Matroid Theory Note

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Introduction

This note is about the Matroid Theory course at San Francisco State in Fall 2023.

Lecture 1

Lecture 2

Definition 1. A *matroid* (E, I) consists of

- a finite set E which is called *ground set*
- a collection $\mathcal{I} \subset 2^E$ of *independent* subsets of E such that

  (I1) $\emptyset \in \mathcal{I}$;

  (I2) If $I \subset J$ and $J \in \mathcal{I}$ then $I \in \mathcal{I}$;

  (I3) If $I, J \in \mathcal{I}$ and $|I| < |J|$ then there exists $j \in J \setminus I$ such that $I \cup \{j\} \in \mathcal{I}$.

Example 1. When $E = \{a, b, c, d, e, f\}$, $\mathcal{I}$ is a matroid when

$$\begin{aligned} \mathcal{I} = \{&\emptyset, a, b, c, d, e, ab, \\ &ac, ad, ae, bc, bd, be, cd, ce, \\ &abc, abd, abe, acd, ace\}. \end{aligned}$$

Historically, the common features of independence is observed in many different fields and

Example 2. Linear matroids.

Example 3. Graphical matroids.

Example 4. Transversal matroids.

Example 5. Algebraic matroids.

Proposition 1. Linear matroids are indeed matroids.

Proof. Let V be a vector space and take a finite set $E \subset V$. Define $\mathcal{I}$ as a collection of linearly independent subsets of E. The first axiom is satisfied. The second axiom is satisfied because any subset of an independent set is independent. For the last one, suppose I, J are linearlt independent subsets of E and $|I| < |J|$.

Let $I = \{v_1, \cdots, v_a\}, J = \{w_1, \cdots, w_b\}$ where $a < b$. By contradiction, assume that $I \cap w_k$ is dependent for all $k = 1, \cdots, b$. Then all w_k can be written as linear combinations of $v_1, \cdots v_a$ because I is independent. Therefore all w_k are in span(I) and span(J) is contained in span(I). It implies that $\dim \operatorname{span}(I) > \dim \operatorname{span}(J)$ which contradicts to that $a < b$. $\square$

Lemma 1. If a set J of edges have no cycles, then $G|_J$ has $n - |J|$ connected components.

Lecture 3

Definition 2 (Graph Terminologies)**.**

- A *path* from u to v is a sequence of edges coming from u to v allowing repetition of vertices but not repetition of edges.
- A *cycle* is a path from u to u.
- A *forest* is a graph with no cycles.
- A *tree* is a connected forest.
- A graph is called *connected* if there is a path between any two vertices v, w.
- We call two vertices u, v are in the same connected component if there is a path between them.

Lemma 2. A forest (V, E) has $|V| - |E|$ connected components.

Proof. Induction on $|E|$. Suppose that $|E| = 0$. Then each vertice is a connected components hence there are $|V|$ connected components. Let $G = (V, E)$ be a forest with k edges. Let's remove an edge e. Then $(V, E-e)$ has $|V|-|E-e| = |V|-|E|+1$ connected components by the induction hypothesis. Let $e = uv$ then u, v are in distinct components. Otherwise, there are a path from u to v other than e and it implies there is a cycle. It contradicts to that G is a forest. Hence adding e back in E merges two distinct connected components into one; hence there are $|V| - |E|$ connected components. □

We can use induction on the case that the math object we are treating has the *size* in $\mathbb{N}$. We have the induction hypothesis for all with size $k - 1$ (or all with size $\leq k - 1$) and show that the same hypothesis for all with size k.

Proposition 2. Let $G = (V, E)$ be a graph, and let $\mathcal{I}$ be a collection of set of edges with no cycles. Then $(E, \mathcal{I})$ is a matroid. It is called the *graphical matroid* of G.

Proof. For the first axiom (I1), $\emptyset$ contains no cycles hence $\emptyset \in \mathcal{I}$. For the second axiom (I2), suppose not. Then there exists some $I \subset J$ such that J is independent but I is not. It induces a contradiction because J contains a cycle if I contains a cycle.

Lastly, we will show (I3). Assume I, J are independent and $|I| < |J|$. Assume that $I \cup j$ is never independent for all $j \in J$. It implies that adding j to I doesn't change the number of connected components; hence $I \cup J$ has $|V| - |E|$ connected components. It induces a contradiction because $I \cup J$ contains J but $I \cup J$ has $|V| - |I|$ connected components which is larger than $|V| - |J|$ connected components of J. Note that adding edges reduces or maintains the number of connected components. $\square$

We can consider $\mathcal{I}$ as a set of *subforests*.

In the previous examples of the linear and graphical matroids,

Definition 3. Two matroids $M_1 = (E_1, \mathcal{I}_1)$ and $M_2 = (E_2, \mathcal{I}_2)$ are isomorphic if there is a bijection $\varphi : E_1 \to E_2$ so that $I_1 \in \mathcal{I}_1$ iff $\varphi(I_2) \in \mathcal{I}_2$.

Definition 4. A *basis* is a maximal independent set. In other words, B is called a *basis* if there exists no independent I having B as its proper subset.

Definition 5. A *circuit* is a minimal independent set. A *loop* is a circuit of size 1. A *coloop* is an element that is in every basis.

Example 6.

Lecture 4

We will talk about 6 different examples.