Physics Reports 783–785 (2018) 1–84

Contents lists available at ScienceDirect

Physics Reports

journal homepage: www.elsevier.com/locate/physrep

Kinetic theory of small-amplitude fluctuations in astrophysical plasmas

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ARTICLE INFO

Article history:
Accepted 15 October 2018
Available online 25 October 2018
Editor: R. Redmer

ABSTRACT

Electromagnetic field fluctuations are arguably among the most fundamental kinetic processes in classical, collisionless plasmas. Considering their two characteristic features, propagation and amplitude modulation, they include such eminent phenomena as waves and instabilities. This work is devoted to an extension of their theoretical understanding with particular emphasis on weakly propagating and aperiodic fluctuations and their spontaneous emission. In this regard, three main results are achieved that contribute to both generic plasma theory as well as astrophysical applications.

Firstly, the fluctuation–dissipation theorem for weakly coupled thermal plasmas is generalized to arbitrary values of the complex frequency in order to rid the previous formulations of their restrictions in this respect.

Secondly, the spontaneously emitted magnetic field fluctuations in the intergalactic medium are addressed. Due to a recently discovered damped and aperiodic mode, the level of magnetic noise is high enough to have the latter serving as a seed field for further amplification processes like magnetohydrodynamic dynamo action, so this spontaneous emission is of major relevance for cosmic magnetogenesis. Here, it is shown that a highly relativistic electron–positron pair beam can trigger a transition of these damped fluctuations into amplified ones in certain wavenumber ranges, thus allowing for an even higher seed level. It is found that only those fluctuations are accessible to amplification whose wavevectors are perpendicular or at least almost perpendicular to the propagation direction of the beam.

Thirdly, the possibilities are investigated to obtain observational evidence for this newly discovered stable branch of the Weibel mode because an empirical confirmation of its existence is still pending today. It is shown that the mode-driven turbulence is incompressible and that, therefore, dispersion measure, rotation measure, and scintillation related techniques are not applicable. The velocity fluctuations generated by the mode, however, are shown to be large enough to qualify line broadening studies as a suitable diagnostic method to detect the mode after all.

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https://doi.org/10.1016/j.physrep.2018.10.003
0370-1573/

1. Introduction: Objects, objectives, and outline of this work

1.1. Fluctuations in plasmas

One of the most outstanding features of plasmas that makes them truly intriguing objects of research is their capability to support waves. Unlike other media with this property such as air or water, there is a huge variety of different wave modes excitable in a plasma under suitable conditions. Mirroring the two constituents, particles and fields, the most important quantities performing the wave motion are the density of electrons or ions on the one hand, and the electric and magnetic field on the other, and for each of these many types of waves exist (Stix, 1992). Apart from the sheer plentitude of modes there are also other exceptional and unique features in this regard. Among them are the possibility of longitudinal electric waves and the occurrence of wave damping that is not associated with particle collisions (Landau, 1946). Appreciating plasmas as nonlinear media unfolds an even richer class of wave phenomena, because then the principle of undisturbed superposition governing the linear approximation is superseded by the possibility of different modes to interact with each other. This includes beat waves, shocks, turbulence, and quasiparticle behavior exhibited in reactions like the merging of a Langmuir and an ion-sound wave into a transversal electromagnetic wave, $\mathrm{L}+\mathrm{S}\rightarrow\mathrm{T}$ (Tsytovich, 1990).

It is a common experience of everyday life that waves are often damped. Sound waves fade away as they travel through air, making it difficult to hear the voice of a distant speaker, and light waves are dimmed inside an opaque medium, so it is getting darker and darker as one dives further into the depth of the sea. Apart from the purely geometrical effect expressed by the inverse square law, the decrease of the intensity is also caused by scattering and absorption losses within the medium. Vice versa, devices like the amplifier of a home stereo, a laser, or a megaphone attest to the opposite case that waves can also be amplified if they are supplied with energy. In a plasma, both phenomena – damping and amplification – can occur as well, albeit their origins and processes are quite different from the previous examples. In some cases the damping can be so strong that the wave cannot even travel a significant distance in space (compared to its wavelength) before it decays. Then, the term wave is hardly appropriate anymore and a new terminology is in order that generalizes the concept of a wave and that is adapted to this situation. The common feature of all the cases mentioned is that a state variable like the density or the electric or magnetic field is subject to a disturbance, i. e., a deviation from the (spatial, temporal, or ensemble) average. Thus, the generalized concept is aptly called *fluctuation*. The term *wave* is usually reserved for the case of weakly damped fluctuations, whereas amplified ones are mostly referred to as *instabilities* (Gary, 1993). In this light, the opening statement can be rephrased as follows: One of the most outstanding features of plasmas is their capability to support a rich class of fluctuation phenomena.

Since fluctuations are a vital ingredient of any plasma, exploring their effects is not a mere purpose in itself pursued in an ivory tower. In fact, they are important for many physical systems encountered in nature and technology: Historically, among the earliest incidents raising awareness of plasma fluctuation phenomena are the investigations of high-frequency oscillations in gas discharges conducted by Tonks and Langmuir (1929) and the long-distance transmission experiments carried out with radio waves. The observation that the latter are reflected at atmospheric layers in high altitudes led to the discovery of the ionosphere, culminating in the Nobel Prize awarded to Appleton (1932) for his prolific contributions to gather conclusive evidence. Afterwards, the further observation that even high-frequency signals in the transparency region above the plasma frequency are reflected was interpreted as scattering from electron density fluctuations and thus led to the additional conclusion that the ionospheric plasma is in a turbulent state (Bailey et al., 1952; Villars and Weisskopf, 1955). Experiments of this kind were also employed to probe laboratory plasmas, and due the subsequent development they still remain firmly established in the contemporary repertoire of diagnostic tools (Froula et al., 2011; Hutchinson, 2002; Bekefi, 1966). The underlying fact that density fluctuations are responsible for the scattering of electromagnetic radiation does not only provide the basis for such sophisticated experimental techniques, it also implies that the radiation traversing the interstellar and intergalactic medium suffers distortions like line broadening or dispersion before it reaches the telescopes detecting it (Rybicki and Lightman, 1979). In the same fashion, fluctuations of the electric and magnetic field have a tremendous impact on the propagation of cosmic rays through the ambient medium because they determine the transport coefficients of a collisionless plasma (Schlickeiser, 2011, 2015). Considering that these cosmic messengers, photons and relativistic charged particles, are the primary sources of information about remote astrophysical objects, a profound understanding of the influences arising from plasma fluctuations is of paramount importance. Moreover, plasma waves are also suspected to play a key role in the coronal heating problem (McIntosh et al., 2011; van Ballegooijen et al., 2011), one of the most pivotal and longstanding topics in astrophysical research. Arduous efforts to address this issue are still ongoing today: Current spacecraft missions such as the recently launched *Parker Solar Probe* (formerly termed *Solar Probe Plus*) are dedicated to providing measurements of, among other parameters, the wave and turbulence profile in the near-Sun environment (Fox et al., 2016; Bale et al., 2016). On a related matter, there are indications that plasma waves are not only responsible for heating the solar corona and accelerating the solar wind, but also for accelerating electrons to suprathermal energies in the Van Allen radiation belts surrounding the Earth's atmosphere (Reeves et al., 2013; Chen et al., 2007; Horne et al., 2005). Controlling and possibly suppressing fluctuations, in particular unstable ones, is one of the chief concerns in technological applications of plasmas such as thermonuclear fusion reactors and plasma accelerators (Zhou, 2017a,b; Mehrling et al., 2017; Vlad et al., 2006). But instabilities are not only detrimental influences hampering the confinement and the efficiency, they can also be utilized: The modulation instability, for instance, can be employed to resonantly excite the driving waves in wakefield accelerators (Siemon et al., 2013). Furthermore, waves and instabilities are

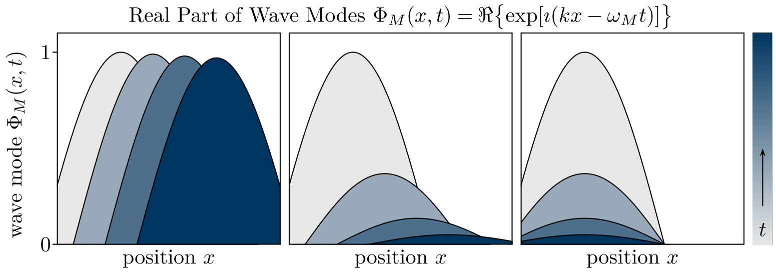

Fig. 1. Propagation and amplitude modulation of one-dimensional damped normal modes with the complex frequency $\omega_M = \Omega_M + \imath\gamma_M$ displaying the three types of limiting cases introduced in the main text: weakly damped modes (left), weakly propagating modes (middle), and aperiodic modes (right).

essential for the understanding of (type III) solar radio bursts (Ziebell et al., 2015; Li et al., 2006; Ginzburg and Zhelezniakov, 1958), and they are involved in the formation of collisionless shock waves (Medvedev and Loeb, 1999; Bret et al., 2013) as well as the associated particle acceleration (Blandford and Eichler, 1987; Ohsawa, 2014; Verkhoglyadova et al., 2015).

Of course, this list of examples is by no means exhaustive, nor is it intended to be. Nevertheless, it already demonstrates the different facets of plasma fluctuations and the broad range of research areas concerned with them, and thus it motivates the need for further investigations and a comprehensive theoretical framework. In order to embrace (i) the collisionless plasmas mostly found in the astrophysical context, (ii) microscopic effects such as the spontaneous emission due to particle discreteness, and (iii) high-frequency processes occurring on short timescales, a kinetic description is inevitable. Bogoliubov, Landau, Vlasov, Klimontovich and other pioneers in the field realized, however, that one cannot simply adopt the well-established kinetic theory of gases, but that a kinetic plasma theory must be developed in its own right. The reason is that, in contrast to neutral gases in which the particles interact with each other only during close encounters, plasmas are dominated by long-range electromagnetic interactions. Maxwell's equations in conjunction with the corresponding kinetic equation define a set of coupled and highly nonlinear equations that constitute the theoretical basis for the computation of field fluctuations in plasmas. Unfortunately, it is virtually impossible to solve these equations in a closed analytical form except in the simplest of cases. The most obvious attempt to tackle this problem consists in a linearization, and this is the approach followed by linear and quasilinear theory. Spurred mostly by the prospect of facilitating nuclear fusion, great efforts were made predominantly in the 1960s and 1970s to improve the theory by including higher order terms as well. In the course of this, weak turbulence theory was developed by Kadomtsev (1965); Sitenko (1967); Sagdeev and Galeev (1969); Davidson (1972); Akhiezer et al. (1975); Tsytovich (1977), and others. Notable further extensions of weak turbulence theory include the semiclassical approach most prominently represented by Tsytovich and Melrose which is somewhat heuristically based on the principle of detailed balance (Tsytovich et al., 2005; Tsytovich, 1995; Melrose, 2008, 2013), and the statistical perturbation expansion scheme advanced by Yoon that is rigorously derived from first principles (Yoon et al., 2016; Yoon, 2006, 2000).

1.2. Weakly propagating and aperiodic fluctuations

The linear or higher-order expansion is commonly carried out in the spectral domain after performing a Fourier transform with respect to the spatial coordinates and a Laplace transform in time. Physically, this procedure corresponds to a normal mode analysis, that is, the attempt to describe the solution in terms of harmonic fluctuations with the spatio-temporal behavior $\propto \exp[\imath(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)]$. Considering that the frequency $\omega = \Omega + \imath\gamma$ is a complex number, this approach aptly matches the two aforementioned characteristics of fluctuations, propagation and amplitude modulation. The two variables Ω and γ quantify these features correspondingly by determining the respective timescales, and the sign of the imaginary part indicates whether the mode is damped or amplified, i. e., whether it is stable or unstable.

From the early pioneering days up to the present day, the research in this field predominantly focused on weakly damped or weakly amplified waves with $|\Omega| \gg |\gamma|$ that travel over a distance of many wavelengths before their amplitude is altered significantly (see Fig. 1). Only little attention was paid to the large class of scenarios in which this assumption is no longer satisfied, including (i) minor deviations from the previous condition, $|\Omega| \lesssim |\gamma|$, as they are exhibited by, for instance, the Buneman instability (Buneman, 1958); (ii) the downright opposite case of weakly propagating fluctuations characterized by strong amplitude modulation, $|\Omega| \ll |\gamma|$; (iii) aperiodic fluctuations with $\Omega = 0$ that do not propagate at all. Nota bene, the latter are by no means a mathematical artifact. On the contrary, they are undisputed and well entrenched phenomena. Amongst them are prominent examples like the firehose and mirror modes in magnetized plasmas and the filamentation and Weibel fluctuations in unmagnetized plasmas (Weibel, 1959; Fried, 1959). It was found quite early that aperiodic thermal fluctuations play an important role in the diffusion in equilibrium plasmas (Okuda and Dawson, 1973). Astonishingly, although these instances are obviously known in the literature, a rigorous generalization of the fundamental theoretical framework has been lacking for a long time and is actually a recent achievement of the past few years. One issue to be addressed in this regard is the computation of the growth rate in terms of the dielectric tensor because the celebrated

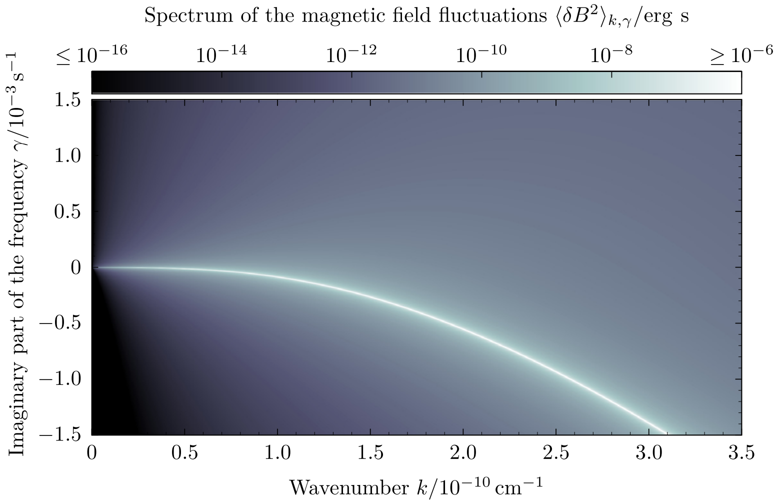

Fig. 2. This color plot visualizes the results obtained by Schlickeiser and Felten (2013) for the equilibrium spectrum of the magnetic field fluctuations in an unmagnetized electron–proton plasma with a common temperature of 10^4 K and a common density of 10^{-7} cm^{-3} for both particle species. The bright line corresponds to the transverse, damped, aperiodic mode $\omega_{\mathrm{M}} = \imath\gamma(k)$ discovered by Felten et al. (2013). It should be noted that the color scale is logarithmic.

Landau formula usually employed is only valid for weakly damped or weakly amplified modes. Yoon (2010) remedied this deficiency to some extent by generalizing weak turbulence theory in this respect, albeit only for an unmagnetized Coulomb plasma. Another restriction that is of a more fundamental nature arises from the resonant denominator entering the equations for the field fluctuations, i. e., from the term $(\omega - \boldsymbol{k} \cdot \boldsymbol{v})^{-1}$ if there is no magnetic background field present or from its counterpart for uniformly magnetized plasmas which contains an integer multiple $(-\ell\Omega_{\mathrm{g}})$ of the gyrofrequency in addition (Sitenko, 1982). The usual procedure is to consider the limit $\gamma \to 0$ that immediately leads to Dirac's δ-distribution $-\imath\pi\,\delta(\Omega - \boldsymbol{k} \cdot \boldsymbol{v})$ by means of the Sokhotski–Plemelj-formula (Nicholson, 1983). Evidently, this approach is only justified if $|\gamma| \ll |\Omega|$, but not if the fluctuations are weakly propagating, let alone aperiodic. The proper generalization to arbitrary complex frequencies is aggravated by the fact that the Laplace integral, which transforms the basic Maxwell–Klimontovich equations into the spectral domain, diverges in the complex half-plane below the abscissa of convergence (Doetsch, 1974). Hence, the form factors and dispersion functions require an analytical continuation to the entire complex plane. Prior to the treatment of collisionless and magnetized plasmas by Schlickeiser and Yoon (2015); Schlickeiser et al. (2015), and Vafin et al. (2016a,b), this procedure has been carried out for the unmagnetized case in a series of papers by Schlickeiser and Yoon (2012); Felten et al. (2013); Lazar et al. (2012); Felten and Schlickeiser (2013a,b,c). There, the electromagnetic fluctuation spectra were computed for arbitrary complex frequencies. Since the relativistic relations for energy, momentum, and velocity were employed, the results are valid within the limits of special relativity without any restrictions on either the particle energies or the phase speeds of the field fluctuations.

In the process, a new collective mode was discovered that appears in every isotropic, unmagnetized plasma. It is damped, aperiodic, and transverse, so it is associated with stable and non-propagating fluctuations of both the electric and the magnetic field. In view of these properties, it can be regarded as the stable branch of the Weibel mode. While the latter is driven by an anisotropy, the former is the counterpart operating in isotropic plasmas. Fig. 2 shows the magnetic fluctuation spectrum computed by Schlickeiser and Felten (2013) for a thermal electron–proton plasma with a non-relativistic temperature of 10^4 K and a density of 10^{-7} cm^{-3} common to both particle species. The mode is clearly discernible as the distinct, bright line indicating fluctuation strengths that are several orders of magnitude above the ambient level. The frequency–wavenumber correlation defined by the curve is a unique feature of collective modes, quantitatively expressed by their dispersion relation $\omega = \omega_M(\boldsymbol{k})$. In this particular case, the mode lies entirely in the lower half-plane $\Im\omega \leq 0$, evincing that it is damped. Owing to this circumstance, it had remained hidden until it was uncovered through the analytical continuation of the fluctuation spectra. The new mode is particularly remarkable because prior to its recognition it was generally accepted that there are only three eigenmodes in an unmagnetized equilibrium plasma (Melrose, 1986): Langmuir waves, ion acoustic waves, and electromagnetic waves corresponding to light. Since the first two are longitudinal, the stable branch of the Weibel mode is the only subluminal, transverse mode in an unmagnetized equilibrium plasma.

In order to appreciate the further consequences of this statement, it is instructive to briefly discuss the origin of fluctuations. As the unruffled surface of a calm sea or the resting bob of a pendulum illustrate, the ability of a system to sustain waves or oscillations does not necessarily imply their actual appearance, because they must be excited in the first place. If the fluctuations happen to be damped, they must even be perpetually re-excited in order to achieve a lasting effect. Triggering oscillations can either occur on account of an external stimulus like a boat traversing the sea or a person pushing the pendulum, or it can occur spontaneously due to the internal constitution of the medium. The spontaneously generated field fluctuations in a plasma are due to its discrete particle structure. This effect is better known as thermal noise, although it is not restricted to thermal equilibrium systems but occurs in arbitrarily distributed plasmas as well (Sitenko, 1982). Being a non-collective phenomenon, spontaneous fluctuations are not constrained by a dispersion relation, but they occur for nearly any combination of wavevector and frequency. Again, this is illustrated in Fig. 2 because the shades of blue covering the entire diagram indicate a non-vanishing fluctuation level. The only constraint that does exist after all is the availability of particles that match the underlying resonance condition. In a uniformly magnetized equilibrium plasma, the corresponding requirement $\omega - \boldsymbol{k} \cdot \boldsymbol{v} = \ell \Omega_{\mathrm{g}}$ poses only a very weak restriction because it will usually be met for suitable multiples of the gyrofrequency. Without a background magnetization, on the other hand, the degree of freedom associated with the integer parameter ℓ does not exist anymore because in this case the resonance condition is given by the relation $\omega - \boldsymbol{k} \cdot \boldsymbol{v} = 0$ or its generalization to complex frequencies. Since special relativity only permits particle velocities below the speed of light, electromagnetic noise in unmagnetized plasmas is limited to subluminal phase speeds. Within the class of these fluctuations, however, spontaneous emission occurs for any pair of ω and $\boldsymbol{k}$, so it is guaranteed that every subluminal eigenmode will be excited, regardless of its dispersion relation. Light and superluminal waves constitute an exception as their self-excitation crucially depends on the presence of a large-scale magnetic field.

Returning to unmagnetized equilibrium plasmas, one can immediately identify the spontaneously excited modes governing the magnetic field fluctuations by contemplating them one by one: Electromagnetic waves are not subluminal, so they are not excited spontaneously; Langmuir and ion acoustic waves, on the other hand, are not transversal, so they are not associated with a magnetic field. Therefore, the stable branch of the Weibel mode has the distinguished property of being the only spontaneously excited mode sustaining magnetic field fluctuations. In view of this novelty, the textbook results concerning magnetic field fluctuations in thermal equilibrium must be revised upwards to an enhanced level.

1.3. The origin of cosmic magnetic fields

The intergalactic medium (IGM) pervading the cosmic voids in the large-scale structure is arguably the most important unmagnetized plasma, so an application of the previous results to this environment presents itself very naturally and turned out to be very valuable. In fact, the aforementioned enhancement of the self-generated noise that exists independently of external drivers like beams or shocks is of considerable consequence in the context of cosmic magnetization. In order to elucidate upon this connection, a brief introduction into this subject will be given here first.

Magnetic fields appear to be ubiquitous in the cosmos. They are observed in stars, in the interplanetary medium, in galaxies and galaxy clusters, and in the filaments of the large-scale structure. Their origin, however, is not satisfactorily understood and remains an active field of research (Subramanian, 2016; Kronberg, 2016; Widrow, 2002; Widrow et al., 2012; Kulsrud and Zweibel, 2008; Ryu et al., 2012). Typical field strengths in galaxies and galaxy clusters are of the order of microgauss, the fields in stars are even several orders of magnitude stronger. There are not too many processes that can generate magnetic fields of such high strengths. The one that is most commonly discussed in this context is the action of a (magnetohydrodynamic) dynamo (Parker, 1955). It converts ordered kinetic energy into magnetic field energy. In particular, the α-ω dynamo is driven by differential rotation (ω) and helical turbulence (α). Other possible generators include flux-conserving compression and plasma instabilities such as the Weibel or filamentation instability. These processes might explain the observed field strengths because they are rather effective, the generated fields grow exponentially according to $B(t) = B_0 \exp(\gamma t)$. However, they cannot create magnetic fields *ex nihilo*. On the contrary, they are only a means of amplification of pre-existing fields B_0, i. e., they require a non-vanishing initial field to start from. A widely accepted paradigm, therefore, is that of a three-stage process (Kulsrud and Zweibel, 2008; Rees, 1987): (1) In the first step, small magnetic seed fields are generated. Depending on their onset and the effectivity of the amplification processes only about 10^{-20} G are required. (2) Once the seed fields are present, they are amplified by dynamo action, plasma instabilities, compression, or other effects. (3) Afterwards or simultaneously, the magnetic fields become coherently ordered and spatially distributed. This may occur, for instance, due to the shock of a supernova explosion.

It is evident in this light that identifying the origin of the magnetic seed fields poses an important task. An early attempt to solve this issue is due to Biermann (1950) who suggested the battery effect named after him: Since electrons and protons have different masses, they are subject to centrifugal forces of different strength in rotating plasmas. The resulting small charge separations constitute currents that generate magnetic fields. This effect, however, is confined to a particular class of environments and has a small volume filling factor because it relies crucially on the presence of rotating objects.

There is a plethora of alternative proposals for processes that might generate the required seed fields. A common categorization catalogs them by the cosmological epoch of their onset, e. g., during inflation, during cosmological phase transitions, during the radiation dominated era, or after the recombination era. Firstly, this categorization reflects the kind of physics involved such as quantum chromodynamics, quantum electrodynamics, or plasma physics for example. Secondly, the onset is also important because the amplification by the previously mentioned processes requires time. The earlier the onset of the seed fields occurs, the stronger are the magnetic fields after a given amplification period. If the field strengths are considered fixed by the values observed today, one can rephrase the statement accordingly: The earlier the onset of the seed field occurs, the weaker are the requirements regarding the effectivity of the amplification mechanisms. Furthermore, the onset also determines the role that magnetic fields played in structure formation. And finally, the cosmological era of the onset is relevant for the further fate of the magnetic fields because they might become erased in subsequent eras. For example, Lesch and Birk (1998) argue that large-scale magnetic fields cannot survive the highly resistive radiation era

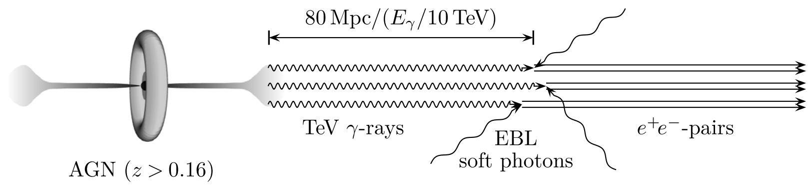

Fig. 3. Schematic illustration of the pair production caused by sufficiently distant Active Galactic Nuclei (AGN): The AGN emit highly energetic gamma rays of some 10 TeV. After a mean free path of several 10 Mpc the latter interact with the low energy photons of the Extragalactic Background Light (EBL) producing electron–positron pairs which are highly energetic themselves and propagate as collimated beams in the forward direction of the TeV-photons.

because of damping, and Schlickeiser (2012) claims that a possible previous magnetization vanished during the long lasting recombination era due to the absence of charged matter in the IGM.

These last two arguments support the suggestion of Tajima et al. (1992); Schlickeiser and Felten (2013); Yoon et al. (2014), and others, that spontaneously emitted field fluctuations in the intergalactic plasma are also suitable sources of seed fields. Such fluctuations are available ever since the IGM is in the plasma state that it still sustains today. Furthermore, they are not restricted by the dependence on particular structures or environments such as rotating gas clouds. They are present in any plasma and hence they have a high volume filling factor. Finally, the occurrence of field fluctuations is a well-established fact in plasma physics, so the proposal to consider them as sources of seed fields does not rely on exotic physics.

In order to gain progress and identify the relevant processes that generate the magnetic seed fields, the latter or at least their remnants must be investigated observationally. The major difficulty in gaining observational data, however, is that these fields are strongly modified by their environment. In particular, they already experienced the amplification process, so that they either have seized to exist in their previous form, or, if they are maintained by their source, they are too weak to be detectable among the superimposed strong fields. These circumstances lead to the significance of extragalactic fields: In the voids of the large-scale structure the magnetic seed fields might still exist in their original state, unaltered by disturbances of their surrounding environment (Neronov and Semikoz, 2009).

Unfortunately, no reliable direct observations of these fields were obtained so far. Using an indirect approach instead, Neronov and Vovk (2010) claim that the strength of magnetic fields in the IGM must be at least $3 \cdot 10^{-16}$ G. Their reasoning is based on the observed gamma ray spectrum from blazars: The latter emit TeV radiation in the direction of the Earth that interacts with the extragalactic background light (EBL) creating electron–positron pairs; this scenario is depicted in Fig. 3. Considering that the mean free path for $\gamma\gamma$-annihilation is about 80 Mpc/(E_γ/10 TeV), the active galactic nucleus (AGN) must have a redshift of at least 0.16 in order to be sufficiently distant for this process to be significant. Regarding the further fate of the highly relativistic pairs one might expect that they lose a fraction of their energy to the cosmic microwave background (CMB) by a cascade of inverse Compton scattering events. Since the resulting GeV photons propagate in the same direction as the initial pairs, the spectrum of every blazar should display a distinct peak in the GeV energy band. However, such a signal is not observed by the Fermi satellite. The conclusion that Neronov and Vovk (2010) draw from this is that the electron–positron pairs must be deflected by extragalactic magnetic fields, implying that also the scattered GeV photons are not collimated in the forward direction of the beam along the line of sight. Thus, the non-detection of the GeV signal is interpreted as an indirect proof for the existence of magnetic fields in the IGM. Based on these arguments the authors derived a lower bound of $B > 3 \cdot 10^{-16}$ G for the field strength in their quantitative analysis. These results rely on the assumption that the gamma-ray emission of blazars occurs continuously over long periods (10^6 years). Weakening this assumption, Dermer et al. (2011) found a more conservative lower bound of 10^{-18} G. But these conclusions are by no means compelling as Broderick et al. (2012) and Schlickeiser et al. (2012b) pointed out. They showed that the relativistic pair beam leads to plasma instabilities in the IGM with much lower growth rates than the inverse Compton cooling time. Hence, the pairs lose their energy by heating the IGM rather than by producing a GeV gamma ray signal. In view of this, the derivation of a lower limit for the intergalactic magnetic fields breaks down and becomes obsolete.

Since robust conclusions about the magnetic fields in the IGM cannot be drawn reliably from observations after all, theoretical predictions become all the more important. Tajima et al. (1992) employed the fluctuation–dissipation theorem to derive both the spectrum and the integrated total strength of the magnetic field fluctuations of a thermal equilibrium plasma. They found that the latter possesses a distinct peak at zero frequency, providing a level of $\langle B_0^2\rangle/8\pi = T(\omega_\mathrm{p}/c)^3/(2\pi^3)$ for the corresponding energy density. However, as discussed in the previous subsection, such an approach excludes complex-valued frequencies from the outset. In particular, aperiodic fluctuations with $\Re\omega = 0$ and with a non-vanishing imaginary part of the frequency are missed. The latter become all the more important in view of the newly discovered stable branch of the Weibel mode. Schlickeiser (2012) and Schlickeiser and Felten (2013) computed the fluctuation level anew taking this aperiodic mode into account. Assuming viscous damping, they derived a field strength of $1.5 \cdot 10^{-16}$ G in cosmic voids with a maximum coherence length of 10^{15} cm (corresponding to $3.2 \cdot 10^{-4}$ pc). Since the semi-empirical treatment of damping losses is not entirely satisfactory, Yoon et al. (2014) refined the calculations by deriving a spectral balance equation from first principles that self-consistently accounts for the competing effects of spontaneous emission, induced emission, and absorption. The resulting Kirchhoff-type radiation law is valid for arbitrary complex frequencies and allows a rigorous inclusion of the newly discovered aperiodic mode. The integrated spectrum of the magnetic field fluctuations yields a value of $6 \cdot 10^{-18}$ G for the IGM shortly after the reionization onset. Thus, spontaneous emission processes in plasmas may certainly be regarded as a possible source of seed fields.

1.4. Objectives of this work

The general intention pursued in this article is to resume, improve, and advance the outlined plasma fluctuation theory with particular emphasis on aperiodic fluctuations. In detail, the following three topics are addressed:

Objective 1. One of the cornerstones of statistical physics, and hence of modern physics in general, is the fluctuation–dissipation theorem (FDT). After its abstract formulation in the context of linear response theory (Callen and Welton, 1951; Kubo, 1957, 1966), it has since then been applied to a large number of rather different many-particle systems. Of course, plasmas are among them as well (Dougherty and Farley, 1960; Rostoker, 1961; Sitenko, 1967). In this context, the theorem establishes a relation between the spectrum of the field fluctuations on the one hand and a response function such as the dielectric or the conductivity tensor on the other hand. This does not only provide a connection between the random fluctuations and the dissipative properties of the plasma, but at the same time also between the microscopic configuration of the many-body system and a macroscopic observable.

The plasma-related version of the FDT found in the literature is restricted to weakly damped or weakly growing oscillations. The reason is that the derivations stated there crucially rely on the assumption of real-valued frequencies corresponding to the limit $\Im\omega \to 0$. As argued before, this approach excludes weakly propagating and aperiodic fluctuations from the outset. The fact that linear response theory makes heavy use of complex analysis, culminating in celebrated results like the Kramers–Kronig relations, is no actual objection to this statement. As long as the frequency is not introduced as a complex number from the beginning by performing a Laplace transform, all spectral quantities have no physical meaning for non-real frequencies, even if they are analytically continued to the complex plane for mathematical convenience. Therefore, a generalization of the FDT, valid for arbitrary complex frequencies, is indispensable. It is the purpose of this work to derive such an improved formulation of the theorem for relativistic equilibrium plasmas by starting from the Laplace transformed equations and by properly treating the frequency as an arbitrary complex number throughout the entire calculations. Both the magnetized case as well as unmagnetized limit will be considered.

Objective 2. It has already been emphasized that the spontaneously emitted field fluctuations in the IGM exist even in the absence of stimulating external means such as beams or shocks. This spontaneous self-excitation qualifies them as an ideal source of magnetic seed fields in the context of cosmic magnetization. Although a relativistic particle beam is not required for the fluctuation level of $6 \cdot 10^{-18}$ G cited above, it may enhance the latter significantly after all. For it is well known that an energetic particle beam can have tremendous effects on the background plasma and vice versa (Sitenko, 1967; Treumann and Baumjohann, 2001; Schlickeiser, 2002; Schlickeiser et al., 2012b; Krakau and Schlickeiser, 2014; Menzler and Schlickeiser, 2015). In particular, the relativistic particles can deposit a considerable fraction of their energy into their ambient medium by exciting instabilities. In this light, the earlier investigations can be developed further by implementing the influence of a highly relativistic electron–positron pair beam. As discussed before, such a scenario occurs quite naturally in the IGM due to the pair production caused by the TeV radiation of an AGN (Fig. 3).

