

# Lectures on Kinetic Theory and Magnetohydrodynamics of Plasmas

(Oxford MMathPhys/MSc in Mathematical and Theoretical Physics)

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These are the notes for my lectures on plasma physics, taught since 2014 as part of the MMathPhys programme at Oxford. Part I contains the lectures on plasma kinetics that formed part of the course on “Kinetic Theory,” taught jointly with Paul Dellar and James Binney (succeeded by Jean-Baptiste Fouvy and then Chris Hamilton). The more advanced sections assembled in Part II cover the material first taught in the “Advanced Topics in Plasma Physics” course during the dreadful Trinity Term of 2020, under the Covid-19 lockdown. Part III is an introduction to magnetohydrodynamics, which was part of the course on “Advanced Fluid Dynamics,” taught by me jointly with Paul Dellar in 2015-21, before I handed over to Caroline Terquem. These notes evolved from two earlier courses: “Advanced Plasma Theory,” taught as a graduate course at Imperial College in 2008, and “Magnetohydrodynamics and Turbulence,” taught three times as a Mathematics Part III course at Cambridge in 2005-06. Extracts from these notes have also been used in (and in part written for) my lectures at successive plasma-physics sessions of École de Physique des Houches in 2017, 2019, and 2023. Finally, Part IV is dedicated to the marriage of kinetics and MHD. It originates from the Les Houches lectures of 2013 and 2015 and has been taught as part of the MMathPhys course on “Collisionless Plasma Physics,” which I took over from Felix Parra in 2022. I will be grateful for any feedback from students, tutors or sympathisers.

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# PART I

## Kinetic Theory of Plasmas

### 1. Kinetic Description of a Plasma

We shall study a gas consisting of charged particles—ions and electrons. In general, there may be many different species of ions, with different masses and charges, and, of course, only one type of electrons.

I will index particle species by  $\alpha$  ( $\alpha = e$  for electrons,  $\alpha = i$  for ions). Each is characterised by its mass  $m_\alpha$  and charge  $q_\alpha = Z_\alpha e$ , where  $e$  is the magnitude of the electron charge and  $Z_\alpha$  is a positive or negative integer (e.g.,  $Z_e = -1$ ).

#### 1.1. Quasineutrality

We shall always assume that plasma is neutral overall:

$$\sum_{\alpha} q_{\alpha} N_{\alpha} = eV \sum_{\alpha} Z_{\alpha} \bar{n}_{\alpha} = 0, \quad (1.1)$$

where  $N_\alpha$  is the number of the particles of species  $\alpha$ ,  $\bar{n}_\alpha = N_\alpha/V$  is their mean number density and  $V$  the volume of the plasma. This condition is known as *quasineutrality*.

#### 1.2. Weak Interactions

Interaction between charged particles is governed by the Coulomb potential:

$$U(|\mathbf{r}_i^{(\alpha)} - \mathbf{r}_j^{(\alpha')}|) = -\frac{q_{\alpha} q_{\alpha'}}{|\mathbf{r}_i^{(\alpha)} - \mathbf{r}_j^{(\alpha')}|}, \quad (1.2)$$

where by  $\mathbf{r}_i^{(\alpha)}$  I mean the position of the  $i$ -th particle of species  $\alpha$ . It is a safe bet that we will only be able to have a nice closed kinetic description if the gas is approximately ideal, i.e., if particles interact weakly, viz.,

$$k_B T \gg U \sim \frac{e^2}{\Delta r} \sim e^2 n^{1/3}, \quad (1.3)$$

where  $k_B$  is the Boltzmann constant, which will henceforth be absorbed into the temperature  $T$ , and  $\Delta r \sim n^{-1/3}$  is the typical interparticle distance. Let us see what this condition means and implies physically.

#### 1.3. Debye Shielding

Let us consider a plasma in thermodynamic equilibrium (as one does in statistical mechanics, I will refuse to discuss, for the time being, how exactly it got there). Take one particular particle, of species  $\alpha$ . It creates an electric field around itself,  $\mathbf{E} = -\nabla\varphi$ ; all other particles sit in this field (Fig. 1)—and, indeed, also affect it, as we will see below. In equilibrium, the densities of these particles ought to satisfy Boltzmann's formula:

$$n_{\alpha'}(\mathbf{r}) = \bar{n}_{\alpha'} e^{-q_{\alpha'} \varphi(\mathbf{r})/T} \approx \bar{n}_{\alpha'} - \frac{\bar{n}_{\alpha'} q_{\alpha'} \varphi}{T}, \quad (1.4)$$

where  $\bar{n}_{\alpha'}$  is the mean density of the particles of species  $\alpha'$  and  $\varphi(\mathbf{r})$  is the electrostatic potential, which depends on the distance  $\mathbf{r}$  from our “central” particle. As  $r \rightarrow \infty$ ,  $\varphi \rightarrow 0$  and  $n_{\alpha'} \rightarrow \bar{n}_{\alpha'}$ . The exponential can be Taylor-expanded provided the weak-interaction condition (1.3) is satisfied ( $e\varphi \ll T$ ).

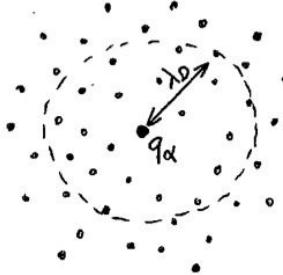


FIGURE 1. A particle amongst particles and its Debye sphere.

By the Gauss–Poisson law, we have

$$\begin{aligned} \nabla \cdot \mathbf{E} = -\nabla^2 \varphi &= 4\pi q_\alpha \delta(\mathbf{r}) + 4\pi \sum_{\alpha'} q_{\alpha'} n_{\alpha'} \\ &\approx 4\pi q_\alpha \delta(\mathbf{r}) + 4\pi \underbrace{\sum_{\alpha'} q_{\alpha'} \bar{n}_{\alpha'}}_{= 0 \text{ by quasineutrality}} - \underbrace{\left( \sum_{\alpha'} \frac{4\pi \bar{n}_{\alpha'} q_{\alpha'}^2}{T} \right)}_{\equiv 1/\lambda_D^2} \varphi. \end{aligned} \quad (1.5)$$

In the first line of this equation, the first term on the right-hand side is the charge density associated with the “central” particle and the second term is the charge density of the rest of the particles. In the second line, I used the Taylor-expanded Boltzmann expression (1.4) for the particle densities and then the quasineutrality (1.1) to establish the vanishing of the second term. The combination that has arisen in the last term as a prefactor of  $\varphi$  has dimensions of inverse square length, so we define the *Debye length* to be

$$\lambda_D \equiv \left( \sum_{\alpha} \frac{4\pi \bar{n}_{\alpha} q_{\alpha}^2}{T} \right)^{-1/2}. \quad (1.6)$$

Using also the obvious fact that the solution of (1.5) must be spherically symmetric, we recast this equation as follows

$$\frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial \varphi}{\partial r} - \frac{1}{\lambda_D^2} \varphi = -4\pi q_\alpha \delta(\mathbf{r}). \quad (1.7)$$

The solution to this that asymptotes to the Coulomb potential  $\varphi \rightarrow q_\alpha/r$  as  $r \rightarrow 0$  and to zero as  $r \rightarrow \infty$  is

$$\varphi = \frac{q_\alpha}{r} e^{-r/\lambda_D}. \quad (1.8)$$

Thus, in a quasineutral plasma, charges are shielded on typical distances  $\sim \lambda_D$ .

Obviously, this statistical calculation only makes sense if the “Debye sphere” has very many particles in it, viz., if

$$n \lambda_D^3 \gg 1. \quad (1.9)$$

Let us check that this is the case:

$$n \lambda_D^3 \sim n \left( \frac{T}{ne^2} \right)^{3/2} = \left( \frac{T}{n^{1/3} e^2} \right)^{3/2} \gg 1, \quad (1.10)$$

provided the weak-interaction condition (1.3) is satisfied. The quantity  $n\lambda_D^3$  is called the *plasma parameter*.

#### 1.4. Micro- and Macroscopic Fields

This calculation tells us something very important about electromagnetic fields in a plasma. Let  $\mathbf{E}^{(\text{micro})}(\mathbf{r}, t)$  and  $\mathbf{B}^{(\text{micro})}(\mathbf{r}, t)$  be the exact *microscopic fields* at a given location  $\mathbf{r}$  and time  $t$ . These fields are responsible for interactions between particles. On distances  $l \ll \lambda_D$ , these will be just the two-particle interactions—binary collisions between particles in vacuo, just like in a neutral gas (except the interparticle potential is the Coulomb potential). In contrast, on distances  $l \gtrsim \lambda_D$ , individual particles' fields are shielded and what remains are fields due to collective influence of large numbers of particles—*macroscopic fields*:

$$\mathbf{E}^{(\text{micro})} = \underbrace{\langle \mathbf{E}^{(\text{micro})} \rangle}_{\equiv \mathbf{E}} + \delta\mathbf{E}, \quad \mathbf{B}^{(\text{micro})} = \underbrace{\langle \mathbf{B}^{(\text{micro})} \rangle}_{\equiv \mathbf{B}} + \delta\mathbf{B}, \quad (1.11)$$

where the macroscopic fields  $\mathbf{E}$  and  $\mathbf{B}$  are averages over some intermediate scale  $l$  such that

$$\Delta r \sim n^{-1/3} \ll l \ll \lambda_D. \quad (1.12)$$

Such averaging (or “coarse-graining”) is made possible by the condition (1.9).

Thus, plasma has a new feature compared to neutral gas: because the Coulomb potential is long-range ( $\propto 1/r$ ), the fields decay on a length scale that is long compared to the interparticle distances [ $\lambda_D \gg \Delta r \sim n^{-1/3}$  according to (1.9)] and so, besides interactions between individual particles, there are also collective effects: interaction of particles with mean macroscopic fields due to all other particles.

Before I use this approach to construct a description of the plasma as a continuum (on scales  $\gg l$ ), let us check that particles travel sufficiently long distances between collisions in order to feel the macroscopic fields, viz., that their mean free path is  $\lambda_{\text{mfp}} \gg \lambda_D$ . The mean free path can be estimated in terms of the collision cross-section  $\sigma$ :

$$\lambda_{\text{mfp}} \sim \frac{1}{n\sigma} \sim \frac{T^2}{ne^4} \quad (1.13)$$

because  $\sigma \sim d^2$  and the effective distance  $d$  by which particles have to approach each other in order to have significant Coulomb interaction is inferred by balancing the Coulomb potential energy (1.2) with the particle temperature,  $e^2/d \sim T$ . Using (1.13) and (1.6), we find

$$\frac{\lambda_{\text{mfp}}}{\lambda_D} \sim \frac{T^2}{ne^4} \left( \frac{ne^2}{T} \right)^{1/2} \sim n\lambda_D^3 \gg 1, \quad \text{q.e.d.} \quad (1.14)$$

Thus, it makes sense to talk about a particle travelling long distances experiencing the macroscopic fields exerted by the rest of the plasma collectively before being deflected by a much larger, but also much shorter-range, microscopic field of another individual particle.

### 1.5. Maxwell's Equations

The exact microscopic fields satisfy *Maxwell's equations* and, since Maxwell's equations are linear, so do the macroscopic fields: by direct averaging,

$$\nabla \cdot \langle \mathbf{E}^{(\text{micro})} \rangle = 4\pi \langle \sigma^{(\text{micro})} \rangle, \quad (1.15)$$

$$\nabla \cdot \langle \mathbf{B}^{(\text{micro})} \rangle = 0, \quad (1.16)$$

$$\nabla \times \langle \mathbf{E}^{(\text{micro})} \rangle + \frac{1}{c} \frac{\partial \langle \mathbf{B}^{(\text{micro})} \rangle}{\partial t} = 0, \quad (1.17)$$

$$\nabla \times \langle \mathbf{B}^{(\text{micro})} \rangle - \frac{1}{c} \frac{\partial \langle \mathbf{E}^{(\text{micro})} \rangle}{\partial t} = \frac{4\pi}{c} \langle \mathbf{j}^{(\text{micro})} \rangle. \quad (1.18)$$

The new quantities here are the averages of the microscopic charge density  $\sigma^{(\text{micro})}$  and the microscopic current density  $\mathbf{j}^{(\text{micro})}$ . How do we calculate them?

Clearly, they depend on where all the particles are at any given time and how fast these particles move. We can assemble all this information in one function:

$$F_\alpha(\mathbf{r}, \mathbf{v}, t) = \sum_{i=1}^{N_\alpha} \delta^3(\mathbf{r} - \mathbf{r}_i^{(\alpha)}(t)) \delta^3(\mathbf{v} - \mathbf{v}_i^{(\alpha)}(t)), \quad (1.19)$$

where  $\mathbf{r}_i^{(\alpha)}(t)$  and  $\mathbf{v}_i^{(\alpha)}(t)$  are the exact phase-space coordinates of particle  $i$  of species  $\alpha$  at time  $t$ , i.e., these are the solutions of the exact equations of motion for all these particles moving in the microscopic fields  $\mathbf{E}^{(\text{micro})}(t, \mathbf{r})$  and  $\mathbf{B}^{(\text{micro})}(t, \mathbf{r})$ . The function  $F_\alpha$  is called the *Klimontovich distribution function*. It is a random object (i.e., it fluctuates on scales  $\ll \lambda_D$ ) because it depends on the exact particle trajectories, which depend on the exact microscopic fields. In terms of this distribution function,

$$\sigma^{(\text{micro})}(\mathbf{r}, t) = \sum_{\alpha} q_{\alpha} \int d\mathbf{v} F_{\alpha}(\mathbf{r}, \mathbf{v}, t), \quad (1.20)$$

$$\mathbf{j}^{(\text{micro})}(\mathbf{r}, t) = \sum_{\alpha} q_{\alpha} \int d\mathbf{v} \mathbf{v} F_{\alpha}(\mathbf{r}, \mathbf{v}, t). \quad (1.21)$$

We now need to average these quantities for use in (1.15) and (1.18). We shall assume that the average over microscales (1.12) and the ensemble average (i.e., the average over many different initial conditions) are the same. The ensemble average of  $F_\alpha$  is an object familiar from the kinetic theory of gases (Dellar 2015), the so-called *one-particle distribution function*:

$$\langle F_{\alpha} \rangle = f_{1\alpha}(\mathbf{r}, \mathbf{v}, t) \quad (1.22)$$

(I shall henceforth omit the subscript 1). If we learn how to compute  $f_{\alpha}$ , then we can average (1.20) and (1.21), substitute them into (1.15) and (1.18), and have the following set of macroscopic Maxwell's equations:

$$\nabla \cdot \mathbf{E} = 4\pi \sum_{\alpha} q_{\alpha} \int d\mathbf{v} f_{\alpha}(\mathbf{r}, \mathbf{v}, t), \quad (1.23)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (1.24)$$

$$\nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = 0, \quad (1.25)$$

$$\nabla \times \mathbf{B} - \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c} \sum_{\alpha} q_{\alpha} \int d\mathbf{v} \mathbf{v} f_{\alpha}(\mathbf{r}, \mathbf{v}, t). \quad (1.26)$$

### 1.6. Vlasov–Landau Equation

We now need an evolution equation for  $f_\alpha(\mathbf{r}, \mathbf{v}, t)$ , hopefully in terms of the macroscopic fields  $\mathbf{E}(\mathbf{r}, t)$  and  $\mathbf{B}(\mathbf{r}, t)$ , so we can couple it to (1.23–1.26) and thus have a closed system of equations describing our plasma.

The process of deriving it starts with Liouville’s theorem and is a direct generalisation of the BBGKY procedure familiar from gas kinetics (e.g., [Dellar 2015](#))<sup>1</sup> to the somewhat more cumbersome case of a plasma:

- many species  $\alpha$ ;

- presence of forces due to the macroscopic fields  $\mathbf{E}$  and  $\mathbf{B}$ ;

- Coulomb potential for interparticle collisions, with some attendant complications to do with its long-range nature: in brief, use Rutherford’s cross section and cut off long-range interactions at  $\lambda_D$  (this is described in many textbooks and plasma-physics courses: see, e.g., [Parra 2019a](#), [Kunz 2021](#), or a shortcut in §11.6 of these Lectures).

The result of this derivation is

$$\frac{\partial f_\alpha}{\partial t} + \{f_\alpha, H_{1\alpha}\} = \left( \frac{\partial f_\alpha}{\partial t} \right)_c. \quad (1.27)$$

The Poisson bracket contains  $H_{1\alpha}$ , the Hamiltonian for a single particle of species  $\alpha$  moving in the *macroscopic* fields  $\mathbf{E}$  and  $\mathbf{B}$ —all the microscopic fields  $\delta\mathbf{E}$ <sup>2</sup> are gone into the collision operator on the right-hand side, of which more will be said shortly (§1.8).

Technically speaking, one ought to be working with canonical variables, but dealing with canonical momenta of charged particles in a magnetic field is an unnecessary complication, so I shall stick to the  $(\mathbf{r}, \mathbf{v})$  representation of the phase space. Then (1.27) takes the form of Liouville’s equation, but with microscopic fields hidden inside the collision operator [see (1.41)]:

$$\frac{\partial f_\alpha}{\partial t} + \frac{\partial}{\partial \mathbf{r}} \cdot (\dot{\mathbf{r}} f_\alpha) + \frac{\partial}{\partial \mathbf{v}} \cdot (\dot{\mathbf{v}} f_\alpha) = \left( \frac{\partial f_\alpha}{\partial t} \right)_c, \quad (1.28)$$

where

$$\dot{\mathbf{r}} = \mathbf{v}, \quad \dot{\mathbf{v}} = \frac{q_\alpha}{m_\alpha} \left( \mathbf{E} + \frac{\mathbf{v} \times \mathbf{B}}{c} \right). \quad (1.29)$$

This gives us the *Vlasov–Landau equation*:

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v} \cdot \nabla f_\alpha + \frac{q_\alpha}{m_\alpha} \left( \mathbf{E} + \frac{\mathbf{v} \times \mathbf{B}}{c} \right) \cdot \frac{\partial f_\alpha}{\partial \mathbf{v}} = \left( \frac{\partial f_\alpha}{\partial t} \right)_c.$$

(1.30)

Any other macroscopic force that the plasma might be subject to (e.g., gravity) can be added to the Lorentz force in the third term on the left-hand side, as long as its divergence in velocity space is  $(\partial/\partial \mathbf{v}) \cdot \text{force} = 0$ . Equation (1.30) is closed by Maxwell’s equations (1.23–1.26).

### 1.7. Klimontovich’s Version of BBGKY

By way of a technical digression, let me outline the (beginning of the) derivation of (1.30) due to [Klimontovich \(1967\)](#). Consider the Klimontovich distribution function (1.19) and calculate

<sup>1</sup>In §1.7, I will sketch Klimontovich’s version of this procedure ([Klimontovich 1967](#)).

<sup>2</sup> $\delta\mathbf{B}$  turns out to be irrelevant as long as the particle motion is non-relativistic,  $v/c \ll 1$ .

its time derivative: by the chain rule,

$$\begin{aligned}\frac{\partial F_\alpha}{\partial t} = & - \sum_i \frac{d\mathbf{r}_i^{(\alpha)}(t)}{dt} \cdot \left[ \frac{\partial}{\partial \mathbf{r}} \delta^3(\mathbf{r} - \mathbf{r}_i^{(\alpha)}(t)) \delta^3(\mathbf{v} - \mathbf{v}_i^{(\alpha)}(t)) \right] \\ & - \sum_i \frac{d\mathbf{v}_i^{(\alpha)}(t)}{dt} \cdot \left[ \frac{\partial}{\partial \mathbf{v}} \delta^3(\mathbf{r} - \mathbf{r}_i^{(\alpha)}(t)) \delta^3(\mathbf{v} - \mathbf{v}_i^{(\alpha)}(t)) \right].\end{aligned}\quad (1.31)$$

First, because  $\mathbf{r}_i^{(\alpha)}(t)$  and  $\mathbf{v}_i^{(\alpha)}(t)$  obviously do not depend on the phase-space variables  $\mathbf{r}$  and  $\mathbf{v}$ , the derivatives  $\partial/\partial\mathbf{r}$  and  $\partial/\partial\mathbf{v}$  can be pulled outside, so the right-hand side of (1.31) can be written as a divergence in phase space. Secondly, the particle equations of motion give us

$$\frac{d\mathbf{r}_i^{(\alpha)}(t)}{dt} = \mathbf{v}_i^{(\alpha)}(t), \quad (1.32)$$

$$\frac{d\mathbf{v}_i^{(\alpha)}(t)}{dt} = \frac{q_\alpha}{m_\alpha} \left[ \mathbf{E}^{(\text{micro})}(\mathbf{r}_i^{(\alpha)}(t), t) + \frac{\mathbf{v}_i^{(\alpha)}(t) \times \mathbf{B}^{(\text{micro})}(\mathbf{r}_i^{(\alpha)}(t), t)}{c} \right], \quad (1.33)$$

which are to be substituted into the right-hand side of (1.31)—*after* it is written in the divergence form. Since the time derivatives of  $\mathbf{r}_i^{(\alpha)}(t)$  and  $\mathbf{v}_i^{(\alpha)}(t)$  inside the divergence multiply delta functions identifying  $\mathbf{r}_i^{(\alpha)}(t)$  with  $\mathbf{r}$  and  $\mathbf{v}_i^{(\alpha)}(t)$  with  $\mathbf{v}$ ,  $\mathbf{r}_i^{(\alpha)}(t)$  may be replaced by  $\mathbf{r}$  and  $\mathbf{v}_i^{(\alpha)}(t)$  by  $\mathbf{v}$  in the right-hand sides of (1.32) and (1.33) when they go into (1.31). This gives (wrapping all the sums of delta functions back into  $F_\alpha$ )

$$\frac{\partial F_\alpha}{\partial t} = -\nabla \cdot (\mathbf{v} F_\alpha) - \frac{\partial}{\partial \mathbf{v}} \cdot \left[ \frac{q_\alpha}{m_\alpha} \left( \mathbf{E}^{(\text{micro})}(\mathbf{r}, t) + \frac{\mathbf{v} \times \mathbf{B}^{(\text{micro})}(\mathbf{r}, t)}{c} \right) F_\alpha \right]. \quad (1.34)$$

Finally, because  $\mathbf{r}$  and  $\mathbf{v}$  are independent variables and the Lorentz force has zero divergence in  $\mathbf{v}$  space,  $F_\alpha$  satisfies *exactly*

$$\boxed{\frac{\partial F_\alpha}{\partial t} + \mathbf{v} \cdot \nabla F_\alpha + \frac{q_\alpha}{m_\alpha} \left( \mathbf{E}^{(\text{micro})} + \frac{\mathbf{v} \times \mathbf{B}^{(\text{micro})}}{c} \right) \cdot \frac{\partial F_\alpha}{\partial \mathbf{v}} = 0}. \quad (1.35)$$

This is the *Klimontovich equation*. There is no collision integral here because microscopic fields are explicitly present. The equation is closed by the microscopic Maxwell's equations:

$$\nabla \cdot \mathbf{E}^{(\text{micro})} = 4\pi \sum_\alpha q_\alpha \int d\mathbf{v} F_\alpha(\mathbf{r}, \mathbf{v}, t), \quad (1.36)$$

$$\nabla \cdot \mathbf{B}^{(\text{micro})} = 0, \quad (1.37)$$

$$\nabla \times \mathbf{E}^{(\text{micro})} + \frac{1}{c} \frac{\partial \mathbf{B}^{(\text{micro})}}{\partial t} = 0, \quad (1.38)$$

$$\nabla \times \mathbf{B}^{(\text{micro})} - \frac{1}{c} \frac{\partial \mathbf{E}^{(\text{micro})}}{\partial t} = \frac{4\pi}{c} \sum_\alpha q_\alpha \int d\mathbf{v} \mathbf{v} F_\alpha(\mathbf{r}, \mathbf{v}, t). \quad (1.39)$$

Now let us separate the microscopic fields into mean (macroscopic) and fluctuating parts according to (1.11); also

$$F_\alpha = \underbrace{\langle F_\alpha \rangle}_{\equiv f_\alpha} + \delta F_\alpha. \quad (1.40)$$

Maxwell's equations are linear, so averaging them gives the same equations for  $\mathbf{E}$  and  $\mathbf{B}$  in terms of  $f_\alpha$  [see (1.23–1.26)] and for  $\delta\mathbf{E}$  and  $\delta\mathbf{B}$  in terms of  $\delta F_\alpha$ . Averaging the Klimontovich equation (1.35) gives the Vlasov–Landau equation:

$$\begin{aligned}\frac{\partial f_\alpha}{\partial t} + \mathbf{v} \cdot \nabla f_\alpha + \frac{q_\alpha}{m_\alpha} \left( \mathbf{E} + \frac{\mathbf{v} \times \mathbf{B}}{c} \right) \cdot \frac{\partial f_\alpha}{\partial \mathbf{v}} \\ = -\frac{q_\alpha}{m_\alpha} \left\langle \left( \delta\mathbf{E} + \frac{\mathbf{v} \times \delta\mathbf{B}}{c} \right) \cdot \frac{\partial \delta F_\alpha}{\partial \mathbf{v}} \right\rangle \equiv \left( \frac{\partial f_\alpha}{\partial t} \right)_c.\end{aligned}\quad (1.41)$$

The macroscopic fields in the left-hand side satisfy the macroscopic Maxwell's equations (1.23–1.26). The microscopic fluctuating fields  $\delta\mathbf{E}$  and  $\delta\mathbf{B}$  inside the average in the right-hand side satisfy microscopic Maxwell's equations with fluctuating charge and current densities expressed in terms of  $\delta F_\alpha$ . Thus, the right-hand side is quadratic in  $\delta F_\alpha$ . In order to close this equation, we need an expression for the correlation function  $\langle \delta F_\alpha \delta F_{\alpha'} \rangle$  in terms of  $f_\alpha$  and  $f_{\alpha'}$ . This is basically what the BBGKY procedure plus truncation of velocity integrals based on an expansion in  $1/n\lambda_D^3$  achieve. The result is the Landau collision operator (or the more precise Lenard–Balescu one; see §§11.5 and 11.6).

Further details are a bit complicated (see Klimontovich 1967), but my aim here was just to show how the fields are split into macroscopic and microscopic ones, with the former appearing explicitly in the kinetic equation and the latter wrapped up inside the collision operator. The presence of the macroscopic fields and the consequent necessity for coupling the kinetic equation with Maxwell's equations for these fields is the main mathematical difference between the kinetics of neutral gases and the kinetics of plasmas.

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### 1.8. Collision Operator

Finally, a few words about the plasma collision operator (or “collision integral”), first derived explicitly by Landau (1936) (the same considerations apply to the the more general operator due to Lenard 1960 and Balescu 1960). It describes two-particle collisions both within the species  $\alpha$  and with other species  $\alpha'$  and so depends both on  $f_\alpha$  and on all other  $f_{\alpha'}$ . Its derivation is left to you as an exercise in BBGKY'ing, calculating cross sections and velocity integrals (or in googling; shortcut: see Parra 2019a, Kunz 2021, or Swanson 2008).<sup>3</sup> In these Lectures, I shall focus on collisionless aspects of plasma kinetics. Whenever a need arises for invoking the collision operator, the important things about it for us will be its properties:

- *conservation of particles* (within each species  $\alpha$ ),

$$\int d\mathbf{v} \left( \frac{\partial f_\alpha}{\partial t} \right)_c = 0; \quad (1.42)$$

- *conservation of momentum*,

$$\sum_\alpha \int d\mathbf{v} m_\alpha \mathbf{v} \left( \frac{\partial f_\alpha}{\partial t} \right)_c = 0 \quad (1.43)$$

(same-species collisions conserve momentum, whereas different-species collisions conserve it only after summation over species—there is friction of one species against another; e.g., the friction of electrons against the ions is the Ohmic resistivity of the plasma, known as “Spitzer resistivity”: see Parra 2019a, Kunz 2021, or Helander & Sigmar 2005);

- *conservation of energy*,

$$\sum_\alpha \int d\mathbf{v} \frac{m_\alpha v^2}{2} \left( \frac{\partial f_\alpha}{\partial t} \right)_c = 0; \quad (1.44)$$

- Boltzmann's *H*-theorem: the kinetic entropy

$$S = - \sum_\alpha \iint d\mathbf{r} d\mathbf{v} f_\alpha \ln f_\alpha \quad (1.45)$$

cannot decrease, and, as  $S$  is conserved by all the collisionless terms in (1.30), the collision

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<sup>3</sup>A somewhat unorthodox derivation of both the Lenard–Balescu and Landau operators will be given in §§11.5 and 11.6, respectively, as a by-product of a discussion of collisionless (*sic*) relaxation.

operator must have the property that

$$\frac{dS}{dt} = - \sum_{\alpha} \iint d\mathbf{r} d\mathbf{v} \left( \frac{\partial f_{\alpha}}{\partial t} \right)_c \ln f_{\alpha} \geq 0, \quad (1.46)$$

with equality obtained if and only if  $f_{\alpha}$  is a local Maxwellian;

- unlike the Boltzmann operator for neutral gases, the Landau operator expresses the cumulative effect of many glancing (rather than “head-on”) collisions (due to the long-range nature of the Coulomb interaction) and so it is a Fokker–Planck operator:<sup>4</sup>

$$\left( \frac{\partial f_{\alpha}}{\partial t} \right)_c = \frac{\partial}{\partial \mathbf{v}} \cdot \sum_{\alpha'} \left( \mathbf{A}_{\alpha\alpha'}[f_{\alpha'}] + \mathbf{D}_{\alpha\alpha'}[f_{\alpha'}] \cdot \frac{\partial}{\partial \mathbf{v}} \right) f_{\alpha}, \quad (1.47)$$

where the drag  $\mathbf{A}_{\alpha\alpha'}$  (vector) and diffusion  $\mathbf{D}_{\alpha\alpha'}$  (matrix) coefficients are integral (in  $\mathbf{v}$  space) functionals of  $f_{\alpha'}$ . The Fokker–Planck form (1.47) of the Landau operator means that it describes diffusion in velocity space and so will erase sharp gradients in  $f_{\alpha}$  with respect to  $\mathbf{v}$ —a property that we will find very important in §5.

### 1.9. So What’s New and What Now?

Let me summarise the new features that have appeared in the kinetic description of a plasma compared to that of a neutral gas.

- First, particles are charged, so they interact via Coulomb potential. The collision operator is, therefore, different: the cross-section is the Rutherford cross-section, most collisions are glancing (with interaction on distances up to the Debye length), leading to diffusion of the particle distribution function in velocity space. Mathematically, this is manifested in the collision operator in (1.30) having the Fokker–Planck structure (1.47).

One can spin out of the Vlasov–Landau equation (1.30) a theory that is analogous to what is done with Boltzmann’s equation in gas kinetics (Dellar 2015): derive *fluid equations*, calculate viscosity, thermal conductivity, Ohmic resistivity, etc., of a collisionally dominated plasma, i.e., of a plasma in which the collision frequency of the particles is much greater than all other relevant time scales. This is done in the same way as in gas kinetics, but now applying the Chapman–Enskog procedure to the Landau collision operator. This is quite a lot of work—and constitutes core textbook material (see Parra 2019a; Kunz 2021). In magnetised plasmas especially, the resulting fluid dynamics of the plasma are quite interesting and quite different from neutral fluids—we shall see some of this in Part III of these Lectures, while the classic treatment of the transport theory can be found in Braginskii (1965); a great textbook on collisional transport is Helander & Sigmar (2005) (see Krommes 2018 for a modernist approach).

- Secondly, Coulomb potential is long-range, so the electric and magnetic fields have a macroscopic (mean) part on scales longer than the Debye length—a particle experiencing these fields is not undergoing a collision in the sense of bouncing off another particle, but is, rather, interacting, via the fields, with the collective of all the other particles. Mathematically, this manifests itself as a Lorentz-force term appearing in the right-hand side of the Vlasov–Landau kinetic equation (1.30). The macroscopic  $\mathbf{E}$  and  $\mathbf{B}$  fields that

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<sup>4</sup>The simplest example that I can think of in which the collision operator is a velocity-space diffusion operator of this kind is the gas of Brownian particles [each with velocity described by Langevin’s equation (6.65)]. This is treated in detail in §6.9 of Schekochihin (2019). In these Lectures, the general Fokker–Planck form (1.47) emerges in (7.35), and then again, from a different angle, in (11.22); the Landau operator is (11.36).

figure in it are determined by the particles via their mean charge and current densities that enter the macroscopic Maxwell's equations (1.23–1.26).

In the case of neutral gas, all the interesting kinetic physics is in the collision operator, hence the focus on transport theory in gas-kinetic literature (see, e.g., the classic monograph by [Chapman & Cowling 1991](#) if you want an overdose of this). In the collisionless limit, the kinetic equation for a neutral gas,

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f = 0, \quad (1.48)$$

simply describes particles with some initial distribution individualistically flying in straight lines along their initial directions of travel. In contrast, for a plasma, even the *collisionless kinetics* (and, indeed, especially the collisionless—or *weakly collisional*—kinetics) is interesting and nontrivial because the particles, via the average properties of their distribution—charge densities and currents,—collectively modify  $\mathbf{E}$  and  $\mathbf{B}$ , which then act on individual particles and thus modify  $f_\alpha$ , etc. This “plasma democracy” is a whole new conceptual world and it is on the effects involving interactions between particles and fields that I shall focus here, in pursuit of maximum novelty.<sup>5</sup>

I shall also be in pursuit of maximum simplicity (well, to use Einstein's dictum, “as simple as possible, but not simpler”!) and so will mostly restrict my considerations to the “*electrostatic approximation*”:

$$\mathbf{B} = 0, \quad \mathbf{E} = -\nabla\varphi. \quad (1.49)$$

This, of course, eliminates a huge number of interesting and important phenomena without which plasma physics would not be the voluminous subject that it is, but I cannot do them justice in just a few lectures (so see [Parra 2019b](#) for a course largely devoted to collisionless magnetised plasmas). Some opportunities for generalising electrostatic theory to electromagnetic one will be provided in Q2 and Q3.

Thus, we shall henceforth consider a simplified kinetic system, called the *Vlasov–Poisson system*:

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v} \cdot \nabla f_\alpha - \frac{q_\alpha}{m_\alpha} (\nabla\varphi) \cdot \frac{\partial f_\alpha}{\partial \mathbf{v}} = 0, \quad (1.50)$$

$$-\nabla^2\varphi = 4\pi \sum_\alpha q_\alpha \int d\mathbf{v} f_\alpha. \quad (1.51)$$

Formally, considering a collisionless plasma<sup>6</sup> would appear to be legitimate as long as the collision frequency is small compared to the characteristic frequencies of any other evolution that might be going on. What are the characteristic time scales (and length scales) in a plasma and what phenomena occur on these scales? These questions bring us to our next theme.

## 2. Equilibrium and Fluctuations

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<sup>5</sup>Similarly interesting things happen when the field tying the particles together is gravity—an even more complicated situation because, while the potential is long-range, rather like the Coulomb potential, gravity is not shielded and so all particles feel each other at all distances. This gives rise to a very sophisticated theory ([Binney 2016; Fouvy 2019](#)).

<sup>6</sup>Or, I stress again, a weakly collisional plasma. The collision operator is dropped in (1.50), but let us not forget about it entirely even if the collision frequency is small; it will make a come back in §5.

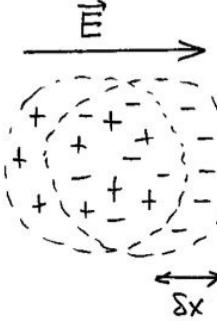


FIGURE 2. A displaced population of electrons will set up a quasineutrality-restoring electric field, leading to plasma oscillations.

### 2.1. Plasma Frequency

Consider a plasma in equilibrium, in a happy quasineutral state. Suppose a population of electrons strays from this equilibrium and upsets quasineutrality a bit (Fig. 2). If they have shifted by distance  $\delta x$ , the restoring force on each electron will be

$$m_e \delta \ddot{x} = -eE = -4\pi e^2 n_e \delta x \quad \Rightarrow \quad \delta \ddot{x} = -\underbrace{\frac{4\pi e^2 n_e}{m_e}}_{\equiv \omega_{pe}^2} \delta x, \quad (2.1)$$

so there will be oscillations at what is known as the (electron) *plasma frequency*:

$$\boxed{\omega_{pe} = \sqrt{\frac{4\pi e^2 n_e}{m_e}}}. \quad (2.2)$$

Thus, we expect *fluctuations* of electric field in a plasma with characteristic frequencies  $\omega \sim \omega_{pe}$  (these are Langmuir waves; I will derive their dispersion relation formally in §3.4). These fluctuations are due to *collective* motions of the particles—so they are still *macroscopic* fields in the nomenclature of §1.4.

The time scale associated with  $\omega_{pe}$  is the scale of restoration of quasineutrality. The distance an electron can travel over this time scale before the restoring force kicks in, i.e., the distance over which quasineutrality can be violated, is (using the *thermal speed*  $v_{the} \sim \sqrt{T/m_e}$  to estimate the electron's velocity)

$$\frac{v_{the}}{\omega_{pe}} \sim \sqrt{\frac{T}{m_e}} \sqrt{\frac{m_e}{e^2 n_e}} = \sqrt{\frac{T}{e^2 n_e}} \sim \lambda_D, \quad (2.3)$$

the Debye length (1.6)—not surprising, as this is, indeed, the scale on which microscopic fields are shielded and plasma is quasineutral (§1.3).

Finally, let us check that the plasma oscillations happen on collisionless time scales. The collision frequency of the electrons is, using (2.3) and (1.14),

$$\nu_e \sim \frac{v_{the}}{\lambda_{mfp}} = \frac{v_{the}}{\omega_{pe}} \frac{\omega_{pe}}{\lambda_{mfp}} \sim \frac{\lambda_D}{\lambda_{mfp}} \omega_{pe} \ll \omega_{pe}, \quad \text{q.e.d.} \quad (2.4)$$

### 2.2. Slow vs. Fast

The plasma frequency  $\omega_{pe}$  is only one of the characteristic frequencies (the largest) of the fluctuations that can occur in plasmas. We will think of the scales of all these

fluctuations as short and of the associated variation in time and space as *fast*. They occur against the background of some *equilibrium state*,<sup>7</sup> which is either constant or varies *slowly* in time and space. The slow evolution and spatial variation of the equilibrium state can be due to slowly changing, large-scale external conditions that gave rise to this state or, as we will discover soon, it can be due to the average effect of a sea of small fluctuations.

Formally, what we are embarking on is an attempt to set up a mean-field theory, separating slow (large-scale) and fast (small-scale) parts of the distribution function:

$$f(\mathbf{r}, \mathbf{v}, t) = f_0(\epsilon^a \mathbf{r}, \mathbf{v}, \epsilon t) + \delta f(\mathbf{r}, \mathbf{v}, t), \quad (2.5)$$

where  $\epsilon$  is some small parameter characterising the scale separation between fast and slow variation (note that this separation need not be the same for spatial and time scales, hence  $\epsilon^a$ ). To avoid clutter, I shall drop the species index where this does not lead to ambiguity.

For simplicity, I will abolish the spatial dependence of the equilibrium distribution altogether and consider *homogeneous* systems:

$$f_0 = f_0(\mathbf{v}, \epsilon t), \quad (2.6)$$

which also means  $\mathbf{E}_0 = 0$  (there is no equilibrium electric field). Equivalently, all our considerations are restricted to scales much smaller than the characteristic system size. Formally, this equilibrium distribution can be defined as the average of the exact distribution over the volume of space that we are considering and over time scales intermediate between the fast and the slow ones:<sup>8</sup>

$$f_0(\mathbf{v}, t) = \langle f(\mathbf{r}, \mathbf{v}, t) \rangle \equiv \frac{1}{\Delta t} \int_{t-\Delta t/2}^{t+\Delta t/2} dt' \int \frac{d\mathbf{r}}{V} f(\mathbf{r}, \mathbf{v}, t'), \quad (2.7)$$

where  $\omega^{-1} \ll \Delta t \ll t_{\text{eq}}$ , where  $t_{\text{eq}}$  is the equilibrium time scale.

### 2.3. Multiscale Dynamics

It is convenient to work in Fourier space:

$$\varphi(\mathbf{r}, t) = \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} \varphi_{\mathbf{k}}(t), \quad f(\mathbf{r}, \mathbf{v}, t) = f_0(\mathbf{v}, t) + \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} \delta f_{\mathbf{k}}(\mathbf{v}, t). \quad (2.8)$$

Then the Poisson equation (1.51) becomes

$$\varphi_{\mathbf{k}} = \frac{4\pi}{k^2} \sum_{\alpha} q_{\alpha} \int d\mathbf{v} \delta f_{\mathbf{k}\alpha} \quad (2.9)$$

and the Vlasov equation (1.50) written for  $\mathbf{k} = 0$  (i.e., the spatial average of the equation) is

$$\frac{\partial f_0}{\partial t} + \frac{\partial \delta f_{\mathbf{k}=0}}{\partial t} = -\frac{q}{m} \sum_{\mathbf{k}} \varphi_{-\mathbf{k}} i\mathbf{k} \cdot \frac{\partial \delta f_{\mathbf{k}}}{\partial \mathbf{v}}, \quad (2.10)$$

<sup>7</sup>Or even just an initial state that is slow to change.

<sup>8</sup>I use angular brackets to denote this average, but it should be clear that this is not the same thing as the average (1.11) that separated the macroscopic fields from the microscopic ones. The latter average was over sub-Debye scales, whereas the new average (2.7) is over scales that are larger than fluctuation scales but smaller than the system size; both fluctuations and equilibrium are “macroscopic” in the language of §1.4.

where we can replace  $\varphi_{-\mathbf{k}} = \varphi_{\mathbf{k}}^*$  because  $\varphi(\mathbf{r}, t)$  is a real field. Averaging over time according to (2.7) eliminates fast variation and gives

$$\frac{\partial f_0}{\partial t} = -\frac{q}{m} \sum_{\mathbf{k}} \left\langle \varphi_{\mathbf{k}}^* i\mathbf{k} \cdot \frac{\partial \delta f_{\mathbf{k}}}{\partial \mathbf{v}} \right\rangle. \quad (2.11)$$

The right-hand side of (2.11) describes the slow evolution of the equilibrium (mean) distribution due to the effect of fluctuations (see §§6 and 11). In practice, the main question is often how the equilibrium evolves and so we need a closed equation for the evolution of  $f_0$ . This should be obtainable at least in principle because the fluctuating fields appearing in the right-hand side of (2.11) themselves depend on  $f_0$ : indeed, writing the Vlasov equation (1.50) for the  $\mathbf{k} \neq 0$  modes, we find the following evolution equation for the fluctuations:

$$\underbrace{\frac{\partial \delta f_{\mathbf{k}}}{\partial t} + i\mathbf{k} \cdot \mathbf{v} \delta f_{\mathbf{k}}}_{\text{particle streaming (phase mixing)}} = \underbrace{\frac{q}{m} \varphi_{\mathbf{k}} i\mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{v}}}_{\text{wave-particle interaction (linear)}} + \underbrace{\frac{q}{m} \sum_{\mathbf{k}'} \varphi_{\mathbf{k}'} i\mathbf{k}' \cdot \frac{\partial \delta f_{\mathbf{k}-\mathbf{k}'}}{\partial \mathbf{v}}}_{\text{nonlinear interactions}}. \quad (2.12)$$

The three terms that control the evolution of the perturbed distribution function in (2.12) represent the three physical effects that I shall focus on in these Lectures. The second term on the left-hand side describes the free ballistic motion of particles (“streaming”). It gives rise to the phenomenon of phase mixing (§5) and, in its interplay with plasma waves, to Landau damping and kinetic instabilities (§3). The first term on the right-hand side contains the interaction of the electric-field perturbations (waves) with the equilibrium particle distribution (§3). The second term on the right-hand side captures the nonlinear interactions between the fluctuating fields and the perturbed distribution—it is negligible only when fluctuation amplitudes are small enough (which, sadly, they rarely are) and is responsible for *plasma turbulence* (§§7.2, 8, and 12.2) and other nonlinear phenomena (e.g., §12.1).

The programme for determining the slow evolution of the equilibrium is “simple”: solve (2.12) together with (2.9), calculate the *correlation function* of the fluctuations,  $\langle \varphi_{\mathbf{k}}^* \delta f_{\mathbf{k}} \rangle$ , as a functional of  $f_0$ , and use it to close (2.11); then proceed to solve the latter. Obviously, this is impossible to do in most cases. But it is possible to construct a hierarchy of approximations to the answer and learn much interesting physics in the process.

## 2.4. Hierarchy of Approximations

### 2.4.1. Linear Theory

Consider first infinitesimal perturbations of the equilibrium. All nonlinear terms can then be ignored, (2.11) turns into  $f_0 = \text{const}$  and (2.12) becomes

$$\frac{\partial \delta f_{\mathbf{k}}}{\partial t} + i\mathbf{k} \cdot \mathbf{v} \delta f_{\mathbf{k}} = \frac{q}{m} \varphi_{\mathbf{k}} i\mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{v}}, \quad (2.13)$$

the *linearised kinetic equation*. Solving this together with (2.9) allows one to find oscillating and/or growing/decaying<sup>9</sup> perturbations of a particular equilibrium  $f_0$ . The theory

<sup>9</sup>We shall see (§5) that growing/decaying linear solutions imply the equilibrium distribution giving/receiving energy to/from the fluctuations.

for doing this is very well developed and contains some of the core ideas that give plasma physics its intellectual shape (§3).

Physically, the linear solutions will describe what happens over short term, viz., on times  $t$  such that

$$\omega^{-1} \ll t \ll t_{\text{eq}} \text{ or } t_{\text{nl}}, \quad (2.14)$$

where  $\omega$  is the characteristic frequency of the perturbations,  $t_{\text{eq}}$  is the time after which the equilibrium starts getting modified by the perturbations via (2.11) [which depends on the amplitude to which they can grow; if perturbations do grow, i.e., the equilibrium is unstable, they can modify the equilibrium by this mechanism so as to render it stable], and  $t_{\text{nl}}$  is the time at which perturbation amplitudes become large enough for nonlinear interactions between individual modes to matter [second term on the right-hand side of (2.12); if perturbations grow, they can saturate by this mechanism].

#### 2.4.2. Quasilinear Theory (QLT)

Suppose

$$t_{\text{eq}} \ll t_{\text{nl}}, \quad (2.15)$$

i.e., *growing perturbations start modifying the equilibrium before they saturate nonlinearly*. Then the strategy is to solve (2.13) [together with (2.9)] for the perturbations, use the result to calculate their correlation function needed in the right-hand side of (2.11), then work out how the equilibrium therefore evolves and hence how large the perturbations must grow in order for this evolution to turn the unstable equilibrium into a stable one. This is a classic piece of theory, important conceptually—I will describe it in detail and do one example in §6. In reality, however, it happens relatively rarely that unstable perturbations saturate at amplitudes small enough for the nonlinear interactions not to matter [i.e., for (2.15) to hold true].

#### 2.4.3. Weak-Turbulence (WT) Theory

Sometimes, one is not lucky enough to get away with QLT (i.e., alas,  $t_{\text{nl}} \ll t_{\text{eq}}$  or  $t_{\text{nl}} \sim t_{\text{eq}}$ ), but is lucky enough to have perturbations saturating nonlinearly at a *small amplitude* such that<sup>10</sup>

$$t_{\text{nl}} \gg \omega^{-1}, \quad (2.16)$$

i.e., *perturbations oscillate linearly faster than they interact nonlinearly* (this can happen, e.g., because propagating wave packets do not stay together long enough to break up completely in one encounter). Because waves are fast compared to nonlinear evolution in this approximation, it is possible to “quantise” them, i.e., to treat a nonlinear turbulent state of the plasma as a cocktail consisting of both “true” particles (ions and electrons) and “quasiparticles” representing electromagnetic excitations (§7).

In this approximation, one can do perturbation theory treating the nonlinear term in (2.12) as small and expanding in the small parameter  $(\omega t_{\text{nl}})^{-1}$ . The resulting *weak (or “wave”) turbulence theory* is quite an analytical *tour de force*—but it is a lot of work to do it properly! I will provide an introduction to it in §§7.2 and 8.4. The classic texts on this are Kadomtsev (1965) (early but lucid) and Zakharov *et al.* (1992) (mathematically definitive); a recent textbook in Zakharov’s tradition is Nazarenko (2011), while the quasiparticle approach, with Feynman diagrams and all that, can be learned from Tsy托ovich (1970, 1972, 1977, 1995) (with difficulty) or from Kingsep (1996) (quite enjoyably, but in Russian).

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<sup>10</sup>Note that the nonlinear time scale is typically inversely proportional to the amplitude; see (2.12).



FIGURE 3. Lev Landau (1908–1968), great Soviet physicist, quintessential theoretician, author of the Book, cult figure. It is a minor feature of his scientific biography that he wrote the two most important plasma-physics papers of all time ([Landau 1936](#), [1946](#)). He also got a Nobel Prize (1962), but not for plasma physics. (a) Cartoon by A. A. Yuzefovich (from [Landau & Lifshitz 1976](#)); the caption says “Dau spake...” (...unto the students, also depicted). (b) Landau’s mugshot from NKVD prison (1938), where he ended up for seditious talk and from whence he was released in 1939 after Peter Kapitsa’s personal appeal to Stalin.

Note that because the nonlinear term couples perturbations at different  $\mathbf{k}$ ’s (scales), this theory will lead to multi-scale (usually, power-law) fluctuation spectra.

#### 2.4.4. Strong-Turbulence Theory

If perturbations manage to grow to a level at which

$$t_{\text{nl}} \sim \omega^{-1}, \quad (2.17)$$

we are facing *strong turbulence*. This is actually what mostly happens (including in WT systems, where turbulence often transitions into the strong regime at small enough, or large enough, scales). Theory of such regimes tends to be of phenomenological/scaling kind, often in the spirit of the classic [Kolmogorov \(1941\)](#) theory of hydrodynamic turbulence. No one really knows how to move very far beyond this sort of approach—and not for lack of trying. I will return to this topic in §§8.6 and 12.2.

### 3. Linear Theory: Waves, Landau Damping and Kinetic Instabilities

Enough idle chatter, let us calculate! In this section, we are concerned with the linearised Vlasov–Poisson system, (2.13) and (2.9):

$$\frac{\partial \delta f_{\mathbf{k}\alpha}}{\partial t} + i\mathbf{k} \cdot \mathbf{v} \delta f_{\mathbf{k}\alpha} = \frac{q_\alpha}{m_\alpha} \varphi_{\mathbf{k}} \mathbf{i} \mathbf{k} \cdot \frac{\partial f_{0\alpha}}{\partial \mathbf{v}}, \quad (3.1)$$

$$\varphi_{\mathbf{k}} = \frac{4\pi}{k^2} \sum_{\alpha} q_{\alpha} \int d\mathbf{v} \delta f_{\mathbf{k}\alpha}. \quad (3.2)$$

For compactness of notation, I will drop both the species index  $\alpha$  and the wave number  $\mathbf{k}$  in the subscripts, unless they are necessary for understanding.

We will discover that electrostatic perturbations in a plasma described by (3.1) and (3.2) oscillate, can pass their energy to particles (damp) or even grow, sucking energy from the particles. We will also discover that it is useful to know some complex analysis.

#### 3.1. Initial-Value Problem

We shall follow Landau’s original paper ([Landau 1946](#)) in considering an initial-value problem—because, as we will see, perturbations can be damped or grow, so it is not

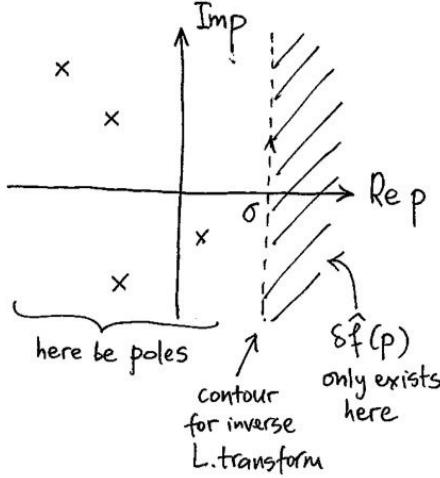


FIGURE 4. Layout of the complex- $p$  plane:  $\hat{\delta}f(p)$  is analytic for  $\text{Re } p \geq \sigma$ . At  $\text{Re } p < \sigma$ ,  $\hat{\delta}f(p)$  may have singularities (poles).

appropriate to think of them over  $t \in [-\infty, +\infty]$  (and—NB!!!—the damped perturbations are *not* pure eigenmodes; see §5.3). So we look for  $\delta f(\mathbf{v}, t)$  satisfying (3.1) with the initial condition

$$\delta f(\mathbf{v}, t=0) = g(\mathbf{v}). \quad (3.3)$$

It is, therefore, appropriate to use the *Laplace transform* to solve (3.1):

$$\boxed{\hat{\delta}f(p) = \int_0^\infty dt e^{-pt} \delta f(t)} . \quad (3.4)$$

It is a mathematical certainty that if there exists a real number  $\sigma > 0$  such that

$$|\delta f(t)| < e^{\sigma t} \text{ as } t \rightarrow \infty, \quad (3.5)$$

then the integral (3.4) exists (i.e., is finite) for all values of  $p$  such that  $\text{Re } p \geq \sigma$ . The inverse Laplace transform, giving us back our distribution function as a function of time, is then

$$\delta f(t) = \frac{1}{2\pi i} \int_{-i\infty+\sigma}^{i\infty+\sigma} dp e^{pt} \hat{\delta}f(p), \quad (3.6)$$

where the integral in the complex plane is along a straight line parallel to the imaginary axis and intersecting the real axis at  $\text{Re } p = \sigma$  (Fig. 4).

Since we expect to be able to recover our desired time-dependent function  $\delta f(\mathbf{v}, t)$  from its Laplace transform, it is worth knowing the latter. To find it, we Laplace-transform (3.1):

$$\begin{aligned} \text{l.h.s.} &= \int_0^\infty dt e^{-pt} \frac{\partial \delta f}{\partial t} = [e^{-pt} \delta f]_0^\infty + p \int_0^\infty dt e^{-pt} \delta f = -g + p \hat{\delta}f, \\ \text{r.h.s.} &= -ik \cdot \mathbf{v} \hat{\delta}f + \frac{q}{m} \hat{\varphi} ik \cdot \frac{\partial f_0}{\partial \mathbf{v}}. \end{aligned} \quad (3.7)$$

Equating these two expressions, we find the solution:

$$\boxed{\hat{\delta}f(p) = \frac{1}{p + ik \cdot \mathbf{v}} \left[ i \frac{q}{m} \hat{\varphi}(p) \mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{v}} + g \right]} . \quad (3.8)$$

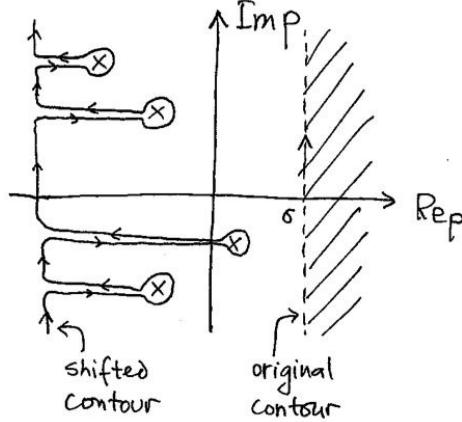


FIGURE 5. New contour for the inverse Laplace transform.

The Laplace transform of the potential,  $\hat{\varphi}(p)$ , itself depends on  $\delta\hat{f}$  via (3.2):

$$\begin{aligned}\hat{\varphi}(p) &= \int_0^\infty dt e^{-pt} \varphi(t) = \frac{4\pi}{k^2} \sum_\alpha q_\alpha \int d\mathbf{v} \delta\hat{f}_\alpha(p) \\ &= \frac{4\pi}{k^2} \sum_\alpha q_\alpha \int d\mathbf{v} \frac{1}{p + i\mathbf{k} \cdot \mathbf{v}} \left[ i \frac{q_\alpha}{m_\alpha} \hat{\varphi}(p) \mathbf{k} \cdot \frac{\partial f_{0\alpha}}{\partial \mathbf{v}} + g_\alpha \right].\end{aligned}\quad (3.9)$$

This is an algebraic equation for  $\hat{\varphi}(p)$ . Collecting terms, we get

$$\underbrace{\left[ 1 - \sum_\alpha \frac{4\pi q_\alpha^2}{k^2 m_\alpha} i \int d\mathbf{v} \frac{1}{p + i\mathbf{k} \cdot \mathbf{v}} \mathbf{k} \cdot \frac{\partial f_{0\alpha}}{\partial \mathbf{v}} \right]}_{\equiv \epsilon(p, \mathbf{k})} \hat{\varphi}(p) = \frac{4\pi}{k^2} \sum_\alpha q_\alpha \int d\mathbf{v} \frac{g_\alpha}{p + i\mathbf{k} \cdot \mathbf{v}}. \quad (3.10)$$

The prefactor in the left-hand side, which I denote  $\epsilon(p, \mathbf{k})$ , is called the *dielectric function*, because it encodes all the self-consistent charge-density perturbations that plasma sets up in response to an electric field. This is going to be an important function, so let us write it out beautifully:

$$\boxed{\epsilon(p, \mathbf{k}) = 1 - \sum_\alpha \frac{\omega_{p\alpha}^2}{k^2} \frac{i}{n_\alpha} \int d\mathbf{v} \frac{1}{p + i\mathbf{k} \cdot \mathbf{v}} \mathbf{k} \cdot \frac{\partial f_{0\alpha}}{\partial \mathbf{v}}}, \quad (3.11)$$

where the plasma frequency of species  $\alpha$  is defined by [cf. (2.2)]

$$\omega_{p\alpha}^2 = \frac{4\pi q_\alpha^2 n_\alpha}{m_\alpha}. \quad (3.12)$$

The solution of (3.10) is

$$\hat{\varphi}(p) = \frac{4\pi}{k^2 \epsilon(p, \mathbf{k})} \sum_\alpha q_\alpha \int d\mathbf{v} \frac{g_\alpha}{p + i\mathbf{k} \cdot \mathbf{v}}. \quad (3.13)$$

To calculate  $\varphi(t)$ , we need to inverse-Laplace-transform  $\hat{\varphi}$ : similarly to (3.6),

$$\varphi(t) = \frac{1}{2\pi i} \int_{-i\infty+\sigma}^{i\infty+\sigma} dp e^{pt} \hat{\varphi}(p). \quad (3.14)$$

How do we do this integral? Recall that  $\delta\hat{f}$  and, therefore,  $\hat{\varphi}$  only exists (i.e., is finite) for

$\text{Re } p \geq \sigma$ , whereas at  $\text{Re } p < \sigma$ , it can have singularities, i.e., poles—let us call them  $p_i$ , indexed by  $i$ . If we analytically continue  $\hat{\varphi}(p)$  everywhere to  $\text{Re } p < \sigma$  except those poles, the result must have the form

$$\hat{\varphi}(p) = \sum_i \frac{c_i}{p - p_i} + A(p), \quad (3.15)$$

where  $c_i$  are some coefficients (residues) and  $A(p)$  is the analytic part of the solution. The integration contour in (3.14) can be shifted to  $\text{Re } p \rightarrow -\infty$  but with the proviso that it cannot cross the poles, as shown in Fig. 5 (this is proven by making a closed loop out of the old and the new contours, joining them at  $\pm i\infty$ , and noting that this loop encloses no poles). Then the contributions to the integral from the vertical segments of the contour are exponentially small,<sup>11</sup> the contributions from the segments leading towards and away from the poles cancel, and the contributions from the circles around the poles can, by Cauchy's formula, be expressed in terms of the poles and residues:

$$\boxed{\varphi(t) = \sum_i c_i e^{p_i t}}. \quad (3.16)$$

Thus, in the long-time limit, *perturbations of the potential will evolve  $\propto e^{p_i t}$ , where  $p_i$  are poles of  $\hat{\varphi}(p)$* . In general,  $p_i = -i\omega_i + \gamma_i$ , where  $\omega_i$  is a real frequency (giving wave-like behaviour of perturbations),  $\gamma_i < 0$  represents damping and  $\gamma_i > 0$  growth of the perturbations (instability).

Note that we need not be particularly interested in what  $c_i$ 's are because, if we set up an initial perturbation with a given  $\mathbf{k}$  and then wait long enough, only the fastest-growing or, failing growth, the slowest-damped mode will survive, with all others having exponentially small amplitudes. Thus, a typical outcome of the linear theory is  $\varphi(t)$  oscillating at some frequency and growing or decaying at some unique rate. Since this is a solution of a linear equation, the prefactor in front of the exponential can be scaled arbitrarily and so does not matter.

Going back to (3.13), we realise that the poles of  $\hat{\varphi}(p)$  are zeros of the dielectric function:

$$\boxed{\epsilon(p_i, \mathbf{k}) = 0} \Rightarrow p_i = p_i(\mathbf{k}) = -i\omega_i(\mathbf{k}) + \gamma_i(\mathbf{k}). \quad (3.17)$$

To find the wave frequencies  $\omega_i$  and the damping/growth rates  $\gamma_i$ , we must solve this equation, which is called the *plasma dispersion relation*.

### 3.2. Calculating the Dielectric Function: the “Landau Prescription”

In order to be able to solve  $\epsilon(p, \mathbf{k}) = 0$ , we must learn how to calculate  $\epsilon(p, \mathbf{k})$  for any given  $p$  and  $\mathbf{k}$ . Before I wrote (3.15), I said that  $\hat{\varphi}$ , given by (3.13), had to be analytically continued to the entire complex plane from the area where its analyticity was guaranteed ( $\text{Re } p \geq \sigma$ ), but I did not explain how this was to be done. In order to do it, we must

<sup>11</sup>They are exponentially small *in time* as  $t \rightarrow \infty$  because the integrand of the inverse Laplace transform (3.14) contains a factor of  $e^{\text{Re } pt}$ , which decays faster than any of the “modes” in (3.16). If  $\hat{\varphi}(p)$  does not grow too fast at large  $p$ , the integral along the vertical part of the contour may also vanish at any finite  $t$ , but that is not guaranteed in general: indeed, looking ahead to the explicit expression (3.27) for  $\hat{\varphi}(p)$ , with the Landau prescription for analytic continuation to  $\text{Re } p < 0$  analogous to (3.20), we see that  $\hat{\varphi}(p)$  will contain a term  $\propto G_\alpha(ip/k)$ , which can be large at large  $\text{Re } p$ , e.g., if  $G_\alpha(v_z)$  is a Maxwellian. Note also that we need the (wildly oscillating in time) integral of  $e^{pt}\hat{\varphi}(p)$  over the horizontal segments with  $\text{Im } p \rightarrow \pm\infty$  to vanish. This is fine provided  $\hat{\varphi}(\pm i\infty) = 0$ , which is usually OK.

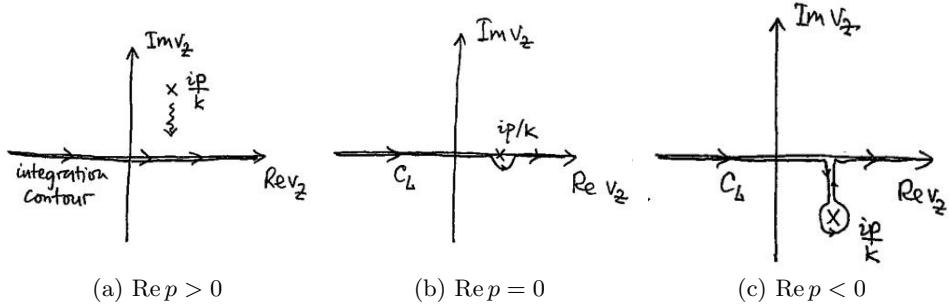


FIGURE 6. The Landau prescription for the contour of integration in (3.18).

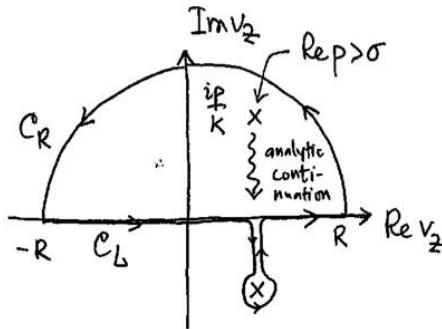


FIGURE 7. Proof of Landau's prescription [see (3.19)].

learn how to calculate the velocity integral in (3.11)—if we want  $\epsilon(p, \mathbf{k})$  and, therefore, its zeros  $p_i$ —and also how to calculate the similar integral in (3.13) containing  $g_\alpha$ .

First of all, let us turn these integrals into a 1D form. Given  $\mathbf{k}$ , we can always choose the  $z$  axis to be along  $\mathbf{k}$ .<sup>12</sup> Then

$$\begin{aligned} \int d\mathbf{v} \frac{1}{p + i\mathbf{k} \cdot \mathbf{v}} \mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{v}} &= \int dv_z \frac{1}{p + ikv_z} k \frac{\partial}{\partial v_z} \underbrace{\int dv_x \int dv_y f_0(v_x, v_y, v_z)}_{\equiv F(v_z)} \\ &= -i \int_{-\infty}^{+\infty} dv_z \frac{F'(v_z)}{v_z - ip/k}. \end{aligned} \quad (3.18)$$

Assuming, reasonably, that  $F'(v_z)$  is a nice (analytic) function everywhere, we conclude that the integrand in (3.18) has one pole,  $v_z = ip/k$ . When  $\text{Re } p \geq \sigma > 0$ , this pole is harmless because, in the complex plane associated with the  $v_z$  variable, it lies above the integration contour, which is the real axis,  $v_z \in (-\infty, +\infty)$ . We can think of analytically continuing the above integral to  $\text{Re } p < \sigma$  as moving the pole  $v_z = ip/k$  down, towards and below the real axis. As long as  $\text{Re } p > 0$ , this can be done with impunity, in the sense that the pole stays above the integration contour, and so the analytic continuation is simply the same integral (3.18), still along the real axis. However, if the pole moves so far down that  $\text{Re } p = 0$  or  $\text{Re } p < 0$ , we must deform the contour of integration in such a way as to keep the pole always above it, as shown in Fig. 6. This is called the *Landau prescription* and the contour thus deformed is called the *Landau contour*,  $C_L$ .

Let me prove that this is indeed an analytic continuation, i.e., that the integral (3.18),

<sup>12</sup>NB: This means that in what follows,  $k \geq 0$  by definition.

adjusted to be along  $C_L$ , is an analytic function for all values of  $p$ . Let us cut the Landau contour at  $v_z = \pm R$  and close it in the upper half-plane with a semicircle  $C_R$  of radius  $R > \sigma/k$  (Fig. 7). Then, with integration running along the truncated  $C_L$  and counterclockwise along  $C_R$ , we get, by Cauchy's formula,

$$\int_{C_L} dv_z \frac{F'(v_z)}{v_z - ip/k} + \int_{C_R} dv_z \frac{F'(v_z)}{v_z - ip/k} = 2\pi i F'\left(\frac{ip}{k}\right). \quad (3.19)$$

Since analyticity is guaranteed for  $\text{Re } p \geq \sigma$ , the integral along  $C_R$  is analytic. The right-hand side is also analytic, by assumption. Therefore, the integral along  $C_L$  is analytic—this is the integral along the Landau contour if we take  $R \rightarrow \infty$ . Q.e.d.

With the Landau prescription, our integral is calculated as follows:

$$\int_{C_L} dv_z \frac{F'(v_z)}{v_z - ip/k} = \begin{cases} \int_{-\infty}^{+\infty} dv_z \frac{F'(v_z)}{v_z - ip/k} & \text{if } \text{Re } p > 0, \\ \mathcal{P} \int_{-\infty}^{+\infty} dv_z \frac{F'(v_z)}{v_z - ip/k} + i\pi F'\left(\frac{ip}{k}\right) & \text{if } \text{Re } p = 0, \\ \int_{-\infty}^{+\infty} dv_z \frac{F'(v_z)}{v_z - ip/k} + i2\pi F'\left(\frac{ip}{k}\right) & \text{if } \text{Re } p < 0, \end{cases} \quad (3.20)$$

where the integrals are again over the real axis and the imaginary bits come from the contour making a half (when  $\text{Re } p = 0$ ) or a full (when  $\text{Re } p < 0$ ) circle around the pole. In the case of  $\text{Re } p = 0$ , or  $ip = \omega$ , the integral along the real axis is formally divergent and so we take its *principal value*, defined as

$$\mathcal{P} \int_{-\infty}^{+\infty} dv_z \frac{F'(v_z)}{v_z - \omega/k} = \lim_{\varepsilon \rightarrow 0} \left[ \int_{-\infty}^{\omega/k - \varepsilon} + \int_{\omega/k + \varepsilon}^{+\infty} \right] dv_z \frac{F'(v_z)}{v_z - \omega/k}. \quad (3.21)$$

The difference between (3.21) and the usual Lebesgue definition of an integral is that the latter would be

$$\int_{-\infty}^{+\infty} dv_z \frac{F'(v_z)}{v_z - \omega/k} = \left[ \lim_{\varepsilon_1 \rightarrow 0} \int_{-\infty}^{\omega/k - \varepsilon_1} + \lim_{\varepsilon_2 \rightarrow 0} \int_{\omega/k + \varepsilon_2}^{+\infty} \right] dv_z \frac{F'(v_z)}{v_z - \omega/k}, \quad (3.22)$$

and this, with, in general,  $\varepsilon_1 \neq \varepsilon_2$ , diverges logarithmically, whereas in (3.21), the divergences neatly cancel.

The  $\text{Re } p = 0$  case in (3.20),

$$\int_{C_L} dv_z \frac{F'(v_z)}{v_z - \omega/k} = \mathcal{P} \int_{-\infty}^{+\infty} dv_z \frac{F'(v_z)}{v_z - \omega/k} + i\pi F'\left(\frac{\omega}{k}\right), \quad (3.23)$$

which tends to be of most use in analytical theory, is a particular instance of *Plemelj's formula*: for a real  $\zeta$  and a well-behaved function  $f$  (no poles on or near the real axis),

$$\lim_{\varepsilon \rightarrow +0} \int_{-\infty}^{+\infty} dx \frac{f(x)}{x - \zeta \mp i\varepsilon} = \mathcal{P} \int_{-\infty}^{+\infty} dx \frac{f(x)}{x - \zeta} \pm i\pi f(\zeta), \quad (3.24)$$

also sometimes written as

$$\lim_{\varepsilon \rightarrow +0} \frac{1}{x - \zeta \mp i\varepsilon} = \mathcal{P} \frac{1}{x - \zeta} \pm i\pi \delta(x - \zeta), \quad (3.25)$$

Finally, armed with Landau's prescription, we are ready to calculate. The dielectric

function (3.11) becomes

$$\boxed{\epsilon(p, \mathbf{k}) = 1 - \sum_{\alpha} \frac{\omega_{p\alpha}^2}{k^2} \frac{1}{n_{\alpha}} \int_{C_L} dv_z \frac{F'_{\alpha}(v_z)}{v_z - ip/k}}, \quad (3.26)$$

and, analogously, our Laplace-transformed solution (3.13) becomes

$$\hat{\varphi}(p) = -\frac{4\pi i}{k^3 \epsilon(p, \mathbf{k})} \sum_{\alpha} q_{\alpha} \int_{C_L} dv_z \frac{G_{\alpha}(v_z)}{v_z - ip/k}, \quad (3.27)$$

where  $G_{\alpha}(v_z) = \int dv_x \int dv_y g_{\alpha}(v_x, v_y, v_z)$ .

### 3.3. Solving the Dispersion Relation: the Limit of Slow Damping/Growth

A particularly analytically solvable and physically interesting case is one in which, for  $p = -i\omega + \gamma$ ,  $\gamma \ll \omega$  and  $\gamma \ll kv_{\text{tha}}$ , i.e., the case of the damping or growth time of the waves being longer than either their period or the time particles take to cross them. In this limit, the dispersion relation (3.17) is

$$\epsilon(p, \mathbf{k}) \approx \epsilon(-i\omega, \mathbf{k}) + i\gamma \frac{\partial}{\partial \omega} \epsilon(-i\omega, \mathbf{k}) = 0. \quad (3.28)$$

Setting the real part of (3.28) to zero gives the equation for the real frequency:

$$\boxed{\text{Re } \epsilon(-i\omega, \mathbf{k}) = 0}. \quad (3.29)$$

Setting the imaginary part of (3.28) to zero gives us the damping/growth rate in terms of the real frequency:

$$\boxed{\gamma = -\text{Im } \epsilon(-i\omega, \mathbf{k}) \left[ \frac{\partial}{\partial \omega} \text{Re } \epsilon(-i\omega, \mathbf{k}) \right]^{-1}}. \quad (3.30)$$

Thus, we now only need  $\epsilon(p, \mathbf{k})$  with  $p = -i\omega$ . Using (3.23), we get

$$\text{Re } \epsilon = 1 - \sum_{\alpha} \frac{\omega_{p\alpha}^2}{k^2} \frac{1}{n_{\alpha}} \mathcal{P} \int dv_z \frac{F'_{\alpha}(v_z)}{v_z - \omega/k}, \quad (3.31)$$

$$\text{Im } \epsilon = - \sum_{\alpha} \frac{\omega_{p\alpha}^2}{k^2} \frac{\pi}{n_{\alpha}} F'_{\alpha}\left(\frac{\omega}{k}\right). \quad (3.32)$$

Let us consider a two-species plasma, consisting of electrons and a single species of ions. There will be two interesting limits:

- “High-frequency” electron waves:  $\omega \gg kv_{\text{the}}$ , where  $v_{\text{the}} = \sqrt{2T_e/m_e}$  is the “thermal speed” of the electrons;<sup>13</sup> this limit will give us Langmuir waves (§3.4), slowly damped or growing (§3.5).
- “Low-frequency” ion waves: a particularly tractable limit will be that of “hot” electrons and “cold” ions, viz.,  $kv_{\text{the}} \gg \omega \gg kv_{\text{thi}}$ , where  $v_{\text{thi}} = \sqrt{2T_i/m_i}$  is the “thermal speed” of the ions; this limit will give us the sound (“ion-acoustic waves”; §3.8), which also can be damped or growing (§3.9).

<sup>13</sup>This is a standard well-defined quantity for a Maxwellian equilibrium distribution  $F_e(v_z) = (n_e/\sqrt{\pi} v_{\text{the}}) \exp(-v_z^2/v_{\text{the}}^2)$ , but if we wish to consider a non-Maxwellian  $F_e$ , let  $v_{\text{the}}$  be some typical speed characterising the width of the equilibrium distribution, defined by, e.g., (3.36).

### 3.4. Langmuir Waves

Consider the limit

$$\frac{\omega}{k} \gg v_{\text{the}}, \quad (3.33)$$

i.e., the phase velocity of the waves is much greater than the typical velocity of a particle from the “thermal bulk” of the distribution. This means that in (3.31), we can expand in  $v_z \sim v_{\text{the}}$  being small compared to  $\omega/k$  (higher values of  $v_z$  are cut off by the “thermal” fall-off of the equilibrium distribution function). Note that  $\omega \gg kv_{\text{the}}$  also implies  $\omega \gg kv_{\text{thi}}$  because

$$\frac{v_{\text{thi}}}{v_{\text{the}}} = \sqrt{\frac{T_i}{T_e} \frac{m_e}{m_i}} \ll 1 \quad (3.34)$$

as long as  $T_i/T_e$  is not huge.<sup>14</sup> Thus, (3.31) becomes

$$\begin{aligned} \text{Re } \epsilon &= 1 + \sum_{\alpha} \frac{\omega_{\text{p}\alpha}^2}{k^2} \frac{1}{n_{\alpha}} \frac{k}{\omega} \mathcal{P} \int dv_z F'_{\alpha}(v_z) \left[ 1 + \frac{kv_z}{\omega} + \left( \frac{kv_z}{\omega} \right)^2 + \left( \frac{kv_z}{\omega} \right)^3 + \dots \right] \\ &= 1 + \sum_{\alpha} \frac{\omega_{\text{p}\alpha}^2}{k\omega} \underbrace{\left[ \frac{1}{n_{\alpha}} \int dv_z F'_{\alpha}(v_z) - \frac{k}{\omega} \frac{1}{n_{\alpha}} \int dv_z F_{\alpha}(v_z) \right]}_{=0} \underbrace{\left[ \frac{1}{n_{\alpha}} \int dv_z v_z F_{\alpha}(v_z) - 3 \frac{k^3}{\omega^3} \frac{1}{n_{\alpha}} \int dv_z v_z^2 F_{\alpha}(v_z) \right]}_{=1} \\ &\quad - 2 \frac{k^2}{\omega^2} \underbrace{\frac{1}{n_{\alpha}} \int dv_z v_z F_{\alpha}(v_z)}_{=0} - 3 \frac{k^3}{\omega^3} \underbrace{\frac{1}{n_{\alpha}} \int dv_z v_z^2 F_{\alpha}(v_z)}_{=v_{\text{th}\alpha}^2/2} + \dots \\ &= 1 - \sum_{\alpha} \frac{\omega_{\text{p}\alpha}^2}{\omega^2} \left[ 1 + \frac{3}{2} \frac{k^2 v_{\text{th}\alpha}^2}{\omega^2} + \dots \right], \end{aligned} \quad (3.35)$$

where we have integrated by parts everywhere, assumed that there are no mean flows,  $\langle v_z \rangle = 0$ , and, in the last term, used

$$\langle v_z^2 \rangle = \frac{v_{\text{th}\alpha}^2}{2}, \quad (3.36)$$

which is indeed the case for a Maxwellian  $F_{\alpha}$  or, if  $F_{\alpha}$  is not a Maxwellian, can be viewed as the definition of  $v_{\text{th}\alpha}$ .

The ion contribution to (3.35) is small because

$$\frac{\omega_{\text{pi}}^2}{\omega_{\text{pe}}^2} = \frac{Z m_e}{m_i} \ll 1, \quad (3.37)$$

so ions do not participate in this dynamics at all. Therefore, to lowest order, the dispersion relation (3.29) becomes

$$\text{Re } \epsilon \approx 1 - \frac{\omega_{\text{pe}}^2}{\omega^2} = 0 \quad \Rightarrow \quad \boxed{\omega^2 = \omega_{\text{pe}}^2 = \frac{4\pi e^2 n_e}{m_e}}. \quad (3.38)$$

This is the [Tonks & Langmuir \(1929\)](#) dispersion relation for what is known as *Langmuir, or plasma, oscillations*. This is the formal derivation of the result that we already had, on less mathematically rigorous, physical grounds, in §2.1.

<sup>14</sup>For hydrogen plasma,  $\sqrt{m_i/m_e} \approx 42$ , the answer to the Ultimate Question of Life, Universe and Everything ([Adams 1979](#)).

We can do a little better if we retain the (small)  $k$ -dependent term in (3.35):

$$\text{Re } \epsilon \approx 1 - \frac{\omega_{\text{pe}}^2}{\omega^2} \left( 1 + \underbrace{\frac{3 k^2 v_{\text{the}}^2}{2 \omega^2}}_{\text{use}} \right) = 0 \quad \Rightarrow \quad \boxed{\omega^2 \approx \omega_{\text{pe}}^2 (1 + 3k^2 \lambda_{\text{De}}^2)}, \quad (3.39)$$

where  $\lambda_{\text{De}} = v_{\text{the}}/\sqrt{2}\omega_{\text{pe}} = \sqrt{T_e/4\pi e^2 n_e}$  is the “electron Debye length” [cf. (1.6)]. Equation (3.39) is the [Bohm & Gross \(1949a\)](#) dispersion relation, describing an upgrade of the Langmuir oscillations to dispersive *Langmuir waves*, which have a non-zero group velocity (this effect is due to the electron pressure perturbation joining the electric field in providing the restoring force for the waves: see Exercise 3.1).

Note that all this is only valid for  $\omega \gg kv_{\text{the}}$ , which we now see is equivalent to

$$k\lambda_{\text{De}} \ll 1 \quad (3.40)$$

(the wave length of the perturbation is long compared to the Debye length).

**Exercise 3.1. Langmuir hydrodynamics.**<sup>15</sup> Starting from the linearised kinetic equation for electrons and ignoring perturbations of the ion distribution function completely, work out the fluid equations for electrons (i.e., the evolution equations for the electron density  $n_e$  and velocity  $u_e$ ) and show that you can recover the Langmuir waves (3.39) if you assume that electrons behave as a 1D adiabatic fluid (i.e., have the equation of state  $p_e n_e^{-\gamma} = \text{const}$  with  $\gamma = 3$ ). You can prove that they indeed do this by calculating their density and pressure directly from the Landau solution for the perturbed distribution function (see §§5.3 and 5.6), ignoring resonant particles. The “hydrodynamic” description of Langmuir waves will reappear in §8.

### 3.5. Landau Damping and Kinetic Instabilities

Now let us calculate the damping rate of Langmuir waves using (3.30):

$$\frac{\partial \text{Re } \epsilon}{\partial \omega} \approx \frac{2\omega_{\text{pe}}^2}{\omega^3}, \quad \text{Im } \epsilon \approx -\frac{\omega_{\text{pe}}^2}{k^2} \frac{\pi}{n_e} F'_e\left(\frac{\omega}{k}\right) \quad \Rightarrow \quad \boxed{\gamma \approx \frac{\pi}{2} \frac{\omega^3}{k^2} \frac{1}{n_e} F'_e\left(\frac{\omega}{k}\right)}, \quad (3.41)$$

where  $\omega$  is given by (3.39) (if  $F_e$  is Maxwellian, the dispersive correction in  $\omega^2$ , when substituted into the exponential, will change  $\gamma$  by a factor of order unity, hence it is worth keeping). Provided  $\omega F'(\omega/k) < 0$  (as would be the case, e.g., for any distribution monotonically decreasing with  $|v_z|$ ; see Fig. 8a),  $\gamma < 0$  and so this is indeed a damping rate, the celebrated *Landau damping* ([Landau 1946](#); it was confirmed experimentally two decades later, by [Malmberg & Wharton 1964](#)).

The same theory also describes a class of *kinetic instabilities*: if  $\omega F'(\omega/k) > 0$ , then  $\gamma > 0$ , so perturbations grow exponentially with time. An iconic example is the *bump-on-tail instability* (Fig. 8b), which arises when a high-energy ( $v_z \gg v_{\text{the}}$ ) electron beam is injected into a plasma<sup>16</sup> and whose quasilinear saturation we will study in great detail in §6.

We see that the damping or growth of plasma waves occur via their interaction with the particles whose velocities coincide with the phase velocity of the wave (“*Landau resonance*”). Because such particles are moving in phase with the wave, its electric field

<sup>15</sup>This is based on the 2017 exam question.

<sup>16</sup>Here we are dealing with the case of a “warm beam” (meaning that it has a finite width). It turns out that there exists also another instability, leading to growth of perturbations with  $\omega/k$  to the right of the bump’s peak, due to a different, “fluid” kind of resonance and possible even for “cold beams” (i.e., beams of particles that all have the same velocity): see §3.7.

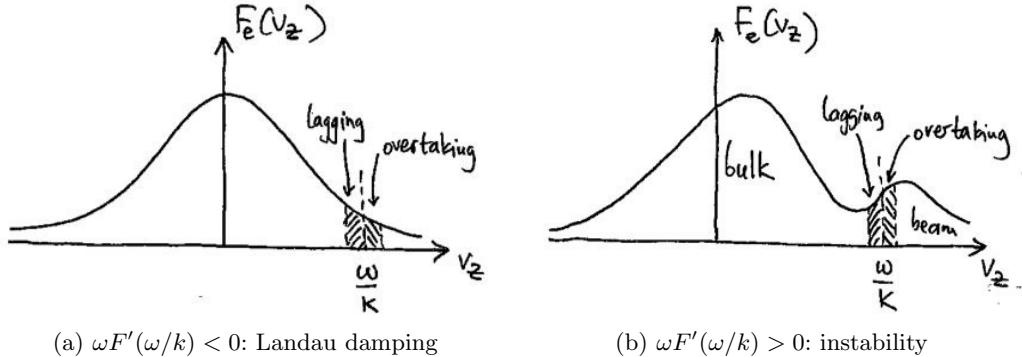


FIGURE 8. The Landau resonance (particle velocities equalling phase speed of the wave  $v_z = \omega/k$ ) leads to damping of the wave if more particles lag just behind than overtake the wave and to instability in the opposite case.

is stationary in their reference frame and so can do work on these particles, giving its energy to them (damping) or receiving energy from them (instability). In contrast, other, out-of-phase, particles experience no mean energy change over time because the field that they “see” is oscillating. It turns out (§3.6) that the process works in the spirit of socialist redistribution: the particles slightly lagging behind the wave will, on average, receive energy from it, damping the wave, whereas those overtaking the wave will have some of their energy taken away, amplifying the wave. The condition  $\omega F'(\omega/k) < 0$  corresponds to the stragglers being more numerous than the strivers, leading to net damping;  $\omega F'(\omega/k) > 0$  implies the opposite, leading to an instability (which then leads to flattening of the distribution; see §6).

Let us note again that these results are quantitatively valid only in the limit (3.33), or, equivalently, (3.40). It makes sense that damping should be slow ( $\gamma \ll \omega$ ) in the limit where the waves propagate much faster than the majority of the electrons ( $\omega/k \gg v_{\text{the}}$ ) and so can interact only with a small number of particularly fast particles (for a Maxwellian equilibrium distribution, it is an exponentially small number  $\sim e^{-\omega^2/k^2 v_{\text{the}}^2}$ ). If, on the other hand,  $\omega/k \sim v_{\text{the}}$ , the waves interact with the majority population and the damping should be strong: *a priori*, we might expect  $\gamma \sim k v_{\text{the}}$ .<sup>17</sup>

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**Exercise 3.2. Stability of isotropic distributions.** Prove that if  $f_{0e}(v_x, v_y, v_z) = f_{0e}(v)$ , i.e., if it is a 3D-isotropic distribution, monotonic or otherwise, the Langmuir waves at  $k\lambda_{De} \ll 1$  are always damped (this is solved in Lifshitz & Pitaevskii 1981; the statement of stability of isotropic distributions is in fact valid much more generally than just for long-wavelength Langmuir waves: see Exercise 4.2).

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Landau’s method of working out waves and damping in collisionless plasmas, and in particular his prescription for dealing with the singularities in the integrals, has always elicited a degree of dissatisfaction in the minds of some mathematically inclined physicists and motivated them to search for alternatives. Perhaps the earliest and best known such alternative is the formalism due to van Kampen. His objective was more mathematical rigour—but even if this is of limited appeal to you, the book by van Kampen & Felderhof (1967) is still a good read and a good chance to question and re-examine your understanding of how it all works.

A key question that preoccupied van Kampen and many of those who re-examined Landau’s solution later on was whether the initial-value problem for the linear evolution of perturbations in a plasma could be solved in the usual way such things are done elsewhere in physics: by decomposing the initial perturbation into some convenient set of normal modes, advancing each

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<sup>17</sup>This is indeed correct. You can confirm it numerically using (3.84) and (3.90).

of them in time, and then reassembling them back into the desired solution. The answer is yes—van Kampen did find a complete set of modes, although they were not eigenfunctions of a Hermitian operator and thus (arguably) were not very user-friendly. In a short paper by [Ramos & White \(2018\)](#), you will find the most recent and the most transparent (in my view) scheme for how to construct normal modes that *are* eigenfunctions of a Hermitian operator.<sup>18</sup> One curious corollary is that it is possible to cook up special initial perturbations that will not decay at the Landau rate and, in fact, can have any time evolution that one cares to specify! If this intrigues or disturbs you, follow the paper trail from [Ramos & White \(2018\)](#) backward in time.<sup>19</sup>

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Landau damping became a *cause célèbre* in the hard-core mathematics community, as well as in the wider science world, with the award of the Fields Medal in 2010 to Cédric Villani, who proved (with C. Mouhot) that, basically, Landau’s solution of the linearised Vlasov equation survived as a solution of the full nonlinear Vlasov equation for small enough and regular enough initial perturbations: see a “popular” account of this by [Villani \(2014\)](#). The regularity restriction is apparently important and the result can break down in interesting ways ([Bedrossian 2016](#)). The culprit is plasma echo, of which more will be said in §12 (without any claim to mathematical rigour).

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### 3.6. Physical Picture of Landau Damping

The following simple argument ([Lifshitz & Pitaevskii 1981](#)) illustrates the physical mechanism of Landau damping.

Consider an electron moving along the  $z$  axis, subject to a wave-like electric field:

$$\frac{dz}{dt} = v_z, \quad (3.42)$$

$$\frac{dv_z}{dt} = -\frac{e}{m_e} E(z, t) = -\frac{e}{m_e} E_0 \cos(\omega t - kz)e^{\varepsilon t}. \quad (3.43)$$

I have given the electric field a slow time dependence,  $E \propto e^{\varepsilon t}$ , but will later take  $\varepsilon \rightarrow +0$ —this describes the field switching on infinitely slowly from  $t = -\infty$ . Let us assume that the amplitude  $E_0$  of the electric field is so small that it changes the electron’s trajectory only a little over several wave periods. Then the equations of motion can be solved perturbatively.

The lowest-order ( $E_0 = 0$ ) solution is

$$v_z(t) = v_0 = \text{const}, \quad z(t) = v_0 t. \quad (3.44)$$

In the next order, let

$$v_z(t) = v_0 + \delta v_z(t), \quad z(t) = v_0 t + \delta z(t). \quad (3.45)$$

Equation (3.43) becomes

$$\frac{d\delta v_z}{dt} = -\frac{e}{m_e} E(z(t), t) \approx -\frac{e}{m_e} E(v_0 t, t) = -\frac{eE_0}{m_e} \text{Re } e^{[i(\omega - kv_0) + \varepsilon]t}. \quad (3.46)$$

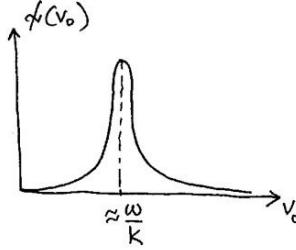
Integrating this gives

$$\begin{aligned} \delta v_z(t) &= -\frac{eE_0}{m_e} \int_0^t dt' \text{Re } e^{[i(\omega - kv_0) + \varepsilon]t'} \\ &= -\frac{eE_0}{m_e} \text{Re} \frac{e^{[i(\omega - kv_0) + \varepsilon]t} - 1}{i(\omega - kv_0) + \varepsilon} \\ &= -\frac{eE_0}{m_e} \frac{\varepsilon e^{\varepsilon t} \cos[(\omega - kv_0)t] - \varepsilon + (\omega - kv_0)e^{\varepsilon t} \sin[(\omega - kv_0)t]}{(\omega - kv_0)^2 + \varepsilon^2}. \end{aligned} \quad (3.47)$$

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<sup>18</sup>This line of thinking has also made its way into galactic dynamics: see [Lau & Binney \(2021a,b\)](#).

<sup>19</sup>Another amusing recent exercise is the paper by [Heninger & Morrison \(2018\)](#), where (following up on [Morrison 1994, 2000](#)), van Kampen’s scheme is recast as a new transform (called “G-transform”) to be used instead of the Laplace transform to solve Landau’s initial-value problem.

FIGURE 9. The function  $\chi(v_0)$  defined in (3.49).

Integrating again, one gets

$$\begin{aligned}
 \delta z(t) &= \int_0^t dt' \delta v_z(t') \\
 &= -\frac{eE_0}{m_e} \int_0^t dt' \operatorname{Re} \frac{e^{[i(\omega-kv_0)+\varepsilon]t'} - 1}{i(\omega-kv_0) + \varepsilon} \\
 &= -\frac{eE_0}{m_e} \left\{ \operatorname{Re} \frac{e^{[i(\omega-kv_0)+\varepsilon]t} - 1}{[i(\omega-kv_0) + \varepsilon]^2} - \frac{\varepsilon t}{(\omega-kv_0)^2 + \varepsilon^2} \right\} \\
 &= -\frac{eE_0}{m_e} \left\{ \frac{[\varepsilon^2 - (\omega - kv_0)^2] \{ e^{\varepsilon t} \cos[(\omega - kv_0)t] - 1 \} + 2\varepsilon(\omega - kv_0)e^{\varepsilon t} \sin[(\omega - kv_0)t]}{[(\omega - kv_0)^2 + \varepsilon^2]^2} \right. \\
 &\quad \left. - \frac{\varepsilon t}{(\omega - kv_0)^2 + \varepsilon^2} \right\}. \tag{3.48}
 \end{aligned}$$

The work done by the field on the electron per unit time, averaged over time, is the power gained by the electron:

$$\begin{aligned}
 \delta P(v_0) &= -e \langle E(z(t), t) v_z(t) \rangle \\
 &\approx -e \left\langle \left[ E(v_0 t, t) + \delta z(t) \frac{\partial E}{\partial z}(v_0 t, t) \right] [v_0 + \delta v_z(t)] \right\rangle \\
 &= -e E_0 e^{\varepsilon t} \underbrace{\left( v_0 \cos[(\omega - kv_0)t] + \delta v_z(t) \cos[(\omega - kv_0)t] \right)}_{\substack{\text{vanishes} \\ \text{under} \\ \text{averaging}}} + \underbrace{\delta z(t) k \sin[(\omega - kv_0)t]}_{\substack{\text{only cos term} \\ \text{from (3.47)} \\ \text{survives} \\ \text{averaging}}} \\
 &= \frac{e^2 E_0^2}{2m_e} e^{2\varepsilon t} \left\{ \frac{\varepsilon}{(\omega - kv_0)^2 + \varepsilon^2} + \frac{2kv_0\varepsilon(\omega - kv_0)}{[(\omega - kv_0)^2 + \varepsilon^2]^2} \right\} \\
 &= \frac{e^2 E_0^2}{2m_e} e^{2\varepsilon t} \frac{d}{dv_0} \underbrace{\frac{\varepsilon v_0}{(\omega - kv_0)^2 + \varepsilon^2}}_{\equiv \chi(v_0)}. \tag{3.49}
 \end{aligned}$$

We see (Fig. 9) that

- if the electron is lagging behind the wave,  $v_0 \lesssim \omega/k$ , then  $\chi'(v_0) > 0 \Rightarrow \delta P(v_0) > 0$ , so energy goes from the field to the electron (the wave is damped);
- if the electron is overtaking the wave,  $v_0 \gtrsim \omega/k$ , then  $\chi'(v_0) < 0 \Rightarrow \delta P(v_0) < 0$ , so energy goes from the electron to the field (the wave is amplified).

Now remember that we have a whole distribution of these electrons,  $F(v_z)$ . So the total power

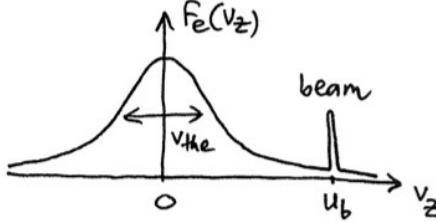


FIGURE 10. Electron distribution with a cold beam; see (3.54).

per unit volume going into (or out of) them is

$$\begin{aligned} P &= \int dv_z F(v_z) \delta P(v_z) = \frac{e^2 E_0^2 e^{2\varepsilon t}}{2m_e} \int dv_z F(v_z) \chi'(v_z) \\ &= -\frac{e^2 E_0^2 e^{2\varepsilon t}}{2m_e} \int dv_z F'(v_z) \chi(v_z). \end{aligned} \quad (3.50)$$

Noticing that, by Plemelj's formula (3.25), in the limit  $\varepsilon \rightarrow +0$ ,

$$\chi(v_z) = \frac{\varepsilon v_z}{(\omega - kv_z)^2 + \varepsilon^2} = -\frac{i v_z}{2} \left( \frac{1}{kv_z - \omega - i\varepsilon} - \frac{1}{kv_z - \omega + i\varepsilon} \right) \rightarrow \pi \frac{\omega}{k^2} \delta(v_z - \frac{\omega}{k}), \quad (3.51)$$

we conclude

$$P = -\frac{e^2 E_0^2}{2m_e k^2} \pi \omega F'\left(\frac{\omega}{k}\right). \quad (3.52)$$

As in §3.5, we find damping if  $\omega F'(\omega/k) < 0$  and instability if  $\omega F'(\omega/k) > 0$ .

Thus, around the wave-particle resonance  $v_z = \omega/k$ , the particles just lagging behind the wave receive energy from the wave and those just overtaking it give up energy to it. Therefore, qualitatively, damping occurs if the former particles are more numerous than the latter. We see that Landau's mathematics in §§3.1–3.5 led us to a result that makes physical sense.

### 3.7. Hot and Cold Beams

Let us return to the unstable situation, when a high-energy beam produces a bump on the tail of the distribution function and thus electrostatic perturbations can suck energy out of the beam and grow in the region of wave numbers where  $v_0 < \omega/k < u_b$ . Here  $v_0$  is the point of the minimum of the distribution in Fig. 8(b) and  $u_b$  is the point of the maximum of the bump, which is the velocity of the beam; we are assuming that  $u_b \gg v_{\text{the}}$ . In view of (3.41), the instability will have a greater growth rate if the bump's slope is steeper, i.e., if the beam is colder (narrower in  $v_z$  space).

Imagine modelling the beam by a little Maxwellian distribution with mean velocity  $u_b$ , tucked onto the bulk distribution:<sup>20</sup>

$$F_e(v_z) = \frac{n_e - n_b}{\sqrt{\pi} v_{\text{the}}} \exp\left(-\frac{v_z^2}{v_{\text{the}}^2}\right) + \frac{n_b}{\sqrt{\pi} v_b} \exp\left[-\frac{(v_z - u_b)^2}{v_b^2}\right], \quad (3.53)$$

where  $n_b$  is the density of the beam,  $v_b$  is its width, and so  $T_b = m_e v_b^2 / 2$  is its “temperature”, just like  $T_e = m_e v_{\text{the}}^2 / 2$  is the temperature of the thermal bulk. A colder beam will have less of a thermal spread around  $u_b$ . It turns out that if the width of

<sup>20</sup>The fact that we are working in 1D means that we are restricting our consideration to perturbations whose wave numbers  $\mathbf{k}$  are parallel to the beam's velocity. In general, allowing transverse wave numbers brings into play the transverse (electromagnetic) part of the dielectric tensor (see Q2). However, for non-relativistic beams, the fastest-growing modes will still be the longitudinal, electrostatic ones (see, e.g., [Alexandrov et al. 1984](#), §32).

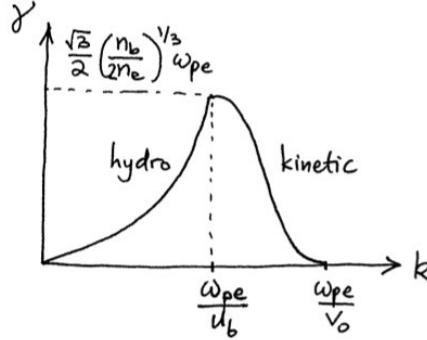


FIGURE 11. Sketch of the growth rate of the hydrodynamic and kinetic beam instabilities: see (3.57) for  $k < \omega_{pe}/u_b$ , (3.59) for  $k = \omega_{pe}/u_b$ , and (3.41) for  $\omega_{pe}/u_b < k < \omega_{pe}/v_0$ , where  $v_0$  is the point of the minimum of the distribution in Fig. 8(b) and  $u_b$  is the point of the maximum of the bump.

the beam is sufficiently small, another instability appears, whose origin is hydrodynamic rather than kinetic. Let us work it out.

Consider a very simple limiting case of the distribution (3.53): let  $v_b \rightarrow 0$  and  $n_b \ll n_e$ . Then (Fig. 10)

$$F_e(v_z) = F_M(v_z) + n_b \delta(v_z - u_b), \quad (3.54)$$

where  $F_M$  is the bulk Maxwellian from (3.53) (with density  $\approx n_e$ , neglecting  $n_b$  in comparison). Let us substitute the distribution (3.54) into the dielectric function (3.26), seek solutions with  $p/k \gg v_{the}$ , expand the part containing  $F_M$  in the same way as we did in §3.4,<sup>21</sup> neglect the ion contribution for the same reason as we did there, and deal with  $\delta'(v_z - u_b)$  in the integrand via integration by parts. The resulting dispersion relation is

$$\epsilon \approx 1 + \frac{\omega_{pe}^2}{p^2} - \frac{n_b}{n_e} \frac{\omega_{pe}^2}{(ku_b - ip)^2} = 0. \quad (3.55)$$

Since  $n_b \ll n_e$ , the last term can only matter for those perturbations that are close to resonance with the beam (this is called the *Cherenkov resonance*):

$$p = -iku_b + \gamma, \quad \gamma \ll ku_b. \quad (3.56)$$

This turns (3.55) into

$$1 - \frac{\omega_{pe}^2}{k^2 u_b^2} + \frac{n_b}{n_e} \frac{\omega_{pe}^2}{\gamma^2} = 0 \quad \Rightarrow \quad \gamma = \pm \sqrt{\frac{n_b}{n_e}} \left( \frac{1}{k^2 u_b^2} - \frac{1}{\omega_{pe}^2} \right)^{-1/2}. \quad (3.57)$$

The expression under the square root is positive and so there is a growing mode only if  $k < \omega_{pe}/u_b$ . This is in contrast to the case of a hot (or warm) beam in §3.5: there, having a kinetic instability required  $\omega F'_e(\omega/k) > 0$ , which was only possible at  $k > \omega_{pe}/u_b$  (the phase speed of the perturbations had to be to the left of the bump's maximum). The new instability that we have found—the *hydrodynamic beam instability*—has the largest growth rate at  $ku_b = \omega_{pe}$ , i.e., when the beam and the plasma oscillations are in resonance, in which case, to resolve the singularity, we need to retain  $\gamma$  in the second

<sup>21</sup>We can treat the Landau contour as simply running along the real axis because we are expecting to find a solution with  $\text{Re } p > 0$  [see (3.20)], for reasons independent of the Landau resonance.

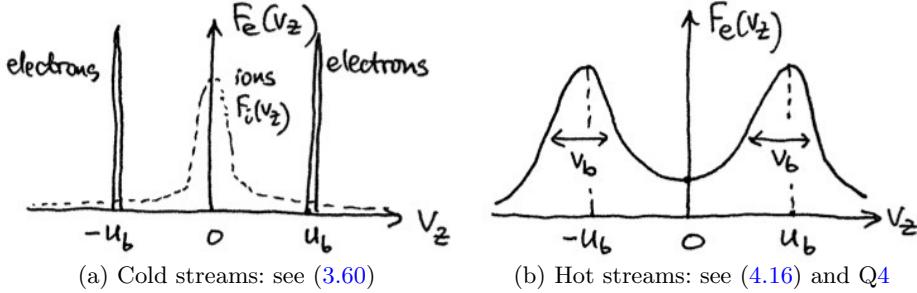


FIGURE 12. Two streams.

term in (3.55). Doing so and expanding in  $\gamma$ , we get

$$\epsilon \approx 1 - \frac{\omega_{pe}^2}{(\omega_{pe} + i\gamma)^2} + \frac{n_b \omega_{pe}^2}{n_e \gamma^2} \approx \frac{2i\gamma}{\omega_{pe}} + \frac{n_b \omega_{pe}^2}{n_e \gamma^2} = 0. \quad (3.58)$$

Solution:

$$\boxed{\gamma = \left( \frac{\pm\sqrt{3} + i}{2}, -i \right) \left( \frac{n_b}{2n_e} \right)^{1/3} \omega_{pe}.} \quad (3.59)$$

The unstable root ( $\text{Re } \gamma > 0$ ) is the interesting one. The growth rate of the combined beam instability, hydrodynamic and kinetic, is sketched in Fig. 11.

---

**Exercise 3.3.** This instability is called “hydrodynamic” because it can be derived from fluid equations (cf. Exercise 3.1) describing cold electrons ( $v_{the} = 0$ ) and a cold beam ( $v_b = 0$ ). Convince yourself that this is the case.

**Exercise 3.4.** Using the model distribution (3.53), work out the conditions on  $v_b$  and  $n_b$  that must be satisfied in order for our derivation of the hydrodynamic beam instability to be valid, i.e., for (3.55) to be a good approximation to the true dispersion relation. What is the effect of finite  $v_b$  on the hydrodynamic instability? Sketch the growth rate of unstable perturbations as a function of  $k$ , taking into account both the hydrodynamic instability and the kinetic one, as well as the Landau damping.

**Exercise 3.5. Two-stream instability.** This is a popular instability<sup>22</sup> that arises, e.g., in a situation where the plasma consists of two cold counter-streams of electrons propagating against a quasineutrality-enforcing background of effectively immobile ions (Fig. 12a). Model the corresponding electron distribution by

$$F_e(v_z) = \frac{n_e}{2} [\delta(v_z - u_b) + \delta(v_z + u_b)] \quad (3.60)$$

and solve the resulting dispersion relation (where the ion terms can be neglected for the same reason as in §3.4). Find the wave number at which perturbations grow fastest and the corresponding growth rate. Find also the maximum wave number at which perturbations can grow. If you want to know what happens when the two streams are warm (have a finite thermal spread  $v_b$ ; Fig. 12b), a nice fully tractable quantitative model of such a situation is the double-Lorentzian distribution (4.16). The dispersion relation for it can be solved exactly: this is done

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<sup>22</sup>It was discovered by engineers (Haeff 1949; Pierce & Hebenstreit 1949) and quickly adopted by physicists (Bohm & Gross 1949b). Buneman (1958) realised that a case with an electron and an ion stream (i.e., with plasma carrying a current) is unstable in a somewhat analogous way (see Q5, and also Q12). The kinetic counterpart to the latter situation is the ion-acoustic instability derived in §3.9 (similarly to the way in which the bump-on-tail instability was the kinetic counterpart to the hydrodynamic beam instability). In §4.4, I will discuss in a more general way the stability of distributions featuring streams.

in Q4. You will again find a hydrodynamic instability, but is there also a kinetic one (due to Landau resonance)? It is an interesting and non-trivial question why not.

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### 3.8. Ion-Acoustic Waves

Let us now see what happens at lower frequencies,

$$v_{\text{the}} \gg \frac{\omega}{k} \gg v_{\text{thi}}, \quad (3.61)$$

i.e., when the waves propagate slower than the bulk of the electron distribution but faster than the bulk of the ion one (Fig. 13). This is another regime in which we might expect to find weakly damped waves: they are out of phase with the majority of the ions, so  $F'_i(\omega/k)$  might be small because  $F_i(\omega/k)$  is small, while as far as the electrons are concerned, the phase speed of the waves is deep in the core of the distribution, perhaps close to its maximum at  $v_z = 0$  (if that is where its maximum is) and so  $F'_e(\omega/k)$  might turn out to be small because  $F_e(v_z)$  changes slowly in that region.

To make this more specific, let us consider Maxwellian electrons:

$$F_e(v_z) = \frac{n_e}{\sqrt{\pi} v_{\text{the}}} \exp \left[ -\frac{(v_z - u_e)^2}{v_{\text{the}}^2} \right], \quad (3.62)$$

where we are, in general, allowing the electrons to have a mean flow (current). We will assume that  $u_e \ll v_{\text{the}}$  but allow  $u_e \sim \omega/k$ . We can anticipate that this will give us an interesting new effect. Indeed,

$$F'_e(v_z) = -\frac{2(v_z - u_e)}{v_{\text{the}}^2} F_e(v_z). \quad (3.63)$$

For resonant particles,  $v_z = \omega/k$ , the prefactor will be small, so we can hope for  $\gamma \ll \omega$ , as anticipated above, but note that its sign will depend on the relative size of  $u_e$  and  $\omega/k$  and so we might (we will!) get an instability (§3.9).

But let us not get ahead of ourselves: we must first calculate the real frequency  $\omega(k)$  of these waves, from (3.29) and (3.31):

$$\begin{aligned} \text{Re } \epsilon &= 1 - \frac{\omega_{\text{pe}}^2}{k^2} \frac{1}{n_e} \mathcal{P} \int dv_z \frac{F'_e(v_z)}{v_z - \omega/k} - \underbrace{\frac{\omega_{\text{pi}}^2}{k^2} \frac{1}{n_i} \mathcal{P} \int dv_z \frac{F'_i(v_z)}{v_z - \omega/k}}_{\approx \frac{\omega_{\text{pi}}^2}{\omega^2} (1 + 3k^2 \lambda_{\text{Di}}^2)} = 0. \end{aligned} \quad (3.64)$$

The last (ion) term in this equation can be expanded in  $k v_z / \omega \ll 1$  in exactly the same way as it was done in (3.35). The expansion is valid provided

$$k \lambda_{\text{Di}} \ll 1, \quad (3.65)$$

and I will retain only the lowest-order term, dropping the  $k^2 \lambda_{\text{Di}}^2$  correction. The second (electron) term in (3.64) is subject to the opposite limit,  $v_z \gg \omega/k$ , so, using (3.63),

$$\frac{\omega_{\text{pe}}^2}{k^2} \frac{1}{n_e} \mathcal{P} \int dv_z \frac{F'_e(v_z)}{v_z - \omega/k} \approx -\frac{\omega_{\text{pe}}^2}{k^2} \frac{1}{n_e} \mathcal{P} \int dv_z \frac{2(v_z - u_e)}{v_{\text{the}}^2 v_z} F_e(v_z) \approx -\frac{2\omega_{\text{pe}}^2}{k^2 v_{\text{the}}^2} = -\frac{1}{k^2 \lambda_{\text{De}}^2}, \quad (3.66)$$

where we have neglected  $u_e \ll v_z$  because this integral is over the thermal bulk of the electron distribution.

With all these approximations, (3.64) becomes

$$\text{Re } \epsilon = 1 + \frac{1}{k^2 \lambda_{\text{De}}^2} - \frac{\omega_{\text{pi}}^2}{\omega^2} = 0. \quad (3.67)$$

The dispersion relation is then

$$\boxed{\omega^2 = \frac{\omega_{\text{pi}}^2}{1 + 1/k^2 \lambda_{\text{De}}^2} = \frac{k^2 c_s^2}{1 + k^2 \lambda_{\text{De}}^2}}, \quad (3.68)$$

where

$$c_s = \omega_{\text{pi}} \lambda_{\text{De}} = \sqrt{\frac{ZT_e}{m_i}} \quad (3.69)$$

is the *sound speed*, called that because, if  $k \lambda_{\text{De}} \ll 1$ , (3.68) describes a wave that is very obviously a sound, or *ion-acoustic, wave*:

$$\boxed{\omega = \pm k c_s}. \quad (3.70)$$

The phase speed of this wave is the sound speed,  $\omega/k = c_s$ . That the expression (3.69) for  $c_s$  combines electron temperature and ion mass is a hint as to the underlying physics of sound propagation in plasma: ions provide the inertia, electrons the restoring pressure force (see Exercise 3.6).

We can now check under what circumstances the condition (3.61) is indeed satisfied:

$$\frac{c_s}{v_{\text{the}}} = \sqrt{\frac{Zm_e}{2m_i}} \ll 1, \quad \frac{c_s}{v_{\text{thi}}} = \sqrt{\frac{ZT_e}{2T_i}} \gg 1, \quad (3.71)$$

with the latter condition requiring that the ions should be colder than the electrons.

**Exercise 3.6. Hydrodynamics of sound waves.**<sup>23</sup> Starting from the linearised kinetic equations for ions and electrons, work out the fluid equations for the plasma, i.e., the evolution equations for its mass density and mass flow velocity. Assuming  $m_e \ll m_i$  (negligible electron inertia) and  $T_i \ll T_e$  (cold ions), show that these equations are

$$\frac{\partial \delta n_i}{\partial t} + n_i \nabla \cdot \mathbf{u}_i = 0, \quad m_i n_i \frac{\partial \mathbf{u}_i}{\partial t} + \nabla \delta p_e = 0, \quad (3.72)$$

and that the sound waves (3.70) with  $c_s$  given by (3.69) are recovered if electrons have the equation of state of an isothermal fluid. Why, and under what assumptions, should they be isothermal physically? Prove mathematically that they indeed are. Why is the equation of state for electrons different in a sound wave than in a Langmuir wave (see Exercise 3.1)? We will revisit ion hydrodynamics in §8.

### 3.9. Damping of Ion-Acoustic Waves and Ion-Acoustic Instability

Are ion acoustic waves damped? Can they grow? We have a standard protocol for answering this question: calculate  $\text{Re } \epsilon$  and  $\text{Im } \epsilon$  and substitute into (3.30). Using (3.67) and (3.32), we find

$$\frac{\partial \text{Re } \epsilon}{\partial \omega} = \frac{2\omega_{\text{pi}}^2}{\omega^3}, \quad \text{Im } \epsilon = -\frac{\omega_{\text{pe}}^2}{k^2} \frac{\pi}{n_e} F'_e\left(\frac{\omega}{k}\right) - \frac{\omega_{\text{pi}}^2}{k^2} \frac{\pi}{n_i} F'_i\left(\frac{\omega}{k}\right). \quad (3.73)$$

The two terms in  $\text{Im } \epsilon$  represent the interaction between the waves and, respectively, electrons and ions. The ion term is small both on account of  $\omega_{\text{pi}} \ll \omega_{\text{pe}}$  and, assuming

<sup>23</sup>The 2020 exam question was based on this.

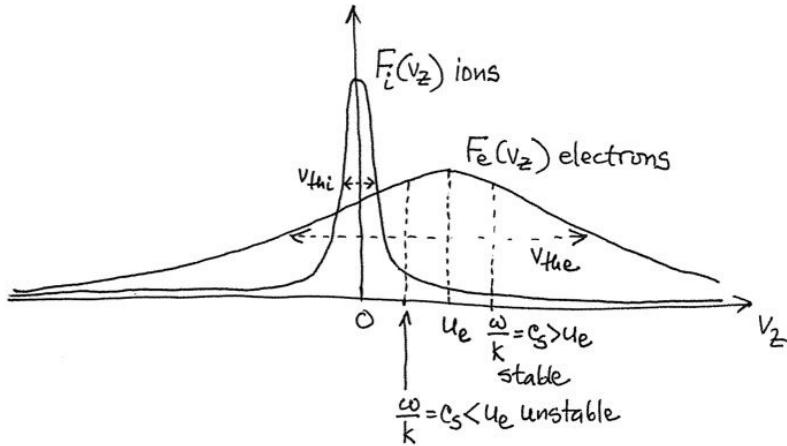


FIGURE 13. Ion-acoustic resonance: damping ( $c_s > u_e$ ) or instability ( $c_s < u_e$ ). Ion Landau damping is weak because  $c_s \gg v_{thi}$ , so in the tail of  $F_i(v_z)$ ; electron damping/instability is also weak because  $u_e, c_s \ll v_{the}$ , so close to the peak  $F_e(v_z)$ .

Maxwellian ions, of the exponential smallness of  $F_i(\omega/k) \propto \exp[-(\omega/kv_{thi})^2]$ . We are then left with

$$\gamma = -\frac{\text{Im } \epsilon}{\partial(\text{Re } \epsilon)/\partial \omega} = -\sqrt{\pi} \frac{\omega^3}{k^2 v_{the}^3 Z m_e} \left( \frac{\omega}{k} - u_e \right), \quad (3.74)$$

where we have used (3.63). In the long-wavelength limit,  $k\lambda_{De} \ll 1$ , we have  $\omega = \pm kc_s$ , and so, for the “+” mode,

$$\gamma = -\sqrt{\frac{\pi Z m_e}{8m_i}} k (c_s - u_e) .$$

(3.75)

If the electron flow is subsonic,  $u_e < c_s$ , this describes the *Landau damping of ion acoustic waves on hot electrons*. If, on the other hand, the electron flow is supersonic, the sign of  $\gamma$  reverses<sup>24</sup> and we discover the *ion-acoustic instability*: excitation of ion acoustic waves by a fast electron current. The instability belongs to the same general class as, e.g., the bump-on-tail instability (§3.5) in that it involves waves sucking energy from particles, but the new conceptual feature here is that such energy conversion can result from a collaboration between different particle species (electrons supplying the energy, ions carrying the wave).

There is a host of related instabilities involving various combinations of electron and ion beams, currents, streams and counter-streams—excellent treatments of them can be found in the textbooks by Krall & Trivelpiece (1973) and by Alexandrov *et al.* (1984) or in the review by Davidson (1983). I shall return to this topic in §4.4.

**Exercise 3.7. Damping of sound waves on ions.**<sup>25</sup> Find the ion contribution to the damping of ion-acoustic waves. Under what conditions does it become comparable to, or larger than, the electron contribution?

**Exercise 3.8.** What happens if  $u_e \gg v_{the}$ ?

<sup>24</sup>Recall that  $k > 0$  by the choice of the  $z$  axis.

<sup>25</sup>The 2016 exam question was loosely based on this.

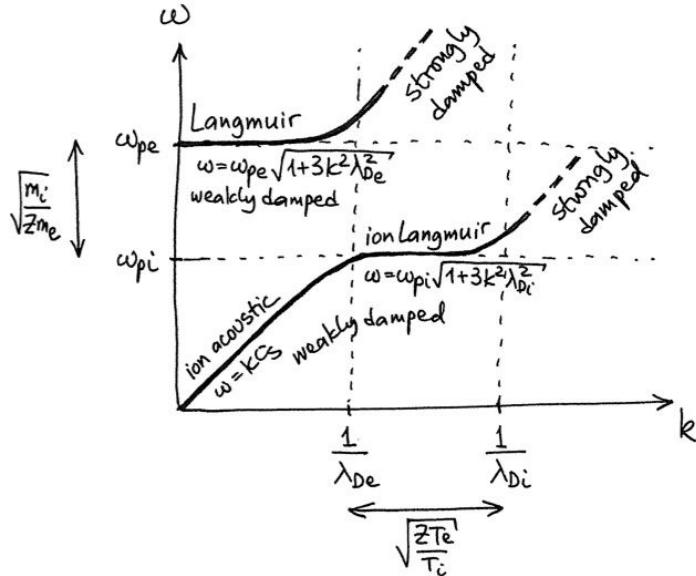


FIGURE 14. Electrostatic (longitudinal) plasma waves.

### 3.10. Ion Langmuir Waves

Note that since

$$\frac{\lambda_{De}}{\lambda_{Di}} = \frac{v_{the}\omega_{pi}}{v_{thi}\omega_{pe}} = \sqrt{\frac{ZT_e}{T_i}}, \quad (3.76)$$

the condition (3.65) need not entail  $k\lambda_{De} \ll 1$  in the limit of cold ions [see (3.71)]—in this case, the size of the Debye sphere (1.6) is set by the ions, rather than by the electrons, and so we can have perfectly macroscopic (in the language of §1.4) perturbations on scales both larger and smaller than  $\lambda_{De}$ . At larger scales, we have found sound waves (3.70). At smaller scales,  $k\lambda_{De} \gg 1$ , the dispersion relation (3.68) gives us *ion Langmuir oscillations*:

$$\omega^2 = \omega_{pi}^2 = \frac{4\pi Z^2 e^2 n_i}{m_i}, \quad (3.77)$$

which are analogous to the electron Langmuir oscillations (3.38) and, like them, turn into dispersive *ion Langmuir waves* if the small  $k^2 \lambda_{Di}^2$  correction in (3.64) is retained, leading to the Bohm–Gross dispersion relation (3.39), but with ion quantities this time.

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**Exercise 3.9.** Derive the dispersion relation for ion Langmuir waves. Investigate their damping/instability.

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### 3.11. Summary of Electrostatic (Longitudinal) Plasma Waves

We have achieved what turns out to be a complete characterisation of electrostatic (also known as “longitudinal”, in the sense that  $\mathbf{k} \parallel \mathbf{E}$ ) waves in an unmagnetised plasma. These are summarised in Fig. 14. In the limit of short wavelengths,  $k\lambda_{De} \gg 1$  and  $k\lambda_{Di} \gg 1$ , the electron and ion branches, respectively, becomes dispersive, their damping rates increase and eventually stop being small. This corresponds to waves having phase speeds that are comparable to the speeds of the particles in the thermal bulk of their

distributions, so a great number of particles are available to have Landau resonance with the waves and absorb their energy—the damping becomes strong.

Note that if the cold-ion condition  $T_i \ll T_e$  is not satisfied, the sound speed is comparable to the ion thermal speed  $c_s \sim v_{thi}$ , and so the ion-acoustic waves are strongly damped at all wave numbers—it is well-nigh impossible to propagate sound through a collisionless hot plasma (in such an environment, no one will hear you scream)!

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Let me digress a little to bring you in contact with a research frontier. In a magnetised plasma, the sound wave looks (and works) essentially the same as (3.70), as long as its wave vector is parallel to the magnetic field—the dispersion relation for such a wave, derived from fluid equations (as in Exercise 3.6) is (15.27). In a *collisionless* such plasma with  $T_i \sim T_e$ , this wave is again heavily damped. Interestingly, however, at high plasma beta [defined by (15.24)], if a sound wave is allowed to have a finite amplitude (i.e., outside linear theory), it becomes “self-sustaining”, by exciting some parasitic micro-instabilities that in turn render the plasma in its path effectively collisional—a rather beautiful phenomenon that has only been worked out recently: read [Kunz et al. \(2020\)](#) if this piques your curiosity.

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**Exercise 3.10. Longitudinal waves in degenerate plasmas.**<sup>26</sup> Sometimes it is meaningful to consider degenerate fermionic plasmas (e.g., electrons in metals or electrons and holes in superconductors). If they are cold enough, their distribution function is simply

$$f_{0\alpha}(\mathbf{v}) = \frac{1}{4} \left( \frac{m_\alpha}{\hbar\pi} \right)^3 H(v_{F\alpha} - v), \quad (3.78)$$

where  $H$  is the Heaviside function ( $= 1$  for positive argument,  $0$  otherwise) and  $v_{F\alpha} = (3\pi^2 n_\alpha)^{1/3} \hbar/m_\alpha$  is the “Fermi velocity” assuming spin-1/2 particles (the Fermi energy is  $m_\alpha v_{F\alpha}^2/2$ ). It is easy to check that (3.78) integrates to  $n_\alpha$ .

