Governing equations in dynamic dialysis (1-d model, based on mass balance)

Lumen side	Shell side
$\rho_1 \frac{\mathrm{d}u_1}{\mathrm{d}z} + u_1 \frac{\mathrm{d}\rho_1}{\mathrm{d}z} = \frac{4}{d_1}(j_{\mathrm{M,A}} - j_{\mathrm{M,B}})$	$\rho_2 \frac{\mathrm{d}u_2}{\mathrm{d}z} + u_2 \frac{\mathrm{d}\rho_2}{\mathrm{d}z} = \frac{4d_1}{d_2^2 - d_1^2}(j_{\mathrm{M,B}} - j_{\mathrm{M,A}})$
$\rho_1 = \omega_{\mathrm{A}1}\rho_{\mathrm{A}} + \omega_{\mathrm{B}1}\rho_{\mathrm{B}}$	$\rho_2 = \omega_{\mathrm{A}2}\rho_{\mathrm{A}} + \omega_{\mathrm{B}2}\rho_{\mathrm{B}}$
$\rho_1 u_1 \frac{\mathrm{d}\omega_{\mathrm{A}1}}{\mathrm{d}z} = \frac{4}{d_1}\left[\omega_{\mathrm{A}1} j_{\mathrm{M,B}} + (1-\omega_{\mathrm{A}1}) j_{\mathrm{M,A}}\right]$	$\rho_2 u_2 \frac{\mathrm{d}\omega_{\mathrm{A}2}}{\mathrm{d}z} = \frac{4d_1}{d_2^2 - d_1^2}\left[(\omega_{\mathrm{A}2}-1) j_{\mathrm{M,A}} - \omega_{\mathrm{A}2} j_{\mathrm{M,B}}\right]$
$\rho_1 u_1 \frac{\mathrm{d}\omega_{\mathrm{B}1}}{\mathrm{d}z} = \frac{4}{d_1}\left[(\omega_{\mathrm{B}1}-1) j_{\mathrm{M,B}} - \omega_{\mathrm{B}1} j_{\mathrm{M,A}}\right]$	$\rho_2 u_2 \frac{\mathrm{d}\omega_{\mathrm{B}2}}{\mathrm{d}z} = \frac{4d_1}{d_2^2 - d_1^2}\left[\omega_{\mathrm{B}2} j_{\mathrm{M,A}} + (1-\omega_{\mathrm{B}2}) j_{\mathrm{M,B}}\right]$
$\omega_{\mathrm{A}1}\big\vert_{z=0} = \omega_{\mathrm{A}10}$ $\omega_{\mathrm{B}1}\big\vert_{z=0} = \omega_{\mathrm{B}10}$ $u_1\big\vert_{z=0} = u_{10}$	$\omega_{\mathrm{A}2}\big\vert_{z=0} = \omega_{\mathrm{A}20}$ $\omega_{\mathrm{B}2}\big\vert_{z=0} = \omega_{\mathrm{B}20}$ $u_2\big\vert_{z=0} = u_{20}$
$j_{\mathrm{M,A}} = \frac{P_{\mathrm{A}}}{\delta}(\omega_{\mathrm{A}2} - \omega_{\mathrm{A}1})$ $j_{\mathrm{M,B}} = \frac{P_{\mathrm{B}}}{\delta}(\omega_{\mathrm{B}1} - \omega_{\mathrm{B}2})$	

General mass balance in lumen side

$$u_1 \frac{\mathrm{d}\rho_1}{\mathrm{d}z} = u_1 \frac{\mathrm{d}\omega_{\mathrm{A}1}}{\mathrm{d}z}\rho_{\mathrm{A}} + u_1 \frac{\mathrm{d}\omega_{\mathrm{B}1}}{\mathrm{d}z}\rho_{\mathrm{B}}$$

$$u_1 \frac{\mathrm{d}\omega_{\mathrm{A}1}}{\mathrm{d}z}\rho_{\mathrm{A}} = \frac{4}{d_1}\frac{\rho_{\mathrm{A}}}{\rho_1}(\omega_{\mathrm{A}1} j_{\mathrm{M,B}} + \omega_{\mathrm{B}1} j_{\mathrm{M,A}})$$

$$u_1 \frac{\mathrm{d}\omega_{\mathrm{B}1}}{\mathrm{d}z}\rho_{\mathrm{B}} = -\frac{4}{d_1}\frac{\rho_{\mathrm{B}}}{\rho_1}(\omega_{\mathrm{A}1} j_{\mathrm{M,B}} + \omega_{\mathrm{B}1} j_{\mathrm{M,A}})$$

$$u_1 \frac{\mathrm{d}\rho_1}{\mathrm{d}z} = \frac{4}{d_1}\frac{\rho_{\mathrm{A}}}{\rho_1}(\omega_{\mathrm{A}1} j_{\mathrm{M,B}} + \omega_{\mathrm{B}1} j_{\mathrm{M,A}}) - \frac{4}{d_1}\frac{\rho_{\mathrm{B}}}{\rho_1}(\omega_{\mathrm{A}1} j_{\mathrm{M,B}} + \omega_{\mathrm{B}1} j_{\mathrm{M,A}})$$

