# hw1

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## 1 Homework 1

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### 1.1 Problem 1

(a) $P(D=1) = P(D=1, C=1, T=1) + P(D=1, C=1, T=0) + P(D=1, C=0, T=1) + P(D=1, C=0, T=0)$
From the table in lecture slides, we can find that
$P(D=1, C=1, T=1) = 0.108$
$P(D=1, C=1, T=0) = 0.072$
$P(D=1, C=0, T=1) = 0.016$ $P(D=1, C=0, T=0) = 0.144$
Therefore,

[1]:
```
print("P(D = 1) = 0.108 + 0.072 + 0.016 + 0.144 = %.3f" % (0.108 + 0.072 + 0.
↪016 + 0.144))
```

```
P(D = 1) = 0.108 + 0.072 + 0.016 + 0.144 = 0.340
```

(b) $P(C=1, D=1) = P(T=1, C=1, D=1) + P(T=0, C=1, D=1)$
From the table in lecture slides, we can find that
$P(D=1, C=1, T=1) = 0.108$
$P(D=1, C=1, T=0) = 0.072$
Therefore,

[2]:
```
print("P(C = 1, D = 1) = 0.108 + 0.072 = %.3f" % (0.108 + 0.072))
```

```
P(C = 1, D = 1) = 0.108 + 0.072 = 0.180
```

(c)

$$P(D=1|C=1) = \frac{P(C=1, D=1)}{P(C=1)}$$

$P(C=1) = P(D=1, C=1, T=1) + P(D=1, C=1, T=0) + P(D=0, C=1, T=1) + P(D=0, C=1, T=0)$
From the table in lecture slides, we can find that
$P(D=1, C=1, T=1) = 0.108$
$P(D=1, C=1, T=0) = 0.072$
$P(D=0, C=1, T=1) = 0.012$
$P(D=0, C=1, T=0) = 0.008$
Thus,

[3]:
```
print("P(C = 1) = 0.108 + 0.072 + 0.012 + 0.008 = %.3f" % (0.108 + 0.072 + 0.
→012 + 0.008))
```

```
P(C = 1) = 0.108 + 0.072 + 0.012 + 0.008 = 0.200
```

Therefore,

[4]:
```
print("P(D = 1|C = 1) = 0.180/0.200 = %.3f" % (0.180/0.200))
```

```
P(D = 1|C = 1) = 0.180/0.200 = 0.900
```

(d)

$$P(C=1|T=1,D=0) = \frac{P(C=1,T=1,D=0)}{P(T=1,D=0)}$$

$P(T=1,D=0) = P(C=1,T=1,D=0) + P(C=0,T=1,D=0)$
From the table in lecture slides, we can find that
$P(C=1,T=1,D=0) = 0.012$
$P(C=0,T=1,D=0) = 0.064$
Thus,

[5]:
```
print("P(T = 1, D = 0) = 0.012 + 0.064 = %.3f" % (0.012 + 0.064))
```

```
P(T = 1, D = 0) = 0.012 + 0.064 = 0.076
```

Therefore,

[6]:
```
print("P(C = 1|T = 1, D = 0) = 0.012/0.076 = %.3f" % (0.012/0.076))
```

```
P(C = 1|T = 1, D = 0) = 0.012/0.076 = 0.158
```

## 1.2 Problem 2

(a) Suppose P(F = 1) is the probability of pulling out a fake coin when picking a coin from the bag randomly, and P(H = 1) is the probability of get a head when fliping that coin pulled out.
Therefore, the answer we are looking for is $P(F=1|H=1)$.

$$P(F=1|H=1) = \frac{P(F=1,H=1)}{P(H=1)}$$

From the question we know that there are n coins in the bag, which $n-1$ of them are normal coins and one is the fake coin which has head on both side.
Thus,
$P(F=1) = \frac{1}{n}$,
$P(H=1) = (\frac{1}{2} \times \frac{n-1}{n}) + (\frac{1}{n}) = \frac{n+1}{2n}$
Since the fake coin has haed on both side,
$P(H=1|F=1) = 1$.
By applying chain rule,
$P(F=1,H=1) = P(H=1|F=1) \times P(F=1) = P(F=1) = \frac{1}{n}$

