Double Oracle Algorithm

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Solving Large Two-Player Zero-Sum Games

1. Matrix Games
2. Classical Approach: Linear Programming
3. Scalable Approach: Double Oracle Algorithm
4. DO Algorithm Beyond Matrix Games

Matrix Games

Two-Player Zero-Sum Games

Two-player zero-sum game is a triple $(\mathcal{A}_1, \mathcal{A}_2, u)$ where

- $\mathcal{A}_1$ and $\mathcal{A}_2$ are strategy sets,
- $u, -u : \mathcal{A}_1 \times \mathcal{A}_2 \to \mathbb{R}$ are utility functions of Player 1 and Player 2, respectively.

Matrix game is a two-player zero-sum game in which $\mathcal{A}_1$ and $\mathcal{A}_2$ are finite.

	r	p	s
r	0	−1	1
p	1	0	−1
s	−1	1	0

Mixed Strategies

For $k = 1, 2$:

Mixed strategy of Player k is a probability distribution $\pi_k \in \Pi_k$ on $\mathcal{A}_k$.

Pure strategy of Player k is an element $a_k \in \mathcal{A}_k$.

Expected utility of Player 1 under a mixed strategy profile (π_1, π_2) is

$$u(\pi_1, \pi_2) := \sum_{a_1 \in \mathcal{A}_1} \sum_{a_2 \in \mathcal{A}_2} u(a_1, a_2) \cdot \pi_1(a_1)\pi_2(a_2).$$

Optimal Strategies

Minimax Theorem

$$\max_{\pi_1 \in \Pi_1} \min_{\pi_2 \in \Pi_2} u(\pi_1, \pi_2) = \min_{\pi_2 \in \Pi_2} \max_{\pi_1 \in \Pi_1} u(\pi_1, \pi_2) \quad =: v.$$

The following are equivalent for a mixed strategy profile (π_1^*, π_2^*).

- (π_1^*, π_2^*) is equilibrium,

$$\min_{\pi_2 \in \Pi_2} u(\pi_1^*, \pi_2) = v = \max_{\pi_1 \in \Pi_1} u(\pi_1, \pi_2^*).$$

- $u(\pi_1^*, \pi_2^*) = v$.

Classical Approach: Linear Programming

Computing Equilibrium

Player 1 solves

$$\max_{\pi_1 \in \Pi_1} \min_{a_2 \in \mathcal{A}_2} u(\pi_1, a_2)$$

as a primal LP:

Maximize v_1

subject to

$$u(\pi_1, a_2) \geq v_1 \quad \forall a_2 \in \mathcal{A}_2$$
$$\pi_1 \in \Pi_1$$
$$v_1 \in \mathbb{R}$$

Player 2 solves

$$\min_{\pi_2 \in \Pi_2} \max_{a_1 \in \mathcal{A}_1} u(a_1, \pi_2)$$

as the dual LP:

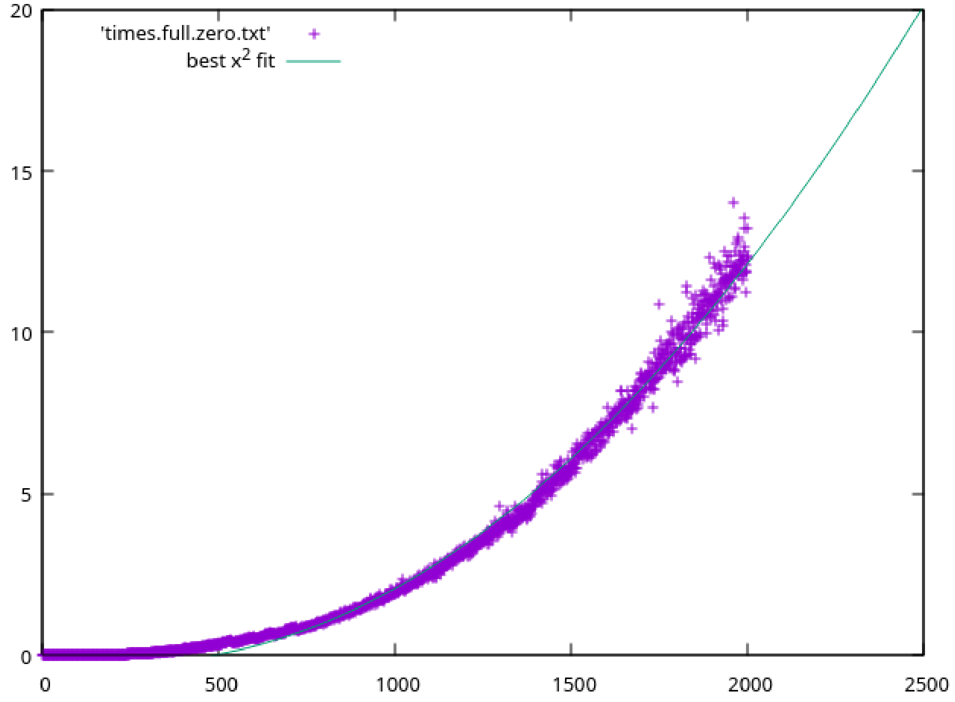
Minimize v_2

subject to

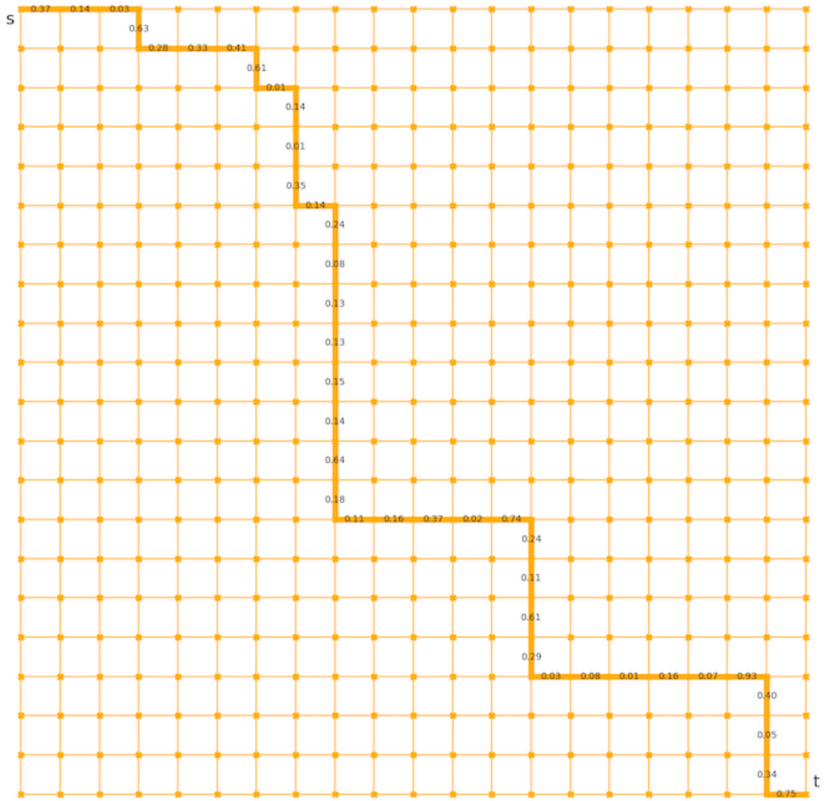
$$u(a_1, \pi_2) \leq v_2 \quad \forall a_1 \in \mathcal{A}_1$$
$$\pi_2 \in \Pi_2$$
$$v_2 \in \mathbb{R}$$

Numerical Experiments

- Julia + JuMP + Gurobi, randomly generated matrix games
- *Number of strategies* vs *Solve time*



Routing-Ambush on a Grid



Routing-Ambush Game (1)

Player 1 chooses a monotone path $p \in \mathcal{A}_1$ from the NW to the SE corner of a 21×21 grid, where each edge e is assigned a detection probability d_e:

$$|\mathcal{A}_1| = \binom{40}{20} \approx 10^{11}$$

Player 2 chooses an edge $e \in \mathcal{A}_2$ of the grid to ambush: $|\mathcal{A}_2| = 2 \cdot 21 \cdot 20 = 840$.

Utility function of Player 1 is

$$u(p, e) := \begin{cases} -d_e & \text{if } e \in p \\ 0 & \text{otherwise} \end{cases}$$

Routing-Ambush Game (2)

The payoff matrix contains approximately 10^{13} entries, making its explicit construction infeasible and the computation of optimal strategies intractable

However, the following expected utilities can be computed efficiently:

- The expected utility of a path $p \in \mathcal{A}_1$ and a mixed strategy π_2,

$$u(p, \pi_2) = -\sum_{e \in p} \pi_2(e) \cdot d_e.$$

- The expected utility of a mixed strategy π_1 and an edge $e \in \mathcal{A}_2$,

$$u(\pi_1, e) = -d_e \cdot \sum_{p \ni e} \pi_1(p).$$

Why LP Can Struggle at Scale

- Large strategy spaces $\Longrightarrow$ huge numbers of constraints/variables
- State-of-the-art LP solvers (Gurobi, MOSEK, CPLEX) can handle extremely large problems with millions of variables in case that the matrix is structured/sparse
- However, many strategies never appear in equilibrium support!

