RESEARCH ARTICLE | MARCH 01 1960

Electrostatic Instabilities of a Uniform Non-Maxwellian Plasma

Oliver Penrose

Phys. Fluids 3, 258–265 (1960)
https://doi.org/10.1063/1.1706024

$$f^{\lambda} = \frac{1}{\pi\theta_1\theta_2} \exp\left[-\frac{(q_\lambda - a_\lambda)^2}{\theta_1^2} - \frac{(p_\lambda - b_\lambda)^2}{\theta_2^2}\right]. \quad (27)$$

The uncertainties in q_λ and p_λ are now θ_1 and θ_2, respectively. Substitution into Eq. (19) gives

$$S_\lambda = K(1 + \log \pi\theta_1\theta_2). \quad (28)$$

We see that S_λ increases as the uncertainties θ_1 and θ_2 increase. If θ_1 and θ_2 are allowed to approach zero, S_λ becomes negatively infinite as it should.

If we imagine a plasma in which initially there is no incoherent radiation present, then it follows from the constancy of the field entropy that no incoherent radiation can exist at a future time. Thus the Vlasov equations are inadequate for the treatment of problems involving the emission of incoherent radiation.

Note that incoherent radiation can not be obtained by adding terms of the Boltzmann collision integral or Fokker-Planck type to the right-hand side of Eq. (14). These terms do cause the particle distribution to approach an equilibrium. However, if Eq. (11) is unchanged, the proof that S_λ is constant goes through as before. What is needed are terms in Eq. (11) which would cause f^λ to approach an equilibrium. Such terms can be obtained from our generalization of the Rosenbluth and Rostoker expansion.[3] The derivation of these terms will be one of the principle objectives of a future paper.

THE PHYSICS OF FLUIDS VOLUME 3, NUMBER 2 MARCH-APRIL, 1960

Electrostatic Instabilities of a Uniform Non-Maxwellian Plasma

OLIVER PENROSE
Imperial College of Science and Technology, London S. W. 7, England
(Received October 9, 1959)

A stability criterion is obtained starting from Vlasov's collision-free kinetic equations. Possible instabilities propagating parallel to an arbitrary unit vector $\mathbf{e}$ are related to a function $F(u) \equiv \Sigma_j \omega_j^2 \int d^3\mathbf{v}\, g_j(\mathbf{v})\, \delta(\mathbf{e}\cdot\mathbf{v} - u)$, where $g_j(\mathbf{v})$ is the normalized unperturbed distribution function, and $\omega_j \equiv (4\pi n_j e_j^2/m_j)^{\frac{1}{2}}$ the plasma frequency, for the jth type of particle. By using a method related to the Nyquist criterion, it is shown that plasma oscillations growing exponentially with time are possible if and only if $F(u)$ has a minimum at a value $u = \xi$ such that $\int_{-\infty}^{\infty} du\,(u - \xi)^{-2}[F(u) - F(\xi)] > 0$. A study of the initial-value problem confirms that the plasma is normally stable if no exponentially growing modes exist; but there is an exceptional class of distribution functions (recognizable by means of an extension of the above criterion) for which linearized stability theory breaks down. The method is applied to several examples, of which the most important is a model of a current-carrying plasma with Maxwell distributions at different temperatures for electrons and ions. The meaning of the mathematical assumptions made is carefully discussed.

I. INTRODUCTION

COLLECTIVE electrostatic effects in an unbounded non-Maxwellian electron gas were first studied by Vlasov.[1] Generalized to a system containing several types of ion, Vlasov's fundamental equations are

$$\frac{\partial f_i}{\partial t} + \mathbf{v}\cdot\frac{\partial f_i}{\partial \mathbf{x}} - \frac{e_i}{m_i}\frac{\partial \phi}{\partial \mathbf{x}}\cdot\frac{\partial f_i}{\partial \mathbf{v}} = 0 \quad (1)$$

$$\nabla^2\phi = -\Sigma_i 4\pi e_i \int d^3\mathbf{v} f_i, \quad (2)$$

where $\mathbf{x}$ and $\mathbf{v}$ are position and velocity vectors, $\phi(\mathbf{x}, t)$ is the electrostatic potential, and e_i, m_i, and $f_i(\mathbf{x}, \mathbf{v}, t)$ are, respectively, the charge (esu), mass, and distribution function for the jth type of ion. Vlasov looked for solutions of the form

$$f_i(\mathbf{x}, \mathbf{v}, t) = n_i[g_i(\mathbf{v}) + h_i(\mathbf{v}) \exp(i\mathbf{k}\cdot\mathbf{x} - i\omega t)], \quad (3)$$

where $\mathbf{k}$, ω, and the n_i are constants, and the g_i are normalized to 1. By linearizing Eqs. (1) and (2) in h_i and ϕ, he showed that for a nontrivial solution, $\mathbf{k}$ and ω must be related by the dispersion formula[2]

$$k^2 = \Sigma_i \frac{4\pi n_i e_i^2}{m_i} \int \frac{\mathbf{k}\cdot\partial g_i/\partial\mathbf{v}}{\mathbf{k}\cdot\mathbf{v} - \omega} d^3\mathbf{v}. \quad (4)$$

In Eqs. (3) and (4), ω may be complex but the other quantities must be real; in particular, $\mathbf{k}$ must be real since otherwise $\exp(i\mathbf{k}\cdot\mathbf{x})$ would not be small enough to justify linearization throughout the unbounded plasma.

[1] A. Vlasov, Zhur. Eksp. i Teoret. Fiz. **8**, 291 (1938).

[2] See I. B. Bernstein, Phys. Rev. **109**, 10 (1958).

