars c_1, c_2 and any vectors ***u***, ***v*** in their domain. (This is just a restatement of Theorem 4.3.2.) Affine transformations with ***f*** nonzero don't have this property.

EXAMPLE 4 Affine Transformations

The mapping

$$S(\mathbf{u}) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\mathbf{u} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

is an affine transformation on R^2. If $\mathbf{u} = (a, b)$, then

$$S(\mathbf{u}) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} b+1 \\ -a+1 \end{bmatrix}$$

The corresponding operation from P_1 to P_1 takes $ax+b$ to $(b+1)x-a+1$.

The relationship between an action on P_n and its corresponding action on the vector of coefficients in R^{n+1}, and the similarities between P_n and R^{n+1}, will be explored in more detail later in this text.

Interpolating Polynomials

Consider the problem of interpolating a polynomial to a set of $n+1$ points $(x_0, y_0), \ldots, (x_n, y_n)$. That is, we seek to find a curve $p(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0$ of minimum degree that goes through each of these data points (Figure 4.4.2). Such a curve must satisfy

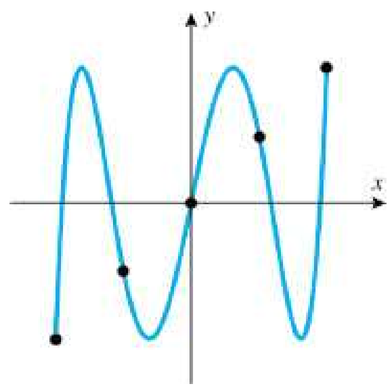

Figure 4.4.2
Interpolation

$$\begin{aligned} y_0 &= a_m x_0^m + a_{m-1}x_0^{m-1} + \cdots + a_1 x_0 + a_0 \\ y_1 &= a_m x_1^m + a_{m-1}x_1^{m-1} + \cdots + a_1 x_1 + a_0 \\ &\vdots \\ y_n &= a_m x_n^m + a_{m-1}x_n^{m-1} + \cdots + a_1 x_n + a_0 \end{aligned}$$

Because the x_i are known, this leads to the following matrix system:

$$\begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^m \\ 1 & x_1 & x_1^2 & \cdots & x_1^m \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & x_{n-1} & x_{n-1}^2 & \cdots & x_{n-1}^m \\ 1 & x_n & x_n^2 & \cdots & x_n^m \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{m-1} \\ a_m \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix}$$

Note that this is a square system when $n = m$. Taking $n = m$ gives the following system for the coefficients of the interpolating polynomial $p(x)$:

$$\begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & x_{n-1} & x_{n-1}^2 & \cdots & x_{n-1}^n \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \\ a_n \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix} \tag{1}$$

The matrix in 1 is known as a ***Vandermonde matrix***; column *j* is the second column raised element wise to the $j-1$ power. The linear system in 1 is said to be a ***Vandermonde system***.

EXAMPLE 5 Interpolating a Cubic

To interpolate a polynomial to the data (−2, 11), (−1, 2), (1, 2), (2, −1), we form the Vandermonde system 1:

$$\begin{bmatrix} 1 & x_0 & x_0^2 & x_0^3 \\ 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

For this data, we have

$$\begin{bmatrix} 1 & -2 & 4 & -8 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 11 \\ 2 \\ 2 \\ -1 \end{bmatrix}$$

The solution, found by Gaussian elimination, is

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

and so the interpolant is $p(x) = -x^3 + x^2 + x + 1$. This is plotted in Figure 4.4.3, together with the data points, and we see that $p(x)$ does indeed interpolate the data, as required.

