Dispersion Relation of Kappa Velocity Distribution Function

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Compute the dielectric function for the Kappa Distribution Function, $\kappa = 1$
Print the results for the Kappa and Kappa Bump-on-Tail as .m Files

Kappa Distribution Function :

$$f_\kappa\ (v;\ \mu_i,\ \Theta_i)\ =\ \frac{1}{\sqrt{\pi \Theta_i^2\ (\kappa - 1\ /\ 2)}}\ \frac{\Gamma\ (\kappa + 1)}{\Gamma\ (\kappa + 1\ /\ 2)}\ \left[1 + \frac{(v - \mu_i)^2}{\Theta_i^2\ (\kappa - 1\ /\ 2)}\right]^{-\kappa - 1}$$

$$f'_\kappa\ (v;\ \mu_i,\ \Theta_i)\ =\ \frac{2\ (v - \mu_i)\ \ (-\kappa - 1)}{\sqrt{\pi}\ \Theta_i^3\ \ (\kappa - 1\ /\ 2)^{3/2}}\ \frac{\Gamma\ (\kappa + 1)}{\Gamma\ (\kappa + 1\ /\ 2)}\ \left[1 + \frac{(v - \mu_i)^2}{\Theta_i^2\ (\kappa - 1\ /\ 2)}\right]^{-\kappa - 2}$$

Bump on − Tail Formula :

$$1 - \frac{\beta}{k^2}\ \int_{-\infty}^{\infty} \frac{f'_\kappa\ (v;\ \mu_1,\ \Theta_1)}{v - \omega\ /\ k}\ d\,v - \frac{1 - \beta}{k^2}\ \int_{-\infty}^{\infty} \frac{f'_\kappa\ (v;\ \mu_2,\ \Theta_2)}{v - \omega\ /\ k}\ d\,v = 0,\ \mathrm{Im}\ (\omega)\ > 0$$

```
κ = 6;
θ = .;
μ = .;
β =.; f =.; g =.; (* Clear variables *)
f[v_,μ_,θ_]:= (Pi*θ^2*(κ-1/2))^(-1/2)*Gamma[κ+1]/Gamma[κ+1/2]*(1+(v-μ)^2/((κ-1/2)*θ^2))^(-κ-1
Df[v_,μ_,θ_]:= D[f[v,μ,θ],v]; (* Differentiate *)
ep[k_,ω_,β_,μ_,θ_]:=β/k^2*Integrate[Df[v,μ,θ]/(v-ω/k),{v,-Infinity,Infinity},Assumptions→Elemen
epR1[k_,ω_,μ_,θ_]:= Residue[Df[v,μ,θ]/(v-ω/k),{v,ω/k}]; (*principle root*)
epR2[k_,ω_,μ_,θ_]:= Residue[Df[v,μ,θ]/(v-ω/k),{v,μ+I*Sqrt[(κ-1/2)*θ^2]}]; (*top pole*)
epResidue[k_,ω_,β_,μ_,θ_]:= FullSimplify[ComplexExpand[β/k^2*(epR1[k,ω,μ,θ]+epR2[k,ω,μ,θ])*2*Pi

ep[k,ω,β,μ,θ] (* Print out result from integration *)
epResidue[k,ω,β,μ,θ] (* Print out result from Residue Theorem *)
Simplify[ComplexExpand[ep[k,ω,β,μ,θ]-epResidue[k,ω,β,μ,θ]],Assumptions→Element[θ,Reals]&&θ>0] (
(*FullSimplify[1-epResidue[k,ω,β,μ1,θ1]-epResidue[k,ω,1-β,μ2,θ2]]*)
```

