Counting Techniques:
Chapter 3

Chapter Outcomes:

- Addition and multiplication rules
- Counting arrangements and permutations
- Counting subsets and combinations
- Number of arrangements when symbols are repeated

- Recall that if we specify a sample space

$S = \{a_1, a_2, \ldots, a_n\}$, where each simple event has probability $1/n$ (i.e. are equally likely), then:

- defining the compound event A which contains r points leads to $P(A) = r/n$.

- To define more complicated probabilities, we first need to review some basic ways to count outcomes from “experiments”.

Basic counting arguments:

1. Addition Rule:

Suppose Job 1 can be done in p ways and Job 2 can be done in q ways.

Then we can do EITHER Job 1 **OR** Job 2 in $p+q$ ways.

- "OR": interpreted as **addition**

2. The Multiplication Rule:

Suppose Job 1 can be done in *p* ways and, **for each of these ways**, Job 2 can be done *q* ways.

Then we can do BOTH Job 1 **AND** Job 2 in *pxq* ways.

- "AND": interpreted as **multiplication**

Sampling “With” and “Without” Replacement

- “With” replacement:

This means that every time an object is selected, it is put back into the pool of possible objects.

Example: Rolling a die, flipping a coin,……

- This method of selection implies that *what we get on the first selection* ***DOES NOT affect*** *what we get on subsequent selections*. More on this idea later in the course!

- “Without” replacement:

This means that every time an object is selected, it is NOT put back into the pool of possible objects.

- This implies that the same object cannot be selected more than once.

- Hence, what we get on the *first selection **WILL affect** what we get on subsequent selections.*

Example: A bag contains 3 blue and 5 red marbles.

a) What is the probability of selecting two blue marbles if the selection process is done ***with*** replacement?

b) What is the probability of selecting two blue marbles if the selection process is done ***without*** replacement?

Example: Suppose a fair die is tossed 3 times. What is the probability that only one of the tosses produced a number greater than 4?

- In many problems, the sample space of interest consists of a set of unique arrangements or sequences (order matters and selection is done without replacement). Often these are called ***permutations***.

Example: How many **different** ordered arrangements of the letters a, b, and c are possible if letters are randomly selected without replacement?

You try:

What if the letters were randomly selected with replacement?

Do you see how with / without replacement matters!

One can employ the following counting principles:

Suppose we have *n* **distinct** symbols, so that we can make:

- $n \times (n-1) \times \cdots \times 1 = \boldsymbol{n!}$ (read "*n* factorial") arrangements of **length *n***, where each symbol is used only once (i.e. the number of permutations of *n* distinct objects).

Notes:

$$n! = n * (n-1) * (n-2) * \cdots .* (2) * (1)$$

Also, $\mathrm{n}! = n \times (n-1)!$ for $n \geq 1$

$0! = 1$

Ex. Suppose I have 10 people to arrange but I have 6 chairs to seat them.

Q. How many possible arrangements can there?

A: $\underline{10}\ \underline{9}\ \underline{8}\ \underline{7}\ \underline{6}\ \underline{5} = 10^{(6)}$

Examples:

4! = 4 x 3 x 2 x 1 = 24

5! = 5 x 4 x 3 x 2 x 1 = 120

As noted above, we can write 5! as 5*4!, or n! as n*(n-1)!

$n^{(k)} = \frac{n!}{(n-k)!}$ $10^{(6)} = \frac{10!}{(10-6)!} = \frac{10!}{4!}$

- $n \times (n-1) \times \cdots \times (n-k+1) = \boldsymbol{n^{(k)}}$ for $k \leq n$ arrangements of **length *k*,** where each symbol is used once (i.e. the number of permutations of *n* distinct objects taken *k* at a time).

