Introduction to the Vlasov-Poisson system

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1 The Vlasov equation

Consider a particle with mass $m > 0$. Let $x(t) \in \mathbb{R}^3$ denote the position of the particle at time $t \in \mathbb{R}$ and $v(t) = \dot{x}(t) = dx(t)/dt$ its velocity. Let $F(t, x(t), v(t))$ be the force acting on the particle at time t. The Newton equations for the particle trajectory are given by

$$\dot{x}(t) = v(t), \qquad \dot{v}(t) = m^{-1}F(t, x(t), v(t)). \tag{1}$$

Now suppose that instead of a single particle we have a collection of a large number N of particles ($N \sim 10^{10}$); the trajectory of each particle satisfies (1), where F is the total force exerted by the remaining particles (we assume that there are no external forces acting on the system). For systems with such a huge number of degrees of freedom the motion of the individual particles is not the physically interesting quantity, since it is impossible to measure it by experiments. Kinetic theory provides a more practical description of the particle system, which is encoded in the *one particle distribution function* $f(t, x, v)$. Given small cube C in phase space centered on (x, v), we may think of the quantity $f(t, x, v)\mathrm{Vol}(C)$ as an approximation for $n(t, x, v)/N$, where $n(t, x, v)$ is the number of particles at time t whose physical state is represented by a point in C. Mathematically $f(t, x, v)$ is a probability measure density; the integral

$$\int_U \int_V f(t, x, v)\, dv\, dx$$

is interpreted as the probability to find a particle at time t in the region $U \times V$ of phase-space. It is clear that f has to satisfy $f \geq 0$ and

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f(t, x, v)\, dv\, dx = 1. \tag{2}$$

In the kinetic description, the 6N ordinary differential equations (1) for the particle system are replaced by a single partial differential equation (PDE) on the distribution function f. To justify the form of this PDE, assume that f is smooth and observe that along each individual trajectory the one particle distribution function must remain constant:

$$\frac{d}{dt} f(t, x(t), v(t)) = 0,$$

where $x(t), v(t)$ is a solution of (1). Provided f is sufficiently regular, the previous equation leads to the following PDE on f:

$$\partial_t f + v \cdot \nabla_x f + \frac{F(t, x, v)}{m} \cdot \nabla_v f = 0, \tag{3}$$

which is known as *Vlasov equation.* Thus the Vlasov equation is a transport PDE on phase-space, whose characteristic system coincide with the Newton equations (1). Since it is an hyperbolic equation of first order in time, (3) must be supplemented with an initial datum $f_{\rm in}(x,v) = f(0,x,v)$, which we assume to satisfy $f_{\rm in} \geq 0$ and the normalization condition (2), i.e.,

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_{\rm in}(x,v)\,dv\,dx = 1. \tag{4}$$

In order for the latter condition to be preserved by the evolution we need to impose a restriction on F. Before getting into this, let us introduce some standard notation and prove a simple lemma. Given $x, v \in \mathbb{R}^3$, and $0 \leq s \leq t$, we denote by $X(s;t,x,v), V(s;t,x,v)$ the solution of the characteristic system (1) that satisfies

$$X(t;t,x,v) = x, \qquad V(t;t,x,v) = v.$$

It is called *backward characteristic.* Moreover we denote $z = (x,v)$ and $Z(s;t,z) = (X,V)(s;t,x,v)$.

Lemma 1. *Let $F \in C^1(\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3)$ such that $\sup\{|F|, (t,x,v) \in [-T,T]\times\mathbb{R}^3\times\mathbb{R}^3\} < \infty$, for all $T > 0$. Then $Z \in C^1(\mathbb{R}\times\mathbb{R}\times\mathbb{R}^6)$ and for all $0 \leq s \leq t$ fixed, the map*

$$Z(s;t,\cdot) : \mathbb{R}^6 \to \mathbb{R}^6$$

is a C^1 diffeomorphism with inverse

$$Z^{-1}(s;t,\cdot) = Z(t;s,\cdot). \tag{5}$$

Moreover

$$\det\left[\nabla_z Z(s;t,z)\right] = 1, \text{ for all } 0 \leq s \leq t \Longleftrightarrow \nabla_v \cdot F = 0. \tag{6}$$

