Poisson Distribution (Section 5.7)

Facts:

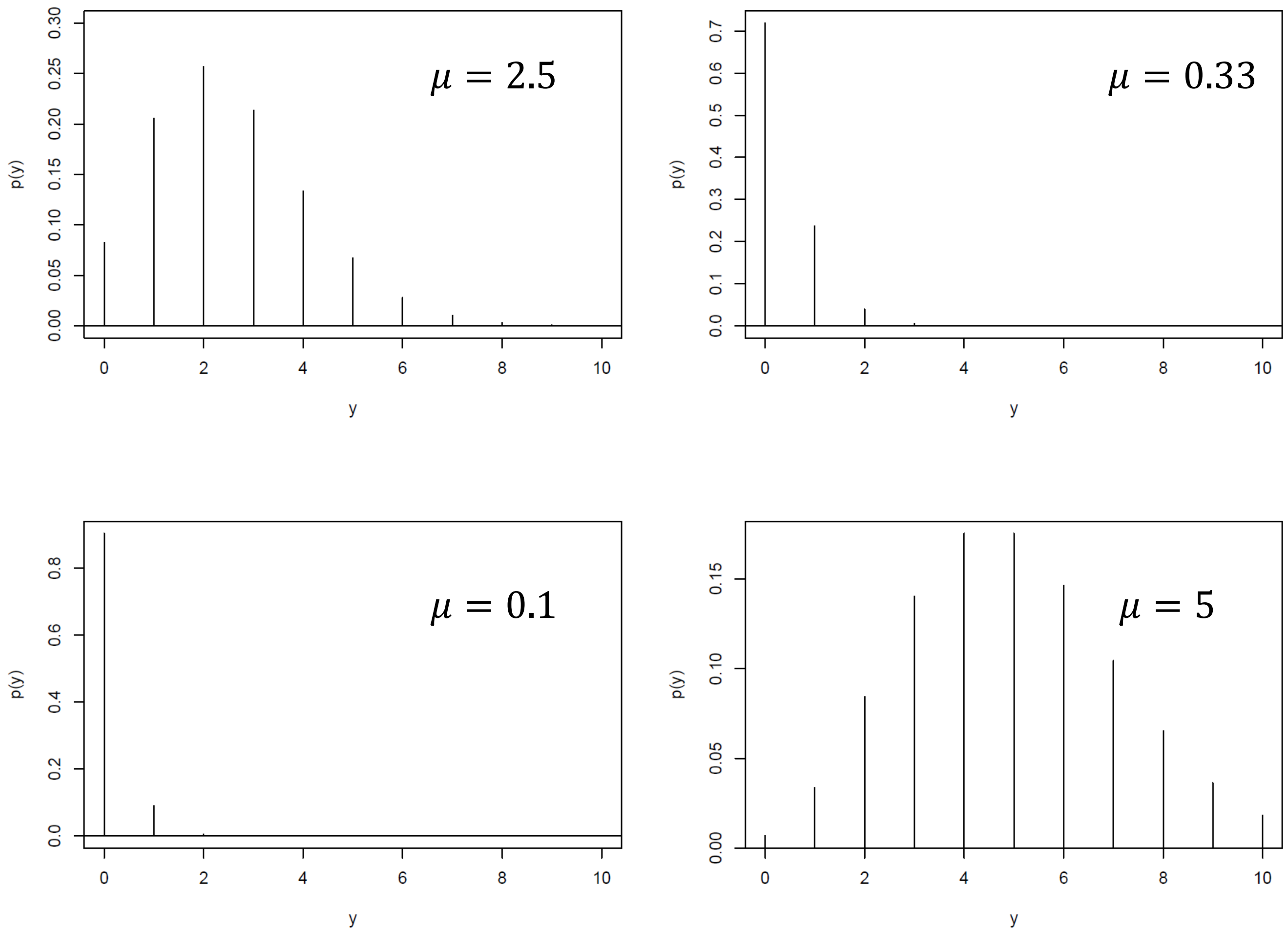
- In this setting, the r.v. X represents the number of events of some type.
- The events occur according to some **rate**, denoted by $\boldsymbol{\mu}$, where $\boldsymbol{\mu > 0}$.
- We write $X \sim Poisson(\mu)$.
- X has p.f. given by

$$\boxed{f(x) = \frac{e^{-\mu}\mu^{x}}{x!} \text{ for } x = 0, 1, 2, \dots .}$$

We have that A = {0, 1, 2,...}.