Opportunity: target only the relevant strategies

Scalable Approach: Double Oracle Algorithm

Double Oracle: Main Ideas

H. B. McMahan, G. J. Gordon, and A. Blum. *Planning in the Presence of Cost Functions Controlled by an Adversary*, Proceedings of the 20th International Conference on Machine Learning (ICML-03). 2003.

- Maintain only restricted sets of strategies
- Solve the restricted game (subgame) to get an equilibrium
- Each player computes a best response to the opponent's mix
- Add violating strategies; repeat until no violations

Subgames

Let $(\mathcal{A}_1, \mathcal{A}_2, u)$ be a matrix game. A subgame of $(\mathcal{A}_1, \mathcal{A}_2, u)$ is a matrix game

$$(\mathcal{A}'_1, \mathcal{A}'_2, u)$$

such that $\mathcal{A}'_1 \subseteq \mathcal{A}_1$ and $\mathcal{A}'_2 \subseteq \mathcal{A}_2$ and u is restricted to $\mathcal{A}'_1 \times \mathcal{A}'_2$.

An example of a 3×2 *subgame of the* 4×5 *matrix game:*

2	9	−5	0	7
8	1	9	2	2
9	−1	0	0	8
−2	1	3	−4	7

Best Response

Given mixed strategies (π_1, π_2), the players can compute their best response maps,

$$\mathrm{BR}_1(\pi_2) := \arg\max_{a_1 \in \mathcal{A}_1} u(a_1, \pi_2) \qquad \text{and} \qquad \mathrm{BR}_2(\pi_1) := \arg\min_{a_2 \in \mathcal{A}_2} u(\pi_1, a_2).$$

Adding best responses to the pure strategies for the row and the column player:

2	9	−5	0	7
8	1	9	2	2
9	−1	0	0	8
−2	1	3	−4	7

2	9	−5	0	7
8	1	9	2	2
9	−1	0	0	8
−2	1	3	−4	7

2	9	−5	0	7
8	1	9	2	2
9	−1	0	0	8
−2	1	3	−4	7

BR Oracles in the Routing-Ambush Games

1. Given a mixed strategy $\pi_2 \in \Pi_2$, Player 1 solves

$$\text{Minimize} \quad \sum_{e \in p} \pi_2(e) \cdot d_e \quad \text{subject to} \;\; p \in \mathcal{A}_1$$

to find the optimal path (Dijkstra's algorithm).

2. Given a mixed strategy $\pi_1 \in \Pi_1$, Player 2 solves

$$\text{Maximize} \quad d_e \cdot \sum_{p \ni e} \pi_1(p) \quad \text{subject to} \;\; e \in \mathcal{A}_2$$

to find the optimal edge (scan over edges).

DO Algorithm

Input: $(\mathcal{A}_1, \mathcal{A}_2, u)$, $\mathcal{A}_1^1 \subseteq \mathcal{A}_1$, $\mathcal{A}_2^1 \subseteq \mathcal{A}_2$, and $i = 0$

---

repeat

  $i \leftarrow i + 1$

  Compute an equilibrium $\left(\pi_1^i, \pi_2^i\right)$ for the subgame $\left(\mathcal{A}_1^i, \mathcal{A}_2^i, u\right)$

  Select best responses $a_1^{i+1} \in \mathrm{BR}_1\left(\pi_2^i\right)$ and $a_2^{i+1} \in \mathrm{BR}_2\left(\pi_1^i\right)$

  $\mathcal{A}_1^{i+1} \leftarrow \mathcal{A}_1^i \cup \left\{a_1^{i+1}\right\}$ and $\mathcal{A}_2^{i+1} \leftarrow \mathcal{A}_2^i \cup \left\{a_2^{i+1}\right\}$

until $a_1^{i+1} \in \mathcal{A}_1^i$ and $a_2^{i+1} \in \mathcal{A}_2^i$

return $\left(\pi_1^i, \pi_2^i\right)$

Bounds for the Value

- At iteration i, compute

$$v_\ell^i := u\left(\pi_1^i, a_2^{i+1}\right) \qquad \text{and} \qquad v_u^i := u\left(a_1^{i+1}, \pi_2^i\right)$$

- Those are lower and upper bounds for the unknown value v of the initial game,

$$v_\ell^i \le v \le v_u^i$$

- Exploitability of $\left(\pi_1^i, \pi_2^i\right)$ is defined as

$$\operatorname{expl}\left(\pi_1^i, \pi_2^i\right) := \frac{1}{2}\left(v_u^i - v_\ell^i\right)$$

Termination Based on Exact Equlibrium Conditions

Proposition

The following are equivalent:

1. $a_1^{i+1} \in \mathcal{A}_1^i$ and $a_2^{i+1} \in \mathcal{A}_2^i$.
2. $\mathcal{A}_1^{i+1} = \mathcal{A}_1^i$ and $\mathcal{A}_2^{i+1} = \mathcal{A}_2^i$
3. $\mathrm{expl}\left(\pi_1^i, \pi_2^i\right) = 0$.

