Lecture 17: EM and HMM
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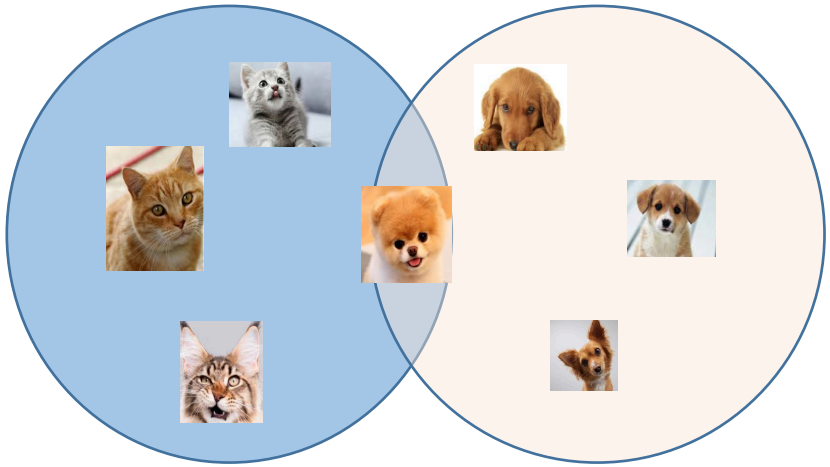
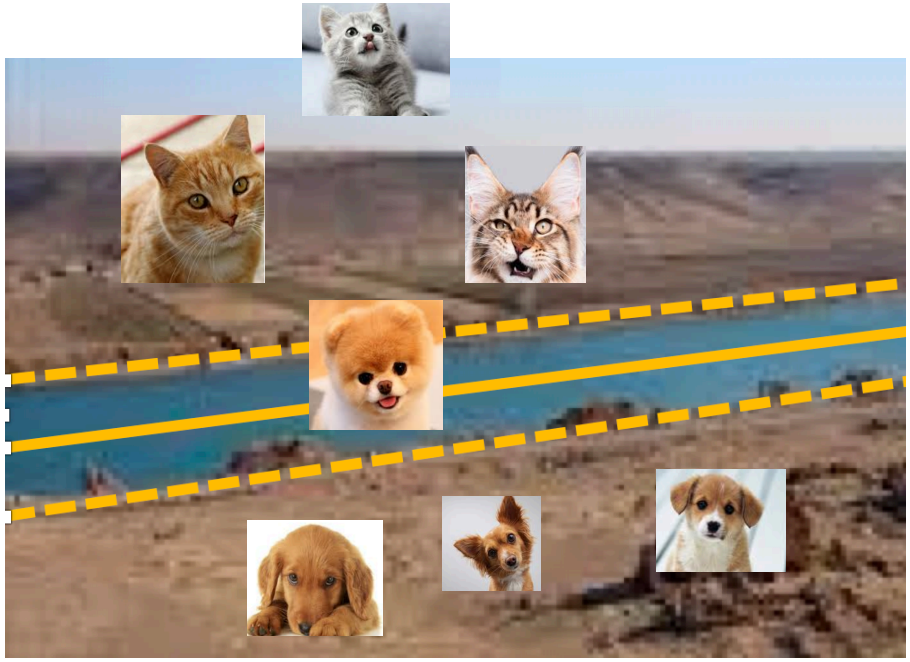
Generative Model v.s. Discriminative model

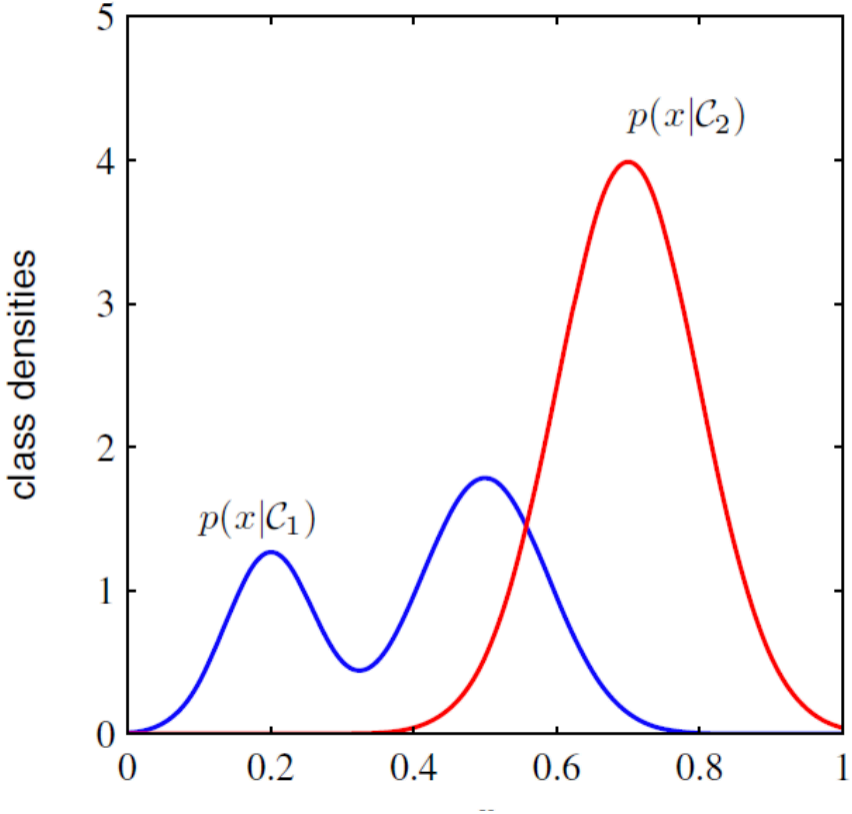




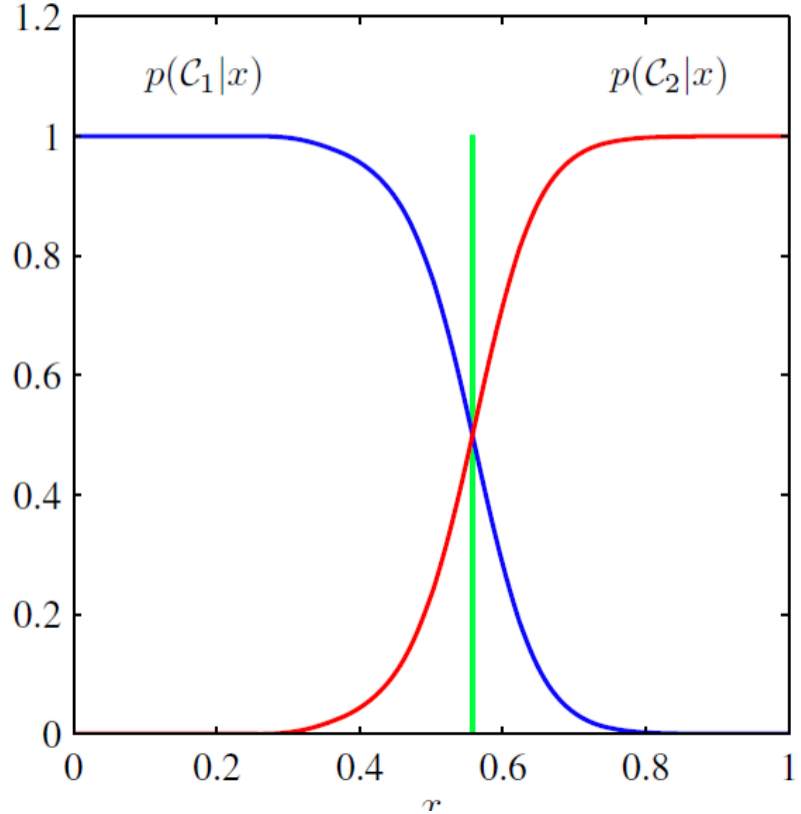
Learn $P(X, Y \mid \Theta)$ or
Learn $P(X \mid Y, \Theta)$ and $P(Y \mid \Theta)$