$n^{(k)} = \frac{n!}{(n-k)!} = {}_nP_k$ (read "*n* to *k* factors")

where $n^{(k)} = \frac{n!}{(n-k)!} = n(n-1)^{(k-1)}$ for $k \geq 1$

- $n \times n \times \cdots \times n = \boldsymbol{n^k}$

arrangements of length k using each symbol more than once.

Note: The arrangements presented, especially $n!$, tend to grow at an extremely fast rate as n gets larger and so **approximations** can be used.

n	0	1	2	3	4	5	6	7	8	9	10
$n!$	1	1	2	6	24	120	720	5040	40320	362880	3628800

Stirling's Approximation:

- For large *n*, this is an approximation to *n*! and it states:

$$n! \text{ is } \textit{asymptotically equivalent} \text{ to } \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

- What do we mean by *asymptotically equivalent*?

This implies that as $n \to \infty$, the approximation becomes better and better (i.e. $\lim\limits_{n\to\infty} \frac{\left(\frac{n}{e}\right)^n \sqrt{2\pi n}}{n!} = 1$).

In other words, the *percentage error* of the approximation approaches zero as *n* increases.

n	1	2	3	4	5	6	7	8	9	10
$n!$	1	2	6	24	120	720	5040	40320	362880	3628800
$\left(\frac{n}{e}\right)^n \sqrt{2\pi n}$	0.9	1.9	5.8	23.5	118.0	710.1	4980.4	39902.4	359536.9	3598695.6

Example: A PIN number of length 4 is formed by randomly selecting 4 digits from the set of digits {0, 1, 2, …, 9}. If selection is done with replacement, find the probability that:

total # of even pins: $10 \cdot 10 \cdot 10 \cdot 5$

a) the PIN number is even. total # of pin numbers: 10^4

$$P(\text{PIN number is even}) = \frac{10^3 \cdot 5}{10^4} = \frac{1}{2}$$

b) the PIN number contains at least one 1.

Aside:

P(at least one …)

= 1 − P(no …)

none

$$P(\text{PIN number contains at least one } 1)$$

$$= 1 - P(\text{PIN number contains no } 1\text{s})$$

$$= 1 - \frac{9^4}{10^4}$$

$= 0.3439$

\# of pins w/no 1s

$\underline{9} \times \underline{9} \times \underline{9} \times \underline{9}$

You Try:

- What happens if selection is done without replacement? Find the probability that:

a) the PIN number is even. $P(\text{Even}) = \frac{1}{2}$

$\underline{10} \times \underline{10} \times \underline{10} \times \underline{5}$

b) the PIN number contains at least one 1.

(d) $\dfrac{10 \times 3! \times 27^{(2)}}{30^{(5)}} = \dfrac{(10 \times 3!) \times 27 \times 26}{30^{(5)}}$

$= 0.0025$

- **Example:** Five separate awards (best scholarship, best leadership qualities, and so on) are to be presented to selected students from a pool of size 30. How many different outcomes are possible if:

a) a student can receive any number of awards?

$\underline{30}\ \underline{30}\ \underline{30}\ \underline{30}\ \underline{30} = 30^5$

b) each student can receive at most one award?

$\#$ of arrangements $= \underline{30}\ \underline{29}\ \underline{28}\ \underline{27}\ \underline{26} = 30^{(5)}$

$= \dfrac{30!}{(30-5)!} = \dfrac{30!}{25!}$

c) Suppose that I have 2 nephews and 1 niece in this pool of 30 students. What is the probability that each of my 3 relatives wins exactly one award under assumption (a)?

# of ways each of the three relatives wins exactly one award

d) What is this same probability under assumption (b)?

