UNIT - 5

Vector Integration

Line Integrals:-

The total work done by $\bar{F}$ during displacement from A to B = $\int_A^B \bar{F}\cdot d\bar{r}$

where $\bar{F} = \nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$

and $d\bar{r} = dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k}$

$\therefore \bar{F}\cdot d\bar{r} = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz.$

Problems:-

1) evaluate $\int_C \bar{F}\cdot d\bar{r}$ where $\bar{F} = x^2\bar{i} + y^3\bar{j}$ and curve 'c' is the arc of the parabola $y = x^2$ in the xy plane from (0,0) to (1,1)

Given $\bar{F} = x^2\bar{i} + y^3\bar{j}$, $d\bar{r} = dx\,\bar{i} + dy\,\bar{j}$

$\bar{F}\cdot d\bar{r} = (x^2\bar{i} + y^3\bar{j})\cdot(dx\,\bar{i} + dy\,\bar{j})$

$= x^2dx + y^3dy$

$\int_C \bar{F}\cdot d\bar{r} = \int_C (x^2dx + y^3dy)$

Now along the curve $y = x^2$

$\Rightarrow dy = 2x\,dx$

The limits are $x = 0$ to $x = 1$

$\int_C \bar{F}\cdot d\bar{r} = \int_{x=0}^{x=1} x^2dx + (x^2)^3 2x\,dx$

$= \int_{x=0}^{1} x^2dx + 2x^7dx$

$$= \int_{x=0}^{1} (x^2 + 2x^7)\,dx$$

$$= \left[\frac{x^3}{3} + 2\frac{x^8}{8}\right]_{x=0}^{1}$$

[upper limit $(x=1)$ − lower limit $(x=0)$]

$$= \left[\frac{1}{3} + \frac{2(1)}{8}\right] - \left[\frac{0}{3} + \frac{2(0)}{8}\right]$$

$$= \frac{1}{3} + \frac{1}{4}$$

$$= \frac{7}{12}$$

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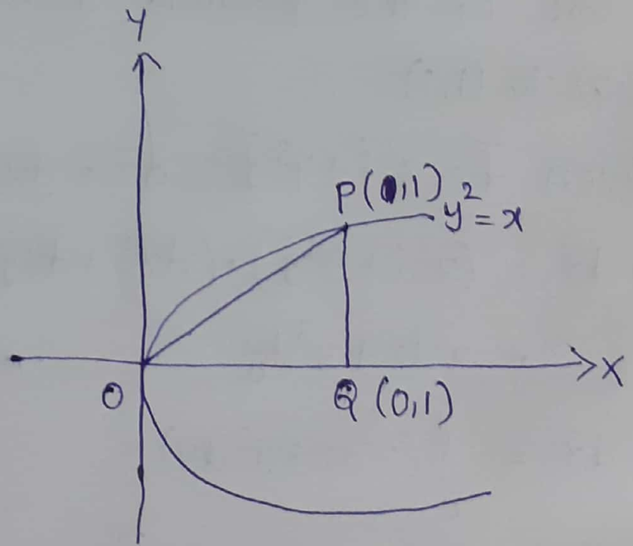
2, Integrate $\bar{f} = x^2\bar{i} + xy\bar{j}$

a) along OP from $O(0,0)$ to $P(0,1)$

b) along X-axis from $x=0$ to $x=1$

c) along the line $x=1$ from $y=0$ to $y=1$

d, along the parabola $y^2 = x$ from $(0,0)$ to $(1,1)$



Given $\bar{f} = x^2\bar{i} + xy\bar{j}$; $d\bar{r} = dx\bar{i} + dy\bar{j}$

then $\bar{f} \cdot d\bar{r} = x^2 dx + xy\,dy$

a) The equation of the line OP is $y = x$

$$\therefore dy = dx$$

$$\int_C \bar{F}\cdot d\bar{r} = \int_0^1 x^2 dx + x\cdot x\,dx$$

$$= \int_0^1 (x^2+x^2)\,dx = \int_0^1 2x^2 dx$$

$$= 2\left[\frac{x^3}{3}\right]_0^1$$

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$$= \frac{2}{3}\left[(1)^3-(0)^3\right]$$

$$= \frac{2}{3}(1) = \frac{2}{3}$$ //

b) The equation of the X-axis is $y=0$

Then $\bar{F}\cdot d\bar{r} = x^2 dx$

$$\int_C \bar{F}\cdot d\bar{r} = \int_{x=0}^{1} x^2 dx = \left[\frac{x^3}{3}\right]_0^1$$

$$= \frac{1}{3} - (0) = \frac{1}{3}$$ //

c) The line $x=1$ from $y=0$ to $y=1$

$x=1 \Rightarrow dx=0$

$\therefore \bar{F}\cdot d\bar{r} = y\,dy$

$$\int_C \bar{F}\cdot d\bar{r} = \int_{y=0}^{1} y\,dy = \left[\frac{y^2}{2}\right]_0^1 = \frac{1}{2} - 0 = \frac{1}{2}$$

d) The equation of the parabola is $y^2 = x$

$2y\,dy = dx$

and y limits are $y=0$ to $y=1$

Then $\bar{F}\cdot d\bar{r} = (y^2)^2\,2y\,dy + y^2\cdot y\,dy$

$= 2y^5 dy + y^3 dy$

$$\int_C \bar{F}\cdot d\bar{r} = \int_{y=0}^{1} 2y^5 dy + y^3 dy = \int_{y=0}^{1} (2y^5+y^3)\,dy$$

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$$= \left[\frac{\cancel{2} y^6}{\cancel{6}3} + \frac{y^4}{4}\right]_0^1 = \left(\frac{1}{3}+\frac{1}{4}\right) - (0)$$

$$= \frac{7}{12} //$$

3, If $\bar{f} = (3x^2+6y)\bar{i} - 14yz\bar{j} + 20xz^2\bar{k}$, Evaluate $\int_C \bar{f}\cdot d\bar{r}$ where c is the straight line joining (0,0,0) to (1,1,1)

The equation of the straight line joining (0,0,0) to (1,1,1)

$$\frac{x-0}{1-0} = \frac{y-0}{1-0} = \frac{z-0}{1-0} = t$$

$\therefore x=t,\ y=t,\ z=t \Rightarrow dx = dy = dz = dt$

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$$d\bar{r} = dx\bar{i} + dy\bar{j} + dz\bar{k}$$
$$= dt\bar{i} + dt\bar{j} + dt\bar{k}$$
$$= (\bar{i}+\bar{j}+\bar{k})\,dt$$

$$\bar{f} = (3t^2+6t)\bar{i} - 14t^2\bar{j} + 20t^3\bar{k}$$

At (0,0,0), t=0 and at (1,1,1), t=1

$$\int_C \bar{f}\cdot d\bar{r} = \int_{t=0}^{1} \left[3t^2+6t-14t^2+20t^3\right] dt$$

$$= \int_{t=0}^{1} (20t^3 - 11t^2 + 6t)\,dt$$

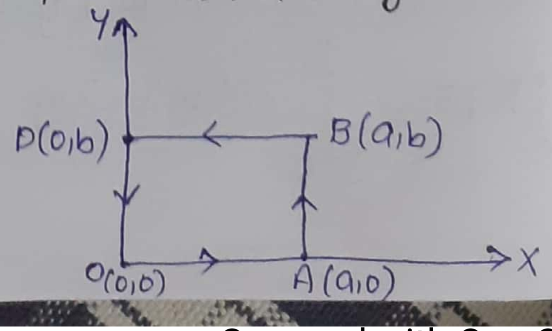
$$= \left[\overset{5}{\cancel{20}}\frac{t^4}{\cancel{4}} - 11\frac{t^3}{3} + \overset{3}{\cancel{6}}\frac{t^2}{\cancel{2}}\right]_0^1 = \left[5(1)^4 - \frac{11}{3}(1)^3 + 3(1)^2\right] - (0)$$

$$= 5 - \frac{11}{3} + 3$$

$$= 8 - \frac{11}{3} = \frac{13}{3} //$$

4, Evaluate $\int_C \bar{F}\cdot d\bar{r}$ where $\bar{F} = (x^2+y^2)\bar{i} - 2xy\bar{j}$ where c is the rectangle in the xy-plane bounded by $y=0,\ x=0,\ x=a,\ y=b$

