# M-1 QUESTION BANK(IMP)

## UNIT-I

## MATRICES

**Short answer questions**

1. Define symmetric,Skew-symmetric and Orthogonal matrices
2. Define Hermitian,Skew-Hermition and Unitary matrices
3. Define rank of a matrix
4. Define echelon form of a Matrix.
5. Define normal form of a matrix
6. State the conditions for consistency of the system of equations AX=B
7. State the conditions for consistency of the system of equations AX=0
8. Define linearly dependent and linearly independent vectors
9. Find the value of k such that rank of A= $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & k & 1 \\ 3 & 1 & 0 & 1 \end{bmatrix}$ is 2
10. Define elementary matrix with an example

**LONG ANSWER QUESTIONS**

1. Find rank of the following matrices by reducing into Echelon form:

i) $\begin{bmatrix} 1 & 4 & 3 & -2 & 1 \\ -2 & -3 & -1 & 4 & 3 \\ -1 & 6 & 7 & 2 & 9 \\ -3 & 3 & 6 & 6 & 12 \end{bmatrix}$ ii) $\begin{bmatrix} 2 & -4 & 3 & -1 & 0 \\ 1 & -2 & -1 & -4 & 2 \\ 0 & 1 & -1 & 3 & 1 \\ 4 & -7 & 4 & -4 & 5 \end{bmatrix}$

2. Find rank of the following matrices by reducing into Normal form:

i) $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$ ii) $\begin{bmatrix} 1 & 2 & -1 & 3 \\ 4 & 1 & 2 & 1 \\ 3 & -1 & 1 & 2 \\ 1 & 2 & 0 & 1 \end{bmatrix}$

3. Determine the non-singular matrices P and Q such that PAQ is in Normal form and find its rank:

i) $\begin{bmatrix} 2 & 1 & -3 & -6 \\ 2 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$ ii) $\begin{bmatrix} 1 & -2 & 3 & 4 \\ -2 & 4 & -1 & -3 \\ -1 & 2 & 7 & 6 \end{bmatrix}$

4. Test for consistency and solve if the equations are consistent:

i) $x + 2y + 2z = 2\,, 3x - 2y - z = 5\,, 2x - 5y + 3z = -4\,,\ x + 4y + 6z = 0$

ii) $x + y + z = 3\,,\ 3x - 5y + 2z = 8\,,\ 5x - 3y + 4z = 14.$

5. Solve the system of equations:

i) $x + y + w = 0\,,\ y + z = 0\,, x + y + z + w = 0\,,\ x + y + 2z = 0$

ii) $2x - y + 3z = 0\,,\ 3x + 2y + z = 0\,,\ x - 4y + 5z = 0$

6. Determine whether the following equations will have non-trivial solutions, if so solve them:

i) $x + 3y - 2z = 0\,,\ 2x - y + 4z = 0\,,\ x - 11y + 14z = 0$

ii) $4x + 2y + z + 3w = 0\,, 6x + 3y + 4z + 7w = 0\,, 2x + y + w = 0$

7. For what values of λ and μ the system of equations:
$x + y + z = 6, x + 2y + 3z = 10, x + 2y + \lambda z = \mu$ has

i) No solution

ii) Unique solutions

iii) Infinite number of solutions

8. Find the value of λ for which the system of equations
$3x - y + 4z = 3\,, x + 2y - 3z = -2\,, 6x + 5y + \lambda z = -3$ will have infinite number of solutions and solve them with that value of λ.

9. Find the values of a and b for which the system of equations
$x + y + z = 3\,, x + 2y + 2z = 6\,, x + 9y + az = b$ will have

i) No solution

ii) Unique solution

iii) Infinitely many solutions

10. Solve the following system using Gauss Elimination method

i) $x - 8y + z = -5\,, x - 2y + 9z = 8\,, 3x + y - z = -8$

ii) $3x + y + 2z = 3\,, 2x - 3y - z = -3\,, x + 2y + z = 4$

11. Solve the following system of equations using Gauss Jordan method:

$10x + y + z = 12\,, 2x + 10y + z + 13\,, x + y + 5z = 7$

# QUESTION BANK

## UNIT-I

## MATRICES

### SHORT ANSWER QUESTIONS

1) Define symmetric, skew symmetric and Orthogonal matrices

Symmetric Matrix: A square matrix $A=[a_{ij}]$ is said to be symmetric if $a_{ij}=a_{ji}$ for every $i$ and $j$

- Thus: A is a symmetric matrix $\Leftrightarrow A=A'$ or $A'=A'$

Skew Symmetric Matrix: A square matrix $A=[a_{ij}]$ is said to be skew-symmetric if $a_{ij}=-a_{ji}$ for every $i$ and $j$

Thus: A is a skew-symmetric matrix $\Leftrightarrow A=-A'$ or $A'=-A$

Orthogonal matrix: A square matrix A is said to be Orthogonal if $AA'=A'A=I$. That is $A^T=A^{-1}$

2) Define Hermitian, skew-Hermition and Unitary matrices

Hermitian Matrix: A square matrix A such that $A^T=\bar{A}$ (or) $(\bar{A})^T=A$ is called a Hermitian matrix

Skew-Hermitian Matrix: A square matrix A such that $A^T=-\bar{A}$ (or) $(\bar{A})^T=-A$ is called a skew-Hermitian Matrix

Unitary Matrix: A square matrix A such that $(\bar{A})^T=\bar{A}^{-1}$

3. Define rank of a Matrix

Let $A$ be an $m \times n$ matrix. If $A$ is a null matrix, we define its rank to be 0.

If $A$ is a non zero matrix, we say that $r$ is the rank of $A$ if

i) every $(r+1)^{th}$ Order minor of $A$ is 0

ii) three exists at least one $r^{th}$ order minor of $A$ which is not zero

Rank of $A$ is denoted by $\rho(A)$

4) Define Echelon form of a matrix

A matrix is said to be echelon form if

i) zero rows, if any, are below any non-zero row is

ii) The first non-zero element in each non-zero row is equal to 1

iii) The number of zero rows before the first non-zero element of a row is less than the number of such zeros in the next row.

5) Define normal form of a Matrix.

Every $m \times n$ matrix of rank $r$ can be reduced to the form $I_r$, $[I_r, 0]$ or $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ by using a finite number of elementary row or column operations, where $I_r$ is the $r$-rowed unit matrix then above form is called "normal form" or "first canonical form"

6) state the conditions for consistency of the system of equations $AX = B$

i) If Rank $A$ = Rank of $[A|B] = n$ then the equation $AX = B$ is Consistent with unique solution

ii) If Rank of $A$ = Rank of $[A|B] \neq n$ then the equation $AX = B$ is consistent with infinite solution

iii) If Rank of $A \neq$ Rank of $[A|B]$ then the equation $AX = B$ have no solution and inconsistent

7) state the conditions for consistency of the system of equations $AX = 0$

i) If Rank of $A$ = Rank of $[A|B] = n$ then the equation $AX = 0$ have trivial solutions

ii) If Rank of $A$ = Rank of $[A|B] < n$ then the equation $AX = 0$ have non trivial solutions

8) Define linearly dependent and linearly independent Vectors

Linearly dependent set: A set $\{x_1, x_2, \cdots x_n\}$ of $r$ vectors is said to be a linearly dependent set, if there exist $r$ scalars $k_1, k_2 \cdots k_r$, not all zero, such that $k_1 x_1 + k_2 x_2 + \cdots + k_r x_r = 0$, where $0$ denotes the $n$ vector with components all zero.

Linearly independent set of vectors: A set $\{x_1 x_2 \cdots x_r\}$ of $r$ vectors is said to be linearly independent set, if the set, is not linearly dependent, i.e, if $k_1 x_1 + k_2 x_2 + \cdots + k_r x_r = 0$ where $0$ denotes the $n$ vector with components all zero

$\Rightarrow k_1 = 0;\ k_2 = 0 \ldots k_r = 0$

9) find the value of $k$ such that the rank of

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & k & 1 \\ 3 & 1 & 0 & 1 \end{bmatrix} \text{ is } 2$$

given Rank$(A) = 2$ $\quad R_2 \to R_2 - R_1,\ R_3 \to R_3 - 3R_1$

$$A = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & -2 & k+1 & -2 \\ 0 & -2 & 3 & -2 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

$$A = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & -2 & k+1 & -2 \\ 0 & 0 & 2-k & 0 \end{bmatrix} \quad \rho(A) = 2$$

$2 - k = 0$

$\boxed{k = 2}$

10) Define elementary matrix with an example

Elementary Matrix: It is a matrix obtained from a unit matrix by a single elementary transformation

Example: $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are the

elementary matrices obtained from $I_3$ by applying the elementary operations $C_1 \leftrightarrow C_2$, $R_3 \to 2R_3$ & $R_1 \to R_1 + 2R_2$

LONG ANSWER QUESTIONS

1) find rank of the following matrices by reducing into Echelon form.

i) $\begin{bmatrix} 1 & 4 & 3 & -2 & 1 \\ -2 & -3 & -1 & 4 & 3 \\ -1 & 6 & 7 & 2 & 9 \\ -3 & 3 & 6 & 6 & 12 \end{bmatrix}$

Apply $R_2 + 2R_1$

$$\begin{matrix} 1 & 4 & 3 & -2 & 1 \\ 0 & 5 & 5 & 0 & 5 \\ -1 & 6 & 7 & 2 & 9 \\ -3 & 3 & 6 & 6 & 12 \end{matrix}$$

Apply $R_3 + R_1$, $R_4 + 3R_1$

$$\begin{bmatrix} 1 & 4 & 3 & -2 & 1 \\ 0 & 5 & 5 & 0 & 5 \\ 0 & 10 & 10 & 0 & 10 \\ 0 & 15 & 15 & 0 & 15 \end{bmatrix}$$

Apply $\frac{R_2}{5}, \frac{R_3}{10}, \frac{R_4}{15}$

$$\begin{bmatrix} 1 & 4 & 3 & -2 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

Apply $R_3 - R_2$, $R_4 - R_2$

$$\begin{bmatrix} 1 & 4 & 3 & -2 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank $(A) = \rho(A)$ = The number of non-zero rows in matrix $= 2$

ii) $$\begin{bmatrix} 2 & -4 & 3 & -1 & 0 \\ 1 & -2 & -1 & -4 & 2 \\ 0 & 1 & -1 & 3 & 1 \\ 4 & -7 & 4 & -4 & 5 \end{bmatrix}$$

given $A = \begin{bmatrix} 2 & -4 & 3 & -1 & 0 \\ 1 & -2 & -1 & -4 & 2 \\ 0 & 1 & -1 & 3 & 1 \\ 4 & -7 & 4 & -4 & 5 \end{bmatrix}$

Interchange $R_1$ and $R_2$

$$\begin{bmatrix} 1 & -2 & -1 & -4 & 2 \\ 2 & -4 & 3 & -1 & 0 \\ 0 & 1 & -1 & 3 & 1 \\ 4 & -7 & 4 & -4 & 5 \end{bmatrix}$$

$R_2 \to 2$ $\quad R_2 \to R_2 - 2R_1 \quad R_4 \to R_4 - 4R_1$

$$\begin{bmatrix} 1 & -2 & -1 & -4 & 2 \\ 0 & 0 & 5 & 7 & -4 \\ 0 & 1 & -1 & 3 & 1 \\ 0 & 1 & 8 & 12 & -3 \end{bmatrix}$$

$R_2 \leftrightarrow R_4$

$$\begin{bmatrix} 1 & -2 & -1 & -4 & 2 \\ 0 & 1 & 8 & 12 & -3 \\ 0 & 1 & -1 & 3 & 1 \\ 0 & 0 & 5 & 7 & -4 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & -2 & -1 & -4 & 2 \\ 0 & 1 & 8 & 12 & -3 \\ 0 & 0 & -9 & -9 & 4 \\ 0 & 0 & 5 & 7 & -4 \end{bmatrix}$$

$R_4 \to 9R_4 + 5R_3$

$$\begin{bmatrix} 1 & -2 & -1 & -4 & 2 \\ 0 & 1 & 8 & 12 & -3 \\ 0 & 0 & -9 & -9 & 4 \\ 0 & 0 & 0 & 18 & -16 \end{bmatrix}$$

Number of Non-zero rows is 4

$\rho(A) = 4$

2) Find rank of the following matrices by reducing into normal form

i) $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

$R_2 \leftrightarrow R_1$

$\begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

Apply $R_2 \to 2R_1$
$R_3 \to 3R_1$
$R_4 \to 6R_1$

$\begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$

Apply $R_4 - R_3$

$\begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 5 & 3 & 7 \end{bmatrix}$

Apply $R_4 - R_2$

$\begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 5 & 3 & 7 \end{bmatrix}$

$\begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Apply $R_2 - R_3$

$\begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & -6 & -3 \\ 0 & 4 & 9 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Apply $R_1 + R_2$

$\begin{bmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & -6 & -3 \\ 0 & 4 & 9 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Apply $R_3 - 4R_2$

$\begin{bmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Apply $C_3 + 8C_1$

$\begin{bmatrix} 1 & 0 & 0 & -7 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Apply $\frac{R_3}{11}$

$$\begin{bmatrix} 1 & 0 & 0 & -7 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{Apply } C_4+7C_1$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{Apply } C_3+6C_2,\ C_4+3C_2$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{Apply } \frac{C_3}{3}, \frac{C_4}{2}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{Apply } C_4-C_3$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is of the form $\begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$. Hence Rank of A is 3

ii) $$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 4 & 1 & 2 & 1 \\ 3 & -1 & 1 & 2 \\ 1 & 2 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 4 & 1 & 2 & 1 \\ 3 & -1 & 1 & 2 \\ 1 & 2 & 0 & k \end{bmatrix}$$

Apply $R_2-4R_1, R_3-3R_1, R_4-R_1$

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 6 & -11 \\ 0 & -7 & 4 & -7 \\ 0 & 0 & 1 & k-3 \end{bmatrix}$$

Apply $7R_1+2R_2, R_3-R_2$

$$\begin{bmatrix} 7 & 0 & 5 & -1 \\ 0 & -7 & 6 & -11 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 1 & k-3 \end{bmatrix}$$

Apply $2R_1+5R_3, R_2+3R_3$

$$\begin{bmatrix} 14 & 0 & 0 & 18 \\ 0 & -7 & 0 & 1 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 1 & k-3 \end{bmatrix}$$

Apply $2R_4+R_3$

$$\begin{bmatrix} 14 & 0 & 0 & 18 \\ 0 & -7 & 0 & 1 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 0 & 2k-2 \end{bmatrix}$$

$\frac{C_1}{14}, \frac{C_2}{-7}, \frac{C_3}{-2}$

$$\begin{bmatrix} 1 & 0 & 0 & 18 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 2k-2 \end{bmatrix}$$

Apply $C_4-18C_1, C_4-C_2, C_4-4C_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2k-2 \end{bmatrix}$$

let $2k-2=0$

$k=1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$

Rank of $A = 3$

3) find non-singular matrices P and Q so that PAQ is of the normal form where $A = \begin{bmatrix} 1 & -2 & 3 & 4 \\ -2 & 4 & -1 & -3 \\ -1 & 2 & 7 & 6 \end{bmatrix}$

we write $A = I_3 A I_4$

i.e $\begin{bmatrix} 1 & -2 & 3 & 4 \\ -2 & 4 & -1 & -3 \\ -1 & 2 & 7 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$R_2+2R_1$ and $R_3+R_1$

$$\begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & 10 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Apply $R_3-2R_1$

$$\begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Applying $\frac{1}{5}R_2$

$$\begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0.4 & 0.2 & 0 \\ 3 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Apply $C_4-C_3$

$$\begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0.4 & 0.2 & 0 \\ -3 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Apply $R_1 - 3R_2$

$$\begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -0.2 & -0.6 & 0 \\ 0.4 & 0.2 & 0 \\ -3 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Apply $C_2 + 2C_1$ & $C_4 - C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -0.2 & -0.6 & 0 \\ 0.4 & 0.2 & 0 \\ -3 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Apply $C_2 \leftrightarrow C_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -0.2 & -0.6 & 0 \\ 0.4 & 0.2 & 0 \\ -3 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

we have $P = \begin{bmatrix} -0.2 & -0.6 & 0 \\ 0.4 & 0.2 & 0 \\ -3 & -2 & 1 \end{bmatrix}$, $Q = \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$$PAQ = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore$ Rank of given matrix is 2

i) *

If $A = \begin{bmatrix} 2 & 1 & -3 & -6 \\ 2 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$ find non-singular matrices such that PAQ is in normal form

we write $A = I_3 A I_4$

$$\begin{bmatrix} 2 & 1 & -3 & -6 \\ 2 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 \leftrightarrow R_1$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & -3 & 1 & 2 \\ 2 & 1 & -3 & -6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to 2R_1$ and $R_3 - 2R_1$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -5 & -1 & -2 \\ 0 & -1 & -5 & -10 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_1 \to R_1 + R_3$

$$\begin{bmatrix} 1 & 0 & -4 & -8 \\ 0 & -5 & -1 & -2 \\ 0 & -1 & -5 & -10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_4 \to C_4 - 2C_3$

$$\begin{bmatrix} 1 & 0 & -4 & 0 \\ 0 & -5 & -1 & 0 \\ 0 & -1 & -5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to C_3 + 4C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -5 & -1 & 0 \\ 0 & -1 & -5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to 5C_3 - C_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & -1 & -24 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 20 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_2 \to 24C_2 - C_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -120 & 0 & 0 \\ 0 & 0 & -24 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & -20 & 20 & 0 \\ 0 & 25 & -1 & 0 \\ 0 & -5 & 5 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\dfrac{R_2}{-120}$ & $\dfrac{R_3}{-24}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1/120 & 1/60 \\ -1/24 & 0 & 1/12 \end{bmatrix} A \begin{bmatrix} 1 & -20 & 20 & 0 \\ 0 & 25 & -1 & 0 \\ 0 & -5 & 5 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1/120 & 1/60 \\ -1/24 & 0 & 1/12 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & -20 & 20 & 0 \\ 0 & 25 & -1 & 0 \\ 0 & -5 & 5 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rank of $A = 3$

4) Test for consistency and solve if the equations are Consistent

i) $x+2y+2z=2$ ; $3x-2y-z=5$ ; $2x-5y+3z=-4$ ; $x+4y+6z=0$

The equation can be written in the matrix form as $AX=B$

$$\begin{bmatrix} 1 & 2 & 2 \\ 3 & -2 & -1 \\ 2 & -5 & 3 \\ 1 & 4 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ -4 \\ 0 \end{bmatrix}$$

The Augmented matrix $[A, B] = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 3 & -2 & -1 & 5 \\ 2 & -5 & 3 & -4 \\ 1 & 4 & 6 & 0 \end{bmatrix}$

Apply $R_2 \to R_2 - 3R_1$ , $R_3 \to R_3 - 2R_1$ , $R_4 \to R_4 - R_1$

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & -8 & -7 & -1 \\ 0 & -9 & -1 & -8 \\ 0 & 2 & 4 & -2 \end{bmatrix}$$

Apply $R_3 \to 8R_3 - 9R_2$ $R_4 \to 4R_4 + R_2$

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & -8 & -7 & -1 \\ 0 & 0 & 55 & -55 \\ 0 & 0 & 9 & -9 \end{bmatrix} \qquad \frac{R_3}{55}, \frac{R_4}{9}$$

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & -8 & -7 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Apply $R_4 \to R_4 - R_3$

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & -8 & -7 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

since Rank of $A = 3$ and Rank of $[A, B] = 3$

we have Rank of $A$ = Rank of $[A, B]$

The given system is consistent and has solution

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & -8 & -7 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -1 \\ 0 \end{bmatrix}$$

$x + 2y + 2z = 2$ — (1) ; $-8y - 7z = -1$ — (2) ; $\boxed{z = -1}$

put $z = -1$ in (2)

$-8y + 7 = -1$ $\boxed{y = 1}$

put $y = 1, z = -1$ in (1)

$x + 2 - 2 = 2$

$\boxed{x = 2}$

$\therefore x = 2, y = 1, z = -1$ is the solution

ii) $x+y+z=3,\ 3x-5y+2z=8,\ 5x-3y+4z=14$

$$A = \begin{matrix} 1 & 1 & 1 \\ 3 & -5 & 2 \\ 5 & -3 & 4 \end{matrix} \quad X = \begin{matrix} x \\ y \\ z \end{matrix} \quad B = \begin{matrix} 3 \\ 8 \\ 14 \end{matrix}$$

$$[A|B] = \begin{matrix} 1 & 1 & 1 & 3 \\ 3 & -5 & 2 & 8 \\ 5 & -3 & 4 & 14 \end{matrix}$$

$R_2 \to R_2 - 3R_1,\ R_3 \to R_3 - 5R_1$

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & -8 & -1 & -1 \\ 0 & -8 & -1 & -1 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & -8 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank of $[A|B] = 2 =$ Rank of $A$

The given solution is consistent and has solution

$$\begin{matrix} 1 & 1 & 1 \\ 0 & -8 & -1 \\ 0 & 0 & 0 \end{matrix} \quad \begin{matrix} x \\ y \\ z \end{matrix} = \begin{matrix} 3 \\ -1 \\ 0 \end{matrix}$$

$$x+y+z = 3$$
$$-8y - z = -1$$

let $z = k$

$$-8y-k=-1$$

$$-8y=k-1$$

$$y=\frac{k-1}{-8}$$

$$x+y+z=3$$

$$x+\frac{k-1}{-8}+k=3$$

$$x=3-\frac{k+1}{-8}-k$$

$$x=\frac{-24-k+1+8k}{-8}$$

$$x=\frac{7k-23}{-8}$$

$\therefore\ x=\frac{7k-23}{-8},\ y=\frac{k-1}{-8},\ z=k$ is the solution.

5) Solve the system of equations:

i) $x+y+w=0$; $y+z=0$; $x+y+z+w=0$; $x+y+2z=0$

The equations can be written in matrix form as

$$AX=0$$

where $A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$ $0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

apply $R_3 \to R_3 - R_1$, $R_4 \to R_4 - R_1$

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & -1 \end{bmatrix}$$

Apply $R_4 - 2R_3$

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Apply $R_1 + R_4$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$\therefore$ Rank$(A) = 4$ & Number of variables $= 4$

Therefore there is no non-zero solution

Hence $x = y = z = w = 0$ is the only solution

ii) $2x - y + 3z = 0$ ; $3x + 2y + z = 0$ ; $x - 4y + 5z = 0$

The given system can be written as $AX = 0$

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & -4 & 5 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad 0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Apply $R_3 \leftrightarrow R_1$

$$\begin{bmatrix} 1 & -4 & 5 \\ 3 & 2 & 1 \\ 2 & -1 & 3 \end{bmatrix}$$

Apply $R_2 \to 3R_1$ and $R_3 - 2R_1$

$$\begin{bmatrix} 1 & -4 & 5 \\ 0 & 14 & -14 \\ 0 & 7 & -7 \end{bmatrix}$$

Apply $2R_3 - R_2$ ; $\frac{R_2}{14}$

$$\begin{bmatrix} 1 & -4 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Rank$(A) = 2$ Number of variables $= 3$

$n - r = 3 - 2 = 1$

$y = z$

let $y = z = k$

$x - 4y + 5z = 0$

$x = 4y - 5z = 4k - 5k = -k$

The solution is given by $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -k \\ k \\ k \end{bmatrix} = k \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

6) Determine whether the following equations will have non-trivial solutions; if so solve them:

i) $x+3y-2z=0,\ 2x-y+4z=0,\ x-11y+14z=0$

Let $A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

then the given system can be written as
$AX = O$

Apply $R_3 - R_1$ and $R_2 - 2R_1$

$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & -14 & 16 \end{bmatrix}$$

Apply $R_3 - 2R_2$

$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & 0 & 0 \end{bmatrix}$$

The rank of matrix is 2 Number of variables is 3
will have one non-zero solution

Let $z = k$ then sub in corresponding equations

$x+3y-2z=0$ and $-7y+8z=0$

$y = \frac{8}{7}k$ ; $x = -3y+2z = \frac{-24}{7}k + 2k$
$= \frac{-10k}{7}$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{-10k}{7} \\ \frac{8}{7}k \\ k \end{bmatrix} = \frac{k}{7}\begin{bmatrix} -10 \\ 8 \\ 7 \end{bmatrix}$$

ii) $4x+2y+z+3w=0,\ 6x+3y+4z+7w=0,\ 2x+y+w=0$

$$AX = \begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = 0$$ (Interchanging the variables x and z)

Apply $R_2 - 4R_1$

$$\begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & -5 & -10 & -5 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

Apply $\frac{R_2}{-5}$

$$\begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

Apply $R_3 - R_2$

$$\begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank (A) = Number of non-zero rows $= 2 < 4$ (unknowns)

2 linearly independent solutions

$$AX = \begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} z \\ y \\ x \\ w \end{bmatrix} = 0$$

$z + 2y + 4x + 3w = 0$

$y + 2x + w = 0$

$x = k_1$ and $w = k_2$ then solve these two equations

$y = -2x - w = -2k_1 - k_2$ and $z = -4x - 2y - 3w = -k_2$

$x = k_1$ and $y = -2k_1 - k_2$, $z = -k_2$ and $w = k_2$

7) For what values of $\lambda$ and $\mu$ the system of equations:
$x+y+z=6$, $x+2y+3z=10$, $x+2y+\lambda z=\mu$ has

i) No solution

ii) Unique solutions

iii) Infinite number of solutions

The matrix form of given system of equations is

$$AX = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix} = B$$

agmented matrix is $[A/B] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{bmatrix}$

Apply $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$[A/B] \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{bmatrix}$$

Case I : Let $\lambda \neq 3$ then rank of $A = 3$ and rank of $[A/B] = 3$
Then the system of equations is consistent. Here the number of unknowns is 3 which is same as the rank of A. The system of equations will have a unique solution

Case II : $\lambda = 3$ and $\mu \neq 10$ then rank of $A = 2$ and rank of $[A/B] = 3$. Since the rank of $A$ and $[A/B]$ are not equal, the system of equations has no solution

Case iii : Let $\lambda = 3$ and $\mu = 10$. Then we have rank of $A$ = rank of $[A/B] = 2$

$\therefore$ The given system of equations will be consistent

Number of unknowns $= 3 >$ rank of $A$

system has infinitely many solutions

8) find the value of $\lambda$ for which the system of eq $3x - y + 4z = 3$, $x + 2y - 3z = -2$, $6x + 5y + \lambda z = -3$ will have infinite number of solutions and solve them with that value of $\lambda$

Matrix form $AX = B$

$$\begin{bmatrix} 3 & -1 & 4 \\ 1 & 2 & -3 \\ 6 & 5 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -3 \end{bmatrix}$$

Augmented matrix $[A, B] = \begin{bmatrix} 3 & -1 & 4 & 3 \\ 1 & 2 & -3 & -2 \\ 6 & 5 & \lambda & -3 \end{bmatrix}$

$R_2 \to 3R_2 - R_1$ ; $R_3 \to R_3 - 2R_1$

$$[A, B] \sim \begin{bmatrix} 3 & -1 & 4 & 3 \\ 0 & 7 & -13 & -9 \\ 0 & 7 & \lambda - 8 & -9 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$[A,B] \sim \begin{bmatrix} 3 & -1 & 4 & 3 \\ 0 & 7 & -13 & -9 \\ 0 & 0 & \lambda+5 & 0 \end{bmatrix}$$

If $\lambda = -5$ Rank of $A = 2$ and Rank of $[A,B] = 2$

Number of unknows $= 3$

Hence when $\lambda = -5$, the given system is consistent and it has an infinite number of solutions

if $\lambda = -5$ $\begin{bmatrix} 3 & -1 & 4 \\ 0 & 7 & -13 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -9 \\ 0 \end{bmatrix}$

