GENERAL BOOK GRAPH AND GENERAL STAIR GRAPH

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Abstract.

Part 1: A strong foundation for modeling complicated structures is provided by graph theory, and in this paper, we investigate the fascinating field of General Book Graphs ($B(p, r)$). These graphs, which are based on how pages are physically arranged in books, present a distinctive viewpoint on how mathematical ideas interact with practical situations. The General Book Graph is defined, its vertex and edge sets are examined, and basic properties like order, size, minimum, and maximum degree are established. Illustrative graphs and other visual tools help to close the gap between abstract theory and concrete depiction. Through this study, we contribute not only to the theoretical side of graph theory but also showcase the applicability of mathematical principles to practical, everyday scenarios.

Part 2: In a parallel exploration, this paper also investigates a "Staircase graph" formed by arranging squares, with each step featuring a number of squares mirroring its position from the top. We present proofs of the consistent square-based ascending pattern, demonstrating its validity. The study contributes to graph theory, shedding light on the intriguing properties of this geometric construct and its potential applications in diverse fields.

1. Introduction

Graph theory is a branch of discrete mathematics, which provides the framework for modeling structures. In this paper, we explore a subclass of graphs known as

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General Book Graphs ($B(p, r)$), which looks like the physical layout of pages from within a book. Our paper uncovers the patterns and properties that are shown. As we dive into the analysis of the General Book Graph, we use principles from graph theory to gain insight into their structures and behaviors. Visual aids, such as the graph depicted in the General Book Graph below, will help us bridge the relationship between the mathematical concept and the representation of a book layout.

Besides, also in graph theory, we investigate an interesting geometric pattern – the ascending square staircase graph. Unlike conventional structures, this graph arranges steps in a unique configuration, with each step containing several squares corresponding to its position from the top. This study aims to inherent properties of the square staircase graph by presenting proof of its ascending square pattern. The investigation lies not only in its contribution to graph theory but also in the potential applications of this geometric construct across various mathematical and computational domains. Investigating this graph is to shed light on the mathematical principles of this staircase graph, providing a foundation for further research on similar geometric patterns within the context of graph theory.

2. Background and Definitions

Definition 2.1. General Book Graph:

Let p be the number of pages, and r be the number of rows on each page, where p, r are positive integers and $r \geq 2$. We define a General Book Graph, denoted $B(p, r)$, as $B(p, r) = (V, E)$.

We assert that $V = V_1 \cup V_2$ where:

$$V_1 = \{v_i : 1 \leq i \leq r\}$$

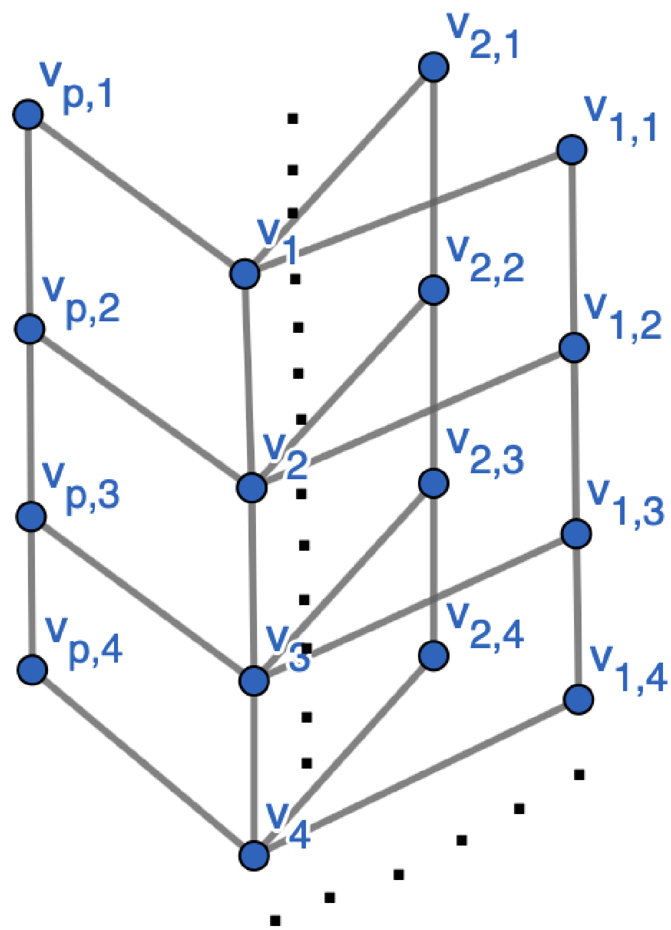
$$V_2 = \{v_{i,j} : 1 \leq i \leq p,\, 1 \leq j \leq r\},$$