Proof. The assertion about the regularity of Z follows from standard ODEs theory. The fact that the inverse of $Z(s;t\cdot)$ is given by (5), i.e.,

$$Z(r;t,Z(t,s,z)) = Z(r;s,z),$$

is also obvious, since both members of the previous equation equal z at the time $r = t$. The claim "$\det\left[\nabla_z Z(s;t,z)\right] = 1$, for all $0 \leq s \leq t$" is equivalent to

$$\frac{d}{ds}\det\left[\nabla_z Z(s;t,z)\right] = 0, \quad \text{for all } 0 \leq s \leq t, \tag{7}$$

for (7) and $Z(t;t,s,z) = 1$ imply $\det\left[\nabla_z Z(s;t,z)\right] = 1$. By the well-known formula from Matrix Calculus,

$$\frac{d\det A}{dt} = \det A \sum_{i,j}(A^{-1})_{ij}\frac{d}{dt}A_{ij}$$

for all time dependent square matrix $A(t) = (A_{ij}(t))$, the solution $Z(s;t,z)$ of the system

$$\dot{z} = G(t,z)$$

satisfies

$$\frac{d}{ds}\det\left[\nabla_z Z(s;t,z)\right] = \nabla_z \cdot G(s, Z(s;t,z))\det\left[\nabla_z Z(s;t,z)\right].$$

In our case, $G = (v,F)$, whence

$$\frac{d}{ds}\det\left[\nabla_z Z(s;t,x,v)\right] = m^{-1}\nabla_v \cdot F(s, Z(s;t,x,v))\det\left[\nabla_z Z(s;t,x,v)\right]$$

and the claim follows. $\square$

Theorem 1. *Let F satisfy the assumptions of Lemma 6 and let $0 \leq f_{\rm in} \in C^1(\mathbb{R}^6)$. There exists a unique global classical solution of the Vlasov equation* (3) *such that $f(0,x,v) = f_{\rm in}(x,v)$ and is given by*

$$f(t,x,v) = f_{\rm in}(X(0;t,x,v), V(0;t,x,v)). \tag{8}$$

In particular, $f_{\rm in} \geq 0 \Rightarrow f \geq 0$, for all (t,x,v). Moreover if (and only if) $\nabla_v \cdot F = 0$ we have

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} C(f)\, dv\, dx = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} C(f_{\rm in})\, dv\, dx, \quad \textit{for all } t \in \mathbb{R}, \tag{9}$$

for all measurable functions $C : [0,\infty) \to \mathbb{R}$. In particular, if $f_{\rm in}$ satisfies (4)*, then the solution $f(t,x,v)$ satisfies* (2)*.*

Proof. For any solution $(X,V)(s)$ of the characteristic system (1) and for all $s,\tau \in \mathbb{R}$ we have

$$f(s, X(s), V(s)) = f(\tau, X(\tau), V(\tau)).$$

Setting $(X,V)(s) = (X,V)(s;t,x,v)$, $s = t$ and $\tau = 0$ yields the formula (8). To prove (9) we write

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} C(f(t,x,v))\, dv\, dx = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} C(f_{\rm in}(X(0;t,x,v), V(0;t,x,v)))\, dv\, dx = \int_{\mathbb{R}^6} f_{\rm in}(Z(0;t,z))\, dz$$

and the claim follows by making the change of variable $z \to Z(0;t,z)$ and using Lemma 6. □

EXERCISE 1: Write down the solution of the Vlasov equation for $F = 0$ (free particles) and $F = F(x) = -kx, x \in \mathbb{R}^3$, $k > 0$ constant.

2 The gravitational and the electric force

Let us consider a physical system composed by two particles P and p. Let $Q \in \mathbb{R}, M > 0$ be the charge and the mass of P and $q \in \mathbb{R}, m > 0$ those of p. Let us fix an orthogonal system of coordinates centered on the position of P and denote by $x = (x_1, x_2, x_3)$ the position of p in these coordinates. Coulomb's law states that the particle P exerts on the particle p a force F_e given by

$$F_e(x) = \frac{qQ}{4\pi\epsilon}\frac{x}{r^3}, \quad r = |x|,$$

where ϵ is a constant, the *permittivity*, which depends on the medium in which the particles are moving (it is assumed that the medium is homogeneous, i.e., its physical properties are the same everywhere in space). In vacuum we have

$$\epsilon \simeq \epsilon_0 = 8,85 \cdot 10^{-12} \text{ (farads per meter)}.$$

Note that the particle P attracts to itself the particle p if $Qq < 0$, i.e., if the charges of the two particles have opposite sign, and conversely P repels p when the two charges have the same sign. By the action-reaction principle, the force exerted by p on P is $-F_e$. At the same fashion, Newton gravitational law states that P exerts on p a gravitational force given by

$$F_g(x) = -\frac{mM}{4\pi G}\frac{x}{r^3}, \quad r = |x|,$$

where G is the gravitational constant, which is given by

$$G \simeq 6,67 \cdot 10^{-11} \ (m^3 kg^{-1} s^{-2}).$$

An important observation is that, as opposed to the electric force F_e, the gravitational force F_g is independent of the medium in which the particles are moving. Moreover the gravitational force is always attractive, since the mass is a positive quantity.