Clearly, $f(x) \geq 0 \; \forall \; x \in A$.

In addition, $\sum_{x \in A} f(x) = \sum_{x=0}^{\infty} f(x) = 1$.



$\mu = 2.5$
$\mu = 0.33$
$\mu = 0.1$
$\mu = 5$
p(y)
y

Some examples of random variables that generally follow a Poisson distribution:

1. The number of misprints on a page (or a group of pages) of a book.

2. The number of people in a community who survive to age 100.

3. The number of wrong telephone numbers that are dialed in a day.

4. The number of customers entering a post office on a given day (or other types of arrivals).

5. The number of α-particles discharged in a fixed period of time from some radioactive material.

6. The number of new potholes in a stretch of highway during the winter months.

7. The number of earthquakes in a region of Canada in a month.

8. The number of lightning strikes in a region of Canada in a month.

Next, we show that the Poisson distribution has connections to a binomial distribution.

Recall that the binomial distribution applies when we have a fixed number of independent trials *n*, each with a constant probability of success *p*.

Imagine now that we do not know the number of trials that are needed, but instead we only know the *average number of successes* (per unit of time).

poisson distrib

In other words, we know the rate of successes (say, per day), but not the number of trials *n*, or the probability of success *p* that led to that rate.

Relationship between Binomial and Poisson Distributions

The Poisson distribution arises from the biomial dist^{n} as n gets large and p gets small.

This tells us that poisson distr. can be used to approximate a binomial prob when n is large and p is small

~ see pg. 93.

General Notes

We can get a good approximation when $n \geq 20$ and $p \leq 0.05$

We can get an excellent approximation when $n \geq 100$, and $np \leq 10$.

We define an average rate:

$$\mu = np$$

Let this be the rate of success (say, per day). That is the number of trials *n* — however many there are — multiplied by the chance of success *p* for each of those trials.

Think of it like this: If the chance of success is *p* and we run *n* trials per day, we will observe *np* successes per day **on average**. That is our average success rate μ.

For example: When we flip a fair coin, we observe a head with probability $p = 0.5$. Hence, if we flip a coin 50 times in a day, our expectation is to see **an average of** $np = 50 \times 0.5 = 25$ heads per day.

Remember that we could observed anywhere from 0 to 50 heads. 25 is the (long-run) average.

Now, recall that if $X \sim Bin\left(n, p = \frac{\mu}{n}\right)$,

then we have:

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \binom{n}{x} \left(\frac{\mu}{n}\right)^x \left(1 - \frac{\mu}{n}\right)^{n-x}$$

for $x = 0, 1, 2, \ldots, n$.

Under this setting, we then consider the **limiting case** of the binomial distribution.

In other words, we want to know what happens to $f(x)$ *as* $n \to \infty$ *(so that* $p = \frac{\mu}{n} \to 0$*)?*

In summary, a Poisson distribution arises from a binomial distribution when $n \to \infty$ and $p \to 0$.

In other words, **a Poisson distribution approximates a binomial distribution when $\boldsymbol{n}$ is very large and $\boldsymbol{p}$ (the probability of success) is small**.

**That is, a Poisson distribution could be used to approximate a binomial probability.

The idea is to keep the product *np* ***fixed*** at some value, μ, and then let $n \to \infty$. This automatically makes $p \to 0$.

Example: A local restaurant is running a contest. A customer receives a ticket each time they purchase a combo (i.e. sandwich, side, and a drink). They claim that 1 in 9 tickets are winners. Say you buy 100 combos!

Assuming that the trials are independent, let's use the **Poisson approximation to the binomial** to solve for the probability that you get no more than 10 winning tickets.

(We will compare the approximate probability with the exact probability to see how well the approximation worked).

Claim: 1 in 9 tickets wins, on average.

The customer buys 100 combos.
We will use the Poisson approx. to the Binomial and compare the approx. prob w/ the exact probability

Let X = the # of winning tickets.