Learn $P(Y \mid X, \Theta)$

Generative Model's view

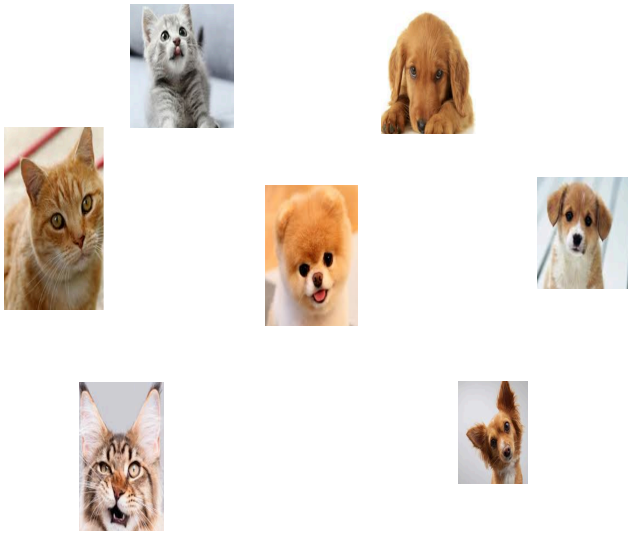


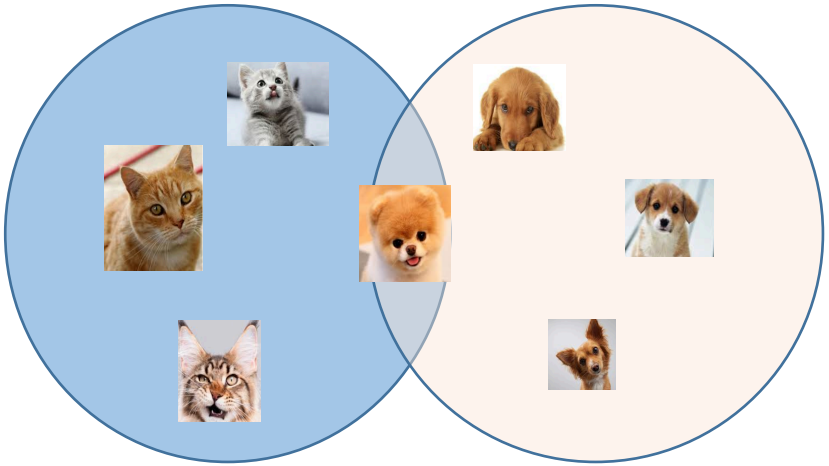
Discriminative Model's view



How about unsupervised learning

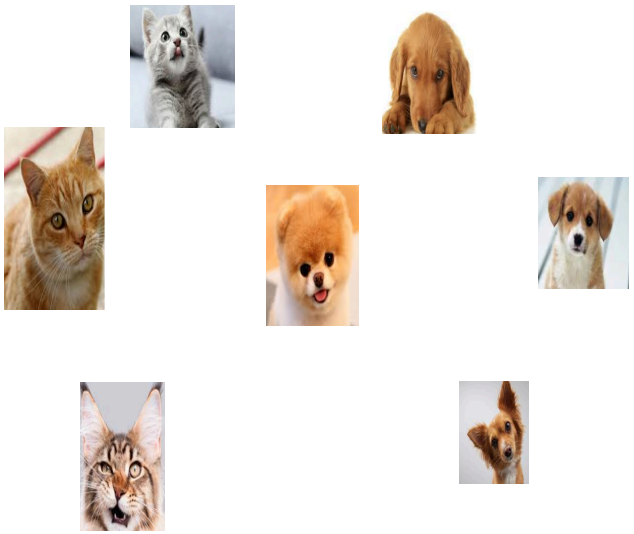
- In unsupervised learning, we only observed input distribution $\tilde{P}(X)$

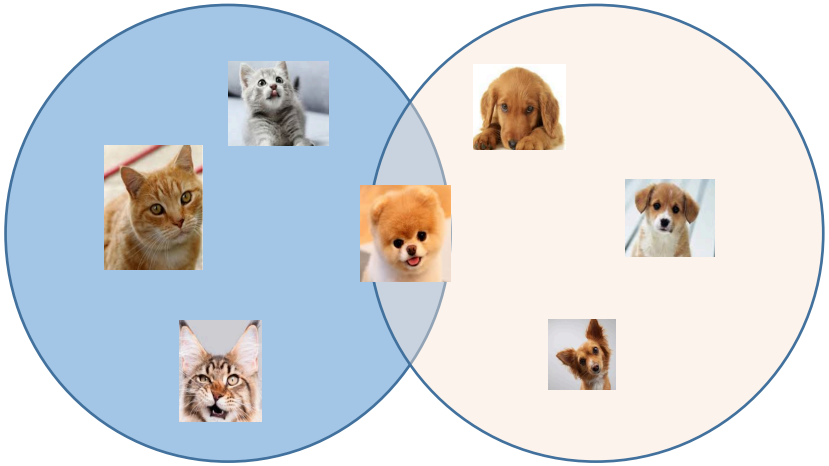




MLE in unsupervised learning

- We only have observation of $\tilde{P}(X)$
- In generative model, we have $P(X, Y \mid \Theta)$
- In discriminative model, we have $P(Y \mid \Theta, X)$
- Which model is more suitable for unsupervised learning?





MLE in unsupervised learning

- We only have observation of $\tilde{P}(X)$
- In generative model, we have $P(X, Y \mid \Theta)$
- We know $P(X \mid \Theta) = \sum_Y P(X, Y \mid \Theta)$
- Therefore, MLE is
  $\text{argmax}_\Theta P(X \mid \Theta) =$
  $\text{argmax}_\Theta \sum_Y P(X, Y \mid \Theta)$

EM algorithm

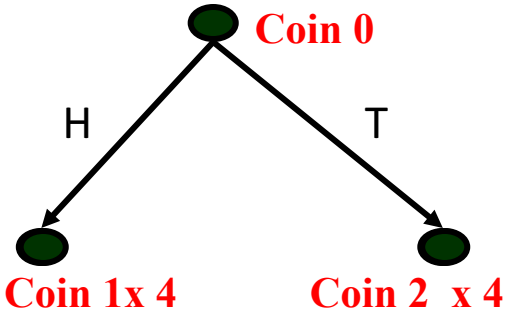
- EM algorithm solves $\text{argmax}_\Theta \sum_Y P(X, Y \mid \Theta)$ by iteratively updating Θ
- In general, known to converge to a local maximum of the maximum likelihood function

Three Coins Example

- We observe a series of coin tosses generated in the following way:
- A person has three coins.
  - Coin 0: probability of Head is α
  - Coin 1: probability of Head p
  - Coin 2: probability of Head q

- Consider the following coin-tossing scenarios:

Scenario I



- Toss coin 0.
  If Head – toss coin 1; o/w – toss coin 2

Observing the sequence
HHHHT, THTHT, HHHHT, HHTTH

produced by Coin 0 , Coin1 and Coin2

Question: Estimate most likely values for p, q (the probability of H in each coin) and the probability to use each of the coins (a)

Scenario I

Supervised Learning

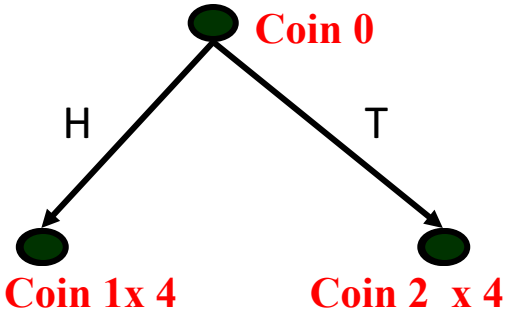
- Toss coin 0.
  If Head – toss coin 1; o/w – toss coin 2

  Observing the sequence
  HHHHT, THTHT, HHHHT, HHTTH

  produced by Coin 0 , Coin1 and Coin2

  Question: Estimate most likely values for p, q (the probability of H in each coin) and the probability to use each of the coins (a)

Scenario II



- Toss coin 0.
  If Head – toss coin 1; o/w – toss coin 2

Observing the sequence
HHHT, HTHT, HHHT, HTTH

produced by ~~Coin 0~~, Coin1 and/or Coin2

Question: Estimate most likely values for p, q (the probability of H in each coin) and the probability to use each of the coins (a)

Intuition of EM algorithm

- Use an iterative approach for estimating the parameters:
  - Guess the probability that a given data point came from Coin 1 or 2; Generate fictional labels, weighted according to this probability.
  - Now, compute the most likely value of the parameters. [recall the scenario I]
  - Compute the likelihood of the data given this model.
  - Re-estimate the initial parameter setting: set them to maximize the likelihood of the data.

