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Energetic ion distribution resulting from neutral beam injection in tokamaks

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The Fokker–Planck equation is studied for an energetic ion beam injected into a magnetized plasma consisting of Maxwellian ions and electrons with $v_{\mathrm{th}_i} \ll v_b \ll v_{\mathrm{th}_e}$. The time evolution of the fast ion distribution is given in terms of an infinite sum of Legendre polynomials for distributions that are axisymmetric about the magnetic field. The effect of charge exchange is included. The resulting ion distribution is somewhat isotropic for velocities much less than the injection velocity, however, the distribution is sharply peaked in both energy and pitch angle for velocities near the injection velocity. Approximate asymptotic expressions are given for the distribution in the vicinity of the injected beam and for velocities greater than the injection velocity. The effect of a weak parallel electric field is also given.

1. Introduction

Currently there is considerable interest in the heating of magnetically confined plasmas by the injection of high energy ion beams (Cordey & Houghton 1973; Stix 1973; Cordey & Core 1974, Rome *et al.* 1975; Berk *et al.* 1975; Goldston 1975). The injected beam distributions are sharply peaked in both energy and pitch angle in velocity space. Coulomb collisions cause the ion beams to lose energy and diffuse in velocity space. The purpose of this work is to find the distribution function of the injected beams using a simplified Fokker–Planck equation.

In §2 we derive the simplified Fokker–Planck equation for the beam distribution. The steady state solution to this equation is obtained in §3 by expanding the distribution in Legendre polynomials, as first suggested by Rosenbluth *et al.* (1957) for distributions that are axisymmetric about the magnetic field. The time evolution of the distribution is given in §4. Charge exchange is discussed in §5. The properties of this distribution are discussed in §6 and a comparison is made with the work of others. Asymptotic expressions are given for the distribution for velocities near the injection velocity in §7 and for velocities greater than the injection velocity in §8. The effect of a weak electric field is treated in §9. The conclusions of this work are given in §10.

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2. Fokker–Planck equation for fast ion beams

The effect of Coulomb collisions on the distribution function is given by the Fokker–Planck equation

$$\frac{\partial f}{\partial t}+\mathbf{v}.\nabla f+\frac{e}{m}\left(\mathbf{E}+\frac{\mathbf{v}}{c}\times\mathbf{B}\right).\frac{\partial f}{\partial \mathbf{v}}=\mathscr{I}(f,f), \tag{1}$$

where the distribution function f is interpreted as a vector with components representing the various species, the ion beams, the background ions, and the electrons. The collision operator $\mathscr{I}(f,f)$ couples the components and makes the Fokker–Planck equation nonlinear.

We use the Landau (1936) form of the Fokker–Planck collision operator (Thompson 1964). The effect on species i of collisions with species j is

$$\mathscr{I}(f_i,f_j)=2\pi e_i^2e_j^2\ln\Lambda\frac{1}{m_i}\frac{\partial}{\partial\mathbf{v}_i}.\int d^3v_j\,\boldsymbol{\omega}.\left(\frac{1}{m_i}\frac{\partial}{\partial\mathbf{v}_i}-\frac{1}{m_j}\frac{\partial}{\partial\mathbf{v}_j}\right)f_if_j, \tag{2}$$

where $$\boldsymbol{\omega}=\frac{\partial^2 g}{\partial\mathbf{v}_i\partial\mathbf{v}_i}=\frac{1}{g^3}(g^2\mathsf{I}-\mathbf{g}\mathbf{g}),\quad g=|\mathbf{g}|=|\mathbf{v}_i-\mathbf{v}_j|$$

is the relative velocity, and $\ln\Lambda$ is the usual Coulomb cut-off factor. Because the entire collision term (2) is a divergence in velocity space, its form is more convenient than that of Rosenbluth, MacDonald & Judd (1957).

Integrating the second term in (2) by parts and using the relation

$$\frac{\partial}{\partial\mathbf{v}_i}.\boldsymbol{\omega}=-\frac{\partial}{\partial\mathbf{v}_j}.\boldsymbol{\omega},$$

$$\mathscr{I}(f_i,f_j)=\frac{2\pi e_i^2e_j^2\ln\Lambda}{m_i^2}\frac{\partial}{\partial\mathbf{v}_i}.\left[\frac{\partial f_i}{\partial\mathbf{v}_i}.\int d^3v_j\,\boldsymbol{\omega}f_j-\frac{m_i}{m_j}f_i\frac{\partial}{\partial\mathbf{v}_i}.\int d^3v_j\,\boldsymbol{\omega}f_j\right]. \tag{3}$$

Now using the definition

$$\boldsymbol{\omega}=\frac{1}{g^3}(g^2\mathsf{I}-\mathbf{g}\mathbf{g})=\frac{\partial^2 g}{\partial\mathbf{v}_i\partial\mathbf{v}_i},$$

$$\mathscr{I}(f_i,f_j)=\frac{2\pi e_i^2e_j^2n_{0_j}\ln\Lambda v_{\mathrm{th}_j}}{m_i^2}\frac{\partial}{\partial\mathbf{v}_i}.\left[\frac{\partial f_i}{\partial\mathbf{v}_i}.\frac{\partial^2F(x_{ij})}{\partial\mathbf{v}_i\partial\mathbf{v}_i}-\frac{m_i}{m_j}f_i\frac{\partial}{\partial\mathbf{v}_i}.\frac{\partial^2F(x_{ij})}{\partial\mathbf{v}_i\partial\mathbf{v}_i}\right], \tag{4}$$

where $$F(x_{ij})=\frac{1}{n_{0_j}v_{\mathrm{th}_j}}\int d^3v_jf_jg_{ij}, \tag{5}$$

$x_{ij}=v_i/v_{\mathrm{th}_j}$, v_{th_j} is the thermal velocity, and n_{0_j} is the density of jth species. The Rosenbluth potentials (Rosenbluth *et al.* 1957; Montgomery & Tidman 1964) can be expressed in terms of the function $F(x_{ij})$ given in (5):

$$G_{ij}=n_{0_j}v_{\mathrm{th}_j}\frac{e_j^2}{e_i^2}F(x_{ij})$$

$$H_{ij}=\tfrac{1}{2}n_{0_j}v_{\mathrm{th}_j}\frac{e_j^2}{e_i^2}\frac{(m_i+m_j)}{m_j}\frac{\partial}{\partial\mathbf{v}_i}.\frac{\partial}{\partial\mathbf{v}_i}F(x_{ij}).$$

Taking the indicated derivatives in (4),

$$\mathscr{I}(f_i,f_j) = \frac{2\pi e_i^2 e_j^2 n_{0_j} \ln \Lambda v_{\mathrm{th}_j}}{m_i^2} \frac{\partial}{\partial \mathbf{v}_i} \cdot \left[\frac{\partial f_i}{\partial \mathbf{v}_i} \cdot \left(\frac{\partial^2 x_{ij}}{\partial \mathbf{v}_i \partial \mathbf{v}_i} F'(x_{ij}) + \frac{\partial x_{ij}}{\partial \mathbf{v}_i} \frac{\partial x_{ij}}{\partial \mathbf{v}_i} F''(x_{ij}) \right) \right.$$

$$- \frac{m_i}{m_j} f_i \left(\frac{\partial}{\partial \mathbf{v}_i} \cdot \frac{\partial^2 x_{ij}}{\partial \mathbf{v}_i \partial \mathbf{v}_i} F'(x_{ij}) + \frac{\partial}{\partial \mathbf{v}_i} \cdot \frac{\partial x_{ij}}{\partial \mathbf{v}_i} \frac{\partial x_{ij}}{\partial \mathbf{v}_i} F''(x_{ij}) + \frac{\partial x_{ij}}{\partial \mathbf{v}_i} \cdot \frac{\partial^2 x_{ij}}{\partial \mathbf{v}_i \partial \mathbf{v}_i} F''(x_{ij}) \right.$$

$$\left. \left. + \frac{\partial x_{ij}}{\partial \mathbf{v}_i} \cdot \frac{\partial x_{ij}}{\partial \mathbf{v}_i} \frac{\partial x_{ij}}{\partial \mathbf{v}_i} F'''(x_{ij}) \right) \right]. \tag{6}$$

In Appendix B we have evaluated the derivatives of x_{ij}

$$\frac{\partial x_{ij}}{\partial \mathbf{v}_i} = \frac{1}{v_{\mathrm{th}_j}} \frac{\mathbf{v}_i}{v_i}, \qquad \frac{\partial}{\partial \mathbf{v}_i} \cdot \frac{\partial x_{ij}}{\partial \mathbf{v}_i} = \frac{2}{v_{\mathrm{th}_j} v_i},$$

$$\frac{\partial^2 x_{ij}}{\partial \mathbf{v}_i \partial \mathbf{v}_i} = \frac{1}{v_{\mathrm{th}_j} v_i^3} (v_i^2 \mathsf{I} - \mathbf{v}_i \mathbf{v}_i), \qquad \frac{\partial}{\partial \mathbf{v}_i} \cdot \frac{\partial^2 x_{ij}}{\partial \mathbf{v}_i \partial \mathbf{v}_i} = -\frac{2}{v_{\mathrm{th}_j}} \frac{\mathbf{v}_i}{v_i^3},$$

$$\frac{\partial x_{ij}}{\partial \mathbf{v}_i} \cdot \frac{\partial^2 x_{ij}}{\partial \mathbf{v}_i \partial \mathbf{v}_i} = 0. \tag{7}$$

Substituting the expressions given in (7) into (6) gives

$$\mathscr{I}(f_i,f_j) = \Gamma_{ij} \frac{\partial}{\partial \mathbf{v}_i} \cdot \left[\frac{\partial f_i}{\partial \mathbf{v}_i} \cdot \left(\frac{\partial^2 v_i}{\partial \mathbf{v}_i \partial \mathbf{v}_i} F'(x_{ij}) + \frac{\mathbf{v}_i \mathbf{v}_i}{v_i^3} x_{ij} F''(x_{ij}) \right) \right.$$

$$\left. + 2 \frac{m_i}{m_j} f_i \frac{\mathbf{v}_i}{v_i^3} \left(F'(x_{ij}) - x_{ij} F''(x_{ij}) - \frac{x_{ij}^2}{2} F'''(x_{ij}) \right) \right], \tag{8}$$

where

$$\Gamma_{ij} = \frac{2\pi e_i^2 e_j^2 n_{0_j} \ln \Lambda}{m_i^2}.$$

The collision operator (8) is in general a three dimensional, fourth order, partial differential operator. In order to proceed, we must make some simplifying assumptions. We take the beam density to be small compared to the density of the background plasma, so that collisions between beam ions are negligible compared to collisions with the background ions and electrons. The further assumption that the injected beam is axisymmetric about the magnetic field greatly simplifies the Fokker–Planck equation, as observed by Rosenbluth *et al.* (1957). The function F defined in (5) can be evaluated analytically for Maxwellian distributions (Glasser 1972; Chandrasekhar 1943). The details of this calculation are given in Appendix A.