In the xy-plane $z=0$



so $\bar{r} = x\bar{i} + y\bar{j}$ and $d\bar{r} = dx\bar{i} + dy\bar{j}$

The path of Integration 'c' consists of the lines OA, AB, BD, DO

$$\int_c \bar{F}\cdot d\bar{r} = \int_c \left[(x^2+y^2)\bar{i} - 2xy\,\bar{j}\right]\cdot\left[dx\bar{i} + dy\bar{j}\right]$$

$$= \int_c (x^2+y^2)\,dx - 2xy\,dy$$

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Along OA, $y=0$, $dy=0$ and x varies from 0 to a

Along AB, $x=a$, $dx=0$ and y varies from 0 to b

Along BD, $y=b$, $dy=0$ and x varies from a to 0

Along DO, $x=0$, $dx=0$ and y varies from b to 0

$$\therefore \int_c \bar{F}\cdot d\bar{r} = \int_{OA} \bar{F}\cdot d\bar{r} + \int_{AB} \bar{F}\cdot d\bar{r} + \int_{BD} \bar{F}\cdot d\bar{r} + \int_{DO} \bar{F}\cdot d\bar{r}$$

$$= \int_{x=0}^{a} \left[(x^2+0)dx - 2x(0)\right] + \int_{y=0}^{b} (a^2+y^2)(0) - 2ay\,dy +$$

$$\int_{x=a}^{0} \left[(x^2+b^2)\,dx - 2xb(0)\right] + \int_{y=b}^{0} (0+y^2)(0) - 2(0)y\,dy$$

$$= \int_{x=0}^{a} x^2\,dx + \int_{y=0}^{b} -2ay\,dy + \int_{x=a}^{0} (x^2+b^2)\,dx + \int_{y=b}^{0} 0\,dy$$

$$= \left[\frac{x^3}{3}\right]_0^a + \left[-\cancel{2}a\frac{y^2}{\cancel{2}}\right]_0^b + \left[\frac{x^3}{3} + b^2x\right]_a^0 + 0$$

$$= \left(\frac{a^3}{3} - 0\right) - \left(ab^2 - 0\right) + \left(0 - \left\{\frac{a^3}{3} + ab^2\right\}\right)$$

$$= \cancel{\frac{a^3}{3}} - ab^2 - \cancel{\frac{a^3}{3}} - ab^2 = \underline{-2ab^2}$$

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5, Find the work done when a force $\bar{F} = (x^2-y^2+x)\bar{i} - (2xy+y)\bar{j}$ moves a particle in the xy plane from (0,0) to (1,1) along

a) The parabola $y^2 = x$

b, The straight line $y = x$

The work done $W = \int_C \bar{F}\cdot d\bar{r}$

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$$W = \int_C (x^2-y^2+x)dx - (2xy+y)dy$$

a) The equation of the parabola $y^2 = x$

$$dx = 2y\,dy$$

y varies from 0 to 1

$\therefore$ The work done $W = \int_{y=0}^{1} [(y^2)^2 - \cancel{y^2} + \cancel{y^2}]\, 2y\,dy - (2y^2y+y)dy$

$$= \int_{y=0}^{1} (2y^5 - 2y^3 - y)\,dy$$

$$= \left[\cancel{2}\frac{y^6}{\cancel{6}\,3} - \cancel{2}\frac{y^4}{\cancel{4}\,2} - \frac{y^2}{2}\right]_0^1$$

$$= \left[\frac{1}{3} - \frac{1}{2} - \frac{1}{2}\right] - (0)$$

$$= \frac{1}{3} - 1 = \frac{-2}{3}$$

b) For the straight line $y = x$

$$dy = dx$$

x varies from $x=0$ to $x=1$

$\therefore$ The work done $W = \int_{x=0}^{1} (\cancel{x^2} - \cancel{x^2} + x)dx - (2x\cdot x + x)dx$

$$= \int_{x=0}^{1} [\cancel{x} - 2x^2 - \cancel{x}]\,dx = \int_{x=0}^{1} -2x^2\,dx$$

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$$= -2\left[\frac{x^3}{3}\right]_0^1 = -2\left[\frac{1}{3}-0\right] = \frac{-2}{3}$$

Note:- If the field is conservative the condition is $\nabla\times\bar{F}=0$

6, Show that $\bar{F}=(y^2\cos x+z^3)\bar{i}+(2y\sin x-4)\bar{j}+(3xz^2+2)\bar{k}$ is a conservative field. Find the work done in moving a particle in this field from $(0,1,-1)$ to $\left(\frac{\pi}{2},-1,2\right)$

The field is conservative if $\nabla\times\bar{F}=0$

$$\nabla\times\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2\cos x+z^3 & 2y\sin x-4 & 3xz^2+2 \end{vmatrix}$$

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$$= \bar{i}\left[\frac{\partial}{\partial y}(3xz^2+2)-\frac{\partial}{\partial z}(2y\sin x-4)\right]-\bar{j}\left[\frac{\partial}{\partial x}(3xz^2+2)-\frac{\partial}{\partial z}(y^2\cos x+z^3)\right]+\bar{k}\left[\frac{\partial}{\partial x}(2y\sin x-4)-\frac{\partial}{\partial y}(y^2\cos x+z^3)\right]$$

$$= \bar{i}[0-0]-\bar{j}[\cancel{3z^2}-\cancel{3z^2}]+\bar{k}[\cancel{2y\cos x}-\cancel{2y\cos x}]$$

$$= 0$$

$\therefore \nabla\times\bar{f}=0$

$\therefore \bar{f}$ is conservative.

$$\text{work done} = \int_C \bar{f}\cdot d\bar{r}$$

$$= \int_C (y^2\cos x+z^3)\,dx+(2y\sin x-4)\,dy+(3xz^2+2)\,dz$$

$$= \int_C (y^2\cos x+z^3)\,dx+\int_C (2y\sin x-4)\,dy+\int_C (3xz^2+2)\,dz$$

$\therefore$ The limits are $(0,1,-1)$ to $\left(\frac{\pi}{2},-1,2\right)$

$$= \int_{(0,1,-1)}^{(\frac{\pi}{2},-1,2)} (y^2\cos x + z^3)\,dx + \int_{(0,1,-1)}^{(\frac{\pi}{2},-1,2)} (2y\sin x - 4)\,dy + \int_{(0,1,-1)}^{(\frac{\pi}{2},-1,2)} (3xz^2+2)\,dz$$

$$= \left[-y^2\sin x + z^3 x\right]_{(0,1,-1)}^{(\frac{\pi}{2},-1,2)} + \left[\cancel{2}\sin x \cdot \frac{y^2}{\cancel{2}} - 4y\right]_{(0,1,-1)}^{(\frac{\pi}{2},-1,2)}$$

$$+ \left[\cancel{3}x\frac{z^3}{\cancel{3}} + 2z\right]_{(0,1,-1)}^{(\frac{\pi}{2},-1,2)}$$

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$$= \left[\left(-(-1)^2\sin\frac{\pi}{2} + 8\frac{\pi}{2}\right) - \left(-1\sin 0^\circ + (-1)^3(0)\right)\right] +$$

$$\left[\left((-1)^2\sin\frac{\pi}{2} - 4(-1)\right) - \left((1)^2\sin 0 - 4(1)\right)\right] +$$

$$\left[\left(\frac{\pi}{2}(2)^3 + 2(2)\right) - \left(0(-1)^3 + 2(-1)\right)\right]$$

$$= \left[(-1+4\pi) - (0)\right] + \left[(1+4) - (1-4)\right] + \left[(4\pi+4) - (-2)\right]$$

$$= -1 + 4\pi + 5 + 3 + 4\pi + 4 + 2$$

$$= 8\pi + 13$$

Try it:-

1) If $\vec{F} = y\hat{i} - x\hat{j}$, evaluate $\int_C \vec{F}\cdot d\vec{r}$ from $(0,0)$ to $(1,1)$ along the path c

(i) The parabola $y = x^2$

(ii) The straight line from $(0,0)$ to $(1,0)$ and then to $(1,1)$

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iii, The straight line joining (0,0) to (1,1)

Q, Calculate the work done when a force $\bar{F} = 3xy\bar{i} - y^2\bar{j}$ moves a particle in xy plane from (0,0) to (1,2) along the parabola $y = 2x^2$

3, If ~~show that~~ $\bar{F} = (axy - z^3)\bar{i} + x^2(a-2)\bar{j} + (1-a)xz^2\bar{k}$ is a conservative field the find the value of constant and also find work done in moving a particle from (1,2,−3) to (1,−4,2)