Step 1: initialization

Coin 0: probability of Head is α
Coin 1: probability of Head p
Coin 2: probability of Head q

- Guess the probability that a given data point came from Coin 1 or 2; Generate fictional labels, weighted according to this probability.

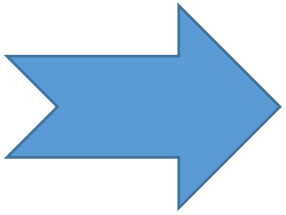
	coin 1=H	coin 1=T
HHHT	100%	0 %
HTHT	100%	0%
HHHT	100%	0%
HTTH	0%	100%

Step 2: Maximum Conditional Likelihood

Coin 0: probability of Head is α
Coin 1: probability of Head p
Coin 2: probability of Head q

- Now, compute the most likely value of the parameters. [recall the scenario I]

	coin 1=H	coin 1=T
HHHT	100%	0 %
HTHT	100%	0%
HHHT	100%	0%
HTTH	0%	100%



HHHHT
HHTHT
HHHHT
THTTH

Step 2: Maximum Conditional Likelihood

Coin 0: probability of Head is α
Coin 1: probability of Head p
Coin 2: probability of Head q

- Now, compute the most likely value of the parameters. [recall the scenario I]

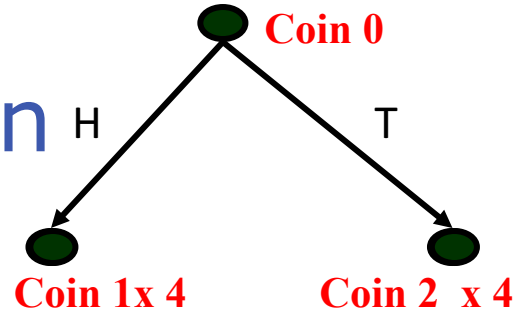
HHHHT
HHTHT
HHHHT
THTTH

$$\alpha_1 = \frac{3}{3+1} = \frac{3}{4}$$

$$p_1 = \frac{8}{8+4} = \frac{2}{3}$$

$$q_1 = \frac{2}{2+2} = \frac{1}{2}$$

Step3: Likelihood Estimation



- Compute the likelihood of the data given this model

$\alpha_1 p_1^{\#H}(1-p_1)^{\#T}$ $\quad$ $(1-\alpha_1)q_1^{\#H}(1-q_1)^{\#T}$

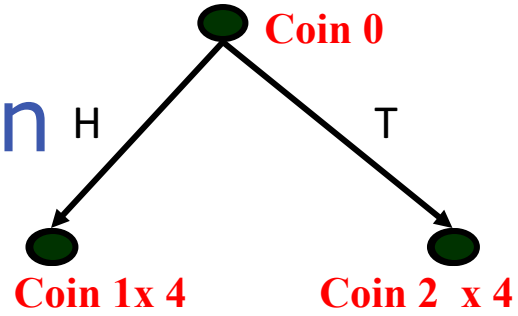
	coin 1=H	coin 1=T
HHHT	$\frac{3}{4}\left(\frac{2}{3}\right)^3\frac{1}{3}$	$\frac{1}{4}\left(\frac{1}{2}\right)^3\frac{1}{2}$
HTHT	$\frac{3}{4}\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^2$	$\frac{1}{4}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2$
HHHT	$\frac{3}{4}\left(\frac{2}{3}\right)^3\frac{1}{3}$	$\frac{1}{4}\left(\frac{1}{2}\right)^3\frac{1}{2}$
HTTH	$\frac{3}{4}\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^2$	$\frac{1}{4}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2$

$$\alpha_1 = \frac{3}{4}$$

$$p_1 = \frac{2}{3}$$

$$q_1 = \frac{1}{2}$$

Step3: Likelihood Estimation



- Compute the likelihood of the data given this model

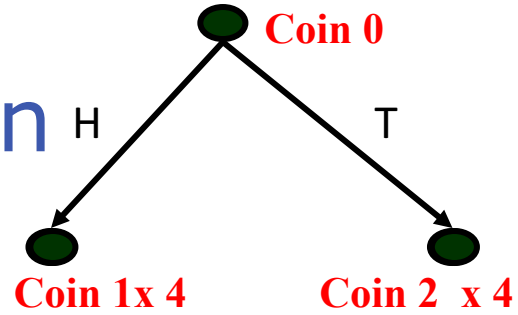
	$\alpha_1 p_1^{\#H}(1-p_1)^{\#T}$ coin 1=H	$(1-\alpha_1)q_1^{\#H}(1-q_1)^{\#T}$ coin 1=T
HHHT	0.074	0.0156
HTHT	0.037	0.0156
HHHT	0.074	0.0156
HTTH	0.037	0.0156

$$\alpha_1 = \frac{3}{4}$$

$$p_1 = \frac{2}{3}$$

$$q_1 = \frac{1}{2}$$

Step3: Likelihood Estimation



- Compute the likelihood of the data given this model

$\alpha_1 p_1^{\#H}(1-p_1)^{\#T}$ $(1-\alpha_1) q_1^{\#H}(1-q_1)^{\#T}$

	coin 1=H	coin 1=T
HHHT	82.6%	17.4%
HTHT		29.7%
HHHT	82.6%	17.4%
HTTH	*70.3%*	29.7%

$$\alpha_1 = \frac{3}{4}$$

$$p_1 = \frac{2}{3}$$

$$q_1 = \frac{1}{2}$$

e.g., $\frac{0.074}{0.074+0.0156} = 82.6\%$

Step 2: Maximum Conditional Likelihood

Coin 0: probability of Head is α
Coin 1: probability of Head p
Coin 2: probability of Head q

- Now, compute the most likely value of the parameters. [recall the scenario I]

	coin 1=H	coin 1=T
HHHT	82.6%	17.4%
HTHT		29.7%
HHHT	82.6%	17.4%
HTTH	*70.3%*	29.7%

HHHHT 82.6%
THHHT 17.4%
HHTHT 70.3%
THTHT 29.7%
HHHHT 82.6%
THHHT 17.4%
HHTTH 70.3%
THTTH 29.7%

Step 2: Maximum Conditional Likelihood

Coin 0: probability of Head is α
Coin 1: probability of Head p
Coin 2: probability of Head q

- Now, compute the most likely value of the parameters. [recall the scenario I]

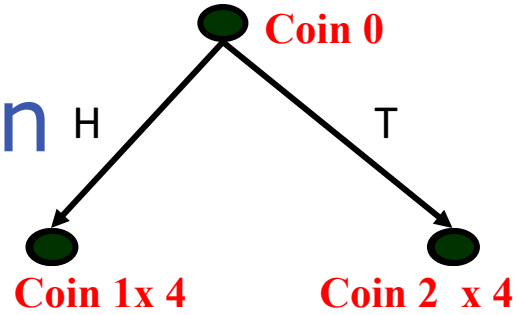
HHHHT 82.6%
HHTHT 70.3%
HHHHT 82.6%
HHTTH 70.3%
THHHT 17.4%
THTHT 29.7%
THHHT 17.4%
THTTH 29.7%

$$\alpha_2 = \frac{82.6\times 2 + 70.3\times 2}{400} = 76.5\%$$

$$p_2 = \frac{82.6\times 6 + 70.3\times 4}{82.6\times 8 + 70.3\times 8} = 63.5\%$$

$$q_2 = \frac{17.4\times 6 + 29.7\times 4}{17.4\times 8 + 29.7\times 8} = 59.2\%$$