Work out all longitudinal waves for such plasmas. You will find Langmuir waves, ion sound, etc., but there will also be a new feature, the so-called “zero-point sound” at  $k\lambda_{De} \gg 1$ , where the new Debye length is  $\lambda_{De} = \sqrt{3}v_{Fe}/\omega_{pe}$ , a wave whose frequency is

$$\omega = \pm k v_{Fe} \left( 1 + 2e^{-2-2k^2\lambda_{De}^2/9} \right). \quad (3.79)$$

Work out also whether any of the waves that you have found are Landau damped, and if they are, at what rate. Sketch a version of Fig. 14 for degenerate plasmas. If you find this calculation difficult, you can look up some of it in [Lifshitz & Pitaevskii \(1981\)](#) and all of it in [Alexandrov et al. \(1984\)](#).

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### 3.12. Plasma Dispersion Function: Putting Linear Theory on Industrial Basis

Clearly, we have entered the realm of practical calculation—it is now easy to imagine an industry of solving the plasma dispersion relation

$$\epsilon(p, \mathbf{k}) = 1 - \sum_\alpha \frac{\omega_{p\alpha}^2}{k^2} \frac{1}{n_\alpha} \int_{C_L} dv_z \frac{F'_\alpha(v_z)}{v_z - ip/k} = 0 \quad (3.80)$$

and similar dispersion relations arising from, e.g., considering electromagnetic perturbations (see Q2), magnetised plasmas (see [Parra 2019b](#)), different equilibria  $F_\alpha$  (see Q3 and Q4), etc.

A Maxwellian equilibrium is obviously an extremely important special case because that is, after all, the distribution towards which plasma is pushed by collisions on long time scales:

$$f_{0\alpha}(\mathbf{v}) = \frac{n_\alpha}{(\pi v_{th\alpha}^2)^{3/2}} e^{-v^2/v_{th\alpha}^2} \Rightarrow F_\alpha(v_z) = \frac{n_\alpha}{\sqrt{\pi} v_{th\alpha}} e^{-v_z^2/v_{th\alpha}^2}. \quad (3.81)$$

For this case, we would like to introduce a new “special function” that would incorporate the Landau prescription for calculating the velocity integral in (3.80) and that we could in some sense “tabulate” once and for all.

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<sup>26</sup>This is based on the 2023 exam question.

Taking  $F_\alpha$  to be (3.81) and letting  $u = v_z/v_{\text{th}\alpha}$  and  $\zeta_\alpha = ip/kv_{\text{th}\alpha}$ , we can rewrite the velocity integral in (3.80) as follows

$$\frac{1}{n_\alpha} \int_{C_L} dv_z \frac{F'_\alpha(v_z)}{v_z - ip/k} = -\frac{2}{\sqrt{\pi} v_{\text{th}\alpha}^2} \int_{C_L} du \frac{u e^{-u^2}}{u - \zeta_\alpha} = -\frac{2}{v_{\text{th}\alpha}^2} [1 + \zeta_\alpha Z(\zeta_\alpha)], \quad (3.82)$$

where the *plasma dispersion function* is defined to be

$$Z(\zeta) = \frac{1}{\sqrt{\pi}} \int du \frac{e^{-u^2}}{u - \zeta}. \quad (3.83)$$

In these terms, the plasma dispersion relation (3.80) becomes

$$\epsilon = 1 + \sum_\alpha \frac{1 + \zeta_\alpha Z(\zeta_\alpha)}{k^2 \lambda_{D\alpha}^2} = 0. \quad (3.84)$$

Note that if the Maxwellian distribution (3.81) has a mean flow, as it did, e.g., in (3.62), this amounts to a shift by some mean velocity  $u_\alpha$  and all one needs to do to adjust the above results is to shift the argument of  $Z$  accordingly:

$$\zeta_\alpha \rightarrow \zeta_\alpha - \frac{u_\alpha}{v_{\text{th}\alpha}}. \quad (3.85)$$

### 3.12.1. Some Properties of $Z(\zeta)$

It is not hard to see that

$$Z'(\zeta) = -\frac{1}{\sqrt{\pi}} \int du e^{-u^2} \frac{\partial}{\partial u} \frac{1}{u - \zeta} = -\frac{2}{\sqrt{\pi}} \int du \frac{u e^{-u^2}}{u - \zeta} = -2 [1 + \zeta Z(\zeta)]. \quad (3.86)$$

Let us treat this identity as a differential equation: the integrating factor is  $e^{\zeta^2}$ , so

$$e^{\zeta^2} Z(\zeta) = -2 \int_0^\zeta dt e^{t^2} + Z(0). \quad (3.87)$$

We know the boundary condition at  $\zeta = 0$  from (3.23): for real  $\zeta$ ,

$$\begin{aligned} \frac{1}{\sqrt{\pi}} \int du \frac{e^{-u^2}}{u - \zeta} &= \frac{1}{\sqrt{\pi}} \underbrace{\mathcal{P} \int_{-\infty}^{+\infty} du \frac{e^{-u^2}}{u - \zeta}}_{= 0 \text{ for } \zeta = 0} + i\sqrt{\pi} e^{-\zeta^2} \Rightarrow Z(0) = i\sqrt{\pi}. \\ &\text{because} \\ &\text{integrand is} \\ &\text{odd} \end{aligned} \quad (3.88)$$

Using this in (3.87) and changing the integration variable  $t = -ix$ , we find

$$Z(\zeta) = e^{-\zeta^2} \left( i\sqrt{\pi} + 2i \int_0^{i\zeta} dx e^{-x^2} \right) = 2i e^{-\zeta^2} \int_{-\infty}^{i\zeta} dx e^{-x^2}. \quad (3.89)$$

This turns out to be a uniformly valid expression for  $Z(\zeta)$ : our function is simply a complex erf!<sup>27</sup> Here is a MATHEMATICA script for calculating it:

$$Z[\text{zeta}_-] := \text{I Sqrt}[\text{Pi}] \text{Exp}[-\text{zeta}_-^2] (1 + \text{I Erfi}[\text{zeta}_-]). \quad (3.90)$$

You can use this to code up (3.84) and explore, e.g., the strongly damped solutions ( $\zeta \sim 1$ ,  $\gamma \sim \omega$ ).

<sup>27</sup>When written in this form, and with  $i\sqrt{\pi}$  factored out,  $w(\zeta) = Z(\zeta)/i\sqrt{\pi} \equiv \text{erfcx}(-i\zeta)$  is called the Faddeeva function, after the Soviet mathematician Vera Faddeeva (Fig. 15), who published the first book of tables of it (Faddeeva & Terent'ev 1954), before those American upstarts Fried & Conte (1961). People meant business in those days. In the 21st century, we have an easier life: just teach a computer to compute our function and make an app.

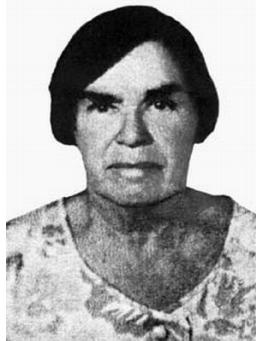


FIGURE 15. Vera Faddeeva (1906–1983), the mother of the plasma dispersion function.

### 3.12.2. Asymptotics of $Z(\zeta)$

If you are a devotee of the ancient art of asymptotic theory, you will find most useful (as, effectively, we did in §§3.4–3.9) various limiting forms of  $Z(\zeta)$ . At small argument  $|\zeta| \ll 1$ , the Taylor series is

$$Z(\zeta) = i\sqrt{\pi} e^{-\zeta^2} - 2\zeta \left( 1 - \frac{2\zeta^2}{3} + \frac{4\zeta^4}{15} - \frac{8\zeta^6}{105} + \dots \right). \quad (3.91)$$

At large argument,  $|\zeta| \gg 1$ ,  $|\operatorname{Re} \zeta| \gg |\operatorname{Im} \zeta|$ , the asymptotic series is

$$Z(\zeta) = i\sqrt{\pi} e^{-\zeta^2} - \frac{1}{\zeta} \left( 1 + \frac{1}{2\zeta^2} + \frac{3}{4\zeta^4} + \frac{15}{8\zeta^6} + \dots \right). \quad (3.92)$$

All the results (for a Maxwellian equilibrium) that I derived in §§3.4–3.10 can be readily obtained from (3.84) by using the above limiting cases (see Q1). It is, indeed, a general practical strategy for studying this and similar plasma dispersion relations to look for solutions in the limits  $\zeta_\alpha \rightarrow 0$  or  $\zeta_\alpha \rightarrow \infty$ , then check under what physical conditions the solutions thus obtained are valid (i.e., that they indeed satisfy  $|\zeta_\alpha| \ll 1$  or  $|\zeta_\alpha| \gg 1$ ,  $|\operatorname{Re} \zeta| \gg |\operatorname{Im} \zeta|$ ), and then fill in the non-asymptotic blanks in the same way that an experienced hunter espying antlers sticking out above the shrubbery can reconstruct, in contour outline, the rest of the hiding deer.

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**Exercise 3.11.** Work out the Taylor series (3.91). A useful step might be to prove this interesting formula (which also turns out to be handy in other calculations; see, e.g., Q8):

$$\frac{d^m Z}{d\zeta^m} = \frac{(-1)^m}{\sqrt{\pi}} \int_{C_L} du \frac{H_m(u) e^{-u^2}}{u - \zeta}, \quad (3.93)$$

where  $H_m(u)$  are Hermite polynomials [defined in (6.74)].

**Exercise 3.12.** Work out the asymptotic series (3.92) using the Landau prescription (3.20) and expanding the principal-value integral similarly to the way it was done in (3.35). Work out also (or look up; e.g., Fried & Conte 1961) other asymptotic forms of  $Z(\zeta)$ , relaxing the condition  $|\operatorname{Re} \zeta| \gg |\operatorname{Im} \zeta|$ .

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## 4. Linear Theory: General Stability Theory

In §3, we learned how to perturb some given equilibrium distribution  $f_{0\alpha}$  infinitesimally and work out whether this perturbation will decay, grow, oscillate, and how quickly. Let me now pose the question in a more general way. In a collisionless plasma, there can be infinitely many

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The Wikipedia article [https://en.wikipedia.org/wiki/Faddeeva\\_function](https://en.wikipedia.org/wiki/Faddeeva_function) offers some useful links to numerical algorithms for doing it, more sophisticated than (3.90).

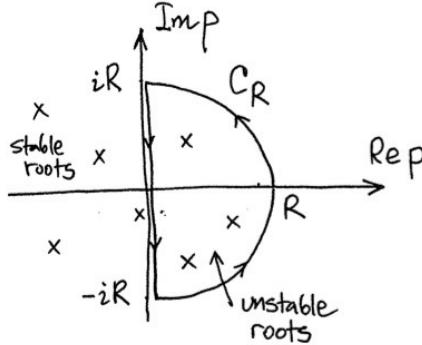


FIGURE 16. Integration contour for (4.4).

possible equilibria, including quite complicated ones. If we set one up, will it persist, i.e., is it stable? If it is not stable, what modification do we expect it to undergo in order to become stable? Other than solving the dispersion relation (3.17) to answer the first question and developing various types of nonlinear theories to answer the second (along the lines advertised in §2.4 and developed in §6 and subsequent sections), both of which can be quite complicated and often intractable technical challenges, do we have at our disposal any general principles that allow us to pronounce on stability? Is there a general insight that we can cultivate as to what sort of distributions are likely to be stable or unstable and to what sorts of perturbations?

We have had glimpses of such general principles already. For example, in §3.5, it was ascertained, by an explicit calculation, that one could encounter a situation with a (small) growth rate if the equilibrium distribution had a positive derivative somewhere along the direction ( $z$ ) of the wave number of the perturbation, viz.,  $v_z F'_e(v_z) > 0$ . I developed this further in §3.7, finding that not only hot but also cold beams and streams triggered instabilities. In Exercise 3.2, I dropped a hint that general statements could perhaps be made about certain general classes of distributions: 3D-isotropic equilibria could be proven stable (we shall prove this again, by a different method, in Exercise 4.2). How general are such statements? Are they sufficient or also necessary criteria? Is there a universal stability litmus test? Let us attack the problem of kinetic stability with an aspiration to generality—although still, for now, for electrostatic perturbations only. We shall also, for now, limit our ambition to determining *linear* stability of generic equilibria, i.e., their stability against infinitesimal perturbations. *Nonlinear* stability will have to wait till §9.

#### 4.1. Nyquist's Method

The problem of linear stability comes down to the question of whether the dispersion relation (3.17) has any unstable solutions: roots with growth rates  $\gamma_i(\mathbf{k}) > 0$ .

It is going to be useful to write the dielectric function (3.26) as follows

$$\epsilon(p, \mathbf{k}) = 1 - \frac{\omega_{pe}^2}{k^2} \int_{C_L} dv_z \frac{\bar{F}'(v_z)}{v_z - ip/k}, \quad (4.1)$$

$$\bar{F} = \frac{1}{n_e} \sum_{\alpha} Z_{\alpha}^2 \frac{m_e}{m_{\alpha}} F_{\alpha} = \frac{F_e}{n_e} + \frac{Z m_e}{m_i} \frac{F_i}{n_i}, \quad (4.2)$$

where the last expression in (4.2) is for the case of a two-species plasma. Thus, the distribution functions of different species come into the linear problem additively, weighted by their species' charges and (inverse) masses.

Let us develop a method (due to Nyquist 1932) for counting zeros of  $\epsilon(p)$  (I will henceforth suppress  $\mathbf{k}$  in the argument) in the half-plane  $\text{Re } p > 0$  (the unstable roots of the dispersion relation). Observe that  $\epsilon(p)$  is analytic (by virtue of our efforts in §3.2 to make it so) and that if  $p = p_i$  is its zero of order  $N_i$ , then in its vicinity,

$$\epsilon(p) = \text{const} (p - p_i)^{N_i} + \dots \Rightarrow \frac{\partial \ln \epsilon(p)}{\partial p} = \frac{N_i}{p - p_i} + \dots, \quad (4.3)$$

so zeros of  $\epsilon(p)$  are poles of  $\partial \ln \epsilon(p) / \partial p$ ; the latter function has no other poles because  $\epsilon(p)$  is analytic. If we now integrate this function over a closed contour  $C_R$  running along the imaginary axis (and just to the right of it:  $p = -i\omega + 0$ ) in the complex  $p$  plane from  $iR$  to  $-iR$  and then along a semicircle of radius  $R$  back to  $iR$  (Fig. 16), we will, in the limit  $R \rightarrow \infty$ , capture all these poles:

$$\lim_{R \rightarrow \infty} \int_{C_R} dp \frac{\partial \ln \epsilon(p)}{\partial p} = 2\pi i \sum_i N_i = 2\pi i N, \quad (4.4)$$

where  $N$  is the total number of zeros of  $\epsilon(p)$  in the half-plane  $\text{Re } p > 0$ . It turns out (as I shall prove in a moment) that the contribution to the integral over  $C_R$  from the semicircle vanishes at  $R \rightarrow \infty$  and so we need only integrate along the imaginary axis:

$$2\pi i N = \int_{+i\infty+0}^{-i\infty+0} dp \frac{\partial \ln \epsilon(p)}{\partial p} = \ln \frac{\epsilon(-i\infty)}{\epsilon(+i\infty)}. \quad (4.5)$$

**Proof.** All we need to show is that

$$|p| \frac{\partial \ln \epsilon(p)}{\partial p} \rightarrow 0 \quad \text{as} \quad |p| \rightarrow \infty, \quad \text{Re } p > 0. \quad (4.6)$$

Indeed, using (4.1) and the Landau integration rule (3.20), we have in this limit:

$$\epsilon(p) = 1 - \frac{\omega_{pe}^2}{k^2} \int_{-\infty}^{+\infty} dv_z \bar{F}'(v_z) \frac{ik}{p} \left( 1 - \frac{ikv_z}{p} + \dots \right) \approx 1 + \frac{1}{p^2} \sum_{\alpha} \omega_{p\alpha}^2, \quad (4.7)$$

where I have integrated by parts and used  $\int dv_z F_{\alpha} = n_{\alpha}$ . Manifestly, the condition (4.6) is satisfied.

Note that, along the imaginary axis  $p = -i\omega$ , by the same expansion and using also the Plemelj formula (3.23), we have

$$\epsilon(-i\omega) \approx 1 - \frac{1}{\omega^2} \sum_{\alpha} \omega_{p\alpha}^2 - i\pi \frac{\omega_{pe}^2}{k^2} \bar{F}'\left(\frac{\omega}{k}\right) \rightarrow 1 \mp i0 \quad \text{as} \quad \omega \rightarrow \mp\infty. \quad (4.8)$$

This is going to be useful shortly.

In view of (4.8) and of our newly proven formula (4.5), as the function  $\epsilon(-i\omega)$  runs along the real line in  $\omega$ , it changes from

$$\epsilon(i\infty) = 1 - i0 \quad \text{at} \quad \omega = -\infty, \quad (4.9)$$

where I have arbitrarily fixed its phase, to

$$\epsilon(-i\infty) = e^{2\pi i N} + i0 \quad \text{at} \quad \omega = +\infty, \quad (4.10)$$

where  $N$  is the number of times the function

$$\epsilon(-i\omega) = 1 - \frac{\omega_{pe}^2}{k^2} \left[ \mathcal{P} \int_{-\infty}^{+\infty} dv_z \frac{\bar{F}'(v_z)}{v_z - \omega/k} + i\pi \bar{F}'\left(\frac{\omega}{k}\right) \right] \quad (4.11)$$

circles around the origin in the complex  $\epsilon$  plane. Since  $N$  is also the number of unstable roots of the dispersion relation, this gives us a way to count these roots by sketching  $\epsilon(-i\omega)$ —this sketch is called the *Nyquist diagram*. Two examples of Nyquist diagrams implying stability are given in Fig. 17: the curve  $\epsilon(-i\omega)$  departs from  $1 - i0$  and comes back to  $1 + i0$  via a path that, however complicated, never makes a full circle around zero. Two examples of unstable situations appear in Fig. 19(b,d): in these cases, zero is circumnavigated, implying that the equilibrium distribution  $\bar{F}$  is unstable (at a given value of  $k$ ).

In order to work out whether the Nyquist curve circles zero (and how many times), all one needs to do is find  $\text{Re } \epsilon(-i\omega)$  at all points  $\omega$  where  $\text{Im } \epsilon(-i\omega) = 0$ , i.e., where the curve intersects the real line, and hence sketch the Nyquist diagram. We shall see in a moment, with the aid of some important examples, how this is done, but let us do a little bit of preparatory work first.

It follows immediately from (4.11) that these crossings happen whenever  $\omega/k = v_*$  is a velocity

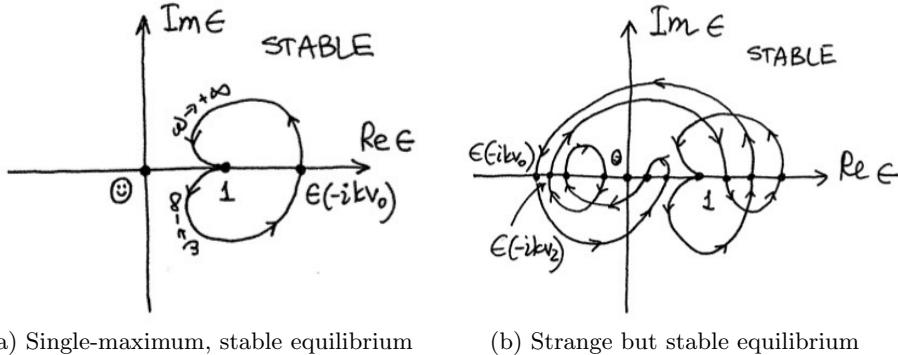


FIGURE 17. Two examples of Nyquist diagrams showing stability (because failing to circle zero): (a) the case of a monotonically decreasing distribution (§4.2, Fig. 18a); (b) another stable case, even though very complicated (it also illustrates the argument in §4.3).

at which  $\bar{F}(v_z)$  has an extremum,  $\bar{F}'(v_*) = 0$ . At these points, the dielectric function (4.11) is real and can be expressed so:

$$\boxed{\epsilon(-ikv_*) = 1 + \frac{\omega_{pe}^2}{k^2} P(v_*)} . \quad (4.12)$$

Here  $P(v_*)$  is (minus) the principal-value integral in (4.11), which can be manipulated as follows:

$$\begin{aligned} P(v_*) &= -\mathcal{P} \int_{-\infty}^{+\infty} dv_z \frac{\bar{F}'(v_z)}{v_z - v_*} = -\mathcal{P} \int_{-\infty}^{+\infty} dv_z \frac{1}{v_z - v_*} \frac{\partial}{\partial v_z} [\bar{F}(v_z) - \bar{F}(v_*)] \\ &= \int_{-\infty}^{+\infty} dv_z \frac{\bar{F}(v_*) - \bar{F}(v_z)}{(v_z - v_*)^2}, \end{aligned} \quad (4.13)$$

where I have integrated by parts; the additional term  $\bar{F}(v_*)$  was inserted under the derivative in order to eliminate the boundary terms arising in this integration by parts around the pole  $v_z = v_*$ .<sup>28</sup>

Now we are ready to analyse particular (and, as we shall see, also generic) equilibrium distributions  $\bar{F}(v_z)$ .

#### 4.2. Stability of Monotonically Decreasing Distributions

Consider first a distribution function that has a single maximum at  $v_z = v_0$  and monotonically decays in both directions away from it (Fig. 18a):  $\bar{F}'(v_0) = 0$ ,  $\bar{F}''(v_0) < 0$ . This means that, besides at  $\omega = \mp\infty$ ,  $\text{Im } \epsilon(-i\omega) \propto \bar{F}'(\omega/k)$  also vanishes at  $\omega = kv_0$ . It is then clear that

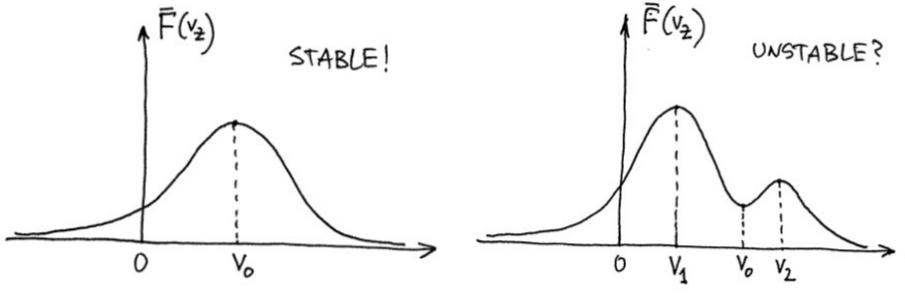
$$\epsilon(-ikv_0) = 1 + \frac{\omega_{pe}^2}{k^2} P(v_0) > 1 \quad (4.14)$$

because  $\bar{F}(v_0) > \bar{F}(v)$  for all  $v_z$  and so  $P(v_0) > 0$ . Thus, the Nyquist curve departs from  $1 - i0$  at  $\omega = -\infty$ , intersects the real line once at  $\omega = kv_0$  and then comes back to  $1 + i0$  without circling zero; the corresponding Nyquist diagram is sketched in Fig. 17(a). Conclusion:

*Monotonically decreasing distributions are stable against electrostatic perturbations.*

We do not in fact need all this mathematical machinery just to prove the stability of monotonically decreasing distributions (in §9.5, we shall see that this is a very robust result)—but it will come handy when dealing with less simple cases. Parenthetically, let us work out

<sup>28</sup>Note that in the final expression in (4.13), there is no longer a need for principal-value integration because,  $v_*$  being a point of extremum of  $\bar{F}$ , the numerator of the integrand is quadratic in  $v_z - v_*$  in the vicinity of  $v_*$ .



(a) Single-maximum distribution (§4.2)      (b) Single-minimum distribution (§4.3)

FIGURE 18. Two examples of equilibrium distributions.

some direct proofs of stability.

**Exercise 4.1. Direct proof of linear stability of monotonically decreasing distributions.** (a) Consider the dielectric function (4.1) with  $p = -i\omega + \gamma$  and assume  $\gamma > 0$  (so the Landau contour is just the real axis). Work out the real and imaginary parts of the dispersion relation  $\epsilon(p) = 0$  and show that it can never be satisfied if  $v_z \bar{F}'(v_z) \leq 0$ , i.e., that any equilibrium distribution that has a maximum at zero and decreases monotonically on both sides of it is stable against electrostatic perturbations.<sup>29</sup>

(b) What if the maximum is at  $v_z = v_0 \neq 0$ ?

**Exercise 4.2. Direct proof of linear stability of isotropic distributions.** (a) Recall Exercise 3.2 and show that all homogeneous, 3D-isotropic (in velocity) equilibria are stable against electrostatic perturbations (with no need to assume long wave lengths).

(b) Prove, in the same way, that isotropic equilibria are also stable against *electromagnetic* perturbations. You will need to derive the transverse dielectric function in the same way as in Q2 or Q3, but for a general equilibrium distribution  $f_{0\alpha}(v_x, v_y, v_z)$ ; failing that, you can look it up in a book, e.g., Krall & Trivelpiece (1973) or Davidson (1983).

### 4.3. Penrose's Instability Criterion

It would be good to learn how to test for stability generic distributions that have multiple minima and maxima: the simplest of them is shown in Fig. 18b, evoking the bump-on-tail situation discussed in §3.5 and thus posing a risk (but, as we are about to see, not a certainty!) of being unstable.

The Nyquist curve  $\epsilon(-i\omega)$  departs from  $1 - i0$  at  $\omega = -\infty$ , then crosses the real line for the first time at  $\omega = kv_1$ , corresponding to the leftmost maximum of  $\bar{F}$ .<sup>30</sup> This crossing is upwards, from the lower to the upper half-plane, and it is not hard to see that a maximum will always correspond to such an upward crossing and a minimum to a downward one, from the upper to the lower half-plane: this follows directly from the change of sign of  $\text{Im } \epsilon$  in Eq. (4.11) because  $\bar{F}'(\omega/k)$  goes from positive to negative at any point of maximum and vice versa at any minimum. After a few crossings back and forth, corresponding to local minima and maxima (if any), the Nyquist curve will come to the downward crossing corresponding to the *global* minimum (other than at  $v_z = \pm\infty$ ) of the distribution function at, say,  $\omega = kv_0$ . If at this point  $P(v_0) > 0$ , then  $\epsilon(-ikv_0) > 1$  and the same is true at all other crossing points  $v_*$  because  $v_0$  is the global minimum of  $\bar{F}$  and so  $P(v_*) > P(v_0) > 0$  for all other extrema. In this situation, illustrated in Fig. 19(a), the Nyquist curve never circumnavigates zero and, therefore,  $P(v_0) > 0$  is a sufficient condition of stability. It is also the necessary one, which is proved in the following way.

<sup>29</sup>This kind of argument can also be useful in stability considerations applying to more complicated situations, e.g., magnetised plasmas (Bernstein 1958).

<sup>30</sup>For the distribution sketched in Fig. 18(b), this maximum is global, so  $P(v_1) > 0$  and, therefore,  $\epsilon(-ikv_1) > 1$ . This is the rightmost such crossing when  $v_1$  is the global maximum.

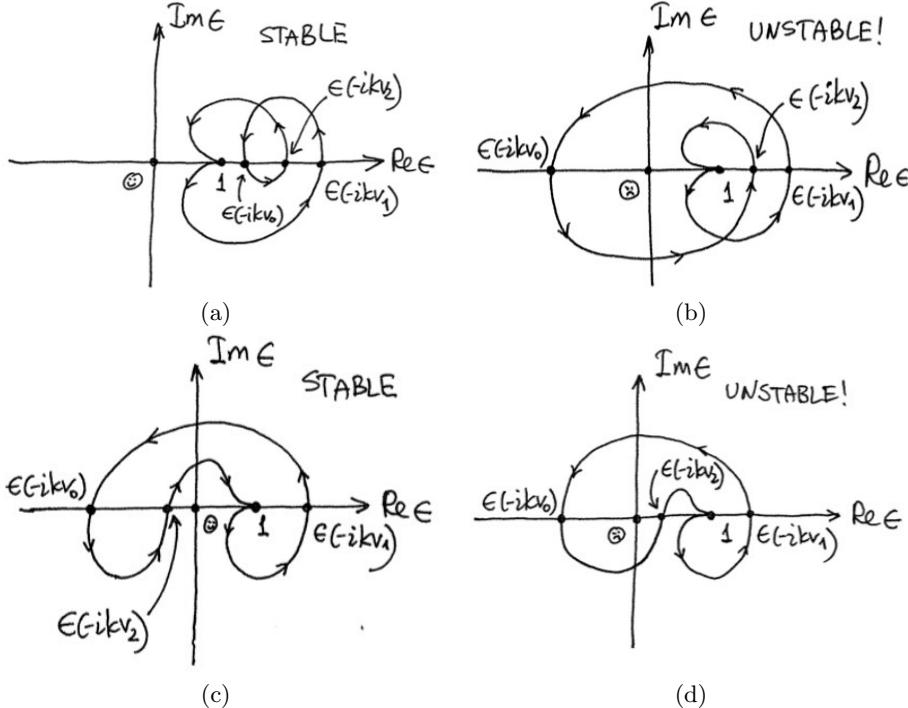


FIGURE 19. Various possible forms of the Nyquist diagram for a single-minimum distribution sketched in Fig. 18b: (a)  $\epsilon(-ikv_0) > 1$ , stable; (b)  $\epsilon(-ikv_0) < 0$ ,  $\epsilon(-ikv_2) > 1$ , unstable; (c)  $\epsilon(-ikv_0) < \epsilon(-ikv_2) < 0$ , stable; (d)  $\epsilon(-ikv_0) < 0 < \epsilon(-ikv_2) < 1$ , unstable.

Suppose  $P(v_0) < 0$ . Then, in (4.12), we can always find a range of  $k$  that are small enough that  $\epsilon(-ikv_0) < 0$ , so the downward crossing at  $v_0$  happens on the negative side of zero in the  $\epsilon$  plane. After this downward crossing, the Nyquist curve will make more crossings, until it finally comes to rest at  $1 + i0$  as  $\omega = +\infty$ . Let us denote by  $v_2 > v_0$  the point of extremum for which the corresponding crossing occurs at a point on the  $\text{Re } \epsilon$  axis that is closest to (but always will be to the right of)  $\epsilon(-ikv_0) < 0$ . If  $\epsilon(-ikv_2) > 0$ , then there is no way back, zero has been fully circumnavigated and so there must be at least one unstable root (see Fig. 19b,d). If  $\epsilon(-ikv_2) < 0$ , there is in principle some wiggle room for the Nyquist curve to avoid circling zero (see Fig. 19c for a single-minimum distribution of Fig. 18b—or Fig. 17b for some serious wiggles). However, since  $P(v_2) > P(v_0)$  for any  $v_2$  (because  $v_0$  is the global minimum of  $\bar{F}$ ), we can always increase  $k$  in (4.12) just enough so  $\epsilon(-ikv_2) > 0$  even though  $\epsilon(-ikv_0) < 0$  still (this corresponds to turning Fig. 19c into Fig. 19d). Thus, if  $P(v_0) < 0$ , there will always be some range of  $k$  inside which there is an instability.

We have obtained a sufficient and necessary condition of instability of an equilibrium  $\bar{F}(v_z)$  against electrostatic perturbations: if  $v_0$  is the point of global minimum of  $\bar{F}$ ,<sup>31</sup>

$$\boxed{P(v_0) = \int_{-\infty}^{+\infty} dv_z \frac{\bar{F}(v_0) - \bar{F}(v_z)}{(v_z - v_0)^2} < 0 \quad \Leftrightarrow \quad \bar{F} \text{ is unstable}}. \quad (4.15)$$

This is the famous *Penrose's instability criterion* (the famous criterion, not the famous Penrose; it was proved by Oliver Penrose 1960, in a stylistically somewhat different way than I did it here). Note that considerations of the kind presented above can be used to work out the wave-number intervals, corresponding to various troughs in  $\bar{F}$ , in which instabilities exist.

Intuitively, the criterion (4.15) says that, in order for a distribution to be unstable, it needs to

<sup>31</sup> Another way of putting this is: *a distribution  $\bar{F}$  is unstable iff it has a minimum at some  $v_0$  for which  $P(v_0) < 0$ .* Obviously, if  $P(v_0) < 0$  at some minimum, it is also negative at the global minimum.

have a trough and this trough must be deep enough. Thus, if  $F(v_0) = 0$ , i.e., if the distribution has a “hole”, it is always unstable (an extreme example of this is the two-stream instability; see Exercise 3.5). Another corollary is that you cannot stabilise a distribution by just adding some particles in a narrow interval around  $v_0$ , as this would create two minima nearby, which, the filled interval being narrow, are still going to render the system unstable. To change that, you must fill the trough substantially with particles—hence the tendency to flatten bumps into plateaux, which we will discover in §6 (this answers, albeit in very broad strokes, the question posed at the beginning of §4 about the types of stable distributions towards which the unstable ones will be pushed as the instabilities saturate).

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**Exercise 4.3.** Consider a single-minimum distribution like the one in Fig. 18(b), but with the global maximum on the right and the lesser maximum on the left of the minimum. Draw various possible Nyquist diagrams and convince yourself that Penrose’s criterion works. If you enjoy this, think of a distribution that would give rise to the Nyquist diagram in Fig. 17(b).

**Exercise 4.4.** What happens if the distribution function  $\bar{F}$  has an inflection point, i.e.,  $\bar{F}(v_0) = 0$ ,  $\bar{F}'(v_0) = 0$ ,  $\bar{F}''(v_0) = 0$ ?

**Exercise 4.5.** What happens if the distribution function has a trough with a flat bottom (i.e., a flat minimum over some interval of velocities)?

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#### 4.4. Bumps, Beams, Streams and Flows

An elementary example of the use of Penrose’s criterion is the two-stream instability, first introduced in Exercise 3.5. The case of two cold streams, represented by (3.60) and Fig. 12(a), is obviously unstable because there is a gaping hole in this distribution. What if we now give these streams some thermal width? This can be modeled by the double-Lorentzian distribution (Fig. 12b)

$$F_e(v_z) = \frac{n_e v_b}{2\pi} \left[ \frac{1}{(v_z - u_b)^2 + v_b^2} + \frac{1}{(v_z + u_b)^2 + v_b^2} \right], \quad (4.16)$$

which is particularly easy to handle analytically. For the moment, we will consider the ions to be infinitely heavy, so  $\bar{F} = F_e$ .

Since the distribution (4.16) is symmetric, it can only have its minimum at  $v_0 = 0$ . Asking that it should indeed be a minimum, rather than a maximum, i.e.,  $\bar{F}'(0) > 0$ , one finds that the condition for this is

$$u_b > \frac{v_b}{\sqrt{3}}. \quad (4.17)$$

Otherwise, the two streams are too wide (in velocity space) and the distribution is monotonically decreasing, so, according to §4.2, it is stable.

If the condition (4.17) is satisfied, the distribution has two bumps, but is this enough to make it unstable? Substituting this distribution into Penrose’s criterion (4.15) and doing the integral exactly<sup>32</sup> we get the necessary and sufficient instability condition:

$$P(0) = -\frac{u_b^2 - v_b^2}{(u_b^2 + v_b^2)^2} < 0 \quad \Leftrightarrow \quad \boxed{u_b > v_b}. \quad (4.18)$$

Thus, if the streams are sufficiently fast and/or their thermal spread is sufficiently narrow, an instability will occur, but it is not quite enough just to have a little trough. Note, by the way, that Penrose’s criterion does not differentiate between hydrodynamic (cold) and kinetic (hot) instability mechanisms (§3.7).

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**Exercise 4.6.** Use Nyquist’s method to work out the range of wave numbers at which perturbations will grow for the two-stream instability (you will find the answer in Jackson 1960—yes, that

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<sup>32</sup>The easiest way to do it is to turn the integration path along the real axis into a loop by completing it with a semicircle at positive or negative complex infinity, where the integrand vanishes, and use Cauchy’s formula.

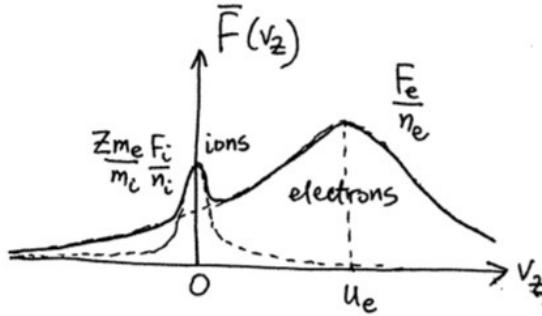


FIGURE 20. Combined distribution (4.2) for cold ions and hot electrons (cf. Fig. 13).

Jackson). Convince yourself that this is all in accord with the explicit solution of the dispersion relation obtained in Q4.

It is obvious how these considerations can be generalised to more complicated situations, e.g., to cases where the streams have different velocities, where one of them is, in fact, the thermal bulk of the distribution and the other is a little bump on its tail (§3.7), where there are more than two streams, etc. The streams also need not be composed of the particles of the same species: indeed, as we saw in (4.1), in the linear theory, the distributions of all species are additively combined into  $\bar{F}$  with weights that are inversely proportional to their masses [see (4.2)]. Thus, the ion-acoustic instability (§3.9) is also just a kind of two-stream—or, if you like, bump-on-tail—instability, with the entire hot and mighty electron distribution making up a magnificent bump on the tail of the cold,  $m_e/m_i$ -weighted ion one (Fig. 20).<sup>33</sup> When the streams/beams have thermal spreads, they are more commonly thought of as mean flows—or currents, if the electron flows are not compensated by the ion ones.

**Exercise 4.7.** Construct an equilibrium distribution to model your favorite plasma system with flows and/or beams and investigate its stability: find the growth rate as a function of wave number, instability conditions, etc.

#### 4.5. Anisotropies

So we have found that various holes, bumps, streams, beams, flows, currents and other such nonmonotonic features in the (combined, multispecies) equilibrium distribution present an instability risk, unless they are sufficiently small, shallow, wide and/or close enough to the thermal bulk. All of these are, of course, *anisotropic* features—indeed, as we saw in Exercise 4.2, 3D-isotropic distributions are harmless, instability-wise. It turns out that anisotropies of the distribution function in velocity space are dangerous even when the distribution decays monotonically in all directions.<sup>34</sup> However, the instabilities that occur in such situations are *electromagnetic*, rather than electrostatic, and so require an investigation into the properties of the transverse dielectric function of the kind derived in Q2 or Q3, but for a general equilibrium. A nice treatment of anisotropy-driven instabilities can be found in Krall & Trivelpiece (1973) and an even more thorough one in Davidson (1983). In §§9.2.1 and 9.5.2, I will show in quite a simple way that, at least in principle, there is always energy available to be extracted from anisotropic distributions.

**Exercise 4.8. Criterion of instability of anisotropic distributions.** This is an independent-study topic. Consider linear stability of general distribution functions to electromagnetic perturbations and work out the stability criterion in the spirit of §4. You

<sup>33</sup>In fact, when the two species' temperatures are the same, there is still an instability, whose criterion can again be obtained by the Nyquist-Penrose method: see Jackson (1960).

<sup>34</sup>In Q3, you have an opportunity to derive the most famous of all instabilities triggered by anisotropy.

should discover that anisotropic distributions such as, e.g., the bi-Maxwellian (6.45), tend to be unstable. Krall & Trivelpiece (1973, §9.10) would be a good place to read about it, but do range beyond.

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## 5. Energy, Entropy, Heating, Irreversibility, and Phase Mixing

While we are done with the “calculational” part of linear theory (calculating whether the field perturbations oscillate, decay or grow, and at what rates), we are not yet done with the “conceptual” part: what exactly is going on, mathematically and physically? The plan of addressing this question in this section is as follows.

- I will show that Landau damping of perturbations of a plasma in thermal equilibrium leads to the heating of this equilibrium—basically, that energy is conserved. This is not a surprise, but it is useful to see explicitly how this works (§5.1).

- I will then ask how it is possible to have heating (an irreversible process) in a plasma that was assumed collisionless and must conserve entropy. In other words, why, physically, is Landau damping a damping? This will lead us to consider entropy evolution in our system and to introduce an important concept of free energy (§5.2).

- In the above context, we will examine (§§5.3 and 5.6) the Laplace-transform solution (3.8) for the perturbed distribution function and establish the phenomenon of *phase mixing*—emergence of fine structure in velocity (phase) space. This will allow collisions and, therefore, irreversibility back in (§5.5). We will also see that the Landau-damped solutions are *not* eigenmodes (while growing solutions can be), and so conclude that it made sense to insist on using an initial-value-problem formalism.

### 5.1. Energy Conservation and Heating

Let us go back to the full, nonlinear Vlasov–Poisson system, with the collision term restored:

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v} \cdot \nabla f_\alpha - \frac{q_\alpha}{m_\alpha} (\nabla \varphi) \cdot \frac{\partial f_\alpha}{\partial \mathbf{v}} = \left( \frac{\partial f_\alpha}{\partial t} \right)_c, \quad (5.1)$$

$$-\nabla^2 \varphi = 4\pi \sum_\alpha q_\alpha \int d\mathbf{v} f_\alpha. \quad (5.2)$$

Let us calculate the rate of change of the electric energy:

$$\begin{aligned} \frac{d}{dt} \int d\mathbf{r} \frac{E^2}{8\pi} &= \underbrace{\int d\mathbf{r} \frac{\nabla \varphi}{4\pi} \cdot \frac{\partial(\nabla \varphi)}{\partial t}}_{\text{by parts}} = - \underbrace{\int d\mathbf{r} \frac{\varphi}{4\pi} \frac{\partial}{\partial t} \nabla^2 \varphi}_{\text{use (5.2)}} = \sum_\alpha q_\alpha \underbrace{\iint d\mathbf{r} d\mathbf{v} \varphi \frac{\partial f_\alpha}{\partial t}}_{\text{use (5.1)}} \\ &= \sum_\alpha q_\alpha \underbrace{\iint d\mathbf{r} d\mathbf{v} \varphi \left[ -\mathbf{v} \cdot \nabla f_\alpha + \frac{q_\alpha}{m_\alpha} (\nabla \varphi) \cdot \frac{\partial f_\alpha}{\partial \mathbf{v}} \right]}_{\text{by parts}} + \underbrace{\left( \frac{\partial f_\alpha}{\partial t} \right)_c}_{\text{vanishes because } f(\pm\infty) = 0} \\ &= \sum_\alpha q_\alpha \iint d\mathbf{r} d\mathbf{v} f_\alpha \mathbf{v} \cdot \nabla \varphi = - \int d\mathbf{r} \mathbf{E} \cdot \mathbf{j}, \end{aligned} \quad (5.3)$$

where  $\mathbf{j}$  is the current density. So the rate of change of the electric field is minus the rate at which electric field does work on the charges, a.k.a. Joule heating—not a surprising

result. The energy goes into accelerating particles, of course: the rate of change of their kinetic energy is

$$\begin{aligned}
 \frac{d\mathcal{K}}{dt} &= \sum_{\alpha} \iint d\mathbf{r} d\mathbf{v} \underbrace{\frac{m_{\alpha} v^2}{2} \frac{\partial f_{\alpha}}{\partial t}}_{\text{use (5.1)}} \\
 &= \sum_{\alpha} \iint d\mathbf{r} d\mathbf{v} \frac{m_{\alpha} v^2}{2} \left[ \underbrace{-\mathbf{v} \cdot \nabla f_{\alpha}}_{\substack{\text{vanishes} \\ \text{because} \\ \text{full} \\ \text{divergence}}} + \underbrace{\frac{q_{\alpha}}{m_{\alpha}} (\nabla \varphi) \cdot \frac{\partial f_{\alpha}}{\partial \mathbf{v}}}_{\substack{\text{by parts in } \mathbf{v} \\ \text{vanishes}}} + \underbrace{\left( \frac{\partial f_{\alpha}}{\partial t} \right)_c}_{\substack{\text{vanishes} \\ \text{because} \\ \text{energy is} \\ \text{conserved}}} \right] \\
 &= - \sum_{\alpha} q_{\alpha} \iint d\mathbf{r} d\mathbf{v} f_{\alpha} \mathbf{v} \cdot \nabla \varphi = \int d\mathbf{r} \mathbf{E} \cdot \mathbf{j}.
 \end{aligned} \tag{5.4}$$

Combining (5.3) and (5.4) gives us the law of energy conservation:

$$\boxed{\frac{d}{dt} \left( \mathcal{K} + \int d\mathbf{r} \frac{E^2}{8\pi} \right) = 0}. \tag{5.5}$$

**Exercise 5.1.** Demonstrate energy conservation for the more general case in which magnetic-field perturbations are also allowed.

Thus, if the perturbations are damped, the energy of the particles must increase—this is usually called *heating*. Strictly speaking, however, heating is a slow, irreversible increase in the mean temperature of the thermal equilibrium. Let us make this statement quantitative. Consider a Maxwellian plasma, homogeneous in space but possibly with some slow dependence on time (cf. §2):

$$f_{0\alpha} = \frac{n_{\alpha}}{(\pi v_{\text{th}\alpha}^2)^{3/2}} e^{-v^2/v_{\text{th}\alpha}^2} = n_{\alpha} \left( \frac{m_{\alpha}}{2\pi T_{\alpha}} \right)^{3/2} e^{-m_{\alpha} v^2/2T_{\alpha}}. \tag{5.6}$$

In a homogeneous system with a fixed volume, the density  $n_{\alpha}$  is constant in time because the number of particles is constant:  $d(Vn_{\alpha})/dt = 0$ . The temperature, however, is allowed to change:  $T_{\alpha} = T_{\alpha}(t)$ . The total kinetic energy of the particles is

$$\begin{aligned}
 \mathcal{K} &= V \sum_{\alpha} \underbrace{\int d\mathbf{v} \frac{m_{\alpha} v^2}{2} f_{0\alpha}}_{\substack{\\ = \frac{3}{2} n_{\alpha} T_{\alpha}}} + \sum_{\alpha} \iint d\mathbf{r} d\mathbf{v} \frac{m_{\alpha} v^2}{2} \delta f_{\alpha}.
 \end{aligned} \tag{5.7}$$

Let us average this over time, as per (2.7): the perturbed part vanishes and we have

$$\langle \mathcal{K} \rangle = V \sum_{\alpha} \frac{3}{2} n_{\alpha} T_{\alpha}. \tag{5.8}$$

Averaging also (5.5) and using (5.8), we get

$$\sum_{\alpha} \frac{3}{2} n_{\alpha} \frac{dT_{\alpha}}{dt} = - \frac{d}{dt} \frac{1}{V} \int d\mathbf{r} \frac{\langle E^2 \rangle}{8\pi}, \tag{5.9}$$

so the heating rate of the equilibrium equals the rate of decrease of the mean energy of the perturbations.

We saw that the perturbations evolve according to (3.16). If we wait for a while, only the slowest-damped mode will matter, with all others exponentially small in comparison. Let us call its frequency and its damping rate  $\omega_{\mathbf{k}}$  and  $\gamma_{\mathbf{k}} < 0$ , respectively, so  $\mathbf{E}_{\mathbf{k}} \propto e^{-i\omega_{\mathbf{k}}t + \gamma_{\mathbf{k}}t}$ . If  $|\gamma_{\mathbf{k}}| \ll \omega_{\mathbf{k}}$ , the time average (2.7) can be defined in such a way that  $\omega_{\mathbf{k}}^{-1} \ll \Delta t \ll |\gamma_{\mathbf{k}}|^{-1}$ . Then (5.9) becomes

$$\sum_{\alpha} \frac{3}{2} n_{\alpha} \frac{dT_{\alpha}}{dt} = - \sum_{\mathbf{k}} 2\gamma_{\mathbf{k}} \frac{|\mathbf{E}_{\mathbf{k}}|^2}{8\pi} > 0. \quad (5.10)$$

The Landau damping rate of the electric-field perturbations is the heating rate of the equilibrium.<sup>35</sup>

This result, while at first glance utterly obvious, might, on reflection, appear to be paradoxical: surely, the heating of the equilibrium implies increasing entropy—but the damping that is leading to the heating is collisionless and, in a collisionless system, in view of the  $H$ -theorem, how can the entropy change?

### 5.2. Entropy and Free Energy

The kinetic entropy for each species of particles is defined to be

$$S_{\alpha} = - \iint d\mathbf{r} d\mathbf{v} f_{\alpha} \ln f_{\alpha}. \quad (5.11)$$

This quantity [or, indeed, the full-phase-space integral of any quantity that is a function only of  $f_{\alpha}$ ; see (9.8)] can only be changed by collisions and, furthermore, the plasma-physics version of Boltzmann's  $H$ -theorem says that

$$\frac{d}{dt} \sum_{\alpha} S_{\alpha} = - \sum_{\alpha} \iint d\mathbf{r} d\mathbf{v} \left( \frac{\partial f_{\alpha}}{\partial t} \right)_c \ln f_{\alpha} \geq 0, \quad (5.12)$$

where equality is achieved iff all  $f_{\alpha}$  are Maxwellian with the same temperature  $T_{\alpha} = T$ .

Thus, *if collisions are ignored, the total entropy stays constant and everything that happens is, in principle, reversible. So how can there be net damping of waves and, worse still, net heating of the equilibrium particle distribution?!* Presumably, any damping solution can be turned into a growing solution by reversing all particle trajectories—so should the overall perturbation level not stay constant?

As I already noted in §5.1, strictly speaking, heating is the increase of the equilibrium temperature—and, therefore, of the equilibrium entropy. Indeed, for each species, the equilibrium entropy is

$$\begin{aligned} S_0 &= - \iint d\mathbf{r} d\mathbf{v} f_0 \ln f_0 = - \iint d\mathbf{r} d\mathbf{v} f_0 \left\{ \ln \left[ n \left( \frac{m}{2\pi} \right)^{3/2} \right] - \frac{3}{2} \ln T - \frac{mv^2}{2T} \right\} \\ &= V \left\{ -n \ln \left[ n \left( \frac{m}{2\pi} \right)^{3/2} \right] + \frac{3}{2} n \ln T + \frac{3}{2} n \right\}, \end{aligned} \quad (5.13)$$

where I have used  $\int d\mathbf{v} (mv^2/2)f_0 = (3/2)nT$ . Since  $n = \text{const}$ ,

$$T \frac{dS_0}{dt} = V \frac{3}{2} n \frac{dT}{dt}, \quad (5.14)$$

so heating is indeed associated with an increase of  $S_0$ .

Since, according to (5.10), this can be successfully accomplished by collisionless damping and since entropy overall can only increase due to collisions, we must search for the

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<sup>35</sup>Obviously, the damping of waves on particles of species  $\alpha$  increases only the temperature of that species.

“missing entropy” (or, rather, for the missing *decrease* of entropy) in the perturbed part of the distribution. The mean entropy associated with the perturbed distribution is<sup>36</sup>

$$\begin{aligned}\langle \delta S \rangle &= \langle S - S_0 \rangle = - \iint d\mathbf{r} d\mathbf{v} \langle (f_0 + \delta f) \ln(f_0 + \delta f) - f_0 \ln f_0 \rangle \\ &= - \iint d\mathbf{r} d\mathbf{v} \left\langle (f_0 + \delta f) \left[ \ln f_0 + \frac{\delta f}{f_0} - \frac{\delta f^2}{2f_0^2} + \dots \right] - f_0 \ln f_0 \right\rangle \\ &\approx - \iint d\mathbf{r} d\mathbf{v} \frac{\langle \delta f^2 \rangle}{2f_0},\end{aligned}\quad (5.15)$$

after expanding to second order in small  $\delta f/f_0$  and using  $\langle \delta f \rangle = 0$ . The total mean entropy of each species,  $\langle S \rangle = S_0 + \langle \delta S \rangle$ , can only be changed by collisions, so, if collisions are ignored, any heating of a given species, i.e., any increase in its  $S_0$  [see (5.14)] must be compensated by a *decrease* in its  $\langle \delta S \rangle$ . The latter can only be achieved by *increasing*  $\langle \delta f^2 \rangle$ : indeed, using (5.14) and (5.15), we find<sup>37</sup>

$$\begin{aligned}T \left( \frac{dS_0}{dt} + \frac{d\langle \delta S \rangle}{dt} \right) &= V \frac{3}{2} n \frac{dT}{dt} - \frac{d}{dt} \iint d\mathbf{r} d\mathbf{v} \frac{T \langle \delta f^2 \rangle}{2f_0} \\ &= - \iint d\mathbf{r} d\mathbf{v} T \left\langle \left( \frac{\partial f}{\partial t} \right)_c \ln f \right\rangle.\end{aligned}\quad (5.16)$$

If the right-hand side is ignored,  $T$  can only increase if  $\langle \delta f^2 \rangle$  increases too.

It is useful to work out the collision term in (5.16) in terms of  $f_0$  and  $\delta f$ : using the fact that  $\langle \delta f \rangle = 0$  by definition and that the number of particles is conserved by the collision operator, we get

$$\begin{aligned}\iint d\mathbf{r} d\mathbf{v} T \left\langle \left( \frac{\partial f}{\partial t} \right)_c \ln f \right\rangle &\approx \iint d\mathbf{r} d\mathbf{v} \left[ T \left( \frac{\partial f_0}{\partial t} \right)_c \ln f_0 + \left\langle \frac{T \delta f}{f_0} \left( \frac{\partial \delta f}{\partial t} \right)_c \right\rangle \right] \\ &= V \int d\mathbf{v} \frac{mv^2}{2} \left( \frac{\partial f_0}{\partial t} \right)_c + \iint d\mathbf{r} d\mathbf{v} \left\langle \frac{T \delta f}{f_0} \left( \frac{\partial \delta f}{\partial t} \right)_c \right\rangle.\end{aligned}\quad (5.17)$$

The second term is the collisional damping of  $\delta f$ , of which more will be said soon. The first term is the collisional energy exchange between the equilibrium distributions of different species (intra-species collisions conserve energy, but inter-species ones do not, because there is friction between species). If the species under consideration is  $\alpha$ , this energy exchange can be represented as  $\sum_{\alpha'} \nu_{\alpha\alpha'} (T_\alpha - T_{\alpha'})$  (see, e.g., Helander & Sigmar 2005) and will act to equilibrate temperatures between species as the system strives toward thermal equilibrium. If the collision frequencies  $\nu_{\alpha\alpha'}$  are small, this is a slow effect. Due to overall energy conservation, the energy-exchange terms vanish exactly if (5.17) is summed over species.

Finally, let us sum (5.16) over species and use (5.9) to relate the total heating to the rate

<sup>36</sup>As an aside, note that this piece of the calculation is entirely independent of what  $f_0$  is. It simply demonstrates that the entropy of an averaged distribution  $f_0$  is always larger than that of the exact distribution  $f$ , as long as  $\delta f \ll f_0$ . If the average is reinterpreted as a coarse graining over phase space, this argument is sometimes viewed as a kind of “proof” (or illustration) of the second law of thermodynamics. Indeed, take  $f_0$  and  $f$  to be the same at some initial time  $t$ . Then  $S_0(t) = \langle S(t) \rangle$ . Now advance to time  $t + \delta t$ . Some small  $\delta f$  arises, but coarse graining “deletes” the information contained in it and (5.15) shows that  $S_0(t + \delta t) > \langle S(t + \delta t) \rangle$  (cf. the general statistical-mechanical argument to the same effect: Schekochihin 2019, §§12.4 and 13.4).

<sup>37</sup>In the second term,  $T$  can be brought inside the time derivative because  $\langle \delta f^2 \rangle / f_0 \ll f_0$ .

of change of the electric-perturbation energy:

$$\boxed{\frac{d}{dt} \left[ \underbrace{\sum_{\alpha} \iint d\mathbf{r} d\mathbf{v} \frac{T_{\alpha} \langle \delta f_{\alpha}^2 \rangle}{2f_{0\alpha}} + \int d\mathbf{r} \frac{\langle E^2 \rangle}{8\pi}}_{\equiv \mathcal{F}} \right] = \sum_{\alpha} \iint d\mathbf{r} d\mathbf{v} \left\langle \frac{T_{\alpha} \delta f_{\alpha}}{f_{0\alpha}} \left( \frac{\partial \delta f_{\alpha}}{\partial t} \right)_{\text{c}} \right\rangle \leqslant 0}, \quad (5.18)$$

where we used (5.17) in the right-hand side (with the total equilibrium collisional energy-exchange terms vanishing upon summation over species). The right-hand side must be non-positive-definite because collisions cannot decrease entropy [see (5.12)].

Equation (5.18) is a way to express the idea that, except for the effect of collisions, the change in the electric-perturbation energy ( $= -\text{heating}$ ) must be compensated by the change in  $\langle \delta f^2 \rangle$ , in terms of a *conservation law* of a quadratic positive-definite quantity,  $\mathcal{F}$ , that measures the total amount of perturbation in the system (a quadratic norm of the perturbed solution).<sup>38</sup> It is not hard to realise that this quantity is the *free energy* of the perturbed state, comprising the entropy of the perturbed distribution and the energy of the electric field:

$$\boxed{\mathcal{F} = \mathcal{E} - \sum_{\alpha} T_{\alpha} \langle \delta S_{\alpha} \rangle, \quad \mathcal{E} = \int d\mathbf{r} \frac{\langle E^2 \rangle}{8\pi}} \quad (5.19)$$

(to be precise, it is the difference—always positive!—between the free energy of the perturbed state,  $\mathcal{E} - \sum_{\alpha} T_{\alpha} \langle S_{\alpha} \rangle$ , and the free energy of the equilibrium,  $- \sum_{\alpha} T_{\alpha} S_{0\alpha}$ ).

It is quite a typical situation in non-equilibrium systems that there is an energy-like (quadratic in the relevant fields and positive definite) quantity, which is conserved except for dissipation. For example, in hydrodynamics, the motions of a fluid are governed by the Navier–Stokes equation:

$$\rho \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \mu \Delta \mathbf{u}, \quad (5.20)$$

where  $\mathbf{u}$  is velocity,  $\rho$  mass density ( $\rho = \text{const}$  for an incompressible fluid),  $p$  pressure and  $\mu$  the dynamical viscosity of the fluid. The conservation law is

$$\frac{d}{dt} \int d\mathbf{r} \frac{\rho u^2}{2} = -\mu \int d\mathbf{r} |\nabla \mathbf{u}|^2 \leqslant 0. \quad (5.21)$$

The conserved quadratic quantity is kinetic energy and the negative-definite dissipation (leading to net entropy production) is viscous heating.

You will find a similar conservation law for incompressible MHD if, in §13.10, you work out the time evolution of  $\int d\mathbf{r} (\rho u^2 / 2 + B^2 / 8\pi)$  assuming  $\rho = \text{const}$  and  $\nabla \cdot \mathbf{u} = 0$ ; cf. (15.70).

**Exercise 5.2. Free energy and kinetic energy of mean plasma flow.** Suppose the perturbation  $\delta f$  contains a mean flow of particles, with velocity  $\mathbf{u}$ . Show that it is then always formally possible to decompose

$$\delta f = \frac{2\mathbf{u} \cdot \mathbf{v}}{v_{\text{th}}^2} f_0 + h, \quad (5.22)$$

<sup>38</sup>Note that the existence of such a quantity implies that the Maxwellian equilibrium is stable: if a quadratic norm of the perturbed solution cannot grow, clearly there cannot be any exponentially growing solutions. This is known as Newcomb's theorem, first communicated to the world in the paper by Bernstein (1958, Appendix I). A generalisation of this principle to general isotropic distributions is the subject of Q6(c) and of §9.4, where the conserved quantity  $\mathcal{F}$  will reemerge in a different way, confirming its status as a Platonic entity that cannot be avoided.

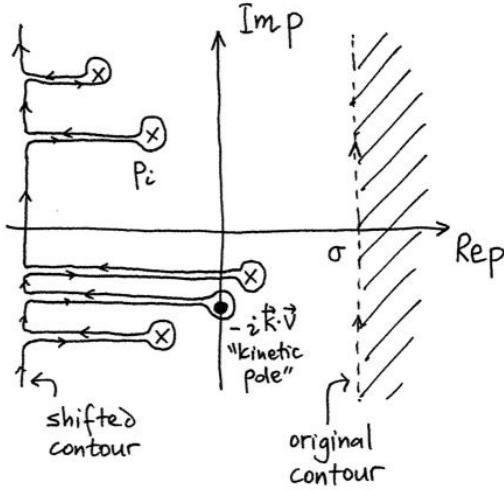


FIGURE 21. Shifting the integration contour in (5.25). This is analogous to Fig. 5 but note the additional “kinetic” pole.

where  $\int d\mathbf{v} \mathbf{v} h = 0$ . Hence show that

$$\iint d\mathbf{r} d\mathbf{v} \frac{T\langle \delta f^2 \rangle}{2f_0} = \int d\mathbf{r} \frac{mn\langle u^2 \rangle}{2} + \iint d\mathbf{r} d\mathbf{v} \frac{T\langle h^2 \rangle}{2f_0}, \quad (5.23)$$

i.e., the entropic part of the free energy is equal to the kinetic energy of the mean plasma flow plus the free entropy associated with the part of the perturbed distribution that has no mean velocity. This means that “fluid” energy budgets such as (5.21) and (15.70) are not just analogs, but particular cases, of the free-energy budget (5.18). In §5.7, I will show how these ideas play out for Langmuir waves.

Thus, as the electric perturbations decay via Landau damping, the perturbed distribution function must grow. This calls for going back to our solution for it (§3.1) and analysing carefully the behaviour of  $\delta f$ .

### 5.3. Structure of Perturbed Distribution Function

Start with our solution (3.8) for  $\delta f(p)$  and substitute into it the solution (3.15) for  $\varphi(p)$ :

$$\hat{\delta f}(p) = \underbrace{\frac{1}{p + i\mathbf{k} \cdot \mathbf{v}}}_{\text{“kinetic” (ballistic) pole}} \left\{ i \frac{q}{m} \underbrace{\left[ \sum_i \frac{c_i}{p - p_i} + A(p) \right] \mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{v}} + g}_{\substack{\text{poles} \\ \text{representing} \\ \text{linear} \\ \text{modes, see} \\ (3.17)}} \right\}. \quad (5.24)$$

To compute the inverse Laplace transform (3.6), we adopt the same method as in §3.1 (Fig. 5), viz., shift the integration contour to large negative  $\text{Re } p$  as shown in Fig. 21 and

use Cauchy's formula:

$$\delta f(t) = \frac{1}{2\pi i} \int_{-i\infty+\sigma}^{i\infty+\sigma} dp e^{pt} \delta \hat{f}(p) = i \underbrace{\frac{q}{m} \sum_i \frac{c_i e^{p_i t}}{p_i + i\mathbf{k} \cdot \mathbf{v}}}_{\substack{\text{eigenmode-like} \\ \text{solution, comes from} \\ \text{the poles of } \varphi(p)}} \mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{v}}$$

$$+ e^{-i\mathbf{k} \cdot \mathbf{v} t} \underbrace{\left\{ g - i \frac{q}{m} \left[ \sum_i \frac{c_i}{p_i + i\mathbf{k} \cdot \mathbf{v}} + A(-i\mathbf{k} \cdot \mathbf{v}) \right] \mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{v}} \right\}}_{\substack{\text{ballistic response,} \\ \text{comes from} \\ p = -i\mathbf{k} \cdot \mathbf{v}}} . \quad (5.25)$$

A perceptive reader has spotted that this formula does not seem to satisfy  $\delta f(t=0) = g$  unless  $A(-i\mathbf{k} \cdot \mathbf{v}) = 0$ . This is because, as explained in footnote 11, the method for calculating the inverse Laplace transform that involves discarding the integral along the vertical part of the shifted contour in Fig. 21 only works in the limit of long times. It is an amusing exercise in complex analysis to show that, in the (overly restrictive) case of  $\varphi(p)$  decaying quickly at  $\text{Re } p \rightarrow -\infty$ , the solution (5.25) is also valid at finite  $t$  and, accordingly,  $A(-i\mathbf{k} \cdot \mathbf{v}) = 0$ , i.e.,  $A(p)$  vanishes for any purely imaginary  $p$ .

The solution (5.25) teaches us two important things.

1) First, *the Landau-damped solution is not an eigenmode*. Even though the evolution of the potential, given by (3.16), does look like a sum of damped eigenmodes of the form  $\varphi \propto e^{p_i t}$ ,  $\text{Re } p_i < 0$ , the full solution of the Vlasov–Poisson system does *not* decay: there is a part of  $\delta f(t)$ , the “ballistic response”  $\propto e^{-i\mathbf{k} \cdot \mathbf{v} t}$ , that oscillates without decaying—in fact, we shall see in §5.6 that  $\delta f$  even has a growing part! It is this part that is responsible for keeping free energy conserved, as per (5.18) without collisions (§5.7). Thus, you may think of Landau damping as a process of transferring (free) energy from the electric-field perturbations to the perturbations of the distribution function.

In contrast to the case of damping, a growing solution ( $\text{Re } p_i > 0$ ) can be viewed as an eigenmode because, after a few growth times, the first term in (5.25) will be exponentially larger than the ballistic term. This will allow us to ignore the latter in our treatment of QLT (§6.1)—a handy, although not necessary (see Q9), simplification. Note that reversibility is not an issue for the growing solutions: so, there may be (and often are) damped solutions as well, so what? We only care about the growing modes because they will be all that is left if we wait long enough.

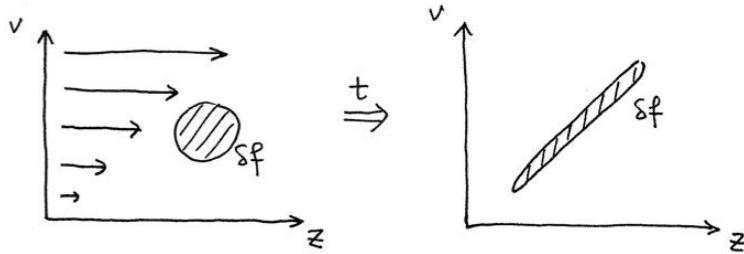
2) Secondly, the  $\delta f$  perturbations have *fine structure in velocity (phase) space*. This structure gets finer with time: roughly speaking, if  $\delta f \propto e^{-ikvt}$ , then

$$\frac{1}{\delta f} \frac{\partial \delta f}{\partial v} \sim ikt \rightarrow \infty \quad \text{as} \quad t \rightarrow \infty. \quad (5.26)$$

This phenomenon is called *phase mixing*. You can think of the basic mechanism responsible for it as a shearing in phase space: the homogeneous part of the linearised kinetic equation,

$$\frac{\partial \delta f}{\partial t} + v \frac{\partial \delta f}{\partial z} = \dots, \quad (5.27)$$

describes advection of  $\delta f$  by a linear shear flow in the the  $(z, v)$  plane. This turns any  $\delta f$  structure in this plane into long thin filaments, with large gradients in  $v$  (Fig. 22).

FIGURE 22. Shearing of  $\delta f$  in phase space.

#### 5.4. Landau Damping Is Phase Mixing

Phase mixing helps us make sense of the notion that, even though  $\varphi$  is the velocity integral of  $\delta f$ , the former can be decaying while the latter is not:

$$\varphi = \frac{4\pi}{k^2} \sum_{\alpha} q_{\alpha} \underbrace{\int d\mathbf{v} \delta f_{\alpha}}_{\substack{\text{fine} \\ \text{structure} \\ \text{cancels}}} \propto e^{-\gamma t} \rightarrow 0. \quad (5.28)$$

The velocity integral over the fine structure increasingly cancels as time goes on—a perturbation initially “visible” as  $\varphi$  phase-mixes away, disappearing into the negative entropy associated with the fine velocity dependence of  $\delta f$  [see (5.15)].

More generally speaking, one can similarly argue that the refinement of velocity dependence of  $\delta f$  causes lower velocity moments of  $\delta f$  (density, flow velocity, pressure, heat flux, and so on) to decrease with time, transferring free energy to higher moments (ever higher as time goes on). One way to formalise this statement neatly is in terms of Hermite moments: since Hermite polynomials are orthogonal, the free energy of the perturbed distribution can be written as a sum of “energies” of the Hermite moments [see (6.77)]. It is then possible to represent the Landau-damped perturbations as having a broad spectrum in Hermite space, with the majority of the free energy residing in high-order moments—infinitely high in the formal limit of zero collisionality and infinite time (see Q8 and [Kanekar et al. 2015](#)).<sup>39</sup>

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Since the  $m$ th-order Hermite moment can, for  $m \gg 1$ , be asymptotically represented as a cosine function in  $v$  space oscillating with the “frequency”  $\sqrt{2m}/v_{\text{th}}$  [see (6.78)], (5.26) implies that the typical order of the moment in which the free energy resides grows with time as  $m \sim (kv_{\text{th}}t)^2$ .

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Taking Hermite (or other kind of) moments of the kinetic equation is essentially the procedure for deriving “fluid” equations for the plasma—or, rather, plasma becomes a fluid if this procedure can be stopped after a few moments (e.g., in the limit of strong collisionality, this happens at the third moment; see [Dellar 2015](#), [Parra 2019a](#), and [Kunz 2021](#)). Since Landau damping is a long-time effect of this phase-mixing process, it cannot be captured by any fluid approximation to the kinetic system involving a truncation of the hierarchy of moment equations at some finite-order moment—it is an essentially kinetic effect “beyond all orders”.

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One useful way to see this is by examining the structure of Langmuir hydrodynamics, which was

<sup>39</sup>An approach involving a Fourier, rather than Hermite, transform in velocity space will be presented in §12.2.2.

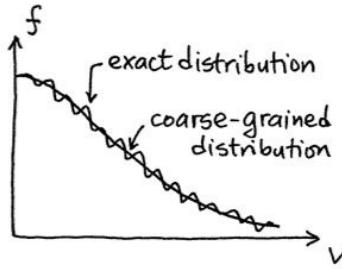


FIGURE 23. Coarse graining of the distribution function.

the subject of Exercise 3.1. The moment hierarchy can be truncated by assuming  $kv_{\text{the}}/\omega \gg 1$ , but one can never capture Landau damping however many moments one keeps: indeed, the Landau damping rate (3.41) for, say, a Maxwellian plasma will be  $\gamma \propto \exp(-\omega^2/k^2 v_{\text{the}}^2)$ , all coefficients in the Taylor expansion of which in powers of  $kv_{\text{the}}/\omega$  are zero.

### 5.5. Role of Collisions

As ever larger velocity-space gradients emerge, it becomes inevitable that at some point they will become so large that *collisions can no longer be ignored*. This is because the Landau collision operator is a Fokker–Planck (diffusion) operator in velocity space [see (1.47)] and so it will eventually wipe out the fine structure in  $\mathbf{v}$ , however small is the collision frequency  $\nu$ . Let us estimate how long this takes.

The size of the velocity-space gradients of  $\delta f$  due to ballistic response is given by (5.26). Then the collision term is

$$\left( \frac{\partial \delta f}{\partial t} \right)_c \sim \nu v_{\text{th}}^2 \frac{\partial^2 \delta f}{\partial \mathbf{v}^2} \sim -\nu v_{\text{th}}^2 k^2 t^2 \delta f. \quad (5.29)$$

Solving for the time evolution of the perturbed distribution function due to collisions, we get

$$\frac{\partial \delta f}{\partial t} \sim -\nu (kv_{\text{th}}t)^2 \delta f \quad \Rightarrow \quad \delta f \sim \exp\left(-\frac{1}{3} \nu k^2 v_{\text{th}}^2 t^3\right) \equiv e^{-(t/t_c)^3}. \quad (5.30)$$

Therefore, the characteristic collisional decay time is

$t_c \sim \frac{1}{\nu^{1/3} (kv_{\text{th}})^{2/3}}$

(5.31)

Note that  $t_c \ll \nu^{-1}$  provided  $\nu \ll kv_{\text{th}}$ , i.e.,  $t_c$  is within the range of times over which our “collisionless” theory is valid. After time  $t_c$ , “collisionless” damping becomes irreversible because the part of  $\delta f$  that is fast-varying in velocity space is lost (entropy has grown) and so it is no longer possible, even in principle, to invert all particle trajectories, have the system retrace back its steps, “phase-unmix” and thus “undamp” the damped perturbation.

In a sufficiently collisionless system, phase unmixing is, in fact, possible if nonlinearity is allowed—giving rise to the beautiful phenomenon of *plasma echo*, in which perturbations can first appear to be damped away but then come back from phase space (§12.1). This effect is a source of much preoccupation to pure mathematicians (Villani 2014; Bedrossian 2016): indeed the validity of the linearised Vlasov equation (3.1) as a sensible approximation to the full nonlinear one (2.12) is in question if the velocity derivative  $\partial \delta f / \partial \mathbf{v}$  in the last term of the latter starts growing uncontrollably. Phase unmixing has also recently turned out to have interesting consequences for the role of Landau damping in plasma turbulence (§§12.2–12.3).

Some rather purist theoreticians sometimes choose to replace collisional estimates of the type discussed above by a stipulation that  $\delta f(\mathbf{v})$  must be “*coarse-grained*” beyond some suitably chosen scale in  $\mathbf{v}$  (Fig. 23)—this is equivalent to saying that the formation of the fine-structured phase-space part of  $\delta f$  constitutes a loss of information and so leads to growth of entropy (i.e., loss of negative entropy associated with  $\langle \delta f^2 \rangle$ ). Somewhat non-rigorously, this means that we can just consider the ballistic term in (5.25) to have been wiped out and use the coarse-grained (i.e., velocity-space-averaged) version of  $\delta f$ :<sup>40</sup>

$$\overline{\delta f} = i \frac{q}{m} \sum_i \frac{c_i e^{p_i t}}{p_i + i\mathbf{k} \cdot \mathbf{v}} \mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{v}}. \quad (5.32)$$

We can check that the correct solution (3.16) for the potential can be recovered from this:

$$\begin{aligned} \varphi &= \frac{4\pi}{k^2} \sum_{\alpha} q_{\alpha} \int d\mathbf{v} \overline{\delta f}_{\alpha} \\ &= \sum_i c_i e^{p_i t} \underbrace{\left[ \sum_{\alpha} i \frac{4\pi q_{\alpha}^2}{m_{\alpha} k^2} \int d\mathbf{v} \frac{1}{p_i + i\mathbf{k} \cdot \mathbf{v}} \mathbf{k} \cdot \frac{\partial f_{0\alpha}}{\partial \mathbf{v}} - 1 + 1 \right]}_{= -\epsilon(p_i, \mathbf{k}) = 0 \text{ by definition of } p_i} = \sum_i c_i e^{p_i t}. \end{aligned} \quad (5.33)$$

If you are wondering how this works without the coarse-graining kludge, read on.

### 5.6. Further Analysis of $\delta f$ : Case–van Kampen Mode

Having given a rather qualitative analysis of the structure and consequences of the solution (5.25), I anticipate a degree of dissatisfaction from a perceptive reader. Yes, there is a non-decaying piece of  $\delta f$ . But conservation of free energy in a collisionless system in the face of Landau damping in fact requires  $\langle \delta f^2 \rangle$  to grow, not just to fail to decay [see (5.18)]. How do we see that this does indeed happen? The analysis that follows addresses this question. These considerations are not really necessary for most practical plasma-physics calculations (see, however, Q9), but it may be necessary for your peace of mind and greater comfort with this whole conceptual framework.

Let us rearrange the solution (5.25) as follows:

$$\delta f(t) = i \frac{q}{m} \sum_i c_i \frac{e^{p_i t} - e^{-i\mathbf{k} \cdot \mathbf{v} t}}{p_i + i\mathbf{k} \cdot \mathbf{v}} \mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{v}} + (g + \dots) e^{-i\mathbf{k} \cdot \mathbf{v} t}.$$

(5.34)

The second term is the ballistic evolution of perturbations (particles flying apart in straight lines at different velocities)—a homogeneous solution of the kinetic equation (3.1). This develops a lot of fine-scale velocity-space structure, but obviously does not grow. The first term, a particular solution arising from the (linear) wave-particle interaction, is more interesting, especially around the resonances  $\text{Re } p_i + \mathbf{k} \cdot \mathbf{v} = 0$ .

Consider one of the modes,  $p_i = -i\omega + \gamma$ , and assume  $\gamma \ll \mathbf{k} \cdot \mathbf{v} \sim \omega$ . This allows us to introduce “intermediate” times:

$$\frac{1}{\mathbf{k} \cdot \mathbf{v}} \ll t \ll \frac{1}{\gamma}. \quad (5.35)$$

This means that the wave has had time to oscillate, phase mixing has got underway, but the perturbation has not yet been damped away significantly. We have then, for the

<sup>40</sup>With an understanding that any integral involving the resonant denominator must be taken along the Landau contour (see Q9). If you adopt this shorthand, you can, nonrigorously but often expeditiously, use Fourier transforms into frequency space, rather than Laplace transforms.

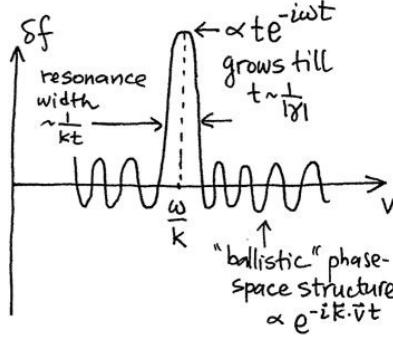


FIGURE 24. Emergence of the Case–van Kampen mode.

relevant piece of the perturbed distribution (5.34),

$$\delta f \propto \frac{e^{p_i t} - e^{-i\mathbf{k} \cdot \mathbf{v} t}}{p_i + i\mathbf{k} \cdot \mathbf{v}} = -ie^{-i\omega t} \frac{e^{\gamma t} - e^{-i(\mathbf{k} \cdot \mathbf{v} - \omega)t}}{\mathbf{k} \cdot \mathbf{v} - \omega - i\gamma} \approx -ie^{-i\omega t} \frac{1 - e^{-i(\mathbf{k} \cdot \mathbf{v} - \omega)t}}{\mathbf{k} \cdot \mathbf{v} - \omega}, \quad (5.36)$$

with the last, approximate, expression valid at the intermediate times (5.35), assuming also that, even though we might be close to the resonance, we shall not come closer than  $\gamma$ , viz.,  $|\mathbf{k} \cdot \mathbf{v} - \omega| \gg \gamma$ . Respecting this ordering, but taking  $|\mathbf{k} \cdot \mathbf{v} - \omega| \ll 1/t$ , we find

$$\delta f \propto t e^{-i\omega t}. \quad (5.37)$$

Thus,  $\delta f$  has a peak that *grows with time*, emerging from the sea of fine-scale but constant-amplitude structures (Fig. 24). The width of this peak is obviously  $|\mathbf{k} \cdot \mathbf{v} - \omega| \sim 1/t$  and so  $\delta f$  around the resonance develops a sharp structure, which, in the formal limit  $t \rightarrow \infty$  (but respecting  $\gamma t \ll 1$ , i.e., with infinitesimal damping), tends to a delta function:

$$\boxed{\delta f \propto -ie^{-i\omega t} \frac{1 - e^{-i(\mathbf{k} \cdot \mathbf{v} - \omega)t}}{\mathbf{k} \cdot \mathbf{v} - \omega} \rightarrow e^{-i\omega t} \pi \delta(\mathbf{k} \cdot \mathbf{v} - \omega)}$$

as  $t \rightarrow \infty$ . (5.38)

Here is a “formal” proof:

$$\frac{1 - e^{-ixt}}{x} = \underbrace{\frac{1 - \cos xt}{x}}_{\substack{\text{finite as} \\ t \rightarrow \infty, \\ \text{even at} \\ x = 0}} + i \underbrace{\frac{\sin xt}{x}}_{\substack{= it \text{ at} \\ x = 0, \\ \text{so dominant}}} \approx \frac{e^{ixt} - e^{-ixt}}{2x} = \frac{i}{2} \int_{-t}^t dt' e^{ixt'} \rightarrow i\pi \delta(x) \quad \text{as } t \rightarrow \infty. \quad (5.39)$$

The  $\delta$ -function solution (5.38) is an instance of a *Case–van Kampen mode* (van Kampen 1955; Case 1959)—an object that belongs to the mathematical realms briefly alluded to at the end of §3.5. Note that writing the solution in the vicinity of the resonance in this form is tantamount to stipulating that any integral taken with respect to  $\mathbf{v}$  (or  $\mathbf{k}$ ) and involving  $\delta f$  must always be done along the Landau contour, circumventing the pole from below [cf. (3.23)]. We will find the representation (5.38) of  $\delta f$  useful in working out the quasilinear theory of Landau damping in Q9.

If we restore finite damping, all this goes on until  $t \sim 1/\gamma$ , with the delta function reaching the height  $\propto 1/\gamma$  and width  $\propto \gamma$ . In the limit  $t \gg 1/\gamma$ , the damped part of the solution decays,  $e^{\gamma t} \rightarrow 0$ , and we are left with just the ballistic part, the second term in (5.25).

### 5.7. Free-Energy Conservation for Landau-Damped Langmuir Waves

Finally, let us convince ourselves that, if we ignore collisions, we can recover (5.18) with a zero right-hand side from the full collisionless Landau-damped solution given by (3.16) and (5.34). For simplicity, let us consider the case of electron Langmuir waves and prove that

$$\boxed{\frac{d}{dt} \int d\mathbf{v} \frac{T|\delta f_{\mathbf{k}}|^2}{2f_0} = -2\gamma_{\mathbf{k}} \frac{|\mathbf{E}_{\mathbf{k}}|^2}{8\pi} = -\frac{d}{dt} \frac{|\mathbf{E}_{\mathbf{k}}|^2}{8\pi}.} \quad (5.40)$$

In (5.34), let the relevant root of the dispersion relation be  $p_i = -i\omega_{pe} + \gamma_{\mathbf{k}}$ , where  $\gamma_{\mathbf{k}}$  is given by (3.41), and assume a Maxwellian  $f_0$ . Based on the discussion §5.6, we should expect  $\delta f_{\mathbf{k}}$  to develop a growing  $\delta$ -like peak around the resonance  $\mathbf{k} \cdot \mathbf{v} \approx \omega_{pe}$ . In this region of velocity space, the distribution function (5.34) for electrons ( $q = -e$ ) is

$$\delta f_{\mathbf{k}}^{(\text{res})} \approx \underbrace{\frac{e}{m_e} c_i e^{p_i t}}_{=\varphi_{\mathbf{k}}} \frac{1 - e^{-i(\mathbf{k} \cdot \mathbf{v} - ip_i)t}}{\mathbf{k} \cdot \mathbf{v} - ip_i} \frac{2\mathbf{k} \cdot \mathbf{v}}{v_{\text{the}}^2} f_0 \approx \frac{e\varphi_{\mathbf{k}}}{T} \mathbf{k} \cdot \mathbf{v} \underbrace{\frac{1 - e^{-i(\mathbf{k} \cdot \mathbf{v} - \omega_{pe})t}}{\mathbf{k} \cdot \mathbf{v} - \omega_{pe}}}_{\approx i\pi\delta(\mathbf{k} \cdot \mathbf{v} - \omega_{pe})} f_0. \quad (5.41)$$

We are going to have to compute  $|\delta f_{\mathbf{k}}|^2$  and squaring delta functions is a dangerous game belonging to the class of games that one must play *veery carefully*.<sup>41</sup> Here is how:

$$\frac{\partial}{\partial t} \left| \frac{1 - e^{-ixt}}{x} \right|^2 = \frac{\partial}{\partial t} \frac{4}{x^2} \sin^2 \frac{xt}{2} = \frac{2 \sin xt}{x} \xrightarrow[t \rightarrow \infty]{} 2\pi\delta(x) \Rightarrow \left| \frac{1 - e^{-ixt}}{x} \right|^2 \xrightarrow[t \rightarrow \infty]{} 2\pi t\delta(x). \quad (5.42)$$

Using this prescription,

$$\begin{aligned} \int d\mathbf{v} \frac{T|\delta f_{\mathbf{k}}^{(\text{res})}|^2}{2f_0} &= \int d\mathbf{v} \frac{e^2 |\varphi_{\mathbf{k}}|^2}{2T} (\mathbf{k} \cdot \mathbf{v})^2 2\pi t \delta(\mathbf{k} \cdot \mathbf{v} - \omega_{pe}) f_0 \\ &= t |\varphi_{\mathbf{k}}|^2 \frac{2\pi e^2 \omega_{pe}^2}{m_e v_{\text{the}}^2 k} F\left(\frac{\omega_{pe}}{k}\right) \\ &= 4t \frac{k^2 |\varphi_{\mathbf{k}}|^2}{8\pi} \underbrace{\frac{\omega_{pe}^4}{k^3} \frac{\pi}{n_e v_{\text{the}}^2}}_{=-\gamma_{\mathbf{k}} > 0; \text{ see (3.41)}} F\left(\frac{\omega_{pe}}{k}\right) \\ &= -4\gamma_{\mathbf{k}} t \frac{|\mathbf{E}_{\mathbf{k}}|^2}{8\pi}. \end{aligned} \quad (5.43)$$

Thus, the entropic part of the free energy grows secularly with time (assuming still  $\gamma_{\mathbf{k}} t \ll 1$ ). Its time derivative is

$$\frac{d}{dt} \int d\mathbf{v} \frac{T|\delta f_{\mathbf{k}}^{(\text{res})}|^2}{2f_0} \approx -4\gamma_{\mathbf{k}} \frac{|\mathbf{E}_{\mathbf{k}}|^2}{8\pi} = -2 \frac{d}{dt} \frac{|\mathbf{E}_{\mathbf{k}}|^2}{8\pi}. \quad (5.44)$$

Despite what it looks like, the extra factor of 2 in (5.44) compared to (5.40) is a feature, not a bug. If you have done Exercise 3.1 (or even just paid attention in §2.1), you know that a Langmuir oscillations involve some mean (oscillating) flows of the plasma, and so a sloshing of energy between potential,  $|\mathbf{E}_{\mathbf{k}}|^2/8\pi$ , and kinetic,  $n_e m_e |\mathbf{u}_{\mathbf{k}}|^2/2$ , where  $\mathbf{u}_{\mathbf{k}} = (1/n_e) \int d\mathbf{v} \mathbf{v} \delta f_{\mathbf{k}}$ . These flows are contained in the non-resonant (“thermal”) part of  $\delta f_{\mathbf{k}}$ , i.e., in  $\delta f_{\mathbf{k}}$  at velocities such that  $\mathbf{k} \cdot \mathbf{v} \ll \omega_{pe}$ . In this region of velocity space, let us write the distribution function (5.34) as follows:

$$\delta f_{\mathbf{k}}^{(\text{th})} = \underbrace{\frac{e}{m_e} c_i e^{p_i t}}_{=\varphi_{\mathbf{k}}} \frac{1}{\mathbf{k} \cdot \mathbf{v} - ip_i} \frac{2\mathbf{k} \cdot \mathbf{v}}{v_{\text{the}}^2} f_0 + e^{-i\mathbf{k} \cdot \mathbf{v}t} h_{\mathbf{k}} \approx -\frac{e\varphi_{\mathbf{k}}}{T} \frac{\mathbf{k} \cdot \mathbf{v}}{\omega_{pe}} f_0 + e^{-i\mathbf{k} \cdot \mathbf{v}t} h_{\mathbf{k}}, \quad (5.45)$$

<sup>41</sup>I am grateful to Glenn Wagner for making me practice what I preach and work out this derivation correctly, with all the meaningful factors of 2.

where  $h_{\mathbf{k}}$  denotes everything in (5.34) that multiplies  $e^{-i\mathbf{k}\cdot\mathbf{v}t}$ . This should remind you of Exercise 5.2. The first term in (5.45) is precisely the plasma flow:

$$\mathbf{u}_{\mathbf{k}} = \frac{1}{n_e} \int d\mathbf{v} \mathbf{v} \delta f_{\mathbf{k}}^{(\text{th})} = -\frac{e\varphi_{\mathbf{k}}}{T\omega_{\text{pe}} n_e} \mathbf{k} \cdot \underbrace{\int d\mathbf{v} \mathbf{v} \mathbf{v} f_0}_{= \ln_e v_{\text{the}}^2 / 2} = \frac{e\mathbf{E}_{\mathbf{k}}}{i\omega_{\text{pe}} m_e} \quad (5.46)$$

The contribution from the second term in (5.45) to the velocity integral has vanished in the limit  $\mathbf{k} \cdot \mathbf{v}t \gg 1$ . Note that (5.46) just says that  $m_e \dot{\mathbf{u}}_{\mathbf{k}} = -e\mathbf{E}_{\mathbf{k}}$ , as indeed is the case in a Langmuir oscillation. It is not hard to check that  $n_e m_e |\mathbf{u}_{\mathbf{k}}|^2 / 2 = |\mathbf{E}_{\mathbf{k}}|^2 / 8\pi$ .

Let us now work out the contribution of (5.45) to the free energy [cf. (5.23)]:

$$\begin{aligned} \int d\mathbf{v} \frac{T|\delta f_{\mathbf{k}}^{(\text{th})}|^2}{2f_0} &= \int d\mathbf{v} \left[ \frac{e^2 |\varphi_{\mathbf{k}}|^2}{2T\omega_{\text{pe}}^2} (\mathbf{k} \cdot \mathbf{v})^2 f_0 + \frac{T|h_{\mathbf{k}}|^2}{2f_0} + e^{-i\mathbf{k}\cdot\mathbf{v}t} (\dots) + e^{i\mathbf{k}\cdot\mathbf{v}t} (\dots)^* \right] \\ &= \frac{|\mathbf{E}_{\mathbf{k}}|^2}{8\pi} + \int d\mathbf{v} \frac{T|h_{\mathbf{k}}|^2}{2f_0}, \end{aligned} \quad (5.47)$$

where the velocity integral has been done in the same way as in (5.46) and the contribution from the terms that oscillate in  $\mathbf{v}$  has been integrated away. The salient property of  $h_{\mathbf{k}}$  is that it does not depend on time. Therefore, its contribution to the time derivative of the free energy vanishes and we get

$$\frac{d}{dt} \int d\mathbf{v} \frac{T|\delta f_{\mathbf{k}}^{(\text{th})}|^2}{2f_0} = \frac{d}{dt} \frac{|\mathbf{E}_{\mathbf{k}}|^2}{8\pi}, \quad (5.48)$$

i.e., the kinetic energy of the Langmuir oscillations decays at the same rate as their potential (electric) energy.

Finally, adding (5.44) and (5.48), we get (5.40), q.e.d.

**Exercise 5.3. Free-energy conservation for sound waves.** Consider an ion-acoustic wave (§3.8) damped on electrons according to (3.75) (with  $u_e = 0$ ). Work out the contributions to free energy from ions and from electrons. Check that free energy is conserved.

## 6. Quasilinear Theory

### 6.1. General Scheme of QLT

In §§3 and 5, I discussed at length the structure of the linear solution corresponding to a Landau-damped initial perturbation. This could be adequately done for a Maxwellian plasma and the result was that, after some interesting transient time-dependent phase-space dynamics, perturbations damped away and their energy turned into heat, increasing somewhat the temperature of the equilibrium (see, however, Q9).

Let us now turn to a different problem: an *unstable* (and so decidedly non-Maxwellian) equilibrium distribution giving rise to exponentially growing perturbations. The specific example on which we shall focus is the *bump-on-tail instability*, which involves generation of unstable Langmuir waves with phase velocities corresponding to instances of positive derivative of the equilibrium distribution function (Fig. 25). The energy of the waves grows exponentially:

$$\frac{\partial |\mathbf{E}_{\mathbf{k}}|^2}{\partial t} = 2\gamma_{\mathbf{k}} |\mathbf{E}_{\mathbf{k}}|^2, \quad \gamma_{\mathbf{k}} = \frac{\pi}{2} \frac{\omega_{\text{pe}}^3}{k^2} \frac{1}{n_e} F' \left( \frac{\omega_{\text{pe}}}{k} \right), \quad (6.1)$$

where  $F(v_z) = \int d\mathbf{v}_x \int d\mathbf{v}_y f_0(\mathbf{v})$  [see (3.41)]. In the absence of collisions, the only way for the system to achieve a nontrivial steady state (i.e., such that  $|\mathbf{E}_{\mathbf{k}}|^2$  is not just zero everywhere) is by adjusting the equilibrium distribution so that

$$\gamma_{\mathbf{k}} = 0 \quad \Leftrightarrow \quad F' \left( \frac{\omega_{\text{pe}}}{k} \right) = 0 \quad (6.2)$$

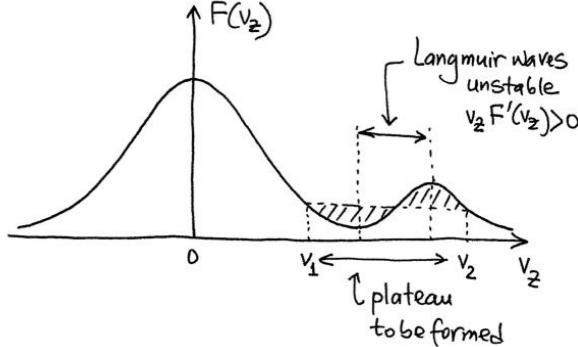


FIGURE 25. An unstable distribution with a bump on its tail.

at all  $k$  where  $|\mathbf{E}_k|^2 \neq 0$ , say,  $k \in [k_2, k_1]$ . If we translate this range into velocities,  $v = \omega_{pe}/k$ , we see that the equilibrium must develop a flat spot:

$$F'(v) = 0 \quad \text{for } v \in [v_1, v_2] = \left[ \frac{\omega_{pe}}{k_1}, \frac{\omega_{pe}}{k_2} \right]. \quad (6.3)$$

This is called a *quasilinear plateau* (§6.4). Obviously, the rest of the equilibrium distribution may (and will) also be modified in some, to be determined, way (§§6.6, 6.7).

These modifications of the original (initial) equilibrium distribution can be accomplished by the growing fluctuations via the feedback mechanism already discussed in §2.3, namely, the equilibrium distribution will evolve slowly according to (2.11):

$$\frac{\partial f_0}{\partial t} = -\frac{q}{m} \sum_{\mathbf{k}} \left\langle \varphi_{\mathbf{k}}^* i\mathbf{k} \cdot \frac{\partial \delta f_{\mathbf{k}}}{\partial \mathbf{v}} \right\rangle. \quad (6.4)$$

The time averaging here [see (2.7)] is over  $\omega_{pe}^{-1} \ll \Delta t \ll \gamma_k^{-1}$ .

The general scheme of QLT is:

- start with an unstable equilibrium  $f_0$ ,
- use the linearised equations (3.1) and (3.2) to work out the linear solution for the growing perturbations  $\varphi_{\mathbf{k}}$  and  $\delta f_{\mathbf{k}}$  in terms of  $f_0$ ,
- use this solution in (6.4) to evolve  $f_0$ , leading, if everything works as it is supposed to, to an ever less unstable equilibrium.

We shall keep only the fastest growing mode (all others are exponentially small after a while), and so the solution (3.16) for the electric perturbations is

$$\varphi_{\mathbf{k}} = c_{\mathbf{k}} e^{(-i\omega_{\mathbf{k}} + \gamma_{\mathbf{k}})t}. \quad (6.5)$$

In the solution (5.25) for the perturbed distribution function, we may ignore the ballistic term because the exponentially growing piece (the first term) will eventually leave all this velocity-space structure behind,<sup>42</sup> so

$$\delta f_{\mathbf{k}} = i \frac{q}{m} \frac{c_{\mathbf{k}} e^{(-i\omega_{\mathbf{k}} + \gamma_{\mathbf{k}})t}}{-i\omega_{\mathbf{k}} + \gamma_{\mathbf{k}} + i\mathbf{k} \cdot \mathbf{v}} \mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{v}} = \frac{q}{m} \frac{\varphi_{\mathbf{k}}}{\mathbf{k} \cdot \mathbf{v} - \omega_{\mathbf{k}} - i\gamma_{\mathbf{k}}} \mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{v}}. \quad (6.6)$$

<sup>42</sup>See, however, Q9 on how to avoid having to wait for this to happen: in fact, the results below are valid for  $\gamma_k t \lesssim 1$  as well.

Substituting (6.6) into (6.4), we get

$$\frac{\partial f_0}{\partial t} = -\frac{q^2}{m^2} \sum_{\mathbf{k}} |\varphi_{\mathbf{k}}|^2 i\mathbf{k} \cdot \frac{\partial}{\partial \mathbf{v}} \frac{1}{\mathbf{k} \cdot \mathbf{v} - \omega_{\mathbf{k}} - i\gamma_{\mathbf{k}}} \mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{D}(\mathbf{v}) \cdot \frac{\partial f_0}{\partial \mathbf{v}}. \quad (6.7)$$

This is a diffusion equation in velocity space, with a velocity-dependent diffusion matrix

$$\begin{aligned} \mathbf{D}(\mathbf{v}) &= -\frac{q^2}{m^2} \sum_{\mathbf{k}} i\mathbf{k}\mathbf{k} |\varphi_{\mathbf{k}}|^2 \frac{1}{\mathbf{k} \cdot \mathbf{v} - \omega_{\mathbf{k}} - i\gamma_{\mathbf{k}}} \\ &= -\frac{q^2}{m^2} \sum_{\mathbf{k}} i\mathbf{k}\mathbf{k} |\varphi_{\mathbf{k}}|^2 \frac{1}{2} \left( \frac{1}{\mathbf{k} \cdot \mathbf{v} - \omega_{\mathbf{k}} - i\gamma_{\mathbf{k}}} + \underbrace{\frac{1}{-\mathbf{k} \cdot \mathbf{v} - \omega_{-\mathbf{k}} - i\gamma_{-\mathbf{k}}}}_{\text{here I changed variables } \mathbf{k} \rightarrow -\mathbf{k}} \right) \\ &= -\frac{q^2}{m^2} \sum_{\mathbf{k}} \frac{\mathbf{k}\mathbf{k}}{k^2} |\mathbf{E}_{\mathbf{k}}|^2 \frac{i}{2} \left( \frac{1}{\mathbf{k} \cdot \mathbf{v} - \omega_{\mathbf{k}} - i\gamma_{\mathbf{k}}} - \frac{1}{\mathbf{k} \cdot \mathbf{v} - \omega_{\mathbf{k}} + i\gamma_{\mathbf{k}}} \right) \\ &= \frac{q^2}{m^2} \sum_{\mathbf{k}} \frac{\mathbf{k}\mathbf{k}}{k^2} |\mathbf{E}_{\mathbf{k}}|^2 \operatorname{Im} \frac{1}{\mathbf{k} \cdot \mathbf{v} - \omega_{\mathbf{k}} - i\gamma_{\mathbf{k}}} \\ &= \frac{q^2}{m^2} \sum_{\mathbf{k}} \frac{\mathbf{k}\mathbf{k}}{k^2} |\mathbf{E}_{\mathbf{k}}|^2 \frac{\gamma_{\mathbf{k}}}{(\mathbf{k} \cdot \mathbf{v} - \omega_{\mathbf{k}})^2 + \gamma_{\mathbf{k}}^2}. \end{aligned} \quad (6.8)$$

To obtain these expressions, I used the fact that the wave-number sum could just as well be over  $-\mathbf{k}$  instead of  $\mathbf{k}$  and that  $\omega_{-\mathbf{k}} = -\omega_{\mathbf{k}}$ ,  $\gamma_{-\mathbf{k}} = \gamma_{\mathbf{k}}$  [because  $\varphi_{-\mathbf{k}} = \varphi_{\mathbf{k}}^*$ , where  $\varphi_{\mathbf{k}}$  is given by (6.5)]. The matrix  $\mathbf{D}$  is manifestly positive definite—this adds credence to our *a priori* expectation that a plateau will form: diffusion will smooth the bump in the equilibrium distribution function.

The question of validity of the QL approximation is quite nontrivial and rife with subtle issues, all of which I have swept under the carpet. They mostly have to do with whether coupling between waves [the last term in (2.12)] truly remains unimportant throughout the quasilinear evolution, especially as the plateau regime is approached and the growth rate of the waves becomes infinitesimally small. If you wish to investigate further—and in the process gain a finer appreciation of nonlinear plasma theory,—the article by Besse et al. (2011) (as far as I know, the most recent substantial contribution to the topic) is a good starting point, from which you can follow the paper trail backwards in time and decide for yourself whether you trust the QLT.

## 6.2. Conservation Laws

When we get to the stage of solving a specific problem (§6.3), we shall see that paying attention to energy and momentum budgets leads one to important discoveries about the QL evolution of the particle distribution. With this prospect in mind, as well as by way of a consistency check, let us check that the quasilinear kinetic equation (6.7) conserves energy and momentum.

### 6.2.1. Energy Conservation

The rate of change of the particle energy associated with the equilibrium distribution is

$$\begin{aligned}
\frac{d\mathcal{K}}{dt} &\equiv \frac{d}{dt} \sum_{\alpha} \iint d\mathbf{r} d\mathbf{v} \frac{m_{\alpha} v^2}{2} f_{0\alpha} = V \sum_{\alpha} \int d\mathbf{v} \frac{m_{\alpha} v^2}{2} \frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{D}_{\alpha}(\mathbf{v}) \cdot \frac{\partial f_{0\alpha}}{\partial \mathbf{v}} \\
&= -V \sum_{\alpha} \int d\mathbf{v} m_{\alpha} \mathbf{v} \cdot \mathbf{D}_{\alpha}(\mathbf{v}) \cdot \frac{\partial f_{0\alpha}}{\partial \mathbf{v}} \\
&= -V \sum_{\alpha} \frac{q_{\alpha}^2}{m_{\alpha}} \sum_{\mathbf{k}} \frac{|\mathbf{E}_{\mathbf{k}}|^2}{k^2} \int d\mathbf{v} \underbrace{\text{Im} \frac{\mathbf{k} \cdot \mathbf{v}}{\mathbf{k} \cdot \mathbf{v} - \omega_{\mathbf{k}} - i\gamma_{\mathbf{k}}} \mathbf{k} \cdot \frac{\partial f_{0\alpha}}{\partial \mathbf{v}}}_{\substack{\text{add and subtract} \\ \omega_{\mathbf{k}} + i\gamma_{\mathbf{k}} \text{ in the} \\ \text{numerator}}} \\
&= -V \sum_{\mathbf{k}} \frac{|\mathbf{E}_{\mathbf{k}}|^2}{4\pi} \underbrace{\text{Im} \left[ (\omega_{\mathbf{k}} + i\gamma_{\mathbf{k}}) \sum_{\alpha} \frac{\omega_{\text{p}\alpha}^2}{k^2} \frac{1}{n_{\alpha}} \int d\mathbf{v} \frac{1}{\mathbf{k} \cdot \mathbf{v} - \omega_{\mathbf{k}} - i\gamma_{\mathbf{k}}} \mathbf{k} \cdot \frac{\partial f_{0\alpha}}{\partial \mathbf{v}} \right]}_{\substack{= 1 - \epsilon(-i\omega_{\mathbf{k}} + \gamma_{\mathbf{k}}, \mathbf{k}) = 1 \\ \text{because } -i\omega_{\mathbf{k}} + i\gamma_{\mathbf{k}} \text{ is a solution of} \\ \text{dispersion relation } \epsilon = 0}} \\
&= -V \sum_{\mathbf{k}} 2\gamma_{\mathbf{k}} \frac{|\mathbf{E}_{\mathbf{k}}|^2}{8\pi} = -\frac{d}{dt} \int d\mathbf{r} \frac{E^2}{8\pi}, \quad \text{q.e.d.,}
\end{aligned} \tag{6.9}$$

viz., the total energy  $\mathcal{K} + \int d\mathbf{r} E^2 / 8\pi = \text{const.}$  This will motivate §6.6.

### 6.2.2. Momentum Conservation

Since unstable distributions like the one with a bump on its tail can carry net momentum, it is useful to calculate its rate of change:

$$\begin{aligned}
\frac{d\mathcal{P}}{dt} &\equiv \frac{d}{dt} \sum_{\alpha} \iint d\mathbf{r} d\mathbf{v} m_{\alpha} \mathbf{v} f_{0\alpha} = V \sum_{\alpha} \int d\mathbf{v} m_{\alpha} \mathbf{v} \frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{D}_{\alpha}(\mathbf{v}) \cdot \frac{\partial f_{0\alpha}}{\partial \mathbf{v}} \\
&= -V \sum_{\alpha} \int d\mathbf{v} m_{\alpha} \mathbf{D}_{\alpha}(\mathbf{v}) \cdot \frac{\partial f_{0\alpha}}{\partial \mathbf{v}} \\
&= -V \sum_{\alpha} \frac{q_{\alpha}^2}{m_{\alpha}} \sum_{\mathbf{k}} \frac{\mathbf{k} |\mathbf{E}_{\mathbf{k}}|^2}{k^2} \int d\mathbf{v} \text{Im} \frac{1}{\mathbf{k} \cdot \mathbf{v} - \omega_{\mathbf{k}} - i\gamma_{\mathbf{k}}} \mathbf{k} \cdot \frac{\partial f_{0\alpha}}{\partial \mathbf{v}} \\
&= -V \sum_{\mathbf{k}} \frac{\mathbf{k} |\mathbf{E}_{\mathbf{k}}|^2}{4\pi} \underbrace{\text{Im} \sum_{\alpha} \frac{\omega_{\text{p}\alpha}^2}{k^2} \frac{1}{n_{\alpha}} \int d\mathbf{v} \frac{1}{\mathbf{k} \cdot \mathbf{v} - \omega_{\mathbf{k}} - i\gamma_{\mathbf{k}}} \mathbf{k} \cdot \frac{\partial f_{0\alpha}}{\partial \mathbf{v}}}_{\substack{= 1 - \epsilon(-i\omega_{\mathbf{k}} + \gamma_{\mathbf{k}}, \mathbf{k}) = 1}} = 0, \quad \text{q.e.d.,}
\end{aligned} \tag{6.10}$$

so momentum can only be redistributed between particles. This will motivate §6.7.

---

## 6.3. Quasilinear Equations for the Bump-on-Tail Instability in 1D

What follows is the iconic QL calculation due to [Vedenov et al. \(1962\)](#) and [Drummond & Pines \(1962\)](#).

These two papers, published in the same year, are a spectacular example of the “great minds think alike” principle. They both appeared in the Proceedings of the 1961 IAEA conference in Salzburg, one of those early international gatherings in which the Soviets (grudgingly allowed out) and the Westerners (eager to meet them) were telling each other about their achievements in the recently declassified controlled-nuclear-fusion research. The entire Proceedings are now online (<http://www-naweb.iaea.org/napc/physics/FEC/1961.pdf>)—a remarkable historical document and a great read, containing, besides the papers (in three languages), a record of the discussions that were held. The [Vedenov et al. \(1962\)](#) paper is in Russian, but you will

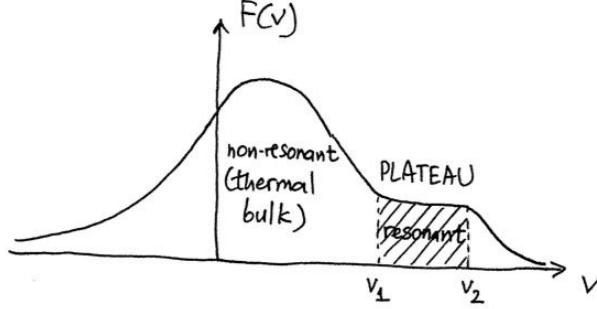


FIGURE 26. Quasilinear plateau.

find a very similar exposition in English in the review by [Vedenov \(1963\)](#) published shortly thereafter. Two other lucid accounts of quasilinear theory belonging to the same historical (and historic!) period are in the books by [Kadomtsev \(1965\)](#) and by [Sagdeev & Galeev \(1969\)](#).

As promised in §6.1, I shall consider electron Langmuir oscillations in 1D, triggered by the bump-on-tail instability, so  $\mathbf{k} = k\hat{z}$ ,  $\omega_k = \omega_{pe}$ ,  $\gamma_k$  is given by (6.1), and the QL diffusion equation (6.7) becomes

$$\boxed{\frac{\partial F}{\partial t} = \frac{\partial}{\partial v} D(v) \frac{\partial F}{\partial v}}, \quad (6.11)$$

where  $F(v)$  is the 1D version of the distribution function,  $v = v_z$  and the diffusion coefficient, now a scalar, is given by

$$\boxed{D(v) = \frac{e^2}{m_e^2} \sum_k |E_k|^2 \text{Im} \frac{1}{kv - \omega_{pe} - i\gamma_k}}. \quad (6.12)$$

As I explained when discussing (6.1), if the fluctuation field has reached a steady state, it must be the case that

$$\frac{\partial |E_k|^2}{\partial t} = 2\gamma_k |E_k|^2 = 0 \quad \Leftrightarrow \quad |E_k|^2 = 0 \quad \text{or} \quad \gamma_k = 0, \quad (6.13)$$

i.e., either there are no fluctuations or there is no growth (or damping) rate. The result is a non-zero spectrum of fluctuations in the interval  $k \in [k_2, k_1]$  and a plateau in the distribution function in the corresponding velocity interval  $v \in [v_1, v_2] = [\omega_{pe}/k_1, \omega_{pe}/k_2]$  [see (6.3) and Fig. 26]. The particles in this interval are *resonant* with Langmuir waves; those in the (“thermal”) bulk of the distribution outside this interval are *non-resonant*. We will have solved the problem completely if we find

- $F^{\text{plateau}}$ , the value of the distribution function in the interval  $[v_1, v_2]$ ,
- the extent of the plateau  $[v_1, v_2]$ ,
- the functional form of the spectrum  $|E_k|^2$  in the interval  $[k_2, k_1]$ ,
- any modifications of the distribution function  $F(v)$  of the nonresonant particles.

#### 6.4. Resonant Region: QL Plateau and Spectrum

Consider first the velocities  $v \in [v_1, v_2]$  for which  $|E_{k=\omega_{pe}/v}|^2 \neq 0$ . If  $L$  is the linear size of the system, the wave-number sum in (6.12) can be replaced by an integral according to

$$\sum_k = \sum_k \frac{\Delta k}{2\pi/L} = \frac{L}{2\pi} \int dk. \quad (6.14)$$

Defining the continuous energy spectrum of the Langmuir waves<sup>43</sup>

$$W(k) = \frac{L}{2\pi} \frac{|E_k|^2}{4\pi}, \quad (6.15)$$

we rewrite the QL diffusion coefficient (6.12) in the following form:

$$D(v) = \frac{e^2}{m_e^2} \frac{1}{v} \operatorname{Im} \int dk \frac{4\pi W(k)}{k - \omega_{pe}/v - i\gamma_k/v} = \frac{e^2}{m_e^2} \frac{4\pi^2}{v} W\left(\frac{\omega_{pe}}{v}\right). \quad (6.16)$$

The last expression is obtained by applying Plemelj's formula (3.25) to the wave-number integral taken in the limit  $\gamma_k/v \rightarrow +0$ .<sup>44</sup> Substituting now this expression into (6.11) and using also (6.1) to express

$$\gamma_k = \frac{\pi}{2} \frac{\omega_{pe}^3}{k^2} \frac{1}{n_e} F'\left(\frac{\omega_{pe}}{k}\right) \Rightarrow \frac{\partial F}{\partial v} = \left[ \frac{2}{\pi} \frac{k^2}{\omega_{pe}^3} n_e \gamma_k \right]_{k=\omega_{pe}/v}, \quad (6.17)$$

we get

$$\frac{\partial F}{\partial t} = \frac{\partial}{\partial v} \frac{e^2}{m_e^2} \frac{4\pi^2}{v} W\left(\frac{\omega_{pe}}{v}\right) \left[ \frac{2}{\pi} \frac{k^2}{\omega_{pe}^3} n_e \gamma_k \right]_{k=\omega_{pe}/v} = \frac{\partial}{\partial v} \frac{\omega_{pe}}{m_e v^3} \underbrace{2\gamma_{\omega_{pe}/v} W\left(\frac{\omega_{pe}}{v}\right)}_{= \partial W/\partial t}. \quad (6.18)$$

Rearranging, we arrive at

$$\frac{\partial}{\partial t} \left[ F - \frac{\partial}{\partial v} \frac{\omega_{pe}}{m_e v^3} W\left(\frac{\omega_{pe}}{v}\right) \right] = 0. \quad (6.19)$$

Thus, during QL evolution, the expression in the square brackets stays constant in time. Since at  $t = 0$ , there are no waves,  $W = 0$ , we find

$$F(0, v) + \frac{\partial}{\partial v} \frac{\omega_{pe}}{m_e v^3} W\left(t, \frac{\omega_{pe}}{v}\right) = F(t, v) \rightarrow F^{\text{plateau}} \quad \text{as } t \rightarrow \infty. \quad (6.20)$$

In the saturated state ( $t \rightarrow \infty$ ),  $W(\omega_{pe}/v) = 0$  outside the interval  $v \in [v_1, v_2]$ . Therefore, (6.20) gives us two implicit equations for  $v_1$  and  $v_2$ :

$$F(0, v_1) = F(0, v_2) = F^{\text{plateau}} \quad (6.21)$$

and, after integration over velocities, also an equation for  $F^{\text{plateau}}$ :<sup>45</sup>

$$\int_{v_1}^{v_2} dv [F^{\text{plateau}} - F(0, v)] = 0 \Rightarrow F^{\text{plateau}} = \frac{1}{v_2 - v_1} \int_{v_1}^{v_2} dv F(0, v). \quad (6.22)$$

Finally, integrating (6.20) with respect to  $v$  and using the boundary condition  $W(\omega_{pe}/v_1) = 0$ , we get, at  $t \rightarrow \infty$ ,

$$W\left(\frac{\omega_{pe}}{v}\right) = \frac{m_e v^3}{\omega_{pe}} \int_{v_1}^v dv' [F^{\text{plateau}} - F(0, v')]. \quad (6.23)$$

<sup>43</sup>Why the prefactor is  $1/4\pi$ , rather than  $1/8\pi$ , will become clear at the end of §6.5.

<sup>44</sup>In fact, the wave-number integral must be taken along the Landau contour (i.e., keeping the contour below the pole) regardless of the sign of  $\gamma_k$ : see Q9, where you get to work out the QL theory for Landau-damped, rather than growing, perturbations.

<sup>45</sup>This is somewhat reminiscent of the “Maxwell construction” in thermodynamics of real gases: the plateau sits at such a level that the integral under it, i.e., the number of particles involved, stays the same as it was for the same velocities in the initial state; see Fig. 25.

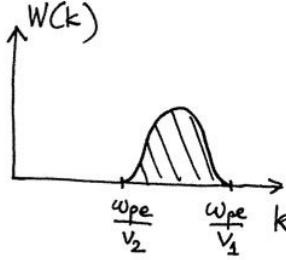


FIGURE 27. Quasilinear spectrum.

Hence the spectrum is

$$W(k) = \frac{m_e \omega_{pe}^2}{k^3} \int_{v_1}^{\omega_{pe}/k} dv [F^{\text{plateau}} - F(0, v)] \quad \text{for } k \in \left[ \frac{\omega_{pe}}{v_2}, \frac{\omega_{pe}}{v_1} \right] \quad (6.24)$$

and  $W(k) = 0$  everywhere else (Fig. 27).

Thus, we have completed the first three items of the programme formulated at the end of §6.3. What about the particle distribution outside the resonant region? How is it modified by the quasilinear evolution? Is it modified at all? The following calculation shows that it must be.

### 6.5. Energy of Resonant Particles

Since feeding the instability requires extracting energy from the resonant particles, their energy must change. We calculate this change by taking the  $m_e v^2/2$  moment of (6.20):

$$\begin{aligned} \mathcal{H}_{\text{res}}(\infty) - \mathcal{H}_{\text{res}}(0) &= \int_{v_1}^{v_2} dv \frac{m_e v^2}{2} [F^{\text{plateau}} - F(0, v)] \\ &= \int_{v_1}^{v_2} dv \frac{m_e v^2}{2} \frac{\partial}{\partial v} \frac{\omega_{pe}}{m_e v^3} W\left(\frac{\omega_{pe}}{v}\right) \\ &= -\omega_{pe} \int_{v_1}^{v_2} dv \frac{1}{v^2} W\left(\frac{\omega_{pe}}{v}\right) \\ &= - \int_{\omega_{pe}/v_2}^{\omega_{pe}/v_1} dk W(k) = -2 \sum_k \frac{|E_k|^2}{8\pi} \equiv -2 \mathcal{E}(\infty). \end{aligned} \quad (6.25)$$

Thus, only half of the energy lost by the resonant particles goes into the electric-field energy of the waves,

$$\mathcal{E}(\infty) = \frac{\mathcal{H}_{\text{res}}(0) - \mathcal{H}_{\text{res}}(\infty)}{2}. \quad (6.26)$$

Since the energy must be conserved overall [see (6.9)], we must account for the missing half: this is easy to do physically, as, obviously, the electric energy of the waves is their potential energy, which is half of their total energy—the other half being the kinetic energy of the oscillatory plasma motions associated with the wave (in §5.7, this was worked out explicitly). These oscillations are enabled by the non-resonant, “thermal-bulk” particles, and so we must be able to show that, as a result of QL evolution, these particles pick up the total of  $\mathcal{E}(\infty)$  of energy—one might say that the plasma is *heated*.

### 6.6. Heating of Non-Resonant Particles

Consider the thermal bulk of the distribution,  $v \ll v_1$  (assuming that the bump is indeed far out in the tail of the distribution). The QL diffusion coefficient (6.12) becomes, assuming now  $\gamma_k, kv \ll \omega_{pe}$  and using the last expression in Eq. (6.8),

$$\begin{aligned} D(v) &= \frac{e^2}{m_e^2} \sum_k |E_k|^2 \frac{\gamma_k}{(kv - \omega_{pe})^2 + \gamma_k^2} \approx \frac{e^2}{m_e^2} \sum_k |E_k|^2 \frac{\gamma_k}{\omega_{pe}^2} \\ &= \frac{e^2}{m_e^2 \omega_{pe}^2} \sum_k \frac{1}{2} \frac{\partial |E_k|^2}{\partial t} = \frac{4\pi e^2}{m_e^2 \omega_{pe}^2} \frac{d}{dt} \sum_k \frac{|E_k|^2}{8\pi} = \frac{1}{m_e n_e} \frac{d\mathcal{E}}{dt}, \end{aligned} \quad (6.27)$$

independent of  $v$ . The QL evolution equation (6.11) for the bulk distribution is then<sup>46</sup>

$$\frac{\partial F}{\partial t} = \frac{1}{m_e n_e} \frac{d\mathcal{E}}{dt} \frac{\partial^2 F}{\partial v^2}. \quad (6.28)$$

Equation (6.28) describes slow diffusion of the bulk distribution, i.e., as the wave field grows, the bulk distribution gets a little broader (which is what heating is). Namely, the “thermal” energy satisfies

$$\frac{d\mathcal{K}_{th}}{dt} = \frac{d}{dt} \int dv \frac{m_e v^2}{2} F = \frac{1}{m_e n_e} \frac{d\mathcal{E}}{dt} \underbrace{\int dv \frac{m_e v^2}{2} \frac{\partial^2 F}{\partial v^2}}_{= m_e n_e} = \frac{d\mathcal{E}}{dt}. \quad (6.29)$$

(by parts twice)

Integrating this with respect to time, we find that the missing half of the energy lost by the resonant particles indeed goes into the thermal bulk:

$$\mathcal{K}_{th}(\infty) - \mathcal{K}_{th}(0) = \mathcal{E}(\infty) = \frac{\mathcal{K}_{res}(0) - \mathcal{K}_{res}(\infty)}{2}.$$

(6.30)

Overall, the energy is, of course, conserved:

$$\mathcal{K}_{th}(\infty) + \mathcal{K}_{res}(\infty) + \mathcal{E}(\infty) = \mathcal{K}_{th}(0) + \mathcal{K}_{res}(0), \quad (6.31)$$

as it should be, in accordance with (6.9).

Equation (6.28) can be explicitly solved: changing the time variable to  $\tau = \mathcal{E}(t)/m_e n_e$  turns it into a simple diffusion equation

$$\frac{\partial F}{\partial \tau} = \frac{\partial^2 F}{\partial v^2}. \quad (6.32)$$

If we let the initial distribution be a Maxwellian and ignore the bump on its tail, the solution is

$$\begin{aligned} F(\tau, v) &= \int dv' F(0, v') \frac{e^{-(v-v')^2/4\tau}}{\sqrt{4\pi\tau}} = \int dv' \frac{n_e}{\sqrt{\pi v_{the}^2 4\pi\tau}} \exp \left[ -\frac{v'^2}{v_{the}^2} - \frac{(v-v')^2}{4\tau} \right] \\ &= \frac{n_e}{\sqrt{\pi(v_{the}^2 + 4\tau)}} \exp \left( -\frac{v^2}{v_{the}^2 + 4\tau} \right). \end{aligned} \quad (6.33)$$

Since

$$v_{the}^2 + 4\tau = \frac{2T_e}{m_e} + \frac{4\mathcal{E}(t)}{m_e n_e} = \frac{2}{m_e} \left[ T_e + \frac{2\mathcal{E}(t)}{n_e} \right], \quad (6.34)$$

one concludes that an initially Maxwellian bulk stays Maxwellian but its temperature grows as

<sup>46</sup>Note that this implies  $d/dt \int dv F(v) = 0$ , so the number of these particles is conserved, there is no exchange between the non-resonant and resonant populations.

