UNIT-2

Linear Differential Equations of Higher order.

A linear Diff eqⁿ of order 'n' is given by

$$\frac{d^n y}{dx^n} + P_1(x)\cdot\frac{d^{n-1} y}{dx^{n-1}} + P_2(x)\cdot\frac{d^{n-2} y}{dx^n} + \cdots + P_n(x)y = Q(x) \quad \text{--- (1)}$$

where $P(x)$ & $Q(x)$ are real values continuous functions

operator:

'D' is called operator and it is defined by $D = \frac{d}{dx}$.

$$D^2 = \frac{d^2}{dx^2}$$

$$D^3 = \frac{d^3}{dx^3} \quad \cdots\cdots \quad D^n = \frac{d^n}{dx^n}.$$

Solving higher order Diff eqⁿs:

Case i): $f(D)\cdot y = 0$

$$\boxed{G.S = C.F + 0}$$

where C.F = Complementary function.

Case(iii) :- $f(D) \cdot y = \phi(x)$

$$\boxed{G \cdot S = C \cdot F + P \cdot I}$$

where C·F = complementary function.

P·I = particular Integral

To Find Complementary function (C·F):-

→ write the operator form of the given diff eqn by putting $\frac{d}{dx} = D$

i.e $\boxed{f(D) \cdot y = 0}$

→ Now find auxilary eqn (A·E) by putting $\boxed{D = m}$ in operator & Equate to zero

i.e $\boxed{f(m) = 0}$

→ Find the roots of Auxilary eqn using basic method and write the complementary function Depending on types of roots

Types of Roots & C.F :-

Case(i) :- The roots are Real & Distinct :-

$m = m_1, m_2, m_3$

$$\boxed{C.F = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x}}$$

Eg :- ① $m = 2, 3$ & -4

$$C.F = c_1 e^{2x} + c_2 e^{3x} + c_3 e^{-4x}$$

$D^2 = 4$

$D \; 2^2$

② $m = -2, 2$ & 3 $\quad D = \pm 2$

$$C.F = c_1 e^{-2x} + c_2 e^{2x} + c_3 e^{3x}$$

Case(ii) :- The roots are Real and Repeated

→ $m = m_1$ & m_1

$$C.F = (c_1 + c_2 x) e^{m_1 x}$$

eg :- $m = 2, 2$

$$C.F = (c_1 + c_2 x) e^{2x}$$

→ $m = m_1, m_1$ & m_1

$$C.F = (c_1 + c_2 x + c_3 x^2) e^{m_1 x}$$

Eg :- $m = 3, 3$ & 3

$$C.F = (c_1 + c_2 x + c_3 x^2) e^{3x}$$

** $m = m_1, m_2, m_3, m_3$

$$C.F = c_1 e^{m_1 x} + c_2 e^{m_2 x} + (c_1 + c_2 x) e^{m_3 x}.$$

Case (iii): The roots are complex

$m = \alpha \pm i\beta$

$\alpha, \beta \rightarrow$ Real roots

$i\beta \rightarrow$ Purely imag

$$\boxed{C.F = e^{\alpha x}[c_1 \cos\beta x + c_2 \sin\beta x]}$$

1. Solve $[D^2 - 5D + 6]y = 0$

Sol: The given diff eq^n is of the form

$f(D) \cdot y = 0$

$f(D) \cdot y = 0$

$$\boxed{G.S = C.F}$$

i) find C.F

$(D^2 - 5D + 6)y = 0$

$f(D) \cdot y = 0$

A. Eq^n: put $[D = m]$ in operator of equate to zero

$m^2 - 5m + 6 = 0$

$m^2 - 2m - 3m + 6 = 0$

$m(m-2) - 3(m-2) = 0$

$$\boxed{m = 2, 3}$$

The roots are real and Distinct

$$\boxed{C.F = C_1 e^{2x} + C_2 e^{3x}}$$

$G.S = C.F$

$$\boxed{G.S = C_1 e^{2x} + C_2 e^{3x}}$$

(2) solve $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$

Sol:- $D = \frac{d}{dx}$

$D^2y + Dy + y = 0$

$[D^2 + D + 1]\, y = 0$

$f(D) \cdot y = 0$

$$\boxed{G.S = C.F}$$

To find C.F:- $(D^2 + D + 1)\, y = 0$

$m^2 + m + 1 = 0$

$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{-1 \pm \sqrt{3i^2}}{2}$

$= \frac{-1 \pm \sqrt{3}\,i}{2}$

$$\boxed{m = \frac{-1}{2} \pm \sqrt{\frac{3}{2}}\, i}$$

$\alpha \pm i\beta$

$G.S = e^{(-1/2)x} \left[C_1 \cos \frac{\sqrt{3}}{2}x + C_2 \sin \frac{\sqrt{3}}{2}x \right]$

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Method ①:- To find G.S of $f(D)\cdot y = Q(x)$

where $Q(x) = e^{ax}$

$$\{ f(x)\cdot y = e^{ax} \}$$

procedure:-

1) To Find C.F

2) To find $P.I = \frac{1}{f(D)} \cdot Q(x)$

$$= \frac{1}{f(D)} e^{ax}$$ put $D = a$.

$$\boxed{P.I = \frac{1}{f(a)} e^{ax}}$$

3. To find $\boxed{G.S = C.F + P.I}$.

Note:-

If put $D = a$ we get failure case then multiply x to the numerator & derivative of the denominator w.r.to D w.r.to Continue the same process until we get the particular Integral (P.I)

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1. Solve $(D^2-4D+13)y = e^{2x}$.