Step3: Likelihood Estimation



- Compute the likelihood of the data given this model

$\alpha_2 p_2^{\#H}(1-p_2)^{\#T}$ coin 1=H

$(1-\alpha_2)q_2^{\#H}(1-q_2)^{\#T}$ coin 1=T

HHHT
HTHT
HHHT
HTTH

$$\alpha_2 = 76.5\%$$

$$p_2 = 63.5\%$$

$$q_2 = 59.2\%$$

Intuition of EM algorithm

- Use an iterative approach for estimating the parameters:
  - Guess the probability that a given data point came from Coin 1 or 2; Generate fictional labels, weighted according to this probability.
  - Now, compute the most likely value of the parameters. [recall the scenario I]
  - Compute the likelihood of the data given this model.
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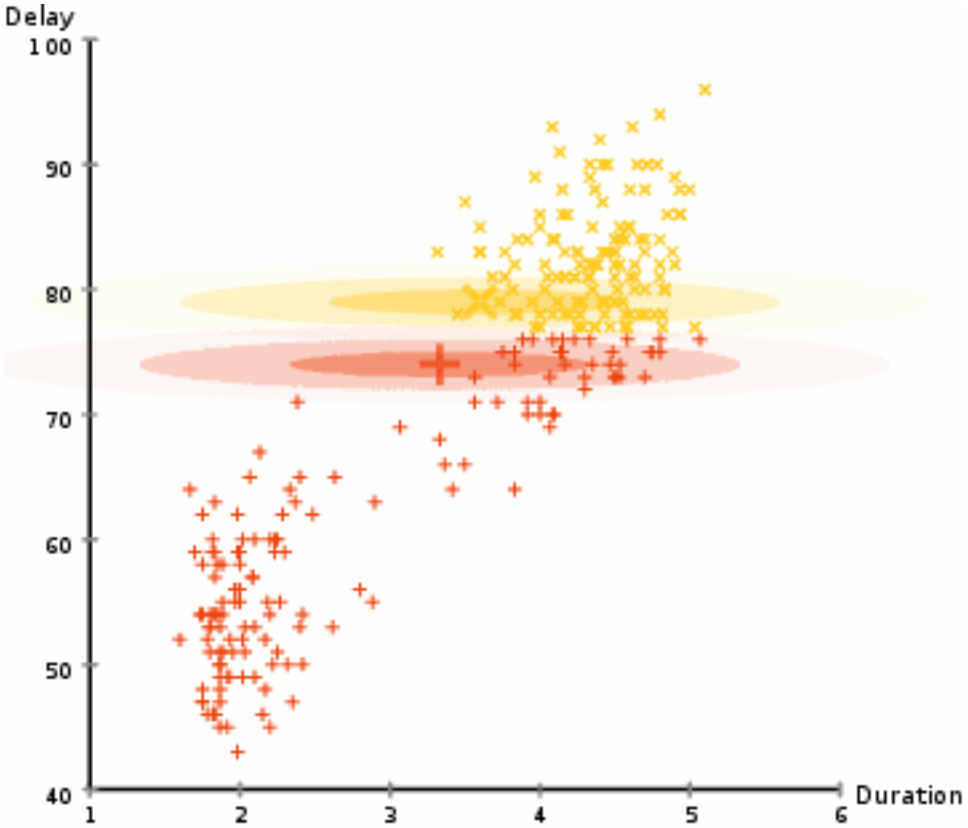
Real world Example

GMM clustering of Old Faithful eruption data



GMM as the marginal distribution P(x) of a joint distribution P(x, z)

- Remember, in GMM, we model the marginal probability as

$$p(\boldsymbol{x}) = \sum_{k=1}^{K} \omega_k N(\boldsymbol{x}|\boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k)$$

$$\omega_k = p(z = k)$$

Iterative procedure

- LetType equation here. θ represent all parameters $\{\omega_k, \mu_k, \Sigma_k\}$

Step 0: initialize $\boldsymbol{\theta}$ with some values (random or otherwise)

Step 1: compute γ_{nk} using the current $\boldsymbol{\theta}$

Step 2: update $\boldsymbol{\theta}$ using the just computed γ_{nk}

Step 3: go back to Step 1

γ_{nk}: given a data point x_n how likely it belongs to k^{th} cluster

$$\omega_k = \frac{\sum_n \gamma_{nk}}{\sum_k \sum_n \gamma_{nk}}, \quad \boldsymbol{\mu}_k = \frac{1}{\sum_n \gamma_{nk}} \sum_n \gamma_{nk} \boldsymbol{x}_n$$

$$\boldsymbol{\Sigma}_k = \frac{1}{\sum_n \gamma_{nk}} \sum_n \gamma_{nk} (\boldsymbol{x}_n - \boldsymbol{\mu}_k)(\boldsymbol{x}_n - \boldsymbol{\mu}_k)^{\mathrm{T}}$$

Since γ_{nk} is binary, the previous solution is nothing but

- For ω_k: count the number of data points whose z_n is k and divide by the total number of data points (note that $\sum_k \sum_n \gamma_{nk} = N$)
- For $\boldsymbol{\mu}_k$: get all the data points whose z_n is k, compute their mean
- For $\boldsymbol{\Sigma}_k$: get all the data points whose z_n is k, compute their covariance matrix

Estimate γ_{nk}

- γ_{nk} the assignment of instance n to cluster k, can be defined as $\gamma_{nk} = P(z_n = k|x_n)$
- Can be computed vis the posterior probability

$$p(z_n = k|\boldsymbol{x}_n) = \frac{p(\boldsymbol{x}_n|z_n = k)p(z_n = k)}{p(\boldsymbol{x}_n)} = \frac{p(\boldsymbol{x}_n|z_n = k)p(z_n = k)}{\sum_{k'=1}^{K} p(\boldsymbol{x}_n|z_n = k')p(z_n = k')}$$

$N(x|\mu_k, \Sigma_k)$ ω_k

Parameter estimation for GMMs

- If cluster assignments are observed $\{z_n\}$ are given
  - We know the cluster of each point
  - Let $\gamma_{nk} = 1$ if instance n belongs to cluster k, otherwise $\gamma_{nk} = 0$
- Then the maximum likelihood estimation is

$$\omega_k = \frac{\sum_n \gamma_{nk}}{\sum_k \sum_n \gamma_{nk}}, \quad \boldsymbol{\mu}_k = \frac{1}{\sum_n \gamma_{nk}} \sum_n \gamma_{nk} \boldsymbol{x}_n$$

$$\boldsymbol{\Sigma}_k = \frac{1}{\sum_n \gamma_{nk}} \sum_n \gamma_{nk} (\boldsymbol{x}_n - \boldsymbol{\mu}_k)(\boldsymbol{x}_n - \boldsymbol{\mu}_k)^{\mathrm{T}}$$

Hidden Markov Model

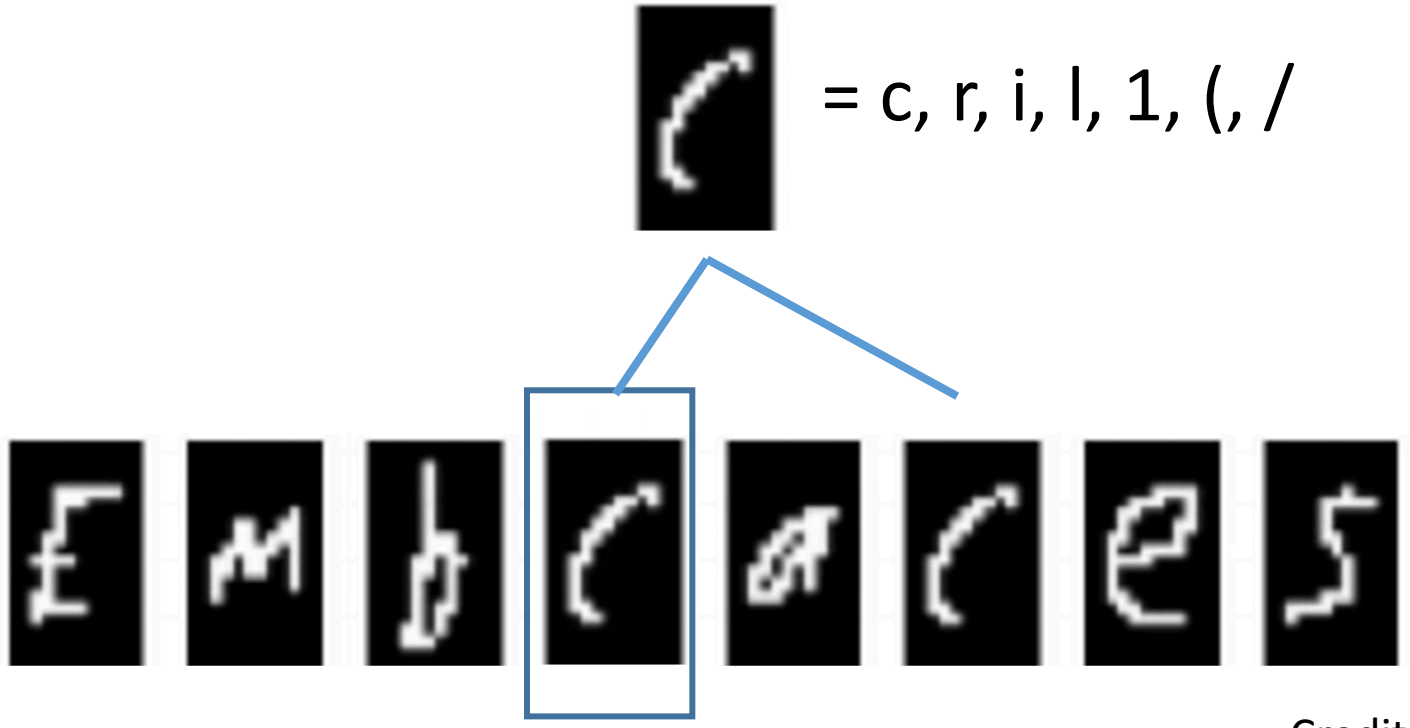
-- Go beyond binary/multiclass classification

