Master of Science
Sustainable Management and Technology

SUSTAINABLE AND ENTREPRENEURIAL FINANCE

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SUSTAINABLE FINANCE

Lecture 2: Fundamentals of Asset Management

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Objectives of the lecture

This lecture is an introduction to **Investment** (or Asset Management or Portfolio Management).

- **Optimal portfolio allocation** (Markowitz, 1952): The optimization problem reduces to a simple mean-variance criterion

- **Capital Asset Pricing Model (CAPM)** (Sharpe, 1964, Lintner, 1965): At equilibrium, there is a unique factor explaining cross-section differences between firms' expected return, i.e., their **beta**

- **Market efficiency** (Fama, 1965): If markets are efficient, abnormal returns should be unpredictable

We now discuss these 3 steps

We also discuss how to measure **inputs** of the allocation process?

Major Steps in Asset Management

What is the optimal decision of individual investors?

Optimal portfolio allocation

(Optimal weights given expected excess returns)

→ Equilibrium

What makes a difference between stock returns today?

Capital Asset Pricing Model / Factor Models

(Cross-section heterogeneity)

μ_i^e for all firms

Can we predict stock returns for tomorrow?

Efficient market hypothesis

(Time-series dynamics)

Unexpected returns $\varepsilon_{i,t+1}$ are unpredictable

Complete description of excess returns properties

$$R_{i,t+1} - R_{f,t} = \mu_i^e + \varepsilon_{i,t+1}$$

Objectives of the lecture

Readings

Fama, E.F. (1965), The Behavior of Stock-Market Prices, *Journal of Business*, 38(1), 34-105.

Lintner, J. (1965), "The valuation of risk assets and the selection of risky investments in stock portfolios and capital budgets", *Review of Economics and Statistics*, 47(1), 13-37.

Markowitz, H. (1952), Portfolio Selection, *Journal of Finance*, 7(1), 77-91.

Sharpe, W.F. (1964), "Capital asset prices: A Theory of Market Equilibrium under Conditions of Risk", *Journal of Finance*, 19(3), 425-442

Tobin, J. (1958), "Liquidity Preference as Behavior Towards Risk", *Review of Economic Studies*, 25.1(2), 65-86

Objectives of the lecture

➔ **Optimal Allocation**

- **Capital Asset Pricing Model**
- **Market Efficiency**
- **Predicting Risk Premia and the Covariance Matrix**

Notations

Assume that we have N risky securities.

bonds, stoks, edge funds, something you want to invent on

Securities returns: $R_{t+1} = (R_{1,t+1}, \dots, R_{N,t+1})'$

Expected returns: $\mu_{t+1} = E[R_{t+1}] = (\mu_{1,t+1}, \dots, \mu_{N,t+1})'$ (measured in t)

we can have expectations about the return, but we do not know with certainty

Covariance matrix: $\Sigma_{t+1} = E[(R_{t+1} - \mu_{t+1})(R_{t+1} - \mu_{t+1})'] = \begin{pmatrix} \sigma_{1,t+1}^2 & \cdots & \sigma_{1N,t+1} \\ \vdots & \ddots & \vdots \\ \sigma_{1N,t+1} & \cdots & \sigma_{N,t+1}^2 \end{pmatrix}$

how risky is my prediction

allocation decision

Portfolio weights: $\alpha_t = (\alpha_{1,t}, \dots, \alpha_{N,t})'$, with $\alpha_t' e = \sum_{i=1}^{N} \alpha_{i,t} = 1$ $\quad e = (1, \dots, 1)'$

Ex-post portfolio return: $R_{p,t+1} = \alpha_t' R_{t+1} = \sum_{i=1}^{N} \alpha_{i,t} R_{i,t+1}$

Portfolio expected return: $\mu_{p,t+1} = E\left[R_{p,t+1}\right] = \alpha_t' \mu_{t+1} = \sum_{i=1}^{N} \alpha_{i,t} \mu_{i,t+1}$

Ex-ante portfolio variance: $\sigma_{p,t+1}^2 = V\left[R_{p,t+1}\right] = \alpha_t' \Sigma_{t+1} \alpha_t = \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_{i,t} \alpha_{j,t} \sigma_{ij,t+1}$

Mean-Variance Case (Markowitz, 1952)

On the efficient frontier, there are many portfolios. Are we indifferent between all these portfolios? It depends on our **risk aversion** and therefore on how we trade-off risk and return.

We denote by λ the risk aversion parameter

When there is no risk-free asset, we must take care of the constraint $\alpha_t' e = 1$.

The solution is the portfolio weight α_t^* that maximizes:

$$\begin{cases} \max_{\alpha_t} \mu_{p,t+1} - \dfrac{\lambda}{2}\sigma_{p,t+1}^2 = \alpha_t'\mu_{t+1} - \dfrac{\lambda}{2}\alpha_t'\Sigma_{t+1}\alpha_t \\ \text{subject to } \alpha_t' e = 1 \end{cases}$$

γ: Lagrange multiplier

Remark: From now on, we use $\alpha = \alpha_t$, $\mu = \mu_{t+1}$, $\Sigma = \Sigma_{t+1}$.