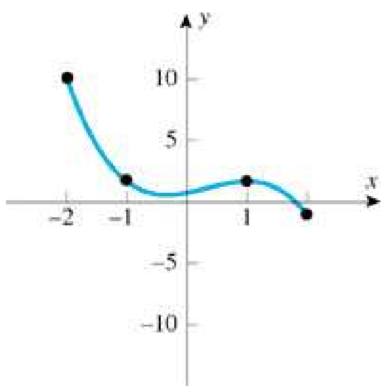

Figure 4.4.3 The interpolant of Example 4

Newton Form

The interpolating polynomial $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ is said to be written in its natural, or standard, form. But there is convenience in using other forms. For example, suppose we seek a cubic interpolant to the data (x_0, y_0), (x_1, y_1), (x_2, y_2), (x_3, y_3). If we write

$$p(x) = a_3 x^3 + a_2 x^2 + a_1 x + a_0 \tag{2}$$

in the equivalent form

$$p(x) = a_3(x - x_0)^3 + a_2(x - x_0)^2 + a_1(x - x_0) + a_0$$

then the interpolation condition $p(x_0)=y_0$ immediately gives $a_0=y_0$. This reduces the size of the system that must be solved from $(n+1)\times(n+1)$ to $n\times n$. That is not much of a savings, but if we take this idea further, we may write 2 in the equivalent form

$$p(x)=b_3(x-x_0)(x-x_1)(x-x_2)+b_2(x-x_0)(x-x_1)+b_1(x-x_0)+b_0 \tag{3}$$

which is called the ***Newton form*** of the interpolant. Set $h_i=x_i-x_{i-1}$ for $i=1$, 2, 3. The interpolation conditions give

$$\begin{aligned}
&p(x_0)=b_0\\
&p(x_1)=b_1h_1+b_0\\
&p(x_2)=b_2(h_1+h_2)h_2+b_1(h_1+h_2)+b_0\\
&p(x_3)=b_3(h_1+h_2+h_3)(h_2+h_3)h_3+b_2(h_1+h_2+h_3)(h_2+h_3)+b_1(h_1+h_2+h_3)+b_0
\end{aligned}$$

that is,

$$\begin{bmatrix} 1 & 0 & 0 & 0\\ 1 & h_1 & 0 & 0\\ 1 & h_1+h_2 & (h_1+h_2)h_2 & 0\\ 1 & h_1+h_2+h_3 & (h_1+h_2+h_3)(h_2+h_3) & (h_1+h_2+h_3)(h_2+h_3)h_3 \end{bmatrix}\begin{bmatrix} b_0\\ b_1\\ b_2\\ b_3\end{bmatrix}=\begin{bmatrix} y_0\\ y_1\\ y_2\\ y_3\end{bmatrix} \tag{4}$$

Unlike the Vandermonde system 1, this system has a lower triangular coefficient matrix. This is a much simpler system. We may solve for the coefficients very easily and efficiently by ***forward-substitution***, in analogy with back-substitution. In the case of equally spaced points arranged in increasing order, we have $h_i=h>0$, so 4 becomes

$$\begin{bmatrix} 1 & 0 & 0 & 0\\ 1 & h & 0 & 0\\ 1 & 2h & 2h^2 & 0\\ 1 & 3h & 6h^2 & 6h^3 \end{bmatrix}\begin{bmatrix} b_0\\ b_1\\ b_2\\ b_3\end{bmatrix}=\begin{bmatrix} y_0\\ y_1\\ y_2\\ y_3\end{bmatrix}$$

Note that the determinant of 4 is nonzero exactly when h_i is nonzero for each *i*, so there exists a unique interpolant whenever the x_i are distinct. Because the Vandermonde system computes a different form of the same interpolant, it too must have a unique solution exactly when the x_i are distinct.