4) If $\bar{F} = (3x^2+6y)\bar{i} - 14yz\bar{j} + 20xz^2\bar{k}$, then evaluate $\int_C \bar{F}\cdot d\bar{r}$ from (0,0,0) to (1,1,1) along the curve $x = t,\ y = t^2,\ z = t^3$

Surface Integrals:-

1) If the surface S is projected on yz-plane then surface Integral $\iint_S \bar{F}\cdot\bar{n}\,ds = \iint_R \bar{F}\cdot\bar{n}\dfrac{dy\,dz}{|\bar{n}\cdot\bar{i}|}$

2) If the surface S is projected on zx-plane then Surface integral $\iint_S \bar{F}\cdot\bar{n}\,ds = \iint_R \bar{F}\cdot\bar{n}\dfrac{dz\,dx}{|\bar{n}\cdot\bar{j}|}$

3) If the surface S is projected on xy-plane then Surface Integral $\iint_S \bar{F}\cdot\bar{n}\,ds = \iint_R \bar{F}\cdot\bar{n}\dfrac{dx\,dy}{|\bar{n}\cdot\bar{k}|}$

where R is the projection of S

∴ Surface Integral is $\iint_S \bar{F}\cdot\bar{n}\,ds$

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1) Evaluate $\iint_S \bar{F}\cdot\bar{n}\,ds$ where $\bar{F} = yz\bar{i}+zx\bar{j}+xy\bar{k}$ and S is that part of the surface of the sphere $x^2+y^2+z^2=1$ which lies in the first octant

Vector normal to the surface $\phi(x,y,z) = x^2+y^2+z^2=1$ is given by $\nabla\phi = \nabla[x^2+y^2+z^2]$

$$= \bar{i}\frac{\partial}{\partial x}(x^2+y^2+z^2)+\bar{j}\frac{\partial}{\partial y}(x^2+y^2+z^2)+\bar{k}\frac{\partial}{\partial z}(x^2+y^2+z^2)$$

$$= \bar{i}(2x)+\bar{j}(2y)+\bar{k}(2z)$$

$$\nabla\phi = 2x\bar{i}+2y\bar{j}+2z\bar{k}$$

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$\bar{n}$ = a unit normal vector $= \frac{\nabla\phi}{|\nabla\phi|}$

$$= \frac{2x\bar{i}+2y\bar{j}+2z\bar{k}}{\sqrt{4x^2+4y^2+4z^2}}$$

$$= \frac{2x\bar{i}+2y\bar{j}+2z\bar{k}}{2\sqrt{x^2+y^2+z^2}} \quad [\because x^2+y^2+z^2=1]$$

$$= x\bar{i}+y\bar{j}+z\bar{k}$$

Let R be the projection of the surface S on the xy-plane The region R is bounded by x-axis, y-axis and the circle $x^2+y^2=1,\ z=0$

$$\iint_S \bar{F}\cdot\bar{n}\,ds = \iint_R \bar{F}\cdot\bar{n}\frac{dx\,dy}{|\bar{n}\cdot\bar{k}|}$$

$$\bar{F}\cdot\bar{n} = (yz\bar{i}+zx\bar{j}+xy\bar{k})\cdot(x\bar{i}+y\bar{j}+z\bar{k})$$

$$= xyz+xyz+xyz = 3xyz$$

$$\bar{n}\cdot\bar{k} = (x\bar{i}+y\bar{j}+z\bar{k})\cdot\bar{k} = z$$

$$\iint_S \bar{F}\cdot\bar{n}\,ds = \iint_R 3xy\cancel{z}\cdot\frac{dx\,dy}{\cancel{z}}$$

$= 3\iint_R xy\,dx\,dy$

By changing into polar coordinates $x = r\cos\theta$, $y = r\sin\theta$

$dx\,dy = r\,d\theta\,dr$

The limits are $\theta \to 0$ to $\frac{\pi}{2}$

$r \to 0$ to 1

$$= 3\int_0^{\pi/2}\int_0^1 (r\cos\theta)(r\sin\theta)\, r\,d\theta\,dr$$

$$= 3\int_0^{\pi/2}\int_0^1 r^3\cos\theta\sin\theta\,d\theta\,dr$$

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$$= 3\int_0^{\pi/2}\cos\theta\sin\theta\left[\int_0^1 r^3\,dr\right]d\theta = 3\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta\left[\frac{r^4}{4}\right]_0^1$$

$$= 3\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta\left(\frac{1}{4}\right)$$

$$= \frac{3}{4}\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta$$

$$= \frac{3}{4}\left(\frac{1}{2}\right) = \frac{3}{8}$$

Q, Evaluate $\iint_S \bar{F}\cdot\bar{n}\,ds$ where $\bar{F} = y\bar{i} + 2x\bar{j} - z\bar{k}$ and S is the surface of the plane $2x+y=6$ in the first octant cut off by the plane $z=4$

Vector normal to the surface $\phi = 2x+y-6$ is given by

$$\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

$$= \bar{i}\frac{\partial}{\partial x}[2x+y] + \bar{j}\frac{\partial}{\partial y}[2x+y] + \bar{k}\frac{\partial}{\partial z}[2x+y]$$

$$= 2\bar{i} + \bar{j}$$

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$\bar{n}$ = a unit normal vector $= \frac{\nabla\phi}{|\nabla\phi|} = \frac{2\bar{i}+\bar{j}}{\sqrt{4+1}} = \frac{1}{\sqrt{5}}(2\bar{i}+\bar{j})$

Let R be the projection of S on the xz-plane

The region R is bounded x-axis, z-axis and the plane $2x+y=0$, $y=0$, $z=4$

$$\iint_S \bar{F}\cdot\bar{n}\,ds = \iint_R \bar{F}\cdot\bar{n}\,\frac{dx\,dz}{|\bar{n}\cdot\bar{j}|}$$

$$\bar{F}\cdot\bar{n} = (y\bar{i}+2x\bar{j}-z\bar{k})\cdot\frac{1}{\sqrt{5}}(2\bar{i}+\bar{j})$$

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$$= \frac{1}{\sqrt{5}}[2y+2x]$$

$$\bar{n}\cdot\bar{j} = \frac{1}{\sqrt{5}}(2\bar{i}+\bar{j})\cdot\bar{j} = \frac{1}{\sqrt{5}}$$

$$\iint_S \bar{F}\cdot\bar{n}\,ds = \iint_R \frac{1}{\cancel{\sqrt{5}}}(2y+2x)\cdot\cancel{\sqrt{5}}\,dx\,dz$$

$$= 2\iint_R (y+x)\,dx\,dz$$

$$= 2\iint_R (6-2x+x)\,dx\,dz \qquad \because y = 6-2x$$

$$= 2\iint_R (6-x)\,dx\,dz$$

$$= 2\int_{z=0}^{4}\int_{x=0}^{3}(6-x)\,dx\,dz$$

$$= 2\int_{x=0}^{3}(6-x)[z]_0^4\,dx = 2\int_{x=0}^{3}(6-x)(4-0)\,dx$$

$$= 8\left[\int_{x=0}^{3}(6-x)\,dx\right]$$

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$$= 8\left[6x - \frac{x^2}{2}\right]_0^3$$

$$= 8\left[\left(18 - \frac{9}{2}\right) - 0\right]$$

$$= \cancel{8}^{4}\left(\frac{27}{\cancel{2}}\right) = \underline{108}$$

3, evaluate $\iint_S \bar{F}\cdot\bar{n}\,ds$ where $\bar{F} = z^2\bar{i} + x^2\bar{j} - y^2 z\bar{k}$ and S is the surface of the cylinder $x^2+y^2=16$ included in the first octant between $z=0$ and $z=5$

Vector normal to the surface $\phi = x^2+y^2$

$$\nabla\phi = \bar{i}\frac{\partial}{\partial x}(x^2+y^2) + \bar{j}\frac{\partial}{\partial y}(x^2+y^2) + \bar{k}\frac{\partial}{\partial z}(x^2+y^2)$$

$$= 2x\bar{i} + 2y\bar{j} + 0\bar{k} = 2x\bar{i} + 2y\bar{j}$$

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$\bar{n}$ = a unit normal vector $= \dfrac{\nabla\phi}{|\nabla\phi|}$

$$= \frac{2x\bar{i} + 2y\bar{j}}{\sqrt{4x^2+4y^2}}$$

$$= \frac{\cancel{2}[x\bar{i}+y\bar{j}]}{\cancel{2}\sqrt{x^2+y^2}} = \frac{x\bar{i}+y\bar{j}}{\sqrt{16}}$$