(c)

lets call the awards A, B, C, D, E

3! arrangement for each arrangement

R_1 R_2 R_3

ABC, ABD, ABE, BCD, BCE, ACD, ACE, BDE, ADE, CDE ← 10 arrangements

# of possible arrangements where relatives have exactly one

$P(\ldots) = \frac{10(3!)\,27^2}{\ldots}$

award $= 10 \times 3! \times 27^2$

total arrangements

# of arrangements per arrangement

$P(A) = \frac{10(3!)27^2}{30^5}$

R_1	R_2	R_3	27	27
A	B	C	D	E

You try:

a) In how many ways can 3 boys and 3 girls sit in a row?

b) In how many ways can 3 boys and 3 girls be seated if the boys and the girls are seated together?

c) In how many ways can 3 boys and 3 girls be seated if only the boys must sit together?

d) In how many ways can 3 boys and 3 girls be seated if no boys and no girls are allowed to sit together?

- Sometimes the outcomes in the sample space are subsets of a **fixed size**:
- **Example:** Suppose we randomly select a subset of 3 digits from {0,1,2,…,9}, giving rise to

$$S=\{(0,1,2),(0,1,3),\ldots,(7,8,9)\}.$$

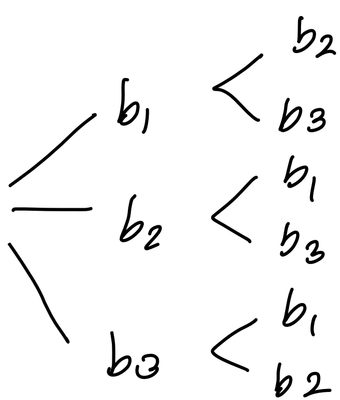
- We are now interested in finding the **number of subsets of size *k*** that can be formed from a **set of *n* distinct objects** when **order does not matter** and selection is done without replacement.

permutations, combinations → choosing

- Let us start with a slightly simpler example:
- Suppose we have **three books**: b_1, b_2, b_3.
- We choose **two** of the books to read.
- In how many ways can the two books be chosen if:

(permutation

1. The order the books are read matters:



$(b_1, b_2), (b_1, b_3)$
$(b_2, b_1)(b_2, b_3)$
$(b_3, b_1)(b_3, b_2)$

$$3^{(2)} = \frac{3!}{(3-2)!} = \frac{3!}{2!} = 6$$

Combinations

2. The order the books are read does not matter:

$(b_1, b_2), (b_1, b_3), (b_2, b_3) = 3$ outcomes

$$\binom{3}{2} = \frac{3!}{(3-2!)2!} = \frac{3!}{2!} = 3$$

- Now, let's return to our earlier example and adopt a similar strategy to find the total number of outcomes in *S*.

- **Recall:** Suppose we randomly select a subset of 3 digits from {0,1,2,…,9} when order does not matter.

(non-distinct)

If order mattered: $10^{(3)} = \frac{10!}{(10-3)!} = \frac{10!}{7!} = 10 \times 9 \times 8$

(distinct)

If order did not matter: $\binom{10}{8} = \frac{10!}{(10-3)!\,3!} = \frac{10!}{7!\,3!}$

Since order does not matter, we need to account for

Aside:

Ex. (1,2,3) only needs to be count once.

the over-counting → Recall: there would be 3! ways to arrange

- The previous results can be generalized as follows:
- The number of subsets of size *k* that can be chosen from a set of *n* distinct objects is given by:

$\binom{n}{k}$ is NOT the fraction $\frac{n}{k}$

$$\boxed{{}_nC_k = \binom{n}{k} = \frac{n^{(k)}}{k!} = \frac{n!}{k!(n-k)!}}$$

- $\binom{n}{k}$ is read as "*n* choose *k*" and represents the *combinatorial symbol*.

- Now, let’s use this idea to calculate some probabilities. Continuing with the previous example:
- Recall that we randomly select a subset of 3 digits from the set {0,1,2,…,9}.

- We found that 120 subsets of 3 digits can be formed. Therefore, if we randomly select a subset, then each subset has the same probability of selection, namely 1/120.

Example:

Find the probability that:

a) all the digits in the selected subset are even.

