Part 1: Assignment 1

Character count is around 12.000 without tables

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1 Introduction

Firms are frequently assumed to achieve proportional increases in output by proportionally increasing inputs. This specific relationship between inputs and output is known as constant returns to scale. This paper empirically investigates the Cobb-Douglas production function, analyzing the influence of capital and labor in French manufacturing firms. We examine the input mix of manufacturing firms by employing fixed-effects and first-difference estimations. However, we are unable to develop a consistent estimator that allows us to formally assess the hypothesis of constant returns to scale. As if the estimator is consistent, we evaluate the hypothesis using first-difference estimation and reject it.

2 Econometric Theory

We formulate the relationship in the log-transformed Cobb-Douglass production function for firm i at time t as

$$y_{it} = \beta_K k_{it} + \beta_L l_{it} + c_i + u_{it}. \tag{1}$$

Here y_{it} denotes log of adjusted sales, k_{it} is log of adjusted capital stock and l_{it} is the log of employment. Hence, $\beta_K + \beta_L$ are input elasticities.

We employ a panel dataset of $N = 441$ French manufacturing firms observed at the odd years from 1968 to 1979. Given our panel data structure, we allow some time-invariant c_i to affect output that may vary across firms. As introduced, the objective of this analysis is to investigate the return to scale for capital and labor. In order to do so, we test whether $\beta_K + \beta_L = 1$, which would constitute constant return to scale.

In the following section, we introduce the estimators used for β_K and β_L. We present the required assumptions for the estimator(s) to be consistent and the implications for our estimator(s). Later, we formulate how these imposed assumptions can be formally tested. Finally, we present how we test whether there is evidence in the data for constant return to scale.

2.1 Estimation method

We seek to derive estimates of (β_K, β_L) by applying FE and FD estimation for equation (1). In the empirical analysis we outline our reasoning for the use of these estimators rather than POLS or RE estimation. The FE-estimator is given by,

$$\hat{\boldsymbol{\beta}}_{FE} = (\hat{\beta}_K, \hat{\beta}_L)'_{FE} = \boldsymbol{(\ddot{X}'\ddot{X})^{-1}\ddot{X}'\ddot{y}} \tag{2}$$

where $\ddot{\boldsymbol{X}}$ $(NT \times 2)$ are the time-demeaned regressors (l_{it}, k_{it}) stacked as columns, and $\ddot{\boldsymbol{y}}$ with dimensions $NT \times 1$ is the time-demeaned dependent variable y_{it} stacked over t and i. The FD-estimator is

$$\hat{\boldsymbol{\beta}}_{FD} = (\hat{\beta}_K, \hat{\beta}_L)'_{FD} = \boldsymbol{(\Delta X'\Delta X)^{-1}\Delta X'\Delta y} \tag{3}$$

where $(\boldsymbol{\Delta X, \Delta y})$ denotes (l_{it}, k_{it}) first-differenced and stacked as columns over t and i and y_{it} first-differenced and stacked over t and then i. Notice the first-differenced regressors imply that the dimensions of $\boldsymbol{\Delta X}$ is now $N(T-1) \times 2$ since we loose an observation when taking the first difference.

2.2 Consistency and asymptotic normality

For consistent estimators $\hat{\beta}_{FD}, \hat{\beta}_{FE} \xrightarrow{p} (\beta_K, \beta_L)$, we require that FE.1-2 and FD.1-2 hold. FE.1 imposes that $E[u_{it}|(k_{i1}, l_{i1}), (k_{i2}, l_{i2}), ..., (k_{iT}, l_{iT}), c_i] = 0, \forall t = 0, 1, .., T$, while FD.1 similarly requires $E[u_{it}|(k_{it}, l_{it}), c_i], \forall t = 0, 1, ..., T$. Hence, FE.1 and FD.1 are similar in nature and require exogeneity of the regressors, which we test formally. Although FE.1 is more strict than FD.1, we expect that FD.1 is also violated if FE.1 is rejected.

FE.2 requires that $rank(E[\boldsymbol{\ddot{X}_i'\ddot{X}_i}]) = 2$ while similarly for FD.2, $rank(E[\boldsymbol{\Delta X_i'\Delta X_i}]) = 2$, where $\boldsymbol{\ddot{X}_i}$ is $(T \times 2)$ and $\boldsymbol{\Delta X_i}$ is $((T-1) \times 2)$. These are merely necessary to identify $\hat{\beta}_{FE}$ and $\hat{\beta}_{FE}$ from equations (2) and (3). The following section will outline how we test the validity of these assumptions. Under FD.1 and FD.2 we have that

$$\sqrt{N}(\hat{\boldsymbol{\beta}}_{FD} - \boldsymbol{\beta}) \xrightarrow{d} \mathcal{N}\left(0, \boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{A}^{-1}\right),$$

where $\boldsymbol{A} = E\left(\boldsymbol{\Delta X}_i'\boldsymbol{\Delta X}_i\right)$, $\boldsymbol{B} = E\left(\boldsymbol{\Delta X}_i'\boldsymbol{e}_i\boldsymbol{e}_i'\boldsymbol{\Delta X}_i\right)$ and $\boldsymbol{e}_i = \boldsymbol{\Delta u_i}$ is the $((T-1) \times 1)$ error vector in FD-estimation. Similar asymptotic property holds for the FE-estimator under FE.1 and FE.2.