$$F(x) = \left(x + \frac{1}{2x} \right) \Phi(x) + \tfrac{1}{2} \Phi'(x), \tag{9}$$

where $\Phi(x)$ is the error function and $\Phi'(x)$ is its derivative,

$$\Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt, \quad \Phi'(x) = \frac{d\Phi}{dx} = \frac{2}{\sqrt{\pi}} e^{-x^2}. \tag{10}$$

The asymptotic forms of the error function are (Abramowitz & Stegun 1966)

$$\Phi(x) \approx \frac{2x}{\sqrt{\pi}} \left(1 - \frac{x^2}{3} + \frac{x^4}{10} - \dots \right) \quad \text{for} \quad x \ll 1,$$

$$\Phi(x) \approx 1 - \frac{e^{-x^2}}{\sqrt{\pi} x} \left(1 - \frac{1}{2x^2} + \frac{3}{4x^4} - \dots \right) \quad \text{for} \quad x \gg 1. \tag{11}$$

Substituting (11) into (9), the asymptotic forms of $F(x)$ are given by

$$F(x) = \frac{2}{\sqrt{\pi}}(1+\tfrac{1}{3}x^2-\tfrac{1}{30}x^4+\dots) \quad \text{for} \quad x \ll 1, \tag{12}$$

$$F(x) = x+\frac{1}{2x}-\frac{e^{-x^2}}{2\sqrt{\pi}x^4}\left(1-\frac{3}{x^2}+\dots\right) \quad \text{for} \quad x \gg 1. \tag{13}$$

We now make the assumption that the injected beam velocity is between the ion and electron thermal velocities, $v_{\text{th}_i} \ll v_b \ll v_{\text{th}_e}$. This assumption will be satisfied if the beam energy is greater than the ion temperature, but not a factor M_i/m_e greater than the electron temperature. The collision term (8) for the beam ions with the background electrons simplifies considerably.

For the collisions of the ion beams with the background electrons, we use (12) in (8).

$$\mathscr{I}(f_b,f_e) = \frac{4x_{be}}{3\sqrt{\pi}v}\Gamma_{be}\frac{\partial}{\partial \mathbf{v}}\cdot\left[\frac{\partial f_b}{\partial \mathbf{v}}+\frac{2M_b x_{be}^2}{m_e}\frac{\mathbf{v}}{v^2}f_b\right]. \tag{14}$$

Neglecting the first term in (14) compared to the second term which is a factor $2M_b x_{be}^2/m_e$ larger (we will check this ordering below)

$$\mathscr{I}(f_b,f_e) = \frac{8x_{be}^3}{3\sqrt{\pi}v^3}\frac{M_b}{m_e}\Gamma_{be}\left(v\frac{\partial f_b}{\partial v}+3f_b\right). \tag{15}$$

For the collisions of the ion beams with the background ions, we use (13) in (8),

$$\mathscr{I}(f_b,f_i) = \Gamma_{bi}\frac{\partial}{\partial \mathbf{v}}\cdot\left[\frac{\partial f_b}{\partial \mathbf{v}}\cdot\left(\frac{\partial^2 v}{\partial \mathbf{v}\,\partial \mathbf{v}}\left(1-\frac{1}{2x_{bi}^2}\right)+\frac{\mathbf{v}\mathbf{v}}{v^3}\frac{1}{x_{bi}^2}\right)+\frac{2M_b}{M_i}f_b\frac{\mathbf{v}}{v^3}\right]. \tag{16}$$

Neglecting terms in (16) of order x_{bi}^{-2} compared to terms of order unity (we will check this ordering below),

$$\mathscr{I}(f_b,f_i) = \Gamma_{bi}\frac{\partial}{\partial \mathbf{v}}\cdot\left[\frac{\partial^2 v}{\partial \mathbf{v}\,\partial \mathbf{v}}\cdot\frac{\partial f_b}{\partial \mathbf{v}}+\frac{2M_b \mathbf{v}}{M_i v^3}f_b\right]. \tag{17}$$

The complicated derivatives in the first term of (17) can be greatly simplified if we assume f_b is axisymmetric about the magnetic field. We introduce spherical coordinates and transform to the variable $\xi = \cos\theta = (\mathbf{v}/v).\hat{e}_3$, where $\hat{e}_3$ is a unit vector along the magnetic field. In this coordinate system (17) can be written in the amazingly simple form (the details of this transformation are given in appendix B),

$$\mathscr{I}(f_b,f_i) = \frac{\Gamma_{bi}}{v^3}\left(\frac{\partial}{\partial \xi}(1-\xi^2)\frac{\partial f_b}{\partial \xi}+\frac{2M_b v}{M_i}\frac{\partial f_b}{\partial v}\right). \tag{18}$$

Combining the collision terms (15) and (18), the Fokker–Planck collision operator for the ion beam becomes

$$\left(\frac{\partial f_b}{\partial t}\right)_{\text{coll}} = \frac{1}{v^3}\left[\frac{8\Gamma_{be}}{3\sqrt{\pi}}\frac{M_b x_{be}^3}{m_e}\left(v\frac{\partial f_b}{\partial v}+3f_b\right)+\sum_i \Gamma_{bi}\left(\frac{2M_b}{M_i}v\frac{\partial f_b}{\partial v}+\frac{\partial}{\partial \xi}(1-\xi^2)\frac{\partial f_b}{\partial \xi}\right)\right], \tag{19}$$

where the sum on i extends over all ion species. The drag exerted by the background electrons on the beam will equal that of the background ions on the beam when the beam velocity is

$$v_c = \left(\frac{3\sqrt{\pi}m_e Z_1}{4M_{ib}}\right)^{\frac{1}{3}} v_{\text{th}_e} = 0{\cdot}09\left(\frac{M_H}{M_b}Z_1\right)^{\frac{1}{3}} v_{\text{th}_e}, \quad \text{where} \quad Z_1 = \sum_i \frac{n_i Z_i^2 M_b}{n_e M_i}.$$

For beam velocities below the crossover velocity, the ion drag will dominate and for velocities above v_c the electron drag will dominate. Most present injection schemes have beam velocities of order v_c.

In deriving (15) we neglected a term of order unity compared to a term of order $2M_b x_{be}^2/m_e$ which is about 40 for v near v_c. Similarly, in deriving (17) we neglected a term of order $x_{b_i}^{-2}$ which is about $(1/20)(T_i/T_e) \ll 1$, for v near v_c.

In terms of the crossover velocity v_c, we write (19) in the form

$$\left(\frac{\partial f_b}{\partial t}\right)_{\text{coll}} = \frac{1}{\tau_s v^3}\left[v\frac{\partial}{\partial v}(v^3+v_c^3)f_b + Z_2\frac{v_c^3}{2}\frac{\partial}{\partial \xi}(1-\xi^2)\frac{\partial f_b}{\partial \xi}\right], \tag{20}$$

where the Spitzer slowing down time is given by (Spitzer 1962; Montgomery & Tidman 1964)

$$\tau_s = \frac{v_c^3}{2\Gamma_{be}Z_1} = \frac{M_b^2 v_c^3}{4\pi e^2 e_b^2 Z_1 n_e \ln \Lambda}$$

and the effective charge of the background ions by $Z_2 = \sum_i n_i Z_i^2/n_e Z_1$. The first term in (20) gives the background electron drag on the ion beam and the second term gives the background ion drag on the ion beam. The rate of energy transfer to the background electrons is equal to the rate of energy transfer to the background ions when the ion beam has velocity v_c. The third term gives the perpendicular diffusion of the ion beam in pitch angle due to scattering off the background ions.

3. Steady state solution of the Fokker–Planck equation

We will now look for the time-asymptotic solution to the spatially homogeneous Fokker–Planck equation. Having assumed that the beam distribution is axisymmetric about the magnetic field, so that the velocity dependent term in the Boltzmann equation

$$\frac{e_b}{M_b c}(\mathbf{v}\times\mathbf{B}_0)\cdot\frac{\partial f_b}{\partial \mathbf{v}} = -\Omega_b\frac{\partial f_b}{\partial \phi} = 0,$$

the appropriate Fokker–Planck equation is given by (20).

First, we consider the homogeneous equation

$$\frac{\partial f_b}{\partial t} = \frac{1}{\tau_s v^3}\left[v\frac{\partial}{\partial v}(v^3+v_c^3)f_b + Z_2\frac{v_c^3}{2}\frac{\partial}{\partial \xi}(1-\xi^2)\frac{\partial f_b}{\partial \xi}\right] = 0. \tag{21}$$

This is a second order partial differential equation in the variables v and ξ. We note, however, that the second order differential operator in ξ is just Legendre's operator. Following the procedure first suggested by Rosenbluth *et al.* (1957), we expand f_b in a series of Legendre polynomials in ξ,

$$f_b(v,\xi) = \sum_{l=0}^{\infty} f_l(v)\, P_l(\xi) \tag{22}$$

where $$f_l(v) = \frac{2l+1}{2}\int_{-1}^{x} f_b(v,\xi)\, P_l(\xi)\, d\xi.$$

Substituting the expansion (22) into the Fokker–Planck equation (21), we obtain a decoupled, although infinite, set of first order ordinary differential equations for the coefficients $f_l(v)$,

$$v\frac{df_l}{dv}+\frac{[3v^3-\frac{1}{2}Z_2 l(l+1)v_c^3]f_l(v)}{v^3+v_c^3}=0. \tag{23}$$

The solution of (23) is

$$f_l(v)=\frac{A_l}{v^3+v_c^3}\left(\frac{v^3}{v^3+v_c^3}\right)^{\frac{1}{6}l(l+1)Z_2}, \tag{24}$$

where the constants A_l are determined from the boundary values.

