A Additional Proofs

Lemma A.1. *(Lemma 2.3) With probability $1 - O\left(\frac{1}{m}\right)$, the threshold k is such that $|\pi_1 \cup \cdots \cup \pi_k| \in \left[\rho n, \rho n + \frac{24 \log m}{\epsilon}\right]$.*

Proof. Let A be the last index such that $f_A \leq \rho n$ and let B be the first index such that $f_B \geq \rho n + \frac{24 \log m}{\epsilon}$. Suppose k doesn't satisfy the requirements in the theorem statement; then either (i) for some $1 \leq i \leq A$, we have $\gamma_i \geq \hat{T}$ or (ii) for some $B \leq j \leq m$, we have $\gamma_j \leq \hat{T}$. We will bound the probability that these events occur. Let $1 \leq i \leq A$; we have the following

$$\Pr[\gamma_i \geq \hat{T}] = \Pr\left[\text{Lap}(2/\epsilon) + \text{Lap}(4/\epsilon) \geq \frac{12 \log m}{\epsilon}\right]$$
$$\leq \Pr\left[\text{Lap}(2/\epsilon) \geq \frac{4 \log m}{\epsilon}\right] + \Pr\left[\text{Lap}(4/\epsilon) \geq \frac{8 \log m}{\epsilon}\right] \leq \frac{2}{m^2}$$

Then by the union bound, the probability that (i) occurs is at most $O(\frac{1}{m})$. The bound for (ii) is similar (and in fact, symmetric) so the lemma follows directly. □

Lemma A.2. *(Lemma 2.7) There exists an $(2\epsilon, \delta)$-differentially private algorithm for the maximum coverage problem such that for some constant C, if we have*

$$k \leq \frac{C\epsilon_0}{\ln^2(n)} \cdot \mathit{OPT},$$

then the algorithm is a 0.15-approximation with probability $1 - O(\frac{1}{n})$, where $\epsilon_0 = \frac{\epsilon}{2\ln(e\ln(n)/\delta \ln(1+\alpha))}$.

Proof. Let $\alpha < 1 - \frac{1}{e}$ be a small constant and take $C = (1 - \frac{1}{e} - \alpha)\ln(1+\alpha)/2$. By algebra, we see that if we k is not too large as in the lemma statement, then Theorem 2.6 implies that there exists an (ϵ', δ')-differentially private algorithm for the maximum coverage problem which is an α-approximation to the optimal solution in expectation, where $\epsilon' = \frac{\ln(1+\alpha)}{\ln n}\epsilon$ and $\delta' = \frac{\ln(1+\alpha)}{\ln n}\delta$. Note that the current approximation guarantee for the algorithm is in expectation, but we will need something slightly stronger.

To convert the approximation guarantee from to a guarantee in expectation to one with high probability, we can simply repeat the algorithm $T = \frac{\ln n}{\ln(1+\alpha)}$ times and choose the solution which cover the most elements (via the exponential mechanism). Note that repeating the algorithm T times is (ϵ, δ)-differentially private by basic composition; since the exponential mechanism is ϵ-differentially private, our entire mechanism is $(2\epsilon, \delta)$-differentially private, as desired.

Next, we analyze the utility of our proposed mechanism. Let $X_1, \ldots, X_T$ be the (random) number of elements covered by the sets chosen by the algorithm in T independent runs. Let $i \in [T]$ be arbitrary; by Markov's inequality, we have

$$\Pr[\text{OPT} - X_i \geq (1+\alpha)\mathbb{E}[\text{OPT} - X_i]] \leq \frac{1}{1+\alpha}.$$

Since $\mathbb{E}[X_i] \geq \alpha \cdot \text{OPT}$, we have $\mathbb{E}[\text{OPT} - X_i] \leq (1-\alpha)\text{OPT}$ so we can claim

$$\Pr[\text{OPT} - X_i \geq (1+\alpha)(1-\alpha)\text{OPT}] = \Pr[\text{OPT} - X_i \geq (1-\alpha^2)\text{OPT}] \leq \frac{1}{1+\alpha}.$$

Moving terms around, we can rewrite the above as

$$\Pr[X_i \leq \alpha^2 \cdot \text{OPT}] \leq \frac{1}{1+\alpha}. \tag{1}$$

Using this, we can conclude

$$\Pr[\max_{i \in [T]} X_i > \alpha^2 \cdot \text{OPT}] = 1 - \Pr[\max_{i \in [T]} X_i \leq \alpha^2 \cdot \text{OPT}] = 1 - \prod_{i=1}^{T} \Pr[X_i \leq \alpha^2 \cdot \text{OPT}] \geq 1 - \frac{1}{n},$$

where the final inequality follows by (1). Finally, we need to apply the exponential mechanism on these T families of sets to guarantee privacy. Let X be the number of elements covered by the chosen set; by the utility guarantees of the exponential mechanism, we have

$$\Pr\left[X \leq \max_{i \in [T]} X_i - \frac{4 \ln n}{\epsilon}\right] \leq \frac{1}{n} \tag{2}$$

Note that by our assumption on k, we have $\text{OPT} \geq \frac{k \ln^2(n)}{C\epsilon_0}$. For even moderately large n, this implies $0.1 \cdot \text{OPT} \geq \frac{4 \ln n}{\epsilon}$, so we have

$$\Pr\left[X \geq (\alpha^2 - 0.1)\text{OPT}\right] \geq 1 - O\left(\frac{1}{n}\right). \tag{3}$$

Taking $\alpha = 0.5$ suffices to give us a 0.15-approximation algorithm with high probability. □

Lemma A.3. *(Lemma 3.3) Under* (ϵ, δ)*-differential privacy, Partial Set Cover cannot be solved exactly (with high probability).*

Proof. Let's consider a Partial Vertex Cover instance. Let S_n denote the star graph on n vertices (i.e., there are $n-1$ nodes all connected to a single center node). Our graph G will consist of two star graphs $S_{n/2}$ (centered at nodes u and v) connected together with a single edge (u, v). Let the covering requirement be $\rho = \frac{1}{2}$. Clearly, the optimal Partial Vertex Cover has size 1 by choosing u or v; assume for contradiction either u or v is output with probability $1 - o(1)$. Now consider any graph G' with 3 additional edges in G. Now the covering constraint ρ requires our solution to cover at least $\frac{n}{2} + 2$ edges, so only choosing u or v is insufficient. But by group privacy, the algorithm will still choose either u or v with probability at least $e^{-3}(1 + o(1))$, a contradiction. $\square$