Mean-Variance Case (Markowitz, 1952)

The solution without a risk-free asset

The Lagrangian is: $\mathcal{L} = \alpha'\mu - \frac{\lambda}{2}\alpha'\Sigma\,\alpha - \gamma(\alpha' e - 1)$

The derivatives are:

$$\frac{\partial L}{\partial \alpha} = \mu - \lambda\Sigma\alpha - \gamma e = 0 \qquad \text{and} \qquad \frac{\partial L}{\partial \gamma} = \alpha' e - 1 = 0$$

Step 1: $\mu - \lambda\Sigma\alpha - \gamma e = 0 \quad \rightarrow \quad \alpha = \frac{1}{\lambda}\Sigma^{-1}(\mu - \gamma e)$

Step 2: $e'\alpha = \frac{1}{\lambda}e'\Sigma^{-1}(\mu - \gamma e) = 1 \quad \rightarrow \quad e'\Sigma^{-1}\mu - \gamma e'\Sigma^{-1}e = \lambda$

$$\rightarrow \quad \gamma = \frac{e'\Sigma^{-1}\mu - \lambda}{e'\Sigma^{-1}e}$$

Step 3: $\alpha = \frac{1}{\lambda}\Sigma^{-1}(\mu - \gamma e) = \frac{1}{\lambda}\Sigma^{-1}\mu - \frac{1}{\lambda}\Sigma^{-1}e\,\frac{e'\Sigma^{-1}\mu - \lambda}{e'\Sigma^{-1}e}$

$$\alpha = \frac{1}{\lambda}\Sigma^{-1}\mu + \frac{\Sigma^{-1}e}{e'\Sigma^{-1}e} - \frac{1}{\lambda}\Sigma^{-1}e\,\frac{e'\Sigma^{-1}\mu}{e'\Sigma^{-1}e}$$

Mean-Variance Case (Markowitz, 1952)

The solution without a risk-free asset

Proposition: When there is no risk-free asset and no restriction on portfolio weights, the weights of the optimal portfolio are:

$$\alpha^* = \boxed{\frac{\cancel{\Sigma^{-1}}e}{e'\cancel{\Sigma^{-1}}e}} + \frac{1}{\lambda}\Sigma^{-1}\left(\mu - \frac{\cancel{e'}\cancel{\Sigma^{-1}}\mu}{\cancel{e'}\cancel{\Sigma^{-1}}e}e\right) = \alpha^*_{\text{gmv}} + \frac{1}{\lambda}\Sigma^{-1}\left(\mu - \frac{e'\Sigma^{-1}\mu}{e'\Sigma^{-1}e}e\right)$$

where α_{gmv} is independent from expected returns

$$\boxed{\alpha^* = \left(1 - \frac{e'\Sigma^{-1}\mu}{\lambda}\right)\alpha^*_{\text{gmv}} + \left(\frac{e'\Sigma^{-1}\mu}{\lambda}\right)\alpha^*_{\text{spec}}}$$

We obtain the well-known **Mutual Fund Separation Theorem**. Investors invest in:

- **the global minimum-variance portfolio:** $\alpha^*_{\text{gmv}} = \frac{\Sigma^{-1}e}{e'\Sigma^{-1}e}$ with weight $\left(1 - \frac{e'\Sigma^{-1}\mu}{\lambda}\right)$

- **the speculative portfolio:** $\alpha^*_{\text{spec}} = \frac{\Sigma^{-1}\mu}{e'\Sigma^{-1}\mu}$ with weight $\left(\frac{e'\Sigma^{-1}\mu}{\lambda}\right)$

Mean-Variance Case (Markowitz, 1952)

Alternative derivations

An investor minimizing the portfolio variance subject to a return constraint or maximizing the portfolio return subject to a volatility constraint will find the same solution (with no particular value for the risk aversion parameter):

use for the project to build the efficient frontier

$$\begin{cases} \min\limits_{\alpha} \sigma_p^2 = \alpha' \Sigma\, \alpha \\ \text{s.t.} \quad \mu_p \geq \tilde{\mu}_p \ \text{ and } \ \alpha' e = 1 \end{cases} \quad \text{or} \quad \begin{cases} \max\limits_{\alpha} \mu_p = \alpha' \mu \\ \text{s.t.} \quad \sigma_p \leq \tilde{\sigma}_p \ \text{ and } \ \alpha' e = 1 \end{cases}$$

Proposition: When there are no restrictions on the portfolio weights, the weights of the **minimum variance portfolio (MVP)** for a required return $\tilde{\mu}_p$ are:

$$\alpha^* = \Lambda_1 + \Lambda_2 \tilde{\mu}_p$$

where $\Lambda_1 = \frac{\Sigma^{-1}}{D}[Be - A\mu]$ and $\Lambda_2 = \frac{\Sigma^{-1}}{D}[C\mu - Ae]$

with $A = e'\Sigma^{-1}\mu$, $B = \mu'\Sigma^{-1}\mu$, $C = e'\Sigma^{-1}e$ and $D = BC - A^2$.

The collection of MVPs for various $\tilde{\mu}_p$ gives the **mean-variance efficient frontier**

Mean-Variance Case (Markowitz, 1952)

The efficient frontier

The efficient frontier is given by the relation:

$$\tilde{\mu}_p = \frac{A}{C} + \sqrt{\frac{D}{C}\left(\tilde{\sigma}_p^2 - \frac{1}{C}\right)}$$

Remarks:

- Any portfolio of MVPs is also an MVP.