Next, we consider the inhomogeneous equation, including the source term,

$$\left(\frac{\partial f_b}{\partial t}\right)_{\text{coll}}=\frac{1}{\tau_s v^3}\left[v\frac{\partial}{\partial v}(v^3+v_c^3)f_b+Z_2\frac{v_c^3}{2}\frac{\partial}{\partial \xi}(1-\xi^2)\frac{\partial f_b}{\partial \xi}\right]+S(v,\xi)=0. \tag{25}$$

As before, expanding in a series of Legendre polynomials decouples (25)

$$v\frac{df_l}{dv}+\frac{[3v^3-\frac{1}{2}Z_2 l(l+1)v_c^3]f_l(v)}{v^3+v_c^3}=-\frac{v^3}{v^3+v_c^3}\tau_s S_l(v), \tag{26}$$

where
$$S_l(v)=\frac{2l+1}{2}\int_{-1}^{1}S(v,\xi)P_l(\xi)\,d\xi. \tag{27}$$

The equations (26) for the coefficients $f_l(v)$ are first order, inhomogeneous, ordinary differential equations which have an exact solution for any source $S_l(v)$,

$$f_l(v)=\frac{-\tau_s}{v^3+v_c^3}\left(\frac{v^3}{v^3+v_c^3}\right)^{\frac{1}{6}l(l+1)Z_2}\int^{v}S_l(v)\left(\frac{v^3+v_c^3}{v^3}\right)^{\frac{1}{6}l(l+1)Z_2}v^2\,dv. \tag{28}$$

If we take the source to be a delta function in both energy and angle,

$$S(v,\xi)=\frac{S^0}{v^2}\delta(v-v_b)\,\delta(\xi-\xi_b), \tag{29}$$

the particular solution will be

$$f_l(v)=-\frac{(2l+1)}{2}\frac{\tau_s S^0}{v^3+v_c^3}P_l(\xi_b)\left[\left(\frac{v^3}{v_b^3}\right)\left(\frac{v_b^3+v_c^3}{v^3+v_c^3}\right)\right]^{\frac{1}{6}l(l+1)Z_2}U(v-v_b), \tag{30}$$

where $U(v-v_b)$ is a unit step function.

The solution for the beam distribution is

$$f_b(v,\xi)=\frac{S^0\tau_s}{v^3+v_c^3}\sum_{l=0}^{\infty}\frac{(2l+1)}{2}P_l(\xi_b)P_l(\xi)\left[\left(\frac{v^3}{v_b^3}\right)\left(\frac{v_b^3+v_c^3}{v^3+v_c^3}\right)\right]^{\frac{1}{6}l(l+1)Z_2}U(v_b-v), \tag{31}$$

where we have chosen the coefficients $A_l=\frac{1}{2}(2l+1)S^0\tau_s P_l(\xi_b)\left((v_b^3+v_c^3)/v_c^3\right)^{\frac{1}{6}l(l+1)Z_2}$ in the homogeneous solution (24). Thus, $f_b(v,\xi)$ given in (31) has no ions travelling faster than the injection velocity v_b because in our Fokker–Planck equation (20) we have neglected the parallel diffusion terms compared to the ion perpendicular diffusion term. The number density is given by

$$N_b=\int_0^{\infty}dv\,v^2\int_{-1}^{1}d\xi\,f_b(v,\xi).$$

Using the beam distribution (31)

$$N_b = S_0 \tau_s \int_0^{v_b} \frac{v^2 dv}{v^3 + v_c^3} = \frac{S^0 \tau_s}{3} \ln\left(\frac{v_b^3 + v_c^3}{v_c^3}\right) = S_0 \tau_0(v_b).$$

4. Time evolution of the beam distribution

The time evolution of the beam distribution is given by (25), including the time derivative,

$$\tau_s \frac{\partial f_b}{\partial t}(v, \xi, t) = \frac{1}{v^2} \frac{\partial}{\partial v}(v^3 + v_c^3) f_b(v, \xi, t) + \frac{Z_2 v_c^3}{2v^3} \frac{\partial}{\partial \xi}(1 - \xi^2) \frac{\partial f_b}{\partial \xi}(v, \xi, t) + \tau_s S(v, \xi, t). \tag{32}$$

Comparing the first two terms, we find that for velocities v greater than the crossover velocity v_c the beam slows down with little scattering in pitch angle. However, for $v < v_c$ the beam slows down and scatters in pitch angle at comparable rates and hence the distribution becomes isotropic. Increasing the effective charge Z_2 enhances the isotropy of the beam.

Decomposing the beam distribution and source into a series of Legendre polynomials as in (22),

$$\tau_s \frac{\partial f_l}{\partial t}(v, t) = \frac{1}{v^2} \frac{\partial}{\partial v}(v^3 + v_c^3) f_l(v, t) - \frac{l(l+1) v_c^3 Z_2}{2v^3} f_l(v, t) + \tau_s S_l(v, t). \tag{33}$$

Laplace transforming (33) in time, we obtain

$$\frac{1}{v^2} \frac{\partial}{\partial v}(v^3 + v_c^3) \tilde{f}_l(v, p) - \frac{l(l+1) v_c^3 Z_2}{2v^3} \tilde{f}_l(v, p) - \tau_s p \tilde{f}_l(v, p) = -\tau_s \tilde{S}_l(v, p), \tag{34}$$

where tildes denote the transformed functions of variable p and we have taken $f_l(v, t = 0) = 0$ as the initial value.

Integrating the first order ordinary differential equations (34), we obtain

$$\tilde{f}_l(v, p) = \frac{\tau_s}{v^3 + v_c^3} \left(\frac{v^3}{v^3 + v_c^3}\right)^{\frac{1}{6} l(l+1) Z_2} (v^3 + v_c^3)^{\frac{1}{3} p \tau_s} \int_v^\infty \tilde{S}_l(v', p) \left(\frac{v'^3 + v_c^3}{v'^3}\right)^{\frac{1}{6} l(l+1) Z_2} \frac{v'^2 dv'}{(v'^3 + v_c^3)^{\frac{1}{3} p \tau_s}}. \tag{35}$$

Inverting the Laplace transform and interchanging the Laplace integration with the velocity integration gives

$$f_l(v, t) = \frac{1}{2\pi i} \frac{\tau_s}{v^3 + v_c^3} \left(\frac{v^3}{v^3 + v_c^3}\right)^{\frac{1}{6} l(l+1) Z_2} \int_v^\infty dv' v'^2 \left(\frac{v'^3 + v_c^3}{v'^3}\right)^{\frac{1}{6} l(l+1) Z_2}$$
$$\times \int_{-i\infty}^{i\infty} dp \tilde{S}_l(v', p) \exp\left\{p\left[t - \frac{\tau_s}{3} \ln\left(\frac{v'^3 + v_c^3}{v^3 + v_c^3}\right)\right]\right\} \tag{36}$$

Recognizing that

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} dp \tilde{S}_l(v', p) \exp\left\{p\left[t - \frac{\tau_s}{3} \ln\left(\frac{v'^3 + v_c^3}{v^3 + v_c^3}\right)\right]\right\} = S_l\left[v', t - \frac{\tau_s}{3} \ln\left(\frac{v'^3 + v_c^3}{v^3 + v_c^3}\right)\right],$$

the expression (36) can be written in the form

$$f_l(v,t) = \frac{\tau_s}{v^3+v_c^3}\left(\frac{v^3}{v^3+v_c^3}\right)^{\frac{1}{6}l(l+1)Z_2}\int_v^\infty S_l\left[v', t-\frac{\tau_s}{3}\ln\left(\frac{v'^3+v_c^3}{v^3+v_c^3}\right)\right]\left(\frac{v'^3+v_c^3}{v'^3}\right)^{\frac{1}{6}l(l+1)Z_2} v'^2 dv'. \tag{37}$$

This is the generalization of (28) including time dependence. The particles which reach velocity v at time t were injected at velocity v' a time

$$\tfrac{1}{3}\tau_s \ln((v'^3+v_c^3)/(v^3+v_c^3)) = \tau_0(v')-\tau_0(v)$$

ago, where $\tau_0(v) = \frac{1}{3}\tau_s \ln((v^3+v_c^3)/v_c^3)$ is the time required for a particle of velocity v to slow down to $v = 0$ (Stix 1972).

For a source that is a delta function in pitch angle and energy, switched on at time $t = 0$ and held constant thereafter,

$$S(v,\xi,t) = (S^0/v^2)\,\delta(v-v_b)\,\delta(\xi-\xi_b)\,U(t),$$

the integral in (37) can be performed. The resulting beam distribution is

$$f_b(v,\xi,t) = \frac{S^0\tau_s}{v^3+v_c^3}U\left[t-\tfrac{1}{3}\tau_s\ln\left(\frac{v_b^3+v_c^3}{v^3+v_c^3}\right)\right]\sum_{l=0}^{\infty}\tfrac{1}{2}(2l+1)P_l(\xi)P_l(\xi_b)$$
$$\times\left[\frac{v^3}{v_b^3}\times\left(\frac{v_b^3+v_c^3}{v^3+v_c^3}\right)\right]^{\frac{1}{6}l(l+1)Z_2}U(v_b-v). \tag{38}$$

All of the injected particles slow down along the same characteristic trajectory because energy diffusion has been neglected. This produces a travelling front in the distribution function with no particles with velocity

$$v < v_{\min}(t) = [(v_b^3+v_c^3)\exp(-3t/\tau_s)-v_c^3]^{\frac{1}{3}}$$

for $t < \tau_0(v_b) = \frac{1}{3}\tau_s\ln((v_b^3+v_c^3)/v_c^3)$. As in (31), there are no particles with $v > v_b$.