This dispersion formula can be used[3-5] to discuss the stability of a uniform non-Maxwellian plasma with velocity distribution functions $g_i(\mathbf{v})$. If Eq. (4) has a solution for which Im $\omega > 0$, so that $|\exp(i\omega t)| \to \infty$ as $t \to \infty$, then a small perturbation in the distribution function can grow exponentially until the linear approximation breaks down. Physically, this means that the uniform state described by the $g_i(\mathbf{v})$'s is unstable against the self-excitation of plasma oscillations. On the other hand, if (4) has no solution for which Im $\omega > 0$, no exponentially growing plasma oscillations are possible. This is normally taken to imply that the uniform state is stable, though it could conceivably be unstable, for example, against perturbations increasing linearly with time.

The purpose of this paper is to obtain a simple criterion of stability for a uniform non-Maxwellian plasma. The above discussion shows that two separate questions are involved: first, to obtain a method for discovering whether or not (4) has solutions with Im $\omega > 0$, and secondly, to verify that the plasma is stable when there are no such solutions. These questions will be discussed in Secs. II and III, respectively, and then the resulting stability criterion will be applied to some special cases.

To simplify the notation, we shall write (4) in the form

$$k^2 = Z(\omega/k), \tag{5}$$

where

$$Z(\zeta) \equiv \int_{-\infty}^{\infty} (u - \zeta)^{-1}(dF/du)\, du, \tag{6}$$

$$F(u) \equiv \Sigma_i (4\pi n_i e_i^2/m_i) G_i(u), \tag{7}$$

and

$$G_i(u) \equiv \int g_i(\mathbf{v})\, \delta(u - \mathbf{e}\cdot\mathbf{v})\, d^3\mathbf{v}. \tag{8}$$

Here ζ is a complex variable which can be interpreted as phase velocity, k means the length of the vector $\mathbf{k}$, $\mathbf{e}$ means the unit vector $\mathbf{k}/k$, and $F(u)$ is a weighted sum of the distribution functions for the components of velocity along $\mathbf{e}$. Note that $F(u)$ depends on $\mathbf{e}$ but not on k. If exponentially growing plasma waves can propagate along $\mathbf{e}$, then (5) has a solution with Im $\omega > 0$, and this implies, since $k > 0$, that the function $Z(\zeta)$ takes a real positive value somewhere in the upper half of the ζ plane. Conversely, if $Z(\zeta)$ takes a real positive value at some point ζ_0 in the upper half plane, then (4) can be satisfied by taking $k = \sqrt{Z(\zeta_0)}$ and $\omega = \zeta_0 k$, so that Im $\omega = k$ Im $\zeta_0 > 0$, and exponentially growing modes do exist. Thus, exponentially growing modes exist if and only if the function $Z(\zeta)$ takes a real positive value somewhere in the upper half of the ζ plane.

The analysis to be given in Secs. II and III will hold only for fairly smooth $G_i(u)$ with suitable behavior at infinity. In addition to the trivial consequences of (8),

$$G_i(u) \geq 0, \qquad \int_{-\infty}^{\infty} G_i(u)\, du = 1,$$

we shall require, for all j, that

$$\int_{-\infty}^{\infty} |G_j(u)|^2\, du < \infty,$$
$$\int_{-\infty}^{\infty} |G_{1j}(u)|^2\, du < \infty, \tag{9}$$
$$\int_{-\infty}^{\infty} |G''(u)|^2\, du < \infty,$$

where

$$G_{1j}(u) \equiv \int |\mathbf{e}\cdot\partial g_j/\partial\mathbf{v}|\, \delta(\mathbf{e}\cdot\mathbf{v} - u)\, d^3\mathbf{v} \geq |G_j'(u)|, \tag{10}$$

and also that $G_j''(u)$ is bounded.

II. STABILITY CRITERION

It was shown in the foregoing that exponentially growing modes exist if, and only if, the function

$$Z(\zeta) = \int_{-\infty}^{\infty} \frac{F'(u)\, du}{u - \zeta} = \int_{-\infty}^{\infty} \frac{F(u)\, du}{(u - \zeta)^2} \tag{11}$$

takes a real positive value somewhere in the upper half plane Im $\zeta > 0$. By (11), Z is an analytic function of ζ, regular in the ζ-plane cut along the real axis. Its behavior on the upper side of the cut is given[6] by

$$Z(\xi + i0) = \lim_{\eta\to 0} \int_{-\infty}^{\infty} du F'(u)\, \frac{u - \xi + i\eta}{(u - \xi)^2 + \eta^2}$$
$$= P\int_{-\infty}^{\infty} (u - \xi)^{-1} F'(u)\, du + i\pi F'(\xi), \tag{12}$$

where P denotes the Cauchy principal value.

[3] O. Buneman, Phys. Rev. Letters 1, 8 (1958).
[4] P. L. Auer, Phys. Rev. Letters 1, 411 (1958).
[5] F. D. Kahn, Astrophys. J. 129, 468 (1959).

[6] S. G. Mikhlin, *Integral Equations* (Pergamon Press, London, England, 1957), pp. 115–116. Theorems 1 and 3 of this reference show that $Z(\xi + i0)$ is bounded and continuous, since the boundedness of G_j'' imply a Lipschitz condition on $F'(u)$.