$3x - y + 4z = 3$ —(1) $\quad 7y - 13z = -9$ —(2)

let $z = k$

$7y - 13k = -9 \Rightarrow y = \frac{13k - 9}{7}$

Substituting the value of $y$ in (1), we get

$3x - \frac{1}{7}(13k - 9) + 4k = 3 \Rightarrow 3x = \frac{13}{7}k - 4k + 3 - \frac{9}{7}$

$3x = -\frac{15}{7}k + \frac{12}{7} \Rightarrow x = \frac{1}{7}(-5k + 4)$

$\therefore$ The solution is $x = \frac{1}{7}(-5k+4)$, $y = \frac{1}{7}(13k - 9)$, $z = k$

9) Find the values of a and b for which the equations $x+y+z=3$, $x+2y+2z=6$, $x+9y+az=b$ have

i) No solution ii) A unique solution iii) Infinite no. of sol

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 9 & a \end{bmatrix} \quad X = \begin{matrix} x \\ y \\ z \end{matrix} \quad B = \begin{matrix} 3 \\ 6 \\ b \end{matrix}$$

$AX = B$

Augmented matrix $[A,B] = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 6 \\ 1 & 9 & a & b \end{bmatrix}$ $R_2 - R_1 ; R_3 - R_1$

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 8 & a-1 & b-3 \end{bmatrix}$$

$R_1 \to R_1 - R_2 ; R_3 \to R_3 - 8R_2$

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & a-17 & b-27 \end{bmatrix}$$

Case I : Let $a = 17, b = 27$

The rank of A = rank of $[A,B] = 2$

No. of variables = 3

rank(A) = rank $[A|B] = 2 < 3$

∴ The system is consistant and will have infinite solutions

Case II : Let $a = 17, b \neq 27$

Rank of A = 2 and Rank of $[A,B] = 3$

∴ The system is inconsistant (No solution)

Case (iii) : $a \neq 17$, $b \neq 27$

Rank $A$ = Rank $[A, B] = 3$ = no. of variables

$\therefore$ The system will be consistent and there will be unique solution

) solve the following system using Gauss elimination method

) $x-8y+z=-5,\ x-2y+9z=8,\ 3x+y-z=-8$

$$A=\begin{bmatrix}1&-8&1\\1&-2&9\\3&1&-1\end{bmatrix}\quad X=\begin{bmatrix}x\\y\\z\end{bmatrix}\quad B=\begin{bmatrix}-5\\8\\-8\end{bmatrix}$$

$$[A|B]=\begin{bmatrix}1&-8&1&-5\\1&-2&9&8\\3&1&-1&-8\end{bmatrix}$$

$R_2\to R_2-R_1,\ R_3\to R_3-3R_1$

$$\begin{bmatrix}1&-8&1&-5\\0&6&8&13\\0&25&-4&7\end{bmatrix}$$

$R_3\to 6R_3-25R_2$

$$\begin{bmatrix}1&-8&1&-5\\0&6&8&13\\0&0&-224&-283\end{bmatrix}$$

Rank of $[A|B]=3$

$$\begin{bmatrix}1&-8&1\\0&6&8\\0&0&-224\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-5\\13\\-283\end{bmatrix}$$

$x-8y+z=-5$

$6y+8z=13$

$-224z=-283$

$$-224z = -283$$

$$z = \frac{283}{224}$$

$$6y + 8\left(\frac{283}{224}\right) = 13$$

$$6y = 13 - 8\left(\frac{283}{224}\right)$$

$$6y = 13 - \frac{283}{28}$$

$$6y = \frac{81}{28}$$

$$y = \frac{81}{168}$$

$$x - 8y + z = -5$$

$$x = -5 + 8y - z$$

$$x = -5 + \frac{27}{7} - \frac{283}{224}$$

$$x = -5 + \frac{83}{32}$$

$$= -\frac{77}{32}$$

$$X = \begin{bmatrix} -77/32 \\ 81/168 \\ 283/224 \end{bmatrix}$$

$3x+y+2z=3, \quad 2x-3y-z=-3, \quad x+2y+z=4$

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix}$$

$$[A|B] = \begin{bmatrix} 3 & 1 & 2 & 3 \\ 2 & -3 & -1 & -3 \\ 1 & 2 & 1 & 4 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 2 & 1 & 4 \\ 2 & -3 & -1 & -3 \\ 3 & 1 & 2 & 3 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1 ; \quad R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & -7 & -3 & -11 \\ 0 & -5 & -1 & -9 \end{bmatrix}$$

$R_3 \to 7R_3 - 5R_2$

$$\begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & -7 & -3 & -11 \\ 0 & 0 & 8 & -8 \end{bmatrix} \quad \frac{R_3}{8}$$

$$\begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & -7 & -3 & -11 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Rank of $[A|B] = 3$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -7 & -3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -11 \\ -1 \end{bmatrix}$$

$$x + 2y + z = 4$$

$$-7y - 3z = -11$$

$$\boxed{z = -1}$$

$$-7y - 3(-1) = -11$$

$$-7y + 3 = -11$$

$$-7y = -11 - 3$$

$$\cancel{-}7y = \cancel{-}14$$

$$y = 14/7$$

$$\boxed{y = 2}$$

$$x + 2(2) - 1 = 4$$

$$x + 4 - 1 = 4$$

$$x + 3 = 4$$

$$x = 4 - 3$$

$$\boxed{x = 1}$$

$$X = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

ii) $10x+y+z=12,\ 2x+10y+z=13,\ x+y+5z=7$.

$$A=\begin{bmatrix}10 & 1 & 1\\ 2 & 10 & 1\\ 1 & 1 & 5\end{bmatrix} \quad X=\begin{bmatrix}x\\ y\\ z\end{bmatrix} \quad B=\begin{bmatrix}12\\ 13\\ 7\end{bmatrix}$$

$$[A|B]=\begin{bmatrix}10 & 1 & 1 & 12\\ 2 & 10 & 1 & 13\\ 1 & 1 & 5 & 7\end{bmatrix}$$

$R_1 \leftrightarrow R_3$

$$\begin{bmatrix}1 & 1 & 5 & 7\\ 2 & 10 & 1 & 13\\ 10 & 1 & 1 & 12\end{bmatrix}$$

$R_2 \to R_2-2R_1$ ; $R_3 \to R_3-10R_1$

$$\begin{bmatrix}1 & 1 & 5 & 7\\ 0 & 8 & -9 & -1\\ 0 & -9 & -49 & -58\end{bmatrix}$$

$R_3 \to 8R_3+9R_2$

$$\begin{bmatrix}1 & 1 & 5 & 7\\ 0 & 8 & -9 & -1\\ 0 & 0 & -473 & -473\end{bmatrix} \quad \frac{R_3}{-473}$$

$$\begin{bmatrix}1 & 1 & 5 & 7\\ 0 & 8 & -9 & -1\\ 0 & 0 & 1 & 1\end{bmatrix}$$

$$\begin{bmatrix}1 & 1 & 5\\ 0 & 8 & -9\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}7\\ -1\\ 1\end{bmatrix}$$

$x+y+5z=7$

$8y-9z=-1$

$\boxed{z=1}$

$8y-9=-1$

$8y=-1+9$

$8y=8$

$\boxed{y=1}$

$x+1+5=7$

$x+6=7$

$x=7-6$

$\boxed{x=1}$

$$X=\begin{bmatrix}1\\1\\1\end{bmatrix}$$

## UNIT-II

## EIGEN VALUES,EIGEN VECTORS AND QUADRATIC FORMS

**Short answer questions**

1. Define Eigen values and eigen vectors of a matrix
2. State the condition of diagonalizability of a matix
3. State Caley –Hamilton theorem and write its applications
4. Write any five properties of eigen values
5. If A= $\begin{bmatrix} 5 & 2 & 3 \\ 0 & -1 & 4 \\ 0 & 0 & 6 \end{bmatrix}$ , find eigen values of i) 5A ii)$A^4$ iii) adjA iv) $A^{-1}$
6. Prove that Eigen values of unitary matrix are of constant modulus
7. Define Modal matrix and Spectral matrix
8. i) If $\lambda$ is an eigen values of an Orthogonal matrix then $1/\lambda$ is also its Eigen value
   ii) Prove that if $\lambda$ is an Eigen value of a matrix A then $\lambda$+KI is Eigen value of A+KI
9. Define Quadratic form, canonical form, index, signature and nature of quadratic form
10. Find index, signature and nature of quadratic form 2xy+2yz+2zx
11. Define linear Transformation
12. If $\lambda$ is an Eigen value of a matrix A then prove that $\lambda^2$ is an Eigen value of $A^2$
13. Define Algebraic multiplicity and Geometric multiplicity of an Eigen value of a matrix
14. Show that Eigen values of unitary matrix are of unit modulus.

**LONG ANSWER QUESTIONS**

1. Find the Eigen values and the corresponding Eigen vectors of the matrices: $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ $\begin{bmatrix} 3 & 2 & 2 \\ 1 & 2 & 2 \\ -1 & -1 & 0 \end{bmatrix}$
2. Prove that the Eigen values of Hermitian matrix are all real.
3. Prove that the Eigen values of real symmetric matrix are all real.
4. Diagonalize the matrix A= $\begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$
5. Diagonalize the matrix $\begin{bmatrix} 9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$ Hence find $A^{-1.}$
6. If A= $\begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$ verify Caley-Hamiltton theorem, Hence find $A^{-1}$.
7. Disscuss the nature of the Quadratic form, Find its Rank, Index, and Signature.
   (1) $x^2+4xy+6xz-y^2+2yz+4z^2$
   (2) $2x^2+2y^2+2z^2+2yz$
8. Reduce the Quadratic form $10x^2+2y^2+5z^2-10zx+6yz-4xy$ the Canonical form. Also find the transformation.

9) Reduce the Quadratic form $2x^2+5y^2+3z^2+4xy$ to Canonical form. Also find the transformation.

10) Reduce the Quadratic form $3x^2+2y^2+3z^2-2xy-2zx$ to the Canonical form by Orthogonal transformation. Also find the transformation.

11) Reduce the Quadratic form $3x^2+5y^2+3z^2-2yz+2zx-2xy$ to the Canonical form by Orthogonal transformation. Also find the transformation.

12) Express the Hermitian matrix $A=\begin{bmatrix} 1 & -i & 1+i \\ i & 0 & 2-3i \\ 1-i & 2+3i & 2 \end{bmatrix}$

As P+iQ where P is a Real symmetric matrix and Q is a Real skew-symmetric matrix.

QUESTION BANK
UNIT - II

EIGEN VALUES AND EIGEN VECTORS

## Short Answer questions.

1. Define Eigen values and eigen vectors of a matrix.

Ans: Eigen values:- The roots of characterstic eqn's are known as eigen values (or) characterstic roots

Eigen vectors:- If $AX = \lambda X$ $(X \neq 0)$ then X is called eigen vector corresponding to eigen value $\lambda$.

2. state the condition of diagonalizability of a matrix

Ans:- A matrix A is diagonalizable if there exists an invertible matrix P such that $P^{-1}AP = D$ where D is a diagonal matrix. Also the matrix P is said to diagonalize A or transform A to diagonal form.

3. state caley-Hamilton theorem and write its applications

Ans:- Every square matrix statisfies its own characteristic equation that is if chaacteristic eqn of A is

$$\lambda^n - a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_{n-1}\lambda - a_n = 0$$

Then $A^n - a_1A^{n-1} + a_2A^{n-2} + \cdots + a_{n-1}A - a_nI = 0$

Applications:-

The important applications of cayley-Hamilton theorem are

1. To find the inverse of a matrix
2. To find higher power of the matrix.

4. Write any five properties of eigen values.

(i) The sum of eigen values of a square matrix is equal to its trace and product of the eigen values is equal to its determinant.

(ii) If $\lambda$ is an eigen value of $A$ corresponding to the eigen vector $x$, then $\lambda^n$ is an eigen value of $A^n$ corresponding to eigen vector $x$.

(iii) A square matrix $A$ and its transpose $A^T$ have the same eigen values.

(iv) If $\lambda$ is an eigen value of a matrix $A$ then $k\lambda$ is an eigen value of the matrix $kA$ corresponding to same eigen vector $x$.

(v) If $x$ is an eigen vector of $A$ then $x$ cannot correspond to more than one eigen value.

5. if $A = \begin{bmatrix} 5 & -2 & 3 \\ 0 & -1 & 4 \\ 0 & 0 & 6 \end{bmatrix}$. find eigen values of i) $5A$ ii) $A^4$ iii) $Adj^{\cdot} A$ iv) $A^{-1}$

$S_1 = 5 - 1 + 6 = 10$

$S_2 = \begin{vmatrix} -1 & 4 \\ 0 & 6 \end{vmatrix} + \begin{vmatrix} 5 & 3 \\ 0 & 6 \end{vmatrix} + \begin{vmatrix} 5 & -2 \\ 0 & -1 \end{vmatrix}$

$= -6 + 30 - 5 = 19.$

$S_3 = -30$

$\Rightarrow \lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$\lambda^3 - 10\lambda^2 + 19\lambda + 30 = 0$

$\lambda = -1, 6, 5.$

(i) If $\lambda_1, \lambda_2, \lambda_3 \cdots \lambda_n$ are eigen values of $A$ then $K\lambda_1, K\lambda_2, \cdots K\lambda_n$ are eigen values of $KA$.

For $5A$, the eigen values are $5\lambda_1, 5\lambda_2, 5\lambda_3$

$= -1(5), \quad 6(5), \quad 5(5)$

$= -5, 30, 25.$

(ii) If $\lambda$ is eigen value of $A$, then $\lambda^n$ is eigen value of $A^n$.

For $A^4 \Rightarrow \lambda^n = \lambda^4$

$= (-1)^4, (6)^4, (5)^4$

$= 1, 1296, 625.$

(iii) If $\lambda$ is eigen value of matrix, then $\frac{|A|}{\lambda}$ is eigen value of Adj A.

For Adj A, eigen values are

$\frac{|A|}{-1}, \frac{|A|}{6}, \frac{|A|}{5} = \frac{-30}{-1}, \frac{-30}{6}, \frac{-30}{5}$

$= 30, -5, -6.$

(iv) If $\lambda$ is eigen value of matrix $A$, then $\lambda^{-1}$ is eigen value of $A^{-1}$.

For $A^{-1}$, eigen values are

$(-1)^{-1}, (6)^{-1}, (5)^{-1}$

$= -1, \frac{1}{6}, \frac{1}{5}.$

6) Prove that eigen values of unitary matrix are of constant modulus.

Let A be a square unitary matrix whose eigen value is $\lambda$ with corresponding eigen vector $x$.

Then we have $Ax = \lambda x$ —— (1) $\Rightarrow \bar{A}\bar{x} = \bar{\lambda}\bar{x} = \bar{x}^T\bar{A}^T = \bar{\lambda}\bar{x}^T$ —— (2)

Since A is unitary, we have $(\bar{A})^T A = I$ —— (3)

(1) and (2) gives, $\bar{x}^T\bar{A}^T \cdot Ax = \lambda\bar{\lambda}\bar{x}^T x$

(i.e.,) $\bar{x}^T = \lambda\bar{\lambda}\bar{x}^T . x$ by (3) $\Rightarrow \bar{x}^T x(1-\lambda\bar{\lambda}) = 0$

Since $\bar{x}^T x \neq 0$, we must have $1-\lambda\bar{\lambda} = 0 \Rightarrow \lambda\bar{\lambda} = 1$

Since $|\lambda| = |\bar{\lambda}|$ we must have $|\lambda| = 1$

Cor 1: The characteristic root of an orthogonal matrix is unit modudulus". If the element of unitary matrix are all real then it is an Orthogonal matrix.

Cor 2: The only eigen values of unitary matrix can be $+1$ or $-1$

7) Define Modal Matrix and Spectral matrix

Modal Matrix:

It is used in the diagonalization process involving eigen values and eigenvectors

Modal Matrix M for the matrix A is the $n \times n$ matrix formed with the eigen vectors of A as columns in M. It is utilized in the similarity transformation

Spectral matrix:

Where D is an $n \times n$ diagonal matrix with the eigen values of A on the main diagonal of D and zeros elsewhere. The matrix D is called the spectral matrix of A

8. i) If $\lambda$ is an eigen values of an orthogonal matrix then $1/\lambda$ is also its Eigen value.

An:- Proof:- We know that if $\lambda$ is an eigen value of a matrix $A$, then $1/\lambda$ is an eigen value of $A^{-1}$. Since $A$ is an orthogonal matrix, therefore $A^{-1} = A'$

$\therefore \frac{1}{\lambda}$ is an eigen value of $A'$

But the matrices $A$ and $A'$ have the same eigen values, since the determinants $|A-\lambda I|$ and $|A'-\lambda I|$ are same

Hence $\frac{1}{\lambda}$ is also an eigen value of $A$.

ii) Prove that if $\lambda$ is an Eigen value of a matrix $A$ then $\lambda + KI$ is Eigen value of $A+KI$

An: Proof:- Let $\lambda$ be an eigen value of $A$ and $x$ the corresponding eigen vector.

Then, by definition $AX = \lambda X$ --- (1)

Now $(A+KI)X = AX + KIX = \lambda X + KX = (\lambda+k)x$ [by 1] -- (2)

From (2), we see that the scalar $\lambda + k$ is an eigen values of the matrix $A+KI$ and $x$ is a corresponding eigen vector.

9. Define Quadratic form, canonical form, Index, signature and nature of Quadratic form.

An: Quadratic form:- A homogeneous expression of second degree any no. of variables is called Quadratic form.

Canonical form :- It is a Quadratic form without cross product terms

Ex :- $x^2 + 25x_2^2 + 28x_3^2$

Index :- The no. of +ve terms in canonical form and it is denoted by "p"

Signature :- The difference of "-ve" terms and "+ve" terms of canonical form denoted by s

Nature of Quadratic form :-

→ positive definite :- If all terms are +ve (or) all eigen values are '+ve'

→ positive semi definite :- all terms [eigen values] are '+ve' with at least 1 zero

→ Negative definite :- All terms of eigen values are negative

→ Negative semidefinite :- If all terms of eigen values are negative with atleast one zero

→ Indefinite :- If some are +ve and some are -ve

10. Find index, signature and nature of Quadratic form

$2xy + 2yz + 2zx$

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$\Rightarrow |A - \lambda I| = 0 \Rightarrow \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0$

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

~~$S_1 = 0$~~ ~~$S_2 = 0$~~ ~~$S_3 = 2$~~.

~~$\lambda^3 - 2 = 0$~~

$$-\lambda^3 + 3\lambda + 2 = 0$$

$$\lambda = -1, -1, 2.$$

$\therefore$ Indefinite. rank $r = 3$.

Index $s = 1$

Signature $2s - r = 2 - 3 = -1$.

11. Define linear Transformation?

let the point $P(x, y)$ with respect to a set of rectangle axes $OX$ and $OY$ be transformed to the point $P'(x', y')$ with respect to a set of rectangular axes $OX'$, and $OY'$ by folloing relation

$$x' = a_1 x + a_2 y \qquad y' = b_1 x + b_2 y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ b_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

such a transformation is called Linear transformation

12. If $\lambda$ is an Eigen Value of a matrix $A$ then prove that $\lambda^2$ is an Eigen value of $A^2$

Proof: Since $\lambda$ is an eigen Value of $A$ corresponding to the eigen vector $x$, we have

$$AX = \lambda x \quad \text{——— (1)}$$

Premultiply (1) by $A$. $\quad A(AX) = A(\lambda x)$

i.e., $(AA)x = \lambda(Ax)$ i.e., $A^2 x = \lambda \cdot \lambda x = \lambda^2 x$

Hence $\lambda^2$ is eigen value of $A^2$ with $x$ itself as the corresponding eigen vector. Thus the theorem is true to $n=2$. Let the result be true for $n=k$

Then $A^k x = \lambda^k x$

Premultiplying this by $A$ and using $Ax = \lambda x$,

we get

$$A^{k+1} x = \lambda^{k+1} x$$

This implies that $\lambda^{k+1}$ is eigen value of $A^{k+1}$ with $X$ itself as the corresponding eigen vector. Hence by the principle of mathematical induction, the theorem is true for all positive integers $n$.

18) Define Algebraic multiplicity and Geometric multiplicity of an Eigen Value of a matrix.

Algebraic multiplicity:

Suppose $A$ is $n \times n$ matrix. If $\lambda_1$ is a characteristics equation of $A$, then ~~it~~ $t$ is called the algebraic multiplicity of $\lambda_1$

In other words, Algebraic multiplicity of an eigen value is the number of times of repetition of an eigen value

ie, if $|A - \lambda I| = 0 = (\lambda - \lambda_1)^r$ then the algebraic multiplicity of eigen value $\lambda_1$ is $r$.

Geometric multiplicity:

If $s$ is the number of linearly independent characteristic vectors corresponding to the repeated eigen value $\lambda$, then $s$ is called the geometric multiplicity of $\lambda$

That is, geometric multiplicity of an eigen value is the number of linearly independent eigen vectors corresponding to the repeated eigen value $\lambda$

14) Show that eigen values of unitary matrix are of unit modulus.

Let $A$ be a square unitary matrix whose eigen value is $\lambda$ with corresponding eigen vector $x$

Then we have $Ax = \lambda x$ —— ①

$\bar{A}\bar{x} = \bar{\lambda}\bar{x} \Rightarrow \bar{x}^T \bar{A}^T = \bar{\lambda}\bar{x}^T$ —— ②

Since $A$ is unitary, we have $(\bar{A})^T A = I$ —— ③

① and ② gives, $\bar{x}^T \bar{A}^T A x = \lambda\bar{\lambda}\bar{x}^T x$

(i.e.,) $\bar{x}^T = \lambda\bar{\lambda}\,\bar{x}^T . x$ by (3) $\Rightarrow \bar{x}^T x(1-\lambda\bar{\lambda}) = 0$

Since $\bar{x}^T x \neq 0$, we must have $1-\lambda\bar{\lambda} = 0 \Rightarrow \lambda\bar{\lambda} = 1$

Since $|\lambda| = |\bar{\lambda}|$ we must have $|\lambda| = 1$

Cor 1 : The characteristic root of an orthogonal matrix is unit modulus". If the element of unitary matrix are all real then it is an orthogonal matrix

Cor 2 : The only eigen values of unitary matric can be $+1$ or $-1$

## LONG ANSWER QUESTIONS

1) Find the Eigen Values and the corresponding Eigen vectors of the matrices :

i)
$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

Characteristic equation of A is $|A-\lambda I|=0$

i.e.,
$$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$$8-\lambda\ ((7-\lambda)-(3-\lambda)) + 6[-6(3-\lambda)+8] + 2[24-2(7-\lambda)] = 0$$

$$8-\lambda\ (21-7\lambda+3\lambda+\lambda^2) + 6[-18+6\lambda+8] + 2[24-14+2\lambda] = 0$$

$$8-\lambda\ (21-4\lambda+\lambda^2) + 6(-10+6\lambda) + 2(10+2\lambda) = 0$$

$$-\lambda^3+18\lambda^2-45\lambda = 0$$

$$-\lambda(\lambda^2-18\lambda+45)=0$$

$$-\lambda(\lambda^2-15\lambda-3\lambda+45)=0$$

$$-\lambda(\lambda(\lambda-15)-3(\lambda-15))=0$$

$$-\lambda((\lambda-15)(\lambda-3))=0$$

$$\lambda=0,\ 15,\ 3$$

Case (i): Let $\lambda=0$

To find $X$

$$(A-\lambda I)X=0$$

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$

$$\begin{bmatrix} 2 & -4 & 3 \\ -6 & 7 & -4 \\ 8 & -6 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 : R_2 + 3R_1 \quad ; \quad R_3 : R_3 - 4R_1$

$$\begin{bmatrix} 2 & -4 & 3 \\ 0 & -5 & 5 \\ 0 & 10 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 : R_3 + 2R_2$

$$\begin{bmatrix} 2 & -4 & 3 \\ 0 & -5 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad 2x_1 - 4x_2 + 3x_3 = 0$ —— (1)

$-5x_2 + 5x_3 = 0$

$x_2 = x_3$

$\therefore$ Sub $\boxed{x_2 = x_3}$ in (1)

$2x_1 - 4x_3 + 3x_3 = 0$

$2x_1 - x_3 = 0$

$2x_1 = x_3$

$x_1 = \frac{x_3}{2}$

let $x_3 = K$

$x_2 = K$

$x_1 = \frac{K}{2}$

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} K/2 \\ K \\ K \end{bmatrix} = K/2 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$$X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

Case-2:

Let $\lambda = 15$

$(A - \lambda I)X = 0$

$$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$

$$\begin{pmatrix} 2 & -4 & -12 \\ -6 & -8 & -4 \\ -7 & -6 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$R_1 \Rightarrow \frac{R_1}{2} \qquad R_2 \to -\frac{R_2}{2}$

$$\begin{pmatrix} 1 & -2 & -6 \\ 3 & 4 & 2 \\ -7 & -6 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$R_2 : R_2 - 3R_1 \qquad R_3 : R_3 + 7R_1$

$$\begin{pmatrix} 1 & -2 & -6 \\ 0 & 10 & 20 \\ 0 & -20 & -40 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$R_3 : R_3 + 2R_2$

$$\begin{pmatrix} 1 & -2 & -6 \\ 0 & 10 & 20 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow x_1 \cancel{+} - 2x_2 - 6x_3 = 0$ —— (1)

$$10x_2 + 20x_3 = 0$$

$$x_2 = -2x_3$$

Sub $x_2$ value in (1)

$$x_1 + 4x_3 - 6x_3 = 0$$

$$x_1 - 2x_3 = 0$$

$$x_1 = 2x_3$$

let $x_3 = K$ Then $x_2 = -2K$ $x_1 = 2K$

$$X_2 = \begin{pmatrix} 2K \\ -2K \\ K \end{pmatrix} = K \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}$$

$$X_2 = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$$

Case-III Let $\lambda = 3$

$(A-\lambda I)x = 0$

$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 2 & -4 & 0 \\ -6 & 4 & -4 \\ 5 & -6 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\cancel{R_2 : R_2 + 3R_1}$

$$\cancel{\begin{bmatrix} 2 & -4 & 0 \\ 0 & & \\ & & \end{bmatrix}}$$

$R_1 : R_1/2 \quad R_2 : R_2/2$

$$\begin{bmatrix} 1 & -2 & 0 \\ -3 & 2 & -2 \\ 5 & -6 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 : R_2 + 3R_1 \quad R_3 : R_3 - 5R_1$

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & -4 & -2 \\ 0 & 4 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 : R_3 + R_2$

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & -4 & -2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rightarrow x_1 - 2x_2 = 0$ ——— (1)

$-4x_2 - 2x_3 = 0$

$\boxed{x_2 = -\tfrac{1}{2}x_3}$ sub in (1)

$x_1 - 2\left(-\tfrac{1}{2}\right)x_3 = 0$

$\boxed{x_1 = -\cancel{\tfrac{1}{2}}x_3}$

let $x_3 = k$

$$X_3 = \begin{bmatrix} -k \\ -\tfrac{1}{2}k \\ k \end{bmatrix} = \frac{k}{2}\begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix} \quad \therefore X_3 = \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$$

(iii) $\begin{pmatrix} 3 & 2 & 2 \\ 1 & 2 & 2 \\ -1 & -1 & 0 \end{pmatrix}$

$$A = \begin{pmatrix} 3 & 2 & 2 \\ 1 & 2 & 2 \\ -1 & -1 & 0 \end{pmatrix}$$

Characteristic equation of $A$ is $|A-\lambda I| = 0$

i.e., $\begin{vmatrix} 3-\lambda & 2 & 2 \\ 1 & 2-\lambda & 2 \\ -1 & -1 & -\lambda \end{vmatrix} = 0$

$$3-\lambda\,[(2-\lambda)(-\lambda)+2] - 2[-\lambda+2] + 2[-1+(2-\lambda)] = 0$$

$$3-\lambda\,[-2\lambda+\lambda^2+2] + 2\lambda - 4 + 2[-1+2-\lambda] = 0$$

$$-6\lambda + 3\lambda^2 + \cancel{6} + 2\lambda^2 - \lambda^3 - \cancel{2\lambda} + \cancel{2\lambda} - 4 - \cancel{2} + 4 - 2\lambda = 0$$

$$-\lambda^3 + 5\lambda^2 - 8\lambda + 4 = 0$$

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

$$\therefore \lambda = 2, 2, 1$$

<u>Case 1</u> let $\lambda = 2$

$$(A-\lambda I)X = 0$$

$$\begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & 2 \\ -1 & -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 : R_2 - R_1 \qquad R_3 : R_3 + R_1$

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & -2 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad x_1 + 2x_2 + 2x_3 = 0$ ——— (i)

$-2x_2 = 0 \Rightarrow x_2 = 0$

$x_2 = 0$ Sub in (i)

$$x_1 + 2x_3 = 0$$

$$x_1 = -2x_3$$

let $\boxed{x_3 = k}$

$$x_1 = -2k$$

$$x_2 = 0$$

$$X_1 = \begin{pmatrix} -2k \\ 0 \\ k \end{pmatrix} = k\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

$$X_1 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

<u>Case-2</u> let $\lambda = 1$

$$\begin{pmatrix} 2 & 2 & 2 \\ 1 & 1 & 2 \\ -1 & -1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$R_1 \leftrightarrow R_2$

$$\begin{pmatrix} 1 & 1 & 2 \\ 2 & 2 & 2 \\ -1 & -1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$R_2 : R_2 - 2R_1 \qquad R_3 : R_3 + R_2$

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & -2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\Rightarrow \quad x_1 + x_2 + 2x_3 = 0 \quad \text{——— (1)}$$

$$-2x_3 = 0$$

$$x_3 = 0$$

let $\boxed{x_2 = k}$ $\quad x_1 = -k$

$$\therefore X_2 = \begin{pmatrix} -k \\ k \\ 0 \end{pmatrix} = -k\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$X_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

2) Prove that the Eigen Values of Hermitian matrix are all real

Proof :

Let $A$ be a Hermitian matrix. If $x$ be the eigen Vector Corresponding to the eigen value $\lambda$ of $A$, then

$$Ax = \lambda x \quad \text{—— (1)}$$

Multiply both sides of (1) by $x^{\theta}$, we get

$$x^{\theta}Ax = \lambda x^{\theta}x \quad \text{—— (2)}$$

Taking Conjugate transpose of bothsides of (2), we get

$$(x^{\theta}Ax)^{\theta} = (\lambda x^{\theta}x)^{\theta}$$

i.e., $$x^{\theta}A^{\theta}(x^{\theta})^{\theta} = \bar{\lambda}x^{\theta}(x^{\theta})^{\theta}$$

$[\because (ABC)^{\theta} = C^{\theta}B^{\theta}A^{\theta} \text{ and } (KA)^{\theta} = \bar{K}A^{\theta}]$

Or $$x^{\theta}A^{\theta}x = \bar{\lambda}x^{\theta}x \quad [\because (x^{\theta})^{\theta} = x,\ (A^{\theta})^{\theta} = A] \quad \text{—— (3)}$$

From (2) & (3)

$$\lambda x^{\theta}x = \bar{\lambda}x^{\theta}x$$

i.e.; $$(\lambda - \bar{\lambda})x^{\theta}x = 0$$

$$\lambda - \bar{\lambda} = 0$$

$$\therefore \lambda = \bar{\lambda}$$

Hence $\lambda$ is real

Thus the eigen values of a Hermitian matrix are all real.