It is the second objective of this work, therefore, to analyze how the spontaneously emitted aperiodic fluctuations in the IGM are affected by the $e^{\pm}$-beam induced by an AGN. The goal is to derive the corresponding spectral balance equation for the disturbed background plasma. It will be shown that the previous results are modified due to symmetry breaking effects of the beam and due to the free energy it provides such that the absorption coefficient becomes negative in certain wavenumber ranges of the spectrum. There, the field fluctuations are actually amplified. Considering the close link between the absorption coefficient and the growth rate, the occurrence of negative absorption can be interpreted as the partial transformation of the formerly stable mode into an instability. This may have deep implications for the total fluctuation level because one might expect it to be significantly higher than the previous estimates. Furthermore, the fluctuation spectrum computed in this article is of great importance for the transport of cosmic rays in the IGM due to wave–particle interactions. Mathematically, this connection becomes evident because the transport coefficients entering the Fokker–Planck equation are determined by the spectral distribution of the field fluctuations (Schlickeiser, 2011, 2015).

Objective 3. All in all, the previous discussions revealed that the stable branch of the Weibel mode is of considerable interest both from a purely theoretical point of view and from a more application oriented perspective. It is all the more a most disagreeable state of affairs that this mode has never been observed, yet. For the most part, the difficulties in this respect arise from the fact that unmagnetized plasmas are almost impossible to maintain in the laboratory. Moreover, the naturally occurring plasmas accessible to *in situ* measurements are, by and large, permeated by a magnetic field that affects the dynamics of the plasma to such an extent that the precondition of the mode is violated. In view of these circumstances, the only remaining alternative is to recourse to remote environments in the cosmos that are able to host the mode. Thus, the voids between the filaments of the large-scale structure gain relevance once more.

Regarding the pursuant investigations of the IGM, the most obvious source of information to resort to is electromagnetic radiation originating from a distant emitter. While the rays are traversing the turbulent field fluctuations, the latter might leave a characteristic footprint on the radiation profile that provides valuable insights into the microphysics of the IGM.

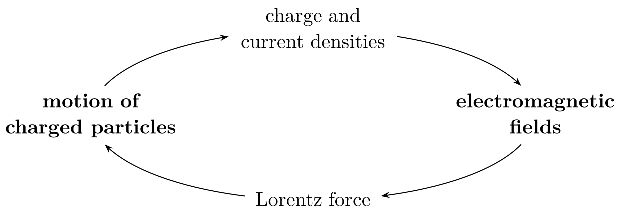

Fig. 4. Illustration of the fundamental complexity encountered in plasma physics due to the coupling of particles and fields: The motion of charged particles is caused by and causes electromagnetic fields; vice versa, the fields are generated by and generate the motion of charged particles.

Suitable diagnostic techniques in this respect are line broadening, dispersion and rotation measure as well as scintillation data.

In order to remedy the unsatisfactory predicament that an empirical confirmation of the stable Weibel mode is still pending and in order to pave the way for affirmative observational evidence, therefore, this article examines the applicability of the these methods by calculating the relevant density and velocity fluctuations in the IGM that are driven by the mode. It will be confirmed that transverse modes in isotropic and unmagnetized plasmas inevitably generate incompressible turbulence, i. e., these modes are not associated with density fluctuations. Consequently, dispersion measure and scintillation data measurements are not applicable in this case, and for all practical purposes the same also holds for the rotation measure as will be shown as well. Assuming that the neutrals in the IGM adopt the velocity profile of the protons by mutual charge exchange and elastic collisions (ambipolar diffusion), the feasibility of line broadening studies becomes the main focus of attention in this context. The *conditio sine qua non* for significantly enhanced broadening of the spectral line is that the mode-driven proton velocity fluctuations exceed the thermal level considerably. In view of this, augmented importance comes to the third major objective of this article: computing the wavenumber spectrum of these velocity fluctuations – which is also valuable in its own right as it characterizes the state of the microturbulence – and integrating the spectrum so as to determine if the resulting total value is high enough to qualify line broadening as a gateway to conclusive and compelling evidence for the mode.

1.5. Outline of this work

In order to ensure a coherent description, the pursuit of the designated objectives must be preceded by a complementary exposition of the underlying first principle equations, their premises, and a concise motivation of the concepts involved. Such an endeavor is aggravated by the fact that it is faced with an entangled conglomerate of concepts stemming from different branches of physics, each of them equipped with its own terminology. Moreover, another complication is the fundamental complexity that is inherent to any plasma and that manifests itself mathematically in the coupling of the Maxwell–Klimontovich system of equations. Physically, it can be attributed to the causal chain in which particles and fields are related to each other (see Fig. 4): Electromagnetic fields *induce* the motion of electric charges by means of the Lorentz force, while they themselves are also *induced by* the motion of charged particles. Vice versa, particle motion generates and is generated by electromagnetic fields. This twofold role of being both cause and effect (which is no *circulus in probando* due to the required initial conditions) constitutes the fundamental complexity encountered in plasma physics.

The structure of the first part of this work is guided by the task of disentangling these interwoven threads. This approach appears to be quite instructive, because it reveals which step in the derivation of the fluctuation theory arises in connection with electrodynamics, which one is of a statistical origin, and which one must be attributed to the unification of both. For instance, the notion of a mode arises in the context of electrodynamics, whereas the spontaneous fluctuations are an entirely statistical effect. Therefore, it is appropriate to introduce each concept in its native framework so as to obtain a more evident exposition, unburdened by the remaining topics.

Section 2 is devoted to the computation of electromagnetic fields in conductive media. The idea behind this outset is to demonstrate the generic procedure and motivate the underlying concepts rather than obtaining the fields of a specific plasma configuration. In particular, the notion of linear eigenmodes is introduced which is of vital importance for the remainder of this work. There, the fields under consideration will be fluctuating random fields in a plasma, and since these are governed by the same set of equations, the results obtained in Section 2 can immediately be adopted. This isolated treatment of the electromagnetic fields necessarily involves a certain degree of abstraction as no particular plasma model is specified, such as the magnetohydrodynamic or the kinetic one. The only assumption entering the calculations is that the fields are sufficiently small in order to justify a linear response approximation. However, the linear response coefficient, i. e., the conductivity tensor, appears as an unknown parameter that cannot be specified any further from within this isolated electrodynamic framework, but it must be determined by a microscopic model of matter. At this stage, therefore, it becomes evident very naturally why the initially omitted kinetic plasma model must eventually be included after all.

Section 3 addresses the relevant aspects of non-equilibrium statistical mechanics to this end. As far as the description of plasma fluctuations is concerned, the formalism developed by Klimontovich (1967, 1972, 1982, 1997) has proven to be very powerful and elegant, so the main purpose in this section is to introduce his approach to kinetic theory. It is instructive, however, to contrast his method of moments with the more common framework centered around the reduced s-particle distribution functions that enter the BBGKY-hierarchy. In the course of these elaborations, it will also be highlighted what assumptions and preconditions are adopted in order to achieve statistical closure by truncating the chain of equations. Finally, the origin of spontaneous fluctuations will be elucidated and traced back to the discrete particle structure of a plasma.

Section 4 unifies the preceding results into a coherent theory of plasma fluctuations. This entails important improvements: Firstly, the linear response relation between currents and fields, that was adopted in Section 2 solely on account of a certain plausibility, can now be justified and placed on solid ground because the statistical equations derived in Section 3 state exactly the required linear relation. Additionally, they even express the response tensor, a formally unspecified quantity, in terms of the distribution function. Secondly, the statistical formalism is improved by accounting for the full relativistic dynamics of the electromagnetic fields instead of restricting the theory to non-relativistic Coulomb potentials. A further feature that was not yet addressed up to this point are spatial and temporal symmetries of the correlation functions. In this regard, the notions of homogeneous and stationary turbulence will be discussed in order to provide the corresponding equations for the inverse Fourier–Laplace transform that respects these symmetries.

Section 5 is dedicated to the objective of deriving a generalized formulation of the fluctuation–dissipation theorem. The results are valid for arbitrary values of the complex frequency, so they apply to aperiodic and weakly propagating fluctuations as well. Moreover, the computations abide the laws of special relativity. This does not only admit the description of highly relativistic particle distributions, but also the proper analysis of fluctuations with phase speeds comparable to the speed of light in vacuum. The theorem is established for the more general case of a magnetized plasma first and the limit of a vanishing magnetic background field will be considered afterwards. In order to provide further support for the additional terms entering the theorem, the dissipated energy losses are computed. Since the latter are not only determined by the anti-hermitian part of the Maxwell tensor but also by its hermitian part if the frequency is complex, this confirms the necessity for additional terms in the fluctuation–dissipation theorem.

Section 6 contains the analysis of the spontaneously emitted field fluctuations in the IGM. In order to investigate the impact of a highly relativistic electron–positron pair on the background plasma, the corresponding spectral balance equation is derived. The underlying ratio between beam and background density amounts to some 10^{-15}. While this value poses too large of an obstacle for numerical simulations, it provides an excellent foundation for the application of perturbation theory. For the sake of utmost generality, no assumptions are made regarding the relative orientation of wavevector and beam. The resulting Kirchhoff-type radiation law takes the simultaneous effects of spontaneous emission and induced absorption into account. On that basis, the absorption coefficient is inspected for negative values. Effectively, this negative absorption corresponds to an amplification of the fluctuations and, thus, constitutes an entirely new feature in comparison with the unperturbed case that is solely due to the presence of the beam. In order to answer this question in a more profound way than just a simple yes-or-no statement, the wavenumbers compatible with negative absorption are located in the spectrum, and a tangible criterion for the occurrence of amplification is derived. Despite its originally quantitative form, the subsequent discussion shows that the criterion is accessible to an illustrative geometrical interpretation.

Section 7 elucidates the prospects of gathering observational evidence for the stable branch of the Weibel mode. It is proven anew that a transverse mode operating in an isotropic unmagnetized plasma is not associated with density fluctuations. This immediately excludes experimental techniques employing dispersion measure and scintillation data. By estimating the corresponding rotation measure, it is shown that this observable must be discarded as well. In this light, the focus of attention shifts all the more to line broadening effects induced by the mode. Thus, the spectrum of the mode-driven proton velocity fluctuations is computed. Since this spectrum also offers some valuable insight into the state of the microturbulence, the power law spectral indices for the relevant wavenumber ranges are computed by analyzing the asymptotic behavior. Finally, the contributions from the entire spectrum are integrated in order to infer the velocity fluctuations in real space, because their magnitude determines whether line broadening studies are a suitable means to obtain an empirical confirmation for the existence of the mode.

Section 8 summarizes the most important results attained in conjunction with the three research objectives, gives an outlook on the questions arising from them, and discusses the opportunities for expedient further developments and investigations.

2. Electromagnetic fields in the linear response approximation

Outline. This section is devoted to a brief review of plasma electrodynamics. The primary purpose is to motivate the notion of an eigenmode, starting from first principles. This will also provide the relevant equations required in subsequent parts of this work. The main procedure followed here is to carry out a formal linear response expansion that yields a relation between the current density and the fields, i. e., Ohm's law. Complemented by the latter, Maxwell's equations become a closed set of equations that can readily be solved by performing a Fourier–Laplace transform.

References. The arguments brought forward in this section are to a large extent inspired by the monograph by Melrose and McPhedran (1991), but the exposition also benefited from the books by Akhiezer et al. (1975); Alexandrov et al. (1984); Ichimaru (1992), and Sitenko (1982).

2.1. Electromagnetic field equations

The very foundation of the entire theory of classical electrodynamics are Maxwell's equations for the electric field $\boldsymbol{E}_{\mathbf{r},t}$ and the magnetic field $\boldsymbol{B}_{\mathbf{r},t}$. They relate the fields to each other and to their material sources, the charge density $\rho_{\mathbf{r},t}$ and the current density $\boldsymbol{J}_{\mathbf{r},t}$:

$$\nabla \times \boldsymbol{E}_{\mathbf{r},t} = -\frac{1}{c}\frac{\partial \boldsymbol{B}_{\mathbf{r},t}}{\partial t}\,, \tag{1}$$

$$\nabla \times \boldsymbol{B}_{\mathbf{r},t} = \frac{4\pi}{c}\boldsymbol{J}_{\mathbf{r},t} + \frac{1}{c}\frac{\partial \boldsymbol{E}_{\mathbf{r},t}}{\partial t}\,, \tag{2}$$

$$\nabla \cdot \boldsymbol{E}_{\mathbf{r},t} = 4\pi\rho_{\mathbf{r},t}\,, \tag{3}$$

$$\nabla \cdot \boldsymbol{B}_{\mathbf{r},t} = 0\,. \tag{4}$$

These equations hold in vacuum as well as in the presence of media. In the latter case, $\rho_{\mathbf{r},t}$ and $\boldsymbol{J}_{\mathbf{r},t}$ do not only describe the free, experimentally controllable charges but also the ones that constitute the medium and give rise to possible polarization and magnetization effects (Melrose and McPhedran, 1991; Sitenko, 1967). In any case, charge and current density are not independent of each other but are related due to the conservation of electric charge as expressed by the continuity equation

$$\partial\rho_{\mathbf{r},t}/\partial t + \nabla \cdot \boldsymbol{J}_{\mathbf{r},t} = 0\,. \tag{5}$$

The relationship between the electromagnetic fields and the electric charge carriers giving rise to charge and current densities is twofold: On the one hand, the charges generate fields as can be seen by Maxwell's equations above. On the other hand, they are also the recipients of the effects emanating from electromagnetic fields because they are subject to the Lorentz force. Since in a plasma there are highly movable charge carriers available, any electromagnetic field will inevitably influence their motion by means of the Lorentz force and thereby induce further charge and current densities in the medium. For this reason, the electromagnetic fields on the one hand and the charge and current densities on the other hand are coupled to each other, usually in a highly nonlinear manner. From a theoretical point of view, this poses the major difficulty in solving Maxwell's equations. The theory of electrodynamics in plasmas has to take these circumstances into account that charge carriers and fields perpetually influence each other.

However, not all of the electromagnetic fields and currents are due to the plasma itself. In fact, one is usually faced with situations in which additionally there are also external influences present which are (at least approximately) independent of the plasma behavior. Examples are the electromagnetic fields imposed on laboratory plasmas or the magnetic field of the Sun pervading the heliosphere. The task then is to compute the ensuing total field consisting of the external stimulus as well as the plasma response. Although the considerations in this section are sufficiently general to be independent of any particular plasma model, the plasma is nevertheless assumed to be fully ionized. As a consequence, the presence of bound charges can be excluded.

As pointed out by Melrose and McPhedran (1991), there are two customary approaches to describe the electromagnetic response in media: One possibility is to introduce polarization and magnetization vectors. By definition, they are the induced electric and magnetic dipole moment per unit volume, respectively. However, this is only suitable for static responses as these entities cannot even be defined in an unambiguous way for fluctuating fields, and, more importantly, they become ill-defined for spatially dispersive media such as plasmas (ibid.). Since the electromagnetic processes in a plasma are highly time varying, a different approach is appropriate. In short, its main idea is to describe the plasma response in terms of the induced current density and to specify it as a functional of the fields, $\boldsymbol{J}_{\text{ind}} = \boldsymbol{J}_{\text{ind}}\{\boldsymbol{E}, \boldsymbol{B}\}$. External influences simply appear as a field-independent contribution in this generalized Ohm's law, $\boldsymbol{J}\{\boldsymbol{E}, \boldsymbol{B}\} = \boldsymbol{J}_{\text{ext}} + \boldsymbol{J}_{\text{ind}}\{\boldsymbol{E}, \boldsymbol{B}\}$. For temporally and spatially dispersive media such as plasmas, this dependency of the currents on the fields can be formulated much easier and much more elegantly in Fourier–Laplace space.

2.2. The fields in Fourier–Laplace space

Intuitively, the most obvious attempt to gain knowledge about the electromagnetic fields in the plasma is a normal mode analysis, i. e., the search for solutions of Maxwell's equations having the exponential space and time dependency $\exp[\imath(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)]$. However, this approach leads to singularities, most prominently in form of the factor $(\omega - \boldsymbol{k}\cdot\boldsymbol{v})^{-1}$ corresponding to Cerenkov resonance. Although a proper and meaningful treatment of these singularities is possible (see van Kampen, 1955), it remains somewhat tedious. A more natural way to handle singularities is to apply the tools provided by complex analysis. In this spirit, Landau (1946) refined the normal mode analysis by performing a Fourier transform in the spatial variable and a Laplace transform in time. Although the procedures used by van Kampen and Landau are equivalent (Case, 1959), for the present purpose of examining aperiodic modes it is advantageous to follow Landau here.

In order to remain as close as possible to the normal mode formalism, the kernel of the Laplace transform will be denoted by $\exp(\imath\omega t)$, making the complex number ω the conjugate variable with respect to time rather than the more conventional choice $s = -\imath\omega$ predominantly found in the mathematical literature. Accordingly, the Fourier–Laplace transform of a field $f_{\boldsymbol{r},t}$ will be denoted by $f_{\boldsymbol{k},\omega}$. A concise definition including prefactor conventions and other technicalities as well as a brief discussion of some basic properties to be employed later on are summarized in Appendix A.1. Applying the transformation rules (A.3)–(A.6), Maxwell's equations as well as the continuity equation can readily be converted into the corresponding algebraic equations in Fourier–Laplace space, yielding

$$\boldsymbol{k} \times \boldsymbol{E}_{\boldsymbol{k},\omega} = -(\imath/2\pi c)\boldsymbol{B}_{\boldsymbol{k},t=0} + (\omega/c)\boldsymbol{B}_{\boldsymbol{k},\omega}\,, \tag{6}$$

$$\boldsymbol{k} \times \boldsymbol{B}_{\boldsymbol{k},\omega} = -(4\pi\imath/c)\boldsymbol{J}_{\boldsymbol{k},\omega} + (\imath/2\pi c)\boldsymbol{E}_{\boldsymbol{k},t=0} - (\omega/c)\boldsymbol{E}_{\boldsymbol{k},\omega}\,, \tag{7}$$

$$\boldsymbol{k} \cdot \boldsymbol{E}_{\boldsymbol{k},\omega} = -4\pi\imath\rho_{\boldsymbol{k},\omega}\,, \tag{8}$$

$$\boldsymbol{k} \cdot \boldsymbol{B}_{\boldsymbol{k},\omega} = 0\,, \tag{9}$$

and

$$(\imath/2\pi)\rho_{\boldsymbol{k},t=0} - \omega\rho_{\boldsymbol{k},\omega} + \boldsymbol{k}\cdot\boldsymbol{J}_{\boldsymbol{k},\omega} = 0\,. \tag{10}$$

Taking the scalar product of the first two equations with the wavevector and making use of the last three equations leads to

$$\boldsymbol{k} \cdot \boldsymbol{B}_{\boldsymbol{k},t=0} = 0 \tag{11}$$

$$\boldsymbol{k} \cdot \boldsymbol{E}_{\boldsymbol{k},t=0} = -4\pi\imath\rho_{\boldsymbol{k},t=0}\,. \tag{12}$$

This is the Fourier–Laplace space formulation of the well known fact that Maxwell's divergence equations simply have the status of initial conditions. Vice versa, it can now be seen that equations (8) and (9) are obsolete as they are implied by (6)–(7) and the initial conditions. Furthermore, the continuity equation (10) can be regarded as a mere definition of the charge density $\rho_{\boldsymbol{k},\omega}$. Therefore, only the two remaining Eqs. (6)–(7) are needed for the further development of the theory. They can be decoupled by solving Faraday's law of induction for the magnetic field so as to eliminate it in Ampère's law and to keep the electric field as the only unknown function:

$$\boldsymbol{B}_{\boldsymbol{k},\omega} = \frac{c\boldsymbol{k}\times\boldsymbol{E}_{\boldsymbol{k},\omega}}{\omega} + \frac{\imath\boldsymbol{B}_{\boldsymbol{k},t=0}}{2\pi\omega} \overset{(11)}{=} \frac{c}{\omega}\boldsymbol{k}\times\left(\boldsymbol{E}_{\boldsymbol{k},\omega} - \frac{\imath\boldsymbol{k}\times\boldsymbol{B}_{\boldsymbol{k},t=0}}{2\pi ck^2}\right). \tag{13}$$

This reduction to only one vector-valued variable resembles the implementation of electrodynamic potentials in the *temporal gauge*, characterized by the condition that the scalar potential vanishes identically (Melrose and McPhedran, 1991). But relation (13) does not only express $\boldsymbol{B}_{\boldsymbol{k},\omega}$ in terms of $\boldsymbol{E}_{\boldsymbol{k},\omega}$, thus allowing one to eliminate the former and thereby to decouple the set of equations, it also states that the magnetic field is purely transversal even inside a conductive medium. While the electric field is also transversal in vacuum, this is no longer true in plasmas as will become evident shortly. The only remaining equation (7) provides the necessary connection between currents and fields. Eliminating the magnetic field by virtue of (13) yields

$$\left[\hat{\mathbf{1}} - \frac{k^2c^2}{\omega^2}\left(\hat{\mathbf{1}} - \frac{\boldsymbol{k}\otimes\boldsymbol{k}}{k^2}\right)\right]\cdot\boldsymbol{E}_{\boldsymbol{k},\omega} = \boldsymbol{E}^{\mathrm{ini}}_{\boldsymbol{k},\omega} - \frac{4\pi\imath}{\omega}\boldsymbol{J}_{\boldsymbol{k},\omega}\,, \tag{14}$$

where the initial fields were collected in the new entity

$$\boldsymbol{E}^{\mathrm{ini}}_{\boldsymbol{k},\omega} = \frac{\imath}{2\pi\omega}\left(\boldsymbol{E}_{\boldsymbol{k},t=0} - \frac{c}{\omega}\boldsymbol{k}\times\boldsymbol{B}_{\boldsymbol{k},t=0}\right). \tag{15}$$

Thus far, the coupled set of differential equations was transformed into a single algebraic equation for the electric field only. As outlined in the beginning, the remaining task is to account for the polarization and magnetization effects due to the presence of the plasma by specifying the relation $\boldsymbol{J} = \boldsymbol{J}(\boldsymbol{E})$. Once this is accomplished, the above equation can be solved for the field $\boldsymbol{E}$.

2.3. Plasma response

The particulars about the medium have not entered the derivations so far. In this sense, the theory presented here is independent of the underlying model of the medium. Surprisingly, to a large extent this will still be the case after writing down an explicit equation for the functional dependency $\boldsymbol{J} = \boldsymbol{J}(\boldsymbol{E})$ in this subsection. It is not before the next section that a plasma model will be specified, which in this work will be a kinetic one.

The main idea is to perform a Taylor expansion of the current density around $\boldsymbol{E} = 0$. Applying the summation convention, it reads

$$\begin{aligned}\boldsymbol{J}(\boldsymbol{E}) &= \sum_{n=0}^{\infty}\frac{1}{n!}\,\frac{\partial^n\boldsymbol{J}(\boldsymbol{E}=0)}{\partial E_{i_1}\cdots\partial E_{i_n}}\,E_{i_1}\cdots E_{i_n}\\ &= \underbrace{\boldsymbol{J}(\boldsymbol{E}=0)}_{\substack{\text{spontaneous}\\ \text{current}}} + \underbrace{\frac{\partial\boldsymbol{J}(\boldsymbol{E}=0)}{\partial E_i}\,E_i}_{\substack{\text{linear response}\\ \text{current}}} + \underbrace{\frac{1}{2}\,\frac{\partial^2\boldsymbol{J}(\boldsymbol{E}=0)}{\partial E_i\,\partial E_j}\,E_iE_j + \cdots}_{\substack{\text{nonlinear response}\\ \text{current}}}\,.\end{aligned} \tag{16}$$

The zeroth-order term is independent of the stimulus and describes, in this sense, spontaneous, non-induced currents. These can be due to external influences, for example, or the random noise occurring in every many-particle system. The first-order term describes the linear response of the medium to the applied field. If the latter is not too strong, it is plausible (although by no means compelling), to assume that the terms of higher order $n > 1$, i. e., the nonlinear response terms, become smaller and smaller with increasing order and can thus be neglected above a finite maximum order N. In the context of turbulent plasma fields, this is the basic assumption of weak turbulence theory. In the linear approximation the maximum order is $N = 1$, which leads to a generalized formulation of Ohm's law.

Up to this point, only the dependency of the current on the electric field driving it was considered. Now, attention must be paid to the dependence on time and position. It is clear that, in general, the response depends on the temporal behavior of the stimulus. For example, if the stimulus is oscillating, then the response will probably be different to the one induced by a constant stimulus. Furthermore, the response at some time will also depend on the stimuli at earlier times. The water waves excited by a stone thrown into a lake, for instance, will still be present some time after the stone has hit the surface. Similar arguments also hold for the spatial coordinates because the response at some point in space can be excited by a stimulus occurring at some other location. Again, the water waves excited by a stone are an illustrative example.

According to the principle of causality, the effect cannot precede the cause, implying that the current $\boldsymbol{J}(t)$ depends only on values of the field $\boldsymbol{E}(t')$ for times $t' \leq t$. In view of this restriction, Eq. (16) must be rewritten in the more precise form

$$\begin{aligned} \boldsymbol{J}_{\boldsymbol{r},t} &= \boldsymbol{J}^{\mathrm{sp}}_{\boldsymbol{r},t} + \int \frac{\mathrm{d}^3 r'}{(2\pi)^3} \int_{-\infty}^{t} \frac{\mathrm{d}t'}{2\pi}\, \boldsymbol{\sigma}^{(1)i}_{\boldsymbol{r}\boldsymbol{r}'tt'} E^i_{\boldsymbol{r}'t'} \\ &\quad + \int \frac{\mathrm{d}^3 r'}{(2\pi)^3} \int \frac{\mathrm{d}^3 r''}{(2\pi)^3} \int_{-\infty}^{t} \frac{\mathrm{d}t'}{2\pi} \int_{-\infty}^{t} \frac{\mathrm{d}t''}{2\pi}\, \boldsymbol{\sigma}^{(2)ij}_{\boldsymbol{r}\boldsymbol{r}'\boldsymbol{r}''tt't''} E^i_{\boldsymbol{r}'t'} E^j_{\boldsymbol{r}''t''} + \cdots \quad . \end{aligned} \tag{17}$$

This time, the unknown derivatives of $\boldsymbol{J}$ with respect to E_i were expressed as generic coefficients $\boldsymbol{\sigma}^{(n)ij\dots}$, the response tensors. These quantities contain the entire information about the electromagnetic properties of the plasma. In a macroscopic theory they appear as material parameters which cannot be specified any further, but a microscopic plasma model must determine them in terms of appropriate small scale parameters. The higher the values of the response tensors become, the higher is the resulting current density. Therefore, the response tensors are called conductivity tensors in this context.

If the system is not aging, i. e., if the material properties are not changing with time, then it is homogeneous in time. This means that it does not matter at what time an observation takes place, all that matters is the *time difference* between the application of the stimulus and the occurrence of the response: $\boldsymbol{\sigma}(t, t') = \boldsymbol{\sigma}(t - t')$. Counter examples are roasting metals or materials that break after they have been bent elastically for a couple of times. In the same fashion, in a spatially uniform plasma the response does not depend on the absolute position in space but only on its relative location with respect to the stimulus: $\hat{\boldsymbol{\sigma}}(\boldsymbol{r}, \boldsymbol{r}') = \hat{\boldsymbol{\sigma}}(\boldsymbol{r} - \boldsymbol{r}')$. Due to these symmetries of the conductivity tensors, their arguments can be substituted as the new integration variables, thereby passing the entire $(\boldsymbol{r}, t)$-dependence on to the powers of the electric field. The Fourier–Laplace transform of the latter can be carried out by employing the respective convolution theorem for Fourier and Laplace transforms as well as the shift theorems for translations (Papoulis, 1962; Doetsch, 1974):

$$\begin{aligned} \boldsymbol{J}_{\boldsymbol{k},\omega} &= \boldsymbol{J}^{\mathrm{sp}}_{\boldsymbol{k},\omega} + \boldsymbol{\sigma}^{(1)i}_{\boldsymbol{k},\omega} E^i_{\boldsymbol{k},\omega} \\ &\quad + \int \mathrm{d}^3 k' \int_{-\infty+\imath s}^{\infty+\imath s} \mathrm{d}\omega'\, \boldsymbol{\sigma}^{(2)ij}_{\boldsymbol{k}',\boldsymbol{k}-\boldsymbol{k}',\omega',\omega-\omega'} E^i_{\boldsymbol{k}'\omega'} E^j_{\boldsymbol{k}-\boldsymbol{k}',\omega-\omega'} + \cdots \quad . \end{aligned} \tag{18}$$

The wavevector- and frequency-dependent susceptibilities appearing here are defined as their respective counterpart from real space, on which a Fourier–Laplace transform was performed for every space–time coordinate individually, i. e.,

$$\boldsymbol{\sigma}^{(2)ij}_{\boldsymbol{k}\boldsymbol{k}'\omega\omega'} \equiv \int \frac{\mathrm{d}^3 r}{(2\pi)^3} \int \frac{\mathrm{d}^3 r'}{(2\pi)^3} \int_0^\infty \frac{\mathrm{d}t}{2\pi} \int_0^\infty \frac{\mathrm{d}t'}{2\pi}\, \mathrm{e}^{-\imath\boldsymbol{k}\cdot\boldsymbol{r}} \mathrm{e}^{-\imath\boldsymbol{k}'\cdot\boldsymbol{r}'} \mathrm{e}^{\imath\omega t} \mathrm{e}^{\imath\omega' t'} \boldsymbol{\sigma}^{(2)ij}_{\boldsymbol{r}\boldsymbol{r}'tt'}\,, \tag{19}$$

and accordingly for all other orders. Comparing (17) and (18) it becomes manifest that the formulation of the functional dependency $\boldsymbol{J}(\boldsymbol{E})$ is indeed, as claimed before, much simpler in the spectral domain.

Since the nth-order term in the response expansion is the n-fold product of space- and time-dependent field components, it becomes an n-fold convolution in Fourier–Laplace space. As such it can be interpreted as the resonant interaction of n different wave modes because one can always write

$$\int \mathrm{d}^3 k'\, f(\boldsymbol{k}') g(\boldsymbol{k} - \boldsymbol{k}') = \int \mathrm{d}^3 k' \int \mathrm{d}^3 k''\, \delta(\boldsymbol{k} - \boldsymbol{k}' - \boldsymbol{k}'') f(\boldsymbol{k}') g(\boldsymbol{k}'') \tag{20}$$

and similar expressions for multiple convolutions. The right-hand side describes the occurrence of two independent wave modes $\boldsymbol{k}'$ and $\boldsymbol{k}''$, and the resonance condition $\boldsymbol{k}' + \boldsymbol{k}'' = \boldsymbol{k}$ for the beat wave corresponds to the conservation of momentum (Diamond et al., 2010). In the same fashion, a resonance condition for the frequencies appears, too, describing energy conservation.

This coupling of different wave modes is one of the key features of nonlinear theory, but does not occur in the linear regime. There, on the contrary, the principle of undisturbed superposition holds. It was already pointed out above that the linear approximation neglects the terms of order $N > 1$ on account of the assumption that the field strength $E_{\boldsymbol{k},\omega}$ and, hence, the wave energy density $w_{\boldsymbol{k},\omega} \propto E^2_{\boldsymbol{k},\omega}$ are sufficiently low. The corresponding linear response relation is a generalized formulation of Ohm's law that allows for spontaneous currents

$$\boldsymbol{J}_{\boldsymbol{k},\omega} = \boldsymbol{J}^{\mathrm{sp}}_{\boldsymbol{k},\omega} + \hat{\boldsymbol{\sigma}}_{\boldsymbol{k},\omega} \cdot \boldsymbol{E}_{\boldsymbol{k},\omega} \,. \tag{21}$$

Here, the more common notation for the tensor product was used instead of the expression $\boldsymbol{\sigma}^{(1)i}_{\boldsymbol{k},\omega} E^i_{\boldsymbol{k},\omega}$.

The results of this subsection were obtained solely by reasoning about general relations between stimulus and response and their temporal and spatial dependency. A more exhaustive justification of the response expansion and its approximative truncation as well as a specification of the conductivity tensor and the spontaneous current density must be provided by a particular plasma model. In this work, a kinetic model is employed, and since the current density is expressed in terms of the particle distribution function within this framework, the response expansion becomes an expansion of the distribution function rather than the current density itself. This is the formulation predominantly found in plasma physics (e. g., Sitenko, 1982; Davidson, 1972; Yoon, 2006), whereas an expansion of the polarization vector is the standard procedure in nonlinear optics (Boyd, 2008; Shen, 1984). For the present purposes, however, the goal of describing the polarization effects in the plasma is achieved: The response is described in terms of the induced current density $\hat{\boldsymbol{\sigma}}_{\boldsymbol{k},\omega} \cdot \boldsymbol{E}_{\boldsymbol{k},\omega}$.