$$E = E_1 \cup E_2 \cup E_3 \text{ where:}$$

$$E_1 = \{v_i, v_{i+1} : 1 \leq i \leq r-1\}$$

$$E_2 = \{v_i, v_{j,i} : 1 \leq j \leq p,\ 1 \leq i \leq r\}$$

$$E_3 = \{v_{j,i}, v_{j,i+1} : 1 \leq j \leq p,\ 1 \leq i \leq r-1\}.$$



Property:

Order: For any values of p and r, the number of vertices in the general book graph, induced by its spine, is r, and the number of vertices induced by pages and rows along each page is $r \times p$. Thus the order is defined as:

$$r + r \times p = r \times (p + 1).$$

Size: For any values of p and r, the number of edges in the general book graph, induced by its spine, is $r - 1$. The number of edges induced by horizontal lines in pages is $r \times p$, and by vertical lines is $(r - 1) \times p$. Thus the size is defined as:

$$(r - 1) + (r \times p) + (r - 1) \times p = r - 1 + r \times p + r \times p - p = 2rp + r - p - 1.$$

Max Degree:

Case 1: $r \geq 3$: The vertex with a max degree is the vertex in the middle of the book's spine. In the spine of the book, it will be connected to 2 other vertices. On each page, it will be connected to another point. Thus, the max degree is $2 + p$.

Case 2: $r = 2$: The vertex with a max degree is the vertex in the book's spine; it will be connected to another vertex in the spine. On each page, it will be connected to another point. Thus, the max degree is $1 + p$.

Min Degree:

For any value of p and r, there are at least 2 edges incident to each vertex. The vertex with the minimum degree is situated to the right of the first or last row of the book's page, and its degree is 2. Therefore, the minimum degree is 2.

Definition 2.2. General Stair Graph:

Let S be the general stair graph created by the blocks attached. Let f be the number of floors of the stair, $f \geq 1$. We define a General Stair Graph, denoted $S(f)$, as $S(f) = (V, E)$.

We assert that

$$V = \sum_{k=1}^{f} V_k \cup V_0$$

where:

$$V_k = \{v_{k,i} : 1 \leq i \leq k + 1\}$$

$$V_0 = \{v_{f+1,i} : 1 \leq i \leq f + 1\}$$

and

$$E = \sum_{k=1}^{f} E_k \cup E_{01} \cup \sum_{k=1}^{f} E'_k \cup E_{02}$$

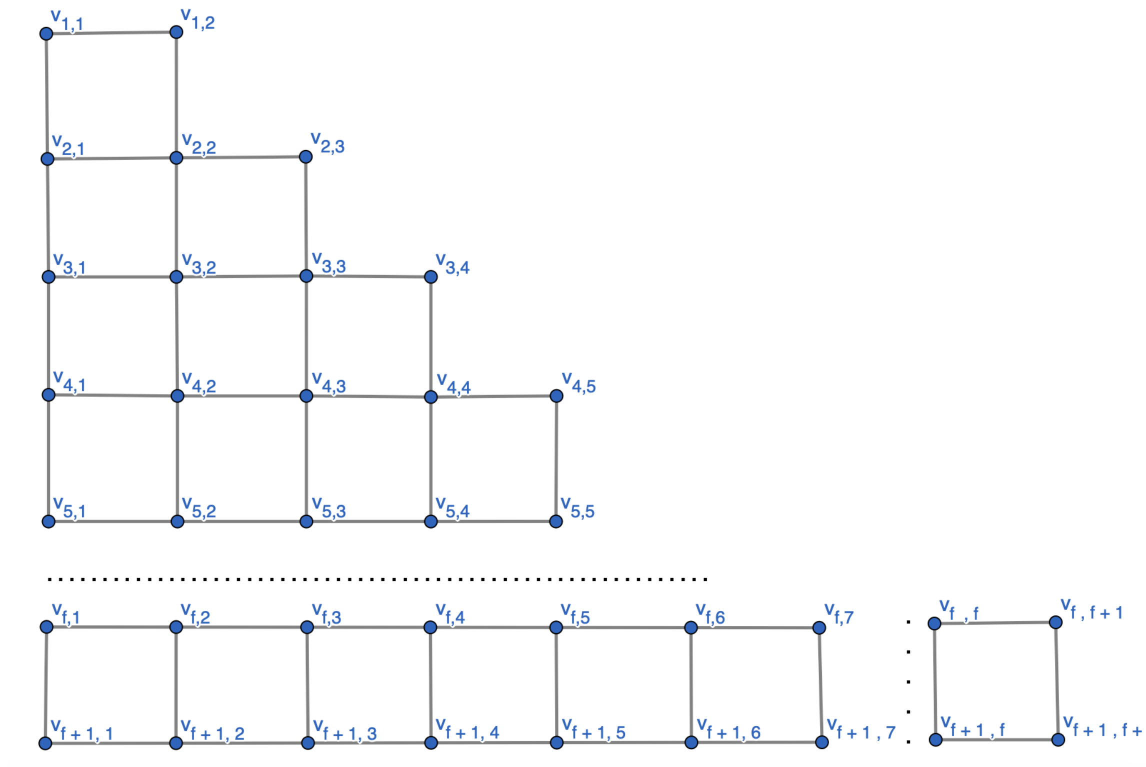
where:

$$E_k = \{v_{k,i}, v_{k,i+1} : 1 \leq i \leq k\}$$

$$E_{01} = \{v_{f+1,i}, v_{f+1,i+1} : 1 \leq i \leq f\}$$

$$E'_k = \{v_{i,k+1}, v_{i+1,k+1} : k \leq i \leq f\}$$

$$E_{02} = \{v_{i,1}, v_{i+1,1} : 1 \leq i \leq f\}.$$



Property:

Order: The number of vertices in the general book graph is induced by the number of floors (f). Thus the order can be defined as:

$$\frac{f^2 + 5f}{2} + 1$$

Size: The number of edges in the general book graph is induced by the number of floors (f). Thus the size can be defined as:

$$f^2 + 3f$$

Max Degree: The maximum degree of the graph is 4, evident in the second column of vertices from the left. Notably, some of the vertices in the second column of vertices are interconnected with 4 other vertices. Therefore, the maximum degree is 4.

Min Degree: The minimum degree of the graph is 2, if you look at any of the vertices at the corner of each "step" we see that there are only 2 other vertices connected to it. Therefore, the minimum degree is 2.

3. Results

Theorem 3.1. *The clique number of the general book graph is 2.*

Proof. Let us consider a general Book graph $B(p, r)$.

Part 1. Proof that Clique Number is at least 2:

Let us consider the set of vertices $\{v_1, v_2\}$ from the vertex Set V_1 in a book graph.

These vertices form a complete subgraph (clique) because there is an edge between v_1 and v_2 (from E_1).

Additionally, Let us consider the set $\{v_1, v_{j,1}\}$ for any j.

These vertices also form a complete subgraph because there is an edge between v_1 and $v_{j,1}$ (from E_2).