For a delta function source that is switched on at $t = 0$ and switched off at $t = t_1$,

$$S(v,\xi,t) = (S^0/v^2)\,\delta(v-v_b)\,\delta(\xi-\xi_b)\,[U(t)-U(t-t_1)],$$

the beam distribution is

$$f_b(v,\xi,t) = \frac{S^0\tau_s}{v^3+v_c^3}\left\{U\left[t-\tfrac{1}{3}\tau_s\ln\left(\frac{v_b^3+v_c^3}{v^3+v_c^3}\right)\right]-U\left[t-t_1-\tfrac{1}{3}\tau_s\ln\left(\frac{v_b^3+v_c^3}{v^3+v_c^3}\right)\right]\right\}$$
$$\times\sum_{l=0}^{\infty}\tfrac{1}{2}(2l+1)P_l(\xi)P_l(\xi_b)\left[\frac{v^3}{v_b^3}\left(\frac{v_b^3+v_c^3}{v^3+v_c^3}\right)\right]^{\frac{1}{6}l(l+1)Z_2}U(v_b-v). \tag{39}$$

The distribution (39) has a second travelling front with no particles with velocity

$$v > v_{\max}(t) = [(v_b^3+v_c^3)\exp(-3(t-t_1)/\tau_s)-v_c^3]^{\frac{1}{3}} \quad \text{for} \quad t > t_1.$$

The beam number density is given by

$$N_b(t) = \int_0^\infty dv\,v^2\int_{-1}^{1}d\xi f_b(v,\xi,t). \tag{40}$$

Using the distribution (38) in (40)

$$N_b(t) = S^0\int_0^{\tau_0(v_b)}U[t+\tau_0(v)-\tau_0(v_b)]\,d\tau_0(v) = \begin{cases}S^0\tau_0(v_b) & \text{for} \quad t \geqslant \tau_0(v_b),\\ S^0 t & \text{for} \quad t \leqslant \tau_0(v_b).\end{cases} \tag{41}$$

where again $\tau_0(v) = \frac{1}{3}\tau_s \ln((v^3+v_c^3)/v_c^3)$. The expression (41) for $N_b(t)$ saturates after $t \geqslant \tau_0(v_b)$ because we have neglected the particles at $v = 0$ by assuming $v > v_i$ in deriving (16). We can include the beam particles with $v \lesssim v_i$ as another species of background ions.

5. Charge exchange

Including the effect of charge exchange in (32) gives,

$$\tau_s \frac{\partial f_b}{\partial t} = \frac{1}{v^2}\frac{\partial}{\partial v}(v^3+v_c^3)f_b + \frac{Z_2 v_c^3}{2v^3}\frac{\partial}{\partial \xi}(1-\xi^2)\frac{\partial f_b}{\partial \xi} - \frac{\tau_s}{\tau_{cx}(v)}f_b + \tau_s S, \tag{42}$$

where $\tau_{cx}(v)$ is the mean charge exchange lifetime of a particle of velocity v. Equation (42) can be solved by the same procedure that led from (32) to (38); the additional term $-\tau_s f_b/\tau_{cx}(v)$ multiplies the solution by the integrating factor

$$\exp\left\{-\tau_s \int_v^{v_b} \frac{v^2}{v^3+v_c^3}\frac{dv}{\tau_{cx}(v)}\right\}. \tag{43}$$

If we take $\tau_{cx}(v)$ to be a constant independent of velocity, the solution of (42) for a delta-function source will be

$$f_b(v,\xi,t) = \frac{S^0\tau_s}{v^3+v_c^3}\left(\frac{v^3+v_c^3}{v_b^3+v_c^3}\right)^{\tau_s/3\tau_{cx}} U\left[t-\frac{\tau_s}{3}\ln\left(\frac{v_b^3+v_c^3}{v^3+v_c^3}\right)\right]\sum_{l=0}^{\infty}\frac{2l+1}{2}P_l(\xi)P_l(\xi_b)$$
$$\times\left[\frac{v^3}{v_b^3}\left(\frac{v_b^3+v_c^3}{v^3+v_c^3}\right)\right]^{\frac{1}{6}l(l+1)Z_2} U(v_b-v). \tag{44}$$

The effect of charge exchange is important when $\tau_{cx} \leqslant \frac{1}{3}\tau_s$. In typical ORMAK discharges $\tau_s/\tau_{cx} \sim$ 0–5 and charge exchange effects can be significant (Callen *et al.* 1974). In ATC, $\tau_{cx} \approx 10$ msec for neutral densities $n_0 = 10^9\ \mathrm{cm^{-3}}$ and $\tau_s \approx$ 10–80 msec (Goldston 1975).

The beam number density $N_b(t)$ is given by (40). Substituting the beam distribution (44) in (40) gives

$$N_b(t) = S^0\tau_{cx}\begin{cases}1-\exp(-\tau_0(v_b)/\tau_{cx}) & \text{for} \quad t \geqslant \tau_0(v_b),\\ 1-\exp(-t/\tau_{cx}) & \text{for} \quad t \leqslant \tau_0(v_b).\end{cases} \tag{45}$$

For $\tau_{cx} \geqslant \tau_0(v_b) \geqslant t$, (45) reduces to (41) which was calculated neglecting charge exchange.

The approximation, made in (44), that $\tau_{cx}(v)$ is independent of the particle energy is not very realistic. The charge exchange cross section is constant at low energy but falls off as $1/E$ for high energy E. Hamasaki & Hui, in a private communication, have fitted the experimental charge exchange cross section for hydrogen (Rose & Clark 1961) by the simple expression

$$\sigma_{cx}(v) = \frac{21\times10^{-16}\ \mathrm{cm}^2}{(0{\cdot}2)(\frac{1}{2}Mv^2)+1}, \quad \text{where} \quad \tfrac{1}{2}Mv^2 \quad \text{is in keV.}' \tag{46}$$

A more rigorous treatment of charge exchange has been given by Sigmar & Clarke (1976). The charge exchange rate is $\tau_{cx}^{-1}(v) = n_n\sigma_{cx}(v)\,v$, where n_n is the neutral particle density. Using the expression (46) in (43) allows one to perform the integration in terms of elementary functions. For $E_b = \frac{1}{2}M_b v_b^2 \gg 5$ keV, the constant 1 in the denominator of (46) is negligible and the beam distribution is given by

$$f_b(v,\xi,t) = \frac{S^0\tau_s}{v^3+v_c^3}\left[\left(\frac{v^3+v_c^3}{v_b^3+v_c^3}\right)^{\frac{1}{2}}\left(\frac{v_b+v_c}{v+v_c}\right)^{\frac{3}{2}}\right]^{\tau_s/3\tau_{cx}}$$
$$\times\exp\left\{-\frac{\tau_s}{3\tau_{cx}}\left[\sqrt{3}\tan^{-1}\left(\frac{2v_b-v_c}{\sqrt{3}v_c}\right)-\sqrt{3}\tan^{-1}\left(\frac{2v-v_c}{\sqrt{3}v_c}\right)\right]\right\}U\left[t-\frac{\tau_s}{3}\ln\left(\frac{v_b^3+v_c^3}{v^3+v_c^3}\right)\right]$$
$$\times\sum_{l=0}^{\infty}\frac{2l+1}{2}P_l(\xi)P_l(\xi_b)\left[\frac{v^3}{v_b^3}\left(\frac{v_b^3+v_c^3}{v^3+v_c^3}\right)\right]^{\frac{1}{6}l(l+1)Z_2}U(v_b-v). \quad (47)$$

6. Properties of the solution to the Fokker–Planck equation

The distribution given by (47) converges rapidly for velocities $v \ll v_b$. In this region of velocity space the distribution is rather spread out and only the first few Legendre polynomials contribute. For velocities near the injection velocity v_b, the distribution is sharply peaked in pitch angle around $\theta_b = \cos^{-1}\xi_b$ and the series given by (47) converges very slowly because all of the Legendre polynomials contribute about equally. Indeed, for $v \approx v_b$, the coefficient

$$\left[\frac{v^3}{v_b^3}\left(\frac{v_b^3+v_c^3}{v^3+v_c^3}\right)\right]^{\frac{1}{6}l(l+1)Z_2} \approx 1$$

and the sum
$$\sum_{l=0}^{\infty}\frac{2l+1}{2}P_l(\xi_b)P_l(\xi) = \delta(\xi_b-\xi)$$

which is just the completeness relation for the Legendre polynomials. Thus, for velocities in the vicinity of v_b, the distribution given by (47) becomes

$$f_b(v,\xi) \approx \frac{S^0\tau_s}{v^3+v_c^3}\delta(\xi_b-\xi)\,U(v_b-v). \quad (48)$$

The steady state beam distribution (31) as well as the more complete beam distribution (47) have been discussed previously by the author (Gaffey 1974; Gaffey & Thompson 1974; Gaffey 1975). The beam distribution has been calculated numerically from (47) for $v \ll v_b$ and from the asymptotic solutions given in the following sections for $v \approx v_b$ and $v > v_b$. The contour plot of the distribution, displayed in figure 2, clearly shows the singular nature of the distribution near the source.

A steady state beam distribution similar to (31) has been obtained recently by Berk *et al.* (1975).

Rome *et al.* (1975) have obtained a beam distribution similar to (44) by expanding the drift kinetic equation in multiple time scales and averaging over a bounce period. They find that at low velocities the beam distribution is broad in pitch angle but for velocities near the injection velocity the beam distribution is quite anisotropic in pitch angle with about half the beam ions lying between the angle of injection and the magnetic axis for the case of nearly parallel injection. Cordey & Core (1974) have solved the steady state Fokker–Planck equation using a WKBJ technique. Expanding their solution to zeroth order yields a distribution similar to (44) without the time dependence.

Ohkawa (1970), Cordey & Houghton (1973), and Stix (1973) have obtained beam distributions similar to (48). Cordey & Houghton started with a Fokker–Planck equation similar to (20) and for velocities greater than the crossover velocity, they neglected the perpendicular diffusion term in (20) and obtained (48) directly. Similarly, Stix assumed $f_b(v,\xi) = f(v)\,\delta(\cos\theta - 1)$ for injection along B_0 and solved the one-dimensional Fokker–Planck equation obtaining a result similar to (48). However, Stix has used the distribution (48) for $v = 0{\cdot}526 v_c$ where $v^3 \ll v_c^3$, and the approximation $v^3 \approx v_c^3$ used in deriving (48) is no longer valid.

