Number Theory Note

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Fall 2023

Lecture 1

Introduction

This note is from the Number Theory class held at UC Berkeley in Fall 2023. The prerequisites for this course is Math 250 A, in particular, the following:

- integrality of an element of a ring over a subring;
- integral ring extensions;
- separable and purely inseparable (algebraic) field extensions;
- Galois theory;
- noetherian rings and modules;
- localization (inverting a multiplicative subset of a ring).

In this course, we will cover the following chapters of *Algebraic Number Theory* of Neukirch.

- Ch 1: Algebraic integers (all the sections but 12, 13, 14)
- Ch 2: Valuations (all the sections but 6, parts of 7, 9, 10)

- Ch 3: Primes, different, discriminant (1, 2, and parts of 3)
- Ch 7: Zeta functions and L-series (a thin subset)
- Ch 6: Class field theory (Section 12, a few other parts)

Overview

The following is the overview of the courses. Let us define a number field.

Definition 1. A number field is a finite (field) extension of $\mathbb{Q}$.

For example, $\mathbb{Q}(\sqrt{2})$ is a number field. We often work with one of the following situations:

$$\begin{array}{ccc} A & \subseteq & K \\ | & & | \\ \mathbb{Z} & \subseteq & \mathbb{Q} \end{array} \qquad \begin{array}{ccc} B & \subseteq & L \\ | & & | \\ A & \subseteq & K \end{array}$$

where K, L are number fields. Here is an example to be proved later. If $K = \mathbb{Q}(\sqrt{2})$, in the left-hand diagram, then $A = \mathbb{Z}[\sqrt{2}]$.

In Chapter 1 Algebraic integers, we will consider a question. Which properties of $\mathbb{Z}$ remain true in A?

$\mathbb{Z}$	A
PID	Usually not PID but non-principality is determined by a finite group

Lecture 2

Lecture 3

Lecture 4