Sol:- The given D.eqⁿ is of the form

$$f(D)\cdot y = Q(x)$$

where $Q(x) = e^{2x}$.

$$\boxed{G.S = C.F + P.I}$$

Case(i):- To find C.F

$$(D^2-4D+13)y = 0$$

$$f(D)\cdot y = 0$$

A.Eqⁿ is:- $m^2-4m+13=0$

$$m = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{4 \pm \sqrt{16-52}}{2a}$$

$$= \frac{4 \pm \sqrt{-36}}{2} \Rightarrow \frac{4 \pm 6i}{2}$$

$$2 \pm 3i \Rightarrow \alpha \pm i\beta$$

$$\boxed{C.F = e^{2x}\left[C_1 \cos 3x + C_2 \sin 3x\right]}$$

Case(ii):- To find P.I:-

$$P.I = \frac{1}{f(D)}\cdot Q(x)$$

$$= \frac{1}{D^2-4D+13} e^{2x}$$

put $D=2$

$$= \frac{1}{(2)^2-4(2)+13} e^{2x}$$

$$= \frac{1}{9} e^{2x}$$

$$\boxed{P.I = \frac{e^{2x}}{9}}$$

$$\boxed{G.S = e^{2x}[c_1 \cos 3x + c_2 \sin 3x] + \frac{e^{2x}}{9}}$$

(9) Solve $(D^2+2D+1)y = e^{-x}$.

Sol:- The given D. Eqn is of the form

$$f(D) \cdot y = Q(x) \qquad Q(x) = e^{-x}$$

$$\boxed{G.s = C.F + P.I} \qquad \& \ a = -1$$

Case i) To find C.F:-

$$(D^2+2D+1)y = 0$$

A. Eqn: $m^2+2m+1=0$

$$m(m+1)+1(m+1)=0$$

$$(m+1)^2=0$$

$$m = -1, -1$$

The roots are real & repeated

$$C.F = (c_1+c_2x)e^{-x}$$

Case(ii) To find P.I

$$P.I = \frac{1}{f(D)} Q(x)$$

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$$P.I = \frac{1}{D^2+2D+1} e^{-x}$$

Put $D = -1$ we are getting failure case $\{(-1)^2+2(-1)+1\}$ $-2+3=0$

$$= \frac{x}{\frac{d}{dD}(D^2+2D+1)} e^{-x}.$$

$$= \frac{x}{2D+2}.$$

Again put $D=-1$ we get failure

Case.

$$= \frac{x \cdot x}{\frac{d}{dD}(2D+2)} e^{-x} = \frac{x^2}{2} e^{-x}$$

$$\boxed{P.I = \frac{x^2}{2} e^{-x}.}$$

$$\boxed{G.S = (C_1 + C_2 x) e^{-x} + \frac{x^2}{2} \cdot e^{-x}}$$

Method - 3 :-

To find G.S of $f(D)\cdot y = Q(x)$

where $Q(x) = x^K$.

'K' is any non-negative

i.e $\boxed{f(D)\cdot y = x}$

procedure :-

1) To find C.F

2) To find P.I $= \frac{1}{f(D)} \cdot Q(x)$

use formula, eliminate $D \cdot D^2$ - - .

3.) To find $\boxed{G.S = C.F + P.I}$

formula :-

1. $(1+D)^{-1} = 1 - D + D^2 - D^3 + D^4 - D^5 + \cdots$

2. $(1-D)^{-1} = 1 + D + D^2 + D^3 + \cdots$

3. $(1-D)^{-2} = 1 + 2D + 3D^2 + 4D^3 + \cdots$

4. $(1+D)^{-2} = 1 - 2D + 3D^2 - 4D^3 + \cdots$

5. $(1-D)^{-3} = 1 + 3D + 6D^2 + 10D^3 + \cdots$

6. $(1+D)^{-3} = 1 - 3D + 6D^2 - 10D^3 + \cdots$

① Solve $(D^2-4)y = x^3+2$

Sol: The given Diff eq^n is of the form

$$f(D)\cdot y = Q(x)$$

where $Q(x) = x^3+2$

i.e $\boxed{f(D)\cdot y = x^K}$

$$\boxed{G.S = C.F + P.I}$$

Case(i) To find C.F :-

$$(D^2-4)y=0$$

A. Eq^n : $m^2-4=0$

$m^2=4 \Rightarrow m = \pm 2$

$m = 2, -2$

The roots are real and distinct

$$\boxed{C.F = c_1 e^{2x} + c_2 e^{-2x}}$$

Case(ii) To find P.I

$$P.I = \frac{1}{f(D)} \cdot Q(x)$$

$$= \frac{1}{D^2-4} x^3+2.$$

$$= \frac{1}{-4\left[1-\frac{D^2}{4}\right]} (x^3+2)$$

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$$= \frac{-1}{4}\left[1-\frac{D^2}{4}\right]^{-1}(x^3+2)$$