$[\because x^2+y^2=16]$

$$= \frac{x\bar{i}+y\bar{j}}{4}$$

Let R be the projection of S on the xz-plane
The region R is bounded x-axis, z-axis and the cylinder $x^2+y^2=16$ and $z=0$ to $z=5$

$$\iint_S \bar{F}\cdot\bar{n}\,ds = \iint_R \bar{F}\cdot\bar{n}\,\frac{dx\,dz}{|\bar{n}\cdot\bar{j}|}$$

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$$\bar{F}\cdot\bar{n} = (z^2\bar{i}+x^2\bar{j}-y^2z\bar{k})\cdot\left[\frac{x\bar{i}+y\bar{j}}{4}\right]$$

$$= \frac{z^2x+x^2y}{4}$$

and $\bar{n}\cdot\bar{j} = \left(\frac{x\bar{i}+y\bar{j}}{4}\right)\cdot\bar{j} = \frac{y}{4}$

$$\iint_S \bar{F}\cdot\bar{n}\,ds = \iint_R \frac{z^2x+x^2y}{4}\,\frac{dx\,dz}{y/4}$$

$$= \iint_R \frac{z^2x+x^2y}{\cancel{4}} \times \frac{\cancel{4}}{y}\,dx\,dz$$

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$$= \iint_R \left[\frac{z^2x+x^2y}{y}\right] dx\,dz$$

The limits are $z=0$ to $z=5$

$x=0$ to $x=4$

$$= \int_{z=0}^{z=5}\int_{x=0}^{x=4}\left[\frac{xz^2}{y}+x^2\right]dx\,dz$$

$$= \int_{z=0}^{5}\int_{x=0}^{4}\left[\frac{xz^2}{\sqrt{16-x^2}}+x^2\right]dx\,dz \qquad \left[\because x^2+y^2=16,\ y^2=16-x^2,\ y=\sqrt{16-x^2}\right.$$

$$= \int_{z=0}^{5}$$

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Volume Integrals:-

The volume integral is denoted by $\iiint_V \bar{F}\,dv$

① Evaluate $\iiint_V \nabla\cdot\bar{F}\,dv$ where $\bar{F} = (2x^2-4z)\bar{i} - 2xy\bar{j} - 8x^2\bar{k}$ and V is bounded by the planes $x=0, y=0, z=0$ and $x+y+z=1$

Given $\bar{F} = (2x^2-4z)\bar{i} - 2xy\bar{j} - 8x^2\bar{k}$

$$\nabla\cdot\bar{F} = \frac{\partial}{\partial x}(2x^2-4z) + \frac{\partial}{\partial y}[-2xy] + \frac{\partial}{\partial z}(-8x^2)$$

$$= 4x - 2x = 2x$$

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$$\iiint_V \nabla\cdot\bar{F}\,dv = \iiint_V 2x\,dx\,dy\,dz$$

$$= 2\int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{z=1-x-y} x\,dz\,dy\,dx$$

$$= 2\int_{x=0}^{1}\int_{y=0}^{1-x} x\,[z]_0^{1-x-y}\,dy\,dx$$

$$= 2\int_{x=0}^{1}\int_{y=0}^{1-x} x(1-x-y)\,dy\,dx$$

$$= 2\int_{x=0}^{1}\int_{y=0}^{1-x} [x-x^2-xy]\,dy\,dx$$

$$= 2\int_{x=0}^{1}\left[xy-x^2y-x\frac{y^2}{2}\right]_0^{1-x} dx$$

$$= 2\int_{x=0}^{} \left[x(1-x) - x^2(1-x) + \frac{x}{2}(1-x)^2\right] dx$$

$$= 2\int_{x=0}^{1} \left[x - x^2 - x^2 + x^3 - \frac{x}{2}(1+x^2-2x)\right] dx$$

$$= 2\int_{x=0}^{1} \left[x - 2x^2 + x^3 - \frac{x}{2} - \frac{x^3}{2} + x^2\right] dx$$

$$= 2\int_{x=0}^{1} \left[\frac{x^3}{2} - x^2 + \frac{x}{2}\right] dx$$

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$$= 2\left[\frac{x^4}{8} - \frac{x^3}{3} + \frac{x^2}{4}\right]_0^1$$

$$= 2\left(\frac{1}{8} - \frac{1}{3} + \frac{1}{4} - (0)\right) = 2\left[\frac{1}{24}\right] = \frac{1}{12}$$

2, Evaluate $\iiint_V \phi \, dv$ where $\phi = 45x^2y$ and V is the closed region bounded by the planes $4x+2y+z=8$ $x=0$, $y=0$, $z=0$.

Given $\phi = 45x^2y$

$$\iiint_V \phi \, dv = \int_{x=0}^{2} \int_{y=0}^{4-2x} \int_{z=0}^{8-4x-2y} 45x^2y \, dx\,dy\,dz$$

$$= \int_{x=0}^{2} \int_{y=0}^{4-2x} 45x^2y \, [z]_0^{8-4x-2y} \, dy\,dx$$

$$= \int_{x=0}^{2} \int_{y=0}^{4-2x} 45x^2y \, [8-4x-2y] \, dy\,dx$$

$$= 45 \int_{x=0}^{2} \int_{y=0}^{4-2x} \left(8x^2 y - 4x^3 y - 2x^2 y^2\right) dy\, dx$$

$$= 45 \int_{x=0}^{2} \left[\cancel{8}^{4} x^2 \frac{y^2}{\cancel{2}} - \cancel{4}^{2} x^3 \frac{y^2}{\cancel{2}} - 2x^2 \frac{y^3}{3} \right]_0^{4-2x} dx$$

$$= 45 \int_{x=0}^{2} \left[4x^2(4-2x)^2 - 2x^3(4-2x)^2 - \frac{2}{3} x^2 (4-2x)^3 \right] dx$$

$$= 45 \int_{x=0}^{2} 4x^2(16+4x^2-16x) - 2x^3(16+4x^2-16x) - \frac{2}{3} x^2 \left(64 - 8x^3 - 96x + 48x^2\right) \Big] dx$$

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$$= 45 \int_{x=0}^{2} \Big[64x^2 + 16x^4 - 64x^3 - 32x^3 - 8x^5 + 32x^4 - \frac{128}{3} x^2 + \frac{16}{3} x^5 + \frac{192}{3} x^3 - \frac{96}{3} x^4 \Big]$$

$$= 45 \int_{x=0}^{2} \left(\frac{16}{3} x^5 - 8x^5\right) + \left(48x^4 - \cancel{\frac{96}{3}}^{32} x^4\right) + \left(-96x^3 + \cancel{\frac{192}{3}}^{64} x^3\right) + \left(64x^2 - \frac{128}{3} x^2\right) dx$$

$$= 45 \int_{x=0}^{2} \left[\frac{-8}{3} x^5 + 16x^4 - 32x^3 + \frac{64}{3} x^2 \right] dx$$

$$= 45 \left[\frac{-8}{3} \frac{x^6}{6} + 16 \frac{x^5}{5} - 32 \frac{x^4}{4} + \frac{64}{3} \frac{x^3}{3} \right]_{x=0}^{2}$$

$$= 45 \left[\frac{-512}{18} + \frac{512}{5} - \frac{512}{4} + \frac{512}{9} \right]$$

$$= \underline{\underline{128}}$$

H-W

1. If $\bar{F} = (2x^2 - 3z)\bar{i} - 2xy\bar{j} - 4x\bar{k}$ then evaluate $\iiint_V \nabla \cdot \bar{F}\, dv$ where V is the closed region bounded by the planes $x=0, y=0, z=0$ and $2x+2y+z=4$

Ans: $\frac{8}{3}$

2. Evaluate $\iint_S \bar{F} \cdot \bar{n}\, ds$ where $\bar{F} = 18z\bar{i} - 12\bar{j} + 3y\bar{k}$ and S is the surface of the plane $2x+3y+6z=12$ in the first octant.