Poisson approx. $f(x) = P(X=x) = \dfrac{e^{-\frac{100}{9}} \cdot (100/9)^x}{x!}$

$\left(\mu = np = 100\left(\frac{1}{9}\right) = \frac{100}{9}\right)$

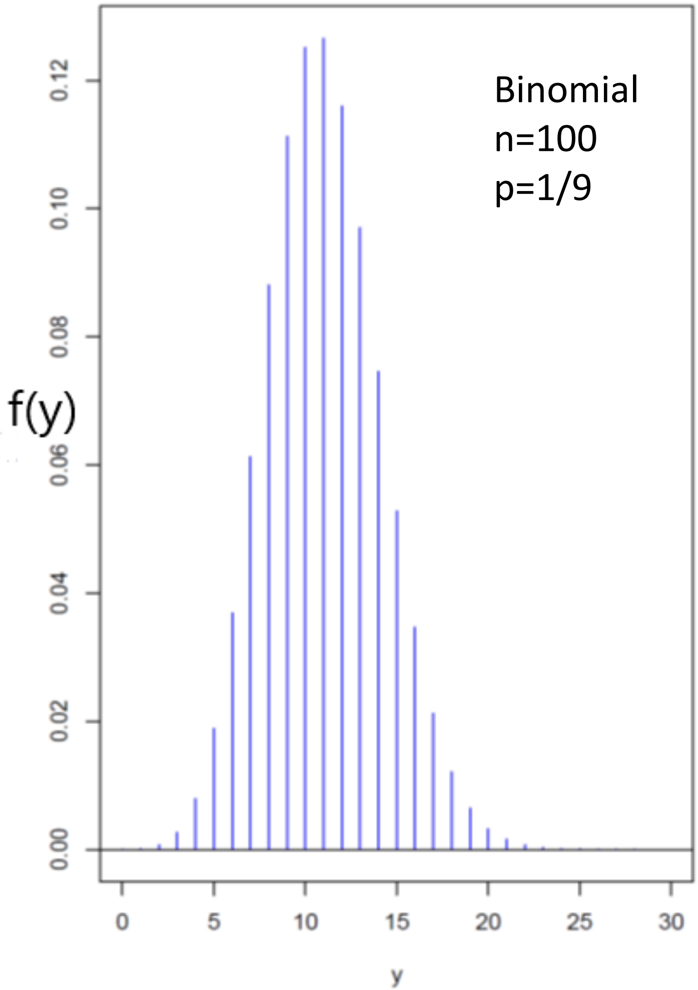
We want $P(X \le 10) = P(X=0) + P(X=1) + \cdots + P(X=10)$
$= 0.4466935$ (approx. prob)

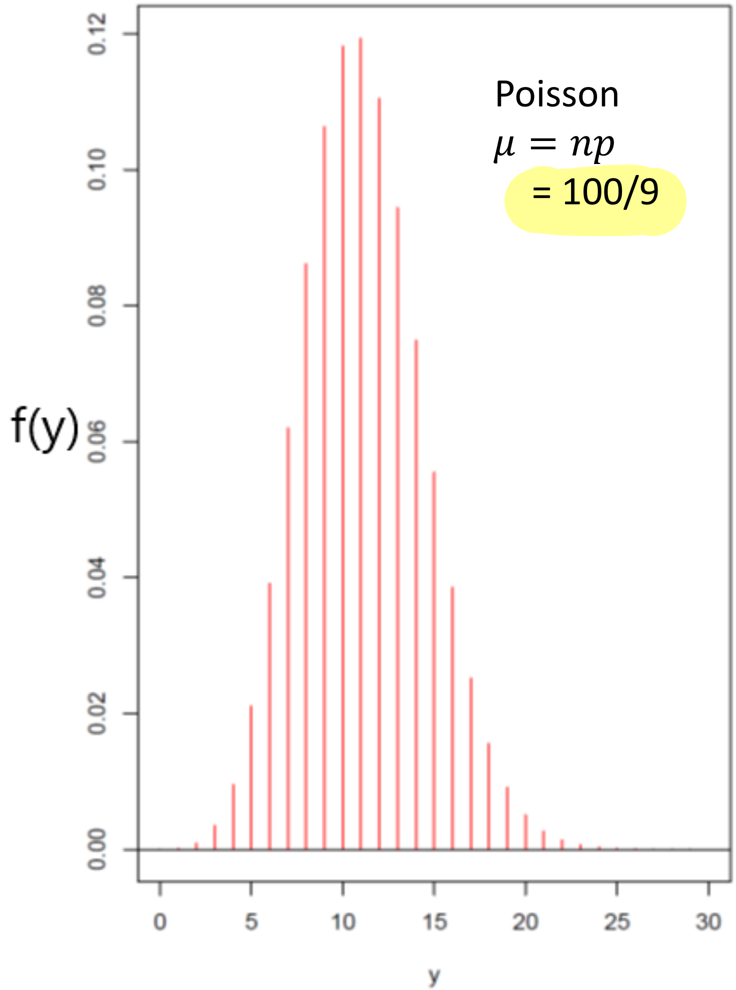
Using R: $P(X \le 10) = \text{ppois}(10, 100/9)$ Aside, p is used for cdf, d is used for pf