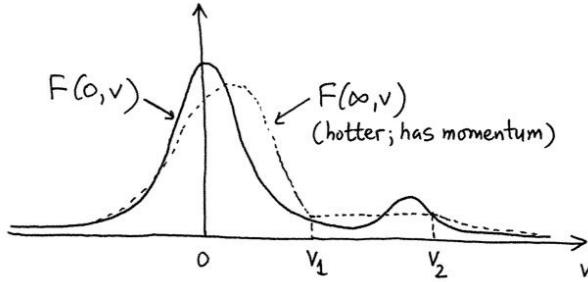


FIGURE 28. The initial distribution and the final outcome of the QL evolution: its bulk hotter and shifted towards the plateau in the tail.

the wave energy grows, reaching in saturation

$$\boxed{T_e(\infty) = T_e(0) + \frac{2\mathcal{E}(\infty)}{n_e}}. \quad (6.35)$$


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### 6.7. Momentum Conservation

The bump-on-tail configuration is in general asymmetric in  $v$  and so the particles in the bump carry a net mean momentum. Let us find out whether this momentum changes. Taking the  $m_e v$  moment of (6.20), we calculate the total momentum lost by the resonant particles:

$$\begin{aligned} \mathcal{P}_{\text{res}}(\infty) - \mathcal{P}_{\text{res}}(0) &= \int_{v_1}^{v_2} dv m_e v [F^{\text{plateau}} - F(0, v)] \\ &= \int_{v_1}^{v_2} dv m_e v \frac{\partial}{\partial v} \frac{\omega_{pe}}{m_e v^3} W\left(\frac{\omega_{pe}}{v}\right) \\ &= -\omega_{pe} \int_{v_1}^{v_2} dv \frac{1}{v^3} W\left(\frac{\omega_{pe}}{v}\right) \\ &= - \int_{\omega_{pe}/v_2}^{\omega_{pe}/v_1} dk \frac{kW(k)}{\omega_{pe}} < 0. \end{aligned} \quad (6.36)$$

This is negative, so momentum is indeed lost. Since it cannot go into electric field [see (6.10)], it must all get transferred to the thermal particles. Let us confirm this.

Going back to the QL diffusion equation (6.28) for the non-resonant particles, at first glance, we have a problem: the diffusion coefficient is independent of  $v$  and so momentum is conserved. However, one should never take zero for an answer when dealing with asymptotic expansions—indeed, it turns out here that we ought to work to higher order in our calculation of  $D(v)$ . Keeping next-order terms in (6.27), we get

$$\begin{aligned} D(v) &= \frac{e^2}{m_e^2} \sum_k |E_k|^2 \frac{\gamma_k}{(kv - \omega_{pe})^2 + \gamma_k^2} = \frac{e^2}{m_e^2} \sum_k |E_k|^2 \frac{\gamma_k}{\omega_{pe}^2} \left(1 + \frac{2kv}{\omega_{pe}} + \dots\right) \\ &\approx \frac{4\pi e^2}{m_e^2 \omega_{pe}^2} \frac{d}{dt} \left[ \sum_k \frac{|E_k|^2}{8\pi} + v \sum_k \frac{k|E_k|^2}{4\pi \omega_{pe}} \right] = \frac{1}{m_e n_e} \frac{d}{dt} \left[ \mathcal{E} + v \int dk \frac{kW(k)}{\omega_{pe}} \right]. \end{aligned} \quad (6.37)$$

Thus, there is a wave-induced drag term in the QL diffusion equation (6.11), which indeed turns out to impart to the thermal particles the small additional momentum

that, according to (6.36), the resonant particles lose when rearranging themselves to produce the QL plateau:

$$\begin{aligned}\frac{d\mathcal{P}_{\text{th}}}{dt} &= \frac{d}{dt} \int dv m_e v F = \int dv m_e v \frac{\partial}{\partial v} D(v) \frac{\partial F}{\partial v} = -m_e \int dv D(v) \frac{\partial F}{\partial v} \\ &= - \left[ \frac{d}{dt} \int dk \frac{kW(k)}{\omega_{pe}} \right] \frac{1}{n_e} \int dv v \frac{\partial F}{\partial v} = \frac{d}{dt} \int dk \frac{kW(k)}{\omega_{pe}},\end{aligned}\quad (6.38)$$

whence, integrating and comparing with (6.36),

$$\mathcal{P}_{\text{th}}(\infty) - \mathcal{P}_{\text{th}}(0) = \int dk \frac{kW(k)}{\omega_{pe}} = \mathcal{P}_{\text{res}}(0) - \mathcal{P}_{\text{res}}(\infty).$$

(6.39)

This means that *the thermal bulk of the final distribution is not only slightly broader (hotter) than that of the initial one (§6.6), but it is also slightly shifted towards the plateau* (Fig. 28).

In a collisionless plasma, this is the steady state. However, as this steady state is approached,  $\gamma_k \rightarrow 0$ , so the QL evolution becomes ever slower and even a very small collision frequency can become important. Eventually, collisions will erode the plateau (even if with great difficulty) and return the plasma to a global Maxwellian equilibrium—which is the fate of all things.

## Plasma Kinetics Problem Set

**1. Industrialised linear theory with the  $Z$  function.** Consider a two-species plasma close to Maxwellian equilibrium. Rederive all the results obtained in §§3.4, 3.5, 3.8, 3.9, 3.10 starting from (3.84) and using the asymptotic expansions (3.91) and (3.92) of the plasma dispersion function.

Namely, consider the limits  $\zeta_e \gg 1$  or  $\zeta_e \ll 1$  and  $\zeta_i \gg 1$ , find solutions in these limits and establish the conditions on the wave number of the perturbations and on the equilibrium parameters under which these solutions are valid.

In particular, for the case of  $\zeta_e \ll 1$  and  $\zeta_i \gg 1$ , obtain general expressions for the wave frequency and damping without assuming  $k\lambda_{De}$  to be either small or large. Recover from your solution the cases considered in §§3.8, 3.9 and 3.10.

Find also the ion contribution to the damping of the ion acoustic and Langmuir waves and comment on the circumstances in which it might be important to know what it is.

Convince yourself that you believe the sketch of longitudinal plasma waves in Fig. 14. If you feel computationally inclined, solve the plasma dispersion relation (3.84) numerically [using, e.g., (3.90)] and see if you can reproduce Fig. 14.

You may wish to check your results against some textbook: e.g., Krall & Trivelpiece (1973) and Alexandrov *et al.* (1984) give very thorough treatments of the linear theory (although in rather different styles than I did).

**2. Transverse plasma waves.** Go back to the Vlasov–Maxwell, rather than Vlasov–Poisson, system and consider *electromagnetic* perturbations in a Maxwellian unmagnetised plasma (unmagnetised in the sense that in equilibrium,  $\mathbf{B}_0 = 0$ ):

$$\frac{\partial \delta f_\alpha}{\partial t} + i\mathbf{k} \cdot \mathbf{v} \delta f_\alpha + \frac{q_\alpha}{m_\alpha} \left( \mathbf{E} + \frac{\mathbf{v} \times \mathbf{B}}{c} \right) \cdot \frac{\partial f_{0\alpha}}{\partial \mathbf{v}} = 0, \quad (6.40)$$

where  $\mathbf{E}$  and  $\mathbf{B}$  satisfy Maxwell's equations (1.23–1.26) with charge and current densities determined by the perturbed distribution function  $\delta f_\alpha$ .

(a) Consider an initial-value problem for such perturbations and show that the equation for the Laplace transform of  $\mathbf{E}$  can be written in the form<sup>47</sup>

$$\hat{\epsilon}(p, \mathbf{k}) \cdot \hat{\mathbf{E}}(p) = \begin{pmatrix} \text{terms associated with initial} \\ \text{perturbations of } \delta f_\alpha, \mathbf{E} \text{ and } \mathbf{B} \end{pmatrix}, \quad (6.41)$$

where the *dielectric tensor*  $\hat{\epsilon}(p, \mathbf{k})$  is, in tensor notation,

$$\epsilon_{ij}(p, \mathbf{k}) = \left( \delta_{ij} - \frac{k_i k_j}{k^2} \right) \epsilon_{TT}(p, k) + \frac{k_i k_j}{k^2} \epsilon_{LL}(p, k) \quad (6.42)$$

and the *longitudinal dielectric function*  $\epsilon_{LL}(p, k)$  is the familiar electrostatic one, given by (3.84), while the *transverse dielectric function* is

$$\epsilon_{TT}(p, k) = 1 + \frac{1}{p^2} \left[ k^2 c^2 - \sum_\alpha \omega_{p\alpha}^2 \zeta_\alpha Z(\zeta_\alpha) \right]. \quad (6.43)$$

<sup>47</sup>In Q3, dealing with the Weibel instability, you will have to do essentially the same calculation, but with a non-Maxwellian equilibrium. To avoid doing the work twice, you could do that question first and then specialise to a Maxwellian  $f_{0\alpha}$ . However, the algebra is a bit hairier for the non-Maxwellian case, so it may be useful to do the simpler case first, to train your hand—and also to have a way to cross-check the later calculation.

(b) Hence solve the transverse dispersion relation,  $\epsilon_{\text{TT}}(p, k) = 0$ , and show that, in the high-frequency limit ( $|\zeta_e| \gg 1$ ), the resulting waves are simply the light waves, which, at long wave lengths, turn into plasma oscillations, viz.,

$$\omega^2 = k^2 c^2 + \omega_{pe}^2. \quad (6.44)$$

What is the wave length above which light can “feel” that it is propagating through plasma?—this is called the *plasma (electron) skin depth*,  $d_e$ . Are these waves damped?

(c) In the low-frequency limit ( $|\zeta_e| \ll 1$ ), show that perturbations are aperiodic (have zero frequency) and damped. Find their damping rate and show that this result is valid for perturbations with wave lengths longer than the plasma skin depth ( $k d_e \ll 1$ ). Explain physically why these perturbations fail to propagate.

*Do one of Q3, Q4, or Q5 (but do not pick Q5 if you are planning to do Q12).*

**3. Weibel instability.** Weibel (1958) realised that transverse plasma perturbations can go unstable if the equilibrium distribution is anisotropic with respect to some special direction  $\hat{\mathbf{n}}$ , namely if  $f_{0\alpha} = f_{0\alpha}(v_\perp, v_\parallel)$ , where  $v_\parallel = \mathbf{v} \cdot \hat{\mathbf{n}}$ ,  $v_\perp = |\mathbf{v}_\perp|$  and  $\mathbf{v}_\perp = \mathbf{v} - v_\parallel \hat{\mathbf{n}}$ . The anisotropy can be due to some beam or stream of particles injected into the plasma, it also arises in collisionless shocks or, generically, when plasma is sheared or non-isotropically compressed by some external force. The simplest model for an anisotropic distribution of the required type is a *bi-Maxwellian*<sup>48</sup>:

$$f_{0\alpha} = \frac{n_\alpha}{\pi^{3/2} v_{\text{th}\perp\alpha}^2 v_{\text{th}\parallel\alpha}} \exp\left(-\frac{v_\perp^2}{v_{\text{th}\perp\alpha}^2} - \frac{v_\parallel^2}{v_{\text{th}\parallel\alpha}^2}\right), \quad (6.45)$$

where, formally,  $v_{\text{th}\perp\alpha} = \sqrt{2T_{\perp\alpha}/m_\alpha}$  and  $v_{\text{th}\parallel\alpha} = \sqrt{2T_{\parallel\alpha}/m_\alpha}$  are the two “thermal speeds” in a plasma characterised by two effective temperatures  $T_{\perp\alpha}$  and  $T_{\parallel\alpha}$  (for each species).

(a) Using exactly the same method as in Q2, consider electromagnetic perturbations in a bi-Maxwellian plasma, assuming their wave vectors to be parallel to the direction of anisotropy,  $\mathbf{k} \parallel \hat{\mathbf{n}}$ . Show that the dielectric tensor again has the form (6.42) and the longitudinal dielectric function is again given by (3.84), while the transverse dielectric function is

$$\epsilon_{\text{TT}}(p, k) = 1 + \frac{1}{p^2} \left[ k^2 c^2 + \sum_\alpha \omega_{p\alpha}^2 \left( 1 - \frac{T_{\perp\alpha}}{T_{\parallel\alpha}} [1 + \zeta_\alpha Z(\zeta_\alpha)] \right) \right]. \quad (6.46)$$

(b) Show that in one of the tractable asymptotic limits, this dispersion relation has a zero-frequency, purely growing solution with the growth rate

$$\gamma = \frac{kv_{\text{th}\parallel e}}{\sqrt{\pi}} \frac{T_{\parallel e}}{T_{\perp e}} (\Delta_e - k^2 d_e^2), \quad (6.47)$$

where  $\Delta_e = T_{\perp e}/T_{\parallel e} - 1$  is the fractional temperature anisotropy, which must be positive in order for the instability to occur. Find the maximum growth rate and the corresponding

<sup>48</sup>In Exercise 4.8, you need the dielectric tensor in terms of a general equilibrium distribution  $f_{0\alpha}(v_x, v_y, v_z)$ . If you are planning to do that exercise, it may save time (at the price of a very slight increase in algebra) to do the derivation with a general  $f_{0\alpha}$  and then specialise to the bi-Maxwellian (6.45). You can check your algebra by looking up the result in Krall & Trivelpiece (1973) or in Davidson (1983).

wave number. Under what condition(s) is the asymptotic limit in which you worked indeed a valid approximation for this solution?

(c) Are there any other unstable solutions? (cf. [Weibel 1958](#))

(d) What happens if the electrons are isotropic but ions are not?

(e\*\*) If you want a challenge and a test of stamina, work out the case of perturbations whose wave number is *not* necessarily in the direction of the anisotropy ( $\mathbf{k} \times \hat{\mathbf{n}} \neq 0$ ). Are the  $\mathbf{k} \parallel \hat{\mathbf{n}}$  or some oblique perturbations the fastest growing? This is a lot of algebra, so only do it if you enjoy this sort of thing. The dispersion relation for this general case appears to be in the Appendix of [Buyer et al. \(2015\)](#), but they only solve it numerically; no one seems to have looked at asymptotic limits. This could be the start of a dissertation.

**4. Two-stream instability.**<sup>49</sup> Consider one-dimensional, electrostatic perturbations in a two-species (electron-ion) plasma. Let the electron distribution function with respect to velocities in the direction ( $z$ ) of the spatial variation of perturbations be a “double Lorentzian” consisting of two counterpropagating beams with velocity  $u_b$  and width  $v_b$ , viz.,

$$F_e(v_z) = \frac{n_e v_b}{2\pi} \left[ \frac{1}{(v_z - u_b)^2 + v_b^2} + \frac{1}{(v_z + u_b)^2 + v_b^2} \right] \quad (6.48)$$

(see Fig. 12b), while the ions are Maxwellian with thermal speed  $v_{thi} \ll u_b$ . Assume also that the phase velocity ( $p/k$ ) will be of the same order as  $u_b$  and hence that the ion contribution to the dielectric function (3.26) is negligible.

(a) By integrating by parts and then choosing the integration contour judiciously, or otherwise, calculate the dielectric function  $\epsilon(p, k)$  for this plasma and hence show that the dispersion relation is

$$\sigma^4 + (2u_b^2 + v_p^2)\sigma^2 + u_b^2(u_b^2 - v_p^2) = 0, \quad (6.49)$$

where  $\sigma = v_b + p/k$  and  $v_p = \omega_{pe}/k$ .

(b) In the long-wavelength limit, viz.,  $k \ll \omega_{pe}/u_b$ , find the condition for an instability to exist and calculate the growth rate of this instability. Is the nature of this instability kinetic (due to Landau resonance) or hydrodynamic?

(c) Consider the case of cold beams,  $v_b = 0$ . Without making any *a priori* assumptions about  $k$ , calculate the maximum growth rate of the instability. Sketch the growth rate as a function of  $k$ .

(d) Allowing warm beams,  $v_b > 0$ , show that the system is unstable provided

$$u_b > v_b \quad \text{and} \quad k < \omega_{pe} \frac{\sqrt{u_b^2 - v_b^2}}{u_b^2 + v_b^2}. \quad (6.50)$$

What is the effect that a finite beam width has on the stability of the system and on the kind of perturbations that can grow?

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In §4.4, you might find it instructive to compare the results that you have just obtained by solving the dispersion relation (6.49) directly with what can be inferred via Penrose’s criterion and Nyquist’s method.

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<sup>49</sup>This is based on the 2019 exam question.

**5. Buneman (1958) instability.**<sup>50</sup> Consider a plasma consisting of Maxwellian ions and a cold electron beam (velocity  $u_e \gg v_{\text{the}}$ —much greater than the width of the distribution).

(a) Starting from the standard expression for the plasma dielectric function describing infinitesimal electrostatic perturbations with wave vector  $k$  in the direction of the beam and  $\propto e^{pt}$  (where  $p$  is, in general, complex), assume  $|p/k| \gg v_{\text{thi}}$  and show that the dispersion relation is

$$1 + \frac{\omega_{\text{pi}}^2}{p^2} - \frac{\omega_{\text{pe}}^2}{(ku_e - ip)^2} = 0. \quad (6.51)$$

(b) Looking for solutions with  $\omega_{\text{pe}} \gg |p| \gg \omega_{\text{pi}}$ , show that there is an instability that attains its maximum growth rate at  $k = \omega_{\text{pe}}/u_e$ . *Hint.* You can do this either by identifying the dominant balance in the dispersion relation or by looking for a solution in the form  $p = |p|e^{i\theta}$  and maximising the growth rate with respect to  $\theta$ .

(c) Show that the maximum growth rate is

$$\gamma = \frac{\sqrt{3}}{2^{4/3}} \omega_{\text{pe}}^{1/3} \omega_{\text{pi}}^{2/3}. \quad (6.52)$$

Is this instability kinetic or hydrodynamic? (I.e., do Landau resonances with either ions or electrons play a role?)

**6. Free energy and stability.** (a) Starting from the linearised Vlasov–Poisson system and assuming a Maxwellian equilibrium, show by direct calculation from the equations, rather than via expansion of the entropy function and the use of energy conservation (as was done in §5.2), that free energy is conserved:

$$\frac{d}{dt} \int d\mathbf{r} \left[ \sum_{\alpha} \int d\mathbf{v} \frac{T_{\alpha} \delta f_{\alpha}^2}{2f_{0\alpha}} + \frac{|\nabla \varphi|^2}{8\pi} \right] = 0. \quad (6.53)$$

This is an exercise in integrating by parts.

(b) Now consider the full Vlasov–Maxwell equations and prove, again for a Maxwellian plasma plus small perturbations,

$$\frac{d}{dt} \int d\mathbf{r} \left[ \sum_{\alpha} \int d\mathbf{v} \frac{T_{\alpha} \delta f_{\alpha}^2}{2f_{0\alpha}} + \frac{|\mathbf{E}|^2 + |\mathbf{B}|^2}{8\pi} \right] = 0. \quad (6.54)$$

(c) Consider the same problem, this time with an equilibrium that is not Maxwellian, but merely isotropic, i.e.,  $f_{0\alpha} = f_{0\alpha}(v)$ , or, in what will prove to be a more convenient form,

$$f_{0\alpha} = f_{0\alpha}(\varepsilon_{\alpha}), \quad (6.55)$$

where  $\varepsilon_{\alpha} = m_{\alpha}v^2/2$  is the particle energy. Find an integral quantity quadratic in perturbed fields and distributions that is conserved by the Vlasov–Maxwell system under these circumstances and that turns into the free energy (6.54) in the case of a Maxwellian equilibrium (if in difficulty, you will find the answer in, e.g., Davidson 1983 or in Kruskal

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<sup>50</sup>This is based on the 2021 exam question. The first half of Q12 is very similar to this (if a little easier algebraically), so doing both of them might be tedious for you.

& Oberman 1958, which appears to be the original source). Argue that

$$\frac{\partial f_{0\alpha}}{\partial \varepsilon_\alpha} < 0 \quad (6.56)$$

is a sufficient condition for stability of small ( $\delta f_\alpha \ll f_{0\alpha}$ , but not necessarily infinitesimal) perturbations in such a plasma.

**7. Fluctuation-dissipation relation for a collisionless plasma.** Let us consider a linear kinetic system in which perturbations are stirred up by an external force, which we can think of as an imposed (time-dependent) electric field  $\mathbf{E}_{\text{ext}} = -\nabla\chi$ . The perturbed distribution function then satisfies

$$\frac{\partial \delta f_\alpha}{\partial t} + \mathbf{v} \cdot \nabla \delta f_\alpha - \frac{q_\alpha}{m_\alpha} (\nabla \varphi_{\text{tot}}) \cdot \frac{\partial f_{0\alpha}}{\partial \mathbf{v}} = 0, \quad (6.57)$$

where  $\varphi_{\text{tot}} = \varphi + \chi$  is the total potential, whose self-consistent part,  $\varphi$ , obeys the usual Poisson equation

$$-\nabla^2 \varphi = 4\pi \sum_\alpha q_\alpha \int d\mathbf{v} \delta f_\alpha \quad (6.58)$$

and the equilibrium  $f_{0\alpha}$  is assumed to be Maxwellian.

(a) By considering an initial-value problem for (6.57) and (6.58) with zero initial perturbation, show that the Laplace transforms of  $\varphi_{\text{tot}}$  and  $\chi$  are related by

$$\hat{\varphi}_{\text{tot}}(p) = \frac{\hat{\chi}(p)}{\epsilon(p)}, \quad (6.59)$$

where  $\epsilon(p)$  is the dielectric function given by (3.84).

(b) Consider a time-periodic external force,

$$\chi(t) = \chi_0 e^{-i\omega_0 t}. \quad (6.60)$$

Working out the relevant Laplace transforms and their inverses [see (3.14)], show that, after transients have decayed, the total electric field in the system will oscillate at the same frequency as the external force and be given by

$$\varphi_{\text{tot}}(t) = \frac{\chi_0 e^{-i\omega_0 t}}{\epsilon(-i\omega_0)}. \quad (6.61)$$

(c) Now consider the *plasma-kinetic Langevin problem*: assume the external force to be a white noise, i.e., a random process with the time-correlation function

$$\langle \chi(t) \chi^*(t') \rangle = 2D \delta(t - t'). \quad (6.62)$$

Show that the resulting steady-state mean-square fluctuation level in the plasma will be

$$\langle |\varphi_{\text{tot}}(t)|^2 \rangle = \frac{D}{\pi} \int_{-\infty}^{+\infty} \frac{d\omega}{|\epsilon(-i\omega)|^2}.$$

(6.63)

This is a kinetic *fluctuation-dissipation relation*: given a certain level of external stirring, parametrised by  $D$ , this formula predicts the fluctuation energy in terms of  $D$  and of the internal dissipative properties of the plasma, encoded by its dielectric function.

(d) For a system in which the Landau damping is weak,  $|\gamma| \ll kv_{\text{th}\alpha}$ , calculate the

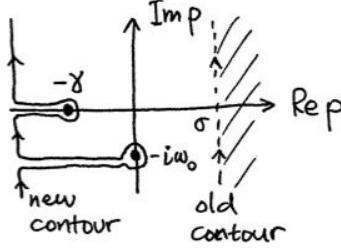


FIGURE 29. Shifting the integration contour in (6.68).

integral (6.63) using Plemelj's formula (3.25) to show that

$$\langle |\varphi_{\text{tot}}(t)|^2 \rangle = D \sum_i \frac{1}{|\gamma_i|} \left[ \frac{\partial \operatorname{Re} \epsilon(-i\omega)}{\partial \omega} \right]_{\omega=\omega_i}^{-2}, \quad (6.64)$$

where  $p_i = -i\omega_i + \gamma_i$  are the weak-damping roots of the dispersion relation.

Here is a reminder of how the *standard Langevin problem* can be solved using Laplace transforms. The Langevin equation is

$$\boxed{\frac{\partial \varphi}{\partial t} + \gamma \varphi = \chi(t)}, \quad (6.65)$$

where  $\varphi$  describes some quantity, e.g., the velocity of a Brownian particle, subject to a damping rate  $\gamma$  and an external force  $\chi$ . In the case of a Brownian particle,  $\chi$  is assumed to be a white noise, as per (6.62). Assuming  $\varphi(t=0) = 0$ , the Laplace-transform solution of (6.65) is

$$\hat{\varphi}(p) = \frac{\hat{\chi}(p)}{p + \gamma}. \quad (6.66)$$

Considering first a non-random oscillatory force (6.60), we have

$$\hat{\chi}(p) = \int_0^\infty dt e^{-pt} \chi(t) = \frac{\chi_0}{p + i\omega_0} \Rightarrow \hat{\varphi}(p) = \frac{\chi_0}{(p + \gamma)(p + i\omega_0)}. \quad (6.67)$$

The inverse Laplace transform of  $\hat{\varphi}$  is calculated by shifting the integration contour to large negative  $\operatorname{Re} p$  while not allowing it to cross the two poles,  $p = -\gamma$  and  $p = -i\omega_0$ , in a manner analogous to that explained in §3.1 (Fig. 5) and shown in Fig. 29. The integral is then dominated by the contributions from the poles:

$$\varphi(t) = \frac{1}{2\pi i} \int_{-i\infty+\sigma}^{i\infty+\sigma} dp e^{pt} \hat{\varphi}(p) = \chi_0 \left( \frac{e^{-i\omega_0 t}}{-i\omega_0 + \gamma} + \frac{e^{-\gamma t}}{-\gamma + i\omega_0} \right) \rightarrow \frac{\chi_0 e^{-i\omega_0 t}}{-i\omega_0 + \gamma} \quad \text{as } t \rightarrow \infty, \quad (6.68)$$

which is quite obviously the right solution of (6.65) with a periodic force (the second term in the brackets is the decaying transient needed to enforce the zero initial condition).

In the more complicated case of a white-noise force [see (6.62)],

$$\begin{aligned} \langle |\varphi(t)|^2 \rangle &= \frac{1}{(2\pi)^2} \left\langle \left| \int_{-i\infty+\sigma}^{i\infty+\sigma} dp e^{pt} \frac{\hat{\chi}(p)}{p + \gamma} \right|^2 \right\rangle \\ &= \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} d\omega \int_{-\infty}^{+\infty} d\omega' e^{[-i(\omega-\omega') + 2\sigma]t} \frac{\langle \hat{\chi}(-i\omega + \sigma) \hat{\chi}^*(-i\omega' + \sigma) \rangle}{(-i\omega + \sigma + \gamma)(i\omega' + \sigma + \gamma)}, \end{aligned} \quad (6.69)$$

where we have changed variables  $p = -i\omega + \sigma$  and similarly for the second integral. The correlation function of the Laplace-transformed force is, using (6.62),

$$\langle \hat{\chi}(p) \hat{\chi}^*(p') \rangle = \int_0^\infty dt \int_0^\infty dt' e^{-pt - p'^* t'} \langle \chi(t) \chi^*(t') \rangle = 2D \int_0^\infty dt e^{-(p+p'^*)t} = \frac{2D}{p + p'^*}, \quad (6.70)$$

provided  $\text{Re } p > 0$  and  $\text{Re } p' > 0$ . Then (6.69) becomes

$$\begin{aligned} \langle |\varphi(t)|^2 \rangle &= \frac{D}{2\pi^2} \int_{-\infty}^{+\infty} d\omega \int_{-\infty}^{+\infty} d\omega' \frac{e^{-i(\omega-\omega')+2\sigma]t}}{[-i(\omega-\omega')+2\sigma](-i\omega+\sigma+\gamma)(i\omega'+\sigma+\gamma)} \\ &= \frac{D}{\pi} \int_{-\infty}^{+\infty} d\omega' \frac{e^{(i\omega'+\sigma)t}}{(i\omega'+\sigma+\gamma)} \frac{1}{2\pi i} \int_{-i\infty+\sigma}^{i\infty+\sigma} dp \frac{e^{pt}}{(p+i\omega'+\sigma)(p+\gamma)} \\ &= \frac{D}{\pi} \int_{-\infty}^{+\infty} d\omega' \frac{e^{(i\omega'+\sigma)t}}{(i\omega'+\sigma+\gamma)} \left[ \frac{e^{-(i\omega'+\sigma)t}}{-i\omega'-\sigma+\gamma} + \frac{e^{-\gamma t}}{\gamma+i\omega'+\sigma} \right], \end{aligned} \quad (6.71)$$

where we have reverted to the  $p$  variable in one of the integrals and then performed the integration by the same manipulation of the contour as in (6.68). We now note that, since there are no exponentially growing solutions in this system,  $\sigma > 0$  can be chosen arbitrarily small. Taking  $\sigma \rightarrow +0$  and neglecting the decaying transient in (6.71), we get, in the limit  $t \rightarrow \infty$ ,

$$\langle |\varphi(t)|^2 \rangle = \frac{D}{\pi} \int_{-\infty}^{+\infty} \frac{d\omega'}{|-i\omega'+\gamma|^2} = \frac{D}{\pi} \int_{-\infty}^{+\infty} \frac{d\omega'}{\omega'^2 + \gamma^2} = \frac{D}{\gamma}. \quad (6.72)$$

Note that, while the integral in (6.72) is doable exactly, it can, for the case of weak damping, also be computed via Plemelj's formula.

Equation (6.72) is the standard Langevin fluctuation-dissipation relation. It can also be obtained without Laplace transforms, either by directly integrating (6.65) and correlating  $\varphi(t)$  with itself or by noticing that

$$\frac{\partial}{\partial t} \frac{\langle \varphi^2 \rangle}{2} + \gamma \langle \varphi^2 \rangle = \langle \chi(t) \varphi(t) \rangle = \left\langle \chi(t) \int_0^t dt' [-\gamma \varphi(t') + \chi(t')] \right\rangle = D, \quad (6.73)$$

where we have used (6.62) and the fact that  $\langle \chi(t) \varphi(t') \rangle = 0$  for  $t' \leq t$ , by causality. Equation (6.72) is the steady-state solution to the above, but (6.73) also teaches us that, if we interpret  $\langle \varphi^2 \rangle/2$  as energy,  $D$  is the power that is injected into the system by the external force. Thus, fluctuation-dissipation relations such as (6.72) tells us what fluctuation *energy* will persist in a dissipative system if a certain amount of *power* is pumped in.

## 8.\* Phase-mixing spectrum.

Here we study the velocity-space structure of the perturbed distribution function  $\delta f$  derived in Q7.

In order to do this, we need to review the *Hermite transform*:

$$\delta f_m = \frac{1}{n} \int dv_z \frac{H_m(u) \delta f(v_z)}{\sqrt{2^m m!}}, \quad u = \frac{v_z}{v_{\text{th}}}, \quad H_m(u) = (-1)^m e^{u^2} \frac{d^m}{du^m} e^{-u^2}, \quad (6.74)$$

where  $H_m$  is the Hermite polynomial of (integer) order  $m$ . We are only concerned with the  $v_z$  dependence of  $\delta f$  (where  $z$ , as always, is along the wave number of the perturbations—in this case set by the wave number of the force); all  $v_x$  and  $v_y$  dependence is Maxwellian and can be integrated out. The inverse transform is given by

$$\delta f(v_z) = \sum_{m=0}^{\infty} \frac{H_m(u) F(v_z)}{\sqrt{2^m m!}} \delta f_m, \quad F(v_z) = \frac{n}{\sqrt{\pi} v_{\text{th}}} e^{-u^2}. \quad (6.75)$$

Because  $H_m(u)$  are orthogonal polynomials, viz.,

$$\frac{1}{n} \int dv_z H_m(u) H_{m'}(u) F(v_z) = 2^m m! \delta_{mm'}, \quad (6.76)$$

they have a Parseval theorem and so the contribution of the perturbed distribution function to the free energy [see (5.18)] can be written as

$$\int dv \frac{T |\delta f|^2}{2f_0} = \frac{nT}{2} \sum_m |\delta f_m|^2. \quad (6.77)$$

In a plasma where perturbations are constantly stirred up by a force, Landau damping must

be operating all the time, removing energy from  $\varphi$  to provide “dissipation” of the injected power. The process of phase mixing that accompanies Landau damping must then lead to a certain fluctuation level  $\langle |\delta f_m|^2 \rangle$  in the Hermite moments of  $\delta f$ . Lower  $m$ 's correspond to “fluid” quantities: density ( $m = 0$ ), flow velocity ( $m = 1$ ), temperature ( $m = 2$ ). Higher  $m$ 's correspond to finer structure in velocity space: indeed, for  $m \gg 1$ , the Hermite polynomials can be approximated by trigonometric functions,

$$H_m(u) \approx \sqrt{2} \left( \frac{2m}{e} \right)^{m/2} \cos \left( \sqrt{2m} u - \frac{\pi m}{2} \right) e^{u^2/2}, \quad (6.78)$$

and so the Hermite transform is somewhat analogous to a Fourier transform in velocity space with “frequency”  $\sqrt{2m}/v_{\text{th}}$ .

- (a) Show that in the kinetic Langevin problem described in Q7(c), the mean square fluctuation level of the  $m$ -th Hermite moment of the perturbed distribution function is given by

$$\langle |\delta f_m(t)|^2 \rangle = \frac{q^2 D}{T^2 \pi 2^m m!} \int_{-\infty}^{+\infty} d\omega \left| \frac{\zeta Z^{(m)}(\zeta)}{\epsilon(-i\omega)} \right|^2, \quad \zeta = \frac{\omega}{kv_{\text{th}}}, \quad (6.79)$$

where  $Z^{(m)}(\zeta)$  is the  $m$ -th derivative of the plasma dispersion function [note (3.93)].

- (b\*\*) Show that, assuming  $m \gg 1$  and  $\zeta \ll \sqrt{2m}$ ,

$$Z^{(m)}(\zeta) \approx \sqrt{2\pi} i^{m+1} \left( \frac{2m}{e} \right)^{m/2} e^{i\zeta\sqrt{2m}-\zeta^2/2} \quad (6.80)$$

and, therefore, that

$$\langle |\delta f_m(t)|^2 \rangle \approx \frac{\text{const}}{\sqrt{m}}.$$

(6.81)

Thus, the *Hermite spectrum* of the free energy is shallow and, in particular, the total free energy diverges—it has to be regularised by collisions. This is a manifestation of a copious amount of fine-scale structure in velocity space (note also how this shows that Landau-damped perturbations involve all Hermite moments, not just the “fluid” ones).

Deriving (6.80) is a (reasonably hard) mathematical exercise: it involves using (3.93) and (6.78) and manipulating contours in the complex plane. This is a treat for those who like this sort of thing. Getting to (6.81) will also require the use of Stirling’s formula.

The Hermite order at which the spectrum (6.81) must be cut off due to collisions can be quickly deduced as follows. We saw in §5.5 that the typical velocity derivative of  $\delta f$  can be estimated according to (5.26) and the time it takes for this perturbation to be wiped out by collisions is given by (5.31). But, in view of (6.78), the velocity gradients probed by the Hermite moment  $m$  are of order  $\sqrt{2m}/v_{\text{th}}$ . The collisional cut off  $m_c$  in Hermite space can then be estimated so:

$$m_c \sim v_{\text{th}}^2 \frac{\partial^2}{\partial v^2} \sim (kv_{\text{th}} t_c)^2 \sim \left( \frac{kv_{\text{th}}}{\nu} \right)^{2/3}. \quad (6.82)$$

Therefore, the total free energy stored in phase space diverges: using (6.77) and (6.81),

$$\frac{1}{n} \int d\mathbf{v} \frac{\delta f^2}{2f_0} = \frac{1}{2} \sum_m \langle |\delta f_m|^2 \rangle \sim \int^{m_c} dm \frac{\text{const}}{\sqrt{m}} \propto \nu^{-1/3} \rightarrow \infty \quad \text{as } \nu \rightarrow +0. \quad (6.83)$$

In contrast, the total free-energy dissipation rate is finite, however small is the collision frequency: estimating the right-hand side of (5.18), we get

$$\frac{1}{n} \int d\mathbf{v} \frac{\delta f}{f_0} \left( \frac{\partial \delta f}{\partial t} \right)_c \sim -\nu \sum_m m \langle |\delta f_m|^2 \rangle \propto \nu \int^{m_c} dm \sqrt{m} \sim kv_{\text{th}}. \quad (6.84)$$

Thus, the kinetic system can collisionally produce entropy at a rate that is entirely independent of the collision frequency.

If you find phase-space turbulence and generally life in Hermite space as fascinating as I do, you can learn more from [Kanekar et al. \(2015\)](#) (on fluctuation-dissipation relations and Hermite spectra) and from §§[12.2–12.3](#) (on what happens when nonlinearity strikes).

*Do one of Q9, Q10, Q11, or Q12 (but do not pick Q12 if you have already done Q5).*

**9. QL theory of Landau damping.** In §6, we discussed the QL theory of an unstable system, in which, whatever the size of the initial electric perturbations, they eventually grow large enough to affect the equilibrium distribution and modify it so as to suppress further growth. In a *stable* equilibrium, any initial perturbations will be Landau-damped, but, if they are sufficiently large to start with, they can also affect  $f_0$  quasilinearly in a way that will slow down this damping.

Consider, in 1D, an initial spectrum  $W(0, k)$  of plasma oscillations (waves) excited in the wave-number range  $[k_2, k_1] = [\omega_{pe}/v_2, \omega_{pe}/v_1] \ll \lambda_{De}^{-1}$ , with total electric energy  $\mathcal{E}(0)$ . Modify the QL theory of §6 to show the following.

(a) A steady state can be achieved in which the distribution develops a plateau in the velocity interval  $[v_1, v_2]$  (Fig. 30). Find  $F^{\text{plateau}}$  in terms of  $v_1$ ,  $v_2$  and the initial distribution  $F(0, v)$ . What is the energy of the waves in this steady state? What is the lower bound on initial electric energy  $\mathcal{E}(0)$  below which the perturbations would just decay without forming a fully-fledged plateau?

(b) Derive the evolution equation for the thermal (nonresonant) bulk of the distribution and show that it cools during the QL evolution, with the total thermal energy declining by the same amount as the electric energy of the waves:

$$\mathcal{H}_{\text{th}}(t) - \mathcal{H}_{\text{th}}(0) = -[\mathcal{E}(0) - \mathcal{E}(t)]. \quad (6.85)$$

Identify where all the energy lost by the thermal particles and the waves goes and thus confirm that the total energy in the system is conserved. Why, physically, do thermal particles lose energy?

(c) Show that we must have

$$\frac{\mathcal{E}(0)}{n_e T_e} \gg \frac{\gamma_k}{\omega_{pe}} \frac{\delta v}{v} \quad (6.86)$$

in order for the wave energy to change only by a small fraction before saturating and

$$\frac{\mathcal{E}(0)}{n_e T_e} \gg \frac{\gamma_k}{\omega_{pe}} \left( \frac{\delta v}{v} \right)^3 \frac{1}{(k \lambda_{De})^2} \quad (6.87)$$

in order for the QL evolution to be faster than the damping. Here  $\delta v = v_2 - v_1$  and  $v \sim v_1 \sim v_2$ .

This question requires some nuance in handling the calculation of the QL diffusion coefficient. In §6.1, we used the expression (6.6) for  $\delta f_k$  in which only the eigenmode-like part was retained, while the phase-mixing terms were dropped on the grounds that we could always just wait long enough for them to be eclipsed by the term containing an exponentially growing factor  $e^{\gamma_k t}$ . When we are dealing with *damped* perturbations, there is no point in waiting because the exponential term is getting smaller, while the phase-mixing terms do not decay (except by collisions, see §§5.3 and 5.5, but we are not prepared to wait for that).

Let us, therefore, bite the bullet and use the full expression (5.25) for the perturbed distribution function, where we single out the slowest-damped mode and assume that all others, if any,

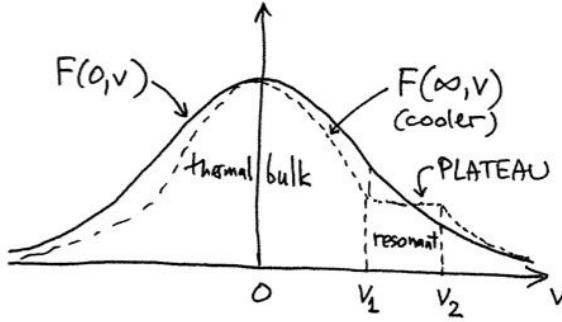


FIGURE 30. Initial stable equilibrium distribution and the final outcome of the QL evolution of a system with Landau-damped electric perturbations.

will be damped fast enough never to produce significant QL effects:

$$\delta f_{\mathbf{k}} = \frac{q}{m} \varphi_{\mathbf{k}} \frac{1 - e^{-i(\mathbf{k} \cdot \mathbf{v} - \omega_{\mathbf{k}})t - \gamma_{\mathbf{k}} t}}{\mathbf{k} \cdot \mathbf{v} - \omega_{\mathbf{k}} - i\gamma_{\mathbf{k}}} \mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{v}} + e^{-i\mathbf{k} \cdot \mathbf{v} t} (g_{\mathbf{k}} + \dots), \quad (6.88)$$

where “...” stand for any possible undamped, phase-mixing remnants of other modes. When the solution (6.88) is substituted into (6.4), where it is multiplied by  $\varphi_{\mathbf{k}}^*$  and time averaged [according to (2.7)], the second term vanishes because, for resonant particles ( $\mathbf{k} \cdot \mathbf{v} \approx \omega_{\mathbf{k}}$ ), it contains no resonant denominators and so is smaller than the first term, whereas for the nonresonant particles, it is removed by time averaging (check that this works at least for  $|\gamma_{\mathbf{k}}|t \lesssim 1$  and indeed beyond that). Keeping only the first term in the expression (6.88), substituting it into (6.4) and going through a calculation analogous to that given in (6.8), we find that the diffusion matrix is (check this)

$$\mathbf{D}(\mathbf{v}) = \frac{q^2}{m^2} \sum_{\mathbf{k}} \frac{\mathbf{k}\mathbf{k}}{k^2} |\mathbf{E}_{\mathbf{k}}|^2 \text{Im} \left\langle \frac{1 - e^{-i(\mathbf{k} \cdot \mathbf{v} - \omega_{\mathbf{k}})t - \gamma_{\mathbf{k}} t}}{\mathbf{k} \cdot \mathbf{v} - \omega_{\mathbf{k}} - i\gamma_{\mathbf{k}}} \right\rangle, \quad (6.89)$$

which is a generalisation of the penultimate line of (6.8). For nonresonant particles, the phase-mixing term is eliminated by time averaging and we end up with the old result: the last line of (6.8). For resonant particles, assuming  $|\gamma_{\mathbf{k}}| \ll |\mathbf{k} \cdot \mathbf{v} - \omega_{\mathbf{k}}| \ll \omega_{\mathbf{k}} \sim \mathbf{k} \cdot \mathbf{v}$  and  $|\gamma_{\mathbf{k}}|t \ll 1$ , we may adopt the approximation (5.39), which we have previously used to analyse the structure of  $\delta f$ . This gives us

$$\mathbf{D}(\mathbf{v}) = \frac{q^2}{m^2} \sum_{\mathbf{k}} \frac{\mathbf{k}\mathbf{k}}{k^2} |\mathbf{E}_{\mathbf{k}}|^2 \pi \delta(\mathbf{k} \cdot \mathbf{v} - \omega_{\mathbf{k}}), \quad (6.90)$$

which is the same result as (6.16)—including, importantly, the sign, which we would have gotten wrong had we just mechanically applied Plemelj's formula to (6.12) with  $\gamma_{\mathbf{k}} < 0$ . This is equivalent to saying that the  $k$  integral in (6.16) should be taken along the Landau contour, rather than simply along the real line.

Note that the above construction was done assuming  $|\gamma_{\mathbf{k}}|t \ll 1$ , i.e., all the QL action has to occur before the initial perturbations decay away (which is reasonable). Note also that there is nothing above that would not apply to the case of unstable perturbations ( $\gamma_{\mathbf{k}} > 0$ ) and so we conclude that results of §6, derived formally for  $\gamma_{\mathbf{k}}t \gg 1$ , in fact also hold on shorter time scales ( $\gamma_{\mathbf{k}}t \ll 1$ , but, obviously, still  $\omega_{\mathbf{k}}t \gg 1$ ).

**10. QL theory of Weibel instability.** (a) Starting from the Vlasov equations including magnetic perturbations, show that the slow evolution of the equilibrium distribution function is described by the following diffusion equation:

$$\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{D}(\mathbf{v}) \cdot \frac{\partial f_0}{\partial \mathbf{v}}, \quad (6.91)$$

where the QL diffusion matrix is

$$\mathbf{D}(\mathbf{v}) = \frac{q^2}{m^2} \sum_{\mathbf{k}} \frac{1}{i(\mathbf{k} \cdot \mathbf{v} - \omega_{\mathbf{k}}) + \gamma_{\mathbf{k}}} \left( \mathbf{E}_{\mathbf{k}}^* + \frac{\mathbf{v} \times \mathbf{B}_{\mathbf{k}}^*}{c} \right) \left( \mathbf{E}_{\mathbf{k}} + \frac{\mathbf{v} \times \mathbf{B}_{\mathbf{k}}}{c} \right) \quad (6.92)$$

and  $\omega_{\mathbf{k}}$  and  $\gamma_{\mathbf{k}}$  are the frequency and the growth rate, respectively, of the fastest-growing mode.

(b) Consider the example of the low-frequency electron Weibel instability with wave numbers  $\mathbf{k}$  parallel to the anisotropy direction [see (6.47)]. Take  $\mathbf{k} = k\hat{z}$  and  $\mathbf{B}_{\mathbf{k}} = B_{\mathbf{k}}\hat{y}$  and, denoting  $\Omega_{\mathbf{k}} = eB_{\mathbf{k}}/m_e c$  (the Larmor frequency associated with the perturbed magnetic field), show that (6.91) becomes

$$\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial v_x} \left( D_{xx} \frac{\partial f_0}{\partial v_x} + D_{xz} \frac{\partial f_0}{\partial v_z} \right) + \frac{\partial}{\partial v_z} D_{zz} \frac{\partial f_0}{\partial v_z}, \quad (6.93)$$

where the coefficients of the QL diffusion tensor are

$$D_{xx} = \sum_k \frac{\gamma_k}{k^2} |\Omega_k|^2, \quad D_{xz} = - \sum_k \frac{2\gamma_k v_x v_z}{k^2 v_z^2 + \gamma_k^2} |\Omega_k|^2, \quad D_{zz} = \sum_k \frac{\gamma_k v_x^2}{k^2 v_z^2 + \gamma_k^2} |\Omega_k|^2. \quad (6.94)$$

(c) Suppose the electron distribution function  $f_0$  is initially the bi-Maxwellian (6.45) with  $0 < T_{\perp}/T_{\parallel} - 1 \ll 1$  (as should be the case for this instability to work). As QL evolution starts, we may define the temperatures of the evolving distribution according to

$$T_{\perp} = \frac{1}{n} \int d\mathbf{v} \frac{m(v_x^2 + v_y^2)}{2} f_0, \quad T_{\parallel} = \frac{1}{n} \int d\mathbf{v} m v_z^2 f_0. \quad (6.95)$$

Show that initially, viz., before  $f_0$  has time to change shape significantly so as no longer to be representable as a bi-Maxwellian, the two temperatures will evolve approximately (using  $\gamma_k \ll kv_{\text{th}}$ ) according to

$$\frac{\partial T_{\perp}}{\partial t} = -\lambda T_{\perp}, \quad \frac{\partial T_{\parallel}}{\partial t} = 2\lambda T_{\perp}, \quad \text{where} \quad \lambda(T_{\perp}, T_{\parallel}) = \sum_k \frac{2\gamma_k |\Omega_k|^2}{k^2 v_{\text{th}\parallel}^2}. \quad (6.96)$$

Thus, QL evolution will lead, at least initially, to the reduction of the temperature anisotropy, thus weakening the instability (these equations should not be used to trace  $T_{\perp}/T_{\parallel} - 1$  all the way to zero because there is no reason why the QL evolution should preserve the bi-Maxwellian shape of  $f_0$ ).

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Note that, even modulo the caveat about the bi-Maxwellian not being a long-term solution, this does not give us a way to estimate (or even guess) what the saturated fluctuation level will be. The standard Weibel lore is that saturation occurs when the approximations that were used to derive the linear theory (Q3) break down, namely, when magnetic field becomes strong enough to magnetise the plasma, rendering the Larmor scale  $\rho_e = v_{\text{the}}/\Omega_k$  associated with the fluctuations small enough to be comparable to the latter's wavelengths  $\sim k^{-1}$ . Using the typical values of  $k$  from (6.47), we can write this condition as follows

$$\Omega_k \sim kv_{\text{the}} \sim \sqrt{\Delta_e} \frac{v_{\text{the}}}{d_e} \quad \Leftrightarrow \quad \frac{1}{\beta_e} \equiv \frac{B^2/8\pi}{n_e T_e} \sim \Delta_e. \quad (6.97)$$

Thus, Weibel instability will produce fluctuations the ratio of whose magnetic-energy density to the electron-thermal-energy density (customarily referred to as the inverse of “plasma beta,”  $1/\beta_e$ ) is comparable to the electron pressure anisotropy  $\Delta_e$ . Because at that point the fluctuations will be relaxing this pressure anisotropy at the same rate as they can grow in the first place [in (6.96),  $\lambda \sim \gamma_k$ ], the QL approach is not valid anymore.

These considerations are, however, usually assumed to be qualitatively sound and lead people to believe that, even in collisionless plasmas, the anisotropy of the electron distribution must be largely self-regulating, with unstable Weibel fluctuations engendered by the anisotropy quickly acting to isotropise the plasma (or at least the electrons).

This is all currently very topical in the part of the plasma-astrophysics world preoccupied with collisionless shocks, origin of the cosmic magnetism, hot weakly collisional environments such as the intergalactic medium (in galaxy clusters) or accretion flows around black holes and many other interesting subjects.

(d) Equations (6.96) say that the total mean kinetic energy,

$$\int d\mathbf{v} \frac{mv^2}{2} f_0 = n \left( T_{\perp} + \frac{T_{\parallel}}{2} \right), \quad (6.98)$$

does not change. But fluctuations are generated and grow at the rate  $\gamma_k$ ! Without much further algebra, can you tell whether you should therefore doubt the result (6.96)?

**11. QL theory of stochastic acceleration.**<sup>51</sup> Consider a population of particles of charge  $q$  and mass  $m$ . Assume that collisions are entirely negligible. Assume further that an electrostatic fluctuation field  $\mathbf{E} = -\nabla\varphi$  (with zero spatial mean) is present and that this field is given and externally determined, i.e., it is unaffected by the particles that are under consideration. This might happen physically if, for example, the particles are a low-density admixture in a plasma consisting of some more numerous species of ions and electrons, which dominate the plasma's dielectric response.

As usual, we assume that the distribution function can be represented as  $f = f_0(t, \mathbf{v}) + \delta f(t, \mathbf{r}, \mathbf{v})$ , where  $f_0$  is spatially homogeneous and changes slowly in time compared to the perturbed distribution  $\delta f \ll f_0$ . Its evolution is described by (2.11), where angle brackets again denote the time average over the fast variation of the fluctuation field.

(a) Assume that  $\varphi$  is sufficiently small for it to be possible to determine  $\delta f$  from the linearised kinetic equation. Let  $\delta f = 0$  at  $t = 0$ . Show that  $f_0$  satisfies a QL diffusion equation with the diffusion matrix

$$\mathbf{D}(\mathbf{v}) = \frac{q^2}{m^2} \sum_{\mathbf{k}} \mathbf{k}\mathbf{k} \frac{1}{2\pi i} \int dp \frac{1}{p + i\mathbf{k} \cdot \mathbf{v}} \int_{-\infty}^t d\tau e^{p\tau} C_{\mathbf{k}}(\tau), \quad (6.99)$$

where the  $p$  integration is along a contour appropriate for an inverse Laplace transform and  $C_{\mathbf{k}}(t - t') = \langle \varphi_{\mathbf{k}}^*(t)\varphi_{\mathbf{k}}(t') \rangle$  is the correlation function of the fluctuation field (which is taken to be statistically stationary, so  $C_{\mathbf{k}}$  depends only on the time difference  $t - t'$ ).

(b) Let the correlation function have the form

$$C_{\mathbf{k}}(\tau) = A_{\mathbf{k}} e^{-\gamma_{\mathbf{k}}|\tau|}, \quad (6.100)$$

i.e.,  $\gamma_{\mathbf{k}}^{-1}$  is the correlation time of the fluctuation field and  $A_{\mathbf{k}}$  its spectrum; assume  $\gamma_{-\mathbf{k}} = \gamma_{\mathbf{k}}$ . Do the integrals in (6.99) and show that, at  $t \gg \gamma_{\mathbf{k}}^{-1}$ ,

$$\mathbf{D}(\mathbf{v}) = \frac{q^2}{m^2} \sum_{\mathbf{k}} \mathbf{k}\mathbf{k} \frac{\gamma_{\mathbf{k}} A_{\mathbf{k}}}{\gamma_{\mathbf{k}}^2 + (\mathbf{k} \cdot \mathbf{v})^2}. \quad (6.101)$$

(c) Restrict consideration to 1D in space and to the limit in which  $\gamma_{\mathbf{k}} \gg kv$  for typical wave numbers of the fluctuations and typical particle velocities (i.e., the fluctuation field is short-time correlated). Assuming that  $f_0$  at  $t = 0$  is a Maxwellian with temperature  $T_0$ , predict the evolution of  $f_0$  with time. Discuss what physically is happening to the

<sup>51</sup>Except for part (d), this is based on the 2018 exam question.

particles. Discuss the validity of the short-correlation-time approximation and of the assumption of slow evolution of  $f_0$ . What is, roughly, the condition on the amplitude and the correlation time of the fluctuation field that makes these assumptions compatible?

(d) When the distribution “heats up” sufficiently, the short-correlation-time approximation will be broken. Staying in 1D, consider the opposite limit,  $\gamma_k \ll kv$ . Show that the resulting QL equation admits a subdiffusive solution, with

$$f_0(t, v) \propto \frac{e^{-v^4/\alpha t}}{t^{1/4}}, \quad \alpha = 16 \frac{q^2}{m^2} \sum_{\mathbf{k}} \gamma_{\mathbf{k}} A_{\mathbf{k}}. \quad (6.102)$$

In view of this result and of (c), discuss qualitatively how an initially “cold” particle distribution would evolve with time.

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The original, classic paper on stochastic acceleration is [Sturrock \(1966\)](#). Note that the velocity dependence of the diffusion matrix (6.101) is determined by the functional form of  $C_{\mathbf{k}}(\tau)$ , so interesting  $\tau$  dependences of the latter can lead to all kinds of interesting distributions  $f_0$  of the accelerated particles.

**12. Linear and QL theory of ion-electron streaming instability.**<sup>52</sup> Consider a plasma consisting of Maxwellian electrons and a cold ion beam (velocity  $u_i \gg v_{\text{the}}$ —much greater than the width of the electron distribution; cf. Q5, where the two species’ roles were reversed). The corresponding distribution functions, which can be considered one-dimensional (in the direction of the beam), are

$$F_e(v) = \frac{n_e}{\sqrt{\pi} v_{\text{the}}} e^{-v^2/v_{\text{the}}^2}, \quad F_i(v) = n_i \delta(v - u_i). \quad (6.103)$$

(a) Starting, as usual, with the dispersion relation that determines the complex increment  $p$  of linear perturbations in an electrostatic plasma and anticipating that an instability will be possible for  $p/k \gg v_{\text{the}}$ , show that, in this limit, the dispersion relation is

$$1 + \frac{\omega_{\text{pe}}^2}{p^2} - \frac{\omega_{\text{pi}}^2}{(ku_i - ip)^2} = 0. \quad (6.104)$$

(b) Show that there is an instability that attains its fastest growth at  $k = \omega_{\text{pe}}/u_i$  and produces, at that wavenumber, waves with the real frequency and growth rate

$$\omega_k = \left[ 1 - \frac{1}{2} \left( \frac{Zm_e}{2m_i} \right)^{1/3} \right] \omega_{\text{pe}}, \quad \gamma_k = \frac{\sqrt{3}}{2} \left( \frac{Zm_e}{2m_i} \right)^{1/3} \omega_{\text{pe}}. \quad (6.105)$$

(c) The equations of QLT for the distribution functions and the wave spectrum are (see §6.1)

$$\frac{\partial F_{\alpha}}{\partial t} = \frac{\partial}{\partial v} \left[ \frac{q_{\alpha}^2}{m_{\alpha}^2} \sum_k \frac{\gamma_k |E_k|^2}{(kv - \omega_k)^2 + \gamma_k^2} \right] \frac{\partial F_{\alpha}}{\partial v}, \quad \frac{\partial |E_k|^2}{\partial t} = 2\gamma_k |E_k|^2, \quad (6.106)$$

where, as usual,  $E_k$  is the electric-field amplitude at wavenumber  $k$ ,  $\omega_k$  is the real frequency of the waves, and  $\gamma_k$  is their growth rate at that wavenumber. Assuming that, at least in the initial stage of the quasilinear evolution, the distribution functions will preserve their functional form (6.103), but  $T_e$  and  $u_i$  will change slowly with time,

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<sup>52</sup>This is based on the 2022 exam question.

show that the kinetic-energy densities of the electrons and ions will satisfy

$$\frac{d}{dt} \int dv \frac{m_e v^2}{2} F_e = \frac{d\mathcal{E}}{dt}, \quad \frac{d}{dt} \frac{m_i n_i u_i^2}{2} = -2 \frac{d\mathcal{E}}{dt}, \quad (6.107)$$

where  $\mathcal{E} = \sum_k |E_k|^2 / 8\pi$  is the energy density of the electric field. Despite their rather innocuous appearance, deriving these equations (especially the second) does require some mildly nontrivial calculations. In carrying them out, you may assume that the wavenumber sums in (6.106) are dominated by the fastest-growing mode described by (6.105).

- (d) Based on (6.107), describe the energy flows in the system and, qualitatively, the evolution of the distribution functions. How do the mean momenta of these distributions evolve? (Only twiddle algebra—rough estimates—are called for here.)
- (e) Make an educated qualitative guess about the typical size of  $\mathcal{E}$  at which the instability saturates, i.e., the distribution becomes stable. The answer is

$$\mathcal{E} \sim \frac{m_e n_e u_i^2}{2}. \quad (6.108)$$

If you are struggling with any of this, especially the qualitative arguments at the end of the question, you may find inspiration in [Krall & Trivelpiece \(1973\)](#).

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## PART II

# Advanced Kinetic Theory

In this “advanced” part of these kinetic-theory lectures, I will try to make some inroads into nonlinear theory beyond QLT—this means not (entirely) neglecting the last term in (2.12), which is responsible for the nonlinear interactions between the perturbed electric field ( $\varphi_{\mathbf{k}'}$ ) and the perturbed distribution function ( $\delta f_{\mathbf{k}-\mathbf{k}'}$ ). Nonlinear theory of anything is, of course, hard—indeed, in most cases, intractable. These days, an impatient researcher’s answer to being faced with a hard question is to outsource it to a computer. This sometimes leads to spectacular successes, but also, somewhat more frequently, to spectacular confusion about how to interpret the output. In dealing with a steady stream of data produced by ever more powerful machines, one is often helped by the residual memory of analytical results obtained in the prehistoric era when computation was harder than theory and plasma physicists had to find ingenious ways to solve nonlinear problems “by hand”—which usually required finding ingenious ways of posing problems that were solvable. These could be separated into three broad categories: interesting particular cases of nonlinear behaviour involving usually just a few interacting waves (a pretty example of this is §12.1), looking at systems of very many waves amenable to some approximate statistical treatment (§§7, 8, 11, and 12.2), and applying general principles, such as stability (§9) or maximum entropy (§10).

## 7. Quasiparticle Kinetics

### 7.1. QLT in the Language of Quasiparticles

First I would like to outline a neat way of reformulating the QL theory, which both sheds some light on the meaning of what was done in §6 and opens up promising avenues for theorising further about nonlinear plasma states.

Let us re-imagine our system of particles and waves as a mixture of two interacting gases: “true” particles (electrons) and *quasiparticles*, or *plasmons*, which will be the “quantised” version of Langmuir waves. If each of these plasmons has momentum  $\hbar\mathbf{k}$  and energy  $\hbar\omega_{\mathbf{k}}$ , we can declare

$$N_{\mathbf{k}} = \frac{V|\mathbf{E}_{\mathbf{k}}|^2/4\pi}{\hbar\omega_{\mathbf{k}}} \quad (7.1)$$

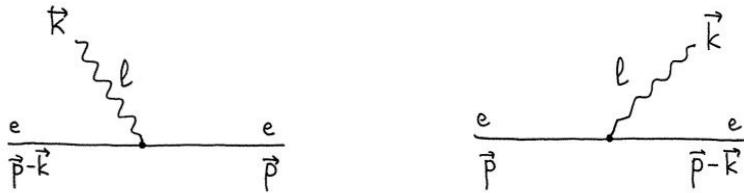
to be the mean occupation number of plasmons with wave number  $\mathbf{k}$  in a box of volume  $V$ . The total energy of these plasmons is then

$$\sum_{\mathbf{k}} \hbar\omega_{\mathbf{k}} N_{\mathbf{k}} = V \sum_{\mathbf{k}} \frac{|\mathbf{E}_{\mathbf{k}}|^2}{4\pi}, \quad (7.2)$$

twice the total electric energy in the system (twice because it includes the energy of the mean oscillatory motion of electrons within a wave; see discussion at the end of §6.5 and/or in §5.7). Similarly, the total momentum of the plasmons is

$$\sum_{\mathbf{k}} \hbar\mathbf{k} N_{\mathbf{k}} = V \sum_{\mathbf{k}} \frac{\mathbf{k}|\mathbf{E}_{\mathbf{k}}|^2}{4\pi\omega_{\mathbf{k}}}. \quad (7.3)$$

This is indeed in line with our previous calculations [see (6.39)]. Note that the role of  $\hbar$  here is simply to define a splitting of wave energy into individual plasmons—this can be done in an arbitrary way, provided  $\hbar$  is small enough to ensure  $N_{\mathbf{k}} \gg 1$ . Since there



(a) absorption of a plasmon by an electron (b) emission of a plasmon by an electron

FIGURE 31. Diagrams for (7.6).

is nothing quantum-mechanical about our system, all our results will in the end have to be independent of  $\hbar$ , so we will use  $\hbar$  as an arbitrarily small parameter, in which it will be convenient to expand, while expecting it eventually to cancel out in all physically meaningful relationships.

We may now think of the QL evolution (or indeed generally of the nonlinear evolution) of our plasma in terms of interactions between plasmons and electrons. These are *resonant* electrons; the thermal bulk only participates via its supporting role of enabling oscillatory plasma motions associated with plasmons. The electrons are described by their distribution function  $f_0(\mathbf{v})$ , which we can, to make our formalism nicely uniform, recast in terms of occupation numbers: if the wave number corresponding to velocity  $\mathbf{v}$  is  $\mathbf{p} = m_e \mathbf{v}/\hbar$ , then its occupation number is

$$n_{\mathbf{p}} = \left( \frac{2\pi\hbar}{m_e} \right)^3 f_0(\mathbf{v}) \quad \Rightarrow \quad \sum_{\mathbf{p}} n_{\mathbf{p}} = \frac{V}{(2\pi)^3} \int d\mathbf{p} n_{\mathbf{p}} = V \int d\mathbf{v} f_0(\mathbf{v}) = V n_e. \quad (7.4)$$

It is understood that  $n_{\mathbf{p}} \ll 1$  (our electron gas is non-degenerate).

The QL evolution of the plasmon and electron distributions is controlled by two processes: absorption or emission of a plasmon by an electron (known as Cherenkov absorption/emission). Diagrammatically, these can be depicted as shown in Fig. 31. As we know from §6.2, they are subject to momentum conservation,  $\mathbf{p} = \mathbf{k} + (\mathbf{p} - \mathbf{k})$ , and energy conservation:

$$\begin{aligned} 0 &= \varepsilon_{\mathbf{p}}^e - \varepsilon_{\mathbf{k}}^l - \varepsilon_{\mathbf{p}-\mathbf{k}}^e = \frac{\hbar^2 p^2}{2m_e} - \hbar\omega_{\mathbf{k}} - \frac{\hbar^2 |\mathbf{p} - \mathbf{k}|^2}{2m_e} = \hbar \left( -\omega_{\mathbf{k}} + \frac{\hbar \mathbf{p} \cdot \mathbf{k}}{m_e} - \frac{\hbar k^2}{2m_e} \right) \\ &= \hbar(\mathbf{k} \cdot \mathbf{v} - \omega_{\mathbf{k}}) + O(\hbar^2). \end{aligned} \quad (7.5)$$

This is the familiar resonance condition  $\mathbf{k} \cdot \mathbf{v} - \omega_{\mathbf{k}} = 0$ . The superscripts  $e$  and  $l$  stand for electrons and (Langmuir) plasmons.

### 7.1.1. Plasmon Distribution

We may now write an equation for the evolution of the plasmon occupation number:<sup>53</sup>

$$\frac{\partial N_{\mathbf{k}}}{\partial t} = \sum_{\mathbf{p}} \left[ - \underbrace{\mathfrak{w}_{\mathbf{p}-\mathbf{k}, \mathbf{k} \rightarrow \mathbf{p}} \delta(\varepsilon_{\mathbf{p}-\mathbf{k}}^e + \varepsilon_{\mathbf{k}}^l - \varepsilon_{\mathbf{p}}^e) n_{\mathbf{p}-\mathbf{k}} N_{\mathbf{k}}}_{\text{Fig. 31(a)}} \right. \\ \left. + \underbrace{\mathfrak{w}_{\mathbf{p} \rightarrow \mathbf{k}, \mathbf{p}-\mathbf{k}} \delta(\varepsilon_{\mathbf{p}}^e - \varepsilon_{\mathbf{k}}^l - \varepsilon_{\mathbf{p}-\mathbf{k}}^e) n_{\mathbf{p}} (N_{\mathbf{k}} + 1)}_{\text{Fig. 31(b)}} \right], \quad (7.6)$$

where  $\mathfrak{w}$  are the probabilities of absorption and emission and must be equal:

$$\mathfrak{w}_{\mathbf{p}-\mathbf{k}, \mathbf{k} \rightarrow \mathbf{p}} = \mathfrak{w}_{\mathbf{p} \rightarrow \mathbf{k}, \mathbf{p}-\mathbf{k}} \equiv w(\mathbf{p}, \mathbf{k}). \quad (7.7)$$

The first term in the right-hand side of (7.6) describes the absorption of one of (indistinguishable)  $N_{\mathbf{k}}$  plasmons by one of  $n_{\mathbf{p}-\mathbf{k}}$  electrons, the second term describes the emission by one of  $n_{\mathbf{p}}$  electrons of one of  $N_{\mathbf{k}} + 1$  plasmons. The  $+1$  is, of course, a small correction to  $N_{\mathbf{k}} \gg 1$  and can be neglected, although sometimes, in analogous but more complicated calculations, it has to be kept because lowest-order terms cancel (see, e.g., §§7.2.1 and 7.2.2). Using (7.7), (7.5) and (7.4), we find

$$\begin{aligned} \frac{\partial N_{\mathbf{k}}}{\partial t} &\approx \sum_{\mathbf{p}} w(\mathbf{p}, \mathbf{k}) \delta(\varepsilon_{\mathbf{p}}^e - \varepsilon_{\mathbf{k}}^l - \varepsilon_{\mathbf{p}-\mathbf{k}}^e) (n_{\mathbf{p}} - n_{\mathbf{p}-\mathbf{k}}) N_{\mathbf{k}} \\ &= V \int d\mathbf{v} w\left(\frac{m_e \mathbf{v}}{\hbar}, \mathbf{k}\right) \delta(\hbar(\mathbf{k} \cdot \mathbf{v} - \omega_{\mathbf{k}})) \left[ f_0(\mathbf{v}) - f_0\left(\mathbf{v} - \frac{\hbar \mathbf{k}}{m_e}\right) \right] N_{\mathbf{k}} \\ &\approx V \int d\mathbf{v} w\left(\frac{m_e \mathbf{v}}{\hbar}, \mathbf{k}\right) \frac{1}{\hbar} \delta(\mathbf{k} \cdot \mathbf{v} - \omega_{\mathbf{k}}) \frac{\hbar}{m_e} \mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{v}} N_{\mathbf{k}} \\ &= \frac{V}{m_e} w\left(\frac{m_e \omega_{pe}}{\hbar k}, k\right) F'\left(\frac{\omega_{pe}}{k}\right) N_{\mathbf{k}} \equiv 2\gamma_{\mathbf{k}} N_{\mathbf{k}} \end{aligned} \quad (7.8)$$

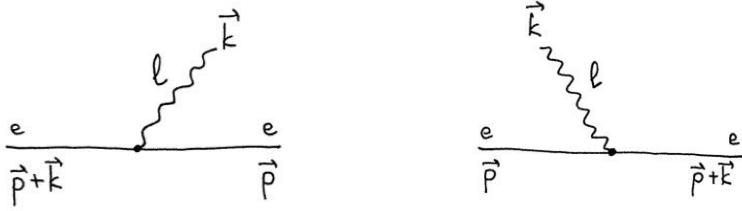
Note that, as expected,  $\hbar$  has disappeared from our equations, after having been used as an expansion parameter.

Since  $N_{\mathbf{k}} \propto |\mathbf{E}_{\mathbf{k}}|^2$  [see (7.1)], the prefactor in (7.8) is clearly just the (twice) growth or damping rate of the waves. Comparing with (6.1), we read off the expression for the absorption/emission probability:

$$w\left(\frac{m_e \omega_{pe}}{\hbar k}, k\right) = \frac{\pi m_e \omega_{pe}^3}{V n_e k^2}. \quad (7.9)$$

Thus, our calculation of Landau damping in §3.5 could be thought of as a calculation of this probability. Whether there is damping or an instability is decided by whether it is absorption or emission of plasmons that occurs more frequently—and that depends on whether, for any given  $\mathbf{k}$ , there are more electrons that are slightly slower or slightly faster than the plasmons with wave number  $\mathbf{k}$ . Note that getting the correct sign of the damping rate is automatic in this approach, since the probability  $w$  must obviously be positive.

<sup>53</sup>Technically speaking, the first term in this equation should be multiplied by  $(1 - n_{\mathbf{p}})$  and the second by  $(1 - n_{\mathbf{p}-\mathbf{k}})$  because electrons are fermions and if our electron, having absorbed or emitted a plasmon, tries to emerge with a momentum that another electron already has, the probability of such an event must be zero. However, since our plasma is classical,  $n_{\mathbf{p}} \ll 1$  and so these extra factors can be approximated by unity. The same comment applies to calculations in §§7.1.2, 7.2.4, and 7.2.5.



(a) emission of a plasmon by an electron (b) absorption of a plasmon by an electron

FIGURE 32. Additional diagrams for (7.10).

### 7.1.2. Electron Distribution

The evolution equation for the occupation number of electrons can be derived in a similar fashion, if we itemise the processes that lead to an electron ending up in a state with a given wave number  $\mathbf{p} = m_e \mathbf{v}/\hbar$  or moving from this state to one with a different wave number. The four relevant diagrams are the two in Fig. 31 and the additional two shown in Fig. 32. The absorption and emission probabilities are the same as before and so are the energy conservation conditions. Therefore,

$$\begin{aligned}
 \frac{\partial n_{\mathbf{p}}}{\partial t} &= \sum_{\mathbf{k}} \left[ \underbrace{\mathfrak{w}_{\mathbf{p}-\mathbf{k}, \mathbf{k} \rightarrow \mathbf{p}} \delta(\varepsilon_{\mathbf{p}-\mathbf{k}}^e + \varepsilon_{\mathbf{k}}^l - \varepsilon_{\mathbf{p}}^e) n_{\mathbf{p}-\mathbf{k}} N_{\mathbf{k}}}_{\text{Fig. 31(a)}} \right. \\
 &\quad + \underbrace{\mathfrak{w}_{\mathbf{p}+\mathbf{k} \rightarrow \mathbf{k}, \mathbf{p}} \delta(\varepsilon_{\mathbf{p}+\mathbf{k}}^e - \varepsilon_{\mathbf{k}}^l - \varepsilon_{\mathbf{p}}^e) n_{\mathbf{p}+\mathbf{k}} (N_{\mathbf{k}} + 1)}_{\text{Fig. 32(a)}} \\
 &\quad - \underbrace{\mathfrak{w}_{\mathbf{p}, \mathbf{k} \rightarrow \mathbf{p}+\mathbf{k}} \delta(\varepsilon_{\mathbf{p}}^e + \varepsilon_{\mathbf{k}}^l - \varepsilon_{\mathbf{p}+\mathbf{k}}^e) n_{\mathbf{p}} N_{\mathbf{k}}}_{\text{Fig. 32(b)}} \\
 &\quad \left. - \underbrace{\mathfrak{w}_{\mathbf{p} \rightarrow \mathbf{k}, \mathbf{p}-\mathbf{k}} \delta(\varepsilon_{\mathbf{p}}^e - \varepsilon_{\mathbf{k}}^l - \varepsilon_{\mathbf{p}-\mathbf{k}}^e) n_{\mathbf{p}} (N_{\mathbf{k}} + 1)}_{\text{Fig. 31(b)}} \right] \\
 &\approx \sum_{\mathbf{k}} [w(\mathbf{p} + \mathbf{k}, \mathbf{k})(n_{\mathbf{p}+\mathbf{k}} - n_{\mathbf{p}}) - w(\mathbf{p}, \mathbf{k})(n_{\mathbf{p}} - n_{\mathbf{p}-\mathbf{k}})] N_{\mathbf{k}} \\
 &\approx \frac{\partial}{\partial \mathbf{p}} \cdot \sum_{\mathbf{k}} w(\mathbf{p}, \mathbf{k}) N_{\mathbf{k}} \mathbf{k} \mathbf{k} \cdot \frac{\partial n_{\mathbf{p}}}{\partial \mathbf{p}}, \tag{7.10}
 \end{aligned}$$

where I have expanded twice in small  $\mathbf{k}$  (i.e., in  $\hbar$ ) and abbreviated  $w(\mathbf{p}, \mathbf{k}) = w(\mathbf{p}, \mathbf{k}) \delta(\varepsilon_{\mathbf{p}}^e - \varepsilon_{\mathbf{k}}^l - \varepsilon_{\mathbf{p}-\mathbf{k}}^e)$ . This is a diffusion equation in  $\mathbf{p}$  (or, equivalently, in  $\mathbf{v} = \hbar \mathbf{p}/m_e$ ) space. In view of (7.4), (7.10) has the same form as (6.7), viz.,

$$\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{D}(\mathbf{v}) \cdot \frac{\partial f_0}{\partial \mathbf{v}}, \tag{7.11}$$

where the diffusion matrix is

$$\mathbf{D}(\mathbf{v}) = \sum_{\mathbf{k}} \mathbf{k} \mathbf{k} \frac{\hbar N_{\mathbf{k}}}{m_e^2} w\left(\frac{m_e \mathbf{v}}{\hbar}, \mathbf{k}\right) \delta(\mathbf{k} \cdot \mathbf{v} - \omega_{\mathbf{k}}) = \frac{e^2}{m_e^2} \sum_{\mathbf{k}} \frac{\mathbf{k} \mathbf{k}}{k^2} |\mathbf{E}_{\mathbf{k}}|^2 \pi \delta(\mathbf{k} \cdot \mathbf{v} - \omega_{\mathbf{k}}). \tag{7.12}$$

The last expression is identical to the resonant form of the QL diffusion matrix (6.8) [cf. (6.16) and (6.90)]. To derive it, we used the definition (7.1) of  $N_{\mathbf{k}}$  and the absorption/emission probability (7.7), already known from linear theory.

Thus, we are able to recover the (resonant part of the) QL theory from our new

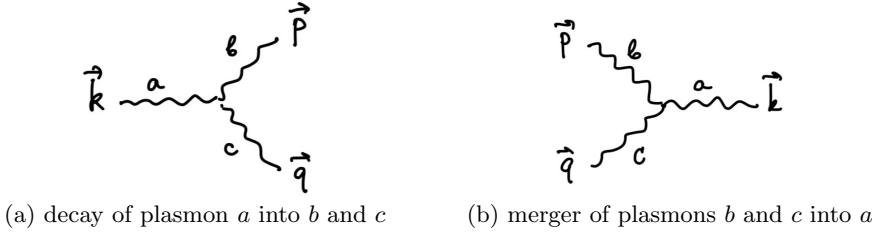


FIGURE 33. Diagrams for (7.15) (three-wave interaction).

electron-plasmon interaction approach. There is more to this approach than a pretty “field-theoretic” reformulation of already-derived earlier results. The diagram technique and the interpretation of the nonlinear state of the plasma as arising from interactions between particles and quasiparticles can be readily generalised to situations in which the nonlinear interactions in (2.12) cannot be neglected and/or more than one type of waves is present. In this new language, the nonlinear interactions would be manifested as interactions between plasmons (rather than only between plasmons and electrons) contributing to the rate of change of  $N_{\mathbf{k}}$ . There are many possibilities: four-plasmon interactions, interactions between plasmons and phonons (sound waves), as well as between the latter and electrons and/or ions, etc. A comprehensive monograph on this subject is Tsytovich (1995).<sup>54</sup> The lectures by Kingsep (1996) are a more human-scale, and humane-style, pedagogical exposition, but are, alas, only available in the original Russian. In §7.2, I will attempt a basic introduction and give a few examples.

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I have introduced the language of kinetics of quasiparticles and their interactions with “true” particles as a reformulation of QLT for plasmas. The method is much more general and originates, as far as I know, from condensed-matter physics, the classic problem being the kinetics of electrons and phonons in metals—the founding texts on this subject are Peierls (1955) and Ziman (1960).

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## 7.2. Weak Turbulence

If we can think about plasmon-electron interactions in terms of occupation numbers and probabilities of absorption/emission, we can also think this way of interactions between plasmons, not necessarily involving electrons or ions directly.

### 7.2.1. Three-Wave Interactions

A plasmon with energy  $\varepsilon_{\mathbf{k}}^a = \hbar\omega_{\mathbf{k}}^a$  and momentum  $\hbar\mathbf{k}$  might “decay” into two other plasmons, with energies and momenta  $\varepsilon_{\mathbf{p}}^b = \hbar\omega_{\mathbf{p}}^b$ ,  $\hbar\mathbf{p}$  and  $\varepsilon_{\mathbf{q}}^c = \hbar\omega_{\mathbf{q}}^c$ ,  $\hbar\mathbf{q}$  (Fig. 33a), or, conversely, the latter two plasmons might merge into one (Fig. 33b). The superscripts  $a$ ,  $b$ , and  $c$  are there to indicate that these plasmons can be of different kinds: e.g., longitudinal Langmuir waves considered in §7.1, sound waves (*phonons*; see §3.8), transverse electromagnetic waves (see Q2). Both the decay and merger processes must conserve

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<sup>54</sup>It is a terrifying remake of his two earlier, equally deadly, books Tsytovich (1970, 1977). He also has a short book of lectures, Tsytovich (1972), which are useful as a kind of extended abstract of his canon, but are not fully self-contained. Two excellent texts on WT that speak a slightly different mathematical language are Zakharov *et al.* (1992) and Nazarenko (2011) (my §§7.3, 7.4, 8.3 and 8.4 were largely inspired by the first of these and the papers that it was based on). Two early-day classics written by Founding Fathers are Kadomtsev (1965) and Sagdeev & Galeev (1969), documenting how it all started. Specifically on weak turbulence of Langmuir waves, there is a long, mushy review by Musher *et al.* (1995).

momentum and energy, viz.,

$$\mathbf{k} = \mathbf{p} + \mathbf{q}, \quad \omega_{\mathbf{k}}^a = \omega_{\mathbf{p}}^b + \omega_{\mathbf{q}}^c, \quad (7.13)$$

and occur with equal probabilities

$$\mathfrak{w}_{\mathbf{k} \rightarrow \mathbf{p}, \mathbf{q}}^{a \rightarrow b+c} = \mathfrak{w}_{\mathbf{p}, \mathbf{q} \rightarrow \mathbf{k}}^{b+c \rightarrow a} \equiv w^{a \leftrightarrow b+c}(\mathbf{k}, \mathbf{p}). \quad (7.14)$$

These constraints are enough for us to infer the general architecture of the evolution equations for the plasmon occupation numbers. Assuming  $a$ ,  $b$ , and  $c$  are all different, the rate of decrease of  $N_{\mathbf{k}}^a$  comes from the first of these processes and the rate of its increase from the second, so, analogously to (7.6),

$$\frac{\partial N_{\mathbf{k}}^a}{\partial t} = \sum_{\mathbf{p}, \mathbf{q}} \delta_{\mathbf{k}, \mathbf{p}+\mathbf{q}} \delta(\omega_{\mathbf{k}}^a - \omega_{\mathbf{p}}^b - \omega_{\mathbf{q}}^c) \underbrace{[-\mathfrak{w}_{\mathbf{k} \rightarrow \mathbf{p}, \mathbf{q}}^{a \rightarrow b+c} N_{\mathbf{k}}^a (N_{\mathbf{p}}^b + 1)(N_{\mathbf{q}}^c + 1)]}_{\text{Fig. 33a}}$$

$$+ \underbrace{\mathfrak{w}_{\mathbf{p}, \mathbf{q} \rightarrow \mathbf{k}}^{b+c \rightarrow a} N_{\mathbf{p}}^b N_{\mathbf{q}}^c (N_{\mathbf{k}}^a + 1)}_{\text{Fig. 33b}}. \quad (7.15)$$

Fig. 33b

The rates are proportional to the products of the occupation numbers of all the participants in the three-plasmon interaction, so they are cubic. The plasmon numbers are not conserved, and, just like in (7.6), the  $+1$ 's in (7.15) appear where the occupation numbers are incremented as a result of the interaction. However, unlike in (7.6), these small corrections are not negligible because if they were neglected, the right-hand of (7.15) would vanish. In view of (7.14), we have (assuming all occupation numbers to be  $\gg 1$ )<sup>55</sup>

$$\boxed{\frac{\partial N_{\mathbf{k}}^a}{\partial t} \approx \sum_{\mathbf{p}} w^{a \leftrightarrow b+c}(\mathbf{k}, \mathbf{p}) \delta(\omega_{\mathbf{k}}^a - \omega_{\mathbf{p}}^b - \omega_{\mathbf{k}-\mathbf{p}}^c) (N_{\mathbf{p}}^b N_{\mathbf{k}-\mathbf{p}}^c - N_{\mathbf{k}}^a N_{\mathbf{p}}^b - N_{\mathbf{k}}^a N_{\mathbf{k}-\mathbf{p}}^c).} \quad (7.16)$$

This is a *kinetic equation* for plasmons of species  $a$  in a system where three-wave interactions are permitted between this species and two others.

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**Exercise 7.1.** (a) Derive the kinetic equations for  $N_{\mathbf{p}}^b$  and  $N_{\mathbf{q}}^c$ , still assuming that  $a, b, c$  are all different. Why does this assumption matter? What are the kinetic equations if  $a = b \neq c$ ?

(b) Confirm that the total energy and momentum of this gas of plasmons, defined analogously to (7.2) and (7.3), respectively, are conserved by the kinetic equations that you have derived.

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When would a contribution to the rate of change of  $N_{\mathbf{k}}^a$  such as the right-hand side of (7.16) be important, in comparison with, e.g., the quasilinear rate (7.8)? Obviously, when the quasilinear rate is small in comparison, e.g., when the waves interact with very few resonant particles in the tail of the distribution,  $\gamma_{\mathbf{k}}$  in (7.8) is exponentially small, and/or  $N_{\mathbf{k}}^a$  fails to saturate quasilinearly before the right-hand side of (7.16) gets larger than  $\gamma_{\mathbf{k}} N_{\mathbf{k}}^a$ . This should be checked in each particular case—I shall talk later on about what kind of particular cases are possible.

It is clear that the right-hand side of (7.16) must originate from the nonlinear term in the kinetic equation (2.12) because it involves coupling between field perturbations at different wave numbers—indeed, the momentum conservation law (7.13) is just the

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<sup>55</sup>Note that if  $N_{\mathbf{k}}^a$  is much larger than all other occupation numbers, the first term in (7.16) can be neglected and the equation takes the form  $\partial N_{\mathbf{k}}^a / \partial t = -\gamma_{\mathbf{k}} N_{\mathbf{k}}^a$ . This is known as a “decay instability”.

restriction on wave numbers arising in the Fourier representation of a quadratic nonlinearity.

How does one complete the derivation of (7.16), i.e., calculate  $w^{a \leftrightarrow b+c}(\mathbf{k}, \mathbf{p})$ ? I am afraid the only way to do it is perturbatively, starting from the Vlasov–Poisson system (2.12) and (2.9) (or some approximation thereto), expanding  $\varphi_{\mathbf{k}}$  into the linear part plus at least one iteration involving the nonlinear term, then working out the rate of change of  $N_{\mathbf{k}}^a \propto k^2 |\varphi_{\mathbf{k}}|^2$ , comparing the result with (7.16) and reading off  $w^{a \leftrightarrow b+c}(\mathbf{k}, \mathbf{p})$ —just as we did in (7.9). What then, might you ask, is the point of the quasiparticle formalism if in the end we must revert back to laborious perturbation theory? Well, apart from giving one copious amounts of physical insight and a pleasing impression of doing field-theoretic calculations, this approach actually reduces the amount of hard labour involved in that perturbation theory: having applied the constraints imposed by the conservation laws and by the principle of equal probabilities (7.14), we were able to establish the general structure of the answer, viz., that the coefficients in front of the three terms in (7.16) must be the same—the same coefficients also turn up in the kinetic equations for  $N_{\mathbf{p}}^b$  and  $N_{\mathbf{q}}^c$  derived in Exercise 7.1. This means that, in ploughing through the perturbation theory, you only need to keep track of contributions that give rise to just one of these terms, thus cutting down the amount of algebra by a factor of 9. Once you try these calculations, you will be very grateful for such a concession!<sup>56</sup>

Since the nonlinearity in (2.12) is quadratic, does this mean that the rate of change of  $N_{\mathbf{k}}^a$  must always be quadratic in plasmon occupation numbers? Not at all, because the probability  $w^{a \leftrightarrow b+c}(\mathbf{k}, \mathbf{p})$  of any particular three-wave process can easily turn out to be zero. Indeed, the conservation laws (7.13) are not necessarily that easy to satisfy. For example, consider the case of three long-wavelength Langmuir plasmons, all with  $k\lambda_{De} \ll 1$ . Their frequency is given by (3.39), viz.,

$$\omega_{\mathbf{k}} \approx \omega_{pe} \left( 1 + \frac{3}{4} k^2 \lambda_{De}^2 \right). \quad (7.17)$$

Clearly, the frequency condition in (7.13) is unfulfillable—you cannot couple two long-wavelength Langmuir plasmons and get a third such plasmon. What happens then, in a gas of Langmuir plasmons? There are two ways of getting a non-zero rate of change of  $N_{\mathbf{k}}$  in such a situation.

**Exercise 7.2.** It turns out that a three-plasmon process involving three ion acoustic waves (§3.8) is also impossible. Why is this?

### 7.2.2. Four-Wave Interactions

The first possibility is to go to next order: while three plasmons cannot couple, four can:

$$\mathbf{k} + \mathbf{k}' = \mathbf{p} + \mathbf{p}', \quad \omega_{\mathbf{k}} + \omega_{\mathbf{k}'} = \omega_{\mathbf{p}} + \omega_{\mathbf{p}'}. \quad (7.18)$$

The probability of this “plasmon-scattering” process, or its inverse, is

$$\mathfrak{w}_{\mathbf{k}, \mathbf{k}' \rightarrow \mathbf{p}, \mathbf{p}'} = \mathfrak{w}_{\mathbf{p}, \mathbf{p}' \rightarrow \mathbf{k}, \mathbf{k}'} \equiv w(\mathbf{k}, \mathbf{p}, \mathbf{p}'). \quad (7.19)$$

<sup>56</sup>I will provide one such calculation in §8.4 and ask you to do another one in Exercises 8.7 and 8.9 (if you are brave, do also the open-ended Exercise 8.10). My vehicle for those will be a simplified set of equations describing the coupled dynamics of plasmons, phonons, and ions. An ultimate repository (one might say graveyard) of many such calculations, done in a fully general setting, is Tsytovich (1995).

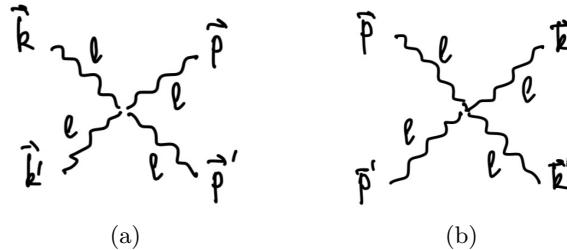


FIGURE 34. Diagrams for (7.20) (four-wave interaction): plasmons scattering off each other.

The kinetic equation is a sum of the two diagrams in Fig. 34:

$$\begin{aligned} \frac{\partial N_{\mathbf{k}}}{\partial t} &= \sum_{\mathbf{k}', \mathbf{p}, \mathbf{p}'} \delta_{\mathbf{k} + \mathbf{k}' \mathbf{p} + \mathbf{p}'} \delta(\omega_{\mathbf{k}} + \omega_{\mathbf{k}'} - \omega_{\mathbf{p}} - \omega_{\mathbf{p}'}) w(\mathbf{k}, \mathbf{p}, \mathbf{p}') \\ &\quad \times \underbrace{[-N_{\mathbf{k}} N_{\mathbf{k}'} (N_{\mathbf{p}} + 1) (N_{\mathbf{p}'} + 1) + N_{\mathbf{p}} N_{\mathbf{p}'} (N_{\mathbf{k}} + 1) (N_{\mathbf{k}'} + 1)]}_{= N_{\mathbf{p}} N_{\mathbf{p}'} (N_{\mathbf{k}} + N_{\mathbf{k}'}) - N_{\mathbf{k}} N_{\mathbf{k}'} (N_{\mathbf{p}} + N_{\mathbf{p}'})}. \end{aligned} \quad (7.20)$$

The lowest-order terms have again cancelled out. The algebra discount for doing the perturbation theory is buy one—get three free (see Exercise 8.7).

### 7.2.3. Langmuir-Sound Turbulence

The second possibility is that a Langmuir plasmon decays into something else. One obviously attractive, and realisable, option is that it should decay into a phonon ( $s$ ) and another, slightly lower-frequency Langmuir plasmon ( $l$ ) (Fig. 35a):

$$\omega_{\mathbf{p}}^l = \omega_{\mathbf{k}}^s + \omega_{\mathbf{p}-\mathbf{k}}^l. \quad (7.21)$$

Indeed this turns out to be the only three-wave process allowed for longitudinal plasma waves. Even this is only allowed—obviously—provided sound waves can actually exist, i.e., if  $T_i \ll T_e$  (see §3.8).<sup>57</sup>

Note that this process is formally very similar to the one that was considered in §7.1: there, an electron emitted or absorbed a plasmon; here, a plasmon emits or absorbs a phonon. The difference is that in §7.1, we could assume that the electron occupation numbers were small but electron momenta large in comparison with the plasmons, whereas here the occupation numbers and momenta of the plasmons and the phonons can be in any relationship to each other (although the plasmons' energies are, of course, much greater than the phonons').

**Exercise 7.3.** (a) Work out under what assumption the energy conservation condition (7.21) can be turned into the wave-resonance condition  $\omega_k^s = \mathbf{k} \cdot \mathbf{v}_k^l$ , where  $\mathbf{v}_k^l$  is the group velocity of the Langmuir wave.

(b) Show, in fact, that the “optimal” phonon for a plasmon with momentum  $\hbar\mathbf{p}$  to emit has  $\mathbf{k} \approx 2\mathbf{p}$ . Thus, phonons are pushed to larger  $k$ 's (smaller scales).

Let the probability of the plasmon-decay/phonon-emission process, or its inverse (Fig. 35b), be  $w(\mathbf{p}, \mathbf{k})$ , and, for compactness of notation, let

$$w(p, k) = w(p, k)\delta(\omega_p^l - \omega_{p-k}^l - \omega_k^s). \quad (7.22)$$

<sup>57</sup>If sound waves are strongly damped (e.g., at  $T_i = T_e$ ), four-wave interactions (§7.2.2) and/or induced-scattering processes of the kind considered in §7.2.4 have to be invoked to describe Langmuir turbulence; see §8.2.7 and Exercises 8.7 and 8.10.

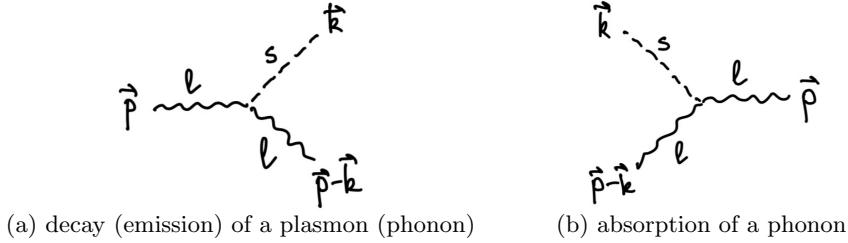


FIGURE 35. Diagrams for (7.23).

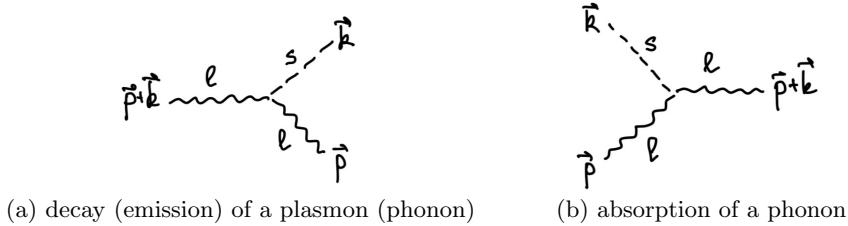


FIGURE 36. Additional diagrams for (7.24).

Tallying up the relevant diagrams gets us the results that were, in fact, already anticipated in Exercise 7.1.

The two diagrams in Fig. 35 give us the rate of change of the number of phonons:

$$\frac{\partial N_{\mathbf{k}}^s}{\partial t} = \sum_{\mathbf{p}} w(\mathbf{p}, \mathbf{k}) (N_{\mathbf{p}}^l N_{\mathbf{p}-\mathbf{k}}^l + N_{\mathbf{p}}^l N_{\mathbf{k}}^s - N_{\mathbf{p}-\mathbf{k}}^l N_{\mathbf{k}}^s) \equiv S_{\mathbf{k}} + \gamma_{\mathbf{k}}^s N_{\mathbf{k}}^s. \quad (7.23)$$

Thus, in the initial absence of phonons, there is a source of them ( $S_{\mathbf{k}} > 0$ ), due to plasmon decay. In their presence, there is a certain rate at which they might be reabsorbed. The sign  $\gamma_{\mathbf{k}}^s$  is not obviously fixed, but it is clear that it should be negative if there is to be a steady state. If we assumed  $N_{\mathbf{p}}^l \ll N_{\mathbf{k}}^s$  and  $k \ll p$ , we would end up with a very similar situation to (7.8): a “Landau damping” of phonons on plasmons (see Exercise 8.13)—or an instability, depending on the sign of  $\gamma_{\mathbf{k}}^s$ , which is determined by the functional form of the plasmon spectrum  $N_{\mathbf{p}}^l$ .

To work out the rate of change of  $N_{\mathbf{p}}^l$ , we need to take into account the diagrams that went into (7.23) plus two more, shown in Fig. 36,—entirely analogously to the derivation of the QL equation for electrons in §7.1.2 (cf. Fig. 32). The difference now is that, in general, the occupation numbers of plasmons are not much smaller than those of the phonons, and also their momenta  $\mathbf{p}$  and  $\mathbf{k}$  are of the same order, rather than  $k \ll p$ . The result is

$$\begin{aligned} \frac{\partial N_{\mathbf{p}}^l}{\partial t} = & \sum_{\mathbf{k}} \underbrace{[w(\mathbf{p}, \mathbf{k}) (N_{\mathbf{p}-\mathbf{k}}^l N_{\mathbf{k}}^s - N_{\mathbf{p}-\mathbf{k}}^l N_{\mathbf{p}}^l - N_{\mathbf{k}}^s N_{\mathbf{p}}^l)]}_{\text{Fig. 35}} \\ & + \underbrace{w(\mathbf{p} + \mathbf{k}, \mathbf{k}) (N_{\mathbf{p}+\mathbf{k}}^l N_{\mathbf{k}}^s + N_{\mathbf{p}+\mathbf{k}}^l N_{\mathbf{p}}^l - N_{\mathbf{k}}^s N_{\mathbf{p}}^l)}_{\text{Fig. 36}} \end{aligned} \equiv M_{\mathbf{p}} + \gamma_{\mathbf{p}}^l N_{\mathbf{p}}^l. \quad (7.24)$$

In the initial absence of phonons, we find an exponential change, at the rate  $\gamma_{\mathbf{p}}^l$ , in the number of plasmons at a given  $\mathbf{p}$ , due to emission of phonons—the counterpart to the source term in (7.23). In the presence of phonons, this rate is modified (downwards—the sign of this contribution *is* definite), to account for the reabsorption of the phonons,

and there is also a mode-coupling term  $M_{\mathbf{p}}$  (containing the  $N_{\mathbf{p}-\mathbf{k}}^l N_{\mathbf{k}}^s$  and  $N_{\mathbf{p}+\mathbf{k}}^l N_{\mathbf{k}}^s$  contributions). Again, assuming  $N_{\mathbf{p}}^l \ll N_{\mathbf{k}}^s$  and  $k \ll p$  would turn (7.24) into something very similar to the QL diffusion equation (7.10) (see Exercise 8.14), but this is not, generally speaking, the right regime this time (I will return to it in §8.5).

The Langmuir–sound system will be my workhorse example, so, to reduce clutter, let me simplify notation a bit:

$$N_{\mathbf{p}}^l = N_{\mathbf{p}}, \quad N_{\mathbf{k}}^s = n_{\mathbf{k}}, \quad \omega_{\mathbf{p}}^l = \omega_{\mathbf{p}}, \quad \omega_{\mathbf{k}}^s = \Omega_{\mathbf{k}}. \quad (7.25)$$

The kinetic equations (7.23) and (7.24) can be written rather compactly as follows

$$\frac{\partial n_{\mathbf{k}}}{\partial t} = \sum_{\mathbf{p}} T_{\mathbf{p},\mathbf{k}}, \quad \frac{\partial N_{\mathbf{p}}}{\partial t} = \sum_{\mathbf{k}} (T_{\mathbf{p}+\mathbf{k},\mathbf{k}} - T_{\mathbf{p},\mathbf{k}}) ,$$

(7.26)

where  $T_{\mathbf{p},\mathbf{k}}$  is the expression under the wave-number sum in (7.23), viz.,

$$T_{\mathbf{p},\mathbf{k}} = w(\mathbf{p}, \mathbf{k}) \delta(\omega_{\mathbf{p}} - \omega_{\mathbf{p}-\mathbf{k}} - \Omega_{\mathbf{k}}) [N_{\mathbf{p}} N_{\mathbf{p}-\mathbf{k}} + (N_{\mathbf{p}} - N_{\mathbf{p}-\mathbf{k}}) n_{\mathbf{k}}]. \quad (7.27)$$

I will call  $T_{\mathbf{p},\mathbf{k}}$  the *transfer function* as it quantifies the shuffling of energy (and of plasmon number) between wave numbers.

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**Exercise 7.4.** Confirm by direct calculation the fact (obvious from the diagrams) that the kinetic equations for the Langmuir–sound turbulence (§7.2.3) and for the Langmuir turbulence with four-wave interactions (§7.2.2) conserve the total number of plasmons  $N = \sum_{\mathbf{p}} N_{\mathbf{p}}$ , as well as the total energy and the total momentum of the quasiparticle gas.

The quasiparticle formalism expounded above gives “kinetic theory” an extended meaning—kinetics both of “real” particles and the waves. Interestingly, for the interactions that involve exclusively waves, the underlying dynamics are in fact fluid, at least approximately: one does not need to worry about weak kinetic damping effects on Langmuir or sound waves in order to derive the “kinetic equations” (7.23) and (7.24) (I will show this formally in §8). But not all nonlinear effects, and not all WT, are fluid-dynamical in this sense. There are nonlinear interactions that can be described by WT and that involve “true” kinetic effects—i.e., wave-particle resonances. The best known example of these is presented in the next section.

#### 7.2.4. Wave–Particle Interactions: Induced Scattering

In §7.1, we saw how to handle the interactions involving one plasmon and one “real” particle (an electron). There is a nonlinear version of these interactions, involving two plasmons: a plasmon with momentum  $\hbar\mathbf{k}$  meets a particle with momentum  $\hbar\mathbf{p} = m\mathbf{v}$ , gets absorbed, particle gets excited, cannot contain itself, immediately emits another plasmon with momentum  $\hbar\mathbf{k}'$ , and speeds away with momentum  $\hbar\mathbf{p}'$  (Fig. 37a). This is called *induced scattering* (or *stimulated emission*). The conservation laws are [cf. (7.5)]<sup>58</sup>

$$\mathbf{k} + \mathbf{p} = \mathbf{k}' + \mathbf{p}' \Rightarrow \mathbf{p}' = \mathbf{p} + \mathbf{k} - \mathbf{k}', \quad (7.28)$$

$$0 = \varepsilon_{\mathbf{k}}^l + \varepsilon_{\mathbf{p}}^e - \varepsilon_{\mathbf{k}'}^l - \varepsilon_{\mathbf{p}'}^e = \hbar\omega_{\mathbf{k}} + \frac{\hbar^2 p^2}{2m} - \hbar\omega_{\mathbf{k}'} - \frac{\hbar^2 |\mathbf{p} + \mathbf{k} - \mathbf{k}'|^2}{2m} \approx \hbar [\omega_{\mathbf{k}} - \omega_{\mathbf{k}'} - (\mathbf{k} - \mathbf{k}') \cdot \mathbf{v}]. \quad (7.29)$$

The probability of this process, and of its inverse (Fig. 37b), is  $w(\mathbf{p}, \mathbf{k}, \mathbf{k}')$ , and I shall again abbreviate  $w(\mathbf{p}, \mathbf{k}, \mathbf{k}') = w(\mathbf{p}, \mathbf{k}, \mathbf{k}') \delta(\varepsilon_{\mathbf{k}}^l + \varepsilon_{\mathbf{p}}^e - \varepsilon_{\mathbf{k}'}^l - \varepsilon_{\mathbf{p}+\mathbf{k}-\mathbf{k}'}^e)$ .

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<sup>58</sup>There is nothing in this consideration that requires the particle to be an electron, it can perfectly well be an ion. Indeed, for Langmuir waves, the latter possibility is a more relevant one, and will be considered in Exercise 8.10.

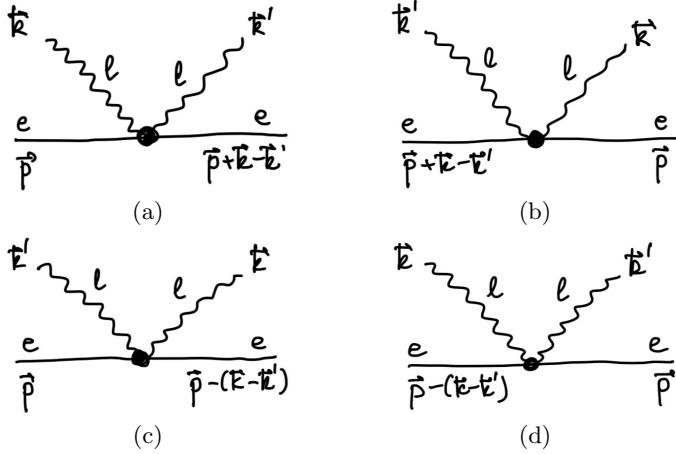


FIGURE 37. Diagrams for (7.30) and (7.32): induced scattering.

We are ready to calculate. The diagrams in Fig. 37(a,b) give us

$$\begin{aligned} \frac{\partial N_{\mathbf{k}}}{\partial t} &= \sum_{\mathbf{p}, \mathbf{k}'} w(\mathbf{p}, \mathbf{k}, \mathbf{k}') [-N_{\mathbf{k}} n_{\mathbf{p}}(N_{\mathbf{k}'} + 1) + N_{\mathbf{k}'} n_{\mathbf{p} + \mathbf{k} - \mathbf{k}'}(N_{\mathbf{k}} + 1)] \\ &\approx \sum_{\mathbf{p}, \mathbf{k}'} w(\mathbf{p}, \mathbf{k}, \mathbf{k}') (\mathbf{k} - \mathbf{k}') \cdot \frac{\partial n_{\mathbf{p}}}{\partial \mathbf{p}} N_{\mathbf{k}'} N_{\mathbf{k}} \equiv 2\gamma_{\mathbf{k}}^{\text{nl}} N_{\mathbf{k}}. \end{aligned} \quad (7.30)$$

This is rather similar to the Landau-damping equation (7.8), except with a new rate, which, expressed in terms of  $f_0(\mathbf{v})$  using (7.4), works out to be

$$2\gamma_{\mathbf{k}}^{\text{nl}} = \sum_{\mathbf{k}'} N_{\mathbf{k}'} \int d\mathbf{v} \frac{V}{m} w\left(\frac{m\mathbf{v}}{\hbar}, \mathbf{k}, \mathbf{k}'\right) \delta(\omega_{\mathbf{k}} - \omega_{\mathbf{k}'} - (\mathbf{k} - \mathbf{k}') \cdot \mathbf{v}) (\mathbf{k} - \mathbf{k}') \cdot \frac{\partial f_0}{\partial \mathbf{v}}. \quad (7.31)$$

This is sometimes called “nonlinear Landau damping”—confusingly, because it has nothing to do with the suppression of Landau damping by particle trapping (O’Neil 1965; Mazitov 1965), usually called that.

Finally, adding up the diagrams in Fig. 37(a,b) and two similar ones in Fig. 37(c,d), we get the equation for the electron distribution function: denoting  $\mathbf{k}'' = \mathbf{k} - \mathbf{k}'$ ,

$$\begin{aligned} \frac{\partial n_{\mathbf{p}}}{\partial t} &\approx \sum_{\mathbf{k}, \mathbf{k}'} N_{\mathbf{k}} N_{\mathbf{k}'} [w(\mathbf{p}, \mathbf{k}, \mathbf{k}')(n_{\mathbf{p} + \mathbf{k}''} - n_{\mathbf{p}}) - w(\mathbf{p} - \mathbf{k}'', \mathbf{k}, \mathbf{k}')(n_{\mathbf{p}} - n_{\mathbf{p} - \mathbf{k}''})] \\ &\approx \frac{\partial}{\partial \mathbf{p}} \cdot \sum_{\mathbf{k}, \mathbf{k}'} w(\mathbf{p}, \mathbf{k}, \mathbf{k}') N_{\mathbf{k}} N_{\mathbf{k}'} \mathbf{k}'' \mathbf{k}'' \cdot \frac{\partial n_{\mathbf{p}}}{\partial \mathbf{p}} \quad \Leftrightarrow \quad \frac{\partial f_0}{\partial t} = \frac{\partial}{\partial \mathbf{v}} \cdot D^{\text{nl}}(\mathbf{v}) \cdot \frac{\partial f_0}{\partial \mathbf{v}}. \end{aligned} \quad (7.32)$$

This is a diffusion equation, like (7.11), but now with a “nonlinear” diffusion coefficient that depends quadratically on the electric-field spectrum. Note that since this effect is second order in  $N_{\mathbf{k}}$ , it should, formally, be subdominant to the QL diffusion (7.11), which is first order. However, the resonance (7.29) is much easier to achieve because, rather than matching  $\omega_{\mathbf{k}}$  to  $\mathbf{k} \cdot \mathbf{v}$ , which might push you far into the tail of the distribution, here all you need to do is match the frequency difference  $\omega_{\mathbf{k}} - \omega_{\mathbf{k}'}$  for some pair of wave numbers to  $(\mathbf{k} - \mathbf{k}') \cdot \mathbf{v}$  (so, roughly, you are matching the particle’s velocity not to the wave’s phase velocity but to its group velocity). This accommodating nature of induced scattering is its claim to ubiquitous relevance.

There is an important observation that one can make without knowing  $w(\mathbf{p}, \mathbf{k}, \mathbf{k}')$ . Since (7.32) is a diffusion equation, it should cause  $f_0(\mathbf{v})$  to broaden with time, i.e., the particle distribution will heat up (cf. Q11)—this is called *turbulent heating*. Since energy is conserved, the plasmon population should on average lose energy, shifting to lower frequencies and lower wave numbers—this is corroborated by quantitative calculations (see Tsytovich’s books or the original papers by Pikelner & Tsytovich 1969 and Liperovskii & Tsytovich 1970, as well as Musher *et al.* 1995, who review further interesting complications). Therefore, WT will push waves into a region of

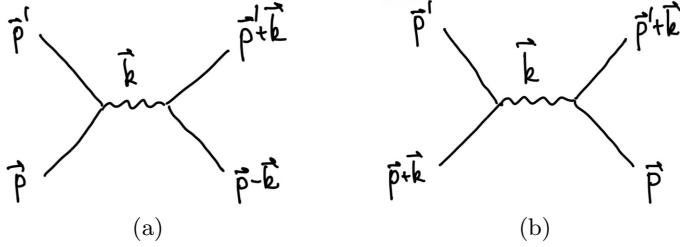


FIGURE 38. Diagrams for (7.34): binary collisions.

$\mathbf{k}$  space where the WT approximation ceases to be valid and the turbulence becomes strong—I will come back to this point in §7.5.

Working out  $w(\mathbf{p}, \mathbf{k}, \mathbf{k}')$  for this problem is a bit of a nightmare—or, depending on your attitude, a character-building endeavour (Exercise 8.10). You will find the answer in Kadomtsev (1965), Kingsep (1996), or, alongside 1001 similar calculations of probabilities of WT processes, in Tsy托ovich (1995),

### 7.2.5. A Digression: “Real” Collisions

Finally, let me give you another example of the use of the diagram technique, which, while not properly a WT calculation, illustrates the unifying nature of the formalism. If we can talk about interactions between waves and particles, why not use the same language for interactions between particles and thus work out the general form of the collision integral previewed in (1.47).

A binary Coulomb collision can be thought of as two particles meeting, exchanging a plasmon with a certain energy and momentum, and parting ways (Fig. 38a). For simplicity, consider two particles of the same species. The energy conservation requirement is

$$0 = \varepsilon_{\mathbf{p}} + \varepsilon_{\mathbf{p}'} - \varepsilon_{\mathbf{p}-\mathbf{k}} - \varepsilon_{\mathbf{p}+\mathbf{k}} = \frac{\hbar^2}{2m} (p^2 + p'^2 - |\mathbf{p} - \mathbf{k}|^2 - |\mathbf{p}' + \mathbf{k}|^2) \approx \frac{\hbar^2}{m} \mathbf{k} \cdot (\mathbf{p} - \mathbf{p}'), \quad (7.33)$$

assuming at the last step that  $k \ll p, p'$  (glancing collisions). Let the probability of this process be  $w(\mathbf{p}, \mathbf{p}', \mathbf{k})$  and abbreviate  $w(\mathbf{p}, \mathbf{p}', \mathbf{k}) = w(\mathbf{p}, \mathbf{p}', \mathbf{k}) \delta(\varepsilon_{\mathbf{p}} + \varepsilon_{\mathbf{p}'} - \varepsilon_{\mathbf{p}-\mathbf{k}} - \varepsilon_{\mathbf{p}+\mathbf{k}})$ . Adding up the diagrams in Fig. 38, and their inverses, we get

$$\begin{aligned} \frac{\partial n_{\mathbf{p}}}{\partial t} &= \sum_{\mathbf{k}, \mathbf{p}'} [w(\mathbf{p}, \mathbf{p}', \mathbf{k}) (-n_{\mathbf{p}} n_{\mathbf{p}'} + n_{\mathbf{p}-\mathbf{k}} n_{\mathbf{p}'+\mathbf{k}}) + w(\mathbf{p} + \mathbf{k}, \mathbf{p}', \mathbf{k}) (n_{\mathbf{p}+\mathbf{k}} n_{\mathbf{p}'} - n_{\mathbf{p}} n_{\mathbf{p}'+\mathbf{k}})] \\ &\approx \sum_{\mathbf{k}, \mathbf{p}'} \left[ w(\mathbf{p}, \mathbf{p}', \mathbf{k}) \left( n_{\mathbf{p}} \mathbf{k} \cdot \frac{\partial n_{\mathbf{p}'}}{\partial \mathbf{p}'} - n_{\mathbf{p}'} \mathbf{k} \cdot \frac{\partial n_{\mathbf{p}}}{\partial \mathbf{p}} \right) \right. \\ &\quad \left. + w(\mathbf{p} + \mathbf{k}, \mathbf{p}', \mathbf{k}) \left( n_{\mathbf{p}'} \mathbf{k} \cdot \frac{\partial n_{\mathbf{p}+\mathbf{k}}}{\partial \mathbf{p}} - n_{\mathbf{p}} \mathbf{k} \cdot \frac{\partial n_{\mathbf{p}'} }{\partial \mathbf{p}'} \right) \right] \\ &\approx \frac{\partial}{\partial \mathbf{p}} \cdot \sum_{\mathbf{k}, \mathbf{p}'} w(\mathbf{p}, \mathbf{p}', \mathbf{k}) \mathbf{k} \mathbf{k} \cdot \left( n_{\mathbf{p}'} \frac{\partial n_{\mathbf{p}}}{\partial \mathbf{p}} - n_{\mathbf{p}} \frac{\partial n_{\mathbf{p}'} }{\partial \mathbf{p}'} \right). \end{aligned} \quad (7.34)$$

Rewriting this in terms of  $f_0(\mathbf{v})$ , we get

$$\begin{aligned} \frac{\partial f_0(\mathbf{v})}{\partial t} &= \frac{\partial}{\partial \mathbf{v}} \cdot \int d\mathbf{v}' \sum_{\mathbf{k}} \frac{V\hbar}{m^2} w \left( \frac{m\mathbf{v}}{\hbar}, \frac{m\mathbf{v}'}{\hbar}, \mathbf{k} \right) \delta(\mathbf{k} \cdot (\mathbf{v} - \mathbf{v}')) \\ &\quad \mathbf{k} \mathbf{k} \cdot \left[ f_0(\mathbf{v}') \frac{\partial f_0(\mathbf{v})}{\partial \mathbf{v}} - f_0(\mathbf{v}) \frac{\partial f_0(\mathbf{v}')}{\partial \mathbf{v}'} \right]. \end{aligned} \quad (7.35)$$

This is indeed the general Fokker–Planck form (1.47) of the collision integral. The probability  $w$  will be worked out in §§11.5 and 11.6.

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**Exercise 7.5. Multispecies collisions.** Generalise this construction to binary collisions between particles of different species.

You will find a diagrammatic derivation of the collision integral for gravitating systems, analogous to (7.35), in a recent paper by [Hamilton \(2021\)](#), who claims to have been inspired by these lectures—from classroom to research frontier in one quick leap, a good example to follow!

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### 7.3. Statistical Mechanics of Quasiparticles

#### 7.3.1. Entropy

The expressions for the rates of change of quasiparticle occupation numbers are a kind of collision integrals for the quasiparticles (the analogy reinforced by §7.2.5). It is not hard to see that they have an entropy and an  $H$ -theorem. The quasiparticles are indistinguishable bosons (no exclusion principle), so their entropy ought to be

$$S = - \sum_{\mathbf{k}} [N_{\mathbf{k}} \ln N_{\mathbf{k}} - (1 + N_{\mathbf{k}}) \ln(1 + N_{\mathbf{k}})] \approx \sum_{\mathbf{k}} \ln N_{\mathbf{k}}. \quad (7.36)$$

The simple approximate expression arises because all occupation numbers are  $N_{\mathbf{k}} \gg 1$ . In other words, the entropy of the quasiparticle gas is just the sum of Boltzmann entropies associated with each  $\mathbf{k}$ .

Let us prove that this entropy cannot decrease: e.g., for the Langmuir–sound system (7.26),

$$\begin{aligned} \frac{dS}{dt} &= \sum_{\mathbf{k}} \frac{1}{n_{\mathbf{k}}} \frac{\partial n_{\mathbf{k}}}{\partial t} + \sum_{\mathbf{p}} \frac{1}{N_{\mathbf{p}}} \frac{\partial N_{\mathbf{p}}}{\partial t} = \sum_{\mathbf{k}, \mathbf{p}} \left( \frac{T_{\mathbf{p}, \mathbf{k}}}{n_{\mathbf{k}}} + \frac{T_{\mathbf{p+k}, \mathbf{k}}}{N_{\mathbf{p}}} - \frac{T_{\mathbf{p}, \mathbf{k}}}{N_{\mathbf{p}}} \right) \\ &= \sum_{\mathbf{k}, \mathbf{p}} T_{\mathbf{p}, \mathbf{k}} \left( \frac{1}{n_{\mathbf{k}}} + \frac{1}{N_{\mathbf{p-k}}} - \frac{1}{N_{\mathbf{p}}} \right) \\ &= \sum_{\mathbf{k}, \mathbf{p}} w(\mathbf{p}, \mathbf{k}) \frac{[N_{\mathbf{p}} N_{\mathbf{p-k}} + (N_{\mathbf{p}} - N_{\mathbf{p-k}}) n_{\mathbf{k}}]^2}{n_{\mathbf{k}} N_{\mathbf{p}} N_{\mathbf{p-k}}} \geq 0, \quad \text{q.e.d.} \end{aligned} \quad (7.37)$$

---

**Exercise 7.6.** Prove that the  $H$ -theorem holds for the four-wave kinetic equation (7.20).

You might be wondering—legitimately—how it happened that, treating a collisionless and, therefore, reversible, system, we have ended up with an  $H$ -theorem and, therefore, irreversibility. Obviously, an implicit assumption has been made that must be the quasiparticle equivalent of the hypothesis of molecular chaos—in WT, this is the so called “random-phase approximation”, which I will introduce in §8.4.2, once I have a specific example of underlying dynamical equations to work with.

#### 7.3.2. Thermodynamical Equilibrium Distributions

Where there is entropy increase, there is thermodynamical equilibrium. The standard operating procedure is to maximise entropy subject to holding constant whatever invariants the system has. The Langmuir–sound system has two: the total energy  $\mathcal{E}$  and the total number of plasmons  $N$  (see Exercise 7.4).<sup>59</sup> Therefore, we must solve the maximisation problem

$$S - \beta \left( \sum_{\mathbf{k}} \hbar \Omega_{\mathbf{k}} n_{\mathbf{k}} + \sum_{\mathbf{p}} \hbar \omega_{\mathbf{p}} N_{\mathbf{p}} - \mathcal{E} \right) + \beta \mu \left( \sum_{\mathbf{p}} N_{\mathbf{p}} - N \right) \rightarrow \max, \quad (7.38)$$

---

<sup>59</sup> As usual in statistical mechanics, we do not need to worry about the total momentum because we can always transfer ourselves into the reference frame moving with the total momentum of our system.

where  $\beta = 1/T$  and  $\mu$  are Lagrange multipliers (temperature and chemical potential). Varying the above expression with respect to  $n_{\mathbf{k}}$  and  $N_{\mathbf{p}}$ , we get the equilibrium distributions of the plasmon and phonon fields:

$$\boxed{n_{\mathbf{k}} = \frac{T}{\hbar\Omega_{\mathbf{k}}}, \quad N_{\mathbf{p}} = \frac{T}{\hbar\omega_{\mathbf{p}} - \mu}}. \quad (7.39)$$

The distributions (7.39) are *Rayleigh–Jeans distributions*, familiar from the statistical mechanics of radiation.

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**Exercise 7.7.** Verify by direct calculation that the Rayleigh–Jeans distributions (7.39) are indeed stationary solutions of the kinetic equations (7.26).

**Exercise 7.8.** An attempt to integrate these distributions to find  $T$  and  $\mu$  in terms of  $\mathcal{E}$  and  $N$  produces divergent results. Is this a problem? How can it be fixed?

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#### 7.4. Stationary Nonequilibrium Distributions

I have kept using the word “turbulence”, but I actually I have not done any turbulence so far—just derived generic kinetic equations for weakly interacting quasiparticles (§7.2) and shown that they support some fairly bland thermal equilibria (§7.3). The interesting applications are indeed to turbulence—and that means to systems that are *not* in thermal equilibrium.

Since our kinetic equations conserve energy, that means that their right-hand sides cannot either generate or dissipate any net energy—they can only move energy around between different wave numbers or exchange it between different species of waves. Not worrying about the latter for a moment, assuming wave-number isotropy, and defining the  $k$ -shell-integrated total-energy spectrum for the Langmuir–sound system as<sup>60</sup>

$$\bar{E}_k = \frac{V k^2}{2\pi^2} (\hbar\Omega_{\mathbf{k}} n_{\mathbf{k}} + \hbar\omega_{\mathbf{k}} N_{\mathbf{k}}), \quad \int dk \bar{E}_k = \mathcal{E}, \quad (7.40)$$

it must be possible to write the evolution equation for it in terms of an energy flux  $\varepsilon_k$ :

$$\frac{\partial \bar{E}_k}{\partial t} = \frac{V k^2}{2\pi^2} \sum_{\mathbf{p}} [\hbar\Omega_{\mathbf{k}} T_{\mathbf{p},\mathbf{k}} + \hbar\omega_{\mathbf{k}} (T_{\mathbf{k+p,p}} - T_{\mathbf{k,p}})] \equiv -\frac{\partial \varepsilon_k}{\partial k}. \quad (7.41)$$

Since the plasmon number  $N = \sum_{\mathbf{p}} N_{\mathbf{p}} \equiv \int dp \bar{N}_p$  is also conserved,  $\bar{N}_p = (V p^2 / 2\pi^2) N_{\mathbf{p}}$  satisfies

$$\frac{\partial \bar{N}_p}{\partial t} = \frac{V p^2}{2\pi^2} \sum_{\mathbf{k}} (T_{\mathbf{p+k,k}} - T_{\mathbf{p,k}}) \equiv -\frac{\partial \Gamma_p}{\partial p}, \quad (7.42)$$

where  $\Gamma_p$  is the plasmon-number flux (also known as the “wave-action flux”). Stationary solutions can be of two flavours: those with zero flux and those with a constant flux. The Rayleigh–Jeans solutions (7.39) are of the former kind: nothing flows anywhere because there are no sources or sinks of either energy or particles. If there were, there would not be an equilibrium, and a (statistically) stationary state would have to feature the invariants constantly flowing from sources to sinks. This is exactly what a turbulent system is: a chaotic system in which energy and other invariants, if they are there, are constantly injected at some scales (i.e., some  $k$ 's) and flow to other scales where they can be dissipated. The sources and sinks are not in the kinetic equations that we have derived—they must be added as they represent some additional physics, not covered

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<sup>60</sup>The prefactor comes from  $\sum_{\mathbf{k}} = V \int d\mathbf{k}/(2\pi)^3 = V \int dk 4\pi k^2/(2\pi)^3 = (V/2\pi^2) \int dk k^2$ .

by nonlinear interactions, which are conservative. Injection can be (and usually is in plasma physics) done by some instability; dissipation is ultimately always by collisions, but, in “collisionless” (i.e., weakly collisional) systems, those are usually accessed via phase mixing (Landau damping)—this was discussed quite thoroughly in §5, although to what extent that discussion applies to turbulent systems is a tricky question (see §12). If the injection and dissipation are concentrated at sufficiently disparate wave numbers, there will be scale-invariant  $k$ -space intervals in between that energy and other invariants must flow through in order to get dissipated—it is in these intervals that constant-flux solutions will materialise. Working out these solutions, by setting

$$\varepsilon_k = \varepsilon = \text{const}, \quad \Gamma_p = \Gamma = \text{const}, \quad (7.43)$$

and the directions of the corresponding fluxes (the signs of  $\varepsilon$  and  $\Gamma$ ) is what we shall need the kinetic equations for (see §§8.4.3–8.4.6).