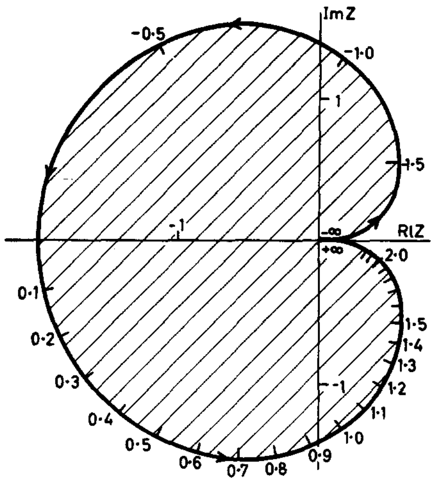

FIG. 1. The curve $Z(R)$ for the Maxwell distribution $F(u) = (\omega^2/\alpha\sqrt{\pi})\exp(-u^2/\alpha^2)$. The axes are marked in units of ω^2/α^2. Values of u/α are shown on the curve itself. The image of the upper half plane is shaded, and includes no positive real values.

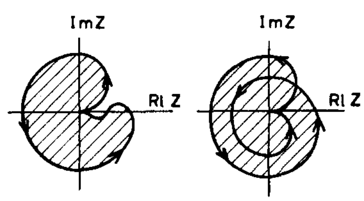

FIG. 2. Possible $Z(R)$ curves which do enclose positive real values. The image of the upper half plane is shaded.

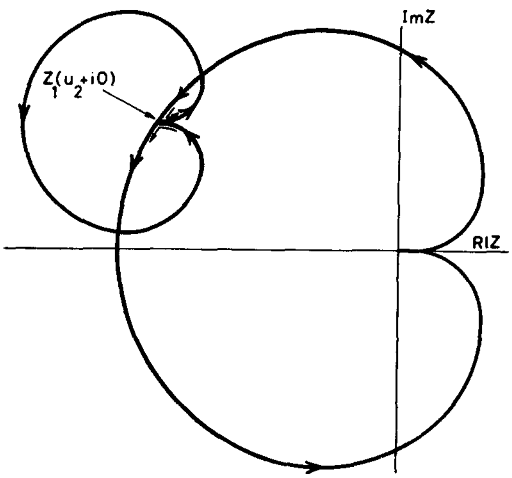

FIG. 3. Possible $Z(R)$ curve for a double Maxwellian distribution [Eq. (27)]. Both loops are the same shape as the curve of Fig. 1.

To exhibit the relevant properties of the function $Z(\xi + i0)$ we shall use certain diagrams in the Z plane. The function $Z(\zeta)$ assigns to each point in the ζ plane a unique 'image' in the Z plane. Suppose now that the point ζ moves along the upper side of the real axis from $-\infty$ to $+\infty$. Since $Z(\xi + i0)$ is bounded and continuous,[6] the image $Z(\zeta)$ will trace out a directed curve[7] (see Figs. 1–3) which we shall call the image of the real axis, or $Z(R)$. This curve starts and finishes at the origin $Z = 0$, since $Z(\infty) = 0$. Now, if Z_0 is any point not lying on $Z(R)$, then, by the argument principle,[8] the point $Z(\xi + i0)$ moving along $Z(R)$ winds round Z_0 anticlockwise the same number of times as $Z(\zeta)$ takes the value Z_0 in the upper half plane. Hence the image of any point in the upper half plane (of the ζ plane) must be either inside $Z(R)$ or upon it; therefore the image of the upper half plane is the interior of $Z(R)$. Consequently, $Z(\zeta)$ takes positive values somewhere in this half plane if and only if the curve $Z(R)$ encloses part of the positive real Z axis. This in turn can happen if and only if the curve $Z(R)$ crosses the positive real Z axis; moreover, since the point $Z(\xi + i0)$ moving along $Z(R)$ encloses the points inside in an anticlockwise sense, it must be moving upwards on its right-hand-most crossing of the positive real Z axis. Consequently, $Z(\zeta)$ takes positive values in the upper half plane if and only if there is a point where Re $Z(\xi + i0)$ is positive and Im $Z(\xi + i0)$ changes sign from $-$ to $+$. Now, by (12), such a sign change of Im $Z(\xi + i0) = \pi F'(\xi)$ corresponds to a minimum of $F(u)$. Also, Re $Z(\xi + i0)$ can, by (12), be expressed in the form[9]

$$\begin{aligned}\operatorname{Re} Z(\xi + i0) &= \lim_{\epsilon\to 0}\left\{\int_{-\infty}^{\xi-\epsilon} + \int_{\xi+\epsilon}^{\infty}\right\}\frac{d[F(u) - F(\xi)]}{u - \xi}\\ &= \lim\left\{\int + \int\right\}(u-\xi)^{-2}[F(u) - F(\xi)]\,du\\ &\quad + \lim\,[2F(\xi) - F(\xi-\epsilon) - F(\xi+\epsilon)]/\epsilon\\ &= P\int_{-\infty}^{\infty}(u-\xi)^{-2}[F(u) - F(\xi)]\,du. \qquad (13)\end{aligned}$$

If $F(u)$ has a minimum at $u = \xi$, the principal value in (13) is unnecessary.

Accordingly we may deduce the following criterion[10] of stability: *Exponentially growing modes exist if, and only if, there is a minimum of $F(u)$ at a value $u = \xi$ such that $\int_{-\infty}^{\infty}(u-\xi)^{-2}[F(u) - F(\xi)]\,du > 0$.* Auer[4] noted that a minimum of $F(u)$ was

[7] This curve is closely related to the Nyquist diagram of servomechanism theory, which has been applied for plasmas by E. G. Harris [Phys. Rev. Letters 2, 34 (1959)]. N. G. Van Kampen [Physica 21, 949 (1955)] has used a similar method.

[8] E. T. Copson, *Theory of Functions of a Complex Variable* (Oxford University Press, Oxford, England, 1935), p. 119.

[9] W. K. Hayman (private communication, 1959).