How well did we do w/ the approx?
$f(x) = P(X=x) = \binom{100}{x}\left(\frac{1}{9}\right)^x \left(\frac{8}{9}\right)^{(100-x)}$

Let X = # of winning tickets $X \sim \text{Bin}(100, 1/9)$
We want $P(X \le 10) = P(X=0) + P(X=1) + \cdots + P(X=10)$
$= 0.438889$

In R: $P(X \le 10) = \text{pbinom}(10, 100, 1/9)$





In R: $P(X=x)$ = dbinom(x,n,p)

$= \text{dpois}(x, \text{mean})$
(x, np)

You Try:

If you buy a lottery ticket in 50 lotteries, in each of which your chance of winning a prize is 1/100, what is the (approximate) probability that you will win a prize:

a) at least once?

b) exactly once?

c) at least twice?

Notes:

1. The same idea can be applied when p is close to 1. So, if p is close to 1, simply interchange the labels of "success" and "failure". In doing so, now the probability of "success" is close to 0, and you can proceed with the Poisson approximation to the binomial.

2. Historically, the Poisson approximation was a useful tool as it was easier to work with computationally. However, with the advent of better computers these days, calculating exact probabilities is no longer an issue under any circumstances.

Poisson Distribution from the Poisson Process (Section 5.8)

- In order to discuss the **Poisson process**, we first need to introduce some notation:
- Define the so-called *"order"* notation:

$$g(\Delta t) = o(\Delta t) \; as \; \Delta t \to 0.$$

- This means that **the function g approaches 0 faster than Δt does as Δt approaches 0**.

i.e. $$\frac{g(\Delta t)}{\Delta t} \to 0 \; as \; \Delta t \to 0$$

For example:

1. $g(x) = x^2$

$\Delta t = 1 : g(1) = 1^2 = 1$

$\Delta t = \frac{1}{2} : g(\frac{1}{2}) = (\frac{1}{2})^2 = \frac{1}{4}$

$\Delta t = \frac{1}{10} : g(\frac{1}{10}) = (\frac{1}{10})^2 = \frac{1}{100}$

$\Delta t = \frac{1}{100} : g(\frac{1}{100}) = (\frac{1}{100})^2 = \frac{1}{10000}$

g is approaching 0 faster that Δt as $\Delta t \to 0$

2. $g(x) = \sqrt{x}$

$g(x) = \sqrt{x} = x^{\frac{1}{2}}$

$\Delta t = 1 \quad g(1) = \sqrt{1} = 1$

$\Delta t = \frac{1}{2} \quad g(\frac{1}{2}) = \sqrt{(\frac{1}{2})} = 0.7071$

$\Delta t = 0.1 \quad g(\frac{1}{10}) = 0.3162$

$= \sqrt{\frac{1}{10}} =$

→ we see that this is not the case here

Physical setup: ** Poisson : rate !!