2.4. The wave equation

The essence of Maxwell's equations in the spectral domain is given by (14). The difficulty in solving it for the fields is that, due to the polarization effects in a plasma, the current density implicitly depends on the fields. This issue has been solved in the last subsection by specifying an explicit equation for this dependency. Plugging the generalized Ohm's law (21) for the linear response into (14) results in the wave equation

$$\hat{\boldsymbol{\Lambda}}_{\boldsymbol{k},\omega} \cdot \boldsymbol{E}_{\boldsymbol{k},\omega} = \boldsymbol{E}^{\mathrm{ini}}_{\boldsymbol{k},\omega} - (4\pi\imath/\omega)\boldsymbol{J}^{\mathrm{sp}}_{\boldsymbol{k},\omega} \,, \tag{22}$$

where

$$\hat{\boldsymbol{\Lambda}}_{\boldsymbol{k},\omega} \equiv \hat{\mathbf{1}} - (k^2c^2/\omega^2)(\hat{\mathbf{1}} - \boldsymbol{k} \otimes \boldsymbol{k}/k^2) + (4\pi\imath/\omega)\hat{\boldsymbol{\sigma}}_{\boldsymbol{k},\omega} \tag{23}$$

is the Maxwell tensor describing the electromagnetic properties of the medium as they are predicted by the underlying plasma model. Since, in contrast to the total current $\boldsymbol{J}\{\boldsymbol{E}\}$ appearing in (14), neither the Maxwell tensor nor the initial fields nor the spontaneous current density depend on the field $\boldsymbol{E}_{\boldsymbol{k},\omega}$, the above equation allows a direct computation of the generated waves.

In order to solve the wave equation it is advantageous to introduce the adjugate $\hat{\boldsymbol{\lambda}}$ of the Maxwell tensor. (Temporarily, the $(\boldsymbol{k}, \omega)$-subscripts are suppressed here for the sake of readability.) By definition, it is the transpose of the cofactor matrix. The (i, j)th element of the latter can be found by deleting the ith row and the jth column of $\hat{\boldsymbol{\Lambda}}$, computing the determinant of the resulting matrix, and multiplying by $(-1)^{i+j}$, which leads to

$$\hat{\boldsymbol{\lambda}} \equiv \begin{pmatrix} \Lambda_{22}\Lambda_{33} - \Lambda_{32}\Lambda_{23} & \Lambda_{32}\Lambda_{13} - \Lambda_{12}\Lambda_{33} & \Lambda_{12}\Lambda_{23} - \Lambda_{22}\Lambda_{13} \\ \Lambda_{31}\Lambda_{23} - \Lambda_{21}\Lambda_{33} & \Lambda_{11}\Lambda_{33} - \Lambda_{31}\Lambda_{13} & \Lambda_{21}\Lambda_{13} - \Lambda_{11}\Lambda_{23} \\ \Lambda_{21}\Lambda_{32} - \Lambda_{31}\Lambda_{22} & \Lambda_{31}\Lambda_{12} - \Lambda_{11}\Lambda_{32} & \Lambda_{11}\Lambda_{22} - \Lambda_{21}\Lambda_{12} \end{pmatrix} . \tag{24}$$

It should be noted that the adjugate matrix exists and is properly defined irrespective of whether $\hat{\boldsymbol{\Lambda}}$ is invertible or singular, i. e., no matter whether the determinant of the Maxwell tensor vanishes or not. In this sense, the adjugate generalizes the concept of an inverse because

$$\hat{\boldsymbol{\Lambda}} \cdot \hat{\boldsymbol{\lambda}} = \hat{\boldsymbol{\lambda}} \cdot \hat{\boldsymbol{\Lambda}} = (\det \hat{\boldsymbol{\Lambda}})\hat{\mathbf{1}} \tag{25}$$

holds even in the case $\det \hat{\boldsymbol{\Lambda}} = 0$ (Shafarevich and Remizov, 2013). If, however, the invertibility criterion $\det \hat{\boldsymbol{\Lambda}} \neq 0$ is met, then relation (25) implies that the inverse of the Maxwell tensor can be written in terms of its adjugate and its determinant, $\hat{\boldsymbol{\Lambda}}^{-1} = \hat{\boldsymbol{\lambda}}/(\det \hat{\boldsymbol{\Lambda}})$. Remembering that the Maxwell tensor is a function of the wavevector and the complex frequency it becomes evident that the determinant of $\hat{\boldsymbol{\Lambda}}$ also depends on $\boldsymbol{k}$ and ω:

$$\Lambda(\boldsymbol{k}, \omega) \equiv \det(\hat{\boldsymbol{\Lambda}}_{\boldsymbol{k},\omega}) \,. \tag{26}$$

As a consequence, the question whether or not $\hat{\boldsymbol{\Lambda}}_{\boldsymbol{k},\omega}$ is invertible also depends on wavenumber and frequency. In fact, one can interpret the non-invertibility as a singularity of the inverse as a function of $\boldsymbol{k}$ and ω:

$$\hat{\boldsymbol{\Lambda}}^{-1}_{\boldsymbol{k},\omega} = \hat{\boldsymbol{\lambda}}_{\boldsymbol{k},\omega}/\Lambda(\boldsymbol{k}, \omega) \,. \tag{27}$$

According to these considerations, it is possible to solve the wave equation by applying the inverse Maxwell operator on both sides, leading to

$$\boldsymbol{E}_{\boldsymbol{k},\omega} = \frac{\hat{\boldsymbol{\lambda}}_{\boldsymbol{k},\omega} \cdot [\boldsymbol{E}^{\mathrm{ini}}_{\boldsymbol{k},\omega} - (4\pi\imath/\omega)\boldsymbol{J}^{\mathrm{sp}}_{\boldsymbol{k},\omega}]}{\Lambda(\boldsymbol{k}, \omega)} \,. \tag{28}$$

This equation states that there are two possible reasons for the presence of waves in the plasma: (1) There are spontaneous currents that generate waves, e. g., a charged test particle traversing the plasma or the currents associated with the particle discreteness. Hence, *new waves are generated by spontaneous currents*. (2) The other reason for the presence of waves is that they have already been there before, as stated by the initial conditions. In this case, the wave equation describes the *propagation (and modification) of pre-existing waves* through the medium. As one would intuitively expect, the occurring fields – in particular: their propagation and their extinction or amplification – also depend on the electromagnetic properties of the plasma, which are mathematically represented here by the inverse Maxwell tensor.

2.5. Normal modes

After computing the Fourier–Laplace transforms of the fields in the last subsection, the next step is to perform the inverse transforms in order to go back from the spectral domain to real space. To this end, it is assumed that the Laplace transforms possess a meromorphic continuation in fulfillment of the condition required by the inversion formula (A.11). Apart from a transient contribution, therefore, the electric field is given by

$$\boldsymbol{E}_{\boldsymbol{r},t} = -2\pi\imath \sum_{\varpi\in\mathbb{P}} \int \mathrm{d}^3k\ \mathrm{e}^{\imath(\boldsymbol{k}\cdot\boldsymbol{r}-\varpi t)}\, \mathrm{Res}_{\varpi}\left\{\boldsymbol{E}_{\boldsymbol{k},\omega}\right\}, \tag{29}$$

where equation (A.7) was applied to invert the Fourier transform. The symbol $\mathbb{P}$ denotes the set of all poles of the integrand. (For simplicity, no special notation was used to distinguish between $\boldsymbol{E}_{\boldsymbol{k},\omega}$ and its meromorphic continuation.) An analogue equation also holds for the magnetic field. Since $\boldsymbol{B}_{\boldsymbol{k},\omega}$ is given by Faraday's law of induction, viz. (13), the point $\omega = 0$ appears as a first-order pole. As such, the corresponding residue can be computed by means of the generic formula

$$\mathrm{Res}_{\varpi}\frac{f(\omega)}{g(\omega)} = \left.\frac{f(\omega)}{g'(\omega)}\right|_{\omega=\varpi}, \tag{30}$$

where $f(\omega)$ is supposed to be analytic and ϖ is a first order root of $g(\omega)$ with $g'(\varpi) \neq 0$. This procedure leads to

$$\begin{aligned}\boldsymbol{B}_{\boldsymbol{r},t} &= -2\pi\imath \sum_{\substack{\varpi\in\mathbb{P}\\ \varpi\neq 0}} \int \mathrm{d}^3k\ \mathrm{e}^{\imath(\boldsymbol{k}\cdot\boldsymbol{r}-\varpi t)}\, \mathrm{Res}_{\varpi}\left\{\frac{c}{\omega}\boldsymbol{k}\times\boldsymbol{E}_{\boldsymbol{k},\omega} + \frac{\imath\boldsymbol{B}_{\boldsymbol{k},t=0}}{2\pi\omega}\right\}\\ &\quad -2\pi\imath \int \mathrm{d}^3k\ \mathrm{e}^{\imath\boldsymbol{k}\cdot\boldsymbol{r}}\left(c\boldsymbol{k}\times\boldsymbol{E}_{\boldsymbol{k},\omega=0} + \frac{\imath\boldsymbol{B}_{\boldsymbol{k},t=0}}{2\pi}\right).\end{aligned} \tag{31}$$

According to (6), the two terms inside the parentheses of the second integral add up to zero. The initial value term in the first integral is holomorphic everywhere except at the origin $\omega = 0$ and, hence, its residue at $\varpi \neq 0$ vanishes. Therefore, the analogue to (29) for the magnetic field reads

$$\boldsymbol{B}_{\boldsymbol{r},t} = -2\pi\imath \sum_{\substack{\varpi\in\mathbb{P}\\ \varpi\neq 0}} \int \mathrm{d}^3k\ \mathrm{e}^{\imath(\boldsymbol{k}\cdot\boldsymbol{r}-\varpi t)}\, \mathrm{Res}_{\varpi}\left\{(c/\omega)\boldsymbol{k}\times\boldsymbol{E}_{\boldsymbol{k},\omega}\right\}. \tag{32}$$

Now that the electric and the magnetic field are expressed in terms of the residues of $\boldsymbol{E}_{\boldsymbol{k},\omega}$, the solution of the wave equation must be incorporated. In this regard, it is common practice in plasma theory to assume that both the numerator and the denominator in (28) are entire functions of the complex frequency (Montgomery and Tidman, 1964). *A priori*, there is no indication for this hypothesis to be true, so it must be validated once the spontaneous currents and the Maxwell operator have been specified by a particular plasma model. The merit of this assumption is that it entails two important consequences: Firstly, it is guaranteed that $\boldsymbol{E}_{\boldsymbol{k},\omega}$ possesses a meromorphic continuation in compliance with the requirements of the inversion formulas. Secondly, the poles of this meromorphic continuation are given by the roots of the denominator:

$$\Lambda(\boldsymbol{k},\omega) = 0\,. \tag{33}$$

Due to this equation the formerly independent variables ω and $\boldsymbol{k}$ become coupled. Hence, the solutions of the so-called dispersion relation (33) can be formulated by expressing the frequency as a function of the wavenumber:

$$\omega = \omega_M(\boldsymbol{k})\,. \tag{34}$$

Usually, there are several different functions $\omega_M(\boldsymbol{k})$ solving equation (33) and the index M is used to label these modes, here. For a more detailed discussion of a great variety of wave modes the reader is referred to the monographs of Gary (1993); Melrose (1986), and Stix (1992). By virtue of the inversion formulas (29) and (32) the electromagnetic fields in the medium can be expressed as

$$\boldsymbol{E}_{\boldsymbol{r},t} = \sum_M \int \mathrm{d}^3k\ \mathrm{e}^{\imath(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_M(\boldsymbol{k})t)}\, \boldsymbol{E}^{(M)}_{\boldsymbol{k}}, \tag{35}$$

$$\boldsymbol{B}_{\boldsymbol{r},t} = \sum_M \int \mathrm{d}^3k\ \mathrm{e}^{\imath(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_M(\boldsymbol{k})t)}\, (c/\omega_M(\boldsymbol{k}))\boldsymbol{k}\times\boldsymbol{E}^{(M)}_{\boldsymbol{k}}, \tag{36}$$

where the corresponding spectral distribution of the amplitudes is given by

$$\boldsymbol{E}_{\boldsymbol{k}}^{(M)} = \mathrm{Res}_{\omega_M(\boldsymbol{k})}\left\{\frac{2\pi\imath\hat{\boldsymbol{\lambda}}_{\boldsymbol{k},\omega}\cdot[(4\pi\imath/\omega)\boldsymbol{J}_{\boldsymbol{k},\omega}^{\mathrm{sp}} - \boldsymbol{E}_{\boldsymbol{k},\omega}^{\mathrm{ini}}]}{\Lambda(\boldsymbol{k},\omega)}\right\}. \tag{37}$$

If the modes happen to be simple roots of the dispersion function $\Lambda(\boldsymbol{k},\omega)$, then they are simple poles of $\boldsymbol{E}_{\boldsymbol{k},\omega}$ so that the residues can be computed by means of (30) again:

$$\boldsymbol{E}_{\boldsymbol{k}}^{(M)} = \left.\frac{2\pi\imath\hat{\boldsymbol{\lambda}}_{\boldsymbol{k},\omega}\cdot[(4\pi\imath/\omega)\boldsymbol{J}_{\boldsymbol{k},\omega}^{\mathrm{sp}} - \boldsymbol{E}_{\boldsymbol{k},\omega}^{\mathrm{ini}}]}{\partial\Lambda(\boldsymbol{k},\omega)/\partial\omega}\right|_{\omega=\omega_M(\boldsymbol{k})}. \tag{38}$$

It is instructive to consider not only the total fields but also the isolated modes. The contribution of a given mode M to the overall electric field is

$$\boldsymbol{E}_{\boldsymbol{r},t}^{(M)} = \int \mathrm{d}^3k\, \mathrm{e}^{\imath(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_M(\boldsymbol{k})t)}\boldsymbol{E}_{\boldsymbol{k}}^{(M)}, \tag{39}$$

and accordingly for the magnetic field. One can immediately infer the Fourier transform which simply reads

$$\boldsymbol{E}_{\boldsymbol{k},t}^{(M)} = \mathrm{e}^{-\imath\omega_M(\boldsymbol{k})t}\boldsymbol{E}_{\boldsymbol{k}}^{(M)}. \tag{40}$$

Since the electric field is a real-valued function, its Fourier transform obeys the symmetry relation $\boldsymbol{E}_{\boldsymbol{k},t}^{(M)*} = \boldsymbol{E}_{-\boldsymbol{k},t}^{(M)}$. In particular, this must be fulfilled at $t = 0$, implying that the symmetry must already hold for $\boldsymbol{E}_{\boldsymbol{k}}^{(M)}$. Going back to non-vanishing times t again, one can further conclude that the phase obeys the symmetry relation as well:

$$\left(\boldsymbol{E}_{\boldsymbol{k}}^{(M)}\right)^* = \boldsymbol{E}_{-\boldsymbol{k}}^{(M)}, \qquad \omega_M^*(\boldsymbol{k}) = -\omega_M(-\boldsymbol{k}). \tag{41}$$

These equations are of no particular importance at this stage but they will be required later on so they are only presented here for further reference.[1]

Eqs.(35)–(37) are the final results for the fluctuating fields in the linear approximation. They state that the electromagnetic fields in the plasma are a superposition of plane waves of independent modes. The magnetic field is always transversal, whereas in general there is no such restriction regarding the orientation of the electric field. Furthermore, the previous finding that the plasma waves are either excited by spontaneous currents or otherwise are the remnants of already excited waves, still manifests itself in these equations. Once excited, the further fate of the waves is governed by the electromagnetic properties of the plasma. In a kinetic model, for instance, a positive slope of the distribution function with respect to the momentum variable may lead to an exponential amplification of the waves, granted that a sufficient supply of free energy is available. This situation is the classical counterpart of the quantum mechanical population inversion that enables the dominance of stimulated emission in lasers.

In order to confirm that the Maxwell tensor determines whether a wave is absorbed or amplified, one should remember that the roots of its determinant specify the dispersion relation $\omega = \omega_{(M)}(\boldsymbol{k})$ of the modes. Since the mode frequency $\omega_M = \Omega_M + \imath\gamma_M$ is a complex number, the waves are subject to an exponential amplitude modification over time because

$$\exp[\imath(\boldsymbol{k}\cdot\boldsymbol{r} - \omega_M(\boldsymbol{k})t)] = \exp[\gamma_M(\boldsymbol{k})t]\,\exp[\imath(\boldsymbol{k}\cdot\boldsymbol{r} - \Omega_M(\boldsymbol{k})t)]. \tag{42}$$

While the real part of the Laplace variable is the angular frequency of the wave, the imaginary part describes the amplitude modulation: Its sign determines whether the wave is amplified, $\gamma_M > 0$, or damped, $\gamma_M < 0$, and the absolute value $|\gamma_M|$ is the corresponding growth or decay rate. The mode with the greatest rate is the dominating one. These considerations also contain the stability criterion of linear theory: The plasma is stable if all of the modes are damped ones, because if only a single mode was amplified the field strength would grow beyond all boundaries. In practice, of course, infinitely large field strengths cannot occur, so this means that, as the field strengths become larger, the system evolves beyond the linear regime, and hence nonlinear effects become important that stop the exponential growth eventually.

According to these considerations, the two key features of normal modes are propagation and amplitude modulation. In order to classify the behavior of fluctuations and assess their impact on the plasma properties, it is instructive to distinguish

[1] In the context of symmetries one should also mention the Onsager relations for the response tensor (Onsager, 1931; Vanwormhoudt, 1965). They are a consequence of the reversibility exhibited by the dynamics at a microscopic level and they can be derived from the invariance of the dynamic equation, e. g. the classical equation of motion $\mathrm{d}\boldsymbol{p}/\mathrm{d}t = e[\boldsymbol{E} + (\boldsymbol{v}/c)\times\boldsymbol{B}]$, under the simultaneous inversion of time, momentum, and magnetic field (Melrose and McPhedran, 1991). In the Fourier domain, time inversion corresponds to frequency inversion because for any function $g(t)$ and its time-reversed counterpart $g_{\mathrm{rev}}(t) = g(-t)$ one obtains the following relation between their respective Fourier transforms:

$$\mathcal{F}\{g_{\mathrm{rev}}\}(\omega) = \int_{-\infty}^{\infty}\frac{\mathrm{d}t}{2\pi}\,\mathrm{e}^{\imath\omega t}g(-t) = \int_{-\infty}^{\infty}\frac{\mathrm{d}\tau}{2\pi}\,\mathrm{e}^{-\imath\omega\tau}g(\tau) = \mathcal{F}\{g\}(-\omega).$$

This allows for an easy reformulation of the symmetry displayed by the dynamic equation in terms of spectral quantities and eventually leads to the Onsager relations. In the case of a (unilateral) Laplace transform, however, an expression similar to the one above cannot be derived on account of the asymmetric integration limits 0 and ∞. In a broader perspective, the Laplace transform ignores the past by design because only the behavior of $g(t)$ for $t > 0$ enters $\mathcal{L}\{g\}$. The transform of $g_{\mathrm{rev}}(t) = g(-t)$, on the other hand, is completely determined by $g(t < 0)$, which is independent of $g(t > 0)$. Thus, one cannot expect to find a general relation between $\mathcal{L}\{g\}$ and $\mathcal{L}\{g_{\mathrm{rev}}\}$, so Laplace transforms are arguably inappropriate for the description of reversibility and, hence, the Onsager relations.

the limiting cases that one feature is much more pronounced than the other (see Fig. 1): Weakly damped or weakly amplified fluctuations are characterized by $|\Omega_M| \gg |\gamma_M|$, so they are waves that travel a distance of many wavelengths before their amplitude alters significantly. In the opposite case of weakly propagating waves with $|\gamma_M| \gg |\Omega_M|$, on the other hand, either growth or decay is the dominant effect in the sense that the amplitude varies considerably before the first oscillation period elapses. This case can degenerate to a situation in which there occurs no propagation anymore at all: $\Omega_M = 0$. Such aperiodic modes with a purely imaginary Laplace variable $\omega_M \in \imath\mathbb{R}$ describe the growth or decay of a field at rest.

3. Kinetic theory of charged particles

Outline. This section provides the basic results from non-equilibrium statistical mechanics that will be employed throughout the rest of this work. The main purpose here is to introduce the formalism developed by Klimontovich because it is particularly well suited for the description of plasma fluctuations. In order to motivate and comprehend his method of moments it is advantageous to contrast it with the more common formulation in terms of reduced s-particle distribution functions underlying the BBGKY-hierarchy. Just as the latter must be truncated in order to obtain a closed set of equations, the requirement of achieving statistical closure also arises in the analogous chain of equations derived by Klimontovich. The discussion of this issue introduces an approximation adequate for collisionless plasmas and, thus, reveals the corresponding preconditions and the restrictions they entail with respect to the range of validity. It is also shown that an ensemble of discrete point particles is always associated with natural statistical fluctuations and that these result in spontaneous noise.

References. The presentation of the theory in this section is mostly influenced by Balescu (1975, 1997); Klimontovich (1967, 1982, 1986, 1997), and Ichimaru (1992).

3.1. Ensemble averages

The system under consideration is a collection of N classical point particles of the same kind, so the distinction between different particle species will be postponed to the next section for the sake of simplicity. The position and momentum vector of the ith particle are unified in the entity $x_i = (\boldsymbol{r}_i, \boldsymbol{p}_i)$. Then, in classical mechanics, the state of the system as a whole can be described by specifying a point $X = (x_1, \ldots, x_N)$ in the $6N$-dimensional phase space. Owing to the dynamical evolution of the system, the state vector is a function of time. All physical entities describing the system, such as density, kinetic energy, or angular momentum, they all are dynamical functions, i. e., they are functions of the phase space variables because they depend on the system state. Among those dynamical functions there is a distinguished one, the Hamiltonian $H(X, t)$. It determines the temporal evolution of the system because

$$\dot{b} = \partial b/\partial t + [\![b, H]\!] \tag{43}$$

holds for any dynamical function $b(X, t)$, where $[\![\cdot\,,\cdot]\!]$ denotes the Poisson bracket. In particular, by successively setting b to be every one of the canonical coordinates and momenta, one obtains Hamilton's equations of motion and, hence, an equation for the evolution of the state vector $X(t)$ itself. In this fashion, the future fate of the system is determined exactly for all times in terms of the $6N$ initial conditions.

Although this description is valid for arbitrarily high N, it is of no direct use for macroscopic systems with $N \gg 1$, such as an astrophysical plasma containing a vast amount of electrons and protons or the huge number of molecules constituting the objects of everyday life. (In the literature one usually mentions Avogadro's number $N_{\mathrm{A}} = 6 \cdot 10^{23}$ as a reference value at this point.) Basically, there are three reasons for this lack of practical usability: (1) It is not possible to gather the entire set of initial conditions for every single particle. (2) It is not possible to solve the coupled system of all $6N$ equations of motion, neither analytically nor numerically. (3) Even if the equations of motion could be solved, it would not be possible to gain any physical insight from them in this form. In sum, it is neither possible nor desirable to solve the exact microscopic equations of classical mechanics.

What is desirable, in contrast, is the deduction of a macroscopic description in terms of a few degrees of freedom such as temperature, pressure, or density. It is clear from the outset, that this requires an intended loss of microscopic information. For example, the macroscopic specification of a temperature is compatible with a great many constellations of the particles at the microscopic level. Surprisingly, it is possible to obtain a consistent theory at the macroscopic level of reduced information. For example, it suffices to specify the temperature distribution at one instant of time and to deduce the temperature distribution at any later time. The great success of macroscopic theories such as thermodynamics or hydrodynamics proves this. The question thus arises how to link both levels of description, i. e., how to process the information loss. Mathematically, it is an operation that assigns the macroscopic pendant B to a given microscopic dynamical function $b(X)$, thereby erasing the dependence on the microscopic variables: $b(X) \mapsto B = \langle b \rangle$. It should be noted, however, that this mapping does not always work in the opposite direction as there are macroscopic variables without a microscopic pendant. The most prominent example is entropy (Balescu, 1975).

In order to find an explicit expression for the averaging operator $\langle . . \rangle$, it is helpful to remember the previous statement that there are lots of different microscopic configurations in accord with a given macroscopic description of the system state. This means that the point in phase space describing the system state degenerates and becomes smeared out. Conceptually this means that all points in phase space must be treated on an equal footing, and quantitatively this can be implemented by means of a phase space distribution function $\varrho(X)$ that attributes a statistical weight to every point in phase space (Balescu, 1975). The average value of a dynamical function $b(X)$ is then given by the sum of all weighted contributions,

$$\langle b\rangle \equiv \int \mathrm{d}X\ \varrho(X)b(X) = \int \mathrm{d}x_1 \cdot\cdot \mathrm{d}x_N\ \varrho(x_1,\,..,x_N)b(x_1,\,..,x_N)\,. \tag{44}$$

This definition fulfills the requirement that the macroscopic entity $\langle b\rangle$ no longer depends on the microscopic state X. It can, however, still depend on other variables such as position or time. In order to obtain meaningful results one must demand that $\varrho(X)$ is non-negative and that it is normalized according to

$$\int \mathrm{d}X\ \varrho(X) = \int \mathrm{d}x_1 \cdot\cdot \mathrm{d}x_N\ \varrho(x_1,\,..,x_N) = 1\,. \tag{45}$$

Due to these equations, $\varrho(X)$ can be interpreted as a probability density, i. e., $\varrho(X)\,\mathrm{d}X$ is the probability that the system is in a state that is located inside the volume $\mathrm{d}X$ around X in phase space. In this statistical perspective, the notion of the system state has changed (ibid.). It is no longer specified by a point in phase space but rather by specifying the phase space distribution function $\varrho(X)$. Nevertheless, in principle, an exact description of the system is still possible: If its mechanical state is given by $X'(t) = (x_1'(t),\,..,x_N'(t))$, then the corresponding phase space distribution function reads

$$\varrho(X,t) = \delta\big(X - X'(t)\big) = \prod_{i=1}^{N} \delta\big(x_i - x_i'(t)\big)\,. \tag{46}$$

But this is just a formal result because the exact trajectory $X'(t)$ is unknown in general. What is known, however, is the smeared out macroscopic state, and, as outlined above, this reduced information is the purpose of introducing the concept of a phase space distribution function in the first place. The equilibrium distribution of the microcanonical ensemble treated in thermodynamics may serve as an example. There, the prescription of the total internal energy is a macroscopic constraint implying that the distribution function vanishes everywhere except on a hypersurface corresponding to the given energy. In addition, the principle of equal a priori probabilities dictates that the value of ϱ must be constant for every point on this surface (see Fig. 5).

In view of the large number of particles and the purposely ignored exact configuration, the phase space distribution function is assumed to be invariant with respect to the permutation of particles even in this classical, non-quantum framework (Balescu, 1975):

$$\varrho(\,..,x_i,\,..,x_j,\,..) = \varrho(\,..,x_j,\,..,x_i,\,..)\,. \tag{47}$$

As outlined above, the phase space density establishes a link between the microscopic and the macroscopic level of description by equipping any dynamical function with its average that implements the required information loss. Consequently, there is a deviation of the average from the exact value, called *fluctuation*:

$$\delta b(X) \equiv b(X) - \langle b\rangle\,. \tag{48}$$

Vice versa, any dynamical function can be decomposed into its average and a fluctuating component, $b = \langle b\rangle + \delta b$. The definition of the averaging operator immediately implies the following relations for dynamical functions $a(X)$ and $b(X)$ as well as constant numbers λ and μ:

$$\langle \lambda a + \mu b\rangle = \lambda\langle a\rangle + \mu\langle b\rangle \qquad \textit{linearity}, \tag{49}$$
$$\langle 1\rangle = 1 \qquad \textit{normalization}, \tag{50}$$
$$\langle\langle a\rangle\rangle = \langle a\rangle \qquad \textit{tower property}, \tag{51}$$
$$\langle \delta a\rangle = 0 \qquad \textit{fluctuation average}, \tag{52}$$
$$\langle \delta a\,\delta b\rangle = \langle ab\rangle - \langle a\rangle\langle b\rangle \qquad \textit{product rule}. \tag{53}$$

Since the average of a fluctuation always vanishes and since the fluctuation itself can usually be regarded as a random function, one uses the variance $\langle(\delta b)^2\rangle$ or the standard deviation $\langle(\delta b)^2\rangle^{1/2}$ as a quantitative measure in this context. In particular, the latter will be the relevant entity when the electromagnetic field fluctuations are computed in the subsequent sections.

The fact that the state of the system is described by the phase space distribution function implies that the latter also contains the temporal evolution. Since the dynamics are entirely governed by the Hamiltonian, the differential equation for ϱ involves H; this is the Liouville equation

$$\partial\varrho/\partial t = [\![H,\varrho]\!] \equiv \hat{L}\varrho\,, \tag{54}$$

where the Liouvillian $\hat{L} = [\![H,\ \cdot\,]\!]$ describes the particular interactions of the system under consideration (Balescu, 1975). In the literature, the Liouvillian is quite often defined with the imaginary unit as an additional prefactor in order to obtain identical equations for the classical and quantum case. In view of (43) one can interpret the Liouville equation as the conservation of the phase space density along the system trajectory, $\dot{\varrho} = 0$. By analogy with the Schrödinger equation

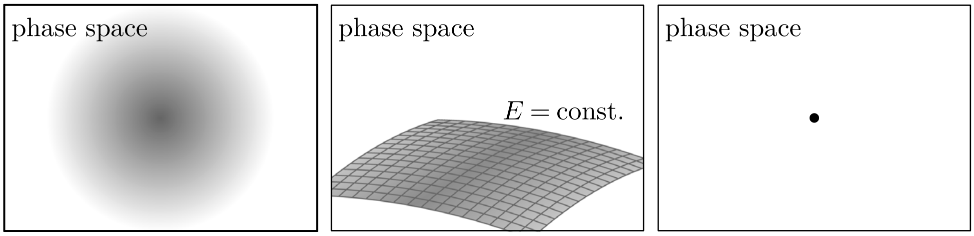

Fig. 5. Schematic illustration of the Liouville distribution function assigning a probability to every point in phase space (left panel). In the microcanonical ensemble every point on the hypersurface of the prescribed energy has the same probability (middle panel). If the microscopic configuration is known exactly, it defines a point in phase space and the corresponding distribution function is a δ-peak (right panel).

in quantum mechanics, a formal solution of Liouville's equation is given by $\varrho(X,t) = \exp(\hat{L}t)\varrho(X,0)$ if the Liouvillian does not depend on time explicitly (otherwise a time ordering operator would be involved). This is just a formal solution because in general the exponential of an operator is rather difficult to compute. However, by means of the same arguments as in quantum mechanics, it allows the definition of a Schrödinger and a Heisenberg type picture for the temporal evolution,

$$\langle b\rangle(t) = \int \mathrm{d}X\; b(X)\varrho(X,t) = \int \mathrm{d}X\; \tilde{b}(X,t)\tilde{\varrho}(X)\,, \tag{55}$$

depending on whether $\exp(\hat{L}t)$ is associated with the dynamical functions or the phase space distribution (ibid.). For the upcoming computations of certain averages it is useful to be equipped with both pictures.

3.2. The BBGKY hierarchy

Due to its huge dimensionality, the Liouville equation is not well suited for practical purposes and can only be solved in the plainest of cases. Thus, the need for deliberate simplifications arises, and along with it comes the necessity to employ reasonable approximations. In this regard, the following observation is useful: The average of many important dynamical functions can be expressed in terms of the *reduced one-particle distribution function*

$$f_1(x_1,t) \equiv N \int \mathrm{d}x_2 \cdot\cdot \mathrm{d}x_N\; \varrho(x_1,\ldots,x_N,t)\,. \tag{56}$$

For example, the knowledge of f_1 suffices to determine the average of the microscopic number density because

$$\begin{aligned}
n(\boldsymbol{r},t) &\equiv \left\langle \sum_{i=1}^{N} \delta(\boldsymbol{r}-\boldsymbol{r}_i)\right\rangle \\
&= \int \mathrm{d}x_1 \cdot\cdot \mathrm{d}x_N\; \varrho(x_1,\ldots,x_N,t) \sum_{i=1}^{N} \delta(\boldsymbol{r}-\boldsymbol{r}_i) \\
&= \sum_{i=1}^{N} \int \mathrm{d}x_1 \cdot\cdot \mathrm{d}x_{i-1}\; \mathrm{d}^3p_i\; \mathrm{d}x_{i+1} \cdot\cdot \mathrm{d}x_N\; \varrho(x_1,\ldots,x_N,t)\Big|_{x_i=(\boldsymbol{r},\boldsymbol{p}_i)} \\
&= \sum_{i=1}^{N} \int \mathrm{d}^3p_1\; \mathrm{d}x_2 \cdot\cdot \mathrm{d}x_N\; \varrho(\boldsymbol{r},\boldsymbol{p}_1,x_2,\ldots,x_N,t) \\
&= \sum_{i=1}^{N} \int \mathrm{d}^3p_1\; \frac{1}{N} f_1(\boldsymbol{r},\boldsymbol{p}_1,t) \\
&= \int \mathrm{d}^3p\, f_1(\boldsymbol{r},\boldsymbol{p},t)\,,
\end{aligned} \tag{57}$$

where Eq. (47) was employed. Obviously, the reduced one-particle distribution function $f_1(x,t)$ is much simpler than the full phase space distribution function $\varrho(X,t)$ because the information contained in all but one variables is lost due to the integration. On account of the invariance of ϱ with respect to the permutation of arguments it does not matter which variable is left out in the integration process so one might as well pick some other one than x_1. By definition, $N^{-1}f_1(x,t)\,\mathrm{d}x$ is the probability that at time t *any* one particle has phase space coordinates inside the volume element $\mathrm{d}x$ around x regardless of the positions and momenta of all other particles. The definition of f_1 can readily be generalized to higher orders $s \le N$. The reduced s-particle distribution function f_s is obtained by integrating over all but s phase space variables:

$$f_s(x_1, \ldots, x_s, t) \equiv \frac{N!}{(N-s)!} \int \mathrm{d}x_{s+1} \cdots \mathrm{d}x_N \; \varrho(x_1, \ldots, x_N, t)\,. \tag{58}$$

It should be noted that the normalization conventions used in the literature are not standardized and may differ from the one employed here (e.g., Balescu, 1975; Klimontovich, 1982). Apart from the corresponding constant prefactors, $f_s(x_1, \ldots, x_s, t)\,\mathrm{d}x_1 \cdots \mathrm{d}x_s$ represents the probability that at time t any s particles of the system have phase space coordinates inside the volume $\mathrm{d}x_1 \cdots \mathrm{d}x_s$ around $(x_1, \ldots, x_s)$ irrespective of the coordinates of the remaining particles. Thus, ignoring the prefactors again, the final function is identical to the phase space distribution function: $f_N = N!\,\varrho$. By definition, f_s inherits the invariance under permutation of particles from ϱ. The full set $\{f_1, \ldots, f_N\}$ of all reduced distribution functions completely determines the system and the average of any dynamical function.