EXAMPLE 6 Interpolating a Cubic in Newton Form

To interpolate a polynomial in Newton form to the data (−2, 11), (−1, 2), (1, 2), (2, −1) of Example 5, we form the system 4:

$$\begin{bmatrix} 1 & 0 & 0 & 0\\ 1 & 1 & 0 & 0\\ 1 & 3 & 6 & 0\\ 1 & 4 & 12 & 12 \end{bmatrix}\begin{bmatrix} b_0\\ b_1\\ b_2\\ b_3\end{bmatrix}=\begin{bmatrix} 11\\ 2\\ 2\\ 1\end{bmatrix}$$

The solution, found by forward-substitution, is

$$\begin{aligned}
b_0 &= 11 & &\\
b_0+b_1 &= 2 & b_1&=-9\\
b_0+3b_1+6b_2 &= 2 & b_2&=3\\
b_0+4b_1+12b_2+12b_3 &= -1 & b_3&=-1
\end{aligned}$$

and so, from 3, the interpolant is

$$\begin{aligned}
p(x)&=-1\cdot(x+2)(x+1)(x-1)+3\cdot(x+2)(x+1)+(-9)\cdot(x+2)+11\\
&=-(x+2)(x+1)(x-1)+3(x+2)(x+1)-9(x+2)+11
\end{aligned}$$

Converting between Forms

The Newton form offers other advantages, but now we turn to the following question: If we have the coefficients of the interpolating polynomial in Newton form, what are the coefficients in the standard form? For example, if we know the coefficients in

$$p(x) = b_3(x-x_0)(x-x_1)(x-x_2) + b_2(x-x_0)(x-x_1) + b_1(x-x_0) + b_0$$

because we have solved 4 in order to avoid having to solve the more complicated Vandermonde system 1, how can we get the coefficients in 2,

$$p(x) = a_3x^3 + a_2x^2 + a_1x + a_0$$

from b_0, b_1, b_2, b_3? Expanding the products in 3 gives

$$\begin{aligned} p(x) &= b_3(x-x_0)(x-x_1)(x-x_2) + b_2(x-x_0)(x-x_1) + b_1(x-x_0) + b_0 \\ &= b_3x^3 + (b_2 - b_3(x_0+x_1+x_2))x^2 \\ &\quad + (b_1 - b_2(x_0+x_1) + b_3(x_0x_1 + x_0x_2 + x_1x_2))x \\ &\quad + b_0 - x_0b_1 + x_0x_1b_2 - x_0x_1x_2b_3 \end{aligned}$$

so

$$\begin{aligned} a_0 &= b_0 - x_0b_1 + x_0x_1b_2 - x_0x_1x_2b_3 \\ a_1 &= b_1 - b_2(x_0+x_1) + b_3(x_2x_1 + x_0x_2 + x_1x_2) \\ a_2 &= b_2 - b_3(x_0+x_1+x_2) \\ a_3 &= b_3 \end{aligned}$$

This can be expressed as

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 & -x_0 & x_0x_1 & -x_0x_1x_2 \\ 0 & 1 & -(x_0+x_1) & x_0x_1 + x_0x_2 + x_1x_2 \\ 0 & 0 & 1 & -(x_0+x_1+x_2) \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix} \tag{5}$$

This is an important result! Solving the Vandermonde system 1 by Gaussian elimination would require us to form an $n \times n$ matrix that might have no nonzero entries and then to solve it using a number of arithmetic operations that grows in proportion to n^3 for large *n*. But solving the lower triangular system 4 requires an amount of work that grows in proportion to n^2 for large *n*, and using 5 to compute the coefficients a_0, a_1, a_2, a_3 also requires an amount of work that grows in proportion to n^2 for large *n*. Hence, for large *n*, the latter approach is an order of magnitude more efficient. The two-step procedure of solving 4 and then using the linear transformation 5 is a superior approach to solving 1 when *n* is large (Figure 4.4.4).

