Grace Chang

Multidimensional Optimizer Math

Note, this is under the function

$$y = -b\mathbf{X} + \frac{1}{2}\mathbf{X}'\mathbf{A}\mathbf{X}$$

Gradient

eg)

$b = \begin{bmatrix} 5 \\ -10 \end{bmatrix}$ and $A1 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ and $A2 = \begin{bmatrix} 1 & -2 \\ -2 & 20 \end{bmatrix}$. x is just a 1×2 matrix of whatever I want.

Then

$$\begin{aligned}
f(x_1, x_2) &= -\vec{b} \cdot \vec{x} + \frac{1}{2}\vec{x}^{\top} A \vec{x} \\
&= -b_1 x_1 - b_2 x_2 + \frac{1}{2}\begin{bmatrix} x_1 & x_2 \end{bmatrix} A \begin{bmatrix} x_1 \\ x_0 \end{bmatrix} \\
&= -b_1 x_1 - b_2 x_2 + \frac{1}{2}\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
&= -b_1 x_1 - b_2 x_2 + \frac{1}{2}\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{bmatrix} \\
&= -b_1 x_1 - b_2 x_2 + \frac{1}{2}\begin{bmatrix} a_{11}x_1^2 + a_{12}x_1x_2 + a_{21}x_1x_2 + a_{22}x_2^2 \end{bmatrix}
\end{aligned}$$

Thus we get the gradient

$$\nabla f(x_1, x_2) = \left(-b_1 + \frac{1}{2}(2a_{11}x_1 + a_{12}x_2 + a_{21}x_2), -b_2 + \frac{1}{2}(a_{12}x_1 + a_{21}x_1 + 2a_{22}x_2)\right)$$

Hessian Calculations

$$\begin{aligned}
H_{ij} &= \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2}(2a_{11}) & \frac{1}{2}(a_{12} + a_{21}) \\ \frac{1}{2}(a_{12} + a_{21}) & \frac{1}{2}(2a_{22}) \end{bmatrix} \\
&= \begin{bmatrix} a_{11} & \frac{1}{2}(a_{12} + a_{21}) \\ \frac{1}{2}(a_{12} + a_{21}) & a_{22} \end{bmatrix}
\end{aligned}$$

Because

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Thus

$$H^{-1} = \frac{1}{a_{11}a_{22} - \frac{1}{4}\left(a_{12} + a_{21}\right)^2} \begin{bmatrix} a_{22} & -\frac{1}{2}\left(a_{12} + a_{21}\right) \\ -\frac{1}{2}\left(a_{12} + a_{21}\right) & a_{11} \end{bmatrix}$$

Nonlinear Least Squares Optimizer

Using this exponential decay function:

$$f\left(x, a, k\right) = a \cdot \exp\left(-k \cdot x\right)$$

Thus

$$\begin{aligned} \frac{\partial f\left(x\left(i\right)\right)}{\partial a} &= \exp\left(-k \cdot x\right) \\ \frac{\partial f\left(x\left(i\right)\right)}{\partial k} &= -ax\exp\left(-k \cdot x\right) \end{aligned}$$

Thus the gradient components are:

$$\begin{aligned} G\left(1\right) &= sum\left\{\left[a\exp\left(-k \cdot x_i\right) - y_i\right]\left[\exp\left(-k \cdot x\right)\right]\right\} \\ G(2) &= sum\left\{\left[a\exp\left(-k \cdot x_i\right) - y_i\right]\left[-ax\exp\left(-k \cdot x\right)\right]\right\} \end{aligned}$$

For the Hessian:

$$\begin{aligned} H(1,1) &= sum\left\{\left[\exp\left(-k \cdot x\right)\right]^2\right\} \\ H(2,2) &= sum\left\{\left(-a \cdot x_i \cdot \exp\left(-k \cdot x_i\right)\right)^2\right\} \\ H\left(1,2\right) = H(2,1) &= sum\left\{\exp\left(-k \cdot x_i\right)\left(-a \cdot x_i\exp\left(-k \cdot x_i\right)\right)\right\} \end{aligned}$$

Next State

$$\begin{bmatrix} a_{i+1} \\ k_{i+1} \end{bmatrix} = \begin{bmatrix} a_i \\ k_i \end{bmatrix} - \left[H + \lambda \cdot DiagH\right]^{-1} G$$