If one of the above conditions is satisfied at iteration i, the current strategy pair $\left(\pi_1^i, \pi_2^i\right)$ is an equilibrium of the initial matrix game $(\mathcal{A}_1, \mathcal{A}_2, u)$.

Approximate Equilibrium

Let $\varepsilon \geq 0$. A strategy pair (π_1^*, π_2^*) is an ε-equilibrium if

$$\max_{\pi_1 \in \Pi_1} u(\pi_1, \pi_2^*) - \varepsilon \leq u(\pi_1^*, \pi_2^*) \qquad \text{and} \qquad u(\pi_1^*, \pi_2^*) \leq \min_{\pi_2 \in \Pi_2} u(\pi_1^*, \pi_2) + \varepsilon.$$

Equivalently:

- $\max_{a_1 \in \mathcal{A}_1} u(a_1, \pi_2^*) - \varepsilon \leq u(\pi_1^*, \pi_2^*) \qquad \text{and} \qquad u(\pi_1^*, \pi_2^*) \leq \min_{a_2 \in \mathcal{A}_2} u(\pi_1^*, a_2) + \varepsilon.$
- $\mathrm{expl}(\pi_1^*, \pi_2^*) \leq \varepsilon$

DO Algorithm for Approximate Equilibrium

Input: $(\mathcal{A}_1, \mathcal{A}_2, u)$, $\mathcal{A}_1^1 \subseteq \mathcal{A}_1$, $\mathcal{A}_2^1 \subseteq \mathcal{A}_2$, $i = 0$, and $\varepsilon > 0$

---

repeat

  $i \leftarrow i + 1$

  Compute the optimal strategies $\left(\pi_1^i, \pi_2^i\right)$ for the subgame $\left(\mathcal{A}_1^i, \mathcal{A}_2^i, u\right)$

  Select best responses $a_1^{i+1} \in \mathrm{BR}_1\left(\pi_2^i\right)$ and $a_2^{i+1} \in \mathrm{BR}_2\left(\pi_1^i\right)$

  $\mathcal{A}_1^{i+1} \leftarrow \mathcal{A}_1^i \cup \left\{a_1^{i+1}\right\}$ and $\mathcal{A}_2^{i+1} \leftarrow \mathcal{A}_2^i \cup \left\{a_2^{i+1}\right\}$

until $\mathrm{expl}\left(\pi_1^i, \pi_2^i\right) \leq \varepsilon$

return $\left(\pi_1^i, \pi_2^i\right)$

Termination Based on Exploitability

Proposition

DO algorithm terminates in finitely-many steps and returns an ε-equilibrium of the original matrix games.

- Worst-case additions of strategies $\leq |\mathcal{A}_1| + |\mathcal{A}_2|$
- The worst-case scenario — corresponding to the solution of the original matrix game — can actually occur...