As mentioned above, in many important cases the knowledge of the reduced one-particle distribution function suffices to compute the relevant macroscopic quantities. In fact, the entire theory of hydrodynamics is built around the moments of f_1 alone. Moreover, the average potential energy can be expressed solely in terms of the reduced two-particle distribution function because the description of binary interactions involves the coordinates of two particles. Thus, in cases such as these the computation of the high-dimensional distribution function $\varrho(X, t)$ can be circumvented altogether. It is reasonable, therefore, to seek dynamic equations for the first few functions $f_1, \ldots, f_s$ separately that do not contain the higher-order functions, let alone ϱ. Unfortunately, such a closed set of equations does not exist for $s < N$. It is possible, however, to gain a set of coupled equations that is accessible to approximations providing the desired closure. The derivations usually presented in the literature rely heavily on the assumption that the internal interactions of the system can be expressed as the sum of two-particle interactions due to central potentials which excludes fully electromagnetic Lorentz forces from the outset (Balescu, 1975; Liboff, 2003; Klimontovich, 1982; Sitenko, 1982). Although relativistic formulations were published (see Hakim (2011) for an overview), they are beyond the scope of this work so that temporarily a Coulomb plasma is considered here. In this case, there exist integro-differential operators $\hat{O}_s$ and $\hat{O}'_s$ containing the interaction potentials such that (Liboff, 2003)

$$(\partial/\partial t + \hat{O}_s) f_s = \hat{O}'_s f_{s+1}\,. \tag{59}$$

This so-called BBGKY-hierarchy constitutes a chain of equations with the distinctive feature that the dynamical equation for every reduced s-particle distribution function is coupled to its successor: the equation for f_1 also contains f_2; in turn, the equation for f_2 involves f_3, and so forth. The entire set of recursively coupled equations for $\{f_1, \ldots, f_N\}$ is equivalent to the Liouville equation for ϱ. Its theoretical benefit is that it provides a good starting point for approximations. The usual procedure is to invoke additional assumptions about the system and to truncate the hierarchy in an approximate sense after a small number of equations. Ideally, it is possible to express f_2 in terms of f_1 and thus to obtain a *kinetic equation*, i.e., a closed equation for the reduced one-particle distribution function alone. It is clear from the outset that, due to the truncation, this can only be valid in an approximate sense. A generic formulation of such a kinetic equation is the generalized Boltzmann equation (Klimontovich, 1986)

$$\hat{D}_{x,t} f_1(x, t) = \hat{C}_{x,t} f_1(x, t)\,, \tag{60}$$

where

$$\hat{D}_{x,t} \equiv \hat{D}_{\boldsymbol{r},\boldsymbol{p},t} \equiv \partial/\partial t + \boldsymbol{v} \cdot \nabla + \langle \boldsymbol{F}_{x,t} \rangle \cdot \nabla_p\,, \tag{61}$$

and where $\langle \boldsymbol{F}_{x,t} \rangle$ describes the average of the microscopic forces. The left-hand side of the Boltzmann equation describes the dynamical evolution of the distribution under the influence of the mean field generated collectively by all particles; the right-hand side contains the modification of f_1 due to the interactions of every particle with a smaller part of the system. In the kinetic theory of gases, therefore, the term *collision operator* was coined for $\hat{C}_{x,t}$, which is also used in the case of plasmas.

Determining a collision operator that matches the particular system at hand by means of a truncation of the BBGKY-hierarchy based on suitable approximations is one of the main tasks of kinetic theory. Usually, this involves a smallness parameter allowing for a perturbation analysis. Examples are the number density of dilute gases, the coupling parameter of weakly coupled systems, or the plasma parameter of ionized gases. The outcomes are kinetic equations like the ones of Vlasov, Boltzmann, Landau, or Balescu–Lenard (Balescu, 1997).

In this light, it is evident that the successful implementation of simplifying approximations is a crucial step in the derivation of collision integrals. With respect to this task, correlation functions proved to be a powerful concept of great assistance. In order to illustrate the basic idea it is useful to remember that $f_2(x_1, x_2)$ is the probability density that two particles have phase space coordinates x_1 and x_2, respectively. If the two particles were statistically independent of each other, their individual probabilities would simply multiply to give the joint probability: $f_2(x_1, x_2) = f_1(x_1) f_1(x_2)$. However, if both particles are correlated, then there must be an additional term,

$$f_2(x_1, x_2) = f_1(x_1) f_1(x_2) + g_2(x_1, x_2)\,. \tag{62}$$

By definition, this additional term g_2 describes the deviation from the uncorrelated case. Thus, it is a measure for the correlation of two particles. Similarly, a function $g_3(x_1, x_2, x_3)$ can be defined that describes the correlation of three particles.

In order account for the possibility that only two particles are correlated while the third one is independent of the others, additional terms appear:

$$f_3(x_1,x_2,x_3)=\prod_{i=1}^{3}f_1(x_i)+\sum_{\pi(i,j,k)}f_1(x_i)g_2(x_j,x_k)+g_3(x_1,x_2,x_3)\,, \tag{63}$$

where the summation extends over all cyclic permutations of $\{1,2,3\}$. In the same fashion, irreducible correlation functions $g_s(x_1,\dots,x_s)$ can be defined for higher orders (Klimontovich, 1982; Balescu, 1975; Krall and Trivelpiece, 1973). In this *Mayer cluster representation*, every s-particle distribution function f_s is replaced by the irreducible correlation function g_s of the respective order for $s\ge 2$. Therefore, in addition to the phase space distribution function ϱ and the set $\{f_1,\dots,f_N\}$ of multi-particle distribution functions, a third alternative for a complete description of the system state has appeared: the set $\{f_1,g_2,\dots,g_N\}$. Its advantage is that it allows a truncation of the hierarchy under suitable conditions, e. g., for weakly coupled systems such as dilute gases or plasmas.

3.3. Klimontovich's method of moments

An alternative formulation of kinetic theory was developed by Klimontovich (1967, 1982, 1986, 1997). It is particularly well suited for the computation of fluctuations. In fact, in Klimontovich's approach they play the same central role that the irreducible correlation functions have in the canonical theory of Bogoliubov, Vlasov and others that was outlined in the last subsection. The very core of this new formulation is the Klimontovich function

$$N_{x,t}\equiv N_{\boldsymbol{r},\boldsymbol{p},t}\equiv\sum_{i=1}^{N}\delta\big(x-x_i(t)\big)\equiv\sum_{i=1}^{N}\delta\big(\boldsymbol{r}-\boldsymbol{r}_i(t)\big)\,\delta\big(\boldsymbol{p}-\boldsymbol{p}_i(t)\big)\,, \tag{64}$$

where $x_i(t)$ describes the exact orbit of the ith particle. Within the limits of classical, non-quantum theory, this function can be interpreted as the microscopically exact phase space density that completely determines the state of the system. In contrast to the Liouville distribution ϱ defined in (46), it is a density with respect to a 6-dimensional volume element $\mathrm{d}x$ instead of a $6N$-dimensional volume element $\mathrm{d}X=\mathrm{d}x_1\,..\,\mathrm{d}x_N$. Moreover, it is not normalized to one, but to the total number of particles, thus representing a number density rather than a probability density:

$$\int \mathrm{d}x\;N_{x,t}=\int \mathrm{d}^3r\;\mathrm{d}^3p\;N_{\boldsymbol{r},\boldsymbol{p},t}=N\,. \tag{65}$$

In order to gain insight into Klimontovich's framework it is instructive to compute the average of $N_{x,t}$. According to the previous discussion, this can be conducted in the Schrödinger type picture that assigns the time dependency to the Liouville distribution. Making use of the invariance of $\varrho(X,t)$ under the permutation of particles, viz. (47), and the definition of the reduced one-particle distribution function, equation (56), one obtains

$$\begin{aligned}\langle N_{x,t}\rangle &=\sum_{i=1}^{N}\int \mathrm{d}x_1\,..\,\mathrm{d}x_N\;\varrho(x_1,\dots,x_N,t)\,\delta(x-x_i)\\ &=\sum_{i=1}^{N}\int \mathrm{d}x_1\,..\,\mathrm{d}x_{i-1}\,\mathrm{d}x_{i+1}\,..\,\mathrm{d}x_N\;\varrho(x_1,\dots,x_{i-1},x,x_{i+1},\dots,x_N,t)\\ &=\sum_{i=1}^{N}\int \mathrm{d}x_2\,..\,\mathrm{d}x_N\;\varrho(x,x_2,\dots,x_N,t)\\ &=f_1(x,t)\,.\end{aligned} \tag{66}$$

Surprisingly, the reduced one-particle distribution function has a microscopic counterpart, namely the Klimontovich function, such that it can be interpreted as the ensemble average of the latter (see Fig. 6). An analogous relation also holds for the reduced two-particle distribution function $f_2(x_1,x_2,t)$. The derivation is very similar, only this time it starts by computing the second moment of $N_{x,t}$. Applying the shorthand notation $\int \mathrm{d}X/\mathrm{d}x_i$ for the integration over all phase space variables $x_1,\dots,x_N$ but the ith one yields

$$\begin{aligned}\langle N_{x,t}N_{x',t}\rangle &=\int \mathrm{d}X\;\varrho(X,t)\sum_{i=1}^{N}\delta(x-x_i)\sum_{j=1}^{N}\delta(x'-x_j)\\ &=\sum_{i=1}^{N}\int \mathrm{d}X\;\varrho(X,t)\,\delta(x-x_i)\Bigg[\sum_{\substack{j=1\\ j\neq i}}^{N}\delta(x'-x_j)+\delta(x-x')\Bigg]\end{aligned}$$

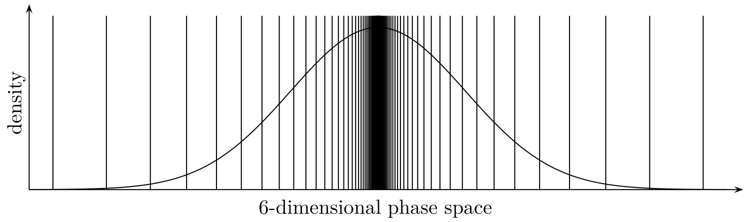

Fig. 6. Schematic illustration of the microscopic Klimontovich function $N_{x,t}$ (vertical lines) and its average $\langle N_{x,t}\rangle = f_1(x,t)$, the reduced one-particle distribution function (curved line). The latter is a smooth function because it does not depend on the microscopic particle variables X anymore due to the averaging. This diagram is a remake of a figure published in the monograph by Ichimaru (1992).

$$
\begin{aligned}
&= \sum_{\substack{i,j=1\\ i\neq j}}^{N} \int \frac{\mathrm{d}X}{\mathrm{d}x_i\,\mathrm{d}x_j}\,\varrho(X,t)\Big|_{\substack{x_i=x\\ x_j=x'}} + \sum_{i=1}^{N}\delta(x-x')\int\frac{\mathrm{d}X}{\mathrm{d}x_i}\,\varrho(X,t)\Big|_{x_i=x}\\
&= \sum_{\substack{i,j=1\\ i\neq j}}^{N} \int \frac{\mathrm{d}X}{\mathrm{d}x_1\,\mathrm{d}x_2}\,\varrho(X,t)\Big|_{\substack{x_1=x\\ x_2=x'}} + \sum_{i=1}^{N}\delta(x-x')\int\frac{\mathrm{d}X}{\mathrm{d}x_1}\,\varrho(X,t)\Big|_{x_1=x}\\
&= f_2(x,x',t) + \delta(x-x')f_1(x,t)\,.
\end{aligned} \tag{67}
$$

In the same fashion, the third, fourth, and higher order moments can be computed. Summarizing the previous findings and supplementing them with the corresponding relation for the next order reveals the general pattern (Klimontovich, 1982; Sitenko, 1982):

$$
\begin{aligned}
\langle N_{x,t}\rangle &= f_1(x,t)\,,\\
\langle N_{x,t}N_{x',t}\rangle &= \delta(x-x')f_1(x,t) + f_2(x,x',t)\,,\\
\langle N_{x,t}N_{x',t}N_{x'',t}\rangle &= \delta(x-x')\,\delta(x-x'')f_1(x,t)\\
&\quad + \delta(x'-x'')f_2(x,x',t) + \delta(x-x')f_2(x',x'',t)\\
&\quad + \delta(x-x'')f_2(x,x',t) + f_3(x,x',x'',t)\,.
\end{aligned} \tag{68}
$$

Thus, the ith moment of the Klimontovich function is given by the first i reduced many-particle distribution functions. Vice versa, the s-particle distribution function can be expressed in terms of the first s moments of $N_{x,t}$. Since the moments play the role that the reduced distribution functions had in the canonical theory outlined in the last subsection, Klimontovich named his scheme the *method of moments* (Klimontovich, 1982, 1986). This designation is all the more appropriate as the relevance of moments goes even further. In order to substantiate this assertion, it is instructive to compute the moments of the fluctuations $\delta N_{x,t}$. Making use of the product rule (53) and of definition (62) of the irreducible correlation function g_2, one obtains in view of the last results

$$
\begin{aligned}
\langle \delta N_{x,t}\,\delta N_{x',t}\rangle &= \langle N_{x,t}N_{x',t}\rangle - \langle N_{x,t}\rangle\langle N_{x',t}\rangle\\
&= \delta(x-x')f_1(x,t) + f_2(x,x',t) - f_1(x,t)f_1(x',t)\\
&= \delta(x-x')f_1(x,t) + g_2(x,x',t)\,.
\end{aligned} \tag{69}
$$

Similarly, higher order relations can be deduced from equations such as (68) for the corresponding moments of $N_{x,t}$. Since the average of any fluctuation vanishes, viz. (52), the first moment is trivial and the beginning of the pattern reads (Klimontovich, 1982)

$$
\begin{aligned}
\langle \delta N_{x,t}\rangle &= 0\,,\\
\langle \delta N_{x,t}\,\delta N_{x',t}\rangle &= \delta(x-x')f_1(x,t) + g_2(x,x',t)\,,\\
\langle \delta N_{x,t}\,\delta N_{x',t}\,\delta N_{x'',t}\rangle &= \delta(x-x')\,\delta(x-x'')f_1(x,t)\\
&\quad + \delta(x'-x'')g_2(x,x',t) + \delta(x-x')g_2(x',x'',t)\\
&\quad + \delta(x-x'')g_2(x,x',t) + g_3(x,x',x'',t)\,.
\end{aligned} \tag{70}
$$

The most important conclusion to be drawn from this is that the moments of the fluctuations, or rather the correlations of $N_{x,t}$, replace the irreducible correlation functions in Klimontovich's formalism. Instead of the set $\{f_1, g_2, \ldots, g_N\}$ the system can also be described in terms of $\langle N_{x,t}\rangle$ and the entirety of the moments, $\{\langle \delta N_{x_1,t}\cdots\delta N_{x_i,t}\rangle\}_{i=1}^{N}$.

3.4. Natural statistical fluctuations

Before turning to the dynamics of these entities, some distinctive features of this formalism should be appreciated regarding the case of a system of entirely uncorrelated particles. Setting all irreducible correlation functions equal to zero and applying the relation $f_1(x, t) = \langle N_{x,t} \rangle$, one obtains

$$\langle \delta N^{\mathrm{uc}}_{x,t}\, \delta N^{\mathrm{uc}}_{x',t} \rangle = \delta(x - x')\langle N^{\mathrm{uc}}_{x,t} \rangle \,, \tag{71}$$

$$\langle \delta N^{\mathrm{uc}}_{x,t}\, \delta N^{\mathrm{uc}}_{x',t}\, \delta N^{\mathrm{uc}}_{x'',t} \rangle = \delta(x - x')\,\delta(x - x'')\langle N^{\mathrm{uc}}_{x,t} \rangle \,, \tag{72}$$

and so forth for higher orders. First of all, it should be noted that the third central moment is non-zero. As pointed out by Rose (1979), this non-Gaussianity of the fluctuations is immanent in this formalism. The reason is that it is based on the discreteness inherent in the Klimontovich function. A further point worth noticing is the non-vanishing result for the second central moment as it appears counter-intuitive at first glance. But, to begin with, it can be made at least plausible when comparing it with the well-known result $\langle (\delta N)^2 \rangle \propto \langle N \rangle$ from classical statistics (Landau and Lifshitz, 1980). A more profound understanding can be obtained by retracing the derivation of the term $\delta(x - x')f_1(x, t)$ in (69), which ultimately leads back to Eq. (67). There, the term in question emerged due to the separation of the self-term $i = j$ in the double sum $\sum_{i,j}$. Therefore, the above result can be interpreted as the description of self-correlations that are present even if different particles are uncorrelated (Tolias et al., 2015). The appearance of the δ-function supports this interpretation. Again, these self-correlations must be attributed to the particle discreteness expressed by the Klimontovich function. If the reduced one-particle distribution function happens to be a thermal one, the phenomenon is known as *thermal noise*. In the general case of an arbitrary distribution function, the effect is referred to as *natural statistical fluctuations* (Tolias et al., 2015; Tsytovich, 1995), *zero fluctuations* (Tsytovich et al., 2008), or, in order to emphasize that it is the classical counterpart of quantum fluctuations, *zero-point fluctuations* (Tsytovich, 1989). Eq. (69) shows that they are also present in a system of correlated particles, only there an additional term appears that describes the interparticle correlations. In this light one can also resolve the ambiguity in the term "correlations" that is used both for the central moments $\langle \delta N \cdot \cdot \delta N \rangle$ and for the functions g_s: The latter refer to the *interparticle* correlations, whereas the former also contain the self-correlations. Since the correlations between different particles are due to the force fluctuations, a significant feature of these self-correlations is that they exist independently of possible fluctuations $\delta \boldsymbol{F}_{x,t}$ just due to the randomness of the fluctuations in the density of discrete particles.

3.5. The hierarchy in Klimontovich's formalism

In order to develop the formalism further, a dynamical equation for the density function $N_{x,t}$ is required. Starting from its definition (64) and making use of the equations of motion for the particles one obtains the Klimontovich equation (Klimontovich, 1967, 1982; Nicholson, 1983)

$$\big(\partial/\partial t + \boldsymbol{v} \cdot \nabla + \boldsymbol{F}_{\boldsymbol{r},\boldsymbol{p},t} \cdot \nabla_p\big) N_{\boldsymbol{r},\boldsymbol{p},t} = 0 \,, \tag{73}$$

where $\boldsymbol{F}_{\boldsymbol{r},\boldsymbol{p},t}$ denotes the sum of all internal and external microscopic forces acting at time t upon a particle with momentum $\boldsymbol{p}$ at the position $\boldsymbol{r}$. Within the underlying classical framework that does not account for quantum effects or particle creation or annihilation, this equation is exact on a microscopic level of description. No approximations or averaging processes have been employed in its derivation. In order to determine the solution uniquely, an initial condition needs to be specified:

$$N_{\boldsymbol{r},\boldsymbol{p},t_0} = \sum_{i=1}^{N} \delta(\boldsymbol{r} - \boldsymbol{r}_{i,0})\,\delta(\boldsymbol{p} - \boldsymbol{p}_{i,0}) \,. \tag{74}$$

The Klimontovich equation states that, starting from this initial value, the density $N_{\boldsymbol{r},\boldsymbol{p},t}$ remains constant along the 6-dimensional phase space trajectory. Therefore, the method of characteristics provides an alternative proof that the initial value problem (73)–(74) describes the dynamics of the function (64) correctly.

Since a kinetic equation describes the temporal evolution of $f_1(x, t) = \langle N_{x,t} \rangle$, its derivation basically means to apply the averaging operator on the Klimontovich equation. In the course of this it is important to notice that the momentum divergence of the force vanishes, $\nabla_p \cdot \boldsymbol{F}_{\boldsymbol{r},\boldsymbol{p},t} = 0$, even in the case of the Lorentz force which depends on the velocity $\boldsymbol{v} = \boldsymbol{p}/\Gamma m$. (Velocity dependent forces other than the Lorentz force are not relevant in this context.) Taking the average shows that the mean force inherits this property and, therefore, the same applies to its fluctuation:

$$\nabla_p \cdot \boldsymbol{F}_{\boldsymbol{r},\boldsymbol{p},t} = 0 \,, \qquad \nabla_p \cdot \langle \boldsymbol{F}_{\boldsymbol{r},\boldsymbol{p},t} \rangle = 0 \,, \qquad \nabla_p \cdot \delta \boldsymbol{F}_{\boldsymbol{r},\boldsymbol{p},t} = 0 \,. \tag{75}$$

As a consequence, ∇_p and the force (regarded as an operator) commute. With this property at disposal, one can take the average of the Klimontovich equation and apply the product rule (53) to find

$$\big(\partial/\partial t + \boldsymbol{v} \cdot \nabla + \langle \boldsymbol{F}_{\boldsymbol{r},\boldsymbol{p},t} \rangle \cdot \nabla_p\big) \langle N_{\boldsymbol{r},\boldsymbol{p},t} \rangle = -\nabla_p \cdot \big\langle \delta \boldsymbol{F}_{\boldsymbol{r},\boldsymbol{p},t}\, \delta N_{\boldsymbol{r},\boldsymbol{p},t} \big\rangle \,. \tag{76}$$

Comparing this result with the generalized Boltzmann equation (60) shows that this is almost a kinetic equation. The subtle difference is that (76) is not a closed equation for $\langle N_{\boldsymbol{r},\boldsymbol{p},t} \rangle$ because the collision integral on the right-hand side depends on the fluctuations. This is reasonable because so far no approximations have been employed and, hence, equation (76) is exact, whereas a kinetic equation in the sense of (60) can only hold approximately.

The examination of the collision integral can be significantly simplified if the underlying internal interactions are due to a potential such that

$$\boldsymbol{F}_{\boldsymbol{r},\boldsymbol{p},t} = \boldsymbol{F}^{\mathrm{ext}}_{\boldsymbol{r},\boldsymbol{p},t} - \int \mathrm{d}^3r'\, \mathrm{d}^3p'\, N_{\boldsymbol{r}',\boldsymbol{p}',t} \nabla_r V(\boldsymbol{r},\boldsymbol{r}')\,. \tag{77}$$

This is certainly fulfilled by the Coulomb force, but not by the more general electromagnetic interactions involving magnetic fields. Nevertheless, for the present purpose of clarifying the structure underlying Klimontovich's formalism, the forces will be assumed to be of the form (77) for the sake of simplicity. This condition will be weakened in the next section where the functional dependency between particles and fields will be examined for the full electromagnetic scenario. Since a possible external force $\boldsymbol{F}^{\mathrm{ext}}_{\boldsymbol{r},\boldsymbol{p},t}$ does not depend on the system coordinates $X = (x_1, \ldots, x_N)$, its fluctuation vanishes and the expression entering the collision integral reads

$$\delta\boldsymbol{F}_{x,t} = -\int \mathrm{d}x'\, \delta N_{x',t} \nabla_r V(\boldsymbol{r},\boldsymbol{r}')\,. \tag{78}$$

The important point to note here is that the collision integral appearing in the dynamic equation for $\langle N_{x,t}\rangle$ is a functional of the moment $\langle \delta N_{x,t}\, \delta N_{x',t}\rangle$. This raises the necessity to derive a dynamic equation for the fluctuations. To this end, both the force and the phase space density are decomposed into their average and fluctuating component in the Klimontovich equation (73) according to $b = \langle b\rangle + \delta b$. Afterwards, equation (76) for the average is subtracted and the property (75) of the force is applied once more. This procedure leads to

$$\hat{D}_{x,t}\, \delta N_{x,t} = -\delta\boldsymbol{F}_{x,t} \cdot \nabla_p \langle N_{x,t}\rangle + \nabla_p \cdot \Big(\langle \delta\boldsymbol{F}_{x,t}\, \delta N_{x,t}\rangle - \delta\boldsymbol{F}_{x,t}\, \delta N_{x,t} \Big)\,, \tag{79}$$

where the short-hand notation $x = (\boldsymbol{r},\boldsymbol{p})$ and the operator $\hat{D}_{x,t}$ defined in (61) have been employed again for the sake of improved readability. The required equation for the second moment can be derived as follows: First, one multiplies (79) by $\delta N_{x',t}$. Then, the resulting equation is rewritten with x and x' interchanged. Finally, one adds both equations and takes their average:

$$\begin{aligned} &\left(\frac{\partial}{\partial t} + \sum_{x\leftrightarrow x'} \Big(\boldsymbol{v}\cdot\nabla + \langle \boldsymbol{F}_{x,t}\rangle \cdot \nabla_p \Big) \right) \langle \delta N_{x,t}\, \delta N_{x',t}\rangle \\ &= -\sum_{x\leftrightarrow x'} \Big(\langle \delta N_{x',t}\, \delta\boldsymbol{F}_{x,t}\rangle \cdot \nabla_p \langle N_{x,t}\rangle + \nabla_p \cdot \langle \delta N_{x',t}\, \delta\boldsymbol{F}_{x,t}\, \delta N_{x,t}\rangle \Big)\,. \end{aligned} \tag{80}$$

The summation is to be taken over both permutations of x and x'. Since the force fluctuation can be traced back to the density fluctuation, viz. (78), the following pattern becomes evident now: The dynamic equation for the average density $\langle N_{x,t}\rangle$ involves the second moment $\langle \delta N_{x,t}\, \delta N_{x',t}\rangle$. In turn, the dynamic equation for the second moment contains the third moment, and so forth. Just as the set $\{f_1, g_2, \ldots, g_N\}$ was replaced by $\langle N_{x,t}\rangle$ and by the moments of $\delta N_{x,t}$ in Klimontovich's formalism, the BBGKY-hierarchy is replaced by a hierarchy of equations for the respective functions. Thus, the recipe for dealing with this situation is the very same: The properties of the particular system at hand must be employed in such a way that the chain of coupled equations can be truncated in an approximate sense. In this fashion it is possible to derive the kinetic equations of Vlasov, Landau, and Balescu–Lenard within the method of moments (Klimontovich, 1982).

Up to this point, the fluctuations have only been considered during the derivation of a kinetic equation for the one-particle distribution function. The reason that they play a key role in this context is that they constitute the collision integral in the exact dynamical equation (76). In addition, they are also of the utmost importance in different, yet related fields of interest: They characterize the dissipative properties of the plasma, most prominently formulated in the fluctuation–dissipation theorem (Callen and Welton, 1951; Schlickeiser and Kolberg, 2015); they describe plasma turbulence in a statistical framework (Tsytovich, 1977; Yoon et al., 2016); and they determine the transport coefficients appearing in the Fokker–Planck equation (Schlickeiser, 2011, 2015). However, in these areas the two-time correlation function $\langle \delta N_{x,t}\, \delta N_{x',t'}\rangle$ is the relevant entity rather than the corresponding one-time function encountered before. The reason is that the former does not only take the spatial randomness into account but also considers the fluctuations as random variables in time. It describes both the correlations at two positions x and x' and the correlations at two instants t and t' of time. A dynamical equation for this function can readily be found by multiplying the exact fluctuation equation (79) by $\delta N_{x',t'}$ and taking the average:

$$\hat{D}_{x,t} \langle \delta N_{x,t}\, \delta N_{x',t'}\rangle = -\langle \delta\boldsymbol{F}_{x,t}\, \delta N_{x',t'}\rangle \cdot \nabla_p \langle N_{x,t}\rangle - \nabla_p \cdot \langle \delta\boldsymbol{F}_{x,t}\, \delta N_{x,t}\, \delta N_{x',t'}\rangle\,. \tag{81}$$

Although the two-time correlation function is the more general concept in comparison with the corresponding one-time function, this equation is much simpler than (80). However, due to the initial condition

$$\langle \delta N_{x,t}\, \delta N_{x',t'}\rangle\Big|_{t=t'} = \langle \delta N_{x,t'}\, \delta N_{x',t'}\rangle \tag{82}$$

the single-time correlation function needs to be computed after all. In practice, therefore, it is often more convenient to perform all computations in terms of the fluctuations $\delta N_{x,t}$ and compute their moments afterwards rather than to start with the moment equations in the first place. But one should always be aware that the fluctuations themselves are microscopic entities and as such not accessible to macroscopic observations. They must be treated as random variables and only their moments, that is, the correlations, are physically meaningful.

Similarly to the derivation of (82), equations for the many-time correlations of higher order can be found. It is evident that these will constitute a hierarchy again in the same fashion as before in the case of the one-time moments. Thus, once more it becomes necessary to truncate the coupled chain of equations in an approximate sense by exploiting the specific properties of the system at hand.

3.6. Closure approximations

In the last two subsections, alternatives to the Liouville distribution ϱ were introduced for the statistical description of a non-equilibrium many-particle system: the system of reduced s-particle distribution functions, the irreducible correlation functions, the moments of $N_{x,t}$, and the moments of the fluctuations $\delta N_{x,t}$. In all of these cases, the dynamics were governed by a coupled chain of equations, either the BBGKY-hierarchy or Klimontovich's cumulant hierarchy. It has already been pointed out that these equations are not yet useful as they stand, but that their theoretical benefit in comparison with the Liouville equation lies in the possibility to truncate the hierarchies by appropriate closure conditions and to obtain a suitable description in an approximative way. This issue is addressed in the current subsection.

There is a plethora of closure schemes known in kinetic theory whose applicability is usually tailored for the particular system at hand and for the description of its effects under investigation. In order to give a non-exhaustive overview, the categorization of Bonitz (1998) is mentioned here that distinguishes three basic types: (1) closure provided by perturbation theory with respect to internal parameters of the system such as the gas density, the coupling parameter or the plasma parameter; (2) closure due to a perturbative treatment with respect to an external parameter like the strength of an external potential or an external magnetic field; (3) non-perturbative closure relying on topological criteria.

The first attempts at a kinetic description of a plasma adopted the well-studied theory of dilute gases. In the simplest case, the latter can be modeled as a hard sphere by considering a binary interaction potential that is constant for distances below a characteristic length r_0 and that vanishes beyond. A low gas density n in this case means that the average distance between particles, $r_{\rm av} = n^{-1/3}$, is much longer than the interaction radius so that $\varepsilon_{\rm gas} = nr_0^3 \ll 1$ naturally presents itself as a smallness parameter for perturbation theory. The zeroth-order approximation with $\varepsilon_{\rm gas} \simeq 0$ defines the ideal gas of non-interacting particles and the first-order approximation leads to the well-known Boltzmann collision integral (Klimontovich, 1982).

However, the implication of the hard sphere model that interactions only occur during close encounters of a few particles is violated in a plasma due to the long-range nature of the interactions between charged particles. On the contrary, a characteristic feature of a plasma is its collective behavior, i. e., the simultaneous interaction of many particles. Since the wide outreach of the interactions also implies that they are not very strong in many cases, a weak coupling approximation suggests itself. In order to quantify the strength of the particle interactions one usually resorts to the ratio of the interaction energy at the average interparticle distance $r_{\rm av} = n^{-1/3}$ to the average kinetic energy. If the system is in or near equilibrium, the average kinetic energy is $\sim k_{\rm B}T$, where $k_{\rm B}$ is Boltzmann's constant and T the temperature. Considering only Coulomb interactions, the ratio in question is $\varepsilon_{\rm wc} = e^2/r_{\rm av}k_{\rm B}T \ll 1$. Starting from the Boltzmann equation, Landau (1936) performed a perturbation analysis based on the smallness of this coupling parameter resulting in the collision integral named after him.