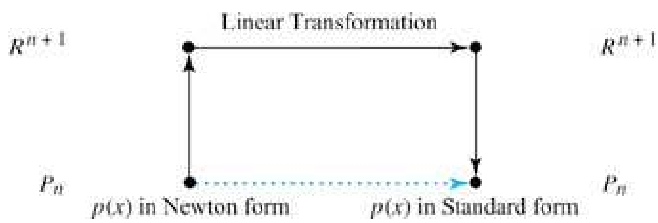

Figure 4.4.4

Indirect route to conversion from Newton form to standard form

EXAMPLE 7 Changing Forms

In Example 4 we found that $a_0 = 1, a_1 = 1, a_2 = 1, a_3 = -1$, whereas in Example 5 we found that $b_0 = 11, b_1 = -9, b_2 = 3, b_3 = -1$ for the same data. From 5, with $x_0 = -2, x_1 = -1, x_1 = 1$, we expect that

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 & -2 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 11 \\ -9 \\ 3 \\ -1 \end{bmatrix}$$

which checks.

There is another approach to solving 1, based on the Fast Fourier Transform, that also requires an amount of work proportional to n^2. The point for now is to see that the use of linear transformations on R^{n+1} can help us perform computations involving polynomials. The original problem—to fit a polynomial of minimum degree to a set of data points—was not couched in the language of linear algebra at all. But rephrasing it in those terms and using matrices and the notation of linear transformations on R^{n+1} has allowed us to see when a unique solution must exist, how to compute it efficiently, and how to transform it among various forms.

Exercise Set 4.4

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1. Identify the operations on polynomials that correspond to the following operations on vectors. Give the resulting polynomial.

(a) $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} - 2\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$

(b) $5\begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} + 6\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ 2 \\ 1 \\ -2 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \\ 0 \\ -2 \\ 0 \end{bmatrix}$

(d) $\pi\begin{bmatrix} 4 \\ -3 \\ 7 \\ 1 \end{bmatrix}$

2.

(a) Consider the operation on P_2 that takes $ax^2 + bx + c$ to $cx^2 + bx + a$. Does it correspond to a linear transformation from R^3 to R^3? If so, what is its matrix?

(b) Consider the operation on P_3 that takes $ax^3 + bx^2 + cx + d$ to $cx^3 - bx^2 - ax + d$. Does it correspond to a linear transformation from R^3 to R^3? If so, what is its matrix?

3.

(a) Consider the transformation of $ax^2 + bx + c$ in P_2 to $|a|$ in P_0. Show that it does not correspond to a linear transformation by showing that there is no matrix that maps (a, b, c) in R^3 to $|a|$ in *R*.

(b) Does the transformation of $ax^2 + bx + c$ in P_2 to *a* in P_0 correspond to a linear transformation from R^3 to *R*?

4.

(a) Consider the operation $M\colon P_2 \longrightarrow P_3$ that takes $p(x)$ in P_2 to $xp(x)$ in P_3. Does this correspond to a linear transformation from R^3 to R^4? If so, what is its matrix?

(b) Consider the operation $N\colon P_2 \longrightarrow P_3$ that takes $p(x)$ in P_n to $(x-1)p(x)$ in P_{n+1}. Does this correspond to a linear transformation from R^3 to R^4? If so, what is its matrix?

(c) Consider the operation $W\colon P_2 \longrightarrow P_3$ that takes $p(x)$ in P_n to $xp(x) + 1$ in P_{n+1}. Does this correspond to a linear transformation from R^3 to R^4? If so, what is its matrix?

5. (For Readers Who Have Studied Calculus) What matrix corresponds to differentiation in each case?

(a) $D\colon P_3 \longrightarrow P_2$

(b) $D\colon P_4 \longrightarrow P_3$

(c) $D\colon P_5 \longrightarrow P_4$

6. **(For Readers Who Have Studied Calculus)** What matrix corresponds to differentiation in each case, assuming we represent $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ as the vector $(a_0, a_1, \ldots, a_{n-1}, a_n)$?

Note This is the opposite of the ordering of coefficients we have been using.

(a) $D\colon P_3 \longrightarrow P_2$

(b) $D\colon P_4 \longrightarrow P_3$

(c) $D\colon P_5 \longrightarrow P_4$

7. Consider the following matrices. What is the corresponding transformation on polynomials? Indicate the domain P_i and the codomain P_j.