0, 1, 2, 3, 4, 5, 6, 7, 8, 9

5 evens and 5 odds.

probability is equal to $\dfrac{\binom{5}{3}}{\binom{10}{3}} = \dfrac{10}{120} = \dfrac{1}{12}$

b) at least one of the digits in the selected subset is less than or equal to 5.

We want P(<u>at least one</u> of the digits is ≤ 5)

$= 1 - P(\text{no digits are} \leq 5)$

$= 1 - P(\text{all digits are} \geq 6)$

$$= 1 - \frac{\binom{4}{3}}{\binom{10}{3}} = 1 - \frac{4}{120} = 1 - \frac{1}{30} = \frac{29}{30}$$

Event A

Sample S

order does not matter, hence we use combinations $\binom{n}{k}$

Example: A forest contains 30 moose, of which six were captured, tagged, and released. A certain time later, five of the 30 moose are randomly captured.

a) How many samples of size five are possible? $= \binom{30}{5}$

b) How many samples of size five, which include two of the six originally tagged moose, are possible? $= \binom{6}{2}\binom{24}{3}$

c) If the five captured moose represent a simple random sample drawn from the 30 moose (six of which are tagged), find the probability that (i) two of the five captured moose are tagged and (ii) none of the five captured moose is tagged.

(i) $\dfrac{\binom{6}{2}\binom{24}{3}}{\binom{30}{\ }}$

$\binom{30}{6}$

(ii) $\dfrac{\binom{6}{0}\binom{24}{6}}{\binom{30}{6}}$

- Properties of $\binom{n}{k}$:

1. $\binom{n}{k} = \dfrac{n!}{k!(n-k)!} = \dfrac{n^{(k)}}{k!}$

2. **Symmetry property:** $\binom{n}{k} = \binom{n}{n-k}$

for all $k = 0,1,\dots,n$ (i.e. $\binom{n}{k}$ is equal to the k^{th} entry in the n^{th} row of *Pascal's Triangle*.)

Pascal's Triangle:

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1 \\
 & & & & \vdots & & & &
\end{array}
$$

3. When *n* is not a non-negative integer $\geq k, \binom{n}{k}$ loses its physical meaning. If *n* is a real number and *k* is a non-negative integer, then we use

$$\binom{n}{k} = \frac{n^{(k)}}{k!} = \frac{n(n-1)\dots(n-k+1)}{k!}$$

Note also that when *n* and *k* are non-negative integers and $k > n$, then

$$\binom{n}{k} = \frac{n^{(k)}}{k!} = \frac{n(n-1)\dots(1)(0)\dots(n-k+1)}{k!} = 0$$

4. Recalling that 0! = 1, then

$$\binom{n}{0} = \binom{n}{n} = 1$$

Mathematically: $\binom{n}{0} = \binom{n}{n} = \frac{n!}{0!n!} = 1$

Logically: There is only one way to choose either none of the objects or all of the objects.

5. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

6. $(1+x)^n = \binom{n}{0}x^0 + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n,$ $x \, \varepsilon \, \mathbb{R}$

(This is the well-known ***Binomial Theorem***)

Next……

How do we determine the number of *ordered* arrangements when symbols are repeated (i.e. they are NOT all distinct)?

3S, 3T, 2I, 1A, 1C

Example: Suppose the letters of the word STATISTICS are arranged at random. Let's define G to be the event that the arrangement begins and ends with S. Find P(G).