$$u_1 \frac{\mathrm{d}\rho_1}{\mathrm{d}z} = \frac{4}{d_1}\frac{\rho_{\mathrm{A}} - \rho_{\mathrm{B}}}{\rho_1}(\omega_{\mathrm{A}1} j_{\mathrm{M,B}} + \omega_{\mathrm{B}1} j_{\mathrm{M,A}})$$

$$\rho_1 \frac{\mathrm{d}u_1}{\mathrm{d}z} + \frac{4}{d_1}\frac{\rho_{\mathrm{A}} - \rho_{\mathrm{B}}}{\rho_1}(\omega_{\mathrm{A}1} j_{\mathrm{M,B}} + \omega_{\mathrm{B}1} j_{\mathrm{M,A}}) = \frac{4}{d_1}(j_{\mathrm{M,A}} - j_{\mathrm{M,B}})$$

$$\rho_1 \frac{\mathrm{d}u_1}{\mathrm{d}z} + \frac{4}{d_1}\frac{\rho_{\mathrm{A}}\omega_{\mathrm{A}1} j_{\mathrm{M,B}} - \rho_{\mathrm{B}}\omega_{\mathrm{A}1} j_{\mathrm{M,B}} + \rho_{\mathrm{A}}\omega_{\mathrm{B}1} j_{\mathrm{M,A}} - \rho_{\mathrm{B}}\omega_{\mathrm{B}1} j_{\mathrm{M,A}}}{\rho_1} = \frac{4}{d_1}\frac{\rho_{\mathrm{A}}\omega_{\mathrm{A}1} + \rho_{\mathrm{B}}\omega_{\mathrm{B}1}}{\rho_1}(j_{\mathrm{M,A}} - j_{\mathrm{M,B}})$$

$$\rho_1 \frac{\mathrm{d}u_1}{\mathrm{d}z} + \frac{4}{d_1}\frac{(\rho_{\mathrm{A}}\omega_{\mathrm{B}1} - \rho_{\mathrm{B}}\omega_{\mathrm{B}1}) j_{\mathrm{M,A}} + (\rho_{\mathrm{A}}\omega_{\mathrm{A}1} - \rho_{\mathrm{B}}\omega_{\mathrm{A}1}) j_{\mathrm{M,B}}}{\rho_1} = \frac{4}{d_1}\frac{(\rho_{\mathrm{A}}\omega_{\mathrm{A}1} + \rho_{\mathrm{B}}\omega_{\mathrm{B}1}) j_{\mathrm{M,A}} - (\rho_{\mathrm{A}}\omega_{\mathrm{A}1} + \rho_{\mathrm{B}}\omega_{\mathrm{B}1}) j_{\mathrm{M,B}}}{\rho_1}$$

$$\rho_1 \frac{\mathrm{d}u_1}{\mathrm{d}z} = \frac{4}{d_1}\frac{[\rho_{\mathrm{A}}(\omega_{\mathrm{A}1} - \omega_{\mathrm{B}1}) + 2\rho_{\mathrm{B}}\omega_{\mathrm{B}1}] j_{\mathrm{M,A}} - [\rho_{\mathrm{B}}(\omega_{\mathrm{B}1} - \omega_{\mathrm{A}1}) + 2\rho_{\mathrm{A}}\omega_{\mathrm{A}1}] j_{\mathrm{M,B}}}{\rho_1}$$

$$\frac{\mathrm{d}u_1}{\mathrm{d}z} = \frac{4}{d_1}\frac{[\rho_{\mathrm{A}}(\omega_{\mathrm{A}1} - \omega_{\mathrm{B}1}) + 2\rho_{\mathrm{B}}\omega_{\mathrm{B}1}] j_{\mathrm{M,A}} - [2\rho_{\mathrm{A}}\omega_{\mathrm{A}1} - \rho_{\mathrm{B}}(\omega_{\mathrm{A}1} - \omega_{\mathrm{B}1})] j_{\mathrm{M,B}}}{\rho_1^2}$$