7. Asymptotic solution near the source

To find the distribution near the source, we return to the Fokker–Planck equation in the form (42)

$$\tau_s \frac{\partial f_b}{\partial t} = \frac{1}{v^2}\frac{\partial}{\partial v}(v^3+v_c^3) f_b + \frac{v_c^3 Z_2}{2v^3}\frac{\partial}{\partial \xi}(1-\xi^2)\frac{\partial f_b}{\partial \xi} - \frac{\tau_s}{\tau_{cx}} f_b + \tau_s S. \tag{49}$$

The time derivatives can be incorporated into the drag term using the characteristic $v(t) = [(v_b^3+v_c^3)\exp(-3t/\tau_s) - v_c^3]^{\frac{1}{3}}$. The charge exchange term multiplies the distribution by the exponential factor (43) which is of order unity near the source. We assume a delta function source, as in (29), with $\xi_b = 1$ (parallel injection). Near the source $v \lesssim v_b$ and $\xi \lesssim 1$ and we introduce dimensionless local variables $w = (v_b - v)/v_b \ll 1$ and $\mu = 1 - \xi \ll 1$. To lowest order in w and μ (49) becomes

$$\frac{\partial f_b}{\partial w} = \frac{v_c^3 Z_2}{v_b^3 + v_c^3}\frac{\partial}{\partial \mu}\left(\mu \frac{\partial f_b}{\partial \mu}\right) + \frac{\tau_s S^0}{v_b^3+v_c^3}\delta(\mu)\,\delta(w). \tag{50}$$

Equation (50) is in the form of a diffusion equation and the solution is

$$f_b(w,\mu) = \frac{\tau_s S^0}{v_c^3 Z_2}\frac{1}{w}\exp\left[-\left(\frac{v_b^3+v_c^3}{v_c^3 Z_2}\right)\frac{\mu}{w}\right]U(w), \tag{51}$$

where U is a step function. In our original variables v and ξ,

$$f_b(v,\xi) = \frac{\tau_s S^0}{v_c^3 Z_2}\frac{v_b}{v_b - v}\exp\left[-\left(\frac{v_b^3+v_c^3}{v_c^3 Z_2}\right)\frac{v_b(1-\xi)}{v_b - v}\right]U(v_b - v), \tag{52}$$

and in terms of the pitch angle $\theta = \cos^{-1}\xi$

$$f_b(v,\theta) = \frac{S^0 \tau_s}{v_c^3 Z_2}\frac{v_b}{v_b - v}\exp\left[-2\left(\frac{v_b^3+v_c^3}{Z_2 v_c^3}\right)\frac{v_b \sin^2\frac{1}{2}\theta}{v_b - v}\right]U(v_b - v). \tag{53}$$

For small pitch angles $\theta \ll 1$, the distribution (53) is nearly Gaussian in $\theta \approx v_\perp/v$. We also note that as v approaches v_b, (52) and (53) become delta functions,

$$\operatorname*{Limit}_{v\to v_b} f_b(v,\xi) = \frac{S^0 \tau_s}{v_b^3+v_c^3}\delta(1-\xi)\,U(v_b - v), \tag{54}$$

which is in agreement with the distribution (48). For nearly parallel injection, $\xi_b \lesssim 1$,

$$f_b(v,\xi) = \frac{S^0 \tau_s}{v_c^3 Z_2}\frac{v_b}{v_b - v}\exp\left[-\left(\frac{v_b^3+v_c^3}{v_c^3 Z_2}\right)\left(\frac{2-\xi_b-\xi}{v_b - v}\right)v_b\right] \times I_0\left[2\left(\frac{v_b^3+v_c^3}{v_c^3 Z_2}\right)\frac{(1-\xi_b)^{\frac{1}{2}}(1-\xi)^{\frac{1}{2}}}{(v_b - v)}v_b\right]U(v_b - v), \tag{52'}$$

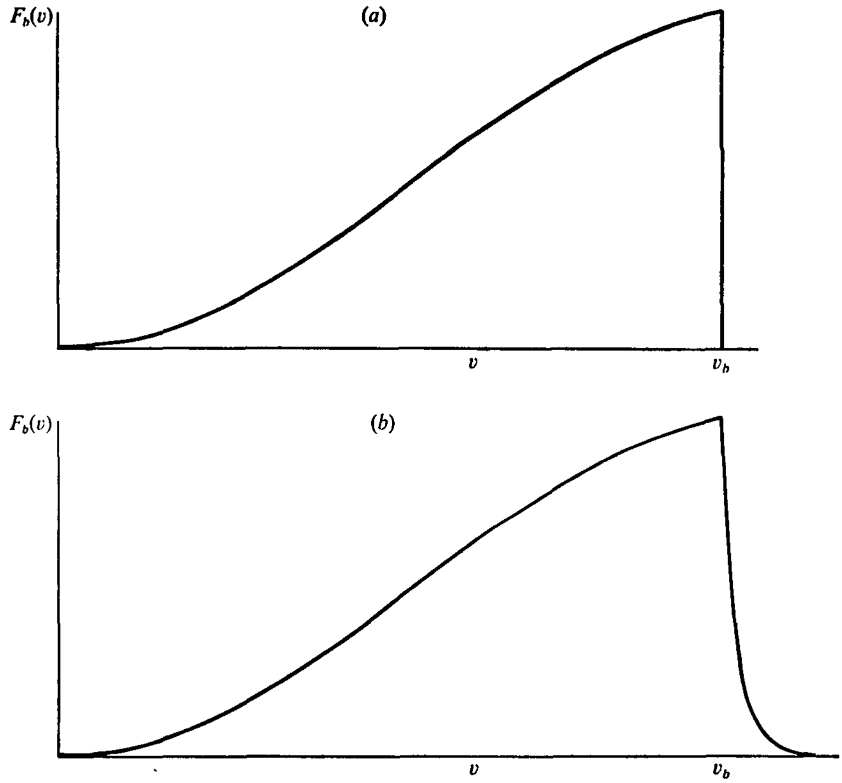

FIGURE 1. Plot of the one-dimensional distribution $F_b(v) = v^2 f_b(v)$ for $v_b = v_c$ and $T_e/T_i = 4$. Shown in (a) is the distribution (48), without energy diffusion, cut off at $v = v_b$. Shown in (b) is the distribution (60) and (61), including energy diffusion, with a high energy tail for $v > v_b$.

where I_0 is a modified Bessel function. As v approaches v_b,

$$\operatorname*{Limit}_{v \to v_b} f_b(v, \xi) = \frac{S^0 \tau_s}{v_b^3 + v_c^3} \delta(\xi_b - \xi)\, U(v_b - v), \tag{54'}$$

which is in agreement with (48).

8. Energy diffusion and the high energy tail

Owing to collisions with faster moving electrons, some of the energetic ions will have velocities greater than the injection velocity v_b and the energetic ion distribution will have a high energy tail rather than being sharply cut off at $v = v_b$, as shown in figure 1. We now include energy diffusion in our model Fokker–Planck equation. Returning to the collision operator for beam-electron collisions (14) and for beam-ion collisions (16), we keep the terms proportional to $\partial^2 f/\partial v^2$, which were neglected in the preceding sections, and the steady-state Fokker–Planck equation becomes

$$\tau_s \frac{\partial f_l}{\partial t} = \frac{1}{2v^3}\left(\frac{m_e}{M_b} v_e^2 v^3 + v_b^2 v_c^3 Z_3\right)\frac{\partial^2 f_l}{\partial v^2} + \frac{1}{v^2}\frac{\partial}{\partial v}(v^3 + v_c^3) f_l$$
$$- \frac{l(l+1)\, v_c^3 Z_2}{2v^3} f_l - \frac{\tau_s}{\tau_{cx}} f_l + \frac{(2l+1)}{2} S^0 \tau_s P_l(\xi_b) \frac{1}{v^2} \delta(v_b - v) = 0, \tag{55}$$

where $$Z_3 = \sum_i \frac{n_i Z_i^2 v_i^2}{n_e Z_1 v_b^2},$$

and, as in (29), we have assumed a delta function source in velocity and pitch angle, expanded the distribution in Legendre polynomials, and included the charge exchange term.

For velocities less than the injection velocity v_b, the energy diffusion term has negligible effect and we find a solution of the form (44) for the homogeneous equation

$$f_b(v,\xi) = \frac{S^0 \tau_s}{v^3+v_c^3}\left(\frac{v^3+v_c^3}{v_b^3+v_c^3}\right)^{\tau_s/3\tau_{cx}} \sum_{l=0}^{\infty} \frac{(2l+1)P_l(\xi_b)P_l(\xi)}{1+(1+4AC_l/B^2)^{\frac{1}{2}}+2AC_l/B^2}\left[\frac{v^3}{v_b^3}\left(\frac{v_b^3+v_c^3}{v^3+v_c^3}\right)\right]^{\frac{1}{6}l(l+1)Z_2} \quad \text{for} \quad v \leqslant v_b, \tag{56}$$

where τ_{cx} has been assumed to be independent of velocity, the time dependence has been omitted, and A, B, and C_l are defined below.

For velocities greater than the injection velocity v_b, we define a dimensionless inner variable $w = (v-v_b)/v_b \ll 1$ and look for a boundary layer solution. To lowest order in w, (55) becomes

$$A\frac{\partial^2 f_l}{\partial w^2} + B\frac{\partial f_l}{\partial w} - C_l f_l = -\frac{\tau_s S_l}{v_b^3}\delta(w), \tag{57}$$

where
$$A = \frac{1}{2}\left(\frac{m_e v_e^2}{M_b v_b^2} + \frac{v_c^3 Z_3}{v_b^3}\right) = \frac{1}{2}\frac{T_e}{E_b}\left[1+\frac{E_b}{T_e}\left(\frac{E_c}{E_b}\right)^{\frac{3}{2}} Z_3\right] \lesssim \tfrac{1}{30}$$

$$B = \frac{v_b^3+v_c^3}{v_b^3} \approx 1, \quad C_l = 3\left[\frac{l(l+1)v_c^3 Z_2}{6v_b^3} + \frac{\tau_s}{3\tau_{cx}} - 1\right] \sim 1$$

and $S_l = \frac{1}{2}(2l+1)S^0 P_l(\xi_b)$. Since (57) is an ordinary differential equation with constant coefficients, we obtain an exponential solution for the homogeneous equation. In our original variables,

$$f_{b+}(v,\xi) = \frac{S^0\tau_s}{v_b^3+v_c^3}\sum_{l=0}^{\infty}\frac{(2l+1)P_l(\xi_b)P_l(\xi)}{1+(1+4AC_l/B^2)^{\frac{1}{2}}+2AC_l/B^2}\exp\left\{-\frac{1}{2A}\left[\frac{(v-v_b)B}{v_b} + \frac{(v-v_b)}{v_b}(B^2+4AC_l)^{\frac{1}{2}}\right]\right\} \quad \text{for} \quad v > v_b, \tag{58}$$

where we have chosen the positive square root to ensure that $f_{b+} \to 0$ as $v \to \infty$. In a recent preprint, Goldston has obtained exponential boundary layer solutions for the distribution for $v > v_b$ and $v < v_b$ which are similar to (58) for $v > v_b$. These distributions are in good agreement with the charge exchange neutral measurements on ATC (Goldston 1975) for velocities less than but nearly equal to v_b and for velocities greater than v_b; however, the agreement is not as good for velocities considerably less than v_b where the polynomial distribution (56) may give a better fit than an exponential distribution.