There are three further nuances. First, valid constant-flux solutions will only be obtainable from kinetic equations without the need to include sources and sinks if the interactions are *local* in  $k$ , i.e., if the sources and the sinks cannot couple to each other directly (otherwise energy might just “leap” directly from sources to sinks). There will be a scheme for checking whether that is true (see the end of §8.4.6). Secondly, stationary solutions of the kinetic equations are only of any use if they are *stable*. That too must be, and can be, checked within the WT formalism (I shall not deal with this; see [Zakharov et al. 1992](#)). Thirdly, the  $k$ -space *isotropy* assumption is not always good, and there is an interesting zoo of non-isotropic possibilities (I will ignore them all; see, e.g., [Musher et al. 1995](#)).

Let us keep all this in mind, but this is about as far as one can get without actually calculating  $w(\mathbf{p}, \mathbf{k})$ . In §8.4, I shall calculate it for the Langmuir-sound turbulence (§7.2.3), after I have derived the dynamical equations for it. Once this is done, I will show you how one can solve the WT kinetic equations and obtain steady-state, constant-flux, nonequilibrium distributions and spectra.

### 7.5. Validity of WT Approximation

Since I have not yet fully revealed what the WT approximation consists of, I cannot just yet give you a full account of the conditions for its validity. These will emerge piecemeal here and in §§8.1 and 8.4, as some specific meat is put on the generic bones that have emerged so far.

What should be clear at this point is that we have confined ourselves to a situation where we must be able to think of our plasma as a gas of monochromatic waves with definite wave numbers (momenta) and frequencies (energies), which interact subject to conservation rules on both, e.g., (7.13) or (7.18). Matching wave numbers is a straightforward requirement that mathematically is traceable to the simple fact that the nonlinear terms in the underlying dynamical equations [e.g., (2.12)], when Fourier transformed, turn into convolution sums over wave numbers. In contrast, the matching of frequencies is a more nontrivial proposition. At the very least, in order to be meaningful, it requires that the wave frequencies be identifiable, i.e., that the waves survive (much) longer than their own period of oscillation. In other words, the typical nonlinear interaction time is long compared to the wave period:

$$\boxed{\omega_{\mathbf{k}} t_{\text{nl}} \gg 1} , \quad (7.44)$$

as promised in §2.4.3. It is perhaps intuitive then that the delta functions constraining the WT interaction probabilities, such as  $\delta(\omega_{\mathbf{p}} - \omega_{\mathbf{p}-\mathbf{k}} - \Omega_{\mathbf{k}})$  in (7.27), in fact have an effective width of order  $\sim t_{\text{nl}}^{-1}$ , which is assumed small compared to any of the frequencies.

How well this assumption is satisfied depends on the amplitudes of the perturbations (i.e., on the overall fluctuation level), which must remain sufficiently small. This has to be checked for each particular system—see, e.g., §8.4.7,<sup>61</sup> where I will finally bid farewell to WT and move on to stronger stuff.

It happens quite often—indeed, more often than not—that a WT system drives itself towards such (low) wave numbers, frequencies, and/or (high) amplitudes that the WT applicability condition (7.44) gets broken, i.e., the frequency-matching delta functions are broadened to the point that they no longer constrain interactions (there was a glimpse of this at the end of §7.2.4). This turns the system from a weakly coupled to a strongly coupled one, from weak turbulence to strong turbulence. In such a regime, it is no longer true that monochromatic plasma waves are the natural elementary building blocks of the nonlinear system, and some other entities, each formally composed of many Fourier modes, emerge to claim pre-eminence. What they are is not always (indeed, not usually) clear. I will attempt to illustrate this point in a specific way in §§8.5 and 8.6.

## 8. Langmuir Turbulence

This section is dedicated to one particular kind of plasma turbulence—probably the best-, or at any rate the earliest-, studied kind: turbulence of coupled Langmuir and sound waves. This is not the most relevant situation for most fusion (except perhaps inertial-fusion) or astrophysical plasmas, but it has the advantage of involving the two most basic plasma waves that we have already studied above, at length, and it also gives me an opportunity to illustrate some concepts and methods that find their application in many other plasma problems: how separation of scales enables one to derive reduced systems of equations (§§8.1 and 8.2),<sup>62</sup> how they turn out to have Hamiltonian structure (§8.3), how one can use them to calculate probabilities of basic WT processes and thus complete, and solve, the kinetic equations derived in §7.2 (§8.4), how the WT approximation breaks down and waves (quasiparticles) cease to be the basic actors of plasma dynamics, ceding ground to essentially nonlinear structures and how then a strongly turbulent regime might be tackled, with great difficulty but also great ingenuity (§§8.5 and 8.6).

### 8.1. Zakharov's Equations

So let me pick up where I left off in §7.2.3 and ask what dynamical equations are satisfied by the plasmons and the phonons—so I can use these equations to calculate  $w(\mathbf{p}, \mathbf{k})$  (and, it will turn out, do much more besides). First, I would like to derive these equations in a physically intuitive, if non-rigorous, way—I will clean up my act in §8.2.

#### 8.1.1. Langmuir Waves

In Exercise 3.1, you had an opportunity to derive the linearised fluid equations for Langmuir waves—they are just the density and velocity moments of the electron kinetic equation, ignoring ions entirely:

$$\frac{\partial \delta \tilde{n}_e}{\partial t} + \nabla \cdot (\bar{n}_e \tilde{\mathbf{u}}_e) = 0, \quad (8.1)$$

$$m_e \bar{n}_e \frac{\partial \tilde{\mathbf{u}}_e}{\partial t} = -e \bar{n}_e \tilde{\mathbf{E}} - \nabla \delta \tilde{p}_e. \quad (8.2)$$

---

<sup>61</sup>Another key assumption that enables one to write the kinetic equations in the form I did above is the random-phase approximation, already alluded to at the end of §7.3.1 to excuse entropy nonconservation and introduced properly in §8.4.2.

<sup>62</sup>In a different setting, this theme will be taken up again in §§15.2 and 15.3.

The overtildes mark the quantities that will turn out to oscillate as a Langmuir wave. The overbars designate the fields that are much slower than that—in the above,  $\bar{n}_e$  so far just means the equilibrium density on top of which the perturbations occur. The last term in (8.2) contains the electron pressure perturbation, which, in Exercise 3.1, you found to be

$$\delta\tilde{p}_e = 3T_e\delta\tilde{n}_e = \frac{3}{2}m_ev_{\text{the}}^2\delta\tilde{n}_e \quad (8.3)$$

at long wavelengths, i.e., if  $kv_{\text{the}} \ll \omega_{\text{pe}}$  (and so kinetic effects could be ignored). The electric field is  $\tilde{\mathbf{E}} = -\nabla\tilde{\varphi}$ , and the Poisson equation tells us that  $4\pi e\delta\tilde{n}_e = \nabla^2\tilde{\varphi}$ . Taking the time derivative of (8.1) and using (8.2) and (8.3), we thus arrive at a closed equation for  $\tilde{\varphi}$ :

$$\frac{\partial^2}{\partial t^2}\nabla^2\tilde{\varphi} + \nabla \cdot \left( \frac{4\pi e^2\bar{n}_e}{m_e}\nabla\tilde{\varphi} \right) - \frac{3}{2}v_{\text{the}}^2\nabla^4\tilde{\varphi} = 0. \quad (8.4)$$

In the linear problem, the dimensional factor in the second term is just  $\omega_{\text{pe}}^2$ , and then from (8.4) we can promptly read off the dispersion relation (3.39) for a Langmuir wave:

$$\omega^2 = \omega_{\text{pe}}^2 + \frac{3}{2}k^2v_{\text{the}}^2. \quad (8.5)$$

Let me now posit that in a nonlinear system where the dominant interactions are between Langmuir and sound waves, the nature of these interactions must be “modulational”: sound waves are much slower than Langmuir waves and so the latter will “see” the former simply as slow density modulation on top of the equilibrium. Mathematically this is captured if we let

$$\bar{n}_e = n_{0e} + \delta\bar{n}_e, \quad (8.6)$$

where  $n_{0e}$  is the homogeneous equilibrium density and  $\delta\bar{n}_e$  the density perturbation associated with sound. This gives us just the right kind of nonlinearity: (8.4) becomes

$$\boxed{\left( \frac{\partial^2}{\partial t^2} + \omega_{\text{pe}}^2 - \frac{3}{2}v_{\text{the}}^2\nabla^2 \right) \nabla^2\tilde{\varphi} = -\omega_{\text{pe}}^2 \nabla \cdot \left( \frac{\delta\bar{n}_e}{n_{0e}} \nabla\tilde{\varphi} \right)}. \quad (8.7)$$

In order to close the system, we need an equation for  $\delta\bar{n}_e$ , the electron-density perturbation associated with sound waves.

### 8.1.2. Sound Waves

Let us work in the limit of cold ions,  $T_i \ll T_e$ , when sound waves can thrive without being bothered by Landau damping (§3.8). In this limit, they too can be described by simple linearised fluid equations, derived in Exercise 3.6:

$$\frac{\partial\delta n_i}{\partial t} + n_{0i}\nabla \cdot \mathbf{u}_i = 0, \quad (8.8)$$

$$m_i n_{0i} \frac{\partial \mathbf{u}_i}{\partial t} = -\nabla\delta\bar{p}_e. \quad (8.9)$$

Since ions are cold, only the electron pressure matters. We do not need to have bars over the ion quantities because they do not have Langmuir-frequency variation, but the electron pressure here is the slow, averaged part that the ions can “see”. If you did Exercise 3.6, you know that electrons are isothermal at this slow time scale, viz.,  $\delta\bar{p}_e = T_e\delta\bar{n}_e$ . Also, the Poisson equation for these perturbations just turns into a statement of quasineutrality:  $Z\delta n_i = \delta\bar{n}_e$  (there cannot be any deviations from that on time scales longer than the inverse plasma frequency; see §2.1); the same relationship is also of

course true for the equilibrium densities:  $Zn_{0i} = n_{0e}$ . The equation for sound waves follows immediately:

$$\frac{\partial^2 \delta \bar{n}_e}{\partial t^2} = c_s^2 \nabla^2 \delta \bar{n}_e. \quad (8.10)$$

Now we need to find a way to modify this equation to account for the emission (or absorption) of sound waves by Langmuir waves. The answer is intuitive: the average effect of a fast-oscillating electric field on particles amounts to an additional effective pressure equal to the energy density of the field—this is known as the *ponderomotive force*. Here is a quick derivation. An electron's equation of motion is

$$m_e \ddot{\mathbf{r}} = -e \tilde{\mathbf{E}}(\mathbf{r}). \quad (8.11)$$

Now split this motion into a fast-oscillating part and a slowly drifting part,  $\mathbf{r} = \bar{\mathbf{r}} + \tilde{\mathbf{r}}$ , and assume  $|\tilde{\mathbf{r}}| \ll |\bar{\mathbf{r}}|$ . The fast-oscillating part satisfies

$$m_e \ddot{\tilde{\mathbf{r}}} \approx -e \tilde{\mathbf{E}}(\bar{\mathbf{r}}) \Rightarrow \tilde{\mathbf{r}} \approx \frac{e}{m_e \omega_{pe}^2} \tilde{\mathbf{E}}(\bar{\mathbf{r}}), \quad (8.12)$$

assuming that  $\tilde{\mathbf{E}}$  oscillates at the single frequency  $\omega_{pe}$ . Time-averaging (8.11) over the fast oscillations and expanding  $\tilde{\mathbf{E}}(\bar{\mathbf{r}} + \tilde{\mathbf{r}})$  in small  $\tilde{\mathbf{r}}$ , we get

$$m_e \ddot{\tilde{\mathbf{r}}} \approx -e \left[ \overline{\tilde{\mathbf{E}}(\bar{\mathbf{r}})} + \tilde{\mathbf{r}} \cdot \nabla \overline{\tilde{\mathbf{E}}(\bar{\mathbf{r}})} \right] = -\frac{e^2}{m_e \omega_{pe}^2} \overline{\tilde{\mathbf{E}} \cdot \nabla \tilde{\mathbf{E}}} = -\frac{1}{n_{0e}} \nabla \frac{|\nabla \tilde{\varphi}|^2}{8\pi}. \quad (8.13)$$

Adding this effective electron pressure  $|\nabla \tilde{\varphi}|^2 / 8\pi$  (or, equivalently, an effective electric potential) to (8.9) and retracing the route that led to (8.10), we get the desired equation featuring a nonlinear coupling of sound waves to Langmuir waves:

$$\boxed{\left( \frac{\partial^2}{\partial t^2} - c_s^2 \nabla^2 \right) \delta \bar{n}_e = c_s^2 \nabla \frac{|\nabla \tilde{\varphi}|^2}{8\pi T_e}}. \quad (8.14)$$

Together with (8.7), this forms a closed system of equations.

**Exercise 8.1.** Is there a ponderomotive contribution to the effective ion pressure and need it be taken into account?

### 8.1.3. Final Form of Zakharov's Equations

We are one step away from the final form of the [Zakharov \(1972\)](#) equations. A further simplification is achieved if, in (8.16), we take out the  $\omega_{pe}$  oscillations:

$$\tilde{\varphi} = \frac{1}{2} (\psi e^{-i\omega_{pe}t} + \psi^* e^{i\omega_{pe}t}), \quad (8.15)$$

where  $\psi$  varies on a time scale much longer than  $\omega_{pe}^{-1}$ . Substituting (8.15) into (8.7) and neglecting  $\partial_t^2 \psi \ll i\omega_{pe} \partial_t \psi$ , dividing through by  $-\omega_{pe} e^{-i\omega_{pe}t}$ , and averaging out oscillatory terms, we get Zakharov's first equation:

$$\boxed{\nabla^2 \left( i\omega_{pe}^{-1} \frac{\partial \psi}{\partial t} + \frac{3}{2} \lambda_{De}^2 \nabla^2 \psi \right) = \frac{1}{2} \nabla \cdot \left( \frac{\delta \bar{n}_e}{n_{0e}} \nabla \psi \right)}. \quad (8.16)$$

Zakharov's second equation is the same as (8.14), but with the substitution of (8.15):

$$\boxed{\left( \frac{\partial^2}{\partial t^2} - c_s^2 \nabla^2 \right) \delta \bar{n}_e = c_s^2 \nabla^2 \frac{|\nabla \psi|^2}{16\pi T_e}}. \quad (8.17)$$

We have our closed system—and even a cursory glance at the structure of these equations confirms that they should capture just the kind of interactions anticipated in §7.2.3: the right-hand side of (8.17) is quadratic in  $\psi$  and so describes two plasmons coupling to generate a phonon (or, rather, a plasmon emitting/absorbing a phonon and becoming a different plasmon); the right-hand side of (8.16) contains a product of  $\delta\bar{n}_e$  and  $\psi$  and so describes the modification of a plasmon by the emission or absorption of a phonon.

**Exercise 8.2. Secondary instabilities of a Langmuir wave.** That plasmons emit phonons—i.e., that decay instabilities exist—can be demonstrated directly from the dynamical equations (8.16) and (8.17). Consider a monochromatic Langmuir wave as the lowest-order solution of Zakharov's equations in the absence of nonlinearity, then perturb around it, linearise, and work out the *secondary instabilities* of the perturbations: what are the growth rates, the peak-growth wave numbers, various parameter regimes, etc.? The best treatment of this topic in the literature that I am aware of is the review by [Thornhill & ter Haar \(1978, §3\)](#) (although these results already appear, presented with extreme pith, in [Zakharov 1972](#)). This is not really kinetic theory, more fluid dynamics, but it is a challenging yet doable calculation that one can sink one's teeth into and enjoy. Note that some of the results of this calculation directly anticipate §8.5.2.

#### 8.1.4. Quasistatic Limit

The linear frequency of the variation of  $\psi$  in (8.16) is

$$\delta\omega_{\mathbf{k}} = \frac{3}{2} k^2 \lambda_{De}^2 \omega_{pe} = \frac{3}{4} \frac{k^2 v_{the}^2}{\omega_{pe}}. \quad (8.18)$$

This frequency will still be greater than the sound frequency, enabling the ponderomotive source term in the right-hand side of (8.17) to excite sound waves, provided

$$\frac{\delta\omega_{\mathbf{k}}}{kc_s} \sim k\lambda_{De} \frac{v_{the}}{c_s} \gg 1 \quad \Leftrightarrow \quad k\lambda_{De} \gg \sqrt{\frac{m_e}{m_i}}. \quad (8.19)$$

In the opposite limit,

$$k\lambda_{De} \ll \sqrt{\frac{m_e}{m_i}}, \quad (8.20)$$

the source term is quasistatic, no sound waves can be excited and (8.17) simply enforces a balance between the electron pressure and the ponderomotive force.<sup>63</sup>

$$\delta\bar{n}_e = -\frac{|\nabla\psi|^2}{16\pi T_e}. \quad (8.21)$$

Putting this into (8.16) gets us a closed equation for  $\psi$ :<sup>64</sup>

$$\boxed{\nabla^2 \left( i\omega_{pe}^{-1} \frac{\partial\psi}{\partial t} + \frac{3}{2} \lambda_{De}^2 \nabla^2 \psi \right) = -\frac{\nabla \cdot (|\nabla\psi|^2 \nabla\psi)}{32\pi n_0 e T_e}}. \quad (8.22)$$

The nonlinearity is cubic, so this is actually an example of a situation where four-wave interactions will dominate nonlinear dynamics (see §7.2.2).

Let us check that the nonlinear term in (8.22) does not spoil the assumption that  $\psi$  changes slowly compared to sound. The characteristic nonlinear time is

$$t_{nl}^{-1} \sim \omega_{pe} W, \quad W = \frac{\overline{|\tilde{\mathbf{E}}|^2}}{8\pi n_e T_e}, \quad (8.23)$$

<sup>63</sup>One might call this the incompressible limit of Zakharov's equations.

<sup>64</sup>A mathematically identical equation, but with  $T_e$  replaced by  $T_e + T_i/Z$  in the right-hand side, arises in the limit of hot ions—hot enough to stream quickly and thus cause the ion density to have a simple Boltzmann response. This approximation is derived in §8.2.7.

where  $W$  is the nondimensionalised mean square amplitude of the Langmuir oscillations. Then

$$t_{\text{nl}} k c_s \gg 1 \quad \Leftrightarrow \quad W \ll k \lambda_{\text{De}} \sqrt{\frac{m_e}{m_i}}. \quad (8.24)$$

If we wanted to break this restriction, i.e., if  $\psi$  were to be allowed to saturate at higher amplitudes than this, we would have to go back to (8.17).

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If you are completely happy with the derivation in §§8.1.1 and 8.1.2, skip the next section, but if you feel that the approximations and assumptions hoisted upon you are difficult to accept at this level of (non-)rigour, read on: in §8.2, I provide a formal perturbative derivation of the [Zakharov \(1972\)](#) equations (8.16) and (8.17), which is surprisingly difficult to find in the literature.

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## 8.2. Formal Derivation of Zakharov's Equations

### 8.2.1. Scale Separations

The problem has four characteristic timescales: the plasma oscillation frequency, the electron streaming rate, the ion sound frequency and the ion streaming rate:

$$\omega_{\text{pe}} \gg k v_{\text{the}} \gg k c_s \sim k v_{\text{thi}}. \quad (8.25)$$

The relative size of these frequencies is controlled by the following three independent parameters:

$$\frac{k v_{\text{the}}}{\omega_{\text{pe}}} \sim k \lambda_{\text{De}} \ll 1, \quad \frac{k c_s}{k v_{\text{the}}} \sim \sqrt{\frac{m_e}{m_i}} \ll 1, \quad \frac{k v_{\text{thi}}}{k c_s} \sim \sqrt{\frac{T_i}{T_e}} \sim 1. \quad (8.26)$$

The scale separation between ions and electrons is non-negotiable as the mass ratio is always small. As long as  $k \lambda_{\text{De}} \ll 1$ , which we will assume here, the electron Landau damping is exponentially small and the electrons will be fluid (as we will see shortly; it is no surprise, given what we know from §3.5). Ions too behave as a fluid if they are cold ( $T_i \ll T_e$ ; cf. §3.8), which is the limit most often considered in the context of Zakharov's equations, if not necessarily one that is most relevant physically.

### 8.2.2. Electron Kinetics and Ordering

As anticipated in §8.1, let us split the electron distribution function and the electrostatic potential into two parts: the time-averaged (“slow”, denoted by overbars) and fluctuating (“fast”, denoted by overtildees):

$$f_e = \bar{f}_e + \tilde{f}_e, \quad \varphi = \bar{\varphi} + \tilde{\varphi}. \quad (8.27)$$

The time average is taken over time scales longer than both  $\omega_{\text{pe}}^{-1}$  and  $(k v_{\text{the}})^{-1}$  but shorter than  $(k c_s)^{-1}$  or  $(k v_{\text{thi}})^{-1}$ , i.e.,  $\bar{f}_e$  and  $\bar{\varphi}$  are the electron distribution and potential that the ions can “see”. The slow part of the electron distribution is assumed to consist of a homogeneous Maxwellian equilibrium (5.6) plus a perturbation:

$$\bar{f}_e = f_{0e} + \delta \bar{f}_e. \quad (8.28)$$

The slow and fast distribution functions satisfy the following equations, which are obtained by time averaging the Vlasov equation (1.50) for electrons ( $\alpha = e$ ,  $q_\alpha = -e$ ) and subtracting the average from the exact equation:

$$\mathbf{v} \cdot \nabla \delta \bar{f}_e + \frac{e}{m_e} (\nabla \bar{\varphi}) \cdot \frac{\partial \bar{f}_e}{\partial \mathbf{v}} + \frac{e}{m_e} \overline{(\nabla \tilde{\varphi}) \cdot \frac{\partial \tilde{f}_e}{\partial \mathbf{v}}} = 0, \quad (8.29)$$

$$\frac{\partial \tilde{f}_e}{\partial t} + \mathbf{v} \cdot \nabla \tilde{f}_e + \frac{e}{m_e} (\nabla \tilde{\varphi}) \cdot \frac{\partial \tilde{f}_e}{\partial \mathbf{v}} + \frac{e}{m_e} (\nabla \tilde{\varphi}) \cdot \widetilde{\frac{\partial \tilde{f}_e}{\partial \mathbf{v}}} + \frac{e}{m_e} (\nabla \tilde{\varphi}) \cdot \frac{\partial \tilde{f}_e}{\partial \mathbf{v}} = 0, \quad (8.30)$$

where all time evolution on ion scales is neglected because  $k v_{\text{the}}$  is large. The slow and fast parts

of the Poisson equation (1.51) are

$$-\nabla^2 \bar{\varphi} = 4\pi e(Z\delta n_i - \delta \bar{n}_e) = 4\pi e \left( Z \int d\mathbf{v} \delta f_i - \int d\mathbf{v} \delta \bar{f}_e \right), \quad (8.31)$$

$$-\nabla^2 \tilde{\varphi} = -4\pi e \tilde{n}_e = -4\pi e \int d\mathbf{v} \tilde{f}_e, \quad (8.32)$$

where  $\delta f_i$  is the perturbed ion distribution function and  $\delta n_i$  its density. We shall solve (8.30) and (8.32) for  $\tilde{\varphi}$  and  $\tilde{f}_e$ , use that to calculate the last term in (8.29), which will give rise to the ponderomotive force, then solve (8.29) for  $\tilde{f}_e$  in terms of  $\tilde{\varphi}$ , and finally use that solution in (8.31) to get an expression for  $\tilde{\varphi}$  in terms of  $f_i$ . The latter can then be coupled with the ion Vlasov–Landau equation (5.1) ( $\alpha = i$ ,  $q_\alpha = Ze$ ), giving rise to a closed “hybrid” system for kinetic ions and “fluid” electrons.

In order to implement this plan, we shall carry out a perturbation expansion of the above equations in the small parameter

$$\varepsilon = k\lambda_{De}. \quad (8.33)$$

The algebra becomes more compact if we first make the following ansatz, designed to remove the third (the largest, as we will see) term in (8.30):<sup>65</sup>

$$\tilde{f}_e = -\mathbf{u} \cdot \frac{\partial \bar{f}_e}{\partial \mathbf{v}} + h, \quad (8.34)$$

where  $\mathbf{u}$  is, by definition, the velocity associated with the plasma oscillation [cf. (5.46)]:

$$\frac{\partial \mathbf{u}}{\partial t} = \frac{e}{m_e} \nabla \tilde{\varphi}. \quad (8.35)$$

Then the fast electron kinetic equation (8.30) becomes

$$\begin{aligned} \frac{\partial h}{\partial t} &= \underbrace{\mathbf{v} \cdot \nabla \left( \mathbf{u} \cdot \frac{\partial \bar{f}_e}{\partial \mathbf{v}} \right)}_{\varepsilon^2} - \underbrace{\mathbf{v} \cdot \nabla h}_{\varepsilon^3} + \underbrace{\frac{e}{m_e} (\nabla \bar{\varphi}) \cdot \frac{\partial}{\partial \mathbf{v}} \mathbf{u} \cdot \frac{\partial \bar{f}_e}{\partial \mathbf{v}}}_{\varepsilon^4} - \underbrace{\frac{e}{m_e} (\nabla \bar{\varphi}) \cdot \frac{\partial h}{\partial \mathbf{v}}}_{\varepsilon^5} \\ &\quad + \underbrace{\frac{e}{m_e} (\nabla \tilde{\varphi}) \cdot \frac{\partial}{\partial \mathbf{v}} \mathbf{u} \cdot \frac{\partial \bar{f}_e}{\partial \mathbf{v}}}_{\varepsilon^2} - \underbrace{\frac{e}{m_e} (\nabla \tilde{\varphi}) \cdot \frac{\partial h}{\partial \mathbf{v}}}_{\varepsilon^3}, \end{aligned} \quad (8.36)$$

where the ordering of each term in the small parameter (8.33) has been determined, as indicated above, based on the following assumptions. The plasma-oscillation velocity (8.35) is

$$\frac{u}{v_{the}} \sim \frac{ke\tilde{\varphi}}{m_e v_{the} \omega_{pe}} \sim k\lambda_{De} \frac{e\tilde{\varphi}}{T_e} \sim \varepsilon, \quad (8.37)$$

if, in general,<sup>66</sup>

$$\frac{e\tilde{\varphi}}{T_e} \sim 1. \quad (8.38)$$

Anticipating that the ponderomotive “potential” [see (8.13)] will enter on equal footing with the slow potential and that the slow perturbed electron distribution will express the Boltzmann response to the latter modified by the former, let us adopt the ordering

$$\frac{\delta \bar{f}_e}{f_{0e}} \sim \frac{e\bar{\varphi}}{T_e} \sim \frac{e^2 |\tilde{\mathbf{E}}|^2}{m_e \omega_{pe}^2 T_e} \sim (k\lambda_{De})^2 \left( \frac{e\tilde{\varphi}}{T_e} \right)^2 \sim \varepsilon^2. \quad (8.39)$$

<sup>65</sup>This is equivalent to splitting the electron distribution function into fast and slow parts using as the velocity variable of  $\tilde{f}_e$  the peculiar velocity of the particle around a centre oscillating with velocity  $\mathbf{u}$  (cf. DuBois *et al.* 1995): namely, set  $f_e = \tilde{f}_e(\mathbf{r}, \mathbf{v} - \mathbf{u}(t, \mathbf{r})) + h(t, \mathbf{r}, \mathbf{v})$  and expand in small  $\mathbf{u}$ .

<sup>66</sup>Note that this ordering means that the equations that are being derived will be, in principle, suitable both for weak- and strong-turbulence regimes (see §8.4.7).

Since the inhomogeneous terms in (8.36) are, thus,  $O(\varepsilon^2)$ , it follows that  $h \sim \varepsilon^2 f_{0e}$  and, since the first term in (8.34) has no density moment,  $\bar{n}_e \sim \varepsilon^2 n_{0e}$ .

From (8.36), to lowest order,

$$\begin{aligned} \frac{\partial h^{(2)}}{\partial t} &= \mathbf{v} \cdot \nabla \mathbf{u} \cdot \frac{\partial f_{0e}}{\partial \mathbf{v}} + \frac{\partial \mathbf{u}}{\partial t} \cdot \widetilde{\frac{\partial}{\partial \mathbf{v}} \mathbf{u}} \cdot \frac{\partial f_{0e}}{\partial \mathbf{v}} \\ &= -\frac{2}{v_{\text{the}}^2} \left[ v_i v_j \partial_i u_j + \left( \delta_{ij} - \frac{2v_i v_j}{v_{\text{the}}^2} \right) \frac{\partial u_i}{\partial t} u_j \right] f_{0e}, \end{aligned} \quad (8.40)$$

where I have used (8.35) and the fact that  $f_{0e}$  is a Maxwellian.

### 8.2.3. Ponderomotive Response

With (8.40) in hand, we are now in a position to calculate the last term in (8.29). First, using (8.34) and (8.35) and keeping terms of order  $\varepsilon^2$  and  $\varepsilon^3$ ,

$$\begin{aligned} \overline{\frac{e}{m_e} (\nabla \tilde{\varphi}) \cdot \frac{\partial \tilde{f}_e}{\partial \mathbf{v}}} &= \overline{\frac{\partial \mathbf{u}}{\partial t} \cdot \frac{\partial}{\partial \mathbf{v}} \left( -\mathbf{u} \cdot \frac{\partial f_{0e}}{\partial \mathbf{v}} + h^{(2)} \right)} \\ &= \frac{2}{v_{\text{the}}^2} \overline{\left( \delta_{ij} - \frac{2v_i v_j}{v_{\text{the}}^2} \right) \frac{\partial u_i}{\partial t} u_j f_{0e}} + \overline{\frac{\partial \mathbf{u}}{\partial t} \cdot \frac{\partial h^{(2)}}{\partial \mathbf{v}}} \\ &= \overline{\frac{\partial}{\partial t} \left\{ \frac{2}{v_{\text{the}}^2} \left[ \frac{u^2}{2} - \frac{(\mathbf{u} \cdot \mathbf{v})^2}{v_{\text{the}}^2} \right] f_{0e} + \mathbf{u} \cdot \frac{\partial h^{(2)}}{\partial \mathbf{v}} \right\}} - \overline{\mathbf{u} \cdot \frac{\partial}{\partial \mathbf{v}} \frac{\partial h^{(2)}}{\partial t}}. \end{aligned} \quad (8.41)$$

The first term is a full time derivative and so vanishes under averaging, whereas the second term can be calculated using (8.40):

$$\begin{aligned} -\overline{\mathbf{u} \cdot \frac{\partial}{\partial \mathbf{v}} \frac{\partial h^{(2)}}{\partial t}} &= \frac{2}{v_{\text{the}}^2} \left[ v_j u_l \partial_l u_j + v_i u_l \partial_i u_l - \frac{2v_i v_j v_l}{v_{\text{the}}^2} u_l \partial_i u_j \right. \\ &\quad \left. - \frac{2}{v_{\text{the}}^2} \left( v_l u_l \frac{\partial u_i}{\partial t} u_i + v_j u_l \frac{\partial u_l}{\partial t} u_j + v_i u_l \frac{\partial u_i}{\partial t} u_l - \frac{2v_i v_j v_l}{v_{\text{the}}^2} u_l \frac{\partial u_i}{\partial t} u_j \right) \right] f_{0e} \\ &= \frac{2}{v_{\text{the}}^2} \left\{ \overline{\mathbf{u} \cdot \nabla \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \nabla \left[ \frac{u^2}{2} - \frac{(\mathbf{u} \cdot \mathbf{v})^2}{v_{\text{the}}^2} \right]} \right. \\ &\quad \left. - \frac{2}{v_{\text{the}}^2} \frac{\partial}{\partial t} \left[ u^2 \mathbf{u} \cdot \mathbf{v} - \frac{2(\mathbf{u} \cdot \mathbf{v})^3}{3v_{\text{the}}^2} \right] \right\} f_{0e} \\ &= 2\mathbf{v} \cdot \nabla \left[ \frac{u^2}{v_{\text{the}}^2} - \frac{(\mathbf{u} \cdot \mathbf{v})^2}{v_{\text{the}}^4} \right] f_{0e}. \end{aligned} \quad (8.42)$$

The last expression was obtained after noticing that any full time derivative vanishes under averaging and that,  $\mathbf{u}$  defined by (8.35) being a potential field, one could rewrite  $\mathbf{u} \cdot \nabla \mathbf{u} = \nabla |\mathbf{u}|^2/2$ .

Note that (8.42) is  $O(\varepsilon^3)$ , as are the other two terms in (8.29). Inserting (8.42) into (8.29), we obtain the following solution for the slow part of the perturbed electron distribution:

$$\delta \bar{f}_e = \left\{ \frac{e\bar{\varphi}}{T_e} - 2 \left[ \frac{u^2}{v_{\text{the}}^2} - \frac{(\mathbf{u} \cdot \mathbf{v})^2}{v_{\text{the}}^4} \right] \right\} f_{0e}. \quad (8.43)$$

The first term is the Boltzmann response, the second the ponderomotive one. The resulting electron density perturbation is

$$\frac{\delta \bar{n}_e}{n_{0e}} = \frac{e\bar{\varphi}}{T_e} - \frac{\overline{u^2}}{v_{\text{the}}^2}. \quad (8.44)$$

### 8.2.4. Electron Fluid Dynamics

In order to obtain the evolution equation for  $\tilde{\varphi}$ , we will need to solve (8.36), coupled to (8.32), to higher order than the lowest, namely, up to  $\varepsilon^4$ . Rather than solving the kinetic equation (8.36)

order by order, it turns out to be a faster procedure to take moments of it exactly and then close the resulting hierarchy by calculating the second moment using  $h^{(2)}$  given by (8.40).

The zeroth (density) moment of (8.36) is

$$\frac{\partial \tilde{n}_e}{\partial t} + \nabla \cdot (\bar{n}_e \mathbf{u}) + \nabla \cdot \int d\mathbf{v} \mathbf{v} h = 0, \quad (8.45)$$

the continuity equation. The first moment is

$$\frac{\partial}{\partial t} \int d\mathbf{v} \mathbf{v} h = -\nabla \cdot \int d\mathbf{v} (\mathbf{u}\mathbf{v} + \mathbf{v}\mathbf{u}) \bar{f}_e - \nabla \cdot \int d\mathbf{v} \mathbf{v} \mathbf{v} h + \frac{e}{m_e} \tilde{n}_e \nabla \bar{\varphi} + \frac{e}{m_e} \widetilde{\tilde{n}_e \nabla \tilde{\varphi}}. \quad (8.46)$$

The first term on the right-hand side is zero to all orders up to at least  $\varepsilon^4$  because, according to (8.43),  $\bar{f}_e$  is even in  $\mathbf{v}$  up to second order. The remaining terms are  $O(\varepsilon^3)$ , except the penultimate one, which is  $O(\varepsilon^5)$  and can be safely dropped. Combining (8.45) with (8.46) and using (8.35), we have

$$\frac{\partial^2 \tilde{n}_e}{\partial t^2} + \nabla \cdot \left( \frac{e}{m_e} \tilde{n}_e \nabla \tilde{\varphi} \right) - \nabla \nabla : \int d\mathbf{v} \mathbf{v} \mathbf{v} h + \nabla \cdot \left( \frac{e}{m_e} \widetilde{\tilde{n}_e \nabla \tilde{\varphi}} \right) = 0. \quad (8.47)$$

This equation is valid up to and including terms of order  $\varepsilon^4$ .

Note that, in order to maintain this level of precision, we need to keep the lowest-order contribution to  $h$  in the second velocity moment. This satisfies (8.40), which it is now convenient to rewrite as

$$\frac{\partial h^{(2)}}{\partial t} = -\frac{2}{v_{\text{the}}^2} \left[ v_i v_j \partial_i u_j + \frac{\partial}{\partial t} \left( \frac{u^2}{2} - \frac{u_i u_j v_i v_j}{v_{\text{the}}^2} \right) \right] f_{0e}. \quad (8.48)$$

The stress tensor satisfies

$$\frac{\partial}{\partial t} \int d\mathbf{v} v_i v_j h = -\frac{n_{0e} v_{\text{the}}^2}{2} (\partial_i u_j + \partial_j u_i + \delta_{ij} \nabla \cdot \mathbf{u}) + \frac{\partial}{\partial t} n_{0e} u_i u_j. \quad (8.49)$$

Therefore,

$$\frac{\partial}{\partial t} \left( \nabla \nabla : \int d\mathbf{v} \mathbf{v} \mathbf{v} h \right) = -\frac{3}{2} v_{\text{the}}^2 \nabla^2 \nabla \cdot (n_{0e} \mathbf{u}) + \frac{\partial}{\partial t} n_{0e} \nabla \nabla : \mathbf{u} \mathbf{u}. \quad (8.50)$$

From (8.45), to lowest order,  $\nabla \cdot (n_{0e} \mathbf{u}) = -\partial \tilde{n}_e / \partial t$  and so the above equation can be integrated in time:

$$\nabla \nabla : \int d\mathbf{v} \mathbf{v} \mathbf{v} h = \frac{3}{2} v_{\text{the}}^2 \nabla^2 \tilde{n}_e + n_{0e} \nabla \nabla : \widetilde{\mathbf{u} \mathbf{u}}. \quad (8.51)$$

Inserting (8.51) into (8.47) and using also the fast Poisson equation (8.32) to express  $\tilde{n}_e$  via  $\tilde{\varphi}$ , we obtain

$$\frac{\partial^2}{\partial t^2} \nabla^2 \tilde{\varphi} + \nabla \cdot \left( \omega_{pe}^2 \frac{\bar{n}_e}{n_{0e}} \nabla \tilde{\varphi} \right) - \frac{3}{2} v_{\text{the}}^2 \nabla^4 \tilde{\varphi} = 4\pi e n_{0e} \nabla \nabla : \widetilde{\mathbf{u} \mathbf{u}} - \nabla \cdot \left[ \frac{e}{m_e} (\nabla^2 \tilde{\varphi}) \widetilde{\nabla \tilde{\varphi}} \right]. \quad (8.52)$$

The left-hand side of this equation is what was intuited in (8.7). The terms on the right-hand side of (8.52), not captured in (8.7), are nonlinear interactions between Langmuir waves, which will disappear in a moment.

The dominant  $\omega_{pe}$  oscillation in (8.52) can now be taken out in the way that was already anticipated in §8.1.3, by introducing, via (8.15), the slow-varying complex amplitude  $\psi$ . Then, from (8.35), to lowest order in  $\varepsilon$ ,

$$\mathbf{u} = i \frac{e}{2m_e \omega_{pe}} \nabla \left( \psi e^{-i\omega_{pe}t} - \psi^* e^{i\omega_{pe}t} \right). \quad (8.53)$$

Substituting (8.15) and (8.53) into (8.52), we get Zakharov's first equation (8.16) in the same way as we did in §8.1.3. The right-hand side of (8.52) disappears because we can average out the oscillatory terms with frequencies  $\omega_{pe}$  and  $2\omega_{pe}$ . Finally, substituting (8.53) into (8.44), we have, for the slow density perturbation,

$$\frac{\delta \bar{n}_e}{n_{0e}} = \frac{e \bar{\varphi}}{T_e} - \frac{|\nabla \psi|^2}{16\pi n_{0e} T_e}. \quad (8.54)$$

To get  $\bar{\varphi}$ , we need to bring in the ions.

### 8.2.5. Ion Kinetics

Since the left-hand side of the slow Poisson equation (8.31) is  $O(\varepsilon^4)$ , while its right-hand side is  $O(\varepsilon^2)$ , (8.31) predictably turns into the quasineutrality equation

$$\delta\bar{n}_e = Z\delta n_i. \quad (8.55)$$

Combined with (8.54), this becomes

$$\frac{e\bar{\varphi}}{T_e} = \frac{|\nabla\psi|^2}{16\pi n_{0e} T_e} + \frac{1}{n_{0i}} \int d\mathbf{v} \delta f_i, \quad (8.56)$$

where  $\psi$  obeys (8.16). The first term in (8.56) is the effective ponderomotive “potential” (in the fluid-dynamical language, it can also be thought of, perhaps more physically, as the effective ponderomotive “pressure”: see §8.1.2). The ion distribution function  $f_i = f_{0i} + \delta f_i$  is found from the ion Vlasov–Landau equation (5.1) with the slow potential  $\bar{\varphi}$ :

$$\boxed{\frac{\partial f_i}{\partial t} + \mathbf{v} \cdot \nabla f_i - \frac{Ze}{m_i} (\nabla\bar{\varphi}) \cdot \frac{\partial f_i}{\partial \mathbf{v}} = \left( \frac{\partial f_i}{\partial t} \right)_c}. \quad (8.57)$$

Together with (8.16), (8.57) and (8.56) make up a closed hybrid system describing kinetic ions and fluid electrons. The electrons affect the ions via the ponderomotive nonlinearity in (8.56), while the ions modulate the plasma frequency and thereby the dynamics of the electrons.

### 8.2.6. Ion Fluid Dynamics

Let me now show how ions can become fluid, giving rise to the second equation (8.17) in the classic [Zakharov \(1972\)](#) system.

The zeroth and first moments of (8.57) are

$$\frac{\partial \delta n_i}{\partial t} + \nabla \cdot \int d\mathbf{v} \mathbf{v} \delta f_i = 0, \quad (8.58)$$

$$\frac{\partial}{\partial t} \int d\mathbf{v} \mathbf{v} \delta f_i + \nabla \cdot \int d\mathbf{v} \mathbf{v} \mathbf{v} \delta f_i = -\frac{Ze}{m_i} n_i \nabla \bar{\varphi} = -c_s^2 n_i \nabla \left( \frac{\delta n_i}{n_{0i}} + \frac{|\nabla\psi|^2}{16\pi n_{0e} T_e} \right), \quad (8.59)$$

where the last expression was obtained with the aid of (8.56). Combining these two equations and keeping only the lowest-order terms, both in  $\varepsilon$  and in  $T_i/T_e$ , which is now assumed small so as to allow the ion pressure (stress) tensor in the left-hand side of (8.59) to be neglected, and replacing  $\delta n_i/n_{0i}$  with  $\delta\bar{n}_e/n_{0e}$  (by quasineutrality), we get Zakharov’s second equation (8.17), describing sound waves excited by the ponderomotive force.

### 8.2.7. Boltzmann Quasistatics

When the ions are not cold ( $T_i/T_e$  is not small), (8.17) regains the ion pressure term, via which it couples to the rest of the moments of  $\delta f_i$ . This is a dissipation channel for the sound waves, via Landau damping, at a typical rate  $\sim k v_{\text{thi}}$ . In the WT language, this is a regime in which induced scattering of plasmons by ions (§7.2.4) should become the dominant process.

There is, however, a very simple limiting case where kinetic physics again becomes irrelevant. The characteristic frequency  $\delta\omega_{\mathbf{k}}$  of  $\psi$  is given by (8.18). Consider the limit in which the ion streaming rate greatly exceeds that frequency:

$$k v_{\text{thi}} \gg \delta\omega_{\mathbf{k}} \sim k^2 \lambda_{\text{De}}^2 \omega_{\text{pe}} \quad \Leftrightarrow \quad k \lambda_{\text{De}} \ll \sqrt{\frac{T_i}{T_e} \frac{m_e}{m_i}}. \quad (8.60)$$

This is best achievable (obviously) when ions are hot, the opposite limit to the one considered in §8.2.6. In this limit, the  $\partial f_i/\partial t$  term in the ion kinetic equation (8.57) can be neglected. It is then easy to verify, by direct substitution, that the resulting equation is solved by the Maxwell–Boltzmann distribution:

$$f_i = \frac{n_{0i}}{(\pi v_{\text{thi}}^2)^{3/2}} \exp\left(-\frac{v^2}{v_{\text{thi}}^2} - \frac{Ze\bar{\varphi}}{T_i}\right) \Rightarrow \delta f_i = -\frac{Ze\bar{\varphi}}{T_i} f_{0i} \Rightarrow \frac{\delta n_i}{n_{0i}} = -\frac{Ze\bar{\varphi}}{T_i}. \quad (8.61)$$

Thus, the ion response is simply Boltzmann. So is the (slow) electron response, except with an

additional effective potential associated with the ponderomotive force—we already know this from (8.54). Combining (8.61) with that and with quasineutrality, via (8.56), we get

$$\boxed{\delta\bar{n}_e = -\frac{|\nabla\psi|^2}{16\pi(T_e + T_i/Z)}}. \quad (8.62)$$

Interestingly, this is mathematically the same result as (8.21) was for *cold* ions and  $k\lambda_{De} \ll \sqrt{m_e/m_i}$ . Just like in that cold-ion quasistatic limit, in this, hot-ion one, everything is now wrapped up in a single Zakharov equation—(8.22) with  $T_e$  replaced by  $T_e + T_i/Z$ ,—which describes Langmuir dynamics controlled by four-wave interactions (§7.2.2).

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### 8.3. Hamiltonian Form of Zakharov's Equations

In order to make a transition from the dynamical equations for the fields  $\psi$  and  $\delta\bar{n}_e$  to kinetic equations for the occupation numbers of plasmons and phonons, we need to recast the former in terms of variables that we can directly connect to the energies and, therefore, occupation numbers, of our “particles”, in the way it was done in (7.2) for the quasiparticle formulation of QLT. Thus, for the plasmons, we again have the total energy<sup>67</sup>

$$H^l = \sum_{\mathbf{p}} \hbar\omega_{\mathbf{p}} N_{\mathbf{p}} = V \sum_{\mathbf{p}} \frac{|\tilde{\mathbf{E}}_{\mathbf{p}}|^2}{4\pi} = V \sum_{\mathbf{p}} \frac{p^2 |\psi_{\mathbf{p}}|^2}{8\pi} \equiv \sum_{\mathbf{p}} \hbar\omega_{\mathbf{p}} a_{\mathbf{p}}^* a_{\mathbf{p}}, \quad (8.63)$$

where a new variable

$$\boxed{a_{\mathbf{p}} = p \sqrt{\frac{V}{8\pi\hbar\omega_{\mathbf{p}}}} \psi_{\mathbf{p}}} \quad (8.64)$$

has been introduced. It makes (8.63) look like a Hamiltonian of a system of particles created or annihilated by the “operators”  $a_{\mathbf{p}}^*$  and  $a_{\mathbf{p}}$ . We shall see in a moment that this is not a coincidence.

Let us now write the first Zakharov equation (8.16) in wave-number space:

$$p^2 \left( i\omega_{pe}^{-1} \frac{\partial}{\partial t} - \frac{3}{2} p^2 \lambda_{De}^2 \right) \psi_{\mathbf{p}} = \frac{1}{2} \mathbf{p} \cdot \sum_{\mathbf{k}} (\mathbf{p} - \mathbf{k}) \psi_{\mathbf{p}-\mathbf{k}} \xi_{\mathbf{k}}, \quad \xi_{\mathbf{k}} \equiv \left( \frac{\delta\bar{n}_e}{n_{0e}} \right)_{\mathbf{k}}, \quad (8.65)$$

or, in terms of  $a_{\mathbf{p}}$ ,

$$i \frac{\partial a_{\mathbf{p}}}{\partial t} - \delta\omega_{\mathbf{p}} a_{\mathbf{p}} = \frac{\omega_{pe}}{2} \sum_{\mathbf{k}, \mathbf{q}} \delta_{\mathbf{p}, \mathbf{k}+\mathbf{q}} \frac{\mathbf{p} \cdot \mathbf{q}}{pq} a_{\mathbf{q}} \xi_{\mathbf{k}}. \quad (8.66)$$

I have made use of the previously introduced notation (8.18) for the “residual” frequency  $\delta\omega_{\mathbf{p}}$  and also approximated  $\omega_{\mathbf{p}} \approx \omega_{\mathbf{p}-\mathbf{k}} \approx \omega_{pe}$  in the nonlinear term.

Now let us turn to the second Zakharov equation (8.17), describing sound waves. In fact, I need to walk back to the two fluid equations from which it originated: the ion continuity equation (8.8) and the ion momentum equation (8.9), the former recast in terms of  $\xi = \delta\bar{n}_e/n_{0e}$  using quasineutrality and the latter with  $\delta\bar{p}_e = T_e \delta\bar{n}_e +$  an “effective pressure” term representing the ponderomotive force [see (8.13)]:

$$\frac{\partial \xi_{\mathbf{k}}}{\partial t} = -i\mathbf{k} \cdot \mathbf{u}_{i,\mathbf{k}}, \quad \frac{\partial \mathbf{u}_{i,\mathbf{k}}}{\partial t} = -i\mathbf{k} (c_s^2 \xi_{\mathbf{k}} + \chi_{\mathbf{k}}), \quad (8.67)$$

<sup>67</sup>If the appearance of  $\hbar$  in a purely classical calculation triggers you, you may safely set  $\hbar = 1$  everywhere. I carry it in order to keep track of dimensions, and for the uniformity of the formalism between here and §7.

where, expanding the quadratic expression for the ponderomotive force as a Fourier convolution and using (8.64),

$$\chi_{\mathbf{k}} \equiv \frac{\overline{(|\nabla\psi|^2)}_{\mathbf{k}}}{16\pi m_i n_{0i}} = - \sum_{\mathbf{p}} \frac{\mathbf{p} \cdot (\mathbf{k} - \mathbf{p}) \psi_{\mathbf{p}} \psi_{\mathbf{k}-\mathbf{p}}^*}{16\pi m_i n_{0i}} = \frac{\hbar\omega_{pe}}{2m_i n_{0i} V} \sum_{\mathbf{p}, \mathbf{q}} \delta_{\mathbf{k}, \mathbf{p}-\mathbf{q}} \frac{\mathbf{p} \cdot \mathbf{q}}{pq} \overline{a_{\mathbf{p}} a_{\mathbf{q}}^*}. \quad (8.68)$$

At the last step, I made use of the fact that  $\psi_{\mathbf{q}} = \psi_{-\mathbf{q}}$ —this follows from (8.15) and the reality condition  $\tilde{\varphi}_{-\mathbf{q}} = \tilde{\varphi}_{\mathbf{q}}^*$ . Now, clearly, the flow is potential,  $\mathbf{u}_{i,\mathbf{k}} = ik\phi_{\mathbf{k}}$ , so our equations become

$$\frac{\partial \xi_{\mathbf{k}}}{\partial t} = k^2 \phi_{\mathbf{k}}, \quad \frac{\partial \phi_{\mathbf{k}}}{\partial t} = -c_s^2 \xi_{\mathbf{k}} - \chi_{\mathbf{k}}. \quad (8.69)$$

The linear part of these equations is Hamiltonian, in canonical variables  $(\xi_{\mathbf{k}}, \phi_{\mathbf{k}})$ , with the Hamiltonian being (up to a multiplicative constant) the energy of the sound waves (cf. Exercise 5.3):

$$H^s = \int d\mathbf{r} \left( \frac{m_i n_{0i} u_i^2}{2} + \frac{n_{0e} T_e}{2} \frac{\delta \bar{n}_e^2}{n_{0e}^2} \right) = \frac{m_i n_{0i} V}{2} \sum_{\mathbf{k}} (k^2 \phi_{\mathbf{k}}^2 + c_s^2 \xi_{\mathbf{k}}^2) \equiv \sum_{\mathbf{k}} \hbar \Omega_{\mathbf{k}} b_{\mathbf{k}}^* b_{\mathbf{k}}, \quad (8.70)$$

where  $\Omega_{\mathbf{k}} = kc_s$ . The last step is accomplished by the standard transformation

$$b_{\mathbf{k}} = \sqrt{\frac{m_i n_{0i} V}{2\hbar}} \left( \sqrt{\frac{c_s}{k}} \xi_{\mathbf{k}} + i \sqrt{\frac{k}{c_s}} \phi_{\mathbf{k}} \right). \quad (8.71)$$

Note that, while  $\xi_{\mathbf{k}}$  and  $\phi_{\mathbf{k}}$  are Fourier transforms of real fields (so  $\xi_{-\mathbf{k}} = \xi_{\mathbf{k}}^*$  and  $\phi_{-\mathbf{k}} = \phi_{\mathbf{k}}^*$ ), the new variable  $b_{\mathbf{k}}$  is genuinely complex and so contains full information about both fields, which can be extracted back from it:

$$\xi_{\mathbf{k}} = \sqrt{\frac{\hbar k}{2m_i n_{0i} V c_s}} (b_{\mathbf{k}} + b_{-\mathbf{k}}^*), \quad \phi_{\mathbf{k}} = -i \sqrt{\frac{\hbar c_s}{2m_i n_{0i} V k}} (b_{\mathbf{k}} - b_{-\mathbf{k}}^*). \quad (8.72)$$

The two equations (8.69) can, therefore, be wrapped into one:

$$i \frac{\partial b_{\mathbf{k}}}{\partial t} - \Omega_{\mathbf{k}} b_{\mathbf{k}} = \sqrt{\frac{m_i n_{0i} V k}{2\hbar c_s}} \chi_{\mathbf{k}} = \sum_{\mathbf{p}, \mathbf{q}} M_{\mathbf{k}\mathbf{p}\mathbf{q}} \delta_{\mathbf{k}, \mathbf{p}-\mathbf{q}} \overline{a_{\mathbf{p}} a_{\mathbf{q}}^*}, \quad (8.73)$$

where the mode-coupling coefficients (“matrix elements”) are

$$M_{\mathbf{k}\mathbf{p}\mathbf{q}} = \frac{\omega_{pe}}{2} \sqrt{\frac{\hbar k}{2m_i n_{0i} V c_s}} \frac{\mathbf{p} \cdot \mathbf{q}}{pq}. \quad (8.74)$$

Using (8.72) in (8.66), we can get the first Zakharov equation into a form that features the same coupling coefficients:

$$i \frac{\partial a_{\mathbf{p}}}{\partial t} - \delta \omega_{\mathbf{p}} a_{\mathbf{p}} = \sum_{\mathbf{k}, \mathbf{q}} M_{\mathbf{k}\mathbf{p}\mathbf{q}} \delta_{\mathbf{p}, \mathbf{k}+\mathbf{q}} a_{\mathbf{q}} (b_{\mathbf{k}} + b_{-\mathbf{k}}^*) = \sum_{\mathbf{k}, \mathbf{q}} M_{\mathbf{k}\mathbf{p}\mathbf{q}} a_{\mathbf{q}} (b_{\mathbf{k}} \delta_{\mathbf{p}, \mathbf{q}+\mathbf{k}} + b_{\mathbf{k}}^* \delta_{\mathbf{p}, \mathbf{q}-\mathbf{k}}). \quad (8.75)$$

It is now manifest that (8.75) and (8.73) are Hamiltonian equations in the form they are usually written for creation-annihilation variables:

$$i\hbar \frac{\partial a_{\mathbf{p}}}{\partial t} = \frac{\partial H}{\partial a_{\mathbf{p}}^*}, \quad i\hbar \frac{\partial b_{\mathbf{k}}}{\partial t} = \frac{\partial \bar{H}}{\partial b_{\mathbf{k}}^*}, \quad (8.76)$$

where the Hamiltonian consists of the free-plasmon energy (8.63) (with the constant

part of the frequency  $\omega_{pe}$  taken out), the free-phonon energy (8.70), and the interaction Hamiltonian describing the emission/absorption of phonons by plasmons:

$$H = \hbar \sum_{\mathbf{k}} (\delta\omega_{\mathbf{k}} a_{\mathbf{k}}^* a_{\mathbf{k}} + \Omega_{\mathbf{k}} b_{\mathbf{k}}^* b_{\mathbf{k}}) + \hbar \sum_{\mathbf{k}, \mathbf{p}, \mathbf{q}} \delta_{\mathbf{p}, \mathbf{k}+\mathbf{q}} M_{\mathbf{k}\mathbf{p}\mathbf{q}} (b_{\mathbf{k}}^* a_{\mathbf{q}}^* a_{\mathbf{p}} + a_{\mathbf{p}}^* a_{\mathbf{q}} b_{\mathbf{k}}) . \quad (8.77)$$

The two terms in the interaction Hamiltonian manifestly correspond to the two diagrams in Fig. 35.

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**Exercise 8.3. Plasmon number conservation.** Prove (and note) that our equations also conserve the total number of plasmons  $N = \sum_{\mathbf{p}} a_{\mathbf{p}}^* a_{\mathbf{p}}$  (which is why it was OK to take the constant-frequency part out of the Hamiltonian). In Exercise 7.4, you already showed that the kinetic equation for  $N_{\mathbf{p}}$  also had this property. Note also that, unlike energy,  $N$  is not an invariant of (8.7), so it is an *adiabatic invariant* whose conservation depends on the assumption  $\partial/\partial t \ll \omega_{pe}$ , which got us from (8.7) to (8.16).

That our system ought to be Hamiltonian, and that the Hamiltonian for the three-wave interactions described by the diagrams in Fig. 35 should have the general form (8.77), could have been argued *a priori*, so the added value from the above derivations is that we have also worked out the coupling coefficients  $M_{\mathbf{k}\mathbf{p}\mathbf{q}}$  and the correspondence between the creation-annihilation variables  $a_{\mathbf{k}}$ ,  $b_{\mathbf{k}}$  and the physical fields  $\tilde{\varphi}$ ,  $\delta\bar{n}_e$ . If you like the Hamiltonian approach, you will find a very extensive treatment of its application to plasma turbulence in the review by [Zakharov et al. \(1985\)](#)—and an even more general (not specially focused on plasma physics), and more pedagogical, exposition in the now-classic book by [Zakharov et al. \(1992\)](#).

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**Exercise 8.4. Hamiltonian for four-wave interactions.** Consider the quasistatic version (8.22) of the Zakharov system. By deriving the evolution equation for  $a_{\mathbf{k}}$ , or otherwise, show that the plasmon Hamiltonian with four-wave interactions is

$$H = \hbar \sum_{\mathbf{k}} \delta\omega_{\mathbf{k}} a_{\mathbf{k}}^* a_{\mathbf{k}} + \hbar \sum_{\mathbf{k}, \mathbf{k}', \mathbf{p}, \mathbf{p}'} \delta_{\mathbf{k}+\mathbf{k}', \mathbf{p}+\mathbf{p}'} M_{\mathbf{k}\mathbf{k}'\mathbf{p}\mathbf{p}'} a_{\mathbf{p}}^* a_{\mathbf{p}'}^* a_{\mathbf{k}} a_{\mathbf{k}'} , \quad (8.78)$$

and calculate the coupling coefficients  $M_{\mathbf{k}\mathbf{k}'\mathbf{p}\mathbf{p}'}$ . You will find the answers to this Exercise, as well as to Exercises 8.5, 8.7, 8.9, and 8.11 in the literature cited above, or in the original, famous paper by [Zakharov \(1972\)](#)—but following his derivations will not necessarily be easier than working them out by yourself.

#### 8.4. Weak Langmuir Turbulence

Let me now show you how to derive (§§8.4.1–8.4.2) and solve (§§8.4.3–8.4.6) the kinetic equations (7.23) and (7.24) starting from the dynamical equations (8.73) and (8.75). As far as I know, the example of WT considered here was first worked out by [Zakharov & Kuznetsov \(1978\)](#) and became a minor classic of the WT genre.

Note that because our starting point is a fairly generic Hamiltonian system (with the nature of the specific problem hidden in  $M_{\mathbf{k}\mathbf{p}\mathbf{q}}$ ), the basic strategies and ideas laid out below will have much more general applicability than to the specific, narrow, and perhaps even irrelevant, problem of Langmuir–sound turbulence at  $T_i \ll T_e$ .

##### 8.4.1. Perturbation Theory

We need evolution equations for  $N_{\mathbf{p}} = \langle |a_{\mathbf{p}}|^2 \rangle$  and  $n_{\mathbf{k}} = \langle |b_{\mathbf{k}}|^2 \rangle$ , so let us multiply (8.75) and (8.73) by  $a_{\mathbf{k}}^*$  and  $b_{\mathbf{k}}^*$ , respectively, subtract the complex conjugates of the

same, average<sup>68</sup> and see what happens. In fact, we only need to do one equation, e.g., the sound-wave one (8.73), because we know that it will have the structure (7.23) and, once we work out  $w(\mathbf{p}, \mathbf{k})$ , we will be able to write the Langmuir-wave kinetic equation (7.24) without any further effort.

Thus, we have for the phonon occupation number:<sup>69</sup>

$$\frac{\partial n_{\mathbf{k}}}{\partial t} = 2 \operatorname{Im} \sum_{\mathbf{p}, \mathbf{q}} \delta_{\mathbf{k}, \mathbf{p}-\mathbf{q}} M_{\mathbf{k}\mathbf{p}\mathbf{q}} \langle b_{\mathbf{k}}^* a_{\mathbf{p}} a_{\mathbf{q}}^* \rangle. \quad (8.79)$$

As always in nonlinear problems, the second-order correlator depends on the third-order one—the so-called *closure problem*. The perturbative solution to this problem is to assume the nonlinearity to be weak and so to truncate the expansion for the fields in the higher-order correlator at the linear level. As we shall see shortly, if we do this at the level of the third-order correlator, everything will vanish, so we need to iterate once more.

Using (8.75) for  $a_{\mathbf{p}}$  and  $a_{\mathbf{q}}^*$  and (8.73) for  $b_{\mathbf{k}}^*$ , we get

$$\begin{aligned} \left[ \frac{\partial}{\partial t} + i(\omega_{\mathbf{p}} - \omega_{\mathbf{q}} - \Omega_{\mathbf{k}}) \right] \langle b_{\mathbf{k}}^* a_{\mathbf{p}} a_{\mathbf{q}}^* \rangle &= i \sum_{\mathbf{k}', \mathbf{k}''} [M_{\mathbf{k}\mathbf{k}'\mathbf{k}''} \delta_{\mathbf{k}, \mathbf{k}' - \mathbf{k}''} \langle a_{\mathbf{k}'}^* a_{\mathbf{k}''} a_{\mathbf{p}} a_{\mathbf{q}}^* \rangle \\ &\quad - M_{\mathbf{k}'\mathbf{p}\mathbf{k}''} (\delta_{\mathbf{p}, \mathbf{k}'' + \mathbf{k}'} \langle b_{\mathbf{k}}^* b_{\mathbf{k}'} a_{\mathbf{k}''} a_{\mathbf{q}}^* \rangle + \delta_{\mathbf{p}, \mathbf{k}'' - \mathbf{k}'} \langle b_{\mathbf{k}}^* b_{\mathbf{k}'} a_{\mathbf{k}''} a_{\mathbf{q}}^* \rangle) \\ &\quad + M_{\mathbf{k}'\mathbf{q}\mathbf{k}''} (\delta_{\mathbf{q}, \mathbf{k}'' + \mathbf{k}'} \langle b_{\mathbf{k}}^* b_{\mathbf{k}'} a_{\mathbf{p}} a_{\mathbf{k}''}^* \rangle + \delta_{\mathbf{q}, \mathbf{k}'' - \mathbf{k}'} \langle b_{\mathbf{k}}^* b_{\mathbf{k}'} a_{\mathbf{p}} a_{\mathbf{k}''}^* \rangle)] \equiv i A_{\mathbf{k}\mathbf{p}\mathbf{q}}. \end{aligned} \quad (8.80)$$

Note that, since only the difference of Langmuir frequencies enters, I have, for brevity of notation, replaced  $\delta\omega_{\mathbf{p}}$  with  $\omega_{\mathbf{p}}$ . We do not need to go any further in perturbation theory and so can solve the above equation treating the right-hand side as a constant (being an average, it changes slowly in time compared to any of the waves' oscillations):

$$\langle b_{\mathbf{k}}^* a_{\mathbf{p}} a_{\mathbf{q}}^* \rangle = \frac{1 - e^{-i(\omega_{\mathbf{p}} - \omega_{\mathbf{q}} - \Omega_{\mathbf{k}})t}}{\omega_{\mathbf{p}} - \omega_{\mathbf{q}} - \Omega_{\mathbf{k}}} A_{\mathbf{k}\mathbf{p}\mathbf{q}} \rightarrow i\pi\delta(\omega_{\mathbf{p}} - \omega_{\mathbf{q}} - \Omega_{\mathbf{k}}) A_{\mathbf{k}\mathbf{p}\mathbf{q}} \quad (8.81)$$

as  $t \rightarrow \infty$  (i.e., after many wave periods), by the same token as in (5.39)—there will be a lot of these delta functions in kinetic theory!

Now we need to calculate the fourth-order correlation functions inside  $A_{\mathbf{k}\mathbf{p}\mathbf{q}}$ . That will require another conceptual step.

#### 8.4.2. Random-Phase Approximation

Our dynamical equations (8.73) and (8.75), with their small nonlinearities, describe the evolution of amplitudes and phases of the dynamical variables:  $a_{\mathbf{k}} = |a_{\mathbf{k}}|(t)e^{i\theta_{\mathbf{k}}(t)}$  and similarly for  $b_{\mathbf{k}}$ . The amplitudes evolve slowly (to lowest order in the nonlinearity, not at all) and phases quickly:  $\theta_{\mathbf{k}}(t) \approx \omega_{\mathbf{k}}t$ . This is not unlike (indeed, exactly analogous) to ballistic motion of particles in an ideal gas. When nonlinearity is introduced, even if small, the trajectories are perturbed and phases quickly stochasticised—this too is analogous to molecular chaos arising even from infrequent collisions between particles.<sup>70</sup>

Thus, we shall assume the randomness of phases and consequent vanishing (to lowest order in the perturbation theory) of correlations between the dynamical variables except

<sup>68</sup>What “average” means will be explained in §8.4.2.

<sup>69</sup>I have removed the time averaging (denoted by overbars) that was needed to iron out variation of the Langmuir electric fields on time scales shorter than the sound frequency. Any sensible averaging here will take care of that automatically.

<sup>70</sup>The fact that the wave frequencies are dispersive helps: as time goes on, different  $\mathbf{k}$ 's decorrelate—this is similar to phase mixing.

when their phases manage to cancel exactly.<sup>71</sup> To wit:

$$\langle a_{\mathbf{k}} \rangle = \langle |a_{\mathbf{k}}| e^{i\theta_{\mathbf{k}}} \rangle = 0, \quad (8.82)$$

$$\langle a_{\mathbf{k}} a_{\mathbf{k}'} \rangle = \langle |a_{\mathbf{k}}| |a_{\mathbf{k}'}| e^{i(\theta_{\mathbf{k}} + \theta_{\mathbf{k}'})} \rangle = 0, \quad (8.83)$$

$$\langle a_{\mathbf{k}} a_{\mathbf{k}'}^* \rangle = \langle |a_{\mathbf{k}}| |a_{\mathbf{k}'}| e^{i(\theta_{\mathbf{k}} - \theta_{\mathbf{k}'})} \rangle = N_{\mathbf{k}} \delta_{\mathbf{k}, \mathbf{k}'}. \quad (8.84)$$

By the same token, all odd-order correlators vanish, as do all even-order ones that do not have an equal number of occurrences of  $a_{\mathbf{k}}$  and  $a_{\mathbf{k}}^*$ . Thus, taking, e.g., one of the fourth-order correlators from (8.80):

$$\langle a_{\mathbf{k}'}^* a_{\mathbf{k}''} a_{\mathbf{p}} a_{\mathbf{q}}^* \rangle = N_{\mathbf{k}'} N_{\mathbf{p}} (\delta_{\mathbf{k}', \mathbf{k}''} \delta_{\mathbf{p}, \mathbf{q}} + \delta_{\mathbf{p}, \mathbf{k}'} \delta_{\mathbf{q}, \mathbf{k}''}). \quad (8.85)$$

Also, in a system where frequencies of different types of waves are well separated ( $\omega_{\mathbf{p}} \gg \Omega_{\mathbf{k}}$  always), only correlations between the same species of waves survive because otherwise phases cannot be matched. Let us do two other correlators in (8.80):

$$\langle b_{\mathbf{k}}^* b_{\mathbf{k}'} a_{\mathbf{k}''} a_{\mathbf{q}}^* \rangle = n_{\mathbf{k}} N_{\mathbf{q}} \delta_{\mathbf{k}, \mathbf{k}'} \delta_{\mathbf{q}, \mathbf{k}''}, \quad \langle b_{\mathbf{k}}^* b_{\mathbf{k}'} a_{\mathbf{p}} a_{\mathbf{k}''}^* \rangle = n_{\mathbf{k}} N_{\mathbf{p}} \delta_{\mathbf{k}, \mathbf{k}'} \delta_{\mathbf{p}, \mathbf{k}''}. \quad (8.86)$$

The rest of the correlators in (8.80) are zero.

**Exercise 8.5.** Work out, under the same assumptions, the 6th-order correlator  $\langle a_{\mathbf{k}} a_{\mathbf{p}} a_{\mathbf{q}} a_{\mathbf{k}'}^* a_{\mathbf{p}'}^* a_{\mathbf{q}'}^* \rangle$  in terms of the second-order ones ( $N_{\mathbf{k}} = \langle |a_{\mathbf{k}}|^2 \rangle$ ). This will come useful in Exercise 8.7.

We are done: the right-hand side of (8.80) is

$$\begin{aligned} A_{\mathbf{k}pq} &= M_{\mathbf{k}pq} N_{\mathbf{p}} N_{\mathbf{q}} - M_{\mathbf{k}pq} n_{\mathbf{k}} N_{\mathbf{q}} + M_{\mathbf{kqp}} n_{\mathbf{k}} N_{\mathbf{p}} \\ &= M_{\mathbf{k}pq} [N_{\mathbf{p}} N_{\mathbf{q}} + (N_{\mathbf{p}} - N_{\mathbf{q}}) n_{\mathbf{k}}]. \end{aligned} \quad (8.87)$$

Note that the first term in (8.85) did not contribute because  $M_{0\mathbf{k}'\mathbf{k}'} = 0$ , and that  $M_{\mathbf{k}pq} = M_{\mathbf{kqp}}$  was used to simplify things. Putting (8.87) back into (8.79), via (8.81), we recover, triumphantly, the anticipated kinetic equation in the form (7.23):

$$\frac{\partial n_{\mathbf{k}}}{\partial t} = 2\pi \sum_{\mathbf{p}} M_{\mathbf{k}, \mathbf{p}, \mathbf{p}-\mathbf{k}}^2 \delta(\omega_{\mathbf{p}} - \omega_{\mathbf{p}-\mathbf{k}} - \Omega_{\mathbf{k}}) [N_{\mathbf{p}} N_{\mathbf{p}-\mathbf{k}} + (N_{\mathbf{p}} - N_{\mathbf{p}-\mathbf{k}}) n_{\mathbf{k}}] \equiv \sum_{\mathbf{p}} T_{\mathbf{p}, \mathbf{k}}. \quad (8.88)$$

Manifestly, by comparison with (7.23) [or (7.27)],

$$w(\mathbf{p}, \mathbf{k}) = 2\pi M_{\mathbf{k}, \mathbf{p}, \mathbf{p}-\mathbf{k}}^2 = w_0 k \left( \frac{\mathbf{p}}{p} \cdot \frac{\mathbf{p} - \mathbf{k}}{|\mathbf{p} - \mathbf{k}|} \right)^2, \quad w_0 = \frac{\pi \hbar \omega_{\text{pe}}^2}{4m_i n_{0i} V c_s}. \quad (8.89)$$

In fact, to calculate this, we did not even need to follow all of the terms in (8.80), it was enough just to work out one of them, e.g.,  $\langle a_{\mathbf{k}'}^* a_{\mathbf{k}''} a_{\mathbf{p}} a_{\mathbf{q}}^* \rangle \propto N_{\mathbf{p}} N_{\mathbf{p}-\mathbf{k}}$ . With (8.89) in hand, we also have the kinetic equation (7.24) for the plasmons: to write it in a short form anticipated in (7.26),

$$\frac{\partial N_{\mathbf{p}}}{\partial t} = \sum_{\mathbf{k}} (T_{\mathbf{p}+\mathbf{k}, \mathbf{k}} - T_{\mathbf{p}, \mathbf{k}}), \quad (8.90)$$

where  $T_{\mathbf{p}, \mathbf{k}}$  is the transfer function (7.27), now fully specified by the expression under the wave-number sum in (8.88).

The rules (8.82–8.86), and similar, for calculating correlators are generally known as

<sup>71</sup>“Exactly” means *exactly* here: this is not just about matching linear frequencies!

the *random-phase approximation (RPA)*. Since they amount to splitting higher-order correlators into second-order ones, they are mathematically equivalent to assuming the fields to be Gaussian to lowest order in the perturbation theory (the dynamical variables are an accumulation of a sequence of random, independent kicks due to nonlinear interactions with each other—an instance of Central Limit Theorem, one might argue). Recalling the calculations in §7.2, it is now obvious that this assumption was implicitly already made when the rates of change of occupation numbers were written as products of these numbers—a quasiparticle version of Boltzmann’s (in fact, Maxwell’s and Ehrenfest’s) *Stosszahlansatz* (no wonder then that the standard form of the collision integral emerged so easily in §7.2.5). Just how valid, or otherwise, this assumption might be, is, as often with such things, a subtler question than it looks. I will sweep it resolutely under the carpet, while referring you to extended discussions of the matter in the textbooks by [Zakharov et al. \(1992\)](#) and [Nazarenko \(2011\)](#), and move on happily to play with the kinetic equations (8.88) and (8.90).

**Exercise 8.6.** To practice perturbation theory and the use of RPA, write, starting from (8.75), the evolution equation for  $N_p$  in terms of third-order correlators, then solve for the latter in terms of fourth-order ones, split the averages, and verify that the kinetic equation for the plasmons is indeed (8.90).

**Exercise 8.7. Kinetic equation for weak Langmuir turbulence with four-wave interactions.** Starting from the Hamiltonian derived in Exercise 8.4, work out, along the same lines as done above, the kinetic equation for the plasmon occupation number in Langmuir turbulence with four-wave interactions.

#### 8.4.3. Further Simplifications and Approximations

As I promised in §7.4, I shall look for *isotropic* solutions of the kinetic equations (8.88) and (8.90), so  $N_p$  depends only on  $p$  and  $n_k$  on  $k$ . This is sensible because the frequencies only depend on the magnitudes of the wave numbers,  $\omega_p = \omega_p$ ,  $\Omega_k = \Omega_k$ —there are no special directions in the system.<sup>72</sup> Since both  $\omega_p$  and  $\Omega_k$  are simple power functions,

$$\omega_p = \omega_{pe} + \frac{3v_{\text{the}}^2}{4\omega_{pe}} p^2 \equiv \omega_{pe} + \alpha p^2 \equiv \omega_{pe} + \delta\omega_p, \quad \Omega_k = c_s k, \quad (8.91)$$

there are one-to-one correspondences between wave numbers and frequencies, a feature that will greatly simplify calculations. Indeed, we can consider the occupation numbers to be functions of the frequencies only:  $N_p = N(\omega_p)$  and  $n_k = n(\Omega_k)$ . Then the transfer function is also a function of frequencies only:

$$\begin{aligned} T_{p,k} &= w_0 k \left( \frac{p^2 - \mathbf{p} \cdot \mathbf{k}}{p|\mathbf{p} - \mathbf{k}|} \right)^2 \delta(\omega_p - \omega_{|\mathbf{p}-\mathbf{k}|} - \Omega_k) [N_p N_{\mathbf{p}-\mathbf{k}} + (N_p - N_{\mathbf{p}-\mathbf{k}}) n_k] \\ &\equiv \delta(\omega_p - \omega_{|\mathbf{p}-\mathbf{k}|} - \Omega_k) \tilde{T}(\omega_p, \Omega_k). \end{aligned} \quad (8.92)$$

Everything inside  $\tilde{T}(\omega_p, \Omega_k)$  that depends on  $\omega_{|\mathbf{p}-\mathbf{k}|}$  is reconstructible in terms of  $\omega_p$  and  $\Omega_k$  via the delta function. This includes  $\mathbf{p} \cdot \mathbf{k}$ , which appears in the prefactor: indeed, the argument of the delta function is<sup>73</sup>

$$\alpha p^2 - \alpha |\mathbf{p} - \mathbf{k}|^2 - \Omega_k = \alpha(2\mathbf{p} \cdot \mathbf{k} - k^2) - \Omega_k = 0 \quad \Rightarrow \quad \mathbf{p} \cdot \mathbf{k} = \frac{k^2}{2} + \frac{\Omega_k}{\alpha}. \quad (8.93)$$

<sup>72</sup>This will become spectacularly untrue in systems with a background magnetic field: see §§15.3 and 15.4.

<sup>73</sup>So an “optimal” phonon for a plasmon to emit is one with  $\mathbf{k} \approx 2\mathbf{p}$  (this was Exercise 7.3b).

A very attractive further simplification is now possible if we assume  $\delta\omega_p \gg \Omega_k$  (which is true outside the quasistatic limit of §8.1.4):

$$\begin{aligned} T_{\mathbf{p}+\mathbf{k},\mathbf{k}} &= \delta(\omega_{|\mathbf{p}+\mathbf{k}|} - \omega_p - \Omega_k) \tilde{T}(\omega_{|\mathbf{p}+\mathbf{k}|}, \Omega_k) \\ &\approx \delta(\omega_{|\mathbf{p}+\mathbf{k}|} - \omega_p - \Omega_k) \left[ \tilde{T}(\omega_p, \Omega_k) + \underbrace{(\omega_{|\mathbf{p}+\mathbf{k}|} - \omega_p)}_{= \Omega_k} \frac{\partial \tilde{T}}{\partial \omega_p} \right] \end{aligned} \quad (8.94)$$

In the same vein, expanding  $N_{\mathbf{p}-\mathbf{k}} = N(\omega_{|\mathbf{p}-\mathbf{k}|})$  in (8.92) in  $\omega_p - \omega_{|\mathbf{p}-\mathbf{k}|} = \Omega_k$  and using (8.93) in the prefactor, we find

$$\tilde{T}(\omega_p, \Omega_k) \approx w_0 k \left( 1 - \frac{k^2}{2p^2} \right)^2 \left( N_{\mathbf{p}}^2 + \Omega_k n_{\mathbf{k}} \frac{\partial N_{\mathbf{p}}}{\partial \omega_p} \right). \quad (8.95)$$

Note that  $\Omega_k \partial N_{\mathbf{p}} / \partial \omega_p = (c_s k / 2\alpha p) \partial N_{\mathbf{p}} / \partial p$ .

From (8.95), we see immediately that for the Rayleigh–Jeans distributions (7.39),  $\tilde{T}(\omega_p, \Omega_k) = 0$ , so they are (still) legitimate stationary solutions of our equations. It makes sense that these solutions should be there, but they are not what we are after—we should be looking for the constant-flux solutions promised in §7.4.

#### 8.4.4. Plasmon Flux

Let us prepare to use (8.94) and (8.92) in the plasmon kinetic equation (8.90), where we can simplify the  $\mathbf{k}$  sum by noticing that the dependence on the direction of  $\mathbf{k}$  (with respect to  $\mathbf{p}$ ) is only left inside the delta functions:

$$\begin{aligned} \sum_{\mathbf{k}} \delta(\alpha(2\mathbf{p} \cdot \mathbf{k} \pm k^2) - c_s k) &= \frac{V}{(2\pi)^3} \int_0^\infty dk k^2 2\pi \int_{-1}^1 d\cos \theta \delta(\alpha(2pk\cos \theta \pm k^2) - c_s k), \\ &= \frac{V}{4\pi^2} \int_0^{2p \pm c_s/\alpha} \frac{dk k}{2\alpha p}, \end{aligned} \quad (8.96)$$

where “+” applies to the integral of (8.94) and “−” to the integral of (8.92). Note that  $c_s/\alpha \ll 2p$  because  $\Omega_p \ll \delta\omega_p$ . We can now turn the plasmon kinetic equation (8.90) into the conservative form (7.42) by defining, as was done in §7.4,  $\bar{N}_p = (Vp^2/2\pi^2)N_{\mathbf{p}}$  and using (8.96):

$$\begin{aligned} \frac{\partial \bar{N}_p}{\partial t} &\approx \left( \frac{V}{2\pi^2} \right)^2 \frac{p}{4\alpha} \left\{ \int_0^{2p+c_s/\alpha} dk \left[ k \tilde{T}(\omega_p, \Omega_k) + k \Omega_k \frac{\partial \tilde{T}}{\partial \omega_p} \right] - \int_0^{2p-c_s/\alpha} dk k \tilde{T}(\omega_p, \Omega_k) \right\} \\ &\approx \left( \frac{V}{2\pi^2} \right)^2 \frac{p}{4\alpha} \left( \frac{2c_s}{\alpha} 2p \tilde{T} + \int_0^{2p} dk \frac{c_s k^2}{2\alpha p} \frac{\partial \tilde{T}}{\partial p} \right) \\ &= \left( \frac{V}{2\pi^2} \right)^2 \frac{c_s}{8\alpha^2} \left( 8p^2 \tilde{T} + \int_0^{2p} dk k^2 \frac{\partial \tilde{T}}{\partial p} \right) = -\frac{\partial \Gamma_p}{\partial p}, \end{aligned} \quad (8.97)$$

where the plasmon-number flux is

$$\begin{aligned} \Gamma_p &= - \left( \frac{V}{2\pi^2} \right)^2 \frac{c_s}{8\alpha^2} \int_0^{2p} dk k^2 \tilde{T}(\omega_p, \Omega_k) \\ &= - \left( \frac{V}{2\pi^2} \right)^2 \frac{w_0 c_s}{8\alpha^2} \int_0^{2p} dk k^3 \left( 1 - \frac{k^2}{2p^2} \right)^2 \left( N_{\mathbf{p}}^2 + \frac{c_s k n_{\mathbf{k}}}{2\alpha p} \frac{\partial N_{\mathbf{p}}}{\partial p} \right). \end{aligned} \quad (8.98)$$

I have used the approximate expression (8.95) for  $\tilde{T}(\omega_p, \Omega_k)$ . This approximation is called the “diffusion approximation” because under it, (8.97) has turned into a diffusion equation for  $\bar{N}_p$  in wave-number space.<sup>74</sup>

So this is what I promised you in §7.4: a specific expression for a flux, setting which to be constant will yield stationary nonequilibrium distributions. Here is how to find them. Let us look for power-law solutions:

$$N_p = Ap^{-x}, \quad n_{\mathbf{k}} = Bk^{-y}. \quad (8.99)$$

Putting these into (8.98) and rescaling  $p$  out of the integral by introducing a new integration variable  $\xi = k/p$ , we get

$$\Gamma_p = - \left( \frac{V}{2\pi^2} \right)^2 \frac{w_0 c_s A^2}{8\alpha^2} p^{4-2x} \underbrace{\int_0^2 d\xi \xi^3 \left( 1 - \frac{\xi^2}{2} \right)^2 \left( 1 - \frac{c_s B x}{2\alpha A} \xi^{-y+1} p^{x-y-1} \right)}_{\equiv I_1(p)}. \quad (8.100)$$

This can only be independent of  $p$  if the powers of  $p$  in both terms vanish, giving  $x = 2$  and  $y = x - 1 = 1$ , so

$$N_p = A_{\Gamma} p^{-2}, \quad n_{\mathbf{k}} = B_{\Gamma} p^{-1}. \quad (8.101)$$

I have equipped the constants with a subscript  $\Gamma$  to emphasise that they are specific to the regime in which the distributions are determined by the constancy of the plasmon flux.

Substituting (8.101) back into (8.100), we get, after doing the integral  $I_1 = 4/3$ :

$$\Gamma = - \left( \frac{V}{2\pi^2} \right)^2 \frac{w_0 c_s A_{\Gamma}^2}{6\alpha^2} \left( 1 - \frac{c_s B_{\Gamma}}{\alpha A_{\Gamma}} \right). \quad (8.102)$$

The sign of this expression is not definite, so we do not yet have a way of knowing whether the plasmon number flows to smaller or larger  $p$ . The answer is that it will flow to smaller  $p$ , but we will only be able to deduce this formally once we know what happens with the energy flux.

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<sup>74</sup>That this simplification should be possible is a particular feature of the problem at hand, not a generic property of WT. Neither is it essential to have such an approximation in order to be able to find constant-flux, power-law solutions as I shall do below. Such solutions (can) also exist in cases where the rates of change of quasiparticle occupation numbers are only explicitly expressible as wave-number integrals—a famous example is Exercise 8.9.

### 8.4.5. Energy Flux

Let us turn to the phonon kinetic equation (8.88). This contains a similar integral to those we encountered above:

$$\begin{aligned}
\frac{\partial n_{\mathbf{k}}}{\partial t} &= \sum_{\mathbf{p}} \delta(\omega_p - \omega_{|\mathbf{p}-\mathbf{k}|} - \Omega_k) \tilde{T}(\omega_p, \Omega_k) \\
&= \frac{V}{(2\pi)^3} \int_0^\infty dp p^2 \tilde{T}(\omega_p, \Omega_k) 2\pi \int_{-1}^1 d\cos\theta \delta(\alpha(2pk\cos\theta - k^2) - c_s k) \\
&\approx \frac{V}{2\pi^2} \frac{w_0}{4\alpha} \int_{k/2}^\infty dp p \left(1 - \frac{k^2}{2p^2}\right)^2 \left(N_{\mathbf{p}}^2 + \frac{c_s k n_{\mathbf{k}}}{2\alpha p} \frac{\partial N_{\mathbf{p}}}{\partial p}\right) \\
&= \frac{V}{2\pi^2} \frac{w_0 A^2}{4\alpha} k^{2-2x} \underbrace{\int_{1/2}^\infty d\zeta \zeta^{1-2x} \left(1 - \frac{1}{2\zeta^2}\right)^2 \left(1 - \frac{c_s B x}{2\alpha A} \zeta^{x-2} k^{x-y-1}\right)}_{\equiv I_2(k)}. \quad (8.103)
\end{aligned}$$

At the last step, I substituted the power-law solutions (8.99) and rescaled  $k$  out of the integral by changing the integration variable to  $\zeta = p/k$ .

We are now ready to construct an equation for the evolution of the total-energy spectrum

$$\bar{E}_k = \frac{V k^2}{2\pi^2} (\hbar\Omega_k n_{\mathbf{k}} + \hbar\delta\omega_k N_{\mathbf{k}}). \quad (8.104)$$

This was already defined in (7.40), but here I have taken out the bit corresponding to the constant part of the plasmon frequency,  $\hbar\omega_{pe}N_{\mathbf{k}}$ , because that has the same evolution as the plasmon number, governed by (8.97). I shall call  $\bar{E}_k$  ‘‘residual energy’’. Using (8.103), (8.97) and (8.100), we get

$$\begin{aligned}
\frac{\partial \bar{E}_k}{\partial t} &= \left(\frac{V}{2\pi^2}\right)^2 \frac{w_0 A^2}{4\alpha} \hbar\Omega_k k^{4-2x} I_2(k) - \hbar\delta\omega_k \frac{\partial \Gamma_k}{\partial k} \\
&= \left(\frac{V}{2\pi^2}\right)^2 \frac{\hbar w_0 c_s A^2}{4\alpha} \underbrace{\left[k^{5-2x} I_2(k) + \frac{1}{2} k^2 \frac{\partial}{\partial k} k^{4-2x} I_1(k)\right]}_{= k^{5-2x} [I_2 + (2-x)I_1]} \equiv -\frac{\partial \varepsilon_k}{\partial k}. \quad (8.105)
\end{aligned}$$

This expression can only be scale-invariant if the  $k$  dependence in  $I_1$  and  $I_2$  disappears. The condition for that is

$$y = x - 1. \quad (8.106)$$

In fact, this could have perhaps been guessed *a priori*: it is just the statement that the contributions to  $\bar{E}_k$  from the phonons and the plasmons have the same scaling:  $\Omega_k n_{\mathbf{k}} \sim \delta\omega_k N_{\mathbf{k}}$ .

Adopting this assumption, we conclude immediately that a stationary solution requires

$$I_2 + (2-x)I_1 = 0, \quad (8.107)$$

where the integrals, which now depend only on the exponent  $x$  and on the constants  $A$

and  $B$ , are

$$I_1 = \int_0^2 d\xi \xi^3 \left(1 - \frac{\xi^2}{2}\right)^2 \left(1 - \frac{c_s B x}{2\alpha A} \xi^{-x+2}\right), \quad (8.108)$$

$$\begin{aligned} I_2 &= \int_{1/2}^{\infty} d\zeta \zeta^{1-2x} \left(1 - \frac{1}{2\zeta^2}\right)^2 \left(1 - \frac{c_s B x}{2\alpha A} \zeta^{x-2}\right) \\ &= \int_0^2 d\xi \xi^{2x-3} \left(1 - \frac{\xi^2}{2}\right)^2 \left(1 - \frac{c_s B x}{2\alpha A} \xi^{-x+2}\right). \end{aligned} \quad (8.109)$$

The last transformation of  $I_2$  was accomplished by changing the integration variable to  $\xi = 1/\zeta$ , mapping the integration domain of  $I_2$  onto that of  $I_1$ . The two integrals become identical at  $x = 3$ , which then conveniently solves (8.107). Thus, we have obtained the constant-residual-energy-flux solutions:

$$\boxed{N_k = A_\varepsilon k^{-3}, \quad n_k = B_\varepsilon k^{-2}}. \quad (8.110)$$

As in (8.101), the constants have acquired a subscript indicating that they pertain to the constant- $\varepsilon$  regime only.

Let us now look at the residual-energy flux. Unlike for  $\Gamma_p$  in (8.98), we do not have a nice general expression for  $\varepsilon_k$  in terms of  $N_k$  and  $n_k$ . For power-law solutions, however, we can easily get  $\varepsilon_k$  from (8.105) by direct integration:

$$\varepsilon_k = - \left(\frac{V}{2\pi^2}\right)^2 \frac{\hbar w_0 c_s A^2}{4\alpha} \frac{I_2 + (2-x)I_1}{6-2x} k^{6-2x}. \quad (8.111)$$

The  $k$  dependence does indeed disappear at  $x = 3$ , but the remaining expression has a 0/0 indeterminacy. By L'Hôpital's rule,

$$-\left[\frac{I_2 + (2-x)I_1}{6-2x}\right]_{x=3} = \frac{1}{2} \left[ \frac{\partial I_2}{\partial x} + (2-x) \frac{\partial I_1}{\partial x} - I_1 \right]_{x=3} = c_1 + c_2 \frac{c_s B_\varepsilon}{\alpha A_\varepsilon}, \quad (8.112)$$

where the numerical coefficients, arising from the ugly mêlée of fractions and logs that is the exact calculation of  $I_1$ ,  $I_2$ , and their derivatives, are

$$c_1 = \frac{4(3 \ln 2 - 1)}{9} \approx 0.035, \quad c_2 = \frac{3754}{3675} - \frac{44 \ln 2}{35} \approx 0.150. \quad (8.113)$$

For the record,

$$I_1 = \frac{4}{3} + \frac{2^{5-x} x [32 + x(x-10)]}{(x-6)(x-8)(x-10)} \frac{c_s B}{\alpha A}, \quad (8.114)$$

$$I_2 = \frac{2^{2x-3} [4 + x(x-3)]}{x(x^2-1)} - \frac{2^{x-1} [8 + x(x-2)]}{(x+2)(x+4)} \frac{c_s B}{\alpha A}, \quad (8.115)$$

hence, after differentiation and substitution of  $x = 3$ , the values (8.113).

The useful takeaway is that the coefficients (8.113) are positive, and so then is the residual-energy flux (8.111), always:

$$\boxed{\varepsilon = \left(\frac{V}{2\pi^2}\right)^2 \frac{\hbar w_0 c_s A_\varepsilon^2}{4\alpha} \left(c_1 + c_2 \frac{c_s B_\varepsilon}{\alpha A_\varepsilon}\right) > 0}. \quad (8.116)$$

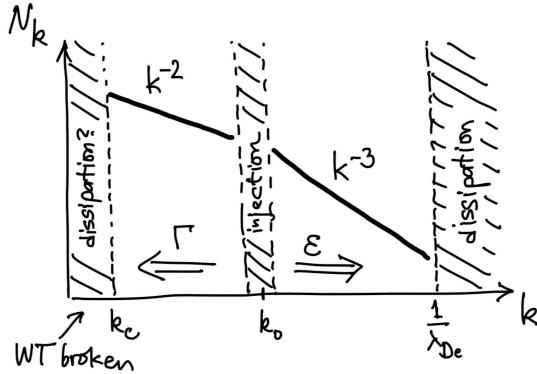


FIGURE 39. Direct and inverse cascades. Life in the shaded regions is not described by the scale-invariant WT calculation, but the spectra and cascades in between are. The dissipation region at small scales starts at  $k\lambda_{De} \sim 1$ , where Langmuir waves start getting heavily Landau damped (cf. Fig. 14). The (presumed) dissipation region at large scales, where WT breaks down, should start around  $k \sim k_c$  given by (8.127).

#### 8.4.6. Direct and Inverse Cascades

Thus, if energy is injected into a plasmon–phonon gas at some wave number  $k_0$  and at the rate  $\varepsilon$ , it will be *cascaded directly*, i.e., flow to larger wave numbers (smaller scales)  $k > k_0$ , leaving in its wake the distributions (8.110). The energy here is the residual energy (8.104), from which the constant part of the plasmon frequency has been taken out.

The latter (in fact, dominant) part of the total energy,  $\sum_p \hbar\omega_{pe} N_p$ , is proportional to the plasmon number  $N$  and conserved separately. It cannot also have a direct cascade. Indeed, the plasmon flux is given by (8.100) and we can easily calculate it for the spectra (8.110), i.e., for  $x = 3$  and  $y = 2$ : using (8.114),

$$\Gamma_k = - \left( \frac{V}{2\pi^2} \right)^2 \frac{w_0 c_s A_\varepsilon^2}{8\alpha^2} \left( \frac{4}{3} - \frac{44}{35} \frac{c_s B_\varepsilon}{\alpha A_\varepsilon} \right) k^{-2} \rightarrow 0 \quad \text{as } k \rightarrow \infty. \quad (8.117)$$

So it peters out at small scales. The natural conclusion is, therefore, that the plasmon number will *cascade inversely*, i.e., flow to smaller wave numbers (larger scales)  $k < k_0$ . The expression (8.102) must then be negative—not being sign-definite, it can be, implying  $c_s B_\Gamma < \alpha A_\Gamma$  for the distributions (8.101).

The dual-cascade picture just described is illustrated in Fig. 39.

**Exercise 8.8.** What is the residual-energy flux at  $k < k_0$ ? Does it matter in this region?

There is a very simple dimensional argument, invented by Fjørtoft (1953) in the context of 2D turbulence, that allows one to predict which invariant will flow to small scales and which to large. Dimensionally, the constant-flux distributions must satisfy

$$\frac{k \bar{N}_k}{\tau_k} \sim \Gamma, \quad \frac{k \bar{E}_k}{\tau_k} \sim \frac{\hbar \alpha k^3 \bar{N}_k}{\tau_k} \sim \varepsilon, \quad (8.118)$$

where  $\tau_k$  is the effective rate at which nonlinear interactions happen at the scale  $k^{-1}$ . At the injection wave number  $k_0$ , this implies that

$$\varepsilon \sim \hbar \alpha k_0^2 \Gamma. \quad (8.119)$$

If there were a direct cascade of plasmon number, this would imply, at  $k \gg k_0$ , an arriving

residual-energy flux

$$\frac{k\bar{E}_k}{\tau_k} \sim \hbar\omega^2\Gamma \gg \varepsilon. \quad (8.120)$$

No such supply of residual energy is available from the injection scale, so this scenario cannot be realised. In contrast, if the plasmon number cascades inversely, to  $k \ll k_0$ , the amount of residual energy arriving there is minuscule, so everything is fine. The basic conclusion is that the invariant that has more powers of  $k$  cascades directly and the one that has fewer inversely.

Let me summarise our formal progress. We have found the scaling of the distributions, the direction of cascades, and the relationships (8.102) and (8.116) between the constant fluxes corresponding to these cascades and the constants  $A_\Gamma$ ,  $B_\Gamma$ ,  $A_\varepsilon$ , and  $B_\varepsilon$  in the distributions (8.101) and (8.110). Thus, we have a viable solution.<sup>75</sup> Note that the fact that the integrals (8.108) and (8.109) that went into the calculation of the fluxes converged is the confirmation of *locality* of fluxes—a blow up would have indicated that the whole approach of solving in scale-invariant ranges of  $k$  away from injection and dissipation scales had, in fact, been inapplicable.

There is one outstanding task that I will not take up. Since the constants  $A_\Gamma$ ,  $B_\Gamma$  and  $A_\varepsilon$ ,  $B_\varepsilon$  are in general different, we have four constants to determine and so far only two equations relating them to the fluxes: (8.102) and (8.116). The additional constraints will come from matching the fluxes to the injection region around  $k_0$ , where the (in general, nonuniversal) physics of energy injection will connect  $\Gamma$  and  $\varepsilon$  to the perturbation amplitudes—and thus to  $k_0$  and the constants.

An elegant way to sort this out would be to demand that the plasmon flux (8.117) at  $k > k_0$  and the energy flux at  $k < k_0$  (calculated in Exercise 8.8) be exactly zero. The former condition gives a nontrivial relationship between  $A_\varepsilon$  and  $B_\varepsilon$ , but in trying to enforce the latter, one runs into the problem that setting  $\varepsilon_k = 0$  also makes  $\Gamma = 0$  at  $k < k_0$ . This exhumes from under the rug a subtlety going rather beyond the range of issues that I wish to engage with here. The problem at hand has an annoying degeneracy: the constant- $\Gamma$  scalings (8.101) actually coincide with the scalings of the equilibrium Rayleigh–Jeans distributions (7.39). Does this mean that they cannot support anything other than zero fluxes? This wrinkle was ironed out by [Kanashov & Rubenchik \(1980\)](#), whose solution is reproduced at the end of §3.2.2 of [Zakharov et al. \(1992\)](#): the gist of it is that the true solutions are, in fact, ever so slightly different from the power laws (8.101) (there are logarithmic corrections) and can support non-zero fluxes after all. Do investigate if you are intrigued.

This is as much as I shall say about the formal WT theory. The subject is much vaster than this, and full of deep insights, subtle nuances, clever tricks, twists and turns. I have chosen to work through one complete calculation, which features many of the key ideas and methods of the WT theory, but if you wish to become an expert, read the textbooks by [Zakharov et al. \(1992\)](#) and [Nazarenko \(2011\)](#) and the rest of the literature cited above.<sup>76</sup> Either of Exercises 8.9 or 8.10 offers a proactive independent-study path that should prove both challenging and exciting.

**Exercise 8.9. Spectra of Langmuir turbulence with four-wave interactions.** Starting from the kinetic equation derived in Exercise 8.7, work out the Langmuir-wave spectra arising from four-wave interactions. Your conclusion should be that plasmons will flow to larger scales,

<sup>75</sup>It is perhaps worth spelling out that this is *a* solution. No proof has been provided, and there is none, that the isotropic solution is the only one possible. Indeed, anisotropic solutions do exist. I again refer you to [Musher et al. \(1995\)](#) for a taste of them.

<sup>76</sup>The example that I have chosen is not, in fact, (in my view) particularly clearly presented in [Zakharov et al. \(1992\)](#), or in the original paper by [Zakharov & Kuznetsov \(1978\)](#), so I hope this section has provided some added value.

leaving behind a distribution  $N_{\mathbf{k}} \propto k^{-7/3}$ . The answers are in [Zakharov \(1972\)](#) or [Zakharov et al. \(1992\)](#). In getting there, you will learn about *Zakharov transformations*, an inspired feat of analytical skulduggery, of which the mapping  $\zeta = 1/\xi$  in §8.4.5 was an almost trivial example. You will also experience locality not being always guaranteed.

**Exercise 8.10. Induced scattering of Langmuir waves on ions.** This is an independent-study topic. Consider the regime ( $T_i \sim T_e$ ) in which sound waves are strongly damped but not so strongly as for the hot-ion quasistatic limit (§8.2.7) to apply. That is, the system is described by the first Zakharov equation (8.16) coupled via  $\delta\bar{n}_e$  to the ion kinetic equation (8.57). In this regime, the dominant nonlinear process is the induced scattering of plasmons on ions (analogous to what was considered in §7.2.4). Work out what the scattering probability is, what, therefore, are the kinetic equations for the plasmons and the ions, and study what kind of solutions they might have. In the literature, the common monicker for this regime is “isothermal Langmuir turbulence” (“isothermal” in the sense that  $T_i = T_e$ ; the Langmuir-sound turbulence discussed above is, accordingly, “nonisothermal Langmuir turbulence”). In navigating the literature, you may choose to follow the Tsytovich approach, featuring heavy-duty perturbative kinetic calculations (see the books cited at the end of §7.1 or the original papers cited in §7.2.4), or (or and?) the Zakharov one, with a more Hamiltonian flavour (see reviews by [Zakharov et al. 1985](#) and [Musher et al. 1995](#), the latter focussing specially on WT). If you plough through even some of it, you have my respect, but you may well emerge with an enduring hatred of the Soviet scientific writing style.

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#### 8.4.7. Breakdown of WT Approximation

Let me focus on a key finding of the above WT calculation: both the plasmons’ number  $N$  and, to lowest approximation, their energy  $\hbar\omega_{pe}N$  are pushed by the WT interactions to ever larger scales. As phonons have a shallower distribution at these scales than the plasmons,  $n_{\mathbf{k}} \propto kN_{\mathbf{k}}$  (and an even shallower energy spectrum  $\hbar k c_s n_{\mathbf{k}}$ ), everything is dominated by the plasmons there—a phenomenon sometimes referred to as the formation of a “Langmuir condensate”. I shall now argue that the WT approximation will break down for this condensate, below a certain critical wave number.

The condition for a system to be in the WT regime is that quasiparticles endure over a number of interactions, i.e., their frequencies must be large compared to the characteristic rate at which the nonlinearity acts—in §7.5, I wrote this as (7.44). Let us check when this is satisfied for Zakharov’s equations. For the first of them, (8.16), describing the evolution of the plasmon field, this requirement takes the form

$$\delta\omega_{\mathbf{k}} = \frac{3}{2} k^2 \lambda_{De}^2 \omega_{pe} \gg t_{nl}^{-1} \sim \omega_{pe} \frac{\delta\bar{n}_e}{n_{0e}} \quad \Leftrightarrow \quad \boxed{\frac{\delta\bar{n}_e}{n_{0e}} \ll k^2 \lambda_{De}^2}, \quad (8.121)$$

an upper limit on the perturbed density amplitude, or, equivalently, a lower limit on the wave number.

Now consider the second Zakharov equation (8.17). The condition for the nonlinear term in this equation to be small compared to the linear terms is

$$\frac{\delta\bar{n}_e}{n_{0e}} \gg W = \frac{|\tilde{\mathbf{E}}|^2}{8\pi n_e T_e}, \quad (8.122)$$

where  $W$  is the ratio of electric-field energy density to the electron thermal pressure—a measure of the energy in the plasmon field. Obviously, in order for the conditions (8.122) and (8.121) to be realisable simultaneously, it must be the case that

$$\boxed{W \ll k^2 \lambda_{De}^2}. \quad (8.123)$$

Note that since  $\tilde{\mathbf{E}} = -\nabla\tilde{\varphi}$ , this condition is equivalent to

$$\frac{e\tilde{\varphi}}{T_e} \ll 1, \quad (8.124)$$

whereas Zakharov's equations are perfectly valid when  $e\tilde{\varphi}/T_e \sim 1$  [see (8.38)], so they are able to describe non-WT dynamics.

As Langmuir excitations move to smaller  $k$ 's, the condition (8.123) of WT's validity for those excitations at those scales, viz.,

$$\frac{1}{n_e T_e} \int_0^k dp \hbar\omega_{pe} \bar{N}_p \ll k^2 \lambda_{De}^2, \quad (8.125)$$

becomes ever more stringent and eventually untenable. I have integrated the plasmon energy only up to  $k$  so as to include just the larger-scale excitations. Let us estimate when (8.125) becomes untenable. Since, according to (8.101),  $N_p \propto p^{-2}$ , we have<sup>77</sup>

$$\bar{N}_p \propto p^2 N_p \propto p^0 \Rightarrow N \sim \int_0^{k_0} dp \bar{N}_p \sim k_0 \bar{N}_p \Rightarrow \int_0^k dp \hbar\omega_{pe} \bar{N}_p \sim \frac{k H^l}{k_0}, \quad (8.126)$$

where  $H^l \approx \hbar\omega_{pe} N \equiv W n_e T_e$  is the total plasmon energy, with  $W$  as defined in (8.122). Therefore, the critical wave number  $k_c$  at which WT breaks down is

$$k_c \lambda_{De} \sim \frac{W}{k_0 \lambda_{De}}.$$

(8.127)

Thus, WT drives itself into a strong-turbulence regime. Something strongly nonlinear must happen at  $k \lesssim k_c$  for these scales to receive the plasmon flux supplied to them by the WT cascade and, if there is to be a stationary state, to dissipate it. What that might be is the subject to which I shall now turn.

**Exercise 8.11. Breakdown of WT approximation for Langmuir turbulence in the quasistatic limit.** In §8.1.4, we saw that at  $k\lambda_{De} \ll \sqrt{m_e/m_i}$ , the ponderomotive force arising from the plasmon field becomes too slow to generate sound waves and the ion dynamics instead become quasistatic. Then there are no longer sound waves, and the inequality (8.122) turns into equality. Together with (8.121), it again gives (8.123) as the validity condition for WT—but this is a different WT, one in which four-wave interactions dominate. As shown in Exercise 8.9, this too will support an inverse plasmon cascade (indeed it is the sole physically realisable constant-flux solution for that kind of turbulence). Suppose the transition to this WT regime happens before the critical wave number (8.127) is reached. Estimate the critical wave number at which WT will break down in this case.

**Exercise 8.12. Weak MHD turbulence.** If you have already followed §15.3, this is a good place to do Exercise 15.12. Note in particular the rather subtle set of issues around the breakdown of WT approximation for weak RMHD turbulence—you can think about that on your own and/or read Schekochihin (2022, §4 and Appendix A).

<sup>77</sup>The contribution to  $N$  from  $p > k_0$  is not, strictly speaking, negligible. Since  $N_p \propto p^{-3}$  in that range [see (8.110)], the shell-integrated distribution is  $\bar{N}_p \propto p^{-1}$ . This will give an additional contribution to  $N$  of the order of  $\ln(1/k_0 \lambda_{De})$  (assuming dissipation kicks in at  $k \lambda_{De} \sim 1$ ). This is a log of a largish number, but it is not a big crime to view it as order unity. Note that unlike the plasmon energy, the energy of the phonons is very heavily dominated by the smallest scales ( $\lambda_{De}$ ) because, at  $k > k_0$ ,  $n_k \propto k^{-2}$  and so the corresponding 1D energy spectrum is  $\propto \Omega_k k^2 n_k \propto k$ .

### 8.5. Kinetics and Fluid Dynamics of Langmuir Condensate

The WT inverse cascade has filled large scales with a Langmuir condensate, which we no longer assume satisfies the WT assumptions. Phonons are scarce. What can be done to describe this situation? Here is another flavour of quasiparticle kinetics that is both useful and interesting to play with.

#### 8.5.1. Spatially Inhomogeneous Quasiparticle Kinetics

Let us consider the interaction of the condensate with a very slow and very large-scale (much larger than the plasmon scales) density field—a gentle modulation of the plasmon frequency. Zakharov's equations (8.16) and (8.17) are still valid. We shall still want to treat the Langmuir oscillations as quasiparticles but now let us think of them as living in a spatially varying density field (equivalently, potential) determined by the ion motions, with the latter not necessarily quantisable into phonons—meaning that the nonlinearity in (8.17) no longer needs to be small. This breaks the restriction (8.122) and, therefore, (8.123) no longer needs to be satisfied, even though (8.121) can still hold (and so plasmons are still plasmons). This partial liberation from WT will allow us to derive again the kinetic equation for the plasmons by working out an evolution equation for their average energy at each wave number, but this time treating the slow density field  $\delta\bar{n}_e$  as non-random, and thus not involved in the averaging. This approach and the results of it that I will show below (§8.5.2 and Exercises 8.13 and 8.14), were pioneered, early in the WT game, by Vedenov & Rudakov (1965) and pedagogically expanded in the follow-up paper by Vedenov *et al.* (1967).

Mathematically, the new trick is to introduce a two-point correlation function of the plasmon field

$$\langle \psi(\mathbf{r})\psi^*(\mathbf{r}') \rangle = C(\mathbf{R}, \boldsymbol{\rho}), \quad \mathbf{R} = \frac{\mathbf{r} + \mathbf{r}'}{2}, \quad \boldsymbol{\rho} = \mathbf{r} - \mathbf{r}', \quad (8.128)$$

and treat it as a function not of  $\mathbf{r}$  and  $\mathbf{r}'$  but the two new variables  $\mathbf{R}$  and  $\boldsymbol{\rho}$  introduced above. We shall assume that  $\nabla_{\boldsymbol{\rho}}C \gg \nabla_{\mathbf{R}}C$ , i.e., that the  $\boldsymbol{\rho}$  variable captures the smaller scales at which plasmons exist, while  $\mathbf{R}$  picks up the spatial inhomogeneity associated with the gentle variation of  $\delta\bar{n}_e$  (this is the same as taking the limit, which I have so far avoided, of phonon wave numbers being small compared to the plasmon ones,  $k \ll p$  in the notation of §7.2.3; this limit should be recoverable from our new approach). In terms of the new variables,

$$\mathbf{r} = \mathbf{R} + \frac{\boldsymbol{\rho}}{2}, \quad \mathbf{r}' = \mathbf{R} - \frac{\boldsymbol{\rho}}{2}, \quad \nabla_{\mathbf{r}} = \frac{1}{2}\nabla_{\mathbf{R}} + \nabla_{\boldsymbol{\rho}}, \quad \nabla_{\mathbf{r}'} = \frac{1}{2}\nabla_{\mathbf{R}} - \nabla_{\boldsymbol{\rho}}. \quad (8.129)$$

The average energy density of the plasmon field *at any given point  $\mathbf{R}$  in space* is

$$\begin{aligned} \frac{\langle |\nabla\psi(\mathbf{R})|^2 \rangle}{8\pi} &= \left[ \frac{\nabla_{\mathbf{r}} \cdot \nabla_{\mathbf{r}'} \langle \psi(\mathbf{r})\psi^*(\mathbf{r}') \rangle}{8\pi} \right]_{\mathbf{r}=\mathbf{r}'=\mathbf{R}} \approx \left[ -\frac{\nabla_{\boldsymbol{\rho}}^2 C(\mathbf{R}, \boldsymbol{\rho})}{8\pi} \right]_{\boldsymbol{\rho}=0} \\ &= \sum_{\mathbf{p}} \frac{p^2 C_{\mathbf{p}}(\mathbf{R})}{8\pi} \equiv \sum_{\mathbf{p}} \hbar\omega_{\mathbf{p}} f_{\mathbf{p}} \quad \Rightarrow \quad \boxed{f_{\mathbf{p}}(\mathbf{R}) \approx \frac{p^2 C_{\mathbf{p}}(\mathbf{R})}{8\pi \hbar\omega_{\text{pe}}}}, \end{aligned} \quad (8.130)$$

where  $C_{\mathbf{p}}(\mathbf{R})$  is the Fourier transform of  $C(\mathbf{R}, \boldsymbol{\rho}) = \sum_{\mathbf{p}} e^{i\mathbf{p}\cdot\boldsymbol{\rho}} C_{\mathbf{p}}(\mathbf{R})$  with respect to  $\boldsymbol{\rho}$ . The new definition of the plasmons' *occupation-number density*  $f_{\mathbf{p}}$  is obvious; what I previously called their occupation number is the integral of this new quantity over all space:  $N_{\mathbf{p}} = \int d\mathbf{R} f_{\mathbf{p}}(\mathbf{R})$ , but the new feature now is that  $f_{\mathbf{p}}$  is allowed to vary (slowly) in space.

Let us make an enlightened guess as to what the kinetic equation for  $f_{\mathbf{p}}(\mathbf{R})$  must be. This is a distribution function of quasiparticles whose one-particle Hamiltonian is

$$H_1(\mathbf{R}, \mathbf{p}) = \hbar\omega_{\mathbf{p}}(\mathbf{R}) \approx \hbar\omega_{pe} \left[ 1 + \frac{1}{2} \frac{\delta\bar{n}_e(\mathbf{R})}{n_{0e}} + \frac{3}{2} p^2 \lambda_{De}^2 \right]. \quad (8.131)$$

Its dependence on the quasiparticle momentum  $\hbar\mathbf{p}$  comes from the usual wave-number dispersion in the frequency and its dependence on the particle position  $\mathbf{R}$  from the fact that the frequency is slowly modulated in space by  $\delta\bar{n}_e$ . Then, by Liouville's theorem, the distribution function of these particles must satisfy (with  $\hbar$ 's helpfully cancelling) an equation that is quite familiar to the practitioners of "ray-tracing" optics:

$$\frac{\partial f_{\mathbf{p}}}{\partial t} = -\dot{\mathbf{R}} \cdot \nabla_{\mathbf{R}} f_{\mathbf{p}} - \dot{\mathbf{p}} \cdot \frac{\partial f_{\mathbf{p}}}{\partial \mathbf{p}} = -\frac{\partial \omega_{\mathbf{p}}}{\partial \mathbf{p}} \cdot \nabla_{\mathbf{R}} f_{\mathbf{p}} + (\nabla_{\mathbf{R}} \omega_{\mathbf{p}}) \cdot \frac{\partial f_{\mathbf{p}}}{\partial \mathbf{p}}. \quad (8.132)$$

Denoting by  $\mathbf{v}_{\mathbf{p}} = 3\omega_{pe}\lambda_{De}^2 \mathbf{p}$  the group velocity of a Langmuir wave and resuscitating the §8.3 notation  $\xi = \delta\bar{n}_e/n_{0e}$ , we get

$$\frac{\partial f_{\mathbf{p}}}{\partial t} + \mathbf{v}_{\mathbf{p}} \cdot \nabla_{\mathbf{R}} f_{\mathbf{p}} - \frac{\omega_{pe}}{2} (\nabla_{\mathbf{R}} \xi) \cdot \frac{\partial f_{\mathbf{p}}}{\partial \mathbf{p}} = 0.$$

(8.133)

This looks exactly like the electrostatic Vlasov equation (1.50), with the wave's group velocity playing the part of quasiparticle's velocity and the ion-scale (electron) pressure providing the effective potential. The proof that this really works is given below, but the easily convinced and the impatient can leap over it.

**Derivation of (8.133).** The starting point must be Zakharov's first equation (8.16), which I would like to rewrite as follows

$$\nabla_{\mathbf{r}}^2 \left( i\omega_{pe}^{-1} \frac{\partial \psi}{\partial t} + \frac{3}{2} \lambda_{De}^2 \nabla_{\mathbf{r}}^2 \psi - \frac{\xi \psi}{2} \right) = -\frac{1}{2} \nabla_{\mathbf{r}} \cdot (\psi \nabla_{\mathbf{r}} \xi). \quad (8.134)$$

Let us multiply this equation by  $\nabla_{\mathbf{r}'}^2 \psi^*(\mathbf{r}')$  and subtract the same equation but complex-conjugated and with  $\mathbf{r}$  and  $\mathbf{r}'$  swapped:

$$\begin{aligned} \nabla_{\mathbf{r}}^2 \nabla_{\mathbf{r}'}^2 \left[ i\omega_{pe}^{-1} \frac{\partial}{\partial t} + \frac{3}{2} \lambda_{De}^2 (\nabla_{\mathbf{r}}^2 - \nabla_{\mathbf{r}'}^2) - \frac{\xi(\mathbf{r}) - \xi(\mathbf{r}')}{2} \right] \psi(\mathbf{r}) \psi^*(\mathbf{r}') = \\ -\frac{1}{2} \left\{ \nabla_{\mathbf{r}'}^2 \nabla_{\mathbf{r}} \cdot [\psi(\mathbf{r}) \psi^*(\mathbf{r}') \nabla_{\mathbf{r}} \xi(\mathbf{r})] - \nabla_{\mathbf{r}}^2 \nabla_{\mathbf{r}'} \cdot [\psi(\mathbf{r}) \psi^*(\mathbf{r}') \nabla_{\mathbf{r}'} \xi(\mathbf{r}')] \right\}. \end{aligned} \quad (8.135)$$

Now average this equation, leaving  $\xi$  outside the averages, as promised above, and use (8.129) to transform everything to the  $(\mathbf{R}, \rho)$  variables. Assuming slow variation of  $\xi$ , we may approximate

$$\xi(\mathbf{r}) \approx \xi(\mathbf{R}) + \frac{\rho}{2} \cdot \nabla_{\mathbf{R}} \xi(\mathbf{R}), \quad \xi(\mathbf{r}') \approx \xi(\mathbf{R}) - \frac{\rho}{2} \cdot \nabla_{\mathbf{R}} \xi(\mathbf{R}) \quad \Rightarrow \quad \nabla_{\mathbf{r}} \xi(\mathbf{r}) \approx \nabla_{\mathbf{r}'} \xi(\mathbf{r}') \approx \nabla_{\mathbf{R}} \xi(\mathbf{R}) \quad (8.136)$$

and neglect  $\nabla_{\mathbf{R}} C$  compared to  $\nabla_{\rho} C$  wherever opportune. The result is

$$\nabla_{\rho}^4 \left[ i\omega_{pe}^{-1} \frac{\partial}{\partial t} + 3\lambda_{De}^2 \nabla_{\rho} \cdot \nabla_{\mathbf{R}} - \frac{1}{2} (\rho \cdot \nabla_{\mathbf{R}} \xi) \right] C(\mathbf{R}, \rho) = -\nabla_{\rho}^2 [(\nabla_{\mathbf{R}} \xi) \cdot \nabla_{\rho} C(\mathbf{R}, \rho)]. \quad (8.137)$$

Finally, Fourier transform in  $\rho$  and divide through by  $-ip^2$ :

$$\left( \omega_{pe}^{-1} \frac{\partial}{\partial t} + 3\lambda_{De}^2 \mathbf{p} \cdot \nabla_{\mathbf{R}} \right) p^2 C_{\mathbf{k}} = \frac{p^2}{2} (\nabla_{\mathbf{R}} \xi) \cdot \frac{\partial C_{\mathbf{p}}}{\partial \mathbf{p}} + (\nabla_{\mathbf{R}} \xi) \cdot \mathbf{p} C_{\mathbf{p}} = \frac{1}{2} (\nabla_{\mathbf{R}} \xi) \cdot \frac{\partial}{\partial \mathbf{p}} p^2 C_{\mathbf{p}}. \quad (8.138)$$

Since  $C_{\mathbf{p}}$  is  $f_{\mathbf{p}}$  times a constant [see (8.130)], (8.138) only needs to be divided by  $8\pi\hbar$  to become the anticipated kinetic equation (8.133), q.e.d.

To (8.133) must be appended an equation for  $\xi(\mathbf{R})$ , which is just the second Zakharov

equation (8.17), whereby  $\xi$  is coupled to the plasmon distribution via the ponderomotive force:

$$\left( \frac{\partial^2}{\partial t^2} - c_s^2 \nabla_{\mathbf{R}}^2 \right) \xi(\mathbf{R}) = \nabla_{\mathbf{R}}^2 \frac{\hbar \omega_{pe}}{2m_i n_{0i}} \sum_{\mathbf{p}} f_{\mathbf{p}}(\mathbf{R}). \quad (8.139)$$

In our new kinetic system, where  $f_{\mathbf{p}}(\mathbf{R})$  is coupled to the field  $\xi(\mathbf{R})$ , (8.139) plays the role of the Poisson equation (1.51).

We can do with the Vedenov–Rudakov system (8.133) and (8.139) all the same things that we know how to do with the Vlasov–Poisson system (1.50) and (1.51), e.g., derive the “Landau damping” of a sound wave in a plasmon gas (Exercise 8.13) or work out the QL diffusion of the latter in a random field of sound waves (Exercise 8.14), but the really interesting new phenomenon for the sake of which this has all been done is that the plasmon gas—the Langmuir condensate pumped up by the WT cascade—is unstable. This is an effect in which the ability to handle spatially inhomogeneous quasiparticle distributions provided to us by this new formalism turns out to be essential.

**Exercise 8.13. Damping of a sound wave in a plasmon gas.** (a) Via a calculation analogous to those in §3, work out the dispersion relation for a weakly damped sound wave propagating through plasmon gas and calculate its Landau-damping rate.

(b) Show that this is the same result as can be derived from the phonon kinetic equation (7.23) under appropriate assumptions. What is, therefore,  $w(\mathbf{p}, \mathbf{k})$  and does it agree with (8.89)?

**Exercise 8.14. QL diffusion of plasmon gas.** (a) Construct a QL theory, analogously to §6, of plasmon diffusion in a stochastic field of sound waves.

(b) Show that this is also derivable from the plasmon kinetic equation (7.24) under appropriate assumptions. Again make sure that you have calculated  $w(\mathbf{p}, \mathbf{k})$  correctly.

**Exercise 8.15. Plasmon–ion kinetics.** Lifting the cold-ion approximation, i.e., allowing  $T_i \sim T_e$ , write a closed set of equations for coupled dynamics of plasmons and ions.