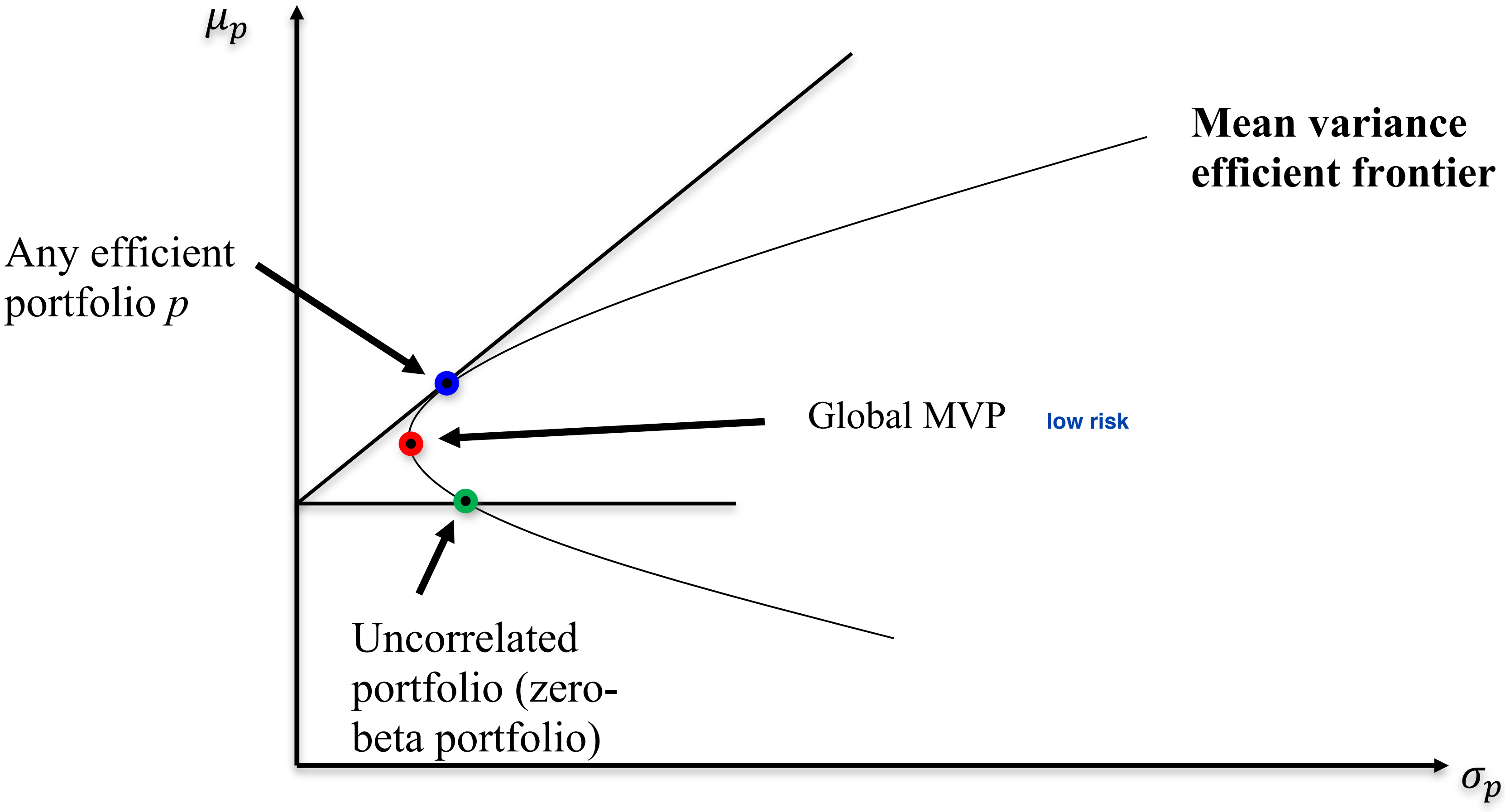
- The **Global Minimum Variance Portfolio** (Global MVP) is given by

$$\alpha_{\text{gmv}} = \frac{\Sigma^{-1}e}{C} \qquad \text{with} \qquad \mu_{\text{gmv}} = \frac{A}{C} \qquad \text{and} \qquad \sigma_{\text{gmv}}^2 = \frac{1}{C}$$

- The covariance of any asset or portfolio return R_p with the Global MVP is $Cov\left[R_{\text{gmv}}, R_p\right] = \frac{1}{C}$.

Mean-Variance Case (Markowitz, 1952)

Mean-variance efficient frontier



Mean-Variance Case (Tobin, 1958)

The solution with a risk-free asset

There is a risk-free asset, which the investor can borrow or lend with unlimited amount. The solution is the portfolio weight α^* that maximizes:

$$\mu_p - \frac{\lambda}{2}\sigma_p^2 = [\alpha'\mu + (1 - \alpha'e)R_f] - \frac{\lambda}{2}\alpha'\Sigma\alpha$$

Proposition: If there is a risk-free asset, the weights of the **optimal portfolio** are:

$\alpha^* = \frac{1}{\lambda}\Sigma^{-1}(\mu - R_f e)$ in risky assets

$1 - e'\alpha^*$ in the risk-free asset

The sum of the α^* does not necessarily sum to 1, because the investor can hold some amount of risk-free asset. The set of all portfolios when λ varies is called the **Capital Allocation Line**.

Remark: If there is one risky asset only (μ_m, σ_m^2), then $\alpha^* = (\mu_m - R_f)/(\lambda\sigma_m^2)$

Mean-Variance Case (Tobin, 1958)

The solution with a risk-free asset – Alternative derivation

An investor minimizing the portfolio variance subject to a return constraint will find the same solution:

$$\begin{cases} \min\limits_{\alpha} \ \sigma_p^2 = \alpha' \Sigma \, \alpha \\ \text{s.t.} \ \ \mu_p = \alpha' \mu + (1 - e'\alpha) R_f \geq \tilde{\mu}_p \end{cases} \quad \text{or} \quad \begin{cases} \max\limits_{\alpha} \ \mu_p = \alpha' \mu + (1 - e'\alpha) R_f \\ \text{s.t.} \quad \sigma_p = \sqrt{\alpha' \Sigma \, \alpha} \leq \tilde{\sigma}_p \end{cases}$$

Proposition: If there is a risk-free asset, the weights of the risky assets in the **optimal portfolio** for a required return $\tilde{\mu}_p$ are:

$$\alpha^* = \frac{\tilde{\mu}_p - R_f}{\mu^{e\prime} \Sigma^{-1} \mu^e} \Sigma^{-1} \mu^e = \frac{\gamma}{2} \Sigma^{-1} \mu^e$$

where $\mu^e = \mu - R_f e$ is the vector of excess returns and γ is the Lagrange multiplier.