Although the Landau collision integral is very useful for many practical purposes, it has a major disadvantage from a theoretical point of view: It diverges logarithmically for both short and long interparticle distances. The first divergence is caused by the infinite value of the Coulomb potential for $r \to 0$. It can be remedied either by an *infinite* series in the coupling parameter (since the assumption of weak interactions breaks down if two charges are located within close vicinity of each other) or, more rigorously, with a quantum mechanical treatment accounting for repulsive effects at short distances (Balescu, 1997). In order to reveal the origin of the second divergence one has to take the collective behavior into consideration again. Because of the long range of the Coulomb potential, every particle interacts simultaneously with many others that, in turn, screen the first particle from the influences of all other charges that are even more distant. This *Debye screening* leads to an effective interaction potential that displays an exponential cutoff of the Coulomb potential: $V_{\rm eff}(r) = e^2 \exp(-r/\lambda_{\rm D})/r$ (Nicholson, 1983). Thus, the *Debye length*

$$\lambda_{\rm D} \equiv (k_{\rm B}T/4\pi ne^2)^{1/2} \tag{83}$$

describes an effective interaction range. Particles with a distance $r \lesssim \lambda_{\rm D}$ are correlated due to the Coulomb interaction whereas particles with $r \gtrsim \lambda_{\rm D}$ are screened from their mutual influence. So the reason for the divergence of the Landau collision integral at large distances is that this cutoff was missed by assuming an unscreened Coulomb potential. In order to fix this shortcoming, a different smallness parameter is needed. According to the previous considerations, the number of particles inside a Debye sphere,

$$N_{\rm D} \equiv (4/3)\pi n\lambda_{\rm D}^3\,, \tag{84}$$

corresponds to the number of particles any given charge interacts with simultaneously. As mentioned before, a characteristic feature of a plasma is that this number is very high so that a suitable smallness parameter for a perturbation expansion is given by its inverse, the plasma parameter $\varepsilon_{\rm p}$:

$$\varepsilon_{\rm p} \equiv 1/N_{\rm D} \ll 1\,. \tag{85}$$

These definitions imply that the plasma parameter is, apart from a factor 3, equal to the ratio $e^2/\lambda_D k_B T$ of the interaction energy at distance λ_D (*not* at distance r_{av} as before) to the average kinetic energy. This property confirms that ε_p is, as intended, an appropriate perturbation parameter for weakly coupled systems displaying significant collective behavior that dominates over the interactions of few-particle interactions. Therefore, it is intuitively clear that the correlations of two, three, or even more particles are related to the plasma parameter. At least for equilibrium systems one obtains (Krall and Trivelpiece, 1973)

$$|f_1| \sim \varepsilon_p^0\,, \qquad |g_2| \sim \varepsilon_p^1\,, \qquad |g_3| \sim \varepsilon_p^2\,, \qquad |g_4| \sim \varepsilon_p^3\,, \qquad \dots\,. \tag{86}$$

Balescu (1997) points out that it is reasonable to assume the validity of these relations for near-equilibrium systems as well because at least one two-particle interaction is involved in binary correlations, two interaction processes are involved in ternary correlations (for instance, 1–2 and 1–3), and so forth. Since these interactions scale with ε_p, the pattern (86) seems to be justified even in non-equilibrium situations.

In the zeroth order limit $\varepsilon_p \simeq 0$ that defines an ideal or collisionless plasma, the correlations are neglected altogether: $g_s \simeq 0$. This *mean field approximation* is the simplest closure condition and leads to the Vlasov equation (Vlasov, 1938; Liboff, 2003)

$$\hat{D}_{x,t}\langle N_{x,t}\rangle \equiv \big(\partial/\partial t + \boldsymbol{v}\cdot\nabla + \langle \boldsymbol{F}_{\boldsymbol{r},\boldsymbol{p},t}\rangle\cdot\nabla_p\big)\langle N_{\boldsymbol{r},\boldsymbol{p},t}\rangle = 0\,, \tag{87}$$

where $f_1(x,t) = \langle N_{x,t}\rangle$ was used. It describes an entirely uncoupled system without any correlations. This does not mean that the particles are not interacting at all, but it means that they are only interacting collectively instead of binarily, ternarily, and so forth. Their collective influences (as well as possible external fields) are unified in a mean field $\langle \boldsymbol{F}_{x,t}\rangle$ that appears as a quasi-external force in the sense that, due to the averaging process, it does not depend on the particle orbits $X = (x_1, \dots, x_N)$. Comparing (87) with the generic Boltzmann equation (60) confirms that the Vlasov equation is a kinetic equation in which the collision integral has been neglected altogether. It is worth emphasizing that, although the Vlasov equation (87) and the Klimontovich equation (73) are formally very similar, there are two distinctive differences. Firstly, the Klimontovich equation involves the microscopically resolved entities $\boldsymbol{F}_{x,t}$ and $N_{x,t}$, whereas the Vlasov equation describes their averages. Secondly, the Klimontovich equation is exact; the Vlasov equation, however, is an approximation that neglects the particle correlations contained in the collision integral.

A characteristic property of the Vlasov equation its reversibility, i. e., the invariance under time reversals. As a consequence, it conserves the entropy of the system (Klimontovich, 1982). Therefore, the mean field approximation cannot describe dissipation and the evolution of a plasma towards equilibrium. This is plausible in view of the total neglect of correlations and collisions. In order to overcome this shortcoming, the effects of the next order in the plasma parameter must be taken into account. The result is the Balescu–Lenard kinetic equation for weakly coupled plasmas (Liboff, 2003) that was independently derived from Balescu (1960) and Lenard (1960).

Now the question arises how to translate these closure conditions into Klimontovich's scheme. According to Section 3.3, suitable formulations must involve the moments of the fluctuations. Since the smallness of the plasma parameter $\varepsilon_p \equiv 1/N_D$ corresponds to a large number of particles inside a Debye sphere, a reasonable closure approximation is that the fluctuations of the phase space density are small:

$$\langle \delta N_{x,t}\,\delta N_{x',t}\rangle \simeq 0\,. \tag{88}$$

Indeed, in view of (78) this assumption transforms (76) into the Vlasov equation. Another way to obtain the latter in Klimontovich's formalism is to set $\delta \boldsymbol{F}_{x,t} = 0$ in (76). This is in agreement with the earlier interpretation that, in this approximation, the particles do not interact with each other but only with the mean field $\langle \boldsymbol{F}_{x,t}\rangle$. To next order, instead of (88) the triple moments are neglected:

$$\langle \delta N_{x,t}\,\delta N_{x',t}\,\delta N_{x'',t}\rangle \simeq 0\,. \tag{89}$$

In order to compute the collision integral in this approximation one must solve equation (80). In view of the closure assumption it becomes

$$\left(\frac{\partial}{\partial t} + \sum_{x\leftrightarrow x'}\big(\boldsymbol{v}\cdot\nabla + \langle \boldsymbol{F}_{x,t}\rangle\cdot\nabla_p\big)\right)\langle \delta N_{x,t}\,\delta N_{x',t}\rangle = -\sum_{x\leftrightarrow x'}\langle \delta N_{x',t}\,\delta \boldsymbol{F}_{x,t}\rangle\cdot\nabla_p\langle N_{x,t}\rangle\,. \tag{90}$$

If one is interested in the two-time moments rather than the one-time correlations appearing in the collision integral, equation (81) must be consulted. In this approximation it reads

$$\hat{D}_{x,t}\langle \delta N_{x,t}\,\delta N_{x',t'}\rangle = -\langle \delta \boldsymbol{F}_{x,t}\,\delta N_{x',t'}\rangle\cdot\nabla_p\langle N_{x,t}\rangle\,. \tag{91}$$

It has already been mentioned in Section 3.5 that it is much more convenient in practice to compute the fluctuations $\delta N_{x,t}$ first and to deduce the corresponding moments afterwards. In this case, (90) and (91) both can be obtained from the simpler equation

$$\hat{D}_{x,t}\,\delta N_{x,t} = -\delta \boldsymbol{F}_{x,t}\cdot\nabla_p\langle N_{x,t}\rangle\,. \tag{92}$$

Table 1
Closures for weakly correlated systems leading to the common kinetic equations of Vlasov, Landau, Lenard–Balescu–Guernsey (LBG) and Boltzmann. This table is an excerpt from the one published in the article by Daligault (2011).

Closure	$\langle \delta N_{x,t}\, \delta N_{x',t} \rangle$	$\langle \delta N_{x,t}\, \delta N_{x',t}\, \delta N_{x'',t} \rangle$
Vlasov	0	0
Landau	$\delta(x-x')f_1(x,t)$	0
LBG	$\delta(x-x')f_1(x,t)+g_2(x,x',t)$	0
Boltzmann	$\delta(x-x')f_1(x,t)$	$\delta(x-x')\,g_2(x',x'',t)+$ cyclic permut. $+\,\delta(x-x')\,\delta(x'-x'')f_1(x,t)$

The proof is carried out along the lines of the derivations leading to the exact equations (80) and (81). One must keep in mind, however, that only the correlations are non-random quantities. In particular, only the latter can provide meaningful initial conditions. As one would expect, it is possible to derive the Balescu–Lenard collision integral on the basis of the equation above (Klimontovich, 1982). Table 1 summarizes several closure conditions in the Klimontovich formulation (Daligault, 2011).

As a final remark, Rostoker's superposition principle for *dressed particles* is mentioned here (Rostoker, 1964). It relates the problem of finding an approximate solution for the hierarchy equations to a test particle problem. In essence it states that, to first order in the plasma parameter, the plasma can be treated as an ensemble of uncorrelated quasi-particles that are equipped (or *dressed*) with the mean field response (Krommes, 1976). The latter can be interpreted as a shielding cloud that surrounds every individual test charge and that stems from the combined influence of all other particles. This test particle approach to kinetic theory can also be generalized to higher orders in the plasma parameter (Matsuda, 1970).

3.7. Characteristics of the evolution operator

The preceding subsections revealed a common mathematical structure underlying the dynamic equations for all the different functions of relevance, namely the one-particle distribution function, the fluctuations of the phase space density, and the many-time correlations. According to the exact equations (76) and (79), and (81), as well as the approximations (87) and (91)–(92), the time evolution is always determined by an initial value problem of the first-order hyperbolic form

$$\hat{D}_{x,t}\, u(x,t) = b(x,t,u)\,, \qquad u(x,t_0) = u_{\text{ini}}(x)\,, \tag{93}$$

that is governed by the operator

$$\hat{D}_{x,t} = \partial/\partial t + a(x,t)\partial/\partial x = \partial/\partial t + \boldsymbol{v}\cdot\nabla + \langle \boldsymbol{F}_{\boldsymbol{r},\boldsymbol{p},t}\rangle \cdot \nabla_p\,. \tag{94}$$

Here, the notation $x = (\boldsymbol{r}, \boldsymbol{p})$ was employed again and the abbreviation $a(x,t) = (\boldsymbol{v}, \langle \boldsymbol{F}_{x,t}\rangle)$ was introduced. Associated with this spatio-temporal differential operator $\hat{D}_{x,t}$ are curves $x_c(\tau)$ in phase space called its *characteristics* (Meister and Struckmeier, 2002). By definition, they are solutions of the ordinary differential equation

$$\partial x_c(\tau)/\partial\tau = a(x_c(\tau),\tau)\,. \tag{95}$$

Of course, there is a whole family of such characteristics due to the yet unspecified constant of integration. For every point $\tilde{x}$ in phase space there is exactly one characteristic $x_c(\tilde{x},\tau)$ fulfilling the initial condition $x_c(\tilde{x},t_0)=\tilde{x}$. Hence, both $\tilde{x}$ and τ are free variables and one can define a coordinate transform from (x,t) to $(\tilde{x},\tau)$ by setting $x = x_c(\tilde{x},\tau)$ and $t=\tau$. The inverse transform can be obtained by solving $x = x_c(\tilde{x},t)$ for $\tilde{x}$. By design, this transform possesses the property

$$\frac{\partial}{\partial\tau} = \frac{\partial x(\tilde{x},\tau)}{\partial\tau}\frac{\partial}{\partial x} + \frac{\partial t(\tilde{x},\tau)}{\partial\tau}\frac{\partial}{\partial t} = a(x,t)\frac{\partial}{\partial x} + \frac{\partial}{\partial t} = \hat{D}_{x,t}\,. \tag{96}$$

Thus, the partial differential operator $\hat{D}_{x,t}$ transforms into an ordinary differential operator in only one single variable. As a consequence, the initial value problem (93) reduces to the ordinary problem $\partial u/\partial\tau = b$. Once it is solved, the coordinate transform must be inverted. Although this method of characteristics leads to an explicit solution even if b depends on u, one can also perform the τ-integration in a formal sense in order to obtain an integral equation for u. In the end, the solution is expressed as the sum of two contributions:

$$u(x,t) = u_0(x,t) + u_1(x,t)\,, \tag{97}$$

$$u_0(x,t) = u_{\text{ini}}\big(\tilde{x}(x,t)\big)\,, \tag{98}$$

$$u_1(x,t) = \Big[\int_{t_0}^{t} \mathrm{d}\tau\; b\Big(x_c(\tilde{x},\tau),\tau,u\big(x_c(\tilde{x},\tau),\tau\big)\Big)\Big]_{\tilde{x}=\tilde{x}(x,t)}\,. \tag{99}$$

The functions u_0 and u_1 can be identified as the homogeneous solution meeting the required initial condition and the inhomogeneous solution with vanishing initial value, respectively:

$$\hat{D}_{x,t}\, u_0(x,t) = 0\,, \qquad u_0(x,t_0) = u_{\rm ini}(x)\,, \tag{100}$$

$$\hat{D}_{x,t}\, u_1(x,t) = b(x,t,u)\,, \qquad u_1(x,t_0) = 0\,. \tag{101}$$

In view of these general results, the remaining task is to compute the characteristics associated with the particular operator (94) by solving the equations of motion $\dot{\boldsymbol{p}} = \langle \boldsymbol{F}_{x,t}\rangle$ and $\dot{\boldsymbol{r}} = \boldsymbol{v} = \boldsymbol{p}/\Gamma m$ in accordance with the initial conditions $\boldsymbol{p}(t_0) = \tilde{\boldsymbol{p}}$ and $\boldsymbol{r}(t_0) = \tilde{\boldsymbol{r}}$. Once the characteristics are found in this fashion, one solves these equations for the initial vectors $\tilde{\boldsymbol{r}}$ and $\tilde{\boldsymbol{p}}$ as outlined above. Considering this procedure it is immediately evident that the solution depends crucially on the underlying mean field. The two most important cases in the astrophysical context are the unmagnetized plasma, $\langle \boldsymbol{F}_{x,t}\rangle = 0$, and the magnetized plasma being subject to a constant and homogeneous magnetic field: $\langle \boldsymbol{F}_{x,t}\rangle = (e/c)\boldsymbol{v} \times \langle \boldsymbol{B}\rangle$. Average electric fields can usually be neglected because they would be short-circuited at once on account of the high conductivity of the plasma. For unmagnetized plasmas one readily obtains

$$\left.\begin{array}{l} \boldsymbol{p} = \tilde{\boldsymbol{p}} \\ \boldsymbol{r} = \tilde{\boldsymbol{r}} + \tilde{\boldsymbol{v}}(t-t_0) \end{array}\right\} \quad \longleftrightarrow \quad \left\{\begin{array}{l} \tilde{\boldsymbol{p}} = \boldsymbol{p} \\ \tilde{\boldsymbol{r}} = \boldsymbol{r} - \boldsymbol{v}(t-t_0)\,. \end{array}\right. \tag{102}$$

Thus, the characteristics describe hypothetical particles streaming freely along straight lines. In the case of a magnetized plasma the computation is more involved because the particles are gyrating around the magnetic field lines. Denoting the unit vector in the direction of the field by $\boldsymbol{e}_\parallel = \boldsymbol{B}/B$, the characteristics are given by

$$\boldsymbol{p} = \tilde{\boldsymbol{p}}_\parallel + \tilde{\boldsymbol{p}}_\perp \cos[\Omega_{\rm g}(t-t_0)] - (\boldsymbol{e}_\parallel \times \tilde{\boldsymbol{p}}) \sin[\Omega_{\rm g}(t-t_0)]\,, \tag{103}$$

$$\begin{aligned} \boldsymbol{r} = \tilde{\boldsymbol{r}} &- \boldsymbol{e}_\parallel \times (\tilde{\boldsymbol{v}}/\Omega_{\rm g}) + \tilde{\boldsymbol{v}}_\parallel(t-t_0) + (\tilde{\boldsymbol{v}}_\perp/\Omega_{\rm g}) \sin[\Omega_{\rm g}(t-t_0)] \\ &+ (\boldsymbol{e}_\parallel \times \tilde{\boldsymbol{v}}/\Omega_{\rm g}) \cos[\Omega_{\rm g}(t-t_0)]\,, \end{aligned} \tag{104}$$

where the parallel and perpendicular component of a vector $\boldsymbol{a}$ with respect to the magnetic field are denoted by $\boldsymbol{a}_\parallel = (\boldsymbol{e}_\parallel \cdot \boldsymbol{a})\boldsymbol{e}_\parallel$ and $\boldsymbol{a}_\perp = \boldsymbol{a} - \boldsymbol{a}_\parallel$, respectively. These results can readily be verified by plugging them into the equations of motion. The magnitude of the momentum vector is a constant and, as a consequence, so are the Lorentz factor Γ and the relativistic gyro-frequency $\Omega_{\rm g} = eB/\Gamma mc$. Solving these coupled equations for the initial vectors yields

$$\tilde{\boldsymbol{p}} = \boldsymbol{p}_\parallel + \boldsymbol{p}_\perp \cos[\Omega_{\rm g}(t-t_0)] + (\boldsymbol{e}_\parallel \times \boldsymbol{p}) \sin[\Omega_{\rm g}(t-t_0)]\,, \tag{105}$$

$$\begin{aligned} \tilde{\boldsymbol{r}} = \boldsymbol{r} &- \boldsymbol{e}_\parallel \times (\boldsymbol{v}/\Omega_{\rm g}) - \boldsymbol{v}_\parallel(t-t_0) - (\boldsymbol{v}_\perp/\Omega_{\rm g}) \sin[\Omega_{\rm g}(t-t_0)] \\ &+ (\boldsymbol{e}_\parallel \times \boldsymbol{v}/\Omega_{\rm g}) \cos[\Omega_{\rm g}(t-t_0)]\,, \end{aligned} \tag{106}$$

in agreement with the findings of Klimontovich (1982). As a first application, these results can be used to solve the Vlasov equation (87) for the reduced one-particle distribution function $f_1(x,t) = \langle N_{x,t}\rangle$. Since it is inherently homogeneous, the solution simply reads $f_1(x,t) = f_{\rm ini}(\tilde{x}(x,t))$. Thus, in the Vlasov picture the particles behave as if they were isolated, single particles, solely moving under the influence of the mean field. This outcome reproduces the assumption of negligible interparticle correlations under which the Vlasov equation was derived.

3.8. *Spontaneous and induced fluctuations*

The rather formal considerations conducted in the last subsection provide some important insights into the general structure of plasma fluctuations. It is advantageous to turn to the two-time correlations first. According to the preceding discussions, they are determined by the initial value problem (81)–(82) or its approximation (91). Hence, their dynamics are governed by the operator $\hat{D}_{x,t}$,

$$\hat{D}_{x,t} \langle \delta N_{x,t}\, \delta N_{x',t'}\rangle = b(x,t)\,, \qquad \langle \delta N_{x,t}\, \delta N_{x',t'}\rangle\Big|_{t=t'} = \langle \delta N_{x,t'}\, \delta N_{x',t'}\rangle \tag{107}$$

with

$$\begin{aligned} b(x,t) &\equiv -\langle \delta \boldsymbol{F}_{x,t}\, \delta N_{x',t'}\rangle \cdot \nabla_p \langle N_{x,t}\rangle - \nabla_p \cdot \langle \delta \boldsymbol{F}_{x,t}\, \delta N_{x,t}\, \delta N_{x',t'}\rangle \\ &\simeq -\langle \delta \boldsymbol{F}_{x,t}\, \delta N_{x',t'}\rangle \cdot \nabla_p \langle N_{x,t}\rangle\,. \end{aligned} \tag{108}$$

This is a hyperbolic partial differential equation in the variables x and t. The primed quantities are merely parameters, here. The solutions of initial value problems of this kind were derived in the last subsection:

$$\langle \delta N_{x,t}\, \delta N_{x',t'}\rangle = \langle \delta N_{x,t}\, \delta N_{x',t'}\rangle_0 + \langle \delta N_{x,t}\, \delta N_{x',t'}\rangle_{\rm ind}\,. \tag{109}$$

The second term on the right-hand side denotes the inhomogeneous solution with vanishing initial condition:

$$\hat{D}_{x,t} \langle \delta N_{x,t}\, \delta N_{x',t'}\rangle_{\rm ind} = b(x,t)\,, \qquad \langle \delta N_{x,t}\, \delta N_{x',t'}\rangle_{\rm ind}\Big|_{t=t'} = 0\,. \tag{110}$$

Accordingly, the first term is the solution of the corresponding homogeneous equation. It also implements the required initial conditions:

$$\hat{D}_{x,t}\langle\delta N_{x,t}\,\delta N_{x',t'}\rangle_0 = 0\,, \qquad \langle\delta N_{x,t}\,\delta N_{x',t'}\rangle_0\Big|_{t=t'} = \langle\delta N_{x,t'}\,\delta N_{x',t'}\rangle\,. \tag{111}$$

In view of these equations it is evident that the dynamics of the term labeled "0" are not affected by the field fluctuations. Indeed, this term is the solution one would obtain for the case $\delta\boldsymbol{F}_{x,t} = 0$ describing a system without any interparticle correlations that is only influenced by the mean field $\langle\boldsymbol{F}_{x,t}\rangle$ entering the operator $\hat{D}_{x,t}$. Therefore, one can identify this term with the natural statistical correlations discussed in Section 3.4. Assuming that they determine the initial state as well, one infers from (71) that

$$\langle\delta N_{x,t}\,\delta N_{x',t'}\rangle_0\Big|_{t=t'} = \delta(x-x')\langle N_{x',t'}\rangle\,. \tag{112}$$

In this approximation, the initial value problem for the homogeneous solution can readily be solved by applying the general results obtained in the last subsection:

$$\langle\delta N_{x,t}\,\delta N_{x',t'}\rangle_0 = \delta\big(\tilde{x}(x,t)-x'\big)\langle N_{x',t'}\rangle\,. \tag{113}$$

Since these correlations are completely independent of $\delta\boldsymbol{F}_{x,t}$ they are, in this sense, spontaneous. The effect of the field fluctuations is entirely contained in the inhomogeneous solution, characterizing the latter as induced correlations. Hence the label "ind". The spontaneous and induced terms contributing to the two-time correlations (109) correspond to the decomposition of its one-time counterpart (69) into self-correlations and interparticle correlations.

It has already been mentioned before that it is more convenient for practical purposes to work with the fluctuations itself rather than with the correlations. Since in both cases the dynamics are governed by the operator $\hat{D}_{x,t}$, the previous procedure can be repeated. Although the following formulation is to some extent simpler and clearer and hence predominantly found in the literature (Ichimaru, 1977, 1992; Klimontovich, 1997; Tolias et al., 2015), it was still necessary to consider the correlations first. The reason is that the bare microscopic fluctuations are not accessible to observations, it is therefore not possible to specify initial values for them. Only the correlations are physically meaningful entities that can serve as initial conditions. Keeping this in mind, the results of Section 3.7 imply that the fluctuations can be decomposed into two parts:

$$\delta N_{x,t} = \delta N^0_{x,t} + \delta N^{\mathrm{ind}}_{x,t}\,. \tag{114}$$

The first term on the right-hand side denotes the homogeneous solution of (79) or its approximation (92). It satisfies the same initial conditions as the total fluctuations. Based on the observation that the dynamics of $\delta N^0_{x,t}$ are independent of the field fluctuations, this term can be ascribed to the natural statistical fluctuations again. Since the latter are defined by the second central moment, one must formulate the initial condition as

$$\langle\delta N^0_{x,t}\,\delta N^0_{x',t'}\rangle = \delta\big(\tilde{x}(x,t)-x'\big)\langle N_{x',t'}\rangle\,. \tag{115}$$

In the case of an unmagnetized plasma, for instance, the characteristics of the evolution operator are given by Eq. (102), implying

$$\langle\delta N^0_{\boldsymbol{r},\boldsymbol{p},t}\,\delta N^0_{\boldsymbol{r}',\boldsymbol{p}',t'}\rangle = \delta(\boldsymbol{r}-\boldsymbol{r}'-\boldsymbol{v}(t-t'))\,\delta(\boldsymbol{p}-\boldsymbol{p}')\langle N_{\boldsymbol{r}',\boldsymbol{p}',t'}\rangle\,. \tag{116}$$

The last term in (114) describes the part of the fluctuations that is induced by the field fluctuations and that constitutes the interparticle correlations. Mathematically, it is the inhomogeneous solution of (79) or (92), respectively, with vanishing initial value:

$$\hat{D}_{x,t}\,\delta N^{\mathrm{ind}}_{x,t} = b'(x,t)\,, \qquad \delta N^{\mathrm{ind}}_{x,t_0} = 0\,, \tag{117}$$

where

$$\begin{aligned} b'(x,t) &\equiv -\delta\boldsymbol{F}_{x,t}\cdot\nabla_p\langle N_{x,t}\rangle + \nabla_p\cdot\big(\langle\delta\boldsymbol{F}_{x,t}\,\delta N_{x,t}\rangle - \delta\boldsymbol{F}_{x,t}\,\delta N_{x,t}\big) \\ &\simeq -\delta\boldsymbol{F}_{x,t}\cdot\nabla_p\langle N_{x,t}\rangle\,. \end{aligned} \tag{118}$$

The remaining task of kinetic fluctuation theory consists in solving this differential equation. In the course of this, the functional dependency of the field fluctuations on the density fluctuations must be resolved. This self-consistent approach will explicitly be carried out in the next section. There, it will be more convenient to consider the initial value problem in the slightly modified formulation

$$\hat{D}_{x,t}\big(\delta N_{x,t}-\delta N^0_{x,t}\big) \simeq -\delta\boldsymbol{F}_{x,t}\cdot\nabla_p f_1(x,t)\,, \qquad \delta N_{x,t_0} = \delta N^0_{x,t_0}\,. \tag{119}$$

Here, the induced fluctuations were replaced according to (114) and the relation $\langle N_{x,t}\rangle = f_1(x,t)$ was applied once more.

It is instructive to take the formal solution (99) found in the previous subsection into consideration:

$$\delta N_{x,t} = \delta N^0_{x,t} - \left[\int_{t_0}^{t}\mathrm{d}\tau\,\big(\delta\boldsymbol{F}_{x,\tau}\cdot\nabla_p f_1(x,\tau)\big)_{x=x_c(\tilde{x},\tau)}\right]_{\tilde{x}=\tilde{x}(x,t)}\,. \tag{120}$$

Now one can see that the solution of the initial value problem (119) is a linear response expansion of $\delta N_{x,t}$ in terms proportional to the strength of the field fluctuations. As such, this result is the required microscopic justification of the generalized Ohm's law (21) because the density fluctuations are directly related to fluctuations of the current density as will be seen in the next section. The onset of the field fluctuations is set to the initial time t_0, because at this instant only spontaneous fluctuations occur.

This initial value provided by the self-correlations is important in view of the mutual effect of particles and fields on each that was already discussed before (see Fig. 4) and that also appears in regard to the fluctuations. At first glance, this might seem tautologic: The field fluctuations are caused by the density fluctuations, and the density fluctuations are caused by the field fluctuations. It is *not* a circular argumentation, however, if an initial excitation, a *prima causa*, is guaranteed. Mathematically this corresponds to the specification of suitable initial conditions. Physically this is not satisfactory, however, because one needs to explain where this initial excitation of fluctuations comes from and why $\delta N_{x,t} = 0$ and $\delta \boldsymbol{F}_{x,t} = 0$ is not an acceptable solution for the cycle of mutual induction of fluctuations. This is where the natural statistical fluctuations come into play. Since they occur independently of field fluctuations in any plasma, they are the required source that initiates the cyclic interplay of particle and field fluctuations: The spontaneous fluctuations $\delta N^0_{x,t}$ induce field fluctuations which in turn lead to induced fluctuations $\delta N^{\mathrm{ind}}_{x,t}$ of the phase space density, and so forth. In order to follow the response even further than in the equations derived above, nonlinear terms must be taken into account. Such a procedure is usually carried out by expanding the fluctuations $\delta N_{x,t}$ in powers of the field fluctuations (Yoon et al., 2016).

4. Plasma fluctuation theory

Outline. After the separate discussions of the statistical and the electrodynamic aspects relevant for this article, both parts will be unified in this section yielding a self-consistent theory of plasma fluctuations that takes the full coupling of particles and fields into account. This consolidation is very fruitful because both sides complement each other: Kinetic theory provides a sound justification for the linear response relation underlying the electrodynamic eigenmode analysis and it also determines the conductivity tensor. In turn, the full electromagnetic dynamics covered by Maxwell's equations are the proper generalization of the non-relativistic Coulomb interactions that were assumed for the sake of simplicity in parts of the previous section. Since the Laplace transform of the kinetic fluctuation equation leads to a convolution integral if the distribution function is time dependent, a timescale analysis is introduced in order to simplify the resulting equations. Further new features are the symmetries of the second order correlation functions, in particular their invariance with respect to spatial or temporal translations, that entail a simplified inversion formula for the Fourier–Laplace transform. The latter is not only a requisite needed for the subsequent investigations but it also simplifies the computations carried out at the end of this section addressing the spectrum of the natural statistical fluctuations.

References. The topics covered here are discussed in more detail by Akhiezer et al. (1975); Alexandrov et al. (1984); Klimontovich (1982, 1997); Oberman and Williams (1983), and Sitenko (1967, 1982).

4.1. Microscopic level of description

The system under consideration is a fully ionized plasma consisting of electrons and ions of the same kind. The position and momentum vectors of the ith particle of species a are denoted by $x_{ia} = (\boldsymbol{r}_{ia}, \boldsymbol{p}_{ia})$. Considering that the electric charge e_a and the rest mass m_a are the same for every particle of a given species, the equations of motion in this notation read

$$\dot{\boldsymbol{r}}_{ia} = \boldsymbol{v}_{ia} = \boldsymbol{p}_{ia}/\Gamma_{ia} m_a \,, \tag{121}$$

$$\dot{\boldsymbol{p}}_{ia} = \left(\boldsymbol{F}^{\mathrm{ext},a}_{x,t} + \boldsymbol{F}^a_{x,t}\right)_{x=x_{ia}} = \left[\boldsymbol{F}^{\mathrm{ext},a}_{\boldsymbol{r},\boldsymbol{p},t} + e_a\left(\boldsymbol{E}_{\boldsymbol{r},t} + \frac{\boldsymbol{v}}{c} \times \boldsymbol{B}_{\boldsymbol{r},t}\right)\right]_{x=x_{ia}} . \tag{122}$$

Here, $\Gamma_{ia} = (1 + \boldsymbol{p}^2_{ia}/m^2_a c^2)^{1/2}$ denotes the Lorentz factor and $\boldsymbol{F}^{\mathrm{ext},a}_{x,t}$ accounts for possible external forces. In order to obtain a self-consistent model of the plasma, this mechanical description of the particle dynamics must be complemented by the electrodynamic theory of the fields. This is achieved by employing Maxwell's equations,

$$\nabla \times \boldsymbol{E}_{\boldsymbol{r},t} = -(1/c)\partial \boldsymbol{B}_{\boldsymbol{r},t}/\partial t \,, \tag{123}$$

$$\nabla \times \boldsymbol{B}_{\boldsymbol{r},t} = (4\pi/c)\boldsymbol{J}_{\boldsymbol{r},t} + (1/c)\partial \boldsymbol{E}_{\boldsymbol{r},t}/\partial t \,, \tag{124}$$

$$\nabla \cdot \boldsymbol{E}_{\boldsymbol{r},t} = 4\pi \rho_{\boldsymbol{r},t} \,, \tag{125}$$

$$\nabla \cdot \boldsymbol{B}_{\boldsymbol{r},t} = 0 \,. \tag{126}$$

The equations established so far are closed by the relations between the motion of the particles and the charge and current densities:

$$\rho_{\boldsymbol{r},t} = \sum_a \sum_i e_a \,\delta\big(\boldsymbol{r} - \boldsymbol{r}_{ia}(t)\big) \,, \tag{127}$$

$$\boldsymbol{J}_{\boldsymbol{r},t} = \sum_a \sum_i e_a \boldsymbol{v}_{ia}(t)\,\delta\big(\boldsymbol{r} - \boldsymbol{r}_{ia}(t)\big) \,. \tag{128}$$

The whole set of equations clearly reveals the fundamental coupling between particles and fields that is characteristic for a plasma (see Fig. 4). In order to implement the statistical tools provided in the last section, the Klimontovich function for every particle species a is introduced:

$$N^a_{x,t} \equiv N^a_{\boldsymbol{r},\boldsymbol{p},t} \equiv \sum_i \delta\big(x - x_{ia}(t)\big) \equiv \sum_i \delta\big(\boldsymbol{r} - \boldsymbol{r}_{ia}(t)\big)\, \delta\big(\boldsymbol{p} - \boldsymbol{p}_{ia}(t)\big)\,. \tag{129}$$

The rationale behind distinguishing the particle species at this point instead of simply defining one density for the entire system is more than just an ease of notation resulting from the common mass and charge values shared by the respective population of electrons and ions. Rather, it becomes necessary because the invariance of the distribution functions under the permutation of particles was a crucial assumption in the course of the last section. This condition inevitably leads to the requirement that all particles accounted for by a given distribution function must belong to the same species. The next task regarding the transition to a statistical description is to substitute the particle variables by the corresponding expression containing $N^a_{x,t}$. Thus, the equations of motion are replaced by the Klimontovich equation analogous to (73),

$$\big[\partial/\partial t + \boldsymbol{v}\cdot\nabla + \big(\boldsymbol{F}^{\mathrm{ext},a}_{\boldsymbol{r},\boldsymbol{p},t} + \boldsymbol{F}^a_{\boldsymbol{r},\boldsymbol{p},t}\big)\cdot\nabla_p\big] N^a_{\boldsymbol{r},\boldsymbol{p},t} = 0\,. \tag{130}$$

Moreover, the charge and current densities must also be formulated in terms of $N_{x,t}$. From the definition (129) one readily obtains

$$\rho_{\boldsymbol{r},t} = \sum_a e_a \int \mathrm{d}^3p\; N^a_{\boldsymbol{r},\boldsymbol{p},t}\,, \qquad \boldsymbol{J}_{\boldsymbol{r},t} = \sum_a e_a \int \mathrm{d}^3p\; \boldsymbol{v} N^a_{\boldsymbol{r},\boldsymbol{p},t}\,. \tag{131}$$

Thus, the microscopic state of the plasma can be described completely in terms of the electromagnetic fields and the Klimontovich function by the closed set of Eqs. (123)–(126) and (130)–(131).