The coefficients in (56) and (58) were chosen to satisfy the boundary conditions at the source:

(*a*) that the distribution be continuous at v_b

$$f_{b-}(v_b,\xi) = f_{b+}(v_b,\xi), \tag{59a}$$

(*b*) that the discontinuity in the first derivative at v_b be equal to the source strength

$$Av_b^2 \frac{\partial f_l}{\partial v}\bigg|_{v_b-\delta v}^{v_b+\delta v} = -\frac{\tau_s S_l}{v_b^2}. \tag{59b}$$

The boundary condition (59*b*) is obtained by integrating the differential equation (55) from $v_b - \delta v$ to $v_b + \delta v$ across the source. This condition is analogous to the boundary condition that without energy diffusion the discontinuity of the distribution (31) at v_b be equal to the source strength.

For typical tokamak injection schemes $A \approx \frac{1}{2}(T_e/E_b) \lesssim \frac{1}{30}$, $B \approx 1$ and $C_l \sim 1$ (for small l) so that we can neglect $AC_l/B^2 \ll 1$. With this approximation, the distribution (56) is identical to (44), omitting the time dependence. For $v \lesssim v_b$ (56) becomes

$$f_{b-}(v,\xi) \approx \frac{S^0 \tau_s}{v^3 + v_c^3}\delta(\xi_b - \xi), \quad v \lesssim v_b, \tag{60}$$

in agreement with (48). For $v > v_b$ (58) becomes

$$f_{b+}(v,\xi) \approx \frac{S^0\tau_s}{v_b^3+v_c^3}\,\delta(\xi_b-\xi)\exp\left\{-\frac{2(v-v_b)\,(v_b^3+v_c^3)}{(m_e/M_b)\,v_e^2 v_b^2 + v_c^2 v_b^2 Z_3}\right\}, \quad v > v_b. \tag{61}$$

The one dimensional distributions $F_b(v) = v^2 f_b(v)$ corresponding to (60) and (61) are plotted in figure 1.

Far out in the high energy tail $v \gg v_b \gtrsim v_c$, the pitch angle scattering term is small and the steady state Fokker–Planck equation (55) becomes

$$\tau_s \frac{\partial f_b}{\partial t} = \epsilon v_b^2 \frac{\partial^2 f_b}{\partial v^2} + v\frac{\partial f_b}{\partial v} + 3f_b = 0, \tag{62}$$

where $\epsilon = m_e v_e^2/2M_b v_b^2 = \frac{1}{2}(T_e/E_b)$ and terms of order v_c^3/v^3 have been neglected. Equation (62) has the general solution

$$\begin{aligned} f_b(v,\xi) &= \frac{S^0\tau_s}{v_b^3+v_c^3}\delta(\xi_b-\xi)\,\epsilon v_b^2\frac{d^2}{dv^2}\exp\left(-\frac{v^2}{2\epsilon v_b^2}\right)\left[c_1 + c_2\int^{v/\sqrt{\epsilon v_b}} e^{\frac{1}{2}t^2}dt\right] \\ &= \frac{S^0\tau_s}{v_b^3+v_c^3}\delta(\xi_b-\xi)\left\{\left(\frac{v^2}{\epsilon v_b^2}-1\right)\exp\left(-\frac{v^2}{2\epsilon v_b^2}\right)\left[c_1+c_2\int^{v/\sqrt{\epsilon v_b}} e^{\frac{1}{2}t^2}dt\right] - \frac{c_2 v}{\sqrt{\epsilon v_b}}\right\}, \\ &\qquad \text{for} \quad v \gg v_b \gtrsim v_c. \end{aligned} \tag{63}$$

The first factor in (63) is a Hermite polynomial of order two. The integral in (63) is Dawson's integral (Abramowitz & Stegun 1966) and it may also be expressed as an error function of imaginary argument or as the real part of the plasma dispersion function (Fried & Conte 1961) of real argument. The arbitrary constants c_1 and c_2 are chosen to make f_b and $\partial f_b/\partial v$ continuous when matching the boundary layer solution (61) to the expression (63).

9. Electric field

Including an electric field parallel to the magnetic field in the model Fokker–Planck equation (42) gives

$$\tau_s\left\{\frac{\partial f_b}{\partial t}+\frac{eE_{\text{eff}\parallel}}{M_b}\left[\xi\frac{\partial f_b}{\partial v}+\frac{1}{v}(1-\xi^2)\frac{\partial f_b}{\partial \xi}\right]\right\}$$
$$=\frac{1}{v^2}\frac{\partial}{\partial v}(v^3+v_c^3)f_b+\frac{v_c^3 Z_2}{2v^3}\frac{\partial}{\partial \xi}(1-\xi^2)\frac{\partial f_b}{\partial \xi}-\frac{\tau_s}{\tau_{cx}}f_b+\tau_s S, \quad (64)$$

where
$$E_{\text{eff}\parallel}=E_\parallel\left(1-\frac{Z_b}{Z_1 Z_2}\right)$$

is the effective electric field, including the neoclassical correction due to electron motion opposite to the ion motion in the electric field (Furth & Rutherford 1972). The second term on the left-hand side of (64) is acceleration due to the electric field. The third term is the distortion of the distribution caused by the electric field.

We expand the distribution in Legendre polynomials and, as in (29), we assume a delta function source in velocity and pitch angle. The time derivative in (64) can be incorporated into the drag term by using the characteristic

$$v(t)=[(v_b^3+v_c^3)\,e^{-3t/\tau_s}-v_c^3]^{\frac{1}{3}}$$

and it multiplies the distribution by the step function $U[v-v(t)]$ as in (38). The charge exchange term multiplies the distribution by the exponential factor given by (43). These terms will not be discussed further in this section.

Using the recurrence relations for Legendre polynomials (Abramowitz & Stegun 1966)

$$(2l+1)\,\xi P_l(\xi)=(l+1)\,P_{l+1}(\xi)+lP_{l-1}(\xi),$$
$$(2l+1)\,P_l(\xi)=\frac{dP_{l+1}}{d\xi}-\frac{dP_{l-1}}{d\xi}$$

and the differential equation

$$\frac{d}{d\xi}(1-\xi^2)\frac{dP_l}{d\xi}+l(l+1)\,P_l(\xi)=0,$$

we find that the electric field couples the lth Legendre component to the $(l+1)$th and $(l-1)$th components.

$$\frac{eE_{\text{eff}\parallel}\tau_s}{M_b v}\left[\left(\frac{l+1}{2l+3}\frac{v\,df_{l+1}}{dv}+\frac{(l+1)(l+2)}{2l+3}f_{l+1}\right)+\left(\frac{l}{2l-1}\frac{v\,df_{l-1}}{dv}-\frac{l(l-1)}{2l-1}f_{l-1}\right)\right]$$
$$=\frac{1}{v^2}\frac{d}{dv}(v^3+v_c^3)f_l-\frac{l(l+1)\,v_c^3 Z_2}{2v^3}f_l+\tau_s S_l\frac{1}{v^2}\delta(v_b-v) \quad (65)$$

where $S_l=S^0\,\frac{1}{2}(2l+1)\,P_l(\xi_b)$. Thus, (65) is an infinite, tridiagonal set of first order, linear ordinary differential equations, whereas the set (26) without the electric field were decoupled.

For beam velocities near the injection velocity v_b, the coupling parameter in (65) is

$$\frac{eE_{\text{eff}\parallel}\tau_s}{M_b v_b}=\frac{E_{\text{eff}\parallel}M_b v_c^3}{4\pi n_e e^3 Z_b^2 Z_1 v_b \ln\Lambda}=\frac{14\cdot 8}{Z_b^2 Z_1^{\frac{1}{3}}}\left(\frac{M_b}{M_H}\right)^{\frac{1}{3}}\frac{E_{\text{eff}\parallel}}{E_D}\frac{v_c}{v_b},$$

where the Dreicer (1960) runaway electric field $E_D = 4\pi n_e e^3 \ln\Lambda/m_e v_{\mathrm{th}_e}^2$. The electric field is usually much less than the Dreicer field in typical tokamak experiments and we can expand the distribution as

$$f_l(v) = f_l^0(v) + f_l^1(v)$$

where f_l^1/f_l^0 is of order $E_{\mathrm{eff}\parallel}/E_D \ll 1$. The zeroth-order equations are identical to (26) for a delta function source and the solution is given by (31),

$$f_l^0(v) = \frac{\tau_s S_l}{v^3+v_c^3}\left[\frac{v^3}{v_b^3}\left(\frac{v_b^3+v_c^3}{v^3+v_c^3}\right)\right]^{\frac{1}{3}l(l+1)Z_2} U(v_b - v). \tag{66}$$

The first order equations are decoupled.

$$\frac{1}{v^2}\frac{d}{dv}(v^3+v_c^3)f_l^1 - \frac{l(l+1)v_c^3 Z_2}{2v^3}f_l^1 = \frac{eE_{\mathrm{eff}\parallel}\tau_s}{M_b v}\left[\frac{l+1}{2l+3}\left(\frac{v\,df_{l+1}^0}{dv}+(l+2)f_{l+1}^0\right)\right.$$
$$\left.+\frac{l}{2l-1}\left(\frac{v\,df_{l-1}^0}{dv}-(l-1)f_{l-1}^0\right)\right]. \tag{67}$$

These differential equations are similar in form to (26) and they are solved using the integrating factor given in (28) with the right hand side of (67) as the source function.