Mean-Variance Case (Tobin, 1958)

The solution with a risk-free asset – Alternative derivation

Tobin's Two-fund Separation Theorem: Every mean-variance efficient portfolio is a combination of the risk-free asset and the **tangency portfolio** with weight:

$$\alpha_T = \frac{\alpha^*}{e'\alpha^*} = \frac{\Sigma^{-1}\mu^e}{e'\Sigma^{-1}\mu^e} \qquad \text{where} \qquad \mu^e = \mu - R_f\, e$$

The tangency portfolio is also characterized by:

$$\mu_T - R_f = \alpha_T'\mu^e = \frac{\mu^{e\prime}\Sigma^{-1}\mu^e}{e'\Sigma^{-1}\mu^e} \qquad \text{and} \qquad \sigma_T^2 = \alpha_T'\Sigma\,\alpha_T = \frac{\mu^{e\prime}\Sigma^{-1}\mu^e}{(e'\Sigma^{-1}\mu^e)^2}$$

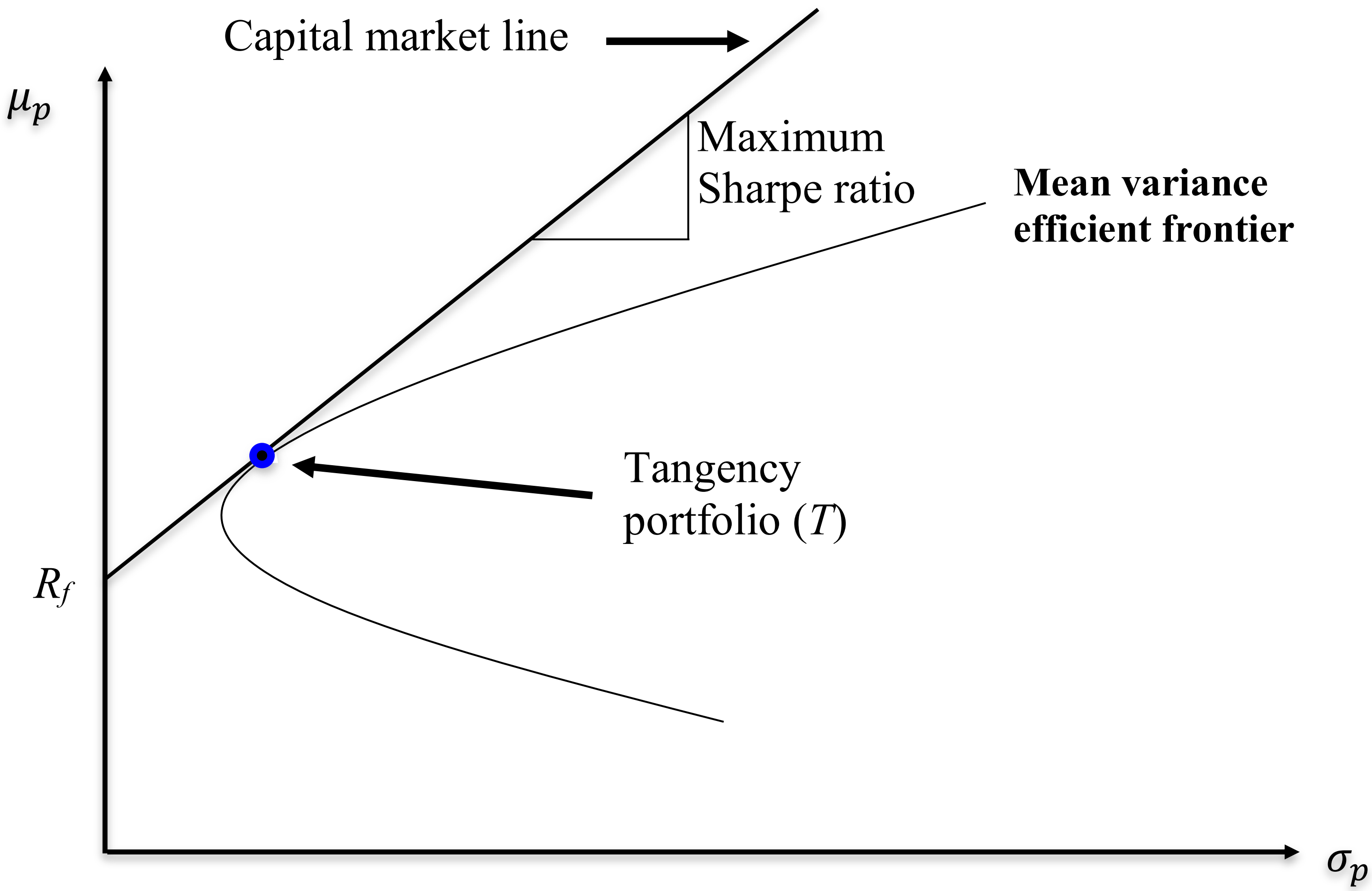
The risk-adjusted performance (Sharpe ratio) is

$$\frac{\mu_T - R_f}{\sigma_T} = \sqrt{\left(\mu - R_f e\right)'\Sigma^{-1}(\mu - R_f e)}$$

Remark: The tangency portfolio is the risky portfolio with the maximum Sharpe ratio

Mean-Variance Case (Tobin, 1958)

Mean-variance efficient frontier



Remark: The tangency portfolio is the risky portfolio with the maximum Sharpe ratio. It is often referred to as the **market portfolio** (true at equilibrium).

Objectives of the Lecture

- Optimal Allocation

➔ **Capital Asset Pricing Model**

- Market Efficiency

- Predicting Risk Premia and the Covariance Matrix

Market Equilibrium and CAPM

In the previous analysis, expected excess returns are given and assumed to be known by all investors.