Therefore, there exist cliques of size 2 in a general book graph.

Part 2. Proof that Clique Number is at most 2:

Let us suppose there exists a complete subgraph of size 3 in $B(p, r)$.

This would imply the existence of three vertices v_a, v_b, v_c forming a complete subgraph.

1. If all three vertices reside in the spine V_1, the resulting shape forms a triangle, characterized by three edges: $v_i - v_{i+1}$, $v_{i+1} - v_{i+2}$, and $v_{i+2} - v_i$. However, considering the edges in set $E_1 = \{v_i, v_{i+1} : 1 \leq i \leq r - 1\}$ in the context of the

spine, there is no $v_{i+2} - v_i$ edge. Consequently, it is impossible for all three vertices to be contained exclusively in V_1.

2. If three vertices do not reside on the same page, the corresponding subgraph is incomplete. This is because the vertices situated on the outer edges of the pages are not connected to each other of other pages. Consequently, when three vertices are not co-located on the same page, a complete graph cannot be formed, resulting in no clique number.

3. If three vertices reside on the same page, there will be three cases:

Case 1: All three vertices are in the outer edge of the page. Three edges will be in $E_3 = \{v_{j,i}, v_{j,i+1} : 1 \leq j \leq p,\, 1 \leq i \leq r-1\}$. Because they shape a complete graph, three vertices should shape to be a triangle, which means three edges will be $v_{j,i} - v_{j,i+1}, v_{j,i+1} - v_{j,i+2}, v_{j,i} - v_{j,i+2}$. This thing is out of E_3's definition. So it is impossible.

Case 2: There are two vertices on the outer edge of the book and one vertex on the spine of the book. The formation of a complete graph is precluded in this case, as the vertex on the spine can only connect to a single vertex on the outer edge of each page. Consequently, there is no clique number of 3 in this particular configuration.

Case 3: There are two vertices on the spice of the book and one vertex on the outer edge of the book. The formation of a complete graph is precluded in this case, as the vertex on the spine can only connect to a single vertex on the outer edge of each page. Consequently, there is no clique number of 3 in this particular configuration.

Therefore, the size of the largest complete subgraph cannot be greater than 2.

Conclusion: Combining these arguments, the size of the largest complete subgraph in $B(p,r)$ is exactly 2. Therefore, the clique number of $B(p,r)$ is 2.

$\square$

Theorem 3.2. *The smallest cycle of the general book graph is 4.*

Proof. By definition, a 4-cycle in a graph is a closed path of length 4, such that no cycle exists.

Consider the General Book Graph $B(p,r)$. The vertices are divided into two sets, V_1 and V_2. The set V_1 consists of vertices v_i where $1 \leq i \leq r$, representing the spine of the book. The set V_2 consists of vertices $v_{i,j}$ where $1 \leq i \leq p$ and $1 \leq j \leq r$, representing the pages and rows on each page.

Assume that there exists a cycle of length 3 among the points in $B(p,r)$.

Consider the possible cycles in the general book graph:

Case 1: Cycles involving only vertices in V_1: It is impossible that a cycle involves only vertices from V_1, because in V_1, the spine is a line connected by vertices so it does not form a closed path. Thus, cycles of length 3 are not possible in V_1.

Case 2: Cycles involving only vertices in V_2: It is impossible that a cycle involves only vertices from V_2, because V_2 represents the outer edge of pages. First, these vertices are lie on the outer edge of the page which is a line so it does not form a closed path within V_2. Second, there is no edge connected vertices from outer edges of pages to others so it also cannot form a closed path. Thus, cycles of length 3 are not possible in V_3.

Case 3: Cycles involving vertices from both V_1 and V_2: Any cycle involving both V_1 and V_2 will have a minimum length of 4. This is because you need at least two vertices from V_1 and two vertices from V_2 (or vice versa) to form a closed path.