Previous Lectures

- Binary linear classification
  - Perceptron, SVMs, Logistic regression, Naïve Bayes
  - Output: $y \in \{1, -1\}$
- Multi-class classification
  - Multiclass Perceptron, Multiclass SVM…
  - Output: $y \in \{1,2,3, \ldots K\}$

Why is structure important?
Hand written recognition example

- What is this English letter?



Credit: Ben Taskar

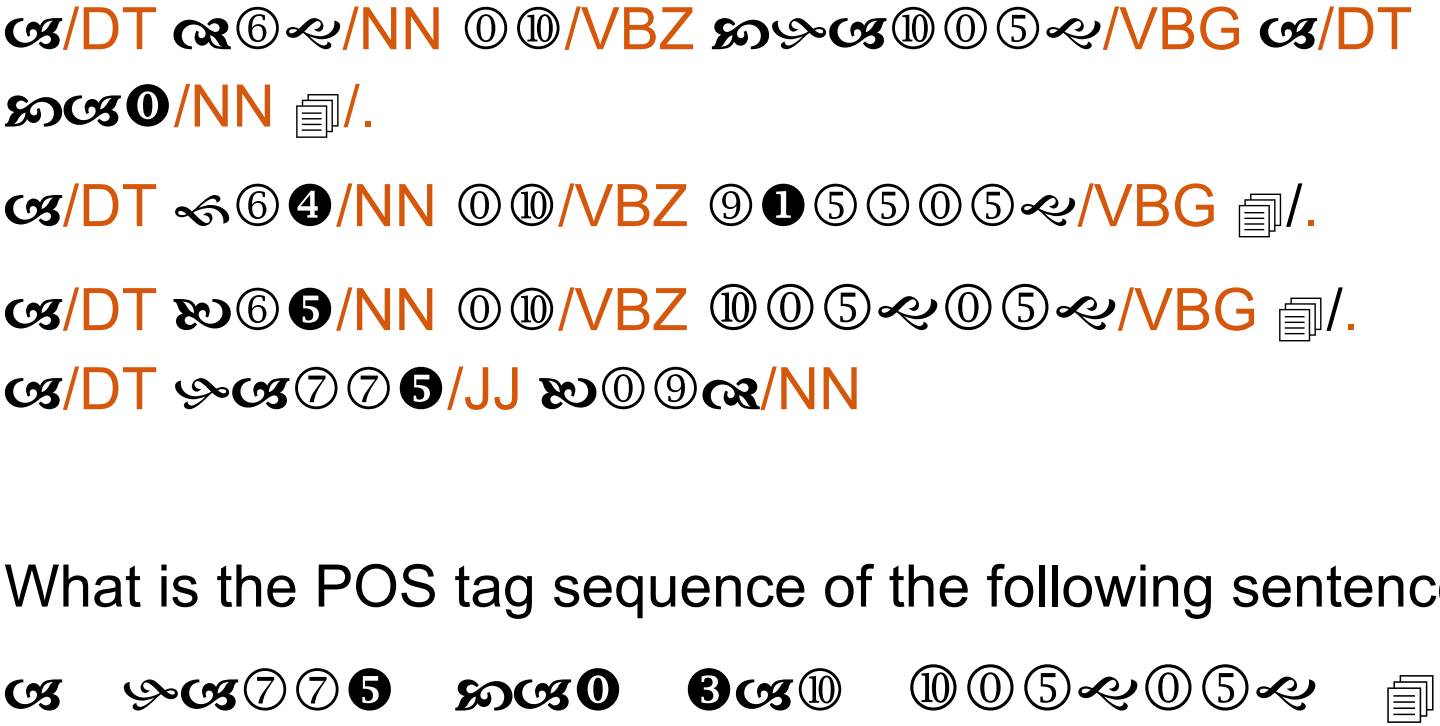
Sequential tagging

- The process of assigning a part-of-speech to each word in a collection (sentence).

WORD	tag
the	**DET**
koala	**N**
put	**V**
the	**DET**
keys	**N**
on	**P**
the	**DET**
table	**N**

Let's try

Don't worry! There is no problem with your eyes or computer.



What is the POS tag sequence of the following sentence?

Let’s try

- 𝒢/DT 𝒸⑥𝓈/NN ⓪⑩/VBZ 𝓃𝒽𝒢⑩⓪⑤𝓈/VBG 𝒢/DT 𝓃𝒽𝒢❶/NN 🗐/.
  a/DT dog/NN is/VBZ chasing/VBG a/DT cat/NN ./.
- 𝒢/DT 𝒻⑥❹/NN ⓪⑩/VBZ ⑨❶⑤⑤⓪⑤𝓈/VBG 🗐/.
  a/DT fox/NN is/VBZ running/VBG ./.
- 𝒢/DT 𝓫⑥❺/NN ⓪⑩/VBZ ⑩⓪⑤𝓈⓪⑤𝓈/VBG 🗐/.
  a/DT boy/NN is/VBZ singing/VBG ./.
- 𝒢/DT 𝒽𝒢⑦⑦❺/JJ 𝓫⓪⑨𝒸/NN
  a/DT happy/JJ bird/NN
- 𝒢 𝒽𝒢⑦⑦❺ 𝓃𝒢❶ ❸𝒢⑩ ⑩⓪⑤𝓈⓪⑤𝓈 🗐
  a happy cat was singing .

How you predict the tags?

- Two types of information are useful
  - Relations between words and tags
  - Relations between tags and tags
    - DT NN, DT JJ NN…
    - Fed in "The Fed" is a Noun because it follows a Determiner
- One possible model
  - Each output label is dependent on its neighbors in addition to the input

Outline

- *Sequence models*
- Hidden Markov models
  - Inference with HMM
  - (Supervised) Learning for HMM

Sequences

- Sequences of states
  - Text is a sequence of words or even letters
  - A video is a sequence of frames
- Our goal (for now):
  Define probability distributions over sequences
- If $x_1, x_2, \cdots, x_n$ is a sequence that has n tokens, we want to be able to define

$$P(x_1, x_2, \cdots, x_n)$$

A history-based model

$$P(x_1, x_2, \cdots, x_n) = \prod_{i=1}^{n} P(x_i | x_1, x_2, \cdots, x_{i-1})$$

- Each token is dependent on all the tokens that came before it
  - Simple conditioning
  - Each $P(x_i \mid \ldots)$ is a multinomial probability distribution over the tokens

Example: A Language model

It was a bright cold day in April.

$$P(\text{It was a bright cold day in April}) =$$
$$P(\text{It}) \times \quad \leftarrow \text{Probability of a word starting a sentence}$$
$$P(\text{was}|\text{It}) \times \quad \leftarrow \text{Probability of a word following “It”}$$
$$P(\text{a}|\text{It was}) \times \quad \leftarrow \text{Probability of a word following “It was”}$$
$$P(\text{bright}|\text{It was a}) \times \quad \leftarrow \text{Probability of a word following “It was a”}$$
$$P(\text{cold}|\text{It was a bright}) \times$$
$$P(\text{day}|\text{It was a bright cold}) \times \cdots$$

A history-based model

$$P(x_1, x_2, \cdots, x_n) = \prod_{i=1}^{n} P(x_i | x_1, x_2, \cdots, x_{i-1})$$

- Each token is dependent on all the tokens that came before it
  - Simple conditioning
  - Each $P(x_i \mid \ldots)$ is a multinomial probability distribution over the tokens
- What is the problem here?
  - How many parameters do we have?
    - Grows with the size of the sequence!