Ans: 24

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3. If $\bar{F} = 2y\bar{i} - z\bar{j} + x^2\bar{k}$ and 'S' is the surface of the parabolic cylinder $y^2 = 8x$ in the first octant bounded by the plane $y=4$ and $z=6$ and then evaluate $\iint_S \bar{F} \cdot \bar{n}\, ds$

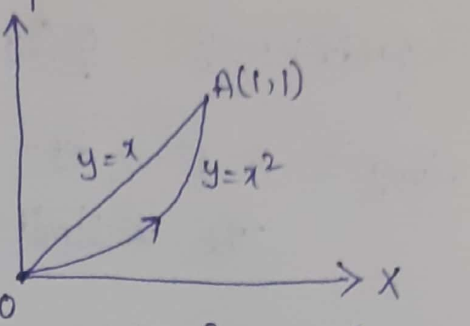
Vector Integral Theorems:-

1. Green's theorem in the plane:-

Let M and N are continuous functions of x and y having continuous real derivatives $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ in a closed region R bounded by curve C then

$$\oint_C M\,dx + N\,dy = \iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] dx\,dy$$

1) Verify Green's theorem in the plane for $\oint_C (xy+y^2)\,dx + x^2 dy$ where C is the closed curve of the region bounded by $y=x$ and $y=x^2$



By Green's theorem, we have

$$\oint_C M\,dx+N\,dy = \iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] dx\,dy \quad \text{--- (1)}$$

The two curves $y=x$ and $y=x^2$ intersecting at $O(0,0)$ and $A(1,1)$

$x^2 = x \Rightarrow x^2 - x = 0$

$x(x-1)=0$

$x=0,\ x=1$

Here $M = xy+y^2$ and $N = x^2$

$\cancel{\oint_C M\,dx + N\,dy =}$

In eq (1) $\quad R.H.S = \iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] dx\,dy$ 19

$$= \iint_R \left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(xy+y^2)\right] dx\,dy$$

$$= \iint_R [2x - x - 2y]\, dx\,dy$$

$$= \iint_R (x-2y)\, dx\,dy$$

$$= \int_{x=0}^{1} \int_{y=x^2}^{y=x} [x-2y]\, dx\,dy$$

$$= \int_{x=0}^{1} \left[xy - \cancel{2}\frac{y^2}{\cancel{2}}\right]_{y=x^2}^{y=x} dx$$

$$= \int_{x=0}^{1} [x(x)-(x)^2] - [x(x^2)-(x^2)^2]\, dx$$

$$= \int_{x=0}^{1} (\cancel{x^2}-\cancel{x^2}) - (x^3-x^4)\, dx$$

$$= \int_{x=0}^{1} (x^4 - x^3)\,dx = \left[\frac{x^5}{5} - \frac{x^4}{4}\right]_0^1 = \frac{1}{5} - \frac{1}{4} = \frac{-1}{20} \quad \text{— (2)}$$

In eq (1) ~~cond~~ consider L.H.S

$$\oint_C M\,dx + N\,dy = \oint_C (xy + y^2)\,dx + x^2\,dy$$

Along 'c' $y = x^2 \Rightarrow dy = 2x\,dx$

The x limits are $x = 0$ to $x = 1$ (by figure)

$$\oint_C (xy+y^2)\,dx + x^2\,dy = \oint_C \left[x(x^2) + (x^2)^2\right]dx + x^2(2x\,dx)$$

$$= \int_0^1 \left[x^3 + x^4 + 2x^3\right]dx$$

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$$= \int_0^1 (3x^3 + x^4)\,dx$$

$$= \left[3\frac{x^4}{4} + \frac{x^5}{5}\right]_0^1 = \frac{3}{4} + \frac{1}{5} = \frac{19}{20} \quad \text{— (3)}$$

Along 'c' $y = x \Rightarrow dy = dx$

The limits are $x = 1$ to $x = 0$ (by figure)

$$\oint_C (xy+y^2)\,dx + x^2\,dy = \int_1^0 \left[x(x) + x^2\right]dx + x^2\,dx$$

$$= \int_1^0 (2x^2 + x^2)\,dx = \int_1^0 3x^2\,dx$$

$$= \left[\cancel{3}\frac{x^3}{\cancel{3}}\right]_1^0 = (0-1) = -1 \quad \text{— (4)}$$

Add (3) & (4) $\Rightarrow$

$$\text{L.H.S} = \oint_C (xy+y^2)\,dx + x^2\,dy = \frac{19}{20} - 1 = \frac{-1}{20} \quad \text{— (5)}$$

$\therefore$ From Eq ④ & ⑤

L.H.S = R.H.S

$\therefore$ The Green's theorem is verified

2) Evaluate by Green's theorem $\oint_C e^{-x}\sin y\,dx + e^{-x}\cos y\,dy$ where c is the rectangle whose vertices are $(0,0)$, $(\pi,0)$, $(\pi,\frac{\pi}{2})$, $(0,\frac{\pi}{2})$

$M = e^{-x}\sin y$, $N = e^{-x}\cos y$

$\frac{\partial M}{\partial y} = e^{-x}\cos y$, $\frac{\partial N}{\partial x} = -e^{-x}\cos y$

and $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -e^{-x}\cos y - e^{-x}\cos y$

$= -2e^{-x}\cos y$ 21

$$\oint_C e^{-x}\sin y\,dx + e^{-x}\cos y\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy$$

$$= \iint_R \left(-2e^{-x}\cos y\right)dx\,dy$$

The Given limits are $x = 0$ to $x = \pi$

$y = 0$ to $y = \pi/2$

$$= -2\int_{x=0}^{\pi}\int_{y=0}^{\pi/2}(e^{-x}\cos y)\,dx\,dy$$

$$= -2\int_{x=0}^{\pi} e^{-x}dx \int_{y=0}^{\pi/2}\cos y\,dy$$

$$= -2\int_{x=0}^{\pi} e^{-x}dx\,(\sin y)_0^{\pi/2}$$

$$= -2\int_{x=0}^{\pi} e^{-x}dx\,[\sin\pi/2 - \sin 0]$$

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$$= -2\int_{x=0}^{\pi} e^{-x}\,dx\,[1-0]$$

$$= +2\left[\frac{e^{-x}}{-1}\right]_0^{\pi} = 2\left[e^{-\pi}-e^{0}\right]$$

$$= 2(e^{-\pi}-1)$$

3) A vector field is given by $\bar{F} = \sin y\,\bar{i} + x(1+\cos y)\bar{j}$ evaluate the line integral over the circular path $x^2+y^2=a^2,\ z=0$

Given $\bar{F} = \sin y\,\bar{i} + x(1+\cos y)\,\bar{j}$

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$$\oint_C \bar{F}\cdot d\bar{r} = \oint_C \sin y\,dx + x(1+\cos y)\,dy$$

By Green's theorem $= \iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] dx\,dy$

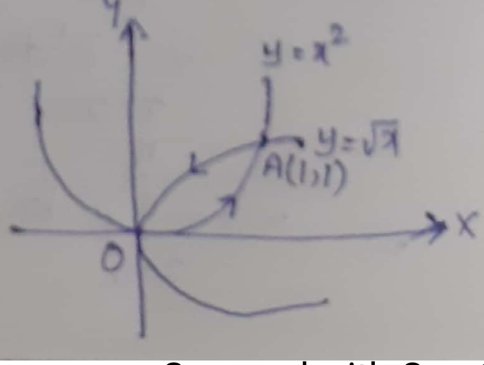
$$= \iint_R \left\{\frac{\partial}{\partial x}\left[x(1+\cos y)\right] - \frac{\partial}{\partial y}(\sin y)\right\} dx\,dy$$

$$= \iint_R \left[1+\cancel{\cos y}-\cancel{\cos y}\right] dx\,dy$$

$$= \iint_R dx\,dy = \iint_R dA = A$$

$=$ Area of the circle $x^2+y^2=a^2 = \pi a^2$

4) verify Green's theorem in the plane for $\oint_C (3x^2-8y^2)\,dx + (4y-6xy)\,dy$ where c is the region defined by $y=\sqrt{x}$, $y=x^2$



By Green's theorem, we have

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy \quad \text{—} \; ①$$

The given two curves $y = \sqrt{x}$, $y = x^2$ intersecting at $O(0,0)$ ~~to~~ & $A(1,1)$

Now consider R.H.S of eq ①

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R \left[\frac{\partial}{\partial x}\left[4y - 6xy\right] - \frac{\partial}{\partial y}\left(3x^2 - 8y^2\right)\right] dx\,dy$$

$$= \iint_R (-6y + 16y)\, dx\,dy$$

$$= \iint_R (10y)\, dx\,dy = \int_{x=0}^{1} \int_{y=x^2}^{y=\sqrt{x}} (10y)\, dx\,dy$$

$$= \int_{x=0}^{1} \overset{11}{\cancel{-22}} \left[\frac{y^2}{7}\right]_{y=x^2}^{y=\sqrt{x}}$$

$$= -11 \int_{x=0}^{1} (x - x^4)\, dx$$

$$= -11 \left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1 = -11\left(\frac{1}{2} - \frac{1}{5}\right)$$

$$= -11\left(\frac{3}{10}\right) = \frac{-33}{10}$$

eq ②

$$\therefore \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \frac{-33}{10} \quad \text{—} \; ②$$