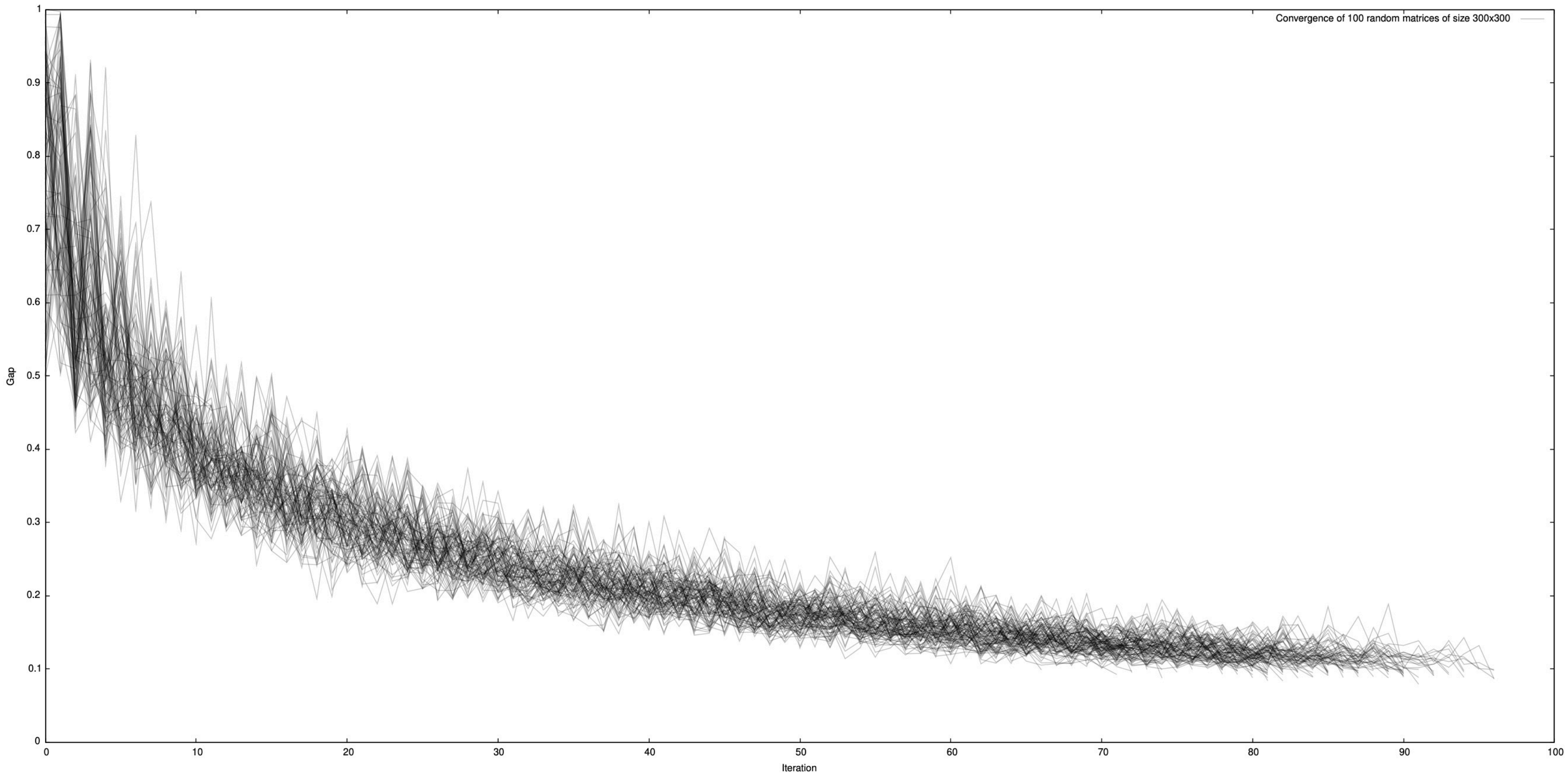
Worst-case Example

- Both players pick an integer from 1 to n. The higher number wins, and a tie occurs if they choose the same.
- The game has a unique ε-equilibrium "pick n" for any $0 \leq \varepsilon < 1$.