3) Prove that the Eigen Values of real symmetric matrix are all real.

If the elements of a Hermitian matrix $A$ are all real, then $A$ is a real symmetric matrix. Thus a real symmetric matrix is Hermitian

→ Continue proof as Question 2 ←

4b Diagonalize the matrix $A = \begin{pmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{pmatrix}$

Given $A = \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$

The characteristic equation of A is $|A-\lambda I| = 0$

i.e., $\begin{vmatrix} 8-\lambda & -8 & -2 \\ 4 & -3-\lambda & -2 \\ 3 & -4 & 1-\lambda \end{vmatrix} = 0$

$\Rightarrow (8-\lambda)\left[(-3-\lambda)(1-\lambda)-8\right] + 8\left[4(1-\lambda)+6\right] - 2\left\{-16-3(-3-\lambda)\right\} = 0$

$\Rightarrow -\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0$

$\Rightarrow \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

$\therefore \lambda = 3, 1, 2$

$\therefore$ Diagonal matrix is $\begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

Case-1 let $\lambda = 3$

$(A-\lambda I)X = 0$

i.e., $\begin{bmatrix} 8-3 & -8 & -2 \\ 4 & -3-3 & -2 \\ 3 & -4 & 1-3 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{pmatrix} 5 & -8 & -2 \\ 4 & -6 & -2 \\ 3 & -4 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$R_2 : 5R_2 - 4R_1 \qquad R_3 : 5R_3 - 3R_1$

$\begin{pmatrix} 5 & -8 & -2 \\ 0 & 2 & -2 \\ 0 & 4 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$R_3 : R_3 - 2R_2$

$$\begin{pmatrix} 5 & -8 & -2 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$5x_1 - 8x_2 - 2x_3 = 0 \longrightarrow (1)$$

$$2x_2 - 2x_3 = 0 \longrightarrow (2)$$

$$\boxed{x_2 = x_3}$$

let $x_3 = k$

$x_2 = k$

sub values in (1)

$$5x_1 - 8k - 2k = 0$$

$$5x_1 - 10k = 0$$

$$5x_1 = 10k$$

$$x_1 = 2k$$

$$X_1 = \begin{pmatrix} 2k \\ k \\ k \end{pmatrix} = k \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \qquad X_1 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

Case 2 $\lambda = 1$

$$(A - \lambda I)X = 0$$

$$\begin{pmatrix} 7 & -8 & -2 \\ 4 & -4 & -2 \\ 3 & -4 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$R_2 : 7R_2 - 4R_1$ $\quad R_3 : 7R_3 - 3R_1$

$$\begin{pmatrix} 7 & -8 & -2 \\ 0 & 4 & -6 \\ 0 & -4 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$R_3 : R_3 + R_2$

$$\begin{pmatrix} 7 & -8 & -2 \\ 0 & 4 & -6 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$7x_1 - 8x_2 - 2x_3 = 0 \quad \text{——} (1)$$

$$4x_2 - 6x_3 = 0$$

$$4x_2 = 6x_3$$

$$\rightarrow x_2 = \tfrac{3}{2} x_3$$

let $\boxed{x_3 = k}$ $\quad x_2 = \tfrac{3}{2} k$ sub in (1)

$$7x_1 - 8\left(\tfrac{3}{2}\right)k - 2k = 0$$

$$14x_1 - 24k - 4k = 0$$

$$14x_1 - 28k = 0$$

$$14x_1 = 28k$$

$$\boxed{x_1 = 2k}$$

$$X_2 = \begin{bmatrix} 2k \\ \tfrac{3}{2}k \\ k \end{bmatrix} = \frac{k}{2}\begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$$

Case-3 $\lambda = 2$

$$(A - \lambda I)X = 0$$

$$\begin{bmatrix} 8-2 & -8 & -2 \\ 4 & -3-2 & -2 \\ 3 & -4 & 1-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 6 & -8 & -2 \\ 4 & -5 & -2 \\ 3 & -4 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 : 6R_2 - 4R_1 \qquad R_3 : 2R_3 - R_1$

$$\begin{bmatrix} 6 & -8 & -2 \\ 0 & 2 & -4 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rightarrow \quad 6x_1 - 8x_2 - 2x_3 = 0 \quad \text{——} (1)$

$2x_2 - 4x_3 = 0 \quad \text{——} (2)$

$2x_2 = 4x_3$

$x_2 = 2x_3$

let $x_3 = k \qquad x_2 = 2k$ sub in (1)

$6x_1 - 16k - 2k = 0$

$6x_1 - 18k = 0$

$6x_1 = 18k$

$x_1 = 3k$

$$X_3 = \begin{bmatrix} 3k \\ 2k \\ k \end{bmatrix} = k \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$X_3 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

Thus $P = [X_1 \; X_2 \; X_3] = \begin{bmatrix} 2 & 4 & 3 \\ 1 & 3 & 2 \\ 1 & 2 & 1 \end{bmatrix}$

$$P^{-1}AP = D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & \cancel{2} \\ 0 & 0 & 2 \end{bmatrix}$$

5b. Diagonalize the matrix $\begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$ Hence find $A^{-1}$

$$A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$$

Characteristic eq$^n$ $|A - \lambda I| = 0$

$$\begin{vmatrix} -9-\lambda & 4 & 4 \\ -8 & 3-\lambda & 4 \\ -16 & 8 & 7-\lambda \end{vmatrix} = 0$$

$-9-\lambda \left( (3-\lambda)(7-\lambda) - 32 \right) - 4 \left( (-8)(7-\lambda) + 64 \right) + 4 \left( -64 - (-16)(3-\lambda) \right) = 0$

$-9-\lambda \left( 21 - 3\lambda - 7\lambda + \lambda^2 - 32 \right) - 4 \left( -56 + 8\lambda + 64 \right) + 4 \left( -64 + 48 - 16\lambda \right) = 0$

$$-9-\lambda(-11-10\lambda+\lambda^2)-4(8\lambda+8)+4(-16-16\lambda)=0$$

$$99+90\lambda-9\lambda^2+11\lambda+10\lambda^2-\lambda^3-32\lambda-32-64-64\lambda=0$$

$$-\lambda^3+\lambda^2+5\lambda+3=0$$

$$\lambda^3-\lambda^2-5\lambda-3=0$$

$$\lambda=3,-1,-1$$

Case-1

Let $\lambda=3$

$$\begin{bmatrix}-12 & 4 & 4\\ -8 & 0 & 4\\ -16 & 8 & 4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=0$$

$R_2: 12R_2-8R_1$

$R_3: 12R_3-16R_1$

$$\begin{bmatrix}-12 & 4 & 4\\ 0 & -32 & 16\\ 0 & 32 & 16\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=0$$

$R_3:-R_3+R_2$

$$\begin{bmatrix}-12 & 4 & 4\\ 0 & -32 & 16\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=0$$

$\frac{R_2}{16}$, $\frac{R_1}{4}$

$$\begin{bmatrix}-3 & 1 & 1\\ 0 & -2 & 1\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=0$$

$$-3x_1+x_2+x_3=0 \quad (1)$$

$$-2x_2+x_3=0 \quad (2)$$

from (2)

$$-2x_2=-x_3$$

$$x_2=\tfrac{1}{2}x_3$$

let $x_3=k$ $\quad x_2=k/2$

sub $x_2$ and $x_3$ in eq (1)

$$-3x_1 + \tfrac{1}{2}\cancel{x_3}K + K = 0$$

$$-\cancel{6}x_1 + 3K = 0$$

$$-6x_1 = -3K$$

$$x_1 = K/2$$

$$X_1 = \begin{pmatrix} K/2 \\ K/2 \\ K \end{pmatrix} = \frac{K}{2}\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

$$X_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

Case 2 let $\lambda = -1$

$$\begin{bmatrix} -\cancel{4}8 & 4 & 4 \\ -8 & 4 & 4 \\ -16 & 8 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$R_2 : R_2 - 2R_1$

$R_3 : R_3 - 2R_1$

$$\begin{bmatrix} \cancel{4}8 & 4 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$R_1 \to R/4$

$$\begin{bmatrix} -2 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$-2x_1 + x_2 + x_3 = 0 \quad \text{—— (1)}$$

$n = 3 \quad r = 1$

$n - r = 2$

let $x_2 = k_1$

$x_3 = k_2$

$$-2x_1 + k_1 + k_2 = 0$$

$$-2x_1 = \underline{-(k_1 + k_2)}$$

$$x_1 = \frac{k_1 + k}{2}$$

$$X_2 = \begin{bmatrix} \frac{k_1 + k_2}{2} \\ k_1 \\ k_2 \end{bmatrix}$$

$$X = \frac{k_1}{2} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \frac{k}{2} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \quad X_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$\therefore \quad X_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \quad X_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \quad X_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

Thus $P(x_1\ x_2\ x_3) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$

$P^{-1}AP = D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

6b. If $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$ verify Caley-Hamilton theorem. Hence find $A^{-1}$

Given $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$

Characteristic eqn of A is given by $|A-\lambda I| = 0$

$\Rightarrow \begin{vmatrix} 1-\lambda & 2 & -1 \\ 2 & 1-\lambda & -2 \\ 2 & -2 & 1-\lambda \end{vmatrix} = 0$

$1-\lambda\left[(1-\lambda)(1-\lambda)-4\right] - 2\left[2(1-\lambda)+4\right] - 1\left[-4-(2)(1-\lambda)\right] = 0$

$1-\lambda\left[1-\lambda-\lambda+\lambda^2-4\right] - 2\left[2-2\lambda+4\right] - 1\left[-4-2+2\lambda\right] = 0$

$1-\lambda\left[-3-2\lambda+\lambda^2\right] - 2\left[6-2\lambda\right] - 1\left[-6+2\lambda\right] = 0$

$-3-\cancel{2\lambda}+\lambda^2+3\lambda+2\lambda^2-\lambda^3 - 12+\cancel{4\lambda}+6-\cancel{2\lambda} = 0$

$-\lambda^3+3\lambda^2+3\lambda-9 = 0$

$\lambda^3-3\lambda^2-3\lambda+9 = 0.$

By Cayley-Hamilton theorem, Matrix A should satisfy its characteristic eqⁿ

$$A^3 - 3A^2 - 3A + 9I = 0 \quad \text{——— (2)}$$

Now
$$A^2 = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 6 & -6 \\ 0 & 9 & -6 \\ 0 & 0 & 3 \end{bmatrix}$$

and
$$A^3 = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 6 & -6 \\ 0 & 9 & -6 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 24 & -21 \\ 6 & 21 & -24 \\ 6 & -6 & 3 \end{bmatrix}$$

$\therefore$ $A^3 - 3A^2 - 3A + 9I$

$$= \begin{bmatrix} 3 & 24 & -21 \\ 6 & 21 & -24 \\ 6 & -6 & 3 \end{bmatrix} - 3\begin{bmatrix} 3 & 6 & -6 \\ 0 & 9 & -6 \\ 0 & 0 & 3 \end{bmatrix} - 3\begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} + 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence Cayley Hamilton is Verified

To find $A^{-1}$

Multiply eq (2) with $A^{-1}$ on both sides

$$A^{-1}\left[A^3 - 3A^2 - 3A + 9I\right] = A^{-1}$$

$$A^2 - 3A - 3I + 9A^{-1} = 0$$

$$9A^{-1} = 3A + 3I - A^2$$

$$A^{-1} = \frac{1}{9}\left\{\begin{bmatrix} 3 & 6 & -3 \\ 6 & 3 & -6 \\ 6 & -6 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 6 & -6 \\ 0 & 9 & -6 \\ 0 & 0 & 3 \end{bmatrix}\right\}$$

$$= \frac{1}{9}\begin{bmatrix} 3 & 0 & 3 \\ 6 & -3 & 0 \\ 6 & -6 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & 0 \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

7) Discuss the nature of the Quadratic form. Find its Rank, Index and Signature

1) $x^2 + 4xy + 6xz - y^2 + 2yz + 4z^2$

This can be written in matrix form

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 4 \end{bmatrix}$$

Characteristic eqn of A is $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & -1-\lambda & 1 \\ 3 & 1 & 4-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\left[(-1-\lambda)(4-\lambda) - 1\right] - 2\left[2(4-\lambda) - 3\right] + 3\left[2 - 3(-1-\lambda)\right] = 0$$

$$(1-\lambda)\left[-4+\lambda-4\lambda+\lambda^2-1\right] - 2\left[8-2\lambda-3\right] + 3\left[2+3+3\lambda\right] = 0$$

$$(1-\lambda)\left[-5-3\lambda+\lambda^2\right] - 2\left[5-2\lambda\right] + 3\left[5+3\lambda\right] = 0$$

$$-5-3\lambda+\lambda^2+5\lambda+3\lambda^2-\lambda^3 - 10+4\lambda+15+9\lambda = 0$$

$$-\lambda^3 + 4\lambda^2 + 15\lambda = 0$$

$$\lambda^3 - 4\lambda^2 - 15\lambda = 0$$

$$\lambda(\lambda^2 - 4\lambda - 15) = 0$$

$$\Rightarrow \lambda = 0 \text{ (or) } \lambda^2 - 4\lambda - 15 = 0$$

$$\lambda = 0, \quad \lambda = 2+\sqrt{19}, \quad 2-\sqrt{19}$$

Quadratic form is indefinite.

Rank = 2.

Index = 2

Signature = $\cancel{4}$-2 = 2

(2) $2x^2+2y^2+2z^2+2yz$.

This can be write in a matrix

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

Characteristic equation of A is $|A-\lambda I|=0$

$$\begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 2-\lambda & \phi \\ 0 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)\left[(2-\lambda)(2-\lambda)-\phi\right] - 0 + 0 = 0$$

$$2-\lambda\left[4-2\lambda-2\lambda+\lambda^2-1\right] = 0$$

$$2-\lambda\left[4-4\lambda+\lambda^2-1\right] = 0$$

$$8-8\lambda+2\lambda^2-4\lambda+4\lambda^2-\lambda^3-2+\lambda = 0$$

$$-\lambda^3+6\lambda^2-11\lambda+6 = 0$$

$$\lambda^3-6\lambda^2+11\lambda-6 = 0$$

$$\lambda = 1,\ 3,\ 2$$

Quadratic form is finite

Rank = 3

Index = 3

Signature = 3-0 = 3

86

Reduce the Quadratic form $10x^2+2y^2+5z^2-10zx+6yz-4xy$ the Canonical form. Also find the tranformation.

Given $10x^2+2y^2+5z^2-10zx+6yz-4xy$.

This can be written in matrix $A = \begin{bmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{bmatrix}$

$A = IAI$

i.e., $\begin{bmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$R_2 : 5R_2 + R_1 \qquad R_3 : 2R_3 + R_1$

$\begin{bmatrix} 10 & -2 & 5 \\ 0 & 8 & 10 \\ 0 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 5 & 0 \\ 1 & 0 & 2 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$C_2 : 5C_2 + C_1 \qquad C_3 : 2C_3 + C_1$

$\begin{bmatrix} 10 & 0 & 0 \\ 0 & 40 & 20 \\ 0 & 20 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 5 & 0 \\ 1 & 0 & 2 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 1 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$R_3 : 2R_3 - R_2$

$\begin{bmatrix} 10 & 0 & 0 \\ 0 & 40 & 20 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 5 & 0 \\ 1 & -5 & 4 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 1 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$C_3 : 2C_3 - C_2$

$\begin{bmatrix} 10 & 0 & 0 \\ 0 & 40 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 5 & 0 \\ 1 & -5 & 4 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 1 \\ 0 & 5 & -5 \\ 0 & 0 & 4 \end{bmatrix}$

Given form is reduced into normal form

i.e., $D = P^T A P$ where

$$D = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 40 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 5 & -5 \\ 0 & 0 & 4 \end{bmatrix}$$

The linear transformation is

$$X = PY$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 5 & -5 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$x = y_1 + y_2 + y_3 \; ; \quad y = 5y_2 - 5y_3 \; ; \quad z = 4y_3$$

The given form is reduced to

$$Y^T D Y = [y_1 \; y_2 \; y_3] \begin{bmatrix} 10 & 0 & 0 \\ 0 & 40 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= 10y_1^2 + 40y_2^2$$

96 Reduce the Quadratic form $2x^2 + 5y^2 + 3z^2 + 4xy$ to Canonical form. Also find the transformation.

Given form is

$$2x^2 + 5y^2 + 3z^2 + 4xy$$

This can be written in Matrix A form is

$$A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$A = IAI$$

$$\begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 : R_2 - R_1$

$$\begin{bmatrix} 2 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 : C_2 - C_1$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\frac{R_1}{\sqrt{2}}, \frac{C_1}{\sqrt{2}}$ ; $\frac{R_2}{\sqrt{3}}, \frac{C_2}{\sqrt{3}}$ ; $\frac{R_3}{\sqrt{3}}, \frac{C_3}{\sqrt{3}}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 0 & 0 \\ -1/\sqrt{3} & 1/\sqrt{3} & 0 \\ 0 & 0 & 1/\sqrt{3} \end{bmatrix} A \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{3} & 0 \\ 0 & 1/\sqrt{3} & 0 \\ 0 & 0 & 1/\sqrt{3} \end{bmatrix}$$

It is in the form of $D = P^T A P$

$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a diagonal matrix and

$$P^T = \begin{bmatrix} 1/\sqrt{2} & 0 & 0 \\ -1/\sqrt{3} & 1/\sqrt{3} & 0 \\ 0 & 0 & 1/\sqrt{3} \end{bmatrix}$$

The canonical form is $y_1^2 + y_2^2 + y_3^2$ which is given by $X = PY$ where $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

10b. Reduce the Quadratic form $3x^2 + 2y^2 + 3z^2 - 2xy - 2yz$ to the canonical form by orthogonal transformation. Also find the transformation

$$A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

Charecterestic of equation A is $|A - \lambda I| = 0$

$$\begin{vmatrix} 3-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{vmatrix} = 0$$

$(3-\lambda)((2-\lambda)(3-\lambda)-0)+1((-1)(3-\lambda)+0)-1(0-(-1)(2-\lambda)) = 0$

$3-\lambda(6-2\lambda-3\lambda+\lambda^2)+1(-3+\lambda)-1(0-(-2+\lambda)) = 0$

$3-\lambda \quad (6-5\lambda+\lambda^2)+3-\lambda-2+2\lambda = 0$

$18-15\lambda+3\lambda^2-6\lambda+5\lambda^2-\lambda^3+3-\lambda-2+2\lambda = 0$

$-\lambda^3-24\lambda+8\lambda^2+19 = 0$

$-\lambda^3+8\lambda^2-22\lambda+23 = 0$

$\lambda^3-8\lambda^2+22\lambda-17 = 0$

$S_1 : 3+2+3 = 8$

$S_2 : 5+9+5 = 19$

$S_3 : 12$

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$\lambda^3 - 8\lambda^2 + 19\lambda - 12 = 0$

$\lambda = 1, 3, 4.$

when $\lambda = 1$

$$[A-\lambda I] = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$R_2 \to 2R_2 + R_1$

$$\begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & -2 \\ 0 & -1 & 2 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2x - y = 0$

$y - 2z = 0$

let $z = k$

$y - 2k = 0 \Rightarrow y = 2k.$

$$2x - 2k = 0$$
$$x = k.$$

$$X_1 = \begin{bmatrix} k \\ 2k \\ k \end{bmatrix} = k \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

when $\lambda = 3$

$$[A - \lambda I] = \begin{bmatrix} 0 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$R_1 \leftrightarrow R_2 \Rightarrow \begin{bmatrix} -1 & -1 & -1 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

$$R_3 \to R_3 - R_2 \Rightarrow \begin{bmatrix} -1 & -1 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x - y - z = 0$$
$$-y = 0$$
$y = 0$ let $z = k$
$$-x - 0 - k = 0$$
$$x = -k.$$

$$X_2 = \begin{bmatrix} -k \\ 0 \\ k \end{bmatrix} = k \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

when $\lambda = 4$

$$[A - \lambda I] = \begin{bmatrix} -1 & -1 & 0 \\ -1 & -2 & -1 \\ 0 & -1 & -1 \end{bmatrix}$$

$$R_2 \to R_2 - R_1 \Rightarrow \begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix}$$

$$R_3 \to R_3 - R_2 \Rightarrow \begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x - y = 0$$
$$-y - z = 0$$

let $z = k$ $\Rightarrow$ $-y - k = 0$, $y = -k$ $\Rightarrow$ $-x - (-k) = 0$, $x = k.$

$$x_3 = \begin{bmatrix} k \\ -k \\ k \end{bmatrix} = k \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\|x_1\| = \sqrt{(1)^2+(2)^2+(1)^2} = \sqrt{6}$$

$$\|x_2\| = \sqrt{(-1)^2+(0)^2+(1)^2} = \sqrt{2}$$

$$\|x_3\| = \sqrt{(1)^2+(-1)^2+(1)^2} = \sqrt{3}$$

$$e_1 = \frac{x_1}{\|x_1\|} = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix} \quad e_2 = \frac{x_2}{\|x_2\|} = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \quad e_3 = \frac{x_3}{\|x_3\|} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$P = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$P^T A P = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} = D$$

$$Y^T D Y = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= y_1^2 + 3y_2^2 + 4y_3^2$$

Index = 3

Rank = 3

Signature = 2(3) − 3 = 3.

Nature = +ve definite.

11.