### 8.5.2. Modulational Instability of Plasmon Gas

Let us take the analogy between quasiparticle kinetics and “real” kinetics another logical step further and derive the “fluid dynamics” of the plasmon gas. Namely, define the density and velocity of the “plasmon flow”,

$$N(\mathbf{R}) = \sum_{\mathbf{p}} f_{\mathbf{p}}(\mathbf{R}), \quad \mathbf{U}(\mathbf{R}) = \frac{1}{N(\mathbf{R})} \sum_{\mathbf{p}} \mathbf{v}_{\mathbf{p}} f_{\mathbf{p}}, \quad (8.140)$$

and work out the evolution equations for them by taking moments of (8.133). As usual,

$$\frac{\partial N}{\partial t} + \nabla \cdot (N \mathbf{U}) = 0, \quad (8.141)$$

$$N \left( \frac{\partial \mathbf{U}}{\partial t} + \mathbf{U} \cdot \nabla \mathbf{U} \right) = -\nabla \cdot \mathbf{P} - \frac{3}{2} \omega_{pe}^2 \lambda_{De}^2 N \nabla \xi, \quad (8.142)$$

where the “plasmon pressure”

$$\mathbf{P} = \sum_{\mathbf{p}} (\mathbf{v}_{\mathbf{p}} - \mathbf{U})(\mathbf{v}_{\mathbf{p}} - \mathbf{U}) f_{\mathbf{p}} \quad (8.143)$$

will be neglected, subject to *a posteriori* confirmation in Exercise 8.16.

This assumption of “cold” plasmons gives us a closed set of the plasmon-fluid equations, which we can now linearise around a homogeneous, static state with a constant density

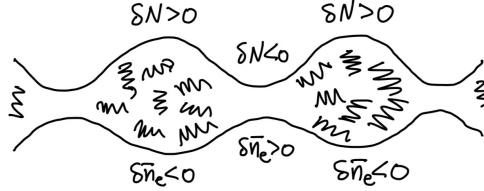


FIGURE 40. Modulational instability creates “caverns” filled with Langmuir oscillations ( $\delta N > 0$ ), expelling plasma particles ( $\delta \bar{n}_e < 0$ ).

$N_0$  and velocity  $\mathbf{U}_0 = 0$ :

$$\frac{\partial \delta N}{\partial t} + N_0 \nabla \cdot \mathbf{U} = 0, \quad (8.144)$$

$$\frac{\partial \mathbf{U}}{\partial t} = -\frac{3}{4} v_{\text{the}}^2 \nabla \xi, \quad (8.145)$$

$$\frac{\partial^2 \xi}{\partial t^2} - c_s^2 \nabla^2 \xi = \frac{\hbar \omega_{\text{pe}}}{2m_i n_{0i}} \nabla^2 \delta N, \quad (8.146)$$

the last equation being (8.139). Assuming perturbations  $\propto e^{-i\Omega t + i\mathbf{k} \cdot \mathbf{R}}$  converts these equations into the dispersion relation

$$\Omega^4 - k^2 c_s^2 \Omega^2 - k^4 v_0^4 = 0, \quad v_0^4 = \frac{3\hbar \omega_{\text{pe}} N_0 v_{\text{the}}^2}{8m_i n_{0i}} = \frac{3}{8} W v_{\text{the}}^2 c_s^2, \quad (8.147)$$

where  $W = \hbar \omega_{\text{pe}} N_0 / n_{0e} T_e$  is the dimensionless Langmuir excitation level, already defined in (8.122). The solution of this dispersion relation is

$$\boxed{\Omega_k^2 = \frac{k^2 c_s^2}{2} \left( 1 \pm \sqrt{1 + 4 \frac{v_0^4}{c_s^4}} \right)}. \quad (8.148)$$

The “+” root is the sound wave with a nonlinear modification of the frequency due to finite  $W$  (i.e., due to the plasmon gas supplying additional background pressure). The much more exciting “−” root is always unstable ( $\Omega_k^2 < 0$ )—this is called the *modulational instability*. The two interesting limits are the subsonic and the supersonic:

$$v_0 \ll c_s \Leftrightarrow W \ll \frac{m_e}{m_i} \Rightarrow \Omega_k \approx i \frac{k v_0^2}{c_s}, \quad (8.149)$$

$$v_0 \gg c_s \Leftrightarrow W \gg \frac{m_e}{m_i} \Rightarrow \Omega_k \approx i k v_0. \quad (8.150)$$

The nature of the unstable perturbation becomes transparent if we work out the perturbation of the density of plasmons vs. that of ions and electrons: using (8.144) and (8.145), for  $\Omega_k^2 < 0$ ,

$$\frac{\delta N_{\mathbf{k}}}{N_0} = -\frac{3k^2 v_{\text{the}}^2}{4|\Omega_k^2|} \left( \frac{\delta \bar{n}_e}{n_{0e}} \right)_{\mathbf{k}}. \quad (8.151)$$

Thus, the system breaks up into regions of low plasma density with high concentration of plasmons and vice versa (Fig. 40). The mechanism for the instability is quite simple: since  $\xi$  has the opposite sign to  $\delta N$ , the force in (8.145) is a negative pressure (on plasmons). An initial upward perturbation of the plasmon density in a region will act to suck in more plasmons while expelling plasma particles from that region by ponderomotive force. In other words, regions of high electric-energy (plasmon) density push out the plasma and

regions of high plasma density push out the plasmons. Thus, the system breaks up into so-called “caverns” filled with Langmuir-oscillating electric fields. Inside these caverns, the plasma frequency is smaller than outside (because  $\delta\bar{n}_e < 0$ ), so Langmuir waves will have a tendency to get trapped in them when the perturbation amplitude becomes finite and the modulational instability saturates.

Thus, the Langmuir condensate is not made up of monochromatic waves freely running around the system but rather of spatially localised plasmon bunches, known as *cavitons*. It is tempting to think of them as the new quasiparticles—and indeed in 1D you can do that because stable 1D soliton solutions to Zakharov’s equations exist (§8.5.3). In 3D, however, the situation is trickier (§8.5.4).

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**Exercise 8.16. Validity of the cold-plasmon approximation.** Work out the condition under which the results above are consistent with the neglect of the plasmon pressure in (8.142). You should find

$$\Delta p^2 \lambda_{De}^2 \ll \begin{cases} W & \text{if } W \ll \frac{m_e}{m_i}, \\ \sqrt{W \frac{m_e}{m_i}} & \text{if } W \gg \frac{m_e}{m_i}, \end{cases} \quad (8.152)$$

where  $\Delta p^2$  is the mean square width of the plasmon distribution [cf. the WT validity condition (8.123)].

**Exercise 8.17. Kinetic modulational instability.** Relax the cold-plasmon assumption and work out a kinetic theory of the modulational instability by linearising (8.133) and (8.139) directly, without recourse to fluid equations (this is actually the same calculation as in Exercise 8.13, just with a different focus).

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### 8.5.3. Langmuir Solitons

Coming soon, but for now, see Gorev *et al.* (1976); Thornhill & ter Haar (1978); Kingsep (1996).

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### 8.5.4. Langmuir Collapse

In 3D, caverns filled with plasmons (cavitons) will collapse self-similarly, in finite time. Here is the simplest, although nonrigorous, way to see this.

Let us imagine that modulational instability has resulted in the Langmuir condensate breaking up in many isolated cavitons, so we can treat each as a separate entity. Zakharov’s equation (8.16) conserves the plasmon number, and we shall assume (somewhat cavalierly) that it does so in each individual caviton. Then, whatever the caviton does, it must be the case that

$$N \sim E^2 l^d \sim \text{const} \Rightarrow E^2 \sim \frac{1}{l^d}, \quad (8.153)$$

where  $E^2/8\pi$  is the energy density of the Langmuir electric fields inside the caviton,  $l$  its characteristic size and  $d$  its dimension. This implies that if the caviton is not stationary, either  $E$  inside it will grow while its size diminishes, or vice versa. The possibility that  $E$  decays is not interesting because it would take us back to a homogeneous Langmuir condensate, which we know is unstable.

Let us then explore what kind of self-similar solutions of Zakharov’s equations might exist that accommodate growing  $E$ . The two equations (8.16) and (8.17) can be written schematically in a “scaling” form keeping only the powers of the characteristic scale  $l$

and characteristic time  $\tau$  associated with each term, viz.,

$$\nabla^2 \left( i\omega_{pe}^{-1} \frac{\partial \psi}{\partial t} + \frac{3}{2} \lambda_{De}^2 \nabla^2 \psi \right) = \frac{1}{2} \nabla \cdot (\xi \nabla \psi) \Rightarrow \left[ \frac{E}{\tau} \right] + \left[ \frac{E}{l^2} \right] = \left[ \xi E \right], \quad (8.154)$$

$$\left( \frac{\partial^2}{\partial t^2} - c_s^2 \nabla^2 \right) \xi = c_s^2 \nabla^2 \frac{|\nabla \psi|^2}{16\pi n_{0e} T_e} \Rightarrow \left[ \frac{\xi}{\tau^2} \right] + \left[ \frac{\xi}{l^2} \right] = \left[ \frac{E^2}{l^2} \right], \quad (8.155)$$

where  $E \sim \nabla \psi$  and I continue using the notation  $\xi = \delta \bar{n}_e / n_{0e}$ . Each of these two equations will boil down to a balance between two of its terms, with the third one negligible. At least one term involving  $\tau$  must survive somewhere if there is to be a time-dependent solution. One can systematically go through all possible options as to which terms to neglect, hence work out the scalings of  $E$ ,  $\xi$ , and  $\tau$  with  $l$  and then check that the neglected terms do indeed vanish in comparison with the retained ones as  $l \rightarrow 0$ . It is not hard to convince oneself that the only viable possibility is to drop the first term on the left-hand side of (8.154) (i.e., the first Zakharov equation turns into static relation) and the second term on the left-hand side of (8.155) (i.e., the dynamics are supersonic).

**Exercise 8.18.** Check this. Show that the physical regime in which these approximations hold is

$$\boxed{W \gg \frac{m_e}{m_i}}, \quad (8.156)$$

the supersonic limit that already made an appearance in (8.150).

The remaining balances then imply

$$\xi \sim \frac{1}{l^2}, \quad E^2 \sim \frac{\xi l^2}{\tau^2} \sim \frac{1}{\tau^2} \Rightarrow \tau \sim l^{d/2}, \quad (8.157)$$

where (8.153) was used at the last step. Knowing this, we see that it was fine to neglect the first term in (8.154) if

$$\frac{l^2}{\tau} \rightarrow 0 \quad \text{as} \quad l \rightarrow 0 \quad \Leftrightarrow \quad d < 4, \quad (8.158)$$

and the second term in (8.155) if

$$\frac{\tau}{l} \rightarrow 0 \quad \text{as} \quad l \rightarrow 0 \quad \Leftrightarrow \quad d > 2. \quad (8.159)$$

You can see why  $d$  matters. At  $d = 3$ , we have a perfectly legitimate possibility.

Finally, what is the relationship of  $\tau$  to time? Since  $l \rightarrow 0$  with advancing time, we must have  $\tau \rightarrow 0$  as well. The only way to achieve that is  $\tau = t_0 - t$ , where  $t_0$  is some fixed time. Then

$$\boxed{E^2 \sim \frac{1}{(t_0 - t)^2}, \quad \frac{\delta \bar{n}_e}{n_{0e}} \sim \frac{1}{(t_0 - t)^{4/3}}, \quad l \sim (t_0 - t)^{2/3}}. \quad (8.160)$$

Thus, the caviton collapses to a point and fields inside it go to infinity in finite time. The collapse is explosive.

This is, of course, not actually a solution of anything yet, only a piece of initial back-of-the-matchbox scoping out. At the very least, three further things must be done.

(i) The scalings (8.160) can be used to design the self-similar solution of Zakharov's equations (8.154) and (8.155), where the spatial dependence can be made spherically symmetric (since we have implicitly assumed the caviton to have only one scale), the new similarity variable is  $r/(t_0 - t)^{2/3}$ , and the terms anticipated to be negligible should

be neglected (note, however, that  $\psi$  is a complex field and can have a time-dependent phase, which we have learned nothing about in the above consideration, but which can be chosen to have such a time dependence that self-similarity is not destroyed). One could then see if the resulting equations have a reasonable solution.

(ii) What if  $W \ll m_e/m_i$ , i.e., as you saw in Exercise 8.18, the supersonic regime does not apply? It turns out that one can show that there is still a collapse, which will eventually take the plasmon energy beyond the supersonic limit, returning the above solution to relevance.

(iii) How inevitable is the collapsing solution? Can one demonstrate formally that it cannot be avoided? It turns out that one can.

Zakharov (1972) did all of those things, so I refer you to his paper for further mathematical enlightenment.

Why does Langmuir collapse matter? It matters because it provides a scenario for how energy pumped by WT from small scales into the Langmuir condensate at large ones might find a way to get dissipated. If the condensate breaks up into cavitons, which then collapse in finite time down to scales as small as  $\lambda_{De}$  (which is the limit of validity of the equations predicting the collapse), then it is reasonable to think that the electric energy trapped in the cavitons will then be deposited into electrons via Landau damping, or some nonlinear version thereof (also a tricky subject: see §§12.2–12.3). Thus, the nonlinear physics finally accomplishes, by a circuitous route, what nonlinear physics is supposed to accomplish: bring energy from injection to thermalisation.

## 8.6. Strong Langmuir Turbulence

The task of the strong-turbulence theory then is to work out the distribution of energy in  $\mathbf{k}$  space that is left in the wake of (or as signature of) its journey from cavitons formed by the modulational instability out of the Langmuir condensate at large scales, through their collapse to small scales, and finally to electron heat. There is no textbook-level certainty about the answers here, but there are many clever ideas.

### 8.6.1. Soliton Turbulence

Coming soon; see Kingsep *et al.* (1973), Gorev *et al.* (1976).

### 8.6.2. Caviton Turbulence

Coming soon. See reviews by Gorev *et al.* (1976), Thornhill & ter Haar (1978), Rudakov & Tsytovich (1978), Goldman (1984), Zakharov *et al.* (1985), and Robinson (1997), which document the accumulation and evolution of views over time, until saturation was reached.

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**Exercise 8.19. Statistical mechanics of Langmuir condensate.** Pelletier (1980a,b) made a clever attempt to deploy the machinery of condensed matter physics to describe the statistical ensemble of *strongly interacting* plasmons with the Hamiltonian (8.78). If you like that sort of cross-cultural spirit, work through his calculation and see what you think.

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Much of what I have dealt with in §§7.2–8.6 has been basically fluid turbulence—the kinetics was of quasiparticles. The fluctuating fields were perturbations on top of an equilibrium, which was hovering in the background and assumed to be Maxwellian whenever it needed to be specified. At no point after §7.1, with the brief exceptions of §§7.2.4, 7.2.5 and Exercise 8.10, did I engage with the question of the evolution of  $f_0$ ,

promised in §2 to be amongst the central ones in these Lectures. I am now going to come back to it from a number of very different angles than I have presented so far.

## 9. Nonlinear Stability and Thermodynamics of Collisionless Plasma

Let me go back to the generalist agenda first articulated at the beginning of §4: What kind of equilibria are stable?<sup>78</sup> Are there universal distributions to which a collisionless plasma will relax? This time I shall ask the stability question while eschewing any recourse to linear theory. Later on, this will nudge us towards certain distributions that will turn out to make some sense statistical-mechanically (§10) and that I will then show can be obtained via QLT (§11).

I will start by presenting this material as a formal template for proofs of nonlinear stability (§9.1), but, after a revealing example (§9.2), it will become clear why I keep referring to it as “thermodynamics” (§9.3).

### 9.1. Nonlinear Stability Theory: Thermodynamical Method

The general idea of the method is to find, for a given initial equilibrium distribution  $f(0)$ , an upper bound on the amount of energy that might be transferred into electromagnetic perturbations (not necessarily small). If that bound is zero, the system is stable; if it is not zero but is sharp enough to be nontrivial, it gives us a constraint on the amplitude of the perturbations in the saturated state.

Here is how it is done.<sup>79</sup> Let us introduce a functional

$$\mathcal{F} = \underbrace{\int d\mathbf{r} \frac{E^2 + B^2}{8\pi}}_{\equiv \mathcal{E}} + \underbrace{\int d\mathbf{Q} [A(\mathbf{Q}, f) - A(\mathbf{Q}, f_*)]}_{\equiv \mathcal{A}[f, f_*]} = \mathcal{E} + \mathcal{A}[f, f_*], \quad (9.1)$$

where  $\mathbf{Q} = (\mathbf{r}, \mathbf{v})$  is the phase-space variable and  $f_*$  is some trial distribution, which will sometimes represent our best guess about the properties of the stable distribution towards which the system wants to evolve and/or in the general vicinity of which we are interested in investigating stability. The function  $A(\mathbf{Q}, f)$  is chosen in such a way that, for any  $f$ ,

$$\mathcal{A}[f, f_*] \geq 0, \quad (9.2)$$

so, obviously,  $f_*$  is then the minimiser of  $\mathcal{A}[f, f_*]$ . If  $\mathcal{A}[f, f_*]$  is also chosen so that  $\mathcal{F}$  is conserved by the (collisionless) Vlasov–Maxwell equations, then  $\mathcal{F}(t) = \mathcal{F}(0)$  and the inequality (9.2) gives us an upper bound on the field energy at time  $t$ :

$$\boxed{\mathcal{E}(t) - \mathcal{E}(0) = \mathcal{A}[f(0), f_*] - \mathcal{A}[f(t), f_*] \leq \mathcal{A}[f(0), f_*]}, \quad (9.3)$$

where  $f(0)$  is the initial ( $t = 0$ ) equilibrium whose stability is under investigation. The bound (9.3) implies stability if  $\mathcal{A}[f(0), f_*] = 0$ ,<sup>80</sup> i.e., certainly for  $f(0) = f_*$ . This

<sup>78</sup>In Q6, isotropic, monotonically decreasing equilibria were found to be stable not just against infinitesimal (linear), electrostatic perturbations, but also against small but finite electromagnetic ones, giving us a taste of a powerful nonlinear constraint.

<sup>79</sup>These ideas appear to have crystallised in the papers by T. K. Fowler in the early 1960s (see his review, [Fowler 1968](#); his reminiscences and speculations on the subject 50 years later can be found in [Fowler 2016](#)). A number of founding fathers of plasma physics were thinking along these lines around the same time (references are given in opportune places below).

<sup>80</sup>This statement is based on the assumption that if the total electromagnetic energy decreases, that corresponds to initial perturbations decaying. You might wonder what happens if  $\mathcal{E}(0)$

guarantees stability of any  $f_*$  for which a functional  $\mathcal{A}[f, f_*]$  satisfying (9.2) and giving a conserved  $\mathcal{F}$  can be produced.

Physically, the above construction is nontrivial if the bound (9.3) is smaller than the total initial kinetic energy of the particles:

$$\mathcal{A}[f(0), f_*] < \sum_{\alpha} \int d\mathbf{Q} \frac{m_{\alpha}v^2}{2} f_{\alpha}(0) \equiv \mathcal{K}(0). \quad (9.4)$$

It is obvious that one cannot extract from a distribution more energy than  $\mathcal{K}(0)$ , but the above tells us that, in fact, one might only be able to extract substantially less.  $\mathcal{A}[f(0), f_*]$  is an upper bound on the *available energy* of the distribution  $f(0)$ . The sharper it can be made, the closer we are to learning something useful. Thus, the idea is to identify some suitable functional  $\mathcal{A}[f, f_*]$  for which  $\mathcal{F}$  is conserved, and some class of trial distributions  $f_*$  for which (9.2) holds, then minimise  $\mathcal{A}[f(0), f_*]$  within that class, subject to whatever physical constraints one can reasonably expect to hold: e.g., conservation of particles, momentum, and/or any other (possibly approximate) invariants of the system (e.g., its adiabatic invariants; see [Helander 2017, 2020](#)).<sup>81</sup>

To make some steps towards a practical implementation of this programme, let us first investigate how to choose  $A$  in such way as to ensure conservation of  $\mathcal{F}$ :

$$\frac{d\mathcal{F}}{dt} = \frac{d\mathcal{E}}{dt} + \sum_{\alpha} \int d\mathbf{Q} \frac{\partial A}{\partial f_{\alpha}} \frac{\partial f_{\alpha}}{\partial t} = \sum_{\alpha} \int d\mathbf{Q} \left( \frac{\partial A}{\partial f_{\alpha}} - \frac{m_{\alpha}v^2}{2} \right) \frac{\partial f_{\alpha}}{\partial t} = 0. \quad (9.5)$$

The second equality was obtained by using the conservation of total energy,

$$\frac{d}{dt}(\mathcal{E} + \mathcal{K}) = 0, \quad \mathcal{K} = \sum_{\alpha} \int d\mathbf{Q} \frac{m_{\alpha}v^2}{2} f_{\alpha}, \quad (9.6)$$

where  $\mathcal{K}$  is the kinetic energy of the particles. Now (9.5) tells us how to choose  $A$ :

$$A(\mathbf{Q}, f) = \sum_{\alpha} \left[ \frac{m_{\alpha}v^2}{2} f_{\alpha} + G_{\alpha}(f_{\alpha}) \right], \quad (9.7)$$

where  $G_{\alpha}(f_{\alpha})$  are arbitrary functions of  $f_{\alpha}$ . These can be added to  $A$  because Vlasov's equation has an infinite number of invariants: for any (sufficiently smooth)  $G_{\alpha}(f_{\alpha})$ ,

$$\frac{d}{dt} \int d\mathbf{Q} G_{\alpha}(f_{\alpha}) = 0. \quad (9.8)$$

This follows from the fact that, in the absence of collisions, the kinetic equation (1.30) expresses the conservation of phase volume in  $(\mathbf{r}, \mathbf{v})$  space (the flow in this phase space is divergence-free).

contains some equilibrium magnetic field and if that equilibrium is unstable: can the equilibrium field's energy be tapped and transferred partially into unstable perturbations of kinetic energy in such a way that  $\mathcal{E}(t) < \mathcal{E}(0)$  even though the system is unstable? I do not know how to isolate formally the set of conditions under which this is impossible (you may wish to think about this question; §17 might help). To avoid this problem, we could just restrict applicability of all considerations in this section to unmagnetised initial equilibria.

<sup>81</sup>[Krall & Trivelpiece \(1973\)](#) comment with a slight air of resignation that, with the rules of the game much vaguer than in linear theory, the thermodynamical approach to stability is “more art than science”. In the Russian translation of their textbook, this statement provokes a disapproving footnote from the scientific editor (A. M. Dykhne), who observes that the *right* way to put it would have been “more art than craft”.

**Exercise 9.1.** Prove the conservation law (9.8), assuming that the system is isolated.

These conserved quantities are sometimes referred to as Casimir invariants. The existence of an infinite number of conservation laws suggests that the evolution of a collisionless system in phase space is much more constrained than that of a collisional one, when the evolution is constrained only by conservation of particles, momentum and energy and the requirement (5.12) that entropy must not decrease. These constraints will play a major role in the discussion of collisionless relaxation in §§10 and 11. Keep in mind, however—and continue keeping in mind throughout that discussion—that it is an open question how long these constraints, in fact, survive in a real plasma. We saw in §5.5 that, in the presence of perturbations with  $k \neq 0$ , phase mixing would activate collisions after a time  $t \propto \nu^{-1/3}$ —much shorter than the collision time  $\nu^{-1}$ , but, formally, still asymptotically long in the limit  $\nu \rightarrow +0$ . However, in turbulent plasmas, it is likely that the collisional phase-space scales are, in fact, reached over times that are independent of  $\nu$  as  $\nu \rightarrow +0$ —just like in hydrodynamic turbulence, viscous (“Kolmogorov”) scales are reached on timescales independent of viscosity as the latter tends to  $+0$ , a property known as the “dissipative anomaly”. For now, I shall push on while ignoring this issue, but will return to it in §12.2.

A quick sanity check is to try  $G_\alpha(f_\alpha) = 0$ . The inequality (9.2),  $\mathcal{A}[f, f_*] \geq 0$ , is then certainly satisfied for  $f_* \propto \delta(\mathbf{v})$  and the bound (9.3) becomes

$$\mathcal{E}(t) - \mathcal{E}(0) \leq \mathcal{K}(0), \quad (9.9)$$

i.e., one cannot extract any more energy than the total energy contained in the distribution—indeed, one cannot. Let us now see how to learn something more nontrivial.

## 9.2. Fowler's Thermodynamics

The formula (9.7) guarantees that the functional  $\mathcal{F}$  given by (9.1) will be conserved by a collisionless plasma, but we still have the challenge of choosing  $G(f)$  and  $f_*$  in such a way that (9.2) is satisfied, i.e., that  $f_*$  is the maximiser of  $-\int d\mathbf{Q} A(\mathbf{Q}, f)$ . It is hard not to think immediately, as Fowler (1963, 1968) did, of thermodynamic entropy. As we know from statistical mechanics, the solution to the problem of maximising the entropy (5.11) subject to fixed kinetic energy and particle number,

$$-\int d\mathbf{Q} f \ln f - \beta \left( \int d\mathbf{Q} \frac{mv^2}{2} f - \mathcal{K} \right) - \lambda \left( \int d\mathbf{Q} f - N \right) \rightarrow \max, \quad (9.10)$$

is the Maxwellian distribution

$$f_* = e^{-1-\lambda-\beta mv^2/2} \equiv C \exp \left( -\frac{mv^2}{2T} \right), \quad (9.11)$$

where I have renamed the Lagrange multipliers  $\beta = 1/T$ ,  $\lambda = -1 - \ln C$ , and dropped the species index (all of this works species-by-species). It is then quite obviously the case that if we let

$$G(f) = Tf \left( \ln \frac{f}{C} - 1 \right),$$

(9.12)

then, for any  $f$  and any constants  $C$  and  $T$ ,

$$\mathcal{A}[f, f_*] = \int d\mathbf{Q} \left[ \frac{mv^2}{2} (f - f_*) + G(f) - G(f_*) \right] \geq 0. \quad (9.13)$$

We have got our available-energy function! Note that this argument also implies stability of any Maxwellian distribution (9.11) to finite perturbations (no available energy in  $f_*$ ).

With this choice of  $\mathcal{A}[f, f_*]$ , (9.3) now provides an upper bound on the energy of the electromagnetic fields that can be extracted from any given initial distribution  $f(0)$ . In

order to make this bound as sharp as possible, one picks the constants  $C$  and  $T$  (and, therefore, determines  $f_*$ ) so as to minimise  $\mathcal{A}[f(0), f_*]$ .

As far as  $C$  is concerned, it is not hard to see that

$$\frac{\partial \mathcal{A}[f(0), f_*]}{\partial C} = \frac{T}{C} \int d\mathbf{Q} [f_* - f(0)] = 0 \Rightarrow C = \frac{N}{V} \left( \frac{m}{2\pi T} \right)^{3/2} \equiv C_*(T), \quad (9.14)$$

i.e., the number of particles  $N = \int d\mathbf{Q} f(0)$  (of each species) must be the same for the initial distribution and for  $f_*$ .

Setting  $C = C_*(T)$ , let us now minimise  $\mathcal{A}[f(0), f_*]$  with respect to  $T$ . Observe first that (9.13) can be slightly simplified:

$$\mathcal{A}[f(0), f_*] = \mathcal{K}(0) + T \int d\mathbf{Q} f(0) \ln \frac{f(0)}{C_*(T)}. \quad (9.15)$$

Minimising this gives us

$$\frac{\partial \mathcal{A}[f(0), f_*]}{\partial T} = \int d\mathbf{Q} f(0) \ln \frac{f(0)}{C_*(T)} + \frac{3}{2} N = 0. \quad (9.16)$$

The root  $T_*$  of this equation is straightforward to extract. It is then rather illuminating to observe that, with  $T = T_*$ , (9.15) becomes simply

$$\min_{C,T} \mathcal{A}[f(0), f_*] = \mathcal{K}(0) - \frac{3}{2} NT_* = \mathcal{K}(0) - \mathcal{K}_*, \quad (9.17)$$

i.e., just the difference between the kinetic energies of the initial distribution and the target Maxwellian distribution, whose temperature is set by (9.16).

### 9.2.1. Anisotropic Equilibria

To test-drive Fowler's method, let me consider an anisotropic initial distribution—the case that, at the end of §4, I had to relegate to Exercise 4.8 as it needed substantial extra work if it were to be handled by the method developed there. The *bi-Maxwellian distribution* is a useful and certainly the simplest model for anisotropic equilibria:

$$f(0) = \bar{C} \exp \left( -\frac{mv_\perp^2}{2T_\perp} - \frac{mv_\parallel^2}{2T_\parallel} \right), \quad \bar{C} = n \left( \frac{m}{2\pi \bar{T}} \right)^{3/2}, \quad (9.18)$$

where  $\bar{T} = T_\perp^{2/3} T_\parallel^{1/3}$  and  $T_\perp$  and  $T_\parallel$  are the “temperatures” of particles' motion perpendicular and parallel to some special direction. Is this distribution unstable? (Yes: see Q3.) To obtain an upper bound on the energy available for extraction from it, compute

$$\int d\mathbf{Q} f(0) \ln \frac{f(0)}{C_*(T)} = \frac{3}{2} N \left( \ln \frac{T_*}{\bar{T}} - 1 \right). \quad (9.19)$$

Substituting this into (9.16), we see that the latter equation is satisfied for  $T_* = \bar{T}$ . The resulting estimate (9.17) of the available energy is

$$\mathcal{E}(t) - \mathcal{E}(0) \leq \min_{C,T} \mathcal{A}[f(0), f_*] = \frac{3}{2} N \left( \frac{2}{3} T_\perp + \frac{1}{3} T_\parallel - T_\perp^{2/3} T_\parallel^{1/3} \right). \quad (9.20)$$

The bound is zero when  $T_\perp = T_\parallel$  and is always positive otherwise (because it is the difference between an arithmetic and a geometric mean of the two temperatures). We do not, of course, have any way of knowing how good an approximation this is to the true

saturated level of whatever instability (if any) might exist here in any particular physical regime,<sup>82</sup> but this does show that temperature anisotropy is a source of free energy.

Further examples of such calculations can be found in Krall & Trivelpiece (1973, §9.14) and Fowler (1968). A certain further development of the methodology discussed above allows one to derive upper bounds not just on the energy of perturbations but also on their growth rates (Fowler 1964, 1968). If you feel that Fowler's privileging of the Maxwellian is arbitrary and unjustified, Exercise 10.1 is for you, but first do this one.

**Exercise 9.2. Fowler's thermodynamics for beams.** For some conveniently chosen model of a two-beam distribution (see §§3.7, 4.4, 10.3, and Q4 for inspiration), calculate the upper bound on the available energy by Fowler's method.

### 9.3. Why This is Thermodynamics

As already hinted by my language and notation,  $\mathcal{F}$  is the generalised version of the free energy (5.19), which was, in fact, the difference between the free energy of a given state of plasma and the free energy of a state characterised by  $f_*$ : in thermodynamic notation, we have

$$\boxed{\mathcal{F} = \mathcal{E} + \delta\mathcal{K} - \sum_{\alpha} T_{\alpha} \delta S_{\alpha}}, \quad (9.21)$$

where  $\delta\mathcal{K} = \mathcal{K} - \mathcal{K}_*$ ,  $\delta S_{\alpha} = S_{\alpha}[f_{\alpha}] - S_{\alpha}[f_{*\alpha}]$ , and  $S_{\alpha}[f_{\alpha}]$  is the entropy associated with the distribution  $f_{\alpha}$ . Comparing this with (9.1), we see that

$$\mathcal{A}[f, f_*] = \mathcal{K} - \mathcal{K}_* - \sum_{\alpha} T_{\alpha} (S_{\alpha}[f_{\alpha}] - S_{\alpha}[f_{*\alpha}]), \quad (9.22)$$

where, from (9.7), our choice of entropy is

$$S_{\alpha}[f_{\alpha}] = -\frac{1}{T_{\alpha}} \int d\mathbf{Q} G_{\alpha}(f_{\alpha}). \quad (9.23)$$

Thus, picking  $G(f)$  is equivalent to picking an “effective entropy” function for the collisionless system and thereby choosing a particular “thermodynamics”. If one then computes  $f_*$  by maximising this entropy subject to fixed  $\mathcal{K}$ , the inequality (9.13) is guaranteed. All Fowler did was use the standard thermodynamic entropy (5.11) in this argument. Clearly, this was begging to be done, but, in fact, why should this be the entropy for a collisionless plasma? Let us see if we can do anything more general, or better justified.

This may be an opportune place to pause and explain why entropy in a collisionless plasma is not “obviously” always just  $-\int d\mathbf{Q} f \ln f$ . Indeed, is this not the expression that Shannon (1948) and Jaynes (2003) teach us that we must always go for, as we did in (5.11)? The crucial thing to realise is that the distribution function  $f(\mathbf{Q})$  is *not* the probability that is supposed to go into Shannon's entropy, it is just the continuous version of the occupation number of a single-particle state given by the phase-space location  $\mathbf{Q}$  (this is textbook material: see, e.g.,

<sup>82</sup>Nominally, this calculation applies with equal validity to many different instabilities that can be triggered by temperature anisotropy in both unmagnetised and magnetised plasmas—and indeed also to some anisotropic distributions that are, in fact, stable (which is common in magnetised plasmas where the externally imposed magnetic field is sufficiently large). Well, the fact that some energy is available does not mean that it will always be extracted. Indeed, Exercises 9.2, 9.3, 10.1 and 11.10 suggest that there are situations of interest in which none of the available-energy bounds derived in §9 represents an adequate estimate of what will actually happen.

([Schekochihin 2019](#)). The complete state of a box of classical gas or plasma is, in general, described by the *joint* probability of having a particle at  $\mathbf{Q}_1$ , another particle at  $\mathbf{Q}_2$ , and so on for all  $N$  particles in the system. Let me call this object  $f_N(\mathbf{Q}_1, \dots, \mathbf{Q}_N)$ , a probability-density function in a  $6N$ -dimensional space. The Shannon entropy of this system is

$$S = - \int \cdots \int d\mathbf{Q}_1 \dots d\mathbf{Q}_N f_N(\mathbf{Q}_1, \dots, \mathbf{Q}_N) \ln f_N(\mathbf{Q}_1, \dots, \mathbf{Q}_N). \quad (9.24)$$

Now, since we assume, for a non-degenerate, classical plasma that particles do not know about each other and have identical distributions,

$$f_N(\mathbf{Q}_1, \dots, \mathbf{Q}_N) = f(\mathbf{Q}_1) \cdots f(\mathbf{Q}_N), \quad (9.25)$$

we can immediately conclude that

$$S = -N \int d\mathbf{Q} f(\mathbf{Q}) \ln f(\mathbf{Q}), \quad (9.26)$$

recovering (5.11) up to an irrelevant multiplicative constant. For a collisionless plasma, no such simplification is possible. Its state is characterised, by definition, by some particular instantiation of the distribution function (phase density)  $f(\mathbf{Q})$ , which, in a highly phase-mixed, turbulent system, is a random variable. In principle, if we imagine an ensemble of such systems, there is a probability measure on the space of these functions,  $P[f]$ . Then the Shannon entropy for the system is

$$S = - \int Df P[f] \ln P[f], \quad (9.27)$$

functionally integrated over all possible  $f$ . This is not very helpful. The reason this does not break up into anything immediately obvious in terms of  $\mathbf{Q}$ -space integrals of functions of  $f$  is that a collisionless system is, in a manner of speaking, quantum—it conserves phase volume and so different bits of phase space cannot just be filled independently of each other, there is a kind of Pauli exclusion principle for phase volumes. This property will become explicit in §10. In §11.8, I will formulate a generalised kinetic theory (“hyperkinetics”) in which a version of (9.27) does indeed emerge as the right kind of entropy.

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#### 9.4. Kruskal–Oberman Thermodynamics of Small Perturbations

There is a neat development (due, it seems, to [Kruskal & Oberman 1958](#)) of the formalism presented at the beginning of this section that allows one to dispense with Maxwellian equilibria at the price of keeping departures from the equilibrium small.

Let us investigate the stability of *isotropic* distributions with respect to small (but not necessarily infinitesimal) perturbations, i.e., take  $f(t) = f_* + \delta f(t)$ ,  $\delta f \ll f(0)$ . Let also  $f(0) = f_*$ , so the available-energy bound (9.3) will imply stability if we can find such a  $G(f)$  that (9.2) holds.

In (9.7), we expand

$$G(f) = G(f_*) + G'(f_*)\delta f + G''(f_*)\frac{\delta f^2}{2} + \dots \quad (9.28)$$

and use this to obtain, keeping terms up to second order,

$$\mathcal{A}[f(t), f_*] = \int d\mathbf{Q} \left\{ \left[ \frac{mv^2}{2} + G'(f_*) \right] \delta f + G''(f_*) \frac{\delta f^2}{2} \right\}. \quad (9.29)$$

Suppose we contrive to pick  $G(f_*)$  in such a way that

$$G'(f_*) = -\frac{mv^2}{2} \equiv -\varepsilon, \quad (9.30)$$

obliterating the first-order term in (9.29). Fowler’s effective entropy (9.12) certainly has this property, but, hopefully, it is not the only one that does. Noting that  $f_* = f_*(\varepsilon)$  by

assumption (it is isotropic) and assuming also that it is a monotonic function of  $\varepsilon$ , we can differentiate (9.30) with respect to  $f_*$  and get

$$G''(f_*) = -\frac{1}{\partial f_*/\partial \varepsilon} \Rightarrow \mathcal{A}[f(t), f_*] = \int d\mathbf{Q} \frac{\delta f^2}{2(-\partial f_*/\partial \varepsilon)}. \quad (9.31)$$

We see that  $\mathcal{A}[f(t), f_*] \geq 0$  and, therefore, (9.3) with  $f_* = f(0)$  implies stability if (for every species)

$$\frac{\partial f_*}{\partial \varepsilon} < 0 \Rightarrow \text{stability}.$$

(9.32)

Besides stability, this construction has given us a conserved quadratic free energy for our system:

$$\mathcal{F} = \mathcal{E} + \mathcal{A}[f, f_*] = \int d\mathbf{r} \frac{E^2 + B^2}{8\pi} + \sum_{\alpha} \iint d\mathbf{r} d\mathbf{v} \frac{\delta f_{\alpha}^2}{2(-\partial f_{*\alpha}/\partial \varepsilon_{\alpha})}.$$

(9.33)

The stability condition (9.32) makes  $\mathcal{F}$  positive definite and so no wonder the system is stable: perturbations around  $f_*$  have a conserved norm! For a Maxwellian equilibrium,  $-\partial f_*/\partial \varepsilon = f_*/T$ , so this  $\mathcal{F}$  becomes (the electromagnetic version of) the free energy (5.19). Then, in the spirit of §9.3, (9.33) is a natural generalisation of that free energy to non-Maxwellian, collisionless plasmas.

In Q6, the results of this section are obtained in a more straightforward way, directly from the Vlasov–Maxwell equations.

This style of thinking has been having a revival: see, e.g., the discussion of firehose and mirror stability of a magnetised plasma in Kunz *et al.* (2015). Generalised free-energy invariants like  $\mathcal{F}$  are important not just for stability calculations, but also for theories of kinetic turbulence in weakly collisional environments, e.g., the solar wind (see, e.g., Schekochihin *et al.* 2009).

### 9.5. Gardner's Theorem

The stability condition (9.32), viz., the statement that *if the equilibrium distributions of all species depend only on the particle energy and decrease monotonically with it, then the system is stable*, is, in fact an extremely general and fundamental result—as was shown by Gardner (1963) in a classic two-page paper.<sup>83</sup> The proof is as follows.

For every species (suppressing species indices), let me take  $G(f) = 0$  in (9.7), but construct a nontrivial  $f_*$  that satisfies (9.2),  $\mathcal{A}[f(t), f_*] \geq 0$ , at every time  $t$  since the beginning of the evolution of  $f(t)$  from the initial distribution  $f(0)$ . This amounts to adopting a “zero-entropy thermodynamics”, i.e., one that minimises energy.

For any given  $f(0)$ , define  $f_*$  to be a monotonically decreasing function of  $v^2$  (i.e., of energy), such that for any  $\eta > 0$ , the volume of the region in the phase space  $\mathbf{Q} = (\mathbf{r}, \mathbf{v})$  where  $f_* > \eta$  is the same as the volume of the phase-space region where  $f(0) > \eta$ . Then  $f_*$  is the distribution with the smallest kinetic energy, denoted here by  $\mathcal{K}_*$ , that can be

<sup>83</sup>The stability of *Maxwellian* equilibria against small perturbations was first proved by W. Newcomb, whose argument was published as Appendix I of Bernstein (1958). He was followed by Fowler (1963), who proved stability against finite perturbations in the way described in §9.2. Gardner (1963) attributes the first appearance of the stability condition (9.32) to an obscure 1960 report by M. N. Rosenbluth, although the same condition was derived also by Kruskal & Oberman (1958), more or less in the manner described in §9.4. Clearly, many great minds were thinking alike.

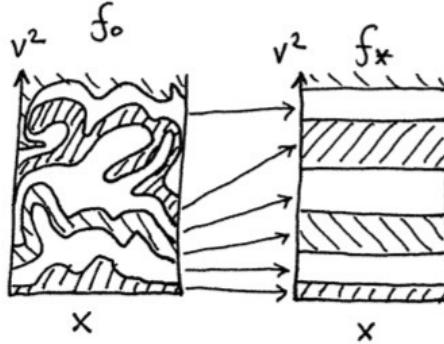


FIGURE 41. Gardner's restacking of the distribution, conserving phase-space volume.

reached from  $f_0$  while preserving phase-space volume:

$$\mathcal{K}(t) \geq \mathcal{K}_*. \quad (9.34)$$

Indeed, while the phase-space volume occupied by any given value of the probability density is the same for  $f(0)$  and for  $f_*$ , the corresponding energy is always lower for  $f_*$  than for  $f(0)$  or for any other  $f$  that can evolve from it, because in  $f_*$ , the values of the probability density are rearranged in such a way as to put the largest of them at the lowest values of  $v^2$ , thus minimising the kinetic energy. A vivid analogy is to think of the evolution of  $f$  under the collisionless kinetic equation (1.28) as the evolution of a mixture of “fluids” of different densities (values of  $f$ ) advected in a 6D phase-space  $\mathbf{Q} = (\mathbf{r}, \mathbf{v})$  by a divergence-free flow  $\dot{\mathbf{Q}} = (\dot{\mathbf{r}}, \dot{\mathbf{v}})$ . The lowest-energy state is the one in which these fluids are arranged in layers of density decreasing with increasing  $v^2$ , the heaviest at the bottom, the lightest at the top (Fig. 41). This rearrangement of the distribution function is known as “Gardner restacking”.

In view of (9.34), and since  $A$  is given by (9.7) with  $G(f) = 0$ ,

$$\mathcal{A}[f, f_*] = \mathcal{K}(t) - \mathcal{K}_* \geq 0, \quad (9.35)$$

so (9.2) holds and the bound (9.3) on the available energy follows. When  $f(0) = f_*$ , i.e., the equilibrium distribution satisfies (9.32), this equilibrium is stable, q.e.d. For a general initial distribution  $f(0)$ , the available energy is

$$\boxed{\mathcal{A}[f(0), f_*] = \mathcal{K}(0) - \mathcal{K}_*}. \quad (9.36)$$

Finally, note that the condition (9.32) is sufficient for stability, but not necessary, as we already know from, e.g., Exercise 4.2.

### 9.5.1. Gardner vs. Fowler

In the proof of Gardner's theorem, I set  $G(f) = 0$ . In fact, there was no need to do this because Gardner restacking preserves phase volume and, therefore, cannot change the  $\int d\mathbf{Q} G(f)$  part of the available energy. This simple observation enables us to assess the relative quality of Fowler's (or any Fowler-like) and Gardner's methods for estimating the available energy in a distribution. Let me temporarily rename the Gardner-restacked distribution  $f_G$  and imagine having chosen some non-zero  $G(f)$  and some target equilibrium  $f_*$ . Then, since  $\int d\mathbf{Q} G(f(0)) = \int d\mathbf{Q} G(f_G)$ , the available energy (9.13) in  $f(0)$  is

$$\mathcal{A}[f(0), f_*] = \mathcal{K}(0) - \mathcal{K}_G + \mathcal{A}[f_G, f_*] \geq \mathcal{K}(0) - \mathcal{K}_G = \mathcal{A}[f(0), f_G], \quad (9.37)$$

since  $\mathcal{A}[f_G, f_*] \geq 0$ . Thus, Gardner's bound on the available energy is always higher than Fowler's or his ilk's. Intuitively, not increasing the entropy of the distribution gives you the best estimate of the energy that can be released from it. This result was pointed out to me by Robbie Ewart (2021), and is, as far as I know, original.

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### 9.5.2. Helander's Method of Gardner Restacking

So, Gardner restacking is as well as we can do in calculating the available energy, so it is worth thinking about how might work out Gardner-restacked distributions and hence calculate the available energy—a fancy way of thinking of this is as of seeking “ground states” (the states of minimum energy) of Vlasov's equation. In a recent paper, [Helander \(2017\)](#) developed an elegant scheme for doing this.

The idea is to look for a distribution  $f_*$  such that the kinetic energy of any distribution evolving from it cannot increase. So, let us set  $f(0) = f_*$  and evolve  $f(t)$  forward a short time  $\delta t$ . The collisionless kinetic equation can be written simply as [cf. (1.28)]<sup>84</sup>

$$\frac{\partial f}{\partial t} + \dot{\mathbf{Q}} \cdot \frac{\partial f}{\partial \mathbf{Q}} = 0 \quad \Rightarrow \quad f(\delta t) \approx f_* - \delta t \dot{\mathbf{Q}} \cdot \frac{\partial f_*}{\partial \mathbf{Q}}. \quad (9.38)$$

The first-order (in  $\delta t$ ) kinetic-energy change from  $f_*$  to  $f(\delta t)$  is then

$$\delta \mathcal{H}[\delta \mathbf{Q}] = - \int d\mathbf{Q} \varepsilon(\mathbf{Q}) \delta \mathbf{Q} \cdot \frac{\partial f_*}{\partial \mathbf{Q}}, \quad (9.39)$$

where  $\varepsilon(\mathbf{Q}) = mv^2/2$  and  $\delta \mathbf{Q} = \delta t \dot{\mathbf{Q}}$ . We want to minimise  $\mathcal{H}$ , so we need  $\delta \mathcal{H} = 0$ . This will be achieved for  $f_*$  such that  $\delta \mathcal{H}[\delta \mathbf{Q}] = 0$  for any phase-space vector  $\delta \mathbf{Q}$  that behaves appropriately (vanishes) at the boundaries and satisfies  $(\partial/\partial \mathbf{Q}) \cdot \delta \mathbf{Q} = 0$ . The latter condition is imposed because the phase-space velocity field in (9.38) must be divergence-free:  $(\partial/\partial \mathbf{Q}) \cdot \dot{\mathbf{Q}}$  (the system is Hamiltonian). This last condition can be enforced by means of a Langrange multiplier  $\lambda(\mathbf{Q})$ :

$$\delta \mathcal{H}[\delta \mathbf{Q}] - \int d\mathbf{Q} \lambda(\mathbf{Q}) \frac{\partial}{\partial \mathbf{Q}} \cdot \delta \mathbf{Q} = 0 \quad \Leftrightarrow \quad \varepsilon(\mathbf{Q}) \frac{\partial f_*}{\partial \mathbf{Q}} = \frac{\partial \lambda}{\partial \mathbf{Q}} \quad \Rightarrow \quad \frac{\partial \varepsilon}{\partial \mathbf{Q}} \times \frac{\partial f_*}{\partial \mathbf{Q}} = 0. \quad (9.40)$$

Therefore,  $f_* = f_*(\varepsilon(\mathbf{Q}))$ —the desired minimum-energy distribution must be a function of the particle energy only, as anticipated by Gardner.

Thus, any such  $f_*$  is a minimum-energy state, but we now must find one that is accessible from a given initial distribution  $f(0)$  via collisionless evolution, i.e., conserving phase-space volumes. This condition can be written in the form of the conservation law (9.8) with  $G(f) = H(f(\mathbf{Q}) - \eta)$ , where  $H$  is the Heaviside function, picking out the volume of phase space where  $f > \eta$ :

$$\Gamma[f, \eta] \equiv \int d\mathbf{Q} H(f(\mathbf{Q}) - \eta) = \text{const.} \quad (9.41)$$

Since this is conserved, for any  $f_*$  accessible from a given  $f(0)$ ,  $\Gamma[f_*, \eta] = \Gamma[f(0), \eta]$ . Notice now that if  $\partial f_*/\partial \varepsilon < 0$ , as it should be if  $f_*$  is a Gardner function,  $H(f_*(\varepsilon) - \eta) = H(\varepsilon_\eta - \varepsilon)$ , where  $\varepsilon_\eta$  is the energy for which  $f_*(\varepsilon_\eta) = \eta$ . Therefore,

$$\Gamma[f_*, \eta] = \int d\mathbf{Q} H(\varepsilon_\eta - \varepsilon(\mathbf{Q})) \equiv \Omega(\varepsilon_\eta), \quad (9.42)$$

where the function  $\Omega(\varepsilon_\eta)$  is entirely independent of  $f_*$ , being just the integrated density of states

<sup>84</sup>The phase-space variables do not have to be  $\mathbf{Q} = (\mathbf{r}, \mathbf{v})$ , as, e.g., they are not in such reductions of kinetic theory as gyrokinetics (see, e.g., the reviews by [Howes et al. 2006](#) and [Abel et al. 2013](#)). These more complicated phase spaces usually describe systems in which particle motion is constrained by some adiabatic invariants (in gyrokinetics, it is the particles' first adiabatic invariant—the angular momentum of their Larmor gyration), so, effectively, the dimensionality of the phase space where particles can freely roam is lower than 6. Everything in §9.5.2 can be done (and is done in Helander's paper) for general phase spaces, where the formula for  $\varepsilon(\mathbf{Q})$  might be more complicated than  $mv^2/2$ .

corresponding to the energy  $\varepsilon_\eta$ : since  $\varepsilon(\mathbf{Q}) = mv^2/2$ ,

$$\Omega(\varepsilon) = \frac{4\pi V}{3} \left( \frac{2\varepsilon}{m} \right)^{3/2}. \quad (9.43)$$

Collecting all these relations, we conclude that  $\Omega(\varepsilon_\eta) = \Gamma[f_*, \eta] = \Gamma[f(0), \eta] = \Gamma[f(0), f_*(\varepsilon_\eta)]$ , or, for any  $\varepsilon$ ,

$$\boxed{\Gamma[f(0), f_*(\varepsilon)] = \Omega(\varepsilon)}. \quad (9.44)$$

This is an integral equation for the Gardner function  $f_*(\varepsilon)$  accessible from the initial distribution  $f(0)$ . In the form  $\Omega'(\varepsilon) = f'_*(\varepsilon) \partial \Gamma[f(0), f_*(\varepsilon)] / \partial f_*$ , it was first derived by [Dodin & Fisch \(2005\)](#), by a somewhat more circuitous route.

### 9.5.3. Anisotropic Equilibria

Let me give an example of the use of Helander's scheme for the bi-Maxwellian distribution (9.18). To work out the Gardner distribution corresponding to it, observe that the volume  $\Gamma[f(0), \eta]$  of the part of phase space where  $f(0) > \eta$  is  $V$  times the volume of the velocity-space ellipsoid

$$\frac{mv_\perp^2}{2T_\perp} + \frac{mv_\parallel^2}{2T_\parallel} = \ln \frac{\bar{C}}{\eta} \quad \Rightarrow \quad \Gamma[f(0), \eta] = \frac{4\pi V}{3} \left( \frac{2\bar{T}}{m} \ln \frac{\bar{C}}{\eta} \right)^{3/2}. \quad (9.45)$$

Letting  $\eta = f_*(\varepsilon)$  and, according to (9.44), equating  $\Gamma[f(0), f_*(\varepsilon)]$  to (9.43), we find

$$f_*(\varepsilon) = \bar{C} \exp \left( -\frac{\varepsilon}{\bar{T}} \right). \quad (9.46)$$

This is an interesting, if perhaps somewhat rigged, example of a Maxwellian equilibrium having a special significance even in the absence of collisions. Unsurprisingly, therefore, the upper bound on the available energy

$$\mathcal{A}[f(0), f_*] = \mathcal{K}(0) - \mathcal{K}_* = \frac{3}{2} V n \left( \frac{2}{3} T_\perp + \frac{1}{3} T_\parallel - \bar{T} \right) \quad (9.47)$$

is exactly the same as (9.20), which I found by Fowler's method. Obviously, in general, the Gardner distribution will not coincide with Fowler's target Maxwellian—and, according to §9.5.1, will give a better bound.

In [Helander \(2017, 2020\)](#), you will find other examples, e.g., a nice demonstration that Maxwellian equilibria with spatially dependent density and temperature have available energy.

**Exercise 9.3. Gardner distribution for beams.** For some conveniently chosen model of a two-beam distribution, compute the Gardner distribution by Helander's method and calculate the available energy. You may wish to re-use the model that you chose in Exercise 9.2 and confirm, in line with (9.37), that the bound on the available energy based on Gardner restacking is tighter than what you can get from Fowler.

## 10. Statistical Mechanics of Collisionless Plasma

The core step of the “thermodynamical” method of §9 was to find a stable trial distribution  $f_*$  in some sense related to the initial state  $f(0)$ . If  $f(0) = f_*$ , then stability was guaranteed; if not, one could estimate how much energy might be releasable if  $f(0)$  were to go unstable—but was there an implication that the system might actually relax to such a state? For a closed system, the answer is no. Indeed, consider the case of Gardner's distribution (§9.5). It is accessible from the initial distribution in the sense that phase-volume conservation is respected by Gardner restacking. However, it manifestly has lower energy. In a closed system, this energy can only go into fluctuating fields,

but, being spatially homogeneous, Gardner's distribution can support no such fields self-consistently.<sup>85</sup>

Let me stress that Gardner's distribution cannot be interpreted as some mean equilibrium, with the fluctuating fields hidden in a  $\delta f$ , because it is obtained under the requirement of phase-volume conservation, which applies only to the exact total distribution function. As I have noted already, Gardner's thermodynamics is zero-entropy thermodynamics, so it makes sense that no separation of  $f$  into a coarse-grained and a fluctuating distribution is allowed.<sup>86</sup>

The result of §9.4, viz., conservation of the generalised free energy (9.33), suggests that a state featuring a stable Gardner-like distribution, i.e., a monotonically decreasing  $f_*(\varepsilon)$ , plus small perturbations, is a viable one, but it does not offer a procedure for calculating  $f_*$  if one starts with  $f(0)$  somewhere far away from it. Fowler's construction (§9.2) dodges this problem by simply postulating that  $f_*$  is a Maxwellian, which allows one to use the standard formula for entropy— $G_\alpha(f_\alpha)$  in (9.12)—and prove the stability of a Maxwellian with respect to finite perturbations, but does not explain what is so special or inevitable about a Maxwellian in a collisionless plasma.

In order to determine what stable distributions are accessible to a collisionless plasma starting from a given state  $f(0)$ , and, therefore, what distribution it will (or may) relax to, one must do one of two things: either figure out what the right entropy function is and maximise it subject to appropriate constraints (e.g., energy conservation), or derive an evolution equation for the mean distribution function directly from the Vlasov–Maxwell equations (i.e., derive a “collisionless collision integral”). The first of these approaches is explored in this section. The second will be taken up in §11.

### 10.1. Lynden-Bell Statistics

Is there a way to determine universal, stable collisionless equilibria without regard to the precise nature of initial conditions, mimicking the way in which Maxwellian, Fermi–Dirac and Bose–Einstein distributions emerge in statistical mechanics as universal equilibrium states (see, e.g., Exercise 16.1 of Schekochihin 2019)? Let us work though a statistical argument proposed originally by Lynden-Bell (1967) in the context of collisionless relaxation of kinetic systems of mutually gravitating objects (e.g., stars in a galaxy)—and quickly realised by Kadomtsev & Pogutse (1970) to be equally relevant to collisionless plasmas (this is one of the few historical occasions when plasma physics followed in the footsteps of galactic dynamics, rather than vice versa).

Let us start by discretising the phase space into a very large number of micro-cells, each with phase volume  $\delta\Gamma$ . Let us assume also (in what is a drastic simplifying step)

<sup>85</sup>The situation is different if the problem of energy extraction is formulated as it is in Dodin & Fisch (2005): an initial distribution is put into some external wave field for a while, then the waves leave the plasma, carrying with them some energy. Then one can imagine the Gardner state being accessible from the initial state, because there is no requirement that the total energy in the system be conserved.

<sup>86</sup>For example, the distribution with a plateau into which an initial bump-on-tail distribution relaxes quasilinearly (§6.3) is *not* the Gardner-restacked version of the latter (and neither does the QL diffusion of the mean distribution  $f_0$  conserve phase volume; indeed, besides  $f_0$ , the system also has  $\delta f$  that supports a broad spectrum of electric perturbations that help create the plateau in the first place). If you are curious, you will find an explicit comparison between the two in the paper by Kolmes & Fisch (2020), who were interested in what kind of distributions could be achieved by Gardner restacking vs. by diffusion (the latter broadly interpreted). It turns out that, by tailoring bespoke external wave fields that extract energy from particle distributions, it is possible to devise an artificial “diffusive algorithm” that recovers Gardner's distribution, but it is not one that a freely relaxing QL plasma actually follows.

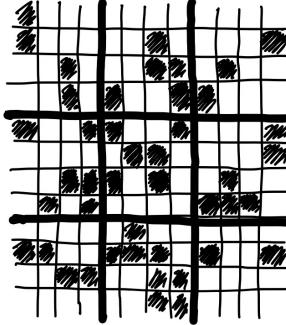


FIGURE 42. Coarse-graining micro-cells into macro-cells ( $\mathcal{M} = 16$ ).

that the exact distribution function in each of these micro-cells is equal to either zero or some constant, the same for all micro-cells:

$$f(\mathbf{Q}) = \eta \quad \text{or} \quad 0, \quad (10.1)$$

where, like in §9.1,  $\mathbf{Q} = (\mathbf{r}, \mathbf{v})$  is the phase-space variable. This is known as a *waterbag distribution*—a constant probability density in a finite subvolume of the phase space. Then

$$\int d\mathbf{Q} f = \eta \delta \Gamma \mathcal{N} = N, \quad (10.2)$$

where  $\mathcal{N}$  is the number of micro-cells with non-zero particle density,  $N$  is the total number of particles, and, naturally, we assume  $N \gg \mathcal{N} \gg 1$ . We are going to think of our plasma as a statistical-mechanical system of  $\mathcal{N}$  phase-density elements, which are allowed, under collisionless evolution, to move around phase space subject to the usual constraints: conservation of energy and conservation of phase volume. The latter constraint in this language means that phase-density elements can never occupy the same micro-cell, i.e., that they obey a Pauli-like exclusion principle. Thus, they are fermion-like particles, but distinguishable by their initial position in phase space.

Let us now coarse-grain our phase space into macro-cells, each containing  $\mathcal{M} \gg 1$  micro-cells (Fig. 42). Let  $\mathcal{N}_i \leq \mathcal{M}$  be the occupation number of the  $i$ -th macro-cell, i.e., the number in it of micro-cells with non-zero content. Then the coarse-grained distribution  $\bar{f}$  can be discretised in terms of the particle density in the  $i$ -th macro-cell:

$$\bar{f}_i = \frac{\eta \mathcal{N}_i}{\mathcal{M}} \leq \eta. \quad (10.3)$$

The total number of ways of setting up a particular such distribution is

$$W = \frac{\mathcal{N}!}{\prod_i \mathcal{N}_i!} \prod_i W_i, \quad W_i = \frac{\mathcal{M}!}{(\mathcal{M} - \mathcal{N}_i)!}. \quad (10.4)$$

Here the first factor is the number of ways of distributing  $\mathcal{N}$  phase-density elements amongst the macro-cells and  $W_i$  is the number of ways to distribute  $\mathcal{N}_i$  distinguishable elements between the micro-cells in the  $i$ -th macro-cell. Assuming that  $\mathcal{N} \gg \mathcal{M} \gg \mathcal{N}_i \gg 1$ , we can use Stirling's formula ( $\ln \mathcal{N}! \approx \mathcal{N} \ln \mathcal{N} - \mathcal{N}$ ) to find the Boltzmann entropy for

our system:

$$\begin{aligned} S = \ln W &\approx \mathcal{N}(\ln \mathcal{N} - 1) - \mathcal{M} \sum_i \left[ \frac{\mathcal{N}_i}{\mathcal{M}} \ln \frac{\mathcal{N}_i}{\mathcal{M}} + \left(1 - \frac{\mathcal{N}_i}{\mathcal{M}}\right) \ln \left(1 - \frac{\mathcal{N}_i}{\mathcal{M}}\right) \right] \\ &= \mathcal{N}(\ln \mathcal{N} - 1) - \frac{1}{\delta \Gamma} \int d\mathbf{Q} \left[ \bar{f} \ln \frac{\bar{f}}{\eta} + \left(1 - \frac{\bar{f}}{\eta}\right) \ln \left(1 - \frac{\bar{f}}{\eta}\right) \right], \end{aligned} \quad (10.5)$$

where  $\int d\mathbf{Q} = \mathcal{M} \delta \Gamma \sum_i$ . This  $S$  is to be maximised under the constraints of a fixed number of particles  $N$  and energy  $\mathcal{K}$  in the distribution. The problem is exactly the same as for a Fermi gas<sup>87</sup> and its solution is the Fermi–Dirac distribution:

$$\boxed{\bar{f} = \frac{\eta}{e^{(\varepsilon-\mu)/T} + 1}}, \quad (10.6)$$

where  $\varepsilon(\mathbf{Q}) = mv^2/2$  is the particle energy corresponding to the given macro-cell, and  $T$  (“temperature”) and  $\mu$  (“chemical potential”) are Lagrange multipliers that are determined by fixing  $N$  and  $\mathcal{K}$ :

$$N = \int d\mathbf{Q} \bar{f}, \quad \mathcal{K} = \int d\mathbf{Q} \varepsilon \bar{f}. \quad (10.7)$$

There is an important nuance here that needs discussing. I derived (10.6) subject to a fixed total *kinetic* energy  $\mathcal{K}$  associated with the coarse-grained distribution  $\bar{f}$ , so I need to know  $\mathcal{K}$  in order to fix the parameters of (10.6). Generally speaking,  $\mathcal{K}$  is not a conserved quantity because there is energy also in the electric (and magnetic) fields. For simplicity, I consider a (statistically) homogeneous system and let each phase-space macro-cell contain the entire position space. Consequently,  $\bar{f}$  is independent of position. In an electrostatic system,  $\bar{f}$  then gives rise to no fields, so all of its energy is kinetic energy.<sup>88</sup> Does this mean that our fixed  $\mathcal{K}$  is, in fact, conserved and so is equal to the energy of the initial state?—Only if, during the evolution of the plasma towards equilibrium, either no energy transfer from  $\bar{f}$  to electric fluctuations (associated with  $\delta f = f - \bar{f}$ ) is allowed, i.e., the initial  $\bar{f}$  is stable, or it is not stable, but, in the process of relaxing to a stable state (e.g., quasilinearly, as in §6), it transfers to fluctuations an amount of energy that is negligible compared to  $\mathcal{K}$ .

Note that since the Lynden-Bell distribution (10.6) is a maximiser of an entropy, it must be stable, by the same argument as Fowler made for a Maxwellian (§9.2). The same will be true about the more general case considered in §10.4.

## 10.2. Non-Degenerate Limit

Just like in the case of Fermi–Dirac statistics, the Maxwellian (non-degenerate) limit is recovered when the initial waterbag distribution is sparse in phase space—i.e., when it is sufficiently spread out around the part of phase space that is accessible subject to

<sup>87</sup>The distinguishability of the phase-density elements turns out not to matter: in (10.4), the factor of  $1/\mathcal{N}_i!$  that would appear in  $W_i$  for indistinguishable fermions is recovered in the prefactor that expresses the number of ways of populating the macro-cells. For a tutorial on Fermi gases, see Schekochihin (2019, §§16–17).

<sup>88</sup>In this respect, my calculation is different from that of Lynden-Bell (1967), who considers a self-gravitating system, which cannot be spatially homogeneous, and so  $\bar{f}$  does give rise to a gravitational potential, which then gives each particle a potential energy that must be included in  $\varepsilon$ . In a 3D plasma, it is actually possible to prove that such inhomogeneous solutions with  $\bar{f}$  that depends only on  $\varepsilon$  do not exist (Ng & Bhattacharjee 2005).

given energy  $\mathcal{K}$ . Mathematically, this limit is achieved by letting

$$e^{-\mu/T} \gg 1 \quad \Rightarrow \quad \bar{f} \approx \eta e^{\mu/T} e^{-\varepsilon/T}. \quad (10.8)$$

Then, from (10.7), after doing the integrals (with  $d\mathbf{Q} = drd\mathbf{v}$  and  $\varepsilon = mv^2/2$ ),

$$T = \frac{2\mathcal{K}}{3N}, \quad e^{\mu/T} = \frac{n}{(2\pi T/m)^{3/2}\eta}. \quad (10.9)$$

Thus, the non-degenerate, Maxwellian limit (10.8) is

$$\frac{n}{(2\pi T/m)^{3/2}} \ll \eta \quad \Rightarrow \quad \boxed{\bar{f} = \frac{n}{(2\pi T/m)^{3/2}} e^{-\varepsilon/T}}. \quad (10.10)$$

Of course it is not a particular surprise that a Maxwellian distribution emerges from a statistical mechanics of completely randomised objects with fixed overall energy. This might appear to be a argument in favour of universality in collisionless plasmas—but that hinges on the validity of the assumption that the phase-density elements of which the distribution consists are independent and free to sample the entirety of phase space without fear or prejudice.<sup>89</sup>

### 10.3. Warm and Cold Beams

Let me offer a very simple example of a physically plausible waterbag distribution: a pair of beams (streams) homogeneous in position space with density  $n_b$ , with velocities  $\pm u_b$  in some direction, and constant velocity-space density over a region of width  $v_b$  around  $\pm u_b$ , in 3D (a 1D cut of this might look like Fig. 43). This is a waterbag with

$$\eta \sim \frac{n_b}{v_b^3}, \quad \mathcal{K} \sim n_b V m u_b^2. \quad (10.11)$$

It is non-degenerate if  $v_b \ll u_b$ , i.e., if the beams are “cold”—as is obvious from (10.10). One therefore expects to see a distribution close to a Maxwellian with  $T \sim mu_b^2$  at the end of the relaxation process (Fig. 43a). In contrast, “warm” beams,  $v_b \sim u_b$ , should tend to a Fermi–Dirac flat-topped equilibrium with Fermi energy  $\varepsilon_F \sim mu_b^2$  (Fig. 43b). Remarkably, some recent numerical evidence may be showing something like this happening ([Skoutnev et al. 2019](#)).

**Exercise 10.1. Lynden-Bell thermodynamics and Lynden-Bell equilibria for beams.** Adapt Fowler’s thermodynamics (§9.2) to accommodate Lynden-Bell’s distribution as  $f_*$  and investigate the nonlinear stability of a distribution consisting of two square beams (Fig. 43). Work out and plot (or at least sketch) the parameters of the “target” Lynden-Bell distribution:  $T$  and  $\mu$  as functions of  $u_b$  and  $v_b$ . In the limits  $v_b \ll u_b$  or  $T \ll u_b, v_b$ , everything should be doable analytically (in the latter case, via Sommerfeld expansion; see, e.g., §17.3.3 of [Schekochihin 2019](#)). How does the available energy in the initial distribution, liable to be transferred to the electric fluctuations, compare to  $\mathcal{K}$ ?

If you have done Exercises 9.2, 9.3 or 10.1 for cold beams ( $v_b \ll u_b$ ), you know that their available energy is most of their energy, leaving a stable distribution of width  $\sim v_b$ . You might then wonder why it was meaningful to talk about a Lynden–Bell distribution for them, derived keeping  $\mathcal{K}$  constant and ending up with  $T \sim mu_b^2$ . The answer is

<sup>89</sup>One way in which this might not be true would be if our system had further invariants that constrained its dynamics on the time scales of interest. An example is a magnetised plasma in which particles conserve adiabatic invariants (cf. [Helander 2017, 2020](#)). Another is galactic dynamics—Lynden-Bell’s original preoccupation—where stars are approximately tied to the Keplerian ellipses they follow around the central black hole ([Binney 2016; Fouvry 2019](#)).

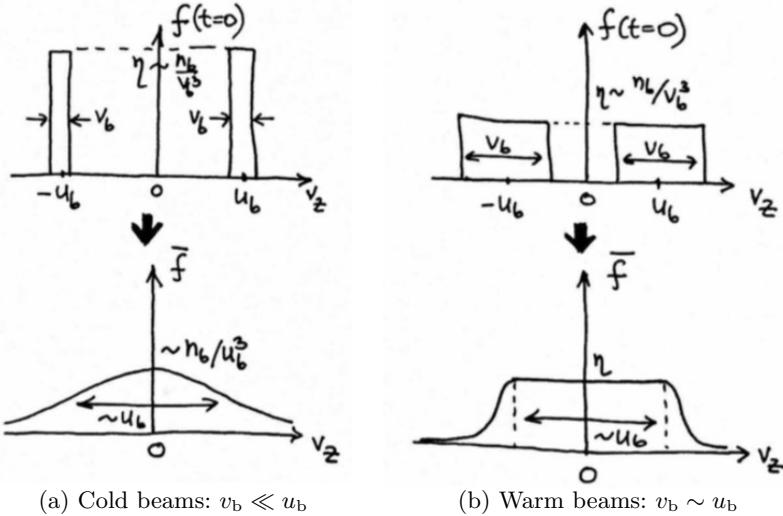


FIGURE 43. Waterbag beams and their Lynden-Bell equilibria.

because the Gardner or Fowler bounds on the available energy for such cases turn out to be extremely poor-quality: in a real collisionless-relaxation process, the amount of energy released will be much smaller, so it is actually not a bad approximation to assume that  $\mathcal{K}$  stays constant. Neither how to prove this nor why, physically, it is true, is obvious—you will have an opportunity to think about this again in Exercise 11.10. Here let me just say that this result means that real relaxation processes are usually not “smart” enough to maximise energy extraction and the electric fields generated by the unstable distribution going unstable, while carrying only a small fraction of its energy, turn out to be sufficient to push the distribution into a stable shape, from which no further energy can be released.

#### 10.4. Multi-Waterbag Statistics

The above construction contained a very restrictive assumption of an initial waterbag distribution. This restriction is, however, not hard to remove. Let us discretise the values that the distribution function can take and index them by  $J$ , so a general distribution function is represented as a superposition of waterbags:

$$f(\mathbf{Q}) = \sum_J f_J(\mathbf{Q}), \quad f_J(\mathbf{Q}) = \eta_J \quad \text{or} \quad 0. \quad (10.12)$$

If there are  $\mathcal{N}_J$  phase elements with density  $\eta_J$ , then  $\delta\Gamma\mathcal{N}_J$  is the phase-space volume occupied by the  $J$ -th waterbag, i.e., the phase-space volume where  $f = \eta_J$ . This is conserved by the collisionless evolution of  $f$ . The corresponding number of particles is  $N_J = \eta_J \delta\Gamma\mathcal{N}_J$ . As before, we may now coarse-grain  $f$  over groups (macro-cells) of  $\mathcal{M}$  microcells and represent the resulting  $\bar{f}$  in terms occupation numbers  $\mathcal{N}_{i,J}$  of the  $i$ -th macro-cell by elements of phase density  $\eta_J$ . If one generalises straightforwardly the one-waterbag derivation presented above, it then turns out that

$$\bar{f} = \sum_J \eta_J p_J(\varepsilon), \quad p_J(\varepsilon) = \frac{e^{-\beta\eta_J(\varepsilon-\mu_J)}}{1 + \sum_{J'} e^{-\beta\eta_{J'}(\varepsilon-\mu_{J'})}}, \quad (10.13)$$

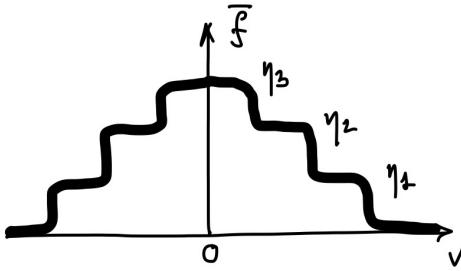


FIGURE 44. A close-to-degenerate 3-waterbag Lynden-Bell equilibrium.

where  $\mu_J$  and  $\beta$  are determined by

$$N_J = \eta_J \int d\mathbf{Q} p_J, \quad \mathcal{K} = \int d\mathbf{Q} \varepsilon \bar{f}. \quad (10.14)$$

**Exercise 10.2. Multi-waterbag equilibrium.** Show that this is true ([Lynden-Bell 1967](#)).

**Exercise 10.3.** Considering the Maxwellian limit, or otherwise, propose a natural way to define the overall temperature ([Lynden-Bell 1967](#) has an answer to this, which does not seem to me to be the most natural one).

Thus, the more general equilibrium distribution is a kind of superposition of many Fermi–Dirac distributions or, in the non-degenerate limit, of Maxwellians, with effective temperatures  $1/\beta\eta_J$ . When degenerate, they are ziggurat-like superpositions of step functions (Fig. 44). Note that, with a large number of steps, any monotonically decreasing function of  $\varepsilon$  can be approximated by such a ziggurat, so any Gardner distribution (§9.5) can be represented as a Lynden-Bell equilibrium.

**Exercise 10.4. Lynden-Bell distributions are Gardner distributions.** Show that any Lynden-Bell distribution (10.13) is a Gardner distribution (i.e.,  $\partial\bar{f}/\partial\varepsilon < 0$ ), and so is stable (this is easy).

**Exercise 10.5. Gardner distributions are Lynden-Bell distributions.** Consider a Gardner distribution, i.e., a monotonically decreasing function of  $\varepsilon$  alone (cf. §9.5). Discretise it to be a multi-waterbag distribution (10.12). Show that, under an appropriate coarse-graining, the corresponding Lynden-Bell equilibrium (10.13) is, in fact, the same distribution, i.e., that Lynden-Bell’s entropy-maximisation procedure does not change Gardner’s distribution. Equivalently, any Gardner distribution can be represented with arbitrary precision by (10.13) (with  $\beta = \infty$ ). Intuitively, this is obvious, as explained above, but it does, in fact, take a bit of work to prove. I owe this result to a conversation with Per Helander (2019) and the MMathPhys dissertation of [Brown \(2021\)](#).

One can make some further useful and aesthetically pleasing steps generalising the multi-waterbag formalism to a continuum of values of  $\eta_J$ —this will be done in §11.7.1 and then in §11.8, where this will usher us into an extended phase space that includes  $\eta$  as a variable. But for now, let me push on from statistical mechanics to kinetic theory.

## 11. Kinetic Theory of Relaxation to Lynden-Bell Equilibria

Just like in the theory of gases, statistical mechanics points us to a maximum-entropy distribution, but we need kinetic theory to see if such a distribution can be reached dynamically, and under what assumptions. To address this, I am going to go back to the

QL scheme and use it to derive a kind of “collisionless collision integral” that evolves the mean distribution function of a collisionless, homogeneous, electrostatic plasma towards some equilibrium that maximises an appropriate entropy and could perhaps be argued to be universal—thus, for an initial single-waterbag distribution, I will recover the Fermi–Dirac distribution (10.6) as a solution of this collision integral. I will then show, in §§11.7–11.8, how to generalise this to the case of multiple waterbags that we encountered in §10.4. Getting there will require some assumptions about the nonlinear behaviour in phase space—assumptions that are neither obvious nor, possibly, always correct, and that will therefore be re-examined in §12, upon injection of some more nonlinear physics.

### 11.1. General Form of Collision Integrals

So the equilibrium (slow, space- and time-averaged) distribution  $f_0$  again evolves according to (2.11):

$$\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial \mathbf{v}} \cdot \left[ -\frac{q}{m} \sum_{\mathbf{k}} i\mathbf{k} \langle \varphi_{\mathbf{k}}^* \delta f_{\mathbf{k}} \rangle \right] = \frac{\partial}{\partial \mathbf{v}} \cdot \left[ -\frac{e}{m} \sum_{\mathbf{k}} \mathbf{k} \operatorname{Im} \langle \varphi_{\mathbf{k}}^* \delta f_{\mathbf{k}} \rangle \right], \quad (11.1)$$

where the imaginary part was distilled out of the  $\mathbf{k}$  integral by splitting the latter in two equal parts and replacing  $\mathbf{k} \rightarrow -\mathbf{k}$  in one of them [cf. (6.8)]; I have also specialised to electrons ( $q = -e$ ) and will assume, for simplicity, that ions play no role apart from providing a homogeneous neutralising background.

Poisson’s law (2.9) in this case is just

$$\varphi_{\mathbf{k}} = -\frac{4\pi e}{k^2} \int d\mathbf{v}' \delta f_{\mathbf{k}}(\mathbf{v}'), \quad (11.2)$$

and, therefore, (11.1) turns into

$$\boxed{\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial \mathbf{v}} \cdot \left[ \frac{4\pi e^2}{m} \sum_{\mathbf{k}} \frac{\mathbf{k}}{k^2} \int d\mathbf{v}' \operatorname{Im} C_{\mathbf{k}}(\mathbf{v}, \mathbf{v}') \right]}, \quad (11.3)$$

where the right-hand side—the prototypical collision integral—is expressed in terms of (the imaginary part of) the correlation function

$$C_{\mathbf{k}}(\mathbf{v}, \mathbf{v}') = \langle \delta f_{\mathbf{k}}(\mathbf{v}) \delta f_{\mathbf{k}}^*(\mathbf{v}') \rangle. \quad (11.4)$$

This establishes an important principle—the evolution of  $f_0$  is determined by the second-order correlator of  $\delta f$ , which, if we want a closed equation, it is now our task to express in terms of  $f_0$ . All derivations of collision integrals (“true” or “collisionless”) are schemes for doing this, usually requiring some “closure assumption(s)” at some stage.

This is, of course, just a re-iteration of the message of §2.3: to know about the equilibrium, we need to understand the fluctuations—we need a theory of phase-space turbulence.

### 11.2. Microgranulation Ansatz

Let us attempt to avoid having such a theory and instead venture an extremely primitive guess at what the correlation function (11.4) might be. Imagine that we start the evolution of our collisionless system with some initial distribution—generally speaking, unstable—and let it proceed for a while. Instabilities might flare up and saturate, particles will stream and phase-mix the distribution, etc., so fairly quickly the exact  $f$  will become stable but extremely chopped up and fine-structured in phase space.<sup>90</sup> The average of this

<sup>90</sup>This is the part of the evolution that Lynden-Bell (1967) calls “violent relaxation”.

distribution over space and over “fast” times (meaning frequencies associated with any plasma processes, e.g.,  $\sim \omega_{pe}$ , and with particle streaming,  $\sim \mathbf{k} \cdot \mathbf{v}$ ), is  $f_0$ . The remainder is  $\delta f = f - f_0$ . Surely it is then reasonable to think of this perturbed distribution as thoroughly mixed in phase space and essentially random?

If this is reasonable, then perhaps we can model  $\delta f$  by a random phase-space density field  $g(\mathbf{r}, \mathbf{v})$  (assigned a different letter for reasons about to transpire) whose correlation function is

$$\langle g(\mathbf{r}, \mathbf{v})g(\mathbf{r}', \mathbf{v}') \rangle = \langle g^2 \rangle(\mathbf{v}) \Delta\Gamma \delta(\mathbf{r} - \mathbf{r}') \delta(\mathbf{v} - \mathbf{v}'), \quad (11.5)$$

where  $\Delta\Gamma$  is the phase-space volume representing the “width” of the two delta functions; note that the one-point average  $\langle g^2 \rangle$  is a function only of  $\mathbf{v}$  because the system is assumed to be statistically homogeneous in position space. I shall call (11.5) (perhaps a bit pompously) the *microgranulation ansatz*. A way to make it *almost* true by construction is to redefine our average as coarse-graining over phase-space macro-cells of at least the volume  $\Delta\Gamma = \Delta r^3 \Delta v^3$ , where  $\Delta r$  and  $\Delta v$  are, respectively, the position- and velocity-space correlation scales of  $g$ . I said “*almost*” because the whole scheme depends on it being possible also to make both  $\Delta r$  and  $\Delta v$  sufficiently small:

$$\Delta r \ll V^{1/3}, \quad \Delta v \ll \left( \frac{\mathcal{K}}{mN} \right)^{1/2} \sim v_{th}, \quad (11.6)$$

where  $V$  is the system’s volume and  $\mathcal{K}$  its total energy. In other words, we are still assuming that, as a result of phase mixing,  $g$  would lose any system-scale correlations in either position or velocity space. Note that we are also assuming that  $\Delta r$  and  $\Delta v$  are sufficiently large for  $\Delta\Gamma$  to contain so many particles that “true” Coulomb collisions are not a player—I will return later to what this means quantitatively.

If we assume that the correlation function (11.4) of  $\delta f$  is well-modelled by (11.5), viz.,

$$\begin{aligned} C_{\mathbf{k}}(\mathbf{v}, \mathbf{v}') &= \langle g_{\mathbf{k}}(\mathbf{v})g_{\mathbf{k}}^*(\mathbf{v}') \rangle = \iint \frac{d\mathbf{r} d\mathbf{r}'}{V^2} e^{-i\mathbf{k} \cdot (\mathbf{r} - \mathbf{r}')} \langle g(\mathbf{r}, \mathbf{v})g(\mathbf{r}', \mathbf{v}') \rangle \\ &= \langle g^2 \rangle(\mathbf{v}) \frac{\Delta\Gamma}{V} \delta(\mathbf{v} - \mathbf{v}'), \end{aligned} \quad (11.7)$$

a bitter disappointment awaits us: since (11.7) is manifestly real, the right-hand side of (11.3) is identically zero. This is a particular case of the general fact that, since, by the property of Fourier transforms of real functions,  $C_{\mathbf{k}}^*(\mathbf{v}, \mathbf{v}') = C_{\mathbf{k}}(\mathbf{v}', \mathbf{v})$ ,  $\text{Im } C_{\mathbf{k}}(\mathbf{v}, \mathbf{v}')$  contains only the antisymmetric part of the correlation function with respect to the transformation  $\mathbf{v} \leftrightarrow \mathbf{v}'$ . Thus, to get anything non-trivial, we must busy ourselves with the extraction of this antisymmetric part.

There are two ways in which  $\text{Im } C_{\mathbf{k}}(\mathbf{v}, \mathbf{v}')$  can be present. The first is that the microgranulation ansatz is plain wrong even approximately and the true correlation function of phase-space turbulence is not symmetric. The second (which is a mild variant of the first) is that microgranulation ansatz is approximately fine, but there is a correction to it that gives rise to a non-zero collision integral. The first possibility will be examined in §12.2, but here and now I will explore the second one, which is, in a certain sense, a direct generalisation of the standard derivation of “true” collision integrals in the classic kinetic theory.

### 11.3. Quasilinear Collision Integrals

The idea is to treat the fully phase-mixed field  $g(\mathbf{r}, \mathbf{v})$  described by (11.5) as an initial condition for  $\delta f$ , then calculate the time evolution of the latter from that initial state over times long compared to streaming times ( $\mathbf{k} \cdot \mathbf{v} t \gg 1$ ) but short compared to the

times over which  $f_0$  changes significantly, and then work out the contribution from that evolved  $\delta f$  to the evolution of  $f_0$ . A further approximation is to assume this evolution to be quasilinear, i.e., to segregate the linear part of the evolution of  $\delta f$  from the nonlinear part and make simplifying assumptions about the latter—as one does in QL theory.

The perturbation  $\delta f = f - f_0$  satisfies (2.12) exactly and its linear part (2.13) approximately (within the QL approximation), viz.,

$$\frac{\partial \delta f_{\mathbf{k}}}{\partial t} + i\mathbf{k} \cdot \mathbf{v} \delta f_{\mathbf{k}} = -i \frac{e}{m} \varphi_{\mathbf{k}} \mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{v}}. \quad (11.8)$$

Given an initial perturbation  $g_{\mathbf{k}}(\mathbf{v})$ , the Laplace-transformed solution of (11.8) is the same as (3.8) and (3.13), viz.,

$$\delta \hat{f}_{\mathbf{k}}(p) = -i \frac{e}{m p + i\mathbf{k} \cdot \mathbf{v}} \hat{\varphi}_{\mathbf{k}}(p) \mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{v}} + \hat{h}_{\mathbf{k}}(p), \quad (11.9)$$

$$\hat{h}_{\mathbf{k}}(p) = \frac{g_{\mathbf{k}}(\mathbf{v})}{p + i\mathbf{k} \cdot \mathbf{v}}, \quad (11.10)$$

$$\hat{\varphi}_{\mathbf{k}}(p) = -\frac{4\pi e}{k^2 \epsilon(p, \mathbf{k})} \int d\mathbf{v} \hat{h}_{\mathbf{k}}(p), \quad (11.11)$$

$$\epsilon(p, \mathbf{k}) = 1 - i \frac{4\pi e^2}{mk^2} \int d\mathbf{v} \frac{1}{p + i\mathbf{k} \cdot \mathbf{v}} \mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{v}}. \quad (11.12)$$

Formally, this is linear theory, but we should keep in mind that if, instead of  $\hat{h}_{\mathbf{k}}(p)$  being given by (11.10), we just treated it as an unknown function, then (11.9) could be interpreted as a decomposition of  $\delta \hat{f}$  into a linear and nonlinear parts, with the latter to be determined. The calculations that follow will admit this possibility until the start of §11.3.1.

The time-dependent solution is recovered from (11.9) and (11.11) via the inverse Laplace transform (3.14), but I now wish to change the integration variable to  $p = -i\omega + \sigma$ :

$$\delta f_{\mathbf{k}}(t) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} e^{(-i\omega + \sigma)t} \delta f_{\mathbf{k}\omega}, \quad \varphi_{\mathbf{k}}(t) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} e^{(-i\omega + \sigma)t} \varphi_{\mathbf{k}\omega}. \quad (11.13)$$

The integration contour now runs along the real line and the Laplace-transformed functions have been rebranded to look almost like Fourier-transformed ones:

$$\delta f_{\mathbf{k}\omega} = \delta \hat{f}_{\mathbf{k}}(-i\omega + \sigma) = \frac{e}{m} \frac{\varphi_{\mathbf{k}\omega}}{\omega - \mathbf{k} \cdot \mathbf{v} + i\sigma} \mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{v}} + h_{\mathbf{k}\omega}, \quad (11.14)$$

$$h_{\mathbf{k}\omega} = \hat{h}_{\mathbf{k}}(-i\omega + \sigma) = \frac{ig_{\mathbf{k}}(\mathbf{v})}{\omega - \mathbf{k} \cdot \mathbf{v} + i\sigma}, \quad (11.15)$$

$$\varphi_{\mathbf{k}\omega} = \hat{\varphi}_{\mathbf{k}}(-i\omega + \sigma) = -\frac{4\pi e}{k^2 \epsilon_{\mathbf{k}\omega}} \int d\mathbf{v} h_{\mathbf{k}\omega}, \quad (11.16)$$

$$\epsilon_{\mathbf{k}\omega} = \epsilon(-i\omega + \sigma, \mathbf{k}) = 1 + \frac{4\pi e^2}{mk^2} \int d\mathbf{v} \frac{1}{\omega - \mathbf{k} \cdot \mathbf{v} + i\sigma} \mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{v}}. \quad (11.17)$$

The reason for this rearrangement is that I now want to assume that  $\hat{\varphi}_{\mathbf{k}}(p)$  and, therefore,  $\delta \hat{f}_{\mathbf{k}}(p)$  have no poles at  $\text{Re } p > 0$ , i.e.,  $f_0$  supports no instabilities, so I may let  $\sigma \rightarrow +0$ , only needing it in the denominators to tell me how to circumvent the ballistic pole  $\omega = \mathbf{k} \cdot \mathbf{v}$  (Fig. 45).

In the context of collisional relaxation of  $f_0$ , this assumption is justified as follows. The initial distribution may well be unstable, but we can always wait for its instabilities to get excited, saturate and change  $f_0$  in such a way as to shut themselves down (e.g.,

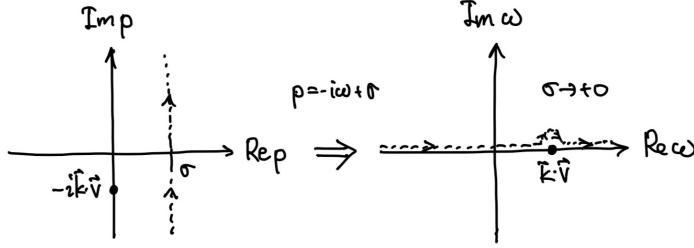


FIGURE 45. Change of variable in the inverse Laplace transform (11.13).

quasilinearly, as they did in §6). After that, we have a distribution function that is stable or, at worst, marginally stable, i.e.,  $\epsilon_{k\omega}$  might have zeros on the real- $\omega$  line and certainly at  $\text{Im } \omega = \text{Re } p \leq 0$ , but not at  $\text{Im } \omega = \text{Re } p > 0$ . This assumption means that we should expect to derive an evolution equation for  $f_0$  that conserves its kinetic energy (since we do not expect new electric fluctuations to be excited). This is a setting that is similar to what the Lynden-Bell-style calculation in §10 described [see discussion after (10.6)]—and thus (10.6) should be (and indeed will be) recoverable in this approach, under the right assumptions. In Exercise 11.10, you will discover that the amount of energy likely to be lost by an initially unstable distribution to the electric-field fluctuations (which are then needed to evolve the stable  $f_0$  in the way that is about to be elaborated) is quite small, provided a host of assumptions that I am about to introduce are correct.

We are ready to roll. Substituting (11.13) and (11.14) into (11.1), we get<sup>91</sup>

$$\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial \mathbf{v}} \cdot \sum_{\mathbf{k}} \mathbf{k} \operatorname{Im} \iint \frac{d\omega d\omega'}{(2\pi)^2} e^{-i(\omega-\omega')t} \left( -\frac{e^2}{m^2} \frac{\langle \varphi_{k\omega} \varphi_{k\omega'}^* \rangle}{\omega - \mathbf{k} \cdot \mathbf{v} + i\sigma} \mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{v}} - \frac{e}{m} \langle h_{k\omega} \varphi_{k\omega'}^* \rangle \right). \quad (11.18)$$

The first term is the familiar QL diffusion (here written for a general-form  $\varphi$ ) whereas the second term turns out to be a kind of drag [like in the Fokker–Planck collision integral (1.47)], whose form we shall now elaborate a little further. Using (11.16), we get

$$\begin{aligned} -\frac{e}{m} \langle h_{k\omega}(\mathbf{v}) \varphi_{k\omega'}^* \rangle &= \frac{4\pi e^2}{mk^2 \epsilon_{k\omega}^*} \int d\mathbf{v}' \langle h_{k\omega}(\mathbf{v}) h_{k\omega'}^*(\mathbf{v}') \rangle \\ &= \frac{4\pi e^2}{mk^2 \epsilon_{k\omega} \epsilon_{k\omega'}^*} \int d\mathbf{v}' \left[ C_{k\omega k\omega'}(\mathbf{v}, \mathbf{v}') + \frac{4\pi e^2}{mk^2} \int d\mathbf{v}'' \frac{C_{k\omega k\omega'}(\mathbf{v}, \mathbf{v}')}{\omega - \mathbf{k} \cdot \mathbf{v}'' + i\sigma} \mathbf{k} \cdot \frac{\partial f_0(\mathbf{v}'')}{\partial \mathbf{v}''} \right], \end{aligned} \quad (11.19)$$

where  $C_{k\omega k\omega'}(\mathbf{v}, \mathbf{v}') = \langle h_{k\omega}(\mathbf{v}) h_{k\omega'}^*(\mathbf{v}') \rangle$  and the last expression was obtained by multiplying and dividing by  $\epsilon_{k\omega}$ . The reason for this seemingly gratuitous manipulation is two-fold.

First, the first term in (11.19) vanishes after it is plugged into (11.18):

$$\operatorname{Im} \iint \frac{d\omega d\omega'}{(2\pi)^2} e^{-i(\omega-\omega')t} \frac{C_{k\omega k\omega'}(\mathbf{v}, \mathbf{v}')}{\epsilon_{k\omega} \epsilon_{k\omega'}^*} = \operatorname{Im} \langle \tilde{h}_{\mathbf{k}}(\mathbf{v}) \tilde{h}_{\mathbf{k}}^*(\mathbf{v}') \rangle = 0, \quad (11.20)$$

provided the correlation function of the “dressed” distribution function  $\tilde{h}_{\mathbf{k}} = (1/2\pi) \int d\omega e^{-i\omega t} h_{k\omega} / \epsilon_{k\omega}$  is symmetric with respect to swapping velocities,  $\mathbf{v} \leftrightarrow \mathbf{v}'$ . If  $h_{k\omega}$  is given by (11.15) with the initial condition  $g_{\mathbf{k}}(\mathbf{v})$  satisfying the microgranulation

<sup>91</sup>A conscientious reader might at this point become worried about what exactly is meant by averages in (11.18). She will find some succour in §11.2.

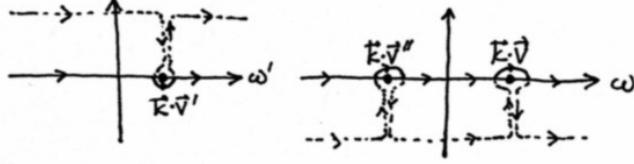


FIGURE 46. The contours for calculating the double integral (11.26).

ansatz (11.5), this is indeed the case. Thus, here, effectively, is the place where the symmetric part of the correlation function (11.4) is got rid of and only the non-vanishing part of the collision integral (11.3) is kept.

Secondly, the first term in (11.18), upon insertion of (11.16), turns into an expression of a similar form to the second term in (11.19):

$$-\frac{e^2}{m^2} \frac{\langle \varphi_{k\omega} \varphi_{k\omega'}^* \rangle}{\omega - \mathbf{k} \cdot \mathbf{v} + i\sigma} \mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{v}} = -\frac{16\pi^2 e^4}{m^2 k^4 \epsilon_{k\omega} \epsilon_{k\omega'}^*} \iint d\mathbf{v}' d\mathbf{v}'' \frac{C_{k\omega\omega'}(\mathbf{v}'', \mathbf{v}')}{\omega - \mathbf{k} \cdot \mathbf{v} + i\sigma} \mathbf{k} \cdot \frac{\partial f_0(\mathbf{v})}{\partial \mathbf{v}}. \quad (11.21)$$

Putting all this together, we end up with (11.18) in the form that suggests that we might be on our way towards something like the particle-collision integral (7.35):

$$\boxed{\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial \mathbf{v}} \cdot \int d\mathbf{v}'' \left[ D(\mathbf{v}, \mathbf{v}'') \cdot \frac{\partial f_0(\mathbf{v})}{\partial \mathbf{v}} - D(\mathbf{v}'', \mathbf{v}) \cdot \frac{\partial f_0(\mathbf{v}'')}{\partial \mathbf{v}''} \right]}, \quad (11.22)$$

where the “diffusion kernel” is

$$D(\mathbf{v}'', \mathbf{v}) = -\frac{16\pi^2 e^4}{m^2} \text{Im} \sum_{\mathbf{k}} \frac{\mathbf{k}\mathbf{k}}{k^4} \iint \frac{d\omega d\omega'}{(2\pi)^2} \frac{e^{-i(\omega-\omega')t}}{\epsilon_{k\omega} \epsilon_{k\omega'}^*} \int d\mathbf{v}' \frac{C_{k\omega\omega'}(\mathbf{v}, \mathbf{v}')}{\omega - \mathbf{k} \cdot \mathbf{v}'' + i\sigma}. \quad (11.23)$$

Note that the QL diffusion coefficient similar to the one appearing in (6.7) is then  $\int d\mathbf{v}'' D(\mathbf{v}, \mathbf{v}'')$ .

### 11.3.1. Quasilinear Approximation

Note that, except in justifying the vanishing of (11.20), I have not yet actually used the QL approximation, i.e., the explicit expression (11.15) for  $h_{k\omega}$  in terms of  $g_{\mathbf{k}}$ . Let me use it now:

$$C_{k\omega\omega'}(\mathbf{v}, \mathbf{v}') = \frac{C_{\mathbf{k}}(\mathbf{v}, \mathbf{v}')}{(\omega - \mathbf{k} \cdot \mathbf{v} + i\sigma)(\omega' - \mathbf{k} \cdot \mathbf{v}' - i\sigma)}, \quad C_{\mathbf{k}}(\mathbf{v}, \mathbf{v}') = \langle g_{\mathbf{k}}(\mathbf{v}) g_{\mathbf{k}}^*(\mathbf{v}') \rangle, \quad (11.24)$$

and, therefore,

$$D(\mathbf{v}'', \mathbf{v}) = -\frac{16\pi^2 e^4}{m^2} \text{Im} \sum_{\mathbf{k}} \frac{\mathbf{k}\mathbf{k}}{k^4} \int d\mathbf{v}' C_{\mathbf{k}}(\mathbf{v}, \mathbf{v}') I_{\mathbf{k}}(\mathbf{v}, \mathbf{v}', \mathbf{v}''), \quad (11.25)$$

$$I_{\mathbf{k}}(\mathbf{v}, \mathbf{v}', \mathbf{v}'') = \iint \frac{d\omega d\omega'}{(2\pi)^2} \frac{e^{-i(\omega-\omega')t}}{\epsilon_{k\omega} \epsilon_{k\omega'}^*(\omega - \mathbf{k} \cdot \mathbf{v} + i\sigma)(\omega' - \mathbf{k} \cdot \mathbf{v}' - i\sigma)(\omega - \mathbf{k} \cdot \mathbf{v}'' + i\sigma)}. \quad (11.26)$$

The remaining work is now in calculating the double integral (11.26). If  $\epsilon_{k\omega}$  has no poles at real  $\omega$  (i.e.,  $f_0$  is a stable distribution), the integral is done by shifting the  $\omega'$  integration contour upwards to  $\text{Im } \omega' \rightarrow +\infty$ , but snagging on the pole at  $\omega' = \mathbf{k} \cdot \mathbf{v}'$  and the  $\omega$  contour downwards to  $\text{Im } \omega \rightarrow -\infty$ , snagging on the poles at  $\omega = \mathbf{k} \cdot \mathbf{v}$  and

$\omega = \mathbf{k} \cdot \mathbf{v}''$  (see Fig. 46). The result is

$$I_{\mathbf{k}}(\mathbf{v}, \mathbf{v}', \mathbf{v}'') = \frac{e^{-i\mathbf{k} \cdot (\mathbf{v} - \mathbf{v}')t}}{\epsilon_{\mathbf{k}, \mathbf{k} \cdot \mathbf{v}} \epsilon_{\mathbf{k}, \mathbf{k} \cdot \mathbf{v}'}^*} \underbrace{\frac{1}{\mathbf{k} \cdot (\mathbf{v} - \mathbf{v}'')} \left( 1 - e^{i\mathbf{k} \cdot (\mathbf{v} - \mathbf{v}'')t} \frac{\epsilon_{\mathbf{k}, \mathbf{k} \cdot \mathbf{v}}}{\epsilon_{\mathbf{k}, \mathbf{k} \cdot \mathbf{v}''}} \right)}_{\rightarrow -i\pi\delta(\mathbf{k} \cdot (\mathbf{v} - \mathbf{v}'')) \text{ as } t \rightarrow \infty}, \quad (11.27)$$

where the  $\delta$ -function approximation is obtained in the same way as in (5.39). Thus, (11.25) becomes

$$\mathbf{D}(\mathbf{v}'', \mathbf{v}) = \frac{16\pi^3 e^4}{m^2} \operatorname{Re} \sum_{\mathbf{k}} \frac{\mathbf{k}\mathbf{k}}{k^4} \delta(\mathbf{k} \cdot (\mathbf{v} - \mathbf{v}'')) \int d\mathbf{v}' \frac{C_{\mathbf{k}}(\mathbf{v}, \mathbf{v}') e^{-i\mathbf{k} \cdot (\mathbf{v} - \mathbf{v}')t}}{\epsilon_{\mathbf{k}, \mathbf{k} \cdot \mathbf{v}} \epsilon_{\mathbf{k}, \mathbf{k} \cdot \mathbf{v}'}^*}. \quad (11.28)$$

Note that with the emergence of  $\delta(\mathbf{k} \cdot (\mathbf{v} - \mathbf{v}''))$ , we have inched another step towards a result that looks like the particle-collision integral (7.35).

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**Exercise 11.1. Inclusion of plasma waves.** What if  $\epsilon_{\mathbf{k}\omega}$  does have poles at real  $\omega$ , or close by? Try your hand at generalising this theory, from here and onwards to §11.5, to this case, which describes the evolution of a marginally stable or even weakly unstable  $f_0$ , and so of a plasma that can support a population of waves. You will discover a bridge between the theory being developed here and the kind of QLT that was presented in §6. You may find some inspiration for this calculation in the classic paper by [Rogister & Oberman \(1968\)](#) (the sequel to it, [Rogister & Oberman 1969](#), brings in the WT processes familiar from §7.2, but stops short of wading into strongly turbulent waters).

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### 11.3.2. Microgranulation Ansatz Re-applied

In order to calculate the diffusion kernel (11.28) and, therefore, the collision integral (11.22), we yet again need to know the correlation function  $C_{\mathbf{k}}(\mathbf{v}, \mathbf{v}')$  of  $g$ . This is the moment to bring back, now with non-trivial pay-off, the microgranulation ansatz (11.7). The result is that (11.22) becomes

$$\begin{aligned} \frac{\partial f_0}{\partial t} = & \frac{16\pi^3 e^4 \Delta\Gamma}{m^2 V} \frac{\partial}{\partial \mathbf{v}} \cdot \int d\mathbf{v}'' \sum_{\mathbf{k}} \frac{\mathbf{k}\mathbf{k}}{k^4} \frac{\delta(\mathbf{k} \cdot (\mathbf{v} - \mathbf{v}''))}{|\epsilon_{\mathbf{k}, \mathbf{k} \cdot \mathbf{v}}|^2} \\ & \cdot \left[ \langle g^2 \rangle(\mathbf{v}'') \frac{\partial f_0(\mathbf{v})}{\partial \mathbf{v}} - \langle g^2 \rangle(\mathbf{v}) \frac{\partial f_0(\mathbf{v}'')}{\partial \mathbf{v}''} \right]. \end{aligned} \quad (11.29)$$

A reader who followed carefully the discussion and derivation of the quasiparticle-collision integrals in §§7 and 8.4.2 has realised that the contortions through which I am going here are an attempt to set up for a collisionless plasma something resembling the WT random-phase approximation or the usual *Stosszahlansatz*. It is the microgranulation ansatz that breaks reversibility and is about to give us a “collision” integral with an  $H$ -theorem and a universal solution.

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**Exercise 11.2. Momentum and energy conservation for “collisionless collision integrals”.** Show that the general integral (11.29) conserves the momentum and kinetic energy of the particle distribution  $f_0$ .

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This is as far as we can go without further closure assumptions: we still need to work out  $\langle g^2 \rangle$  in terms of  $f_0$ . Let us get on with it.

### 11.4. Kadomtsev–Pogutse Collision Integral

With the assumptions and the definition of averaging described above,  $f_0$  is the same as  $\bar{f}$  in §10 and the macro-cells over which it is coarse-grained are the same randomly

(and statistically independently) filled macro-cells as in the Lynden-Bell statistics (so, in the language of §10,  $\Delta\Gamma = \mathcal{M}\delta\Gamma$ ). Unsurprisingly, the same result is about to pop out. Indeed, let us, for maximum simplicity, assume that the exact distribution is a single waterbag (10.1). Then<sup>92</sup>

$$\langle g^2 \rangle = \langle (f - f_0)^2 \rangle = \langle f^2 \rangle - f_0^2 = (\eta - f_0)f_0, \quad (11.30)$$

because, for a single-waterbag distribution,  $\langle f^2 \rangle = \langle \eta f \rangle = \eta f_0$ . With this additional closure, the “proto-collision integral” (11.29) turns into

$$\begin{aligned} \frac{\partial f_0}{\partial t} = & \frac{16\pi^3 e^4 \Delta\Gamma}{m^2 V} \frac{\partial}{\partial \mathbf{v}} \cdot \int d\mathbf{v}'' \sum_{\mathbf{k}} \frac{\mathbf{k}\mathbf{k}}{k^4} \frac{\delta(\mathbf{k} \cdot (\mathbf{v} - \mathbf{v}''))}{|\epsilon_{\mathbf{k}, \mathbf{k} \cdot \mathbf{v}}|^2} \\ & \cdot \left[ (\eta - f_0(\mathbf{v}'')) f_0(\mathbf{v}'') \frac{\partial f_0(\mathbf{v})}{\partial \mathbf{v}} - (\eta - f_0(\mathbf{v})) f_0(\mathbf{v}) \frac{\partial f_0(\mathbf{v}'')}{\partial \mathbf{v}''} \right]. \end{aligned} \quad (11.31)$$

This is the “collisionless collision integral” of Kadomtsev & Pogutse (1970) (whose derivation I have more or less followed). It is not hard to show that it has an  $H$ -theorem with the entropy (10.5) and that this entropy is maximised—and the right-hand-side of (11.31) is annihilated—by the Fermi–Dirac distribution (10.6).

**Exercise 11.3.** Show this (you will find the solution in Adkins 2018).

**Exercise 11.4.** Generalise the above calculation to multiple species (electrons and ions).

The procedure that I have followed above to derive the collision integral (11.31) can be used in many different contexts. For one of the more recent and exotic such applications, see the derivation by Bar-Or *et al.* (2021) of quantum kinetic equation for “fuzzy dark matter”. A more mundane, textbook example is the derivation of the “true” collision integrals for Fermi and Bose gases described in the appendix of Ewart *et al.* (2022).

### 11.5. Lenard–Balescu Collision Integral

Just as in §10, the non-degenerate limit is  $f_0 \ll \eta$ , turning (11.31) into

$$\frac{\partial f_0}{\partial t} = \frac{16\pi^3 e^4 \eta \Delta\Gamma}{m^2 V} \frac{\partial}{\partial \mathbf{v}} \cdot \int d\mathbf{v}'' \sum_{\mathbf{k}} \frac{\mathbf{k}\mathbf{k}}{k^4} \frac{\delta(\mathbf{k} \cdot (\mathbf{v} - \mathbf{v}''))}{|\epsilon_{\mathbf{k}, \mathbf{k} \cdot \mathbf{v}}|^2} \cdot \left[ f_0(\mathbf{v}'') \frac{\partial f_0}{\partial \mathbf{v}} - f_0(\mathbf{v}) \frac{\partial f_0}{\partial \mathbf{v}''} \right].$$

(11.32)

This has a functional form that is identical to the standard Lenard–Balescu collision integral (Lenard 1960; Balescu 1960) and its solution is, obviously, a Maxwellian.

How is it possible that collisionless and collisional behaviour turns out to be the same? Mathematically, this is not hard to understand. Let us recall what is meant

<sup>92</sup>It has not escaped my perceptive reader that a potential mathematical illegality is being perpetrated: is coarse-grain average of a coarse-grain average equal to itself, and is it true that  $\langle ff_0 \rangle = \langle f_0^2 \rangle = f_0^2$ ? In general, no, not when the coarse-graining is done by convolution with some continuous shape function. However, it is fine if we formally do this as we did in §10, by grouping fixed discrete micro-cells into fixed discrete macro-cells. Alternatively, we may hope that the differences between  $f_0$  and  $\langle f_0 \rangle$  and similar quantities are small enough to be ignored or, in a cleaner move, resort to an ensemble average over many possible realisations of  $f$  and assume that it will do the coarse-graining job automatically. Note that our time average (2.7) in fact suffered from the same problem. There too, we could instead have done an ensemble average and assumed that that would automatically have slow time evolution and no spatial dependence (in a homogeneous system).

by collisions in plasma physics. When a plasma (or, rather, a collection of individual particles) is described by its Klimontovich distribution function (1.19), the latter satisfies a “collisionless” Vlasov equation involving the microscopic electromagnetic field—the Klimontovich equation (1.35). Collisions acquire a specific mathematical meaning when this equation is averaged (coarse-grained) over the Debye scale, leading to the kinetic equation (1.41), where the collision integral is defined as the correlation function of the differences between the averaged (“macroscopic”) and the exact (“microscopic”) fields and distribution. This is basically the same procedure as the one that led to (11.1) and thence to the “collisionless collision integral” (11.22) and its descendants (11.31) and (11.32). If we interpret the exact distribution  $f$  as the Klimontovich distribution,

$$f(\mathbf{r}, \mathbf{v}) = \sum_{i=1}^N \delta(\mathbf{r} - \mathbf{r}_i) \delta(\mathbf{v} - \mathbf{v}_i), \quad (11.33)$$

then the difference between it and  $f_0$  will satisfy the microgranulation ansatz (11.5) automatically, because the Klimontovich distribution is only non-zero at the exact phase-space positions  $(\mathbf{r}_i, \mathbf{v}_i)$  of the particles (assumed initially random), so the correlation function of  $f - f_0$  must needs be a delta function. In this interpretation,  $\Delta\Gamma$  is the effective width of the delta functions associated with individual particles, while  $\eta$  is their height (so the Klimontovich distribution is a kind of single waterbag). Clearly, they are related by

$$\eta\Delta\Gamma = 1, \quad (11.34)$$

which is all we need to know in (11.32), finally turning it into the Lenard–Balescu collision integral. It is not surprising that the solution is a Maxwellian because the Klimontovich distribution is extremely non-degenerate: all these delta functions always occupy a negligible fraction of the available phase space.

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### 11.6. Landau’s Collision Integral

While I am at it, let me show for completeness how the Landau (1936) collision integral is recovered from the Lenard–Balescu one. If we want to use (11.32) (with  $\eta\Delta\Gamma = 1$ ) as an expression for *bona fide* collisions, i.e., if  $\delta f$  and  $\varphi$  are interpreted as fluctuating fields below the Debye scale, we must restrict the  $\mathbf{k}$  summation to  $k\lambda_D \gg 1$ . In this approximation,  $|\epsilon_{\mathbf{k}, \mathbf{k} \cdot \mathbf{v}}|^2 \approx 1$ . The  $\mathbf{k}$  sum then becomes tractable: denoting  $\mathbf{v} - \mathbf{v}'' = \mathbf{w}$ , we get

$$\frac{1}{V} \sum_{\mathbf{k}} \frac{\mathbf{k}\mathbf{k}}{k^4} \delta(\mathbf{k} \cdot (\mathbf{v} - \mathbf{v}'')) = \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{\mathbf{k}\mathbf{k}}{k^4} \delta(\mathbf{k} \cdot \mathbf{w}) = \frac{1}{w} \int \frac{d\mathbf{k}_{\perp}}{(2\pi)^3} \frac{\mathbf{k}_{\perp}\mathbf{k}_{\perp}}{k_{\perp}^4} = \frac{1}{8\pi^2 w} \left( \mathbf{I} - \frac{\mathbf{w}\mathbf{w}}{w^2} \right) \int \frac{d\mathbf{k}_{\perp}}{k_{\perp}}, \quad (11.35)$$

where  $\mathbf{k}_{\perp} = \mathbf{k} \cdot (\mathbf{I} - \mathbf{w}\mathbf{w}/w^2)$ . The divergent integral is the Coulomb logarithm  $\Lambda = \ln(\lambda_D/d)$  if the integration is cut off at the distance of closest approach  $d$  [see text after (1.13)] and the Debye scale  $\lambda_D$ . Consequently, (11.32) becomes

$$\boxed{\frac{\partial f_0}{\partial t} = \frac{2\pi e^4 \Lambda}{m^2} \frac{\partial}{\partial \mathbf{v}} \cdot \int \frac{d\mathbf{v}''}{w} \left( \mathbf{I} - \frac{\mathbf{w}\mathbf{w}}{w^2} \right) \cdot \left[ f_0(\mathbf{v}'') \frac{\partial f_0}{\partial \mathbf{v}} - f_0(\mathbf{v}) \frac{\partial f_0}{\partial \mathbf{v}''} \right].} \quad (11.36)$$

This is the Landau (1936) collision integral [cf. (1.47)]. Note that both it and the Lenard–Balescu integral (11.32) have the functional form anticipated in (7.35), which is as it should be—the calculations above are, thus, a way to work out the interaction probability  $w(\mathbf{p}, \mathbf{p}', \mathbf{k})$ .

### 11.7. Multi-Waterbag Collision Integral

The Kadomtsev–Pogutse collision integral can be generalised to multi-waterbag (and, therefore, general) distributions—leading to a new “collisionless collision integral” whose fixed point is the multi-waterbag Lynden-Bell distribution introduced in §10.4. The clever scheme that I

am about to explain can be traced back to Appendix C of the paper by Chavanis (2005) on geophysical turbulence; below, I will follow the derivation of it that was invented by Robbie Ewart in ignorance of that previous work (and published as Ewart *et al.* 2022). While the applicability or validity of Chavanis' and Ewart's integral are as yet unexplored, I want to go through its derivation as it nicely ties up some of the narrative threads that have lead us here.

Let us rewind back to (11.29). For a single waterbag,  $\langle f^2 \rangle = \eta f_0$  and thus a closure of the second-order average in terms of the first-order one,  $f_0$ , is achieved (§11.4). This is not as easy for a multi-waterbag distribution (10.12):

$$\langle f^2 \rangle = \sum_{JJ'} \langle f_J f_{J'} \rangle = \sum_J \langle f_J^2 \rangle = \sum_J \eta_J \langle f_J \rangle \equiv \sum_J \eta_J^2 p_J. \quad (11.37)$$

The second equality here holds because different waterbags cannot occupy the same point in phase space;  $p_J(\mathbf{v})$  is the probability that the point  $\mathbf{v}$  is occupied by the  $J$ -th waterbag (note that averages are assumed to be spatially homogeneous, i.e., I have effectively included the entire position space into each macro-cell). The expression (11.37) cannot be easily expressed in terms of

$$f_0 \equiv \langle f \rangle = \sum_J \langle f_J \rangle = \sum_J \eta_J p_J. \quad (11.38)$$

In principle, if we have  $n$  waterbags, we can calculate  $n$  independent averages  $f_0, \langle f^2 \rangle, \dots, \langle f^n \rangle = \sum_J \eta_J^n p_J$ , and hence calculate all  $p_J$ 's as functions of these averages and  $\eta_J$ 's. Higher-order averages,  $\langle f^{n+1} \rangle, \dots$ , can then be computed in terms of the lower-order ones. Clearly, if we attempted to derive kinetic equations for  $\langle f^k \rangle$  along the same lines as (11.29), the evolution of  $\langle f^k \rangle$  would end up being expressed in terms of  $f_0, \langle f^2 \rangle, \dots, \langle f^{k+1} \rangle$ . If we took this all the way to  $k = n$ , we would then be able to close our system by expressing  $\langle f^{n+1} \rangle$  in terms of the first  $n$  averages.

It is depressingly obvious that this is not a useful plan for large  $n$ —and it is only if we can handle  $n \gg 1$  waterbags that we would be approaching the desired case of a general initial distribution. Instead, Chavanis (2005) and Ewart *et al.* (2022) propose that we admit ignorance and embrace the kind of statistical-inference approach that is commonly used in statistical mechanics to find probability distributions that are guaranteed to reproduce certain quantities correctly (see lecture notes by Schekochihin 2019 or, for a magisterial treatment of this method, the book by Jaynes 2003). In this instance, what we want is a set of  $p_J$ 's that reproduce  $f_0(\mathbf{v})$ , at every  $\mathbf{v}$ , according to (11.38). They also need to sum up to unity (if we include as  $J = 0$  the empty waterbag with  $\eta_0 = 0$ ) and to preserve the number of particles (per unit volume) in each waterbag:

$$\eta_J \int d\mathbf{v} p_J(\mathbf{v}) = \frac{N_J}{V}. \quad (11.39)$$

This is achieved by maximising their Shannon entropy subject to these constraints:

$$\begin{aligned} & - \int d\mathbf{v} \sum_J p_J(\mathbf{v}) \ln p_J(\mathbf{v}) - \int d\mathbf{v} \psi(\mathbf{v}) \left[ \sum_{J \neq 0} \eta_J p_J(\mathbf{v}) - f_0(\mathbf{v}) \right] \\ & - \sum_{J \neq 0} \gamma_J \left[ \int d\mathbf{v} p_J(\mathbf{v}) - \frac{N_J}{V \eta_J} \right] \\ & - \int d\mathbf{v} \lambda(\mathbf{v}) \left[ \sum_J p_J(\mathbf{v}) - 1 \right] \rightarrow \max, \end{aligned} \quad (11.40)$$

where  $\psi(\mathbf{v})$ ,  $\gamma_J$  and  $\lambda(\mathbf{v})$  are Lagrange multipliers. Note that the fact that we have to resort to such desperate measures indicates that, contrary to what I said at the end of §11.2, the microgranulation ansatz is not quite enough to break correlators—thus, Chavanis's and Ewart's scheme, together with the microgranulation ansatz, should be viewed as a kind of extension of the *Stosszahlansatz* to collisionless plasma.

The maximisation (11.40) yields immediately

$$p_J(\mathbf{v}) = \frac{1}{Z(\psi(\mathbf{v}))} e^{-\psi(\mathbf{v})\eta_J - \gamma_J} = -\frac{\partial \ln Z}{\partial \gamma_J}(\mathbf{v}), \quad (11.41)$$

where the “partition function” is

$$Z(\psi(\mathbf{v})) \equiv e^{1+\lambda(\mathbf{v})} = 1 + \sum_{J \neq 0} e^{-\psi(\mathbf{v})\eta_J - \gamma_J}. \quad (11.42)$$

It depends not just on  $\mathbf{v}$  but also on all  $\gamma_J$ 's, which are determined from the system of equations (11.39), where  $N_J$ 's are treated as known—they and  $\eta_J$ 's encode the information that is contained in the initial distribution and cannot be destroyed by collisionless dynamics; the conservation of  $N_J$ 's is the multi-waterbag approximation's version of the infinite number of conserved quantities (Casimir invariants) in a collisionless plasma [see (9.8)]. Finally, the Lagrange multiplier  $\psi(\mathbf{v})$  is determined from the constraint (11.38):

$$f_0(\mathbf{v}) = \frac{1}{Z(\psi(\mathbf{v}))} \sum_J \eta_J e^{-\psi(\mathbf{v})\eta_J - \gamma_J} = -\frac{\partial \ln Z}{\partial \psi}(\mathbf{v}). \quad (11.43)$$

This gives us a correspondence between  $f_0(\mathbf{v})$  and  $\psi(\mathbf{v})$ . If this correspondence is one-to-one, at least within some class of distributions of interest, determining one of these functions will fix the other.

We can now compute the previously unknown second-order average in our collision integral (11.29), according to (11.37) and (11.41):

$$\langle f^2 \rangle = \frac{1}{Z} \sum_J \eta_J^2 e^{-\psi(\mathbf{v})\eta_J - \gamma_J} = \frac{1}{Z} \frac{\partial^2 Z}{\partial \psi^2}. \quad (11.44)$$

Conveniently, this gives

$$\langle g^2 \rangle = \langle f^2 \rangle - f_0^2 = \frac{1}{Z} \frac{\partial}{\partial \psi} Z \frac{\partial \ln Z}{\partial \psi} - \left( \frac{\partial \ln Z}{\partial \psi} \right)^2 = \frac{\partial^2 \ln Z}{\partial \psi^2} = -\frac{\partial f_0}{\partial \psi}. \quad (11.45)$$

Note that since the left-hand side of this equation is always positive,  $f_0$  is always a monotonically decreasing function of  $\psi$ , which is good news from the point of view of solvability of (11.43).

Using (11.45) and  $\partial f_0 / \partial \mathbf{v} = (\partial f_0 / \partial \psi)(\partial \psi / \partial \mathbf{v})$  in (11.29), we get the multi-waterbag collision integral:

$$\frac{\partial f_0}{\partial t} = -\frac{16\pi^3 e^4 \Delta \Gamma}{m^2 V} \frac{\partial}{\partial \mathbf{v}} \cdot \int d\mathbf{v}'' \sum_{\mathbf{k}} \frac{\mathbf{k}\mathbf{k}}{k^4} \frac{\delta(\mathbf{k} \cdot (\mathbf{v} - \mathbf{v}''))}{|\epsilon_{\mathbf{k}, \mathbf{k} \cdot \mathbf{v}}|^2} \cdot \left[ \frac{\partial \psi(\mathbf{v})}{\partial \mathbf{v}} - \frac{\partial \psi(\mathbf{v}'')}{\partial \mathbf{v}''} \right] \frac{\partial f_0}{\partial \psi}(\mathbf{v}) \frac{\partial f_0}{\partial \psi}(\mathbf{v}''), \quad (11.46)$$

a pleasing case of the more general result looking more compact than the less general Kadomtsev–Pogutse integral (11.31). This compactness is, of course, somewhat deceptive: the expression (11.46) has to be complemented by the relation (11.43) between  $f_0$  and  $\psi$ , the definition of the partition function (11.42), and, worst of all, a gigantic system of coupled equations (11.39) for  $\gamma_J$ 's, enforcing phase-volume conservation.

This may suddenly seem rather unwieldy, and perhaps it is, but there is one thing that is very easy to see from (11.46): a steady state is achieved for  $\psi(\mathbf{v}) = \beta m v^2 / 2 = \beta \varepsilon$ , where I have chosen the constant prefactor so as to match the familiar Lynden-Bell multi-waterbag equilibrium (10.13) (note that, in the notation of §10.4,  $\gamma_J = -\beta \eta_J \mu_J$ ).

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**Exercise 11.5.** Check that the Kadomtsev–Pogutse integral (11.31) is readily recovered from (11.46) in the case of a single waterbag. What is  $\psi(\mathbf{v})$  in terms of  $f_0(\mathbf{v})$  in this case?