Solution: Lose the history

Discrete Markov Process

- A system can be in one of K states at a time
- State at time t is x_t
- First-order Markov assumption

  The state of the system at any time is ***independent*** of the full sequence history given the previous state

$$P(x_1, x_2, \cdots, x_n) = \prod_{i=1}^{n} P(x_i | x_{i-1})$$

  - Defined by two sets of probabilities:
    - Initial state distribution: $P(x_1 = S_j)$
    - State transition probabilities: $P(x_i = S_j \mid x_{i-1} = S_k)$

Example: Another language model

It was a bright cold day in April

$$P(\text{It was a bright cold day in April}) =$$
$$P(\text{It}) \times$$
$$P(\text{was}|\text{It}) \times$$
$$P(\text{a}|\text{was}) \times$$
$$P(\text{bright}|\text{a}) \times$$
$$P(\text{cold}|\text{bright}) \times$$
$$P(\text{day}|\text{cold}) \times \cdots$$

Probability of a word starting a sentence

Probability of a word following "It"

Probability of a word following "was"

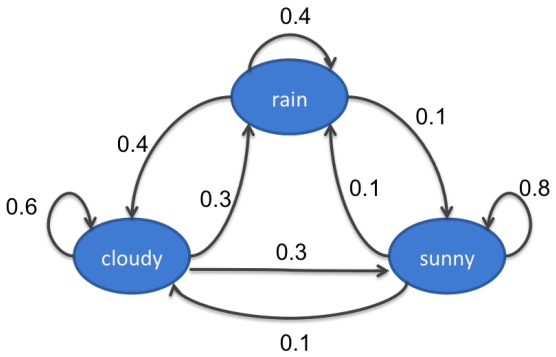
Probability of a word following "a"

If there are K tokens/states, how many parameters do we need? $O(K^2)$

Example: The weather

- Three states: rain, cloudy, sunny

State transitions:



- Observations are Markov chains:

Eg: *cloudy sunny sunny rain*

Probability of the sequence =

P(cloudy) P(sunny|cloudy) P(sunny | sunny) P(rain | sunny)

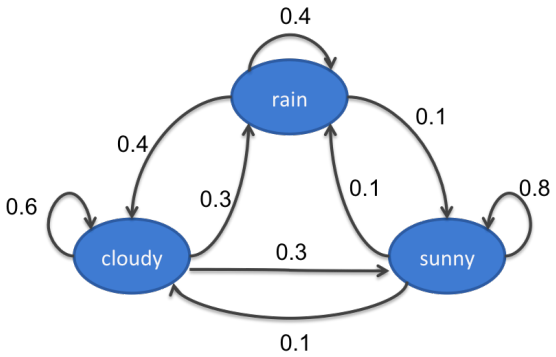
Initial probability

Transition probabilities

Example: The weather

- Three states: rain, cloudy, sunny

State transitions:



- Obse

These probabilities define the model; can find P(any sequence)

Eg: *cl*

Probability of the sequence =

P(cloudy) P(sunny|cloudy) P(sunny | sunny) P(rain | sunny)

Initial probability

Transition probabilities

m^{th} order Markov Model

- A generalization of the first order Markov Model
  - Each state is only dependent on m previous states
  - How many parameters do you need?

Outline

- *Sequence models*
- Hidden Markov models
  - Inference with HMM
  - Supervised Learning for HMM

Hidden Markov Model

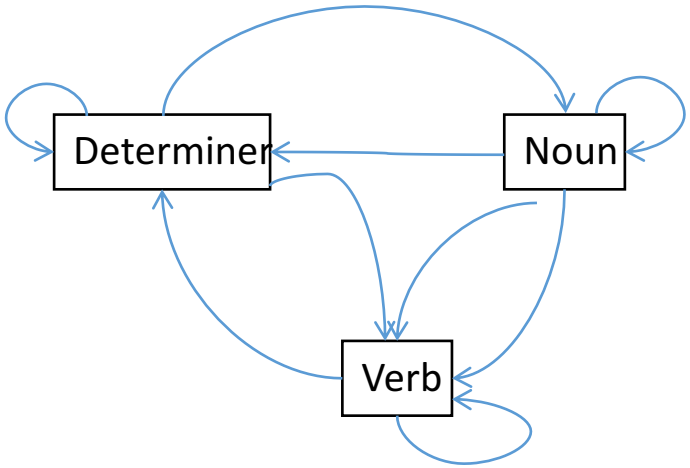
- Discrete Markov Model:
  - States follow a Markov chain
  - *Each state is an observation*

- Hidden Markov Model:
  - States follow a Markov chain
  - States are not observed
  - Each state stochastically emits an observation

Toy part-of-speech example

The Fed raises interest rates

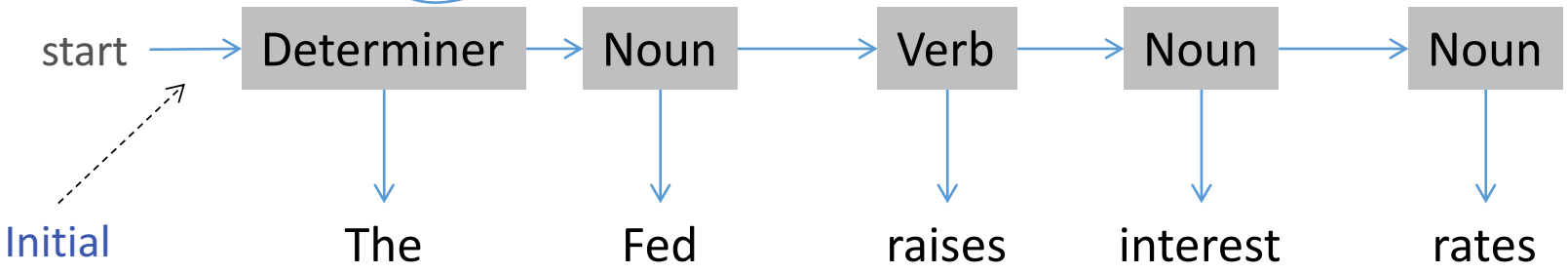
Transitions



Emissions

P(The | Determiner) = 0.5
P(A | Determiner) = 0.3
P(An | Determiner) = 0.1
P(Fed | Determiner) = 0
…

P(Fed| Noun) = 0.001
P(raises| Noun) = 0.04
P(interest| Noun) = 0.07
P(The| Noun) = 0
…



Joint model over states and observations

- Notation
  - Number of states = K, Number of observations = M
  - π: Initial probability over states (K dimensional vector)
  - A: Transition probabilities (K×K matrix)
  - B: Emission probabilities (K×M matrix)

- Probability of states and observations
  - Denote states by $y_1, y_2, \cdots$ and observations by $x_1, x_2, \cdots$

$$P(x_1, x_2, \cdots, x_n, y_1, y_2, \cdots y_n) = P(y_1) \prod_{i=1}^{n-1} P(y_{i+1}|y_i) \prod_{i=1}^{n} P(x_i|y_i)$$

