LOVE AFFAIRS: ROMEO AND JULIET

An application of Differential Equation by Uyen Bui.

Introduction

In old Verona, Italy, Romeo and Juliet, from enemy families, fall in love against the odds. But their happiness does not last because of fights and misunderstandings. Romeo has to leave, and Juliet is forced to marry Paris. This model shows us the complexities of love and the universal longing for connection through the ages as a timeless example of the highs and lows of human emotion.

01 Romeo's feelings without Juliet

The process describes how Romeo's feelings change over time. At first, when there was no Juliet yet, it depends only on Romeo's current feelings.

Let $R(t)$ be Romeo's Love for Juliet at time t.

$$R(t) > 0 \Rightarrow \text{love.}$$

$$R(t) < 0 \Rightarrow \text{hate.}$$

This linear differential equation below represents Romeo's rate of change of love $\frac{dR}{dt}$, which depends on his current level of love R and a constant a that determines how fast his love grows or diminishes.

$$\frac{dR}{dt} = aR$$

$$\frac{dR}{aR} = dt$$

$$\frac{1}{a}\int \frac{dR}{R} = \int dt$$

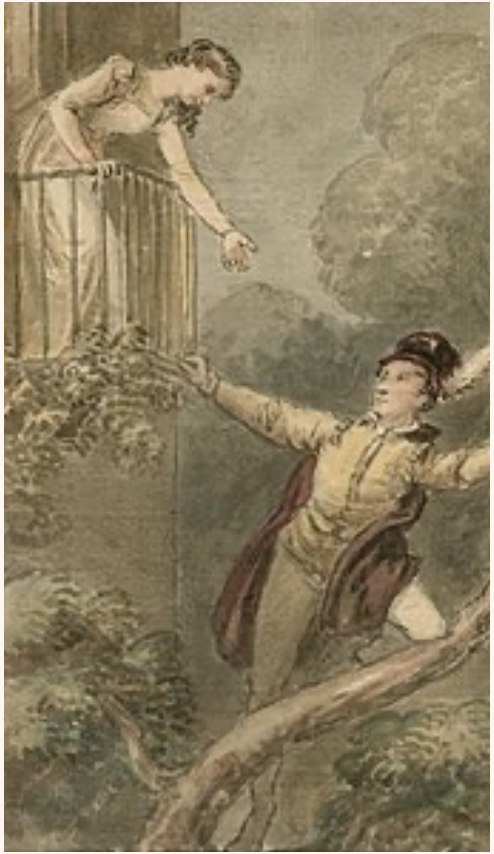
$$\frac{1}{a}(\ln|R| + C_1) = t + C_2$$

$$\frac{\ln|R|}{a} = t - \frac{C_1}{a} + C_2$$

$$\ln|R| = at + C$$

$$R = \pm e^{at+C} = ke^{at}$$

$$R = ke^{at}$$



02 Romeo's feelings with Juliet

Romeo's life gets more complex with Juliet's presence. Romeo's feelings influence Juliet's and vice versa. Let J be Juliet's love for Romeo at time t.

$$\frac{dR}{dt} = aR + bJ.$$

Love's style:

$$\begin{aligned}
\text{Excited Love:} \quad & a > 0, b > 0 \Rightarrow \text{ Romeo and Juliet share happiness together.}\\
\text{One-sided Love:} \quad & a > 0, b < 0 \Rightarrow \text{Romeo likes Juliet, but does not care much about her.}\\
\text{Encouraged Love:} \quad & a < 0, b > 0 \Rightarrow \text{Juliet's encouragement helps Romeo overcome his uncertainty.}\\
\text{Toxic Love:} \quad & a < 0, b < 0 \Rightarrow \text{Both Romeo and Juliet feel unhappy together.}
\end{aligned}$$

03 "Love Equation System"

Using the same model of Romeo for Juliet, now we have a coupled differential equations:

$$\frac{dR}{dt} = aR + bJ$$
$$\frac{dJ}{dt} = cR + dJ$$

$$\begin{pmatrix} \frac{dR}{dt} \\ \frac{dJ}{dt} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} R \\ J \end{pmatrix} = A \begin{pmatrix} R \\ J \end{pmatrix}$$

Suppose A is an $n \times n$ matrix. A represents the dependence between two differential equations:

1. A is diagonal $\iff$ A has exactly n distinct eigenvalues $\Rightarrow$ Separate Evolution.
2. A is non-diagonal $\Rightarrow$ Mutual Influence.

$$\det(A - \lambda I) = \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}$$

$$\Rightarrow (a-\lambda)(d-\lambda) - bc = 0$$

$$\Rightarrow (\lambda)^2 - (a+d)\lambda + ad - bc = 0$$

THEOREM 6 An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

Solve for Δ :

$$\begin{aligned} \Delta &= (a+d)^2 - 4ad + 4bc \\ &= a^2 + 2ad + d^2 - 4ad + 4bc \\ &= a^2 - 2ad + d^2 + 4bc \\ &= (a-d)^2 + 4bc \end{aligned}$$

1. If $\Delta > 0 \implies$ diagonal:

Using the quadratic formula:

$$\lambda_1 = \frac{a+d+\sqrt{(a-d)^2+4bc}}{2}$$

$$\lambda_2 = \frac{a+d-\sqrt{(a-d)^2+4bc}}{2}$$

2. If $\Delta = 0 \implies$ non $-$ diagonal

Since we know the discriminant is zero:

$$\lambda = \frac{(a+d)}{2}$$

Solve for the general solution

Recall:

THEOREM Suppose the matrix $\mathbf{A}$ has a real eigenvalue λ with associated eigenvector $\mathbf{V}$. Then the linear system $d\mathbf{Y}/dt = \mathbf{A}\mathbf{Y}$ has the straight-line solution

$$\mathbf{Y}(t) = e^{\lambda t}\mathbf{V}.$$

Moreover, if λ_1 and λ_2 are distinct, real eigenvalues with eigenvectors $\mathbf{V}_1$ and $\mathbf{V}_2$ respectively, then the solutions $\mathbf{Y}_1(t) = e^{\lambda_1 t}\mathbf{V}_1$ and $\mathbf{Y}_2(t) = e^{\lambda_2 t}\mathbf{V}_2$ are linearly independent and

$$\mathbf{Y}(t) = k_1 e^{\lambda_1 t}\mathbf{V}_1 + k_2 e^{\lambda_2 t}\mathbf{V}_2$$

is the general solution of the system. ■

Matrix (Coupled Differential Equation):

$$\begin{pmatrix} \frac{dR}{dt} \\ \frac{dJ}{dt} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} R \\ J \end{pmatrix} = A \begin{pmatrix} R \\ J \end{pmatrix}$$

Suppose A has 2 distinct eigenvalues (calculated):

$$\lambda_1 = \frac{a + d + \sqrt{(a-d)^2 + 4bc}}{2}$$

$$\lambda_2 = \frac{a + d - \sqrt{(a-d)^2 + 4bc}}{2}$$

General Solution:

Consider v_1 is the eigenvector associated with λ_1 and v_2 is the eigenvector associated with λ_2. Assume that λ_1 and λ_2 are distinct. The general solution for the system would be:

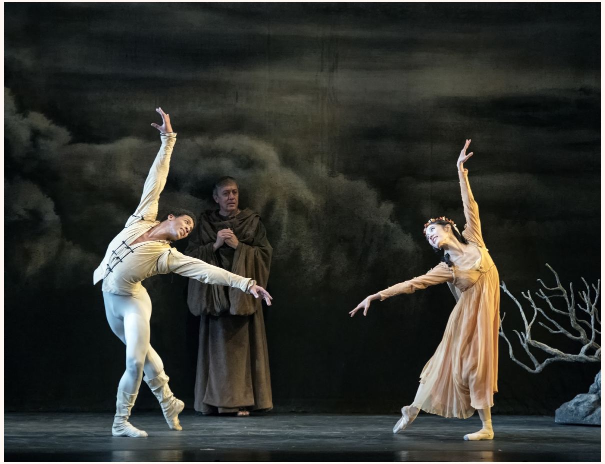
$$Y(t) = k_1 e^{\lambda_1 t} V_1 + k_2 e^{\lambda_2 t} V_2.$$

where k_1 and k_2 are constants. This solution expresses how Romeo and Juliet's feelings interact and evolve at a given time in their relationship.

04 Example

The outcome of the tragedy Romeo and Juliet:

Juliet, forced into a marriage by her family, turns to Friar Laurence for help. He provides a potion to fake her death. Misunderstanding, Romeo fatally fights with Paris, Juliet's intended. Romeo then takes his own life. Awakening to find Romeo dead, Juliet follows suit.



Analyze:

Their deep love amplifies the pain of their deaths, so external pressures turn overwhelmingly negative.

Remind:

$$\frac{dR}{dt} = aR + bJ$$
$$\frac{dJ}{dt} = cR + dJ$$

Condition:

$$a, d > 0; b, c < 0,$$
$$|b| > |a|$$
$$|c| > |d|.$$

"Both-died" equation system

$$\begin{pmatrix} \frac{dR}{dt} \\ \frac{dJ}{dt} \end{pmatrix} = \begin{pmatrix} 2 & -4 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} R \\ J \end{pmatrix}$$

1.

$$\det(A-\lambda I) = \begin{pmatrix} 2-\lambda & -4 \\ -3 & 1-\lambda \end{pmatrix} = 0$$

$$\Rightarrow (2-\lambda)(1-\lambda)-(-3)(-4)=0$$

$$\Rightarrow (\lambda)^2 - 3\lambda + 2 - 12 = 0$$

$$\Rightarrow (\lambda)^2 - 3\lambda - 10 = 0$$

$$\Rightarrow (\lambda+2)(\lambda-5)=0$$

$$\Rightarrow \begin{cases} \lambda_1 = -2 \\ \lambda_2 = 5 \end{cases}$$

2. Solve for eigenvectors:

For: $\lambda_1 = -2$

$$\det(A-\lambda I) = \begin{pmatrix} 4 & -4 \\ -3 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For: $\lambda_2 = 5$

$$\det(A-\lambda I) = \begin{pmatrix} -3 & -4 \\ -3 & -4 \end{pmatrix} \sim \begin{pmatrix} 1 & \frac{4}{3} \\ 0 & 0 \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} -\frac{4}{3} \\ 1 \end{pmatrix}$$

3. The "both-died" general solution would be:

$$Y(t) = k_1 e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + k_2 e^{5t} \begin{pmatrix} -\frac{4}{3} \\ 1 \end{pmatrix}.$$

4. Assume that the initial condition $(R(0), J(0))$ is $(1,2)$. It indicates their initial levels of affection for each other. Solve for the equation:

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} = k_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + k_2 \begin{pmatrix} -\frac{4}{3} \\ 1 \end{pmatrix}.$$

$$\Rightarrow \begin{cases} k_1 + \frac{-4}{3} k_2 = 1 \\ k_1 + k_2 = 2 \end{cases} \Rightarrow \begin{cases} k_1 = \frac{11}{7} \\ k_2 = \frac{3}{7} \end{cases}$$

The final solution would be:

$$Y(t) = \frac{11}{7} e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{3}{7} e^{5t} \begin{pmatrix} -\frac{4}{3} \\ 1 \end{pmatrix}.$$

FINAL MEANING

REFERENCES

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THANK YOU FOR LISTENING!