$$\boxed{(1-D)^{-1} = 1+D+D^2+D^3+\cdots}$$

$$= \frac{-1}{4}\left[1+\frac{D^2}{4}+\left(\frac{D^2}{4}\right)^2+\left(\frac{D^2}{4}\right)^3+\cdots\right]x^3+2$$

Neglecting $\left(\frac{D^2}{4}\right)^2$ - - - on Both sides

$$P.I = \frac{-1}{4}\left[1+\frac{D^2}{4}\right](x^3+2)$$

$$P.I = \frac{-1}{4}\left[(x^3+2)+\frac{1}{4}D^2(x^3+2)\right]$$

$$\boxed{P.I = \frac{-1}{4}\left[(x^3+2)+\frac{1}{4}(6x)\right]}$$

$G.S = C.F + P.I.$

$$\boxed{G.S = C_1e^{-2x}+C_2e^{2x}+\left(\frac{-1}{4}\right)\left[x^3+2+\frac{6x}{4}\right]}$$

(2) $(D^2-9)y = (4x^2+2x)$

Sol: The given D.Eqn is of the form

$$f(D)\cdot y = Q(x)$$

where $Q(x) = 4x^2+2x$

$$\boxed{f(D)\cdot y = x^k}$$

$$\boxed{G \cdot S = C \cdot F + P \cdot I}$$

Case(i) To find C·F

$(D^2-9)y=0$.

A·Eqⁿ $m^2-9=0$

$m^2=\pm 9$

$m=\pm 3 \Rightarrow \boxed{m=3,-3}$

The roots are real and distinct

$$\boxed{C \cdot F = c_1 e^{3x}+c_2 e^{-3x}}$$

Case(ii) To find P·I

$$P \cdot I = \frac{1}{f(D)} Q(x)$$

$$= \frac{1}{D^2-9}(4x^2+2x)$$

$$= \frac{1}{-9\left[1-\frac{D^2}{9}\right]}(4x^2+2x)$$

$$= \frac{-1}{9}\left[1-\frac{D^2}{9}\right]^{-1}(4x^2+2x)$$

$(1-D)^{-1} = 1+D+D^2+D^3+D^4+\ldots$

$$= \frac{-1}{9}\left[1+\frac{D^2}{9}+\cancel{\left(\frac{D^2}{9}\right)^2}+\cancel{\left(\frac{D^2}{9}\right)^3}+\ldots\right]$$

$$= \frac{-1}{9}\left[1+\frac{D^2}{9}\right](4x^2+2x)$$

$$= \frac{-1}{9}\left[\left[4x^2+2x+\frac{1}{9}D^2(4x^2+2x)\right]\right]$$

$$= \frac{-1}{9}\left[4x^2+2x+\frac{1}{9}8\right]$$

$$\boxed{P.I = \frac{-1}{9}\left[4x^2+2x+\frac{8}{9}\right]}$$

$$G.S = C.F + P.I$$

$$G.S = c_1e^{3x}+c_2e^{-3x}+\left(\frac{-1}{9}\right)\left[4x^2+2x+\frac{8}{9}\right]$$

③ $(D^2+1)y = x^2$

Sol:- The given D. Eq^n is of the form

$$f(D)\cdot y = Q(x)$$

where $Q(x) = x^2$

Case(i) To find C.F

$$(D^2+1)y = 0$$

A. Eq^n $m^2+1=0$

$$m^2 = \pm i$$

$$\boxed{m = 0 \pm i} \Rightarrow \boxed{\alpha \pm i\beta}$$

The roots are Complex.

$$\boxed{C.F = e^{0x}\left[c_1\cos x + c_2\sin x\right]}$$

Case(ii) To find P.I

$$P.I = \frac{1}{f(D)} \cdot Q(x)$$

$$= \frac{1}{D^2+1} x^2$$

$$= \frac{1}{1(1+D^2)} \cdot x^2$$

$$= \frac{1}{1}(1+D^2)^{-1} x^2$$

$$\boxed{(1+D)^{-1} = 1 - D + D^2 - D^3 + D^4 + \cdots}$$

$$= 1\left[1 - \left(\frac{D^2}{1}\right) + \left(\frac{D^2}{1}\right)^2 - \left(\frac{D^2}{1}\right)^3 - \cdots\right] x^2$$