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By Green's theorem, we have

$$\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy \quad \text{--- (1)}$$

The given two curves $y = \sqrt{x}$, $y = x^2$ intersecting at $O(0,0)$ & $A(1,1)$

Now consider L.H.S of eq (1)

$$\oint_C M dx + N dy = \oint_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$$

Along C, $y = x^2 \Rightarrow dy = 2x\,dx$

The limits are $x = 0$ to $x = 1$

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$$\oint_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$$

$$= \int_{x=0}^{1} (3x^2 - 8x^4)dx + (4x^2 - 6x^3)\,2x\,dx$$

$$= \int_{x=0}^{1} (3x^2 - 8x^4 + 8x^3 - 12x^4)\,dx$$

$$= \int_{x=0}^{1} (-20x^4 + 8x^3 + 3x^2)\,dx$$

$$= \left[-\cancel{20}^{\,4}\frac{x^5}{\cancel{5}} + \cancel{8}^{\,2}\frac{x^4}{\cancel{4}} + \cancel{3}\frac{x^3}{\cancel{3}}\right]_0^1$$

$$= (-4 + 2 + 1) = -1 \quad \text{--- (2)}$$

Along C, $y = \sqrt{x} \Rightarrow y^2 = x \Rightarrow dx = 2y\,dy$

The limits are $y = 1$ to 0

$$\oint_C (3x^2 - 8y^2)dx + (4y - 6xy)dy =$$

$$\int_{y=1}^{0} (3y^4 - 8y^2)\,2y\,dy + (4y - 6y^3)\,dy$$

$$= \int_1^0 (6y^5 - 16y^3 + 4y - 6y^3)\, dy$$

$$= \int_1^0 (6y^5 - 22y^3 + 4y)\, dy$$

$$= \left[\cancel{6} \frac{y^6}{\cancel{6}} - \overset{11}{\cancel{22}} \frac{y^4}{\cancel{4}_2} + \overset{2}{\cancel{4}} \frac{y^2}{\cancel{2}} \right]_1^0$$

$$= (0) - \left(1 - \frac{11}{2} + 2\right) = -\left(3 - \frac{11}{2}\right) = -\left(\frac{-5}{2}\right) = \frac{5}{2} \quad -(3)$$

Adding (2) & (3) we get the value of L.H.S of Eq (1)

$$\oint_C (3x^2 - 8y^2)\, dx + (4y - 6xy)\, dy = -1 + \frac{5}{2} = \frac{3}{2} \quad -(4)$$

Now consider R.H.S of Eq (1) 25

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R \frac{\partial}{\partial x}[4y - 6xy] - \frac{\partial}{\partial y}[3x^2 - 8y^2]\, dx\,dy$$

$$= \iint_R (-6y + 16y)\, dx\,dy$$

$$= \iint_R 10y\, dx\,dy = \int_{x=0}^{1} \int_{y=x^2}^{y=\sqrt{x}} 10y\, dx\,dy$$

$$= \overset{5}{\cancel{10}} \int_{x=0}^{1} \left[\frac{y^2}{\cancel{2}}\right]_{y=x^2}^{y=\sqrt{x}} dx$$

$$= 5\int_{x=0}^{1} (x - x^4)\, dx$$

$$= 5\left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1$$

$$= 5\left[\frac{1}{2} - \frac{1}{5}\right] = \cancel{5}\left(\frac{3}{\cancel{10}_2}\right) = \frac{3}{2}$$

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$\therefore \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \frac{3}{2}$ — (5)

From eq (4) & (5), we get

L.H.S = R.H.S of eq (1)

$\therefore$ The Green's theorem is verified

5, Evaluate by Green's theorem

$\oint_C (\cos x \sin y - xy)dx + \sin x \cos y\, dy$ where 'c' is the circle $x^2+y^2=1$

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By Green's theorem,

$$\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

where $M = \cos x \sin y - xy$, $N = \sin x \cos y$

$$\frac{\partial M}{\partial y} = \cos x \cos y - x \quad , \quad \frac{\partial N}{\partial x} = \cos x \cos y$$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R (\cancel{\cos x \cos y} - \cancel{\cos x \cos y} + x)\, dx\,dy$$

$$= \iint_R x\, dx\,dy$$

By changing into polar coordinates

$x = r\cos\theta$, $y = r\sin\theta$

$dx\,dy = r\,d\theta\,dr$ The limits are $\theta = 0$ to 2π

$r = 0$ to 1

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} r\cos\theta \; r\,d\theta\,dr$$

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$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} r^2 \cos\theta \, d\theta \, dr$$

$$= \int_{\theta=0}^{2\pi} \left[\frac{r^3}{3}\right]_0^1 \cos\theta \, d\theta = \int_0^{2\pi} \frac{1}{3} \cos\theta \, d\theta$$

$$= \frac{1}{3}\left[\sin\theta\right]_0^{2\pi}$$

$$= \frac{1}{3}\left[\sin 2\pi - \sin 0\right]$$

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$$= \frac{1}{3}[0-0] = \frac{1}{3}(0) = 0$$

2) The Divergence theorem of Gauss:

[Relation b/w surface and volume Integrals]

Let $\bar{F}$ be a vector point function having continuous partial derivatives in the region V bounded by a closed surface S, then

$$\iint_S \bar{F} \cdot \bar{n} \, ds = \iiint_V \nabla \cdot \bar{F} \, dv$$

Problems:-

1) Evaluate $\iint_S \left[(x+z)dydz + (y+z)dzdx + (x+y)dxdy\right]$ where S is the surface of the sphere $x^2+y^2+z^2=4$

By Divergence theorem, the given surface integral is equal to volume integral.

$$\iiint_V \left[\frac{\partial}{\partial x}(x+z) + \frac{\partial}{\partial y}(y+z) + \frac{\partial}{\partial z}(x+y)\right] dv$$

$$= \iiint_V (1+1+0) \, dv = 2V$$

where V is the volume of the sphere $x^2+y^2+z^2=4$

$$= 2\left[\frac{4}{3}\pi r^3\right] = 2\left[\frac{4}{3}\pi(2)^3\right] \quad (\because r=2)$$

$$= 2\left(\frac{4}{3}\pi(8)\right)$$

$$= \frac{64}{3}\pi$$

2) If $\bar{F} = ax\bar{i} + by\bar{j} + cz\bar{k}$, a, b, c are constants show that $\iint_S \bar{F}\cdot\bar{n}\,ds = \frac{4}{3}\pi(a+b+c)$ where S is the surface of a unit sphere

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By Divergence theorem, we have

$$\iint_S \bar{F}\cdot\bar{n}\,ds = \iiint_V (\nabla\cdot\bar{F})\,dv$$

where V is the volume enclosed by S

$$= \iiint_V \nabla\cdot\left[ax\bar{i}+by\bar{j}+cz\bar{k}\right] dv$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(ax) + \frac{\partial}{\partial y}(by) + \frac{\partial}{\partial z}(cz)\right] dv$$

$$= \iiint_V (a+b+c)\,dv = (a+b+c)V$$

$$= (a+b+c)\frac{4\pi}{3}$$

$\because$ [Volume of sphere $\frac{4}{3}\pi a^3$ and $a=1$]

3) Verify Divergence theorem for $\bar{F} = (x^2-yz)\bar{i} + (y^2-zx)\bar{j} + (z^2-xy)\bar{k}$ taken over a rectangular parallelopiped $0\le x\le a,\ 0\le y\le b,\ 0\le z\le c$

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Given $\bar{F} = (x^2 - yz)\bar{i} + (y^2 - zx)\bar{j} + (z^2 - xy)\bar{k}$

Now $\nabla \cdot \bar{F} = \frac{\partial}{\partial x}(x^2 - yz) + \frac{\partial}{\partial y}(y^2 - zx) + \frac{\partial}{\partial z}(z^2 - xy)$

$= 2x + 2y + 2z$

$$\iiint_V \nabla \cdot \bar{F}\, dv = \iiint_V 2(x+y+z)\, dv$$

$$= 2\int_{z=0}^{c} \int_{y=0}^{b} \int_{x=0}^{a} (x+y+z)\, dx\, dy\, dz$$

$$= 2\int_{z=0}^{c} \int_{y=0}^{b} \left(\frac{x^2}{2} + yx + zx\right)_0^a dy\, dz$$