General mass balance in shell side

$$\frac{\mathrm{d}\rho_2}{\mathrm{d}z} = \frac{\mathrm{d}\omega_{\mathrm{A}2}}{\mathrm{d}z}\rho_{\mathrm{A}} + \frac{\mathrm{d}\omega_{\mathrm{B}2}}{\mathrm{d}z}\rho_{\mathrm{B}}$$

$$\frac{\mathrm{d}\rho_2}{\mathrm{d}z} = \frac{4d_1}{d_2^2 - d_1^2}\frac{[-\omega_{\mathrm{B}2} j_{\mathrm{M,A}} - \omega_{\mathrm{A}2} j_{\mathrm{M,B}}]\,\rho_{\mathrm{A}} + [\omega_{\mathrm{B}2} j_{\mathrm{M,A}} + \omega_{\mathrm{A}2} j_{\mathrm{M,B}}]\,\rho_{\mathrm{B}}}{\rho_2 u_2}$$

$$\rho_2 \frac{\mathrm{d}u_2}{\mathrm{d}z} + \frac{4d_1}{d_2^2 - d_1^2}\frac{-\omega_{\mathrm{B}2}(\rho_{\mathrm{A}} - \rho_{\mathrm{B}}) j_{\mathrm{M,A}} - \omega_{\mathrm{A}2}(\rho_{\mathrm{A}} - \rho_{\mathrm{B}}) j_{\mathrm{M,B}}}{\rho_2} = \frac{4d_1}{d_2^2 - d_1^2}(j_{\mathrm{M,B}} - j_{\mathrm{M,A}})$$

$$\frac{\mathrm{d}u_2}{\mathrm{d}z} = -\frac{4d_1}{d_2^2 - d_1^2}\frac{[(\omega_{\mathrm{A}2} - \omega_{\mathrm{B}2})\rho_{\mathrm{A}} + 2\omega_{\mathrm{B}2}\rho_{\mathrm{B}}]\, j_{\mathrm{M,A}} - [2\omega_{\mathrm{A}2}\rho_{\mathrm{A}} - (\omega_{\mathrm{A}2} - \omega_{\mathrm{B}2})\rho_{\mathrm{B}}]\, j_{\mathrm{M,B}}}{\rho_2^2}$$

Rearranged governing equations in explicit ODEs

$$\frac{\mathrm{d}\omega_{\mathrm{A}1}}{\mathrm{d}z} = \frac{4}{d_1}\frac{\omega_{\mathrm{A}1} j_{\mathrm{M,B}} + (1-\omega_{\mathrm{A}1}) j_{\mathrm{M,A}}}{\rho_1 u_1}$$

$$\frac{\mathrm{d}\omega_{\mathrm{A}2}}{\mathrm{d}z} = \frac{4d_1}{d_2^2 - d_1^2}\frac{(\omega_{\mathrm{A}2}-1) j_{\mathrm{M,A}} - \omega_{\mathrm{A}2} j_{\mathrm{M,B}}}{\rho_2 u_2}$$

$$\frac{\mathrm{d}\omega_{\mathrm{B}1}}{\mathrm{d}z} = \frac{4}{d_1}\frac{(\omega_{\mathrm{B}1}-1) j_{\mathrm{M,B}} - \omega_{\mathrm{B}1} j_{\mathrm{M,A}}}{\rho_1 u_1}$$

$$\frac{\mathrm{d}\omega_{\mathrm{B}2}}{\mathrm{d}z} = \frac{4d_1}{d_2^2 - d_1^2}\frac{\omega_{\mathrm{B}2} j_{\mathrm{M,A}} + (1-\omega_{\mathrm{B}2}) j_{\mathrm{M,B}}}{\rho_2 u_2}$$

$$\frac{\mathrm{d}u_1}{\mathrm{d}z} = \frac{4}{d_1}\frac{[\rho_{\mathrm{A}}(\omega_{\mathrm{A}1}-\omega_{\mathrm{B}1}) + 2\rho_{\mathrm{B}}\omega_{\mathrm{B}1}] j_{\mathrm{M,A}} - [2\rho_{\mathrm{A}}\omega_{\mathrm{A}1} - \rho_{\mathrm{B}}(\omega_{\mathrm{A}1}-\omega_{\mathrm{B}1})] j_{\mathrm{M,B}}}{\rho_1^2}$$

$$\frac{\mathrm{d}u_2}{\mathrm{d}z} = -\frac{4d_1}{d_2^2 - d_1^2}\frac{[(\omega_{\mathrm{A}2}-\omega_{\mathrm{B}2})\rho_{\mathrm{A}} + 2\omega_{\mathrm{B}2}\rho_{\mathrm{B}}]\, j_{\mathrm{M,A}} - [2\omega_{\mathrm{A}2}\rho_{\mathrm{A}} - (\omega_{\mathrm{A}2}-\omega_{\mathrm{B}2})\rho_{\mathrm{B}}]\, j_{\mathrm{M,B}}}{\rho_2^2}$$

ICs

$$\omega_{\mathrm{A}1}|_{z=0} = \omega_{\mathrm{A}10}$$
$$\omega_{\mathrm{A}2}|_{z=0} = \omega_{\mathrm{A}20}$$
$$\omega_{\mathrm{B}1}|_{z=0} = \omega_{\mathrm{B}10}$$
$$\omega_{\mathrm{B}2}|_{z=0} = \omega_{\mathrm{B}20}$$
$$u_1|_{z=0} = u_{10}$$
$$u_2|_{z=0} = u_{20}$$