$$f_l^1(v) = \frac{eE_{\mathrm{eff}\parallel}\tau_s}{M_b(v^3+v_c^3)}\left(\frac{v^3}{v^3+v_c^3}\right)^{\frac{1}{6}l(l+1)Z_2}\int^v dv\,v\left(\frac{v^3+v_c^3}{v^3}\right)^{\frac{1}{6}l(l+1)Z_2}$$
$$\times\left[\frac{l+1}{2l+3}\left(\frac{v\,df_{l+1}^0}{dv}+(l+2)f_{l+1}^0\right)+\frac{l}{2l-1}\left(\frac{v\,df_{l-1}^0}{dv}-(l-1)f_{l-1}^0\right)\right]. \tag{68}$$

Using the zeroth order part of the differential equation (65) to express $df_{l\pm1}^0/dv$ in terms of $f_{l\pm1}^0$ and $S_{l\pm1}(1/v^2)\,\delta(v_b-v)$,

$$f_l^1(v) = \frac{eE_{\mathrm{eff}\parallel}\tau_s}{M_b(v^3+v_c^3)}\frac{v_b^2}{v_b^3+v_c^3}\left[\frac{l+1}{2l+3}\tau_s S_{l+1}+\frac{l}{2l-1}\tau_s S_{l-1}\right]\left[\frac{v^3}{v_b^3}\left(\frac{v_b^3+v_c^3}{v^3+v_c^3}\right)\right]^{\frac{1}{6}l(l+1)Z_2}$$
$$\times U(v_b-v) - \frac{eE_{\mathrm{eff}\parallel}\tau_s}{M_b}f_l^0(v)\int_v^{v_b}\frac{dv\,v}{v^3+v_c^3}\frac{1}{f_l^0(v)}\left[\frac{l+1}{2l+3}\left(\frac{(l+1)(l+2)v_c^3 Z_2}{2(v^3+v_c^3)}\right.\right.$$
$$\left.\left.-\frac{3v^3}{v^3+v_c^3}+l+2\right)f_{l+1}^0(v)+\frac{l}{2l-1}\left(\frac{l(l-1)v_c^3 Z_2}{2(v^3+v_c^3)}-\frac{3v^3}{v^3+v_c^3}-(l-1)\right)f_{l-1}^0(v)\right]. \tag{69}$$

The integrated terms in (69) can be absorbed in zeroth order distribution

$$\tilde{f}_l^0(v) = f_l^0(v)\left[1+\frac{eE_{\mathrm{eff}\parallel}\tau_s}{M_b}\frac{v_b^2}{v_b^3+v_c^3}\left(\frac{l+1}{2l+3}\frac{S_{l+1}}{S_l}+\frac{l}{2l-1}\frac{S_{l-1}}{S_l}\right)\right]. \tag{70}$$

Using the expression (66) for $f_l^0(v)$, the integral in (69) can be simplified

$$\tilde{f}_l^1(v) = -\frac{eE_{\mathrm{eff}\parallel}\tau_s}{M_b}f_l^0(v)\int_v^{v_b}\frac{dv\,v}{v^3+v_c^3}\left[\left[\frac{l+1}{2l+3}\left(\frac{(l+1)(l+2)v_c^3 Z_2}{2(v^3+v_c^3)}-\frac{3v^3}{v^3+v_c^3}+l+2\right)\right.\right.$$
$$\times\frac{S_{l+1}}{S_l}\left[\frac{v^3}{v_b^3}\left(\frac{v_b^3+v_c^3}{v^3+v_c^3}\right)\right]^{\frac{1}{3}(l+1)Z_2}+\frac{l}{2l-1}\left(\frac{l(l-1)v_c^3 Z_2}{2(v^3+v_c^3)}-\frac{3v^3}{v^3+v_c^3}-(l-1)\right)$$
$$\left.\left.\times\frac{S_{l-1}}{S_l}\left[\frac{v^3}{v_b^3}\left(\frac{v_b^3+v_c^3}{v^3+v_c^3}\right)\right]^{-\frac{1}{3}lZ_2}\right]\right]. \tag{71}$$

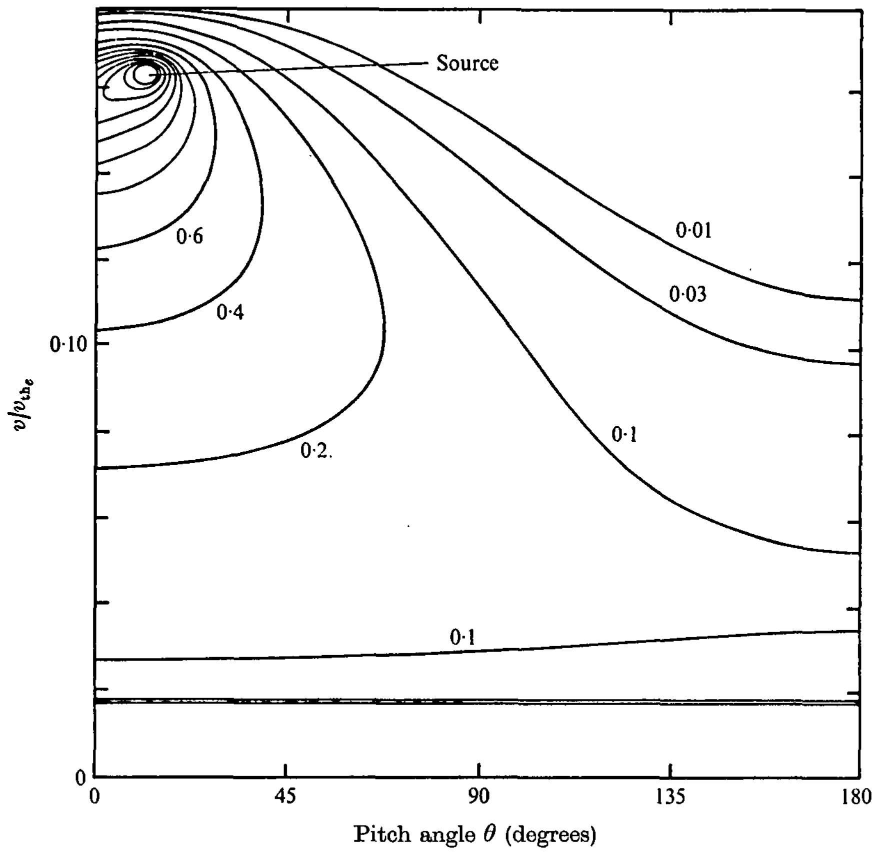

FIGURE 2. Contour plot of the three dimensional distribution (47), (56) and (58), for v vs θ, where $v_b = 1{\cdot}31\ v_c$, $\theta_b = 12°$ and $T_e/T_i = 4$.

The expression (71) does not agree with the expression (11) recently published by Cordey & Core (1974). In particular Cordey & Core have incorrect coefficients depending on l, have neglected the terms $3v^3/(v^3+v_c^3)$, and have incorrect powers of $[v^3/(v^3+v_c^3)]$.

The perturbation to the number density is determined by $f_0^1(v)$. As an example, we have explicitly evaluated $f_0(v) = \tilde{f}_0^0(v) + \tilde{f}_0^1(v)$.

$$f_0(v) = f_0^0(v) + \frac{eE_{\mathrm{eff}\parallel}\tau_s}{3M_b}\frac{v^2}{v^3+v_c^3}f_1^0(v). \tag{72}$$

The effect of the electric field is to increase $f_0(v)$ for parallel injection and to decrease $f_0(v)$ for antiparallel injection.

10. Conclusions

We have derived the appropriate Fokker–Planck equation for a fast ion beam interacting with a background plasma consisting of Maxwellian ions and electrons. With the assumption that the beam distribution is axisymmetric about the magnetic field, the Fokker–Planck equation can be expanded in an infinite sum of Legendre polynomials and the resulting first order linear differential equations can be solved exactly for an arbitrary source function. An explicit soiution has been given for the case of a delta function source. The resulting beam distribution function is somewhat spread out in pitch angle for velocities much less than the injection velocity, but for v near v_b the beam distribution function is sharply peaked in pitch angle. These features are clearly seen in the contour plot given in figure 2. Asymptotic expressions have been given for the distribution in the vicinity of the injected beam and for velocities greater than the injection velocity. The effect of a weak parallel electric field has also been given.

Several recent preprints (Berk *et al.* 1975; Goldston 1975; Rome *et al.* 1976) were received while this work was in progress and a comparison of our results with the work of others has been given. Our energetic ion distribution is in good agreement with measurements of charge exchange neutrals in ATC (Goldston 1975) and in ORMAK (Berry *et al.* 1975; Callen *et al.* 1975). The contour plot of the distribution given in figure 2 is in good qualitative agreement with the numerical solutions of the Fokker–Planck equation (Kulsrud *et al.* 1973) for neutral beam injection (Kulsrud, private communication).

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Appendix A

In this appendix we evaluate the potential

$$F(x_{ij}) = \frac{1}{n_{0_j} v_{\mathrm{th}_j}} \int f(v_j)\, g_{ij}\, d^3v_j \tag{A 1}$$

defined in (5) where $g_{ij} = |\mathbf{v}_i - \mathbf{v}_j|$ and $x_{ij} = v_i/v_{\mathrm{th}_j}$. For a Maxwellian distribution,

$$f(v_j) = \frac{n_{0j}}{\pi^{\frac{3}{2}} v_{\mathrm{th}_j}^3} \exp\left(-v_j^2/v_{\mathrm{th}_j}^2\right), \tag{A 2}$$

$$F(x_{ij}) = \frac{2}{\sqrt{\pi} v_{\mathrm{th}_j}^4} \int_0^\infty dv_j\, v_j^2 \exp\left(-v_j^2/v_{\mathrm{th}_j}^2\right) \int_0^\pi d\theta \sin\theta\, (v_i^2 - 2v_i v_j \cos\theta + v_j^2)^{\frac{1}{2}}, \tag{A 3}$$

where the z axis has been chosen to lie along $\mathbf{v}_i$. Performing the angular integration in (A 3),

$$F(x_{ij}) = \frac{2}{3\sqrt{\pi} v_{\mathrm{th}_j}^4} \int_0^\infty \frac{dv_j\, v_j}{v_i} \exp\left(-v_j^2/v_{\mathrm{th}_j}^2\right) \left[(v_i + v_j)^3 - |v_i - v_j|^3\right]. \tag{A 4}$$