In fact, μ^e should result from the confrontation of supply and demand, i.e., it should be consistent with an equilibrium model.

The CAPM (Capital Asset Pricing Model) provides such an equilibrium model and established where expected excess returns are coming from.

The CAPM has been extended to allow for more factors.

Market Equilibrium and CAPM

Assumptions

A1. Investors are **price-takers** (believe that security prices are unaffected by their own trades)

A2. All investors plan for one **identical holding period**. This behavior is myopic in that it ignores everything that might happen after the end of the single-period horizon

A3. Investments are limited to a **universe of publicly traded financial assets**, such as stocks and bonds, and to risk-free borrowing or lending arrangement. Investors may borrow or lend any amount at the risk-free rate

A4. Investors pay **no taxes** on returns and **no transaction costs** on trades in securities

A5. All investors are **rational mean-variance optimizers**, meaning that they all use the Markowitz/Tobin portfolio selection model.

A6. **Homogeneous expectations**: All investors use the same expected returns and covariance matrix of security returns to generate the efficient frontier and the unique optimal risky portfolio. They may have different aversion to risk.

Market Equilibrium and CAPM

Results

R1. All investors will choose to hold the same **market portfolio** (M), which is a market-value-weighted portfolio of all existing securities

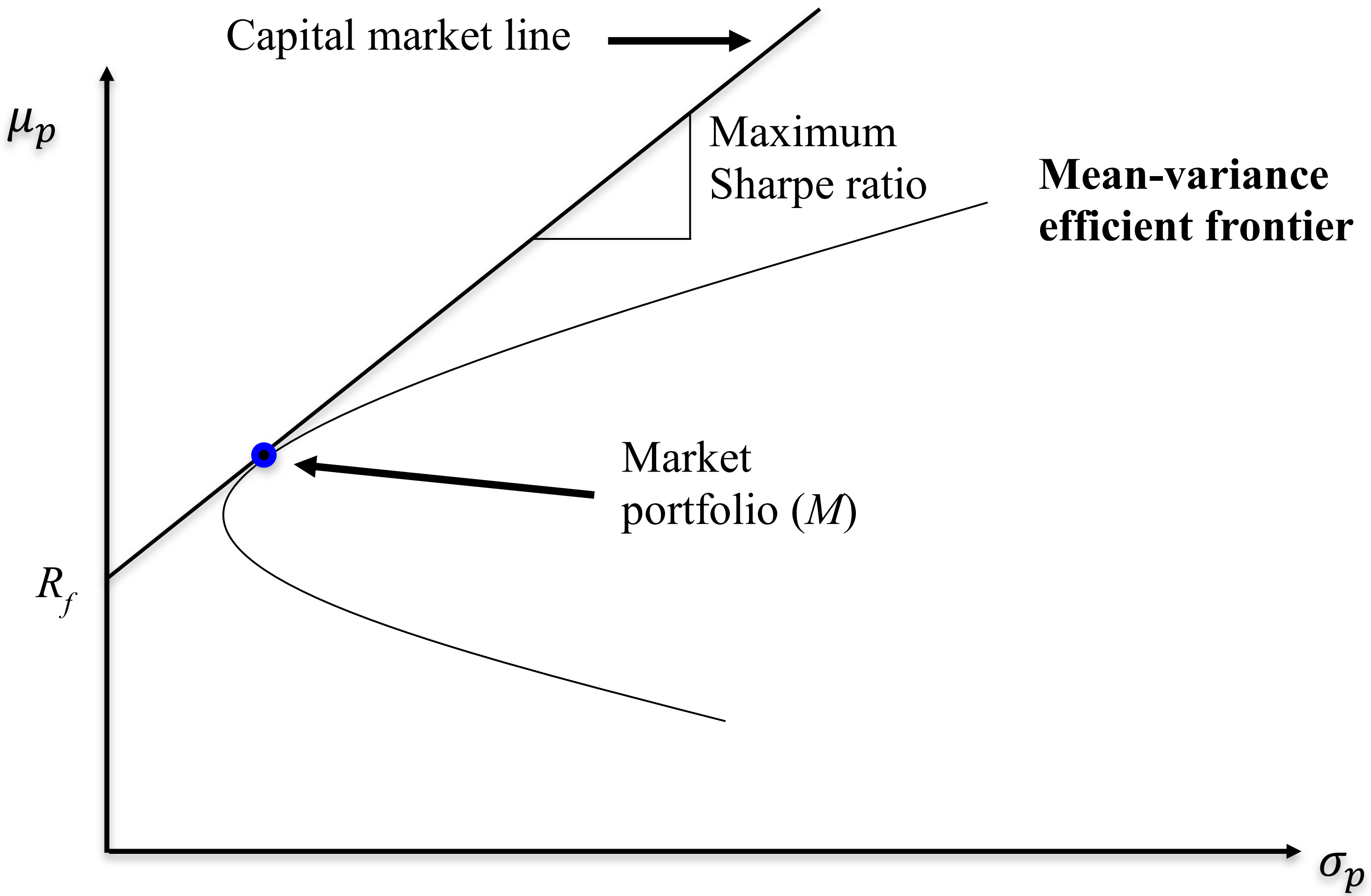
R2. Not only will the market portfolio be on the efficient frontier, but it also will be the tangency portfolio to the optimal **capital allocation line** derived by each investor. As a result, the **capital market line (CML)**, the line from the risk-free rate through the market portfolio, M, is also the best attainable capital allocation line.

All investors hold M as their optimal risky portfolio, differing only in the amount invested in it versus in the risk-free asset.