$$= \left[1 - \left(\frac{D^2}{1}\right)\right] \cdot x^2$$

$$= x^2 - D^2(x^2)$$

$$\boxed{P.I = x^2 - 2}$$

$$\boxed{G.S = C.F + P.I}$$

$$\boxed{G.S = e^{x}\left[C_1 \cos x + C_2 \sin x\right] + (x^2 - 2)}$$

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Method - 4 :- To find G.S of $f(D).y = Q(x)$

where $Q(x) = e^{ax}.V$

where $\boxed{V = \sin ax \,|\, \cos ax}$

or

$\boxed{V = x^K}$

procedure:-

1. To find C.F
2. To find $P.I = \frac{1}{f(D)} . Q(x)$

$$P.I = \frac{1}{f(D)} e^{ax}.V$$

$$\boxed{\text{put } D = D+a}$$

$$P.I = e^{ax}. \frac{1}{f(D+a)} . V$$

use

- Now apply 2nd method or 3rd method until we get particular integral

3. To find $G.S = C.F + P.I$

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① Solve $(D^2-3D+2)y = e^x \cos x$

Sol:- The given diff eqⁿ is in the form $f(D)\cdot y = e^{ax}\cdot V$

$$(D^2-3D+2)y = e^x \cos x$$

$$\boxed{G.S = C.F + P.I}$$

Case (i) : To find C.F :-

$$(D^2-3D+2)y = 0$$

A.E.q $\cdot\ m^2-3m+2=0$

$$m^2-2m-m+2=0$$

$$m(m-2)-(m-2)=0$$

$$(m-1)(m-2)=0$$

$$m=1,2$$

$$\boxed{C.F = C_1e^x + C_2e^{2x}}$$

Case (ii) : To find P.I

$$P.I = \frac{1}{f(D)}\cdot Q(x)$$

$$= \frac{1}{D^2-3D+2}\, e^x \cos x$$

$a=1$

put $D = D+1$

Method - 4 :- To find G.S of $f(D).y = Q(x)$

where $Q(x) = e^{ax}.V$

where $\boxed{V = \sin ax \,|\, \cos ax}$

or

$\boxed{V = x^K}$

procedure:-

1. To find C.F

2. To find $P.I = \frac{1}{f(D)} \cdot Q(x)$

$$P.I = \frac{1}{f(D)} e^{ax}.V$$

$\boxed{\text{put } D = D+a}$

$$P.I = e^{ax}.\frac{1}{f(D+a)}.V$$

use

- Now apply 2nd method or 3rd method until we get particular integral

3. To find $G.S = C.F + P.I$

① Solve $(D^2-3D+2)y = e^x \cos x$

Sol:- The given diff eqⁿ is in the form

$$f(D)\cdot y = e^{ax}\cdot V$$

$$(D^2-3D+2)\,y = e^x \cos x$$

$$\boxed{G\cdot S = C\cdot F + P\cdot I}$$

Case (i) : To find C·F :-

$$(D^2-3D+2)\,y = 0$$

A.Eq . $m^2-3m+2=0$

$$m^2-2m-m+2=0$$

$$m(m-2)-(m-2)=0$$

$$(m-1)(m-2)=0$$

$$m=1,2$$

$$\boxed{C.F = C_1 e^x + C_2 e^{2x}}$$

Case (ii) : To find P.I

$$P.I = \frac{1}{f(D)}\cdot Q(x)$$

$$= \frac{1}{D^2-3D+2}\, e^x \cos x$$

$a=1$

put $D = D+1$

$$= e^{x} \cdot \frac{1}{(D+1)^2 - 3(D+1) + 2} \cos x$$

$$= e^{x} \frac{1}{D^2 + 1 + 2D - 3D - 3 + 2} \cos x.$$

$$= e^{x} \frac{1}{D^2 - D} \cos x \quad (a=1).$$

$$\left\{ \frac{1}{f(D^2)} \cos ax. \text{ use } 2^{nd} \text{ method} \right)$$

put $D^2 = -1^2$.

$$= e^{x} \frac{1}{-1-D} \cos x.$$

$$= e^{x} \frac{-1}{(1+D)} \cos x$$

$$= e^{x} \frac{-1}{(1+D)} \times \frac{(1-D)}{(1-D)} \cos x.$$

$$= -e^{x} \frac{(1-D)}{(1-D^2)} \cos x$$

put $D^2 = -1$

$$= -e^{x} \frac{1-D}{1-(-1)} \cos x$$

$$= \frac{-e^{x}}{2} (1-D) \cos x$$

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$$= \frac{-e^{x}}{2}\left[\cos x - D(\cos x)\right] \qquad D = \frac{d}{dx}$$

$$= \frac{-e^{x}}{2}\left[\cos x - (-\sin x)\right]$$

$$\boxed{P.I = \frac{-e^{x}}{2}\left[\cos x + \sin x\right]}$$

$$G.S = C.F + P.I$$

$$G.S = C_1 e^{x} + C_2 e^{2x} + \left(\frac{-e^{x}}{2}\right)\left[\cos x + \sin x\right]$$