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$$= 2\int_{z=0}^{c} \int_{y=0}^{b} \left(\frac{a^2}{2} + ay + az\right) dy\, dz$$

$$= 2\int_{z=0}^{c} \left(\frac{a^2}{2} y + a\frac{y^2}{2} + azy\right)_0^b dz$$

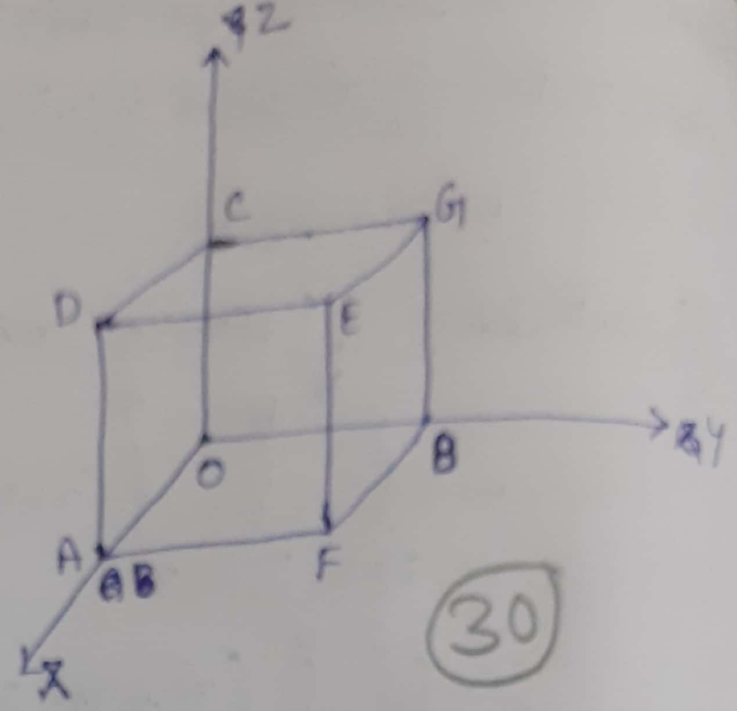
$$= 2\int_{z=0}^{c} \left[\frac{a^2 b}{2} + \frac{ab^2}{2} + abz\right] dz$$

$$= 2\left[\frac{a^2 b}{2} z + \frac{ab^2}{2} z + ab\frac{z^2}{2}\right]_0^c$$

$$= 2\left[\frac{a^2 bc}{2} + \frac{ab^2 c}{2} + \frac{abc^2}{2}\right]$$

$$= \frac{\cancel{2}\, abc}{\cancel{2}}[a+b+c]$$

$$\iiint_V \nabla \cdot \bar{F}\, dv = abc(a+b+c) \quad \text{—— (1)}$$



(30)

To calculate $\iint_S \bar{F}\cdot\bar{n}\, ds$ over the six faces of the rectangular parallelopiped

The faces are DEFA, AGCO, BGEF, OADC, GCDE, AFBO

① over the face DEFA ; $\bar{n} = \bar{i}$, $x = a$

$$\iint_{DEFA} \bar{F}\cdot\bar{n}\, ds = \int_{z=0}^{c}\int_{y=0}^{b} \left[(a^2-yz)\bar{i} + (y^2-za)\bar{j} + (z^2-ay)\bar{k}\right]\cdot\bar{i}\; dy\,dz$$

$$= \int_{z=0}^{c}\int_{y=0}^{b} (a^2-yz)\, dy\, dz$$

$$= \int_{z=0}^{c} \left[a^2y - z\frac{y^2}{2}\right]_0^b dz$$

$$= \int_{z=0}^{c} \left(a^2b - z\frac{b^2}{2}\right) dz = \left[a^2bz - \frac{b^2}{2}\frac{z^2}{2}\right]_0^c$$

$$= a^2bc - \frac{b^2c^2}{4} \quad \text{—— ②}$$

(2) over the face AGCO, $\bar{n} = -\bar{i}$, $x = 0$

$$\iint_{AGCO} F\cdot\bar{n}\,ds = \int_{z=0}^{c}\int_{y=0}^{b}\left[(-yz)\bar{i} + y^2\bar{j} + z^2\bar{k}\right]\cdot(-\bar{i})\,dy\,dz$$

$$= \int_{z=0}^{c}\int_{y=0}^{b} yz\,dy\,dz$$

$$= \int_{z=0}^{c} z\left[\frac{y^2}{2}\right]_0^b dz$$

$$= \int_{z=0}^{c} \frac{z}{2}(b^2)\,dz = \frac{b^2}{2}\left[\frac{z^2}{2}\right]_0^c$$

$$= \frac{b^2c^2}{4} \quad \text{— (3)}$$

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3) over the face BGEF, $\bar{n} = \bar{j}$, $y = b$

$$\iint_{BGEF} F\cdot\bar{n}\,dS = \int_{x=0}^{a}\int_{z=0}^{c}\left[(x^2-bz)\bar{i} + (b^2-zx)\bar{j} + (z^2-bx)\bar{k}\right]\cdot\bar{j}\;dx\,dz$$

$$= \int_{x=0}^{a}\int_{z=0}^{c}(b^2-zx)\,dx\,dz$$

$$= \int_{z=0}^{c}\left[b^2x - z\frac{x^2}{2}\right]_0^a dz$$

$$= \int_{z=0}^{c}\left(b^2a - z\frac{a^2}{2}\right)dz = \left[ab^2z - \frac{a^2}{2}\frac{z^2}{2}\right]_0^c$$

$$= ab^2c - \frac{a^2c^2}{4} \quad \text{— (4)}$$

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4) Over the face OADC, $\bar{n} = -\bar{j}$, $y = 0$

$$\iint_{OADC} \bar{F}\cdot\bar{n}\, ds = \int_{z=0}^{c}\int_{x=0}^{a} (-zx)\bar{j}\cdot(-\bar{j})\, dx\, dz$$

$$= \int_{z=0}^{c}\int_{x=0}^{a} zx\, dx\, dz$$

$$= \int_{z=0}^{c} z\left[\frac{x^2}{2}\right]_0^a dz$$

$$= \frac{1}{2}\int_{z=0}^{c} za^2\, dz = \frac{a^2}{2}\left[\frac{z^2}{2}\right]_0^c$$

$$= \frac{a^2c^2}{4} \quad \text{--- (5)}$$

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5) over the face GCDE, $\bar{n} = \bar{k}$, $z = c$

$$\iint_{GCDE} \bar{F}\cdot\bar{n}\, ds = \int_{x=0}^{a}\int_{y=0}^{b} (c^2 - xy)\bar{k}\cdot\bar{k}\, dx\, dy$$

$$= \int_{x=0}^{a}\int_{y=0}^{b} \left[c^2 - xy\right] dx\, dy$$

$$= \int_{x=0}^{a} \left[c^2y - x\frac{y^2}{2}\right]_0^b dx$$

$$= \int_{x=0}^{a} \left(c^2b - x\frac{b^2}{2}\right) dx$$

$$= \left[c^2bx - \frac{b^2}{2}\frac{x^2}{2}\right]_0^a$$

$$= abc^2 - \frac{a^2b^2}{4} \quad \text{--- (6)}$$

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6, over the face AFBO, $\bar{n} = -\bar{K}$, $z=0$

$$\iint_{AFBO} F\cdot\bar{n}\,ds = \int_{x=0}^{a}\int_{y=0}^{b} (-xy)\bar{K}\cdot(-\bar{K})\,dx\,dy$$

$$= \int_{x=0}^{a}\int_{y=0}^{b} xy\,dx\,dy$$

$$= \int_{x=0}^{a} x\left[\frac{y^2}{2}\right]_0^b dx = \frac{1}{2}\int_{x=0}^{a} xb^2\,dx$$

$$= \frac{b^2}{2}\left[\frac{x^2}{2}\right]_0^a$$

$$= \frac{a^2b^2}{4} \quad — (7)$$

(33)

Adding the six faces of integrals

i.e, (2) + (3) + (4) + (5) + (6) + (7) $\Rightarrow$

$$\iint_S F\cdot\bar{n}\,ds = a^2bc - \cancel{\frac{b^2c^2}{4}} + \cancel{\frac{b^2c^2}{4}} + ab^2c - \cancel{\frac{a^2c^2}{4}} + \cancel{\frac{a^2c^2}{4}}$$

$$+ abc^2 - \cancel{\frac{a^2b^2}{4}} + \cancel{\frac{a^2b^2}{4}}$$

$$= abc[a+b+c] \quad — (8)$$

From (1) & (8), eq^{n}s, we get

$$\iiint_V (\nabla\cdot F)\,dv = \iint_S F\cdot\bar{n}\,ds$$

$\therefore$ The Divergence theorem is verified//

4) If S is any closed surface enclosing a volume V and $\bar{F} = x\bar{i} + 2y\bar{j} + 3z\bar{K}$ P.T $\iint_S F\cdot\bar{n}\,ds = 6V$