Finally, we have FE.3 which states that u_{it} are serially uncorrelated and have constant variance. This is similar for first difference, but here it its assumed for the first differenced errors, e_{it} where $e_{it} \equiv \Delta u_{it}$. These assumptions are important in terms of efficiency. The estimator for the robust variance matrix for the first difference regression model is:

$$\widehat{Avar(\hat{\boldsymbol{\beta}}_{\boldsymbol{FD}})} = (\boldsymbol{\Delta X' \Delta X})^{-1} \left(\sum_{i=1}^{N} \boldsymbol{\Delta X_i}' \hat{\boldsymbol{e}}_{\boldsymbol{i}} \hat{\boldsymbol{e}}_{\boldsymbol{i}}' \boldsymbol{\Delta X_i} \right) (\boldsymbol{\Delta X' \Delta X})^{-1} \tag{4}$$

where $\hat{\boldsymbol{e}}_{\boldsymbol{i}}$ is a $(T-1) \times 1$ vector with residuals of the regression model as defined above. Notice, it is equivalent for the FE regression model but with corresponding regressors and residuals.

2.3 Empirical tests

We test for strict exogeneity by conducting FE estimation on the two models, respectively

$$y_{it} = \beta_K k_{it} + \beta_L l_{it} + \delta_K k_{it+1} + c_i + u'_{it}$$
$$y_{it} = \beta_K k_{it} + \beta_L l_{it} + \delta_L l_{it+1} + c_i + u''_{it}$$

The null hypothesis of the test is that $\delta_K = 0$ or $\delta_L = 0$ against the two-sided alternative by a usual t-test. The test statictic is asymptotically standard normal under the null. If we reject the null hypothesis, we cannot assume strict exogeneity, which invalidates the FD- and FE-estimator.

We test for homoskedasticity by examining any potential autocorrelation in the error term. Specifically, we run the residual regression

$$\hat{u}^j_{it} = \rho^j \hat{u}^j_{it-1} + \epsilon^j_{it} \tag{5}$$

where $\hat{u}^j = \hat{u}$ for FE- and $\hat{u}^j = \hat{e}$ for FD-estimation. We compute standard t-tests with robust standard errors of the null hypothesis of $\rho^j = 0$, with the alternative that $\rho^j \neq 0$. The test statistics are asymptotically normal under the null.

Finally, we are interested in testing linear hypotheses. Formally, we test a null hypothesis of the form $\boldsymbol{R\beta} = \boldsymbol{r}$ where $\boldsymbol{R}$ is a $Q \times K$ matrix with Q being the number of restrictions and K number of regressors (i.e. 2 in our case) and $\boldsymbol{r}$ being a $Q \times 1$ matrix. The alternative hypothesis is $\boldsymbol{R\beta} \neq \boldsymbol{r}$. The Wald statistics for testing the above hypothesis is:

$$W = (\boldsymbol{R}\hat{\boldsymbol{\beta}} - \boldsymbol{r})' \left[\boldsymbol{R} \widehat{Avar(\hat{\boldsymbol{\beta}})} \boldsymbol{R}' \right]^{-1} (\boldsymbol{R}\hat{\boldsymbol{\beta}} - \boldsymbol{r})$$

where $Avar(\hat{\boldsymbol{\beta}})$ is the robust asymptotic variance of the estimated coefficients. The Wald statistics is asymptotically χ^2_Q-distributed with Q degrees of freedom under the null hypothesis. The following section will outline our empirical findings.

3 Empirical Findings

In terms of our choice of estimator we argue that both the POLS and RE estimation will be misleading in this framework. Specifically, if we believe that we have any time-constant effect within firms, i.e. $c_i \neq 0$ in eq. (1), the POLS will suffer from omitted variable bias. This could for instance be patents, if we believe that they are constant over time and affects firms output, which would lead to inconsistent estimates of our parameters. Furthermore, if e.g. firms with patents has higher levels of capital or labor, the RE assumption is violated, i.e. $E(c_i|k,l) \neq 0$, which would also lead to inconsistent estimates of our parameters.

Thus, we choose to estimate eq. (1) using FE and FD, as they allow for time-invariant productivity factors that may be influenced by the levels of capital and labor. We transform our data as stated in section 2.1, and estimate the coefficients by employing equation (2) and (3). The estimated coefficients are presented in Table 1 and 2. Both models are estimated using the estimator for the robust variance matrix from equation (4) as the residuals seem to suffer from autocorrelation in the residuals. The tests are conducted by running the auxiliary regression described by equation (5). For both models, we reject that $\rho^j = 0$ as the t-test statistics are 6.6 and -7.1 for the FE and FD models respectively, see Table 3.