Separating the radial integral into two parts to remove the absolute value in (A 4),

$$F(x_{ij}) = \frac{4}{3\sqrt{\pi}v_{\mathrm{th}_j}^4}\left[\frac{1}{v_i}\int_0^{v_i} dv_j v_j^2(3v_i^2+v_j^2)\exp(-v_j^2/v_{\mathrm{th}_j}^2) + \int_{v_i}^{\infty} dv_j v_j(v_i^2+3v_j^2)\exp(-v_j^2/v_{\mathrm{th}_j}^2)\right]. \quad \text{(A 5)}$$

Defining the dimensionless variable $t = v_j/v_{\mathrm{th}_j}$, the dimensionless parameter $x = v_i/v_{\mathrm{th}_j}$ and suppressing the subscripts i and j, (A 5) becomes

$$F(x) = \frac{4}{3\sqrt{\pi}}\left(\frac{1}{x}\int_0^x dt\, t^2(t^2+3x^2)\,e^{-t^2} + \int_x^{\infty} dt\, t(3t^2+x^2)\,e^{-t^2}\right). \quad \text{(A 6)}$$

The integral from 0 to x can be evaluated in terms of error functions and the integral from x to ∞ is elementary.

$$F(x) = (x+1/2x)\,\Phi(x) + \tfrac{1}{2}\Phi'(x), \quad \text{(A 7)}$$

where the error function $\Phi(x) = (2/\sqrt{\pi})\int_0^x e^{-t^2}dt$ and its derivative

$$\Phi'(x) = (2/\sqrt{\pi})\,e^{-x^2}.$$

The expression (A 7) is the result used in (9).

Appendix B

In this appendix we shall evaluate various expressions used in §2.

We define g to be the magnitude of the relative velocity between species i and species j

$$g = |\mathbf{g}| = |\mathbf{v}_i - \mathbf{v}_j|. \quad \text{(B 1)}$$

$$\frac{\partial g}{\partial \mathbf{v}_i} = \frac{\mathbf{g}}{g} = \hat{g}, \quad \text{a unit vector} \quad \text{(B 2)}$$

$$\frac{\partial^2 g}{\partial \mathbf{v}_i \partial \mathbf{v}_i} = \frac{1}{g^3}(g^2\mathsf{I} - \mathbf{g}\mathbf{g}) \equiv \boldsymbol{\omega}, \quad \text{(B 3)}$$

where I is the unit tensor and $\boldsymbol{\omega}$, which annihilates the component parallel to $\hat{g}$, is the tensor used in the Landau form of the Fokker–Planck collision operator (2).

$$\frac{\partial}{\partial \mathbf{v}_i}\cdot\boldsymbol{\omega} = \frac{\partial}{\partial \mathbf{v}_i}\cdot\left[\frac{1}{g^3}(g^2\mathsf{I} - \mathbf{g}\mathbf{g})\right] = -\frac{2\mathbf{g}}{g^3}, \quad \text{(B 4)}$$

$$\frac{\partial^2}{\partial \mathbf{v}_i \partial \mathbf{v}_i} : \boldsymbol{\omega} = -2\frac{\partial}{\partial \mathbf{v}_i}\cdot\left(\frac{\mathbf{g}}{g^3}\right) = 0. \quad \text{(B 5)}$$

In (5) we defined the quantity $x_{ij} = v_i/v_{\mathrm{th}_j}$.

Proceeding as in (B 2)–(B 5) and suppressing the subscripts i and j,

$$\frac{\partial x}{\partial \mathbf{v}} = \frac{1}{v_{\mathrm{th}}}\frac{\partial v}{\partial \mathbf{v}} = \frac{1}{v_{\mathrm{th}}}\frac{\mathbf{v}}{v}, \quad \text{(B 6)}$$

$$\frac{\partial}{\partial \mathbf{v}}\cdot\frac{\partial x}{\partial \mathbf{v}} = \frac{1}{v_{\mathrm{th}}}\frac{\partial}{\partial \mathbf{v}}\cdot\frac{\partial v}{\partial \mathbf{v}} = \frac{1}{v_{\mathrm{th}}}\frac{2}{v}, \quad \text{(B 7)}$$

$$\frac{\partial^2 x}{\partial \mathbf{v}\partial \mathbf{v}} = \frac{1}{v_{\mathrm{th}}}\frac{\partial^2 v}{\partial \mathbf{v}\partial \mathbf{v}} = \frac{1}{v_{\mathrm{th}}}\left(\frac{\mathsf{I}}{v} - \frac{\mathbf{v}\mathbf{v}}{v^3}\right), \quad \text{(B 8)}$$

which annihilates the component parallel to $\mathbf{v}$.

$$\frac{\partial}{\partial \mathbf{v}} \cdot \frac{\partial^2 x}{\partial \mathbf{v} \partial \mathbf{v}} = \frac{1}{v_{\text{th}}} \frac{\partial}{\partial \mathbf{v}} \cdot \left(\frac{\mathsf{I}}{v} - \frac{\mathbf{v}\mathbf{v}}{v^3} \right) = -\frac{2}{v_{\text{th}}} \frac{\mathbf{v}}{v^3} \tag{B 9}$$

$$\frac{\partial x}{\partial \mathbf{v}} \cdot \frac{\partial^2 x}{\partial \mathbf{v} \partial \mathbf{v}} = \frac{1}{v_{\text{th}}^2} \frac{\mathbf{v}}{v} \cdot \left(\frac{\mathsf{I}}{v} - \frac{\mathbf{v}\mathbf{v}}{v^3} \right) = 0. \tag{B 10}$$

The expressions (B 6)–(B 10) are given in (7), and are used to write the Fokker–Planck collision operator (6) in the form given by (8).

In (17) we encountered the complicated expressions

$$\frac{\partial}{\partial \mathbf{v}} \cdot \left[\frac{\partial f_b}{\partial \mathbf{v}} \cdot \frac{\partial^2 v}{\partial \mathbf{v} \partial \mathbf{v}} \right] \tag{B 11}$$

and

$$\frac{\partial}{\partial \mathbf{v}} \cdot \left[\frac{\mathbf{v}}{v^3} f_b \right]. \tag{B 12}$$

These expressions can be conveniently evaluated by introducing spherical coordinates v, θ, ϕ and transforming to the variable

$$\xi = \cos\theta = \frac{\mathbf{v}}{v} \cdot \hat{e}_3, \tag{B 13}$$

where $\hat{e}_3$ is a unit vector along the magnetic field.

$$\frac{\partial \xi}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}} \frac{\mathbf{v}}{v} \cdot \hat{e}_3 = \frac{\partial^2 v}{\partial \mathbf{v} \partial \mathbf{v}} \cdot \hat{e}_3, \quad \text{using (B 6).} \tag{B 14}$$

For an axisymmetric distribution, $\partial f_b / \partial \phi = 0$ and

$$\frac{\partial f_b}{\partial \mathbf{v}} = \frac{\partial f_b}{\partial v} \frac{\partial v}{\partial \mathbf{v}} + \frac{\partial f_b}{\partial \xi} \frac{\partial \xi}{\partial \mathbf{v}}.$$

Using (B 10) and (B 14)

$$\frac{\partial f_b}{\partial \mathbf{v}} \cdot \frac{\partial^2 v}{\partial \mathbf{v} \partial \mathbf{v}} = \frac{\partial f_b}{\partial \xi} \hat{e}_3 \cdot \frac{\partial^2 v}{\partial \mathbf{v} \partial \mathbf{v}} \cdot \frac{\partial^2 v}{\partial \mathbf{v} \partial \mathbf{v}} = \frac{\partial f_b}{\partial \xi} \frac{\hat{e}_3}{v} \cdot \frac{\partial^2 v}{\partial \mathbf{v} \partial \mathbf{v}}. \tag{B 15}$$

Substituting (B 15) into (B 11) and using the property that $\partial^2 v / \partial \mathbf{v} \partial \mathbf{v}$ annihilates the component parallel to $\mathbf{v}$ (B 10)

$$\frac{\partial}{\partial \mathbf{v}} \cdot \left[\frac{\partial f_b}{\partial \mathbf{v}} \cdot \frac{\partial^2 v}{\partial \mathbf{v} \partial \mathbf{v}} \right] = \left[\frac{\partial^2 f_b}{\partial \xi^2} \frac{\hat{e}_3}{v} \cdot \frac{\partial^2 v}{\partial \mathbf{v} \partial \mathbf{v}} + \frac{1}{v} \frac{\partial f_b}{\partial \xi} \frac{\partial}{\partial \mathbf{v}} \right] \cdot \frac{\partial^2 v}{\partial \mathbf{v} \partial \mathbf{v}} \cdot \hat{e}_3$$

$$= \frac{(1-\xi^2)}{v^3} \frac{\partial^2 f_b}{\partial \xi^2} - \frac{2\xi}{v^3} \frac{\partial f_b}{\partial \xi} = \frac{1}{v^3} \frac{\partial}{\partial \xi} \left[(1-\xi^2) \frac{\partial f_b}{\partial \xi} \right]. \tag{B 16}$$

This second order differential operation in ξ is Legendre's operator.

We next evaluate the expression (B 12)

$$\frac{\partial}{\partial \mathbf{v}} \cdot \left(\frac{\mathbf{v}}{v^3} f_b \right) = \frac{\partial f_b}{\partial \mathbf{v}} \cdot \frac{\mathbf{v}}{v^3} + f_b \frac{\partial}{\partial \mathbf{v}} \cdot \frac{\mathbf{v}}{v^3}$$

$$= \frac{1}{v^2} \frac{\partial f_b}{\partial v} - f_b \frac{\partial}{\partial \mathbf{v}} \cdot \frac{\partial}{\partial \mathbf{v}} \frac{1}{v}. \tag{B 17}$$

The second term in (B 17) is $\nabla_{\mathbf{v}}^2(1/v) = -4\pi\delta(\mathbf{v})$. However, since the volume element is $v^2 dv$, we can neglect the delta function and write

$$\frac{\partial}{\partial \mathbf{v}} \cdot \left(\frac{\mathbf{v}}{v^3} f_b\right) = \frac{1}{v^2} \frac{\partial f_b}{\partial v} \tag{B 18}$$

Finally, substituting the expressions (B 16) and (B 18) into the collision term in (17) produces the result quoted in (18).

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