4.2. Averages and fluctuations

In order to obtain a macroscopic description of the plasma, the ensemble averages of the electromagnetic fields and the phase space density must be computed. Since Maxwell's equations are linear in the microscopic entities, taking their average poses no difficulties:

$$\nabla \times \langle \boldsymbol{E}_{\boldsymbol{r},t}\rangle = -(1/c)\partial\langle \boldsymbol{B}_{\boldsymbol{r},t}\rangle/\partial t\,, \tag{132}$$

$$\nabla \times \langle \boldsymbol{B}_{\boldsymbol{r},t}\rangle = (4\pi/c)\langle \boldsymbol{J}_{\boldsymbol{r},t}\rangle + (1/c)\partial\langle \boldsymbol{E}_{\boldsymbol{r},t}\rangle/\partial t\,, \tag{133}$$

$$\nabla \cdot \langle \boldsymbol{E}_{\boldsymbol{r},t}\rangle = 4\pi\langle \rho_{\boldsymbol{r},t}\rangle\,, \tag{134}$$

$$\nabla \cdot \langle \boldsymbol{B}_{\boldsymbol{r},t}\rangle = 0\,. \tag{135}$$

The solution of these equations is simplified by the characteristic feature of a plasma that it is overall electrically neutral, i. e., its average charge and current density vanish:

$$\langle \rho_{\boldsymbol{r},t}\rangle = \sum_a e_a \int \mathrm{d}^3p\; \langle N^a_{\boldsymbol{r},\boldsymbol{p},t}\rangle = 0\,, \tag{136}$$

$$\langle \boldsymbol{J}_{\boldsymbol{r},t}\rangle = \sum_a e_a \int \mathrm{d}^3p\; \boldsymbol{v}\langle N^a_{\boldsymbol{r},\boldsymbol{p},t}\rangle = 0\,. \tag{137}$$

Unless previously excited waves are present, the boundary conditions allow no other solutions than the trivial ones, that is, on average there are no internal electromagnetic fields, and therefore no internal Lorentz force:

$$\langle \boldsymbol{E}_{\boldsymbol{r},t}\rangle = 0\,, \qquad \langle \boldsymbol{B}_{\boldsymbol{r},t}\rangle = 0\,, \qquad \langle \boldsymbol{F}^a_{\boldsymbol{r},\boldsymbol{p},t}\rangle = 0\,. \tag{138}$$

However, there still may be external fields present. By definition, these are not generated by the charged particles of the plasma itself but by some other means. In the astrophysical context, the most commonly discussed scenarios accessible to an analytical treatment are a constant and uniform magnetic field $\boldsymbol{B}$ on the one hand and unmagnetized plasmas on the other hand. Since the external force does not depend on the phase space variables of the plasma particles by definition, it remains unchanged by the averaging operator:

$$\langle \boldsymbol{F}^{\mathrm{ext},a}_{\boldsymbol{p}}\rangle = \boldsymbol{F}^{\mathrm{ext},a}_{\boldsymbol{p}} = \begin{cases} 0 & \text{(unmagnetized plasma)},\\ (e_a/c)\boldsymbol{v}\times\boldsymbol{B} & \text{(magnetized plasma)}. \end{cases} \tag{139}$$

The distribution function $f^a_1(x,t) = \langle N^a_{x,t}\rangle$ of the plasma particles must be found by solving the averaged Klimontovich function. In the mean field approximation, collisions are neglected on account of the smallness of the plasma parameter and one obtains the Vlasov equation, viz. (87),

$$\big(\partial/\partial t + \boldsymbol{v}\cdot\nabla + \boldsymbol{F}^{\mathrm{ext},a}_{\boldsymbol{p}}\cdot\nabla_p\big) f^a_1(\boldsymbol{r},\boldsymbol{p},t) = 0\,. \tag{140}$$

In the two particular cases under consideration the external force is independent of the spatial coordinates. Therefore, it is reasonable to assume that the distribution function $f_1^a(x,t) = \langle N_{x,t}^a \rangle$ is also symmetric with respect to translations in space. In the remainder of this work, the one-particle distribution function will be assumed to be spatially uniform, i. e., independent of $\boldsymbol{r}$. Under this assumption, however, its normalization becomes a subtle matter that requires special attention. In view of (45) and (56), f_1^a is normalized according to

$$\int \mathrm{d}x\, f_1^a(x,t) \equiv \int \mathrm{d}^3r\, \mathrm{d}^3p\, f_1^a(\boldsymbol{r},\boldsymbol{p},t) = N_a\,. \tag{141}$$

If the distribution function is independent of the spatial coordinate, the corresponding integral diverges. Fortunately, this problem can be dealt with very easily. To this end, one considers the plasma to be contained in a box of finite volume V and computes the limit $V \to \infty$ in the end. Since the volume integral simply gives V, one obtains

$$\int \mathrm{d}^3p\, f_1^a(\boldsymbol{p},t) = N_a/V \equiv n_a\,, \tag{142}$$

where n_a denotes the average particle density of species a. If the latter remains fixed during the limit $V \to \infty$, then the previous equation still applies for infinite volumes. Since it is more convenient to deal with a distribution function normalized to unity, a new function is introduced that meets this requirement:

$$f_{\boldsymbol{p},t}^a \equiv f_1^a(\boldsymbol{p},t)/n_a\,, \qquad \int \mathrm{d}^3p\, f_{\boldsymbol{p},t}^a = 1\,. \tag{143}$$

An important example of such a homogeneous function is the Maxwell–Jüttner distribution. It is the relativistic generalization of the Maxwell distribution function describing a thermal equilibrium plasma and it will be underlying the investigations of the unmagnetized intergalactic medium.

In order to compute the plasma fluctuations, the equations of Section 4.1 containing the full microscopic details must be consulted again. The general procedure is to decompose every dynamical function b into its average and its fluctuating component, $b = \langle b \rangle + \delta b$, and to subtract the corresponding equation for the average afterwards. Due to their linearity, Maxwell's equations hold unaltered for the fluctuating fields,

$$\nabla \times \delta\boldsymbol{E}_{\boldsymbol{r},t} = -(1/c)\partial\, \delta\boldsymbol{B}_{\boldsymbol{r},t}/\partial t\,, \tag{144}$$

$$\nabla \times \delta\boldsymbol{B}_{\boldsymbol{r},t} = (4\pi/c)\delta\boldsymbol{J}_{\boldsymbol{r},t} + (1/c)\partial\, \delta\boldsymbol{E}_{\boldsymbol{r},t}/\partial t\,, \tag{145}$$

$$\nabla \cdot \delta\boldsymbol{E}_{\boldsymbol{r},t} = 4\pi\, \delta\rho_{\boldsymbol{r},t}\,, \tag{146}$$

$$\nabla \cdot \delta\boldsymbol{B}_{\boldsymbol{r},t} = 0\,, \tag{147}$$

and the same applies to the equations for the fluctuating charge and current densities:

$$\delta\rho_{\boldsymbol{r},t} = \sum_a e_a \int \mathrm{d}^3p\, \delta N_{\boldsymbol{r},\boldsymbol{p},t}^a \qquad \delta\boldsymbol{J}_{\boldsymbol{r},t} = \sum_a e_a \int \mathrm{d}^3p\, \boldsymbol{v}\, \delta N_{\boldsymbol{r},\boldsymbol{p},t}^a\,. \tag{148}$$

As argued before, the external force is independent of the microscopic phase space variables. Therefore, it remains invariant under the averaging operator, viz. (139), so it does not fluctuate at all. The only force fluctuations are the internal ones, generated by the fluctuating electromagnetic fields:

$$\delta\boldsymbol{F}_{\boldsymbol{r},\boldsymbol{p},t}^{\mathrm{ext},a} = 0\,, \qquad \delta\boldsymbol{F}_{\boldsymbol{r},\boldsymbol{p},t}^a = e_a\big(\delta\boldsymbol{E}_{\boldsymbol{r},t} + (\boldsymbol{v}/c)\times\delta\boldsymbol{B}_{\boldsymbol{r},t}\big)\,. \tag{149}$$

The above procedure for deriving equations determining the fluctuations does not result in a closed equation for $\delta N_{\boldsymbol{r},\boldsymbol{p},t}^a$ in the case of the Klimontovich function. According to the discussion of the last section, an additional closure relation becomes necessary. If the plasma parameter is small, a suitable approximation is given by (119), complemented by (115). In the context of the current section it reads

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla + \boldsymbol{F}_{\boldsymbol{p}}^{\mathrm{ext},a}\cdot\nabla_p\right)\big(\delta N_{\boldsymbol{r},\boldsymbol{p},t}^a - \delta N_{\boldsymbol{r},\boldsymbol{p},t}^{a0}\big) = -n_a\, \delta\boldsymbol{F}_{\boldsymbol{r},\boldsymbol{p},t}^a\cdot\nabla_p f_{\boldsymbol{p},t}^a \tag{150}$$

with

$$\delta N_{\boldsymbol{r},\boldsymbol{p},t_0}^a = \delta N_{\boldsymbol{r},\boldsymbol{p},t_0}^{a0}\,, \tag{151}$$

$$\langle \delta N_{x,t}^{a0}\, \delta N_{x',t'}^{b0} \rangle = \delta_{ab}\, \delta\big(\tilde{x}(x,t) - x'\big) n_a f_{\boldsymbol{p}',t'}^a\,. \tag{152}$$

Since the last equation describes the *self*-correlations of discrete particles, a non-vanishing value can occur only if the particle species of both fluctuations are identical. This condition is implemented in the Kronecker delta, δ_{ab}. It constitutes a new feature that did not appear in the previous section because only a single particle species was considered there for simplicity. The relevant characteristics $\tilde{x}(x,t)$ are given by (102) and (105)–(106). In the case of an unmagnetized plasma, for instance,

$$\langle \delta N_{\boldsymbol{r},\boldsymbol{p},t}^{a0}\, \delta N_{\boldsymbol{r}',\boldsymbol{p}',t'}^{b0} \rangle = \delta_{ab}\, \delta(\boldsymbol{r} - \boldsymbol{r}' - \boldsymbol{v}(t-t'))\, \delta(\boldsymbol{p} - \boldsymbol{p}') n_a f_{\boldsymbol{p}',t'}^a\,. \tag{153}$$

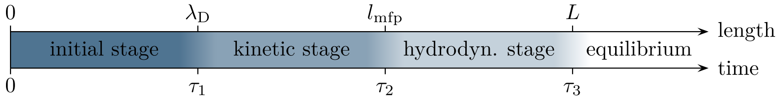

Fig. 7. Ordering of time- and lengthscales according to Bogoliubov's hypothesis and the corresponding stages in the evolution towards equilibrium.

This formulation of the natural statistical fluctuations is the one most commonly found in the plasma literature (Alexandrov et al., 1984; Klimontovich, 1982, 1997; Sitenko, 1982; Tolias et al., 2015; Yoon et al., 2016). Similarly, the corresponding expression for a magnetized plasma can be obtained from the characteristics (105)–(106):

$$\begin{aligned}\langle \delta N^{a0}_{\boldsymbol{r},\boldsymbol{p},t}\, \delta N^{b0}_{\boldsymbol{r}',\boldsymbol{p}',t'}\rangle = {} & \delta_{ab} n_a f^a_{\boldsymbol{p}',t'}\, \delta\big\{\boldsymbol{r}-\boldsymbol{r}'-\boldsymbol{e}_\parallel \times (\boldsymbol{v}/\Omega_g) - \boldsymbol{v}_\parallel (t-t') \\ & - (\boldsymbol{v}_\perp/\Omega_g)\sin[\Omega_g(t-t')] + (\boldsymbol{e}_\parallel \times \boldsymbol{v}/\Omega_g)\cos[\Omega_g(t-t')]\big\} \\ & \delta\big\{\boldsymbol{p}_\parallel + \boldsymbol{p}_\perp \cos[\Omega_g(t-t')] + (\boldsymbol{e}_\parallel \times \boldsymbol{p})\sin[\Omega_g(t-t')] - \boldsymbol{p}'\big\}\,. \end{aligned} \tag{154}$$

Now that the system of basic equations is specified, the subsequent subsections are devoted to the computation of the plasma fluctuations $\delta\boldsymbol{E}_{\boldsymbol{r},t}$, $\delta\boldsymbol{B}_{\boldsymbol{r},t}$ and $\delta N^a_{\boldsymbol{r},\boldsymbol{p},t}$. These calculations follow the *test wave approach* that assumes the particle distribution function $f_{\boldsymbol{p},t}$ to be known (Schlickeiser, 2002). Starting from this prescribed distribution, the fluctuations can be obtained by solving equations (144)–(152). This approach neglects the back-reaction of the fluctuations on the distribution function as specified by the collision integral. Apart from the smallness of the plasma parameter discussed before, a justification for this procedure is given by Bogoliubov's hypothesis stating that the distribution function evolves much slower than the fluctuations. It will be discussed next.

4.3. Timescale analysis

Although Bogoliubov (1962) originally formulated his hypothesis in the context of gas kinetics, it can be transferred to the theory of plasmas all the same (Ichimaru, 1992). Preparatory for its actual formulation, three characteristic lengthscales must be defined: (1) The correlation length corresponds to an effective interaction range of the particles. In a plasma, this is the Debye length λ_D. (2) The second lengthscale is the mean free path l_{mfp} between two successive collisions of an average particle. Strictly speaking, there are several of such lengths considering that there are two particle species, electrons and ions: l_{ee}, l_{ii} and l_{ei}. Since any of these obeys the inequalities to be discussed below, their distinction can be omitted. (3) Lastly, there is a length L characterizing the macroscopic dimension of the system at hand. It can be given by the size of the vessel containing a laboratory plasma, or, more generally, the distance specified by the underlying boundary conditions, or the length along which an external magnetic field can be considered uniform. Associated with these lengthscales λ_D, l_{mfp} and L are corresponding timescales τ_1, τ_2 and τ_3 obtained by multiplication with a characteristic speed such as the thermal electron speed. In a collisionpoor plasma, the collective effects dominate over the influence of collisions, i.e., $\lambda_D \ll l_{mfp}$. A similar relation also holds for the timescales, $\tau_1 \ll \tau_2$. Furthermore, a hydrodynamic description of the plasma assumes a thermalization due to interparticle collisions, requiring $l_{mfp} \ll L$. This assumption is only mentioned here for the sake of completeness as it is dispensable in the present work because the weaker requirement $\lambda_D \ll L$ suffices for a kinetic description. To summarize, the ordering of scales is given by (see Fig. 7)

$$\lambda_D \ll l_{mfp} \ll L\,, \qquad \tau_1 \ll \tau_2 \ll \tau_3\,. \tag{155}$$

Apart from this ordering, Bogoliubov's hypothesis consists in the identification of τ_1, τ_2 and τ_3 with the respective relaxation time of the correlations, of the particle distribution function, and of the hydrodynamic variables. The rationale behind these assumptions is the following. By definition, τ_1 estimates how long two particles remain in their domain of mutual interaction. Once some time $t > \tau_1$ has elapsed, both particles are no longer correlated. Thus, τ_1 characterizes the relaxation time of the correlation function. The next parameter, τ_2, describes the mean time between two successive collisions of a given particle. Assuming that collisions are the equilibrating forces yielding a thermal distribution, τ_2 determines the relaxation time for the distribution function.

In this picture, the evolution towards equilibrium occurs in three consecutive stages (Liboff, 2003; Bonitz, 1998): In the *initial stage* $t < \tau_1$, the correlation functions are still evolving due to collective interactions. The particle distribution function cannot respond all that fast and remains almost constant because collisions do not play a significant role, yet. In the *kinetic stage* $\tau_1 < t < \tau_2$, the correlations have already relaxed. Therefore, they can be expressed as functionals of the distribution function, thus enabling a closure of the kinetic hierarchy. Enough time has elapsed for a significant number of collisions to occur that drive the evolution of the distribution function. Once the latter has relaxed, a (magneto-) hydrodynamic behavior sets in during the final, *hydrodynamic stage* $t > \tau_2$. The distribution function, the correlations, and therefore the entire system state are determined by number density, streaming velocity, and temperature. Treumann (1999) conducts a refined analysis of timescales, distinguishing four phases in the evolution of power and entropy: Starting from the linear regime, the system enters a non-linear stage, followed by a turbulent quasi-equilibrium until the final Boltzmann regime is reached.

At this point, the hypothetical character of Bogoliubov's picture shall be stressed once more. Kinetic instabilities, for example, can alter the behavior of the system drastically if a sufficient supply of free energy is available. The assumptions regarding the hydrodynamic regime are not even needed for the present analysis. A crucial premise for the calculations to come, however, is that *the distribution function evolves on a much longer timescale than the correlations and fluctuations* (Ichimaru, 1992; Klimontovich, 1972; Oberman and Williams, 1983; Sitenko, 1982). Bogoliubov's hypothesis was merely introduced here in order to provide a plausible justification of this postulate.

In view of the above assumption, a possible way to proceed is to neglect the time dependence of the distribution function altogether as far as the evolution of the fluctuations is concerned. This simplifies the further computations enormously because, then, the right-hand side of (150) no longer contains a product of two functions of time. Thus, the Laplace transform of the equation in question does not involve a convolution integral but yields solely algebraic expressions. A disadvantage of this course is, however, that the inferred balance equation for the spectral energy density emitted into a given mode M simply reproduces the well-known result of quasilinear theory, $\partial I^{(M)}_{\boldsymbol{k},t}/\partial t = 2[\Im\omega_M(\boldsymbol{k})]I^{(M)}_{\boldsymbol{k},t}$ (Vedenov et al., 1961, 1962; Drummond and Pines, 1962; Sitenko, 1982). The derivation will be discussed in more detail in Section 6.4. So although spontaneous currents were taken into account, there still appears no term in the balance equation that is independent of $I_{\boldsymbol{k}}$ and thus describes spontaneous emission.

A possible way to overcome this disadvantage is to weaken the assumption of a totally time independent distribution function. Instead, the Bogoliubov hypothesis is employed by introducing two *independent* time variables t_1 and t_2 (Davidson, 1972; Oberman and Williams, 1983; Sitenko, 1982; Belyi, 2002; Yoon et al., 2016). The physically infinitesimal time interval $\mathrm{d}t_1$ is supposed to be short enough to resolve the dynamics of the fluctuations and correlations. In contrast, the timescale t_2 is too coarse-grained to record the rapidly oscillating fluctuations, it can only describe the dynamics of the distribution function and the long-term evolution of the fluctuations. By design, the distribution function depends on t_2, but not on t_1. This corresponds to the desired situation that, on the one hand, the Laplace transform with respect to the fluctuation timescale does not lead to a convolution integral because the distribution function is not affected by the transform. Yet on the other hand, the distribution function contains a time dependence after all, leading to a spectral balance equation that accounts for the spontaneously emitted fluctuations (Yoon et al., 2014).

The technique employed by, e. g., Yoon et al. (2014, 2016) to implement this two-timescale approximation follows Sitenko (1982). First of all, the premise that both timescales are independent of each other implies that

$$\partial/\partial t = \partial/\partial t_1 + \partial/\partial t_2 \,. \tag{156}$$

Due to the vanishing initial conditions a Laplace transform (LT) with respect to the fluctuation timescale t_1 yields

$$\partial/\partial t \xrightarrow{\text{LT}} -\imath\omega + \partial/\partial t_2 = -\imath\hat{\omega} \,, \tag{157}$$

where the derivative with respect to the adiabatic timescale t_2 was absorbed into the frequency operator

$$\hat{\omega} \equiv \omega + \imath\,\partial/\partial t_2 \,. \tag{158}$$

In view of (157), the situation is effectively the same as in the case of a single time variable t with a time independent distribution function. The entire effect is contained in the replacement $\omega \to \hat{\omega}$. The further procedure is to solve the algebraic equations in the spectral domain as if the frequency still was a mere number and to postpone the evaluation of the time derivative until the wave equation has been deduced. This course is justified in view of the assumption that taking the slow time variation of the distribution function into account only provides a small correction to the equilibrium case. In the same spirit, the substitution $\omega \to \hat{\omega}$ is carried out: Functions of operators such as $\hat{\omega}$ only make sense if they are expressed as a power series. Due to the smallness of the additional operator contribution in (158), the infinite series can be reduced to the first order approximation. Thus, the absorbed time derivative is retrieved in the wave equation by the replacement

$$h(\omega) \to h(\hat{\omega}) = h\Big(\omega + \imath\,\frac{\partial}{\partial t_2}\Big) \simeq h(\omega) + \imath\,\frac{\partial h}{\partial\omega}\,\frac{\partial}{\partial t_2} \,, \tag{159}$$

where h denotes an arbitrary function of the frequency. This technique is an abridged and hence more easily applicable version of a multiple timescale analysis. The latter was originally developed by Frieman (1963) and Sandri (1963) in order to avoid secularities (i. e., divergences) occurring within the framework of a conventional perturbation expansion. A detailed explanation of this approach and its application to plasma theory can be found in the monographs of Davidson (1972) and Sitenko (1982).

4.4. The wave equation for fluctuating fields

In order to solve the Maxwell–Klimontovich system of equations, it is transferred into the spectral domain by performing a Fourier transform with respect to the spatial variable $\boldsymbol{r}$ and a Laplace transform with respect to the fluctuation timescale t_1. In view of assumption (151) stating that the system was uncorrelated at first, the *induced* fluctuations of the phase space density possess a vanishing initial value, and the same applies to the field fluctuations as well. Apart from that, the transformation of Maxwell's equations is carried out along the lines of Section 2: Faraday's law of induction specifies the magnetic field in terms of the electric field, and this relation allows one to eliminate the former in the generalized Ampere's law, so analogous to (13)–(14) one obtains

$$\delta \boldsymbol{B}_{\boldsymbol{k},\omega} = (c/\omega)\boldsymbol{k} \times \delta \boldsymbol{E}_{\boldsymbol{k},\omega} \,, \tag{160}$$

$$\left[\hat{\mathbf{1}} - (k^2c^2/\omega^2)(\hat{\mathbf{1}} - \boldsymbol{k}\otimes\boldsymbol{k}/k^2)\right]\cdot \delta \boldsymbol{E}_{\boldsymbol{k},\omega} = -(4\pi\imath/\omega)\,\delta \boldsymbol{J}_{\boldsymbol{k},\omega} \,. \tag{161}$$

Since the dependency on the adiabatic timescale t_2 is currently of no relevance, it was suppressed here in the indices in order not to overburden the notation. The transformations of the current density and force fluctuations, (148) and (149), respectively, are straightforward:

$$\delta \boldsymbol{J}_{\boldsymbol{k},\omega} = \sum_a e_a \int \mathrm{d}^3p\ \boldsymbol{v}\,\delta N^a_{\boldsymbol{k},\boldsymbol{p},\omega} \,, \tag{162}$$

$$\delta \boldsymbol{F}^a_{\boldsymbol{k},\boldsymbol{p},\omega} = e_a\big(\delta \boldsymbol{E}_{\boldsymbol{k},\omega} + (\boldsymbol{v}/c)\times \delta \boldsymbol{B}_{\boldsymbol{k},\omega}\big) \,. \tag{163}$$

Once more, Faraday's law can be used to eliminate the magnetic field in the last equation. In the course of this, a double vector product arises that can be simplified by means of the Grassmann identity, i. e., the "bac-cab rule":

$$\delta \boldsymbol{F}^a_{\boldsymbol{k},\boldsymbol{p},\omega}\cdot\nabla_p f^a_{\boldsymbol{p}} = (e_a/\omega)\,\delta \boldsymbol{E}_{\boldsymbol{k},\omega}\cdot\left[(\omega - \boldsymbol{k}\cdot\boldsymbol{v})\nabla_p f^a_{\boldsymbol{p}} + (\boldsymbol{k}\cdot\nabla_p f^a_{\boldsymbol{p}})\boldsymbol{v}\right] . \tag{164}$$

Here, the scalar product with the momentum gradient of the distribution function was computed because the force fluctuations appear only in this constellation in the dynamic equation (150). The Fourier–Laplace transform of the latter reads

$$\big(-\imath(\omega - \boldsymbol{k}\cdot\boldsymbol{v}) + \boldsymbol{F}^{\mathrm{ext},a}_{\boldsymbol{p}}\cdot\nabla_p\big)\big(\delta N^a_{\boldsymbol{k},\boldsymbol{p},\omega} - \delta N^{a0}_{\boldsymbol{k},\boldsymbol{p},\omega}\big) = -n_a\,\delta \boldsymbol{F}^a_{\boldsymbol{k},\boldsymbol{p},\omega}\cdot\nabla_p f^a_{\boldsymbol{p}} \,. \tag{165}$$

The further course of derivations crucially depends on whether the plasma is magnetized or unmagnetized. Since the second case is the easier one and since it reveals the underlying structure leading to the wave equation more clearly, it will be considered first. By definition, the external force vanishes in an unmagnetized plasma, viz. (139). In this situation, solving the previous equation is merely an algebraic task. Making use of expression (164) found earlier for the scalar product of the force fluctuations and the momentum gradient, one obtains

$$\delta N^a_{\boldsymbol{k},\boldsymbol{p},\omega} = \delta N^{a0}_{\boldsymbol{k},\boldsymbol{p},\omega} - \frac{\imath e_a n_a}{\omega}\left(\nabla_p f^a_{\boldsymbol{p}} + \frac{\boldsymbol{k}\cdot\nabla_p f^a_{\boldsymbol{p}}}{\omega - \boldsymbol{k}\cdot\boldsymbol{v}}\,\boldsymbol{v}\right)\cdot\delta \boldsymbol{E}_{\boldsymbol{k},\omega} \,. \tag{166}$$

This is a first order response expansion of the phase space density fluctuations in powers of the field fluctuations. Inserting it into Eq. (162) for the fluctuations of the current density, one obtains a generalized version of Ohm's law with the same form as (21),

$$\delta \boldsymbol{J}_{\boldsymbol{k},\omega} = \delta \boldsymbol{J}^0_{\boldsymbol{k},\omega} + \hat{\boldsymbol{\sigma}}_{\boldsymbol{k},\omega}\cdot\delta \boldsymbol{E}_{\boldsymbol{k},\omega} \,. \tag{167}$$

Here, the conductivity tensor and the source current density stemming from the natural statistical fluctuations are given by

$$\hat{\boldsymbol{\sigma}}_{\boldsymbol{k},\omega} = -\sum_a \frac{\imath e_a^2 n_a}{\omega}\int \mathrm{d}^3p\ \boldsymbol{v}\otimes\left(\nabla_p f^a_{\boldsymbol{p}} + \frac{\boldsymbol{k}\cdot\nabla_p f^a_{\boldsymbol{p}}}{\omega - \boldsymbol{k}\cdot\boldsymbol{v}}\,\boldsymbol{v}\right) , \tag{168}$$

$$\delta \boldsymbol{J}^0_{\boldsymbol{k},\omega} = \sum_a e_a \int \mathrm{d}^3p\ \boldsymbol{v}\,\delta N^{a0}_{\boldsymbol{k},\boldsymbol{p},\omega} \,. \tag{169}$$

This result verifies the assertion of Section 2 that a kinetic plasma model will justify the linear response approximation and, moreover, that it will specify the conductivity tensor, which previously was an unknown macroscopic parameter, in terms of the kinetic parameters of the plasma. It is already known from Section 2 that Ohm's law is the key to disentangling the coupled equations for particles and fields and that, as such, it is a crucial ingredient in the derivation of the wave equation. Indeed, the latter immediately follows from inserting Ohm's law into (161),

$$\hat{\boldsymbol{\Lambda}}_{\boldsymbol{k},\omega}\cdot\delta \boldsymbol{E}_{\boldsymbol{k},\omega} = -(4\pi\imath/\omega)\,\delta \boldsymbol{J}^0_{\boldsymbol{k},\omega} \,, \tag{170}$$

where the Maxwell tensor is given by

$$\hat{\boldsymbol{\Lambda}}_{\boldsymbol{k},\omega} = \hat{\mathbf{1}} - \frac{k^2c^2}{\omega^2}\Big(\hat{\mathbf{1}} - \frac{\boldsymbol{k}\otimes\boldsymbol{k}}{k^2}\Big) + \frac{4\pi\imath}{\omega}\hat{\boldsymbol{\sigma}}_{\boldsymbol{k},\omega} \,. \tag{171}$$

Thus, the earlier results (22) and (23) are recovered on the kinetic level of description. In order to achieve this for a magnetized plasma as well, the external force (139) exerted by a uniform magnetic field $\boldsymbol{B} = B\boldsymbol{e}_\parallel$ must be taken into account. The basic steps of the derivation remain the same, but establishing the response expansion corresponding to (166) becomes more involved. The computations can be facilitated by employing cylindrical coordinates $p_\perp$, φ and $p_\parallel$ in momentum space. Then, the scalar product of the external force and the momentum gradient appearing in (165) possesses the simple form

$$\boldsymbol{F}^{\mathrm{ext},a}_{\boldsymbol{p}}\cdot\nabla_p = -\Omega_a\,\partial/\partial\varphi \,. \tag{172}$$

The relativistic gyrofrequency introduced here is given by its nonrelativistic counterpart divided by the Lorentz factor Γ_a:

$$\Omega_a \equiv e_a B/\Gamma_a m_a c\,. \tag{173}$$

In comparison with the unmagnetized case considered before, the derivative with respect to φ appearing in the second last equation constitutes a new feature. Due to its presence, solving (165) is no longer a purely algebraic task because a differential equation must be solved now:

$$\frac{\partial}{\partial\varphi}\,\delta N^{\mathrm{ind},a}_{\boldsymbol{k},\boldsymbol{p},\omega} + \frac{\imath(\omega-\boldsymbol{k}\cdot\boldsymbol{v})}{\Omega_a}\,\delta N^{\mathrm{ind},a}_{\boldsymbol{k},\boldsymbol{p},\omega} = \frac{n_a}{\Omega_a}\,\delta\boldsymbol{F}^a_{\boldsymbol{k},\boldsymbol{p},\omega}\cdot\nabla_p f^a_{\boldsymbol{p}}\,. \tag{174}$$

Here, the abbreviation $\delta N^{\mathrm{ind},a}_{\boldsymbol{k},\boldsymbol{p},\omega}$ was adopted for the induced fluctuations $\delta N^a_{\boldsymbol{k},\boldsymbol{p},\omega} - \delta N^{a0}_{\boldsymbol{k},\boldsymbol{p},\omega}$ to ease the notation. This ordinary differential equation can either be solved by the method of variation of constants or, alternatively, by employing an integrating factor (Montgomery and Tidman, 1964). It should be noted that the Lorentz factor Γ_a, and hence also the relativistic gyrofrequency are independent of the angular variable φ. Furthermore, the following result obtained by the variation of constants remains valid despite the fact that the right-hand side of Eq. (174) is a functional of $\delta N^{\mathrm{ind,a}}_{\boldsymbol{k},\boldsymbol{p},\omega}$ due to the coupling of particles and fields:

$$\delta N^{\mathrm{ind},a}_{\boldsymbol{k},\boldsymbol{p},\omega} = \frac{n_a}{\Omega_a}\int_{\infty\,\mathrm{sgn}(e_a\Gamma_a)}^{\varphi} \mathrm{d}\varphi'\,\exp\big[\imath\big(\psi(\varphi')-\psi(\varphi)\big)\big]\big[\delta\boldsymbol{F}^a_{\boldsymbol{k},\boldsymbol{p},\omega}\cdot\nabla_p f^a_{\boldsymbol{p}}\big]_{\varphi'}\,. \tag{175}$$

The function $\psi(\varphi)$ denotes an antiderivative of $(\omega-\boldsymbol{k}\cdot\boldsymbol{v})/\Omega_a$ with respect to the angular variable φ. Since only the difference of ψ evaluated at two different values of the argument appears in the above equation, the constant of integration is irrelevant in this context:

$$\mathrm{d}\psi/\mathrm{d}\varphi = (\omega-\boldsymbol{k}\cdot\boldsymbol{v})/\Omega_a\,. \tag{176}$$

An explicit expression for ψ can readily be specified if the remaining freedom to choose a convenient coordinate system is used to demand that the wavevector be perpendicular with respect to the y-axis, i. e., $\boldsymbol{k} = (k_\perp, 0, k_\parallel)$:

$$\psi(\varphi) = \big[(\omega - k_\parallel v_\parallel)\varphi - k_\perp v_\perp \sin\varphi\big]/\Omega_a\,. \tag{177}$$

Another point that needs to be commented on is the lower integration limit appearing in (175). In order to obtain a unique solution of the differential equation (174), a suitable initial condition must be imposed. The initial condition naturally associated with an angular variable such as φ is periodicity. As argued by Montgomery and Tidman (1964), the lower integration limit specified in the solution guarantees that the latter is 2π-periodic for all values of the complex frequency ω, irrespective of the value of its imaginary part. The results obtained in these references are in agreement with (175). This becomes evident by making use of expression (164) for the field fluctuations:

$$\begin{aligned}\delta N^a_{\boldsymbol{k},\boldsymbol{p},\omega} - \delta N^{a0}_{\boldsymbol{k},\boldsymbol{p},\omega} = {}& \frac{e_a n_a}{\Omega_a\omega}\int_{\infty\,\mathrm{sgn}(e_a\Gamma_a)}^{\varphi}\mathrm{d}\varphi'\,\exp\big[\imath\big(\psi(\varphi')-\psi(\varphi)\big)\big]\\ & \big[(\omega-\boldsymbol{k}\cdot\boldsymbol{v})\nabla_p f^a_{\boldsymbol{p}} + (\boldsymbol{k}\cdot\nabla_p f^a_{\boldsymbol{p}})\boldsymbol{v}\big]_{\varphi'}\cdot\delta\boldsymbol{E}_{\boldsymbol{k},\omega}\,.\end{aligned} \tag{178}$$

This equation is the counterpart of (166) for the case of a magnetized plasma. Since the field fluctuations are independent of the momentum variable, they can be extracted outside the integral, thus allowing the derivation of the response relation along the lines of the unmagnetized case: Inserting the equation above into (162) for the fluctuations of the current density already reproduces Ohm's law (167), albeit with a different conductivity tensor:

$$\begin{aligned}\hat{\boldsymbol{\sigma}}_{\boldsymbol{k},\omega} = {}& \sum_a \frac{e_a^2 n_a}{\Omega_a\omega}\int \mathrm{d}^3p\;\boldsymbol{v}\otimes\int_{\infty\,\mathrm{sgn}(e_a\Gamma_a)}^{\varphi}\mathrm{d}\varphi'\,\exp\big[\imath\big(\psi(\varphi')-\psi(\varphi)\big)\big]\\ & \big[(\omega-\boldsymbol{k}\cdot\boldsymbol{v})\nabla_p f^a_{\boldsymbol{p}} + (\boldsymbol{k}\cdot\nabla_p f^a_{\boldsymbol{p}})\boldsymbol{v}\big]_{\varphi'}\,.\end{aligned} \tag{179}$$

This expression for the linear response tensor is not well suited for practical purposes. Alternative formulations were derived by Schlickeiser (2002, 2010). But still and all, the previous computations meet the present objective to establish the validity of the linear response relation (167) also for magnetized plasmas. The wave equation (170) is the same in both cases, too, since it is a direct consequence of Ohm's law as was shown above. The only difference lies in the conductivity tensor entering the Maxwell tensor.