**Exercise 11.6. H-theorem for the multi-waterbag integral.** Show that the Lynden-Bell entropy for the multi-waterbag case, which you worked out in Exercise 10.2, was, up to irrelevant additive and multiplicative constants,

$$S = - \int d\mathbf{v} \sum_J p_J(\mathbf{v}) \ln p_J(\mathbf{v}), \quad (11.47)$$

where the summation includes  $J = 0$ , for which  $p_0 = 1 - \sum_{J \neq 0} p_J$ . Show that this entropy is never decreased by the multi-waterbag integral (11.46), with a stationary solution achieved if and only if  $f_0$  is the Lynden-Bell multi-waterbag equilibrium (10.13). The key step in this

derivation is to realise that the entropy can be written as

$$S = \int d\mathbf{v} (\psi f_0 + \ln Z) + \sum_{J \neq 0} \gamma_J \frac{N_J}{V \eta_J}. \quad (11.48)$$


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### 11.7.1. Continuous Limit of the Multi-Waterbag Formalism

The reason to be interested in multi-waterbag distributions is, of course, that any distribution can be discretised into that form. It is, therefore, instructive—and, as it turns out, quite straightforward—to recast the results of §11.7 (and of §10.4) in a continuous form. This is accomplished by replacing

$$\sum_J \rightarrow \frac{1}{\Delta\eta} \int_0^{\eta_{\max}} d\eta \quad (11.49)$$

everywhere,  $\eta_{\max}$  being the maximum value that  $f(\mathbf{Q})$  takes and  $\Delta\eta = \eta_{J+1} - \eta_J \rightarrow 0$  the “mesh” size for the waterbag discretisation. The only (important) nuance is in what becomes of the phase-volume-conservation constraints (11.39). Let us summon back the convenient function (9.41),  $\Gamma(\eta) \equiv \int d\mathbf{Q} H(f(\mathbf{Q}) - \eta)$ , which returns the amount of phase volume where  $f(\mathbf{Q}) > \eta$ . The number of particles in a phase-space volume where  $\eta < f(\mathbf{Q}) \leq \eta + \Delta\eta$  is

$$N(\eta) = -\eta \Delta\eta \Gamma'(\eta) = \eta \Delta\eta \rho(\eta), \quad \rho(\eta) = \int d\mathbf{Q} \delta(f(\mathbf{Q}) - \eta). \quad (11.50)$$

Therefore, letting  $N_J = N(\eta_J)$ , we can rewrite the constraint (11.39) as follows:

$$\eta \int d\mathbf{v} \frac{e^{-\psi(\mathbf{v})\eta - \gamma(\eta)}}{Z/\Delta\eta} = \frac{\eta \Delta\eta \rho(\eta)}{V}, \quad Z(\psi(\mathbf{v})) = \int d\eta e^{-\psi(\mathbf{v})\eta - \gamma(\eta)}, \quad (11.51)$$

where the partition function has been redefined without the prefactor of  $1/\Delta\eta$ , which it acquired when (11.49) was applied to (11.42). Happily,  $\Delta\eta$  then falls out of the equation:

$V \int d\mathbf{v} \frac{1}{Z} e^{-\psi(\mathbf{v})\eta - \gamma(\eta)} = \rho(\eta).$

(11.52)

The relationship (11.43) between  $f_0$  and  $\psi$  is also independent of  $\Delta\eta$ :

$$f_0(\mathbf{v}) = \frac{\int d\eta \eta e^{-\psi(\mathbf{v})\eta - \gamma(\eta)}}{\int d\eta e^{-\psi(\mathbf{v})\eta - \gamma(\eta)}} = -\frac{\partial \ln Z}{\partial \psi}. \quad (11.53)$$

Everything else is the same as before and we again have the multi-waterbag integral (11.46), except now it is subject to the integral constraint (11.52), determining  $\gamma(\eta)$  from  $\rho(\eta)$ . The function  $\rho(\eta)$ , defined in (11.50), represents all the information that is retained from the exact distribution function  $f(\mathbf{Q})$  due to phase-volume conservation.

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## 11.8. Hyperkinetic Collision Integral

It is possible to generalise the Kadomtsev–Pogutse integral more or less straightforwardly to general distributions, not limited to a single waterbag or even a discrete set of them—alongside the more baroque scheme presented in §11.7, this too is described in Ewart *et al.* (2022), but the derivation for discrete waterbags goes back to Severne & Luwel (1980) (in application to galactic dynamics). The required formalism is motivated by the continuous limit of the multi-waterbag statistics (§11.7.1), but can be understood without reference to it. Let us introduce a new exact and a new coarse-grained (averaged) distribution functions:

$$P(\mathbf{Q}, \eta) = \delta(f(\mathbf{Q}) - \eta), \quad P_0(\mathbf{v}, \eta) = \langle P(\mathbf{Q}, \eta) \rangle. \quad (11.54)$$

They measure the (exact and average) amount of phase volume occupied by particles with position and velocity  $\mathbf{Q} = (\mathbf{r}, \mathbf{v})$  whose phase density is equal to  $\eta$ .<sup>93</sup> They have the properties

$$\int d\eta P(\mathbf{Q}, \eta) = \int d\eta P_0(\mathbf{v}, \eta) = 1, \quad (11.55)$$

$$\int d\eta \eta P(\mathbf{Q}, \eta) = f(\mathbf{Q}), \quad \int d\eta \eta P_0(\mathbf{v}, \eta) = f_0(\mathbf{v}), \quad (11.56)$$

$$\int d\mathbf{Q} P(\mathbf{Q}, \eta) = V \int d\mathbf{v} P_0(\mathbf{v}, \eta) = \rho(\eta) = \text{const.} \quad (11.57)$$

The constancy of  $\rho(\eta)$  [already introduced in (11.50)], for every  $\eta$ , is the expression of the phase-volume conservation—an infinite number of conservation laws (the Casimir invariants already mentioned at the end of §9.1).

Technically, the function  $P(\mathbf{Q}, \eta)$  lives in an extended 7D phase space  $(\mathbf{r}, \mathbf{v}, \eta)$ , but of course it is, in fact, limited by (11.57) to a 6D hypersurface within this 7D space. Knowing its coarse-grained mean  $P_0(\mathbf{v}, \eta)$  does, however, provide more information than  $f_0(\mathbf{v})$ , because it allows one to calculate, e.g.,

$$\langle f(\mathbf{Q})^2 \rangle = \int d\eta \eta^2 P_0(\mathbf{v}, \eta), \quad (11.58)$$

precisely the quantity that we have needed a scheme for calculating to obtain closed expressions for the “collisionless collision integrals” in §§11.4 and 11.7. This is a change of attitude to the distribution function  $f_0(\mathbf{v})$ : I am now treating it as a “fluid” quantity—the mean density of the “phase fluid” that is fully described by the exact density  $f(\mathbf{Q})$ —and so it is just the first  $\eta$  moment of the “kinetic” quantity  $P_0(\mathbf{v}, \eta)$ . Higher-order correlation functions of  $f(\mathbf{Q})$  are, accordingly, higher  $\eta$  moments of  $P_0(\mathbf{v}, \eta)$ —just like in the derivation of fluid dynamics from kinetics, the closure problem for fluid moments is overcome by determining the kinetic distribution function over a higher-dimensional phase space. This formalism, which I shall call *hyperkinetics*, originates from galactic dynamics and vortex kinetics (for which it is nicely presented in Chavanis *et al.* 1996; the review-style paper by Chavanis 2021 has a treatment of this topic that is quite similar to mine and contains also very many historical cross-references between different fields and lines of thought).

Progress is possible because it is very easy to derive a closed equation for  $P(\mathbf{Q}, \eta)$ : using Vlasov’s equation for  $f$ , we get

$$\begin{aligned} \frac{\partial P}{\partial t} &= \delta'(f - \eta) \frac{\partial f}{\partial t} = -\delta'(f - \eta) \left[ -\mathbf{v} \cdot \nabla f - \frac{e}{m} (\nabla \varphi) \cdot \frac{\partial f}{\partial \mathbf{v}} \right] \\ &= -\mathbf{v} \cdot \nabla P - \frac{e}{m} (\nabla \varphi) \cdot \frac{\partial P}{\partial \mathbf{v}}. \end{aligned} \quad (11.59)$$

Thus,  $P$  just satisfies the usual Vlasov’s equation. Its average  $P_0$ , which we can assume to be spatially homogeneous, obeys, therefore,

$$\frac{\partial P_0}{\partial t} = \frac{\partial}{\partial \mathbf{v}} \cdot \left[ -\frac{e}{m} \langle (\nabla \varphi) \delta P \rangle \right], \quad (11.60)$$

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<sup>93</sup> $P_0(\mathbf{v}, \eta)$  is just the continuous version of the waterbag probabilities  $p_J$  introduced in §11.7: see (11.38). Namely,  $P_0(\mathbf{v}, \eta) = p_J(\mathbf{v})/\Delta\eta$ .

where  $\delta P = P - P_0$  and, since  $\delta f = \int d\eta \eta \delta P$ , Poisson's equation is

$$-\nabla^2 \varphi = 4\pi e \int d\mathbf{v} \int d\eta \eta \delta P. \quad (11.61)$$

Working out the average in the right-hand side of (11.60) is a problem that can be solved entirely analogously to what I did to determine the rate of change of  $f_0$  in (11.1). All the steps are analogous, except with  $f_0 \rightarrow P_0$ ,  $\delta f \rightarrow \delta P$ , and wherever a velocity integral occurs, there must now also be an integral with respect to  $\eta$ , viz.,  $\int d\mathbf{v} \rightarrow \int d\mathbf{v} \int d\eta \eta$ , because all velocity integrals come from Poisson's law.

The first place where something mildly different happens is the microgranulation ansatz: assuming, as in §11.2, that different points in the (coarse-grained) phase space are uncorrelated, we get, instead of (11.5),

$$\langle g(\mathbf{Q}, \eta)g(\mathbf{Q}', \eta') \rangle \approx \langle g(\mathbf{Q}, \eta)g(\mathbf{Q}, \eta') \rangle \Delta\Gamma \delta(\mathbf{Q} - \mathbf{Q}'). \quad (11.62)$$

The correlation function in the prefactor can now be worked out as follows:

$$\begin{aligned} \langle g(\mathbf{Q}, \eta)g(\mathbf{Q}, \eta') \rangle &= \langle [\delta(f(\mathbf{Q}) - \eta) - P_0(\mathbf{v}, \eta)] [\delta(f(\mathbf{Q}) - \eta') - P_0(\mathbf{v}, \eta')] \rangle \\ &= \langle \delta(f(\mathbf{Q}) - \eta) \delta(f(\mathbf{Q}) - \eta') \rangle - P_0(\mathbf{v}, \eta) P_0(\mathbf{v}, \eta') \\ &= [\delta(\eta - \eta') - P_0(\mathbf{v}, \eta')] P_0(\mathbf{v}, \eta). \end{aligned} \quad (11.63)$$

This is the generalisation of the single-waterbag formula (11.30). In what follows, we shall only need the first moment of this expression:

$$\int d\eta' \eta' \langle g(\mathbf{Q}, \eta)g(\mathbf{Q}, \eta') \rangle = (\eta - f_0(\mathbf{v})) P_0(\mathbf{v}, \eta). \quad (11.64)$$

We are ready to write the hyperkinetic collision integral (11.31):

$$\begin{aligned} \frac{\partial P_0}{\partial t} &= \frac{16\pi^3 e^4 \Delta\Gamma}{m^2 V} \frac{\partial}{\partial \mathbf{v}} \cdot \int d\mathbf{v}'' \sum_{\mathbf{k}} \frac{\mathbf{k}\mathbf{k}}{k^4} \frac{\delta(\mathbf{k} \cdot (\mathbf{v} - \mathbf{v}''))}{|\epsilon_{\mathbf{k}, \mathbf{k} \cdot \mathbf{v}}|^2} \\ &\cdot \int d\eta'' \eta'' \left[ (\eta'' - f_0(\mathbf{v}'')) P_0(\mathbf{v}'', \eta'') \frac{\partial P_0(\mathbf{v}, \eta)}{\partial \mathbf{v}} - (\eta - f_0(\mathbf{v})) P_0(\mathbf{v}, \eta) \frac{\partial P_0(\mathbf{v}'', \eta'')}{\partial \mathbf{v}''} \right]. \end{aligned} \quad (11.65)$$

The last line of the above formula can be slightly simplified, sacrificing symmetry but eliciting the form of the diffusion and drag terms:

$$\left[ \int d\eta'' \eta''^2 P_0(\mathbf{v}'', \eta'') - f_0^2(\mathbf{v}'') \right] \frac{\partial P_0(\mathbf{v}, \eta)}{\partial \mathbf{v}} - \left[ (\eta - f_0(\mathbf{v})) \frac{\partial f_0(\mathbf{v}'')}{\partial \mathbf{v}''} \right] P_0(\mathbf{v}, \eta). \quad (11.66)$$

The collision integral (11.65) looks quite attractive in that it is manifestly a generalisation of the Kadomtsev–Pogutse integral (11.31) and does not involve solving transcendental functional equations [see (11.52)] or performing transformations that may or may not always exist [see (11.53)] like Chavanis' and Ewart's integral (§11.7) does.

**Exercise 11.7.** (a) By retracing the steps that led from (11.1) to (11.31), but now keeping track of the  $\eta$  dependences, check that (11.65) is indeed duly recovered.

(b) Check that the integral (11.65) automatically preserves the properties (11.55) and (11.57), and that it conserves the total number of particles, their momentum, and energy.

**Exercise 11.8. H-theorem for the hyperkinetic collision integral.** State it and prove it.

The hyperkinetic collision integral (11.65) has an entropy—just the usual Shannon

(1948) entropy of  $P_0$ ,<sup>94</sup>

$$S = -V \int d\mathbf{v} \int d\eta P_0(\mathbf{v}, \eta) \ln P_0(\mathbf{v}, \eta) \quad (11.67)$$

—and an  $H$ -theorem (Exercise 11.8), so it pushes the extended distribution to the unique fixed point (easily checked to be one by direct substitution)

$$P_0(\mathbf{v}, \eta) = \frac{e^{-\beta\eta\varepsilon-\gamma(\eta)}}{\int d\eta e^{-\beta\eta\varepsilon-\gamma(\eta)}} \Rightarrow f_0(\mathbf{v}) = \frac{\int d\eta \eta e^{-\beta\eta\varepsilon-\gamma(\eta)}}{\int d\eta e^{-\beta\eta\varepsilon-\gamma(\eta)}}, \quad (11.68)$$

where  $\varepsilon = mv^2/2$ ,  $\beta$  is determined by fixing the initial energy of the distribution (conserved by the collision integral) and  $\gamma(\eta) \equiv -\beta\eta\mu(\eta)$  by fixing its “waterbag content”, expressed by  $\rho(\eta)$  in (11.57). The function  $f_0$  in (11.68) is just the continuous, multi-waterbag version of the Lynden-Bell equilibrium (10.13), which, as we know from Exercise 10.5, is general enough to represent any Gardner distribution. Thus, one might say that we have done for Gardner and Lynden-Bell what Boltzmann did for Maxwell and Gibbs: shown that an equilibrium motivated by thermodynamics/statistical mechanics is reached dynamically, and how.

**Exercise 11.9.** Observe that the closure underpinning Chavanis’ and Ewart’s multi-waterbag collision integral of §11.7 was equivalent to assuming that a collisionless system would always quickly maximise the entropy (11.67) *locally* in phase space (i.e., for each  $\mathbf{v}$ ). The relaxation to the global Lynden-Bell equilibrium then happens more slowly, according to Chavanis’ and Ewart’s integral. This is somewhat analogous to standard collisional gases, which relax quickly, at the collision rate, to a local Maxwellian, and then slowly, at the diffusion rate, to the global one (see, e.g., Schekochihin 2019). If one could prove that the hyperkinetic collision integral (11.65) does indeed contain within itself such a two-timescale evolution, that would promote Chavanis’ and Ewart’s integral from a clever closure scheme to a valid approximation. I do not know how to do this, or whether it is true, so I give this problem to you as an exercise.

### 11.9. Where Has This Got Us So Far

What can we conclude from the fact that the Lynden-Bell equilibria can be recovered from kinetic theory? The obvious observation so far is the correspondence between the assumptions of Lynden-Bell’s statistical mechanics and those of the kinetic theory à la Kadomtsev & Pogutse (1974). In order to get the latter, we needed the stability of  $f_0$  [see discussion just before (11.18)], the microgranulation ansatz (11.5), and phase-volume conservation in the form (11.57). The Lynden-Bell equilibria (11.68) and fixed energy (Exercise 11.2) followed. In the statistical-mechanical approach, we needed to fix the energy [see (10.14)] and assume thorough mixing of phase-density elements subject only to the exclusion principle dictated by phase-volume conservation [see discussion after (10.2)]. This gave us an entropy to maximise, and hence an equilibrium, whereas in kinetic theory, the existence of the Lynden-Bell entropy (11.67) was merely a humble property of the collision integral (Exercise 11.8).

Suspending for a moment all the legitimate doubts about the assumptions that have gone into this, let us observe that the progress achieved in §11 in comparison with the statistical-mechanical calculations of §10 is that we can now follow the relaxation of  $f_0$  in time, at least after some initial period during which all the instabilities of an initial state sort themselves out and  $f_0$  becomes stable. The rate at which  $f_0$  relaxes to equilibrium can be read off from (11.31) or (11.65): ignoring any factors of order unity and using (11.35)

<sup>94</sup>This is essentially the same expression as (9.27).

to estimate the size of the  $\mathbf{k}$  sum,<sup>95</sup> we get the effective collision frequency

$$\boxed{\nu_{\text{eff}} \sim \nu \eta \Delta \Gamma}, \quad (11.69)$$

where  $\nu$  is the “true” (Coulomb) collision frequency. In the “true” collisional limit,  $\eta \Delta \Gamma = 1$  [see (11.34)], but in the collisionless regime,  $\eta \Delta \Gamma$  (the number of particles that fit into a fully occupied macro-cell whose size is the correlation volume of  $g$ ) must be large, so the “collisionless collision frequency”  $\nu_{\text{eff}}$  is, generally speaking, much larger than  $\nu$ . Can we estimate  $\nu_{\text{eff}}$  in terms of some physical (measurable) parameters?

Since we are dealing with collisionless dynamics, initial distribution will matter and determine  $\eta$ . One good physical example is some collection of spatially homogeneous beams, all with the same phase density—then, in terms of the beam number density  $n_b$  and width  $v_b$ ,  $\eta \sim n_b/v_b^3$  [see (10.11)]. In the case of multiple waterbags,  $\eta$  in (11.69) must be replaced by some characteristic value: most obviously, in view of (11.29),

$$\eta_{\text{eff}} = \frac{\langle g^2 \rangle}{f_0} \quad (11.70)$$

(the typical deviation of the phase-space density from its mean).

The correlation volume  $\Delta \Gamma = \Delta r^3 \Delta v^3$  is a trickier quantity to make sense of. As the evolution of a collisionless system proceeds, one might argue that  $g$  would get ever more fine-scaled (“phase-mixed”; cf. §5.3), i.e., that  $\Delta v$  (and, due to nonlinear coupling of  $\mathbf{k}$ 's, also  $\Delta r$ ) would decrease with time.<sup>96</sup> Under this scenario,  $\Delta \Gamma(t)$  will decrease with time and so the convergence of  $f_0$  towards equilibrium will slow down as time goes on.

It remains an open research topic exactly how to calculate  $\Delta v(t)$  and  $\Delta r(t)$ , or, indeed, more generally,  $\langle g(\mathbf{r}, \mathbf{v})g(\mathbf{r}', \mathbf{v}') \rangle$ , i.e., whether the microgranulation ansatz (11.5) is at all valid. A key challenge in this context is to go beyond the QL approximation. As I intimated in §11.3, one could reinterpret  $h_{k\omega}$  in (11.14) as containing not just the ballistic evolution (11.15) but also the “nonlinear part” of  $\delta f_{k\omega}$ . The question is what is its correlation function  $C_{k\omega\omega'}(\mathbf{v}, \mathbf{v}')$ , which then goes into the general collision integral (11.22)—and indeed whether (11.22) is still a valid, i.e., if the correlation function will have the symmetry required to achieve (11.20) (in §12, I will alight on a scenario in which this will not be the case). You will find some further work on this topic in the classic papers by Kadomtsev & Pogutse (1971) and Dupree (1972), although I recommend them with some hesitation: neither is particularly transparent (and both are almost certainly technically wrong even if suggestive of ways forward), so you might end up with less clarity rather than more.<sup>97</sup> The basic idea of these papers is that the correlation scales  $\Delta r$  and  $\Delta v$  will decrease in time much less quickly than you might imagine based on naïve phase-mixing estimates—because long-time correlated phase-space “clumps” (or “granulations”) will allegedly form. The “collisionless collision integral” (11.22), or something resembling it, with an appropriate clump correlation function  $C_{k\omega\omega'}(\mathbf{v}, \mathbf{v}')$  then represents some effective scattering of particles by these clumps. The status of all this

<sup>95</sup>This includes the  $1/|\epsilon_{\mathbf{k}, \mathbf{k} \cdot \mathbf{v}}|^2$  factor. For  $k\lambda_D \gg 1$ ,  $|\epsilon_{\mathbf{k}, \mathbf{k} \cdot \mathbf{v}}| \approx 1$  and the contribution to the  $\mathbf{k}$  sum from these wave numbers is  $\sim 1$ ; for  $k\lambda_D \ll 1$ ,  $|\epsilon_{\mathbf{k}, \mathbf{k} \cdot \mathbf{v}}| \sim 1/(k\lambda_D)^2$  and the contribution from these wave numbers is  $\sim 1$ .

<sup>96</sup>Eventually,  $\Delta r$  would be limited by  $\sim \lambda_D$  and  $\Delta v$  by collisions, however small they are—see §5.5—but let us assume for a moment that collisionless dynamics can continue as long as we like, even though, as I already warned at the end of §9.1, this is likely to be a very bad assumption.

<sup>97</sup>The textbook by Diamond *et al.* (2010, Chapter 8) is on an ideological continuum from Dupree (1972), and, while also not a transparent read (imho), is probably the most up-to-date exposition of the current status of that line of thinking.

is rather uncertain: there was a lot of analytical work done in the 1970s and 1980s along and around these lines, but most of it has remained hypothetical, due to the difficulty of nonlinear theory and impossibility at the time of kinetic numerical simulations capable of resolving anything. The latter impediment to progress appears set to be lifted in the near future, so now probably is a good time to revisit the old theories and attempt new ones.

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**Exercise 11.10. Fluctuation energy in a Lynden-Bell plasma** (Brown 2021; Ewart *et al.* 2022). (a) Use Poisson's equation and the microgranulation ansatz to show that, in a 3D plasma that is in a Lynden-Bell equilibrium, either single-waterbag or multi-waterbag, the electric-field fluctuation energy is

$$\mathcal{E} \equiv \frac{V}{8\pi} \sum_{\mathbf{k}} k^2 \langle |\varphi_{\mathbf{k}}|^2 \rangle = \frac{4e^2 \Delta \Gamma k_{\max} V}{\beta m} \int_0^\infty dv f_0(v), \quad (11.71)$$

where  $\beta$  is the parameter of the Lynden-Bell equilibrium and  $k_{\max}$  is some appropriate UV cutoff, needed to evaluate the wave-number integral (cf. §11.6). In the derivation of (11.71), the formula (11.45) should prove handy, with  $\psi = \beta\varepsilon$ , as it is for the Lynden-Bell equilibria.

(b) Show therefore that the ratio of the fluctuation energy to the kinetic energy in the distribution is

$$\frac{\mathcal{E}}{\mathcal{K}} \sim \frac{e^2 \Delta \Gamma k_{\max}}{\beta T^2}, \quad (11.72)$$

where  $T = mv_{\text{th}}^2/2$  and  $v_{\text{th}}$  is the characteristic width of the distribution function  $f_0$ . The ratio (11.72) is clearly small in the degenerate limit ( $\beta \rightarrow \infty$ ). In the non-degenerate limit, use the estimate  $\beta\eta \sim 1/T$  (for a single waterbag) to show that

$$\frac{\mathcal{E}}{\mathcal{K}} \sim \frac{\nu_{\text{eff}}}{\omega_{\text{pe}}} k_{\max} \lambda_D, \quad (11.73)$$

where the effective collision frequency  $\nu_{\text{eff}}$  is defined by (11.69). How safe, therefore, is the assumption that  $\mathcal{E} \ll \mathcal{K}$ ?

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## 12. Plasma Echo, Phase Unmixing, and Phase-Space Turbulence

Since the question of the phase-space, and, in particular, velocity-space, structure of the perturbed distribution functions has acquired new urgency in §11, especially in what concerns the *nonlinear* evolution of fluctuations in plasmas, it is time to revisit phase mixing, but in a nonlinear regime.

In §5.3, we saw that the perturbed distribution function (5.25) associated with a Landau-damped electric perturbation contained an undamped ballistic term proportional to  $e^{-ik \cdot vt}$ . As time went on, this term became ever more oscillatory in  $\mathbf{v}$  and so no longer contributed to  $\varphi$ , but the information contained in it was not destroyed until it hit collisions. It turns out that this information can be retrieved.

Let us think in 1D and imagine sending an electric pulse into plasma (i.e., setting up some initial perturbation) at time  $t_1$ , with wave number  $k_1$ . It will be Landau damped, but the undamped ballistic part of the resulting perturbed distribution function will be

$$\delta f_{k_1} = a_1 e^{-ik_1 v(t-t_1)}, \quad (12.1)$$

where  $a_1$  is the contents of the curly bracket in (5.25) and  $v$  the component of the velocity parallel to the direction of the wave vector. Now let us wait for a while and send in another pulse, with  $k_2 > k_1$  at  $t_2 > t_1$ . This too will be Landau damped, leaving behind

$$\delta f_{k_2} = a_2 e^{-ik_2 v(t-t_2)}. \quad (12.2)$$

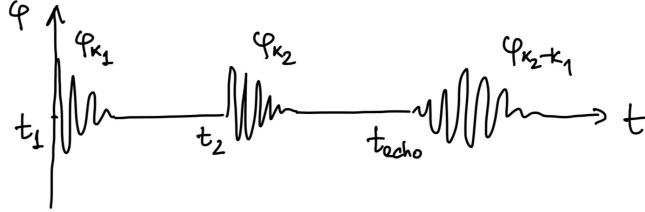


FIGURE 47. Two subsequent pulses, then an echo at time (12.4).

In the linear approximation, these are independent solutions, which simply add up. But if nonlinearity is allowed, they can couple to produce

$$\delta f_{k_2-k_1} \propto \delta f_{k_2} \delta f_{k_1}^* = a_1^* a_2 e^{-ik_2 v(t-t_2) + ik_1 v(t-t_1)} = a_1^* a_2 e^{-i[(k_2 - k_1)t - (k_2 t_2 - k_1 t_1)]v}. \quad (12.3)$$

Around the time

$$t_{\text{echo}} = \frac{k_2 t_2 - k_1 t_1}{k_2 - k_1}, \quad (12.4)$$

the distribution function (12.3) does not oscillate strongly in  $v$  and so can have an associated electric perturbation  $\varphi_{k_2-k_1} \propto \int dv \delta f_{k_2-k_1}$ , which will pop out of nowhere some time after the second pulse has gone in (e.g., if  $k_2 = 2k_1$ ,  $t_{\text{echo}} = 2t_2 - t_1 > t_2$ ; Fig. 47). Physically, what happens is that the second pulse “catches up” with the first pulse in phase space, couples to it, and effectively reverses the direction of phase mixing, producing “phase unmixing” and bringing information back into the electric field. This is called *plasma echo* and is a strange but real thing: having been derived by [Gould et al. \(1967\)](#), it was promptly produced in a laboratory by [Malmberg et al. \(1968\)](#).

The original derivation is a nice exercise in perturbation theory, which I will reproduce in §12.1. But this is not a mere curiosity: for nonlinear plasma systems with many interacting modes (i.e., for plasma turbulence), the implication may be that Landau damping and phase mixing are suppressed, or even altogether disabled, thus rendering nonlinear plasma more “fluid-like” than the linear one is. That in turn may mean that the microgranulation ansatz (11.5) is suspect and the phase-space correlations are a lot longer than linear, quasilinear, or weakly nonlinear theories might lead one to believe. This is a research frontier—I will discuss it starting from §12.2.

### 12.1. Textbook Derivation of Plasma Echo

Let us consider a situation in which only the electron distribution function is perturbed (i.e., frequencies are high) and its perturbation satisfies (2.12), as usual:

$$\frac{\partial \delta f_{\mathbf{k}}}{\partial t} + i\mathbf{k} \cdot \mathbf{v} \delta f_{\mathbf{k}} + i \frac{e}{m} \varphi_{\mathbf{k}} \mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{v}} = -i \frac{e}{m} \sum_{\mathbf{k}'} \varphi_{\mathbf{k}'} \mathbf{k}' \cdot \frac{\partial \delta f_{\mathbf{k}-\mathbf{k}'}}{\partial \mathbf{v}} \equiv N_{\mathbf{k}}(\mathbf{v}, t), \quad (12.5)$$

where  $\varphi_{\mathbf{k}}$  now contains both the self-consistent field arising from  $\delta f_{\mathbf{k}}$  via Poisson’s equation (2.9) and an externally applied field  $\chi_{\mathbf{k}}$  [cf. Q7]:

$$\varphi_{\mathbf{k}} = \chi_{\mathbf{k}} - \frac{4\pi e}{k^2} \int d\mathbf{v} \delta f_{\mathbf{k}}. \quad (12.6)$$

We shall solve an initial-value problem with  $\delta f_{\mathbf{k}}(t=0)=0$  and use  $\chi_{\mathbf{k}}$  to set up our two pulses:

$$\chi(t, \mathbf{r}) = A_1 \delta(t - t_1) \cos k_1 z + A_2 \delta(t - t_2) \cos k_2 z, \quad (12.7)$$

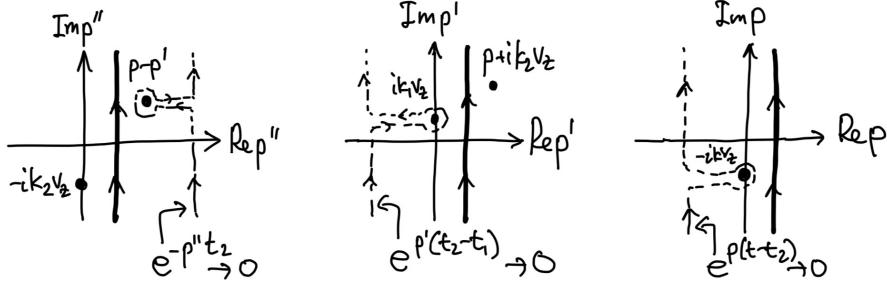


FIGURE 48. Integration contours for the integrals in (12.14) and (12.15).

where  $z$  is the axis along which they propagate and  $t_2 > t_1 > 0$ ,  $k_2 > k_1$  (the case of  $A_2 = 0$ ,  $t_1 = 0$ , and  $N_{\mathbf{k}} = 0$  is exactly equivalent to the standard Landau-damping problem solved in §3).

Laplace transforming everything, we get

$$\hat{\delta f}_{\mathbf{k}}(p) = \frac{1}{p + i\mathbf{k} \cdot \mathbf{v}} \left[ -i \frac{e}{m} \hat{\varphi}_{\mathbf{k}}(p) \mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{v}} + \hat{N}_{\mathbf{k}}(p) \right]. \quad (12.8)$$

This is not really a solution, just a rewriting of (12.5), because the Laplace-transformed nonlinear term  $\hat{N}_{\mathbf{k}}(p)$  contains  $\delta f_{\mathbf{k}}$ :

$$\begin{aligned} \hat{N}_{\mathbf{k}}(p) &= -i \frac{e}{m} \sum_{\mathbf{k}'} \mathbf{k}' \cdot \frac{\partial}{\partial \mathbf{v}} \int_0^\infty dt e^{-pt} \int_{-i\infty+\sigma'}^{+i\infty+\sigma'} \int_{-i\infty+\sigma''}^{+i\infty+\sigma''} \frac{dp' dp''}{(2\pi i)^2} e^{(p'+p'')t} \hat{\varphi}_{\mathbf{k}'}(p') \hat{\delta f}_{\mathbf{k}-\mathbf{k}'}(p'') \\ &= i \frac{e}{m} \sum_{\mathbf{k}'} \mathbf{k}' \cdot \frac{\partial}{\partial \mathbf{v}} \iint \frac{dp' dp''}{(2\pi i)^2} \frac{\hat{\varphi}_{\mathbf{k}'}(p') \hat{\delta f}_{\mathbf{k}-\mathbf{k}'}(p'')}{p' + p'' - p}, \quad \text{Re } p > \sigma' + \sigma'', \end{aligned} \quad (12.9)$$

where I have dropped, but will remember, the integration limits. Keeping this expression in mind, let us combine (12.6) and (12.8):

$$\hat{\varphi}_{\mathbf{k}}(p) = \frac{\hat{\chi}_{\mathbf{k}}(p)}{\epsilon(p, \mathbf{k})} - \frac{4\pi e}{k^2 \epsilon(p, \mathbf{k})} \int d\mathbf{v} \frac{\hat{N}_{\mathbf{k}}(p)}{p + i\mathbf{k} \cdot \mathbf{v}}, \quad (12.10)$$

where  $\epsilon(p, \mathbf{k})$  is the usual dielectric function, most recently written out in (11.12). The first term in (12.10) is the linear response [cf. (6.59) in Q7], while the second one is the nonlinear one. Assuming the latter to be small, we can calculate it perturbatively, i.e., by using the linear approximation to  $\hat{\varphi}$  and  $\hat{\delta f}$  in (12.9). Namely,

$$\begin{aligned} \hat{N}_{\mathbf{k}}(p) &= \frac{e^2}{m^2} \sum_{\mathbf{k}'} \mathbf{k}' \cdot \frac{\partial}{\partial \mathbf{v}} \iint \frac{dp' dp''}{(2\pi i)^2} \frac{\hat{\chi}_{\mathbf{k}'}(p') \hat{\chi}_{\mathbf{k}-\mathbf{k}'}(p'')}{\epsilon(p', \mathbf{k}') \epsilon(p'', \mathbf{k} - \mathbf{k}') [p'' + i(\mathbf{k} - \mathbf{k}') \cdot \mathbf{v}] (p' + p'' - p)} \\ &\quad \times (\mathbf{k} - \mathbf{k}') \cdot \frac{\partial f_0}{\partial \mathbf{v}}. \end{aligned} \quad (12.11)$$

This is as far as we can get without inputting specific information about  $\hat{\chi}_{\mathbf{k}}(p)$ , which we shall need in order to do the  $p'$  and  $p''$  integrals. From (12.7),

$$\hat{\chi}_{\mathbf{k}}(p) = \frac{A_1}{2} e^{-pt_1} (\delta_{\mathbf{k}, \mathbf{k}_1} + \delta_{\mathbf{k}, -\mathbf{k}_1}) + \frac{A_2}{2} e^{-pt_2} (\delta_{\mathbf{k}, \mathbf{k}_2} + \delta_{\mathbf{k}, -\mathbf{k}_2}). \quad (12.12)$$

There will be a lot of terms in  $\hat{\chi}_{\mathbf{k}'}(p') \hat{\chi}_{\mathbf{k}-\mathbf{k}'}(p'')$ , but we only want the ones that are proportional to  $\delta_{\mathbf{k}, \mathbf{k}_2 - \mathbf{k}_1}$  (it is a tedious, but character-building **exercise** to confirm that

there are no echoes in the rest of them). They are

$$\frac{A_1 A_2}{4} \delta_{\mathbf{k}, \mathbf{k}_2 - \mathbf{k}_1} \left( e^{-p' t_1 - p'' t_2} \delta_{\mathbf{k}', -\mathbf{k}_1} + e^{-p' t_2 - p'' t_1} \delta_{\mathbf{k}', \mathbf{k}_2} \right). \quad (12.13)$$

Putting this into (12.11) and swapping integration variables  $p' \leftrightarrow p''$  in the second term, we get, for  $\mathbf{k} = \mathbf{k}_2 - \mathbf{k}_1$ ,

$$\begin{aligned} \hat{N}_{\mathbf{k}}(p) &= -\frac{e^2 A_1 A_2 k_1 k_2}{4m^2} \frac{\partial}{\partial v_z} \iint \frac{dp' dp''}{(2\pi i)^2} \frac{e^{-p' t_1 - p'' t_2}}{\epsilon(p', -k_1) \epsilon(p'', k_2) (p' + p'' - p)} \\ &\quad \times \left( \frac{1}{p'' + ik_2 v_z} + \frac{1}{p' - ik_1 v_z} \right) \frac{\partial f_0}{\partial v_z} \\ &= \frac{e^2 A_1 A_2 k_1 k_2}{4m^2} \frac{\partial}{\partial v_z} \int \frac{dp'}{2\pi i} \frac{e^{p'(t_2 - t_1) - p t_2} (p + ikv_z)}{\epsilon(p', -k_1) \epsilon(p - p', k_2) (p - p' + ik_2 v_z) (p' - ik_1 v_z)} \frac{\partial f_0}{\partial v_z} \\ &= \frac{e^2 A_1 A_2 k_1 k_2}{4m^2} \frac{\partial}{\partial v_z} \frac{e^{ik_1 v_z (t_2 - t_1) - p t_2}}{\epsilon(ik_1 v_z, -k_1) \epsilon(p - ik_1 v_z, k_2)} \frac{\partial f_0}{\partial v_z}. \end{aligned} \quad (12.14)$$

In the last two lines, I did the the  $p''$  and  $p'$  integrals, in that order. The  $p''$  integral is done by pushing the integration contour rightwards to  $\text{Re } p'' \rightarrow \infty$ , but circumventing the pole at  $p'' = p - p'$  clockwise; only the pole's contribution survives because  $e^{-p'' t_2} \rightarrow 0$  at  $\text{Re } p'' \rightarrow \infty$  (Fig. 48). The  $p'$  integral is done by pushing the contour leftwards to  $\text{Re } p' \rightarrow -\infty$ , but circumventing the ballistic pole at  $p' = ik_1 v_z$  anticlockwise; again only the pole's contribution survives because  $e^{p'(t_2 - t_1)} \rightarrow 0$  at  $\text{Re } p' \rightarrow -\infty$  (Fig. 48). The dielectric function is assumed to have no poles at  $\text{Re } p' > 0$  (i.e., there is only damping, no instabilities) and the Landau-damping poles it might have at  $\text{Re } p' < 0$  are ignored because they will give rise to damped solutions, not echoes.

Finally, let us put (12.14) into (12.10), integrate out  $v_x$  and  $v_y$  dependence, so  $f_0(\mathbf{v}) \rightarrow F(v_z)$  [cf. (3.18)], and inverse Laplace transform, to get, for the echo (with  $\mathbf{k} = \mathbf{k}_2 - \mathbf{k}_1$ ),

$$\begin{aligned} \varphi_{\mathbf{k}}^{\text{echo}}(t) &= -\frac{4\pi e}{k^2} \int dv_z \int \frac{dp}{2\pi i} \frac{e^{pt} \hat{N}_{\mathbf{k}}(p)}{\epsilon(p, k) (p + ikv_z)} \\ &= -i \frac{\pi e^3 A_1 A_2 k_1 k_2}{m^2 k} \int dv_z \int \frac{dp}{2\pi i} \frac{e^{p(t-t_2) + ik_1 v_z (t_2 - t_1)}}{\epsilon(p, k) \epsilon(ik_1 v_z, -k_1) \epsilon(p - ik_1 v_z, k_2) (p + ikv_z)^2} \frac{\partial F}{\partial v_z} \\ &= -i(t - t_2) \frac{\pi e^3 A_1 A_2 k_1 k_2}{m^2 k} \int dv_z \frac{e^{-i[k(t-t_2) - k_1(t_2-t_1)]v_z}}{\epsilon(-ikv_z, k) \epsilon(ik_1 v_z, -k_1) \epsilon(-ik_2 v_z, k_2)} \frac{\partial F}{\partial v_z}. \end{aligned} \quad (12.15)$$

In the second line, I integrated by parts with respect to  $v_z$ , producing a second-order ballistic pole at  $p = -ikv_z$ , and in the third line, I profited from this pole by pushing the  $p$ -integration contour leftwards to  $\text{Re } p \rightarrow -\infty$ , using the fact that  $e^{p(t-t_2)} \rightarrow 0$  there, for  $t > t_2$ , and thus being left only with the contribution from the pole, circumnavigated anticlockwise (Fig. 48). Note that, since the pole is second order, I had to differentiate the expression under the integral with respect to  $p$  to obtain the residue—in doing so, I ignored the derivatives of the dielectric functions in comparison with the derivative of  $e^{p(t-t_2)}$ , because the latter was large in  $t - t_2$ . We see that the exponential in (12.15) that oscillates in  $v_z$  does not in fact oscillate very much around precisely the time  $t = t_{\text{echo}}$  given by (12.4)—this is the effect that we were after!

There is a nice, simple answer in the limit of  $k\lambda_{De} \gg 1$ , when all dielectric functions in (12.15) can be approximated by unity (and Landau damping is strong). Assuming  $F(v_z)$  is a Maxwellian, we get

$$\varphi_{\mathbf{k}}^{\text{echo}}(t) \approx (t - t_{\text{echo}})(t - t_2) \frac{\pi e^3 A_1 A_2 k_1 k_2}{m^2} e^{-k^2 v_{\text{th}}^2 (t - t_{\text{echo}})^2 / 4}. \quad (12.16)$$

Thus, the echo pulse arises at  $t = t_{\text{echo}}$ , and then phase-mixes away again.

**Exercise 12.1. Plasma echo for weakly damped waves.** Work out  $\varphi_{\mathbf{k}}^{\text{echo}}(t)$  in the opposite limit,  $k\lambda_{\text{De}} \ll 1$ , when the Landau-damping rates are exponentially small and the  $v_z$  integral in (12.15) can be done by picking up the poles associated with the zeros of the dielectric functions in the denominator. Show in particular that the echo pulse first grows exponentially at the rate  $\gamma_{\mathbf{k}_1}(k_2 - k_1)/k_1$  and then decays at the rate  $\gamma_{\mathbf{k}_2 - \mathbf{k}_1}$ , where  $\gamma_{\mathbf{k}}$  is the Landau-damping rate at the wave number  $\mathbf{k}$ . You will find the solution in [Gould et al. \(1967\)](#).

## 12.2. Phase-Space Turbulence and Stochastic Echo

I shall now show you how perturbations being dragged back from phase space by nonlinear interactions can be described formally in a fairly transparent fashion. In the process, we will learn a new way of looking at phase mixing—and unmixing—and gain some insight into the nature of “phase-space turbulence”.

At the beginning of §11.1, we saw that a crucial role in the “collisionless collision integrals” was played by the correlation function (11.4) of the perturbed distribution function,  $C_{\mathbf{k}}(\mathbf{v}, \mathbf{v}') = \langle \delta f_{\mathbf{k}}(\mathbf{v}) \delta f_{\mathbf{k}}^*(\mathbf{v}') \rangle$ . Here I shall be interested in this function both with a view of deriving a better “collisionless collision integral” than I managed in §11 and because it is interesting in itself, as a descriptor of phase-space turbulence. Phase mixing should manifest itself as  $C_{\mathbf{k}}(\mathbf{v}, \mathbf{v}')$  developing ever smaller-scale dependence on  $\mathbf{v} - \mathbf{v}'$ , and phase unmixing the opposite.

Since

$$\frac{\partial C_{\mathbf{k}}(\mathbf{v}, \mathbf{v}')}{\partial t} = \left\langle \delta f_{\mathbf{k}}^*(\mathbf{v}') \frac{\partial \delta f_{\mathbf{k}}(\mathbf{v})}{\partial t} + \delta f_{\mathbf{k}}(\mathbf{v}) \frac{\partial \delta f_{\mathbf{k}}^*(\mathbf{v}')}{\partial t} \right\rangle, \quad (12.17)$$

to derive the evolution equation for  $C_{\mathbf{k}}(\mathbf{v}, \mathbf{v}')$ , one must multiply (12.5) by  $\delta f_{\mathbf{k}}^*(\mathbf{v}')$  and add to the resulting equation its complex conjugate with  $\mathbf{v}$  and  $\mathbf{v}'$  swapped:

$$\begin{aligned} \frac{\partial C_{\mathbf{k}}(\mathbf{v}, \mathbf{v}')}{\partial t} + i\mathbf{k} \cdot (\mathbf{v} - \mathbf{v}') C_{\mathbf{k}}(\mathbf{v}, \mathbf{v}') &= -i \frac{e}{m} \left[ \langle \varphi_{\mathbf{k}} \delta f_{\mathbf{k}}^*(\mathbf{v}') \rangle \mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{v}} \right. \\ &\quad \left. + \sum_{\mathbf{k}'} \mathbf{k}' \cdot \frac{\partial}{\partial \mathbf{v}} \langle \varphi_{\mathbf{k}'} \delta f_{\mathbf{k}-\mathbf{k}'}(\mathbf{v}) \delta f_{\mathbf{k}}^*(\mathbf{v}') \rangle \right] + (\mathbf{v} \leftrightarrow \mathbf{v}')^* \\ &\equiv S_{\mathbf{k}}(\mathbf{v}, \mathbf{v}') + N_{\mathbf{k}}(\mathbf{v}, \mathbf{v}'), \end{aligned} \quad (12.18)$$

where  $S_{\mathbf{k}}(\mathbf{v}, \mathbf{v}')$  is the “source” term containing the second-order correlators and  $N_{\mathbf{k}}(\mathbf{v}, \mathbf{v}')$  is the nonlinear term containing the third-order ones.

### 12.2.1. Hacking the Closure Problem

With third-order correlators appearing in the evolution equation for the second-order one, we are faced with the usual “closure problem” of nonlinear theory (cf. §8.4.1). In order to make the points I wish to make, I am going to have to ram through some rather brutal simplifications. Namely, I will treat  $\varphi_{\mathbf{k}}$  as a known field, i.e., ignore any self-consistent contribution that  $\delta f_{\mathbf{k}}$  makes to it. This can be interpreted as approximating  $\varphi_{\mathbf{k}} \approx \chi_{\mathbf{k}}$  and ignoring the second term in the “forced Poisson law” (12.6), or as using just the linear approximation in (12.10), or as hoping, non-rigorously, that the effect of correlations between  $\delta f_{\mathbf{k}}(\mathbf{v})$  under the velocity integral in (12.6) and  $\delta f$ 's appearing in the right-hand side of (12.18) can be ignored. Furthermore, I will assume that  $\varphi_{\mathbf{k}}(t)$  is a random and rapidly decorrelating field—more rapidly than  $\delta f$  evolves either linearly or nonlinearly. This is obviously not a great assumption unless we are literally imposing a white-noise field onto our system, but it will have to do if we want some kind of closure.

Just like with the random-phase approximation in §8.4.2, the microgranulation ansatz in §11.3.2, or the *Stosszahlansatz* in the standard particle-collision theory (Parra 2019a; Kunz 2021), one needs to break long-term/long-range correlations in some way to get closed equations.

Thus, the proposed assumption, known in different contexts (Falkovich *et al.* 2001; Rincon 2019) as the Kazantsev–Kraichnan model, or a “short-sudden approximation”, is [cf. (6.62) in Q7]

$$\boxed{\frac{e^2}{m^2} \langle \varphi_{\mathbf{k}}(t) \varphi_{\mathbf{k}'}^*(t') \rangle = 2D_k \delta_{\mathbf{k}, \mathbf{k}'} \delta(t - t')} , \quad (12.19)$$

where  $D_k$  is a known function.<sup>98</sup> The correlators in the right-hand side of the  $C_{\mathbf{k}}(\mathbf{v}, \mathbf{v}')$ -evolution equation (12.18) can then be computed as follows. Let us formally integrate the kinetic equation (12.5) up to time  $t$ :

$$\delta f_{\mathbf{k}}(t) = -i \int_{\text{past}}^t dt' \left\{ \mathbf{k} \cdot \mathbf{v} \delta f_{\mathbf{k}}(t') + \frac{e}{m} \left[ \varphi_{\mathbf{k}}(t') \mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{v}} + \sum_{\mathbf{k}'} \varphi_{\mathbf{k}'}(t') \mathbf{k}' \cdot \frac{\partial \delta f_{\mathbf{k}-\mathbf{k}'}(t')}{\partial \mathbf{v}} \right] \right\}. \quad (12.20)$$

It does not matter from when in the past we integrate, because only a very short period immediately preceding  $t$  will contribute to correlations between  $\delta f(t)$  and  $\varphi(t)$ . Let us now substitute (12.20) into the right-hand side of (12.18). Since  $\delta f(t)$  can only depend on the values of  $\varphi(t')$  in the past,  $t' < t$ , we may split any correlators between  $\delta f(t')$  and  $\varphi(t')$  or  $\varphi(t)$ . Namely:

$$\begin{aligned} -i \frac{e}{m} \langle \varphi_{\mathbf{k}}(t) \delta f_{\mathbf{k}}^*(t, \mathbf{v}') \rangle &= \frac{e}{m} \int^t dt' \left\{ \mathbf{k} \cdot \mathbf{v} \langle \varphi_{\mathbf{k}}(t) \rangle \langle \delta f_{\mathbf{k}}^*(t', \mathbf{v}') \rangle \right. \\ &\quad + \frac{e}{m} \left[ \underbrace{\langle \varphi_{\mathbf{k}}(t) \varphi_{\mathbf{k}'}^*(t') \rangle}_{\text{use (12.19)}} \mathbf{k} \cdot \frac{\partial f_0(\mathbf{v}')}{\partial \mathbf{v}'} \right. \\ &\quad \left. \left. + \sum_{\mathbf{k}'} \langle \varphi_{\mathbf{k}}(t) \varphi_{\mathbf{k}'}^*(t') \rangle \mathbf{k}' \cdot \frac{\partial \langle \delta f_{\mathbf{k}-\mathbf{k}'}^*(t', \mathbf{v}') \rangle}{\partial \mathbf{v}'} \right] \right\} \\ &= D_k \mathbf{k} \cdot \frac{\partial f_0(\mathbf{v}')}{\partial \mathbf{v}'} , \end{aligned} \quad (12.21)$$

so the source term in the  $C_{\mathbf{k}}(\mathbf{v}, \mathbf{v}')$ -evolution equation (12.18) is

$$S_{\mathbf{k}}(\mathbf{v}, \mathbf{v}') = 2D_k \mathbf{k} \mathbf{k} : \frac{\partial f_0(\mathbf{v})}{\partial \mathbf{v}} \frac{\partial f_0(\mathbf{v}')}{\partial \mathbf{v}'} . \quad (12.22)$$

In the same vein, to calculate  $N_{\mathbf{k}}(\mathbf{v}, \mathbf{v}')$ , we may construct an evolution equation for  $\delta f_{\mathbf{k}-\mathbf{k}'}(t, \mathbf{v}) \delta f_{\mathbf{k}}^*(t, \mathbf{v}')$  and formally integrate it:

$$\begin{aligned} \delta f_{\mathbf{k}-\mathbf{k}'}(t, \mathbf{v}) \delta f_{\mathbf{k}}^*(t, \mathbf{v}') &= -i \int^t dt' \left\{ \frac{e}{m} \sum_{\mathbf{k}''} \left[ \varphi_{\mathbf{k}''}(t') \mathbf{k}'' \cdot \frac{\partial \delta f_{\mathbf{k}-\mathbf{k}''}(t', \mathbf{v})}{\partial \mathbf{v}} \delta f_{\mathbf{k}}^*(t', \mathbf{v}') \right. \right. \\ &\quad \left. \left. - \varphi_{\mathbf{k}''}^*(t') \mathbf{k}'' \cdot \frac{\partial \delta f_{\mathbf{k}-\mathbf{k}''}^*(t', \mathbf{v}')}{\partial \mathbf{v}'} \delta f_{\mathbf{k}-\mathbf{k}'}(t', \mathbf{v}) \right] + \dots \right\} . \end{aligned} \quad (12.23)$$

<sup>98</sup>Here  $k = |\mathbf{k}|$ , so statistical isotropy is assumed, as is statistical spatial homogeneity:  $\delta_{\mathbf{k}, \mathbf{k}'}$  in (12.19) enforces the latter, viz., the requirement that  $\langle \varphi(\mathbf{r}, t) \varphi(\mathbf{r}', t') \rangle$  is a function only of  $\mathbf{r} - \mathbf{r}'$  (it is a simple **exercise** to confirm this).

The terms hiding behind “...” will not matter because all odd-order correlators of  $\varphi$  vanish when we calculate

$$\begin{aligned} & -i\frac{e}{m}\langle\varphi_{\mathbf{k}'}(t)\delta f_{\mathbf{k}-\mathbf{k}'}(t, \mathbf{v})\delta f_{\mathbf{k}}^*(t, \mathbf{v}')\rangle \\ &= D_{k'}\left[\mathbf{k}'\cdot\frac{\partial}{\partial\mathbf{v}}\langle\delta f_{\mathbf{k}}(t, \mathbf{v})\delta f_{\mathbf{k}}^*(t, \mathbf{v}')\rangle + \mathbf{k}'\cdot\frac{\partial}{\partial\mathbf{v}'}\langle\delta f_{\mathbf{k}-\mathbf{k}'}^*(t, \mathbf{v}')\delta f_{\mathbf{k}-\mathbf{k}'}(t, \mathbf{v})\rangle\right] \\ &= D_{k'}\mathbf{k}'\cdot\left[\frac{\partial C_{\mathbf{k}}(\mathbf{v}, \mathbf{v}')}{\partial\mathbf{v}} + \frac{\partial C_{\mathbf{k}-\mathbf{k}'}(\mathbf{v}, \mathbf{v}')}{\partial\mathbf{v}'}\right]. \end{aligned} \quad (12.24)$$

This has worked out this way because, when the model  $\varphi_{\mathbf{k}}$  correlator (12.19) was used,  $\mathbf{k}''$  in the first term in (12.23) was set to  $-\mathbf{k}'$  and  $\mathbf{k}''$  in the second term to  $\mathbf{k}'$ . Finally, since  $C_{\mathbf{k}}^*(\mathbf{v}', \mathbf{v}) = C_{\mathbf{k}}(\mathbf{v}, \mathbf{v}')$ , the nonlinear term in the  $C_{\mathbf{k}}(\mathbf{v}, \mathbf{v}')$ -evolution equation (12.18) becomes

$$\begin{aligned} N_{\mathbf{k}}(\mathbf{v}, \mathbf{v}') &= \sum_{\mathbf{k}'} D_{k'}\mathbf{k}'\mathbf{k}'\cdot\left(2\frac{\partial^2 C_{\mathbf{k}-\mathbf{k}'}}{\partial\mathbf{v}\partial\mathbf{v}'} + \frac{\partial^2 C_{\mathbf{k}}}{\partial\mathbf{v}\partial\mathbf{v}'} + \frac{\partial^2 C_{\mathbf{k}}}{\partial\mathbf{v}'\partial\mathbf{v}'}\right) \\ &= D\left(\frac{\partial}{\partial\mathbf{v}} + \frac{\partial}{\partial\mathbf{v}'}\right)^2 C_{\mathbf{k}} + 2\sum_{\mathbf{k}'} D_{k'}\mathbf{k}'\mathbf{k}'\cdot\frac{\partial^2}{\partial\mathbf{v}\partial\mathbf{v}'}(C_{\mathbf{k}-\mathbf{k}'} - C_{\mathbf{k}}), \end{aligned} \quad (12.25)$$

where  $D = \sum_{\mathbf{k}'} k'^2 D_{k'}/3$  is the QL diffusion coefficient (cf. Q11).

We have paid a heavy price both in dodgy assumptions and in algebra, but the evolution equation (12.18) for the phase-space correlation function  $C_{\mathbf{k}}(\mathbf{v}, \mathbf{v}')$  is now closed.

**Exercise 12.2. Furutsu–Novikov formula.** At least one element of dodgyness can, in fact, be removed. The average-splitting scheme that I employed above is, more formally, based on the following mathematical formula, due to [Furutsu \(1963\)](#) and [Novikov \(1965\)](#). If  $\varphi(q)$  is a Gaussian random field that depends on variable(s)  $q$  (which can include time, space, wave number, vector index, etc.) and  $F[\varphi]$  is a functional of  $\varphi$ , then their correlation function can be expressed as a convolution of the second-order correlator of  $\varphi$  and the functional derivative of  $F$  with respect to  $\varphi$ :

$$\boxed{\langle\varphi(q)F[\varphi]\rangle = \int dq'\langle\varphi(q)\varphi(q')\rangle\left\langle\frac{\delta F[\varphi]}{\delta\varphi(q')}\right\rangle}. \quad (12.26)$$

Convince yourself that the results (12.22) and (12.25) can be derived from (12.20) using this formula. If you are unsure how to handle functional derivatives, here are three basic rules:

$$\frac{\delta\varphi(q)}{\delta\varphi(q')} = \delta(q - q'), \quad \frac{\delta}{\delta\varphi(q)} \int dq'\varphi(q')K(q') = K(q), \quad \frac{\delta}{\delta\varphi}FG = \frac{\delta F}{\delta\varphi}G + F\frac{\delta G}{\delta\varphi}. \quad (12.27)$$

Note also that  $\delta F/\delta\varphi(q) = 0$  if  $F$  does not depend on the value of  $\varphi$  at  $q$ , e.g., if  $q$  is time,  $F$  is some function that only depends on the past history of  $\varphi$  (as  $\delta f$  does in the calculation above), and we are trying to take a functional derivative of it with respect to  $\varphi$  taken at a future moment.

**Exercise 12.3. Two-time correlation function.** Consider the two-time correlation function

$$\mathcal{C}_{\mathbf{k}}(t - t', \mathbf{v}, \mathbf{v}') = \langle\delta f_{\mathbf{k}}(t, \mathbf{v})\delta f_{\mathbf{k}}^*(t', \mathbf{v}')\rangle, \quad (12.28)$$

where, without loss of generality,  $t > t'$ , and, in a statistical steady state,  $\mathcal{C}_{\mathbf{k}}$  depends only on the time delay  $\tau = t - t'$ , rather than separately on  $t$  and  $t'$ . Using the same methods, assumptions and approximations as above, derive the following equation:

$$\frac{\partial\mathcal{C}_{\mathbf{k}}}{\partial\tau} + i\mathbf{k}\cdot\mathbf{v}\mathcal{C}_{\mathbf{k}} = D\frac{\partial^2\mathcal{C}_{\mathbf{k}}}{\partial\mathbf{v}^2}. \quad (12.29)$$

Solve this equation to show that

$$\boxed{\mathcal{C}_k(\tau, \mathbf{v}, \mathbf{v}') = \int d\mathbf{v}'' \frac{1}{(4\pi D\tau)^{3/2}} \exp\left(-\frac{|\mathbf{v} - \mathbf{v}''|^2}{4D\tau} - i\mathbf{k} \cdot \frac{\mathbf{v} + \mathbf{v}''}{2}\tau - \frac{Dk^2\tau^3}{12}\right) C_k(\mathbf{v}'', \mathbf{v}')}, \quad (12.30)$$

where  $C_k$  is the one-time correlation function (11.4). What is, therefore, the correlation time of  $\delta f_k$ ?

---

### 12.2.2. Phase Mixing and Unmixing

Let us now discuss what the  $C_k(\mathbf{v}, \mathbf{v}')$ -evolution equation (12.18), with (12.22) for  $S_k$  and (12.25) for  $N_k$ , tells us. For a moment, let us ignore  $N_k$  and solve the linear equation

$$\frac{\partial C_k}{\partial t} + i\mathbf{k} \cdot (\mathbf{v} - \mathbf{v}') C_k = S_k. \quad (12.31)$$

The solution is

$$C_k = S_k \frac{1 - e^{-ik \cdot (\mathbf{v} - \mathbf{v}') t}}{i\mathbf{k} \cdot (\mathbf{v} - \mathbf{v}')} \rightarrow S_k \pi \delta(\mathbf{k} \cdot (\mathbf{v} - \mathbf{v}')) \quad \text{as } t \rightarrow \infty, \quad (12.32)$$

with the  $\delta$ -function limit derived in the same way as it was in (5.39). This tells us that linear physics wants to make  $C_k(\mathbf{v}, \mathbf{v}')$  very fine scaled in phase space, with  $\delta f(\mathbf{v})$  taken at very close neighbouring values of  $\mathbf{v}$  becoming decorrelated [cf. the microgranulation ansatz (11.5)]. This is a manifestation of a familiar phenomenon—phase mixing.

It is then natural to change variables as follows:

$$(\mathbf{v}, \mathbf{v}') \rightarrow (\mathbf{u}, \mathbf{w}), \quad \text{where } \mathbf{u} = \frac{\mathbf{v} + \mathbf{v}'}{2}, \quad \mathbf{w} = \mathbf{v} - \mathbf{v}'. \quad (12.33)$$

In these new variables, phase mixing will look like growth of  $\partial/\partial\mathbf{w}$ , whereas the  $\mathbf{u}$  dependence of the correlation function will just capture the large-scale, inhomogeneous structure of the velocity space—basically,  $f_0$  having a scale  $\sim v_{\text{th}}$ . As a matter of fact, this scale will tend to increase with time via what is recognisably QL diffusion: for  $f_0$ , the version of QLT with an external field was the subject of Q11, while for  $C_k$ , the first term in (12.25) is obviously just diffusion in  $\mathbf{u}$ . Thus, let us assume that  $\partial/\partial\mathbf{w} \gg \partial/\partial\mathbf{u}$  and write out (12.18) with (12.22) and (12.25) in this approximation:

$$\frac{\partial C_k}{\partial t} + i\mathbf{k} \cdot \mathbf{w} C_k \approx S_k + D \frac{\partial^2 C_k}{\partial \mathbf{u}^2} - 2 \sum_{\mathbf{k}'} D_{\mathbf{k}'} \left( \mathbf{k}' \cdot \frac{\partial}{\partial \mathbf{w}} \right)^2 (C_{\mathbf{k}-\mathbf{k}'} - C_k). \quad (12.34)$$

Since our objective is to characterise the emergence, or otherwise, of the small-scale structure in  $\mathbf{w}$ , let us make another formal step and Fourier transform in this variable:

$$C_{\mathbf{k},\mathbf{s}}(\mathbf{u}) = \int \frac{d\mathbf{w}}{(2\pi)^3} C_k(\mathbf{u}, \mathbf{w}) e^{i\mathbf{s} \cdot \mathbf{w}}. \quad (12.35)$$

This “mixed-variable” object, containing information both about the large-scale real-space inhomogeneity and a Fourier-space representation of the small-scale structure, is known in quantum mechanics (and in signal processing) as the *Wigner function* (it already appeared, in a different context, in §8.5). It satisfies

$$\boxed{\frac{\partial C_{\mathbf{k},\mathbf{s}}}{\partial t} + \mathbf{k} \cdot \frac{\partial C_{\mathbf{k},\mathbf{s}}}{\partial \mathbf{s}} \approx S_k(\mathbf{u}, \mathbf{u}) \delta(\mathbf{s}) + D \frac{\partial^2 C_{\mathbf{k},\mathbf{s}}}{\partial \mathbf{u}^2} + 2 \sum_{\mathbf{k}'} D_{\mathbf{k}'} (\mathbf{k}' \cdot \mathbf{s})^2 (C_{\mathbf{k}-\mathbf{k}',\mathbf{s}} - C_{\mathbf{k},\mathbf{s}})}. \quad (12.36)$$

A further approximation deployed above is

$$S_{\mathbf{k}}(\mathbf{v}, \mathbf{v}') \approx S_{\mathbf{k}}(\mathbf{u}, \mathbf{u}) = 2D_k \left[ \mathbf{k} \cdot \frac{\partial f_0(\mathbf{u})}{\partial \mathbf{u}} \right]^2, \quad (12.37)$$

because  $\mathbf{w}$  is assumed small while  $S_{\mathbf{k}}$  only contains large-scale velocity dependence, via  $f_0$ .

The new equation (12.36) gives us a transparent way to see phase mixing: the second term on the left-hand side is just a propagation term in  $s$  space, at “velocity”  $\mathbf{k}$ , while the source ( $S_{\mathbf{k}}$ ) term on the right is a “boundary condition” at small  $s$ . Perturbations seeded by this source travel to larger values of  $s$  provided  $\mathbf{k} \cdot \mathbf{s} > 0$ —this is phase mixing. In principle,  $\mathbf{k} \cdot \mathbf{s} < 0$  would take perturbations from larger to smaller  $s$ —phase unmixing!—but there are no sources at large  $s$  within linear theory. The nonlinear game changer is the last, mode-coupling term on the right-hand side of (12.36): it can easily couple a phase-mixing perturbation with  $(\mathbf{k} - \mathbf{k}') \cdot \mathbf{s} > 0$  to a phase-unmixing one with  $\mathbf{k} \cdot \mathbf{s} < 0$  and thus send a pulse back to low  $s$ . This is a stochastic version of the echo effect: perturbations chase each other into phase space and can turn around if they couple in the right way.<sup>99</sup>

Will they couple in the right way? Even though (12.36) is a closed equation, it is not so easy to solve, and it is also unknown how well it describes, even qualitatively, the systems with a self-consistent electric field. We do know some solutions of it, in 1D and with further simplifying assumptions (Adkins & Schekochihin 2018; Nastac *et al.* 2022)—on this basis, the tentative answer is yes, phase unmixing can indeed change things in a dramatic way. I will show the simplest of these solutions in §12.3.

**Exercise 12.4. Multispecies phase-space turbulence.** Generalise the above developments to the case of multiple species, i.e., derive closed evolution equations for

$$C_{\mathbf{k}\alpha\alpha'}(\mathbf{v}, \mathbf{v}') = \langle \delta f_{\mathbf{k}\alpha}(t, \mathbf{v}) \delta f_{\mathbf{k}\alpha'}^*(t, \mathbf{v}') \rangle, \quad (12.38)$$

$$\mathcal{C}_{\mathbf{k}\alpha\alpha'}(t - t', \mathbf{v}, \mathbf{v}') = \langle \delta f_{\mathbf{k}\alpha}(t, \mathbf{v}) \delta f_{\mathbf{k}\alpha'}^*(t', \mathbf{v}') \rangle, \quad t > t'. \quad (12.39)$$

What is the new version of the  $(\mathbf{u}, \mathbf{w})$  variables?

The equation that you will have derived for  $\mathcal{C}_{\mathbf{k}\alpha\alpha'}(\tau)$  is identical to (12.29) and is solved in the same way. The equation for the one-time function  $C_{\mathbf{k}\alpha\alpha'}$  has not been properly analysed, so it is not known what interesting effects might lurk there, if any. If you investigate, you might find something new.

### 12.3. Adkins’ Solution for Phase-Space Advection in 1D

To turn (12.36) into something solvable, we make three further simplifications:

(i) restrict everything to 1D;

(ii) ignore the  $u$ -space diffusion term on the grounds that  $\partial/\partial u \ll s$ ;

(iii) assume that the spectrum of the electric perturbations,  $D_{k'}$ , is concentrated at larger scales than  $C_{k,s}$ , i.e.,  $k' \ll k$  (in the context of passive-scalar advection, this is known as the Batchelor 1959 regime).

<sup>99</sup>In Schekochihin *et al.* (2016) and Adkins & Schekochihin (2018), you will find two versions of this argument set up in terms of Hermite, rather than Fourier, transforms, which are, of course, equivalent at large Hermite order and large  $s$  (see Q8 for a quick tutorial). In the Hermite formalism, there exists a neat decomposition of the distribution function into phase-mixing and phase-unmixing parts, with the nonlinear term represented as their coupling, but I now nevertheless prefer the Fourier language because it does not shackle one to a Maxwellian  $f_0$ .

The last of these allows us to approximate

$$C_{k-k',s} - C_{k,s} \approx -k' \frac{\partial C_{k,s}}{\partial k} + \frac{k'^2}{2} \frac{\partial^2 C_{k,s}}{\partial k^2}, \quad (12.40)$$

i.e., the coupling between  $k$ 's now happens in small local steps, diffusively, rather than in finite leaps. The first term in (12.40) vanishes under the  $k'$  integration in (12.36), and we arrive at the following rather cute equation:

$$\cancel{\frac{\partial C_{k,s}}{\partial t}} + k \frac{\partial C_{k,s}}{\partial s} = S_k \delta(s) + \gamma s^2 \frac{\partial^2 C_{k,s}}{\partial k^2}, \quad \gamma = \sum_{k'} k'^4 D_{k'}, \quad (12.41)$$

where we shall drop the time derivative, in pursuit of a steady-state solution.

What this equation does is quite easy to see qualitatively. There are two competing effects: phase (un)mixing, i.e., propagation in  $s$  at velocity  $k$ , and mode coupling, which in the Batchelor regime takes the form of  $k$ -space diffusion, with diffusivity  $\gamma s^2$ . At lower  $s$ , phase (un)mixing dominates, but at higher  $s$ , the wave-number diffusion takes over. This change of behaviour takes place when the characteristic time scales associated with the two processes become equal:

$$\frac{k}{s} \sim \frac{\gamma s^2}{k^2} \Rightarrow s \sim \frac{k}{\gamma^{1/3}} \equiv s_{\text{echo}}. \quad (12.42)$$

At  $s \ll s_{\text{echo}}$ , and away from  $s = 0$ , (12.41) reduces to

$$\frac{\partial C_{k,s}}{\partial s} = 0 \Rightarrow C_{k,s} = f(k), \quad (12.43)$$

a flat  $s$  spectrum with some non-obvious  $k$  dependence, decided by how it connects to other parts of the solution.

---

**Exercise 12.5.** This is just the linear phase-mixing spectrum. Show that it is equivalent to the Hermite spectrum (6.81) derived in Q8.

---

At  $s \gg s_{\text{echo}}$ , (12.41) reduces to

$$\frac{\partial^2 C_{k,s}}{\partial k^2} = 0 \Rightarrow C_{k,s} = g(s), \quad (12.44)$$

a flat  $k$  spectrum with some non-obvious  $s$  dependence—the result of wave-number diffusion taking over. This diffusion can take perturbations from positive to negative  $k$ , or vice versa, and thus allow them to go from phase mixing to phase unmixing.

The dynamics that gives rise to these stationary spectra can be schematically described as follows (Fig. 49). Suppose a perturbation is launched at  $k > 0$  and low  $s$ . Phase mixing will move it “vertically” in the  $(k, s)$  plane to  $s \sim s_{\text{echo}}$ . Then diffusion will spread it “horizontally” from  $k > 0$  to  $k < 0$  until it reaches  $s \sim -s_{\text{echo}} \sim -k/\gamma^{1/3}$  (which is positive at negative  $k$ ). There, phase unmixing will take over and push the perturbation to lower  $s$ , towards  $s = 0$ , and then to larger negative  $s$ , until it reaches  $s \sim s_{\text{echo}}$ , again becomes subject to diffusion, spreads back to positive  $k$ , reaches  $s \sim -s_{\text{echo}}$ , and is again phase-unmixed towards  $s = 0$ . The cycle is complete. In fact, we shall see shortly that it is not quite a cycle but an outward unwinding spiral—a property of Adkins’ exact solution spotted by Nastac *et al.* (2022). But first let us derive that exact solution.

### 12.3.1. Self-Similar Phase-Space Spectrum

This intuition can be backed up by an exact calculation, which also allows one to determine the non-obvious spectra  $f(k)$  and  $g(s)$ . The idea is to look for a self-similar

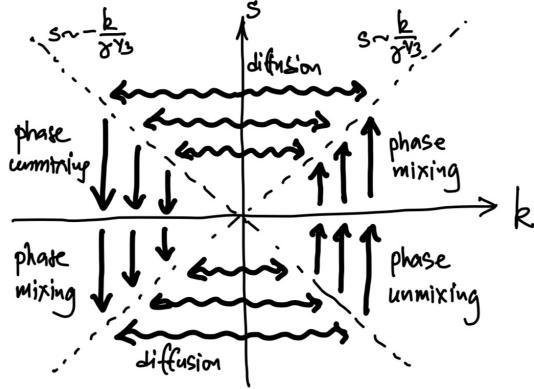


FIGURE 49. Schematic of the circulation in phase space: phase mixing,  $k$ -space diffusion, phase unmixing, etc.

solution of

$$k \frac{\partial C_{k,s}}{\partial s} = \gamma s^2 \frac{\partial^2 C_{k,s}}{\partial k^2} \quad (12.45)$$

in the form

$$C_{k,s} = \frac{1}{s^\alpha} \Phi(\xi), \quad \xi = \frac{k}{\gamma^{1/3} s}. \quad (12.46)$$

We can fix  $\alpha$  by noticing that (12.45) implies

$$\frac{\partial}{\partial s} \int_{-\infty}^{+\infty} dk k C_{k,s} = 0, \quad \int_{-\infty}^{+\infty} dk k C_{k,s} = \gamma^{2/3} s^{2-\alpha} \int_{-\infty}^{+\infty} d\xi \xi \Phi(\xi) \Rightarrow \alpha = 2. \quad (12.47)$$

Substituting (12.46) with  $\alpha = 2$  into (12.45), we get

$$\Phi'' = -\xi^2 \Phi' - 2\xi \Phi = -(\xi^2 \Phi)' \Rightarrow \Phi' + \xi^2 \Phi = c_1 = \text{const.} \quad (12.48)$$

Integrating once again, we get

$$\Phi(\xi) = e^{-\xi^3/3} \left( c_1 \int_{-\infty}^{\xi} d\xi' e^{\xi'^3/3} + c_2 \right) = c_1 \int_0^{\infty} \frac{dz e^{-z}}{|z^3 - 3z|^{2/3}}. \quad (12.49)$$

The second expression was obtained by setting the integration constant  $c_2 = 0$  to prevent blow-up at  $\xi \rightarrow -\infty$  and by changing the integration variable to  $z = (\xi^3 - \xi'^3)/3$ . Finally, putting this back into (12.46), we get

$$C_{k,s} = \text{const} \int_0^{\infty} \frac{dz e^{-z}}{|z^3 - 3\gamma s^3 z|^{2/3}} \rightarrow \begin{cases} \frac{\text{const}}{k^2}, & s \ll \frac{k}{(3\gamma)^{1/3}} \equiv s_{\text{echo}}, \\ \frac{\text{const}}{s^2}, & s \gg \frac{k}{(3\gamma)^{1/3}} \equiv s_{\text{echo}}. \end{cases} \quad (12.50)$$

This solution is sketched in Fig. 50. Asymptotically, it is just the solution anticipated in (12.43) and (12.44), but we now also know how the “phase-space spectrum” scales with  $k$  and  $s$  in all relevant limits.

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**Exercise 12.6.** Spot a mathematical sleight of hand in the above calculation. You will find a (lengthy) fix to it in [Adkins & Schekochihin \(2018\)](#), leading, however, to the same solution.

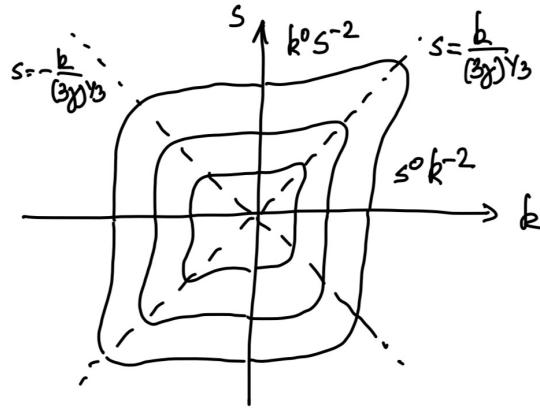


FIGURE 50. Sketch of the contour lines of the solution (12.50).

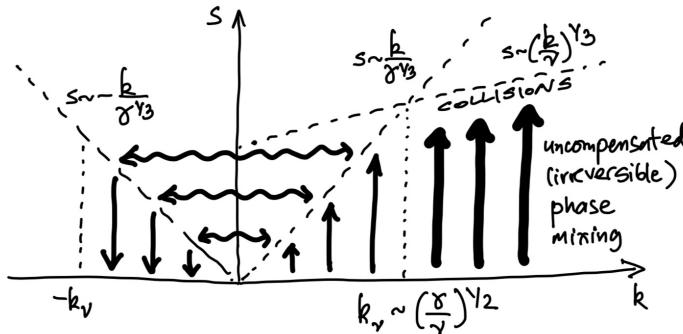


FIGURE 51. Same as the upper half-plane of Fig. 49 but with collisions.

### 12.3.2. Constant Flux

Coming soon—from [Nastac et al. \(2022\)](#).

### 12.3.3. Role of Collisions

In §5.5, my mantra was that, however infrequent particle collisions were, phase mixing in a linear system would always (and fairly quickly) produce sufficiently small velocity-space scales in order for collisions to switch on, thermalise perturbations, and make the system irreversible. What about collisions in the nonlinear, echoing/phase-unmixing regime? The easiest way to model their effect is to add to (12.41) an extra damping term proportional to  $-\nu s^2$ , to reflect the fact that the collision operator is a velocity-space diffusion operator and so, in the crudest approximation, should look like  $\nu \partial^2 / \partial w^2$ :

$$\frac{\partial C_{k,s}}{\partial t} + k \frac{\partial C_{k,s}}{\partial s} = S_k \delta(s) + \gamma s^2 \frac{\partial^2 C_{k,s}}{\partial k^2} - \nu s^2 C_{k,s}, \quad (12.51)$$

where  $\nu$  is the collision frequency.

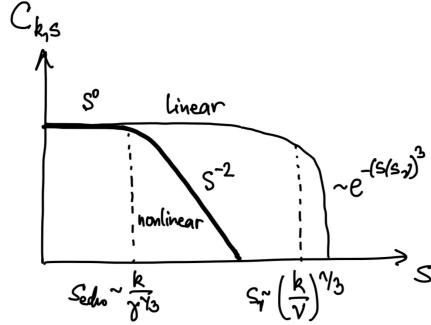


FIGURE 52. Velocity-space spectra  $C_{k,s}$  in the linear regime with a collisional cut off,  $k \gtrsim k_\nu$  [see (12.52)] and in the nonlinear regime,  $k \ll k_\nu$  [see (12.50)].

The stationary ( $\partial/\partial t = 0$ ) solution of (12.51) in the linear ( $\gamma = 0$ ) approximation is

$$C_{k,s} = f(k) e^{-(s/s_\nu)^3}, \quad s_\nu = \left( \frac{3k}{\nu} \right)^{1/3}, \quad (12.52)$$

reflecting the fact, obvious from comparing time scales ( $k/s$  vs.  $\nu s^2$ ), that collisions become competitive with phase mixing when  $s \gtrsim s_\nu$ . If perturbations that phase mix, moving “vertically” in the  $(k, s)$  plane, hit  $s_\nu$  before the echo threshold  $s_{\text{echo}}$ , they will get dissipated and lose their chance to diffuse to negative  $k$ 's and phase unmix (Fig. 51). These are the perturbations with

$$s_\nu \sim \left( \frac{k}{\nu} \right)^{1/3} \lesssim s_{\text{echo}} \sim \frac{k}{\gamma^{1/3}} \quad \Leftrightarrow \quad \boxed{k \gtrsim \left( \frac{\gamma}{\nu} \right)^{1/2} \equiv k_\nu}. \quad (12.53)$$

Thus, there is a  $k$ -space cut off  $k_\nu$ , a kind of “Kolmogorov scale” for kinetic plasmas, beyond which the nonlinearity does not matter and perturbations just “Landau-damp” as they would in a linear system ( $k_\nu$  is amplitude-dependent, via  $\gamma$ , so in a linearised system,  $k_\nu \rightarrow 0$ ).

#### 12.4. Implications for Collisionless Relaxation

To be continued...

## PART III

# Magnetohydrodynamics

### 13. MHD Equations

Like hydrodynamics from gas kinetics, MHD can be derived systematically from the Vlasov–Maxwell–Landau equations for a plasma in the limit of large collisionality + a number of additional assumptions (see, e.g., [Goedbloed & Poedts 2004](#); [Parra 2019a](#); [Kunz 2021](#)). Here I will adopt a purely fluid approach—partly to make these lectures self-consistent and partly because there is a certain beauty in it: we need to know relatively little about the properties of the constituent substance in order to spin out a very sophisticated and complete theory about the way in which it flows. This approach is also more generally applicable because the substance that we will be dealing with need not be gaseous, like plasma—you may also think of liquid metals, various conducting solutions, etc.

So, let us declare an interest in the flow of a conducting fluid and attempt to be guided in our description of it by the very basic things: conservation laws of mass, momentum and energy plus Maxwell’s equations for the electric and magnetic fields. This will prove sufficient for most of our purposes. So we shall consider a fluid characterised by the following quantities:

- $\rho$ —mass density,
- $\mathbf{u}$ —flow velocity,
- $p$ —pressure,
- $\sigma$ —charge density,
- $\mathbf{j}$ —current density,
- $\mathbf{E}$ —electric field,
- $\mathbf{B}$ —magnetic field.

Our immediate objective is to find a set of closed equations that would allow us to determine all of these quantities as functions of time and space within the fluid.

#### 13.1. Conservation of Mass

This is the most standard of all arguments in fluid dynamics (Fig. 53):

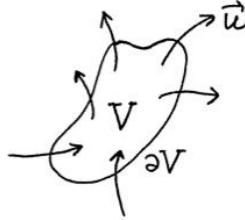
$$\frac{d}{dt} \underbrace{\int_V d\mathbf{r} \rho}_{\substack{\text{mass inside a} \\ \text{volume of} \\ \text{fluid}}} = - \underbrace{\int_{\partial V} (\rho \mathbf{u}) \cdot d\mathbf{S}}_{\substack{\text{mass flux} \\ \text{in/out}}} = - \int_V d\mathbf{r} \nabla \cdot (\rho \mathbf{u}). \quad (13.1)$$

As this equation holds for any  $V$ , however small, it can be converted into a differential relation

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0}.$$

(13.2)

This is the *continuity equation*.

FIGURE 53. Stuff flowing in and out of a volume  $V$  enclosed by surface  $\partial V$ .

## 13.2. Conservation of Momentum

A similar approach:

$$\begin{aligned}
 \frac{d}{dt} \underbrace{\int_V dr \rho \mathbf{u}}_{\text{momentum inside a volume of fluid}} &= - \underbrace{\int_{\partial V} (\rho \mathbf{u} \mathbf{u}) \cdot d\mathbf{S}}_{\substack{\text{Reynolds stress} \\ \text{momentum flux through boundary}}} - \underbrace{\int_{\partial V} p d\mathbf{S}}_{\substack{\text{pressure on boundary} \\ (\text{momentum flux due to internal motion})}} - \underbrace{\int_{\partial V} \boldsymbol{\Pi} \cdot d\mathbf{S}}_{\text{viscous stress}} + \underbrace{\int_V dr \mathbf{F}}_{\substack{\text{all other forces (E&M will come in here)}}} \\
 &= \int_V dr [-\nabla \cdot (\rho \mathbf{u} \mathbf{u}) - \nabla p - \nabla \cdot \boldsymbol{\Pi} + \mathbf{F}].
 \end{aligned} \tag{13.3}$$

In differential form, this becomes

$$\begin{aligned}
 \underbrace{\frac{\partial}{\partial t} \rho \mathbf{u}}_{\substack{= \\ = \rho \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \frac{\partial \rho}{\partial t} \\ = \rho \frac{\partial \mathbf{u}}{\partial t} - \mathbf{u} \cancel{\nabla \cdot (\rho \mathbf{u})}}} &= \underbrace{-\nabla \cdot (\rho \mathbf{u} \mathbf{u}) - \nabla p - \nabla \cdot \boldsymbol{\Pi} + \mathbf{F}}, \\
 &\quad \substack{= -\rho \mathbf{u} \cdot \nabla \mathbf{u} \\ - \mathbf{u} \cancel{\nabla \cdot (\rho \mathbf{u})}}
 \end{aligned} \tag{13.4}$$

and so, finally,

$$\boxed{
 \underbrace{\rho \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right)}_{\substack{\equiv \frac{d\mathbf{u}}{dt} \\ \text{convective derivative}}} = -\nabla p - \nabla \cdot \boldsymbol{\Pi} + \mathbf{F}.
 } \tag{13.5}$$

This is the *momentum equation*.

One part of this equation does have to be calculated from some knowledge of the microscopic properties of the constituent fluid or gas—the viscous stress. For a gas, it is done in kinetic theory (e.g., [Lifshitz & Pitaevskii 1981](#); [Dellar 2015](#); [Schekochihin 2019](#), §6.8):

$$\boldsymbol{\Pi} = -\rho \nu \left[ \nabla \mathbf{u} + (\nabla \mathbf{u})^T - \frac{2}{3} \nabla \cdot \mathbf{u} \mathbf{I} \right], \tag{13.6}$$

where  $\nu$  is the kinematic (Newtonian) viscosity.

In a magnetised plasma (i.e., such that its collision frequency  $\ll$  Larmor frequency of the gyrating charges), the viscous stress is much more complicated and anisotropic with respect to the direction of the magnetic field: because of their Larmor motion, charged particles diffuse differently across and along the field. This gives rise to the so-called [Braginskii \(1965\)](#) stress (see, e.g., [Helander & Sigmar 2005](#); [Parra 2019a](#); [Kunz 2021](#)).

In what follows, we will never require the explicit form of  $\boldsymbol{\Pi}$ .

---

### 13.3. Electromagnetic Fields and Forces

The fact that the fluid is conducting means that it can have distributed charges ( $\sigma$ ) and currents ( $\mathbf{j}$ ) and so the electric ( $\mathbf{E}$ ) and magnetic ( $\mathbf{B}$ ) fields will exert body forces on the fluid. Indeed, for one particle of charge  $q$ , the Lorentz force is

$$\mathbf{f}_L = q \left( \mathbf{E} + \frac{\mathbf{v} \times \mathbf{B}}{c} \right), \quad (13.7)$$

and if we sum this over all particles (or, to be precise, average over their distribution and sum over species), we will get

$$\mathbf{F} = \sigma \mathbf{E} + \frac{\mathbf{j} \times \mathbf{B}}{c}. \quad (13.8)$$

This body force (force density) goes into (13.5) and so we must know  $\mathbf{E}$ ,  $\mathbf{B}$ ,  $\sigma$  and  $\mathbf{j}$  in order to compute the fluid flow  $\mathbf{u}$ .

Clearly it is a good idea to bring in Maxwell's equations:

$$\nabla \cdot \mathbf{E} = 4\pi\sigma \quad (\text{Gauss}), \quad (13.9)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (13.10)$$

$$\frac{\partial \mathbf{B}}{\partial t} = -c \nabla \times \mathbf{E} \quad (\text{Faraday}), \quad (13.11)$$

$$\nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{j} + \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} \quad (\text{Ampère-Maxwell}). \quad (13.12)$$

To these, we append Ohm's law in its simplest form: The electric field in the frame of a fluid element moving with velocity  $\mathbf{u}$  is

$$\mathbf{E}' = \mathbf{E} + \frac{\mathbf{u} \times \mathbf{B}}{c} = \eta \mathbf{j}, \quad (13.13)$$

where  $\mathbf{E}$  is the electric field in the laboratory frame and  $\eta$  is the *Ohmic resistivity*.

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Normally, the resistivity, like viscosity, has to be computed from kinetic theory (see, e.g., [Helander & Sigmar 2005](#); [Parra 2019a](#); [Kunz 2021](#)) or tabulated by assiduous experimentalists. In a magnetised plasma, the simple form (13.13) of Ohm's law is only valid at spatial scales longer than the Larmor radii and time scales longer than the Larmor periods of the particles (see, e.g., [Goedbloed & Poedts 2004](#); [Parra 2019b](#)).

Equations (13.9–13.13) can be reduced somewhat if we assume (quite reasonably for most applications) that our fluid flow is *non-relativistic*. Let us stipulate that all fields evolve on time scales  $\sim \tau$ , have spatial scales  $\sim \ell$  and that the flow velocity is

$$u \sim \frac{\ell}{\tau} \ll c. \quad (13.14)$$

Then, from Ohm's law (13.13),

$$E \sim \frac{u}{c} B \ll B, \quad (13.15)$$

so *electric fields are small compared to magnetic fields*.

In Ampère–Maxwell’s law (13.12),

$$\frac{\left| \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} \right|}{|\nabla \times \mathbf{B}|} \sim \frac{\frac{1}{c} \frac{1}{\tau} \frac{u}{c} B}{\frac{1}{c} \frac{B}{\ell}} \sim \frac{u^2}{c^2} \ll 1, \quad (13.16)$$

so *the displacement current is negligible* (note that at this point we have ordered out light waves; see Q2 in Kinetic Theory). This allows us to revert to the pre-Maxwell form of Ampère’s law:

$$\boxed{\mathbf{j} = \frac{c}{4\pi} \nabla \times \mathbf{B}} . \quad (13.17)$$

Thus, the current is no longer an independent field, there is a one-to-one correspondence  $\mathbf{j} \leftrightarrow \mathbf{B}$ .

Finally, comparing the electric and magnetic parts of the Lorentz force (13.8), and using Gauss’s law (13.9) to estimate  $\sigma \sim E/\ell$ , we get

$$\frac{|\sigma \mathbf{E}|}{\left| \frac{1}{c} \mathbf{j} \times \mathbf{B} \right|} \sim \frac{\frac{1}{\ell} E^2}{\frac{1}{c} \frac{c}{\ell} B^2} \sim \frac{E^2}{B^2} \sim \frac{u^2}{c^2} \ll 1. \quad (13.18)$$

Thus, the MHD body force is

$$\boxed{\mathbf{F} = \frac{\mathbf{j} \times \mathbf{B}}{c} = \frac{(\nabla \times \mathbf{B}) \times \mathbf{B}}{4\pi}} . \quad (13.19)$$

This goes into (13.5) and we note with relief that  $\sigma$ ,  $\mathbf{j}$  and  $\mathbf{E}$  have all fallen out of the momentum equation—we only need to know  $\mathbf{B}$ .

### 13.4. Maxwell Stress and Magnetic Forces

Let us take a break from formal derivations to consider what (13.19) teaches us about the sort of new dynamics that our fluid will experience as a result of being conducting. To see this, it is useful to play with the expression (13.19) in a few different ways.

By simple vector algebra,

$$\mathbf{F} = \underbrace{\frac{\mathbf{B} \cdot \nabla \mathbf{B}}{4\pi}}_{\text{"magnetic tension"} \quad -} - \underbrace{\nabla \frac{B^2}{8\pi}}_{\text{"magnetic pressure"} \quad =} = -\nabla \cdot \underbrace{\left( \frac{B^2}{8\pi} \mathbf{I} - \frac{\mathbf{B}\mathbf{B}}{4\pi} \right)}_{\text{"Maxwell stress"}}, \quad (13.20)$$

where the last expression was obtained with the aid of  $\nabla \cdot \mathbf{B} = 0$ . Thus, the action of the Lorentz force in a conducting fluid amounts to a new form of stress. Mathematically, this “Maxwell stress” is somewhat similar to the kind of stress that would arise from a suspension in the fluid of elongated molecules—e.g., polymer chains, or other kinds of “balls on springs” (see, e.g., Dellar 2017; the analogy can be made rigorous: see Ogilvie & Proctor 2003). Thus, we expect that the magnetic field threading the fluid will impart to it a degree of “elasticity” (you will have an opportunity of a practical engagement with this analogy in Exercise 15.4).

Exactly what this means dynamically becomes obvious if we rewrite the magnetic tension and pressure forces in (13.20) in the following way. Let  $\mathbf{b} = \mathbf{B}/B$  be the unit

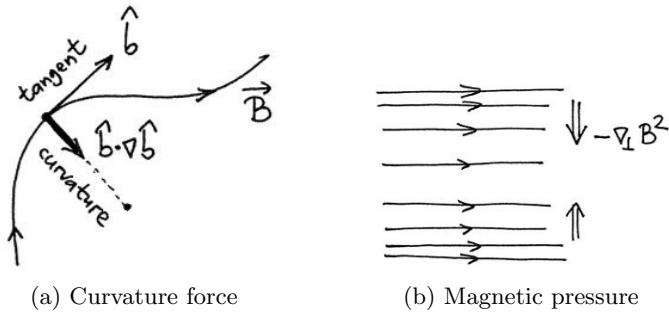


FIGURE 54. Magnetic forces.

vector in the direction of  $\mathbf{B}$  (the unit tangent to the field line). Then

$$\mathbf{B} \cdot \nabla \mathbf{B} = B \mathbf{b} \cdot \nabla(B \mathbf{b}) = B^2 \mathbf{b} \cdot \nabla \mathbf{b} + B \mathbf{b} \cdot \nabla \frac{B^2}{2} \quad (13.21)$$

and, putting this back into (13.20), we get

$$\mathbf{F} = \underbrace{\frac{B^2}{4\pi} \mathbf{b} \cdot \nabla \mathbf{b}}_{\text{"curvature force"}} - \underbrace{(1 - \mathbf{b} \cdot \mathbf{b}) \cdot \nabla \frac{B^2}{8\pi}}_{\substack{\equiv \nabla_{\perp} \\ \text{magnetic pressure force}}} \quad (13.22)$$

Thus, we learn that the Lorentz force consists of two distinct parts (Fig. 54):

- *curvature force*, so called because  $\mathbf{b} \cdot \nabla \mathbf{b}$  is the vector curvature of the magnetic field line—the implication being that field lines, if bent, will want to straighten up;
- *magnetic pressure*, whose presence implies that field lines will resist compression or rarefaction (the field wants to be uniform in strength).

Note that both forces act perpendicularly to  $\mathbf{B}$ , as they must, since magnetic field never exerts a force along itself on a charged particle [see (13.7)].

So this is the effect of the field on the fluid. What is the effect of the fluid on the field?

### 13.5. Evolution of Magnetic Field

Returning to deriving MHD equations, we use Ohm's law (13.13) to express  $\mathbf{E}$  in terms of  $\mathbf{u}$ ,  $\mathbf{B}$  and  $\mathbf{j}$  in the right-hand side of Faraday's law (13.11). We then use Ampere's law (13.17) to express  $\mathbf{j}$  in terms of  $\mathbf{B}$ . The result is

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left( \mathbf{u} \times \mathbf{B} - \frac{c^2 \eta}{4\pi} \nabla \times \mathbf{B} \right). \quad (13.23)$$

After using also  $\nabla \cdot \mathbf{B} = 0$  to get  $\nabla \times (\nabla \times \mathbf{B}) = -\nabla^2 \mathbf{B}$  and renaming  $c^2 \eta / 4\pi \rightarrow \eta$ , the *magnetic diffusivity*, we arrive at the *magnetic induction equation* (due to Hertz):

$$\frac{\partial \mathbf{B}}{\partial t} = \underbrace{\nabla \times (\mathbf{u} \times \mathbf{B})}_{\text{advection}} + \underbrace{\eta \nabla^2 \mathbf{B}}_{\text{diffusion}}. \quad (13.24)$$

Note that if  $\nabla \cdot \mathbf{B} = 0$  is satisfied initially, any solution of (13.24) will remain divergence-free at all times.

### 13.6. Magnetic Reynolds Number

The relative importance of the diffusion term (it is obvious what this does) and the advection term (to be discussed in the next few sections) in (13.24) is measured by a dimensionless number:

$$\frac{|\nabla \times (\mathbf{u} \times \mathbf{B})|}{|\eta \nabla^2 \mathbf{B}|} \sim \frac{\frac{u}{\ell} B}{\frac{\eta}{\ell^2} B} = \frac{u\ell}{\eta} \equiv \text{Rm}, \quad (13.25)$$

called the *magnetic Reynolds number*. In nature, it can take a very broad range of values:

liquid metals in industrial contexts (metallurgy):  $\text{Rm} \sim 10^{-3} \dots 10^{-1}$ ,

planet interiors:  $\text{Rm} \sim 100 \dots 300$ ,

solar convective zone:  $\text{Rm} \sim 10^6 \dots 10^9$ ,

interstellar medium (“warm” phase):  $\text{Rm} \sim 10^{18}$ ,

intergalactic medium (cores of galaxy clusters):  $\text{Rm} \sim 10^{29}$ ,

laboratory “dynamo” experiments:  $\text{Rm} \sim 1 \dots 10^3$ .

Generally speaking, when flow velocities are large/distances are large/resistivities are low,  $\text{Rm} \gg 1$  and it makes sense to consider “ideal MHD,” i.e., the limit  $\eta \rightarrow 0$ . In fact,  $\eta$  often needs to be brought back in to deal with instances of large  $\nabla \mathbf{B}$ , which arise naturally from solutions of ideal MHD equations (see §14.1, Q5, §19.2 and Parra 2019a), but let us consider the ideal case for now to understand what the advective part of the induction equation does to  $\mathbf{B}$ .

### 13.7. Lundquist Theorem

The ideal ( $\eta = 0$ ) version of the induction equation (13.24),

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}), \quad (13.26)$$

implies that *fluid elements that lie on a field line initially will remain on this field line*, i.e., “the magnetic field moves with the flow.”

**Proof.** Unpacking the double vector product in (13.26),

$$\frac{\partial \mathbf{B}}{\partial t} = -\mathbf{u} \cdot \nabla \mathbf{B} + \mathbf{B} \cdot \nabla \mathbf{u} - \mathbf{B} \nabla \cdot \mathbf{u} + \mathbf{u} \nabla \cdot \mathbf{B}, \quad (13.27)$$

or, using the notation for the “convective derivative” [see (13.5)],

$$\frac{d\mathbf{B}}{dt} \equiv \left( \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \mathbf{B} = \mathbf{B} \cdot \nabla \mathbf{u} - \mathbf{B} \nabla \cdot \mathbf{u}. \quad (13.28)$$

The continuity equation (13.2) can be rewritten in a somewhat similar-looking form

$$\frac{d\rho}{dt} \equiv \left( \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \rho = -\rho \nabla \cdot \mathbf{u} \quad \Rightarrow \quad \nabla \cdot \mathbf{u} = -\frac{1}{\rho} \frac{d\rho}{dt}. \quad (13.29)$$

The last expression is now used for  $\nabla \cdot \mathbf{u}$  in (13.28):

$$\frac{d\mathbf{B}}{dt} = \mathbf{B} \cdot \nabla \mathbf{u} + \frac{\mathbf{B}}{\rho} \frac{d\rho}{dt}. \quad (13.30)$$

Multiplying this equation by  $1/\rho$  and noting that

$$\frac{1}{\rho} \frac{d\mathbf{B}}{dt} - \frac{\mathbf{B}}{\rho^2} \frac{d\rho}{dt} = \frac{d}{dt} \frac{\mathbf{B}}{\rho}, \quad (13.31)$$

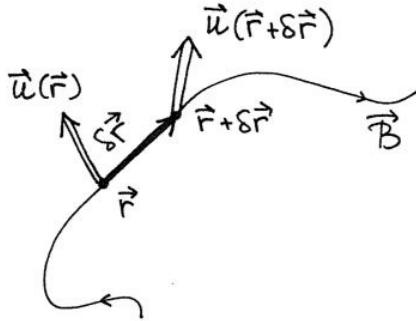


FIGURE 55. Pressure-balanced perturbations.

we arrive at

$$\boxed{\frac{d}{dt} \frac{\mathbf{B}}{\rho} = \frac{\mathbf{B}}{\rho} \cdot \nabla \mathbf{u}} . \quad (13.32)$$

Let us compare the evolution of the vector  $\mathbf{B}/\rho$  with the evolution of an infinitesimal Lagrangian separation vector in a moving fluid: the convective derivative is the Lagrangian time derivative, so

$$\frac{d}{dt} \delta \mathbf{r}(t) = \mathbf{u}(\mathbf{r} + \delta \mathbf{r}) - \mathbf{u}(\mathbf{r}) \approx \delta \mathbf{r} \cdot \nabla \mathbf{u} . \quad (13.33)$$

Thus,  $\delta \mathbf{r}$  and  $\mathbf{B}/\rho$  satisfy the same equation. This means that if two fluid elements are initially on the same field line,

$$\delta \mathbf{r} = \text{const} \frac{\mathbf{B}}{\rho} , \quad (13.34)$$

then they will stay on the same field line, q.e.d.<sup>100</sup>

This means that in MHD, the fluid flow will be entraining the magnetic-field lines with it—and, as we saw in §13.4, the field lines will react back on the fluid:  
—when the fluid tries to bend the field, the field will want to spring back,  
—when the fluid tries to compress or rarefy the field, the field will resist as if it possessed (perpendicular) pressure.

This is the sense in which MHD fluid is “elastic”: it is threaded by magnetic-field lines, which move with it and act as elastic bands.

### 13.8. Flux Freezing

There is an essentially equivalent formulation of the result of §13.7 that highlights the fact that the ideal induction equation (13.26) is a *conservation law*—conservation of *magnetic flux*.

The magnetic flux through a surface  $S$  (Fig. 56a) is, by definition,

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{S} \quad (13.35)$$

( $d\mathbf{S} \equiv \hat{\mathbf{n}} dS$ , where  $\hat{\mathbf{n}}$  is a unit normal pointing out of the surface). *The flux  $\Phi$  depends on the loop  $\partial S$ , but not on the choice of the surface spanning it.* Indeed, if we consider two

<sup>100</sup>Lundquist theorem opens the door to a Lagrangian description of MHD fluid that contains some mathematical and physical delights: I will pick up this thread in §13.12.

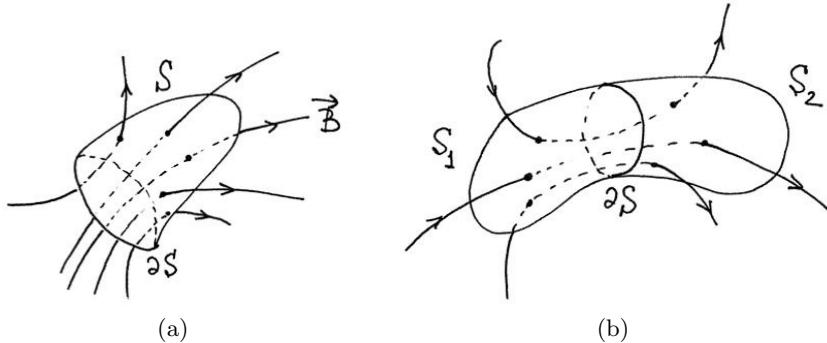


FIGURE 56. Magnetic flux through a loop  $\partial S$  (a) is independent of the surface spanning the loop (b).

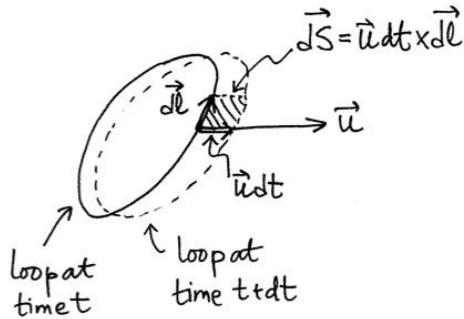


FIGURE 57. A loop moving with the fluid.

surfaces,  $S_1$  and  $S_2$ , spanning the same loop  $\partial S$  (Fig. 56b) and define  $\Phi_{1,2} = \int_{S_{1,2}} \mathbf{B} \cdot d\mathbf{S}$ , then the flux out of the volume  $V$  enclosed by  $S_1 \cup S_2 = \partial V$  is

$$\Phi_2 - \Phi_1 = \int_{\partial V} \mathbf{B} \cdot d\mathbf{S} = \int_V d\mathbf{r} \nabla \cdot \mathbf{B} = 0, \quad \text{q.e.d.} \quad (13.36)$$

**Alfvén's Theorem.** *Flux through any loop moving with the fluid is conserved.*

**Proof.** Let  $S(t)$  be a surface spanning the loop at time  $t$ . If the loop moves with the fluid (Fig. 57), at the slightly later time  $t+dt$  it is spanned (for example) by the surface

$$S(t+dt) = S(t) \cup \begin{array}{l} \text{ribbon traced by the loop} \\ \text{as it moves over time } dt. \end{array} \quad (13.37)$$

Then the flux at time  $t$  is

$$\Phi(t) = \int_{S(t)} \mathbf{B}(t) \cdot d\mathbf{S} \quad (13.38)$$

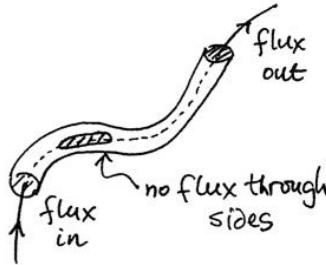


FIGURE 58. Flux tube.

and at the later time,

$$\begin{aligned}
 \Phi(t + dt) &= \int_{S(t+dt)} \mathbf{B}(t + dt) \cdot d\mathbf{S} \\
 &= \underbrace{\int_{S(t)} \mathbf{B}(t + dt) \cdot d\mathbf{S}}_{= \Phi(t)} + \underbrace{\int_{\text{ribbon}} \mathbf{B}(t + dt) \cdot d\mathbf{S}}_{= \mathbf{u} dt \times dl} \\
 &= \underbrace{\int_{S(t)} \mathbf{B}(t) \cdot d\mathbf{S} + dt \int_{S(t)} \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}}_{\text{ideal induction equation (13.26)}} \\
 &= -dt \int_{S(t)} [\nabla \times (\mathbf{u} \times \mathbf{B})] \cdot d\mathbf{S}.
 \end{aligned} \tag{13.39}$$

Therefore,

$$\frac{d\Phi}{dt} = \frac{\Phi(t + dt) - \Phi(t)}{dt} = \int_{S(t)} \underbrace{\left[ \frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{u} \times \mathbf{B}) \right]}_{\text{ideal induction equation (13.26)}} \cdot d\mathbf{S} = 0, \quad \text{q.e.d.} \tag{13.40}$$

This result means that *field lines are frozen into the flow*. Indeed, consider a flux tube enclosing a field line (Fig. 58). As the tube deforms, the field line stays inside it because fluxes through the ends and sides of the tube cannot change.

Note that *Ohmic diffusion breaks flux freezing*, as is obvious from (13.40) if in the integrand one uses the induction equation (13.24) keeping the resistive term.

### 13.9. Amplification of Magnetic Field by Fluid Flow

An interesting physical consequence of these results is that *flows of conducting fluid can amplify magnetic fields*. For example, consider a flow that stretches an initial cylindrical tube of length  $l_1$  and cross section  $S_1$  into a long thin spaghetti of length  $l_2$  and cross section  $S_2$  (Fig. 59). By conservation of flux,

$$B_1 S_1 = B_2 S_2. \tag{13.41}$$

By conservation of mass,

$$\rho_1 l_1 S_1 = \rho_2 l_2 S_2. \tag{13.42}$$

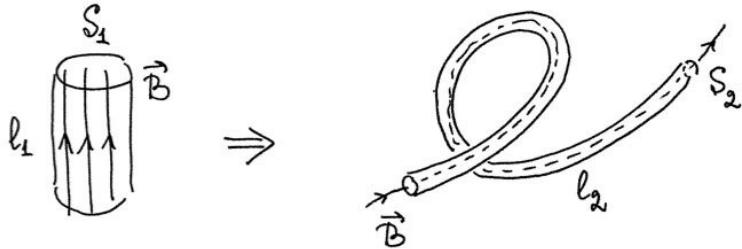


FIGURE 59. Amplification of magnetic field by stretching.

Therefore,

$$\frac{B_2}{\rho_2 l_2} = \frac{B_1}{\rho_1 l_1} \Rightarrow \frac{B_2}{B_1} = \frac{\rho_2 l_2}{\rho_1 l_1}. \quad (13.43)$$

In an incompressible fluid,  $\rho_2 = \rho_1$ , and the field is amplified by a factor  $l_2/l_1$ . In a compressible fluid, the field can also be amplified by compression.

Going back to the induction equation in the form (13.27),

$$\frac{\partial \mathbf{B}}{\partial t} + \underbrace{\mathbf{u} \cdot \nabla \mathbf{B}}_{\text{advection}} = \underbrace{\mathbf{B} \cdot \nabla \mathbf{u}}_{\text{stretching}} - \underbrace{\mathbf{B} \nabla \cdot \mathbf{u}}_{\text{compression}}, \quad (13.44)$$

the three terms in it are responsible for, in order, advection of the field by the flow (i.e., the flow carrying the field around with it), “stretching” (amplification) of the field by velocity gradients that make fluid elements longer and, finally, compression or rarefaction of the field by convergent or divergent flows (unless  $\nabla \cdot \mathbf{u} = 0$ , as it is in an incompressible fluid).

Hence arises the famous problem of *MHD dynamo*: are there fluid flows that lead to sustained amplification of the magnetic fields? The answer is yes—but the flow must be 3D (the absence of dynamo action in 2D is a theorem, the simplest version of which is due to [Zeldovich 1956](#); see Q4). Magnetic fields of planets, stars, galaxies, etc. are all believed to owe their origin and persistence to this effect. This topic requires (and merits) a more detailed treatment (see reading suggestions below), but for now let us flag two important aspects:

- resistivity, however small, turns out to be impossible to neglect because large gradients of  $\mathbf{B}$  appear as the field is advected by the flow (see §14.1);
- the amplification of the field is checked by the Lorentz force once the field is strong enough that it can act back on the flow, viz., when their energy densities become comparable:

$$\frac{B^2}{8\pi} \sim \frac{\rho u^2}{2}. \quad (13.45)$$

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If you wish to educate yourself on the topic of MHD dynamos (as everyone should), a classic (and mostly timeless) text is [Moffatt \(1978\)](#). The new classic is the review by [Rincon \(2019\)](#), which I recommend strongly.

If you want to stick with my notes, you will find a (somewhat outdated) review in my handwritten 2007 Les Houches lectures available here: <http://www-thphys.physics.ox.ac.uk/people/AlexanderSchekochihin/notes/leshouches07.pdf>. A very short printed review is [Schekochihin & Cowley \(2007, §3\)](#). One of the calculations from these notes is reproduced in §14.1. An up-to-date treatment focussing on the nonlinear stages of the dynamo and on yet-unresolved issues is [Schekochihin \(2022, §12\)](#).

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### 13.10. Conservation of Energy

Let us summarise the equations that we have derived so far, namely (13.2), (13.5) and (13.24), expressing conservation of

$$\text{mass} \quad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \quad (13.46)$$

$$\begin{aligned} \text{momentum} \quad \rho \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) &= -\nabla p - \nabla \cdot \boldsymbol{\Pi} + \frac{(\nabla \times \mathbf{B}) \times \mathbf{B}}{4\pi} \\ &= -\nabla \cdot \underbrace{\left[ \left( p + \frac{B^2}{8\pi} \right) \mathbf{I} - \frac{\mathbf{B}\mathbf{B}}{4\pi} + \boldsymbol{\Pi} \right]}_{\text{total stress}}, \end{aligned} \quad (13.47)$$

$$\text{and flux} \quad \frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}. \quad (13.48)$$

To complete the system, we need an equation for  $p$ , which has to come from the one conservation law that we have not yet utilised: *conservation of energy*.

The total *energy density* is

$$\varepsilon = \underbrace{\frac{\rho u^2}{2}}_{\text{kinetic}} + \underbrace{\frac{p}{\gamma - 1}}_{\text{internal}} + \underbrace{\frac{E^2}{8\pi}}_{\text{electric}} + \underbrace{\frac{B^2}{8\pi}}_{\text{magnetic}}, \quad (13.49)$$

where the electric energy can (and, for consistency with §13.3, must) be neglected because  $E^2/B^2 \sim u^2/c^2 \ll 1$ . We follow the same logic as we did in §§13.1 and 13.2:

$$\begin{aligned} \frac{d}{dt} \underbrace{\int_V d\mathbf{r} \varepsilon}_{\text{energy inside a volume of fluid}} &= - \underbrace{\int_{\partial V} \left( \frac{\rho u^2}{2} + \frac{p}{\gamma - 1} \right) \mathbf{u} \cdot d\mathbf{S}}_{\text{fluid flow carrying its kinetic and internal energy through boundary}} - \underbrace{\int_{\partial V} [(p\mathbf{I} + \boldsymbol{\Pi}) \cdot \mathbf{u}] \cdot d\mathbf{S}}_{\text{work done on boundary by pressure and viscous stress}} \\ &\quad - \underbrace{\int_{\partial V} \mathbf{q} \cdot d\mathbf{S}}_{\text{heat flux}} - \underbrace{\int_{\partial V} \frac{c}{4\pi} (\mathbf{E} \times \mathbf{B}) \cdot d\mathbf{S}}_{\text{Poynting flux}}. \end{aligned} \quad (13.50)$$

Like the viscous stress  $\boldsymbol{\Pi}$ , the heat flux  $\mathbf{q}$  must be calculated kinetically (in a plasma) or tabulated (in an arbitrary complicated substance). In a gas,  $\mathbf{q} = -\kappa \nabla T$ , but it is more complicated in a magnetised plasma (see, e.g., Braginskii 1965; Helander & Sigmar 2005; Parra 2019a; Kunz 2021).

Note that the magnetic energy and the work done by the Lorentz force are not included in the first two terms on the right-hand side of (13.50) because all of that must already be correctly accounted for by the Poynting flux. Indeed, since  $c\mathbf{E} = -\mathbf{u} \times \mathbf{B} + \eta \nabla \times \mathbf{B}$

[this is (13.13), with  $\eta$  renamed as in (13.24)], we have

$$\int_{\partial V} \frac{c}{4\pi} (\mathbf{E} \times \mathbf{B}) \cdot d\mathbf{S} = \underbrace{\int_{\partial V} \frac{B^2}{8\pi} \mathbf{u} \cdot d\mathbf{S}}_{\text{magnetic energy flow}} + \underbrace{\int_{\partial V} \left[ \left( \frac{B^2}{8\pi} \mathbf{I} - \frac{\mathbf{B}\mathbf{B}}{4\pi} \right) \cdot \mathbf{u} \right] \cdot d\mathbf{S}}_{\text{work done by Maxwell stress}} \\ + \underbrace{\int_{\partial V} \eta \frac{(\nabla \times \mathbf{B}) \times \mathbf{B}}{4\pi} \cdot d\mathbf{S}}_{\text{resistive slippage accounting for field not being precisely frozen into flow}} . \quad (13.51)$$

After application of Gauss's theorem and shrinking of the volume  $V$  to infinitesimality, we get the differential form of (13.50):

$$\frac{\partial}{\partial t} \left( \frac{\rho u^2}{2} + \frac{p}{\gamma - 1} + \frac{B^2}{8\pi} \right) = -\nabla \cdot \left[ \frac{\rho u^2}{2} \mathbf{u} + \frac{\gamma}{\gamma - 1} p \mathbf{u} + \boldsymbol{\Pi} \cdot \mathbf{u} + \mathbf{q} \right. \\ \left. + \frac{B^2 \mathbf{I} - \mathbf{B}\mathbf{B}}{4\pi} \cdot \mathbf{u} + \eta \frac{(\nabla \times \mathbf{B}) \times \mathbf{B}}{4\pi} \right] . \quad (13.52)$$

It remains to separate the evolution equation for  $p$  by using the fact that we know the equations for  $\rho$ ,  $\mathbf{u}$  and  $\mathbf{B}$  and so can deduce the rates of change of the kinetic and magnetic energies.

### 13.10.1. Kinetic Energy

Using (13.46) and (13.47),

$$\frac{\partial}{\partial t} \frac{\rho u^2}{2} = \frac{u^2}{2} \frac{\partial \rho}{\partial t} + \rho \mathbf{u} \cdot \frac{\partial \mathbf{u}}{\partial t} \\ = -\frac{u^2}{2} \nabla \cdot (\rho \mathbf{u}) - \rho \mathbf{u} \cdot \nabla \frac{u^2}{2} - \mathbf{u} \cdot \left\{ \nabla \cdot \left[ \left( p + \frac{B^2}{8\pi} \right) \mathbf{I} - \frac{\mathbf{B}\mathbf{B}}{4\pi} + \boldsymbol{\Pi} \right] \right\} \\ = -\nabla \cdot \left[ \cancel{\frac{\rho u^2}{2} \mathbf{u}} + p \cancel{\mathbf{u}} + \cancel{\left( \frac{B^2}{8\pi} \mathbf{I} - \frac{\mathbf{B}\mathbf{B}}{4\pi} \right)} \cancel{\cdot \mathbf{u}} + \boldsymbol{\Pi} \cdot \mathbf{u} \right] \\ + \underbrace{p \nabla \cdot \mathbf{u}}_{\text{compressional work}} + \underbrace{\left( \frac{B^2}{8\pi} \mathbf{I} - \frac{\mathbf{B}\mathbf{B}}{4\pi} \right) : \nabla \mathbf{u}}_{\text{energy exchange with magnetic field}} + \underbrace{\boldsymbol{\Pi} : \nabla \mathbf{u}}_{\text{viscous dissipation}} . \quad (13.53)$$

The flux terms (energy flows and work by stresses on boundaries) that have been crossed out cancel with corresponding terms in (13.52) once (13.53) is subtracted from it.

### 13.10.2. Magnetic Energy

Using the induction equation (13.48),

$$\begin{aligned} \frac{\partial}{\partial t} \frac{B^2}{8\pi} &= \frac{\mathbf{B}}{4\pi} \cdot [-\mathbf{u} \cdot \nabla \mathbf{B} + \mathbf{B} \cdot \nabla \mathbf{u} - \mathbf{B} \nabla \cdot \mathbf{u} + \eta \nabla^2 \mathbf{B}] \\ &= -\nabla \cdot \left[ \frac{B^2}{8\pi} \cancel{\mathbf{u}} + \underbrace{\eta \frac{(\nabla \times \mathbf{B}) \times \mathbf{B}}{4\pi}}_{\text{energy exchange with velocity field}} \right] - \underbrace{\left( \frac{B^2}{8\pi} \mathbf{I} - \frac{\mathbf{B}\mathbf{B}}{4\pi} \right) : \nabla \mathbf{u}}_{\text{Ohmic dissipation}} - \underbrace{\eta \frac{|\nabla \times \mathbf{B}|^2}{4\pi}}_{\text{Ohmic dissipation}}. \end{aligned} \quad (13.54)$$

Again, the crossed out flux terms will cancel with corresponding terms in (13.52). The metamorphosis of the resistive term into a flux term and an Ohmic dissipation term is a piece of vector algebra best checked by expanding the divergence of the flux term. Finally, the  $\mathbf{u}$ -to- $\mathbf{B}$  energy exchange term (penultimate on the right-hand side) corresponds precisely to the  $\mathbf{B}$ -to- $\mathbf{u}$  exchange term in (13.53) and cancels with it if we add (13.53) and (13.54).

### 13.10.3. Thermal Energy

Subtracting (13.53) and (13.54) from (13.52), consummating the promised cancellations, and mopping up the remaining  $\nabla \cdot (p\mathbf{u})$  and  $p\nabla \cdot \mathbf{u}$  terms, we end up with the desired evolution equation for the thermal (internal) energy:

$$\underbrace{\frac{d}{dt} \frac{p}{\gamma - 1}}_{\text{advection of internal energy}} = \underbrace{-\nabla \cdot \mathbf{q}}_{\text{heat flux}} - \underbrace{\frac{\gamma}{\gamma - 1} p \nabla \cdot \mathbf{u}}_{\text{compressional heating}} - \underbrace{\boldsymbol{\Pi} : \nabla \mathbf{u}}_{\text{viscous heating}} + \underbrace{\eta \frac{|\nabla \times \mathbf{B}|^2}{4\pi}}_{\text{Ohmic heating}}. \quad (13.55)$$

A further rearrangement and the use of the continuity equation (13.46) to express  $\nabla \cdot \mathbf{u} = -d \ln \rho / dt$  turn (13.55) into

$$\frac{d}{dt} \ln \frac{p}{\rho^\gamma} = \frac{\gamma - 1}{p} \left( -\nabla \cdot \mathbf{q} - \boldsymbol{\Pi} : \nabla \mathbf{u} + \eta \frac{|\nabla \times \mathbf{B}|^2}{4\pi} \right). \quad (13.56)$$

This form of the thermal-energy equation has very clear physical content: the left-hand side represents advection of the entropy of the MHD fluid by the flow—each fluid element behaves adiabatically, except for the sundry non-adiabatic effects on the right-hand side. The latter are the heat flux in/out of the fluid element and the dissipative (viscous and resistive) heating, leading to entropy production. Note that the form of the viscous stress  $\boldsymbol{\Pi}$  ensures that the viscous heating is always positive [see, e.g., (13.6)]. In these Lectures, I will, for the most part, focus on ideal MHD and so use the adiabatic version of (13.56), with the right-hand side set to zero.

Let me reiterate the equations of *ideal MHD*, now complete:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \quad (13.57)$$

$$\rho \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \frac{(\nabla \times \mathbf{B}) \times \mathbf{B}}{4\pi}, \quad (13.58)$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}), \quad (13.59)$$

$$\left( \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \frac{p}{\rho^\gamma} = 0. \quad (13.60)$$

In what follows, we shall study various solutions and symptotic regimes of these rather nice equations.

In the context of plasma physics, ideal MHD equations (13.57–13.60) are a *collisional* approximation: the key place where that comes in is the isotropy of pressure. A more general situation is one where pressure is a tensor,  $\mathbf{P}$ , with  $\nabla p$  in (13.58) replaced by  $\nabla \cdot \mathbf{P}$ . To calculate  $\mathbf{P}$ , one then needs the kinetic equation or some appropriate closure approximation (or model). In the presence of a magnetic field that is strong enough to make the ion Larmor frequency much greater than the collision rate, the pressure tensor on scales that are longer than the Larmor radius becomes diagonal with respect to the local direction of  $\mathbf{B}$ :

$$\mathbf{P} = p_\perp (\mathbf{I} - \mathbf{b}\mathbf{b}) + p_\parallel \mathbf{b}\mathbf{b}. \quad (13.61)$$

A particularly popular and well-known closure model for calculating the perpendicular and parallel pressures is the so-called *double-adiabatic*, or *CGL equations* (after the orginal authors Chew *et al.* 1956; see also Kulsrud 1983):

$$\frac{d}{dt} \frac{p_\perp}{\rho B} = 0, \quad \frac{d}{dt} \frac{p_\parallel B^2}{\rho^3} = 0. \quad (13.62)$$

These equations, which replace the adiabatic law (13.60), express the property of particles on Larmor orbits in magnetic field to conserve certain adiabatic invariants of motion. Plasma dynamics become very different (and often quite counterintuitive) in this and similar pressure-anisotropic approximations than for regular ideal MHD (e.g., it is extremely difficult to amplify magnetic field in such a plasma: this can be shown quite easily—see Helander *et al.* 2016). I shall come back to this topic in Exercise 15.8 and from §22.2.1 (where CGL equations are derived) onwards.