R3. The **risk premium on the market portfolio** is proportional to its risk and the degree of risk aversion of the representative investor:

$$E\left[R_{m,t+1}\right] - R_{f,t} = \lambda\, \sigma_m^2 \qquad (\alpha^* = 1)$$

Market Equilibrium and CAPM



Market Equilibrium and CAPM

Results (cont'd)

R4. The **risk premium on individual assets** is proportional the risk premium on the market portfolio, *M*, and the beta coefficient of the security relative to the market.

- There is only one risk factor for the assets: their **correlation with the market portfolio** (with return $R_{m,t}$)
- This correlation is measured by the **beta parameter**, $\beta_i = \frac{cov[R_{i,t+1},R_{m,t+1}]}{V[R_{m,t+1}]} = \frac{e_i'\Sigma\alpha}{\alpha'\Sigma\alpha}$
- At the equilibrium, the **expected return** (or **risk premium**) of individual asset *i* is (price times quantity of risk)

or risk premium:
reward for thaking the risk

$$\underbrace{E\left[R_{i,t+1}\right] - R_{f,t}}_{\text{Expected excess return of asset } i} = \underbrace{\frac{E\left[R_{m,t+1} - R_{f,t}\right]}{\sigma_m}}_{\text{Price of risk}} \times \underbrace{\frac{cov\left[R_{i,t+1}, R_{m,t+1}\right]}{\sigma_m}}_{\text{Systematic risk of asset } i}$$

$$E\left[R_{i,t+1}\right] - R_{f,t} = \frac{cov\left[R_{i,t+1}, R_{m,t+1}\right]}{V\left[R_{m,t+1}\right]} E\left[R_{m,t+1} - R_{f,t}\right] = \beta_i E\left[R_{m,t+1} - R_{f,t}\right]$$

CAPM: The risk premium of asset *i* is equal to its beta × the excess return of the market portfolio. Assets with high systematic risk (high beta) offer high expected return

CAPM and Multi-factor Models

CAPM was first published by Sharpe (1964) and Lintner (1965)

Limitations to CAPM

- Market Portfolio is not directly observable
- Research shows that other factors affect returns

Fama French Three-Factor Model:

- Market beta
- Size
- Book value relative to market value

Fund managers are often evaluated based on running the adjusted CAPM regression:

$$R_{p,t+1} - R_{f,t} = \alpha_p + \beta_p\left(R_{b,t+1} - R_{f,t}\right) + \varepsilon_{p,t+1}$$

where $R_{p,t+1}$ is the fund return and $R_{b,t+1}$ is the return of the benchmark of the fund

Passive management = beta **versus** **Active management = alpha**

Objectives of the Lecture

- Optimal Allocation
- Capital Asset Pricing Model

➔ **Market Efficiency**

- Predicting Risk Premia and the Covariance Matrix

Efficient Market Hypothesis (EMH)

In an **efficient** capital market, stock prices fully reflect available information (Fama, 1965)

Efficient Market Hypothesis (EMH) has implications for investors and firms:

- Since information is reflected in security prices quickly, knowing information *when it is released* does an investor no good.

- Firms should expect to receive the fair value for securities that they sell. Firms cannot profit from fooling investors in an efficient market.

Definitions of Market Efficiency

Three forms of efficiency:

1. Weak-form Efficiency: The information set includes only the history of prices and volumes. Using past prices or returns (***technical analysis***) does not yield a positive return. Weak-form efficiency is related to the hypothesis of random walk in prices or non-predictability of returns.

2. Semistrong-form Efficiency: The information set includes all information known to all market participants (***publicly available information***). Using public information (fundamental analysis) does not yield a positive abnormal return.

3. Strong-form Efficiency: The information set includes all information known to any market participants (***private information***). Even private information does not yield a positive abnormal return.

Definitions of Market Efficiency

Market efficiency does not imply that the return should be zero. Rather, it means that there is no profit beyond the **normally required return**.

- Normal returns are defined by your preferred asset pricing model. For instance, they may be defined by the CAPM:

$$E\left[R_{i,t+1}\right] = R_{f,t} + \beta_i E\left[R_{m,t+1} - R_{f,t}\right]$$

- Then, **abnormal returns** ($\varepsilon_{i,t+1}$) are computed as the difference between the return on a security and its normal return.

- Forecasts of the abnormal return are made using the chosen information set (for instance, past returns). If the abnormal return is forecastable, then the EMH is rejected.

Tests of the EMH are in fact joint tests because they test (1) the EMH and (2) the model for normal returns. Therefore, if the joint hypothesis is rejected, it may be due to the rejection of the equilibrium model or to the rejection of the EMH.

Implications of Efficient Market Hypothesis

Time-series implications

- A **stock price** (including dividends) is a **random walk** with a positive trend (compensation for risk taking)

- **Expected excess returns** (risk premia) are given by an equilibrium model (e.g., CAPM)

$$\underbrace{R_{i,t+1} - R_{f,t}}_{\text{Excess return}} = \underbrace{E[R_{i,t+1} - R_{f,t}]}_{\text{Expected excess return } (\mu_i^e)} + \underbrace{\varepsilon_{i,t+1}}_{\text{Abnormal return}}$$

- As the flow of information is random, **abnormal returns** are **unpredictable** conditional on a given information set (depending on the version of the EMH)

- There is **no free lunch**: in efficient markets, underpriced or overpriced securities do not exist, so it is not possible to design a strategy that generates a systematic abnormal return

Timing the market does not work

Implications of Efficient Market Hypothesis

Investment implications

- **Fundamental analysis** does not work because the quality of firms is already public. The only form that can work is security analysis: find firms that are better than everyone else's estimated

- Under EMH, active management is wasted effort and costly

- The only worthy strategy is **passive management**:

  - Investors should invest in a buy-and-hold strategy

  - Investors should invest in a highly diversified portfolio (market portfolio)

  - Investment firms should create an index fund and the fund manager should only tailor the portfolio to the needs of the investors

Efficient Market Hypothesis (EMH)

Evidence

1. The weak form of EMH is not rejected by the data

2. The performance of professional managers is broadly consistent with market efficiency (semi-strong form). The amounts by which professional managers as a group beat or are beaten by the market fall within the margin of statistical uncertainty.