Reduce the Quadratic form $3x^2+5y^2+3z^2-2yz+2zx-2xy$ to the canonical form by orthogonal transformation. Also find the transformation

Given form is $3x^2+5y^2+3z^2-2yz+2zx-2xy$

This can be write in Matrix $A = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}$

The characteristics eqn of A is $|A-\lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)[(5-\lambda)(3-\lambda)-1]+1[(-1)(3-\lambda)+1]+1[1-(5-\lambda)]=0$$

$$(3-\lambda)[15-5\lambda-3\lambda+\lambda^2]+[-3+\lambda+1]+1[-4+\lambda]=0$$

$$(3-\lambda)[14-8\lambda+\lambda^2]+[-2+\lambda]-4+\lambda=0$$

$$42-24\lambda+3\lambda^2-14\lambda+8\lambda^2-\lambda^3-2+2\lambda-4=0$$

$$-\lambda^3+11\lambda^2-36\lambda+36=0$$

$$\lambda^3-11\lambda^2+36\lambda-36=0$$

$$\therefore \lambda = 2, 3, 6$$

(i) $\lambda = 2$

$$[A-\lambda I]x = 0$$

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$R_2 : R_2 + R_1 \qquad R_3 \to R_3 - R_1$

$$\begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$x-y+z=0 \quad \text{--- } (1)$$

$$2y=0 \Rightarrow \boxed{y=0}$$

$GM = LI = n - r = 3 - 2 = 1$

let $z = k$

from $x = y - z$

$x = 0 - k$

$\boxed{x = -k}$

$$X = \begin{bmatrix} -k \\ 0 \\ k \end{bmatrix} = k \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

(ii) $\lambda = 3$

$[A - \lambda I]X = 0$

$$\begin{bmatrix} 0 & -1 & 1 \\ -1 & +2 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$-y + z = 0$ —— (1)

$-x + 2y - z = 0$ —— (2)

$x - y = 0$ —— (3)

Solving (1) & (2)

$$\begin{array}{cccccc} \frac{x}{} & & \frac{y}{} & & \frac{z}{} & \\ -1 & 1 & & 0 & & -1 \\ 2 & -1 & & -1 & & 2 \end{array}$$

$$\frac{x_1}{1-2} = \frac{y}{-1+0} = \frac{z}{-1} = k$$

$$\frac{x}{-1} = \frac{y}{-1} = \frac{z}{-1} = k$$

$$X_2 = \begin{bmatrix} -k \\ -k \\ -k \end{bmatrix} = -k \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Case-3 let $\lambda = 6$

$$|A - \lambda I| = 0$$

$$\begin{bmatrix} -3 & -1 & +1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$-3x - y + z = 0 \quad \text{--- (1)}$$

$$-x - y - z = 0 \quad \text{--- (2)}$$

Solving (1) & (2)

$$\begin{matrix} -1 & 1 & -3 & -1 \\ -1 & -1 & -1 & -1 \end{matrix}$$

$$\frac{x}{1+1} = \frac{-y}{-3+1} = \frac{z}{3-1} = k$$

$$\frac{x}{2} = \frac{y}{-4} = \frac{z}{2} = k$$

$$\frac{x}{1} = \frac{y}{-2} = \frac{z}{1} = k$$

$$X_3 = \begin{bmatrix} k \\ -2k \\ k \end{bmatrix} = k \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$\|X_1\| = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$$

$$\|X_2\| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

$$\|X_3\| = \sqrt{(1)^2 + (-2)^2 + 1^2} = \sqrt{6}$$

$$e_1 = \frac{X_1}{\|X_1\|} = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

$$e_2 = \frac{X_2}{\|X_2\|} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$$e_3 = \frac{X_3}{\|X_3\|} = \begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$$

$$P = [e_1 \ e_2 \ e_3] = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}$$

$$P^TAP = \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix} = D$$

$$CF = Y^T D Y$$

$$= [y_1 \ y_2 \ y_3] \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$CF = 2y_1^2 + 3y_2^2 + 6y_3^2$$

Index = 3

Signature = $6 - 3 = 3$

Nature = +ve definite

Rank = 3

124. Express the Hermitian matrix $A = \begin{bmatrix} 1 & -i & 1+i \\ i & 0 & 2-3i \\ 1-i & 2+3i & 3 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & -i & 1+i \\ i & 0 & 2-3i \\ 1-i & 2+3i & 2 \end{bmatrix}$$

Every square matrix can be uniquelly expressed as sum of Hermitian & skew Hermitian matrix

$$A = \left(\frac{A+A^\theta}{2}\right) + \left(\frac{A-A^\theta}{2}\right)$$

$$= P + Q$$

where $P$ is hermitian, $Q$ is skew-hermitian

$$A^{\theta} = (\bar{A})'$$

$$\bar{A} = \begin{bmatrix} 1 & i & 1-i \\ -i & 0 & 2+3i \\ 1+i & 2-3i & 2 \end{bmatrix}$$

$$A^{\theta} = \begin{bmatrix} 1 & -i & 1+i \\ i & 0 & 2-3i \\ 1-i & 2+3i & 2 \end{bmatrix}$$

$$P = \frac{A+A^{\theta}}{2} = \frac{1}{2}\left[\begin{bmatrix} 1 & -i & 1+i \\ i & 0 & 2-3i \\ 1-i & 2+3i & 2 \end{bmatrix} + \begin{bmatrix} 1 & -i & 1+i \\ i & 0 & 2-3i \\ 1-i & 2+3i & 2 \end{bmatrix}\right]$$

$$= \frac{1}{2}\begin{bmatrix} 2 & -2i & 2+2i \\ 2i & 0 & 4-6i \\ 2-2i & 4+6i & 4 \end{bmatrix}$$

$\therefore$ $P$ is hermitian

$$Q = \frac{A-A^{\theta}}{2} = \frac{1}{2}\left[\begin{bmatrix} 1 & -i & 1+i \\ i & 0 & 2-3i \\ 1-i & 2+3i & 2 \end{bmatrix} - \begin{bmatrix} 1 & -i & 1+i \\ i & 0 & 2-3i \\ 1-i & 2+3i & 2 \end{bmatrix}\right]$$

$$= \frac{1}{2}(0)$$

$$= 0$$

$$A = P + iQ$$

$$= \frac{1}{2}\begin{bmatrix} 2 & \cancel{2}-2i & \cancel{4}+2i \\ 2i & 0 & 4-6i \\ 2-2i & 4+6i & 4 \end{bmatrix} + 0$$

$$= \frac{1}{2}\begin{bmatrix} 2 & -2i & 2+2i \\ 2i & 0 & 4-6i \\ 2-2i & 4+6i & 4 \end{bmatrix}$$ is Hermitian matrix of A

$\therefore$

# UNIT-III
# SEQUENCES AND SERIES

## LONG ANSWER QUESTIONS

1. Examine the convergence of $\frac{1}{3}x^2+\frac{1.2}{3.5}x^3+\frac{1.2.3}{3.5.7}x^4+.....,(x>0)$ .

2. Examine the convergence of $\frac{1}{1.3.5}-\frac{1}{3.5.7}+\frac{1}{5.7.9}-\frac{1}{7.9.11}+....$ .

3. Find the interval of convergence of the following series $\frac{1}{1-x}+\frac{1}{2(1-x)^2}+\frac{1}{3(1-x)^3}+.....$

4. Test the convergence of the series $\sum_{n=1}^{\infty}\frac{x^{2n}}{(n+1)\sqrt{n}}$

5. Find the interval of convergence of the series $\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+.......\infty$

6. Test the convergence of the series $\frac{x}{1}+\frac{1.x^3}{2.3}+\frac{1.3x^5}{2.4.5}+\frac{1.3.5x^7}{2.4.6.7}+.......(x>0)$

7.Test whether the following series is absolutely convergent / conditionally convergent

$$\sum_{1}^{\infty}(-1)^{n+1}\left(\sqrt{n+1}-\sqrt{n}\right)$$

8.Find the interval of convergence for the series $\sum_{n=1}^{\infty}(-1)^n\frac{n(x+1)^n}{2^n}$

9 . Test whether the following series is absolutely Cgt or conditionally Cgt

$$\frac{1}{5\sqrt{2}}-\frac{1}{5\sqrt{3}}+\frac{1}{5\sqrt{4}}..........$$

10. Test the convergence of the series $\sum_{n=1}^{\infty}\frac{2n!}{n!(n)}$

11. Test the convergence of the series $\sum\frac{1}{n}\log\left(\frac{n+1}{n}\right)$

12. Examine whether the following series is absolutely convergent or conditionally convergent

$$1-\frac{1}{3!}+\frac{1}{5!}-\frac{1}{7!}+.....$$

13.Test the convergence of the following series $\sum\frac{(n!)^2}{(2n!)!}x^{2n}$

14.Test the convergence of the following series $\sum_{n=1}^{\infty} \frac{1.3.5...(2n+1)}{2.5.8....(3n+2)}$

15.Find the interval of convergence for the following series $\sum \frac{n^2-1}{n^2+1} x^n$

**SHORT ANSWER QUESTIONS**

1.Test the convergence of $\sum\left(\sqrt{n^3+1}-\sqrt{n^3}\right)$ .

2.Test the convergence of $\sum\left(x^n/n^{n-1}\right),\ (x>0)$ .

**3.** Examine the convergence of $\sum 1/n(2n+1)$ .

**4.**Examine the convergence of $\sum 1/\left(n^{3/2}+n+1\right)$.

**5.**Test the convergence of the following series $\sum \sin^2 \frac{1}{n}$.

**6.**Test the convergence of the series $\sqrt{n+1}-\sqrt{n-1}$

7.Find the interval of convergence for the following series $\sum \frac{n^2-1}{n^2+1} x^n$

8.Test the convergence of the following series $\sum \frac{1}{(\log\ \log n)^n}$ .

**9.** Prove that the series $\frac{(-1)^n}{n(\log n)^3}$ converges absolutely.

**10.**Examine the convergence or divergence of $\sum x^{2n}/(n+2)\sqrt{(n+2)},(x>0)$

11. State ratio test, raabe's test, root test, log test, integral test, limit comparision test, Leibnitz's test

12Define conditional convergence and absolute convergence

# QUESTION BANK

## UNIT - III

## SEQUENCE AND SERIES

Short answer questions

1. Test the convergence of $\sum\left(\sqrt{n^3+1}-\sqrt{n^3}\right)$

Let $U_n = \sqrt{n^3+1}-\sqrt{n^3}$

Multiply and divide with $\left(\sqrt{n^3+1}+\sqrt{n^3}\right)$

$$U_n = \frac{\left[\sqrt{n^3+1}-\sqrt{n^3}\right]\left[\sqrt{n^3+1}+\sqrt{n^3}\right]}{\sqrt{n^3+1}+\sqrt{n^3}}$$

$$= \frac{(n^3+1)-n^3}{\sqrt{n^3+1}+\sqrt{n^3}}$$

$$= \frac{1}{\sqrt{n^3}\left[\sqrt{1+\frac{1}{n^3}}+1\right]}$$

Take $V_n = \dfrac{1}{n^{3/2}}$

$$\lim_{n\to\infty} \frac{V_n}{U_n} = \lim_{n\to\infty} \frac{\dfrac{1}{\sqrt{n^3}\left[\sqrt{1+\frac{1}{n^3}}+1\right]}}{\dfrac{1}{\sqrt{n^3}}}$$

$$= \underset{n\to\infty}{Lt} \frac{1}{\sqrt{1+\frac{1}{n^3}}+1}$$

$$= \frac{1}{\sqrt{1+0}+1} = \frac{1}{1+1} = \frac{1}{2} \neq 0$$

$\therefore$ By comparision test, $\Sigma U_n$ and $\Sigma V_n$ converges or diverges together

$$\Sigma V_n = \Sigma \frac{1}{n^{3/2}}$$

$$p = \frac{3}{2} \; (>1)$$

By p-test $\Sigma V_n$ is convergent

$\therefore$ $\Sigma U_n$ also convergent.

2. Test the convergence of $\Sigma (x^n / n^{n-1})$, $(x>0)$

$$U_n = \frac{x^n}{n^{n-1}} = \frac{n \cdot x^n}{n^n}$$

$$(U_n)^{1/n} = n^{1/n} \left(\frac{x^n}{n^n}\right)^{\frac{1}{n}} = n^{1/n} \frac{x}{n}$$

$$\underset{n\to\infty}{Lt} (U_n)^{1/n} = \underset{n\to\infty}{Lt} n^{1/n} \cdot \underset{n\to\infty}{Lt} \frac{x}{n}$$

$$= 1 \times 0 = 0 \quad (<1)$$

$\therefore$ By cauchy's Root test, given Series $\Sigma U_n$ is convergent.

3. Examine the convergence of $\Sigma 1/n(2n+1)$.

Take $U_n = \dfrac{1}{n(2n+1)}$

$$= \frac{1}{n.n\left(2+\frac{1}{n}\right)}$$

$$U_n = \frac{1}{n^2\left(2+\frac{1}{n}\right)}$$

Take $V_n = \dfrac{1}{n^2}$

$$\underset{n\to\infty}{Lt} \frac{U_n}{V_n} = \underset{n\to\infty}{Lt} \frac{\frac{1}{n^2\left(2+\frac{1}{n}\right)}}{\frac{1}{n^2}}$$

$$= \underset{n\to\infty}{Lt} \frac{1}{2+\frac{1}{n}}$$

$$= \frac{1}{2+0} = \frac{1}{2} \quad (\neq 0)$$

$\therefore$ By limit comparision test, $\Sigma U_n$ and $\Sigma V_n$ both converges or diverges together

$$\Sigma V_n = \Sigma \frac{1}{n^2}$$

$$P = 2 \quad (>1)$$

By p-test $\Sigma V_n$ is convergent

$\therefore$ $\Sigma U_n$ is also convergent.

4. Examine the convergence of $\Sigma 1/(n^{3/2}+n+1)$.

$$U_n = \frac{1}{(n^{3/2}+n+1)}$$

$$= \frac{1}{n^{3/2}\left[1+\frac{1}{n^{1/2}}+\frac{1}{n^{3/2}}\right]}$$

Take $V_n = \frac{1}{n^{3/2}}$

$$\underset{n\to\infty}{Lt} \frac{U_n}{V_n} = \underset{n\to\infty}{Lt} \frac{\frac{1}{n^{3/2}\left[1+\frac{1}{n^{1/2}}+\frac{1}{n^{3/2}}\right]}}{\frac{1}{n^{3/2}}}$$

$$= \underset{n\to\infty}{Lt} \frac{1}{1+\frac{1}{n^{1/2}}+\frac{1}{n^{3/2}}}$$

$$= \frac{1}{1+0+0} = 1 \quad (\neq 0)$$

$\therefore$ By limit comparision test $\Sigma U_n$ and $\Sigma V_n$ both converges or diverges together

$$\Sigma V_n = \Sigma \frac{1}{n^{3/2}}$$

$$p = \frac{3}{2} \quad (>1)$$

By p-test $\Sigma V_n$ converges

$\therefore$ $\Sigma U_n$ is also convergent.

5. Test the convergence of the following Series $\Sigma \sin^2 \frac{1}{n}$.

$U_n = \sin^2\left(\frac{1}{n}\right)$

Take $V_n = \frac{1}{n^2}$

$$\underset{n\to\infty}{Lt} \left(\frac{U_n}{V_n}\right) = \underset{n\to\infty}{Lt} \left[\frac{\sin\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)}\right]^2$$

$$= \underset{\theta\to\infty}{Lt} \left[\frac{\sin\theta}{\theta}\right]^2$$

$$= 1^2 = 1 \neq 0$$

By Limit comparision test

$\Sigma U_n$ and $\Sigma V_n$ both converge or diverge together.

$\Sigma V_n = \frac{1}{n^2}$

$p = 2 \; (>1)$

By p-test $\Sigma V_n$ converges

$\therefore$ $\Sigma U_n$ also converges.

6. Test the convergence of the Series $\sqrt{n+1} - \sqrt{n-1}$.

$U_n = \sqrt{n+1} - \sqrt{n-1}$

Multiply and divide with $\sqrt{n+1}+\sqrt{n-1}$

$$= \frac{(\sqrt{n+1})^2 - (\sqrt{n-1})^2}{\sqrt{n+1}+\sqrt{n-1}}$$

$$= \frac{n+1-n+1}{\sqrt{n+1}+\sqrt{n-1}}$$

$$= \frac{2}{\sqrt{n+1}+\sqrt{n-1}}$$

$$= \frac{2}{\sqrt{n}\left[\sqrt{1+\frac{1}{n}}+\sqrt{1-\frac{1}{n}}\right]}$$

Let $v_n = \frac{1}{\sqrt{n}}$

$$\underset{n\to\infty}{Lt} \frac{u_n}{v_n} = \underset{n\to\infty}{Lt} \frac{\frac{2}{\sqrt{n}\left[\sqrt{1+\frac{1}{n}}+\sqrt{1-\frac{1}{n}}\right]}}{\frac{1}{\sqrt{n}}}$$

$$= \frac{2}{1+1} = 1$$

$\therefore$ $\Sigma u_n$ and $\Sigma v_n$ both converge or diverge together by comparision test

$v_n = \frac{1}{n^{1/2}}$ $\quad p = \frac{1}{2} > 1$

$\Sigma v_n$ diverges by p test

$\therefore$ $\Sigma u_n$ also diverges.

7. Find the interval of convergence for the series

$$\sum \frac{n^2-1}{n^2+1} x^n$$

Same as 15 th question in long answer questions.

8. Test the convergence of series $\sum \frac{1}{(\log \log n)^n}$

$$\sum U_n = \sum \frac{1}{(\log \log n)^n}$$

$$(U_n)^{1/n} = \frac{1}{\log(\log n)}$$

$$\underset{n \to \infty}{Lt} (U_n)^{1/n} = 0 \quad (<1)$$

By Cauchy's $n^{th}$ root test, series is convergent.

9. Prove that the series $\sum \frac{(-1)^n}{n(\log n)^3}$ converges absolutely.

$$\sum U_n = \sum \frac{1}{n(\log n)^3}$$

$$\sum \frac{(-1)^n}{n(\log n)^2} = \frac{1}{2(\log 2)^3} - \frac{1}{3(\log 3)^3} + \frac{1}{4(\log 4)^3} - \cdots\cdots$$

This is an alternating series.

$$U_{n+1} = \frac{1}{(n+1)(\log(n+1))^3}$$

$$U_{n+1} < U_n$$

$$\underset{n\to\infty}{Lt} \frac{1}{n(\log n)^3} = 0$$

By Leibnitz's test $\Sigma U_n$ is absolutely convergent.

10. Examine the convergence or divergence of $\Sigma x^{2n} / (n+2)\sqrt{n+2}$

$$U_n = \frac{x^{2n}}{(n+2)\sqrt{n+2}}$$

$$U_{n+1} = \frac{x^{2n+2}}{(n+3)\sqrt{n+3}}$$

$$\underset{n\to\infty}{Lt} \frac{U_{n+1}}{U_n} = \underset{n\to\infty}{Lt} \frac{x^{2n+2}}{(n+3)\sqrt{n+3}} \cdot \frac{(n+2)\sqrt{n+2}}{x^{2n}}$$

$$= \underset{n\to\infty}{Lt} \left[\frac{n+2}{n+3}\left[\frac{n+2}{n+3}\right]^{1/2}\right] x^2$$

$$= \underset{n\to\infty}{Lt} \left[\frac{1+2/n}{1+3/n} \cdot \left[\frac{1+2/n}{1+3/n}\right]^{1/2}\right] x^2$$

$$= \frac{1+0}{1+0} \cdot \left(\frac{1+0}{1+0}\right)^{1/2} \cdot x^2 = x^2$$

$\Sigma U_n$ converges if $x^2 < 1$

$\Sigma U_n$ diverges if $x^2 > 1$

Test fails if $x^2 = 1$

$$U_n = \frac{1}{(n+2)\sqrt{n+2}} = \frac{1}{(n+2)^{3/2}}$$

$$V_n = \frac{1}{n^{3/2}}$$

$$\underset{n\to\infty}{Lt} \frac{u_n}{v_n} = \underset{n\to\infty}{Lt} \frac{n^{3/2}}{n^{3/2}\left(1+\frac{2}{n}\right)^{3/2}}$$

$$= \underset{n\to\infty}{Lt} \frac{1}{\left[1+\frac{2}{n}\right]^{3/2}} = 1$$

By limit comparision test, $\Sigma u_n$ and $\Sigma v_n$ both converge or diverge together.

$$\Sigma v_n = \Sigma \frac{1}{n^{3/2}}$$

$p = 3/2 \ (>1)$

$\Sigma v_n$ converges

$\therefore \ \Sigma u_n$ also converges

$\therefore \ \Sigma u_n$ is convergent if $x^2 \leq 1$

$\Sigma u_n$ is divergent if $x^2 > 1$.

11. state ratio test, raabe's test, root test, log test, integral test, limit comparision test, Leibnitz's test.

Ratio test :-

If $\Sigma u_n$ is a series of positive terms such that

$\underset{n\to\infty}{Lt} \frac{u_n}{u_{n+1}} = l$ then

a) $\Sigma u_n$ is convergent if $l > 1$

b) $\Sigma u_n$ is divergent if $l < 1$

c) Test fails to decide the nature of series if $l = 1$.

## Raabe's test :-

Let $\Sigma U_n$ be a Series of positive terms and let

$$\lim_{n\to\infty} n\left[\frac{U_n}{U_{n+1}} - 1\right] = l . \text{ Then}$$

a) If $l > 1$, the Series Converges

b) if $l < 1$, the Series diverges

c) Test fails when $l = 1$.

## Root test :-

If $\Sigma U_n$ is a Series of positive terms such that

$$\underset{n\to\infty}{Lt}\ U_n^{1/n} = l, \text{ then}$$

a) $\Sigma U_n$ converges if $l < 1$

b) $\Sigma U_n$ diverges if $l > 1$

c) Test fails if $l = 1$.

## Log test :-

If $\Sigma U_n$ is a Series of positive terms such that

$$\underset{n\to\infty}{Lt}\ n \log\left[\frac{U_n}{U_{n+1}}\right] = l, \text{ then}$$

a) Series Converges if $l > 1$

b) Series diverges if $l < 1$

c) Test fails when $l = 1$.

**Integral test :-** Let $\Sigma U_n = u_1 + U_2 + U_3 + \dots + U_n + \dots$

be Series with positive and non-increasing terms i-e

$u_1 \geq U_2 \geq U_3 \geq \dots$

Let $f(x)$ be a non-negative decreasing function such that $f(1) = u_1$, $f(2) = u_2$, .... $f(n) = u_n$ on $[1, \infty)$. Then the Series $\sum_{n=1}^{\infty} f(n)$ converges or diverges according to the improper integral $\int_1^{\infty} f(x)\,dx$ is finite or infinite.

**Limit comparision test :-** If $\Sigma U_n$ and $\Sigma V_n$ are Two Series of positive terms and $\underset{n\to\infty}{Lt} \frac{U_n}{V_n} = l \neq 0$, then the Series $\Sigma U_n$ and $\Sigma V_n$ both converge or diverge.

**Leibnitz's test :-**

If $\{U_n\}$ is a sequence of positive terms such that

a) $u_1 \geq u_2 \geq u_3 \geq \dots \geq u_n \geq U_{n+1} \dots$

b) $\underset{n\to\infty}{Lt} U_n = 0$

then the alternating Series $\sum_{n=1}^{\infty} (-1)^{n-1} u_n$ is convergent.

12. Define conditional convergence and absolute convergence.

Absolute convergence :-

Consider a series $\Sigma U_n$ where $U_n$'s are positive or negative

the series $\Sigma U_n$ is said to be absolutely convergent if $\Sigma |U_n|$ is convergent.

Conditional convergence :-

If $\Sigma U_n$ converges and $\Sigma |U_n|$ diverges, then we say that $\Sigma U_n$ converges conditionally or converges non-absolutely or semi convergent.

## Long Answer Questions

1. Examine the convergence of $\frac{1}{3}x^2 + \frac{1.2}{3.5}x^3 + \frac{1.2.3}{3.5.7}x^4 + \dots$ $(x>0)$.

$$U_n = \frac{1.2.3.\dots n}{3.5.7.\dots(2n+1)} x^{n+1}$$

Apply ratio test

$$U_{n+1} = \frac{1.2.3.\dots.n}{3.5.7.\dots(2n+1)[2(n+1)+1]} \cdot x^{n+1+1}$$

$$= \frac{1.2.3.\dots.n}{3.5.7\dots(2n+1)(2n+3)} x^{n+2}$$

$$\frac{U_n}{U_{n+1}} = \frac{1.2.3.\dots.n \cdot x^{n+1}}{3.5.7.\dots.(2n+1)} \cdot \frac{3.5.7.\dots(2n+1)(2n+3)}{1.2.3.\dots.n \cdot x^{n+2}}$$

$$= \frac{2n+3}{n+1} \cdot \frac{1}{x}$$

$$= \frac{n\left(2+\frac{3}{n}\right)}{n\left(1+\frac{1}{n}\right)x} = \frac{2+\frac{3}{n}}{\left(1+\frac{1}{n}\right)x}$$

$$\underset{n\to\infty}{Lt} \frac{U_n}{U_{n+1}} = \underset{n\to\infty}{Lt} \frac{\left(2+\frac{3}{n}\right)}{\left(1+\frac{1}{n}\right)x} = \frac{2+0}{(1+0)x}$$

$$= \frac{2}{x}$$

$\Sigma U_n$ converges if $\frac{2}{x} > 1 \Rightarrow x < 2$

$\Sigma U_n$ diverges if $\frac{2}{x} < 1 \Rightarrow x > 2$

Test fails if $\frac{2}{x} = 1 \Rightarrow x = 2$

when $x = 2$

Apply Raabe's test

$$\underset{n \to \infty}{Lt}\, n\left[\frac{U_n}{U_{n+1}} - 1\right] = \underset{n \to \infty}{Lt}\, n\left[\frac{2n+3}{(n+1)2} - 1\right]$$

$$= \underset{n \to \infty}{Lt}\, n\left[\frac{2n+3}{2n+2} - 1\right]$$

$$= \underset{n \to \infty}{Lt}\, n\left[\frac{2n+3-2n-2}{2n+2}\right]$$

$$= \underset{n \to \infty}{Lt}\, n\left[\frac{1}{2n+2}\right]$$

$$= \underset{n \to \infty}{Lt}\, n\left[\frac{1}{n\left(2+\frac{2}{n}\right)}\right]$$

$$= \underset{n \to \infty}{Lt}\, \frac{1}{2+\frac{2}{n}}$$

$$= \frac{1}{2+0} = \frac{1}{2} \quad (<1)$$

$\therefore$ $\Sigma U_n$ diverges when $x = 2$

$\therefore$ $\Sigma U_n$ converges if $x < 2$

$\Sigma U_n$ diverges if $x \geq 2$

2. Examine the convergence of Series

$$\frac{1}{1.3.5} - \frac{1}{3.5.7} + \frac{1}{5.7.9} - \frac{1}{7.9.11} + \cdots$$

$$U_n = \frac{(-1)^{n-1}}{(2n-1)(2n+1)(2n+3)}$$

Given Series is alternating Series

$$U_n > U_{n+1}$$

The terms are in decreasing order.

$$\underset{n\to\infty}{Lt} U_n = \underset{n\to\infty}{Lt} \frac{1}{(2n-1)(2n+1)(2n+3)}$$

$$= \underset{n\to\infty}{Lt} \frac{1}{n.n.n\left(2-\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\left(2+\frac{3}{n}\right)}$$

$$= \underset{n\to\infty}{Lt} \frac{1}{n^3\left(2-\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\left(2+\frac{3}{n}\right)}$$

$$= \frac{1}{\infty(2-0)(2+0)(2+0)}$$

$$\underset{n\to\infty}{Lt} U_n = 0$$

By Leibnits test

The Series $\Sigma U_n$ Converges.

3. Find the interval of convergence of the Series

$$\frac{1}{1-x} + \frac{1}{2(1-x)^2} + \frac{1}{3(1-x)^3} + \cdots$$

$$U_n = \frac{1}{n(1-x)^n}$$

Apply Ratio test

$$U_{n+1} = \frac{1}{(n+1)(1-x)^{n+1}}$$

$$\frac{U_n}{U_{n+1}} = \frac{1}{n(1-x)^n} \frac{(n+1)(1-x)^{n+1}}{1}$$

$$= \frac{(1-x)\,n+1}{n}$$

$$\frac{U_n}{U_{n+1}} = \frac{(n+1)(1-x)}{n}$$

$$\underset{n\to\infty}{Lt} \frac{U_n}{U_{n+1}} = \underset{n\to\infty}{Lt} \frac{(n+1)(1-x)}{n}$$

$$= \underset{n\to\infty}{Lt} \frac{n\left(1+\frac{1}{n}\right)(1-x)}{n}$$

$$= \underset{n\to\infty}{Lt} \left[1+\frac{1}{n}\right](1-x)$$

$$= (1+0)(1-x)$$

$$= 1-x$$

$\Sigma U_n$ converges if $1-x > 1 \Rightarrow x<0$

$\Sigma U_n$ diverges if $1-x<1 \Rightarrow x>0$

Test fails if $1-x=1 \Rightarrow x=0$

when $x=0$

$$U_n = \frac{1}{n(1-0)^n} = \frac{1}{n}$$

$p=1$ $(\neq 0)$

By p-test

$\Sigma U_n$ diverges.