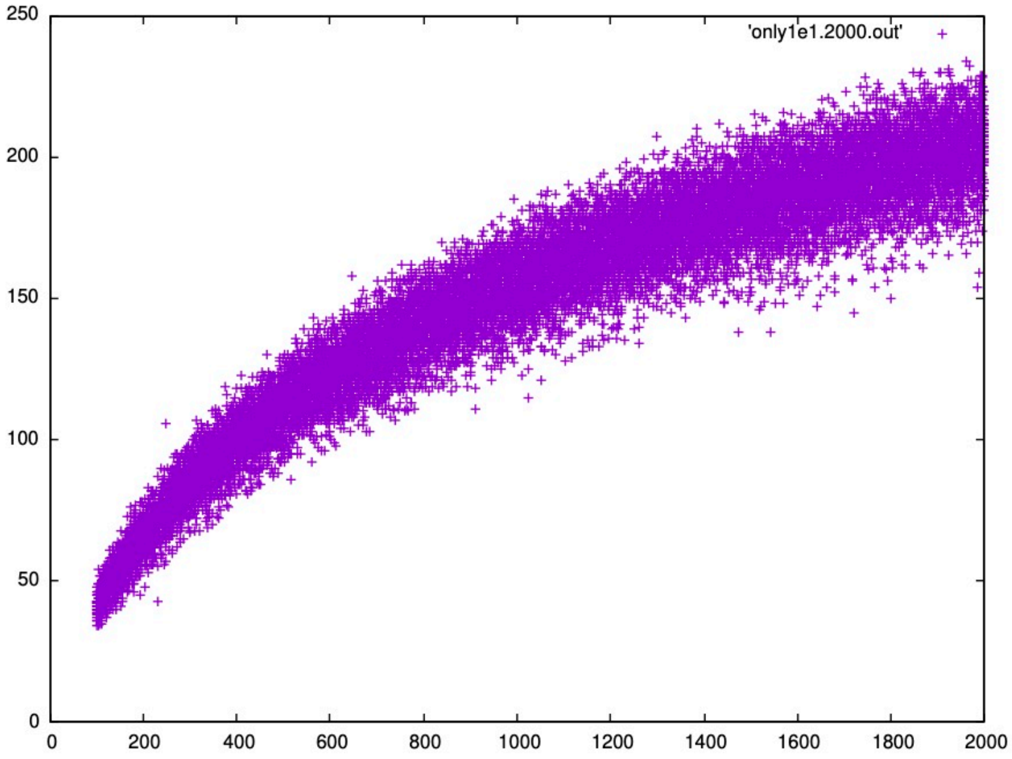
0	−1	−1	−1
1	0	−1	−1
1	1	0	−1
1	1	1	0

- Starting from $a_1 = a_2 = 1$ and selecting best responses by the smallest index, the DO algorithm converges in n steps (R. Horčík).

Iterations and convergence criterion for 300 ×300 games



The size of matrix and iterations



Small-Support ε-Equilibria Exist in Matrix Games

Theorem Let $\varepsilon > 0$. Every $n \times n$ matrix game has an ε-equilibrium in which each player mixes among at most

$$\frac{\log n}{\varepsilon^2} \text{ pure strategies.}$$

- Althöfer, Ingo. *On sparse approximations to randomized strategies and convex combinations.* Linear Algebra and its Applications 199 (1994): 339-355.
- Lipton, Richard J., and Neal E. Young. *Simple strategies for large zero-sum games with applications to complexity theory.* Proceedings of the 26th annual ACM symposium on Theory of computing, 1994.

DO Algorithm Beyond Matrix Games

Beyond Matrix Games

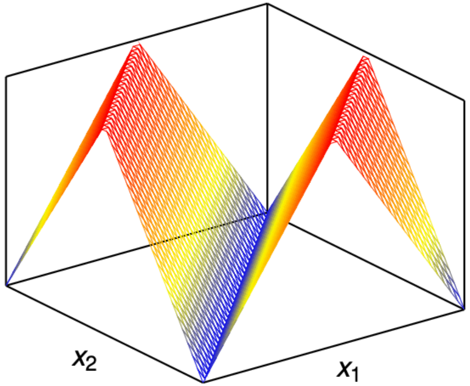
A two-player zero-sum game $(\mathcal{A}_1, \mathcal{A}_2, u)$ can have infinite strategy sets $\mathcal{A}_1$ and $\mathcal{A}_2$:

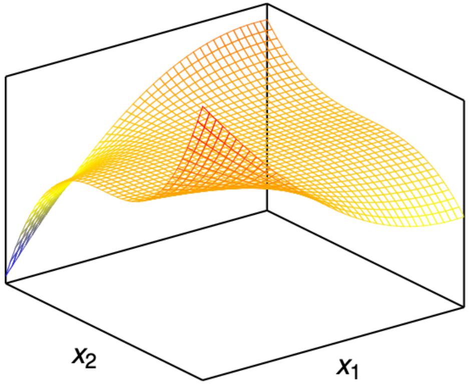
- Resource allocation problems
- Colonel Blotto game
- Routing/Ambush games in a continuous domain
- Normalized auction models
- Adversarial ML problems

Continuous Games

A two-player zero-sum game $(\mathcal{A}_1, \mathcal{A}_2, u)$ is continuous if

- the strategy sets $\mathcal{A}_1 \subseteq \mathbb{R}^m$ and $\mathcal{A}_2 \subseteq \mathbb{R}^n$ are compact and
- the utility function u is continuous over $\mathcal{A}_1 \times \mathcal{A}_2$.





Mixed Strategies in Continuous Games

A mixed strategy of Player k in a continuous game $(\mathcal{A}_1, \mathcal{A}_2, u)$ is a Borel probability measure π_k over $\mathcal{A}_k$, where $k = 1, 2$.

The expected utility $u(\pi_1, \pi_2)$ of Player 1 under a strategy pair (π_1, π_2) is the Lebesgue integral of u over $\mathcal{A}_1 \times \mathcal{A}_2$ with respect to $\pi_1 \times \pi_2$,

$$u(\pi_1, \pi_2) = \int_{\mathcal{A}_1 \times \mathcal{A}_2} u \, d(\pi_1 \times \pi_2).$$

How to Compute the Expected Utility?