Therefore,

$$P(F=1|H=1) = \frac{\frac{1}{n}}{\frac{n+1}{2n}} = \frac{2}{n+1}.$$

(b) Suppose P(H = k) is the probability of having k heads after flipping the picked coin for k times.
Therefore, the answer we are looking for is P(F = 1|H = k).
P(F = 1|H = k) = P(F = 1, H = k)/P(H = k)
From the question we know that the fake coin has heads on both sides.
Thus, by applying chain rule,
P(F = 1, H = k) = P(H = k|F = 1) * P(F = 1) = P(F = 1) = 1/n from part(a).
Since there are n coins in the bag which one of them is the fake coin and the rest are normal coins,
P(H = k) = (P(H = k|F = 1) * P(F = 1)) + (P(H = k|F = 0) * P(F = 0)) = (1/n)+(2^(-k))*(n-1)/n
Therefore,

$$P(F=1|H=k) = \frac{\frac{1}{n}}{\frac{1}{n} + (2^{-k}) \times \frac{n-1}{n}} = \frac{2^k}{2^k + n - 1}$$

## 1.3 Problem 3

(a) Suppose P(H = 1) is the probability of engine becoming too hot, P(L = 1) is the probability of coolant level falls so low and P(W = 1) is the probability of warning light turning on.

(b) From the question we know that the answer we want is $P(L=1|W=1)$.
By Bayes rule we know that

$$P(L=1|W=1) = \frac{P(W=1|L=1)P(L=1)}{P(W=1)} = \frac{P(W=1,L=1)}{P(W=1)}$$

and

$$\begin{aligned} P(W=1) =& P(W=1|H=1,L=1) \times P(H=1,L=1) + P(W=1|H=0,L=0) \times P(H=0,L=0) \\ &+ P(W=1|H=0,L=1) \times P(H=0,L=1) + P(W=1|H=1,L=0) \times P(H=1,L=0) \end{aligned}$$

From the question we can find that

$$P(W=1|H=1,L=1) = 0.9, P(W=1|H=0,L=0) = 0.1, P(W=1|H=0,L=1) = P(W=1|H=1,L$$

Since H and L are independent events,

$$P(H,L) = P(H) \times P(L)$$

and

$$\begin{aligned} P(W=1,L=1) &= P(W=1,L=1)P(H=1) + P(W=1,L=1)P(H=0) \\ &= P(W=1,L=1,H=1) + P(W=1,L=1,H=0) \\ &= P(W=1|L=1,H=1)P(L=1,H=1) + P(W=1|L=1,H=0)P(L=1,H=0) \\ &= 0.9 \times 0.1 \times 0.1 + 0.8 \times 0.9 \times 0.1 \end{aligned}$$

[7]:
```
print("P(W = 1, L = 1) = %.3f" % (0.9*0.1*0.1 + 0.8*0.9*0.1))
```

```
P(W = 1, L = 1) = 0.081
```

Therefore,

$$\begin{aligned} P(W=1) =& 0.9 \times P(H=1)P(L=1) + 0.1 \times P(H=0)P(L=0) \\ &+ 0.8 \times P(H=1)P(L=0) + 0.8 \times P(H=0)P(L=1) \\ =& 0.9 \times 0.1 \times 0.1 + 0.1 \times 0.9 \times 0.9 + 2 \times 0.8 \times 0.1 \times 0.9 \end{aligned}$$

[8]:
```
print("P(W = 1) = %.3f" % (0.9*0.1*0.1 + 0.1*0.9*0.9 + 2*0.8*0.1*0.9))
```

```
P(W = 1) = 0.234
```

Therefore,

$$\begin{aligned} P(L=1|W=1) &= \frac{P(W=1, L=1)}{P(W=1)} \\ &= \frac{0.081}{0.234} \end{aligned}$$

[9]:
```
print("P(L = 1|W = 1) = %.3f" % (0.081*0.234))
```

```
P(L = 1|W = 1) = 0.019
```

(c) From the question we know that the answer we want is $P(L = 1|W = 1, H = 1)$. By Bayes rule we know that

$$\begin{aligned} P(L=1|W=1, H=1) &= \frac{P(W=1, H=1|L=1)P(L=1)}{P(W=1, H=1)} \\ &= \frac{P(W=1, L=1, H=1)}{P(W=1, H=1)} \end{aligned}$$