$$= \pi_{y_1} \prod_{i=1}^{n-1} A_{y_i, y_{i+1}} \prod_{i=1}^{n} B_{y_i, x_i}$$

Jason and his Ice Creams

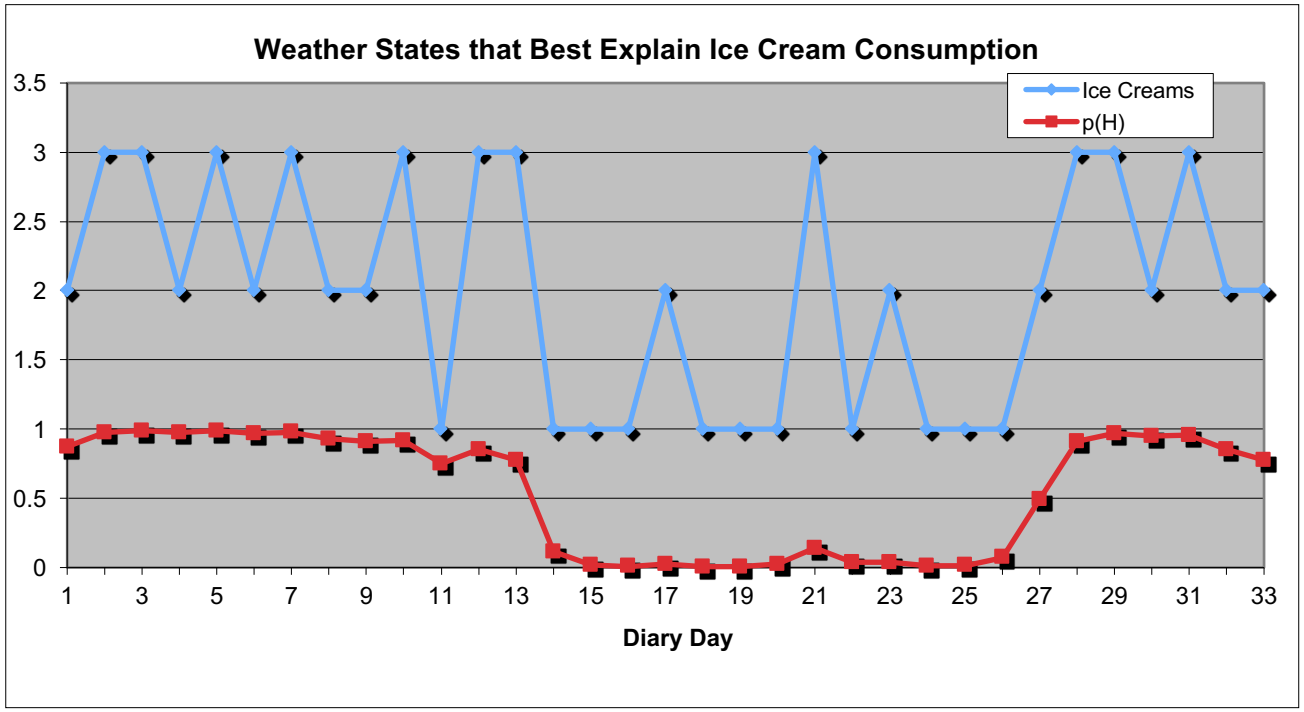
- You are a climatologist in the year 2799
- Studying global warming
- You can't find any records of the weather in Baltimore, MA for summer of 2007
- But you find Jason Eisner's diary
- Which lists how many ice-creams Jason ate every date that summer
- Our job: figure out how hot it was

http://videolectures.net/hltss2010_eisner_plm/
http://www.cs.jhu.edu/~jason/papers/eisner.hmm.xls

(C)old day v.s. (H)ot day

#cones

	p(…|C)	p(…|H)	p(…|START)
(1|…)	0.7	0.1	
(2|…)	0.2	0.2	
(3|…)	0.1	0.7	
(C|…)	0.8	0.1	0.5
(H|…)	0.1	0.8	0.5
…)	0.1	0.1	0



Other applications

- Speech recognition
  - Input: Speech signal
  - Output: Sequence of words
- NLP applications
  - POS Tagging
- Computational biology
  - Aligning protein sequences
  - Labeling nucleotides in a sequence as exons, introns, etc.

Questions?

Three questions for HMMs

[Rabiner 1999]

1. Given an observation sequence, $x_1, x_2, \cdots x_n$ and a model (π, A, B), how to efficiently calculate the probability of the observation?
2. Given an observation sequence, $x_1, x_2, \cdots, x_n$ and a model (π, A, B), how to efficiently calculate the most probable state sequence?

Inference

3. How to calculate (π, A, B) from observations?

Learning

Outline

- *Sequence models*
- Hidden Markov models
  - Inference with HMM
  - Supervised Learning for HMM

Most likely state sequence

- Input:
  - A hidden Markov model (π, A, B)
  - An observation sequence $\mathbf{x} = (x_1, x_2, \cdots, x_n)$

- Output: A state sequence $\mathbf{y} = (y_1, y_2, \cdots, y_n)$ that corresponds to
  - Maximum *a posteriori* inference (MAP inference)

$$\arg\max_{\mathbf{y}} P(\mathbf{y}|\mathbf{x}, \pi, A, B)$$

- Computationally: combinatorial optimization

MAP inference

- We want $\arg\max_{\mathbf{y}} P(\mathbf{y}|\mathbf{x}, \pi, A, B)$
- We have defined

$$P(x_1, x_2, \cdots, x_n, y_1, y_2, \cdots y_n) = P(y_1) \prod_{i=1}^{n-1} P(y_{i+1}|y_i) \prod_{i=1}^{n} P(x_i|y_i)$$

- But $P(\mathbf{y}|\mathbf{x}, \pi, A, B) \propto P(\mathbf{x}, \mathbf{y}|\pi, A, B)$
  - And we don't care about P(**x**) we are maximizing over **y**
- So, $\arg\max_{\mathbf{y}} P(\mathbf{y}|\mathbf{x}, \pi, A, B) = \arg\max_{\mathbf{y}} P(\mathbf{y}, \mathbf{x}|\pi, A, B)$

How many possible sequences?

The	Fed	raises	interest	rates
Determiner	Verb Noun	Verb Noun	Verb Noun	Verb Noun
1	2	2	2	2

List of allowed tags for each word

In this simple case, 16 sequences (1×2×2×2×2)

Naïve approaches

1. Try out every sequence
   - Score the sequence **y** as $P(\mathbf{y}|\mathbf{x}, \pi, A, B)$
   - Return the highest scoring one
   - What is the problem?
     - Correct, but slow, $O(K^n)$
2. Greedy search
   - Construct the output left to right
   - For each i, elect the best y_i using y_{i-1} and x_i
   - What is the problem?
     - Incorrect but fast, $O(nK)$

Solution: Use the independence assumptions

Recall: The first order Markov assumption

The state at token i is only influenced by the previous state, the next state and the token itself

Given the adjacent labels, the others do not matter

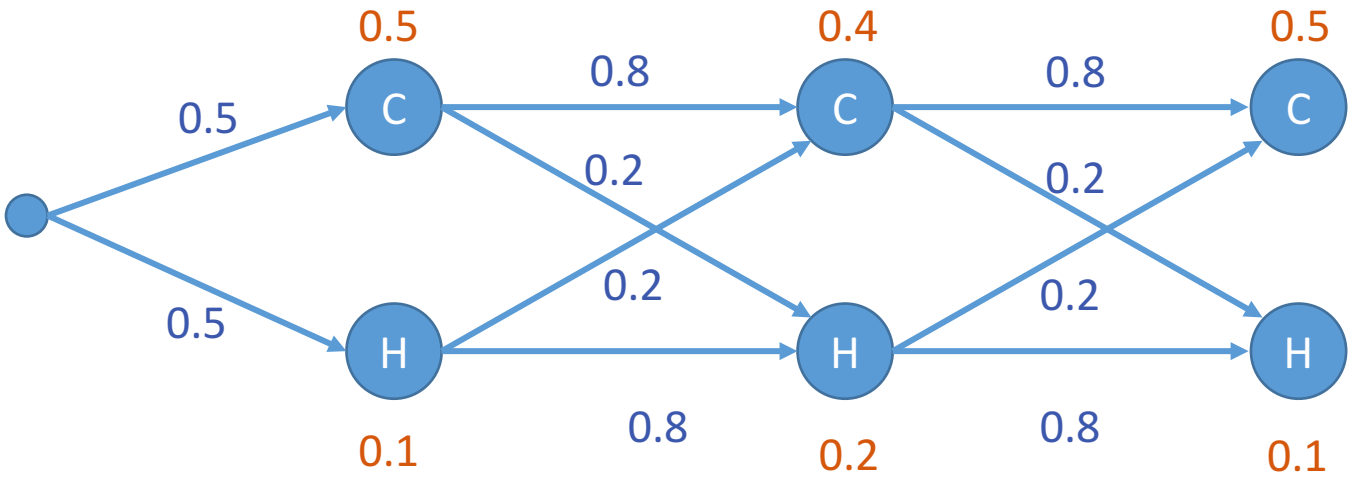
Suggests a recursive algorithm

Jason's ice cream

#cones

	p(…|C)	p(…|H)	p(…|START)
(1|…)	**0.5**	**0.1**	
(2|…)	**0.4**	**0.2**	
(3|…)	**0.1**	**0.7**	
(C|…)	**0.8**	**0.2**	**0.5**
(H|…)	**0.2**	**0.8**	**0.5**

- Best tag sequence for P("1,2,1")?