② solve $(D^2 - 7D + 6)y = e^{2x}(1+x)$

Sol: Let $(D^2 - 7D + 6)y = e^{2x}(1+x)$

$$\cancel{G.S} = f(D).y = e^{ax}V$$

Case(i): To find C.F

$$(D^2 - 7D + 6)y = 0$$

$$\Rightarrow m^2 - 7m + 6 = 0$$

$$m^2 - 6m - m + 6 = 0$$

$$m(m-6) - 1(m-6) = 0$$

$$(m-1)(m-6) = 0$$

$$m = 1, 6$$

$$C.F = C_1 e^{x} + C_2 e^{6x}$$

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Case(ii) To find P.I

$$P.I = \frac{1}{f(D)} Q(x)$$

$$= \frac{1}{D^2-7D+6} e^{2x}(1+x)$$

$$= e^{2x} \frac{1}{(D+2)^2-7(D+2)+6}(1+x)$$

$$= e^{2x} \frac{1}{D^2+4+4D-7D-14+6}(1+x)$$

$$= e^{2x} \frac{1}{D^2-3D-4}(1+x)$$

$\left\{ \frac{1}{f(D)} \cdot x^k \right.$ use 3rd meth

$$= e^{2x} \frac{1}{-4\left[1-\frac{(D^2-3D)}{4}\right]}(1+x)$$

$$= \frac{-1}{4} e^{2x}\left[1-\left(\frac{D^2-3D}{4}\right)\right]^{-1}(1+x)$$

$\left\{ (1-D)^{-1} = 1+D+D^2+D^3+ \cdots \right.$

$$= \frac{-1}{4} e^{2x}\left[1+\left(\frac{D^2-3D}{4}\right)+\cancel{\left(\frac{D^2-3D}{4}\right)^2}+\cdots\right](1+x)$$

Neglecting the higher degree term

$$= \frac{-e^{2x}}{4}\left[1+x+\frac{1}{4}\left\{D^2(1+x)-3D(1+x)\right\}\right]$$

$$= \frac{-e^{2x}}{4}\left[1+x+\frac{1}{4}[(0)-3(1)]\right]$$

$$= \frac{-e^{2x}}{4}\left[1+x-\frac{3}{4}\right]$$

$$G.S = C.F + P.I$$

$$\boxed{G.S = c_1 e^{x} + c_2 e^{6x} + \left(\frac{-e^{2x}}{4}\right)\left[1+x-\frac{3}{4}\right]}$$

③ solve $(D^2+D+1)y = e^{x}\sin 2x$

sol: $(D^2+D+1)y = e^{x}\sin 2x$.

$f(D)\cdot y = e^{ax}V$.

case(i) To find C.F

m^2

1

$(D^2+D+1)y = 0$

$m^2 +$.

$(m^2+m+1) = 0$.

A.Eqn. $m = \frac{-1}{2} \pm \frac{\sqrt{3}}{2}i$

$m = \frac{-1}{2}+\frac{\sqrt{3}}{2}i, \ \frac{-1}{2}-\frac{\sqrt{3}}{2}i$

$C.F = e^{-1/2x}\left[c_1 \cos \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)x\right.$

$$\boxed{C.F = e^{(-1/2)x}\left[c_1 \cos \frac{\sqrt{3}}{2}x + c_2 \sin \frac{\sqrt{3}}{2}x\right]}$$

Case(ii) To find P.I

$$P.I = \frac{1}{f(D)} \cdot G(x)$$

$$= \frac{1}{D^2+D+1} e^{x} \sin 2x \qquad a=1$$

$D = D+1$

$$= e^{x} \frac{1}{(D+1)^2+(D+1)+1} \sin 2x$$

$$= e^{x} \frac{1}{D^2+1+2D+D+1+1} \sin 2x$$

$$= e^{x} \frac{1}{D^2+3D+3} \sin 2x$$

$D^2 = -1.$

$$= e^{x} \frac{1}{-1+3D+3} \sin 2x.$$

$$= e^{x} \frac{1}{3D+2} \sin 2x$$

$$= e^{x} \frac{3D-2}{(3D+2)(3D-2)} \sin 2x$$

$$= e^{x} \frac{3D-2}{9D^2-4} \sin 2x$$

$(3D+2)(3D-2)$

$9D^2 - 6D \ldots$

$9^2 - 6^2$

$$= e^{x} \frac{3D-2}{9D^2-4} \sin 2x \quad [D^2 = -1]$$

$$= e^{x} \frac{3D-2}{9(-1)-4} \sin 2x$$

$$= e^{x} \frac{3D-2}{-9-4} \sin 2x$$

$$= \frac{-e^{x}}{13} (3D \sin 2x - 2 \sin 2x)$$

$$= \frac{-e^{x}}{13} (6 \sin 2x - 2 \sin 2x)$$

$$= \frac{-e^{x}}{13} (4 \sin 2x)$$

$G.S = C.F + P.I$

$$\boxed{G.S = e^{(-1/2)x} \left[C_1 \cos \frac{\sqrt{3}}{2}x + C_2 \sin \frac{\sqrt{3}}{2}x \right] - \frac{e^{x}}{13} (4 \sin 2x)}$$