3. Markets are very efficient, but especially diligent, intelligent, or creative investors may probably make more money than the average investor.

4. Having access to private information is valuable. In general, it is prohibited (insider trading). What about high frequency trading?

Record of U.S. Mutual Funds

Average annual return performance of U.S. equity mutual funds relative to the S&P 500, since inception

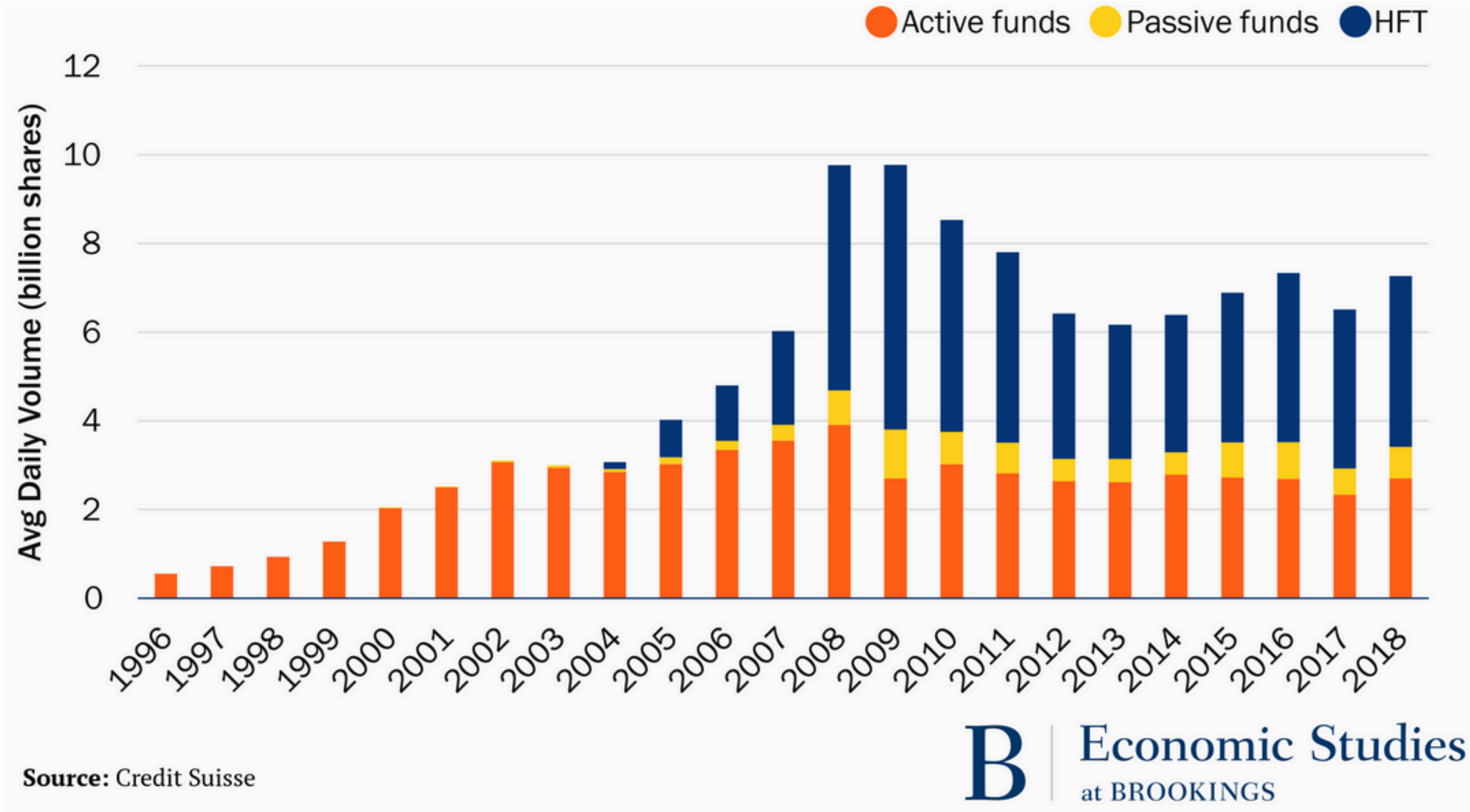
Equity Funds - Life Time Performance						
Inception Years	Age	No. of Funds	Beat SP500		Beat T-Bill	
1924 - 1962	50+	59	26	44%	57	97%
1963 - 1972	40+	58	31	53%	57	98%
1973 - 1982	30+	75	23	31%	74	99%
1983 - 1987	25+	195	91	47%	191	98%
1988 - 1992	20+	277	115	42%	267	96%
1993 - 1997	15+	739	407	55%	715	97%
1998 - 2000	12+	606	411	68%	432	71%
2001 - 2002	10+	324	204	63%	290	90%
2003 - 2005	7+	475	247	52%	420	88%
2006 - 2007	5+	675	240	36%	349	52%
2008 - 2009	3+	592	158	27%	376	64%
2010 - 2011	1+	824	124	15%	467	57%
2012	New	349	99	28%	208	60%
	Total	5248	2176	41%	3903	74%

*Based on monthly returns since inception or Jan '62

http://www.mutualfundobserver.com/2012/12/mutual-funds-that-beat-the-market/

Efficient Market Hypothesis (EMH)

High frequency trading (average daily volume of U.S. equity)