Assume that a certain type of event occurs at random points in time (or space) and satisfies the following conditions:

1. **Independence**: The number of occurrences in non-overlapping intervals are independent.
2. **Individuality:**

$$\mathbf{P}\big(\mathbf{2\ or\ more\ events\ in\ (t, t + \Delta t)}\big) = \mathbf{o(\Delta t)}$$
$$\mathbf{as\ \Delta t \to 0.}$$

In other words, this is telling us that as $\Delta t \to 0$, the probability of 2 or more events occurring in the interval $(t, t + \Delta t)$ is close to zero (i.e. the probability of more than one event occurring during such a small interval is negligible.)

3. **Homogeneity or Uniformity:** Events occur at a homogeneous (uniform) rate λ over time so that

$$\mathbf{P}\left(\mathbf{one\ event\ in\ }(\mathbf{t},\mathbf{t}+\boldsymbol{\Delta}\mathbf{t})\right)=\boldsymbol{\lambda}\boldsymbol{\Delta}\mathbf{t}+\mathbf{o}(\boldsymbol{\Delta}\mathbf{t})$$

for small Δt for any value of t.

These three conditions define what is known as a **Poisson process**.

If we now let X represent the number of event occurrences in a time period of length t, then it can be shown that X has a Poisson distribution with parameter $\mu = \lambda t$.

In a Poisson process with rate of occurrence λ, the number of event occurrences, X, in a time interval of length t has a Poisson distribution with parameter $\mu = \lambda t$, so that the p.f. of X is

$$\boldsymbol{f(x) = \frac{e^{-\lambda t}(\lambda t)^x}{x!} \text{ for } x = 0, 1, 2, \ldots}$$

poisson process, over time

Note that μ can be "scaled up" or "scaled down" accordingly.

For example:

1. Visits to a website during a given time period often follow a Poisson process.

2. Occurrences of certain diseases over time can sometimes be modelled by a Poisson process.

3. The number of customers entering a store or bank is commonly modelled by a Poisson process.

How to Interpret μ and λ:

1. λ refers to the **intensity** or **rate of occurrence**.

For example: A student types on average 60 words a minute. Here, λ = 60.

2. $\mu = \lambda t$ represents the average number of occurrences in t **units** of time.

For example: On average, how many words will the student type in 3 minutes? In 10 seconds?

$$\mu = 60 * (3) = 180.$$
$$\mu = 60 * (1/6) = 10.$$

Example: Suppose earthquakes recorded in Ontario each year follow a Poisson process with an average of 6 per year.

What is the probability that 7 will be recorded in a 2-year period?

Let X represent the number of earthquakes in a two-year period

X has a Poisson distr. w/ $\mu = \lambda t = 6(2) = 12$.

$\lambda = \frac{6 \text{ times}}{\text{year}}$

$t = 2$ years

$$P(X=x) = f(x) = \frac{e^{-12}(12)^x}{x!} \quad x = 0, 1, \ldots ;$$
0 otherwise

We want $P(X=7) = \frac{e^{-12}(12)^7}{7!}$

$= 0.436822$

Using R: dpois(7, 12) $= P(X=7)$

You Try:

At a certain location on Highway 401, the number of cars exceeding the speed limit by more than 10 km/hr is a random variable having a Poisson distribution with $\lambda = 8.4$ per 30 minutes.

What is the probability that, in 1 hour, 10 cars exceed the speed limit by more than 10 km/hr?

The Poisson process also applies when "events" occur randomly in **space**.

X could represent the number of events in a **space (like volume or area) of size $\boldsymbol{\nu}$**. If λ is the average number of events per unit volume (or area), then X has a Poisson distribution with parameter $\mu = \lambda\nu$. (same idea as previously when $\mu = \lambda t$).

So, this model is valid when we replace "time" by "volume" or "area".

Example: In the manufacturing process of commercial carpet, small faults occur at random in the carpet according to a Poisson process at an average rate of 0.95 per 20 m^2. One of the rooms of a new office block has an area of 80 m^2 and has been carpeted using the same commercial carpet described above.

What is the probability that the carpet in that room contains at least 4 faults?

Let X represent the number of faults in the room with $80m^2$ of carpet.