Viterbi algorithm

π: Initial probabilities
A: Transitions
B: Emissions

Max-product algorithm for first order sequences

1. Initial: For each state s, calculate

$$\text{score}_1(s) = P(s)P(x_1|s) = \pi_s B_{x_1,s}$$

1. Recurrence: For i = 2 to n, for every state s, calculate

$$\begin{aligned}\text{score}_i(s) &= \max_{y_{i-1}} P(s|y_{i-1})P(x_i|s)\text{score}_{i-1}(y_{i-1}) \\ &= \max_{y_{i-1}} A_{y_{i-1},s}B_{s,x_i}\text{score}_{i-1}(y_{i-1})\end{aligned}$$

1. Final state: calculate

$$\max_{\mathbf{y}} P(\mathbf{y}, \mathbf{x}|\pi, A, B) = \max_s \text{score}_n(s)$$

This only calculates the max. To get final answer (*argmax*),

- keep track of which state corresponds to the max at each step
- build the answer using these back pointers

Questions?

General idea

- Dynamic programming
  - The best solution for the full problem relies on best solution to sub-problems
  - Memoize partial computation

- Examples
  - Viterbi algorithm
  - Dijkstra's shortest path algorithm
  - …

Complexity of inference

- Complexity parameters
  - Input sequence length: n
  - Number of states: K
- Memory
  - Storing the table: nK (scores for all states at each position)
- Runtime
  - At each step, go over pairs of states
  - $O(nK^2)$

Outline

- *Sequence models*
- Hidden Markov models
  - Inference with HMM
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Learning HMM parameters

- Assume we know the number of states in the HMM
- Two possible scenarios

1. We are given a data set $D = \{<\mathbf{x}_i, \mathbf{y}_i>\}$ of sequences labeled with states

   And we have to learn the parameters of the HMM (π, A, B)

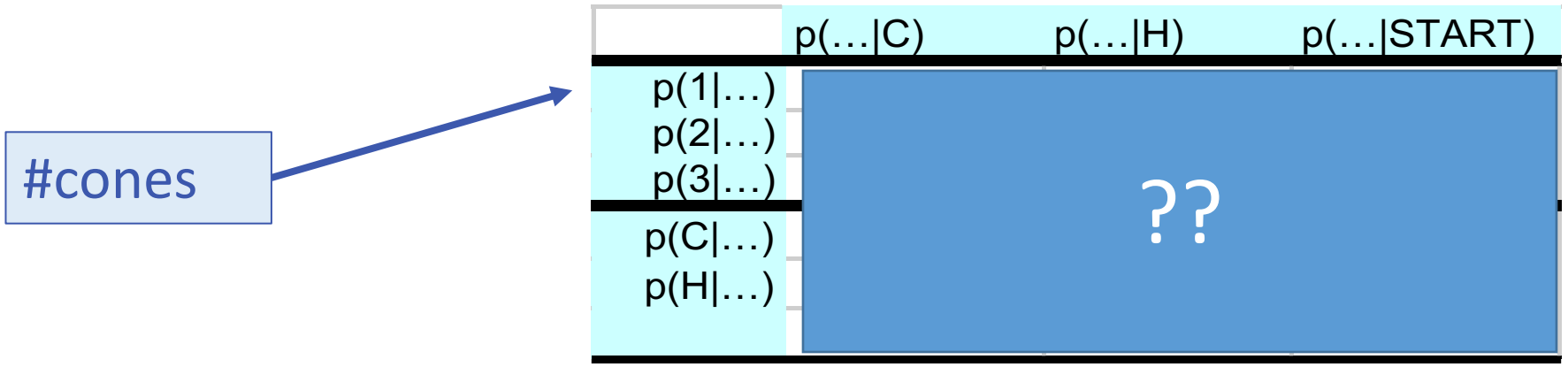
   Supervised learning with complete data

2. We are given only a collection of sequences $D = \{\mathbf{x}_i\}$

   And we have to learn the parameters of the HMM (π, A, B)

   Unsupervised learning, with incomplete data

Jason's ice cream



- Can we figure out (π, A, B) from history records?

#cones	1	2	1	1	2	3	3	3	3	2	3	1	2
Hot/Cold	C	C	C	C	H	H	H	H	C	C	H	H	H

#cones	3	2	2	2	2	3	1	1	1	2	3	2	2
Hot/Cold	H	H	H	C	C	H	H	C	C	H	H	H	C

Supervised learning of HMM

- We are given a dataset D = {<$\mathbf{x}_i$, $\mathbf{y}_i$>}
  - Each $\mathbf{x}_i$ is a sequence of observations and $\mathbf{y}_i$ is a sequence of states that correspond to $\mathbf{x}_i$

  Goal: Learn initial, transition, emission distributions (π, A, B)

- How do we learn the parameters of the probability distribution?
  - The maximum likelihood principle

Where have we seen this before?

$$(\hat{\pi}, \hat{A}, \hat{B}) = \max_{\pi,A,B} P(D|\pi, A, B) = \max_{\pi,A,B} \prod_i P(\mathbf{x}_i, \mathbf{y}_i|\pi, A, B)$$

And we know how to write this in terms of the parameters of the HMM

Supervised learning details

$$(\hat{\pi}, \hat{A}, \hat{B}) = \max_{\pi,A,B} P(D|\pi, A, B) = \max_{\pi,A,B} \prod_i P(\mathbf{x}_i, \mathbf{y}_i|\pi, A, B)$$

π, A, B can be estimated separately just by counting

- Makes learning simple and fast

[Exercise: Derive the following using derivatives of the log likelihood. Requires Lagrangian multipliers.]

Number of instances where the first state is s

$$\pi_s = \frac{\text{count}(\text{start} \rightarrow s)}{n}$$

Number of examples

Initial probabilities

$$A_{s',s} = \frac{\text{count}\,(s \rightarrow s')}{\text{count}\,(s)}$$

Transition probabilities

$$B_{s,x} = \frac{\text{count}\begin{pmatrix} s \\ \downarrow \\ x \end{pmatrix}}{\text{count}\,(s)}$$

Emission probabilities

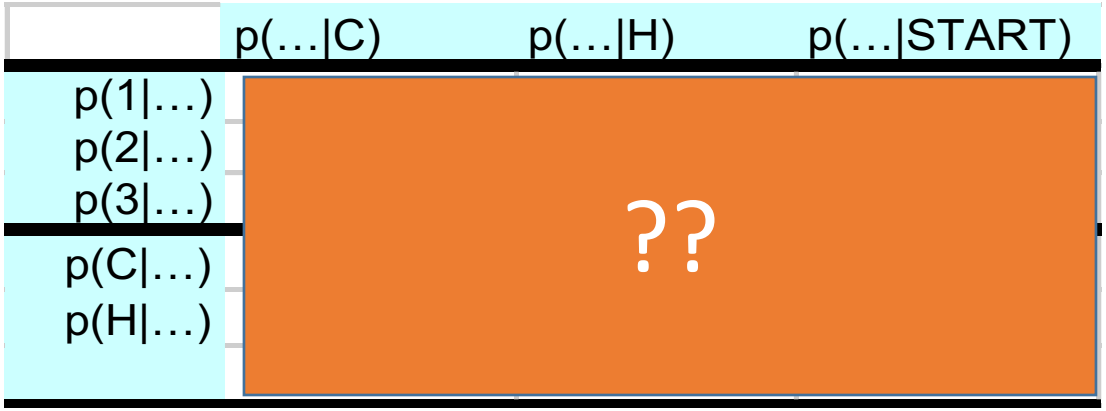
Priors and smoothing

- Maximum likelihood estimation works best with lots of annotated data
  - Never the case
- Priors inject information about the probability distributions
- Effectively additive smoothing
  - Add small constants to the counts

Question I haven't answer

- Can we figure out (π, A, B) from just observation

#cones



#cones	1	2	1	1	2	3	3	3	3	2	3	1	2
Hot/Cold	??												

#cones	3	2	2	2	2	3	1	1	1	2	3	2	2
Hot/Cold	??												

Hidden Markov Models summary

- Predicting sequences
  - As many output states as observations
- Markov assumption helps decompose the score
- Several algorithmic questions
  - Most likely state (decoding)
  - Learning parameters
    - Supervised, Unsupervised