Both the electric and the gravitational forces are *conservative*. This follows by the fact that they are both the gradient of a potential, namely

$$F_e = -q\nabla V_e, \quad F_g = -m\nabla V_g,$$

where

$$V_e = \frac{Q}{4\pi\epsilon}\frac{1}{|x|}, \quad V_g = -\frac{M}{4\pi G}\frac{1}{|x|} \tag{10}$$

are respectively the electric and gravitational potential generated by the particle P. Now, let us recall that the function $(4\pi|x|)^{-1}$ is the fundamental solution of the Poisson equation in three dimensions, that is

$$\Delta\left(\frac{1}{4\pi|x|}\right) = -\delta(0) \quad \text{in } \mathcal{D}'(\mathbb{R}^3). \tag{11}$$

EXERCISE 2: Prove (11).

This observation suggests a natural way to define the electric and gravitational potential induced by a general matter distribution. In fact, the potentials generated by the particle P are distributional solutions of

$$\Delta V_e = -\frac{1}{\epsilon}Q\delta(0), \quad \Delta V_g = \frac{1}{G}M\delta(0). \tag{12}$$

But $Q\delta(0)$, resp. $M\delta(0)$, corresponds to a charge (resp. mass) Q (resp. M) concentrated on the origin, i.e., on the position of P. Now instead of the charge (resp. mass) Q (resp. M) being concentrated on a single particle, assume that it is spread over the whole space with density $\rho_Q(x)$ (resp. $\rho_M(x)$), where

$$\rho_Q \in \mathbb{R}, \quad \int_{\mathbb{R}^3} \rho_Q(x)\,dx = Q; \qquad \rho_M \geq 0, \quad \int_{\mathbb{R}^3} \rho_M(x)\,dx = M.$$

In analogy to (12) we now postulate that the electric/gravitational potentials are solutions of the Poisson equations

$$\Delta V_e = -\frac{1}{\epsilon}\rho_Q, \quad \Delta V_g = \frac{1}{G}\rho_M. \tag{13}$$

The previous equations are valid for $x \in \mathbb{R}^3$ and are complemented with the boundary condition at infinity

$$\lim_{|x|\to\infty} V_e(x) = \lim_{|x|\to\infty} V_g(x) = 0. \tag{14}$$

Up to regularity issues, the solutions of (13) satisfying (14) are given by

$$V_e(x) = \frac{1}{4\pi\epsilon}\int_{\mathbb{R}^3} \frac{\rho_Q(y)}{|x-y|}\,dy, \quad V_g(x) = -\frac{1}{4\pi G}\int_{\mathbb{R}^3} \frac{\rho_M(y)}{|x-y|}\,dy.$$

For the forces exerted by the matter distribution on a particle with charge q and mass m located at $x \in \mathbb{R}^3$ we obtain

$$F_e = -q\nabla V_e = \frac{q}{4\pi\epsilon}\int_{\mathbb{R}^3} \rho_Q(y)\frac{x-y}{|x-y|^3}\,dy, \quad F_g = -m\nabla V_g = -\frac{m}{4\pi G}\int_{\mathbb{R}^3} \rho_M(y)\frac{x-y}{|x-y|^3}\,dy \tag{15}$$

3 The Vlasov-Poisson system

In Section 1 we introduced the Vlasov equation for a system of N particles interacting through a force field F. In Section 2 we have seen the important examples of gravitational and electrical interaction, which for a continuum distribution of matter led us to the Poisson equations (13) for the potentials. In this section we combine the two arguments to derive the Vlasov-Poisson system. For this purpose, we define the charge and mass distribution of a system of identical particles with kinetic distribution f respectively as

$$\rho_Q(t,x) = Q\int_{\mathbb{R}^3} f(t,x,v)\,dv \qquad \rho_M(t,x) = M\int_{\mathbb{R}^3} f(t,x,v)\,dv. \tag{16}$$

Note that by (2),

$$\int_{\mathbb{R}^3} \rho_Q(t,x)\,dx = Q, \qquad \int_{\mathbb{R}^3} \rho_M(t,x) = M.$$