$X \sim Pois(\mu = 0.95 \times 4 = 3.8)$

$\downarrow$

λ - rate of $20 m^2$

So, $P(X=x)$: $f(x) = \dfrac{e^{-\mu}\mu^{x}}{x!}$, $x = 0, 1, 2, \dots$; 0 otherwise

$= \dfrac{e^{-3.8}(3.8)^{x}}{x!}$, $x = 0, 1, 2, \dots$; 0 otherwise

We want P(at least 4 faults) $= 1 - P(X \leq 3)$

$= 1 - [P(X=0) + P(X=1) + P(X=2) + P(X=3)]$

$= 0.526515$

Using R: $P(X \geq 4) = 1 - P(X \leq 3) = 1 - F(3) = 1 -$ ppois(3, 3.8)

Problem 5.8.1 (from the Course Notes):

Suppose that emergency calls to 911 follow a Poisson process with an average of 3 calls per minute. Find the probability there will be:

a) 6 calls in a period of 2.5 minutes.

b) 2 calls in the first minute of a 2.5 minute period, given that 6 calls occur in the entire period.

a) Let X be the number of emergency calls to 911 in 2.5 min.

$X \sim Pois(\mu = 3 \times 2.5 = 7.5)$ (avg. 7.5 calls / 2.5 minutes)

So, $f(x) = P(X=x) = \frac{e^{-\mu}\mu^{x}}{x!} = \frac{e^{-7.5}(7.5)^{x}}{x!}$ $x = 0, 1, \ldots;$ 0 otherwise

We want P(6 calls in a 2.5 minute period)

$= P(X=6) = \frac{e^{-7.5}(7.5)^{6}}{6!} = 0.136718$

Using R: $P(X=6) = dpois(6, 7.5)$

b) We want P(2 calls in first minute of a 2.5 minute period | 6 calls occur in the entire period)

We have to be careful here! Let's define our events:

Let A be the event that there are two calls in the first minute of a 2.5-minute period

Let B be the event that there are 6 calls in total in a 2.5 minute period

$$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{P(A\cap B)}{\left(\frac{e^{-7.5}(7.5)^6}{6!}\right)}$$

$$= \frac{\left(\frac{e^{-\overset{3\times 1}{3}}(3)^2}{2!}\right)\left(\frac{e^{-\overset{3\times 1.5}{4.5}}(4.5)^4}{4!}\right)}{\left(\frac{e^{-7.5}(7.5)^6}{6!}\right)}$$

numerator:

2 calls in the first min ($\mu = 3\times 1 = 3$)

4 calls in the next 1.5 min: ($\mu = 3 \times 1.5 = 4.5$)

$$\approx 0.3110$$

Exercise: Can you simplify this expression and end up with the expression of another probability function?

Combining Other Models with the Poisson Process (Section 5.9)

Sometimes we will need to use two or more probability distributions in a given application.

Example: Server requests come in according to a Poisson process with a rate of 100 requests per minute. A second is defined as “quiet” if it has no requests.

a) Find the probability that a second is "quiet".

Let X represent the number of requests occurring in one second.

$X \sim Pois\left(\mu = \frac{100}{60} = \frac{5}{3}\right)$

$P(X=x) = f(x) = \dfrac{e^{-5/3}\left(\frac{5}{3}\right)^x}{x!}$, $x = 0, 1, \ldots$;

0 otherwise

We want $P(\text{second is "quiet"}) = P(X=0) = \dfrac{e^{-\frac{5}{3}}\cancel{\left(\frac{5}{3}\right)^0}^{\,1}}{\cancel{0!}\;1}$

$= e^{-\frac{5}{3}}$

$= 0.188876$

Using R: $P(X=0) = \text{dpois}(0, 5/3)$

b) Find the probability that we observe 10 "quiet" seconds in a 60-second (1 minute) period.

Let Y = the number of "quiet" seconds in a 60-second period

Y has a binomial distribution with $n = 60$ $p = e^{-5/3}$
$= P(X=0)$ from a)

We want $P(Y=10)$

So $P(Y=y) = \binom{60}{y}\left(e^{-5/3}\right)^{y}\left(1-e^{-5/3}\right)^{60-y}$, $y = 0, 1, \dots, 60$;
0 otherwise

So $P(Y=10) = \binom{60}{10}\left(e^{-5/3}\right)^{10}\left(1-e^{-5/3}\right)^{50}$

$= 0.124013$

Using R: $P(Y=10) =$ dbinom(10, 60, exp(-5/3))

c) Find the probability that we have to wait 30 seconds to get 2 "quiet" seconds.