[10] S. Tamor (unpublished) and others have obtained similar criteria.

necessary for instability but did not give the other part of the criterion.

There is an ambiguity if $F(\xi)$ has a flat minimum occupying a finite range $\xi_1 < \xi < \xi_2$ during which $\mathrm{Im}\, Z(\xi + i0) = 0$ and $\mathrm{Re}\, Z(\xi + i0)$ changes sign. In this case, part of the curve $Z(R)$ lies along the real Z axis. A study of Fig. 4 shows that this part of $Z(R)$ encloses on its left a part of the positive real axis if and only if $\mathrm{Re}\, Z(\xi + i0)$ is positive throughout the range $\xi_1 < \xi < \xi_2$. Thus the criterion given above covers this case too if the condition $\int (u - \xi)^{-2} \cdot [F(u) - F(\xi)]du > 0$ is interpreted to mean that this integral is positive throughout the range $\xi_1 < \xi < \xi_2$.

When the foregoing criterion does indicate that there are no solutions of $k^2 = Z(\zeta)$ in the upper half plane a method for detecting solutions on the boundary of the upper half plane, the upper side of the real ζ axis, can be useful. If such solutions exist, $Z(R)$ touches the real positive Z axis without crossing it, and therefore a suitable criterion is that *in the absence of exponentially growing modes, solutions of $k^2 = Z(\zeta)$ with ζ on the upper side of the real axis occur if and only if there is a value of ξ such that $F'(\xi) = 0$, $F''(\xi) = 0$, and*

$$\int_{-\infty}^{\infty} (u - \xi)^{-2}[F(u) - F(\xi)]\, du > 0.$$

III. STABILITY WHEN NO GROWING MODES EXIST

As was noted in Sec. I, it is not self-evident that the plasma is stable even when $Z(\zeta)$ does not take real positive values anywhere in the upper half plane. In the present section, therefore, we study the stability of the spatially uniform solutions of the fundamental equations (1) and (2), using Vlasov's linearization approximation. Spatial symmetry ensures that we need only consider perturbations proportional to $\exp(i\mathbf{k}\cdot\mathbf{x})$; accordingly we assume

$$\phi(\mathbf{x}, t) = \phi(t) \exp(i\mathbf{k}\cdot\mathbf{x}),$$

and

$$f_i(\mathbf{x}, \mathbf{v}, t) = n_i[g_i(\mathbf{v}) + h_i(\mathbf{v}, t) \exp(i\mathbf{k}\cdot\mathbf{x})],$$

where, to justify linearization,

$$|h_i(\mathbf{v}, t)| \ll g_i(\mathbf{v}) \quad \text{and} \quad |\partial h_i/\partial \mathbf{v}| \ll |\partial g_i/\partial \mathbf{v}|. \tag{14}$$

Substitution of the assumed forms for ϕ and f into (1) and linearization gives an ordinary differential equation whose formal solution is

$$h_i(\mathbf{v}, t) = h_i(\mathbf{v}, 0) \exp(-i\mathbf{k}\cdot\mathbf{v}t) + \frac{e_i}{m_i} i\mathbf{k}\cdot\frac{\partial g_i}{\partial \mathbf{v}} \int_0^t dt'\phi(t') \exp[i\mathbf{k}\cdot\mathbf{v}(t' - t)]. \tag{15}$$

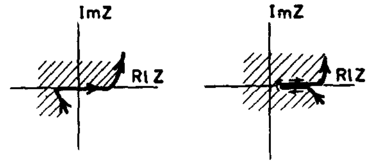

FIG. 4. Possible forms for the part of the $Z(R)$ curve corresponding to a flat minimum. Part of the image of the upper half plane is shaded in each case. The curve on the left "encloses" no positive real values, but the curve on the right does.

Substitution of this into (2) gives a Volterra equation for ϕ,

$$\phi(t) = \psi(t) + \int_0^t K(t - t')\phi(t')\, dt', \tag{16}$$

where

$$\psi(t) \equiv k^{-2}\Sigma_i 4\pi n_i e_i \int d^3\mathbf{v}\, h_i(\mathbf{v}, 0) \exp(-i\mathbf{k}\cdot\mathbf{v}t), \tag{17}$$

and

$$K(t) \equiv ik^{-1} \int_{-\infty}^{\infty} du\, F'(u) \exp(-ikut). \tag{18}$$

The solution of (16) has the form[11]

$$\phi(t) = \psi(t) + \int_0^t \Gamma(t - t')\psi(t')\, dt', \tag{19}$$

where the resolvent Γ satisfies

$$\Gamma(t) = K(t) + \int_0^t K(t - t')\Gamma(t')\, dt'. \tag{20}$$

The solution of (20) is[11]

$$\Gamma(t) = \frac{k}{2\pi} \int_{-\infty + i\eta}^{\infty + i\eta} d\zeta \frac{Z(\zeta)}{k^2 - Z(\zeta)} \exp(-ik\zeta t), \qquad [t > 0], \tag{21}$$

where η is a large positive constant.

In Landau's treatment of plasma oscillations,[12] the asymptotic form of integrals like (21) is estimated by a method involving the analytic continuation of $Z(\zeta)$ into the lower half plane. This procedure seems nonphysical however, since the behavior of the analytically continued $Z(\zeta)$ in the lower half plane is unduly sensitive to small changes in $F(u)$. For example, the two functions $F(u) = (1 + u^2)^{-1}$ and $F(u) = \pi^{-\frac{1}{2}} \alpha^{-1} \int_{-\infty}^{\infty} (1 + u_1^2)^{-1} \cdot \exp[-(u - u_1)^2/\alpha^2]du_1$ are almost identical for small α; yet for the first, the continuation of $Z(\zeta)$ into the lower half plane is $\pi(i + \zeta)^{-2}$, with a pole

[11] E. C. Titchmarsh, *Theory of Fourier Integrals*, (Oxford University Press, Oxford, England, 1937), pp. 311–312.