The way towards a solution of the wave equation has already been paved in Section 2. In the first step, one multiplies (170) by the adjugate $\hat{\boldsymbol{\lambda}}_{\boldsymbol{k},\omega}$ and divides by the determinant $\Lambda(\boldsymbol{k},\omega)$ of the Maxwell tensor:

$$\delta\boldsymbol{E}_{\boldsymbol{k},\omega} = -\frac{4\pi\imath}{\omega}\,\frac{\hat{\boldsymbol{\lambda}}_{\boldsymbol{k},\omega}\cdot\delta\boldsymbol{J}^0_{\boldsymbol{k},\omega}}{\Lambda(\boldsymbol{k},\omega)}\,. \tag{180}$$

In the second step, both the Fourier and Laplace transform must be inverted. This can be achieved by the inversion formulas (A.7) and (A.11). As shown before, this leads to the eigenmode solutions, viz. (35)–(37),

$$\delta\boldsymbol{E}_{\boldsymbol{r},t} = \sum_M \int \mathrm{d}^3k\;\mathrm{e}^{\imath(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_M(\boldsymbol{k})t)}\,\delta\boldsymbol{E}^{(M)}_{\boldsymbol{k}}\,, \tag{181}$$

$$\delta \boldsymbol{B}_{\boldsymbol{r},t} = \sum_M \int \mathrm{d}^3k\, \mathrm{e}^{\imath(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_M(\boldsymbol{k})t)}\,(c/\omega_M(\boldsymbol{k}))\boldsymbol{k}\times\delta\boldsymbol{E}_{\boldsymbol{k}}^{(M)}\,, \tag{182}$$

with

$$\delta\boldsymbol{E}_{\boldsymbol{k}}^{(M)} = 2\pi\imath\,\mathrm{Res}_{\omega_M(\boldsymbol{k})}\left\{\frac{(4\pi\imath/\omega)\hat{\boldsymbol{\lambda}}_{\boldsymbol{k},\omega}\cdot\delta\boldsymbol{J}^0_{\boldsymbol{k},\omega}}{\Lambda(\boldsymbol{k},\omega)}\right\}. \tag{183}$$

A crucial assumption entering the inversion formula for the Laplace transform is that both the numerator and the denominator on the right-hand side of (180) are entire functions of the complex frequency (see Section 2.5). This is necessary to ensure that $\delta\boldsymbol{E}_{\boldsymbol{k},\omega}$ is a meromorphic function whose poles ω_M are the roots of the denominator, $\Lambda(\boldsymbol{k},\omega_M)=0$. Within the kinetic framework underlying the present analysis, this requirement means that the Maxwell tensor, and hence the conductivity tensor, must be entire functions. In view of (168), this amounts to a proper treatment of the singularity appearing in the denominator of the integrand. Mathematically, one needs to find an analytical continuation of a function with the generic form

$$H(z) = \int_{-\infty}^{\infty}\mathrm{d}t\,\frac{h(t)}{z-t}\,, \tag{184}$$

where z is a complex number and $h(z)$ an entire function. Since t is real, the singularity of the integrand lies on the real axis. Consequently, $H(z)$ is holomorphic both in the upper and lower half plane but not on the border $\Im z = 0$. In order to find an analytical continuation $\tilde{H}(z)$, one must apply the Sokhotski–Plemelj theorem (Blanchard, 2003)

$$\lim_{y\to+0}\int_{-\infty}^{\infty}\mathrm{d}t\,\frac{h(t)}{x\pm\imath y-t} = \mathcal{P}\int_{-\infty}^{\infty}\mathrm{d}t\,\frac{h(t)}{x-t}\mp\imath\pi h(x)\,. \tag{185}$$

Here, $\mathcal{P}$ denotes the Cauchy principal value integral. This equation provides the correct treatment of the singularity appearing in the denominator of the integrand. It reveals a discontinuity of $H(z)$ that appears during the transition between the upper and lower half plane. In this light, the analytical continuation of $H(z)$ is given by (Montgomery and Tidman, 1964; Schlickeiser, 2002)

$$\tilde{H}(z) = \begin{cases}\int_{-\infty}^{\infty}\mathrm{d}t\,\frac{h(t)}{z-t} & \text{for } \Im z>0,\\ \int_{-\infty}^{\infty}\mathrm{d}t\,\frac{h(t)}{z-t}-2\pi\imath h(z) & \text{for } \Im z<0.\end{cases} \tag{186}$$

Thus, the assumption that both the numerator and the denominator on the right-hand side of (180) are entire functions can actually be guaranteed by means of an analytical continuation. This justifies the mode representations (181) and (182). At the same time, this result substantiates the assertion from the introductory section that an analytical continuation of the form factors and dispersion functions is an indispensable requirement.

4.5. Homogeneous and stationary turbulence

Given a dynamical function $a_{\boldsymbol{r},t}(X)$ such as the electric or magnetic field, solely the associated fluctuations $\delta a_{\boldsymbol{r},t}(X)$ were considered so far. Due to their random nature, however, only the corresponding moments, that is, the averages of products of several fluctuating quantities, are physically relevant. An important example encountered in Section 3.3 are the moments of the microscopic phase space distribution function. There, it was emphasized that only the full set of all moments allows for a complete description of the stochastic processes. In practice, however, the second moments, i. e., the second order correlations, are the most important ones (see Balescu (2005) and the references therein). In the case of two fluctuating fields $\delta a_{\boldsymbol{r},t}$ and $\delta b_{\boldsymbol{r},t}$ the generic form of the two-point, two-time correlation function is $\langle\delta a_{\boldsymbol{r},t}\,\delta b_{\boldsymbol{r}',t'}\rangle$. Since the latter involves the product of two functions of position and time, possible symmetries with respect to these arguments constitute a new feature that will be addressed next. The presentation given here follows the monographs of Alexandrov et al. (1984); Sitenko (1982) and Balescu (2005).

It is advantageous to attend to the spatial coordinates first, temporarily ignoring the time dependence. The correlator $\langle\delta a_{\boldsymbol{r}}\,\delta b_{\boldsymbol{r}'}\rangle$ is called *homogeneous* if it is invariant under spatial translations, that is, if it is a function of the difference $\boldsymbol{r}-\boldsymbol{r}'$ alone. Formally, one could write $\langle\delta a_{\boldsymbol{r}}\,\delta b_{\boldsymbol{r}'}\rangle = \xi(\boldsymbol{r}-\boldsymbol{r}')$ in this case, but the following notation for the function ξ is more intuitive:

$$\langle\delta a_{\boldsymbol{r}}\,\delta b_{\boldsymbol{r}'}\rangle = \langle\delta a\,\delta b\rangle_{\boldsymbol{r}-\boldsymbol{r}'}\,. \tag{187}$$

However, it should be kept in mind that $\langle\delta a\,\delta b\rangle_{\boldsymbol{r}-\boldsymbol{r}'}$ is just a symbol for a function, so in particular, it is not equal to the correlator $\langle\delta a_{\boldsymbol{r}-\boldsymbol{r}'}\,\delta b_{\boldsymbol{r}-\boldsymbol{r}'}\rangle$. A direct consequence of homogeneity is that the two-point correlations effectively depend on just one variable rather than two. Formally, this can be expressed by substituting either $\boldsymbol{r}$ or $\boldsymbol{r}'$ with $\boldsymbol{R}=\boldsymbol{r}-\boldsymbol{r}'$. Then, spatial homogeneity simply means the independence of the absolute position:

$$\langle\delta a_{\boldsymbol{r}}\,\delta b_{\boldsymbol{r}-\boldsymbol{R}}\rangle = \langle\delta a\,\delta b\rangle_{\boldsymbol{R}} = \langle\delta a_{\boldsymbol{r}'+\boldsymbol{R}}\,\delta b_{\boldsymbol{r}'}\rangle\,. \tag{188}$$

An immediate consequence is that the one-point correlation does not depend on position at all, $\langle \delta a_{\boldsymbol{r}}\, \delta b_{\boldsymbol{r}} \rangle = \langle \delta a\, \delta b \rangle_{\boldsymbol{R}=0}$. This illustrates very clearly why the term *homogeneous* is used for turbulence of this kind. It is also instructive to investigate the symmetry at hand in the spectral domain. Performing Fourier transforms with respect to $\boldsymbol{r}$ and $\boldsymbol{r}'$ yields

$$\begin{aligned}
\langle \delta a_{\boldsymbol{k}}\, \delta b_{\boldsymbol{k}'} \rangle &= \int \frac{\mathrm{d}^3 r}{(2\pi)^3} \int \frac{\mathrm{d}^3 r'}{(2\pi)^3}\, \mathrm{e}^{-\imath \boldsymbol{k}\cdot\boldsymbol{r}} \mathrm{e}^{-\imath \boldsymbol{k}'\cdot\boldsymbol{r}'} \langle \delta a_{\boldsymbol{r}}\, \delta b_{\boldsymbol{r}'} \rangle \\
&= \int \frac{\mathrm{d}^3 R}{(2\pi)^3}\, \mathrm{e}^{\imath \boldsymbol{k}'\cdot\boldsymbol{R}} \langle \delta a\, \delta b \rangle_{\boldsymbol{R}} \int \frac{\mathrm{d}^3 r}{(2\pi)^3}\, \mathrm{e}^{-\imath(\boldsymbol{k}+\boldsymbol{k}')\cdot\boldsymbol{r}} \\
&= \langle \delta a\, \delta b \rangle_{-\boldsymbol{k}'}\, \delta(\boldsymbol{k}+\boldsymbol{k}') \\
&= \langle \delta a\, \delta b \rangle_{\boldsymbol{k}}\, \delta(\boldsymbol{k}+\boldsymbol{k}')\,,
\end{aligned} \tag{189}$$

where

$$\langle \delta a\, \delta b \rangle_{\boldsymbol{k}} = \int \frac{\mathrm{d}^3 R}{(2\pi)^3}\, \mathrm{e}^{-\imath \boldsymbol{k}\cdot\boldsymbol{R}} \langle \delta a\, \delta b \rangle_{\boldsymbol{R}} \tag{190}$$

denotes the Fourier transform of $\langle \delta a\, \delta b \rangle_{\boldsymbol{R}}$ with respect to the relative position $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r}'$. This equation establishes a connection between the Fourier transform of the fluctuations on the one hand and the transform of the correlation function on the other hand. It states that spatially homogeneous turbulence implies the occurrence of the function $\delta(\boldsymbol{k}+\boldsymbol{k}')$ in the spectral domain. Vice versa, whenever the spectral representation of correlations has the form (189), its counterpart in real space is homogeneous. This can be seen by performing the inverse Fourier transform,

$$\begin{aligned}
\langle \delta a_{\boldsymbol{r}}\, \delta b_{\boldsymbol{r}'} \rangle &= \int \mathrm{d}^3 k \int \mathrm{d}^3 k'\, \mathrm{e}^{\imath \boldsymbol{k}\cdot\boldsymbol{r}}\, \mathrm{e}^{\imath \boldsymbol{k}'\cdot\boldsymbol{r}'} \langle \delta a_{\boldsymbol{k}}\, \delta b_{\boldsymbol{k}'} \rangle \\
&= \int \mathrm{d}^3 k\, \mathrm{e}^{\imath \boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}')} \langle \delta a\, \delta b \rangle_{\boldsymbol{k}}\,.
\end{aligned} \tag{191}$$

One can deduce this relation more easily simply by inverting the transform (190). The derivation given above may thus serve as a consistency test showing that (189) is the proper result for the relation between the double Fourier transform with respect to $\boldsymbol{r}$ and $\boldsymbol{r}'$ and the single transform with respect to $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r}'$. If one is interested in the single-point correlations, the integral (191) takes the simpler form

$$\langle \delta a_{\boldsymbol{r}}\, \delta b_{\boldsymbol{r}} \rangle = \int \mathrm{d}^3 k\, \langle \delta a\, \delta b \rangle_{\boldsymbol{k}}\,. \tag{192}$$

Thus, the inverse Fourier transform no longer involves the exponential kernel in this particular situation.

Similar arguments can be applied to the dependence on time. Thus, a two-time correlation is said to be *stationary* if it is invariant under time translations, i. e., if it only depends on the difference of the two arguments:

$$\langle \delta a_t\, \delta b_{t'} \rangle = \langle \delta a\, \delta b \rangle_{t-t'}\,. \tag{193}$$

Again, this property may also be expressed in terms of a new variable $\tau = t - t'$ describing the difference between the two instants under consideration:

$$\langle \delta a_t\, \delta b_{t-\tau} \rangle = \langle \delta a\, \delta b \rangle_{\tau} = \langle \delta a_{t'+\tau}\, \delta b_{t'} \rangle\,. \tag{194}$$

The Fourier transforms with respect to t and t' can readily be computed here in the same manner as in the derivation of (189), and they lead to the corresponding result $\langle \delta a\, \delta b \rangle_{\omega'}\, \delta(\omega+\omega')$. In the case of a Laplace transform, however, the spectrum must have a different form because there is no δ-function for complex arguments in the usual sense. In the following it will be argued that the function

$$\delta_{\mathbb{C}}(\omega - \varpi) \equiv -[2\pi\imath(\omega - \varpi)]^{-1} \tag{195}$$

takes the place of Dirac's δ-function in the complex plane (Balescu, 2005). The first indication for this correspondence arises in the context of integral transforms. The δ-function for real-valued arguments is the Fourier transform of $\exp(-\imath\varpi t)$. Thus, it appears natural to seek the generalization to the complex plane by computing the *Laplace* transform (LT) of this exponential, assuming that ϖ is a complex number:

$$\mathrm{e}^{-\imath\varpi t} \xrightarrow{\mathrm{LT}} \int_0^\infty \frac{\mathrm{d}t}{2\pi}\, \mathrm{e}^{\imath(\omega-\varpi)t} = \delta_{\mathbb{C}}(\omega - \varpi) \qquad \text{for } \Im\omega > \Im\varpi. \tag{196}$$

The integral converges only if the imaginary part of $\omega - \varpi$ is positive. This agrees with the general convergence behavior of Laplace transforms discussed in Appendix A.1 and makes $\Im\varpi$ the abscissa of convergence in this particular case. Nevertheless, the function $\delta_{\mathbb{C}}$ defined in (195) is continuous and even holomorphic in the entire complex plane with exception of the simple pole ϖ, so it constitutes the analytic continuation of the Laplace integral above. An integral representation of this function can be obtained by applying the inverse Laplace transform on $\exp(-\imath\varpi t)$, because the latter is given by the Bromwich integral (A.8):

$$e^{-\imath\varpi t} = \int_{\mathcal{B}} d\omega\, e^{-\imath\omega t}\, \delta_{\mathbb{C}}(\omega - \varpi) \qquad (t > 0). \tag{197}$$

Here, $\mathcal{B}$ denotes a horizontal line above the abscissa of convergence, i. e., the integral runs from $-\infty + \imath s$ to $\infty + \imath s$ for some $s > \Im\varpi$. This equation is the second indication that $\delta_{\mathbb{C}}$ behaves like Dirac's δ-function for complex arguments since the characteristic property of the latter is fulfilled at least for the integrand $\exp(-\imath\varpi t)$.

It should be emphasized that the connection between the δ-distribution and the function $\delta_{\mathbb{C}}$ defined in (195) is deeper than one might expect at first glance and that it goes beyond the mere resemblance between the Fourier and Laplace transforms computed above. This relation can be revealed by attempting to generalize the δ-function to complex arguments not only for $\exp(-\imath\varpi t)$, as before, but for arbitrary integrands. This cannot be achieved simply by multiplying two δ-functions, one for the real part of the argument and one for the imaginary part, because that would yield an *area* density. In complex analysis, however, only *contour* integrals appear. By analogy, the three-dimensional δ-function $\delta^{(3)} = \delta^{(1)}\delta^{(1)}\delta^{(1)}$ only makes sense as an integrand of volume integrals, not line integrals. A hint towards a solution of this problem presents itself in view of Cauchy's integral formula for the circulation along a closed curve C with *clockwise* orientation

$$f(\varpi) = \oint_C d\omega\, f(\omega)\, \frac{1}{-2\pi\imath(\omega - \varpi)} = \oint_C d\omega\, f(\omega)\, \delta_{\mathbb{C}}(\omega - \varpi)\,. \tag{198}$$

Ignoring technical considerations concerning the analyticity of $f(\omega)$ and the like, this equation shows that $\delta_{\mathbb{C}}$ is a suitable candidate for a generalization of the δ-distribution to the complex plane because it possesses exactly the property one would demand of such a function. This candidate is not unique, however, because the residue theorem implies that $-\delta_{\mathbb{C}}^*$ also fulfills the requirement,

$$f(\varpi) = \oint_C d\omega\, f(\omega)\, \frac{1}{-2\pi\imath(\omega - \varpi)^*} = \oint_C d\omega\, f(\omega)\, [-\delta_{\mathbb{C}}(\omega - \varpi)]^*\,. \tag{199}$$

Since a rigorous mathematical treatment is beyond the scope of this work, the reader is referred to the treatise of Antoniou et al. (1999), where it is shown that $\mathrm{sgn}(\Im\varpi)\delta_{\mathbb{C}}(\omega - \varpi^*)$ is the proper generalization in a distributional sense. The sign of the imaginary part was also obtained by the more heuristic derivations by Yoon et al. (2014), where a factor ς was introduced with $\varsigma = 1$ for $\Im\omega > 0$ and $\varsigma = -1$ for $\Im\omega < 0$ during the Laplace transform of the natural statistical fluctuations.

For the present purpose, however, it suffices to establish the previously found properties of the function $\delta_{\mathbb{C}}$. Having those at disposal, the spectral representation of stationary turbulence can be computed as follows, starting from the definition of the Laplace transform:

$$\langle \delta a_\omega\, \delta b_{\omega'} \rangle = \int_0^\infty \frac{dt}{2\pi} \int_0^\infty \frac{dt'}{2\pi}\, e^{\imath\omega t} e^{\imath\omega' t'} \langle \delta a\, \delta b \rangle_{t-t'}\,. \tag{200}$$

In comparison to the previous calculations involving the Fourier transform, the situation is different here due to the asymmetric integration limits. In view of this, before changing the variables of integration, the correlation function is split into the two contributions stemming from positive and negative values of the variable $\tau = t - t'$, respectively:

$$\langle \delta a\, \delta b \rangle_\tau^+ \equiv \begin{cases} 0 & \text{for } \tau < 0, \\ \langle \delta a\, \delta b \rangle_\tau & \text{for } \tau \geq 0, \end{cases} \tag{201}$$

$$\langle \delta a\, \delta b \rangle_\tau^- \equiv \begin{cases} \langle \delta a\, \delta b \rangle_\tau & \text{for } \tau < 0, \\ 0 & \text{for } \tau \geq 0. \end{cases} \tag{202}$$

By design, the sum of these functions gives the total correlator $\langle \delta a\, \delta b \rangle_\tau$. Plugging this representation into the double integral above allows for an adjustment of the integration limits:

$$\begin{aligned} \langle \delta a_\omega\, \delta b_{\omega'} \rangle &= \int_0^\infty \frac{dt}{2\pi} \int_0^\infty \frac{dt'}{2\pi}\, e^{\imath\omega t} e^{\imath\omega' t'} \left(\langle \delta a\, \delta b \rangle_{t-t'}^+ + \langle \delta a\, \delta b \rangle_{t-t'}^- \right) \\ &= \int_0^\infty \frac{dt'}{2\pi} \int_{t'}^\infty \frac{dt}{2\pi}\, e^{\imath\omega t} e^{\imath\omega' t'} \langle \delta a\, \delta b \rangle_{t-t'} \\ &\quad + \int_0^\infty \frac{dt}{2\pi} \int_t^\infty \frac{dt'}{2\pi}\, e^{\imath\omega t} e^{\imath\omega' t'} \langle \delta a\, \delta b \rangle_{t-t'} \\ &= \int_0^\infty \frac{dt'}{2\pi} \int_{t'}^\infty \frac{dt}{2\pi} \left(e^{\imath\omega t} e^{\imath\omega' t'} \langle \delta a\, \delta b \rangle_{t-t'} + e^{\imath\omega t'} e^{\imath\omega' t} \langle \delta a\, \delta b \rangle_{t'-t} \right). \end{aligned} \tag{203}$$

In the last step, the labels of the integration variables were interchanged in the second double integral. Next, t is substituted by the difference $\tau = t - t'$, yielding

$$\langle \delta a_\omega\, \delta b_{\omega'} \rangle = \int_0^\infty \frac{dt'}{2\pi}\, e^{\imath(\omega+\omega')t'} \int_0^\infty \frac{d\tau}{2\pi} \left(e^{\imath\omega\tau} \langle \delta a\, \delta b \rangle_\tau + e^{\imath\omega'\tau} \langle \delta a\, \delta b \rangle_\tau^{\mathrm{rev}} \right), \tag{204}$$

where

$$\langle \delta a\, \delta b \rangle_\tau^{\mathrm{rev}} \equiv \langle \delta a\, \delta b \rangle_{-\tau} \tag{205}$$

denotes the time-reversed correlator. As a result, the two integrals are decoupled and may be evaluated independently. According to (196), the first one can be expressed in terms of the $\delta_{\mathbb{C}}$-function, while the second integral gives the respective Laplace transforms of the two functions inside the parentheses:

$$\langle \delta a_\omega\, \delta b_{\omega'} \rangle = \left(\langle \delta a\, \delta b \rangle_\omega + \langle \delta a\, \delta b \rangle_{\omega'}^{\mathrm{rev}} \right) \delta_{\mathbb{C}}(\omega + \omega') \,. \tag{206}$$

If the turbulence is not only stationary but also isotropic, that is, if the two-time correlation function only depends on the absolute value $|\tau| = |t - t'|$, then a time reversal does not alter the correlations, $\langle \delta a\, \delta b \rangle_\tau^{\mathrm{rev}} = \langle \delta a\, \delta b \rangle_\tau$, and the above equation reproduces the result obtained by Pottier (2003) apart from a different notation. In the general non-isotropic case, equation (206) may be simplified by the further assumption that only non-negative time differences are considered, $\tau \geq 0$. Such a restriction occurs in the context of response theory due to the requirement of causality. Then, $\langle \delta a\, \delta b \rangle_\tau^-$ does not contribute and the time-reversed term disappears:

$$\langle \delta a_\omega\, \delta b_{\omega'} \rangle = \langle \delta a\, \delta b \rangle_\omega\, \delta_{\mathbb{C}}(\omega + \omega') \,. \tag{207}$$

This equation constitutes a relation between the Laplace transform of the fluctuations and the transform of their stationary correlation function. It agrees with the findings of Balescu (2005) if one takes into account that the two-time correlator (193) is considered as a function of $t' - t$ instead of $t - t'$, there. Comparing the above relation with the corresponding result (189) for the Fourier transform shows once more that the function $\delta_{\mathbb{C}}$ has taken the place of Dirac's δ-function due to the complex-valued argument.

The inverse transform can be computed by means of the Bromwich integral. Making use of (207) and, afterwards, of (197) one obtains

$$\begin{aligned}
\langle \delta a_t\, \delta b_{t'} \rangle &= \int_{\mathcal{B}} \mathrm{d}\omega \int_{\mathcal{B}'} \mathrm{d}\omega'\ \mathrm{e}^{-\imath\omega t} \mathrm{e}^{-\imath\omega' t'} \langle \delta a_\omega\, \delta b_{\omega'} \rangle \\
&= \int_{\mathcal{B}} \mathrm{d}\omega\ \mathrm{e}^{-\imath\omega t} \langle \delta a\, \delta b \rangle_\omega \int_{\mathcal{B}'} \mathrm{d}\omega'\ \mathrm{e}^{-\imath\omega' t'}\ \delta_{\mathbb{C}}(\omega + \omega') \\
&= \int_{\mathcal{B}} \mathrm{d}\omega\ \mathrm{e}^{-\imath\omega(t-t')} \langle \delta a\, \delta b \rangle_\omega \,.
\end{aligned} \tag{208}$$

This is consistent with the fact that $\langle \delta a\, \delta b \rangle_\omega$ is defined as the Laplace transform of the left-hand side with respect to $\tau = t - t'$ because the equation above simply formulates the inverse transform. Setting $t' = t$ leads to the one-time correlation function

$$\langle \delta a_t\, \delta b_t \rangle = \int_{\mathcal{B}} \mathrm{d}\omega\ \langle \delta a\, \delta b \rangle_\omega \,. \tag{209}$$

So far the spatial and temporal variables were investigated separately. For the sake of completeness and in order to summarize the previous findings, the case of dynamical functions $a_{\mathbf{r},t}$ and $b_{\mathbf{r},t}$ depending on both position and time is considered next. Their two-point, two-time correlation function is homogeneous and stationary if

$$\langle \delta a_{\mathbf{r},t}\, \delta b_{\mathbf{r}',t'} \rangle = \langle \delta a\, \delta b \rangle_{\mathbf{r}-\mathbf{r}',t-t'} \qquad (t \geq t') \,. \tag{210}$$

Thus, the symmetries of homogeneity and stationarity investigated here do not only provide some insight into the state of the turbulence, but they also simplify the calculations because they reduce the number of independent variables appearing in the correlator from four to two. Accordingly, the spectral representation involves only two integrals rather than four,

$$\langle \delta a_{\mathbf{r},t}\, \delta b_{\mathbf{r}',t'} \rangle = \int \mathrm{d}^3 k \int_{\mathcal{B}} \mathrm{d}\omega\ \mathrm{e}^{\imath[\boldsymbol{k}\cdot(\mathbf{r}-\mathbf{r}') - \omega(t-t')]} \langle \delta a\, \delta b \rangle_{\boldsymbol{k},\omega} \,, \tag{211}$$

where

$$\langle \delta a\, \delta b \rangle_{\boldsymbol{k},\omega} \equiv \int \frac{\mathrm{d}^3 R}{(2\pi)^3} \int_0^\infty \frac{\mathrm{d}\tau}{2\pi}\ \mathrm{e}^{-\imath(\boldsymbol{k}\cdot\boldsymbol{R} - \omega\tau)} \langle \delta a\, \delta b \rangle_{\boldsymbol{R},\tau} \,. \tag{212}$$

In order to compute the correlations, therefore, one needs to know their spectrum $\langle \delta a\, \delta b \rangle_{\boldsymbol{k},\omega}$. The theoretical framework outlined in the last subsections, however, only provides the spectrum of the fluctuations $\delta a_{\boldsymbol{k},\omega}$ and $\delta b_{\boldsymbol{k},\omega}$. Fortunately, both are related to each other in the case of homogeneous and stationary turbulence:

$$\langle \delta a_{\boldsymbol{k},\omega}\, \delta b_{\boldsymbol{k}',\omega'} \rangle = \langle \delta a\, \delta b \rangle_{\boldsymbol{k},\omega}\, \delta(\boldsymbol{k} + \boldsymbol{k}')\, \delta_{\mathbb{C}}(\omega + \omega') \,. \tag{213}$$

The resonance conditions $\boldsymbol{k}' = -\boldsymbol{k}$ and $\omega' = -\omega$ become more intuitive to read without the minus sign. In order to reformulate the previous relation accordingly, one can substitute the variables $\boldsymbol{k}'$ and ω' by their respective negatives. According to Schlickeiser (2002, 2010), the Fourier–Laplace transform of a real-valued function obeys the symmetry relation $\mathrm{FL}(-\boldsymbol{k}, -\omega) = \mathrm{FL}^*(\boldsymbol{k}, \omega)$, so eventually one obtains

$$\langle \delta a_{\boldsymbol{k},\omega}\, \delta b^*_{\boldsymbol{k}',\omega'} \rangle = \langle \delta a\, \delta b \rangle_{\boldsymbol{k},\omega}\, \delta(\boldsymbol{k} - \boldsymbol{k}')\, \delta_{\mathbb{C}}(\omega - \omega') \,. \tag{214}$$

The remaining tasks are to prove that the plasma turbulence under consideration is homogeneous and stationary, indeed, and to compute its spectrum. This issue is addressed in the next subsection.

4.6. Spectra of the spontaneously emitted fluctuations

Since the natural statistical fluctuations are the source generating the field fluctuations, it is reasonable to examine them first. According to (153) and (154) they are homogeneous and stationary, both for the unmagnetized as well as the magnetized plasma. Thus, applying the notation and the results of the last subsection,

$$\langle \delta N^{a0}_{\boldsymbol{k},\boldsymbol{p},\omega}\, \delta N^{b0*}_{\boldsymbol{k}',\boldsymbol{p}',\omega'} \rangle = \langle \delta N^{a0}_{\boldsymbol{p}}\, \delta N^{b0}_{\boldsymbol{p}'} \rangle_{\boldsymbol{k},\omega}\, \delta(\boldsymbol{k}-\boldsymbol{k}')\, \delta_{\mathbb{C}}(\omega-\omega')\,, \tag{215}$$

where the first factor on the right-hand side is given by the Fourier–Laplace transform with respect to $\boldsymbol{r}-\boldsymbol{r}'$ and $t-t'$, respectively:

$$\langle \delta N^{a0}_{\boldsymbol{p}}\, \delta N^{b0}_{\boldsymbol{p}'} \rangle_{\boldsymbol{k},\omega} = \int \frac{\mathrm{d}^3(r-r')}{(2\pi)^3} \int_0^\infty \frac{\mathrm{d}(t-t')}{2\pi} \mathrm{e}^{-\imath[\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}')-\omega(t-t')]} \langle \delta N^{a0}_{\boldsymbol{r},\boldsymbol{p},t}\, \delta N^{b0}_{\boldsymbol{r}',\boldsymbol{p}',t'} \rangle\,. \tag{216}$$

For later reference, the integrations in the previous equation are explicitly carried out for the simpler case of an unmagnetized plasma. Remembering that the distribution function appearing in (153) is independent of the fast timescale, the computation is straightforward in view of property (196) of the $\delta_{\mathbb{C}}$-function:

$$\langle \delta N^{a0}_{\boldsymbol{p}}\, \delta N^{b0}_{\boldsymbol{p}'} \rangle_{\boldsymbol{k},\omega} = (2\pi)^{-3}\, \delta_{ab}\, \delta(\boldsymbol{p}-\boldsymbol{p}')\, \delta_{\mathbb{C}}(\omega-\boldsymbol{k}\cdot\boldsymbol{v}) n_a f^a_{\boldsymbol{p}}\,. \tag{217}$$

The corresponding result usually found in the literature assumes a Fourier transform with respect to time instead of the Laplace transform entering the present analysis. Thus, the frequency ω is a real number, there, and instead of $\delta_{\mathbb{C}}$ the regular δ-distribution appears. Taking these modifications into account, the expression above agrees with the ones of, e. g., Tolias et al. (2015) and Tsytovich (1995).