**Exercise 13.1. MHD with self-gravity.** Consider an MHD system subject to gravity with acceleration  $\mathbf{g} = -\nabla\Phi$ , where the gravitational potential  $\Phi$  satisfies Poisson's equation

$$\nabla^2 \Phi = 4\pi G \rho, \quad (13.63)$$

and  $G$  is the gravitational constant. There will then be an additional body force in the momentum equation (13.5), equal to  $\rho \mathbf{g}$ . Show that, like the magnetic force (13.20), the gravitational force can be written as a divergence of a stress tensor:

$$\rho \mathbf{g} = \nabla \cdot \left( \frac{g^2}{8\pi G} \mathbf{I} - \frac{\mathbf{g}\mathbf{g}}{4\pi G} \right). \quad (13.64)$$

Show also that the total energy of this system is conserved:

$$\frac{d}{dt} \int d\mathbf{r} \left( \frac{\rho u^2}{2} + \frac{p}{\gamma - 1} + \frac{B^2}{8\pi} + \frac{\rho\Phi}{2} \right) = 0, \quad (13.65)$$

where the integration is over the entire system, surface integrals over its boundary are assumed to vanish and

$$\mathcal{E}_G \equiv \int d\mathbf{r} \frac{\rho\Phi}{2} = - \int d\mathbf{r} \frac{g^2}{8\pi G} < 0 \quad (13.66)$$

is the gravitational energy of the system.

### 13.11. Virial Theorem

This is a good place to present a rather nice, exact result that follows directly from MHD equations and helps one decide whether an MHD system can “self-confine”, i.e., whether a blob of plasma (or, more generally, a conducting fluid) can exist without blowing itself apart.

Consider an MHD system whose volume is  $V$ . Its moment of inertia is

$$I = \frac{1}{2} \int d\mathbf{r} \rho r^2. \quad (13.67)$$

Let us see how it evolves with time: using the continuity equation (13.2), we get

$$\frac{dI}{dt} = \frac{1}{2} \int d\mathbf{r} r^2 \frac{\partial \rho}{\partial t} = -\frac{1}{2} \int d\mathbf{r} r^2 \nabla \cdot (\rho \mathbf{u}) = \int d\mathbf{r} \rho \mathbf{u} \cdot \mathbf{r}, \quad (13.68)$$

after integration by parts and assuming  $\mathbf{u} \perp \partial V$  (no in/outflows). Let us take another time derivative of this, this time using the momentum equation in the form (13.4),

$$\frac{\partial}{\partial t} \rho \mathbf{u} = -\nabla \cdot \mathbf{T}, \quad (13.69)$$

where the total stress tensor

$$\mathbf{T} = \rho \mathbf{u} \mathbf{u} + p \mathbf{I} + \boldsymbol{\Pi} + \frac{B^2}{8\pi} \mathbf{I} - \frac{\mathbf{B} \mathbf{B}}{4\pi} - \left( \frac{g^2}{8\pi G} \mathbf{I} - \frac{\mathbf{g} \mathbf{g}}{4\pi G} \right) \quad (13.70)$$

includes the Maxwell stress (13.20) and the gravitational stress (13.64). This gives us

$$\frac{d^2 I}{dt^2} = - \int d\mathbf{r} (\nabla \cdot \mathbf{T}) \cdot \mathbf{r} = - \int d\mathbf{r} [\nabla \cdot (\mathbf{T} \cdot \mathbf{r}) - \mathbf{T} : \nabla \mathbf{r}] = - \int d\mathbf{S} \cdot \mathbf{T} \cdot \mathbf{r} + \int d\mathbf{r} \text{tr } \mathbf{T}. \quad (13.71)$$

Therefore, in steady state, the stresses on the boundary satisfy

$$\int d\mathbf{S} \cdot \mathbf{T} \cdot \mathbf{r} = \int d\mathbf{r} \text{tr } \mathbf{T} = 2\mathcal{E}_{\text{kin}} + 3(\gamma - 1)\mathcal{E}_{\text{th}} + \mathcal{E}_{\text{mag}} + \mathcal{E}_G , \quad (13.72)$$

where  $\mathcal{E}_{\text{kin}}$ ,  $\mathcal{E}_{\text{th}}$ ,  $\mathcal{E}_{\text{mag}}$  and  $\mathcal{E}_G$  are the total kinetic, thermal, magnetic and gravitational energies of the system—the four terms in (13.65); note that  $\text{tr } \boldsymbol{\Pi} = 0$  (any trace is part of  $p$ ). In the absence of gravity, the weighted sum of energies in (13.72) is strictly positive and so it is not possible to have zero stress on the boundary of the system—the system cannot be self-confined. With gravity, since the gravitational energy (13.66) is negative, the right-hand side of (13.72) can be—and, in a steady, self-confined state, is—zero, telling us gravity can confine MHD systems, e.g., stars. The total energy in this case is, as you might have guessed, negative:

$$\mathcal{E} = \mathcal{E}_{\text{kin}} + \mathcal{E}_{\text{th}} + \mathcal{E}_{\text{mag}} + \mathcal{E}_G = -\mathcal{E}_{\text{kin}} - (3\gamma - 4)\mathcal{E}_{\text{th}} < 0. \quad (13.73)$$

A perhaps unexpected corollary of this is that if a star radiates and, therefore, loses energy, then  $\mathcal{E}_{\text{th}}$  must increase (assuming  $\mathcal{E}_{\text{kin}} \ll \mathcal{E}_{\text{th}}$ , i.e., subsonic motions inside)—i.e., “as a star cools, it heats up”.

### 13.12. Lagrangian MHD

There is a Lagrangian formulation of ideal MHD, due to Newcomb (1962, a classic and elegant paper, which I recommend for your reading pleasure). This is both mathematically attractive and sheds some physical light.

Let us label each fluid element’s position at  $t = 0$  by the *Lagrangian coordinate*  $\mathbf{r}_0$ . Then the *Eulerian coordinate*  $\mathbf{r}$  at any given time  $t$  is the position of the same fluid element at that time:

$$\mathbf{r}(t, \mathbf{r}_0) = \mathbf{r}_0 + \boldsymbol{\xi}(t, \mathbf{r}_0), \quad (13.74)$$

where  $\boldsymbol{\xi}$  is the *displacement*. Formally, we shall treat (13.74) as just a coordinate transformation, which can also be inverted: given the Eulerian coordinate  $\mathbf{r}$  of a fluid element at time  $t$ ,

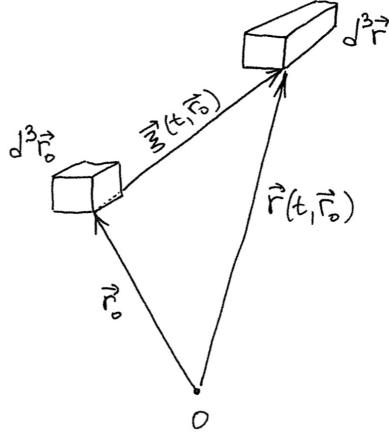


FIGURE 60. Lagrangian displacement of an infinitesimal fluid element.

the Lagrangian coordinate  $\mathbf{r}_0(t, \mathbf{r})$  is where this fluid element was at  $t = 0$ . The coordinate transformation is determined by the history of the fluid flow:

$$\frac{\partial \mathbf{r}(t, \mathbf{r}_0)}{\partial t} = \frac{\partial \xi(t, \mathbf{r}_0)}{\partial t} = \mathbf{u}(t, \mathbf{r}(t, \mathbf{r}_0)) \equiv \mathbf{u}_L(t, \mathbf{r}_0). \quad (13.75)$$

I shall use the subscript “L” to designate Lagrangian fields, i.e., MHD fields as functions of the Lagrangian coordinate and time.

Let us learn how to transform derivatives between Eulerian and Lagrangian variables:

$$\frac{\partial}{\partial r_{0i}} = \frac{\partial r_j}{\partial r_{0i}} \frac{\partial}{\partial r_j} = \left( \delta_{ji} + \frac{\partial \xi_j}{\partial r_{0i}} \right) \frac{\partial}{\partial r_j}, \quad \text{or} \quad \nabla_0 = (\nabla_0 \mathbf{r}) \cdot \nabla = (\mathbf{I} + \nabla_0 \xi) \cdot \nabla, \quad (13.76)$$

$$\frac{\partial}{\partial t_L} \equiv \left( \frac{\partial}{\partial t} \right)_{\mathbf{r}_0} = \left( \frac{\partial}{\partial t} \right)_r + \left( \frac{\partial r_i}{\partial t} \right)_{\mathbf{r}_0} \frac{\partial}{\partial r_i} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla = \frac{d}{dt}. \quad (13.77)$$

Thus, the Lagrangian time derivative is the convective derivative—you knew that! We are now ready to convert the MHD equations (13.57–13.60) to the Lagrangian frame.

### 13.12.1. Density

The continuity equation is an expression of conservation of mass (§13.1). Let us write that for an infinitesimal volume element moving with the fluid:

$$\rho_0(\mathbf{r}_0) d\mathbf{r}_0 = \rho_L(t, \mathbf{r}_0) d\mathbf{r}, \quad (13.78)$$

where  $\rho_0$  and  $d\mathbf{r}_0$  are the density and the volume of the fluid element at  $t = 0$ , and  $\rho_L$  and  $d\mathbf{r}$  are its density and volume at time  $t$  (Fig. 60). So we need to know how volumes change under the coordinate transformation (13.74), i.e., we need the Jacobian of the *strain matrix*  $\partial r_i / \partial r_{0j}$ :

$$d\mathbf{r} = J(t, \mathbf{r}_0) d\mathbf{r}_0, \quad \text{where} \quad J = |\det \nabla_0 \mathbf{r}| = \frac{1}{6} \epsilon_{ijk} \epsilon_{lmn} \frac{\partial r_i}{\partial r_{0l}} \frac{\partial r_j}{\partial r_{0m}} \frac{\partial r_k}{\partial r_{0n}}. \quad (13.79)$$

The continuity equation in the Lagrangian frame is, therefore,

$$\rho_L(t, \mathbf{r}_0) = \frac{\rho_0(\mathbf{r}_0)}{J(t, \mathbf{r}_0)}.$$

(13.80)

### 13.12.2. Pressure

The result (13.80), together with (13.77), makes it really easy to work out the Lagrangian pressure. Indeed, the adiabatic law (13.60) implies that the entropy density of Lagrangian fluid elements never changes:

$$\frac{\partial}{\partial t} \frac{p_L}{\rho_L^\gamma} = \frac{d}{dt} \frac{p}{\rho^\gamma} = 0 \quad \Rightarrow \quad \frac{p_L}{\rho_L^\gamma} = \frac{p_0}{\rho_0^\gamma}. \quad (13.81)$$

Using (13.80), we get

$$\boxed{p_L(t, \mathbf{r}_0) = \frac{p_0(\mathbf{r}_0)}{J^\gamma(t, \mathbf{r}_0)}} . \quad (13.82)$$

### 13.12.3. Magnetic Field

To work out the magnetic field, consider the induction equation in the form (13.32). In Lagrangian variables,

$$\frac{\partial}{\partial t} \frac{\mathbf{B}_L}{\rho_L} = \frac{\mathbf{B}_L}{\rho_L} \cdot \nabla \mathbf{u} = \frac{\mathbf{B}_L}{\rho_L} \cdot (\nabla \mathbf{r}_0) \cdot \nabla_0 \mathbf{u}_L = \frac{\mathbf{B}_L}{\rho_L} \cdot (\nabla \mathbf{r}_0) \cdot \nabla_0 \frac{\partial \mathbf{r}}{\partial t} . \quad (13.83)$$

The solution to this that satisfies  $\mathbf{B}_L/\rho_L = \mathbf{B}_0/\rho_0$  at  $t = 0$  is

$$\frac{\mathbf{B}_L}{\rho_L} = \frac{\mathbf{B}_0}{\rho_0} \cdot \nabla_0 \mathbf{r} . \quad (13.84)$$

Indeed:

$$\frac{\partial}{\partial t} \frac{\mathbf{B}_L}{\rho_L} = \frac{\mathbf{B}_0}{\rho_0} \cdot \nabla_0 \frac{\partial \mathbf{r}}{\partial t} = \frac{\mathbf{B}_L}{\rho_L} \cdot (\nabla_0 \mathbf{r})^{-1} \cdot \nabla_0 \frac{\partial \mathbf{r}}{\partial t} = \frac{\mathbf{B}_L}{\rho_L} \cdot (\nabla \mathbf{r}_0) \cdot \nabla_0 \frac{\partial \mathbf{r}}{\partial t}, \quad \text{q.e.d.} \quad (13.85)$$

Thus, the magnetic field in the Lagrangian frame satisfies (13.84), or, using (13.80),

$$\boxed{\mathbf{B}_L(t, \mathbf{r}_0) = \frac{\mathbf{B}_0(\mathbf{r}_0) \cdot \nabla_0 \mathbf{r}(t, \mathbf{r}_0)}{J(t, \mathbf{r}_0)}} . \quad (13.86)$$

This solution is really just a restatement of the Lundquist theorem: (13.84) says that  $\mathbf{B}/\rho$  transforms in the same way as a vector connecting two infinitesimally close material points in the fluid [see (13.33)].

### 13.12.4. Fluid Flow

Finally, let us deal with the momentum equation (13.58), which in this context is the equation that actually defines the Lagrangian variable transformation (13.74). Using the decomposition (13.20) of the Lorentz force into magnetic pressure and tension, and substituting for  $\rho$ ,  $p$  and  $\mathbf{B}$  from (13.80), (13.82) and (13.86), respectively, we get, noting that  $(\nabla_0 \mathbf{r}) \cdot \nabla = \nabla_0$ ,

$$\boxed{\frac{\rho_0}{J} \frac{\partial^2 \mathbf{r}}{\partial t^2} = -(\nabla_0 \mathbf{r})^{-1} \cdot \nabla_0 \left( \frac{p_0}{J^\gamma} + \frac{|\mathbf{B}_0 \cdot \nabla_0 \mathbf{r}|^2}{8\pi J^2} \right) + \frac{1}{4\pi} \frac{\mathbf{B}_0}{J} \cdot \nabla_0 \frac{\mathbf{B}_0}{J} \cdot \nabla_0 \mathbf{r}} . \quad (13.87)$$

Coupled with the initial condition  $\mathbf{r}(0) = \mathbf{r}_0$  and the formula (13.79) for  $J$ , this equation determines the trajectory  $\mathbf{r}(t, \mathbf{r}_0)$  of each fluid element and hence its Lagrangian displacement  $\boldsymbol{\xi} = \mathbf{r} - \mathbf{r}_0$ , its velocity  $\mathbf{u}_L = \partial \boldsymbol{\xi} / \partial t$ , and the associated density, pressure and magnetic field via (13.80), (13.82) and (13.86). It is not a particularly pretty equation—the price we have paid for being able to integrate explicitly all the other MHD equations.

**Exercise 13.2. Energy in Lagrangian MHD.** Show that the total energy of a volume of MHD fluid in Lagrangian variables is

$$\mathcal{E} = \int d\mathbf{r}_0 \left[ \frac{1}{2} \rho_0 \left( \frac{\partial \mathbf{r}}{\partial t} \right)^2 + \frac{p_0 J^{-(\gamma-1)}}{\gamma-1} + \frac{|\mathbf{B}_0 \cdot \nabla_0 \mathbf{r}|^2}{8\pi J} \right] . \quad (13.88)$$

**Exercise 13.3. Action principle for Lagrangian MHD.** Show that (13.87) can be derived from an action principle,  $\delta S = 0$ , with

$$S = \int_0^t dt \int d\mathbf{r}_0 \mathcal{L}(\mathbf{r}, \dot{\mathbf{r}}), \quad (13.89)$$

where the Lagrangian density is

$$\mathcal{L}(\mathbf{r}, \dot{\mathbf{r}}) = \frac{\rho_0 |\dot{\mathbf{r}}|^2}{2} - \frac{p_0 J^{-(\gamma-1)}}{\gamma-1} - \frac{|\mathbf{B}_0 \cdot \nabla_0 \mathbf{r}|^2}{8\pi J} \quad (13.90)$$

(cf. Kulsrud 2005, §4.7).

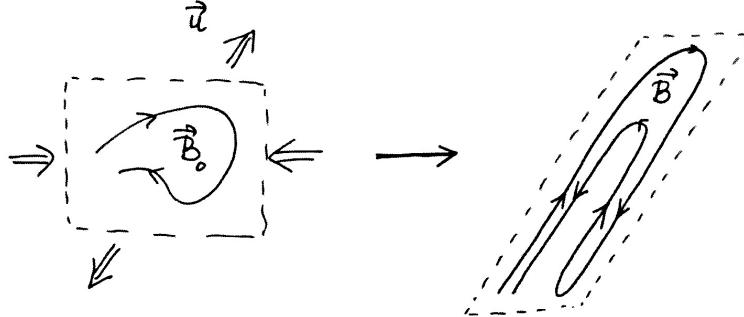


FIGURE 61. Effect of stretching/shearing/compression on magnetic field: “folded” field with small-scale direction reversals.

The Lagrangian formalism, besides shedding conceptual light, turns out to give one some useful analytical tools, e.g., for the treatment of explosive MHD instabilities (Pfirsch & Sudan 1993; Cowley & Artun 1997).

An obvious problem with using this approach to describe the MHD fluid over any significant intervals of time is that it only works for ideal MHD. Even if we restrict ourselves to nice flows that do not have small scales and are thus immune to viscosity, (13.86) tells us that magnetic field can quickly develop small scales and thus access resistivity. An example of how that could happen starting with a fairly generic magnetic configuration is in Q5. An even simpler example is an application of a simple, linear stretching/shearing/compressing transformation (which can be achieved by an arbitrarily large-scale flow) to a curved field line, producing a “folded” field with direction reversals on ever smaller scales (Fig. 61). It is clear that resistivity must kick in, at which point the Lagrangian solution (13.86) can no longer be used. Let me offer two examples of how one might deal with such a situation mathematically and in the process get an idea of what happens physically.

### 13.13. Eyink’s Stochastic Lundquist Theorem

Coming soon...

## 14. Small-Scale Dynamo

### 14.1. Zeldovich’s Dynamo in a Linear Flow

Let us consider the evolution of magnetic field in a linear flow, i.e.,

$$\mathbf{u}(t, \mathbf{r}) = \mathbf{A}(t) \cdot \mathbf{r}, \quad (14.1)$$

where  $\mathbf{A}(t)$  is some matrix that may be a function of time, but not of space. The key simplification that comes with this assumption (admittedly at a heavy price!) is that the strain matrix  $\nabla_0 \mathbf{r}$  is independent of position: indeed, it is the solution of

$$\frac{\partial}{\partial t} \nabla_0 \mathbf{r} = \nabla_0 \mathbf{u}_L = (\nabla_0 \mathbf{r}) \cdot \mathbf{A}^T(t), \quad \nabla_0 \mathbf{r}(t=0) = \mathbf{I}. \quad (14.2)$$

Let us further assume, for simplicity, that the flow is incompressible ( $\nabla \cdot \mathbf{u} = \text{tr } \mathbf{A} = 0$ ). Then  $\rho = \rho_0 = \text{const}$  and so  $J = 1$ .

#### 14.1.1. “Lagrangian” Solution of Induction Equation with Resistivity

The (Lagrangian) magnetic field satisfies the induction equation with resistivity:

$$\frac{\partial \mathbf{B}_L}{\partial t} = \underbrace{\mathbf{B}_L \cdot \nabla \mathbf{u}}_{= \mathbf{A} \cdot \mathbf{B}_L} + \eta \nabla^2 \mathbf{B}_L, \quad (14.3)$$

where the gradients in the right-hand side are still Eulerian. Now let us seek a solution of this equation in the form

$$\mathbf{B}_L(t, \mathbf{r}_0) = \hat{\mathbf{B}}(t)e^{i\mathbf{k}(t)\cdot\mathbf{r}(t, \mathbf{r}_0)}. \quad (14.4)$$

At  $t = 0$ , this is simply a single-Fourier-mode initial field:  $\mathbf{B}_0(\mathbf{r}_0) = \hat{\mathbf{B}}_0 e^{i\mathbf{k}_0 \cdot \mathbf{r}_0}$ , where  $\mathbf{k}_0 = \mathbf{k}(0)$  and  $\hat{\mathbf{B}}_0 = \hat{\mathbf{B}}(0)$ . Let us see if such functions  $\hat{\mathbf{B}}(t)$  and  $\mathbf{k}(t)$  can be found that (14.4) works. Substituting (14.4) into (14.3), one gets

$$\frac{\partial \hat{\mathbf{B}}}{\partial t} + i \left( \frac{\partial \mathbf{k}}{\partial t} \cdot \mathbf{r} + \mathbf{k} \cdot \frac{\partial \mathbf{r}}{\partial t} \right) \hat{\mathbf{B}} = \mathbf{A} \cdot \hat{\mathbf{B}} - \eta k^2(t) \hat{\mathbf{B}}. \quad (14.5)$$

Since  $\dot{\mathbf{r}} = \mathbf{u} = \mathbf{A} \cdot \mathbf{r}$ , the second term on the left-hand side is annihilated if

$$\frac{\partial \mathbf{k}}{\partial t} = -\mathbf{k} \cdot \mathbf{A} \Rightarrow \mathbf{k}(t) = (\nabla \mathbf{r}_0)(t) \cdot \mathbf{k}_0. \quad (14.6)$$

The last expression follows from (14.2) if one works out the time derivative of  $\nabla \mathbf{r}_0$  by differentiating  $(\nabla \mathbf{r}_0) \cdot (\nabla_0 \mathbf{r}) = \mathbf{I}$ . We are left with

$$\frac{\partial \hat{\mathbf{B}}}{\partial t} = \mathbf{A} \cdot \hat{\mathbf{B}} - \eta k^2(t) \hat{\mathbf{B}} \Rightarrow \boxed{\hat{\mathbf{B}}(t) = \hat{\mathbf{B}}_0 \cdot (\nabla_0 \mathbf{r})(t) \exp \left[ -\eta \int_0^t dt' k^2(t') \right]}. \quad (14.7)$$

The solution has been obtained by eliminating the resistive term via an integrating factor and taking care of the rest by using (14.2). The Lagrangian solution (13.86) has re-emerged as a prefactor, now attenuated by resistive decay.

Since the induction equation is linear in  $\mathbf{B}$ , any linear combination of solutions of the form (14.4) is also a solution. Therefore, we can accommodate any initial field  $\mathbf{B}_0$ : it will evolve according to

$$\begin{aligned} \mathbf{B}_L(t, \mathbf{r}_0) &= \sum_{\mathbf{k}_0} \hat{\mathbf{B}}(t, \mathbf{k}_0) e^{i\mathbf{k}(t, \mathbf{k}_0) \cdot \mathbf{r}(t, \mathbf{r}_0)} \\ &= \sum_{\mathbf{k}_0} \hat{\mathbf{B}}_0(\mathbf{k}_0) \cdot (\nabla_0 \mathbf{r})(t) \exp \left[ ik(t, \mathbf{k}_0) \cdot \mathbf{r}(t, \mathbf{r}_0) - \eta \int_0^t dt' k^2(t', \mathbf{k}_0) \right], \end{aligned} \quad (14.8)$$

where  $\hat{\mathbf{B}}_0(\mathbf{k}_0)$  is the Fourier coefficient of the initial field  $\mathbf{B}_0(\mathbf{r}_0) = \sum_{\mathbf{k}_0} \hat{\mathbf{B}}(\mathbf{k}_0) e^{i\mathbf{k}_0 \cdot \mathbf{r}_0}$  and  $k(t, \mathbf{k}_0)$  satisfies (14.6).

### 14.1.2. Is There Dynamo?

An interesting question now is whether the energy of the solution (14.8) grows with time—is the velocity field (14.1) a dynamo? It is not hard to prove a version of the Parseval theorem: the volume average of the magnetic energy is

$$\langle B^2 \rangle(t) \equiv \int \frac{d\mathbf{r}}{V} |\mathbf{B}(t, \mathbf{r})|^2 = \sum_{\mathbf{k}_0} |\hat{\mathbf{B}}(t, \mathbf{k}_0)|^2. \quad (14.9)$$

**Exercise 14.1.** Prove (14.9).

Using (14.7),

$$|\hat{\mathbf{B}}(t, \mathbf{k}_0)|^2 = \hat{\mathbf{B}}_0(\mathbf{k}_0) \cdot \mathbf{g}(t) \cdot \hat{\mathbf{B}}_0^*(\mathbf{k}_0) \exp \left[ -2\eta \int_0^t dt' \mathbf{k}_0 \cdot \mathbf{g}^{-1}(t') \cdot \mathbf{k}_0 \right], \quad (14.10)$$

where  $\mathbf{g}$  is the (covariant) metric tensor associated with the Largangian transformation of variables:

$$g_{ij} = \frac{\partial r_k}{\partial r_{0i}} \frac{\partial r_k}{\partial r_{0j}}, \quad (g^{-1})_{ij} = \frac{\partial r_{0i}}{\partial r_k} \frac{\partial r_{0j}}{\partial r_k}. \quad (14.11)$$

These matrices can (in principle) be calculated via (14.2) for any given  $\mathbf{A}(t)$ .

It is possible to do this with a degree of generality, but a lot of formalism is needed along the

way. I will instead opt for an extremely simple case:

$$\mathbf{A} = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_3 \end{pmatrix} \Rightarrow \mathbf{g} = \begin{pmatrix} e^{2\lambda_1 t} & & 0 \\ & e^{2\lambda_2 t} & \\ 0 & & e^{2\lambda_3 t} \end{pmatrix}, \quad (14.12)$$

where  $\lambda_1 > \lambda_2 > 0 > \lambda_3$  are constants and  $\lambda_1 + \lambda_2 + \lambda_3 = 0$  (incompressibility). These three rates are rates of stretching and compression by the flow. Ignoring exponentially small terms, we get

$$|\hat{\mathbf{B}}(t, \mathbf{k}_0)|^2 \approx |\hat{B}_{01}|^2 \exp \left[ 2\lambda_1 t - \eta \left( \frac{k_{01}^2}{\lambda_1} + \frac{k_{02}^2}{\lambda_2} + \frac{k_{03}^2}{|\lambda_3|} e^{2|\lambda_3|t} \right) \right]. \quad (14.13)$$

This says that for most  $\mathbf{k}_0$ , the corresponding modes will decay superexponentially (after initial transient growth at the “ideal” rate  $2\lambda_1$ ). At any given time  $t \gg \lambda_1^{-1}$ , the domain in the  $\mathbf{k}_0$  space that will dominate the integral (14.9) is one containing modes for which the resistivity has not yet managed to cut the initial growth rate:

$$\eta \left( \frac{k_{01}^2}{\lambda_1} + \frac{k_{02}^2}{\lambda_2} + \frac{k_{03}^2}{|\lambda_3|} e^{2|\lambda_3|t} \right) \lesssim \text{const.} \quad (14.14)$$

This is the interior of an ellipsoid whose volume is  $\propto e^{-|\lambda_3|t}$ . Within this volume,  $|\hat{\mathbf{B}}(t, \mathbf{k}_0)|^2 \sim |\hat{B}_{01}|^2 e^{2\lambda_1 t}$ . Therefore, the integral (14.9) is, roughly,

$$\langle B^2 \rangle(t) \propto e^{(2\lambda_1 - |\lambda_3|)t} = e^{(\lambda_1 - \lambda_2)t}, \quad (14.15)$$

because  $\lambda_3 = -\lambda_1 - \lambda_2$ . Since  $\lambda_1 > \lambda_2$ , the conclusion is that magnetic energy will grow, albeit thanks to an ever shrinking subset of initial modes.

This calculation is due to [Zeldovich et al. \(1984\)](#) (but was cleverly anticipated 20 years previously in a one-page note by [Moffatt & Saffman 1964](#)). The physical interpretation of it was some time in coming—let me explain it the way I understand it ([Schekochihin et al. 2004](#)).

#### 14.1.3. Folded Fields

The three rates  $\lambda_1$ ,  $\lambda_2$  and  $\lambda_3$  represent the flow’s action along the three (Lyapunov) directions locally associated with it: stretching at the rate  $\lambda_1$  along the first direction ( $\hat{\mathbf{e}}_1$ ), compression at the rate  $|\lambda_3|$  along the third ( $\hat{\mathbf{e}}_3$ ) and something along the second, “null” direction ( $\hat{\mathbf{e}}_2$ )—it can indeed be null ( $\lambda_2 = 0$ ), but it can also be stretching or compression at a smaller rate than the other two. It follows from (14.7) and (14.6) that the magnetic field will align with the stretching direction whereas wave vector will align with the compression direction, both exponentially fast:

$$\mathbf{B} \sim B_{01} \hat{\mathbf{e}}_1 e^{\lambda_1 t}, \quad \mathbf{k} \sim k_{03} \hat{\mathbf{e}}_3 e^{|\lambda_3|t}. \quad (14.16)$$

The latter alignment is what makes most modes decay superexponentially—in physical terms, this is the tendency, illustrated by Fig. 61, for the fields to become folded and reverse direction at ever smaller scales. The only modes that survive are those for which the initial wave vector  $\mathbf{k}_0$  was nearly perpendicular to  $\hat{\mathbf{e}}_3$ , with its permitted angular deviation from 90° decaying  $\sim e^{-|\lambda_3|t}$ . Since  $\nabla \cdot \mathbf{B} = 0$ ,  $\mathbf{k}_0 \cdot \hat{\mathbf{B}}_0 = 0$  must be satisfied for the initial field. Thus, the modes that get amplified most are ones for which  $\hat{\mathbf{B}}_0 \parallel \hat{\mathbf{e}}_1$  and  $\mathbf{k}_0 \parallel \hat{\mathbf{e}}_2$ . Such a “winning” fold is sketched in Fig. 62.

---

**Exercise 14.2.** What happens, mathematically and physically, if (a)  $\lambda_2 = 0$ , (b)  $\lambda_2 < 0$ ?

**Exercise 14.3.** Work through an analogous argument in 2D and show that magnetic field always decays (no 2D dynamo; cf. Q4). Does this make sense physically?

---

#### 14.1.4. Further Reading

Considering that the above calculation was done for a preposterously simplistic flow, is it at all useful in understanding real dynamos? It turns out to be surprisingly so. One does see folded fields in generic fluctuation dynamos when they are driven by smooth velocity fields, even random ones ([Schekochihin et al. 2004; Rincon 2019](#)). Locally such fields can be viewed

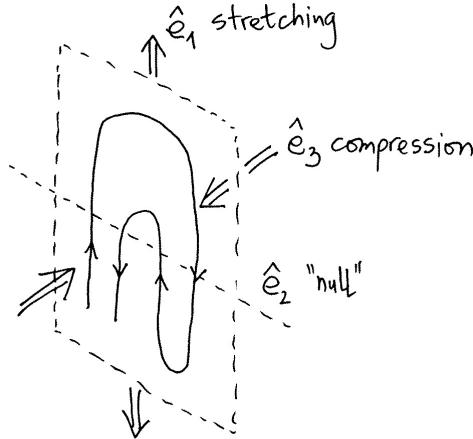


FIGURE 62. A “winning” fold in Zeldovich-type dynamo.

as linear (14.1), but perhaps not constant time. This is OK as the above construction can be generalised to time-dependent fields. Indeed, since the matrix  $\mathbf{g}(t)$  is symmetric, it can at any time be diagonalised by an appropriate rotation:

$$\mathbf{g}(t) = \mathbf{R}^T(t) \cdot \begin{pmatrix} e^{\Lambda_1(t)} & & 0 \\ & e^{\Lambda_2(t)} & \\ 0 & & e^{\Lambda_3(t)} \end{pmatrix} \cdot \mathbf{R}(t), \quad (14.17)$$

where the quantities  $\Lambda_i(t)/2t$  are known as the *finite-time Lyapunov exponents (FTLEs)*. It is possible to prove (Goldhirsch *et al.* 1987) that  $\mathbf{R}(t)$  converges to a constant matrix  $(\hat{e}_1 \hat{e}_2 \hat{e}_3)$  (*Lyapunov basis*) exponentially fast in time and that (more slowly)  $\Lambda_i(t)/2t \rightarrow \lambda_i$  (*Lyapunov exponents*).

When the flow is random, the FTLEs are random functions and then one can prove that (14.15) survives in the form

$$\overline{\langle B^2 \rangle}(t) \propto e^{[\Lambda_1(t) - \Lambda_2(t)]/2}, \quad (14.18)$$

where the overline means averaging over the distribution of the  $\Lambda_i$ 's. This distribution is usually quite hard to calculate for any real flow, so it has only been done for a few very special examples.

If you are intrigued by this line of inquiry, you might find further enlightenment in Zeldovich *et al.* (1984) and then, e.g., in Ott (1998) and Chertkov *et al.* (1999).

## 14.2. Kazantsev Spectrum of Dynamo-Generated Field

Coming soon. In the meanwhile, you may read about this topic in my hand-written notes available here: [http://www-thphys.physics.ox.ac.uk/people/AlexanderSchekochihin/notes/PartIIIMHD/LecturesL05/sec23\\_Spectrum.pdf](http://www-thphys.physics.ox.ac.uk/people/AlexanderSchekochihin/notes/PartIIIMHD/LecturesL05/sec23_Spectrum.pdf).

## 15. MHD in a Straight Magnetic Field

Equations (13.57–13.60) have a very simple static, uniform *equilibrium solution*:

$$\rho_0 = \text{const}, \quad p_0 = \text{const}, \quad \mathbf{u}_0 = 0, \quad \mathbf{B}_0 = B_0 \hat{z} = \text{const}. \quad (15.1)$$

We will turn to more nontrivial equilibria in due course, but first we shall study this one carefully—because it is very generic in the sense that many other, more complicated, equilibria *locally* look just like this.

## 15.1. MHD Waves

If you have an equilibrium solution of any set of equations, your first reflex ought to be to perturb it and see what happens: the system might support waves, instabilities, possibly interesting nonlinear behaviour of small perturbations (e.g., §§6–8).

So we now seek solutions to the MHD equations (13.57–13.60) in the form of

$$\rho = \rho_0 + \delta\rho, \quad p = p_0 + \delta p, \quad \mathbf{u} = \frac{\partial \boldsymbol{\xi}}{\partial t}, \quad \mathbf{B} = \mathbf{B}_0 \hat{\mathbf{z}} + \delta\mathbf{B}, \quad (15.2)$$

where we have introduced the fluid *displacement* field  $\boldsymbol{\xi}$  (cf. §13.12).<sup>101</sup> To start with, we consider all perturbations to be infinitesimal and so linearise the MHD equations (13.57–13.60) as follows.

$$\begin{aligned} \left( \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \rho &= -\rho \nabla \cdot \mathbf{u} \quad \Rightarrow \quad \frac{\partial \delta\rho}{\partial t} = -\rho_0 \nabla \cdot \frac{\partial \boldsymbol{\xi}}{\partial t} \\ &\Rightarrow \boxed{\frac{\delta\rho}{\rho_0} = -\nabla \cdot \boldsymbol{\xi}}, \end{aligned} \quad (15.3)$$

$$\begin{aligned} \left( \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \frac{p}{\rho^\gamma} &= 0 \quad \Rightarrow \quad \frac{\partial}{\partial t} \frac{\delta p}{p_0} = \gamma \frac{\partial}{\partial t} \frac{\delta\rho}{\rho_0} \\ &\Rightarrow \boxed{\frac{\delta p}{p_0} = -\gamma \nabla \cdot \boldsymbol{\xi}}, \end{aligned} \quad (15.4)$$

$$\begin{aligned} \left( \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \mathbf{B} &= \mathbf{B} \cdot \nabla \mathbf{u} - \mathbf{B} \nabla \cdot \mathbf{u} \quad \Rightarrow \quad \frac{\partial \delta\mathbf{B}}{\partial t} = \mathbf{B}_0 \nabla_{\parallel} \frac{\partial \boldsymbol{\xi}}{\partial t} - \hat{\mathbf{z}} \mathbf{B}_0 \nabla \cdot \frac{\partial \boldsymbol{\xi}}{\partial t} \\ &\Rightarrow \frac{\delta\mathbf{B}}{\mathbf{B}_0} = \nabla_{\parallel} \boldsymbol{\xi} - \hat{\mathbf{z}} \nabla \cdot \boldsymbol{\xi} = \nabla_{\parallel} \boldsymbol{\xi}_{\perp} - \hat{\mathbf{z}} \nabla_{\perp} \cdot \boldsymbol{\xi}_{\perp} \\ &\Rightarrow \boxed{\frac{\delta\mathbf{B}_{\perp}}{\mathbf{B}_0} = \nabla_{\parallel} \boldsymbol{\xi}_{\perp}}, \quad \boxed{\frac{\delta\mathbf{B}_{\parallel}}{\mathbf{B}_0} = -\nabla_{\perp} \cdot \boldsymbol{\xi}_{\perp}}, \end{aligned} \quad (15.5)$$

where  $\parallel$  and  $\perp$  denote projections onto the direction ( $z$ ) of  $\mathbf{B}_0$  and onto the plane ( $x, y$ ) perpendicular to it, respectively. Equations (15.5) tell us that parallel displacements produce no perturbation of the magnetic field—obviously not, because the magnetic field is carried with the fluid flow and nothing will happen if you displace a straight uniform field parallel to itself.

The physics of magnetic-field perturbations becomes clearer if we observe that

$$\frac{\delta\mathbf{B}}{\mathbf{B}_0} = \frac{\delta(\mathbf{B}\mathbf{b})}{\mathbf{B}_0} = \delta\mathbf{b} + \hat{\mathbf{z}} \frac{\delta B}{B_0}. \quad (15.6)$$

The perturbed field-direction vector  $\delta\mathbf{b}$  must be perpendicular to  $\hat{\mathbf{z}}$  (otherwise the field direction is unperturbed; formally this is shown by perturbing the equation  $\mathbf{b}^2 = 1$ ). Therefore, the perpendicular and parallel perturbations of the magnetic field are the

<sup>101</sup>Thinking in terms of displacements makes sense in MHD but not so much in (homogeneous) hydrodynamics because in the latter case, just displacing a fluid element produces no back reaction, whereas in MHD, since magnetic fields are frozen into the fluid and are elastic, displacing fluid elements causes magnetic restoring forces to switch on. In other words, an (ideal) MHD fluid “remembers” the state from which it has been displaced, whereas neutral (Newtonian) fluids only “know” about velocities at which they flow.

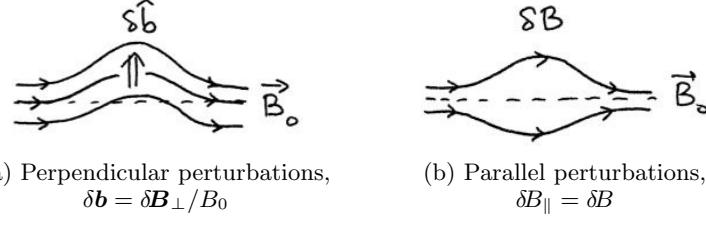


FIGURE 63. Magnetic perturbations.

perturbations of its direction and strength, respectively (Fig. 63):

$$\frac{\delta \mathbf{B}_\perp}{B_0} = \delta \mathbf{b}, \quad \frac{\delta B_\parallel}{B_0} = \delta B. \quad (15.7)$$

Finally, linearising (13.58) gives us

$$\underbrace{\rho \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right)}_{= \rho_0 \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2}} = \underbrace{-\nabla p}_{= -\nabla \delta p \text{ from (15.4)}} - \underbrace{\nabla \frac{B^2}{8\pi}}_{= -\frac{B_0^2}{4\pi} \nabla \frac{\delta B}{B_0}} + \underbrace{\frac{\mathbf{B} \cdot \nabla \mathbf{B}}{4\pi}}_{= \frac{B_0^2}{4\pi} \nabla_\parallel \left( \delta \mathbf{b} + \hat{z} \frac{\delta B}{B_0} \right)} . \quad (15.8)$$

$$\begin{aligned} &= \frac{B_0^2}{4\pi} \left( -\nabla_\perp \frac{\delta B}{B_0} + \nabla_\parallel \delta \mathbf{b} \right) \\ &= \frac{B_0^2}{4\pi} (\nabla_\perp \nabla_\perp \cdot \boldsymbol{\xi}_\perp + \nabla_\parallel^2 \boldsymbol{\xi}_\perp), \end{aligned}$$

Assembling all this, we get

$$\boxed{\frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = c_s^2 \nabla \nabla \cdot \boldsymbol{\xi} + v_A^2 (\nabla_\perp \nabla_\perp \cdot \boldsymbol{\xi}_\perp + \nabla_\parallel^2 \boldsymbol{\xi}_\perp)}, \quad (15.9)$$

where two special velocities have emerged:

$$c_s = \sqrt{\frac{\gamma p_0}{\rho_0}}, \quad v_A = \frac{B_0}{\sqrt{4\pi \rho_0}}, \quad (15.10)$$

the *sound speed* and the *Alfvén speed*, respectively. The former is familiar from fluid dynamics, while the latter is another speed, arising in MHD, at which perturbations can travel. We shall see momentarily how this happens.

---

**Exercise 15.1.** Derive (15.3–15.5) and (15.9) directly from Lagrangian MHD equations (13.80), (13.82), (13.86) and (13.87). A useful starting point is that, for an infinitesimal displacement,  $J = 1 + \nabla_0 \cdot \boldsymbol{\xi}$  [follows from (13.79)].

---

Let us seek wave-like solutions of (15.9),  $\boldsymbol{\xi} \propto \exp(-i\omega t + i\mathbf{k} \cdot \mathbf{r})$ . For such perturbations,

$$\omega^2 \boldsymbol{\xi} = c_s^2 \mathbf{k} \mathbf{k} \cdot \boldsymbol{\xi} + v_A^2 (\mathbf{k}_\perp \mathbf{k}_\perp \cdot \boldsymbol{\xi}_\perp + k_\parallel^2 \boldsymbol{\xi}_\perp). \quad (15.11)$$

Without loss of generality, let  $\mathbf{k} = (k_\perp, 0, k_\parallel)$  (i.e., by definition,  $x$  is the direction of  $\mathbf{k}_\perp$ ; see Fig. 64). Then (15.11) becomes

$$\omega^2 \xi_x = c_s^2 k_\perp (k_\perp \xi_x + k_\parallel \xi_\parallel) + v_A^2 k^2 \xi_x, \quad (15.12)$$

$$\omega^2 \xi_y = v_A^2 k_\parallel^2 \xi_y, \quad (15.13)$$

$$\omega^2 \xi_\parallel = c_s^2 k_\parallel (k_\perp \xi_x + k_\parallel \xi_\parallel). \quad (15.14)$$

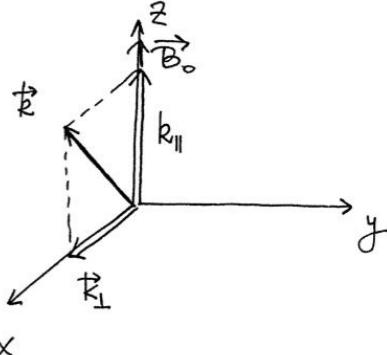


FIGURE 64. Coordinate system for the treatment of MHD waves.

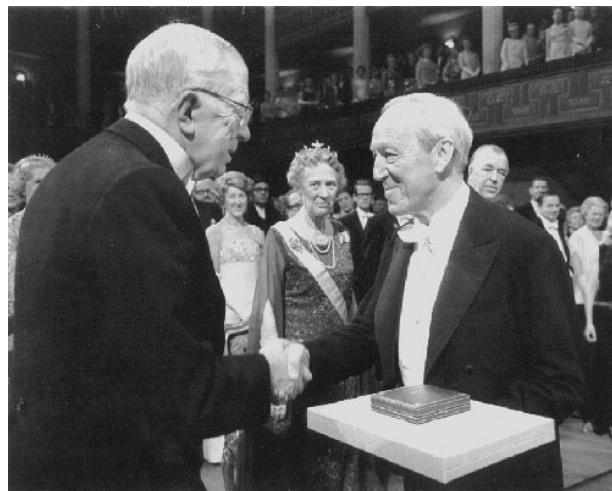


FIGURE 65. Hannes Olof Gösta Alfvén (1908-1995), Swedish electrical engineer and plasma physicist. He was the father of MHD, distrusted religion, computers and Big Bang theory, and got a Nobel Prize “for fundamental work and discoveries in magnetohydrodynamics with fruitful applications in different parts of plasma physics” (1970). In this picture, he is receiving it from King Gustaf VI Adolf of Sweden.

The perturbations of the rest of the fields are

$$\frac{\delta\rho}{\rho_0} = -i\mathbf{k} \cdot \boldsymbol{\xi} = -i(k_{\perp}\xi_x + k_{\parallel}\xi_{\parallel}), \quad (15.15)$$

$$\frac{\delta p}{p_0} = \gamma \frac{\delta\rho}{\rho_0}, \quad (15.16)$$

$$\delta\mathbf{b} = ik_{\parallel}\boldsymbol{\xi}_{\perp} = ik_{\parallel} \begin{pmatrix} \xi_x \\ \xi_y \\ 0 \end{pmatrix}, \quad (15.17)$$

$$\frac{\delta B}{B_0} = -ik_{\perp}\xi_x. \quad (15.18)$$

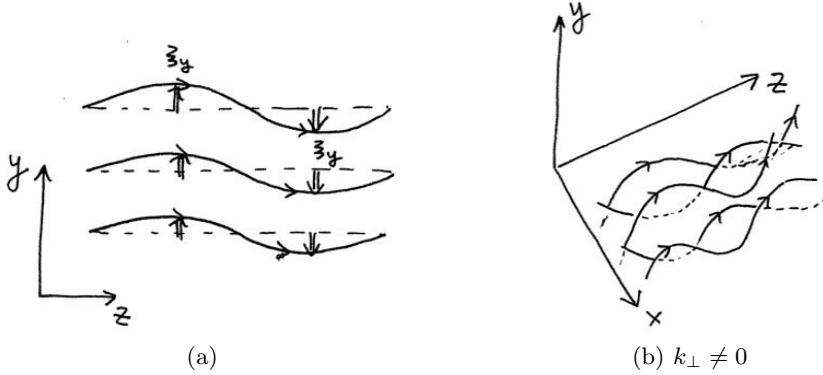


FIGURE 66. Alfvén waves.

### 15.1.1. Alfvén Waves

We start by spotting, instantly, that (15.13) decouples from the rest of the system. Therefore,  $\xi = (0, \xi_y, 0)$  is an eigenvector, with two associated eigenvalues

$$\boxed{\omega = \pm k_{\parallel} v_A} , \quad (15.19)$$

representing *Alfvén waves* propagating parallel and antiparallel to  $\mathbf{B}_0$ . An Alfvénic perturbation is (Fig. 66a)

$$\xi = \xi_y \hat{y}, \quad \delta\rho = 0, \quad \delta p = 0, \quad \delta B = 0, \quad \delta\mathbf{b} = ik_{\parallel} \xi_y \hat{y}, \quad (15.20)$$

i.e., it is incompressible and only involves magnetic field lines behaving as elastic strings, springing back against perturbing motions, due to the restoring curvature force. Note that these waves can have  $k_{\perp} \neq 0$  even though their dispersion relation (15.19) does not depend on  $k_{\perp}$  (Fig. 66b).

### 15.1.2. Magnetosonic Waves

Equations (15.12) and (15.14) form a closed 2D system:

$$\omega^2 \begin{pmatrix} \xi_x \\ \xi_{\parallel} \end{pmatrix} = \begin{pmatrix} c_s^2 k_{\perp}^2 + v_A^2 k^2 & c_s^2 k_{\parallel} k_{\perp} \\ c_s^2 k_{\parallel} k_{\perp} & c_s^2 k_{\parallel}^2 \end{pmatrix} \cdot \begin{pmatrix} \xi_x \\ \xi_{\parallel} \end{pmatrix}. \quad (15.21)$$

The resulting dispersion relation is

$$\omega^4 - k^2(c_s^2 + v_A^2)\omega^2 + c_s^2 v_A^2 k^2 k_{\parallel}^2 = 0. \quad (15.22)$$

This has four solutions:

$$\omega^2 = \frac{1}{2} k^2 \left[ c_s^2 + v_A^2 \pm \sqrt{(c_s^2 + v_A^2)^2 - 4c_s^2 v_A^2 \cos^2 \theta} \right], \quad \cos^2 \theta = \frac{k_{\parallel}^2}{k^2}. \quad (15.23)$$

The two “+” solutions are the “*fast magnetosonic waves*” and the two “−” ones are the “*slow magnetosonic waves*”.

Since both sound and Alfvén speeds are involved, it is obvious that the key parameter demarcating different physical regimes will be their ratio, or, conventionally, the ratio of the thermal to magnetic energies in the MHD medium, known as the *plasma beta*:

$$\beta = \frac{p_0}{B_0^2/8\pi} = \frac{2}{\gamma} \frac{c_s^2}{v_A^2}. \quad (15.24)$$

The magnetosonic waves can be conveniently summarised by the so-called *Friedricks*

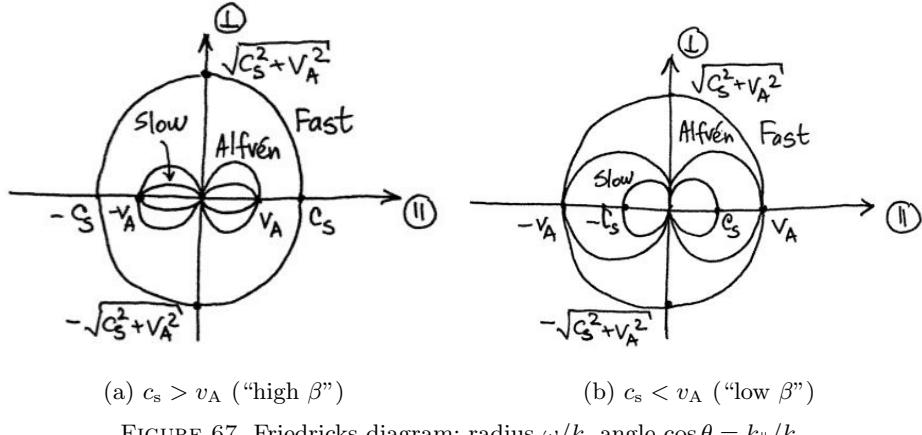
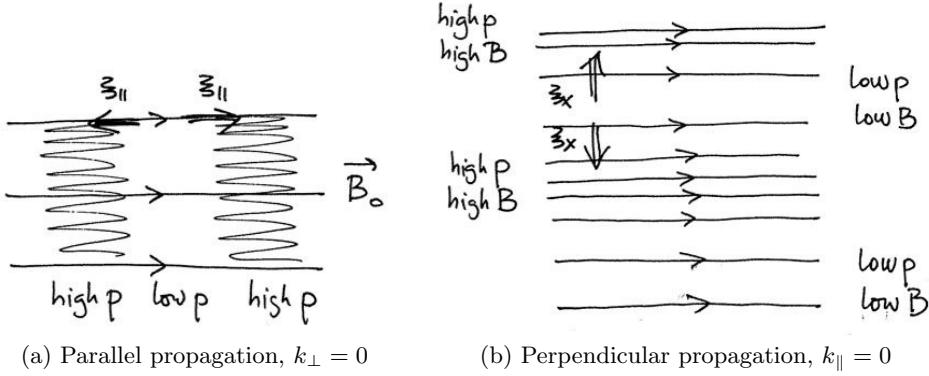
FIGURE 67. Friedricks diagram: radius  $\omega/k$ , angle  $\cos \theta = k_{\parallel}/k$ .

FIGURE 68. Sound waves.

diagram, a graph of (15.23) in polar coordinates where the radius is the phase speed  $\omega/k$  and the angle is  $\theta$ , the direction of propagation with respect to  $\mathbf{B}_0$  (Fig. 67).

Clearly, magnetosonic waves contain perturbations of both the magnetic field and of the “hydrodynamic” quantities  $\rho$ ,  $p$ ,  $\mathbf{u}$ , but working them all out for the case of general oblique propagation ( $\theta \sim 1$ ) is a bit messy. The physics of what is going on is best understood via a few particular cases.

### 15.1.3. Parallel Propagation

Consider  $k_{\perp} = 0$  ( $\theta = 0$ ). Then  $(\xi_x, 0, 0)$  and  $(0, 0, \xi_{\parallel})$  are eigenvectors of the matrix in (15.21) and the two corresponding waves are

- another *Alfvén wave*, this time with perturbation in the  $x$  direction (which, however, is not physically different from the  $y$  direction when  $k_{\perp} = 0$ ):

$$\omega^2 \xi_x = k_{\parallel}^2 v_A^2 \xi_x \quad \Rightarrow \quad \omega = \pm k_{\parallel} v_A , \quad (15.25)$$

$$\xi = \xi_x \hat{x}, \quad \delta\rho = 0, \quad \delta p = 0, \quad \delta B = 0, \quad \delta \mathbf{b} = i k_{\parallel} \xi_x \hat{x} \quad (15.26)$$

(at high  $\beta$ , this is the slow wave, at low  $\beta$ , this is the fast wave);

- the parallel-propagating *sound wave* (Fig. 68a):

$$\omega^2 \xi_{\parallel} = k_{\parallel}^2 c_s^2 \xi_{\parallel} \Rightarrow \boxed{\omega = \pm k_{\parallel} c_s}, \quad (15.27)$$

$$\boldsymbol{\xi} = \xi_{\parallel} \hat{\mathbf{z}}, \quad \frac{\delta\rho}{\rho_0} = -ik_{\parallel} \xi_{\parallel}, \quad \frac{\delta p}{p_0} = \gamma \frac{\delta\rho}{\rho_0}, \quad \delta B = 0, \quad \delta \mathbf{b} = 0 \quad (15.28)$$

(at high  $\beta$ , this is the fast wave, at low  $\beta$ , this is the slow wave); the magnetic field does not participate here at all.

---

Note that this wave is hydrodynamically very similar to the ion-acoustic wave in unmagnetised plasma (cf. Exercise 3.6), but its ability to propagate undamped does hinge on the validity of the fluid approximation, i.e., on the plasma being collisional (except at high  $\beta$  and finite amplitudes, when it can “create” its own “collisionality”—see Kunz *et al.* 2020).

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#### 15.1.4. Perpendicular Propagation

Now consider  $\boxed{k_{\parallel} = 0}$  ( $\theta = 90^\circ$ ). Then  $(\xi_x, 0, 0)$  is again an eigenvector of the matrix in (15.21).<sup>102</sup> The resulting fast magnetosonic wave is again a *sound wave*, but because it is perpendicular-propagating, both thermal and magnetic pressures get involved, the perturbations are compressions/rarefactions in both the fluid and the field, and the speed at which they travel is a combination of the sound and Alfvén speeds (with the latter now representing the magnetic pressure response):

$$\omega^2 \xi_x = k_{\perp}^2 (c_s^2 + v_A^2) \xi_x \Rightarrow \boxed{\omega = \pm k_{\perp} \sqrt{c_s^2 + v_A^2}}, \quad (15.29)$$

$$\boldsymbol{\xi} = \xi_x \hat{\mathbf{x}}, \quad \frac{\delta\rho}{\rho_0} = -ik_{\perp} \xi_x, \quad \frac{\delta p}{p_0} = \gamma \frac{\delta\rho}{\rho_0}, \quad \frac{\delta B}{B_0} = -ik_{\perp} \xi_x, \quad \delta \mathbf{b} = 0. \quad (15.30)$$

Note that the thermal and magnetic compressions are in phase and there is no bending of the magnetic field (Fig. 68b).

#### 15.1.5. Anisotropic Perturbations: $k_{\parallel} \ll k_{\perp}$

Taking  $k_{\parallel} = 0$  in §15.1.4 was perhaps a little radical as we lost all waves apart from the fast one. As we are about to see, a lot of babies were thrown out with this particular bathwater.

So let us consider MHD waves in the limit  $\boxed{k_{\parallel} \ll k_{\perp}}$ . This turns out to be an extremely relevant regime, because, in a strong magnetic field, realistically excitable perturbations, both linear and nonlinear, tend to be highly elongated in the direction of the field. Going back to (15.23) and enforcing this limit, we get

$$\begin{aligned} \omega^2 &= \frac{1}{2} k^2 (c_s^2 + v_A^2) \left[ 1 \pm \sqrt{1 - \frac{4c_s^2 v_A^2}{(c_s^2 + v_A^2)^2} \frac{k_{\parallel}^2}{k^2}} \right] \\ &\approx \frac{1}{2} k^2 (c_s^2 + v_A^2) \left[ 1 \pm 1 \mp \frac{2c_s^2 v_A^2}{(c_s^2 + v_A^2)^2} \frac{k_{\parallel}^2}{k^2} \right]. \end{aligned} \quad (15.31)$$

The upper sign gives the familiar *fast wave*

$$\boxed{\omega = \pm k \sqrt{c_s^2 + v_A^2}}. \quad (15.32)$$

---

<sup>102</sup>As is  $(0, 0, \xi_{\parallel})$ , but with  $\omega = 0$ ; we will deal with this mode in §15.3.4.

This is just the magnetically enhanced sound wave that was considered in §15.1.4. The small corrections to it due to  $k_{\parallel}/k$  are not particularly interesting.

The lower sign in (15.31) gives the *slow wave*

$$\boxed{\omega = \pm k_{\parallel} \frac{c_s v_A}{\sqrt{c_s^2 + v_A^2}}}, \quad (15.33)$$

which is more interesting. Let us find the corresponding eigenvector: from (15.14),

$$\underbrace{(\omega^2 - k_{\parallel}^2 c_s^2)}_{= -k_{\parallel}^2 \frac{c_s^4}{c_s^2 + v_A^2}}, \quad (15.34)$$

from (15.33)

Therefore, the displacements are mostly parallel:

$$\boxed{\frac{\xi_x}{\xi_{\parallel}} = -\frac{k_{\parallel}}{k_{\perp}} \frac{c_s^2}{c_s^2 + v_A^2} \ll 1}. \quad (15.35)$$

Using this equation together with (15.15–15.18), we find that the perturbations in the remaining fields are

$$\frac{\delta\rho}{\rho_0} = -i(k_{\perp}\xi_x + k_{\parallel}\xi_{\parallel}) = -i \frac{v_A^2}{c_s^2 + v_A^2} k_{\parallel}\xi_{\parallel}, \quad (15.36)$$

$$\frac{\delta p}{p_0} = \gamma \frac{\delta\rho}{\rho_0}, \quad (15.37)$$

$$\delta\mathbf{b} = ik_{\parallel}\xi_x \hat{x} = -i \frac{k_{\parallel}}{k_{\perp}} \frac{c_s^2}{c_s^2 + v_A^2} k_{\parallel}\xi_{\parallel} \hat{x} \rightarrow 0, \quad (15.38)$$

$$\frac{\delta B}{B_0} = -ik_{\perp}\xi_x = i \frac{c_s^2}{c_s^2 + v_A^2} k_{\parallel}\xi_{\parallel}. \quad (15.39)$$

Thus, to lowest order in  $k_{\parallel}/k_{\perp}$ , this wave involves no bending of the magnetic field, but has a pressure/density perturbation and a magnetic-field-strength perturbation—the latter in counter-phase to the former (Fig. 69). To be precise, *the slow-wave perturbations are pressure balanced*:

$$\delta \left( p + \frac{B^2}{8\pi} \right) = p_0 \frac{\delta p}{p_0} + \frac{B_0^2}{4\pi} \frac{\delta B}{B_0} = \rho_0 \left( c_s^2 \frac{\delta\rho}{\rho_0} + v_A^2 \frac{\delta B}{B_0} \right) = 0. \quad (15.40)$$

The same is, of course, already obvious from the momentum equation (15.8), where, in the limit  $k_{\parallel} \ll k_{\perp}$  and  $\omega \ll kc_s$  (“incompressible” perturbations; see §15.2), the dominant balance is

$$\nabla_{\perp} \left( p + \frac{B^2}{8\pi} \right) = 0. \quad (15.41)$$

Finally, the Alfvén waves in the limit of anisotropic propagation are just the same as ever (§15.1.1)—they are unaffected by  $k_{\perp}$ , while being perfectly capable of having perpendicular variation (Fig. 66b).

### 15.1.6. High- $\beta$ Limit: $c_s \gg v_A$

Another limit in which high-frequency acoustic response (fast waves) and low-frequency, pressure-balanced Alfvénic response (slow and Alfvén waves) are separated

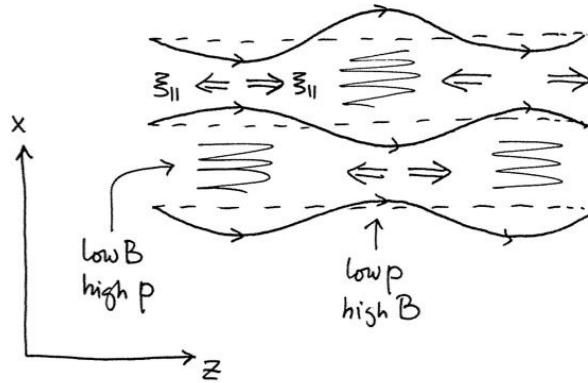


FIGURE 69. Slow wave in the anisotropic limit  $k_{\parallel} \ll k_{\perp}$ : pressure balanced,  $\xi_x \ll \xi_{\parallel}$ .

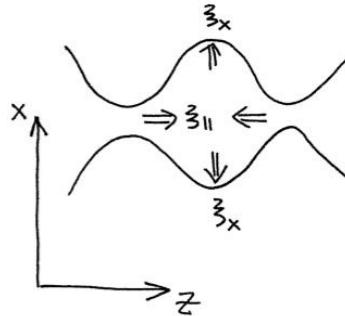


FIGURE 70. Slow wave in the high- $\beta$  limit: pressure balanced,  $\xi_x \sim \xi_{\parallel}$ .

is  $\beta \gg 1 \Leftrightarrow c_s \gg v_A$ .<sup>103</sup> In this limit, the approximate expression (15.31) for the magnetosonic frequencies is still valid, but because  $v_A/c_s$ , rather than  $k_{\parallel}/k$ , is small. The rest of the calculations in §15.1.5 are also valid, with the following simplifications arising from  $v_A$  being negligible compared to  $c_s$ .

The upper sign in (15.31) again gives us the *fast wave*, which, this time, is a pure *sound wave*:

$$\boxed{\omega = \pm kc_s} . \quad (15.42)$$

This is natural because, at high  $\beta$ , the magnetic pressure is negligible compared to thermal pressure and sound can propagate oblivious of the magnetic field.

The lower sign in (15.31) yields the *slow wave*: (15.33) is still valid and becomes, for  $v_A \ll c_s$ ,

$$\boxed{\omega = \pm k_{\parallel}v_A} . \quad (15.43)$$

Because the slow wave's dispersion relation in this limit looks exactly like the dispersion relation (15.19) of an Alfvén wave, it is called the *pseudo-Alfvén wave*. The similarity is deceptive as the nature of the perturbation (the eigenvector) is completely different.

<sup>103</sup>This limit is astrophysically very interesting because magnetic fields locally produced by plasma motions in various astrophysical environments (e.g., interstellar and intergalactic media) can only be as strong energetically as the motions that make them [see (13.45)] and so, the latter being subsonic,  $v_A \sim u \ll c_s$ .

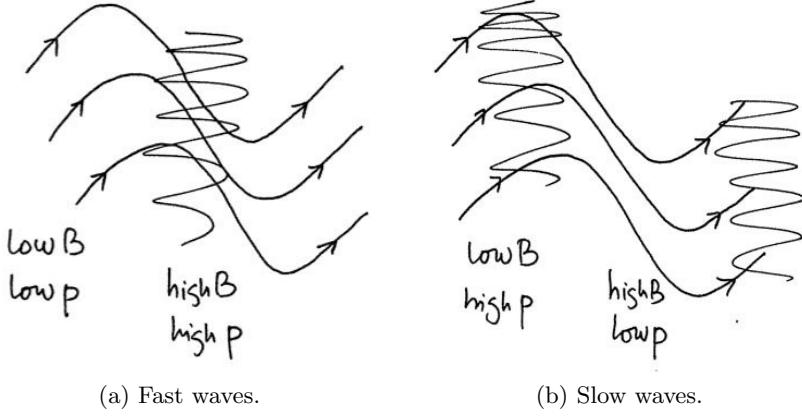


FIGURE 71. General oblique magnetosonic waves.

Substituting  $\omega^2 = k_{\parallel}^2 v_A^2$  into (15.14), we find

$$k_{\perp} \xi_x + k_{\parallel} \xi_{\parallel} = \frac{v_A^2}{c_s^2} k_{\parallel} \xi_{\parallel} \ll k_{\parallel} \xi_{\parallel}. \quad (15.44)$$

This just says that, to lowest order in  $1/\beta$ ,  $\nabla \cdot \boldsymbol{\xi} = 0$ , i.e., the perturbations are incompressible. In contrast to the anisotropic case (15.35), the perpendicular and parallel displacements are now comparable (assuming, in general,  $k_{\parallel} \sim k_{\perp}$ ):

$$\frac{\xi_x}{\xi_{\parallel}} = -\frac{k_{\parallel}}{k_{\perp}}.$$

(15.45)

Also in contrast to the anisotropic case, the density and pressure perturbations are now vanishingly small, but the field can be bent as well as compressed:

$$\frac{\delta\rho}{\rho_0} = -i(k_{\perp} \xi_x + k_{\parallel} \xi_{\parallel}) = -i \frac{v_A^2}{c_s^2} k_{\parallel} \xi_{\parallel} \rightarrow 0, \quad (15.46)$$

$$\frac{\delta p}{p_0} = \gamma \frac{\delta\rho}{\rho_0} \rightarrow 0, \quad (15.47)$$

$$\delta\mathbf{b} = ik_{\parallel} \xi_x \hat{\mathbf{x}} = -i \frac{k_{\parallel}}{k_{\perp}} k_{\parallel} \xi_{\parallel} \hat{\mathbf{x}}, \quad (15.48)$$

$$\frac{\delta B}{B_0} = -ik_{\perp} \xi_x = ik_{\parallel} \xi_{\parallel}. \quad (15.49)$$

The  $\delta B$  and  $\delta\mathbf{b}$  perturbations are in counter-phase, as are  $\xi_{\parallel}$  and  $\xi_x$  (Fig. 70). It is easy to check that pressure balance (15.40) is again maintained by these perturbations.

In the more general case of oblique propagation ( $k_{\parallel} \sim k_{\perp}$ ) and finite beta ( $\beta \sim 1$ ), the fast and slow magnetosonic waves generally have comparable frequencies and contain perturbations of all relevant fields, with the fast waves tending to have the perturbations of the thermal and magnetic pressure in phase and slow waves in counter-phase (Fig. 71).

### 15.2. Subsonic Ordering

Enough linear theory! We shall now occupy ourselves with the behaviour of *finite* (although still small) perturbations of a straight-field equilibrium. While we abandon linearisation (i.e., the neglect of nonlinear terms), much of what the linear theory has

taught us about the basic responses of an MHD fluid remains true and useful. In particular, the linear relations between the perturbation amplitudes of various fields provide us with a guidance as to the *relative size* of finite perturbations of these fields. This makes sense if, while allowing the nonlinearities back in, we do not assume the linear physics to be completely negligible, i.e., if we allow the linear and nonlinear time scales to compete (§15.2.3). We shall see that solutions for which this is the case satisfy self-consistent equations, so can be expected to be realisable (and, as we know from experimental, observational and numerical evidence, *are* realised).

I shall start by constructing nonlinear equations that describe the incompressible limit, i.e., fields and motions that are subsonic: both their phase speeds and flow velocities will be assumed small compared to the speed of sound:

$$\frac{\omega/k}{\sqrt{c_s^2 + v_A^2}} \ll 1, \quad \text{Ma} \equiv \frac{u}{\sqrt{c_s^2 + v_A^2}} \ll 1. \quad (15.50)$$

In this limit, we expect all fast-wave-like perturbations to disappear (in a similar way to the sound waves disappearing in the incompressible Navier–Stokes hydrodynamics) and for the MHD dynamics to contain only Alfvénic and slow-wave-like perturbations. We saw in §§15.1.5 and 15.1.6 that, linearly, fast and slow waves are well separated either in the limit of  $k_{\parallel}/k_{\perp} \ll 1$  or in the limit of  $\beta \gg 1$ . Indeed, comparing the Alfvén frequency (15.19) and slow-wave frequency (15.33) to the sound (fast-wave) frequency (15.32), we get

$$\frac{\omega_{\text{Alfvén}}}{\omega_{\text{fast}}} \sim \frac{k_{\parallel} v_A}{k \sqrt{c_s^2 + v_A^2}} \sim \frac{k_{\parallel}}{k} \frac{1}{\sqrt{1+\beta}}, \quad \frac{\omega_{\text{slow}}}{\omega_{\text{fast}}} \sim \frac{k_{\parallel} c_s v_A}{k (c_s^2 + v_A^2)} \sim \frac{k_{\parallel}}{k} \frac{\sqrt{\beta}}{1+\beta}, \quad (15.51)$$

both of which are small in either of the two limits, satisfying the first of the conditions (15.50).

The second condition (15.50) involves the “*magnetic Mach number*” Ma (generalised to compare the flow velocity to the speed of sound in a magnetised fluid), which measures the size of the perturbations themselves—in the linear theory, this was arbitrarily small, but now we will need to relate it to our other small parameter(s),  $k_{\parallel}/k$  or  $1/\beta$ . This means that we would like to construct an asymptotic *ordering* in which there will be some prescription as to how small, or otherwise, various (fractional) perturbations and small parameters are—not by themselves, i.e., compared to 1, but compared to each other (compared to 1, the small parameters can all formally be taken to be as small as we desire).

The general strategy for ordering perturbations with respect to each other will be to use the linear relations obtained in the two incompressible limits ( $k_{\parallel}/k \ll 1$  or  $\beta \gg 1$ ). If we do not specifically expect one perturbation to be larger or smaller than another on some physical grounds (like the properties of the linear response), we must order them the same; this does not stop us later from constructing subsidiary expansions in which they might be different. For example, MHD equations themselves were an expansion in a number of small parameters, in particular  $u/c$  [see (13.14)]. However, at the time of deriving them, I did not want to rule out sonic or supersonic motions and so, effectively, I ordered  $\text{Ma} \sim 1$ ,  $k_{\parallel}/k \sim 1$  and  $\beta \sim 1$ , as far as the  $u/c$  expansion was concerned, i.e.,  $\text{Ma}$ ,  $k_{\parallel}/k$ ,  $1/\beta \gg u/c$ . Now we are constructing a subsidiary expansion in these other parameters, keeping in mind that they are allowed to be small but not as small as the small parameter already used in the derivation of the MHD equations.<sup>104</sup>

<sup>104</sup>In principle, you should always feel a little paranoid about the question of whether such “nested” asymptotic expansions commute, i.e., whether it matters in which order they are done.

### 15.2.1. Ordering of Alfvénic Perturbations

Since the Alfvénic perturbations decouple completely from the rest (§15.1.1), linear theory does not give us a way to relate  $u_y$  to  $u_{\parallel}$ , so we shall exercise the no-prejudice principle stated above and assume

$$u_y \sim u_{\parallel}, \quad (15.52)$$

i.e., the Mach numbers for the Alfvénic and slow-wave-like motions are comparable. We can, however, relate  $u_y$  to  $\delta\mathbf{b}$ , via the curvature-force response (15.20):

$$|\delta\mathbf{b}| \sim k_{\parallel} \xi_y \sim \frac{k_{\parallel} u_y}{\omega} \sim \frac{u_y}{v_A} \sim \text{Ma} \sqrt{1 + \beta}. \quad (15.53)$$

### 15.2.2. Ordering of Slow-Wave-Like Perturbations

For slow-wave-like perturbations, in either the anisotropic or the high- $\beta$  limit, from (15.14) and (15.33),

$$\nabla \cdot \mathbf{u} \sim \omega(k_{\perp} \xi_x + k_{\parallel} \xi_{\parallel}) \sim \frac{\omega^2}{k_{\parallel} c_s^2} \omega \xi_{\parallel} \sim \frac{v_A^2}{c_s^2 + v_A^2} k_{\parallel} u_{\parallel} \sim \frac{k_{\parallel} u_{\parallel}}{1 + \beta}. \quad (15.54)$$

Thus, the divergence of the flow velocity is small (the dynamics are incompressible) in all three of our (potentially) small parameters:

$$\frac{\nabla \cdot \mathbf{u}}{k \sqrt{c_s^2 + v_A^2}} \sim \frac{k_{\parallel}}{k} \frac{1}{1 + \beta} \text{Ma}. \quad (15.55)$$

From this, we can immediately obtain an ordering for the density and pressure perturbations: using (15.3), (15.4), (15.33) and (15.54) [cf. (15.36) and (15.46)],

$$\frac{\delta\rho}{\rho_0} \sim \frac{\delta p}{p_0} \sim \nabla \cdot \boldsymbol{\xi} \sim \frac{\nabla \cdot \mathbf{u}}{\omega} \sim \frac{\text{Ma}}{\sqrt{\beta}}. \quad (15.56)$$

The magnetic-field-strength (magnetic-pressure) perturbation is, using (15.39) and (15.33) [cf. (15.49)],

$$\frac{\delta B}{B_0} \sim k_{\perp} \xi_x \sim \frac{c_s^2}{c_s^2 + v_A^2} \frac{k_{\parallel} u_{\parallel}}{\omega} \sim \sqrt{\beta} \text{Ma}, \quad (15.57)$$

or, perhaps more straightforwardly, from pressure balance (15.40) and using (15.56),

$$\frac{\delta B}{B_0} = -\frac{\beta}{2} \frac{\delta p}{p_0} \sim \sqrt{\beta} \text{Ma}. \quad (15.58)$$

Finally, in a similar fashion, using (15.17) and (15.57) [cf. (15.38) and (15.48)], we find

$$|\delta\mathbf{b}| \sim k_{\parallel} \xi_x \sim \frac{k_{\parallel}}{k_{\perp}} \sqrt{\beta} \text{Ma} \quad (15.59)$$

for slow-wave-like perturbations. Note that in all interesting limits this is superceded by the Alfvénic ordering (15.53).

### 15.2.3. Ordering of Time Scales

Let us recall that our motivation for using linear relations between perturbations to determine their relative sizes in a nonlinear regime was that linear response will lose

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They usually do commute, but this is not guaranteed and you ought to check if you want to be sure. Another formally justified mathematical worry is whether asymptotic solutions of exact equations are the same as exact solutions of asymptotic equations. This will lead you on a journey to the world of proofs of existence and uniqueness—where I wish you an enjoyable stay.

its exclusive sway but remain non-negligible. In formal terms, this means that we must order the linear and nonlinear time scales to be comparable.<sup>105</sup> The nonlinearities in MHD equations are advective, i.e., they are of the form  $\mathbf{u} \cdot \nabla(\text{stuff})$  and similar, so the rate of nonlinear interaction is  $\sim ku$  (in the case of anisotropic perturbations,  $\sim k_\perp u_\perp$ ). Ordering this to be comparable to the frequencies of the Alfvén and slow waves [see (15.51)] gives us

$$\omega_{\text{Alfvén}} \sim ku \quad \Rightarrow \quad \text{Ma} \sim \frac{k_\parallel}{k} \frac{1}{\sqrt{1+\beta}}, \quad (15.60)$$

$$\omega_{\text{slow}} \sim ku \quad \Rightarrow \quad \text{Ma} \sim \frac{k_\parallel}{k} \frac{\sqrt{\beta}}{1+\beta}. \quad (15.61)$$

Note that the first of these relations supersedes the second in all interesting limits.

#### 15.2.4. Summary of Subsonic Ordering

Thus, the ordering of the time scales determines the size of the perturbations via (15.60). Using this restriction on Ma, we may summarise our subsonic ordering as follows<sup>106</sup>

$$\boxed{\text{Ma} \equiv \frac{u}{\sqrt{c_s^2 + v_A^2}} \sim \frac{|\delta\mathbf{b}|}{\sqrt{1+\beta}} \sim \frac{1}{\sqrt{\beta}} \frac{\delta B}{B} \sim \sqrt{\beta} \frac{\delta\rho}{\rho_0} \sim \sqrt{\beta} \frac{\delta p}{p_0} \sim \frac{k_\parallel}{k} \frac{1}{\sqrt{1+\beta}} \ll 1} \quad (15.62)$$

and  $\omega \sim ku$ . The ordering can be achieved either in the limit of  $k_\parallel/k \ll 1$  or  $1/\beta \ll 1$ , or both. Note that if one of these parameters is small, the other can be order unity or even large (as long as it is not larger than the inverse of the small one).

The case of anisotropic perturbations and arbitrary  $\beta$  applies in a broad range of plasmas, from magnetically confined fusion ones (tokamaks, stellarators) to space (e.g., the solar corona or the solar wind). We shall consider the implications of this ordering in §15.3.

The case of high  $\beta$  applies, e.g., to high-energy galactic and extragalactic plasmas. It is the direct generalisation to MHD of incompressible Navier–Stokes hydrodynamics, i.e., in this case, all one needs to do is solve MHD equations assuming  $\rho = \text{const}$  and  $\nabla \cdot \mathbf{u} = 0$ . We shall consider this case now.

#### 15.2.5. Incompressible MHD Equations

Assuming  $\beta \gg 1$ , our ordering becomes

$$\boxed{\frac{u}{c_s} \sim \frac{\omega}{kc_s} \sim \frac{1}{\sqrt{\beta}} \sim \text{Ma}, \quad |\delta\mathbf{b}| \sim \frac{\delta B}{B_0} \sim \sqrt{\beta} \text{Ma} \sim 1, \quad \frac{\delta\rho}{\rho_0} \sim \frac{\delta p}{p_0} \sim \frac{\text{Ma}}{\sqrt{\beta}} \sim \text{Ma}^2}. \quad (15.63)$$

Thus, the density and pressure perturbations are minuscule, while magnetic perturbations are order unity—magnetic fields are relatively easy to bend (i.e., subsonic motions can tangle the field substantially in this regime). Because of this, it will not make sense to split  $\mathbf{B}$  into  $\mathbf{B}_0$  and  $\delta\mathbf{B}$  explicitly, we will treat the magnetic field as a single field, with no need for a strong mean component.

<sup>105</sup>In the theory of MHD turbulence, this principle, applied at each scale, is known as the *critical balance* (see §15.4).

<sup>106</sup>Note that it is not absolutely necessary to work out the detailed linear theory of a set of equations in order to be able to construct such orderings: it is often enough to know roughly where you are going and simply balance terms representing the physics that you wish to keep (or expect to have to keep). An example of this approach is given in §15.2.8.

Let us examine the MHD equations (13.57–13.60) under the ordering (15.63).

Since  $\omega \sim ku$ , the convective derivative  $d/dt = \partial/\partial t + \mathbf{u} \cdot \nabla$  survives intact in all equations, allowing the advective nonlinearity to enter.

The continuity equation (13.57) simply reiterates our earlier statement that the velocity field is divergenceless to lowest order:

$$\nabla \cdot \mathbf{u} = -\frac{1}{\rho} \frac{d\rho}{dt} \sim \omega \frac{\delta\rho}{\rho_0} \sim Ma^3 k c_s \rightarrow 0. \quad (15.64)$$

The momentum equation (13.58) becomes

$$\left(1 + \frac{\delta\rho}{\rho_0}\right) \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}\right) = -\nabla \underbrace{\left(\frac{c_s^2}{\gamma} \frac{\delta p}{\rho_0} + \frac{B^2}{8\pi\rho_0}\right)}_{\equiv \tilde{p}} + \frac{\mathbf{B} \cdot \nabla \mathbf{B}}{4\pi\rho_0}. \quad (15.65)$$

The density perturbation in the left-hand side is  $\sim Ma^2$  and so negligible compared to unity. The remaining terms in this equation are all the same order ( $\sim Ma^2 k c_s^2$ ) and so they must all be kept. The total “pressure”  $\tilde{p}$  is determined by enforcing  $\nabla \cdot \mathbf{u} = 0$  [see (15.64)]. Namely, our equations are

$$\boxed{\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla \tilde{p} + \mathbf{B} \cdot \nabla \mathbf{B}}, \quad (15.66)$$

where

$$\nabla^2 \tilde{p} = -\nabla \nabla : (\mathbf{u} \mathbf{u} - \mathbf{B} \mathbf{B}) \quad (15.67)$$

and the magnetic field has been rescaled to velocity units,  $\mathbf{B}/\sqrt{4\pi\rho_0} \rightarrow \mathbf{B}$ .

In the induction equation, best written in the form (13.27), all terms are the same order  $\sim kuB \sim Ma k c_s B$  except the one containing  $\nabla \cdot \mathbf{u}$ , which is  $\sim Ma^3 k c_s B$  and so must be neglected. We are left with

$$\boxed{\frac{\partial \mathbf{B}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{B} = \mathbf{B} \cdot \nabla \mathbf{u}}. \quad (15.68)$$

Finally, the internal-energy equation (13.60), which, keeping only the lowest-order terms, becomes

$$\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla\right) \left(\frac{\delta p}{\rho_0} - \gamma \frac{\delta\rho}{\rho_0}\right) = 0, \quad (15.69)$$

can be used to find  $\delta\rho/\rho_0$ , once  $\delta p/p_0 = \gamma(\tilde{p} - B^2/2)/c_s^2$  is calculated from the solution of (15.66–15.68). Note that  $\delta\rho/\rho_0$  is merely a spectator quantity, not required to solve (15.66–15.68), which form a closed set.

Equations (15.66–15.68) are the equations of *incompressible MHD* (let us call it *iMHD*). Note that while they have been obtained in the limit of  $\beta \gg 1$ , all  $\beta$  dependence has disappeared from them—basically, they describe subsonic dynamics on top of an infinite heat bath. This is how it should be: formally, in any good asymptotic theory, it must be possible to make the small parameter arbitrarily small without changing anything in the equations.

**Exercise 15.2.** Show that iMHD conserves the sum of kinetic and magnetic energies,

$$\frac{d}{dt} \int d\mathbf{r} \left(\frac{u^2}{2} + \frac{B^2}{2}\right) = 0. \quad (15.70)$$

**Exercise 15.3.** Check that you can obtain the right waves, viz., Alfvén ([§15.1.1](#)) and pseudo-Alfvén ([§15.1.6](#)), directly from iMHD.

**Exercise 15.4. Magnetoelastic waves.**<sup>107</sup> (a) Show that the iMHD equations can be rewritten as the following closed set describing the evolution of the velocity field  $\mathbf{u}$  and the Maxwell tensor  $M_{ij} = B_i B_j$ :

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial r_j} = -\frac{\partial \tilde{p}}{\partial r_i} + \frac{\partial M_{ij}}{\partial r_j}, \quad (15.71)$$

$$\frac{\partial M_{ij}}{\partial t} + u_n \frac{\partial M_{ij}}{\partial r_n} = M_{nj} \frac{\partial u_i}{\partial r_n} + M_{in} \frac{\partial u_j}{\partial r_n} \quad (15.72)$$

(summation over repeated indices is implied).

(b) Imagine that there is no mean magnetic field, the MHD medium is static, and it is filled with chaotically tangled magnetic fields that are constant in time. Denote their Maxwell tensor  $M_{ij}^{(0)}$ . Assume that these fields have a characteristic scale that is no larger than  $\ell$  and are statistically isotropic, so if we introduce an average (denoted by angle brackets) over scales of order  $\ell$ , then

$$\langle M_{ij}^{(0)} \rangle = v_A^2 \delta_{ij}, \quad (15.73)$$

where (obviously)  $v_A^2 = \langle B^2 \rangle / 3 = \text{const}$ . This is clearly a static ( $u_i = 0$ ) equilibrium solution of (15.71) and (15.72). Consider infinitesimal perturbations  $\delta u_i$  and  $\delta M_{ij}$  around this equilibrium and assume that they vary in space on scales much longer than  $\ell$ , viz.,

$$\langle u_i \rangle = 0 + \delta u_i \ll v_A, \quad \langle M_{ij} \rangle = \langle M_{ij}^{(0)} \rangle + \delta M_{ij}, \quad \delta M_{ij} \ll v_A^2. \quad (15.74)$$

Ignore any possible perturbations of  $u_i$  and  $M_{ij}$  on scales  $\ell$  or smaller. Show that  $\delta u_i$  and  $\delta M_{ij}$  will describe propagating waves, derive their dispersion relation and also the relationship between  $\delta M_{ij}$  and the displacement vector  $\xi_i$  associated with  $\delta u_i = \partial \xi_i / \partial t$ . These are called *magnetoelastic waves*. Think about their physical nature, their similarities with, or differences from, Alfvén waves.

As I already intimated in [§13.4](#), the iMHD equations written in the form of (15.71) and (15.72) are mathematically similar to the equations describing certain kinds of polymer-laden fluids. The intrinsic “elasticity” of the Maxwell stress leads to a kind of isotropic Alfvénic response that gives rise to the magnetoelastic waves. Note, however, that a significant difference between polymer chains and magnetic fields is that the latter have a sign, so there is a distinction between parallel and antiparallel fields, while polymers do not have that. Consequently, the analogy between MHD and polymer fluids becomes very imperfect if dissipation of  $M_{ij}$  is included: for polymers, there is a relaxation term in (15.72) of the form  $-(M_{ij} - v_A^2 \delta_{ij})/\tau$ , describing the polymers’ desire to curl up due to entropic forces; whereas in MHD, the resistive term  $\eta(B_i \nabla^2 B_j + B_j \nabla^2 B_i)$  ([§13.5](#)) cannot be converted into anything that depends only on  $M_{ij}$ —indeed, it would, e.g., heavily damp antiparralel fields that reverse direction on small scales, an effect invisible to  $M_{ij}$ , where the field’s sign cancels out.

### 15.2.6. Elsasser MHD

The iMHD equations possess a remarkable symmetry. Let us introduce *Elsasser (1950) fields*

$$\mathbf{Z}^\pm = \mathbf{u} \pm \mathbf{B} \quad (15.75)$$

and rewrite (15.66) and (15.68) as evolution equations for  $\mathbf{Z}^\pm$ : after trivial algebra,

$$\frac{\partial \mathbf{Z}^\pm}{\partial t} + \mathbf{Z}^\mp \cdot \nabla \mathbf{Z}^\pm = -\nabla \tilde{p} \quad (15.76)$$

<sup>107</sup>This is based on the 2019 exam question. You will find a much more sophisticated version of this calculation in [Hosking et al. \(2020\)](#), where you will learn how to deal with situations in which the dodgy assumption that perturbations of  $u_i$  and  $M_{ij}$  on scales  $\ell$  can be ignored is actually wrong, and also when this assumption is OK.

and, since  $\nabla \cdot \mathbf{Z}^\pm = 0$ ,

$$\nabla^2 \tilde{p} = -\nabla \nabla : \mathbf{Z}^+ \mathbf{Z}^- . \quad (15.77)$$

Thus, one can think of iMHD as representing the evolution of two incompressible “velocity fields” advecting each other.

Let us restore the separation of the magnetic field into its mean and perturbed parts,  $\mathbf{B} = \mathbf{B}_0 + \delta\mathbf{B} = v_A \hat{\mathbf{z}} + \delta\mathbf{B}$  (recall that  $\mathbf{B}$  is in velocity units). Then

$$\mathbf{Z}^\pm = \pm v_A \hat{\mathbf{z}} + \delta\mathbf{Z}^\pm \quad (15.78)$$

and (15.76) becomes

$$\boxed{\frac{\partial \delta\mathbf{Z}^\pm}{\partial t} \mp v_A \nabla_{||} \delta\mathbf{Z}^\pm + \delta\mathbf{Z}^\mp \cdot \nabla \delta\mathbf{Z}^\pm = -\nabla \tilde{p}} . \quad (15.79)$$

Thus,  $\delta\mathbf{Z}^\pm$  are finite, counter-propagating (at the Alfvén speed  $v_A$ ) perturbations—and they interact nonlinearly only with each other, not with themselves. If we let, say,  $\delta\mathbf{Z}^- = 0 \Leftrightarrow \mathbf{u} = \delta\mathbf{B}$ , then  $\delta\mathbf{Z}^+$  satisfies

$$\frac{\partial \delta\mathbf{Z}^+}{\partial t} - v_A \nabla_{||} \delta\mathbf{Z}^+ = 0, \quad (15.80)$$

and similarly for  $\delta\mathbf{Z}^-$  (propagating at  $-v_A$ ) if  $\delta\mathbf{Z}^+ = 0$ . Therefore,

$$\delta\mathbf{Z}^\pm = \mathbf{f}(\mathbf{r} \pm v_A t \hat{\mathbf{z}}), \quad \delta\mathbf{Z}^\mp = 0, \quad (15.81)$$

where  $\mathbf{f}$  is an arbitrary function, are *exact nonlinear solutions* of iMHD. They are called *Elsasser states*. Physically, they are isolated Alfvén-wave packets that propagate along the guide field and never interact (because they all travel at the same speed and so can never catch up with or overtake one another). In order to have any interesting *nonlinear* dynamics, the system must have *counter-propagating* Alfvén-wave packets (see §15.4).

### 15.2.7. Cross-Helicity

Equations (15.76) manifestly support *two* conservation laws:

$$\frac{d}{dt} \int d\mathbf{r} \frac{|\mathbf{Z}^\pm|^2}{2} = 0, \quad (15.82)$$

i.e., the energy of each Elsasser field is individually conserved. This can be reformulated as conservation of the total energy,

$$\frac{d}{dt} \int d\mathbf{r} \frac{1}{2} \left( \frac{|\mathbf{Z}^+|^2}{2} + \frac{|\mathbf{Z}^-|^2}{2} \right) = \frac{d}{dt} \int d\mathbf{r} \left( \frac{u^2}{2} + \frac{B^2}{2} \right) = 0, \quad (15.83)$$

and of a new quantity, known as the *cross-helicity*:

$$\boxed{\frac{d}{dt} \int d\mathbf{r} \frac{1}{2} \left( \frac{|\mathbf{Z}^+|^2}{2} - \frac{|\mathbf{Z}^-|^2}{2} \right)} = \frac{d}{dt} \int d\mathbf{r} \mathbf{u} \cdot \mathbf{B} = 0 . \quad (15.84)$$

In the Elsasser formulation, the cross-helicity is a measure of energy *imbalance* between the two Elssasser fields<sup>108</sup>—this is observed, for example, in the solar wind, where there is significantly more energy in the Alfvénic fluctuations propagating away from the Sun than towards it (see, e.g., Wicks *et al.* 2013).

<sup>108</sup>Cross-helicity can also be interpreted as a topological invariant, counting the linkages between flux tubes and vortex tubes analogously to what magnetic helicity does for the flux tubes alone (see §16.2).

**Exercise 15.5.** To see why we needed incompressibility to get this new conservation law, work out the time evolution equation for  $\int d\mathbf{r} \mathbf{u} \cdot \mathbf{B}$  from the general (compressible) MHD equations and hence the condition under which the cross-helicity is conserved.

### 15.2.8. Stratified MHD

It is quite instructive to consider a very simple example of non-uniform MHD equilibrium: the case of a *stratified atmosphere*. Let us introduce gravity into MHD equations, viz., the momentum equation (13.58) becomes

$$\rho \frac{d\mathbf{u}}{dt} = -\nabla \left( p + \frac{B^2}{8\pi} \right) + \frac{\mathbf{B} \cdot \nabla \mathbf{B}}{4\pi} - \rho g \hat{\mathbf{z}} \quad (15.85)$$

(uniform gravitational acceleration pointing downward, against the  $z$  direction). We wish to consider a static equilibrium inhomogeneous in the  $z$  direction and threaded by a uniform magnetic field (which may be zero):

$$\rho_0 = \rho_0(z), \quad p_0 = p_0(z), \quad \mathbf{u}_0 = 0, \quad \mathbf{B}_0 = B_0 \mathbf{b}_0 = \text{const}, \quad (15.86)$$

where  $\mathbf{b}_0$  is at some general angle to  $\hat{\mathbf{z}}$  and  $p_0(z)$  and  $\rho_0(z)$  are constrained by the vertical force balance:

$$\boxed{\frac{dp_0}{dz} = -\rho_0 g} \quad \Rightarrow \quad g = -\frac{p_0}{\rho_0} \frac{d \ln p_0}{dz} = \frac{c_s^2}{\gamma} \frac{1}{H_p}, \quad (15.87)$$

where it has been opportune to define the pressure scale height  $H_p$ . We shall now seek time-dependent solutions of the MHD equations for which

$$\rho = \rho_0(z) + \delta\rho, \quad \frac{\delta\rho}{\rho_0} \ll 1, \quad p = p_0(z) + \delta p, \quad \frac{\delta p}{p_0} \ll 1, \quad (15.88)$$

and the spatial variation of all perturbations occurs on scales that are small compared to the pressure scale height  $H_p$  or the analogously defined density scale height  $H_\rho = -(d \ln \rho_0 / dz)^{-1}$  (for ordering purposes, we denote them both  $H$ ):

$$\boxed{kH \gg 1}. \quad (15.89)$$

After the equilibrium pressure balance is subtracted from (15.85), this equation becomes, under any ordering in which  $\delta\rho \ll \rho_0$ ,

$$\boxed{\rho_0 \frac{d\mathbf{u}}{dt} = -\nabla \left( \delta p + \frac{B^2}{8\pi} \right) + \frac{\mathbf{B} \cdot \nabla \mathbf{B}}{4\pi} - \delta \rho g \hat{\mathbf{z}}}. \quad (15.90)$$

The last term is the *buoyancy (Archimedes) force*. In order for this new feature to give rise to any nontrivial new physics, it must be ordered comparable to all the other terms in the equation: using (15.87) to express  $g \sim p_0 / \rho_0 H$ , we find

$$\delta \rho g \sim k \delta p \quad \Rightarrow \quad \frac{\delta p}{\rho_0} \sim kH \frac{\delta p}{p_0} \gg \frac{\delta p}{p_0}, \quad (15.91)$$

$$\delta \rho g \sim \frac{k B^2}{4\pi} \quad \Rightarrow \quad \frac{\delta p}{\rho_0} \sim \frac{kH}{\beta} \ll 1 \quad \Rightarrow \quad \beta \gg kH \gg 1. \quad (15.92)$$

So we learn that the density perturbations must now be much larger than the pressure perturbations, but, in order for the former to remain small and for the magnetic field to be in the game,  $\beta$  must be high (it is in anticipation of this that we did not split  $\mathbf{B}$  into  $\mathbf{B}_0$  and  $\delta\mathbf{B}$ , expecting them to be of the same order).

Let us now expand the internal-energy equation (13.60) in small density and pressure perturbations. Denoting  $s = p/\rho^\gamma = s_0(z) + \delta s$  (entropy density) and introducing the entropy scale height

$$\frac{1}{H_s} \equiv \frac{d \ln s_0}{dz} = -\frac{1}{H_p} + \frac{\gamma}{H_\rho} \quad (15.93)$$

(assumed positive), we find<sup>109</sup>

$$\frac{d}{dt} \frac{\delta s}{s_0} = -\frac{u_z}{H_s}, \quad \frac{\delta s}{s_0} = \frac{\delta p}{p_0} - \gamma \frac{\delta \rho}{\rho_0} \approx -\gamma \frac{\delta \rho}{\rho_0}. \quad (15.94)$$

The last, approximate, expression follows from the smallness of pressure perturbations [see (15.91)]. This then gives us

$$\boxed{\frac{d}{dt} \frac{\delta \rho}{\rho_0} = \frac{u_z}{\gamma H_s}}. \quad (15.95)$$

But, on the other hand, the continuity equation (13.57) is

$$\frac{d}{dt} \frac{\delta \rho}{\rho_0} = -\nabla \cdot \mathbf{u} + \frac{u_z}{H_p} \Rightarrow \nabla \cdot \mathbf{u} = u_z \left( \frac{1}{H_\rho} - \frac{1}{\gamma H_s} \right) = \frac{u_z}{\gamma H_p} \Rightarrow \frac{\nabla \cdot \mathbf{u}}{ku} \sim \frac{1}{kH} \ll 1. \quad (15.96)$$

Thus, the dynamics are incompressible again and the role of the continuity equation is to tell us that we must find  $\delta p$  from the momentum equation (15.90) by enforcing  $\nabla \cdot \mathbf{u} = 0$  to lowest order. The difference with iMHD (§15.2.5) is that  $\delta \rho / \rho$  now participates in the dynamics via the buoyancy force and must be found self-consistently from (15.95).

Finally, we rewrite our newly found simplified system of equations for a stratified, high- $\beta$  atmosphere, in the following neat way:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla \tilde{p} + \mathbf{B} \cdot \nabla \mathbf{B} + a \hat{\mathbf{z}}, \quad (15.97)$$

$$\nabla^2 \tilde{p} = -\nabla \nabla : (\mathbf{u} \mathbf{u} - \mathbf{B} \mathbf{B}) + \frac{\partial a}{\partial z}, \quad (15.98)$$

$$\frac{\partial a}{\partial t} + \mathbf{u} \cdot \nabla a = -N^2 u_z, \quad N = \frac{c_s}{\gamma \sqrt{H_s H_p}}, \quad (15.99)$$

$$\frac{\partial \mathbf{B}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{B} = \mathbf{B} \cdot \nabla \mathbf{u}, \quad (15.100)$$

where we have rescaled  $\mathbf{B}/\sqrt{4\pi\rho_0} \rightarrow \mathbf{B}$  and denoted the Archimedes acceleration

$$a = -\frac{\delta \rho}{\rho_0} g = -\frac{\delta \rho}{\rho_0} \frac{c_s^2}{\gamma H_p}, \quad (15.101)$$

a quantity also known as *the buoyancy* of the fluid. We shall call (15.97–15.100) the equations of *stratified MHD (SMHD)*.

A new frequency  $N$ , known as the *Brunt–Väisälä frequency*, has appeared in our equations.<sup>110</sup> In order for all the linear and nonlinear time scales that are present in our equations to coexist legitimately within our ordering, we must demand that the Alfvén, Brunt–Väisälä and nonlinear time scales all be comparable:

$$\boxed{kv_A \sim N \sim ku \Rightarrow \frac{1}{\sqrt{\beta}} \sim \frac{1}{kH} \sim \text{Ma}}. \quad (15.102)$$

This gives us a relative ordering between all the small parameters that have appeared so far, including the new one,  $1/kH$ . Using (15.95) and recalling (15.91), let us summarise the ordering of the perturbations:

$$\frac{u}{c_s} \sim \frac{\delta \rho}{\rho_0} \sim \text{Ma}, \quad \frac{\delta p}{p_0} \sim \text{Ma}^2, \quad |\delta \mathbf{B}| \sim \frac{\delta \mathbf{B}}{B_0} \sim 1. \quad (15.103)$$

The difference with the iMHD high- $\beta$  ordering (15.63) is that the density perturbations have now been promoted to dynamical relevance, thankfully without jeopardising incompressibility

<sup>109</sup>We are able to take equilibrium quantities in and out of spatial derivatives because  $kH \gg 1$  and the perturbations are small.

<sup>110</sup> $N$  is real because we assumed  $H_s > 0$  (a “stably stratified atmosphere”), otherwise the atmosphere becomes convectively unstable—this happens when the equilibrium entropy decreases upwards (cf. §17.3, Q10 and Q6c).

(i.e., still ordering out the sonic perturbations). The ordering (15.103) can be thought of as a generalisation to MHD of the *Boussinesq approximation* in hydrodynamics.

Further investigations of the SMHD equations are undertaken in Q6.

### 15.3. Reduced MHD

We now turn to the anisotropic ordering,  $k_{\parallel}/k \ll 1$  (while  $\beta \sim 1$ , in general), for which we studied the linear theory in §15.1.5. Specialising to this case from our general ordering (15.62), we have

$$\text{Ma} \sim \frac{u_{\perp}}{c_s} \sim \frac{u_{\parallel}}{c_s} \sim |\delta \mathbf{b}| \sim \frac{\delta B}{B_0} \sim \frac{\delta \rho}{\rho_0} \sim \frac{\delta p}{p_0} \sim \frac{\omega}{k_{\perp} c_s} \sim \frac{k_{\parallel}}{k_{\perp}} \ll 1. \quad (15.104)$$

Starting again with the continuity equation (13.57), dividing through by  $\rho_0$  and ordering all terms, we get

$$\underbrace{\left( \frac{\partial}{\partial t} + \mathbf{u}_{\perp} \cdot \nabla_{\perp} + u_{\parallel} \nabla_{\parallel} \right)}_{\text{Ma}} \underbrace{\frac{\delta \rho}{\rho_0}}_{\text{Ma}^2} = - \underbrace{\left( 1 + \frac{\delta \rho}{\rho_0} \right)}_{\text{Ma}} \underbrace{\left( \nabla_{\perp} \cdot \mathbf{u}_{\perp} + \nabla_{\parallel} u_{\parallel} \right)}_{\text{Ma}^2}. \quad (15.105)$$

Thus, to lowest order, the perpendicular velocity field is 2D-incompressible:

$$\mathcal{O}(\text{Ma}) : \boxed{\nabla_{\perp} \cdot \mathbf{u}_{\perp} = 0}. \quad (15.106)$$

In the next order (which we will need in §15.3.2),

$$\mathcal{O}(\text{Ma}^2) : (\nabla \cdot \mathbf{u})_2 = - \left( \frac{\partial}{\partial t} + \mathbf{u}_{\perp} \cdot \nabla_{\perp} \right) \frac{\delta \rho}{\rho_0} = - \frac{d}{dt} \frac{\delta \rho}{\rho_0}, \quad (15.107)$$

where, to leading order, the convective derivative now involves only perpendicular advection.

Equation (15.106) implies that  $\mathbf{u}_{\perp}$  can be written in terms of a *stream function*:

$$\boxed{\mathbf{u}_{\perp} = \hat{\mathbf{z}} \times \nabla_{\perp} \Phi}. \quad (15.108)$$

Similarly, for the magnetic field, we have

$$0 = \nabla \cdot \mathbf{B} = \underbrace{\nabla_{\perp} \cdot \delta \mathbf{B}_{\perp}}_{\text{Ma}} + \underbrace{\nabla_{\parallel} \delta B_{\parallel}}_{\text{Ma}^2} \approx \nabla_{\perp} \cdot \delta \mathbf{B}_{\perp}, \quad (15.109)$$

so  $\delta \mathbf{B}_{\perp}$  is also 2D-solenoidal and can be written in terms of a *flux function*:

$$\boxed{\frac{\delta \mathbf{B}_{\perp}}{\sqrt{4\pi\rho_0}} = \hat{\mathbf{z}} \times \nabla_{\perp} \Psi}. \quad (15.110)$$

Note that  $\Psi = -A_{\parallel}/\sqrt{4\pi\rho_0}$ , the parallel component of the vector potential.

Thus, Alfvénically polarised perturbations,  $\mathbf{u}_{\perp}$  and  $\delta \mathbf{B}_{\perp}$  (see §15.1.1), can be described by two scalar functions,  $\Phi$  and  $\Psi$ . Let us work out the evolution equations for them.

#### 15.3.1. Alfvénic Perturbations

We start with the induction equation, again most useful in the form (13.27). Dividing through by  $B_0$ , we have

$$\frac{d}{dt} \frac{\delta \mathbf{B}}{B_0} = \mathbf{b} \cdot \nabla \mathbf{u} - \mathbf{b} \nabla \cdot \mathbf{u}. \quad (15.111)$$

Throwing out the obviously subdominant  $\delta\mathbf{b}$  contribution in the last term on the right-hand side (i.e., approximating  $\mathbf{b} \approx \hat{\mathbf{z}}$  in that term), then taking the perpendicular part of the remaining equation, we get

$$\frac{d}{dt} \frac{\delta\mathbf{B}_\perp}{B_0} = \mathbf{b} \cdot \nabla \mathbf{u}_\perp. \quad (15.112)$$

As we saw above, the convective derivative is with respect to the perpendicular velocity only and, in view of the stream-function representation (15.108) of the latter, for any function  $f$ , we have, to leading order,

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \mathbf{u}_\perp \cdot \nabla_\perp f = \frac{\partial f}{\partial t} + \hat{\mathbf{z}} \cdot (\nabla_\perp \Phi \times \nabla_\perp f) = \frac{\partial f}{\partial t} + \{\Phi, f\}, \quad (15.113)$$

where the “Poisson bracket” is

$$\{\Phi, f\} = \frac{\partial \Phi}{\partial x} \frac{\partial f}{\partial y} - \frac{\partial \Phi}{\partial y} \frac{\partial f}{\partial x}. \quad (15.114)$$

Similarly, to leading order,

$$\mathbf{b} \cdot \nabla f = \frac{\partial f}{\partial z} + \delta\mathbf{b} \cdot \nabla_\perp f = \frac{\partial f}{\partial z} + \frac{1}{v_A} \hat{\mathbf{z}} \cdot (\nabla_\perp \Psi \times \nabla_\perp f) = \frac{\partial f}{\partial z} + \frac{1}{v_A} \{\Psi, f\}. \quad (15.115)$$

Finally, using (15.113) and (15.115) in (15.112) and expressing  $\delta\mathbf{B}_\perp$  in terms of  $\Psi$  [see (15.110)] and  $\mathbf{u}_\perp$  in terms of  $\Phi$  [see (15.108)], it is a straightforward exercise to show, after “uncurling” (15.112), that

$$\boxed{\frac{\partial \Psi}{\partial t} + \{\Phi, \Psi\} = v_A \frac{\partial \Phi}{\partial z}}. \quad (15.116)$$

Turning now to the momentum equation (13.58), taking its perpendicular part and dividing through by  $\rho \approx \rho_0$ , we get

$$\underbrace{\frac{d\mathbf{u}_\perp}{dt}}_{\text{Ma}^2} = \frac{1}{\rho_0} \left[ -\nabla_\perp \left( p + \frac{B^2}{8\pi} \right) + \frac{\mathbf{B} \cdot \nabla \delta\mathbf{B}_\perp}{4\pi} \right] = -\underbrace{\nabla_\perp \left( \frac{c_s^2}{\gamma} \frac{\delta p}{p_0} + v_A^2 \frac{\delta B}{B_0} \right)}_{\text{Ma}} + \underbrace{v_A^2 \mathbf{b} \cdot \nabla \frac{\delta\mathbf{B}_\perp}{B_0}}_{\text{Ma}^2}. \quad (15.117)$$

To lowest order,

$$\mathcal{O}(\text{Ma}) : \quad \nabla_\perp \left( \frac{c_s^2}{\gamma} \frac{\delta p}{p_0} + v_A^2 \frac{\delta B}{B_0} \right) = 0 \quad \Rightarrow \quad \boxed{\frac{\delta p}{p_0} = -\gamma \frac{v_A^2}{c_s^2} \frac{\delta B}{B_0}}. \quad (15.118)$$

This is a statement of pressure balance, which is physically what has been expected [see (15.41)] and which will be useful in §15.3.2. In the next order, (15.117) contains the perpendicular gradient of the second-order contribution to the total pressure. To avoid having to calculate it, we take the curl of (15.117) and thus obtain

$$\mathcal{O}(\text{Ma}^2) : \quad \nabla_\perp \times \frac{d\mathbf{u}_\perp}{dt} = v_A^2 \nabla_\perp \times \left( \mathbf{b} \cdot \nabla \frac{\delta\mathbf{B}_\perp}{B_0} \right). \quad (15.119)$$

Finally, using again (15.108), (15.110), (15.113) and (15.115) in (15.119), some slightly tedious algebra leads us to

$$\boxed{\frac{\partial}{\partial t} \nabla_\perp^2 \Phi + \{\Phi, \nabla_\perp^2 \Phi\} = v_A \frac{\partial}{\partial z} \nabla_\perp^2 \Psi + \{\Psi, \nabla_\perp^2 \Psi\}}. \quad (15.120)$$

Note that  $\nabla_{\perp}^2 \Phi$  is the vorticity of the flow  $\mathbf{u}_{\perp}$  and so the above equation is the MHD generalisation of the 2D Euler equation.

To summarise the equations (15.120) and (15.116) in their most compact form, we have

$$\frac{d}{dt} \nabla_{\perp}^2 \Phi = v_A \mathbf{b} \cdot \nabla \nabla_{\perp}^2 \Psi, \quad (15.121)$$

$$\frac{d\Psi}{dt} = v_A \frac{\partial \Phi}{\partial z}, \quad (15.122)$$

where the convective time derivative  $d/dt$  and the parallel spatial derivative  $\mathbf{b} \cdot \nabla$  are given by (15.113) and (15.115), respectively. Beautifully, these *nonlinear* equations describing Alfvénic perturbations have decoupled completely from everything else: we do not need to know  $\delta p$ ,  $\delta p$ ,  $u_{\parallel}$  or  $\delta B$  in order to solve for  $\mathbf{u}_{\perp}$  and  $\delta \mathbf{B}_{\perp}$ . *Alfvénic dynamics are self-contained*.

Equations (15.121) and (15.122) are called the Equations of *Reduced MHD (RMHD)*. They were originally derived in the context of tokamak plasmas (Kadomtsev & Pogutse 1974; Strauss 1976) and are extremely popular as a simple paradigm for MHD in a strong guide field—not just in tokamaks, but also in space.<sup>111</sup>

### 15.3.2. Compressive Perturbations

What about the rest of our fields—in the linear language, the slow-wave-like perturbations (§15.1.5)? While we do not need them to compute the Alfvénic perturbations, we might still wish to know them for their own sake.

Returning to the induction equation (15.111) and taking its  $z$  component, we get

$$\frac{d}{dt} \frac{\delta B_{\parallel}}{B_0} = \mathbf{b} \cdot \nabla u_{\parallel} - \nabla \cdot \mathbf{u} \Rightarrow \boxed{\frac{d}{dt} \left( \frac{\delta B}{B_0} - \frac{\delta p}{\rho_0} \right) = \mathbf{b} \cdot \nabla u_{\parallel}}, \quad (15.123)$$

where all terms are  $\mathcal{O}(\text{Ma}^2)$ ,  $\delta B_{\parallel} \approx \delta B$  to leading order and we used (15.107) to express  $\nabla \cdot \mathbf{u}$ . The derivatives  $d/dt$  and  $\mathbf{b} \cdot \nabla$  contain the nonlinearities involving  $\Phi$  and  $\Psi$ , which we already know from (15.121) and (15.122).

To find an equation for  $u_{\parallel}$ , we take the  $z$  component of the momentum equation (13.58):

$$\underbrace{\frac{du_{\parallel}}{dt}}_{\text{Ma}^2} = \frac{1}{\rho_0} \left[ - \underbrace{\frac{\partial}{\partial z} \left( p + \frac{B^2}{8\pi} \right)}_{\text{Ma}^3} + \underbrace{\frac{\mathbf{B} \cdot \nabla \delta B_{\parallel}}{4\pi}}_{\text{Ma}^2} \right] \Rightarrow \boxed{\frac{du_{\parallel}}{dt} = v_A^2 \mathbf{b} \cdot \nabla \frac{\delta B}{B_0}}. \quad (15.124)$$

The parallel pressure gradient is  $\mathcal{O}(\text{Ma}^3)$  because there is pressure balance (15.118) to lowest order.

Finally, let us bring in the energy equation (13.60), as yet unused. To leading order, it is

$$\frac{d}{dt} \frac{\delta s}{s_0} = \frac{d}{dt} \left( \frac{\delta p}{p_0} - \gamma \frac{\delta p}{\rho_0} \right) = 0 \Rightarrow \boxed{\frac{d}{dt} \left( \frac{\delta p}{\rho_0} + \frac{v_A^2}{c_s^2} \frac{\delta B}{B_0} \right) = 0}, \quad (15.125)$$

where, to obtain the final version of the equation, we substituted (15.118) for  $\delta p/p_0$ .

<sup>111</sup>In the latter context, they are used most prominently as a description of Alfvénic turbulence at small scales (see §15.4), for which the RMHD equations can be shown to be the correct description even if the plasma is collisionless and in general requires kinetic treatment (Schekochihin *et al.* 2009; Kunz *et al.* 2015, 2018).

Equations (15.123–15.125) are a complete set of equations for  $\delta B$ ,  $u_{\parallel}$  and  $\delta\rho$ , given  $\Phi$  and  $\Psi$ . These equations are linear in the Lagrangian frame associated with the Alfvénic perturbations, provided the parallel distances are measured along perturbed field lines. Physically, they tell us that *slow waves propagate along perturbed field lines and are passively (i.e., without acting back) advected by the perpendicular Alfvénic flows*.

In what follows, when we refer to RMHD, we will mean all five equations (15.121–15.122) and (15.123–15.125).

**Exercise 15.6.** Check that the linear relationships between various perturbations in a slow wave derived in §15.1.5 are manifest in (15.123–15.125).

**Exercise 15.7.** Show that RMHD equations possess the following exact symmetry:  $\forall \epsilon$  and  $a$ , one can simultaneously scale all perturbation amplitudes by  $\epsilon$ , perpendicular distances by  $a$ , parallel distances and times by  $a/\epsilon$ . This means that parallel and perpendicular distances in RMHD are effectively measured in different units. It also means that the small parameter  $\text{Ma}$  in RMHD can be made arbitrarily small, without any change in the form of the equations, so RMHD is a *bona fide* asymptotic theory (see remark at the end of §15.2.5).

**Exercise 15.8. RMHD and waves in double-adiabatic plasmas.**<sup>112</sup> (a) Recall that in plasmas where collisions are not sufficiently strong to isotropise pressure with respect to the local magnetic-field direction, the scalar pressure is replaced by the diagonal tensor (13.61). Consider a static equilibrium with a constant, uniform magnetic field  $\mathbf{B}_0 = B_0\hat{\mathbf{z}}$ , density  $\rho_0$ , and perpendicular and parallel pressures  $p_{\perp 0} = p_{\parallel 0} = p_0$ . Start from the usual MHD equations, but with  $-\nabla p$  replaced with  $-\nabla \cdot \mathbf{P}$ , and adopt the RMHD ordering. Show that the reduced equations for the Alfvénic perturbations are unaffected by the introduction of the anisotropic pressure, and that the compressive perturbations still satisfy (15.123), but (15.124) is replaced by

$$\rho_0 \frac{du_{\parallel}}{dt} = -\mathbf{b} \cdot \nabla \delta p_{\parallel}. \quad (15.126)$$

Instead of (15.125), work out the reduced version of the CGL closure (13.62).

(b) Use the linearised version of these equations to find the dispersion relation for the slow waves in a double-adiabatic plasma and to show that, in the limit of  $\beta \gg 1$ , these “slow waves” in fact propagate much faster than the Alfvén waves. In what way is the physics of these CGL slow waves different from the physics of the slow waves in standard MHD with isotropic pressure and why, therefore, are they able to propagate faster?

**Exercise 15.9. Firehose instability.**<sup>113</sup> (a) Consider again a plasma with anisotropic pressure (13.61) in a static, uniform equilibrium with a straight magnetic field, but this time allow the equilibrium pressure, not just its perturbations, to be anisotropic:  $p_{\perp 0} \neq p_{\parallel 0}$ . Working in RMHD ordering, show that the Alfvénic perturbations still decouple from the compressive ones and do not depend on  $\delta p_{\perp}$  and  $\delta p_{\parallel}$ .

(b) Derive their dispersion relation and determine the condition under which they become unstable. This is called the *firehose instability*. Can you explain the physical mechanism of this instability? What changes in the feedback to fluid displacements that makes Alfvénic perturbations in a pressure-anisotropic plasma unstable, while when  $p_{\perp 0} = p_{\parallel 0}$ , these perturbations behave as propagating waves?

(c) In the intergalactic medium, the typical pressure anisotropy is  $|p_{\perp} - p_{\parallel}|/p_{\parallel} \sim 10^{-2}$  and plasma beta is  $\beta \sim 10^2$  (or larger). In (certain parts of) fusion devices,  $|p_{\perp} - p_{\parallel}|/p_{\parallel} \sim 10^{-1}$  and  $\beta \sim 10^{-2}$ . Which of these plasmas is likely to suffer from the firehose instability?

<sup>112</sup>This was the 2021 exam question.

<sup>113</sup>The 2016 exam question was based on this.

### 15.3.3. Elsasser Fields and the Energetics of RMHD

The Elsasser approach (§15.2.6) can be adapted to the RMHD system. Defining *Elsasser potentials*

$$\boxed{\zeta^\pm = \Phi \pm \Psi} \quad \Leftrightarrow \quad \delta\mathbf{Z}_\perp^\pm = \mathbf{u}_\perp \pm \frac{\delta\mathbf{B}_\perp}{\sqrt{4\pi\rho_0}} = \hat{\mathbf{z}} \times \nabla_\perp \zeta^\pm, \quad (15.127)$$

it is a straightforward exercise to show that the “vorticities” of the two Elsasser fields,

$$\omega^\pm = \hat{\mathbf{z}} \cdot (\nabla_\perp \times \delta\mathbf{Z}_\perp^\pm) = \nabla_\perp^2 \zeta^\pm \quad (15.128)$$

(fluid vorticities  $\pm$  electric currents), satisfy the following evolution equation

$$\boxed{\frac{\partial\omega^\pm}{\partial t} \mp v_A \frac{\partial\omega^\pm}{\partial z} + \{\zeta^\mp, \omega^\pm\} = \{\partial_j \zeta^\pm, \partial_j \zeta^\mp\}}, \quad (15.129)$$

where summation over the repeated index  $j$  is implied. The main corollary of this equation is the same as in §15.2.6, although here it applies to perpendicular perturbations only: *only counter-propagating Alfvénic perturbations can interact and any finite-amplitude perturbation composed of just one Elsasser field is a nonlinear solution.*

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Some light is perhaps shed on the nature of the interaction between Elsasser fields if we notice that the left-hand side of (15.129) tells us that the Elsasser vorticity  $\omega^\pm$  is propagated along the mean field at the speed  $v_A$  and advected across the field by the Elsasser field  $\delta\mathbf{Z}_\perp^\mp$ . The right-hand side of (15.129) is a kind of vortex-stretching term, implying a tendency for vortices and current layers to be produced in the  $(x, y)$  plane. There is a preference for current layers, as it turns out. The term in the right-hand side of (15.129) has opposite signs for the two Elsasser fields. Therefore, arguably, nonlinear dynamics favour  $\omega^+ \omega^- < 0$ , i.e.,  $|\nabla_\perp^2 \Psi|^2 > |\nabla_\perp^2 \Phi|^2$  (larger currents than vorticities). This is, indeed, what is seen in numerical simulations of MHD turbulence (see Zhdankin *et al.* 2016 and §15.4).

The energies of the two Elsasser fields are individually conserved (cf. §15.2.7),

$$\frac{d}{dt} \int d\mathbf{r} |\nabla_\perp \zeta^\pm|^2 = \frac{d}{dt} \int d\mathbf{r} |\delta\mathbf{Z}_\perp^\pm|^2 = 0, \quad (15.130)$$

i.e., when the two fields do interact, they scatter each other nonlinearly, but do *not* exchange energy.

There is an Elsasser-like formulation for the slow waves as well:<sup>114</sup>

$$\boxed{\delta\mathbf{Z}_\parallel^\pm = u_\parallel \pm \frac{\delta B}{\sqrt{4\pi\rho_0}} \sqrt{1 + \frac{v_A^2}{c_s^2}}}. \quad (15.131)$$

Then, from (15.123–15.125), one gets, after more algebra,

$$\begin{aligned} \frac{\partial \delta Z_\parallel^\pm}{\partial t} \mp \frac{c_s v_A}{\sqrt{c_s^2 + v_A^2}} \frac{\partial \delta Z_\parallel^\pm}{\partial z} = \\ -\frac{1}{2} \left[ \left( 1 \mp \frac{1}{\sqrt{1 + v_A^2/c_s^2}} \right) \{\zeta^\mp, \delta Z_\parallel^\pm\} + \left( 1 \pm \frac{1}{\sqrt{1 + v_A^2/c_s^2}} \right) \{\zeta^\pm, \delta Z_\parallel^\pm\} \right]. \end{aligned} \quad (15.132)$$

Note the (expected) appearance of the slow-wave phase speed [cf. (15.33)] in the left-hand

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<sup>114</sup>At high  $\beta$ ,  $v_A \ll c_s$ , so we recover from (15.131) and (15.127) the Elsasser fields as defined for iMHD in (15.75).

side. Thus, slow waves interact only with Alfvénic perturbations—when  $v_A \ll c_s$ , only with the counterpropagating ones, but at finite  $\beta$ , because the slow waves are slower, a co-propagating Alfvénic perturbation can catch up with a slow one, have its way with it in passing and speed on (it's a tough world).

There is no energy exchange in these interactions: the “+” and “−” slow-wave energies are individually conserved:

$$\frac{d}{dt} \int d\mathbf{r} |\delta Z_{||}^{\pm}|^2 = 0. \quad (15.133)$$

#### 15.3.4. Entropy Mode

There are only two equations in (15.132), whereas we had three equations (15.123–15.125) for our three compressive fields  $\delta B$ ,  $u_{||}$  and  $\delta \rho$ . The third equation, (15.125), was in fact for the entropy perturbation:

$$\boxed{\frac{d\delta s}{dt} = 0}, \quad \frac{\delta s}{s_0} = -\gamma \left( \frac{\delta \rho}{\rho_0} + \frac{v_A^2}{c_s^2} \frac{\delta B}{B_0} \right). \quad (15.134)$$

We see that  $\delta s$  is a decoupled variable, independent from  $\zeta^{\pm}$  or  $\delta Z_{||}^{\pm}$  (because it is the only one that involves  $\delta \rho / \rho_0$ ). Equation (15.134) says that  $\delta s$  is a *passive scalar field*, simply carried around by the Alfvén velocity  $\mathbf{u}_{\perp}$  (via  $d/dt$ ). At high  $\beta$ , this is just a density perturbation.