Out[∘]=

$$\frac{1}{3\,k\,\left(\sqrt{22}\,k\,\theta-2\,i\,k\,\mu+2\,i\,\omega\right)^{8}}16\,i\,\sqrt{\frac{2}{11}}\,\beta\,\theta^{13}\left(\frac{232\,202\,043\,392\,k^{14}\,(k\,\mu-\omega)}{\left(\sqrt{22}\,k\,\theta+2\,i\,k\,\mu-2\,i\,\omega\right)^{8}}+\frac{1}{\theta^{13}}\,k\,\left(k^{6}\left(51\,909\,i\,\sqrt{22}\,\theta^{6}+334\,928\,\theta^{5}\,\mu-45\,386\,i\,\sqrt{22}\,\theta^{4}\,\mu^{2}-78\,144\,\theta^{3}\,\mu^{3}+3660\,i\,\sqrt{22}\,\theta^{2}\,\mu^{4}+2112\,\theta\,\mu^{5}-24\,i\,\sqrt{22}\,\mu^{6}\right)-4\,k^{5}\left(83\,732\,\theta^{5}-22\,693\,i\,\sqrt{22}\,\theta^{4}\,\mu-58\,608\,\theta^{3}\,\mu^{2}+3660\,i\,\sqrt{22}\,\theta^{2}\,\mu^{3}+2640\,\theta\,\mu^{4}-36\,i\,\sqrt{22}\,\mu^{5}\right)\omega+2\,k^{4}\left(-22\,693\,i\,\sqrt{22}\,\theta^{4}-117\,216\,\theta^{3}\,\mu+10\,980\,i\,\sqrt{22}\,\theta^{2}\,\mu^{2}+10\,560\,\theta\,\mu^{3}-180\,i\,\sqrt{22}\,\mu^{4}\right)\omega^{2}+48\,k^{3}\left(1628\,\theta^{3}-305\,i\,\sqrt{22}\,\theta^{2}\,\mu-440\,\theta\,\mu^{2}+10\,i\,\sqrt{22}\,\mu^{3}\right)\omega^{3}+60\,k^{2}\left(61\,i\,\sqrt{22}\,\theta^{2}+176\,\theta\,\mu-6\,i\,\sqrt{22}\,\mu^{2}\right)\omega^{4}-48\,k\,\left(44\,\theta-3\,i\,\sqrt{22}\,\mu\right)\omega^{5}-24\,i\,\sqrt{22}\,\omega^{6}\right)\right)$$

Out[∘]=

$$-\frac{1}{3\left(k^{2}\left(11\,\theta^{2}+2\,\mu^{2}\right)-4\,k\,\mu\,\omega+2\,\omega^{2}\right)^{8}}2\,\beta\left(759\,999\,669\,k^{14}\,\theta^{14}-329\,832\,448\,i\,\sqrt{22}\,k^{14}\,\theta^{12}\,\sqrt{\theta^{2}}\,\mu+329\,832\,448\,i\,\sqrt{22}\,k^{13}\,\theta^{12}\,\sqrt{\theta^{2}}\,\omega-967\,272\,306\,k^{12}\,\theta^{12}\,(-k\,\mu+\omega)^{2}-175\,867\,692\,k^{10}\,\theta^{10}\,(-k\,\mu+\omega)^{4}-31\,975\,944\,k^{8}\,\theta^{8}\,(-k\,\mu+\omega)^{6}-4\,152\,720\,k^{6}\,\theta^{6}\,(-k\,\mu+\omega)^{8}-352\,352\,k^{4}\,\theta^{4}\,(-k\,\mu+\omega)^{10}-17\,472\,k^{2}\,\theta^{2}\,(-k\,\mu+\omega)^{12}-384\,(-k\,\mu+\omega)^{14}\right)$$

Out[∘]=

0

In[ ]:=
```
SetDirectory[NotebookDirectory[]];
<< ToMatlab`
ToMatlab[1-epResidue[k,ω,β,μ1,θ1]-epResidue[k,ω,1-β,μ2,θ2]](* Write the main function in Matlab
g = OpenWrite["dielectricBoT_kappa6.m"];
WriteMatlab[1-epResidue[k,omega,beta,mu1,theta1]-epResidue[k,omega,1-beta,mu2,theta2], g, y]; (*
Close[g];
```