$\therefore$ $\Sigma U_n$ converges if $x<0$

$\Sigma U_n$ diverges if $x \geq 0$

4. Test the convergence of Series $\sum_{n=1}^{\infty} \frac{x^{2n}}{(n+1)\sqrt{n}}$

$$U_n = \frac{x^{2n}}{(n+1)\sqrt{n}}$$

$$U_{n+1} = \frac{x^{2(n+1)}}{(n+1+1)\sqrt{n+1}}$$

$$= \frac{x^{2n+2}}{(n+2)\sqrt{n+1}}$$

Apply Ratio test

$$\frac{U_n}{U_{n+1}} = \frac{x^{2n}}{(n+1)\sqrt{n}} \quad \frac{(n+2)\sqrt{n+1}}{x^{2n+2}}$$

$$= \frac{x^{2n}}{n\left(1+\frac{1}{n}\right)\sqrt{n}} \quad \frac{n\left(1+\frac{2}{n}\right)\sqrt{n+1}}{x^{2n}\cdot x^2}$$

$$= \frac{x^{2n}}{n\left(1+\frac{1}{n}\right)\sqrt{n}} \quad \frac{n\left(1+\frac{2}{n}\right)\sqrt{n}\left(\sqrt{1+\frac{1}{n}}\right)}{x^{2n}\cdot x^2}$$

$$= \frac{\left(1+\frac{2}{n}\right)\left(\sqrt{1+\frac{1}{n}}\right)}{\left(1+\frac{1}{n}\right)\cdot x^2}$$

$$\underset{n\to\infty}{Lt} \frac{U_n}{U_{n+1}} = \underset{n\to\infty}{Lt} \frac{\left(1+\frac{2}{n}\right)\left(\sqrt{1+\frac{1}{n}}\right)}{\left(1+\frac{1}{n}\right)\cdot x^2}$$

$$= \frac{(1+0)(\sqrt{1+0})}{(1+0)\cdot x^2}$$

$$= \frac{1}{x^2}$$

By Ratio test

$\Sigma U_n$ converges if $\frac{1}{x^2} > 1 \Rightarrow x^2 < 1$

$\Sigma U_n$ diverges if $\frac{1}{x^2} < 1 \Rightarrow x^2 > 1$

Test fails if $\frac{1}{x^2} = 1 \Rightarrow x^2 = 1$

when $x^2 = 1$

$$U_n = \frac{1}{\sqrt{n}(n+1)}$$

$$= \frac{1}{\sqrt{n}\, n\left(1+\frac{1}{n}\right)}$$

$$U_n = \frac{1}{n^{3/2}\left(1+\frac{1}{n}\right)}$$

Take $V_n = \frac{1}{n^{3/2}}$

$$\frac{U_n}{V_n} = \frac{\frac{1}{n^{3/2}\left(1+\frac{1}{n}\right)}}{\frac{1}{n^{3/2}}}$$

$$= \frac{1}{1+\frac{1}{n}}$$

$$\underset{n\to\infty}{Lt} \frac{U_n}{V_n} = \underset{n\to\infty}{Lt} \frac{1}{1+\frac{1}{n}}$$

$$= \frac{1}{1+0} = 1 \quad (\neq 0)$$

By Limit Comparision test $\Sigma U_n$ and $\Sigma V_n$ both converges or diverges together

$$\Sigma V_n = \Sigma \frac{1}{n^{3/2}}$$

$$p = \frac{3}{2} \quad (>1)$$

By p-test $\Sigma V_n$ converges

$\therefore \Sigma U_n$ is also convergent.

$\therefore \Sigma U_n$ is convergent if $x^2 \leq 1$

$\Sigma U_n$ is divergent if $x^2 > 1$.

5. Find the interval of convergence of Series

$$\frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \dots \infty$$

Take $U_n = \frac{x^n}{n}$

Apply Ratio test

$$U_{n+1} = \frac{x^{n+1}}{n+1}$$

$$\frac{U_n}{U_{n+1}} = \frac{x^n}{n} \cdot \frac{n+1}{x^{n+1}}$$

$$= \frac{n+1}{n.x}$$

$$\underset{n \to \infty}{Lt} \frac{U_n}{U_{n+1}} = \underset{n \to \infty}{Lt} \frac{n+1}{n.x}$$

$$= \underset{n \to \infty}{Lt} \frac{n\left[1+\frac{1}{n}\right]}{n.x}$$

$$= \underset{n \to \infty}{Lt} \left[1+\frac{1}{n}\right] \frac{1}{x}$$

$$= (1+0)\frac{1}{x}$$

$$= \frac{1}{x}$$

$\Sigma U_n$ converges if $\frac{1}{x} > 1 \Rightarrow x < 1$

$\Sigma U_n$ diverges if $\frac{1}{x} < 1 \Rightarrow x > 1$

Test fails if $\frac{1}{x} = 1 \Rightarrow x = 1$

when $x = 1$

$$U_n = \frac{1^n}{n} = \frac{1}{n}$$

$p = 1$

By p-test $\Sigma U_n$ diverges.

$\therefore \Sigma U_n$ converges if $x < 1$

$\therefore \Sigma U_n$ diverges if $x \geq 1$

6. Test the convergence of Series

$$\frac{x}{1} + \frac{1.x^3}{2.3} + \frac{1.3x^5}{2.4.5} + \frac{1.3.5\,x^7}{2.4.6.7} + \cdots \quad (x>0)$$

Take $U_n = \frac{2n-1}{2n} \frac{x^{2n+1}}{2n+1}$

Apply Ratio test

$$U_{n+1} = \frac{2(n+1)-1}{2(n+1)} \frac{x^{2(n+1)+1}}{2(n+1)+1}$$

$$= \frac{2n+1}{2n+2} \frac{x^{2n+3}}{2n+3}$$

$$\frac{U_n}{U_{n+1}} = \frac{(2n-1)\,x^{2n+1}}{2n(2n+1)} \cdot \frac{(2n+2)(2n+3)}{(2n+1)\,x^{2n+3}}$$

$$= \frac{(2n-1)(2n+2)(2n+3)}{2n(2n+1)(2n+1)} x^{-2}$$

$$\underset{n\to\infty}{Lt} \frac{U_n}{U_{n+1}} = \underset{n\to\infty}{Lt} \frac{(2n-1)(2n+2)(2n+3)}{2n(2n+1)(2n+1)} \frac{1}{x^2}$$

$$= \underset{n\to\infty}{Lt} \frac{n.n.n\left[2-\frac{1}{n}\right]\left[2+\frac{2}{n}\right]\left[2+\frac{3}{n}\right]}{n.n.n(2)\left[2+\frac{1}{n}\right]\left[2+\frac{1}{n}\right]} \frac{1}{x^2}$$

$$= \frac{(2-0)(2+0)(2+0)}{2(2+0)(2+0)} \frac{1}{x^2}$$

$$= \frac{1}{x^2}$$

$\Sigma U_n$ converges if $\frac{1}{x^2} > 1 \Rightarrow x^2 < 1$

$\Sigma U_n$ diverges if $\frac{1}{x^2} < 1 \Rightarrow x^2 > 1$

Test fails if $\frac{1}{x^2} = 1 \Rightarrow x^2 = 1$

~~when~~ $\cancel{x^2 = 1}$

$\cancel{U_n =}$

$\therefore$ The interval of convergence is $(-1, 1)$.

7. Test whether the following Series is absolutely convergent | conditionally convergent.

$$\sum_{1}^{\infty} (-1)^{n+1} \left(\sqrt{n+1} - \sqrt{n}\right).$$

$$U_n = \sqrt{n+1} - \sqrt{n}$$

Multiply and divide with $\left(\sqrt{n+1} + \sqrt{n}\right)$

$$U_n = \sqrt{n+1} + \sqrt{n} \quad \frac{\left(\sqrt{n+1} - \sqrt{n}\right)}{\left(\sqrt{n+1} + \sqrt{n}\right)}$$

$$= \frac{n+1-n}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}}$$

The Series is alternating Series

$$U_n > U_{n+1}$$

$$\underset{n \to \infty}{Lt} U_n = \underset{n \to \infty}{Lt} \frac{1}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\infty} = 0.$$

Terms are in decreasing order

$\therefore$ The Series $\Sigma U_n$ absolutely converges.

8. Find the interval of convergence for the Series

$$\sum_{n=1}^{\infty} (-1)^n \frac{n(x+1)^n}{2^n}.$$

$$U_n = \frac{n(x+1)^n}{2^n}$$

$$U_{n+1} = \frac{(n+1)(x+1)^{n+1}}{2^{n+1}}$$

Apply Ratio test

$$\frac{U_n}{U_{n+1}} = \frac{n(x+1)^n}{2^n} \cdot \frac{2^{n+1}}{(n+1)(x+1)^{n+1}}$$

$$= \frac{n \cdot 2}{(n+1)(x+1)}$$

$$\underset{n\to\infty}{Lt}\left|\frac{U_n}{U_{n+1}}\right| = \underset{n\to\infty}{Lt} \frac{n \cdot 2}{(n+1)|x+1|}$$

$$= \underset{n\to\infty}{Lt} \frac{n \cdot 2}{n\left[1+\frac{1}{n}\right]|x+1|}$$

$$= \underset{n\to\infty}{Lt} \frac{2}{\left[1+\frac{1}{n}\right]|x+1|}$$

$$= \frac{2}{(1+0)|x+1|}$$

$$= \frac{2}{|x+1|}$$

$\therefore$ $\Sigma U_n$ converges if $\frac{2}{|x+1|} > 1 \Rightarrow x < 1$

$\therefore$ $\Sigma U_n$ diverges if $\frac{2}{x+1} < 1 \Rightarrow x > 1$

Test fails if $\frac{2}{x+1} = 0 \Rightarrow x = 1$

$\Sigma U_n$ converges if $|x+1| < 2$

$$-2 < (x+1) < 2$$

$$-3 < x < 1$$

$\therefore$ The interval of convergence is $(-3, 1)$.

9. Test whether the following series is absolutely convergent or conditionally convergent $\frac{1}{5\sqrt{2}} - \frac{1}{5\sqrt{3}} + \frac{1}{5\sqrt{4}} \cdots\cdots$

$$\Sigma U_n = \Sigma (-1)^n \frac{1}{5\sqrt{n}} = \Sigma (-1)^n V_n$$

It is an alternating series

$$V_n = \frac{1}{5\sqrt{n}} > 0$$

$$V_{n+1} = \frac{1}{5\sqrt{n+1}}$$

$$\frac{1}{5\sqrt{n}} > \frac{1}{5\sqrt{n+1}}$$

$$\underset{n\to\infty}{Lt} V_n = \underset{n\to\infty}{Lt} \frac{1}{5\sqrt{n}} = 0$$

By Leibnitz's test, the given series is convergent.

$$|U_n| = \frac{1}{5\sqrt{n}}$$

$$V_n = \frac{1}{\sqrt{n}}$$

$$\underset{n\to\infty}{Lt} \frac{|U_n|}{V_n} = \underset{n\to\infty}{Lt} \frac{1/5\sqrt{n}}{1/\sqrt{n}} = \frac{1}{5} \neq 0$$

By comparision test, both $|U_n|$ and $V_n$ converge or diverge together

$$\Sigma V_n = \Sigma \frac{1}{n^{1/2}}$$

$P = \frac{1}{2} < 1$

By p-test $\Sigma V_n$ diverges

Hence the given Series is conditionally convergent.

10. Test the convergence of the Series $\sum_{n=1}^{\infty} \frac{2n!}{n!(n)}$

$$U_n = \frac{(2n)!}{n!\, n}$$

$$U_{n+1} = \frac{(2n+2)!}{(n+1)!\,(n+1)}$$

$$\frac{U_n}{U_{n+1}} = \frac{(2n)!}{n!\, n} \cdot \frac{(n+1)!\,(n+1)}{(2n+2)!}$$

$$= \frac{(n+1)(n+1)}{n(2n+1)(2n+2)}$$

$$= \frac{(n+1)^2}{n(2n+1)2(n+1)} = \frac{n+1}{2n(2n+1)}$$

$$\underset{n\to\infty}{Lt} \frac{U_n}{U_{n+1}} = 0 \quad (<1)$$

By Ratio test, the given Series is divergent.

11. Examine whether the following Series is absolutely or conditionally convergent.

$$1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \dots$$

$$\Sigma U_n = \Sigma \frac{1}{(2n+1)!}$$

$$\Sigma U_{n+1} = \Sigma \frac{1}{(2n+3)!}$$

$$U_n > U_{n+1}$$

This is alternating Series.

Terms are in decreasing order.

$$\underset{n\to\infty}{Lt} U_n = \underset{n\to\infty}{Lt} \frac{1}{(2n+1)!}$$

$$= 0.$$

$\therefore$ The Series absolutely converges.

12. Test the convergence of Series $\Sigma \frac{1}{n} \log\left(\frac{n+1}{n}\right)$.

$$U_n = \Sigma \frac{1}{n} \log\left(\frac{n+1}{n}\right)$$

Take $V_n = \Sigma \frac{1}{n}$

$$\underset{n\to\infty}{Lt} \frac{U_n}{V_n} = \underset{n\to\infty}{Lt} \frac{\frac{1}{n}\log\left(\frac{n+1}{n}\right)}{\frac{1}{n}}$$

$$= \underset{n\to\infty}{Lt} \log\left(\frac{n+1}{n}\right)$$

$$= \underset{n\to\infty}{Lt} \log\left[\frac{n\left(1+\frac{1}{n}\right)}{n}\right]$$

$$= 0.$$

$$\Sigma V_n = \Sigma \frac{1}{n}$$

By p-test $\Sigma V_n$ diverges

$\therefore$ $\Sigma U_n$ also divergent.

13. Test the convergence of the Series $\sum \frac{(n!)^2}{(2n!)!} x^{2n}$

$$U_n = \frac{(n!)^2}{(2n!)!} x^{2n}$$

$$U_{n+1} = \frac{((n+1)!)^2}{((2n+2)!)!} x^{2n+2}$$

$$\frac{U_n}{U_{n+1}} = \frac{(n!)^2}{(2n!)!} x^{2n} \frac{((2n+2)!)!}{((n+1)!)^2 x^{2n+2}}$$

$$= \frac{(2n+1)(2n+2)}{(n+1)^2} \frac{1}{x^2}$$

$$\underset{n\to\infty}{Lt} \frac{U_n}{U_{n+1}} = \frac{4}{x^2}$$

By Ratio test

$\sum U_n$ converges if $\frac{4}{x^2} > 1 \Rightarrow x^2 < 4$

$\sum U_n$ diverges if $\frac{4}{x^2} < 1 \Rightarrow x^2 > 4$

Test fails if $\frac{4}{x^2} = 1 \Rightarrow x^2 = 4$

when $x^2 = 4$

$$\frac{U_n}{U_{n+1}} = \frac{(2n+1)(2n+2)}{(n+1)^2} \cdot \frac{1}{4} = \frac{2n+1}{2(n+1)}$$

$$\underset{n\to\infty}{Lt} \left[\frac{U_n}{U_{n+1}} - 1\right] = \frac{1}{2} < 1$$

By Raabe's test, series is divergent.

$\therefore$ $\sum U_n$ converges if $x^2 < 4$

$\sum U_n$ diverges if $x^2 \geq 4$.

14. Test the convergence of the Series $\sum_{n=1}^{\infty} \frac{1.3.5\ldots(2n+1)}{2.5.8\ldots(3n+2)}$

$$U_n = \frac{1.3.5\ldots(2n+1)}{2.5.8\ldots(3n+2)}$$

$$U_{n+1} = \frac{1.3.5\ldots(2n+1)(2n+3)}{2.5.8\ldots(3n+2)(3n+5)}$$

$$\frac{U_n}{U_{n+1}} = \frac{1.3.5\ldots(2n+1)}{2.5.8\ldots(3n+2)} \cdot \frac{2.5.8\ldots(3n+2)(3n+5)}{1.3.5\ldots(2n+1)(2n+3)}$$

$$= \frac{3n+5}{2n+3}$$

$$\underset{n\to\infty}{Lt} \frac{U_n}{U_{n+1}} = \underset{n\to\infty}{Lt} \frac{3n+5}{2n+3}$$

$$= \underset{n\to\infty}{Lt} \frac{n\left[3+\frac{5}{n}\right]}{n\left[2+\frac{3}{n}\right]}$$

$$= \frac{3+0}{2+0} = \frac{3}{2} \ (>1)$$

$\therefore$ By Ratio test

$\Sigma U_n$ converges.

15. Find the interval of convergence of Series $\Sigma \frac{n^2-1}{n^2+1} x^n$.

$$U_n = \frac{n^2-1}{n^2+1} x^n$$

$$U_{n+1} = \frac{(n+1)^2-1}{(n+1)^2+1} x^{n+1}$$

$$\frac{U_n}{U_{n+1}} = \frac{n^2-1}{n^2+1} x^n \frac{(n+1)^2+1}{(n+1)^2-1} \frac{1}{x^{n+1}}$$

$$= \frac{n^2\left[1-\frac{1}{n^2}\right]}{n^2\left[1+\frac{1}{n^2}\right]} \frac{n^2\left[\left(1+\frac{1}{n}\right)^2+\frac{1}{n^2}\right]}{n^2\left[\left(1+\frac{1}{n}\right)^2-\frac{1}{n^2}\right]}\left(\frac{1}{x}\right)$$

$$\underset{n\to\infty}{Lt} \frac{U_n}{U_{n+1}} = \frac{1}{x}$$

By Ratio test

$\Sigma U_n$ converges if $\left|\frac{1}{x}\right| > 1 \Rightarrow |x| < 1$

$x \in (-1, 1)$

when $x = 1$, $U_n = \frac{n^2-1}{n^2+1}$

$$\underset{n\to\infty}{Lt} U_n = 1 \neq 0$$

Given Series is divergent

$\therefore$ The interval of convergence of the series is $(-1, 1)$.

# UNIT-IV
# CALCULUS

**Short answer questions:**

1. State Rolle's theorem.
2. Verify Rolle'stheorm for f(x)=$|x|$ in $[\![-1,1]\!]$
3. State Lagrange's mean value theorem
4. Verify mean value theorem for f(x)=$x^{\frac{2}{3}}$ in $[-1,1]$
5. State Cauchy's mean value theorem
6. Verify Cauchy's mean value theorem for f(x)=$x^2$,g(x)=$x^3$ in $[1,2]$
7. Using Taylor,s series expand $(x-2)^4 - 3(x-2)^3 + 4(x-2)^2$ in powers of x
8. Write Taylor's series expansion of f(x) at x=a
9. Find the volume of the the solid generated by revolving the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}$=1 about the major axis
10. Write the volume of the solid generated by y=f(x),x=g(y) formulas
11. Write the Surface areas of y=f(x),x=g(y) formulas
12. Define Beta and Gamma functions.

13.Compute the value of $\Gamma(-\frac{11}{2})$

14.Prove that $\int_0^\infty e^{-y^{\frac{1}{m}}}dy = m\ \Gamma(m)$

15.Explain geometric interpolation of rolle's theorem

**Long answer questions**

1. Verify Rolle's theorem for f(x)=$\frac{sinx}{e^x}$ in $[0,\pi]$
2. Verify Rolle's theorem for f(x)=$log\frac{x^2+ab}{x(a+b0}$ in $[a,b]$
3. Verify Rolle's theorem for f(x)=$\frac{x^2-x-6}{x-1}$ in (-2,3)
4. Verify Lagrange's mean value theorem for f(x)=$\log_e x$ in $[1,e]$
5. Prove that $\frac{\pi}{3} - \frac{1}{5\sqrt{3}}$>$\cos^{-1}\frac{3}{5}$>$\frac{\pi}{3}$-$\frac{1}{8}$ using mean value theorem
6. Prove using mean value theorem $|sinu - sinv| \leq |u - v|$
7. Verify Cauchy's mean value theorem for f(x)=$e^{-x}$,g(x)=$e^x$ in $[2,6]$
8. Obtain the Maclaurin's series expansion of $\log_e(1+x)$
9. Verify Taylor's theorem for f(x)=$(1-x)^{\frac{5}{2}}$ with Lagrange's form of remainder upto 2 terms in the interval [0,1]
10. Expand logcosx about $\frac{\pi}{3}$ using Taylor's expansion

11. The curve $y^2$(a+x)= $x^2(3a-x)$ revolves about the x-axis.Find the volume of the solid generated
12. Find the surface area generated by the revolution of an arc of the catenary y=ccosh$\frac{x}{c}$ about the x-axis from x=0 to x=c
13. Find the surface area of the solid generated by the revolution of the ellipse $\frac{x^2}{a^2}$+$\frac{y^2}{b^2}$=1 about the x-axis
14. State and prove the relation between Beta and Gamma functions
15. To prove that $\Gamma(n)\ \ \Gamma(1-n)=\frac{\pi}{sinn\pi}$
16. Prove that $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$
17. Evaluate $\int_0^{\infty} x^6 e^{-2x}\, dx$
18. Evaluate $\int_0^{\frac{\pi}{2}} sin^5\theta cos^{\frac{7}{2}}\theta d\theta,$
19. 19.Prove B(m, $\frac{1}{2}$) $=2^{2m-1}B(m,m)$

# UNIT - IV
# CALCULS

## Short-Answer Questions:-

1. State Rolle's theorem.

Let $f(x)$ be a function such that

(i) it is continuous in closed interval $[a,b]$;

(ii) it is differentiable in open interval $(a,b)$;

(iii) $f(a) = f(b)$

$\therefore$ There exists $f'(c) = 0$

2. Verify Rolle's theorem for $f(x) = |x|$ in $[-1, 1]$

$$f(x) = |x|$$

i.e $f(x) = x$ for $x \geq 0$

$\quad\quad\quad\; -x$ for $x < 0$

(i) $f(x)$ is continuous for every value of $x$

$\therefore$ $f(x)$ is continuous in closed interval $[-1, 1]$

(ii) $f(x)$ is not derivable at $x = 0$

$$L.H.D = Lf'(0) = \underset{x \to 0}{Lt} \frac{f(x) - f(0)}{x - 0}$$

$$= \underset{x \to 0^-}{Lt} \frac{|x| - 0}{x} = \underset{x \to 0^-}{Lt} \frac{-x}{x}$$

$$= \underset{x \to 0^-}{Lt} (-1) = -1$$

$$R.H.D = Rf'(0) = \underset{x \to 0^+}{Lt} \frac{f(x) - f(0)}{x - 0}$$

$$= \underset{x \to 0^+}{Lt} \frac{|x| - 0}{x} = \underset{x \to 0^+}{Lt} \frac{x}{x} = 1$$

since $Lf'(0) \neq Rf'(0)$ therefore

$f(x)$ is not derivable at $x=0$

Hence Rolle's theorem is not applicable

3. state Lagrange's mean value theorem.

Let $f(x)$ be a function such that

(i) It is continuous in closed interval $[a,b]$ and

(ii) Differentiable in open interval $(a,b)$

Then there exists at least one point c in open interval $(a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

4. Verify mean value theorem for $f(x) = x^{2/3}$ in $[-1,1]$

$f'(x) = \frac{2}{3}x^{-1/3}$

$\therefore$ $f(x)$ is not derivable at $x=0$

$\therefore$ $f(x)$ is not satisified for any mean value theorem.

5 state cauchy's mean value theorem

if $f : [a,b] \to R$, $g : [a,b] \to$ are such that

(i) $f, g$ are continuous on $[a,b]$

(ii) $f, g$ are differentiable on $(a,b)$ and

(iii) $g'(x) \neq 0 \quad \forall \; x \in (a,b)$

Then there exists a point $c \in (a,b)$ such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

6. Verify cauchy's mean value theorem for $f(x) = x^2$, $g(x) = x^3$ in $[1,2]$

we have $f(x) = x^2$, $g(x) = x^3$

$\therefore$ $f, g$ are continuous on $[1,2]$ and $f, g$ are differentiable on $(1,2)$

$g'(x) = 3x^2 \neq 0$

$\therefore$ $f, g$ satisfy the conditions of cauchy's mean value theorem

$$\frac{f(2)-f(1)}{g(2)-g(1)} = \frac{f'(c)}{g'(c)} = \frac{4-1}{8-1} = \frac{2c}{3c^2}$$

$$c = \frac{14}{9} \in (1,2)$$

This verifies cauchy's mean value theorem.

* | Expand $e^x \sin y$ in powers of $x$ and $y$.

1. Using Taylor's theorem express $(x-2)^4 - 3(x-2)^3 + 4(x-2)^2 + 5$

in powers of $x$.

$f(x+h) = (x-2)^4 - 3(x-2)^3 + 4(x-2)^2 + 5$

$f(x) = x^4 - 3x^3 + 4x^2 + 5 \qquad x = -2$

where $f(x) = x^4 - 3x^3 + 4x^2 + 5 \Rightarrow f(-2) = 61$

$f'(x) = 4x^3 - 9x^2 + 8x \Rightarrow f'(-2) = -84$

$f''(x) = 12x^2 - 18x + 8 \qquad f''(-2) = -66.$

substituting these values in Taylor's theorem.

$f(x+h) = f(h) + xf'(h) + \frac{x^2}{2!} f''(h) + \cdots$

$= 61 - 84x + 92\frac{x^2}{2!} - 66\frac{x^3}{3!} + 24\frac{x^4}{4!}$

$f(x-2) = 61 - 84x + 46x^2 - 11x^3 + x^4$

8. Expand Taylor's series expansion of $f(x,y)$ at $[a,b]$

if $f : [a,b] \to R$ is such that

(a) $f^{(n-1)}$ is continuous on $[a,b]$

(b) $f^{(n-1)}$ is derivable on $(a,b)$

$f(b) = f(a) + \frac{b-a}{1!} f'(a) + \frac{(b-a)^2}{2!} f''(a) + \cdots$

9. Find the volume of the solid generated by revolving the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ about the major axis.

Sol- Given equation of ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

when $y=0$, $x=\pm a$

$\therefore$ Major axis of ellipse is $x=-a$ to $x=+a$

The volume of the solid generated by given ellipse revolving about the major axis $= \int_{-a}^{a} \pi y^2 dx = 2\pi\int_{0}^{a} y^2 dx$

$$= 2\pi \int_0^a \left[b^2 - \frac{b^2}{a^2}x^2\right] dx = 2\pi\left[b^2x - \frac{b^2}{a^2}\frac{x^3}{3}\right]_0^a$$

$$= 2\pi\left[b^2a - \frac{b^2}{a^2}\frac{a^3}{3} - (0)\right]$$

$$= 2\pi\left[ab^2 - \frac{ab^2}{3}\right] = \frac{4}{3}\pi ab^2$$

0. Write the volume of the solid generated by $y=f(x)$, $x=g(y)$ formulas.

at $y=f(x) \Rightarrow x$-axis $\quad V=\pi\int_a^b y^2 dx$

at $x=a$, $x=b$

$x=g(x) \Rightarrow y$-axis $\quad V=\int_c^d x^2 dy$

at $y=c$, $y=d$

11. Write the surface area of $y = f(x)$, $x = g(y)$ formulas

at $y = f(x) \Rightarrow x$-axis $S = 2\pi \int_a^b y \sqrt{1+\left(\frac{dy}{dt}\right)^2}\, dx$

at $x = \phi(y) \Rightarrow y$-axis $S = 2\pi \int_c^d x \sqrt{1+\left(\frac{dx}{dt}\right)^2}\, dy$

2. Define Beta and Gamma functions

<u>Beta function</u>: The definite integral $\int_0^1 x^{m-1}(1-x)^{n-1}\, dx$ is called beta function and is denoted by $B(m,n)$

<u>Gamma function</u>: The definite integral $\int_0^\infty e^{-x} x^{n-1}\, dx$ is called Gamma function and is denoted by $\Gamma(n)$

3. Compute the value of $\Gamma\left(\frac{-11}{2}\right)$

$$\Gamma\left(\frac{-11}{2}\right) = \frac{2\cdot2\cdot2\cdot2\cdot2\Gamma(1/2)}{11\cdot9\cdot7\cdot5\cdot3\,(-1/2)} = \frac{2^6}{11\cdot9\cdot7\cdot5\cdot3}\Gamma\left(\frac{1}{2}\right) = \frac{2^6\sqrt{\pi}}{11\cdot9\cdot7\cdot5\cdot3}$$

4. Prove that $\int_0^\infty e^{-y^{\frac{1}{m}}}\, dy = m\Gamma(m)$

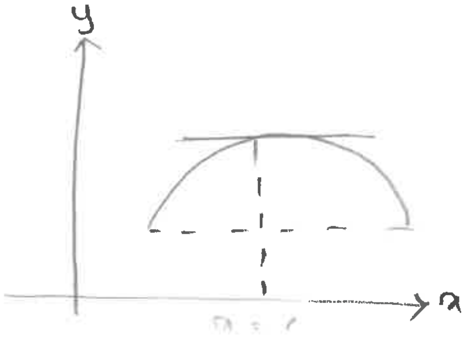
Put $y^{1/m} = x$ i.e $y = x^m$ so that $dy = mx^{m-1}dx$

$$\therefore \int_0^\infty e^{-y^{1/m}}\, dx = \int_0^\infty e^{-x}(mx^{m-1})\, dx = m\int_0^\infty e^{-x} x^{m-1}\, dx = m\Gamma(m)$$

5. Explain geometric interpolation of rolle's theorem.

The curve $y = f(x)$ is continuous in closed interval $[a,b]$

at $x = c$ and $(c, f(c))$



## Long Answer Questions

1. Verify Rolle's theorem for the function $f(x) = \frac{\sin x}{e^x}$ in $[0, \pi]$

Sol- $f(x) = \frac{\sin x}{e^x}$

$$f'(x) = \frac{e^x \cos x - \sin x\, e^x}{(e^x)^2} = \frac{\cos x - \sin x}{e^x}$$

$\therefore f(x)$ is differentiable and continuous at $(0, \pi)$

$\Rightarrow f(0) = \frac{\sin 0}{e^0} = 0 \qquad f(\pi) = \frac{\sin \pi}{e^\pi} = 0$

$\therefore f(0) = f(\pi)$

Hence all conditions of Rolle's theorem are satisfied.