- If π_1 and π_2 are point-supported measures corresponding to pure strategies a_1 and a_2, respectively, then

$$u(\pi_1, \pi_2) = u(a_1, a_2).$$

- If each π_k has finite support $\left\{a_k^1, \ldots, a_k^{\ell(k)}\right\} \subseteq \mathcal{A}_k$, then

$$u(\pi_1, \pi_2) = \sum_{i=1}^{\ell(1)} \sum_{j=1}^{\ell(2)} u\left(a_1^i, a_2^j\right) \cdot \pi_1\left(a_1^i\right)\pi_2\left(a_2^j\right).$$

Equilibria in Continuous Games

Minimax Theorem was extended to continuous games:

Glicksberg's Theorem (1952)

Every continuous game has an equilibrium (π_1^*, π_2^*) in mixed strategies.

Caveat: the equilibrium strategies may have infinite support!

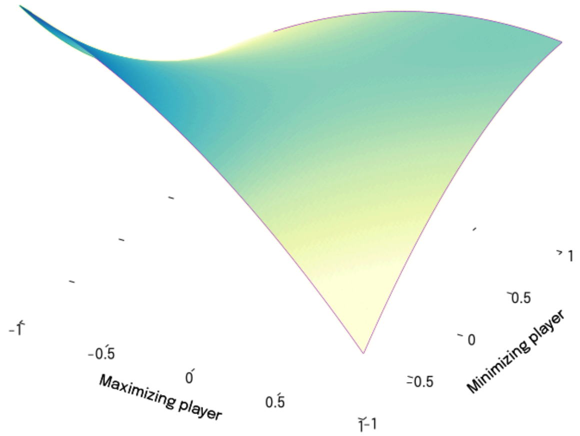
Proposition Let $\varepsilon > 0$. Every continuous game has an ε-equilibrium (π_1^*, π_2^*) in mixed strategies, where both π_1^* and π_2^* have finite supports.

DO Algorithm for Continuous Games

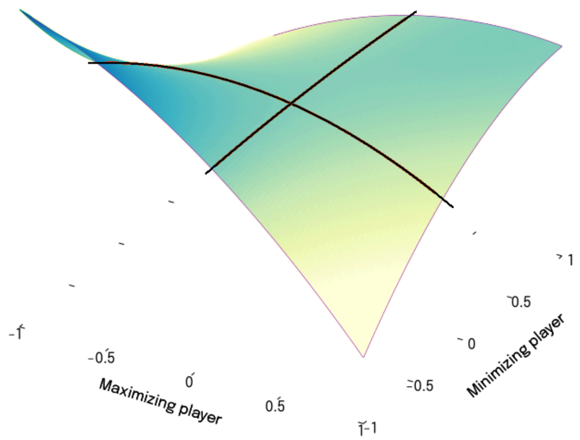
The DO algorithm can be used to approximate equilibria in continuous games with finite supports. Yet:

- Best responses require global optimization in continuous domains.
  - LP
  - MILP
  - Polynomial optimization
- Convergence is harder to analyze due to probability measures living in an infinite-dimensional simplex.

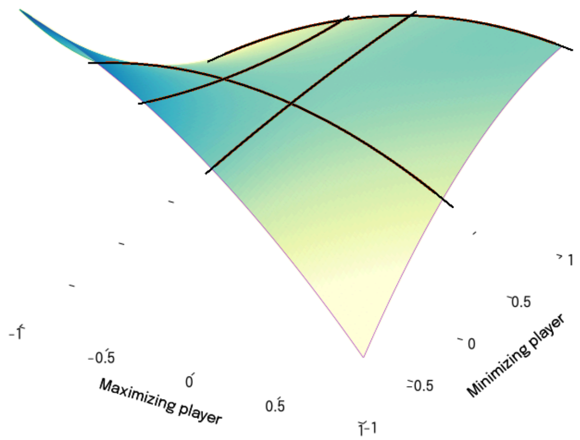
DO Algorithm for Continuous Games: Example



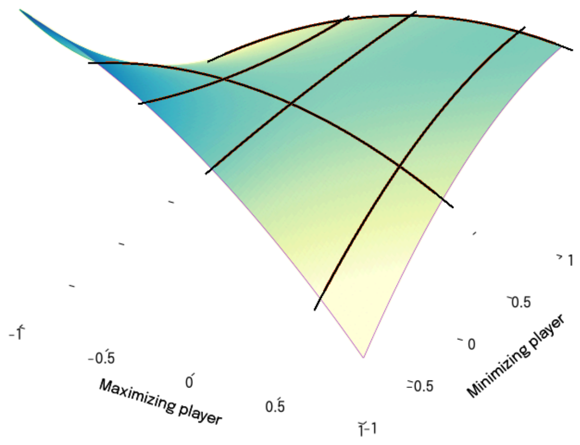
DO Algorithm for Continuous Games: Initialization