[12] L. Landau, J. Phys. (U. S. S. R.) 10, 25 (1946).

at $-i$ and no singularity at infinity, while for the second it is an integral function, with no poles in the finite part of the ζ plane but an essential singularity at infinity. Fortunately, as we shall show, stability can be treated without appealing to analytical continuation.[13]

We shall need the following lemma: if a function and its first derivative are quadratically integrable from $-\infty$ to ∞, then the Fourier transform of the function is absolutely integrable. The proof has been given by Titchmarsh.[14]

According to Eqs. (7) and (9), the lemma applies to the function $F'(u)$, so that, by (18), we have

$$\int_0^\infty |K(t)|\, dt < \infty. \tag{22}$$

Also, according to Eqs. (14), (8), and (9), the lemma applies to the functions

$$\int d^3\mathbf{v}\, h_i(\mathbf{v}, 0)\, \delta(\mathbf{e}\cdot\mathbf{v} - u).$$

Hence, by (17), $\psi(t)$ is a linear combination of absolutely integrable functions, so that

$$\int_{-\infty}^\infty |\psi(t)|\, dt < \infty. \tag{23}$$

Wiener[15] showed that, provided $Z(\zeta) - k^2$ vanishes neither in the upper half plane *nor on the upper side of the real axis*, Eqs. (20) and (22) imply

$$\int_0 |\Gamma(t)|\, dt < \infty. \tag{24}$$

Together with (23) and (19) this implies $\int|\phi(t)|dt < \infty$. By using this fact in (15) we see that $h_i(\mathbf{v},t)$ is bounded as t varies and that its asymptotic form for large t is $\exp(-i\mathbf{k}\cdot\mathbf{v}t)$ times a function of $\mathbf{v}$. Moreover, (14) implies that $h_i(\mathbf{v}, 0)$ is absolutely integrable; therefore by (17) and the Riemann-Lebesgue lemma[16] $\psi(t)$ is bounded and tends to 0 for large t; and therefore by (19) and (24) $\phi(t)$ is bounded and tends to zero (since the convolution of two absolutely integrable functions is bounded and tends to zero). Thus, provided $Z(\zeta)$ is never real and positive for $\text{Im}\,\zeta \geq 0$, the solutions of Vlasov's linearized equations are indeed stable (i.e., bounded). It is plausible that this is also true of the original nonlinear equations, even though $h_i(\mathbf{v}, t)$ does not tend to zero in the linear theory. For the cumulative effect of the neglected nonlinear terms during the time it takes $\phi(t)$ to become negligible can be made arbitrarily small by making the initial perturbation small enough; and since the nonlinear effects are proportional to ϕ, they are certainly negligible after this time.

In the "borderline" cases mentioned at the end of Sec. II, where $Z(\zeta) = k^2$ has solutions on the upper side of the real ζ axis but not in the upper half plane, the foregoing argument fails to prove stability, yet there are no exponentially growing modes. In the simplest such cases, $Z(\zeta) - k^2$ behaves in the upper half plane as if it had a simple zero on the real axis, say at $\zeta = \xi_0$. The asymptotic behavior of $\Gamma(t)$ for large t, determined by the corresponding pole in the integrand of (21), is

$$\Gamma(t) \sim (\text{const}) \exp(-ik\xi_0 t);$$

hence (19) and (15) give

$$\phi(t) \sim (\text{const}) \exp(-ik\xi_0 t)$$

and

$$h_i(\mathbf{v}, t) \sim h_i(\mathbf{v}, 0) \exp(i\mathbf{k}\cdot\mathbf{v}t) + (\text{const})\, (\mathbf{k}\cdot\partial g_i/\partial\mathbf{v}) \cdot [\exp(-i\mathbf{k}\cdot\mathbf{v}t) - \exp(-ik\xi_0 t)]/[\mathbf{k}\cdot\mathbf{v} - k\xi_0]. \tag{25}$$

This shows that, for a range of $\mathbf{v}$ satisfying $\mathbf{k}\cdot\mathbf{v} \simeq k\xi_0$, $h(\mathbf{v}, t)$ ultimately becomes large enough to violate the linearization condition (14); the physical process responsible is the appearance of "trapped particles."[17,18] It appears, then, that linearized theory cannot treat the stability of these borderline cases adequately.

IV. EXAMPLES

The application of these results will be illustrated by some examples. The simplest case is a distribution with a single maximum in $F(u)$; according to the first criterion of Sec. II, such a distribution cannot lead to exponentially growing modes.[19] In particular, if all the $g_i(\mathbf{v})$ are spherically symmetrical, then[20] (7) leads to

$$F'(u) = -\Sigma_i(4\pi n_i e_i^2/m_i)2\pi u g_i(u\mathbf{e}) \tag{26}$$

[13] N. G. van Kampen [Physica **21**, 949 (1955)] also avoids the use of analytic continuation.

[14] See reference 11, pp. 115–116, Theorem (84), with the following modifications: $p = p' = 2$, $\alpha = \beta = 1$, $f(x+h) - f(x-h) \to 2hf'(x)$, $\sin xh \to xh$.

[15] R. E. A. C. Paley and N. Wiener, *Fourier transforms in the complex domain* (American Mathematical Society Colloquim Publications, New York, 1934), Vol. XIX, Sec. 18.

[16] See reference 11, p. 11 (Theorem 1).

[17] D. Bohm and E. P. Gross, Phys. Rev. **75**, 1864 (1949).