① Solve $\frac{d^2y}{dx^2} + y = \operatorname{cosec} x$

parameters of eq^n

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Sol:- Given $R(x) = \operatorname{cosec} x$

$$\boxed{G \cdot S = C \cdot F + P \cdot I} \quad -①$$

Case (i):- To find C·F

$$(D^2y + y) = 0$$

$$(D^2 + 1)y = 0$$

A. Eq^n

$$m^2 + 1 = 0$$

$$m^2 = -1$$

$$m = \pm i$$

$$\boxed{m = 0 \pm i} \rightarrow \text{complex} \quad \alpha \pm i\beta$$

$$\boxed{CF = e^{\alpha x}[C_1 \cos\beta x + C_2 \sin\beta x]}$$

$$\therefore C \cdot F = e^{0x}[C_1 \cos x + C_2 \sin\beta x]$$

$$\boxed{C \cdot F = C_1 \cos x + C_2 \sin x}$$

Case (ii):- To find P·I

$$\boxed{P \cdot I = A\cos x + B\sin x} \quad -②$$

where $u = \cos x$, $v = \sin x$

$\frac{du}{dx} = -\sin x$ | $\frac{dv}{dx} = \cos x$

Now $\phi(x) = u\frac{dv}{dx} - v\frac{du}{dx}$.

$= \cos^2 x + \sin^2 x$

$\boxed{\phi(x) = 1}$

To find A & B

$A = -\int \frac{vR}{\phi(x)}$

$= -\int \frac{\sin x \cdot \operatorname{cosec} x}{1} dx$

$= -\int \sin x \cdot \frac{1}{\sin x} dx$

$\boxed{A = -x}$

& $B = \int \frac{uR}{\phi(x)} dx$

$= \int \frac{\cos x \cdot \operatorname{cosec} x}{1} dx$

$= \int \cos x \cdot \frac{1}{\sin x} dx$

$= \int \cot x \, dx \Rightarrow \boxed{\log(\sin x) = B}$

Put A & B in eqⁿ ②

$P.I = -x\cos x + \log(\sin x)\sin x$.

$\boxed{G.S = C_1\cos x + C_2\sin x - x\cos x + \log(\sin x)\sin x}$

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(2) Solve $(D^2+a^2)y = \tan ax$

Sol:- where $R(x) = \tan ax$.

$$\boxed{G.S = C.F + P.I} \quad \text{--- (1)}$$

Case(i) - To find C.F

$$(D^2+a^2)y = 0$$

A.Eq: $m^2+a^2=0$

$$m^2 = -a^2$$

$$m = \pm ai$$

$$m = 0 \pm ai = \alpha \pm i\beta$$

$$C.F = e^{0x}[c_1 \cos ax + c_2 \sin ax]$$

$$\boxed{C.F = c_1 \cos ax + c_2 \sin ax}$$

Case(ii) To find P.I

$$\boxed{P.I = A\cos ax + B\sin ax} \quad \text{--- (2)}$$

$u = \cos ax$ & $v = \sin ax$.

$$\frac{du}{dx} = (-\sin ax)a \qquad \frac{dv}{dx} = a\cos ax$$

$$\phi(x) = u\frac{dv}{dx} - v\frac{du}{dx}$$

$= a\cos^2 ax + \sin^2 ax \cdot a$

$= a\left[\sin^2 ax + \cos^2 ax\right]$

$\phi(x) = a$

<u>To find A and B :-</u>

where $A = -\int \frac{VR}{\phi(x)}\, dx$

$= -\int \frac{\sin ax \cdot \tan ax}{a}\, dx$

$= \frac{-1}{a}\int \sin ax \cdot \frac{\sin ax}{\cos ax}\, dx$

$= \frac{-1}{a}\int \frac{\sin^2 ax}{\cos ax}\, dx$

$= \frac{-1}{a}\int \frac{1-\cos^2 ax}{\cos ax}\, dx$

$= \frac{-1}{a}\left[\int \sec ax\, dx - \int \cos ax\, dx\right]$

$= \frac{-1}{a}\left[\frac{\log(\sec ax + \tan ax)}{a} - \frac{\sin ax}{a}\right]$

$A = \frac{-1}{a^2}\left[\log(\sec ax + \tan ax) - \sin ax\right]$

& $B = \int \frac{uR}{\phi(x)} dx$

$= \int \frac{\cos ax \cdot \tan ax}{a} dx$

$= \frac{1}{a} \int \cancel{\cos ax} \cdot \frac{\sin ax}{\cancel{\cos ax}} dx$

$= \frac{1}{a} \left[\frac{-\cos ax}{a} \right]$

$\Rightarrow M_2 + m_1 + m_3 + m_4$
$\Rightarrow S_1 + S_2 + S_3 + S_4$

$$\boxed{B = \frac{-1}{a^2} \cos ax}$$

Now put A & B in eq^n ②.