Thus by (13) the electric and gravitational potential generated by the particle system are solutions of

$$\Delta V_e = -\frac{Q}{\epsilon}\int_{\mathbb{R}^3} f\,dv, \qquad \lim_{|x|\to\infty} V_e(x) = 0, \tag{17}$$

$$\Delta V_g = \frac{M}{G}\int_{\mathbb{R}^3} f\,dv, \qquad \lim_{|x|\to\infty} V_g(x) = 0. \tag{18}$$

Let $q = Q/N$, $m = M/N$ be the charge and the mass of each individual particle. Replacing (15) into (3), the Vlasov equation becomes

$$\partial_t f + v\cdot\nabla_x f - \frac{q}{m}\nabla V_e\cdot\nabla_v f = 0, \tag{19}$$

in the case of electric interaction and

$$\partial_t f + v\cdot\nabla_x f - \nabla V_g\cdot\nabla_v f = 0, \tag{20}$$

in the case of gravitational interaction between the particles.

Definition 1. *The system of equations* (17) *and* (19) *is called Vlasov-Poisson system in the plasma physics case, while* (18) *and* (20) *is called Vlasov-Poisson system in the stellar dynamics case.*

The names derive from the fact that (17)-(19) is a widely used model in plasma physics, while (18)-(20) describes the dynamics of the stars in a galaxy[1].

EXERCISE 3: Prove that upon a suitable rescaling

$$\tilde{f}(t,x,v) = \alpha f(at,bx,cv), \quad \tilde{V}(t,x) = \beta V_{e,g}(at,bx),,$$

the pair $(\tilde{f},\tilde{V})$ solves

$$\partial_t\tilde{f} + v\cdot\nabla_x\tilde{f} - \nabla\tilde{V}\cdot\nabla_v\tilde{f} = 0, \tag{21}$$

$$\Delta\tilde{V} = \gamma\int_{\mathbb{R}^3}\tilde{f}\,dv. \tag{22}$$

[1]For the applications, it is appropriate to consider the Vlasov-Poisson system for several species of particles with different charge and mass. Since this generalization does not introduce any new conceptual difficulty for the mathematical problems studied in the sequel, we stick to the case of a single species of particle with mass $m > 0$ and charge $q \in \mathbb{R}$.

where $\gamma = -1$ in the plasma physics case and $\gamma = 1$ in the stellar dynamics case.

The meaning of the previous exercise is that, by a suitable rescaling of the solution (which physically corresponds to choose a set of units for space, time, velocity, mass and charge), all the physical constants in the two Vlasov-Poisson systems can be set to unity. From now we shall work with the system (21) for notational simplicity. Note that $\tilde{f}$ is not subject to the normalization condition (2), i.e., in the new units the solution of the Vlasov-Poisson system (21) is not normalized to a probability measure density. Said this, we now remove the tilde for notational convenience and write the Vlasov-Poisson system in the following final form:

$$\partial_t f + v \cdot \nabla_x f - \nabla V \cdot \nabla_v f = 0, \tag{23a}$$

$$\Delta V = \gamma \int_{\mathbb{R}^3} f \, dv, \qquad \lim_{|x|\to\infty} V(x) = 0, \qquad \gamma = \pm 1. \tag{23b}$$

EXERCISE 4: Prove that (23) is invariant by Galilean transformations (look in the literature for the precise form of the Galilean transformations in three dimensions).

We remark that a solution of (23) consists of the function f only, for as we have seen at the end of Section (2), the solution of the Poisson equation with vanishing boundary condition at infinity is determined by f through the formula

$$V(t,x) = -\frac{\gamma}{4\pi} \int_{\mathbb{R}^3} \left(\int_{\mathbb{R}^3} f \, dv \right) \frac{dy}{|x-y|}. \tag{24}$$

Theorem 2. *Let $0 \le f_{\rm in} \in C_c^1(\mathbb{R}^3 \times \mathbb{R}^3)$. There exists a unique $f \in C^1(\mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3)$ that solves* (23) *and such that $f(0,x,v) = f^{\rm in}(x,v)$. Moreover the induced potential, given by* (24)*, satisfies $V \in C^1(\mathbb{R} \times \mathbb{R}^3)$.*

The second part of the course is devoted to prove the previous theorem.

EXERCISE 5: Prove that for sufficiently regular solutions of the Vlasov-Poisson system, the total energy $\mathcal{E}$ defined as

$$\mathcal{E} = \frac{1}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |v|^2 f(t,x,v) \, dv \, dx - \frac{\gamma}{2} \int_{\mathbb{R}^3} |\nabla V(t,x)|^2 dx \tag{25}$$

is conserved.