[18] I. Bernstein, J. Greene, and M. Kruskal, Phys. Rev. **108**, 546 (1957).

[19] F. Berz [Proc. Phys. Soc. **B69**, 939 (1956)] proved this result by a different method for the case $F(u) = F(-u)$. P. L. Auer [Phys. Rev. Letters **1**, 411 (1958)] extended Berz' result to unsymmetrical $F(u)$.

[20] N. G. van Kampen (see the work cited in footnote 13) has obtained the same result.

so that $F(u)$ has a single maximum at $u = 0$, and there are no exponentially growing modes. If, in addition, the $g_j(\mathbf{v})$ never vanish all together, then by the second criterion of Sec. II there are no modes with real ζ, and the plasma is stable.[20] An important example is thermal equilibrium. The conclusion that the plasma is stable at thermal equilibrium tallies with general principles of statistical mechanics, as remarked by Newcomb.[2]

In another type of spherically symmetrical distribution, there exists a $v_{\max}$ such that all the $g_j(\mathbf{v})$ vanish for $|\mathbf{v}| \geq v_{\max}$ but not for $|\mathbf{v}| < v_{\max}$. Then, by (26), $F(u)$ vanishes for $u \geq v_{\max}$, and the second criterion of Sec. II shows that modes with real ζ do exist. In fact, for any $\xi_0 \geq v_{\max}$ there is a k such that $k^2 = Z(\xi_0)$ holds. The reasoning used to discredit linear theory for such cases in Sec. III applies now only when $\xi_0 = v_{\max}$, since for larger ξ_0 the factor $\partial g_j/\partial \mathbf{v}$ in (25) providentially vanishes wherever $\mathbf{k}\cdot\mathbf{v} \simeq k\xi_0$; but linear theory is still questionable, because any perturbation for which $h_j(\mathbf{v}) \neq 0$ when $|\mathbf{v}| > v_{\max}$ violates the linearization condition (14).

When $F(u)$ has a minimum the plasma may or may not be stable, according to the sign of the expression (13) at the minimum. A simple case arises when $F(\xi) = 0$ but $F(u)$ takes positive values on both sides of $u = \xi$. Then the integral (13) is obviously positive, and the plasma is unstable. This result includes as limiting cases the two-stream[3] and multi-stream[21] instabilities in which $F(u)$ is a linear combination of delta-functions (or very sharply peaked Gaussians).

A similar type of instability comes about when $F(u)$ has a very sharp minimum. Suppose, for example, that F has a minimum at $u = 0$, and that in the range $a \leq u \leq ka$ it satisfies $F(u) \geq F(0) + \mu(u - a)$, where a, k, μ are positive constants. Then the contribution of this range to the integral (13) exceeds $\mu \int_a^{ka} du(u - a)/u^2 = \mu\,[\ln k + k^{-1} - 1]$. For a very sharp minimum the quantities μ, k will be large enough to make this positive contribution dominate the integral, so leading to instability. This type of instability can arise when a beam of electrons is injected into a uniform plasma[17,22] since there is a sharp minimum of F at a value of u just below the velocity of the beam (Fig. 5).

Finally, we consider a generalized two-stream distribution of a form suggested by Fig. 5,

$$F(u) = F_1(u) + F_2(u), \tag{27}$$

[21] R. Q. Twiss, Phys. Rev. **88**, 1392 (1952).
[22] M. Sumi, J. Phys. Soc. Japan **13**, 1476 (1958).

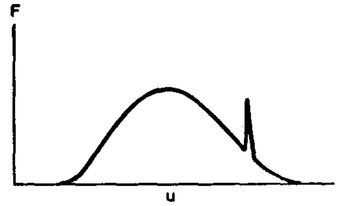

FIG. 5. Possible form for $F(u)$ when a beam of electrons is injected into a plasma.

where

$$F_j(u) \equiv (\omega_j^2/\alpha_j\sqrt{\pi}) \exp\left[-(u - u_j)^2/\alpha_j^2\right] \tag{28}$$

for $j = 1, 2$, and the constants ω_j, α_j, u_j satisfy

$$\omega_1 \gg \omega_2 \tag{29}$$

$$\alpha_1 \gg \alpha_2 \tag{30}$$

$$\alpha_1(\omega_1/\omega_2) \gg \alpha_1 + |u_2 - u_1| \gg (\omega_1/\omega_2)^{\frac{2}{3}}\alpha_2. \tag{31}$$

This F could represent, for example, a current-carrying plasma,[4] in which case $\omega_j^2 = 4\pi n_j\, e_j^2/m_j$ and $\alpha_j^2 = 2kT_j/m_j$, and $j = 1$ for electrons, $j = 2$ for positive ions. Alternatively it could represent a tenuous electron beam ($j = 2$) injected into a denser electron gas ($j = 1$).

The function $Z(\xi + i0)$ used in Sec. II can be written in the form $Z_1 + Z_2$, with $Z_j(\zeta)$ defined by analogy with (11). An estimate of $Z_j(\zeta)$ when $|\zeta - u_j| \gg \alpha_j$ can be obtained by expanding the factor $(\zeta - u)^{-2}$ in (11) in powers of $u - u_j$. It is

$$Z_j(\xi + i0) \simeq \omega_j^2/(\xi - u_j)^2 \quad \text{if} \quad |\xi - u_j| \gg \alpha_j. \tag{32}$$

For smaller $|\xi - u_j|$ the order of magnitude of Z_j is, by (12), (28), and Fig. 1,

$$|Z_j(\xi + i0)| \approx |Z_j(u_j + i0)| \quad \text{for} \quad |\xi - u_j| \approx \text{ or } < \alpha_j$$

$$= \left| P \int (u - u_j)^{-1} F_j'\, du \right| = 2\omega_j^2/\alpha_j^2. \tag{33}$$

By combining (32) and (33), we get a crude estimate of Z_j valid for all ξ:

$$|Z_j(\xi + i0)| \approx \omega_j^2[\alpha_j + |\xi - u_j|]^{-2}, \tag{34}$$

where the sign $\approx$ indicates comparable orders of magnitude.