The associated linear mode is not a wave: its dispersion relation is

$$\boxed{\omega = 0}. \quad (15.135)$$

This is the (famously often forgotten) 7th MHD mode, known as the *entropy mode* (there are 7 equations in MHD, so there must be 7 linear modes: two fast waves, two Alfvén waves, two slow waves and one entropy mode).

**Exercise 15.10.** Go back to §15.1 and find where this mode was overlooked.

Since the entropy mode is decoupled, its “energy” (variance) is individually conserved:

$$\frac{d}{dt} \int d\mathbf{r} |\delta s|^2 = 0. \quad (15.136)$$

Thus, *in RMHD, the (nonlinear) evolution of all perturbations is constrained by 5 separate conservation laws:  $\int d\mathbf{r} |\delta Z_{\perp}^{\pm}|^2$ ,  $\int d\mathbf{r} |\delta Z_{||}^{\pm}|^2$  and  $\int d\mathbf{r} |\delta s|^2$  are all invariants.*

#### 15.3.5. Discussion

Such are the simplifications allowed by anisotropy. Besides greater mathematical simplicity, what is the moral of this story, physically? Let me leave you with two observations.

- In a strong magnetic field, linear propagation is a parallel effect, whilst nonlinearity is a perpendicular effect (advection by  $\mathbf{u}_{\perp}$ , adjustment of propagation direction by  $\delta \mathbf{B}_{\perp}$ ). RMHD equations express the idea that linear and nonlinear physics play equally important role—this becomes the fundamental guiding principle in the theory of MHD turbulence (§15.4). The idea is that complicated nonlinear dynamics that emerge in the perpendicular plane get teased out along the field because propagating waves enforce a degree of parallel spatial coherence. The distances over which this happens are determined by equating linear and nonlinear time scales,  $k_{||} v_A \sim k_{\perp} u_{\perp}$ . Dynamics cannot stay coherent over distances longer than  $\sim k_{||}^{-1}$  determined by this balance because of

causality: points separated by longer parallel distances cannot exchange information quickly enough to catch up with perpendicular nonlinearities acting locally at each of these points. This principle is called *critical balance*.

- Restricting the size of perturbations to be small made the RMHD system, in a certain sense, “less nonlinear” than the full MHD (or than iMHD, where  $\delta B/B_0 \sim 1$  was allowed). This led to the system’s dynamics being constrained by more invariants: the MHD energy invariant got split into 5 individually conserved quadratic quantities.

**Exercise 15.11.** You might find it an interesting exercise to think about properties of the RMHD system in 2D, in the light of the two observations above. How many invariants are there? In what kind of physical circumstances *can* we use 2D RMHD without necessarily expecting parallel coherence of the system to break down by the causality argument?

#### 15.4. MHD Turbulence

RMHD is a good starting point for developing the theory of *MHD turbulence*—a phenomenon observed with great precision in the solar wind and believed ubiquitous in the Universe. Everything that I have to say on this subject can be found in [Schekochihin \(2022\)](#).

**Exercise 15.12. Weak RMHD turbulence.** If you followed §§7.2 and 8.4, try your hand at constructing a WT theory for Alfvén waves in RMHD. A good starting point is the RMHD equations in the form (15.129) as the underlying dynamical equations for the waves. You can check your theory by consulting Appendix A of [Schekochihin \(2022\)](#).

## 16. MHD Relaxation

So far, we have only considered MHD in a straight field against the background of constant density and pressure (except in §15.2.8, where this was generalised slightly). As any more complicated (static) equilibrium will locally look like this, what we have done has considerable universal significance. Now we shall occupy ourselves with a somewhat less universal (i.e., dependent on the circumstances of a particular problem) and more “large-scale” (compared to the dynamics of wavy perturbations) question: *what kind of (static) equilibrium states are there and into which of those states will an MHD fluid normally relax?*

### 16.1. Static MHD Equilibria

Let us go back to the MHD equations (13.57–13.60) and seek static equilibria, i.e., set  $\mathbf{u} = 0$  and  $\partial/\partial t = 0$ . The remaining equations are

$$\boxed{-\nabla p + \frac{\mathbf{j} \times \mathbf{B}}{c} = 0, \quad \mathbf{j} = \frac{c}{4\pi} \nabla \times \mathbf{B}, \quad \nabla \cdot \mathbf{B} = 0} \quad (16.1)$$

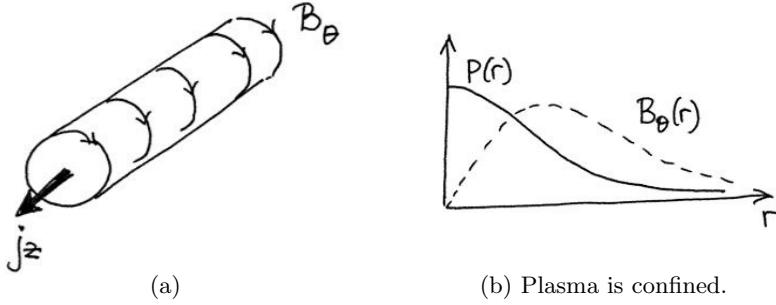
(the force balance, Ampère’s law and the solenoidality-of- $\mathbf{B}$  constraint). These are 7 equations for 7 unknowns ( $p$ ,  $\mathbf{B}$ ,  $\mathbf{j}$ ), so a complete set. Density is irrelevant because nothing moves and so inertia does not matter.

The force-balance equation has two immediate general consequences:

$$\mathbf{B} \cdot \nabla p = 0, \quad (16.2)$$

so *magnetic surfaces are surfaces of constant pressure*, and

$$\mathbf{j} \cdot \nabla p = 0, \quad (16.3)$$

FIGURE 72. *z* pinch.

so currents flow along those surfaces.

Equation (16.2) implies that if magnetic field lines are stochastic and fill the volume of the system, then  $p = \text{const}$  across the system and so the force balance becomes

$$\mathbf{j} \times \mathbf{B} = 0. \quad (16.4)$$

Such equilibria are called *force-free* and turn out to be very interesting, as we shall discover soon (from §16.1.2 onwards).

#### 16.1.1. MHD Equilibria in Cylindrical Geometry

As the simplest example of an inhomogeneous equilibrium, let us consider the case of axial and cylindrical symmetry:

$$\frac{\partial}{\partial \theta} = 0, \quad \frac{\partial}{\partial z} = 0. \quad (16.5)$$

Solenoidality of the magnetic field then rules out it having a radial component:

$$\nabla \cdot \mathbf{B} = \frac{1}{r} \frac{\partial}{\partial r} r B_r = 0 \quad \Rightarrow \quad r B_r = \text{const} \quad \Rightarrow \quad B_r = 0. \quad (16.6)$$

Ampère's law tells us that currents do not flow radially either:

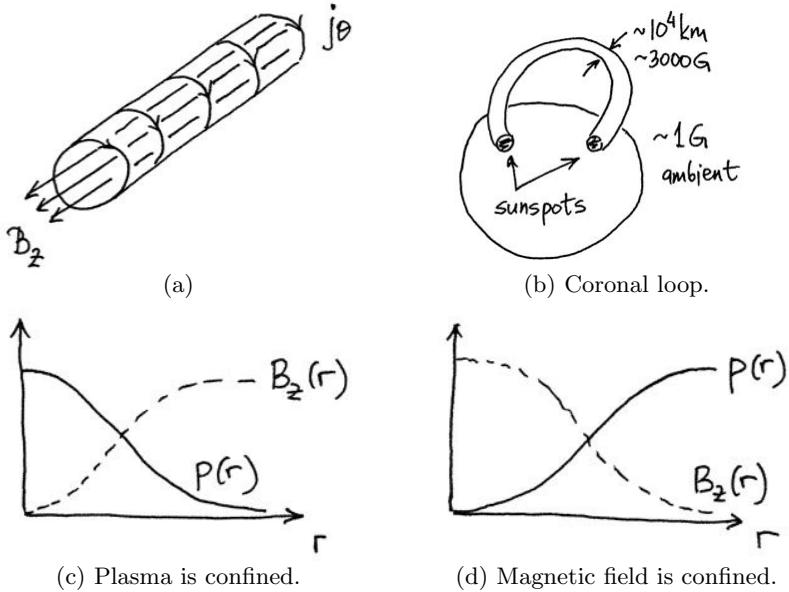
$$\mathbf{j} = \frac{c}{4\pi} \nabla \times \mathbf{B} \quad \Rightarrow \quad \begin{cases} j_r = 0, \\ j_\theta = -\frac{c}{4\pi} \frac{\partial B_z}{\partial r}, \\ j_z = \frac{c}{4\pi r} \frac{1}{r} \frac{\partial}{\partial r} r B_\theta. \end{cases} \quad (16.7)$$

Finally, the radial pressure balance gives us

$$\begin{aligned} \frac{\partial p}{\partial r} &= \frac{(\mathbf{j} \times \mathbf{B})_r}{c} = \frac{j_\theta B_z - j_z B_\theta}{c} = \frac{1}{4\pi} \left( -B_z \frac{\partial B_z}{\partial r} - \frac{B_\theta}{r} \frac{\partial}{\partial r} r B_\theta \right) \\ &= -\frac{\partial}{\partial r} \frac{B_z^2}{8\pi} - \frac{B_\theta^2}{4\pi r} - \frac{\partial}{\partial r} \frac{B_\theta^2}{8\pi} \quad \Rightarrow \quad \boxed{\frac{\partial}{\partial r} \left( p + \frac{B^2}{8\pi} \right) = -\frac{B_\theta^2}{4\pi r}}. \end{aligned} \quad (16.8)$$

This simply says that the total pressure gradient is balanced by the tension force. A general equilibrium for which this is satisfied is called a *screw pinch*.

One simple particular case of this is the *z pinch* (Fig. 72a). This is achieved by letting

FIGURE 73.  $\theta$  pinch.

a current flow along the  $z$  axis, giving rise to an azimuthal field:

$$j_\theta = 0, \quad j_z = \frac{c}{4\pi} \frac{1}{r} \frac{\partial}{\partial r} r B_\theta \quad \Rightarrow \quad B_\theta = \frac{4\pi}{c} \frac{1}{r} \int_0^r dr' r' j_z(r'), \quad B_z = 0. \quad (16.9)$$

Equation (16.8) becomes

$$\boxed{\frac{\partial p}{\partial r} = -\frac{1}{c} j_z B_\theta}. \quad (16.10)$$

The “pinch” comes from magnetic loops and is due to the curvature force: the loops want to contract inwards, the pressure gradient opposes this and so *plasma is confined* (Fig. 72b). This configuration will, however, prove to be very badly unstable (§17.4)—which does not stop it from being a popular laboratory set up for short-term confinement experiments (see, e.g., review by Haines 2011).

Another simple particular case is the  $\theta$  pinch (Fig. 73a). This is achieved by imposing a straight but radially non-uniform magnetic field in the  $z$  direction and, therefore, azimuthal currents:

$$B_\theta = 0, \quad j_z = 0, \quad j_\theta = -\frac{c}{4\pi} \frac{\partial B_z}{\partial r}. \quad (16.11)$$

Equation (16.8) is then just a pressure balance, pure and simple:

$$\boxed{\frac{\partial}{\partial r} \left( p + \frac{B_z^2}{8\pi} \right) = 0}. \quad (16.12)$$

In this configuration, we can confine the plasma (Fig. 73c) or the magnetic flux (Fig. 73d). The latter is what happens, for example, in flux tubes that link sunspots (Fig. 73b). The  $\theta$  pinch is a stable configuration (Q11).

The more general case of a screw pinch (16.8) is a superposition of  $z$  and  $\theta$  pinches, with both magnetic fields and currents wrapping themselves around cylindrical flux surfaces.

The next step in complexity is to assume axial, but not cylindrical symmetry ( $\partial/\partial\theta = 0$ ,  $\partial/\partial z \neq 0$ ). This is explored in Q9.

For a much more thorough treatment of MHD equilibria, the classic textbook is [Freidberg \(2014\)](#).

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### 16.1.2. Force-Free Equilibria

Another interesting and elegant class of equilibria arises if we consider situations in which  $\nabla p$  is negligible and can be completely omitted from the force balance. This can happen in two possible sets of circumstances:

—pressure is the same across the system, e.g., because the field lines are stochastic [a previously mentioned consequence of (16.2)];

— $\beta = p/(B^2/8\pi) \ll 1$ , so thermal energy is negligible compared to magnetic energy and so  $p$  is irrelevant.

A good example of the latter situation is the *solar corona*, where  $\beta \sim 1-10^{-6}$  (assuming  $n \sim 10^9 \text{ cm}^{-3}$ ,  $T \sim 10^2 \text{ eV}$  and  $B \sim 1 - 10^3 \text{ G}$ , the lower value applying in the photosphere, the upper one in the coronal loops; see Fig. 73b)

In such situations, the equilibrium is purely magnetic, i.e., the magnetic field is “*force-free*,” which implies that the current must be parallel to the magnetic field:

$$\boxed{\mathbf{j} \times \mathbf{B} = 0} \quad \Rightarrow \quad \mathbf{j} \parallel \mathbf{B} \quad \Rightarrow \quad \frac{4\pi}{c} \mathbf{j} = \nabla \times \mathbf{B} = \alpha(\mathbf{r}) \mathbf{B}, \quad (16.13)$$

where  $\alpha(\mathbf{r})$  is an arbitrary scalar function. Taking the divergence of the last equation tells us that

$$\mathbf{B} \cdot \nabla \alpha = 0, \quad (16.14)$$

so the function  $\alpha(\mathbf{r})$  is constant on magnetic surfaces. If  $\mathbf{B}$  is chaotic and volume-filling, then  $\alpha = \text{const}$  across the system.

The case of  $\alpha = \text{const}$  is called the *linear force-free field*. In this case, taking the curl of (16.13) and then iterating it once gives us

$$\boxed{-\nabla^2 \mathbf{B} = \alpha \nabla \times \mathbf{B} = \alpha^2 \mathbf{B}} \quad \Rightarrow \quad \boxed{(\nabla^2 + \alpha^2) \mathbf{B} = 0}, \quad (16.15)$$

so the magnetic field satisfies a Helmholtz equation (to solve which, one must, of course, specify some boundary conditions).

Thus, there is, potentially, a large zoo of MHD equilibria. Some of them are stable, some are not, and, therefore, some are more interesting and/or more relevant than others. How does one tell? A good question to ask is as follows. Suppose we set up some initial configuration of magnetic field (by, say, switching on some current-carrying coils, driving currents inside plasma, etc.)—*to what (stable) equilibrium will this system eventually relax?*

In general, any initially arranged magnetic configuration will exert forces on the plasma, these will drive flows, which in turn will move the magnetic fields around; eventually, everything will settle into some static equilibrium. We expect that, normally, some amount of the energy contained in the initial field will be lost in such a relaxation process because the flows will be dissipating, the fields diffusing and/or reconnecting, etc.—the losses occur due to the resistive and viscous terms in the non-ideal MHD equations derived in §13. Thus, one expects that *the final relaxed static state will be a*

*minimum-energy state* and so we must be able to find it by minimising magnetic energy:

$$\int d\mathbf{r} \frac{B^2}{8\pi} \rightarrow \min. \quad (16.16)$$

Clearly, if the relaxation occurred without any constraints, the solution would just be  $\mathbf{B} = 0$ . In fact, there are constraints. These constraints are *topological*: if you think of magnetic field lines as a tangled mess, you will realise that, while you can change this tangle by moving field lines around, you cannot easily undo linkages, knots, etc.—anything that, to be undone, would require the field lines to have “ends”. This intuition can be turned into a quantitative theory once we discover that the induction equation (13.59) has an invariant that involves the magnetic field only and is, in a certain sense, “better conserved” than energy.

## 16.2. Helicity

*Magnetic helicity* in a volume  $V$  is defined as

$$H = \boxed{\int_V d\mathbf{r} \mathbf{A} \cdot \mathbf{B}}, \quad (16.17)$$

where  $\mathbf{A}$  is the vector potential,  $\nabla \times \mathbf{A} = \mathbf{B}$ .

### 16.2.1. Helicity Is Well Defined

This is not obvious because  $\mathbf{A}$  is not unique: a gauge transformation

$$\mathbf{A} \rightarrow \mathbf{A} + \nabla \chi, \quad (16.18)$$

with  $\chi$  an arbitrary scalar function, leaves  $\mathbf{B}$  unchanged and so does not affect physics. Under this transformation, helicity stays invariant:

$$H \rightarrow H + \int_V d\mathbf{r} \mathbf{B} \cdot \nabla \chi = H + \int_{\partial V} d\mathbf{S} \cdot \mathbf{B} \chi = H, \quad (16.19)$$

provided  $\mathbf{B}$  at the boundary is parallel to the boundary, i.e., provided the volume  $V$  encloses the field (nothing sticks out).

### 16.2.2. Helicity Is Conserved

Let us go back to the induction equation (13.23) (in which we retain resistivity to keep track of non-ideal effects, i.e., of the breaking of flux conservation):

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B} - \eta \nabla \times \mathbf{B}). \quad (16.20)$$

“Uncurling” this equation, we get

$$\frac{\partial \mathbf{A}}{\partial t} = \mathbf{u} \times \mathbf{B} - \eta \nabla \times \mathbf{B} + \nabla \chi. \quad (16.21)$$

Using (16.20) and (16.21), we have

$$\begin{aligned} \frac{\partial}{\partial t} \mathbf{A} \cdot \mathbf{B} &= \mathbf{B} \cdot (\mathbf{u} \times \mathbf{B} - \eta \nabla \times \mathbf{B} + \nabla \chi) + \mathbf{A} \cdot [\nabla \times (\mathbf{u} \times \mathbf{B} - \eta \nabla \times \mathbf{B})] \\ &= -\eta \mathbf{B} \cdot (\nabla \times \mathbf{B}) + \nabla \cdot (\mathbf{B} \chi) \\ &\quad - \nabla \cdot [\mathbf{A} \times (\mathbf{u} \times \mathbf{B} - \eta \nabla \times \mathbf{B})] + (\mathbf{u} \times \mathbf{B} - \eta \nabla \times \mathbf{B}) \cdot \underbrace{(\nabla \times \mathbf{A})}_{=\mathbf{B}} \\ &= \nabla \cdot [\mathbf{B} \chi - \mathbf{u} \mathbf{A} \cdot \mathbf{B} + \mathbf{B} \mathbf{A} \cdot \mathbf{u} + \eta \mathbf{A} \times (\nabla \times \mathbf{B})] - 2\eta \mathbf{B} \cdot (\nabla \times \mathbf{B}). \end{aligned} \quad (16.22)$$

Integrating this and using Gauss's theorem, we get

$$\begin{aligned} \frac{\partial}{\partial t} \int_V d\mathbf{r} \mathbf{A} \cdot \mathbf{B} &= \int_{\partial V} dS \cdot [\mathbf{B}\chi - \mathbf{u}\mathbf{A} \cdot \mathbf{B} + \mathbf{B}\mathbf{A} \cdot \mathbf{u} + \eta\mathbf{A} \times (\nabla \times \mathbf{B})] \\ &\quad - 2\eta \int_V d\mathbf{r} \mathbf{B} \cdot (\nabla \times \mathbf{B}). \end{aligned} \quad (16.23)$$

The surface integral vanishes provided both  $\mathbf{u}$  and  $\mathbf{B}$  are parallel to the boundary (no fields stick out and no flows cross). The resistive term in the surface integral can also be ignored either by arranging  $V$  appropriately or simply by taking it large enough so  $\mathbf{B} \rightarrow 0$  on  $\partial V$ , or, indeed, by taking  $\eta \rightarrow +0$ . Thus,

$$\boxed{\frac{dH}{dt} = -2\eta \int d\mathbf{r} \mathbf{B} \cdot (\nabla \times \mathbf{B})}, \quad (16.24)$$

*magnetic helicity is conserved in ideal MHD.*<sup>115</sup>

Furthermore, it turns out that even in resistive MHD, helicity is “better conserved” than energy, in the following sense. As we saw in §13.10.2, the magnetic energy evolves according to

$$\frac{d}{dt} \int d\mathbf{r} B^2 = \left( \begin{array}{c} \text{energy exchange terms} \\ \text{and fluxes} \end{array} \right) - 2\eta \int d\mathbf{r} |\nabla \times \mathbf{B}|^2. \quad (16.25)$$

The first term on the right-hand side contains various fluxes and energy exchanges with the velocity field [see (13.54)], all of which eventually decay as the system relaxes (flows decay by viscosity). The second term represents Ohmic heating. If  $\eta$  is small but the Ohmic heating is finite, it is finite because magnetic field develops fine-scale gradients:  $\nabla \sim \eta^{-1/2}$ , so

$$-2\eta \int d\mathbf{r} |\nabla \times \mathbf{B}|^2 \rightarrow \text{const} \quad \text{as } \eta \rightarrow +0. \quad (16.26)$$

But then the right-hand side of (16.24) is

$$-2\eta \int d\mathbf{r} \mathbf{B} \cdot (\nabla \times \mathbf{B}) = \mathcal{O}(\eta^{1/2}) \rightarrow 0 \quad \text{as } \eta \rightarrow +0. \quad (16.27)$$

Thus, as an initial magnetic configuration relaxes, while its energy can change quickly (on dynamical times), its helicity changes only very slowly in the limit of small  $\eta$ . *The constancy of  $H$  (as  $\eta \rightarrow +0$ ) provides us with the constraint subject to which the energy will need to be minimised.*

Before we use this idea, let us discuss what the conservation of helicity means physically, or, rather, topologically.

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<sup>115</sup>The resistive term in the right-hand side of (16.24) is  $\propto \int d\mathbf{r} \mathbf{B} \cdot \mathbf{j}$ , a quantity known as the *current helicity*.

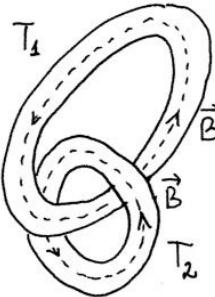


FIGURE 74. Linked flux tubes.

### 16.2.3. Helicity Is a Topological Invariant

Consider two linked flux tubes,  $T_1$  and  $T_2$  (Fig. 74). The helicity of  $T_1$  is the product of the fluxes through  $T_1$  and  $T_2$ :

$$\begin{aligned} H_1 &= \int_{T_1} d\mathbf{r} \mathbf{A} \cdot \mathbf{B} = \int_{T_1} \underbrace{d\mathbf{l}}_{\mathbf{B} d\mathbf{l}} \cdot \underbrace{\frac{d\mathbf{S}}{B dS}}_{\mathbf{A} \cdot \mathbf{B}} \underbrace{\mathbf{A} \cdot \mathbf{B}}_{\mathbf{A} \cdot \mathbf{B}} \\ &= \int_{T_1} \mathbf{A} \cdot \mathbf{B} d\mathbf{l} B \mathbf{B} \cdot d\mathbf{S} = \int_{T_1} \mathbf{A} \cdot d\mathbf{l} \mathbf{B} \cdot d\mathbf{S} = \Phi_1 \underbrace{\int_{T_1} \mathbf{A} \cdot d\mathbf{l}}_{= \int_{S_1} \mathbf{B} \cdot d\mathbf{S}} = \Phi_1 \Phi_2. \\ &\quad = \Phi_{\text{through hole in } T_1} \end{aligned} \quad (16.28)$$

By the same token, in general, in a system of many linked tubes, the helicity of tube  $i$  is

$$H_i = \Phi_i \Phi_{\text{through hole in tube } i} = \Phi_i \sum_j \Phi_j N_{ij}, \quad (16.29)$$

where  $N_{ij}$  is the number of times tube  $j$  passes through the hole in tube  $i$ . The total helicity of this entire assemblage of flux tubes is then

$$H = \sum_{ij} \Phi_i \Phi_j N_{ij}.$$

(16.30)

Thus,  $H$  is the number of linkages of the flux tubes weighted by the field strength in them. It is in this sense that helicity is a topological invariant.

Note that the cross-helicity  $\int d\mathbf{r} \mathbf{u} \cdot \mathbf{B}$  (§15.2.7) can similarly be interpreted as counting the linkages between flux tubes ( $\mathbf{B}$ ) and vortex tubes ( $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ ). The current helicity  $\int d\mathbf{r} \mathbf{B} \cdot \mathbf{j}$  [appearing in the right-hand side of (16.24)] counts the number of linkages between current loops. The latter is not an MHD invariant though.

### 16.3. J. B. Taylor Relaxation

Let us now work out the equilibrium to which an MHD system will relax by *minimising magnetic energy subject to constant helicity*:

$$\delta \int_V d\mathbf{r} (B^2 - \alpha \mathbf{A} \cdot \mathbf{B}) = 0, \quad (16.31)$$

where  $\alpha$  is the Lagrange multiplier introduced to enforce the constant-helicity constraint. Let us work out the two terms:

$$\begin{aligned}\delta \int_V d\mathbf{r} B^2 &= 2 \int_V d\mathbf{r} \mathbf{B} \cdot \delta \mathbf{B} = 2 \int_V d\mathbf{r} \mathbf{B} \cdot (\nabla \times \delta \mathbf{A}) \\ &= 2 \int_V d\mathbf{r} [-\nabla \cdot (\mathbf{B} \times \delta \mathbf{A}) + (\nabla \times \mathbf{B}) \cdot \delta \mathbf{A}] \\ &= -2 \int_{\partial V} d\mathbf{S} \cdot (\mathbf{B} \times \delta \mathbf{A}) + 2 \int_V d\mathbf{r} (\nabla \times \mathbf{B}) \cdot \delta \mathbf{A},\end{aligned}\quad (16.32)$$

$$\begin{aligned}\delta H &= \delta \int_V d\mathbf{r} \mathbf{A} \cdot \mathbf{B} = \int_V d\mathbf{r} (\mathbf{B} \cdot \delta \mathbf{A} + \mathbf{A} \cdot \delta \mathbf{B}) = \int_V d\mathbf{r} [\mathbf{B} \cdot \delta \mathbf{A} + \mathbf{A} \cdot (\nabla \times \delta \mathbf{A})] \\ &= \int_V d\mathbf{r} [\mathbf{B} \cdot \delta \mathbf{A} - \nabla \cdot (\mathbf{A} \times \delta \mathbf{A}) + \underbrace{(\nabla \times \mathbf{A}) \cdot \delta \mathbf{A}}_{= \mathbf{B}}] \\ &= - \int_{\partial V} d\mathbf{S} \cdot (\mathbf{A} \times \delta \mathbf{A}) + 2 \int_V d\mathbf{r} \mathbf{B} \cdot \delta \mathbf{A}.\end{aligned}\quad (16.33)$$

Now, since

$$\frac{\partial \delta \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) = \nabla \times \left( \frac{\partial \boldsymbol{\xi}}{\partial t} \times \mathbf{B} \right) \quad (16.34)$$

for small displacements, we have  $\delta \mathbf{A} = \boldsymbol{\xi} \times \mathbf{B}$ , whence

$$\mathbf{B} \times \delta \mathbf{A} = B^2 \boldsymbol{\xi} - \mathbf{B} \cdot \boldsymbol{\xi} \mathbf{B}, \quad (16.35)$$

$$\mathbf{A} \times \delta \mathbf{A} = \mathbf{A} \cdot \mathbf{B} \boldsymbol{\xi} - \mathbf{A} \cdot \boldsymbol{\xi} \mathbf{B}. \quad (16.36)$$

Therefore, the surface terms in (16.32) and (16.33) vanish if  $\mathbf{B}$  and  $\boldsymbol{\xi}$  are parallel to the boundary  $\partial V$ , i.e., if the volume  $V$  encloses both  $\mathbf{B}$  and the plasma—there are no displacements through the boundary.

This leaves us with

$$\delta \int_V d\mathbf{r} (B^2 - \alpha \mathbf{A} \cdot \mathbf{B}) = 2 \int_V d\mathbf{r} (\nabla \times \mathbf{B} - \alpha \mathbf{B}) \cdot \delta \mathbf{A} = 0, \quad (16.37)$$

which instantly implies that  $\mathbf{B}$  is a linear force-free field:

$$\boxed{\nabla \times \mathbf{B} = \alpha \mathbf{B}} \Rightarrow \boxed{\nabla^2 \mathbf{B} = -\alpha^2 \mathbf{B}}. \quad (16.38)$$

Thus, our system will relax to a linear force-free state determined by (16.38) and system-specific boundary conditions. Here  $\alpha = \alpha(H)$  depends on the (fixed by initial conditions) value of  $H$  via the equation

$$H(\alpha) = \int d\mathbf{r} \mathbf{A} \cdot \mathbf{B} = \frac{1}{\alpha} \int d\mathbf{r} B^2, \quad (16.39)$$

where  $\mathbf{B}$  is the solution of (16.38) (since  $\nabla \times \mathbf{B} = \alpha \mathbf{B} = \alpha \nabla \times \mathbf{A}$ , we have  $\mathbf{B} = \alpha \mathbf{A} + \nabla \chi$  and the  $\chi$  term vanishes under volume integration).

Thus, the prescription for finding force-free equilibria is

- solve (16.38), get  $\mathbf{B} = \mathbf{B}(\alpha)$ , parametrically dependent on  $\alpha$ ,
- calculate  $H(\alpha)$  according to (16.39),
- set  $H(\alpha) = H_0$ , where  $H_0$  is the initial value of helicity, hence calculate  $\alpha = \alpha(H_0)$  and complete the solution by using this  $\alpha$  in  $\mathbf{B} = \mathbf{B}(\alpha)$ .

Note that it is possible for this procedure to return multiple solutions. In that case, the

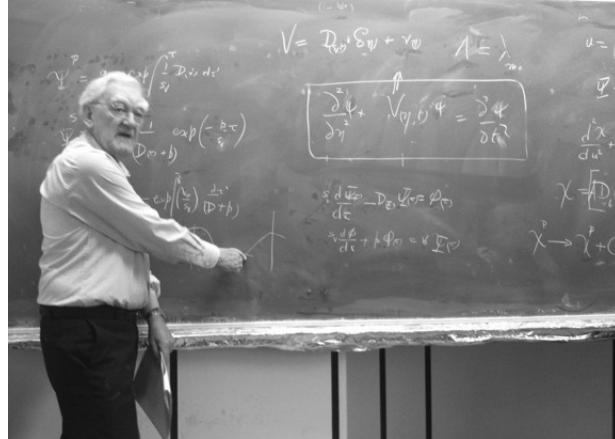


FIGURE 75. John Bryan Taylor (born 1928), one of the founding fathers of modern plasma physics, author of the Taylor relaxation (§16.3), Taylor constraint (in dynamo theory), Chirikov–Taylor map (in chaos theory), the ballooning theory (in tokamak MHD), and many other clever things, including a key role in the design of the UK’s first hydrogen bomb (1957). This picture was taken in 2012 at the Wolfgang Pauli Institute in Vienna.

solution with the smallest energy must be the right one (if a system relaxed to a local minimum, one can always imagine it being knocked out of it by some perturbation and falling to a lower energy).

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**Exercise 16.1. Force-free fields in 2D.**<sup>116</sup> Show that for incompressible MHD confined to the 2D plane  $(x, y)$ , the quantity  $\int d^2r A_z^2$  is conserved, except for resistive dissipation and under suitable assumptions about what happens at the boundaries of the domain (this 2D invariant is sometimes called “anastrophy”). Work out the 2D version of J. B. Taylor relaxation and show that the resulting equilibrium field is a linear force-free field.

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#### 16.4. Relaxed Force-Free State of a Cylindrical Pinch

Let us illustrate how the procedure derived in §16.3 works by considering again the case of cylindrical and axial symmetry [see (16.5)]. The  $z$  component of (16.38) gives us the following equation for  $B_z(r)$ :

$$B_z'' + \frac{1}{r} B_z' + \alpha^2 B_z = 0. \quad (16.40)$$

This is a Bessel equation, whose solution, subject to  $B_z(0) = B_0$  and  $B_z(\infty) = 0$ , is

$$\boxed{B_z(r) = B_0 J_0(\alpha r)}. \quad (16.41)$$

We can now calculate the azimuthal field as follows

$$\alpha B_\theta = (\nabla \times \mathbf{B})_\theta = -B_z' \Rightarrow \boxed{B_\theta(r) = B_0 J_1(\alpha r)}. \quad (16.42)$$

This gives us an interesting twisted field (Fig. 76), able to maintain itself in equilibrium without help from pressure gradients.

Finally, we calculate its helicity according to (16.39): assuming that the length of the

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<sup>116</sup>The 2020 exam question was based on this.

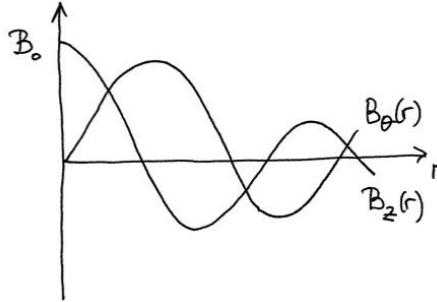


FIGURE 76. Relaxed cylindrical pinch.

cylinder is  $L$ , its radius  $R$  and so its volume  $V = \pi R^2 L$ , we have

$$\begin{aligned} H &= \frac{1}{\alpha} \int d\mathbf{r} B^2 = \frac{2\pi LB_0^2}{\alpha} \int_0^R dr [J_0^2(\alpha r) + J_1^2(\alpha r)] \\ &= \frac{B_0^2 V}{\alpha^2} \left[ J_0^2(\alpha R) + 2J_1^2(\alpha R) + J_2^2(\alpha R) - \frac{2}{\alpha R} J_1(\alpha R) J_2(\alpha R) \right]. \end{aligned} \quad (16.43)$$

If we solve this for  $\alpha = \alpha(H)$ , our solution is complete.

**Exercise 16.2.** Work out what happens in the general case of  $\partial/\partial\theta \neq 0$  and  $\partial/\partial z \neq 0$  and whether the simple symmetric solution obtained above is the correct relaxed, minimum-energy state (not always, it turns out). This is not a trivial exercise. The solution is in [Taylor & Newton \(2015, §9\)](#), where you will also find much more on the subject of J. B. Taylor relaxation, relaxed states and much besides—all from the original source.

There are other useful variational principles—other in the sense that the constraints that are imposed are different from helicity conservation. The need for them arises when one considers magnetic equilibria in domains that do not completely enclose the field lines, i.e., when  $d\mathbf{S} \cdot \mathbf{B} \neq 0$  at the boundary. One example of such a variational principle, also yielding a force-free field (although not necessarily a linear one), is given in Q1(e). A specific example of such a field arises in Q9(f).

### 16.5. Decay of MHD Turbulence

Let me show you how one can use the idea of relaxation at constant helicity to work out, non-rigorously but rather convincingly, I think, the decay law (with time) of some arbitrary initial magnetic configuration. As I argued at the end of §16.1.2, such a configuration would normally go unstable, drive flows, and eventually rearrange itself into a force-free state—without, as we now know, changing its net helicity. Let us imagine that this is all happening in empty space (no interesting boundaries) and that initial magnetic energy is  $\langle B^2 \rangle$  (average over volume), consisting of random, tangled magnetic fields correlated on scale  $\sim \ell_0$ . How will the magnetic energy  $\langle B^2 \rangle$  and the field correlation scale  $\ell$  change as functions of time? This is known as the problem of the decay of MHD turbulence.

Let us do some “twiddle algebra”. As per the relaxation philosophy articulated above, let us assume that the magnetic energy will decay at a rate that is independent of resistivity, to wit, comparable to the inverse dynamical time  $\sim u/\ell$ , where  $u$  is the typical size, and  $\ell$  the typical scale, of the flows in the system:

$$\frac{dB^2}{dt} \sim -\frac{uB^2}{\ell} \sim -\frac{B^3}{\ell}, \quad (16.44)$$

where  $B$  is the typical size of the fields. The last step is reasonable if we assume that the flows are “Alfvénic”,  $u \sim B$  (with the understanding that  $B$  is measured in velocity units, as in §15.2.5), i.e., driven by (or in concert with) the magnetic forces. Conservation of helicity allows

us to relate  $\ell$  to  $B$ : since  $A \sim B\ell$ ,

$$H = V\langle \mathbf{A} \cdot \mathbf{B} \rangle \sim VB^2\ell \sim \text{const} \quad \Rightarrow \quad \ell \propto B^{-2}. \quad (16.45)$$

Using this in (16.44), we get

$$\frac{dB^2}{dt} \propto -B^5 \quad \Rightarrow \quad \langle B^2 \rangle \propto t^{-2/3}, \quad \ell \propto t^{2/3}. \quad (16.46)$$

Thus, we expect the energy to decay as a power-law decay with time, whose exponent is fixed by the assumptions of conserved helicity and Alfvénicity of flows. As their energy decays, the magnetic structures will become larger and larger, also in a power-law fashion. A Taylor state will be reached only when they finally bump into some physical boundaries.

This may all look rather uncontroversial and not very difficult to grasp, but it is, in fact, the tip of a sizeable iceberg of past and current research. I have swept a number of nuances under the rug here, e.g., whether the decay and merger of magnetic structures really happens on an ideal-MHD time scale (not obviously and not always!), and also have not addressed such questions as what might happen if  $H = 0$  but some local links and twists of the magnetic field still impose local topological constraints on its decay. If this topic interests you, read my review ([Schekochihin 2022, §12](#)) and/or the paper by [Hosking & Schekochihin \(2021\)](#), and follow the paper trail from there.

**Exercise 16.3. Decay of 2D MHD turbulence.** Show that in 2D, the decay laws are

$$\langle B^2 \rangle \propto t^{-1}, \quad \ell \propto t^{1/2}. \quad (16.47)$$

### 16.6. Parker's Problem and Topological MHD

Coming soon... On topology in MHD, a very mathematically minded student might enjoy the book by [Arnold & Khesin \(1999\)](#).

## 17. MHD Stability and Instabilities

We now wish to take a more general view of the MHD stability problem: given some static<sup>117</sup> equilibrium (some  $\rho_0$ ,  $p_0$ ,  $\mathbf{B}_0$  and  $\mathbf{u}_0 = 0$ ), will this equilibrium be stable to small perturbations of it, i.e., will these perturbations grow or decay?

There are two ways to answer this question:

1) Carry out the normal-mode analysis, i.e., linearise the MHD equations around the given equilibrium, just as we did when we studied MHD waves in §15.1, and see if any of the frequencies (solutions of the dispersion relation) turn out to be complex, with positive imaginary parts (growth rates). This approach has the advantage of being direct and also of yielding specific information about rates of growth or decay, the character of the growing and decaying modes, etc. However, for spatially complicated equilibria, this is often quite difficult to do and one might be willing to settle for less: just being able to prove that some configuration is stable or that certain types of perturbations might grow. Hence the second approach:

2) Check whether, for a given equilibrium, all possible perturbations will lead to the energy of the system *increasing*. If so, then the equilibrium is stable—this is called *the energy principle* and we shall prove it shortly. If, on the other hand, certain perturbations lead to the energy *decreasing*, that equilibrium is unstable. The advantage of this second

<sup>117</sup>A treatment of the more general case of a dynamic equilibrium,  $\mathbf{u}_0 \neq 0$ , can be found in the excellent textbook by [Davidson \(2016\)](#).

approach is that we do not need to solve the (linearised) MHD equations in order to pronounce on stability, just to examine the properties of the perturbed energy functional.

It should be already quite clear how to do the normal-mode analysis, at least conceptually, so I shall focus on the second approach.

### 17.1. Energy Principle

Recall what the total energy in MHD is (§13.10)

$$\mathcal{E} = \int d\mathbf{r} \left( \frac{\rho u^2}{2} + \frac{B^2}{8\pi} + \frac{p}{\gamma - 1} \right) \equiv \int d\mathbf{r} \frac{\rho u^2}{2} + W. \quad (17.1)$$

As we saw in §15.1, all perturbations of an MHD system away from equilibrium can be expressed in terms of small displacements  $\xi$ —we will work this out shortly for a general equilibrium, but for now, let us accept that this will be true.<sup>118</sup> As  $\mathbf{u} = \partial\xi/\partial t$  by definition of  $\xi$ , we have

$$\mathcal{E} = \int d\mathbf{r} \frac{1}{2} \rho_0 \left| \frac{\partial \xi}{\partial t} \right|^2 + W_0 + \delta W_1[\xi] + \delta W_2[\xi, \xi] + \dots, \quad (17.2)$$

where we have kept terms up to second order in  $\xi$  and so  $W_0$  is the equilibrium part of  $W$  (i.e., its value for  $\xi = 0$ ),  $\delta W_1[\xi]$  is linear in  $\xi$ ,  $\delta W_2[\xi, \xi]$  is bilinear (quadratic), etc. Energy must be conserved to all orders, so

$$\begin{aligned} \frac{d\mathcal{E}}{dt} &= \int d\mathbf{r} \underbrace{\rho_0 \frac{\partial^2 \xi}{\partial t^2} \cdot \frac{\partial \xi}{\partial t}}_{\equiv \mathbf{F}[\xi]} + \delta W_1 \left[ \frac{\partial \xi}{\partial t} \right] + \delta W_2 \left[ \frac{\partial \xi}{\partial t}, \xi \right] + \delta W_2 \left[ \xi, \frac{\partial \xi}{\partial t} \right] + \dots = 0. \end{aligned} \quad (17.3)$$

This must be true at all times, including at  $t = 0$ , when  $\xi$  and  $\partial\xi/\partial t$  can be chosen independently (MHD equations are second-order in time if written in terms of displacements). Therefore, for arbitrary functions  $\xi$  and  $\eta$ ,

$$\int d\mathbf{r} \eta \cdot \mathbf{F}[\xi] + \delta W_1[\eta] + \delta W_2[\eta, \xi] + \delta W_2[\xi, \eta] + \dots = 0. \quad (17.4)$$

In the first order, this tells us that

$$\delta W_1[\eta] = 0, \quad (17.5)$$

which is good to know because it means that  $\delta W_1$  disappears from (17.2) (there are no first-order energy perturbations). In the second order, we get

$$\int d\mathbf{r} \eta \cdot \mathbf{F}[\xi] = -\delta W_2[\eta, \xi] - \delta W_2[\xi, \eta]. \quad (17.6)$$

Let  $\eta = \xi$ . Then (17.6) implies

$$\boxed{\delta W_2[\xi, \xi] = -\frac{1}{2} \int d\mathbf{r} \xi \cdot \mathbf{F}[\xi].} \quad (17.7)$$

This is the part of the perturbed energy in (17.2) that can be both positive and negative. The *Energy Principle* is

$$\boxed{\delta W_2[\xi, \xi] > 0 \text{ for any } \xi \Leftrightarrow \text{equilibrium is stable}} \quad (17.8)$$

<sup>118</sup>In fact, also the fully nonlinear dynamics can be completely expressed in terms of displacements if the MHD equations are written in Lagrangian coordinates (see §13.12).

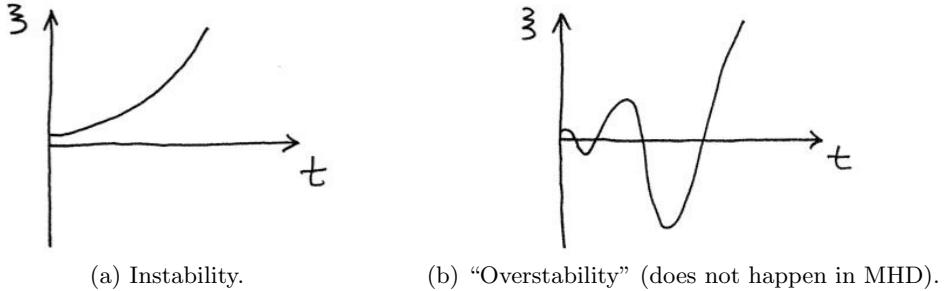


FIGURE 77. MHD instabilities.

(Bernstein *et al.* 1958). Before we are in a position to prove this, we must do some preparatory work.

### 17.1.1. Properties of the Force Operator

Since the right-hand side of (17.6) is symmetric with respect to swapping  $\xi \leftrightarrow \eta$ , so must be the left-hand side:

$$\int d\mathbf{r} \eta \cdot \mathbf{F}[\xi] = \int d\mathbf{r} \xi \cdot \mathbf{F}[\eta]. \quad (17.9)$$

Therefore, operator  $\mathbf{F}[\xi]$  is self-adjoint. Since, by definition,

$$\mathbf{F}[\xi] = \rho_0 \frac{\partial^2 \xi}{\partial t^2}, \quad (17.10)$$

the eigenmodes of this operator satisfy

$$\xi(t, \mathbf{r}) = \xi_n(\mathbf{r}) e^{-i\omega_n t} \Rightarrow \mathbf{F}[\xi_n] = -\rho_0 \omega_n^2 \xi_n. \quad (17.11)$$

As always for self-adjoint operators, we can prove a number of useful statements.

1) *The eigenvalues  $\{\omega_n^2\}$  are real.*

**Proof.** If (17.11) holds, so must

$$\mathbf{F}[\xi_n^*] = -\rho_0 (\omega_n^2)^* \xi_n^*, \quad (17.12)$$

provided  $\mathbf{F}$  has no complex coefficients (we shall confirm this explicitly in §17.2.1). Taking the full scalar products (including integrating over space) of (17.11) with  $\xi_n^*$  and of (17.12) with  $\xi_n$  and subtracting one from the other, we get

$$\begin{aligned} - [\omega_n^2 - (\omega_n^2)^*] \underbrace{\int d\mathbf{r} \rho_0 |\xi_n|^2}_{> 0} &= \int d\mathbf{r} \xi_n^* \cdot \mathbf{F}[\xi_n] - \int d\mathbf{r} \xi_n \cdot \mathbf{F}[\xi_n^*] = 0 \\ \Rightarrow \boxed{\omega_n^2 = (\omega_n^2)^*}, \quad \text{q.e.d.} & \end{aligned} \quad (17.13)$$

This result implies that, if any MHD equilibrium is *unstable*, at least one of the eigenvalues must be  $\omega_n^2 < 0$  and, since it is guaranteed to be real, any MHD instability will give rise to purely growing modes (Fig. 77a), rather than growing oscillations (also known as "overstabilities"; see Fig. 77b).

2) *The eigenmodes  $\{\xi_n\}$  are orthogonal.*

**Proof.** Taking the full scalar products of (17.11) with  $\xi_m$  (assuming  $m \neq n$  and

non-degeneracy of  $\omega_{m,n}^2$ ), and of the analogous equation

$$\mathbf{F}[\boldsymbol{\xi}_m] = -\rho_0 \omega_m^2 \boldsymbol{\xi}_m \quad (17.14)$$

with  $\boldsymbol{\xi}_n$  and subtracting them, we get<sup>119</sup>

$$\begin{aligned} -\underbrace{(\omega_n^2 - \omega_m^2)}_{\neq 0} \int d\mathbf{r} \rho_0 \boldsymbol{\xi}_n \cdot \boldsymbol{\xi}_m &= \int d\mathbf{r} \boldsymbol{\xi}_m \cdot \mathbf{F}[\boldsymbol{\xi}_n] - \int d\mathbf{r} \boldsymbol{\xi}_n \cdot \mathbf{F}[\boldsymbol{\xi}_m] = 0 \\ \Rightarrow \boxed{\int d\mathbf{r} \rho_0 \boldsymbol{\xi}_n \cdot \boldsymbol{\xi}_m = \delta_{nm} \int d\mathbf{r} \rho_0 |\boldsymbol{\xi}_n|^2}, \quad \text{q.e.d.} \end{aligned} \quad (17.15)$$

### 17.1.2. Proof of the Energy Principle

Let us assume completeness of the set of eigenmodes  $\{\boldsymbol{\xi}_n\}$  (not, in fact, an indispensable assumption, but we shall not worry about this nuance here; see [Kulsrud 2005](#), §7.2). Then any displacement at any given time  $t$  can be decomposed as

$$\boldsymbol{\xi}(t, \mathbf{r}) = \sum_n a_n(t) \boldsymbol{\xi}_n(\mathbf{r}). \quad (17.16)$$

The energy perturbation (17.7) is

$$\begin{aligned} \delta W_2[\boldsymbol{\xi}, \boldsymbol{\xi}] &= -\frac{1}{2} \int d\mathbf{r} \boldsymbol{\xi} \cdot \mathbf{F}[\boldsymbol{\xi}] = -\frac{1}{2} \sum_{nm} a_n a_m \int d\mathbf{r} \boldsymbol{\xi}_n \cdot \mathbf{F}[\boldsymbol{\xi}_m] \\ &= \frac{1}{2} \sum_{nm} a_n a_m \omega_m^2 \underbrace{\int d\mathbf{r} \rho_0 \boldsymbol{\xi}_n \cdot \boldsymbol{\xi}_m}_{\text{use (17.15)}} = \frac{1}{2} \sum_n a_n^2 \omega_n^2 \int d\mathbf{r} \rho_0 |\boldsymbol{\xi}_n|^2. \end{aligned} \quad (17.17)$$

By the same token,

$$K[\boldsymbol{\xi}, \boldsymbol{\xi}] \equiv \frac{1}{2} \int d\mathbf{r} \rho_0 |\boldsymbol{\xi}|^2 = \frac{1}{2} \sum_n a_n^2 \int d\mathbf{r} \rho_0 |\boldsymbol{\xi}_n|^2. \quad (17.18)$$

Then, if we arrange  $\omega_1^2 \leq \omega_2^2 \leq \dots$ , the smallest eigenvalue is

$$\omega_1^2 = \min_{\boldsymbol{\xi}} \frac{\delta W_2[\boldsymbol{\xi}, \boldsymbol{\xi}]}{K[\boldsymbol{\xi}, \boldsymbol{\xi}]} \quad (17.19)$$

Therefore,

- condition (17.8) is *sufficient* for stability because, if  $\delta W_2[\boldsymbol{\xi}, \boldsymbol{\xi}] > 0$  for all possible  $\boldsymbol{\xi}$ , then the smallest eigenvalue  $\omega_1^2 > 0$ , and so all eigenvalues are positive,  $\omega_n^2 \geq \omega_1^2 > 0$ ;
- condition (17.8) is *necessary* for stability because, if the equilibrium is stable, then all eigenvalues are positive,  $\omega_n^2 > 0$ , whence  $\delta W_2[\boldsymbol{\xi}, \boldsymbol{\xi}] > 0$  in view of (17.17), q.e.d.

### 17.2. Explicit Calculation of $\delta W_2$

Now that we know that we need the sign of  $\delta W_2$  to ascertain stability (or otherwise), it is worth working out  $\delta W_2$  as an explicit function of  $\boldsymbol{\xi}$ . It is a second-order quantity, but (17.7) tells us that all we need to calculate is  $\mathbf{F}[\boldsymbol{\xi}]$  to first order in  $\boldsymbol{\xi}$ , i.e., we just need to linearise the MHD equations around an arbitrary static equilibrium. The procedure is the same as in §15.1, but without assuming  $\rho_0$ ,  $p_0$  and  $\mathbf{B}_0$  to be spatially homogeneous.

<sup>119</sup>Note that in view of (17.13), we can take  $\{\boldsymbol{\xi}_n\}$  to be real.

### 17.2.1. Linearised MHD Equations

Thus, generalising somewhat the procedure adopted in (15.3–15.5), we have

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 &\Rightarrow \frac{\partial \delta \rho}{\partial t} = -\nabla \cdot \left( \rho_0 \frac{\partial \boldsymbol{\xi}}{\partial t} \right) \\ &\Rightarrow \boxed{\delta \rho = -\nabla \cdot (\rho_0 \boldsymbol{\xi})}, \end{aligned} \quad (17.20)$$

$$\begin{aligned} \left( \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) p = -\gamma p \nabla \cdot \mathbf{u} &\Rightarrow \frac{\partial \delta p}{\partial t} = -\frac{\partial \boldsymbol{\xi}}{\partial t} \cdot \nabla p_0 - \gamma p_0 \nabla \cdot \frac{\partial \boldsymbol{\xi}}{\partial t} \\ &\Rightarrow \boxed{\delta p = -\boldsymbol{\xi} \cdot \nabla p_0 - \gamma p_0 \nabla \cdot \boldsymbol{\xi}}, \end{aligned} \quad (17.21)$$

$$\begin{aligned} \frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) &\Rightarrow \frac{\partial \delta \mathbf{B}}{\partial t} = \nabla \times \left( \frac{\partial \boldsymbol{\xi}}{\partial t} \times \mathbf{B}_0 \right) \\ &\Rightarrow \boxed{\delta \mathbf{B} = \nabla \times (\boldsymbol{\xi} \times \mathbf{B}_0)}. \end{aligned} \quad (17.22)$$

Note that again  $\delta \rho$ ,  $\delta p$  and  $\delta \mathbf{B}$  are all expressed as linear operators on  $\boldsymbol{\xi}$ —and so  $\delta W = \delta \int d\mathbf{r} [B^2/8\pi + p/(\gamma - 1)]$  must also be some operator involving  $\boldsymbol{\xi}$  and its gradients but not  $\partial \boldsymbol{\xi} / \partial t$  (as we assumed in §17.1).

Finally, we deal with the momentum equation (to which we add gravity as this will give some interesting instabilities):

$$\rho \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \frac{(\nabla \times \mathbf{B}) \times \mathbf{B}}{4\pi} + \rho \mathbf{g}. \quad (17.23)$$

This gives us

$$\begin{aligned} \mathbf{F}[\boldsymbol{\xi}] = \rho_0 \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} &= -\nabla \delta p + \frac{(\nabla \times \mathbf{B}_0) \times \delta \mathbf{B}}{4\pi} + \frac{(\nabla \times \delta \mathbf{B}) \times \mathbf{B}_0}{4\pi} + \delta \rho \mathbf{g} \\ &= \nabla (\boldsymbol{\xi} \cdot \nabla p_0 + \gamma p_0 \nabla \cdot \boldsymbol{\xi}) - \mathbf{g} \nabla \cdot (\rho_0 \boldsymbol{\xi}) + \frac{\mathbf{j}_0 \times \delta \mathbf{B}}{c} + \frac{(\nabla \times \delta \mathbf{B}) \times \mathbf{B}_0}{4\pi}, \end{aligned} \quad (17.24)$$

where  $\mathbf{j}_0 = c(\nabla \times \mathbf{B}_0)/4\pi$ , we have used (17.20) and (17.21) for  $\delta \rho$  and  $\delta p$ , respectively, and  $\delta \mathbf{B}$  is given by (17.22).

### 17.2.2. Energy Perturbation

Now we can use (17.24) in (17.7) to calculate explicitly

$$\begin{aligned}
 \delta W_2 &= \frac{1}{2} \int d\mathbf{r} \left[ \underbrace{-\boldsymbol{\xi} \cdot \nabla (\boldsymbol{\xi} \cdot \nabla p_0 + \gamma p_0 \nabla \cdot \boldsymbol{\xi})}_{= (\boldsymbol{\xi} \cdot \nabla p_0) \nabla \cdot \boldsymbol{\xi} + \gamma p_0 (\nabla \cdot \boldsymbol{\xi})^2} + (\mathbf{g} \cdot \boldsymbol{\xi}) \nabla \cdot (\rho_0 \boldsymbol{\xi}) \right. \\
 &\quad \text{after integration by parts} \\
 &= \underbrace{\frac{(\mathbf{j}_0 \times \delta \mathbf{B}) \cdot \boldsymbol{\xi}}{c}}_{= \frac{\mathbf{j}_0 \cdot (\boldsymbol{\xi} \times \delta \mathbf{B})}{c}} \left. - \underbrace{\frac{(\nabla \times \delta \mathbf{B}) \times \mathbf{B}_0}{4\pi} \cdot \boldsymbol{\xi}}_{\text{by parts}} \right] \\
 &= \frac{(\nabla \times \delta \mathbf{B}) \cdot (\boldsymbol{\xi} \times \mathbf{B}_0)}{4\pi} \\
 &= \frac{(\delta \mathbf{B} \times \nabla) \cdot (\boldsymbol{\xi} \times \mathbf{B}_0)}{4\pi} \\
 &= \frac{\delta \mathbf{B} \cdot [\nabla \times (\boldsymbol{\xi} \times \mathbf{B}_0)]}{4\pi} \\
 &= \frac{|\delta \mathbf{B}|^2}{4\pi}, \text{ using (17.22)}
 \end{aligned} \tag{17.25}$$

Thus, we have arrived at a standard textbook (e.g., Kulsrud 2005) expression for the energy perturbation (this expression is non-unique because one can do various integrations by parts):

$$\begin{aligned}
 \delta W_2 &= \frac{1}{2} \int d\mathbf{r} \left[ (\boldsymbol{\xi} \cdot \nabla p_0) \nabla \cdot \boldsymbol{\xi} + \gamma p_0 (\nabla \cdot \boldsymbol{\xi})^2 + (\mathbf{g} \cdot \boldsymbol{\xi}) \nabla \cdot (\rho_0 \boldsymbol{\xi}) \right. \\
 &\quad \left. + \frac{\mathbf{j}_0 \cdot (\boldsymbol{\xi} \times \delta \mathbf{B})}{c} + \frac{|\delta \mathbf{B}|^2}{4\pi} \right], \tag{17.26}
 \end{aligned}$$

where  $\delta \mathbf{B} = \nabla \times (\boldsymbol{\xi} \times \mathbf{B}_0)$ . Note that two of the terms inside the integral (the second and the fifth) are positive-definite and so always stabilising. The terms that are not sign-definite and so potentially destabilising involve equilibrium gradients of pressure, density and magnetic field (currents). It is perhaps not a surprise to learn that Nature, with its fundamental yearning for thermal equilibrium, might dislike gradients—while it is of course not a rule that all such inhomogeneities render the system unstable, we will see that they often do, usually when gradients exceed certain critical thresholds.

All we need to do now is calculate  $\delta W_2$  according to (17.26) for any equilibrium that interests us and see if it can be negative for any class of perturbations (or show that it is positive for all perturbations).

### 17.3. Interchange Instabilities

As the first and simplest example of how one does stability calculations using the Energy Principle, we will (perhaps disappointingly) consider a purely hydrodynamic situation: the stability of a simple hydrostatic equilibrium describing a generic *stratified atmosphere*:

$$\rho_0 = \rho_0(z) \quad \text{and} \quad p_0 = p_0(z) \quad \text{satisfying} \quad \boxed{\frac{dp_0}{dz} = -\rho_0 g} \tag{17.27}$$

(gravity acts downward, against the  $z$  direction,  $\mathbf{g} = -g\hat{\mathbf{z}}$ ).

### 17.3.1. Formal Derivation of the Schwarzschild Criterion

With  $\mathbf{B}_0 = 0$  and the hydrostatic equilibrium (17.27), (17.26) becomes

$$\begin{aligned}\delta W_2 &= \frac{1}{2} \int d\mathbf{r} [\xi_z p'_0 \nabla \cdot \boldsymbol{\xi} + \gamma p_0 (\nabla \cdot \boldsymbol{\xi})^2 - g \xi_z (\rho'_0 \xi_z + \rho_0 \nabla \cdot \boldsymbol{\xi})] \\ &= \frac{1}{2} \int d\mathbf{r} [2p'_0 \xi_z \nabla \cdot \boldsymbol{\xi} + \gamma p_0 (\nabla \cdot \boldsymbol{\xi})^2 - \rho'_0 g \xi_z^2],\end{aligned}\quad (17.28)$$

where we have used  $\rho_0 g = -p'_0$ . We see that  $\delta W_2$  depends on  $\xi_z$  and  $\nabla \cdot \boldsymbol{\xi}$ . Let us treat them as independent variables and minimise  $\delta W_2$  with respect to them (i.e., *seek the most unstable possible situation*):

$$\frac{\partial}{\partial(\nabla \cdot \boldsymbol{\xi})} \left[ \begin{array}{l} \text{integrand} \\ \text{of (17.28)} \end{array} \right] = 2p'_0 \xi_z + 2\gamma p_0 (\nabla \cdot \boldsymbol{\xi}) = 0 \Rightarrow \nabla \cdot \boldsymbol{\xi} = -\frac{p'_0}{\gamma p_0} \xi_z. \quad (17.29)$$

Substituting this back into (17.28), we get

$$\begin{aligned}\delta W_2 &= \frac{1}{2} \int d\mathbf{r} \left( -\frac{p'^2_0}{\gamma p_0} - \rho'_0 g \right) \xi_z^2 = \frac{1}{2} \int d\mathbf{r} \underbrace{\frac{\rho_0 g}{\gamma} \left( \frac{p'_0}{p_0} - \gamma \frac{\rho'_0}{\rho_0} \right)}_{= \frac{d}{dz} \ln \frac{p_0}{\rho'_0}} \xi_z^2.\end{aligned}\quad (17.30)$$

By the Energy Principle, the system is stable iff

$$\delta W_2 > 0 \Leftrightarrow \boxed{\frac{d \ln s_0}{dz} > 0}, \quad (17.31)$$

where  $s_0 = p_0 / \rho_0^\gamma$  is the entropy function. The inequality (17.31) is the *Schwarzschild criterion* for convective stability.<sup>120</sup> If this criterion is broken, there will be an instability, called the *interchange instability*.

This calculation illustrates both the power and the weakness of the method:  
—on the one hand, we have obtained a stability criterion quite quickly and without having to solve the underlying equations,  
—on the other hand, while we have established the condition for instability, we have as yet absolutely no idea what is going on physically.

### 17.3.2. Physical Picture

We can remedy the latter problem by examining what type of displacements give rise to  $\delta W_2 < 0$  when the Schwarzschild criterion is broken. Recalling (17.20) and (17.21) and specialising to the displacements given by (17.29) (as they are the ones that minimise  $\delta W_2$ ), we get

$$\frac{\delta p}{p_0} = -\frac{\boldsymbol{\xi} \cdot \nabla p_0}{p_0} - \gamma \nabla \cdot \boldsymbol{\xi} = -\frac{p'_0}{p_0} \xi_z - \gamma \nabla \cdot \boldsymbol{\xi} = 0, \quad (17.32)$$

$$\frac{\delta \rho}{\rho_0} = -\frac{1}{\rho_0} \nabla \cdot (\rho_0 \boldsymbol{\xi}) = -\frac{\rho'_0}{\rho_0} \xi_z - \nabla \cdot \boldsymbol{\xi} = \frac{1}{\gamma} \left( -\gamma \frac{\rho'_0}{\rho_0} + \frac{p'_0}{p_0} \right) = \frac{1}{\gamma} \underbrace{\frac{d \ln s_0}{dz}}_{< 0} \xi_z. \quad (17.33)$$

(unstable)

<sup>120</sup>We studied perturbations of a stably stratified atmosphere in §15.2.8 and Q6, where we saw that these perturbations indeed did not grow provided the entropy scale length  $1/H_s = d \ln s_0 / dz$  was positive.

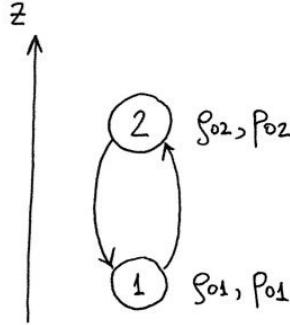


FIGURE 78. Interchange instability.

Thus, the offending perturbations maintain themselves in pressure balance (i.e., they are *not* sound waves) and locally increase or decrease density for blobs of fluid that fall ( $\xi_z < 0$ ) or rise ( $\xi_z > 0$ ), respectively.

This gives us some handle on the situation: if we imagine a blob of fluid slowly rising (slowly, so  $\delta p = 0$ ) from the denser nether regions of the atmosphere to the less dense upper ones, then we can ask whether staying in pressure balance with its surroundings will require the blob to expand ( $\delta\rho < 0$ ) or contract ( $\delta\rho > 0$ ). If it is the latter, it will fall back down, pulled by gravity; if the former, then it will keep rising (buoyantly) and the system will be unstable. The direction of the entropy gradient determines which of these two scenarios is realised.

### 17.3.3. Intuitive Rederivation of the Schwarzschild Criterion

We can use this physical intuition to derive the Schwarzschild criterion directly. Consider two blobs, at two different vertical locations, lower (1) and upper (2), where the equilibrium densities and pressures are  $\rho_{01}$ ,  $p_{01}$  and  $\rho_{02}$ ,  $p_{02}$ . Now interchange these two blobs (Fig. 78). Inside the blobs, the new densities and pressures are  $\rho_1$ ,  $p_1$  and  $\rho_2$ ,  $p_2$ .

Requiring the blobs to stay in pressure balance with their local surroundings gives

$$p_1 = p_{02}, \quad p_2 = p_{01}. \quad (17.34)$$

Requiring the blobs to rise or fall adiabatically, i.e., to satisfy  $p/\rho^\gamma = \text{const}$ , and then using pressure balance (17.34) gives

$$\frac{p_{01}}{\rho_{01}^\gamma} = \frac{p_1}{\rho_1^\gamma} = \frac{p_{02}}{\rho_2^\gamma} \quad \Rightarrow \quad \frac{\rho_1}{\rho_{01}} = \left( \frac{p_{02}}{p_{01}} \right)^{1/\gamma}. \quad (17.35)$$

Requiring that the buoyancy of the rising blob overcome gravity, i.e., that the weight of the displaced fluid be larger than the weight of the blob,

$$\rho_{02}g > \rho_1g, \quad (17.36)$$

gives the condition for instability:

$$\rho_1 < \rho_{02} \quad \Leftrightarrow \quad \frac{\rho_1}{\rho_{02}} = \frac{\rho_{01}}{\rho_{02}} \left( \frac{p_{02}}{p_{01}} \right)^{1/\gamma} < 1 \quad \Leftrightarrow \quad \boxed{\frac{p_{02}}{\rho_{02}^\gamma} < \frac{p_{01}}{\rho_{01}^\gamma}}. \quad (17.37)$$

This is exactly the same as the Schwarzschild condition (17.31) for the interchange instability (and this is why the instability is called that).

Note that, while this is of course a much simpler and more intuitive argument than the application of the Energy Principle, it only gives us a particular example of the

kind of perturbation that would be *unstable* under particular conditions, not any general criterion of what equilibria might be guaranteed to be stable.

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In Q10, we will explore how the above considerations can be generalised to an equilibrium that also features a non-zero magnetic field.

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### 17.4. Instabilities of a Pinch

As our second (also classic) example, we consider the stability of a  $z$ -pinch equilibrium (§16.1.1, Fig. 72):

$$\mathbf{B}_0 = B_0(r)\hat{\theta}, \quad j_0 = j_0(r)\hat{z} = \frac{c}{4\pi r}(rB_0)'\hat{z}, \quad p'_0(r) = -\frac{1}{c}j_0B_0 = -\frac{B_0(rB_0)'}{4\pi r}. \quad (17.38)$$

Since we are going to have to work in cylindrical coordinates, we must first write all the terms in (17.26) in these coordinates and with the equilibrium (17.38):

$$\begin{aligned} (\boldsymbol{\xi} \cdot \nabla p_0)(\nabla \cdot \boldsymbol{\xi}) &= \xi_r p'_0 \left( \frac{1}{r} \frac{\partial}{\partial r} r\xi_r + \cancel{\frac{1}{r} \frac{\partial \xi_\theta}{\partial \theta}} + \frac{\partial \xi_z}{\partial z} \right) \\ &= p'_0 \frac{\xi_r^2}{r} + p'_0 \xi_r \left( \frac{\partial \xi_r}{\partial r} + \frac{\partial \xi_z}{\partial z} \right), \end{aligned} \quad (17.39)$$

$$\gamma p_0 (\nabla \cdot \boldsymbol{\xi})^2 = \gamma p_0 \left( \frac{1}{r} \frac{\partial}{\partial r} r\xi_r + \frac{1}{r} \frac{\partial \xi_\theta}{\partial \theta} + \frac{\partial \xi_z}{\partial z} \right)^2, \quad (17.40)$$

$$\delta \mathbf{B} = \nabla \times (\boldsymbol{\xi} \times \mathbf{B}_0) = \hat{r} \left( \frac{1}{r} \frac{\partial}{\partial \theta} \xi_r B_0 \right) + \hat{\theta} \left( -\frac{\partial}{\partial z} \xi_z B_0 - \frac{\partial}{\partial r} \xi_r B_0 \right) + \hat{z} \left( \frac{1}{r} \frac{\partial}{\partial \theta} \xi_z B_0 \right), \quad (17.41)$$

$$\begin{aligned} \frac{j_0 \cdot (\boldsymbol{\xi} \times \delta \mathbf{B})}{c} &= \underbrace{\frac{j_0}{c}}_{= -\frac{p'_0}{B_0}} (\xi_r \delta B_\theta - \xi_\theta \delta B_r) = p'_0 \left[ \xi_r \left( \frac{\partial \xi_z}{\partial z} + \frac{\partial \xi_r}{\partial r} + \xi_r \frac{B'_0}{B_0} \right) + \cancel{\xi_\theta \frac{1}{r} \frac{\partial \xi_r}{\partial \theta}} \right], \\ &= p'_0 \xi_r \left( \frac{\partial \xi_z}{\partial z} + \frac{\partial \xi_r}{\partial r} \right) + \frac{p'_0 B'_0}{B_0} \xi_r^2, \end{aligned} \quad (17.42)$$

$$\begin{aligned} \frac{|\delta \mathbf{B}|^2}{4\pi} &= \frac{B_0^2}{4\pi r^2} \left[ \left( \frac{\partial \xi_r}{\partial \theta} \right)^2 + \left( \frac{\partial \xi_z}{\partial \theta} \right)^2 \right] + \underbrace{\frac{B_0^2}{4\pi} \left( \frac{\partial \xi_z}{\partial z} + \frac{\partial \xi_r}{\partial r} + \xi_r \frac{B'_0}{B_0} \right)^2}_{= \frac{B_0^2}{4\pi} \left( \frac{\partial \xi_z}{\partial z} + \frac{\partial \xi_r}{\partial r} \right)^2 +} \\ &\quad \frac{2B_0 B'_0}{4\pi} \xi_r \left( \frac{\partial \xi_z}{\partial z} + \frac{\partial \xi_r}{\partial r} \right) + \frac{B_0'^2}{4\pi} \xi_r^2. \end{aligned} \quad (17.43)$$

The terms that are crossed out have been dropped because they combine into a full derivative with respect to  $\theta$  and so, upon substitution into (17.26), vanish under integra-

tion. Assembling all this together, we have

$$\begin{aligned}
\delta W_2 &= \frac{1}{2} \int d\mathbf{r} \left\{ \underbrace{\left( p'_0 + \frac{p'_0 r B'_0}{B_0} + \frac{r B'^2_0}{4\pi} \right) \frac{\xi_r^2}{r}}_{= 2p'_0 + \frac{B_0^2}{4\pi r}} + 2 \underbrace{\left( p'_0 + \frac{B_0 B'_0}{4\pi} \right) \xi_r}_{= -\frac{B_0^2}{4\pi r}} \left( \frac{\partial \xi_z}{\partial z} + \frac{\partial \xi_r}{\partial r} \right) \right. \\
&\quad \left. + \gamma p_0 (\nabla \cdot \boldsymbol{\xi})^2 + \frac{B_0^2}{4\pi r^2} \left[ \left( \frac{\partial \xi_r}{\partial \theta} \right)^2 + \left( \frac{\partial \xi_z}{\partial \theta} \right)^2 \right] + \frac{B_0^2}{4\pi} \left( \frac{\partial \xi_z}{\partial z} + \frac{\partial \xi_r}{\partial r} \right)^2 \right\} \\
&= \frac{1}{2} \int d\mathbf{r} \left\{ 2p'_0 \frac{\xi_r^2}{r} + \frac{B_0^2}{4\pi} \left( \frac{\partial \xi_z}{\partial z} + \frac{\partial \xi_r}{\partial r} - \frac{\xi_r}{r} \right)^2 \right. \\
&\quad \left. + \gamma p_0 (\nabla \cdot \boldsymbol{\xi})^2 + \frac{B_0^2}{4\pi r^2} \left[ \left( \frac{\partial \xi_r}{\partial \theta} \right)^2 + \left( \frac{\partial \xi_z}{\partial \theta} \right)^2 \right] \right\}, \tag{17.44}
\end{aligned}$$

where, in simplifying the first two terms in the integrand, we used the equilibrium equation (17.38):

$$p'_0 = -\frac{B_0^2}{4\pi r} - \frac{B_0 B'_0}{4\pi} \Rightarrow \frac{r B'^2_0}{4\pi} = -\frac{p'_0 r B'_0}{B_0} - \frac{B_0 B'_0}{4\pi} = -\frac{p'_0 r B'_0}{B_0} + p'_0 + \frac{B_0^2}{4\pi r}. \tag{17.45}$$

Finally, after a little further tiding up,

$$\begin{aligned}
\delta W_2 &= \frac{1}{2} \int d\mathbf{r} \left\{ 2p'_0 \frac{\xi_r^2}{r} + \frac{B_0^2}{4\pi} \left( r \frac{\partial}{\partial r} \frac{\xi_r}{r} + \frac{\partial \xi_z}{\partial z} \right)^2 + \gamma p_0 \left( \frac{1}{r} \frac{\partial}{\partial r} r \xi_r + \frac{1}{r} \frac{\partial \xi_\theta}{\partial \theta} + \frac{\partial \xi_z}{\partial z} \right)^2 \right. \\
&\quad \left. + \frac{B_0^2}{4\pi r^2} \left[ \left( \frac{\partial \xi_r}{\partial \theta} \right)^2 + \left( \frac{\partial \xi_z}{\partial \theta} \right)^2 \right] \right\}. \tag{17.46}
\end{aligned}$$

#### 17.4.1. Sausage Instability

Let us first consider axisymmetric perturbations:  $\partial/\partial\theta = 0$ . Then  $\delta W_2$  depends on two variables only:

$$\xi_r \quad \text{and} \quad \eta \equiv \frac{\partial \xi_r}{\partial r} + \frac{\partial \xi_z}{\partial z}. \tag{17.47}$$

Indeed, unpacking all the  $r$  derivatives in (17.46), we get

$$\boxed{\delta W_2 = \frac{1}{2} \int d\mathbf{r} \left[ 2p'_0 \frac{\xi_r^2}{r} + \frac{B_0^2}{4\pi} \left( \eta - \frac{\xi_r}{r} \right)^2 + \gamma p_0 \left( \eta + \frac{\xi_r}{r} \right)^2 \right]}. \tag{17.48}$$

We shall treat  $\xi_r$  and  $\eta$  as independent variables and minimise  $\delta W_2$  with respect to  $\eta$ :

$$\frac{\partial}{\partial \eta} \left[ \begin{array}{l} \text{integrand} \\ \text{of (17.48)} \end{array} \right] = 2 \frac{B_0^2}{4\pi} \left( \eta - \frac{\xi_r}{r} \right) + 2\gamma p_0 \left( \eta + \frac{\xi_r}{r} \right) = 0 \Rightarrow \eta = \frac{1 - \gamma\beta/2}{1 + \gamma\beta/2} \frac{\xi_r}{r}, \tag{17.49}$$

where, as usual,  $\beta = 8\pi p_0/B_0^2$ . Putting this back into (17.48), we get

$$\begin{aligned}
\delta W_2 &= \int d\mathbf{r} p_0 \left[ \frac{rp'_0}{p_0} + \frac{1}{\beta} \left( \frac{\gamma\beta}{1 + \gamma\beta/2} \right)^2 + \frac{\gamma}{2} \left( \frac{2}{1 + \gamma\beta/2} \right)^2 \right] \frac{\xi_r^2}{r^2} \\
&= \int d\mathbf{r} p_0 \left( r \frac{d \ln p_0}{dr} + \frac{2\gamma}{1 + \gamma\beta/2} \right) \frac{\xi_r^2}{r^2}. \tag{17.50}
\end{aligned}$$

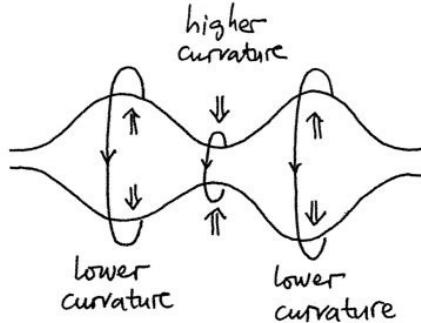


FIGURE 79. Sausage instability.

There will be an instability ( $\delta W_2 < 0$ ) if (but not only if, because we are considering the restricted set of axisymmetric displacements)

$$\boxed{-r \frac{d \ln p_0}{dr} > \frac{2\gamma}{1 + \gamma\beta/2}}, \quad (17.51)$$

i.e., when the pressure gradient is too steep, the equilibrium is unstable.

What sort of instability is this? Recall that the perturbations that we have identified as making  $\delta W_2 < 0$  are axisymmetric, have some radial and axial displacements and are *compressible*: from (17.49),

$$\nabla \cdot \xi = \eta + \frac{\xi_r}{r} = \frac{2}{1 + \gamma\beta/2} \frac{\xi_r}{r}. \quad (17.52)$$

They are illustrated in Fig. 79. The mechanism of this aptly named *sausage instability* is clear: squeezing the flux surfaces inwards increases the curvature of the azimuthal field lines, this exerts stronger curvature force, leading to further squeezing; conversely, expanding outwards weakens curvature and the plasma can expand further.

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**Exercise 17.1.** Convince yourself that the displacements that have been identified cause magnetic perturbations that are consistent with the cartoon in Fig. 79.

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#### 17.4.2. Kink Instability

Now consider non-axisymmetric perturbations ( $\partial/\partial\theta \neq 0$ ) to see what other instabilities might be there. First of all, since we now have  $\theta$  variation,  $\delta W_2$  depends on  $\xi_\theta$ . However, in (17.46),  $\xi_\theta$  only appears in the third term, where it is part of  $\nabla \cdot \xi$ , which enters quadratically and with a positive coefficient  $\gamma p_0$ . We can treat  $\nabla \cdot \xi$  as an independent variable, alongside  $\xi_r$  and  $\xi_z$ , and minimise  $\delta W_2$  with respect to it. Obviously, the energy perturbation is minimal when

$$\nabla \cdot \xi = 0, \quad (17.53)$$

i.e., the most dangerous non-axisymmetric perturbations are incompressible (unlike for the case of the axisymmetric sausage mode in §17.4.1: there we could not—and did not—have such incompressible perturbations because we did not have  $\xi_\theta$  at our disposal, to be chosen in such a way as to enforce incompressibility).

To carry out further minimisation of  $\delta W_2$ , it is convenient to Fourier transform our displacements in the  $\theta$  and  $z$  directions—both are directions of symmetry (i.e., the

equilibrium profiles do not vary in these directions), so this can be done with impunity:

$$\boldsymbol{\xi} = \sum_{m,k} \xi_{mk}(r) e^{i(m\theta + kz)}. \quad (17.54)$$

Then (17.46) (with  $\nabla \cdot \boldsymbol{\xi} = 0$ ) becomes, by Parseval's theorem (the operator  $\mathbf{F}[\boldsymbol{\xi}]$  being self-adjoint; see §17.1.1),

$$\delta W_2 = \frac{1}{2} \sum_{m,k} 2\pi L_z \int_0^\infty dr r \left\{ 2p'_0 \frac{|\xi_r|^2}{r} + \frac{B_0^2}{4\pi} \left[ \left| r \frac{\partial}{\partial r} \frac{\xi_r}{r} + ik\xi_z \right|^2 + \frac{m^2}{r^2} (|\xi_r|^2 + |\xi_z|^2) \right] \right\}. \quad (17.55)$$

As  $\xi_z$  and  $\xi_z^*$  only appear algebraically in (17.55) (no  $r$  derivatives), it is easy to minimise  $\delta W_2$  with respect to them: setting the derivative of the integrand with respect to either  $\xi_z$  or  $\xi_z^*$  to zero, we get

$$-ik \left( r \frac{\partial}{\partial r} \frac{\xi_r}{r} + ik\xi_z \right) + \frac{m^2}{r^2} \xi_z = 0 \quad \Rightarrow \quad \xi_z = \frac{ikr^3}{m^2 + k^2 r^2} \frac{\partial}{\partial r} \frac{\xi_r}{r}. \quad (17.56)$$

Putting this back into (17.55) and assembling terms, we get

$$\begin{aligned} \delta W_2 &= \sum_{m,k} \pi L_z \int_0^\infty dr r \left\{ 2p'_0 \left( \frac{rp'_0}{p_0} + \frac{m^2}{\beta} \right) \frac{|\xi_r|^2}{r^2} \right. \\ &\quad \left. + \frac{B_0^2}{4\pi} \underbrace{\left[ \left( 1 - \frac{k^2 r^2}{m^2 + k^2 r^2} \right)^2 + \frac{m^2 k^2 r^2}{(m^2 + k^2 r^2)^2} \right]}_{= \frac{m^2}{m^2 + k^2 r^2}} \left| r \frac{\partial}{\partial r} \frac{\xi_r}{r} \right|^2 \right\}. \end{aligned} \quad (17.57)$$

The second term here is always stabilising. The most unstable modes will be ones with  $k \rightarrow \infty$ , for which the stabilising term is as small as possible. The remaining term will allow  $\delta W_2 < 0$  and, therefore, an instability, if

$$-r \frac{d \ln p_0}{dr} > \frac{m^2}{\beta}. \quad (17.58)$$

Again, the equilibrium is unstable if the pressure gradient is too steep. The most unstable modes are ones with the smallest  $m$ , viz.,  $m = 1$ .

Note that another way of writing the instability condition (17.58) is

$$-rp'_0 = \frac{B_0^2}{4\pi} + \frac{rB_0 B'_0}{4\pi} > m^2 \frac{B_0^2}{8\pi} \quad \Rightarrow \quad r \frac{d \ln B_0}{dr} > \frac{m^2}{2} - 1, \quad (17.59)$$

where we have used the equilibrium equation (17.38).

What does this instability look like? The unstable perturbations are incompressible:

$$\nabla \cdot \boldsymbol{\xi} = 0 \quad \Rightarrow \quad \frac{1}{r} \frac{\partial}{\partial r} r\xi_r + \frac{im}{r} \xi_\theta + ik\xi_z = 0. \quad (17.60)$$

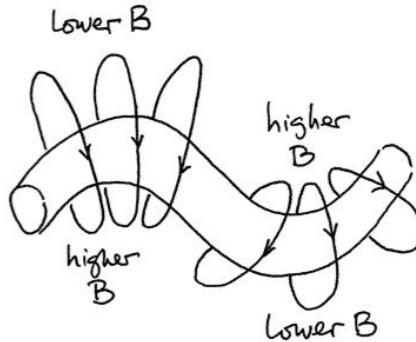


FIGURE 80. Kink instability.

Setting  $m = 1$  and using (17.56), we find

$$i\xi_\theta = -\frac{\partial}{\partial r} r\xi_r + \underbrace{\frac{k^2 r^4}{m^2 + k^2 r^2}}_{\approx r^2} \frac{\partial}{\partial r} \frac{\xi_r}{r} \approx -2\xi_r \quad \text{and} \quad \xi_z \ll \xi_r. \quad (17.61)$$

as  $k \rightarrow \infty$

The basic cartoon (Fig. 80) is as follows: the flux surfaces are bent, with a twist (to remain uncompressed). The bending pushes the magnetic loops closer together and thus increases magnetic pressure in concave parts and, conversely, decreases it in the convex ones. Plasma is pushed from the areas of higher  $B$  to those with lower  $B$ , thermal pressure in the latter (convex) areas becomes uncompensated, the field lines open up further, etc. This is called the *kink instability*.

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Similar methodology can be used to show that, unlike the  $z$  pinch, the  $\theta$  pinch (§16.1.1, Fig. 73) is always stable: see Q11.

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## 18. Magnetic Reconnection

Coming soon...

### 18.1. Tearing Mode

See [http://www-thphys.physics.ox.ac.uk/people/AlexanderSchekochihin/notes/PartIIIMHD/LecturesL05/sec16\\_TearingMode.pdf](http://www-thphys.physics.ox.ac.uk/people/AlexanderSchekochihin/notes/PartIIIMHD/LecturesL05/sec16_TearingMode.pdf), Schekochihin (2022, Appendix D.1), Boldyrev & Loureiro (2018), Parra (2019a), or Kunz (2020).

### 18.2. Sweet–Parker Reconnection

See Schekochihin (2022, Appendix D.4.1).

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**Exercise 18.1. Plasmoid instability.** See Schekochihin (2022, Appendix D.4.2) and Loureiro *et al.* (2007).

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### 18.3. Fast MHD Reconnection

See Schekochihin (2022, Appendix D.5) and Uzdensky *et al.* (2010).

## 19. Further Reading

What follows is *not* a literature survey, but rather just a few pointers for the keen and the curious.

### 19.1. MHD Instabilities

There are very many of these, easily a whole course's worth. They are an interesting topic. A founding text is the old, classic, super-meticulous monograph by [Chandrasekhar \(2003\)](#). In the context of **toroidal (fusion) plasmas**, you want to learn the so-called **ballooning theory**, a tour de force of theoretical plasma physics, which, like the relaxation theory, is associated with J. B. Taylor's name (so his lectures, [Taylor & Newton 2015](#), are a good starting point; the original paper on the subject is [Connor et al. 1979](#)). In the unlikely event that you have an appetite for more **energy-principle calculations** in the style of §17.4, the book by [Freidberg \(2014\)](#) will teach you more than you ever wanted to know. In **astrophysics**, MHD instabilities have been a hot topic since the early 1990s, not least due the realisation by [Balbus & Hawley \(1991\)](#) that the **magnetorotational instability (MRI)** is responsible for triggering turbulence and, therefore, maintaining momentum transport in accretion flows—so the lecture notes by [Balbus \(2015\)](#) are an excellent place to start learning about this subject (this is also an opportunity to learn how to handle equilibria that are *not static*, e.g., most interestingly, featuring **rotating and shear flows**).<sup>121</sup>

As with everything in physics, the frontier in this subject is nonlinear phenomena. One very attractive theoretical topic has been the theory of **explosive instabilities and erupting flux tubes** by S. C. Cowley and his co-workers: the founding (quite pedagogically written) paper was [Cowley & Artun \(1997\)](#), the key recent one is [Cowley et al. \(2015\)](#); follow the paper trail from there for various refinements and applications (from space to tokamaks).

### 19.2. Resistive MHD

Most of our discussion revolved around properties of ideal MHD equations. It is, in fact, quite essential to study resistive effects, even when resistivity is very small, because many ideal solutions have a natural tendency to develop ever smaller spatial gradients, which can only be regularised by resistivity (we touched on this, e.g., in §16.2.2). The key linear result here is the **tearing mode**, a resistive instability associated with the propensity of magnetic-field lines to reconnect—change their topology in such a way as to release some of their energy. This is covered in the lectures by [Parra \(2019a\)](#) and [Kunz \(2020\)](#); other good places to read about it are [Taylor & Newton \(2015\)](#) again, the original paper by [Furth et al. \(1963\)](#), or standard textbooks (e.g., [Sturrock 1994](#), §17).

Here again the frontier is nonlinear: the theory of **magnetic reconnection**: tearing modes, in their nonlinear stage, tend to lead to formation of *current sheets* (which is, in fact, a general tendency of  $X$ -point solutions in MHD), and how reconnection happens after that has been a subject of active research since mid-20th century. Magnetic reconnection is believed to be a key player in a host of plasma phenomena, from solar flares to the so-called “sawtooth crash” in tokamaks, to MHD turbulence. [Kulsrud \(2005, §14\)](#) has a good introduction to the history and the basics of the subject from a live witness and key contributor. There has been much going on in it in the last decade, many of the advances occurring on the collisionless reconnection front requiring kinetic

<sup>121</sup>Another excellent set of lecture notes on **astrophysical fluid dynamics** is [Ogilvie \(2016\)](#), this one originating from Cambridge Part III.

theory (some key names to search for in the extensive recent literature are W. Daughton, J. Drake, J. Egedal), but even within MHD, the discovery of the **plasmoid instability** (amounting to the realisation that current sheets are tearing unstable; see [Loureiro et al. 2007](#)) has led to a new theory of resistive MHD reconnection ([Uzdensky et al. 2010](#)), a development that I (obviously) find important.

Even more recently, magnetic reconnection became intimately intertwined with the theory of MHD turbulence ([§15.4](#))—you will find an account of this in my (hopefully pedagogical) review, [Schekochihin \(2022\)](#). Appendix C of this document also contains a “reconnection primer” covering tearing modes, current sheets and related topics in the most straightforward non-rigorous way that I could manage.

### 19.3. *Dynamo Theory and MHD Turbulence*

These are topics of active research, which one can have full access to with the education provided by these notes, and indeed it is to an extent with these topics in mind (or, at any rate, in my mind) that some of these notes were written. In §§[13.9](#), [14.1](#), [15.4](#), and [16.5](#), further pointers are provided.

### 19.4. *Hall MHD, Electron MHD, Braginskii MHD*

These and other “two-fluid” approximations of plasma dynamics have to do with with (i) what happens at scales where different species (ions and electrons) cannot be considered to move together (**Hall/Electron MHD**; see [Q7](#)) and (ii) how momentum transport (viscosity) and energy transport (heat conduction) operate in a *magnetised* plasma, i.e., a plasma where the Larmor motion of particles dominates over their Coulomb collisions, even though the latter might be faster than the fluid motions (**Braginskii 1965 MHD**). In general, this is a kinetic subject, although certain limits can be treated by fluid approximations. An introduction to these topics is given in [Parra \(2019b\)](#) and [Parra \(2019a\)](#) (see also [Goedbloed & Poedts 2004](#), §3 and the excellent monograph by [Helander & Sigmar 2005](#)).

### 19.5. *Double-Adiabatic MHD and Onwards to Kinetics*

A conceptually interesting and important paradigm, which I have already mentioned parenthetically in several places above, is the **double-adiabatic MHD** ([13.62](#)). This deals with a situation in a magnetised plasma (in the sense defined in [§19.4](#)) when pressure becomes anisotropic, with pressures perpendicular and parallel to the local direction of the magnetic field evolving each according to its own, separate equation, replacing the adiabatic law ([13.60](#)) and based on the conservation of the adiabatic invariants of the Larmor-gyrating particles. The **dynamics of pressure-anisotropic plasma**, based on CGL equations ([13.62](#)) or, which is usually more correct physically, on the full kinetic description (and its reduced versions, e.g., **Kinetic MHD**; see [Parra 2019b](#), also [Kulsrud 1983](#)), are another current frontier, with applications to weakly collisional astrophysical plasmas (from interplanetary to intergalactic). A key feature that makes this topic both interesting and difficult is that pressure anisotropies in high- $\beta$  plasmas trigger small-scale instabilities (e.g., the Alfvén wave becomes unstable—the so-called **firehose instability**; see [Exercise 15.9](#)). These break the fluid approximation and leave us without a good mean-field theory for the description of macroscopic motions in such environments (for a short introduction to these issues, see [Schekochihin et al. 2010](#), although this subject is developing so fast that anything written 10 years ago is at least partially obsolete; you can read [Squire et al. 2017](#) for a taste of how hairy things become in what concerns even such staples as Alfvén waves).

## Magnetohydrodynamics Problem Set

**1. Clebsch Coordinates.** As  $\nabla \cdot \mathbf{B} = 0$ , it is always possible to find two scalar functions  $\alpha(\mathbf{r})$  and  $\beta(\mathbf{r})$  such that

$$\mathbf{B} = \nabla\alpha \times \nabla\beta. \quad (19.1)$$

(a) Argue that any magnetic field line can be described by the equations

$$\alpha = \text{const}, \quad \beta = \text{const}. \quad (19.2)$$

This means that  $(\alpha, \beta, \ell)$ , where  $\ell$  is the distance (arc length) along the field line, are a good set of curvilinear coordinates, known as the *Clebsch coordinates*.

(b) Show that the magnetic flux through any area  $S$  in the  $(x, y)$  plane is

$$\Phi = \int_{\tilde{S}} d\alpha d\beta, \quad (19.3)$$

where  $\tilde{S}$  is the area  $S$  in new coordinates after transforming  $(x, y) \rightarrow (\alpha(x, y, 0), \beta(x, y, 0))$ .

(c) Show that if (19.1) holds at time  $t = 0$  and  $\alpha$  and  $\beta$  are evolved in time according to

$$\frac{d\alpha}{dt} = 0, \quad \frac{d\beta}{dt} = 0, \quad (19.4)$$

where  $d/dt$  is the convective derivative, then (19.1) correctly describes the magnetic field at all  $t > 0$ .

(d) Argue from the above that magnetic flux is frozen into the flow and magnetic field lines move with the flow.

(e\*) Show that the field that minimises the magnetic energy within some domain subject to the constraint that the values of  $\alpha$  and  $\beta$  are fixed at the boundary of this domain (i.e., that the “footpoints” of the field lines are fixed) is a force-free field.<sup>122</sup>

A prototypical example of the kind of fields that arise from the variational principle in (e) is the “arcade” fields describing magnetic loops sticking out of the Sun’s surface, with footpoints anchored at the surface. One such field will be considered in Q9(f) and more can be found in Sturrock (1994, §13).

**2. Uniform Collapse.** A simple model of star formation envisions a sphere of galactic plasma with number density  $n_{\text{gal}} = 1 \text{ cm}^{-3}$  undergoing a gravitational collapse to a spherical star with number density  $n_{\text{star}} = 10^{26} \text{ cm}^{-3}$ . The magnetic field in the galactic plasma is  $B_{\text{gal}} \sim 3 \times 10^{-6} \text{ G}$ . Assuming that flux is frozen, estimate the magnetic field in a star. Find out if this is a good estimate. If not, how, in your view, could we account for the discrepancy?

**3. Flux Concentration.** Consider a simple 2D model of incompressible convective motion (Fig. 81):

$$\mathbf{u} = U \left( -\sin \frac{\pi x}{L} \cos \frac{\pi z}{L}, 0, \cos \frac{\pi x}{L} \sin \frac{\pi z}{L} \right). \quad (19.5)$$

(a) In the neighbourhood of the stagnation point  $(0, 0, 0)$ , linearise the flow, assume vertical magnetic field,  $\mathbf{B} = (0, 0, B(t, x))$  and derive an evolution equation for  $B(t, x)$ ,

<sup>122</sup>This is based on the 2017 exam question.

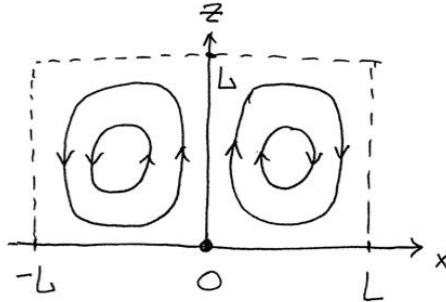


FIGURE 81. Convective cells from Q3.

including both advection by the flow and Ohmic diffusion. Suppose the field is initially uniform,  $B(t = 0, x) = B_0 = \text{const}$ . It should be clear to you from your equation that magnetic field is being swept towards  $x = 0$ . What is the time scale of this sweeping? Given the magnetic Reynolds number  $\text{Rm} = UL/\eta \gg 1$ , show that flux conservation holds on this time scale.

(b) Find the steady-state solution of your equation. Assume  $B(x) = B(-x)$  and use flux conservation to determine the constants of integration (in terms of  $B_0$  and  $\text{Rm}$ ). What is the width of the region around  $x = 0$  where the flux is concentrated? What is the magnitude of the field there?

(c\*) Obtain the time-dependent solution of your equation for  $B$  and confirm that it indeed converges to your steady-state solution. Find the time scale on which this happens.

*Hint.* The following changes of variables may prove useful:  $\xi = \sqrt{\pi \text{Rm}} x/L$ ,  $\tau = \pi U t/L$ ,  $X = \xi e^\tau$ ,  $s = (e^{2\tau} - 1)/2$ .

(d) Can you think of a quick heuristic argument based on the induction equation that would tell you that all these answers were to be expected?

**4. Zeldovich's Antidynamo Theorem.** Consider an arbitrary 2D velocity field:  $\mathbf{u} = (u_x, u_y, 0)$ . Assume incompressibility. Show that, in a finite system (i.e., in a system that can be enclosed within some volume outside which there are no fields or flows), this velocity field cannot be a dynamo, i.e., any initial magnetic field will always eventually decay.

*Hint.* Consider separately the evolution equations for  $B_z$  and for the magnetic field in the  $(x, y)$ -plane. Show that  $B_z$  decays by working out the time evolution of the volume integral of  $B_z^2$ . Then write  $B_x, B_y$  in terms of one scalar function (which must be possible because  $\partial B_x / \partial x + \partial B_y / \partial y = 0$ ) and show that it decays as well.

**5\*. X-Point Collapse.** Consider the following initial magnetic-field configuration:

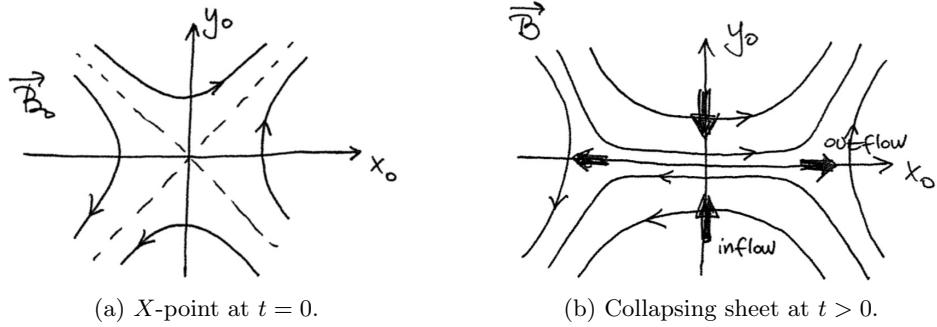
$$\mathbf{B}_0(\mathbf{r}_0) = B_0 \hat{z} + \hat{z} \times \nabla_0 \Psi(x_0, y_0), \quad (19.6)$$

where  $\mathbf{r}_0 = (x_0, y_0, z_0)$ ,  $B_0 = \text{const}$ , and

$$\Psi(x_0, y_0) = \frac{x_0^2 - y_0^2}{2}. \quad (19.7)$$

This is called an *X-point* (Fig. 82a).

(a) Use the Lagrangian MHD equation (13.87), where  $\mathbf{r} = (x, y, z)$ , and seek a solution

FIGURE 82.  $X$ -point collapse.

in the form

$$x = \xi(t)x_0, \quad y = \eta(t)y_0, \quad z = z_0. \quad (19.8)$$

Show that  $\xi$  and  $\eta$  satisfy the following equations

$$\ddot{\xi} = \eta \left( \frac{1}{\eta^2} - \frac{1}{\xi^2} \right), \quad \ddot{\eta} = \xi \left( \frac{1}{\xi^2} - \frac{1}{\eta^2} \right). \quad (19.9)$$

(b) Consider the possibility that, as time goes on,  $\eta(t)$  becomes ever smaller,  $\eta \rightarrow 0$ , while  $\xi(t)$  tends to a constant,  $\xi \rightarrow \xi_c$ . Show that the solution that has this property is

$$\xi(t) \approx \xi_c + \frac{9}{4} \left( \frac{2}{9\xi_c} \right)^{1/3} (t_c - t)^{4/3}, \quad \eta(t) \approx \left( \frac{9\xi_c}{2} \right)^{1/3} (t_c - t)^{2/3} \quad (19.10)$$

as  $t \rightarrow t_c$ , where  $t_c$  is some finite time. This is called the [Syrovatskii \(1971\) solution](#).

(c) Calculate the magnetic field as a function of time and convince yourself that the Syrovatskii solution describes the initial  $X$ -point configuration collapsing explosively to a sheet along the  $x$  axis. What happens after  $t$  reaches  $t_c$ ?

(d) Do a similar calculation, but for incompressible Lagrangian MHD, i.e., assuming  $J = 1 = \text{const}$  (which is now the equation that determines the total pressure; cf. §15.2.5). Show that the solution in this case is

$$\xi(t) = \Lambda(t), \quad \eta(t) = \frac{1}{\Lambda(t)}, \quad (19.11)$$

where  $\Lambda(t)$  is an arbitrary function of time. Take  $\Lambda(t) = e^{\lambda t}$  and show that this solution corresponds to an exponentially collapsing  $X$ -point. This is called the [Chapman & Kendall \(1963\) solution](#). Can this evolution continue forever?

(e) Show that the fluid flow associated with a collapsing solution consists of an inflow (into the “sheet”) and an outflow (from the “sheet”). Going back to the general solution (19.11), assume that the outflow velocity  $u_x$  at a given fixed Lagrangian position  $x_{\text{out}}$  is equal to some known constant  $u_{\text{out}}$  (i.e., as the “sheet” collapses and gets longer, the outflow from its ends is always the same). Find  $\xi(t)$  and  $\eta(t)$  in this case. This solution is due to [Uzdensky & Loureiro \(2016\)](#) (read their paper to find out what the use of it is).

**6. MHD Waves in a Stratified Atmosphere.** The generalisation of iMHD to the case of a stratified atmosphere is explained in §15.2.8. Convince yourself that you understand how the SMHD equations and the SMHD ordering arise and then study them as follows.

(a) Work out all SMHD waves (both their frequencies and the corresponding eigenvectors).

tors). It is convenient to choose the coordinate system in such a way that  $\mathbf{k} = (k_x, 0, k_z)$ , where  $z$  is the vertical direction (the direction of gravity). The mean magnetic field  $\mathbf{B}_0 = B_0 \mathbf{b}_0$  is assumed to be straight and uniform, at a general angle to  $z$ . We continue referring to the projection of the wave number onto the magnetic-field direction as  $k_{\parallel} = \mathbf{k} \cdot \mathbf{b}_0 = k_x b_{0x} + k_z b_{0z}$ . Note that in the case of  $B_0 = 0$ , you are dealing with stratified hydrodynamics, not MHD—the waves that you obtain in this case are the well known *gravity waves*, or “*g-modes*”.

(b) Explain the physical nature of the perturbations (what makes the fluid oscillate) in the special cases (i)  $k_z = 0$  and  $\mathbf{b}_0 = \hat{\mathbf{z}}$ , (ii)  $k_z = 0$  and  $\mathbf{b}_0 = \hat{\mathbf{x}}$ , (iii)  $k_x = 0$ , (iv)  $k_z \neq 0$ ,  $k_x \neq 0$  and  $\mathbf{b}_0 = \hat{\mathbf{z}}$ .

(c) Under what conditions are the perturbations you have found unstable? What is the physical mechanism for the instability? What role does the magnetic field play (stabilising or destabilising) and why? Cross-check your answers with §17.3 and Q10.

(d) Find the conserved energy (a quadratic quantity whose integral over space stays constant) for the full nonlinear SMHD equations (15.97–15.100). Give a physical interpretation of the quantity that you have obtained—why should it be conserved?

*Do either Q7 or Q8.*

**7. Electron MHD.** In certain physical regimes (roughly realised, for example, in the solar-wind and other kinds of astrophysical turbulence at scales smaller than the ion Larmor radius; see Schekochihin *et al.* 2009 or Boldyrev *et al.* 2013), plasma turbulence can be described by an approximation in which the magnetic field is frozen into the electron flow  $\mathbf{u}_e$ , while ions are considered motionless,  $\mathbf{u}_i = 0$ . In this approximation, Ohm’s law becomes<sup>123</sup>

$$\mathbf{E} = -\frac{\mathbf{u}_e \times \mathbf{B}}{c}. \quad (19.12)$$

Here  $\mathbf{u}_e$  can be expressed directly in terms of  $\mathbf{B}$  because the current density in a plasma consisting of motionless hydrogen ions ( $n_i = n_e$ ) and moving electrons is

$$\mathbf{j} = en_e(\mathbf{u}_i - \mathbf{u}_e) = -en_e\mathbf{u}_e, \quad (19.13)$$

but, on the other hand,  $\mathbf{j}$  is known via Ampère’s law. Here  $n_e$  is the electron number density and  $e$  the electron charge.

(a) Using this and Faraday’s law, show that the evolution equation for the magnetic field in this approximation is

$$\frac{\partial \mathbf{B}}{\partial t} = -d_i \nabla \times [(\nabla \times \mathbf{B}) \times \mathbf{B}], \quad (19.14)$$

where the magnetic field has been rescaled to Alfvénic velocity units,  $\mathbf{B}/\sqrt{4\pi m_i n_i} \rightarrow \mathbf{B}$ , and  $d_i = c/\omega_{pi}$  is the ion inertial scale (“ion skin depth”),  $\omega_{pi} = \sqrt{4\pi e^2 n_i / m_i}$ . Equation (19.14) is the equation of *Electron MHD (EMHD)*, completely self-consistent for  $\mathbf{B}$ .

(b) Show that magnetic energy is conserved by (19.14). Is magnetic helicity conserved?

<sup>123</sup>Strictly speaking, the generalised Ohm’s law in this approximation also contains an electron-pressure gradient (see, e.g., Goedbloed & Poedts 2004), but that vanishes upon substitution of  $\mathbf{E}$  into Faraday’s law.

Does J. B. Taylor relaxation work and what kind of field will be featured in the relaxed state? Is it obvious that this field is a good steady-state solution of (19.14)?

(c) Consider infinitesimal perturbations of a straight-field equilibrium,  $\mathbf{B} = B_0 \hat{\mathbf{z}} + \delta\mathbf{B}$ , and show that they are helical waves with the dispersion relation

$$\boxed{\omega = \pm k_{\parallel} v_A k d_i} . \quad (19.15)$$

These are called *Kinetic Alfvén Waves (KAW)*.

(d) Now consider finite perturbations and argue that the appropriate ordering in which linear and nonlinear physics can coexist while perturbations remain small is

$$|\delta\mathbf{b}| \sim \frac{\delta B}{B} \sim \frac{k_{\parallel}}{k} \ll 1. \quad (19.16)$$

Under this ordering, show that the magnetic field can be represented as

$$\frac{\delta\mathbf{B}}{B_0} = \frac{1}{v_A} \hat{\mathbf{z}} \times \nabla_{\perp} \Psi + \hat{\mathbf{z}} \frac{\delta B}{B} \quad (19.17)$$

and the evolution equations for  $\Psi$  and  $\delta B/B_0$  are

$$\boxed{\frac{\partial\Psi}{\partial t} = v_A^2 d_i \mathbf{b} \cdot \nabla \frac{\delta B}{B_0}, \quad \frac{\partial}{\partial t} \frac{\delta B}{B_0} = -d_i \mathbf{b} \cdot \nabla \nabla_{\perp}^2 \Psi} , \quad (19.18)$$

where  $\mathbf{b} \cdot \nabla$  is given by (15.115). These are the equations of *Reduced Electron MHD*.

(e) Check that the conservation of magnetic energy and the KAW dispersion relation (19.15) are recovered from (19.18). Is there any other conservation law?

**8. Hydrodynamics of Rotating Fluid.**<sup>124</sup> Most of this question is not on MHD, but deals with equations describing a somewhat analogous system: also embedded into an external field and supporting anisotropic wave-like perturbations. It is an incompressible fluid rotating at angular velocity  $\boldsymbol{\Omega} = \Omega \hat{\mathbf{z}}$ , where  $\hat{\mathbf{z}}$  is the unit vector in the direction of the  $z$  axis. The velocity field  $\mathbf{u}$  in such a fluid satisfies the following equation

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + 2\mathbf{u} \times \boldsymbol{\Omega}, \quad (19.19)$$

where pressure  $p$  is found from the incompressibility condition  $\nabla \cdot \mathbf{u} = 0$ , the last term on the right-hand side is the Coriolis force, the centrifugal force has been absorbed into  $p$ , and viscosity has been ignored.

(a) Consider infinitesimal perturbations of a static ( $\mathbf{u}_0 = 0$ ), homogeneous equilibrium of (19.19). Show that the system supports waves with the dispersion relation

$$\boxed{\omega = \pm 2\Omega \frac{k_{\parallel}}{k}} . \quad (19.20)$$

These are called *inertial waves*. Here  $\mathbf{k} = (k_{\perp}, 0, k_{\parallel})$  (without loss of generality); the subscripts refer to directions perpendicular and parallel to the axis of rotation.

(b) In the case  $k_{\parallel} \ll k_{\perp}$ , determine the direction of propagation of the inertial waves. Determine also the relationship between the components of the velocity vector  $\mathbf{u}$  associated with the wave. Comment on the polarisation of the wave.

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<sup>124</sup>This is based on the 2018 exam question.

(c) When rotation is strong, i.e., when  $\Omega \gg ku$ , perturbations in a rotating system are anisotropic with  $\epsilon = k_{\parallel}/k_{\perp} \ll 1$ . Order the linear and nonlinear time scales to be similar to each other and work out the ordering of all relevant quantities, namely,  $\mathbf{u}_{\perp}$  (horizontal velocity),  $u_{\parallel}$  (vertical velocity),  $\delta p$  (perturbed pressure),  $\omega$ ,  $\Omega$ ,  $k_{\parallel}$ ,  $k_{\perp}$  with respect to each other and to  $\epsilon$ . Using this ordering, show that the motions of a rotating fluid satisfy the following reduced equations

$$\boxed{\frac{\partial}{\partial t} \nabla_{\perp}^2 \Phi + \{\Phi, \nabla_{\perp}^2 \Phi\} = 2\Omega \frac{\partial u_{\parallel}}{\partial z}, \quad \frac{\partial u_{\parallel}}{\partial t} + \{\Phi, u_{\parallel}\} = -2\Omega \frac{\partial \Phi}{\partial z}}, \quad (19.21)$$

where the “Poisson bracket” is defined by (15.114) and  $\Phi$  is the stream function of the perpendicular velocity, i.e., to the lowest order in  $\epsilon$ ,  $\mathbf{u}_{\perp}^{(0)} = \hat{\mathbf{z}} \times \nabla_{\perp} \Phi$ . Note that, in order to obtain the above equations, you will need to work out  $\nabla_{\perp} \cdot \mathbf{u}_{\perp}$  to both the lowest and next order in  $\epsilon$ , i.e., both  $\nabla_{\perp} \cdot \mathbf{u}_{\perp}^{(0)}$  and  $\nabla_{\perp} \cdot \mathbf{u}_{\perp}^{(1)}$ .

(d) Show that any purely horizontal flows in a strongly rotating fluid must be exactly two-dimensional (i.e., constant along the axis of rotation).

(e) For a strongly rotating, incompressible, highly electrically conducting fluid embedded in a strong uniform magnetic field  $\mathbf{B}_0$  parallel to the axis of rotation, discuss qualitatively under what conditions you would expect anisotropic ( $k_{\parallel} \ll k_{\perp}$ ) Alfvénic and slow-wave-like (pseudo-Alfvénic) perturbations to be decoupled from each other?

There are certain interesting similarities between MHD turbulence and turbulence in rotating fluid systems described by (19.21) and, indeed, also turbulence in stratified environments that we dealt with in §15.2.8 and Q6. If you would like to know more, see [Nazarenko & Schekochihin \(2011\)](#) and follow the paper trail from there.

**9. Grad–Shafranov Equation.** Consider static MHD equilibria (16.1) in cylindrical coordinates  $(r, \theta, z)$  and assume axisymmetry,  $\partial/\partial\theta = 0$ .

(a) Using the solenoidality of the magnetic field, show that any axisymmetric such field can be expressed in the form

$$\mathbf{B} = I \nabla \theta + \nabla \psi \times \nabla \theta, \quad (19.22)$$

where  $I$  and  $\psi$  are functions of  $r$  and  $z$  and  $\nabla \theta = \hat{\theta}/r$  ( $\hat{\theta}$  is the unit basis vector in the  $\theta$  direction). Show that magnetic surfaces are surfaces of  $\psi = \text{const}$ .

(b) Using the force balance, show that  $\nabla I \times \nabla \psi = 0$  and  $\nabla p \times \nabla \psi = 0$  and hence argue that

$$I = I(\psi) \quad \text{and} \quad p = p(\psi) \quad (19.23)$$

are functions of  $\psi$  only (i.e., they are constant on magnetic surfaces).

(c) Again from the force balance, show that  $\psi(r, z)$  satisfies the *Grad–Shafranov equation*

$$\boxed{- \left( \frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} \right) = 4\pi r^2 \frac{dp}{d\psi} + I \frac{dI}{d\psi}}. \quad (19.24)$$

This defines the shape of an axisymmetric equilibrium, given the profiles  $p(\psi)$  and  $I(\psi)$ .

(d) Show that in cylindrical symmetry ( $\partial/\partial\theta = 0, \partial/\partial z = 0$ ), (19.24) reduces to (16.8).

(e\*) Assume  $I(\psi) = \text{const}$  (so the azimuthal field  $B_{\theta} = I/r$  is similar to the magnetic field from a central current) and  $p(\psi) = a\psi$ , where  $a$  is some constant. Find a solution

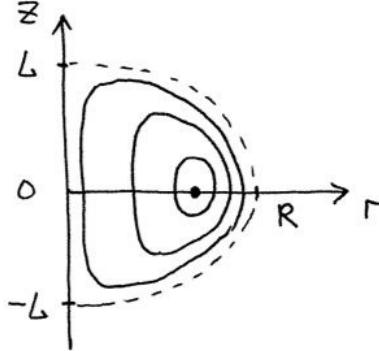


FIGURE 83. A simple equilibrium from Q9, superficially resembling the poloidal cross-section of a tokamak.

of (19.24) that gives rise to magnetic surfaces that resemble nested tori, but with “D-shaped” cross section (Fig. 83; this looks a bit like the modern tokamaks). If you stipulate that  $p$  must vanish at  $r = 0$  and at  $r = R$  along the  $z = 0$  axis and also at  $z = \pm L$  along the  $r = 0$  axis and that the maximum pressure at  $r < R$  is  $p_0$ , show that the corresponding magnetic surfaces are described by

$$\psi = 2\sqrt{\frac{2\pi p_0}{1 + R^2/4L^2}} r^2 \left(1 - \frac{r^2}{R^2} - \frac{z^2}{L^2}\right). \quad (19.25)$$

Where is the (azimuthal) magnetic axis of these surfaces? What is the value of  $a$ ?

(f) Seek solutions to (19.24) that are linear force-free fields. Show that in this case, (19.24) reduces to the Bessel equation (a substitution  $\psi = rf(r, z)$  will prove useful). Set  $B_z(0, 0) = B_0$ . Find solutions of two kinds: (i) ones in a semi-infinite domain  $z \geq 0$ , with the field vanishing exponentially at  $z \rightarrow \infty$ ; (ii) ones periodic in  $z$ . If you also impose the boundary condition  $B_r = 0$  at  $r = R$ , how can this be achieved? Can either of these solutions be the result of J. B. Taylor relaxation of an MHD system? If so, how would one decide whether it is more or less likely to be the correct relaxed state than the solution derived in §16.4?

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You will find the solution of the type (i) in Sturrock (1994, §13) (who also shows how to construct many other force-free fields, useful in various physical and astrophysical contexts). Think of this solution in the context of Q1(e). The solution of type (ii) is a particular case of the general ( $\partial/\partial\theta \neq 0$ ) equilibrium solution derived and discussed in Taylor & Newton (2015, §9). However, the axisymmetric solution is not very useful because, as they show, depending on the values of helicity and of  $R$ , the true relaxed state is either the cylindrically and axially symmetric solution derived in §16.4 or one which also has variation in the  $\theta$  direction.

**10. Magnetised Interchange Instability.** Consider the same set up as in §17.3, but now the stratified atmosphere is threaded by straight horizontal magnetic field (Fig. 84):

$$\rho_0 = \rho_0(z), \quad p_0 = p_0(z), \quad \mathbf{B}_0 = B_0(z)\hat{x}, \quad \frac{d}{dz} \left( p_0 + \frac{B_0^2}{8\pi} \right) = -\rho_0 g. \quad (19.26)$$

We shall be concerned with the stability of this equilibrium.

(a) For simplicity, assume  $\partial\xi/\partial x = 0$ . This rules out any perturbations of the magnetic-field direction,  $\delta\mathbf{b} = 0$ , so there will be no field-line bending, no restoring curvature

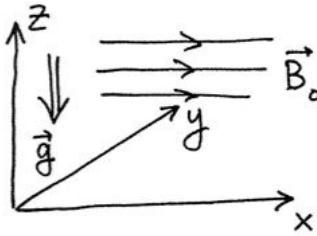


FIGURE 84. Magnetised atmosphere.

forces. For this restricted set of perturbations, work out  $\delta W_2$  and observe that, like in the unmagnetised case considered in §17.3, it depends only on  $\nabla \cdot \xi$  and  $\xi_z$ . Minimise  $\delta W_2$  with respect to  $\nabla \cdot \xi$  and show that

$$\frac{d}{dz} \ln \frac{p_0}{\rho_0^\gamma} + \frac{2}{\beta} \frac{d}{dz} \ln \frac{B_0}{\rho_0} < 0 \quad (19.27)$$

is a sufficient condition for instability (the *magnetised interchange instability*). Would you be justified in expecting stability if the condition (19.27) were not satisfied?

(b) Explain how this instability operates and rederive the condition for instability by considering interchanging blobs (or, rather, flux tubes), in the spirit of §17.3.3.

If field-line bending is allowed ( $\partial \xi / \partial x \neq 0$ ), another instability emerges, the [Parker \(1966\) instability](#). Do investigate.

**11. Stability of the  $\theta$  Pinch.** Consider the following cylindrically and axially symmetric equilibrium:

$$\mathbf{B}_0 = B_0(r)\hat{\mathbf{z}}, \quad \mathbf{j}_0 = j_0(r)\hat{\boldsymbol{\theta}} = -\frac{c}{4\pi} B'_0(r)\hat{\boldsymbol{\theta}}, \quad \frac{d}{dr} \left( p_0 + \frac{B_0^2}{8\pi} \right) = 0 \quad (19.28)$$

(a  $\theta$  pinch; see §16.1.1, Fig. 73). Consider general displacements of the form

$$\xi = \xi_{mk}(r)e^{im\theta+ikz}. \quad (19.29)$$

Show that the  $\theta$  pinch is always stable. Specifically, you should be able to show that

$$\delta W_2 = \pi L_z \int_0^\infty dr r \left\{ \gamma p_0 |\nabla \cdot \xi|^2 + \frac{B_0^2}{4\pi} \left[ k^2 (|\xi_r|^2 + |\xi_\theta|^2) + \left| \frac{\xi_r}{r} + \frac{\partial \xi_r}{\partial r} + \frac{im\xi_\theta}{r} \right|^2 \right] \right\} > 0, \quad (19.30)$$

where  $L_z$  is the length of the cylinder.

## PART IV

## Kinetic Theory of Plasma in a Strong Magnetic Field

**20. Kinetic MHD**20.1. *Peculiar Kinetics*20.2. *Momentum Equations*20.3. *Mass Flow*20.4. *Ideal Ohm's Law*20.5. *Almost MHD*20.6. *Gyrotropic Plasma*20.7. *Firehose Instability*20.8. *Gyroaveraged Kinetic Equation*20.8.1. *Hiding Species Drifts*20.8.2. *Magnetic Moment*20.9. *Kinetic MHD Equations Completed***Exercise 20.1. Energy Budget in KMHD.****21. Kinetics from Particle Motion**This section is heavily influenced by the lecture notes of [Parra \(2019b\)](#).21.1. *Relation Between Particle Motion and Kinetic Equations*21.1.1. *Phase-Space Density*21.1.2. *Phase-Volume Conservation*21.2. *Particle Motion in a Strong Magnetic Field*21.2.1. *Constant Fields*21.2.2. *Slowly Varying Fields*21.3. *High Flow vs. Low Flow*21.4. *High-Flow Drift Kinetics: KMHD Regained*21.5. *Low-Flow Drift Kinetics: Going to Higher Order*21.5.1. *Parallel Kinetics*21.5.2. *General Scheme for Perturbation Theory*21.5.3. *General Scheme Implemented*21.5.4. *Adjusted Variables*

## 22. Kinetic MHD Continued

22.1. *Mirror Instability and Barnes Damping*

22.2. *Origin of Pressure Anisotropy*

22.2.1. *CGL Equations*

22.2.2. *Longitudinal Invariant*

22.2.3. *Pressure Anisotropy*

22.3. *Braginskii MHD*

22.3.1. *Braginskii Viscosity*

22.3.2. *Heat Fluxes*

22.3.3. *Magnetothermal Instability*

The refrain of the KMHD theory is that everything is unstable, on large scales, on small scales, on even smaller scales, etc. It is an almighty mess caused by plasma's general unhappiness with, more or less, anything other than a uniform Maxwellian equilibrium—not really a surprise for the aficionados of thermodynamics! Everything, therefore, is well-nigh-always turbulent, and what one really needs is some sort of effective mean-field theory. We do not have that, so we are still at the stage where we must examine closely the 1001 ways in which unstable configurations can be unstable, with a hope of integrating all this knowledge one day into some general principles of kinetic stability.

22.4. *Free Energy in Kinetic MHD*

**Exercise 22.1. HBI.**

**Exercise 22.2. Magnetoviscous instability.**

## 23. Reduced Kinetic MHD

23.1. *Alfvénic Perturbations*

23.2. *Compressive Perturbations*

23.3. *Free Energy*

**Exercise 23.1. Barnes damping in KRMHD.**

**24. Electrostatic Drift Kinetics**

- 24.1. *Boltzmann Electrons*
- 24.2. *Fluid ITG Instability*
- 24.3. *Kinetic ITG Instability*
- 24.4. *Electrostatic Limit: Formal Derivation*
- 24.5. *Curvature ITG Instability*
- 24.6. *Zonal Flows*

**25. Gyrokinetics****26. Electron Drift Kinetics**

- 26.1. *Electron Drift-Kinetic Equation*
- 26.2. *ETG Instability*
- 26.3. *Kinetic Alfvén Waves*
- 26.4. *Thermo-Alfvénic Instability*
- 26.5. *Collisionless Tearing Mode*

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**Exercise 26.1.** Semicollisional tearing mode.

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