Out[ ]=
```
1+(2/3).*β.*(k.^2.*(11.*θ1.^2+2.*μ1.^2)+(-4).*k.*μ1.*ω+2.*ω.^2).^( ...
  -8).*(759999669.*k.^14.*θ1.^14+(sqrt(-1)*(-329832448)).*22.^(1/2) ...
  .*k.^14.*θ1.^12.*(θ1.^2).^(1/2).*μ1+(sqrt(-1)*329832448).*22.^( ...
  1/2).*k.^13.*θ1.^12.*(θ1.^2).^(1/2).*ω+(-967272306).*k.^12.* ...
  θ1.^12.*((-1).*k.*μ1+ω).^2+(-175867692).*k.^10.*θ1.^10.*((-1).*k.* ...
  μ1+ω).^4+(-31975944).*k.^8.*θ1.^8.*((-1).*k.*μ1+ω).^6+(-4152720).* ...
  k.^6.*θ1.^6.*((-1).*k.*μ1+ω).^8+(-352352).*k.^4.*θ1.^4.*((-1).*k.* ...
  μ1+ω).^10+(-17472).*k.^2.*θ1.^2.*((-1).*k.*μ1+ω).^12+(-384).*((-1) ...
  .*k.*μ1+ω).^14)+(-2/3).*((-1)+β).*(k.^2.*(11.*θ2.^2+2.*μ2.^2)+(-4) ...
  .*k.*μ2.*ω+2.*ω.^2).^(-8).*(759999669.*k.^14.*θ2.^14+(sqrt(-1)*( ...
  -329832448)).*22.^(1/2).*k.^14.*θ2.^12.*(θ2.^2).^(1/2).*μ2+(sqrt( ...
  -1)*329832448).*22.^(1/2).*k.^13.*θ2.^12.*(θ2.^2).^(1/2).*ω+( ...
  -967272306).*k.^12.*θ2.^12.*((-1).*k.*μ2+ω).^2+(-175867692).* ...
  k.^10.*θ2.^10.*((-1).*k.*μ2+ω).^4+(-31975944).*k.^8.*θ2.^8.*((-1) ...
  .*k.*μ2+ω).^6+(-4152720).*k.^6.*θ2.^6.*((-1).*k.*μ2+ω).^8+( ...
  -352352).*k.^4.*θ2.^4.*((-1).*k.*μ2+ω).^10+(-17472).*k.^2.*θ2.^2.* ...
  ((-1).*k.*μ2+ω).^12+(-384).*((-1).*k.*μ2+ω).^14);
```

Solve the Dispersion Relation for the Kappa Distribution Function, $\kappa = 1$

In[ ]:=
```
(*Isol[k_]=Solve[1-ep[k,ω,1,0,1]==0,ω]*)
(*k=0.480;
Isol[k_]=Solve[1-ep[k,ω,0.9,0,1]-ep[k,ω,0.1,4,1]==0,ω];
(*T=Table[Isol[k][[i]],{i,1,12}];
Grid[T,Frame→All] This one works great, I just want it split into real/imaginary parts now*)
For[i=1, i≤12, i++, SIm[i]=ComplexExpand[ReIm[ω/.Isol[k][[i]]][[2]]];
For[i=1, i≤12, i++, SRe[i]=ComplexExpand[ReIm[ω/.Isol[k][[i]]][[1]]];
T = Table[{i,SRe[i],SIm[i]},{i,1,12}];
Grid[T,Frame→All]
Export["MathematicaData/k=0.480.csv", T, "CSV"];
Plot[Evaluate@Table[S[i],{i,1,12}],{k,0,1},PlotLabel→"Kappa Bump-on-Tail Solutions, k=0.325, ϰ=1


karray = {0.1,0.2,0.3};
MyTable = {};
For[j=1,j≤3,j++,
    k=karray[j];
    Isol[k_]=Solve[1-ep[k,ω,0.9,0,1]-ep[k,ω,0.1,4,1]==0,ω];
    For[i=1, i≤12, i++, SIm[i]=ComplexExpand[ReIm[ω/.Isol[k][[i]]][[2]]];
    For[i=1, i≤12, i++, SRe[i]=ComplexExpand[ReIm[ω/.Isol[k][[i]]][[1]]];
    AppendTo[MyTable,{j,SRe[j],SIm[j]}]
]
*)

(*Plot[{S[7],S[12]},{k,0,1},PlotLegends→{"Solution 7, γ = +0.1985i","Solution 12, γ = -0.0784i"
(*Plot[{S4[k],S5[k]},{k,0,1},PlotLegends→{"solution 4","solution 5"},AxesLabel→{k,γ},PlotLabel→
```