Objectives of the lecture

- Optimal Allocation
- Capital Asset Pricing Model
- Market Efficiency

➔ **Predicting Risk Premia and the Covariance Matrix**

Estimating Risk Premia: Unconditional Mean

Given historical prices (including dividends) $\{P_0, P_1, \ldots, P_T\}$, we compute

$$R_{t+1} = \frac{P_{t+1} - P_t}{P_{t+1}} \qquad t = 0, \ldots, T-1$$

Using the log-risk-free rate, $R_{f,t}$, we obtain excess returns: $R^e_{t+1} = R_{t+1} - R_{f,t}$

We then estimate unconditional risk premia as:

$$\hat{\mu}^e = \frac{1}{T}\sum_{t=1}^{T} R^e_t$$

Properties:

- easy to compute and update
- assumes constant risk premia
- with iid normal returns, the central limit theorem states that $\sqrt{T}(\hat{\mu}^e - \mu^e) \sim N(0, \Sigma)$

Estimating Risk Premia: Unconditional Mean

Example:

Assume $\hat{\mu}^e = 6\%$, $\sigma = 15\%$ and $T = 120$ (10 years of monthly data)

The standard error of the sample mean is

$$std[\hat{\mu}^e] = \sigma/\sqrt{T} = 15\%/\sqrt{120} = 1.4\%$$

This implies a 95% confidence interval of

$$\left[\hat{\mu}^e \pm 2\, std[\hat{\mu}^e]\right] = [3.25\% \,;\, 8.75\%]$$

Risk premia are difficult to estimate precisely.

Remark: Most models are designed for daily returns. However, the asset allocation is often performed at the weekly or monthly frequency (→ temporal aggregation).

Estimating Risk Premia: Linear Conditional Mean

Conditional risk premia for next period can be defined as linear functions of current macro-economic variables or firm-specific variables:

- Time series:

$$\hat{\mu}^e_{i,t+1} = a_i + b_i' Z_t \qquad t = 1, \ldots, T$$

where Z_t may include business cycle indicators, such as interest rate, inflation rate, or dividend yield.

- Cross section:

$$\hat{\mu}^e_{i,t+1} = a_i + b_i' Z_{i,t} \qquad t = 1, \ldots, T, i = 1, \ldots, N$$

where $Z_{i,t}$ may be the market beta $\beta_{i,t}$.

Factor Fishing

- **Theory**
  - Market portfolio (CAPM) or Intertemporal hedge portfolios: portfolios maximally correlated with changes in investment opportunities (I-CAPM)

- **Fundamental factors (accounting-based)**
  - Size
  - Book to market
  - Earnings to market
  - Cash-flow to market
  - Dividend yield
  - Industry factors

- **Macroeconomic factors**
  - Default premium
  - Term premium
  - Industrial production
  - Inflation

- **Statistical procedure**
  - Factor analysis or Principal components

Estimating Covariance Matrix: Unconditional Moment

Given excess returns, $\{R_1^e, \dots, R_T^e\}$, the sample covariance matrix can be estimated as:

$$\sigma_i^2 = V[R_i^e] = \frac{1}{T-1}\sum_{t=1}^{T}\left(R_{i,t}^e - \hat{\mu}_i^e\right)^2$$

$$\sigma_{ij} = Cov\left[R_i^e, R_j^e\right] = \frac{1}{T-1}\sum_{t=1}^{T}\left(R_{i,t}^e - \hat{\mu}_i^e\right)\left(R_{j,t}^e - \hat{\mu}_j^e\right)$$

Properties:

- easy to compute and update
- assumes constant return distribution

However:

- the horizon of the allocation is not necessarily the same as the frequency of the data
- we know that volatilities and correlations are time varying.

Estimating Covariance Matrix: Unconditional Moment

For N assets, the number of parameters to estimate is $N(N+1)/2$, the number of observations is $N\,T$

- For $N = 500$ stocks and $T = 600$ months of data (50 years): $N(N+1)/2 = 125'250$ unique parameters and $N\,T = 300'000$ observations, so that each parameter is estimated with 2.4 observations on average.

- The sample covariance matrix is singular when $N > T - 1$.

Estimating Time-Varying Moments: Rolling Window

An alternative approach that accounts for time-variability is the use of **Rolling Windows**

- Consider the first τ observations ($\tau \ll T$)

- Estimate risk premia ($\hat{\mu}_\tau^e$) and the covariance matrix (Σ_τ) using $t = 1, \dots, \tau$

$$\hat{\mu}_\tau^e = \frac{1}{\tau} \sum_{t=1}^{\tau} R_t^e \qquad \Sigma_\tau = \frac{1}{\tau - 1} \sum_{t=1}^{\tau} (R_t^e - \hat{\mu}_\tau^e)(R_t^e - \hat{\mu}_\tau^e)'$$

- Roll the sample by 1 observation

- Estimate risk premia ($\hat{\mu}_{\tau+1}^e$) and the covariance matrix ($\Sigma_{\tau+1}$) using $t = 2, \dots, \tau + 1$

- …

- Estimate risk premia ($\hat{\mu}_T^e$) and the covariance matrix (Σ_T) using $t = T - \tau + 1, \dots, T$

Estimating Time-Varying Moments: EWMA

A slightly better estimator is the **Exponentially Weighted Moving Average (EWMA)**

- Consider a memory parameter ϕ $(0 \leq \phi \leq 1)$, an initial risk premium $(\hat{\mu}_0^e)$, and an initial covariance matrix Σ_0. $\hat{\mu}_0^e$ and Σ_0 can be based on a pre-sample.

- At date τ, compute the risk premium and the covariance matrix as:

$$\hat{\mu}_\tau^e = \phi\, \hat{\mu}_{\tau-1}^e + (1-\phi) R_\tau^e$$

$$\Sigma_\tau = \phi \Sigma_{\tau-1} + (1-\phi)(R_\tau^e - \hat{\mu}_\tau^e)(R_\tau^e - \hat{\mu}_\tau^e)'$$

Remark: Problems related to the number of observations are even more severe.