$$\boxed{P.I = \frac{-1}{a^2} \left[\log(\sec ax + \tan ax) - \sin ax \right] \cos ax + \left(\frac{-1}{a^2} \right) \cos ax \cdot \sin ax}$$

$G.S = C.F + P.I$

$$\boxed{G.S = c_1 \cos ax + c_2 \sin ax + \left(\frac{-1}{a^2} \right) \left[\log(\sec ax + \tan ax) - \sin ax \right] \cos ax + \left(\frac{-1}{a^2} \right) \cdot \cos ax \cdot \sin ax}$$

③ Solve $(D^2y - y) = \frac{2}{1+e^x}$

Sol:- The given diff eqⁿ can be writt

as $(D^2-1)y = \frac{2}{1+e^x}$

where $R(x) = \frac{2}{1+e^x}$

$$\boxed{G.S = C.F + P.I} \quad \text{--- ①}$$

Case(i):- To find C.F

$(D^2-1)y = 0$

$\Rightarrow m^2-1 = 0$

$m^2 = 1 \Rightarrow m = \pm 1$

$m = 1, -1$

The roots are real and disticnt

A.Eq:- $\boxed{C.F = C_1 e^{-x} + C_2 e^{x}}$

Case(ii) To find P.I.

$$\boxed{P.I = A e^{x} + B e^{-x}} \text{--- ②}$$

$u = e^{x}$ $\qquad$ $v = e^{-x}$

$\frac{du}{dx} = e^{x}$ $\qquad$ $\frac{dv}{dx} = -e^{-x}$

$$\phi(x) = u\frac{dv}{dx} - v\frac{du}{dx}$$

$$= e^{x}(-e^{-x}) - e^{-x}(e^{x})$$

$$= -e^{x}e^{-x} - e^{-x}e^{x}$$

$e^{-x} = \frac{1}{e^{x}}$ $\quad = -e^{x-x} - e^{x-x}$

e^{x} $\quad = -e^{0} - e^{0} = -2.$

$$\phi(x) = -2.$$

<u>To find A & B :-</u>

where $A = -\int \frac{VR}{\phi(x)}\, dx.$

$$= -\int \frac{e^{-x}\cdot\frac{2}{1+e^{x}}}{-2}\, dx$$

$$= \frac{+1}{-2}\int e^{-x}\cdot\frac{2}{1+e^{x}}\, dx.$$

$$= \frac{1}{2}\int e^{-x}\cdot\frac{2}{1+e^{x}}\, dx$$

$$= \int \frac{e^{-x}}{1+e^{x}}\, dx \qquad e^{x}(1+ \quad (e^{x}+e^{x^2})$$

$$= \cancel{\log(1+e^{x})}$$

$$\boxed{A = \cancel{\log(1+e^{x})}} \Rightarrow = \int \frac{1}{e^{x}(1+e^{x})}$$

$$= \int \frac{1}{e^{x}} - \frac{1}{e^{x}+1}\, dx$$

$$= \int \frac{1}{e^{x}}\, dx - \int \frac{1}{e^{x}+1}\, dx$$

$$= \int e^{-x}\, dx - \int \frac{1}{1+\frac{1}{e^{-x}}}\, dx$$

$$= -e^{-x} - \int \frac{1}{\frac{e^{-x}+1}{e^{-x}}}\, dx$$

$$= -e^{-x} + \int \frac{(-e^{-x})}{e^{-x}+1}\, dx$$

$$\boxed{A = -e^{-x} + \log(e^{-x}+1)}$$

=

$$\& \quad B = \int \frac{4R}{\phi(x)}\, dx = \int \frac{e^{x}}{-\cancel{x}} \cdot \frac{\cancel{x}}{1+e^{x}}\, dx$$

$$= -\int \frac{e^{x}}{1+e^{x}}\, dx$$

$$\boxed{B = -\log(1+e^{x})}$$

Put A & B in eq (2)

$$P.I = [-e^{-x} + \log(e^{-x}+1)]e^{x} - \log(1+e^{x})e^{-x}$$

Now $G.S = C.F + P.I$

$$\boxed{G.S = c_1 e^{x} + c_2 e^{-x} + \{-e^{-x} + \log(1+e^{-x})\} e^{x} - \log(1+e^{x}) e^{-x}.}$$

~~Chap~~ Chapter-3

Method-1

Homogeneous linear equation:-

Euler-Cauchy's equations:-

Procedure:-

→ An equation of the form

$$x^n \frac{d^n y}{dx^n} + P_1(x) \cdot x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + P_2(x) \cdot x^{n-2} \frac{d^{n-2} y}{dx^2} + \cdots + P_n(x) y = Q(x)$$

where $P_1, P_2 ---- P_n$ are real constants and $Q(x)$ is a function of x is called homogeneous linear eq^n (or) Euler-Cauchy's eq^n.