$\Rightarrow$ There exists $c \in (0, \pi)$ such that $f'(c) = 0$

$$f'(x) = \frac{\cos x - \sin x}{e^x}$$

$$\therefore f'(c) = 0 \Rightarrow \frac{\cos c - \sin c}{e^c} = 0$$

$$\Rightarrow \cos c - \sin c = 0$$

$$\Rightarrow \cos c = \sin c \Rightarrow \tan c = 1$$

$$\therefore \tan c = \tan \pi/4$$

$$c = \pi/4 \in (0, \pi)$$

Hence Rolle's theorem is verified.

2. Verify Rolle's theorem for $f(x) = \log \frac{x^2+ab}{x(a+b)}$ in $[a, b]$

$f(x) = \log(x^2+ab) - \log x - \log(a+b)$

$\Rightarrow$ Since $f(x)$ is a composite function of continuous functions in $[a, b]$. It is continuos in $[a, b]$

$\Rightarrow f'(x) = \frac{1}{x^2+ab}(2x) - \frac{1}{x} = \frac{x^2-ab}{x(x^2+ab)}$

$\therefore$ $f'(x)$ is differntiable at $(a,b)$

$\Rightarrow f(a) = \log\left[\frac{a^2+ab}{a^2+ab}\right] = \log 1 = 0$

$f(b) = \log\left[\frac{b^2+ab}{b^2+ab}\right] = \log 1 = 0$

$\therefore f(a) = f(b)$

Hence all conditions for Rolle's therom are satisfied

$\Rightarrow$ There exists $c \in (a,b)$ such that $f'(c) = 0$

$\Rightarrow \frac{c^2-ab}{c(c^2+ab)} = 0$

$\Rightarrow c^2 = ab \Rightarrow c = \pm\sqrt{ab} \in (a,b)$

Hence Rolle's theorem is verified.

3. Verify Rolle's theorem for $f(x) = \frac{x^2-x-6}{x-1}$ in $(-2,3)$

$\Rightarrow f(x) = \frac{x^2-x-6}{x-1}$

$f'(x) = \frac{(x-1)(2x-1) - (x^2-x-6)(1)}{(x-1)^2} = x^2-2x+7$

clearly for $f(x)$ is not derivable at $x=1$

$\therefore$ $f(x)$ is not derivable in the open interval $(-2,3)$

$\therefore$ Hence Rolle's theorem is not applicable at $f(x)$ in $(-2,3)$

4. Verify Lagrange's mean value theorem for $f(x) = \log_e x$ in $[1, e]$

Sol- $\Rightarrow f(x) = \log_e x$

$f'(x) = \frac{1}{x}$

$\therefore f(x)$ is Derivable and continuous at $(1, e)$

By Lagrange's mean value theorem

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

$$\frac{1}{c} = \frac{\log_e e - \log_e 1}{e-1}$$

$$\frac{1}{c} = \frac{1-0}{e-1} \qquad c = e-1 \in [1, e]$$

Hence Lagrange's mean value theorem is verified

5. Prove that $\frac{\pi}{3} - \frac{1}{5\sqrt{3}} > \cos^{-1}\frac{3}{5} > \frac{\pi}{3} - \frac{1}{8}$ using Lagrange's mean value theorem.

Sol- Let $f(x) = \cos^{-1}(x)$ in interval $[a, b]$

$$f'(x) = \frac{-1}{\sqrt{1-x^2}}$$

By Lagrange's mean value theorem

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

$$\frac{-1}{\sqrt{1-a^2}} = \frac{\cos^{-1}b - \cos^{-1}a}{b-a} \qquad \text{------------} (1)$$

$c \in (a, b) \Rightarrow a < c < b$

$a^2 < c^2 < b^2$

$-a^2 > -c^2 > -b^2$

$1-a^2 > 1-c^2 > 1-b^2$

$$\Rightarrow \frac{1}{\sqrt{1-a^2}} < \frac{1}{\sqrt{1-c^2}} < \frac{1}{\sqrt{1-b^2}}$$

$$\Rightarrow \frac{-1}{\sqrt{1-a^2}} > -\frac{1}{\sqrt{1-c^2}} > -\frac{1}{\sqrt{1-b^2}}$$

$$\Rightarrow \frac{-1}{\sqrt{1-a^2}} > \frac{\cos^{-1}b - \cos^{-1}a}{b-a} > \frac{-1}{\sqrt{1-b^2}}$$

let $a = \frac{1}{2}$ and $b = \frac{3}{5}$. then

$$-\frac{2}{\sqrt{3}} > \frac{\cos^{-1}\frac{3}{5} - \frac{\pi}{3}}{1/10} > -\frac{5}{4}$$

$$-\frac{2}{10\sqrt{3}} > \cos^{-1}\frac{3}{5} - \frac{\pi}{3} > -\frac{5}{4}\cdot\frac{1}{10}$$

$$\Rightarrow \frac{\pi}{3} - \frac{1}{5\sqrt{3}} > \cos^{-1}\frac{3}{5} > \frac{\pi}{3} - \frac{1}{8}$$

Hence the result.

6. Prove using mean value theorem $|\sin u - \sin v| \leq |u-v|$

Sol- If $u=v$, there is nothing to prove

if $u>v$, then consider the function

$f(u) = \sin u$ on $[v,u]$

clearly, $f$ is continuous on $[v,u]$ and derivable on $(v,u)$

$\therefore$ By Lagrange's mean value theorem, there exists $c \in (v,u)$ such that $\frac{f(u)-f(v)}{u-v} = f'(c)$

$$\frac{\sin u - \sin v}{u-v} = \cos c$$

But $|\cos c| \leq 1$ $\therefore$ $\left|\frac{\sin u - \sin v}{u-v}\right| \leq 1$

if $v > u$, then in a similar manner, we have

$\therefore \; |\sin v - \sin u| \leq |v - u|$

$\Rightarrow |\sin u - \sin v| \leq |u - v| \quad [\because |-x| = |x|]$

Hence for all $u, v \in R$

$|\sin u - \sin v| \leq |u - v|$

//.

7 Verify cauchy's mean value theorem for $f(x) = e^{-x}$, $g(x) = e^{x}$ in $[2, 6]$

Sol- given :- $f(x) = e^{x}$ & $g(x) = e^{-x}$

$f(x)$ and $g(x)$ are continuous and derivable for all values of $x$.

$\Rightarrow f(x)$ and $g(x)$ continuous in $[3, 7]$

$f(x)$ and $g(x)$ derivable in $(3, 7)$

Also $g'(x) = e^{-x} \neq 0$ for any $x \in (3, 7)$

Thus $f$ and $g$ satisfy the conditions of cauchy's mean value theorem. Consequently, there exists a point $c \in (3, 7)$ such that

$$\frac{f(7) - f(3)}{g(7) - g(3)} = \frac{f'(c)}{g'(c)}$$

$$\frac{e^{7} - e^{3}}{e^{-7} - e^{-3}} = \frac{e^{c}}{-e^{-c}} \Rightarrow \frac{e^{7} - e^{3}}{\frac{1}{e^{7}} - \frac{1}{e^{3}}} = -e^{2c}$$

$$\Rightarrow -e^{7+3} = -e^{2c}$$

$2c = 10 \Rightarrow c = 5 \in (3, 7)$

Hence cauchy's mean value theorem is verified.

~~Expand let cosy latt $(1, \frac{\pi}{4})$ by taylor's series method.~~

1. Verify Taylor's theorem for $f(x) = (1-x)^{5/2}$ with Lagrange's form of remainder upto 2 terms in interval $[0,1]$

A:- Consider $f(x) = (1-x)^{5/2}$ in $[0,1]$

(i) $f(x), f'(x)$ are continuous in $[0,1]$ &

$f(x)$ satisfies the conditions of Taylor's theorem.

$$f(x) = f(0) + f'(0) + \frac{x^2}{2!} f''(\theta x) \text{ with } 0 < \theta < 1 \quad --(1)$$

Here $n = p = 2$, $a = 0$ and $x = 1$

$f(x) = (1-x)^{5/2} \Rightarrow f(0) = 1$

$f'(x) = \frac{-5}{2}(1-x)^{3/2} \Rightarrow f'(0) = -5/2 \Rightarrow f(0) = 1$

$f''(x) = \frac{45}{4}(1-x)^{1/2} \Rightarrow f''(\theta x) = \frac{15}{4}(1-\theta x)^{1/2}$

$\Rightarrow f''(0) = \frac{15}{4}(1-\theta)^{1/2} \qquad f'(1) = 0$

from (1), we have $f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(\theta x)$

substituting the above values, we get

$$0 = 1 + 1 \cdot \left(\frac{-5}{2}\right) + \frac{1^2}{2!}\frac{15}{4}(1-\theta)^{1/2}$$

$$\theta = \frac{9}{25} = 0.36$$

$\therefore$ $\theta$ lies between 0 and 1

Thus the Taylor's theorem is verified

10. Expand: $\log \cos x$ about $\pi/3$ using Taylor's expansion.

Ans: Let $f(x) = \log \cos x$

Using the formula,

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots \quad \text{--- (1)}$$

we take $a = \pi/3$

$f(x) = \log \cos x \Rightarrow f\left(\frac{\pi}{3}\right) = \log \frac{1}{2} = -\frac{1}{2}$

$f'(x) = -\tan x \Rightarrow f'\left(\frac{\pi}{3}\right) = -\tan \pi/3 = -\sqrt{3}$

$f''(x) = -\sec^2 x \Rightarrow f''\left(\frac{\pi}{3}\right) = -\sec^2 \pi/3 = -4$

$f'''(x) = -2\sec^2 x \tan x \quad f'''\left(\frac{\pi}{3}\right) = -8\sqrt{3}$

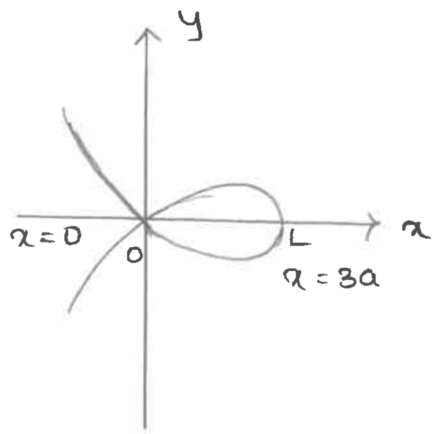
sub in (1) we get

$$f(x) = -\log 2 + \left(x - \frac{\pi}{3}\right)(-\sqrt{3}) + \frac{(x-\pi/3)^2}{2!}(-4) + \frac{\left(x-\frac{\pi}{3}\right)^3}{3!}(-8\sqrt{3}) + \cdots$$

$$f(x) = -\log 2 - \sqrt{3}\left(x - \frac{\pi}{3}\right) - 2\left(x - \frac{\pi}{3}\right)^2 - \frac{4\sqrt{3}}{3}\left(x - \frac{\pi}{3}\right)^3 + \cdots$$

11. The curve $y^2(a+x) = x^2(3a-x)$ revolves above the x-axis. Find the volume of the solid.

Sol| The given curve is symmetrical about the x-axis. when $x=0$; $y=0$ and when $y=0$, $x=3a$. when $x$ varies from 0 to 3a we get upper half of the loop.




$\therefore$ Required loop $= \int_0^{3a} \pi y^2 \, dx$

$$= \pi \int_0^{3a} \frac{x^2(3a-x)}{(a+x)} dx$$

$$= \pi \int_0^{3a} \frac{3ax^2 - x^3}{a+x} dx = \pi \int_0^{3a} \frac{-x^3 + 3ax^2}{a+x} dx$$

$$= \pi \int_0^{3a} \left(-x^2 + 4ax - 4a^2 + \frac{4a^3}{x+a}\right) dx \quad (\text{Dividing Nr by Dr})$$

$$= \pi \left[-\frac{x^3}{3} + 4a \cdot \frac{x^2}{2} - 4a^2x + 4a^3 \cdot \log(x+a)\right]_0^{3a}$$

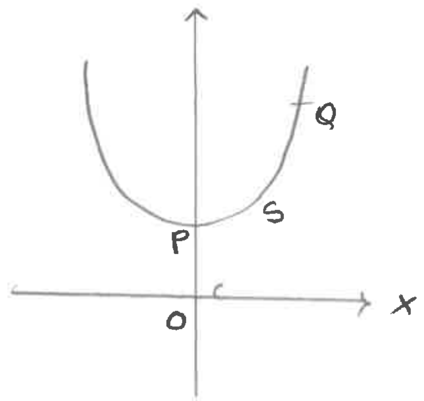
$$= \pi \left[\left\{-\frac{27a^3}{3} + 4a \cdot \frac{9a^2}{2} - 4a^2 \cdot 3a + 4a^3 \log(3a+a)\right\} - \left\{4a^3 \log(0+a)\right\}\right]$$

$$= \pi \left[-\frac{27a^3}{3} + 18a^3 - 12a^3 + 4a^3 \log(4a) - 4a^3 \log(4a) - 4a^3 \log a\right]$$

$$= \pi a^3 \left[-3 + 4\log 4\right] \quad = \pi a^3 \left[8\log 2 - 3\right]$$

12. Find the surface area generated by the revolution of an arc of the catenary $y = c\cosh\frac{x}{c}$ about the x-axis from $x=0$ to $x=c$

Sol: Given that $y = c\cosh\frac{x}{c}$




$$\therefore \frac{dy}{dx} = c\sinh\left(\frac{x}{c}\right)\cdot\left(\frac{1}{c}\right)$$

$$= \sinh\left(\frac{x}{c}\right)$$

Now $$\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$= \sqrt{1+\sinh^2\left(\frac{x}{c}\right)} = \cosh\left(\frac{x}{c}\right)$$

$\therefore$ Required surface area $= \int_0^c 2\pi y \frac{ds}{dx}\cdot dx$

$$= 2\pi\int_0^c c\cosh\left(\frac{x}{c}\right)\cosh\left(\frac{x}{c}\right)$$

$$= 2c\pi\int_0^c \cosh^2\left(\frac{x}{c}\right)dx$$

$$= 2\pi c^2\int_0^1 \cosh^2\theta\, d\theta \quad \left[\text{Put } \frac{x}{c} = \theta \therefore dx = c\,d\theta\right]$$

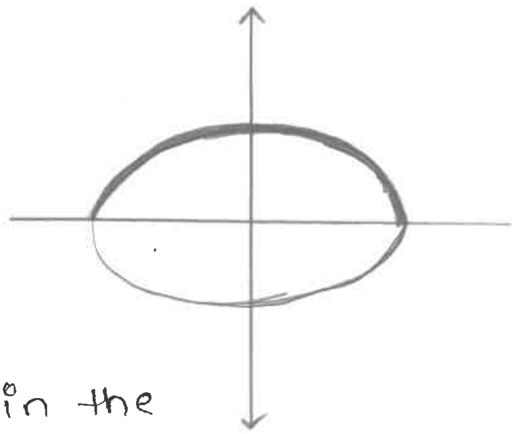
$$= 2\pi c^2\int_0^1 \frac{1+\cosh 2\theta}{2}\, d\theta \quad [\because \cosh 2x = 2\cosh^2 x - 1]$$

$$= \pi c^2\left[\theta + \frac{\sinh 2\theta}{2}\right]_0^1$$

$$= \pi c^2\left[1 + \frac{\sinh 2}{2}\right] = \frac{\pi c^2}{2}(2+\sinh 2)$$

3. Find the surface area of the solid generated by the revolution of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about the $x$-axis.

A. The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



The eq$^n$ of the ellipse can also be written in parametric form as $x = a\cos\theta$ ; $y = b\sin\theta$

Consider the arc of the ellipse in the upper half plane.

Suppose this is revolved about the $x$-axis

Surface area of Solid generated

$$= \int_0^{\pi/2} 2\pi y \, ds$$

$$= 2\int_0^{\pi/2} 2\pi y \left[\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2\right]^{1/2} d\theta$$

$$= 4\pi \int_0^{\pi/2} b\sin\theta \left[(-a\sin\theta)^2 + (b\cos\theta)^2\right]^{1/2} d\theta$$

$$= 4\pi b \int_0^{\pi/2} \sin\theta \left[a^2\sin^2\theta + b^2\cos^2\theta\right]^{1/2} d\theta$$

$$= 4\pi b \int_0^{\pi/2} \sin\theta \left[a^2\sin^2\theta + a^2(1-e^2)\cos^2\theta\right]^{1/2} d\theta$$

$$= 4\pi b \int_0^{\pi/2} \sin\theta \left[a^2(1-\cos^2\theta) + a^2\cos^2\theta - a^2e^2\cos^2\theta\right]^{1/2} d\theta$$

$$= 4\pi b \int_0^{\pi/2} \sin\theta \left[1 - e^2\sin^2\theta\right]^{1/2} d\theta$$

Put $e\cos\theta = \sin t$ then $-e\sin\theta d\theta = \cos t dt$

$\sin\theta d\theta = -\frac{1}{e}\cos t dt$

$\therefore$ Required surfac area $= \int_{\sin^{-1}e}^{0} 4\pi ab \sqrt{1-\sin^2 t}\left(-\frac{1}{e}\right)\cos t dt$

$$= \frac{4\pi ab}{e}\int_0^{\sin^{-1}e} \cos^2 t\, dt$$

$$= \frac{2\pi ab}{e}\int_0^{\sin^{-1}e} (1+\cos 2t)\, dt$$

$$= \frac{2\pi ab}{e}\left[t + \frac{\sin 2t}{2}\right]_0^{\sin^{-1}e}$$

$$= \frac{2\pi ab}{e}\left[t + \sin t\cos t\right]_0^{\sin^{-1}e} = \frac{2\pi ab}{e}\left[t + \sin t\sqrt{(1-\sin^2 t)}\right]_0^{\sin^{-1}e}$$

$$= \frac{2\pi ab}{e}\left[\sin^{-1}e + e\sqrt{1-e^2}\right]$$

$$= 2\pi ab\left[\frac{1}{e}\sin^{-1}e + \sqrt{1-e^2}\right]$$

15) To prove that $\Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}$

Proof :- We know that

$B(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\, dx$ by form 1 of Beta function

Also we have $B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

$$\therefore \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\, dx = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

Taking $m+n=1$ so that $m = 1-n$ we get

$$\int_0^\infty \frac{x^{n-1}}{1+x}\,dx = \frac{\Gamma(1-n)\Gamma(n)}{\Gamma(1)} \quad (\text{or})$$

$$\Gamma n\,\Gamma(1-n) = \int_0^\infty \frac{x^{n-1}}{1+x}\,dx \quad \text{--- (1)}$$

we have $\displaystyle\int_0^\infty \frac{x^{2m}}{1+x^{2n}}\,dx = \frac{\pi}{2n}\operatorname{cosec}\frac{(2m+1)\pi}{2n}$

where $m>0,\ n>0,\ n>m$

Put $x^{2n} = t$ and $\dfrac{2m+1}{2n} = s$

$$\int_0^\infty \frac{t^{(2m/2n)}\,t^{1/2n}}{(2n)(1+t)t}\,dt = \frac{\pi}{2n}\operatorname{cosec} s\pi$$

$$(\text{or})\ \frac{1}{2n}\int_0^\infty \frac{t^{(2m/2n)}\,t^{\frac{1}{2n}-1}}{1+t}\,dt = \frac{\pi}{2n}\operatorname{cosec} s\pi$$

$$(\text{or})\ \int_0^\infty \frac{t^{[(2m+1)/2n]-1}}{1+t}\,dt = \pi\operatorname{cosec} s\pi$$

$$(\text{or})\ \int_0^\infty \frac{t^{s-1}}{1+t}\,dt = \frac{\pi}{\sin s\pi}$$

$$(\text{or})\ \int_0^\infty \frac{x^{n-1}}{1+x}\,dx = \frac{\pi}{\sin n\pi} \quad \text{--- (2)}$$

From (1) and (2), we have

$$\Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}$$

16. Prove that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

Proof:- we know that $B(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

Taking $m = n = \frac{1}{2}$, we have

$$B(\tfrac{1}{2},\tfrac{1}{2}) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}+\frac{1}{2})} = \left[\Gamma(\tfrac{1}{2})\right]^2 \quad [\because \Gamma(1) = 1] \quad \text{---} (1)$$

But $$B(\tfrac{1}{2},\tfrac{1}{2}) = \int_0^1 x^{\frac{1}{2}-1}(1-x)^{\frac{1}{2}-1}\,dx$$

$$= \int_0^1 x^{-1/2}(1-x)^{-1/2}\,dx$$

Put $x = \sin^2\theta\, d\theta$ so that $dx = 2\sin\theta\cos\theta\, d\theta$

Also when $x = 0,\ \theta = 0$; when $x = 1$; $\theta = \pi/2$

$$\therefore B(\tfrac{1}{2},\tfrac{1}{2}) = \int_0^{\pi/2} \frac{1}{\sin\theta}\cdot\frac{1}{\cos\theta}\cdot 2\sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} d\theta = 2(\theta)_0^{\pi/2} = 2\left(\frac{\pi}{2} - 0\right) = \pi \quad \text{---} (2)$$

from (1) & (2) we have

$$\left[\Gamma(\tfrac{1}{2})\right]^2 = \pi \Rightarrow \Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

17. Evaluate $\int_0^\infty x^6 e^{-2x}\,dx$.

Put $2x = y \Rightarrow dx = \frac{1}{2}dy$

$$\therefore \int_0^\infty x^6 e^{-2x}\,dx = \int_0^\infty \left(\frac{y}{2}\right)^6 e^{-y}\cdot\frac{1}{2}dy$$

$$= \frac{1}{2^7}\int_0^\infty e^{-y} y^6\, dy$$

$$= \frac{1}{2^7}\int_0^\infty e^{-y} y^{7-1} dy = \frac{1}{2^7}\Gamma(7)$$

$$= \frac{1}{2^7} \times 6! = \frac{45}{8}$$

18. Evaluate $\int_0^{\pi/2} \sin^5\theta \cos^{\frac{7}{2}}\theta \, d\theta$

we have $\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta = \frac{1}{2} B(m,n)$

Put $2m-1 = 5$ $\qquad\qquad 2n-1 = 7/2$

$m = 3$ $\qquad\qquad n = 9/4$

$$\Rightarrow \int_0^{\pi/2} \sin^5\theta \cos^{7/2} d\theta = \frac{1}{2} B\left(3, \frac{9}{4}\right) = \frac{1}{2} \frac{\Gamma(3)\Gamma\left(\frac{9}{4}\right)}{\Gamma\left(3+\frac{9}{4}\right)}$$

$$= \frac{1}{2} \frac{\Gamma(3)\Gamma\left(\frac{9}{4}\right)}{\Gamma\left(\frac{21}{4}\right)} \qquad \left[\because \Gamma(3) = 2!\right]$$

$$= \frac{\Gamma(9/4)}{\Gamma\left(\frac{21}{4}\right)}$$

$$\therefore \int_0^{\pi/2} \sin^5\theta \cos^{7/2}\theta \, d\theta = \frac{\Gamma(9/4)}{\frac{17}{4}\cdot\frac{13}{4}\cdot\frac{9}{4}\,\Gamma\left(\frac{9}{4}\right)}$$

$$= \frac{64}{1989}$$

19. $B(m, \frac{1}{2}) = 2^{2m-1} B(m,m)$

Writing the above result in terms of Gamma function we have

$$\frac{\Gamma(m)\Gamma(\frac{1}{2})}{\Gamma(m+\frac{1}{2})} = 2^{2m-1} \frac{\Gamma(m)\Gamma(m)}{\Gamma(m+m)} \quad (\text{or}) \quad \frac{\Gamma(\frac{1}{2})}{\Gamma(m+\frac{1}{2})}$$

$$= 2^{2m-1} \frac{\Gamma(m)}{\Gamma(2m)}$$

(or)

$$\frac{\Gamma(m)}{\Gamma(m+\frac{1}{2})} = 2^{mm-1} \frac{\Gamma(m)}{\Gamma(2m)} \quad \text{or} \quad \Gamma(m)\Gamma(m+\frac{1}{2}) = \frac{\Gamma\pi}{2^{2m-1}} \Gamma(2m)$$

4. State and prove the relation between Beta and Gamma function.

$$\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}$$

Proof :- By defination

$$\Gamma n = 2\int_0^\infty e^{-x^2} x^{2n-1}\, dx \quad --- \quad (1)$$

$$\Gamma m = 2\int_0^\infty e^{-y^2} y^{2m-1}\, dy \quad --- \quad (2)$$

$$\Gamma n \Gamma m = 4\int_0^\infty e^{-x^2} x^{2n-1} dx \cdot \int_0^\infty e^{-y^2} 2^{2m-1} dy$$

$$= 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)} x^{2n-1} y^{2m-1}\, dx \cdot dy$$

put $x = r\cos\theta$ $\quad y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

$\theta$ varies from $0$ to $\pi/2$

$r \longrightarrow 0$ to $\infty$

$$= 4\int_0^\infty\int_0^{\pi/2} e^{-r^2} (r\cos\theta)^{2n-1} (r\sin\theta)^{2m-1} \cdot r\,dr\,d\theta$$

$$= 2\int_0^\infty e^{-r^2} r^{2(m+n)-1}\, dr \cdot 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$$

$$= \Gamma{m+n} \cdot \beta(m,n)$$

$$\left[\because 2\int_0^\infty e^{-x^2} x^{2n-1}\, dx = \Gamma n\right]$$

$$\left[\because 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta = \beta(m,n)\right.$$

$$\Rightarrow \Gamma n\,\Gamma m = \Gamma{m+n} \cdot \beta(m,n)$$

$$\beta(m,n) = \frac{\Gamma n\,\Gamma m}{\Gamma{m+n}}$$

Hence Proved

8) Obtain the Maclaurin's series expansion of $\log_e(1+x)$

Sol Let $f(x) = \log_e(1+x)$. then

$f'(x) = \frac{1}{1+x}$; $f''(x) = \frac{-1}{(1+x)^2}$;

$f'''(x) = \frac{(-1)^2 2!}{(1+x)^3}$; $f^4(x) = \frac{(-1)^3 \cdot 3!}{(1+x)^4}$;

Note at $x = -1$, $f'(x)$, $f''(x)$ etc. does not exist.