To find an approximation for $Z(\xi + i0)$ itself, we first choose a quantity α satisfying [compare (29) to (31)]

$$[\alpha_1 + |u_2 - u_1|](\omega_2/\omega_1)^{\frac{2}{3}} \gg \alpha \gg \alpha_2 \tag{35}$$

$$\alpha_1 \gg \alpha \gg (\omega_2/\omega_1)[\alpha_1 + |u_2 - u_1|]. \tag{36}$$

Now, over the range $u_2 - \alpha < \xi < u_2 + \alpha$, the change in Z_1 is roughly $2\,\alpha\,Z_1'$, since $\alpha_1 \gg \alpha$. The order of magnitude of $2\,\alpha\,Z_1'$ is $\alpha\,\omega_1^2\,[\alpha_1 + |u_2 - u_1|]^{-3}$, and that of Z_2 is $\geq \omega_2^2\,[\alpha_2 + \alpha]^{-2}$, by (34). By using both sides of (35), we obtain $Z_2 \gg \alpha\,Z_1'$, and hence, ignoring $\alpha\,Z_1'$,

$$Z(\xi + i0) \simeq Z_1(u_2 + i0) + Z_2(\xi + i0) \qquad \text{if} \quad |\xi - u_2| < \alpha. \tag{37}$$

For the remaining values of ξ, with $|\xi - u_2| \geq \alpha$, we have by (34) and then (29) and the right side of (36),

$$\begin{aligned}\omega_1 Z_1^{-\frac{1}{2}} &\approx |\xi - u_1| + \alpha_1 \leq |\xi - u_2| + |u_2 - u_1| + \alpha_1 \\ &\ll (\omega_1/\omega_2)\,|\xi - u_2| + (\omega_1/\omega_2)\alpha \\ &< 2(\omega_1/\omega_2)[|\xi - u_2| + \alpha_2] \approx \omega_1 Z_2^{-\frac{1}{2}}.\end{aligned}$$

This means $Z_1 \gg Z_2$, so that the approximation is

$$Z(\xi + i0) \simeq Z_1(\xi + i0) \quad \text{for} \quad |\xi - u_2| \geq \alpha. \tag{38}$$

Figure 1 shows the Z-plane diagram for a Maxwell distribution, and Fig. 3, a typical diagram computed from (37) and (38). In Fig. 3, the loop with a cusp at the origin comes from (38); we call it the Z_1 loop. The loop with a cusp at $Z_1(u_2 + i0)$ comes from (37); we call it the Z_2 loop. The Z_1 loop can never enclose positive real values but the Z_2 loop can always do so if it is large enough. The sizes of the Z_1 and Z_2 loops are given [compare (33)] by the squared reciprocal Debye lengths $2(\omega_1/\alpha_1)^2$ and $2(\omega_2/\alpha_2)^2$, respectively; the shapes of the individual loops cannot alter, but their sizes and the direction of the line joining the two cusps can. The direction of this line depends only on $|u_2 - u_1|/\alpha_1$. For fixed $|u_2 - u_1|/\alpha_1$, there is a critical value for $(\omega_2/\alpha_2)^2/(\omega_1/\alpha_1)^2$, the ratio of the sizes of the two loops. Only if the critical value is exceeded will the Z_2 loop be large enough to cross the positive real axis. This critical value has been computed graphically from the curve of Fig. 1. The results are given in Fig. 6.[23]

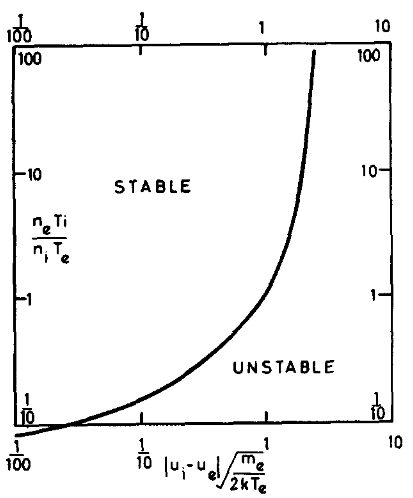

FIG. 6. Stability diagram for a current-carrying plasma, represented by the double Maxwellian distribution (27). Note that

$$|u_i - u_e|\,(m_e/2kT_e)^{\frac{1}{2}} \equiv |u_2 - u_1|/\alpha_1,$$

and that

$$n_e T_i/n_i T_e \equiv (\omega_1/\alpha_1)^2/(\omega_2/\alpha_2)^2.$$

If $e_i \neq e_e$, the ordinate becomes

$$n_e e_e^2 T_i/n_i e_i^2 T_e.$$

This double Maxwellian distribution can be used as a crude model of a current-carrying plasma[4,5]; then the ratio $(\omega_2/\alpha_2)^2/(\omega_1/\alpha_1)^2$ is $n_i e_i^2 T_e/n_e e_e^2 T_i$. Figure 6 shows that when $T_i \approx$ or $> T_e$ electrostatic instabilities set in when $|u_1 - u_2|$ is comparable with the electron thermal speed,[24] but that when $T_i \ll T_e$ they set in at extremely small values of $|u_2 - u_1|$. This result[25] for $T_i \ll T_e$ exemplifies the "sharp minimum" instability discussed in the foregoing. If the relative motion $u_2 - u_1$ is produced by an electric field, the condition for a large fraction of runaway electrons is $|u_2 - u_1| \approx \alpha_1$; thus electrostatic instability seems much more important than runaway electrons when $T_i \ll T_e$ but not when $T_i \approx T_e$.