→ put $x = e^z \Rightarrow \log x = z$

and we have

Sub $xD = \theta$

$x^2 D^2 = \theta(\theta-1)$

$x^3 D^3 = \theta(\theta-1)(\theta-2)$ --- Etc.

where $\frac{d}{dx} = D$ & $\frac{d}{d\theta} = z$

we get the modified diff eqn in θ var

→ To find C.F and P.I and also find G.S

$\boxed{G.S = C.F + P.I}$

problems:-

(1) solve $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = (1+x)^2$

Sol:- The given diff eqn is Euler-Cauchy's equations that is given by

$$x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = (1+x)^2$$

$$x^2 D^2y - 3xDy + 4y = (1+x)^2$$

$$[x^2D^2 - 3xD + 4]y = (1+x)^2 \quad \text{--- (1)}$$

By the method of Euler-Cauchy's

put ~~xD=θ~~ $x = e^z \Rightarrow \log x = z$

$xD = \theta$

$x^2D^2 = \theta(\theta-1)$

~~x^3D^3~~ $=$ ~~$\theta($~~

$$① \Rightarrow [\theta(\theta-1)-3\theta+4]y=(1+e^{z})^2$$

$$[\theta^2-\theta-3\theta+4]y=(1+e^{2z}+2e^{z})$$

$$(\theta^2-4\theta+4)y=(1+e^{2z}+2e^{z}) \quad -②$$

which is Diff in θ variable.

$$G \cdot S = C \cdot F + P \cdot I$$

Case(i) :- To find C·F

$$(\theta^2-4\theta+4)y=0.$$

A·eq) $m^2-4m+4=0$

$$m^2-2m-2m+4=0$$

$$\boxed{m=2,2.}$$

Real and repeated

$$\boxed{C \cdot F=(C_1+C_2 z)e^{2z}}$$

Case(ii) To find P·I :-

$$P \cdot I = \frac{1}{f(\theta)} Q(z)$$

$$= \frac{1}{\theta^2-4\theta+4}(1+e^{2z}+2e^{z})$$

$$= \frac{1}{\theta^2-4\theta+4} \cdot e^{0z} + \frac{1}{\theta^2-4\theta+4}(e^{2z}) + \frac{1}{\theta^2-4\theta+4} 2e^{z}$$

$\left\{ \text{use } 1^{st} \text{ method } \frac{1}{f(D)} e^{ax} \Rightarrow D = a \right\}$

$PI_1 \Rightarrow \theta = 0$

if put $PI_2 \Rightarrow \theta = 2$ then we get failure case

& $PI_3 \Rightarrow \theta = 1$

$$P.I = \frac{1}{4} e^{0z} + \frac{z}{\frac{d}{d\theta}(\theta^2 - 4\theta + 4)} e^{2z} + 2e^{z}$$

$$= \frac{1}{4} + \frac{z}{2\theta - 4} \cdot e^{2z} + 2e^{z}$$

if $\frac{1}{f(D)} e^{ax}$ put $\theta = 2$ in PI_2 we get failure case

$$= \frac{1}{4} + \frac{z \cdot z}{\frac{d}{d\theta}(2\theta - 4)} e^{2z} + 2e^{z}$$

$$\boxed{P.I = \frac{1}{4} + \frac{z^2}{2} e^{2z} + 2e^{z}}$$

$G.S = C.F + P.I$

$$\boxed{G.S = (C_1 + C_2 z) e^{2z} + \frac{1}{4} + \frac{z^2}{2} e^{2z} + 2e^{z}}$$

put $z = \log x$

$$\boxed{G.S = (C_1 + C_2 \log x) e^{2\log x} + \frac{1}{4} + \frac{(\log x)^2 e^{2\log x}}{2} + 2e^{\log x}}$$

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② Solve $(x^3 D^3 + 2x^2 D^2 + 2)y = x + \frac{1}{x}$.

Sol:- The given diff eqⁿ is Eulers-Cauchy's eqⁿ.

from the Euler-Cauchys eqⁿ.

$$(x^3 D^3 + 2x^2 D^2 + 2)y = x + \frac{1}{x} \quad \text{—①}$$

put $xD = \theta$

$$x^2 D^2 = \theta(\theta - 1)$$

$$x^3 D^3 = \theta(\theta-1)(\theta-2)$$

$$① \Rightarrow (\theta(\theta-1)(\theta-2) + 2\theta(\theta-1) + 2)y = x + \frac{1}{x}$$

$$((\theta^2 - \theta)(\theta-2) + 2\theta^2 - 2\theta + 2)y = x + \frac{1}{x}$$

$$(\theta^3 - \cancel{2\theta^2} - \theta^2 + \cancel{2\theta} + \cancel{2\theta^2} - \cancel{2\theta} + 2)y = x + \frac{1}{x}$$

$$\Rightarrow (\theta^3 - \theta^2 + 2)y = \frac{x^2+1}{x} \quad \text{—②}$$

$$G.S = C.F + P.I$$

Case(i): To find P.I.

$$(\theta^3 - \theta^2 + 2)y = 0$$

$$(m^3 - m^2 + 2)y = 0$$

$$m^3 - m^2 + 2 = 0.$$

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