By Divergence theorem

$$\iint_S F\cdot\bar{n}\,ds = \iiint_V \nabla\cdot F\,dv$$

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$$= \iiint_V \nabla \cdot [x\bar{i} + 2y\bar{j} + 3z\bar{k}]\, dv$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(2y) + \frac{\partial}{\partial z}(3z)\right] dv$$

$$= \iiint_V (1+2+3)\, dv$$

$$= 6\iiint_V dv = \underline{\underline{6V}}$$

5, If $\bar{F} = x\bar{i} - y\bar{j} + (z^2-1)\bar{k}$, find the value of $\iint_S \bar{F}\cdot\bar{n}\, ds$ where S is the closed surface bounded by the plane $z=0$, $z=1$ and the cylinder $x^2+y^2=4$

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By Divergence theorem,

$$\iint_S \bar{F}\cdot\bar{n}\, ds = \iiint_V \nabla\cdot\bar{F}\, dv$$

Now $\nabla\cdot\bar{F} = \nabla\cdot[x\bar{i} - y\bar{j} + (z^2-1)\bar{k}]$

$$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(-y) + \frac{\partial}{\partial z}(z^2-1)$$

$$= 1 - 1 + 2z = 2z$$

$$\iiint_V (\nabla\cdot\bar{F})\, dv = \int_{z=0}^{1}\int_{y=-2}^{2}\int_{x=-\sqrt{4-y^2}}^{\sqrt{4-y^2}} 2z\, dx\, dy\, dz$$

$$= \int_{z=0}^{1}\int_{y=-2}^{2} 2z\,[x]_{x=-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\, dy\, dz$$

$$= \int_{z=0}^{1}\int_{y=-2}^{2} 4z\,[x]_{0}^{\sqrt{4-y^2}}\, dy\, dz$$

$$= \int_{z=0}^{1} \int_{y=-2}^{2} 4z\left(\sqrt{4-y^2}\right) dy\, dz$$

$$= \int_{y=-2}^{2} \cancel{4}^{2}\sqrt{4-y^2}\left[\frac{z^2}{\cancel{2}}\right]_0^1 dy$$

$$= \int_{y=-2}^{2} 2\sqrt{4-y^2}\, dy = 2\int_{y=0}^{2} 2\sqrt{4-y^2}\, dy$$

$$= 4\int_{y=0}^{2} \sqrt{4-y^2}\, dy$$

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$$= 4\left[\frac{y}{2}\sqrt{4-y^2} + 2\sin^{-1}\frac{y}{2}\right]_0^2$$

$$= 4\left[\frac{2}{2}\sqrt{4-4} + 2\sin^{-1}(1)\right]$$

$$= 4(2\sin^{-1}1) = 4(\cancel{2})\frac{\pi}{\cancel{2}} = 4\pi$$

6, Evaluate $\iint_S \bar{F}\cdot\bar{n}\, ds$ where $\bar{F} = 2x^2y\,\bar{i} - y^2\bar{j} + 4xz^2\bar{k}$ where S is the surface $y^2+z^2=9$, $x=2$ in the first octant.

By Gauss Divergence theorem,

$$\iint_S \bar{F}\cdot\bar{n}\, ds = \iiint_V (\nabla\cdot\bar{F})\, dv$$

$$= \iiint_V \nabla\cdot\left[2x^2y\,\bar{i} - y^2\bar{j} + 4xz^2\bar{k}\right] dv$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(2x^2y) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(4xz^2)\right] dv$$

$$= \iiint_V \left[4xy - 2y + 8xz\right] dx\, dy\, dz$$

$$= \int_{y=0}^{3} \int_{z=0}^{\sqrt{9-y^2}} \int_{x=0}^{2} (4xy - 2y + 8xz)\, dx\, dy\, dz$$

$$= \int_{y=0}^{3}\int_{z=0}^{\sqrt{9-y^2}} \left[\cancel{4}^{2}y\frac{x^2}{\cancel{2}} - 2yx + \cancel{8}^{4}z\frac{x^2}{\cancel{2}}\right]_0^2 dy\,dz$$

$$= \int_{y=0}^{3}\int_{z=0}^{\sqrt{9-y^2}} (8y - 4y + 16z)\,dy\,dz$$

$$= \int_{y=0}^{3}\int_{z=0}^{\sqrt{9-y^2}} (4y + 16z)\,dy\,dz$$

$$= \int_{y=0}^{3} \left[4yz + \cancel{16}^{8}\frac{z^2}{\cancel{2}}\right]_0^{\sqrt{9-y^2}} dy$$

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$$= \int_{y=0}^{3} \left[4y\sqrt{9-y^2} + 8(9-y^2)\right] dy$$

$$= 4\left[\frac{-1}{2}(9-y^2)^{3/2}\cdot\frac{2}{3} + 18y - \frac{2}{3}y^3\right]_{y=0}^{3}$$

$$\cancel{= 4\left\{\frac{-1}{2}(9-4)^{3/2}\cdot\frac{2}{3} + 36 - \frac{2}{3}(8)\right\} - \left[4\left\{\frac{-1}{2}(9-0)^{3/2}\frac{2}{3}\right\}\right]}$$

$$\cancel{= 180}$$

$$= 4\left[\frac{-1}{2}(0)\cdot\frac{2}{3} + 54 - \frac{2}{\cancel{3}}(\cancel{27}^{9})\right] - 4\left[\frac{-1}{\cancel{2}}(9)^{3/2}\cdot\frac{\cancel{2}}{3}\right]$$

$$= 4(54-18) - 4\left(-\frac{1}{\cancel{3}}(\cancel{27}^{9})\right)$$

$$= 4(36) - 4(-9)$$

$$= 144 + 36 = \underline{180}$$

7, Apply Gauss Divergence theorem, Evaluate $I = \iint_S x^3 dydz + x^2y\,dzdx + x^2z\,dxdy$ where S is the closed surface bounded by the planes $z=0$, $z=b$ and the cylinder $x^2+y^2=a^2$

By Divergence theorem,

Given Integral $I = \iiint_V \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(x^2y) + \frac{\partial}{\partial z}(x^2z)\,dv$

$$= \iiint_V (3x^2 + x^2 + x^2)\,dxdydz$$

$$= \iiint_V 5x^2\,dxdydz$$

$$= \int_{z=0}^{b} \int_{y=-a}^{a} \int_{x=-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} 5x^2\,dxdydz$$

$$= 20\int_{z=0}^{b} \int_{y=0}^{a} \int_{x=0}^{\sqrt{a^2-y^2}} x^2\,dxdydz$$

$$= 20\int_{z=0}^{b} \int_{y=0}^{a} \left[\frac{x^3}{3}\right]_0^{\sqrt{a^2-y^2}} dydz$$

$$= \frac{20}{3}\int_{z=0}^{b} \int_{y=0}^{a} (a^2-y^2)^{3/2}\,dydz$$

$$= \frac{20}{3}\int_{y=0}^{a} (a^2-y^2)^{3/2}\,[z]_0^b\,dy$$

$$= \frac{20}{3}\int_{y=0}^{a} b(a^2-y^2)^{3/2}\,dy$$

put $y = a\sin t$

$dy = a\cos t\,dt$

Limits $y=0 \Rightarrow t=0$

$y=a \Rightarrow a = a\sin t \Rightarrow \sin t = 1$

$t = \pi/2$

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$$= \frac{20}{3} b \int_{t=0}^{\pi/2} (a^2 - a^2 \sin^2 t)^{3/2} a \cos t \, dt$$

$$= \frac{20}{3} b a \int_{t=0}^{\pi/2} (a^2)^{3/2} [\cos^2 t]^{3/2} \cos t \, dt$$

$$= \frac{20}{3} a^4 b \int_{t=0}^{\pi/2} \cos^4 t \, dt$$

$$= \frac{\overset{5}{\cancel{20}}}{\cancel{3}} a^4 b \, \frac{\cancel{3}}{\cancel{4}} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \underline{\underline{\frac{5}{4} \pi a^4 b}}$$

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