$f^4(0) = (-1)^3 \cdot 3! \; \ldots\ldots, f^n(0) = (-1)^{n-1}(n-1)!$

Hence the maclaurian's series expansion of $f(x)$ is

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots \frac{x^n}{n!} f^n(0) + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{f^n(0)}{n!} x^n \qquad \because f(0) = 0$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!}{(n!)} x^n = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

Hence $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

Thus for $-1 < x \leq 1$, $\log(1+x)$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n-1}\frac{x^n}{n} + \cdots$$

# UNIT-V
# PARTIAL DIFFERENTIATION

**Short answer questions:**

1.State Euler's theorem.

2.Write chain rule of partial of differentiation if z=f(x,y) where x= u(r,s) and y= v(r,s)

3.Define Homogeneous function.

4.Define Jacobian and functional dependence

5.If x=uv , y=u/v then find J

6.If x=u(1+v),y=v(1+u) then find $J(\frac{(X,Y)}{(U,V)})$

7.If $u = x^2 - 2y, v = x + y + z, w = x - 2y + 3y$ find $J(\frac{(u,v,w)}{(x,y,z)})$

8.State and prove the necessary and sufficient conditions for extreme of a function `f' of two variables

9.Find the points on the surface $z^2$=xy+1 that are nearest to the origin.

10.Find the stationary points of u (x, y) = Sin x Sin y Sin (x + y) where 0 < x <Π,0 < y <Π and find the maximum u

**Long answer questions**

1. State and prove Euler' theorem of Homogeneous function
2. If $u = f(r,s,t)$, where $r = \frac{x}{y}, s = \frac{y}{z}$ and $t = \frac{z}{x}$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$
3. If $u=\tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$ then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$=sin2u
4. If $x^x y^y z^z$= c, then show that at x = y = z, $\frac{\partial^2 z}{\partial x \partial y} = -\{x\log(ex)\}^{-1}$
5. If μ = log ($x^3+y^3+z^3$-3xyz) prove that

   $\mu_x + \mu_y + \mu_z = 3(x+y+z)^{-1}$ and $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 \mu = \frac{-9}{(x+y+z)^2}$
6. Verify chain rule for Jacobian for the following function $x = u, y = u\tan v, z = w$
7. If $u = \frac{yz}{x}; v = \frac{zx}{y}; w = \frac{xy}{z}$, show that $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 4$.
8. If $x = r\sin\theta\cos\emptyset, y = r\sin\theta\sin\emptyset, z = r\cos\theta$, show that $\frac{\partial(x,y,z)}{\partial(r,\theta,\emptyset)} = r^2\sin\theta$
9. If $u = x^2 - 2y,\ v = x + y + z, w = x - 2y + 3z$ Find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$
10. If $x = u(1-v), y = uv$ prove that $JJ^* = 1$

11. If $x + y + z = u , y + z = uvandz = uvw$ then show that $\frac{\partial(u,v,w)}{\partial(x,y,z)} = u^2 v$

12. If $u = x^2 - y^2, v = 2xywherex = rcos\theta, y = rsin\theta$. Show that $\frac{\partial(u,v)}{\partial(r,\theta)} = 4r^3$

13. If u = f(r,s,t) where r = x / y, s = y / z and t = z / x show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$

14. Show that the functions $u = x + y + z$, $v = x^2+y^2+z^2-2xy-2zx-2yz$ and $w = x^3+y^3+z^3-3xyz$ are functionally related. Find the relation between them.

15. Show that the functions $u = xy + yz + zx$, $v = x^2+y^2+z^2$ and $w = x+y+z$ are functionally related. Find the relation between them.

16. Divide a given positive number n into three parts such that their product is maximum.

17. Locate the stationary points and examine their nature of the following functions :
$u=x^4 + y^4 - 2x^2 + 4xy - 2y^2$, $(x > 0, y > 0)$.

18. A rectangular box open at the top is to have a volume of 32 cubic ft. Find the dimensions of the box, requiring least material for its construction.

19. Find the volume of the greatest rectangular parallelepiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

20. Given that $x + y + z = a$, find the maximum value of $x^m y^n z^p$.

21. Find the minimum value of $x^2 + y^2 + z^2$ given $x + y + z = 3a$.

22. Find the maximum value of $x^2 y^3 z^4$ subject to the condition $2x + 3y + 4z = a$.

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Functional dependence:- Let $u = f(x,y)$ and $v = g(x,y)$ are two given differentiable functions in the independent variables $x, y$

Functional dependence:- If $u, v$ are functions of variables $x, y$. $u, v$ are said to be functionaly dependent if

$$\frac{\partial(u,v)}{\partial(x,y)} = 0$$

5) if $x = uv$, $y = u/v$ then find J

$$\frac{\partial x}{\partial u} = v \qquad \frac{\partial y}{\partial u} = \frac{1}{v}$$

$$\frac{\partial x}{\partial v} = u \qquad \frac{\partial y}{\partial v} = \frac{-u}{v^2}$$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ \frac{1}{v} & \frac{-u}{v^2} \end{vmatrix}$$

$$= v\left(\frac{-u}{v^2}\right) - \frac{1}{u}(u) = \frac{-2u}{v}$$

6) $x = u(1+v)$, $y = v(1+u)$ then find $J\left(\frac{x,y}{u,v}\right)$

$$\frac{\partial x}{\partial u} = 1+v; \quad \frac{\partial y}{\partial u} = v; \quad \frac{\partial x}{\partial v} = u; \quad \frac{\partial y}{\partial v} = 1+u$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1+v & u \\ v & 1+u \end{vmatrix}$$

$$= (1+v)(1+u) - uv$$

$$= 1 + u + v$$

# UNIT - V

# MULTIVARIABLE CALCULUS



## * Short Answer questions:-

1. State Euler's theorem.

A: If $z = f(x, y)$ is a homogeneous function of degree $n$, then

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$$

2. Write chain rule of partial of differentiation if $z = f(x, y)$ where $x = u(r, s)$ and $y = v(r, s)$.

A: $$\Rightarrow \frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$\Rightarrow \frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}$$

3. Define Homogeneous function.

A: A function $f(x, y)$ is said to be homogeneous function of degree 'n' if $f(kx, ky) = k^n f(x, y)$

4. Define Jacobian and Function dependence

Jacobian:- let $u, v$ are functions of two variables $x, y \Rightarrow u = u(x, y)$; $v = v(x, y)$ then

$J\left(\frac{u, v}{x, y}\right) = \frac{\partial(u, v)}{\partial(x, y)}$ is given by $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$

$$\Rightarrow J\left(\frac{u, v}{x, y}\right) = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

7) If $u = x^2 - 2y$ $v = x+y+z$ ; $w = x - 2y + 3y$ find $J\left(\frac{u,v,w}{x,y,z}\right)$

$\frac{\partial u}{\partial x} = 2x$ ; $\frac{\partial u}{\partial y} = -2$ ; $\frac{\partial u}{\partial z} = 0$ ; $\frac{\partial v}{\partial x} = 1$ $\frac{\partial v}{\partial y} = 1$ $\frac{\partial v}{\partial z} = 1$

and $\frac{\partial w}{\partial x} = 1$ ; $\frac{\partial w}{\partial y} = -2$ ; $\frac{\partial w}{\partial z} = 3$

7

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} 2x & -2 & 0 \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix}$$

$$= 2x(3+2) + 2(3-1) = 10x + 4$$

3) State and prove the necessary and sufficient conditions for extreme of a function 'f' of two variables

A! Defination :- let $f(x,y)$ be a function of two variables $x$ and $y$.

At $x = a$; $y = b$, $f(x,y)$ is said to have maximum or minimum value, if $(a,b) > f(a+h, b+h)$ or $f(a,b) < f(a+h, b+k)$ respectively where $h$ and $k$ are small values.

Extreme values : $f(a,b)$ is said to be an extreme values of $f$, If it is a maximum or minimum value.

$\frac{\partial \phi}{\partial x} = P$, $\frac{\partial}{\partial y} = Q$

(i) The necessary condition for $f(x,y)$ to have a maximum or minimum at $(a,b)$ are

$$f_x(a,b) = 0 \quad f_y(a,b) = 0$$

(ii) Sufficient conditions: suppose that $f_x(a,b)=0$, $f_y(a,b)=0$ and let

$$\frac{\partial^2}{\partial x^2} f(a,b) = r ; \quad \frac{\partial^2}{\partial x \partial y} f(a,b) = s ; \quad \frac{\partial^2}{\partial y^2} f(a,b) = t$$

(i) Then $f(a,b)$ is maximum if $rt - s^2 > 0$ and $r < 0$

(ii) Then $f(a,b)$ is minimum if $rt - s^2 > 0$ and $r > 0$

(iii) $f(a,b)$ is not an extreme value if $rt - s^2 < 0$

(iv) If $rt - s^2$ then $f(x,y)$ fails to have maximum or minimum value and its needs further ~~ekkey~~ extreme value is investigation.

(9) Find the points on the surface $z^2 = xy + 1$ that are nearest to the origin.

Sol - let $P(x,y,z)$ be any point on the surface

$$\phi(x,y,z) = z^2 - xy - 1 = 0 \quad --- (1)$$

$$OP = P = \sqrt{x^2+y^2+z^2} \quad [\because \text{shortest distance}]$$

$$P^2 = x^2+y^2+z^2 \quad --- (2)$$

~~[from (1) & (2) we have~~

~~$$P^2 = x^2+y^2+xy+1 \quad --- (3)$$~~

~~let $\frac{\partial^2 P}{\partial x^2} = r \quad \frac{\partial^2 P}{\partial x \partial y} = s \quad \frac{\partial^2 P}{\partial y^2} = t$ ]~~

10: Find the stationary points of $u(x,y) = \sin x \sin y \sin(x+y)$ where $0 < x < \pi$, $0 < y, \pi$ and find maximum

$$f(x,y) = \sin x + \sin y + \sin(x+y)$$

$$\frac{\partial f}{\partial x} = \cos x + \cos(x+y) \quad \text{— (1)} \quad = p$$

$$\frac{\partial f}{\partial y} = \cos y + \cos(x+y) \quad = q$$

$$\frac{\partial^2 f}{\partial x^2} = -\sin x - \sin(x+y) \quad = r$$

$$\frac{\partial^2 f}{\partial y^2} = -\sin y - \sin(x+y) \quad = t$$

$$\frac{\partial^2 f}{\partial x \partial y} = -\sin(x+y) \quad = s$$

equate $p = 0$ & $q = 0$

$$\cos x + \cos(x+y) = 0$$

$$\cos y + \cos(x+y) = 0$$

$$\cos x - \cos y = 0 \qquad \boxed{x = y}$$

sub in (1)

$$\cos x \pm \cos 2x = 0$$

$$2\cos\left(\frac{x+2x}{2}\right)\cos\left(\frac{x-2x}{2}\right) = 0$$

$$2\cos\frac{3x}{2}\cos\frac{x}{2} = 0$$

$$\cos\frac{3x}{2} = 0 \qquad\qquad \cos\frac{x}{2} = 0$$

$$\frac{3x}{2} = \pm\pi/2 \qquad\qquad \frac{x}{2} = \pm\pi/2$$

$$x = \pm\pi/3 \qquad\qquad x = \pm\pi$$

$\therefore$ stationary points are $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ $\left(-\frac{\pi}{3}, \frac{-\pi}{3}\right)$ $(\pi, \pi)$.
$(-\pi, -\pi)$

At $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$

$r = -\sin \pi/3 - \sin \frac{2\pi}{3}$ $\quad t = -\sin y - \sin(x+y)$

$= -\sqrt{3}$ $\quad = -\sqrt{3}$

$s = -\sin\left(\frac{\pi}{3} + \pi/3\right) = -\sqrt{3}/2$

$\therefore rt - s^2 = 9/4 > 0$

$\therefore f$ is maximum at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$

At $\left(-\frac{\pi}{3}, -\frac{\pi}{3}\right)$

$rt - s^2$ is $> 0$

$\therefore f$ is maximum at $\left(-\frac{\pi}{3}, -\frac{\pi}{3}\right)$

At $(\pm\pi, \pm\pi)$

$rt - s^2 = 0$

$\therefore$ There is a need for further investigation.

* **Long Answer questions.** (1)

1. state and prove Euler's theorem of Homogeneous function.

**Theorem:-** If $z = f(x, y)$ is a homogeneous function of degree $n$, then $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$.

**Proof:-** Let $z$ be a homogeneous function of order '$n$'

$$\Rightarrow z = x^n f\left(\frac{y}{x}\right) \quad \text{---- (1)}$$

Differentiating (1) partially w.r.t to $x$ & $y$

$$\Rightarrow \frac{\partial z}{\partial x} = nx^{n-1} f\left(\frac{y}{x}\right) + x^n f'\left(\frac{y}{x}\right)\cdot\left(\frac{-y}{x^2}\right) \quad \text{--- (2)}$$

$$= nx^{n-1} f\left(\frac{y}{x}\right) - y\cdot x^{n-2} f'\left(\frac{y}{x}\right) \quad \left[\because UV = U'V + UV'\right]$$

$$\Rightarrow \frac{\partial z}{\partial y} = x^n f'\left(\frac{y}{x}\right)\cdot\frac{1}{x} = x^{n-1} f'\left(\frac{y}{x}\right) \quad \text{--- (3)}$$

Multiplying (2) and (3) with $x$ & $y$ respectively

$$\Rightarrow x\frac{\partial z}{\partial x} = nx^n f\left(\frac{y}{x}\right) - y\cdot x^{n-1} f'\left(\frac{y}{x}\right) \quad \text{--- (4)}$$

$$\Rightarrow y\frac{\partial z}{\partial y} = x^{n-1}\cdot y f'\left(\frac{y}{x}\right) \quad \text{--- (5)}$$

Adding (4) & (5) eqn

$$\Rightarrow x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nx^n f\left(\frac{y}{x}\right) - \cancel{y\cdot x^{n-1} f'\left(\frac{y}{x}\right)} + \cancel{x^{n-1}\cdot y f'\left(\frac{y}{x}\right)}$$

$$= nx^n f\left(\frac{y}{x}\right) \quad \left[\because \text{from (1)}\right]$$

$$= nz$$

Hence proved Euler's theorem of Homogeneous function $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$.

2. if $u = f(r, s, t)$, where $r = \frac{x}{y}$, $s = \frac{y}{z}$ and $t = \frac{z}{x}$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$

Sol- Given :- $u = f(r, s, t)$

$$r = \frac{x}{y}, \quad s = \frac{y}{z}, \quad t = \frac{z}{x}$$

To find :- $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$

$u = f(r, s, t)$

Partially diff ~~w.r.t x~~ by using "chain rule"

$$\Rightarrow \frac{\partial r}{\partial x} = \frac{1}{y} \qquad \frac{\partial s}{\partial x} = 0 \qquad \frac{\partial t}{\partial x} = -\frac{z}{x^2}$$

$$\frac{\partial r}{\partial y} = \frac{-x}{y^2} \qquad \frac{\partial s}{\partial y} = \frac{1}{z} \qquad \frac{\partial t}{\partial y} = 0$$

$$\frac{\partial r}{\partial z} = 0 \qquad \frac{\partial s}{\partial z} = \frac{-y}{z^2} \qquad \frac{\partial t}{\partial z} = \frac{1}{x}$$

By using chain rule of partial differentiation, we know

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x}$$

$$= \frac{\partial u}{\partial r}\left(\frac{1}{y}\right) + \frac{\partial u}{\partial s}(0) + \frac{\partial u}{\partial t}\left(\frac{-z}{x^2}\right) = \frac{1}{y}\frac{\partial u}{\partial r} - \frac{z}{x^2}\frac{\partial u}{\partial t}$$

$$x\frac{\partial u}{\partial x} = \frac{x}{y}\frac{\partial u}{\partial r} - \frac{z}{x}\frac{\partial u}{\partial t} \quad \text{---- (1)}$$

$$\Rightarrow \frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial y}$$

$$= \frac{\partial u}{\partial r}\left(\frac{-x}{y^2}\right) + \frac{\partial u}{\partial s}\cdot\left(\frac{1}{z}\right) + \frac{\partial u}{\partial t}(0) = -\frac{x}{y^2}\frac{\partial u}{\partial r} + \frac{1}{z}\frac{\partial u}{\partial s}$$

$$y\frac{\partial u}{\partial x} = -\frac{x}{y}\frac{\partial u}{\partial r} + \frac{y}{z}\frac{\partial u}{\partial s} \quad \text{---- (2)}$$

$$\Rightarrow \frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\,\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\,\frac{\partial t}{\partial z}$$

$$= \frac{\partial u}{\partial r}(0) + \frac{\partial u}{\partial s}\left(\frac{-y}{z^2}\right) + \frac{\partial u}{\partial t}\cdot\left(\frac{1}{x}\right) = -\frac{y}{z^2} + \frac{1}{x}\frac{\partial u}{\partial t}$$

$$\therefore z\frac{\partial u}{\partial z} = \frac{-y}{z}\frac{\partial u}{\partial s} + \frac{z}{x}\frac{\partial u}{\partial t} \quad --- \quad (3)$$ ②

Adding (1) & (2) & (3) we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} =$$

$$\left(\frac{x}{y}\cancel{\frac{\partial u}{\partial r}} - \frac{z}{x}\cancel{\frac{\partial u}{\partial t}}\right) + \left(-\frac{x}{y}\cancel{\frac{\partial u}{\partial r}} + \cancel{\frac{y}{z}}\frac{\partial u}{\partial s}\right) + \left(-\cancel{\frac{y}{z}}\frac{\partial u}{\partial s} + \frac{z}{x}\cancel{\frac{\partial u}{\partial t}}\right)$$

$$= 0$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$$

Hence showed.

---

3) If $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$ then prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$$

Sol:- Given :- $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$ which is not a homogeneous function

so, Apply "tan" on both sides we get

$$\Rightarrow \tan u = \frac{x^3+y^3}{x-y} = f(x,y)$$

$\therefore f(x,y)$ homogeneous function of $x$ and $y$ of degree '2'

By using euler's theorem we know

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 2\cdot f \qquad f = \tan u$$
$$---- (1)$$

Partially differentiating $f(x,y)$ with $x$ & $y$

$$\Rightarrow \frac{\partial f}{\partial x} = \sec^2 u\,\frac{\partial u}{\partial x} \qquad \Rightarrow \frac{\partial f}{\partial y} = \sec^2 u\,\frac{\partial u}{\partial y}$$

$$\Rightarrow x\frac{\partial f}{\partial x} = x\cdot\sec^2 u\,\frac{\partial u}{\partial x} \qquad y\frac{\partial f}{\partial y} = y\cdot\sec^2 u\,\frac{\partial u}{\partial y}$$

substitue in (1) eqn

$$\Rightarrow x \cdot \sec^2 u \cdot \frac{\partial u}{\partial x} + y \sec^2 u \frac{\partial u}{\partial y} = 2 \cdot \tan u$$

$$\sec^2 u \left[ x \frac{\partial u}{\partial x} + y \cancel{\sec^2} \frac{\partial u}{\partial y} \right] = 2 \cdot \tan u$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2 \cdot \frac{\tan u}{\sec^2 u}$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2 \cdot \frac{\sin u}{\cos u} \cdot \cos^2 u$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \sin 2u$$

Hence proved.

---

4) if $x^x y^y z^z = c$, then show that at $x = y = z$,

$$\frac{\partial^2 z}{\partial x \partial y} = -\{x \log(ex)\}^{-1}$$

Sol- Given :- $x^x y^y z^z = e$

Apply "log" on both sides

$$\Rightarrow \log[x^x y^y z^z] = \log e \qquad \left[ \because \begin{array}{l} \log a^b = b \log a \\ \log ab = \log a + \log b \end{array} \right]$$

$$\Rightarrow x \log x + y \log y + z \log z = \log e \qquad [\because \log e = 1]$$

$$\Rightarrow z \log z = 1 - x \log x - y \log y$$

Differentiating partially with respect to '$x$'

$$\Rightarrow \left( z \cdot \frac{1}{z} + 1 \cdot \log z \right) \frac{\partial z}{\partial x} = -\left[ x \cdot \frac{1}{x} + \log x \right]$$

$$\Rightarrow (1 + \log z) \frac{\partial z}{\partial x} = -(1 + \log x) \quad - \cancel{(1)}$$

$$\Rightarrow \frac{\partial z}{\partial x} = -\frac{(1 + \log x)}{(1 + \log z)} \quad ---- \quad (7)$$

Given $x = y = z$ then

$$\frac{\partial z}{\partial x} = -\frac{(1+\log x)}{(1+\log x)} = -1 \quad \text{and}$$

Similarly $$\frac{\partial z}{\partial y} = -\frac{(1+\log y)}{1+\log z} \quad --- \quad (2)$$

$$\frac{\partial z}{\partial y} = -\frac{1+\log y}{1+\log y} = -1$$

Now differentiating (2) w.r.t '$x$' we get

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left[-\frac{(1+\log y)}{(1+\log z)}\right]$$

$$= \frac{\partial}{\partial x}\left[-(1+\log y)(1+\log z)^{-1}\right]$$

$$= -(1+\log y)\left[-(1+\log z)^{-2}\frac{1}{z}\frac{\partial z}{\partial x}\right]$$

$$= \frac{1+\log y}{z(1+\log z)^2}\frac{\partial z}{\partial x} \quad --- \quad (3)$$

Given $x = y = z$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{1+\log x}{x(1+\log x)^{2}}(-1) \quad \left[\because \frac{\partial z}{\partial x} = -1\right]$$

$$= -\frac{1}{x(1+\log x)} = -\frac{1}{x(\log e+\log x)} \quad [\because \log e = 1]$$

$$= -\frac{1}{x(\log ex)} \quad [\because \log a + \log b = \log ab]$$

$$= -(x\log ex)^{-1}$$

$$\therefore \frac{\partial^2 z}{\partial x \partial y} = -(x\log ex)^{-1}$$

Hence showed

5) if $\mu = \log(x^3+y^3+z^3-3xyz)$ prove that

$$\mu_x + \mu_y + \mu_z = 3(x+y+z)^{-1}$$ and

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 \mu = \frac{-9}{(x+y+z)^2}$$

<u>Sol-</u> Given $\mu = \log(x^3+y^3+z^3-3xyz)$

differentiating partially w.r.t 'x', 'y' & 'z'

$$\Rightarrow \frac{\partial \mu}{\partial x} = \frac{1}{x^3+y^3+z^3-3xyz}[3x^2-3yz] \quad \text{—— (1)}$$

$$\Rightarrow \frac{\partial \mu}{\partial y} = \frac{1}{x^3+y^3+z^3-3xyz}[3y^2-3xz] \quad \text{—— (2)}$$

$$\Rightarrow \frac{\partial \mu}{\partial z} = \frac{1}{x^3+y^3+z^3-3xyz}[3z^2-3xy] \quad \text{—— (3)}$$

Adding (1) & (2) & (3)

$$\frac{\partial \mu}{\partial x} + \frac{\partial \mu}{\partial y} + \frac{\partial \mu}{\partial z} = \frac{3[x^2+y^2+z^2-xy-yz-zx]}{x^3+y^3+z^3-3xyz}$$

$$= \frac{3(x^2+y^2+z^2-xy-yz-zx)}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)}$$

$$\Rightarrow \frac{\partial \mu}{\partial x} + \frac{\partial \mu}{\partial y} + \frac{\partial \mu}{\partial z} = \frac{3}{(x+y+z)} \quad \text{---- (4)}$$

Now $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 \mu = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial \mu}{\partial x} + \frac{\partial \mu}{\partial y} + \frac{\partial \mu}{\partial z}\right)$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

[$\because$ from (4)]

$$= \frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right) + \frac{\partial}{\partial y}\left(\frac{3}{x+y+z}\right) + \frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right)$$

$$= -\frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} \quad (4)$$

$$= -\frac{9}{(x+y+z)^2}$$

$$\therefore \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$$

Hence proved

(6) Verify chain rule for Jacobian for the following

$x = u, \quad y = u\tan v, \quad z = w$

let $f_1 = x - u = 0 \; ; \; f_2 = y - u\tan v = 0 \; f_3 = z - w = 0$

$\frac{\partial f_1}{\partial x} = 1 \; ; \; \frac{\partial f_1}{\partial u} = -1$ | $\frac{\partial f_2}{\partial x} = 0 \; ; \; \frac{\partial f_2}{\partial u} = -\tan v$ | $\frac{\partial f_3}{\partial x} = 0 \; ; \; \frac{\partial f_3}{\partial u} = 0$

$\frac{\partial f_2}{\partial y} = 0 \; ; \; \frac{\partial f_2}{\partial v} = 0$ | $\frac{\partial f_2}{\partial y} = 1 \; ; \; \frac{\partial f_2}{\partial v} = -u\sec^2 v$ | $\frac{\partial f_3}{\partial y} = 0 \; ; \; \frac{\partial f_3}{\partial v} = 0$

$\frac{\partial f_2}{\partial z} = 0 \quad \frac{\partial f_3}{\partial w} = 0$ | $\frac{\partial f_2}{\partial z} = 0 \; ; \; \frac{\partial f_2}{\partial w} = 0$ | $\frac{\partial f_3}{\partial z} = 1 \; ; \; \frac{\partial f_3}{\partial w} = -1$

By using Implict function

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{\partial(f_1 f_2 f_3)}{\partial(u,v,w)} \div \frac{\partial(f_1 f_2 f_3)}{\partial(u,v,w)} \quad (1)$$

$$\Rightarrow \frac{\partial(f_1 f_2 f_3)}{\partial(x,y,z)} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 = \cancel{\frac{\partial(u,v,w)}{\partial(x,y,}}$$

$$\Rightarrow \frac{\partial(f_1 f_2 f_3)}{\partial(u,v,w)} = \begin{vmatrix} -1 & 0 & 0 \\ \tan v & -u\sec^2 v & 0 \\ 0 & 0 & -1 \end{vmatrix}$$

$$= -1\,(u\sec^2 v)$$

sub in (1)

$$\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)} \div \frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = 1 \div -1\,(u\sec^2 v)$$

$$= \frac{-1}{u\sec^2 v}$$

Now

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} \times \frac{\partial(u,v,w)}{\partial(x,y,z)}$$

$$= -\frac{1}{u\sec^2 v} \times -(u\sec^2 v)$$

$$= 1 = \frac{\partial(x,y,z)}{\partial(x,y,z)}$$

Hence chain rule for Jacobian is verified.