We want P(we have to wait 30 seconds to get 2 quiet seconds)

Let X = # of "non-quiet" seconds before the 2^{nd} "quiet" second.

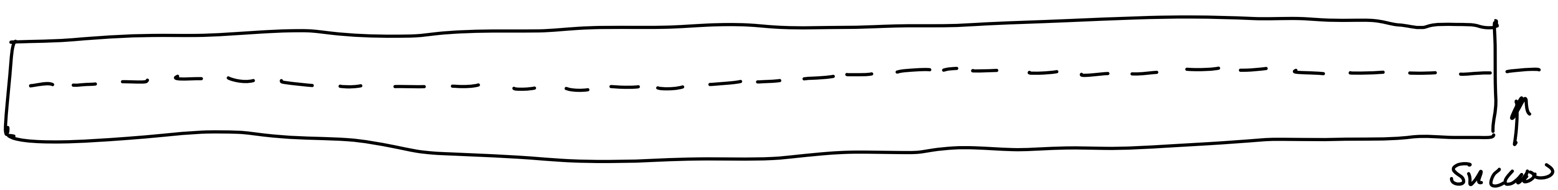
failure

success

X has a negative binomial distribution with $k=2$, $p=e^{-\frac{5}{3}}$

This suggests that, in the previous 29 seconds, there are 28 "non-quiet" seconds, and 1 "quiet" second, and then we have our 2nd "quiet" second.

"visual"



29 previous seconds
- 28 "non-quiet" seconds
- 1 "quiet" second

X has a negative binomial distribution with $k = 2$, $p = e^{-\frac{5}{3}}$

We want $P(X = 28) = \binom{29}{28}\left[1 - e^{-\frac{5}{8}}\right]^{28}\left(e^{-\frac{5}{8}}\right)^{2} = 0.00295$

Using R:

dbinomial (28, 2, exp (−5/3))

$$\begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ x & k & p \end{array}$$

* conditional probability variation

d) If 10 “quiet” seconds occur in 60 seconds, what is the probability that exactly 2 occurred among the first 20 seconds?

We have : 20 second period
40 second period
60 second period

P(2 "quiet" seconds in the first 20 seconds | 10 "quiet" seconds in a 60 second period)

$$= \frac{\left[\binom{20}{2}\cancel{\left(e^{-\frac{5}{3}}\right)^{2}}\cancel{\left(1-e^{-\frac{5}{3}}\right)^{18}}\right]\left[\binom{40}{8}\cancel{\left(e^{-\frac{5}{3}}\right)^{8}}\cancel{\left(1-e^{-\frac{5}{3}}\right)^{32}}\right]}{\left[\binom{60}{10}\cancel{\left(e^{-\frac{5}{3}}\right)^{10}}\cancel{\left(1-e^{-\frac{5}{3}}\right)^{50}}\right]}$$

$$= \frac{\binom{20}{2}\binom{40}{8}}{\binom{60}{10}} \approx 0.193807$$

Summary:

Discrete Distribtion	**Probability Function**	**Range**
Discrete Uniform	$f(x) = \dfrac{1}{b-a+1}$	$x = a, a+1, \ldots, b$
Hypergeometric	$f(x) = \dfrac{\binom{r}{x}\binom{N-r}{n-x}}{\binom{N}{n}}$	$x \geq \max\{0, n-(N-r)\}$ & $x \leq \min\{r, n\}$
Binomial	$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$	$x = 0, 1, 2, \ldots, n$

Discrete Distribution	Probability Function	Range
Negative Binomial (# of failures before the k^th success)	$f(x) = \binom{x+k-1}{x} p^k (1-p)^x$	$x = 0, 1, 2, \ldots$
Geometric (# of failures before the first success)	$f(x) = p(1-p)^x$	$x = 0, 1, 2, \ldots$
Poisson	$f(x) = \frac{e^{-\mu}\mu^x}{x!}$	$x = 0, 1, 2, \ldots$