We conduct test of the assumptions for strict exogeneity, as outlined in section 2.2. The test shows that the coefficients δ_K and δ_L are statistically significant on a 5% percent significance level. This is a major drawback for both models, as we reject consistency, and thus the estimates are not of interest. Thus, it is not possible to test for constant returns to scale. This is our primary finding. Rather than ending our assignment here, we continue as if the assumptions are not violated from section 2.2 to demonstrate the approach.

We argue that the FD estimator is preferred to the FE. Under FE.1-2 and FD.1-2, the choice between these estimators hinges on efficiency. More specifically, how we think the idiosyncratic errors, u_{it}, are distributed. In general, the FE estimator is proven more efficient when the u_{it} is serially uncorrelated, while the FD estimator is preferred when the error follows a random walk. Within the Cobb-Douglass model framework, the unobserved technology that is dependent on time, can be represented by u_{it}. Although we test and reject both assumptions, we argue that technology changes are more likely to follow a random walk rather than being serially uncorrelated. It seems implausible that the level of technology is uncorrelated with the previous level. Therefore, we believe that FD gives a better representation of the error term and is therefore our preferred model. Thus, if the estimator was in fact consistent, we would prefer the FD estimator.

The goal of this assignment is to test whether that product exhibit constant returns to scale. Constant return to scale would imply that the coefficients in the Cobb-Douglas would sum to 1. So in order to test the hypothesis of production exhibiting constant returns to scale, we formulate the null-hypothesis that $\beta_L + \beta_K = 1$, which corresponds to $\boldsymbol{R} = (1, 1)$ and $\boldsymbol{r} = (1)$, and the alternative hypothesis that $\beta_L + \beta_K \neq 1$. The test is conducted as explained in section 2.3. We get a Wald test statistics of 37.35 with a p-value of 0.0. This means that we on a 5% significance level can reject that the test statistics is χ^2_1-distributed and therefore that the $\beta_L + \beta_K = 1$. The economic interpretation is that we reject the hypothesis that production exhibits constant return to scale.

4 Conclusion and discussion

In this assignment, we explored the Cobb-Douglas production function using both fixed effects (FE) and first differences (FD) estimators. Both estimators failed the assumptions of strict exogeneity, which makes them inconsistent and thus we are unable to apply valid inference. This issue might stem from a relationship between the time-dependent total factor productivity (TFP) and the labour and capital stock. If smaller firms can more easily adapt new technology, or alternatively larger firms benefit from increasing know-how in production, the imposed relationship between k, l and u is violated. Additionally, if k_{t+1} is increasing in past profits and thereby TFP-shocks $u_{it}, ..., u_{i0}$, the assumption of strict exogeneity does not hold. We find such a dynamic relationship likely. It is however difficult to combat the issue of exogeneity since TFP shocks are unobservable in nature.

Despite this econometric challenge, we test the hypothesis of constant returns to scale. The empirical results reject this hypothesis, suggesting that production does not exhibit constant returns to scale. However, the reliability of our conclusion depends on the appropriateness of the models. Even though we prefer the FD-estimator in terms of efficiency, neither model fully captures complex factors like technological changes, introducing inconsistency.

First Difference

	Coefficient	*Std. err.*	*test statistic*	*P-value*
β_L	0.73	0.03	21.44	0.00
β_K	0.05	0.03	2.04	0.04
R^2	0.47			

Tabel 1: First difference estimation output. Table presents the estimated coefficients, standard error, t-test stastictics and the P-value (i.e. the probability of the estimated coefficient being insignificant

Fixed Effect

	Coefficient	*Std. err.*	*test statistic*	*P-value*
β_L	0.71	0.03	25.04	0.00
β_K	0.14	0.02	6.36	0.00
R^2	0.47			

Tabel 2: Fixed Effect estimation output. Table presents the estimated coefficients, standard error, t-test stastictics and the P-value (i.e. the probability of the estimated coefficient being insignificant

Hypothesis	Test stastistic	P-value	Conclusion
Strict Exogeneity (lemp)	$t = 2.8$	$P = 0.5\%$	Reject Null (Strict Exogeneity does not hold)
Strict Exogeneity (lcap)	$t = 5.3$	$P = 0\%$	Reject Null (Strict Exogeneity does not hold)
No Serial Autocorrelation (FE Model)	$t = 6.7$	$P = 0\%$	Reject Null (Autocorrelation in FE is present)
No Serial Autocorrelation (FD Model)	$t = -7.1$	$P = 0\%$	Reject Null (Autocorrelation in FD is present)
Constant Returns to Scale ($\beta_L + \beta_K = 1$)	$W = 37.35$	$P = 0\%$	Reject Null (No Constant Returns to Scale)

Tabel 3: The table displays test statistics and p-values for strict exogeneity, serial autocorrelation, and constant returns to scale.