(a) $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 2 & -1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 2 & -1 \\ 2 & 1 & 1 & 3 \end{bmatrix}$

(d) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(e) $\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$

8. Consider the space of all functions of the form $a + b\cos(x) + c\sin(x)$ where *a*, *b*, *c* are scalars.

(a) What matrix, if any, corresponds to the change of variables $x \longrightarrow x - \pi/2$, assuming that we represent a function in this space as the vector (a, b, c)?

(b) What matrix corresponds to differentiation of functions on this space?

9. Consider the space of all functions of the form $a + bt + ce^{t} + de^{-t}$, where *a*, *b*, *c*, *d* are scalars.

(a) What function in the space corresponds to the sum of (1, 2, 3, 4) and (−1, −2, 0, −1), assuming that we represent a function in this space as the vector (a, b, c, d)?

(b) Is $\cosh(t)$ in this space? That is, does $\cosh(t)$ correspond to some choice of *a*, *b*, *c*, *d*?

(c) What matrix corresponds to differentiation of functions on this space?

10. Show that the Principle of Superposition is equivalent to Theorem 4.3.2.

11. Show that an affine transformation with $\boldsymbol{f}$ nonzero is not a linear transformation.

12. Find a quadratic interpolant to the data (−1, 2), (0, 0), (1, 2) using the Vandermonde system approach.

13.

(a) Find a quadratic interpolant to the data (−2, 1), (0, 1), (1, 4) using the Vandermonde system approach from 1.

(b) Repeat using the Newton approach from 4.

14.

(a) Find a polynomial interpolant to the data (−1, 0), (0, 0), (1, 0), (2, 6) using the Vandermonde system approach from 1.

(b) Repeat using the Newton approach from 4.

(c) Use 5 to get your answer in part (a) from your answer in part (b).

(d) Use 5 to get your answer in part (b) from your answer in part (a) by finding the inverse of the matrix.

(e) What happens if you change the data to (−1, 0), (0, 0), (1, 0), (2, 0)?

15.

(a) Find a polynomial interpolant to the data (−2, −10), (−1, 2), (1, 2), (2, 14) using the Vandermonde system approach from 1.

(b) Repeat using the Newton approach from 4.

(c) Use 5 to get your answer in part (a) from your answer in part (b).

(d) Use 5 to get your answer in part (b) from your answer in part (a) by finding the inverse of the matrix.

16. Show that the determinant of the 2×2 Vandermonde matrix

$$\begin{bmatrix} 1 & a \\ 1 & b \end{bmatrix}$$

can be written as $(b - a)$ and that the determinant of the 3×3 Vandermonde matrix

$$\det \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}$$

can be written as $(b - a)(c - a)(c - b)$. Conclude that a unique straight line can be fit through any two points (x_0, y_0),

(x_1, y_1) with x_0 and x_1 distinct, and that a unique parabola (which may be degenerate, such as a line) can be fit through any three points (x_0, y_0), (x_1, y_1), (x_2, y_2) with x_0, x_1, and x_2 distinct.

17.

(a) What form does 5 take for lines?

(b) What form does 5 take for quadratics?

(c) What form does 5 take for quartics?

Discussion & Discovery

18. (For Readers Who Have Studied Calculus)

(a) Does indefinite integration of functions in P_n correspond to some linear transformation from R^{n+1} to R^{n+2}?

(b) Does definite integration (from $x = 0$ to $x = 1$) of functions in P_n correspond to some linear transformation from R^{n+1} to *R*?

19. (For Readers Who Have Studied Calculus)

(a) What matrix corresponds to second differentiation of functions from P_2 (giving functions in P_0)?

(b) What matrix corresponds to second differentiation of functions from P_3 (giving functions in P_1)?

(c) Is the matrix for second differentiation the square of the matrix for (first) differentiation?

20. Consider the transformation from P_2 to P_2 associated with the matrix

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and the transformation from P_2 to P_0 associated with the matrix

$$\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$$

These differ only in their codomains. Comment on this difference. In what ways (if any) is it important?