Total # of arrangements $= \dfrac{10!}{3!\,3!\,2!\,1!\,1!} = \binom{10}{3}\binom{7}{3}\binom{4}{2}\binom{2}{1}\binom{1}{1}$

Ex. S _ _ _ _ _ _ _ _ S # of arrangements $= \frac{8!}{3!\,2!\,1!\,1!\,1!} = \frac{8!}{3!\,2!}$

The results of this STATISTICS example can be generalized in the following way:

- In general, if we have n_i symbols of type i, for $i = 1, 2, \dots, k$ with $n_1 + n_2 + \cdots + n_k = n$, then the number of distinct (i.e., ordered) arrangements using ALL of the symbols is:

$\therefore P(A) = \frac{8! \times 6}{10!} = \frac{1}{15}$

$$\binom{n}{n_1} \times \binom{n-n_1}{n_2} \times \binom{n-n_1-n_2}{n_3} \times \cdots \times \binom{n_k}{n_k}$$

$$= \frac{n!}{n_1!\,n_2!\,n_3!\dots n_k!} = \binom{n}{n_1\ \ n_2\ \dots\ n_k}$$

Example: Find the probability a bridge hand (13 cards randomly dealt from a standard deck of 52 cards without replacement) has:

a) at least 1 ace.

$$= 1 - P(\text{hand has no ace})$$

$$= 1 - \frac{\binom{48}{13}\binom{4}{0}}{\binom{52}{13}} = 0.696182$$

b) 6 spades, 4 hearts, 2 diamonds, and 1 club.

$$P(6\spadesuit, 4\heartsuit, 2\diamondsuit, 1\clubsuit)$$

$$= \frac{\binom{13}{6}\binom{13}{4}\binom{13}{2}\binom{13}{1}}{\binom{52}{13}} = 0.0019592$$

c) a 6-4-2-1 split between the 4 suits.

P(6-4-2-1 split)

$$= 4!\left(\frac{\binom{13}{6}\binom{13}{4}\binom{13}{2}\binom{13}{1}}{\binom{52}{13}}\right)$$

we weren't given a specific suit

d) a 4-3-3-3 split between the 4 suits.

$$P(4\text{-}3\text{-}3\text{-}3 \text{ split})$$

$$= \frac{4!}{1!\,3!}\left(\frac{\binom{13}{6}\binom{13}{4}\binom{13}{2}\binom{13}{1}}{\binom{52}{13}}\right)$$

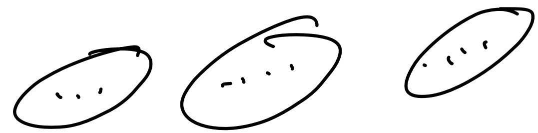
$$= 4\left(\frac{\binom{13}{6}\binom{13}{4}\binom{13}{2}\binom{13}{1}}{\binom{52}{13}}\right)$$

You Try:

$12 - 3 = 9$

- If 12 people are to be divided into 3 committees of respective sizes 3, 4, and 5, how many divisions are possible?

$$\binom{12}{3}\binom{9}{4}\binom{5}{5}$$



Example: A person has 8 friends, of whom 5 will be invited to a party.

a) How many choices are there if 2 friends are feuding and will not attend together?

# of choices = total # of choices − # of choices where they do not attend together

$$\binom{8}{5} - \binom{2}{2}\binom{6}{3} = 36$$

b) How many choices are there if 2 of the friends will only attend together?

$$= \binom{2}{2}\binom{6}{3} = 2$$

Example: There are 5 blue beads and 4 green beads to be arranged in a row on a string. The two ends of the string are not connected. Beads with the same colour are indistinguishable. Find the probability of the following events:

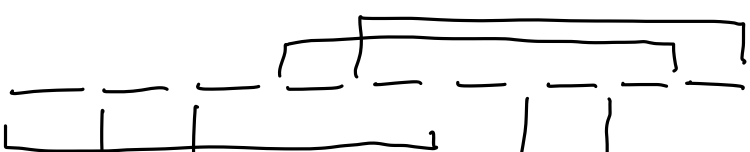
a) A = All 5 blue beads are adjacent to each other

5B, 4G

beads of the same colour are indistinguishable

total # of arrangements: $\binom{9}{5}$

the # of arrangements in A: 5



$P(A) = \frac{5}{\binom{9}{5}}$

b) B = None of the green beads are adjacent to any other green beads