7) If $u = \frac{yz}{x}$ ; $v = \frac{zx}{y}$ ; $w = \frac{xy}{z}$ , show that $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 4$. (5)

Sol:- we have

$$\frac{\partial u}{\partial x} = \frac{-yz}{x^2} \qquad \frac{\partial v}{\partial x} = \frac{z}{y} \qquad \frac{\partial w}{\partial x} = \frac{y}{z}$$

$$\frac{\partial u}{\partial y} = \frac{z}{x} \qquad \frac{\partial v}{\partial y} = \frac{-zx}{y^2} \qquad \frac{\partial w}{\partial y} = \frac{x}{z}$$

$$\frac{\partial u}{\partial z} = \frac{y}{x} \qquad \frac{\partial v}{\partial z} = \frac{x}{y} \qquad \frac{\partial w}{\partial z} = -\frac{xy}{z^2}$$

$$\therefore \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} -\frac{yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & -\frac{zx}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & -\frac{xy}{z^2} \end{vmatrix}$$

$$= \frac{1}{xyz} \begin{vmatrix} \frac{-yzx}{x^2} & \frac{zy}{x} & \frac{yz}{x} \\ \frac{xz}{y} & -\frac{zxy}{y^2} & \frac{xz}{y} \\ \frac{xy}{z} & \frac{xy}{z} & -\frac{xyz}{z^2} \end{vmatrix}$$

$$= \frac{1}{xyz} \cdot \frac{yz}{x} \cdot \frac{xz}{y} \cdot \frac{xy}{z} \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= \frac{x^2y^2z^2}{(xyz)^2}\left[-1(1-1)-1(-1-1)+1(1+1)\right]$$

$$= -1(0)+2+2 = 4$$

$$\therefore \frac{\partial(u,v,w)}{\partial(x,y,z)} = 4$$

---

3) if $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

show that $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta$

Sol - $\frac{\partial x}{\partial r} = \sin\theta\cos\phi$ | $\frac{\partial y}{\partial r} = \sin\theta\sin\phi$ | $\frac{\partial z}{\partial r} = \cos\theta$

$\frac{\partial x}{\partial \theta} = r\cos\theta\cos\phi$ | $\frac{\partial y}{\partial \phi} = r\cos\theta\sin\phi$ | $\frac{\partial z}{\partial \theta} = -r\sin\theta$

$\frac{\partial x}{\partial \phi} = r\sin\theta\sin\phi$ | $\frac{\partial y}{\partial \phi} = r\sin\theta\cos\phi$ | $\frac{\partial z}{\partial \phi} = 0$

$$\therefore \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & \cos\theta \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & -r\sin\theta \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= \cos\theta\left[(r\cos\theta\cos\phi)(r\sin\theta\cos\phi) + (r\cos\theta\sin\phi)(r\sin\theta\sin\phi)\right] + r\sin\theta\left[(\sin\theta\cos\phi)(r\sin\theta\cos\phi) + (\sin\theta\sin\phi)(r\sin\theta\sin\phi)\right]$$

$= \cos\theta\left[r^2\sin\theta\cos\theta(\cos^2\phi+\sin^2\phi)\right] + r\sin\theta\left[r\sin^2\theta\right.$
$\left.(\cos^2\phi+\sin^2\phi)\right]$

$= \cos\theta\left[r^2\sin\theta\cos\theta\right] + r\sin\theta\left[r\sin^2\theta\right]$

$= r^2\sin\theta\cos^2\theta + r^2\sin^3\theta$

$= r^2\sin\theta(\cos^2\theta+\sin^2\theta) = r^2\sin\theta$

since $\dfrac{\partial(x,y,z)}{\partial(r,\theta,\phi)} \cdot \dfrac{\partial(r,\theta,\phi)}{\partial(x,y,z)} = 1$

$$\Rightarrow \frac{\partial(r,\theta,\phi)}{\partial(x,y,z)} = \frac{1}{r^2\sin\theta}$$

---

9) if $u = x^2 - 2y$, $v = x+y+z$, $w = x - 2y + 3z$

find $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}$

Sol- $\dfrac{\partial u}{\partial x} = 2x$ | $\dfrac{\partial v}{\partial x} = 1$ | $\dfrac{\partial w}{\partial x} = 1$

$\dfrac{\partial u}{\partial y} = -2$ | $\dfrac{\partial v}{\partial y} = 1$ | $\dfrac{\partial w}{\partial y} = -2$

$\dfrac{\partial u}{\partial z} = 0$ | $\dfrac{\partial v}{\partial z} = 1$ | $\dfrac{\partial w}{\partial z} = 3$

Now :-

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 2x & -2 & 0 \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix}$$

$= 2x[3+2] + 2[3-1]$

$= 10x + 4$

10) if $x = u(1-v)$, $y = uv$ prove that $JJ^* = 1$

Sol- we have $x = u(1-v)$ $\quad y = uv$ ---- (1)

$$\frac{\partial x}{\partial u} = 1-v \qquad \frac{\partial y}{\partial u} = v$$

$$\frac{\partial x}{\partial y} = -u \qquad \frac{\partial y}{\partial v} = u$$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix}$$

$$= u(1-v) + uv$$

$$= u - u\cancel{v} + u\cancel{v} = u \quad \text{---} \quad (2)$$

eq$^n$ (1) can be written as

$$x = u - uv \qquad v = \frac{y}{u}$$

$$= u - y \qquad = \frac{y}{x+y}$$

$$\Rightarrow u = x+y$$

$$J' = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 \\ -\frac{y}{(x+y)^2} & \frac{x}{(x+y)^2} \end{vmatrix}$$

$$= \frac{x}{(x+y)^2} + \frac{y}{(x+y)^2} = \frac{x+y}{(x+y)^2} = \frac{1}{4}$$

---- (3)

Hence $JJ' = u \cdot \frac{1}{u}$

$= 1$

11) if $x+y+z=u$, $y+z=uv$ and $z=uvw$ then show that $\frac{\partial(u,v,w)}{\partial(x,y,z)} = u^2 v$

Sol:-

$x+y+z=u$

$x+uv=u$

$x=u-vu$

$\frac{\partial x}{\partial u} = 1-v$

$\frac{\partial x}{\partial v} = 0-u$

$\frac{\partial x}{\partial w} = 0$

$y+z=uv$

$y+uvw=uv$

$y=uv-uvw$

$\frac{\partial y}{\partial u} = v-vw$

$\frac{\partial y}{\partial v} = u-uw$

$\frac{\partial y}{\partial w} = -uv$

$z=uvw$

$\frac{dz}{du} = vw$

$\frac{\partial z}{\partial v} = uw$

$\frac{\partial z}{dw} = uv$

Now $\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$

$= \begin{vmatrix} 1-v & -u & 0 \\ v-vw & u-uw & -uv \\ vw & uw & uv \end{vmatrix}$

$= (1-v)[(u-uw)(uv)+(uv)(uw)] + v[(v-vw)uv + uv(vw)]$

$= (1-v)[u^2v - u^2wv + u^2vw] + u[v^2u - uv^2w + uv^2w]$

$= u^2v - u^2\cancel{wv} + u^2\cancel{v}w - u^2v^2 + u^2\cancel{v^2}w - u^2\cancel{v^2}w +$
$v^3u^2 - uv^3w + uv^3w$

$= u^2 v - \cancel{u^2} v^2 + v^{3} u^2$

$= \left[u^2 v\left((u + uv + v^2 u)\right)\right]$

$= u^2 v$

$\therefore \dfrac{\partial(x, y, z)}{\partial(u, v, w)} = u^2 v$

Hence showed

12) If $u = x^2 - y^2$, $v = 2xy$ where $x = r\cos\theta$, $y = r\sin\theta$ show that $\dfrac{\partial(u, v)}{\partial(r, \theta)} = 4r^3$

Sol:- Given

$x = r\cos\theta, \quad y = r\sin\theta$

$u = r^2\cos^2\theta - r^2\sin^2\theta$

$= r^2[\cos^2\theta - \sin^2\theta]$

$= r^2\cos 2\theta$

$v = 2(r\cos\theta)(r\sin\theta)$

$= 2r^2\sin\theta\cos\theta$

$= r^2\sin 2\theta$

$\dfrac{\partial u}{\partial r} = 2r\cos 2\theta$

$\dfrac{\partial v}{\partial r} = 2r\sin 2\theta$

$\dfrac{\partial u}{\partial \theta} = -r^2\sin 2\theta \cdot 2$

$\dfrac{\partial v}{\partial \theta} = r^2\cos 2\theta \cdot 2$

Now, $\dfrac{\partial(u, v)}{\partial(r, \theta)} = \begin{vmatrix} \dfrac{\partial u}{\partial r} & \dfrac{\partial u}{\partial \theta} \\ \dfrac{\partial v}{\partial r} & \dfrac{\partial v}{\partial \theta} \end{vmatrix}$

$= \begin{vmatrix} 2r\cos 2\theta & -r^2\sin 2\theta \cdot 2 \\ 2r\sin 2\theta & r^2\cos 2\theta \cdot 2 \end{vmatrix}$

$= 4r^3\cos^2 2\theta + 4r^3\sin^2 2\theta$

$= 4r^3\left[\cos^2 2\theta + \sin^2 2\theta\right] = 4r^3$

13). Show that the functions $u = x+y+z$, $v = x^2+y^2+z^2-2xy - 2zx - 2yz$ and $w = x^3+y^3+z^3-3xyz$ are functionally related. Find the relation between them

Sol:- Given :- $u = x+y+z$, $v = x^2+y^2+z^2-2xy-2zx-2yz$

$$w = x^3+y^3+z^3-3xyz$$

$u = x+y+z$

$\frac{\partial u}{\partial x} = 1$

$\frac{\partial u}{\partial y} = 1$

$\frac{\partial u}{\partial z} = 1$

$v = x^2+y^2+z^2-2xy-2zx-2yz$

$\frac{\partial v}{\partial x} = 2x - 2y - 2z$

$\frac{\partial v}{\partial y} = 2y - 2x - 2z$

$\frac{\partial v}{\partial z} = 2z - 2x - 2y$

$w = x^3+y^3+z^3-3xyz$

$\frac{\partial w}{\partial x} = 3x - 3yz$

$\frac{\partial w}{\partial y} = 3y - 3xz$

$\frac{\partial w}{\partial y} = 3z - 3xy$

$$\text{Now, } \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 2(x-y-z) & 2(y-x-z) & 2(z-x-y) \\ 3(x^3-yz) & 3(y^2-xz) & 3(z^2-yx) \end{vmatrix}$$

$$= 6\begin{vmatrix} 1 & 1 & 1 \\ x-y-z & y-x-z & z-x-y \\ x^3-yz & y^2-xz & z^2-yx \end{vmatrix}$$

$$C_1 \to C_1 - C_2 \; ; \quad C_2 \to C_2 - C_3$$

$$= 6\begin{vmatrix} 0 & 0 & 1 \\ 2(x-y) & 2(y-z) & z-y-x \\ (x^2-y^2)(x+y+z) & (y-z)(x+y+z) & z^2-yx \end{vmatrix}$$

$$= 6(x-y)(y-z)\begin{vmatrix} 0 & 0 & 1 \\ 2 & 2 & z-y-x \\ x+y+z & x+y+z & z^2-yx \end{vmatrix}$$

$$= 6(x-y)(y-z)\left[1(2x+2y+2z-2x-2y-2z)\right]$$

$$= 6(x-y)(y-z)(0) = 0$$

$\therefore$ $u, v, w$ are functionally related

---

4) Show that the functions $u = xy + yz + zx$ ; $v = x^2+y^2+z^2$ and $w = x+y+z$ are functionally related. Find the relation between them.

Sol:- $u = xy+yz+zx$

$$\frac{\partial u}{\partial x} = y+z \qquad \frac{\partial u}{\partial y} = x+z \qquad \frac{\partial u}{\partial z} = y+x$$

$v = x^2+y^2+z^2$

$$\frac{\partial v}{\partial x} = 2x \qquad \frac{\partial v}{\partial y} = 2y \qquad \frac{\partial v}{\partial z} = 2z$$

$w = x+y+z$

$$\frac{\partial w}{\partial x} = 1 \qquad \frac{\partial w}{\partial y} = 1 \qquad \frac{\partial w}{\partial z} = 1$$

(9)

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} y+z & x+z & y+x \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 2[(y+z)(y-z) - (x+z)(x-z) + (y+x)(x-y)]$$

$$= 2[y^2 - z^2 - x^2 + z^2 + y^2 - x^2]$$

$$= 0$$

$\therefore$ $u, v, w$ are functionally related.

$$(x+y+z)^2 = x^2+y^2+z^2+2xy+2yz+2xz$$

$$\Rightarrow w^2 = v+2u$$

15) Divide a given positive number $n$ into three parts such that their product is maximum

Sol - Let the three numbers be $x, y, z$

given, $x+y+z=n$

Product of three numbers is maximum

$$\Rightarrow f = xyz \quad --- (1)$$

$$\phi = x+y+z-n = 0 \quad --(2)$$

By Lagrange's function

$F = f + \lambda\phi$

$= x+y+z+\lambda(x+y+z-n) \quad \text{---} (3)$

Differentiating partially w.r.t $x, y, z$

$\frac{\partial F}{\partial x} = y+z-\lambda \qquad \frac{\partial F}{\partial y} = x+z-\lambda \qquad \frac{\partial F}{\partial z} = x+y-\lambda$

For maximum (or) minimum value

$\frac{\partial F}{\partial x} = 0; \qquad \frac{\partial F}{\partial y} = 0; \qquad \frac{\partial F}{\partial z} = 0$

$y+z-\lambda = 0 \qquad x+z-\lambda=0 \qquad x+y-\lambda = 0$

$\lambda = y+z \qquad \lambda = x+z \qquad \lambda = x+y$

$\text{---}(4) \qquad \text{---}(5) \qquad \text{---}(6)$

$y+z = x+z \qquad x+z = x+y$

$x = y \qquad y = z$

$\therefore x = y = z$

sub in ② eq$^n$

$\Rightarrow x+y+z = n$

$x+x+x = n$

$3x = n$

$x = n/3 \; ; \; y = n/3 \; ; \; z = y/3$

sub in ①

$\Rightarrow xyz = \left(\frac{n}{3}\right)\left(\frac{n}{3}\right)\left(\frac{n}{3}\right)$

$= \frac{n^3}{27} = \left(\frac{n}{3}\right)^3$

16. Locate the stationary points and examine their nature of the following functions: $u = x^4 + y^4 - 2x^2 + 4xy - 2y^2$ (10)

Sol- we have

$$\frac{\partial u}{\partial x} = 4x^3 - 4x + 4y \qquad \frac{\partial u}{\partial y} = 4y^3 + 4x - 4y$$

$u$ to be maximum or minimum i.e $\frac{\partial u}{\partial x} = 0$, $\frac{\partial u}{\partial y} = 0$

$$\Rightarrow \frac{\partial u}{\partial x} = 4(x^3 - x + y) = 0 \qquad \frac{\partial u}{\partial y} = 4(y^3 + x - y) = 0$$

$$\Rightarrow x^3 - x + y = 0 \qquad = y^3 + x - y = 0 \quad \text{— (2)}$$

$$\Rightarrow x^3 + y = x \quad \text{— (1)}$$

sub (1) in (2)

$$\Rightarrow y^3 + x^3 + \cancel{y} - \cancel{y} = 0$$

$$\Rightarrow x = -y \quad \text{— (4)} \qquad \text{sub (4) in (2)}$$

$$\Rightarrow y^3 - y - y = 0 \Rightarrow y^3 - 2y = 0$$

$$\Rightarrow y^3 = 2y \qquad \Rightarrow y^2 = 2$$

$$\therefore y = \pm\sqrt{2}$$

||ly $x = \pm\sqrt{2}$ [$\because$ from (4)]

$$\therefore x = \sqrt{2}, -\sqrt{2}, 0 \qquad y = -\sqrt{2}, \sqrt{2}, 0$$

$\Rightarrow$ stationary points are $(\sqrt{2}, -\sqrt{2})$ $(-\sqrt{2}, \sqrt{2})$ $(0, 0)$

Now $r = \frac{\partial^2 f}{\partial x^2} = 12x^2 - 4$

$$s = \frac{\partial^2 f}{\partial x \partial y} = 4$$

$$t = \frac{\partial^2 u}{\partial y^2} = 12y^2 - 4$$

$\Rightarrow rt-s^2$ at point $(\sqrt{2},-\sqrt{2})$

$= (12(\sqrt{2})^2-4)(12(\sqrt{2})^2-4)-16$

$= 20\times 20-16 > 0$ & $r>0$

$\therefore$ function has minimum value at $(\sqrt{2},-\sqrt{2})$

$\Rightarrow rt-s^2$ at point $(-\sqrt{2},\sqrt{2})$

$= 20\times 20 - 16 > 0$ & $r>0$

$\therefore$ function has minimum value at $(-\sqrt{2},\sqrt{2})$

$\Rightarrow rt-s^2$ at point $(0,0)$

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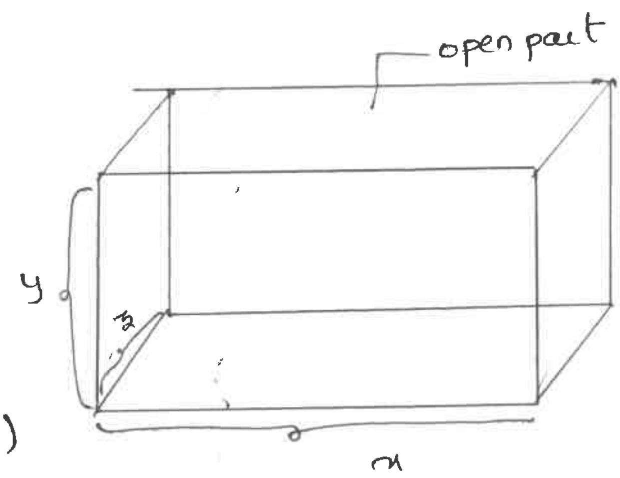
7) A rectangular box open at the Top is to have a volume of 32 cubic ft. Find the dimensions of the box, requiring least material for its constructions.

<u>Sol</u>:- Let $x, y, z$ be the dimensions of the box and $f$ be the surface area of box




$\Rightarrow f = xy + 2yz + 2zx$ ---(1)

let volume be $\phi = xyz - 32 = 0$ ---(2)

By using Lagrange's function

we know

$F = f + \lambda\phi(x,y,z)$

$$F = xy + 2yz + 2zx + \lambda(xyz - 32)$$

differentiating partially with respect to $x, y, z$

For maximum or minimum $\frac{\partial F}{\partial x} = 0$, $\frac{\partial F}{\partial y} = 0$ $\frac{\partial F}{\partial z} = 0$

$$\Rightarrow \frac{\partial F}{\partial x} = y + 2z + \lambda yz = 0 \Rightarrow \lambda = -\frac{(y+2z)}{yz} \quad \text{---} (3)$$

$$\Rightarrow \frac{\partial F}{\partial y} = x + 2z + \lambda xz = 0 \Rightarrow \lambda = -\frac{(x+2z)}{xz} \quad \text{---} (4)$$

$$\Rightarrow \frac{\partial F}{\partial z} = 2y + 2x + \lambda xy = 0 \Rightarrow \lambda = -\frac{2(y+x)}{xy} \quad \text{---} (5)$$

equating (3), (4) & (4, 5) $eq^n$s

$$\frac{y+2z}{y\cancel{z}} = \frac{x+2z}{x\cancel{z}}$$

$$\cancel{xy} + 2xz = \cancel{xy} + 2yz$$

$$x = y$$

$$\frac{x+2z}{\cancel{x}z} = \frac{2(y+x)}{\cancel{x}y}$$

$$xy + 2yz = 2yz + 2zx$$

$$y = 2z$$

$$\Rightarrow x = y = 3/2$$

substitue in (2)

$$xyz = 32$$

$$(x)(x)\left(\frac{x}{2}\right) = 32$$

$$x^3 = 64$$

$$x = 4$$

$$\Rightarrow y = 4, \quad z = 2$$

$\therefore$ dimensions for box are 4, 4, 2

P.T.O

18) Find the volume of the greatest rectangular parallellopiped that can be inscribed in the ellipsoid (12)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Sol:- Length $= 2x$

breadth $= 2y$

Volume $= 2z$

volume $v = (2x)(2y)(2z) = 8xyz$

$$\phi = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \text{--- (1)}$$

$$f = 8xyz \quad \text{--- (2)}$$

By using Lagrange's function

$$F = f + \lambda\phi$$

$$= 8xyz + \lambda\left[\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right]$$

Differentiating partially with respect to $x, y, z$

$$\frac{\partial f}{\partial x} = 8xyz + \frac{2x\lambda}{a^2} = 0 \Rightarrow \frac{yza^2}{x} = -\frac{\lambda}{4} \quad \text{-- (3)}$$

$$\frac{\partial F}{\partial x} = 8xz + \frac{2y\lambda}{b^2} = 0 \Rightarrow \frac{xzb^2}{y} = -\frac{\lambda}{4} \quad \text{--- (4)}$$

$$\frac{\partial F}{\partial z} = 8yz + \frac{2z\lambda}{c^2} = 0 \Rightarrow \frac{yxc^2}{z} = -\frac{\lambda}{4} \quad \text{--- (5)}$$

equating (3) = (4) = (5)

$$\frac{yza^2}{x} = \frac{xzb^2}{y} = \frac{yxc^2}{z} = \frac{-\lambda}{4}$$

$$\frac{yza^2}{x} = \frac{\cancel{x}zb^2}{y} \qquad \frac{\cancel{yz}b^2}{4} = \frac{yxc^2}{z}$$

$$\frac{a^2}{b^2} = \frac{x^2}{y^2} \qquad \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

$$\frac{x^2}{a^2} = \frac{y^2}{b^2} \qquad \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

$$\therefore \quad \frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

sub in (1)

$$\frac{x^2}{a^2} + \frac{x^2}{a^2} + \frac{x^2}{a^2} = 1$$

$$\frac{8x^2}{a^2} = 1 \qquad x = \pm \frac{a}{\sqrt{3}}$$

lly $y = \pm \frac{b}{\sqrt{3}}$ $\qquad z = \pm \frac{c}{\sqrt{3}}$

Taking +ve values as volume At $\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}$

$$\text{volume } V = 8xyz$$

$$= 8\left(\frac{a}{\sqrt{3}}\right)\left(\frac{b}{\sqrt{3}}\right)\left(\frac{c}{\sqrt{3}}\right)$$

$$V = \frac{8abc}{3\sqrt{3}}$$

this is the largest volume of the rectangular parallelopiped inscribed in ellipsoid

19) Given that $x+y+z=a$, find the maximum Value of $x^m y^n z^p$. (13)

**Sol:-** Let $f(x,y,z) = x^m y^n z^p$ ---- (1)

$$\phi(x,y,z) = x+y+z-a=0 \quad ---- (2)$$

Using Lagranges function

$$F = f + \lambda\phi(x,y,z) \quad ---- (3)$$

Differentiating partially w.r.t $x, y, z$

$$F = x^m y^n z^p + \lambda(x+y+z-a)$$

$$\frac{\partial F}{\partial x} = m x^{m-1} y^n z^p + \lambda$$

$$\frac{\partial F}{\partial y} = n x^m y^{n-1} z^p + \lambda$$

$$\frac{\partial F}{\partial z} = p x^m y^n z^{p-1} + \lambda$$

For maximum or minimum $\frac{\partial F}{\partial x}=0, \frac{\partial F}{\partial y}=0, \frac{\partial F}{\partial z}=0$

$$\Rightarrow m x^{m-1} y^n z^p + \lambda = 0 \Rightarrow \frac{m}{x} x^m y^n z^p + \lambda = 0$$

$$\Rightarrow \frac{mu}{x} + \lambda = 0 \Rightarrow x = -\frac{mu}{\lambda} \quad -(4)$$

$\Rightarrow$ similarly $y = -\frac{nu}{\lambda}$ ---(5), $z = -\frac{pu}{\lambda}$ ---- (6)

Substitute $x, y, z$ values in eq 2

$$\Rightarrow -\left[\frac{m}{\lambda} + \frac{n}{\lambda} + \frac{p}{\lambda}\right] u = a$$

$$\Rightarrow -\frac{1}{\lambda}(m+n+p)u = a$$

$$\lambda = -\frac{1}{a}(m+n+p)u$$

substitute these '$\lambda$' value in $u$ we get

$$x = \frac{-mu}{\lambda} = \frac{am}{m+n+p}$$

similarly $y = \frac{-nu}{\lambda} = \frac{an}{m+n+p}$ [from (5)]

$$z = \frac{-pu}{\lambda} = \frac{ap}{m+n+p} \quad \text{[from (6)]}$$

Thus the maximum value is

$$x^m y^n z^p = \left(\frac{am}{m+n+p}\right)^m \left(\frac{an}{m+n+p}\right)^n \left(\frac{ap}{m+n+p}\right)^p$$

$$= \frac{a^{m+n+p} \cdot m^n \cdot n^n \cdot p^p}{(m+n+p)^{m+n+p}}$$

---

10) Find the minimum value of $x^2+y^2+z^2$ given $x+y+z = 3a$.

Sol- $f = x^2+y^2+z^2$ --- (1)

$\phi = x+y+z-3a$ --- (2)

Lagrange's function

$$F = f + \lambda\phi$$

$$= x^2+y^2+z^2+\lambda(x+y+z-3a) \quad \text{--- (3)}$$

differentiating partially with respect to $x, y, z$

$\frac{\partial F}{\partial x} = 0$, $\frac{\partial F}{\partial y} = 0$, $\frac{\partial F}{\partial z} = 0$ for minimum value

$$\frac{\partial F}{\partial x} = 2x + \lambda = 0 \Rightarrow \lambda = -2x \quad --(4)$$

$$\frac{\partial F}{\partial y} = 2y + \lambda = 0 \Rightarrow \lambda = -2y \quad --(5)$$

$$\frac{\partial F}{\partial z} = 2z + \lambda = 0 \Rightarrow \lambda = -2z \quad --(6)$$

$$\Rightarrow \lambda = -2x = -2y = -2z$$

$$\Rightarrow x = y = z ;$$

Sub $x = y = z$ in eq$^n$ (2)

$$x + y + z = 3a \Rightarrow 3x = 3a$$

$$x = a, \quad y = a, \quad z = a$$

$\therefore$ The possible stationary point is $(a, a, a)$

$\therefore$ The minimum value at $(a, a, a)$

Min value $= a^2 + a^2 + a^2 = 3a^2$

21) Find the maximum value of $x^2y^3z^4$ subject to the condition $2x + 3y + 4z = a$.

$$f = x^2y^3z^4 \quad ----(1)$$

$$\phi(x, y, z) = 2x + 3y + 4z - a = 0 \quad ---(2)$$

Using Lagrange's function

$$F = f + \lambda\phi$$

$$F = x^2y^3z^4 + \lambda(2x + 3y + 4z - a) \quad ---(3)$$

Differentiating partially with respect to $x, y, z$

$$\frac{\partial F}{\partial x} = 2xy^3z^4 + 2\lambda \quad ---- 6.$$

$$\frac{\partial F}{\partial y} = 3x^2y^2z^4 + 3\lambda . \qquad \frac{\partial F}{\partial z} = 4x^2y^3z^3 + 4\lambda$$

For maximum or minimum

$$\frac{\partial F}{\partial x} = 0 \qquad \frac{\partial F}{\partial y} = 0 \qquad \frac{\partial F}{\partial z} = 0$$

$$\Rightarrow \frac{\partial F}{\partial x} = 2xy^3z^4 + 2\lambda = 0$$

$$\Rightarrow xy^3z^4 = -\lambda \quad \text{----} \quad (4)$$

$$\Rightarrow \frac{\partial F}{\partial y} = 3x^2y^2z^4 + 3\lambda = 0$$

$$\Rightarrow x^2y^2z^4 = -\lambda \quad \text{———} \quad (5)$$

$$\Rightarrow \frac{\partial f}{\partial z} = 4x^2y^3z^3 + 4\lambda = 0$$

$$\Rightarrow x^2y^3z^3 = -\lambda \quad \text{---} \quad (6)$$

equating

(4) = (5) $\qquad\qquad$ (5) = (6)

$xy^3z^4 = x^2y^2z^4 \qquad\qquad x^2y^2z^4 = x^2y^3z^3$

$x = y \qquad\qquad y = z$

Hence $x = y = z$ --- (7)

substitue (7) in (2)

$2x + 3x + 4x = a \qquad x = a/9$ --- (8)

sub (8) in 1

$$f = (a/9)^2 (a/9)^3 (a/9)^4$$

$$= \left(\frac{a}{9}\right)^9$$

$\therefore$ Maximum value of $f = \left(\frac{a}{9}\right)^9$