21. The third major technique for polynomial interpolation is interpolation using ***Lagrange interpolating polynomials***. Given a set of distinct *x*-values x_0, x_1, … x_n, define the $n + 1$ Lagrange interpolating polynomials for these values by (for $i = 0$, 1, … *n*)

$$L_i(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_0)(x_i-x_1)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}$$

Note that $L_i(x)$ is a polynomial of exact degree *n* and that $L_i(x_j) = 0$ if $i \neq j$, and $L_i(x_i) = 1$. It follows that we can write the polynomial interpolant to (x_0, y_0), …, (x_n, y_n) in the form

$$p(x) = c_0L_0(x) + c_1L_1(x) + \cdots + c_nL_n(x)$$

where $c_i = y_i$, $i = 0$, 1, …, *n*.

(a) Verify that $p(x) = y_0L_0(x) + y_1L_1(x) + \cdots + y_nL_n(x)$ is the unique interpolating polynomial for this data.

(b) What is the linear system for the coefficients c_0, c_1, …, c_n, corresponding to 1 for the Vandermonde approach and to 4 for the Newton approach?

(c) Compare the three approaches to polynomial interpolation that we have seen. Which is most efficient with respect to finding the coefficients? Which is most efficient with respect to evaluating the interpolant somewhere between data points?

22. Generalize the result in Problem 16 by finding a formula for the determinant of an $n \times n$ Vandermonde matrix for arbitrary *n*.

23. The ***norm*** of a linear transformation T_A: $R^n \longrightarrow R^n$ can be defined by

$$\|T\|_E = \max \frac{\|T(\mathbf{x})\|}{\|\mathbf{x}\|}$$

where the maximum is taken over all nonzero ***x*** in R^n. (The subscript indicates that the norm of the linear transformation on the left is found using the Euclidean vector norm on the right.) It is a fact that the largest value is always achieved—that is, there is always some $\mathbf{x}_0$ in R^n such that $\|T\|_E = \max(\|T(\mathbf{x}_0)\| / \|\mathbf{x}_0\|)$. What are the norms of the linear transformations T_A with the following matrices?

(a) $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$

(d) $\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

Chapter 4

Technology Exercises

The following exercises are designed to be solved using a technology utility. Typically, this will be MATLAB, *Mathematica*, Maple, Derive, or Mathcad, but it may also be some other type of linear algebra software or a scientific calculator with some linear algebra capabilities. For each exercise you will need to read the relevant documentation for the particular utility you are using. The goal of these exercises is to provide you with a basic proficiency with your technology utility. Once you have mastered the techniques in these exercises, you will be able to use your technology utility to solve many of the problems in the regular exercise sets.

Section 4.1

T1. **(Vector Operations in R^n)** With most technology utilities, the commands for operating on vectors in R^n are the same as those for operating on vectors in R^2 and R^3, and the command for computing a dot product produces the Euclidean inner product in R^n. Use your utility to perform computations in Exercises 1, 3, and 9 of Section 4.1.

Section 4.2

T1. **(Rotations)** Find the standard matrix for the linear operator on R^3 that performs a counterclockwise rotation of 45° about the *x*-axis, followed by a counterclockwise rotation of 60° about the *y*-axis, followed by a counterclockwise rotation of 30° about the *z*-axis. Then find the image of the point (1, 1, 1) under this operator.

Section 4.3

T1. **(Projections)** Use your utility to perform the computations for $\theta = \pi / 6$ in Example 6. Then project the vectors (1, 1) and (1, −5) . Repeat for $\theta = \pi / 4$, $\pi / 3$, $\pi / 2$, π.

Section 4.4

T1. **(Interpolation)** Most technology utilities have a command that performs polynomial interpolation. Read your documentation, and find the command or commands for fitting a polynomial interpolant to given data. Then use it (or them) to confirm the result of Example 5.