Hence, the graph form cycles of minimum length 4, and no shorter cycle exists among these points in the General Book Graph $B(p, r)$. □

Theorem 3.3. *The clique number of the Stair graph is 2.*

Proof. Let us consider a Stair graph $S(f)$.

Part 1. Proof that Clique Number is at least 2:

Let V_1 be the set of vertices corresponding to the first floor in the stair graph $S(f)$. Since $f \geq 1$, we will consider stair graph when the number of floors is smallest: the set of vertices $\{v_{1,1}, v_{1,2}\}$ from V_1. These vertices form a complete subgraph (clique) because, according to the definition of the stair graph, there is an edge between $v_{1,1}$ and $v_{1,2}$.

Additionally, consider the set $\{v_{1,1}, v_{2,1}\}$. These vertices also form a complete subgraph because, as per the definition, there is an edge between $v_{1,1}$ and $v_{2,1}$.

Therefore, there exist cliques of size 2 in the stair graph $S(f)$.

Part 2. Proof that Clique Number is at most 2:

Suppose there exists a complete subgraph of size greater than 2 in $S(f)$.

This would imply the existence of three vertices v_a, v_b, v_c forming a complete subgraph.

<u>**Case 1:**</u> If all three vertices are on the same floor V_k, then they must form a complete graph (from E_k), which is not possible since the stair graph has a step-wise structure, and there are no edges connecting non-adjacent vertices on the same floor.

<u>**Case 2:**</u> If all three vertices are in different floors. Let v_a be on floor V_i, v_b on floor V_j, and v_c on floor V_l, where $i < j < l$. Since v_a and v_b are on different floors, if there is an edge between them, then the edge must be in $E'i$, $E'i + 1$, ..., $E'j - 1$. Similarly, the edge between v_b and v_c must be in $E'j$, $E'j + 1$, ..., $E'l - 1$.

However, if v_a and v_b, v_b and v_c, which are in different floors, are all adjacent, then they must be straight to be a line. Besides, since v_a be on floor V_i, v_c on floor V_l, and $i < j < l$, v_a and v_c are not in the consecutive floors. And based on stair graph picture, we do not have diagonal line to connect two vertices. Then there is no way that v_a and v_c, two vertices in non-consecutive floors can be directly connected. Therefore, v_a, v_b, and v_c cannot form a complete graph.

Therefore, the assumption that there exists a complete subgraph of size greater than 2 leads to a contradiction.

Conclusion: Combining both arguments, the size of the largest complete subgraph in the stair graph $S(f)$ is exactly 2. Therefore, the clique number of $S(f)$ is 2.

□

Theorem 3.4. *The smallest cycle of the general stair graph is 4.*

Proof. By definition, a 4-cycle in a graph is a closed path of length 4, such that no cycle exists.

Consider the General Stair Graph $S(f)$. The vertices are divided into floors, denoted by $V = \sum_{k=1}^{f} V_k \cup V_0$ where: $V_k = \{v_{k,i} : 1 \leq i \leq k+1\}$ and $V_0 = \{v_{f+1,i} : 1 \leq i \leq f+1\}$.

Consider the possible cycles in the general stair graph:

Case 1: Cycles involving only vertices on the same floor. Each floor is formed by square blocks attached horizontally. To create a closed cycle or a block, we need to include vertices from at least two different consecutive lines. This leads to a 4-cycle as we traverse from one line to the next and back. Hence, the smallest cycle within a single floor is a square block, with a length of 4.

Case 2: The consecutive floors in the stair graph are connected from one edge-by-one edge, implying an increase in the number of edges between them. When we attempt to form a cycle by connecting vertices from non-consecutive floors would result in a larger cycle, given the interconnected nature of consecutive floors. Therefore, the smallest cycle is achieved by connecting vertices within consecutive floors, and this results in a 4-cycle.

In summary, the smallest cycle in the general stair graph is 4. The proof considers both cases of cycles within the same floor and cycles involving vertices on different floors, demonstrating that any cycle smaller than 4 is not possible in the given graph. □

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