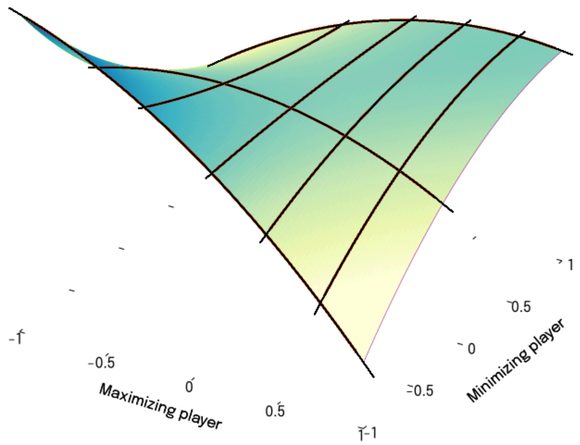
DO Algorithm for Continuous Games: Step 1



DO Algorithm for Continuous Games: Step 2



DO Algorithm for Continuous Games: Step 3



DO Algorithm for Continuous Games: Convergence

Theorem (L. Adam, R. Horčík, T. Kasl, and TK)

- Let $\varepsilon > 0$. The DO algorithm converges to a finitely-supported ε-equilibrium of a continuous game in finitely-many steps.
- Let $\varepsilon = 0$. If the DO algorithm stops at step i, then $\left(\pi_1^i, \pi_2^i\right)$ is an equilibrium.

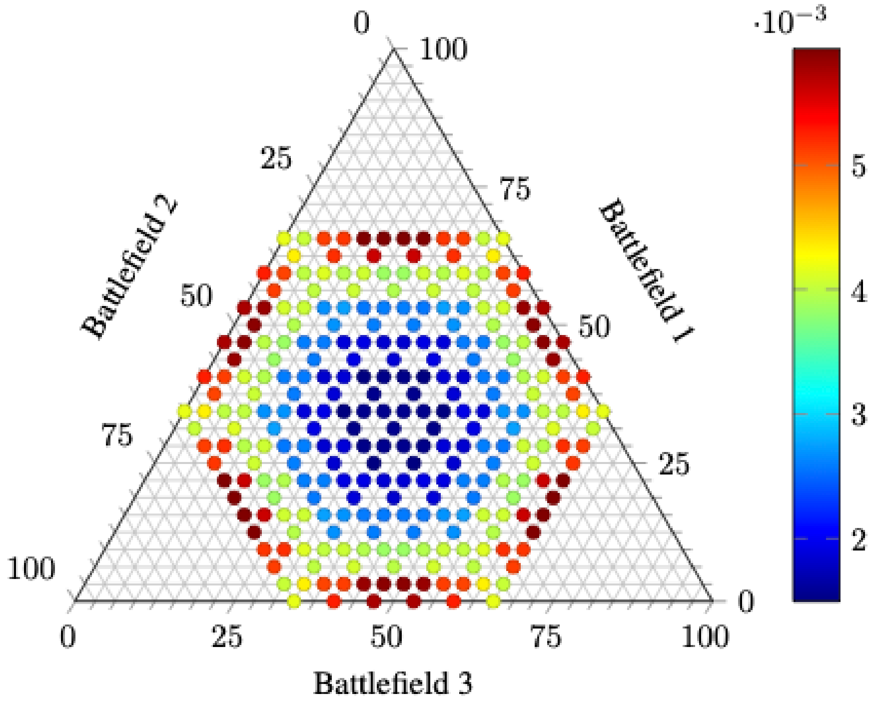
Blotto Games

- Two players simultaneously allocate forces across n battlefields
- Strategy spaces are standard n-simplices
- The utility function is

$$u(\boldsymbol{x}, \boldsymbol{y}) := \sum_{j=1}^{n} a_j \cdot f\left(x_j - y_j\right)$$

  where $a_1, \ldots, a_n > 0$ are weights of battlefields and f is a continuous PL function measuring the performance of the first army on a battlefield
- Best response computation can be formulated as an MILP

Blotto Games: ε-equilibrium



Outlook

- Stochastic best response oracles may improve behavior of DO algorithm for matrix games (R. Horčík)
- The convergence of DO algorithm is an open problem for games in which
  - strategy sets are unbounded (for example, $\mathbb{R}^n$) or
  - utility function u is discontinuous.

Both cases occur naturally: ML models often involve unbounded parameter spaces, and auction models become discontinuous because of tie-breaking.

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