From question (a) we know that
$P(W = 1, L = 1, H = 1) = P(W = 1|L = 1, H = 1)P(L = 1, H = 1) = 0.9 \times 0.1 \times 0.1$
and
$P(W = 1, H = 1) = 0.1 \times 0.8$
Therefore,

$$P(L=1|W=1, H=1) = \frac{0.9 \times 0.1 \times 0.1}{0.1 \times 0.8}$$

[10]:
```
print("P(L = 1|W = 1, H = 1) = %.3f" % ((.9*.1*.1)/(.1*.8)))
```

```
P(L = 1|W = 1, H = 1) = 0.112
```

### 1.4 Problem 4

(a) By Bayes rule we know that

$$p(A=1|B=1, C=1) = \frac{p(B=1, C=1|A=1)p(A=1)}{p(B=1, C=1)}$$

Therefore, set (2) is sufficient for this calculation.

(b) Since we know that $p(B|A,C) = p(B|A)$,

$$\begin{aligned} p(A=1|B=1,C=1) &= \frac{p(B=1,C=1|A=1)p(A=1)}{p(B=1,C=1)} \\ &= \frac{p(B=1|A=1,C=1)p(C=1|A=1)p(A=1)}{p(B=1,C=1)} \\ &= \frac{p(B=1|A=1)p(C=1|A=1)p(A=1)}{p(B=1,C=1)} \end{aligned}$$

Therefore, both set (1) and (2) are sufficient for this calculation.

### 1.5 Problem 5

(a)

[1]:
```
!pip install pyGMs
import numpy as np
import pyGMs as gm
import matplotlib.pyplot as plt
T = gm.Var(0,2)
D = gm.Var(1,2)
C = gm.Var(2,2)
```

```
Collecting pyGMs
  Downloading pyGMs-0.1.0-py3-none-any.whl (157 kB)
     |                  | 157 kB 433 kB/s eta 0:00:01
Installing collected packages: pyGMs
Successfully installed pyGMs-0.1.0
```

(b)

[2]:
```
P = gm.Factor([T,D,C])
P[{T:0,D:0,C:0}] = 0.576
P[{T:0,D:0,C:1}] = 0.008
P[{T:0,D:1,C:0}] = 0.144
P[{T:0,D:1,C:1}] = 0.072
P[{T:1,D:0,C:0}] = 0.064
P[{T:1,D:0,C:1}] = 0.012
P[{T:1,D:1,C:0}] = 0.016
P[{T:1,D:1,C:1}] = 0.108
print(P.table)
```

```
[[[0.576 0.008]
  [0.144 0.072]]

 [[0.064 0.012]
  [0.016 0.108]]]
```

(c) (1)

[3]:
```python
DC = P.condition({T:1})
print(DC.table)
```

```
[[0.064 0.012]
 [0.016 0.108]]
```

(2)

[4]:
```python
TTC = DC.sum([D])
print(TTC.table)
```

```
[0.08 0.12]
```

(3)

[5]:
```python
TT = P.sum([D,C])
TTC[0]/=TT[1]
TTC[1]/=TT[1]
print(TTC.table)
```

```
[0.4 0.6]
```

(d)

$p(C)$ :

[6]:
```python
CC = P.sum([T,D])
print(CC.table)
```

```
[0.8 0.2]
```

$p(D|C)$ :

[7]:
```python
pDgivenC = P.sum([T]) / CC
print(pDgivenC.table)
```

```
[[0.8 0.1]
 [0.2 0.9]]
```

$p(T|C)$ :

[8]:
```python
pTgivenC = P.sum([D]) / CC
print(pTgivenC.table)
```

```
[[0.9 0.4]
 [0.1 0.6]]
```

[10]:
```python
time = CC*pDgivenC*pTgivenC
print(time.table)
print()
print(P.table)
```

```
[[[0.576 0.008]
  [0.144 0.072]]
```

```
 [[0.064 0.012]
  [0.016 0.108]]]

[[[0.576 0.008]
  [0.144 0.072]]

 [[0.064 0.012]
  [0.016 0.108]]]
```

Therefore,

$$p(T, D, C) = p(C)p(D|C)p(T|C)$$