V. DISCUSSION

The main assumptions made in this work are the neglect of collisions in (1), the smoothness conditions (9) and (10), and the linearization of Eq. (1). The neglect of collisions is justified because instability growth rates, being comparable with the plasma frequency, greatly exceed the collision rate. The other two assumptions are closely related to each other. This can be seen by comparing the linearized solution of (1) with the exact solution. For simplicity, we assume $\partial\phi/\partial\mathbf{x}$ depends on t only, and drop the suffix j. Then, if initially

$$f(\mathbf{x}, \mathbf{v}, 0) = g(\mathbf{v}) + h(\mathbf{x}, \mathbf{v}, 0),$$

the linearized solution of (1) [obtainable by Fourier transformation of (15)] is

$$f(\mathbf{x}, \mathbf{v}, t) = g(\mathbf{v}) + h(\mathbf{x} - \mathbf{v}t, \mathbf{v}, 0) - \frac{\partial g}{\partial \mathbf{v}} \cdot \int_0^t \mathbf{a}(t')\,dt', \tag{39}$$

[23] Numerical data used in plotting Figs. 1 and 6 were taken from a paper by Harris [Astrophys. J. 108, 112 (1948)].

[24] S. Tamor (unpublished) has shown for the special case $n_i T_e = n_e T_i$ that the critical value of $|u_2 - u_1|$ is 0.93 $(\alpha_1 + \alpha_2)$ without imposing the conditions (29)–(31).

[25] I. Bernstein (unpublished) also obtained this result for $T_i \ll T_e$.

where

$$\mathbf{a}(t) \equiv -(e/m)\,\partial\phi/\partial\mathbf{x}.$$

The exact solution is

$$f(\mathbf{x}, \mathbf{v}, t) = g\left(\mathbf{v} - \int_0^t \mathbf{a}(t')\,dt'\right) + h\left(\mathbf{x} - \mathbf{v}t + \int_0^t \mathbf{a}(t')t'\,dt\,,\, \mathbf{v} - \int_0^t \mathbf{a}(t')\,dt', 0\right). \quad (40)$$

When we expand this in powers of $\mathbf{a}$ by Taylor's series, we find that the error in (39) is the sum of three terms, one of which is approximately $[\int_0^t \mathbf{a}(t')dt'\cdot\partial/\partial\mathbf{v}]^2\, g(\mathbf{v})$. To make this and its integrals negligible for small $\int \mathbf{a}\,dt$, some boundedness and integrability conditions on $(\mathbf{e}\cdot\partial/\partial\mathbf{v})^2\, g(\mathbf{v})$, closely related to those formulated in (9) and (10), seem inevitable. Thus, for cases where (9) and (10) fail, some improvement on linearized theory is needed before stability can be discussed at all; on the other hand, when (9) and (10) (or some closely related conditions) are satisfied, then the linearized theory used here can probably be justified, as suggested in Sec. III, except for the borderline case where trapped particles become important.

We conclude, then, that the two criteria given in Sec. II, together with a judicious use of Z-plane diagrams, can greatly simplify the work of discovering whether a given spatially uniform plasma is stable or not. For applications to real plasmas, an understanding of the effect of nonuniformities will also be needed.

ACKNOWLEDGMENTS

I am grateful to W. K. Hayman, D. Gabor, S. Doniach, S. Tamor, E. Stringer, and L. Mestel for useful discussions. This work was begun when I was a vacation consultant at the Atomic Energy Research Establishment, Harwell, England.

THE PHYSICS OF FLUIDS VOLUME 3, NUMBER 2 MARCH-APRIL, 1960

Wake of a Satellite Traversing the Ionosphere

S. Rand
Convair, San Diego, California
(Received September 1, 1959)

The particle treatment is applied to a study of the structure of the wake behind a charged body moving supersonically through a low-density plasma. For the case of a body whose dimensions are considerably smaller than a Debye length, a solution is obtained which is very similar in structure to the solution obtained by using the linearized fluid dynamics equation. For the case of a disk whose radial dimensions are much larger than a Debye length, two conical regions are found in the wake. At the surface of each of these cones, over thicknesses of the order of a Debye length, the ion and electron densities are increased over their ambient values. Formulae for the electrohydrodynamic drag on a wire, and on a large disk are obtained.

INTRODUCTION

In a recent paper,[1] the particle treatment was used to determine the potential distribution about a point charge moving slowly through a plasma. The method is now extended to the case of supersonic flow about charged obstacles. In particular, we center our attention on plasma disturbances produced by a satellite traversing the ionosphere. Thus, we consider geometrical structures which may reasonably well represent the antenna and the body of the satellite. The antenna is approximated by an infinite cylindrical structure, considerably less than a Debye length (on the order of a centimeter) in radius. The body is assumed to be a disk with its radial dimensions much greater than a Debye length. This provides a useful approximation to a number of interesting shapes, such as the conical shape of Sputnik III with its direction of motion along the axis. We expect that the structure of the wake is determined primarily by the base of the cone.

It is found that in both cases a wake is produced. The wake behind the antenna, as a two-dimensional analog, is very similar in structure to the wake predicted by Kraus and Watson[2] for plasma flow

[1] S. Rand, Phys. Fluids 2, 649 (1959).

[2] L. Kraus and K. M. Watson, Phys. Fluids 1, 480 (1959).