All following computations are solely based on the spectral representation (215) of the natural statistical fluctuations, so they apply irrespective of whether the plasma is magnetized or unmagnetized. Since this equation states that the source fluctuations are stationary and homogeneous, it is no surprise that all related correlations inherit these symmetries. Firstly, expression (169) for the source currents implies that

$$\langle \delta\boldsymbol{J}^0_{\boldsymbol{k},\omega} \otimes \delta\boldsymbol{J}^{0*}_{\boldsymbol{k}',\omega'} \rangle = \langle \delta\boldsymbol{J}_0 \otimes \delta\boldsymbol{J}_0 \rangle_{\boldsymbol{k},\omega}\, \delta(\boldsymbol{k}-\boldsymbol{k}')\, \delta_{\mathbb{C}}(\omega-\omega')\,, \tag{218}$$

$$\langle \delta\boldsymbol{J}_0 \otimes \delta\boldsymbol{J}_0 \rangle_{\boldsymbol{k},\omega} = \sum_{a,b} e_a e_b \int \mathrm{d}^3p \int \mathrm{d}^3p'\, \boldsymbol{v}\otimes\boldsymbol{v}' \langle \delta N^{a0}_{\boldsymbol{p}}\, \delta N^{b0}_{\boldsymbol{p}'} \rangle_{\boldsymbol{k},\omega}\,. \tag{219}$$

This way of specifying the correlations as a tensor is very convenient because according to the definition of the tensor product, $(\boldsymbol{a}\otimes\boldsymbol{b})_{ij} = a_i b_j$, it contains the full directional information of all Cartesian components. In particular, computing its trace yields the scalar product $\langle \delta\boldsymbol{J}^0_{\boldsymbol{k},\omega} \cdot \delta\boldsymbol{J}^0_{\boldsymbol{k}',\omega'} \rangle$. The definition of the tensor product also implies that the following multiple product rule holds for tensors $\hat{\boldsymbol{A}}$, $\hat{\boldsymbol{B}}$ and vectors $\boldsymbol{a}$, $\boldsymbol{b}$:

$$\begin{aligned}
[(\hat{\boldsymbol{A}}\cdot\boldsymbol{a})\otimes(\hat{\boldsymbol{B}}\cdot\boldsymbol{b})]_{ij} &= (\hat{\boldsymbol{A}}\cdot\boldsymbol{a})_i(\hat{\boldsymbol{B}}\cdot\boldsymbol{b})_j \\
&= A_{ik}a_k B_{jl} b_l \\
&= A_{ik}a_k b_l B^{\mathrm{T}}_{lj} \\
&= A_{ik}(\boldsymbol{a}\otimes\boldsymbol{b})_{kl} B^{\mathrm{T}}_{lj} \\
&= [\hat{\boldsymbol{A}}\cdot(\boldsymbol{a}\otimes\boldsymbol{b})\cdot\hat{\boldsymbol{B}}^{\mathrm{T}}]_{ij}\,,
\end{aligned} \tag{220}$$

where the superscript "T" denotes the transpose of the tensor. This relation is useful in the computation of the field correlations. These are stationary and homogeneous, too, because (180) implies

$$\langle \delta\boldsymbol{E}_{\boldsymbol{k},\omega} \otimes \delta\boldsymbol{E}^*_{\boldsymbol{k}',\omega'} \rangle = \frac{16\pi^2}{\omega\omega'^*} \frac{\hat{\boldsymbol{\lambda}}_{\boldsymbol{k},\omega} \cdot \langle \delta\boldsymbol{J}_0 \otimes \delta\boldsymbol{J}_0 \rangle_{\boldsymbol{k},\omega} \cdot \hat{\boldsymbol{\lambda}}^\dagger_{\boldsymbol{k}',\omega'}}{\Lambda(\boldsymbol{k},\omega)\Lambda^*(\boldsymbol{k}',\omega')}\, \delta(\boldsymbol{k}-\boldsymbol{k}')\, \delta_{\mathbb{C}}(\omega-\omega')\,, \tag{221}$$

where the dagger denotes the adjoint operator, i. e., the complex conjugate of the transpose matrix. In view of the resonance conditions $\boldsymbol{k}'=\boldsymbol{k}$ and $\omega'=\omega$, the primed and unprimed quantities may be interchanged. Therefore, the correlations of the spontaneously emitted field fluctuations are homogeneous and stationary:

$$\langle \delta\boldsymbol{E}_{\boldsymbol{k},\omega} \otimes \delta\boldsymbol{E}^*_{\boldsymbol{k}',\omega'} \rangle = \langle \delta\boldsymbol{E} \otimes \delta\boldsymbol{E} \rangle_{\boldsymbol{k},\omega}\, \delta(\boldsymbol{k}-\boldsymbol{k}')\, \delta_{\mathbb{C}}(\omega-\omega')\,, \tag{222}$$

$$\langle \delta\boldsymbol{E} \otimes \delta\boldsymbol{E} \rangle_{\boldsymbol{k},\omega} = \frac{16\pi^2}{|\omega|^2} \frac{\hat{\boldsymbol{\lambda}}_{\boldsymbol{k},\omega} \cdot \langle \delta\boldsymbol{J}_0 \otimes \delta\boldsymbol{J}_0 \rangle_{\boldsymbol{k},\omega} \cdot \hat{\boldsymbol{\lambda}}^\dagger_{\boldsymbol{k},\omega}}{|\Lambda(\boldsymbol{k},\omega)|^2}\,. \tag{223}$$

This is the desired spectrum of the field correlations that are generated by the natural statistical fluctuations. In its derivation it was tacitly assumed that the well-known relation $f(\boldsymbol{k})\,\delta(\boldsymbol{k}'-\boldsymbol{k}) = f(\boldsymbol{k}')\,\delta(\boldsymbol{k}'-\boldsymbol{k})$ does not only hold for the regular δ-distribution but also for the $\delta_{\mathbb{C}}$-function, i. e., it was assumed that

$$f(\omega)\,\delta_{\mathbb{C}}(\omega'-\omega) = f(\omega')\,\delta_{\mathbb{C}}(\omega'-\omega)\,. \tag{224}$$

It is obvious from the definition (195) that this relation does not hold in an algebraic sense. It is valid, however, if $\delta_{\mathbb{C}}$ is considered as an integrand, very much like the common δ-distribution. This becomes evident by inspecting Cauchy's integral formula (198), which implies that the application of $\oint \mathrm{d}\omega'$ on both sides of (224) yields equal results. Since the inverse Laplace transform involves an integration with respect to the complex frequency, the application of (224) in the derivation of the spectral field correlations is justified.

This insight is also useful for the computation of the magnetic field correlations. As mentioned before, they can be traced back to those of the electric field on account of Faraday's law of induction, (160), which implies

$$\langle \delta\boldsymbol{B}_{\boldsymbol{k},\omega} \otimes \delta\boldsymbol{B}^*_{\boldsymbol{k}',\omega'}\rangle = (c^2/\omega\omega'^*)\langle(\boldsymbol{k}\times\delta\boldsymbol{E}_{\boldsymbol{k},\omega})\otimes(\boldsymbol{k}'\times\delta\boldsymbol{E}^*_{\boldsymbol{k}',\omega'})\rangle\,. \tag{225}$$

In order to apply the findings above, the products must be rearranged such that the tensor product of the field fluctuations appears. To this end, it is useful to rewrite the cross product as a matrix product. This is possible because the cross product is a linear mapping with respect to the second factor. Thus, for two arbitrary vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ one obtains

$$\boldsymbol{a}\times\boldsymbol{b} = \hat{\boldsymbol{a}}_\times\cdot\boldsymbol{b}\,, \tag{226}$$

where the skew-symmetric matrix $\hat{\boldsymbol{a}}_\times$ associated with the vector $\boldsymbol{a}$ is given by the coordinate representation

$$\hat{\boldsymbol{a}}_\times = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}\,. \tag{227}$$

With this formulation at disposal, the multiple products appearing in the magnetic field correlations can be rewritten by means of Eq. (220):

$$\langle \delta\boldsymbol{B}_{\boldsymbol{k},\omega} \otimes \delta\boldsymbol{B}^*_{\boldsymbol{k}',\omega'}\rangle = (c^2/\omega\omega'^*)\hat{\boldsymbol{k}}_\times\cdot\langle\delta\boldsymbol{E}_{\boldsymbol{k},\omega}\otimes\delta\boldsymbol{E}^*_{\boldsymbol{k}',\omega'}\rangle\cdot\hat{\boldsymbol{k}}'^{\mathrm{T}}_\times \tag{228}$$

Now the right-hand side possesses the required form that allows the application of the previous results (222)–(223) for the electric field correlations:

$$\langle \delta\boldsymbol{B}_{\boldsymbol{k},\omega} \otimes \delta\boldsymbol{B}^*_{\boldsymbol{k}',\omega'}\rangle = \langle\delta\boldsymbol{B}\otimes\delta\boldsymbol{B}\rangle_{\boldsymbol{k},\omega}\,\delta(\boldsymbol{k}-\boldsymbol{k}')\,\delta_{\mathbb{C}}(\omega-\omega')\,, \tag{229}$$

$$\langle\delta\boldsymbol{B}\otimes\delta\boldsymbol{B}\rangle_{\boldsymbol{k},\omega} = (c^2/|\omega|^2)\hat{\boldsymbol{k}}_\times\cdot\langle\delta\boldsymbol{E}\otimes\delta\boldsymbol{E}\rangle_{\boldsymbol{k},\omega}\cdot\hat{\boldsymbol{k}}^{\mathrm{T}}_\times\,. \tag{230}$$

Thus, as expected, the magnetic field fluctuations are homogeneous and stationary as well.

5. Generalization of the fluctuation–dissipation theorem

Outline. This section addresses the first research objective defined in the introduction: The fluctuation–dissipation theorem for weakly coupled and uniformly magnetized plasmas is generalized by extending its validity to arbitrary complex frequencies. In order to ascertain the consistency with previous formulations and highlight the new features, the limits $\Im\omega \to 0$ and $\Re\omega \to 0$ corresponding to weakly damped/amplified and aperiodic fluctuations, respectively, are inferred from the general result. Similarly, a formulation for unmagnetized plasmas is derived by considering the limit of a vanishing magnetic background field. In all cases, both the hermitian and the antihermitian part of the Maxwell tensor enter the theorem provided that the frequency is located beyond the real axis. In order to support this finding with further plausibility, the Poynting theorem is transformed into its spectral representation by a Fourier–Laplace transform which confirms that, in contrast to the textbook results obtained after a Fourier–Fourier transform, the dissipation attributed to Ohmic losses is associated with the hermitian and antihermitian part of the response tensor.

Reference. Parts of the results presented in this section were published by Schlickeiser and Kolberg (2015).

5.1. Introduction

Hardly any two formulations of the fluctuation–dissipation theorem (FDT) found in the literature are identical. This remarkable feature makes a precise statement of the theorem quite difficult. But at the same time its different facets also show how powerful a tool it is, being ubiquitous in many different branches of theoretical physics and universally applicable to a manifold of different systems. That being said, it appears advisable to start the introduction to the topic of this section with a qualitative discussion rather than a quantitative formula.

It has already been mentioned before that a microscopic theory of matter must reproduce the macroscopic laws of nature and thereby validate them from a more fundamental level of description. In particular, the parameters that appear in macroscopic formulas and that account for the characteristic properties of the substances under consideration must be explained by the structural model of these substances and traced back to the microscopic parameters entering it.

A very rich and important class of examples is provided by linear response theory. In general, the latter is concerned with an equilibrium system that is exposed to an external disturbance and therefore changes its state (Balescu, 1975; Ichimaru, 1992). A typical example encountered before in Section 2.3 is the (possibly equilibrating) transport of charged particles due to an external field. There, the induced current density constitutes a new observable that did not appear in the equilibrium system and therefore characterizes the reaction of the system to its disturbance in a quantifiable way. If the disturbance is sufficiently weak, the response of the system in terms of such newly emerging observables can often be regarded as a linear functional of the stimulus. The coefficients appearing in these linear relations are called *response functions* (or *susceptibilities*) and they constitute an intrinsic property of the system at hand, characterizing its particular response to external disturbances. Evidently, these parameters are closely related to the atomistic structure of the equilibrium system, so a macroscopic theory cannot determine them any further but only specify them by comparison with observations.

As stated earlier in Section 2.3, the response function entering Ohm's law can be interpreted as a measure for the conductivity or admittance of the plasma. As such, it also determines the impedance or resistivity, i. e., the opposition that an externally stimulated response encounters in the medium. Microscopically, this resistance to a driven, ordered motion of the charge carriers is due to their thermal collisions and haphazard electromagnetic interactions. Thus, a fraction of the energy provided by the external source is converted into unordered and random motion, that is, into heat. This is an example of the general connection between the response function and dissipation. It should be noted that this relation still remains true even in collisionless plasmas because dissipation occurs there as well due to wave–particle interactions (Landau damping). Since the thermal randomness is responsible for the dissipation, there must be a connection between the latter and the fluctuation level of the undisturbed equilibrium system. Quantitatively, this should manifest itself in a relation between the correlation function on the one hand and the response function on the other hand. The FDT expresses this relation and is based on the fact that the interaction of a variable with a random ensemble (a heat bath) is the common process underlying both the fluctuations and the dissipation.

Historically, one of the earliest formulations was given by Nyquist (1928) in the theorem that today is named after him. It states that the thermal noise of a resistor, specified by the mean square voltage fluctuations, is determined by its resistance and temperature. Since then, the theorem was generalized considerably, most prominently by Callen and Welton (1951) and Kubo (1957, 1966). In the course of these generalizations the FDT was placed in the context of linear response theory as outlined before. A generic formulation of the FDT relates the time correlation function of an equilibrium system at temperature T to its generalized susceptibility. Denoting the Fourier transforms of these quantities by S_ω and χ_ω, respectively, the classical or high temperature limit ($\hbar\omega \ll k_{\mathrm{B}}T$) of the theorem reads (Landau and Lifshitz, 1980; Martin, 1968)

$$S_\omega = (2k_{\mathrm{B}}T/\omega)\Im\chi_\omega\,. \tag{231}$$

Following Balescu (1975), three properties of the theorem will be pointed out here that account for its importance in theoretical physics: (1) One striking feature of the FDT is its generality in the sense that only very few assumptions enter its derivation, thus allowing its application to a large class of physical systems. It can even be formulated in a still more general context, for example, by weakening the assumption of a thermal equilibrium or by taking nonlinear effects into account. Vice versa, there are many specialized formulations adapted to a particular system and its characteristic variables. (2) The FDT establishes a link between equilibrium and non-equilibrium statistical mechanics by relating the spectrum of the equilibrium fluctuations to the susceptibility describing the response to a disturbance of equilibrium. (3) The theorem provides a relation between macroscopic and microscopic quantities, and this can be exploited in two ways: If the macroscopic admittance is available from experiments, one can infer the correlation function from the FDT and thereby gain inside into the microscopic structure of the medium. If, on the other hand, a molecular model of a substance is known, then a linear response law can be derived along with a prediction for the value of the corresponding susceptibility.

Although the FDT was successfully applied to plasmas already in the 1960s by Dougherty and Farley (1960); Rostoker (1961), and Sitenko (1967), the subject still remains an active field of research (Kanekar et al., 2015; Belyi, 2002; López et al., 2015). Even for equilibrium plasmas there are several possibilities to express the theorem. To begin with, one may choose the correlation function either of the current density, the electric field, the magnetic field, or the electromagnetic vector potential. For each of these one may take the absolute value $\langle\delta a^2\rangle_{\boldsymbol{k},\omega}$ into account or, more generally, the product of two components, $\langle\delta a_i\,\delta a_j\rangle_{\boldsymbol{k},\omega}$, which may also be represented by the tensor $\langle\delta\boldsymbol{a}\otimes\delta\boldsymbol{a}\rangle_{\boldsymbol{k},\omega}$. Furthermore, the admittance entering the FDT can be taken to be the conductivity $\hat{\boldsymbol{\sigma}}_{\boldsymbol{k},\omega}$, the electric susceptibility $\hat{\boldsymbol{\chi}}_{\boldsymbol{k},\omega} = (4\pi\imath/\omega)\hat{\boldsymbol{\sigma}}_{\boldsymbol{k},\omega}$, the dielectric tensor $\hat{\boldsymbol{\epsilon}}_{\boldsymbol{k},\omega} = \hat{\mathbf{1}} + \hat{\boldsymbol{\chi}}_{\boldsymbol{k},\omega}$, or the Maxwell tensor (171):

$$\hat{\boldsymbol{\Lambda}}_{\boldsymbol{k},\omega} = \hat{\boldsymbol{\Lambda}}^{(0)}_{\boldsymbol{k},\omega} + (4\pi\imath/\omega)\hat{\boldsymbol{\sigma}}_{\boldsymbol{k},\omega}\,, \tag{232}$$

where

$$\hat{\boldsymbol{\Lambda}}^{(0)}_{\boldsymbol{k},\omega} \equiv \hat{\mathbf{1}} - (k^2c^2/\omega^2)(\hat{\mathbf{1}} - \boldsymbol{k}\otimes\boldsymbol{k}/k^2)\,. \tag{233}$$

The present work follows a formulation of Sitenko (1967) that expresses the theorem in terms of the Maxwell tensor and the electric field fluctuations. Adopting equation (2.41) of that monograph to the notation and conventions used here,[2] it

[2] The first and most important adjustment takes the different definitions of the correlation function for homogeneous and stationary fluctuations into account: In contrast to the convention $\langle\delta a_i\,\delta a_j\rangle_{\boldsymbol{R},\tau} = \langle\delta a_i(\boldsymbol{r},t)\,\delta a_j(\boldsymbol{r}-\boldsymbol{R},t-\tau)\rangle$ underlying the present work, Sitenko assumes a plus sign instead of the minus

reads

$$\langle\delta\boldsymbol{E}\otimes\delta\boldsymbol{E}\rangle_{\boldsymbol{k},\omega}\propto\frac{\imath\hbar}{\exp(\hbar\omega/k_{\mathrm{B}}T)-1}(\hat{\boldsymbol{\Lambda}}_{\boldsymbol{k},\omega}^{-1}-\hat{\boldsymbol{\Lambda}}_{\boldsymbol{k},\omega}^{-1\dagger}),\tag{234}$$

where the dagger "†" denotes the adjugate of a matrix, i. e., $M_{ij}^{\dagger}=M_{ji}^{*}$. In view of the different conventions regarding Fourier and Laplace transforms, the prefactors were omitted here. The classical limit $\hbar\to 0$ of this equation corresponds to the high temperature approximation $\hbar\omega\ll k_{\mathrm{B}}T$,

$$\langle\delta\boldsymbol{E}\otimes\delta\boldsymbol{E}\rangle_{\boldsymbol{k},\omega}\propto(\imath k_{\mathrm{B}}T/\omega)(\hat{\boldsymbol{\Lambda}}_{\boldsymbol{k},\omega}^{-1}-\hat{\boldsymbol{\Lambda}}_{\boldsymbol{k},\omega}^{-1\dagger}).\tag{235}$$

Both versions exhibit a prominent feature that is common to many tensor formulations of the FDT (Ichimaru, 1992; Alexandrov et al., 1984): Only the antihermitian part of the Maxwell tensor enters the theorem, whereas the hermitian part does not appear at all. In this sense, the former generalizes the imaginary part of the scalar response function appearing in (231). In order to clarify this terminology it suffices to note that any matrix $\hat{\boldsymbol{M}}$ can be split additively into two contributions such that one of them is hermitian (denoted by a superscript "H") and the other one is antihermitian (superscript "A"):

$$\hat{\boldsymbol{M}}=\hat{\boldsymbol{M}}^{\mathrm{H}}+\hat{\boldsymbol{M}}^{\mathrm{A}},\tag{236}$$

$$\hat{\boldsymbol{M}}^{\mathrm{H}}\equiv\tfrac{1}{2}(\hat{\boldsymbol{M}}+\hat{\boldsymbol{M}}^{\dagger}),\tag{237}$$

$$\hat{\boldsymbol{M}}^{\mathrm{A}}\equiv\tfrac{1}{2}(\hat{\boldsymbol{M}}-\hat{\boldsymbol{M}}^{\dagger}).\tag{238}$$

The non-appearance of the hermitian part in the FDT agrees with the well-known fact that the dissipation occurring in a plasma is solely associated with the antihermitian part of the Maxwell (or dielectric) tensor (Melrose and McPhedran, 1991; Schlickeiser, 2002; Alexandrov et al., 1984). This point of interest will be addressed again later in Section 5.5 and discussed there in more detail.

In the results summarized here in connection with the FDT, the frequency ω was always understood to be the Fourier variable conjugate to time. In particular, it was assumed to be a real number. Since the definition of the response function is a one-sided integral over time due to the principle of causality, it may be continued to complex values of the frequency. The concept of such *generalized susceptibilities* proves very fruitful, in particular concerning their analytical properties, and it culminates in such important results as the celebrated Kramers–Kronig relations (Melrose and McPhedran, 1991). The fact remains, however, that physical significance can only be attributed to complex frequencies if the latter are not introduced by an analytical continuation of Fourier transforms but by performing a Laplace transform from the outset. Only then one is guaranteed that *all* quantities appearing in the theory (not only the susceptibilities, but also the correlation functions) are defined for complex frequencies and that their contributions stemming from complex arguments will also enter the inverse transforms which constitute the actual observables. For it is evident by inspection of Eq. (A.7) that only real-valued arguments of the spectral function contribute to the inverse Fourier transform, irrespective of whether the function is defined for complex arguments or not. It is a structural deficiency, therefore, that aperiodic fluctuations are not accounted for by Fourier transforms but only by Laplace transforms. This motivates the purpose of the present section, and that is to derive a fluctuation–dissipation relation for relativistic equilibrium plasmas – both magnetized and unmagnetized – that is formulated in terms of the respective Laplace transforms of the electric field fluctuations and the Maxwell tensor. In particular, the theorem is supposed to be valid for arbitrary complex frequencies.

Before going into the details of the current derivation, it might be beneficial to briefly outline the procedure usually employed to derive the FDT, or for that matter, to treat plasma emission and absorption in general. Spontaneous emission is mostly tackled by a single-particle approach, that is, one considers the charge density associated with a moving point charge and solves the wave equation to infer the emitted fields. Although this charge is a constituent of the plasma, it may be treated as an external current in the equation because its influence on the dynamics of the plasma is negligible—in this sense it is a test particle. In the next step, the results are used to infer the fluctuations generated by an entire ensemble of emitters. Depending on the trajectories considered, one can compute Cerenkov emission, bremsstrahlung, synchrotron emission, and so forth in this fashion. Absorption and induced emission may be inferred from spontaneous emission by means of the principle of detailed balance and the Einstein coefficients. More commonly, however, absorption is determined by a collective-medium approach based on the response tensor: Due to the restrictions imposed by causality, an infinitesimal imaginary part $\imath 0$ of the frequency is introduced that is determined by the antihermitian part of the response tensor. However, the single-particle approach has the distinctive advantage that it automatically satisfies the conservation of energy and momentum on a microscopic level from the outset as it is based on the principle of detailed balance. This is not the case in a purely classical theory where a radiation reaction force must be added artificially, and even this is not possible in all cases. A further advantage of the single-particle approach is that it also provides an alternative way to compute the dielectric tensor by treating the plasma response as forward scattering off an ensemble of particles. The basic procedure

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sign on the right-hand side, resulting in $\langle\delta a_i^*(\boldsymbol{k},\omega)\,\delta a_j(\boldsymbol{k}',\omega')\rangle\propto\delta(\boldsymbol{k}-\boldsymbol{k}')\,\delta(\omega-\omega')\langle\delta a_i\,\delta a_j\rangle_{\boldsymbol{k},\omega}$ rather than having $\langle\delta a_i(\boldsymbol{k},\omega)\,\delta a_j^*(\boldsymbol{k}',\omega')\rangle$ on the left-hand side. Thus, adopting Sitenko's results to the notation of the present work, one needs to take the complex conjugate or, considering the self-adjointness of the spectral correlation tensor, the transpose. The second adjustment consists in the recovery of Boltzmann's constant k_{B} that does not appear in Sitenko's equations because he sets it equal to unity by choosing appropriate units. The numerical prefactors stemming from different conventions regarding the Fourier transform, on the other hand, were omitted by simply stating proportionality since they have no relevance for the following analysis.

is to compute the disturbance of a particle orbit caused by an electromagnetic field perturbation, because it immediately gives the response current of a single point charge. After averaging over the responses of an entire ensemble of particles, only the macroscopic effect remains that corresponds to forward scattering and that constitutes the collective response of the medium. This method is well known from the optical theorem, which is tellingly also known as the forward-scattering theorem. A more in-depth discussion of the two approaches concerning radiation processes and wave–particle interactions in plasmas is given by Melrose (2008, 1986, 1980).

5.2. Derivation of the theorem

The system investigated here is a thermal equilibrium plasma subject to a uniform magnetic field $\boldsymbol{B} = B\boldsymbol{e}_\parallel$. In order to allow for relativistic energies, a Maxwell–Jüttner distribution function is adopted,

$$f^a_{\boldsymbol{p}} = C_a \exp(-\Gamma_a/\Theta_a)\,, \tag{239}$$

where Γ_a denotes the Lorentz factor common to all particles of species a. Moreover, Θ_a is the relativistic temperature and C_a a constant accounting for the proper normalization of the distribution function:

$$\Theta_a = k_{\mathrm{B}}T/m_a c^2\,, \tag{240}$$

$$C_a = [4\pi m_a^3 c^3 \Theta_a K_2(1/\Theta_a)]^{-1}\,. \tag{241}$$

Here, K_2 denotes the modified Bessel function of the second kind and second order. The assumption of an equilibrium implies that all particle species have the same absolute temperature T. Due to the isotropy of the distribution function, the only preferred direction is specified by the magnetic field. In view of this axial symmetry, it is reasonable to introduce cylindrical coordinates $p_\perp$, φ and $p_\parallel$ in momentum space. Then, the conductivity tensor entering the Maxwell operator (171) is given by (Schlickeiser, 2010; Schlickeiser and Yoon, 2015)

$$\begin{aligned}\hat{\boldsymbol{\sigma}}_{\boldsymbol{k},\omega} = \sum_a \frac{\omega_{\mathrm{p},a}^2 m_a}{4\pi\imath\omega}\int \mathrm{d}^3p\,\Bigg[&\frac{v_\parallel}{p_\perp}\left(p_\perp\frac{\partial f^a_{\boldsymbol{p}}}{\partial p_\parallel} - p_\parallel\frac{\partial f^a_{\boldsymbol{p}}}{\partial p_\perp}\right)(\boldsymbol{e}_\parallel\otimes\boldsymbol{e}_\parallel)\\ &+\sum_{n=-\infty}^{\infty}\frac{\hat{\boldsymbol{T}}^{(n)}_{\boldsymbol{k},\boldsymbol{p}}}{\omega - k_\parallel v_\parallel - n\Omega_a}\left((\omega - k_\parallel v_\parallel)\frac{\partial f^a_{\boldsymbol{p}}}{\partial p_\perp} + k_\parallel v_\perp\frac{\partial f^a_{\boldsymbol{p}}}{\partial p_\parallel}\right)\Bigg]\,,\end{aligned} \tag{242}$$

where $\omega_{\mathrm{p},a}$ denotes the plasma frequency of particle species a and Ω_a the relativistic gyro-frequency $\Omega_a = e_aB/\Gamma_a m_a c$. If the coordinate system is chosen such that the z-axis is parallel to the magnetic field, $\boldsymbol{B} = (0, 0, B)$, and such that the wavevector lies entirely in the x-z-plane, $\boldsymbol{k} = (k_\perp, 0, k_\parallel)$, then the tensor $\hat{\boldsymbol{T}}^{(n)}_{\boldsymbol{k},\boldsymbol{p}}$ can be expressed in terms of Bessel functions of the first kind, J_n, in the following way:

$$\hat{\boldsymbol{T}}^{(n)}_{\boldsymbol{k},\boldsymbol{p}} = \begin{pmatrix} \dfrac{n^2 J_n^2(\xi) v_\perp}{\xi^2} & \dfrac{\imath n J_n(\xi) J_n'(\xi) v_\perp}{\xi} & \dfrac{n J_n^2(\xi) v_\parallel}{\xi} \\ -\dfrac{\imath n J_n(\xi) J_n'(\xi) v_\perp}{\xi} & [J_n'(\xi)]^2 v_\perp & -\imath J_n(\xi) J_n'(\xi) v_\parallel \\ \dfrac{n J_n^2(\xi) v_\parallel}{\xi} & \imath J_n(\xi) J_n'(\xi) v_\parallel & \dfrac{J_n^2(\xi) v_\parallel^2}{v_\perp} \end{pmatrix} \tag{243}$$

with

$$\xi \equiv k_\perp v_\perp/|\Omega_a|\,. \tag{244}$$

The Maxwell–Jüttner distribution function depends on the momentum variable only through the Lorentz factor. Therefore, the distribution is isotropic and its derivatives can readily be found to be

$$\frac{1}{p_\perp}\frac{\partial f^a_{\boldsymbol{p}}}{\partial p_\perp} = \frac{1}{p_\parallel}\frac{\partial f^a_{\boldsymbol{p}}}{\partial p_\parallel} = -\frac{f^a_{\boldsymbol{p}}}{\Theta_a\Gamma_a m_a^2 c^2} = -\frac{f^a_{\boldsymbol{p}}}{k_{\mathrm{B}}T\Gamma_a m_a}\,. \tag{245}$$

In view of this, the expression for the conductivity tensor simplifies further. The first term in the square brackets of (242) vanishes and the rest becomes

$$\hat{\boldsymbol{\sigma}}_{\boldsymbol{k},\omega} = -\sum_a \frac{\omega_{\mathrm{p},a}^2}{4\pi\imath k_{\mathrm{B}}T}\sum_{n=-\infty}^{\infty}\int \mathrm{d}^3p\,\frac{(p_\perp f^a_{\boldsymbol{p}}/\Gamma_a)\hat{\boldsymbol{T}}^{(n)}_{\boldsymbol{k},\boldsymbol{p}}}{\omega - k_\parallel v_\parallel - n\Omega_a}\,. \tag{246}$$

In order to obtain the desired fluctuation–dissipation theorem, one must relate this result with the electromagnetic field fluctuations in a magnetized thermal equilibrium plasma. According to (223) the latter are given by

$$\langle\delta\boldsymbol{E}\otimes\delta\boldsymbol{E}\rangle_{\boldsymbol{k},\omega} = \hat{\boldsymbol{\Lambda}}^{-1}_{\boldsymbol{k},\omega}\cdot\langle\delta\boldsymbol{E}_0\otimes\delta\boldsymbol{E}_0\rangle_{\boldsymbol{k},\omega}\cdot\hat{\boldsymbol{\Lambda}}^{-1\dagger}_{\boldsymbol{k},\omega}\,, \tag{247}$$

in terms of the inverse $\hat{\boldsymbol{\Lambda}}^{-1}_{\boldsymbol{k},\omega} = \hat{\boldsymbol{\lambda}}_{\boldsymbol{k},\omega}/\Lambda(\boldsymbol{k},\omega)$ of the Maxwell tensor and the source fluctuations

$$\langle \delta\boldsymbol{E}_0 \otimes \delta\boldsymbol{E}_0\rangle_{\boldsymbol{k},\omega} = |4\pi/\omega|^2 \langle \delta\boldsymbol{J}_0 \otimes \delta\boldsymbol{J}_0\rangle_{\boldsymbol{k},\omega} \,. \tag{248}$$

Recently, these source fluctuations were computed by Schlickeiser and Yoon (2015) who generalized previous results to relativistic energies and to arbitrary complex frequencies. Rewriting their Eq. (20) without the redundant real part operator leads to

$$\langle \delta\boldsymbol{E}_0 \otimes \delta\boldsymbol{E}_0\rangle_{\boldsymbol{k},\omega} = \sum_a \frac{\imath\omega_{\mathrm{p},a}^2\varsigma}{4\pi^3|\omega|^2} \sum_{n=-\infty}^{\infty} \int \mathrm{d}^3p\, \frac{(p_\perp f_p^a/\Gamma_a)\hat{\boldsymbol{T}}^{(n)}_{\boldsymbol{k},\boldsymbol{p}}}{\omega - k_\parallel v_\parallel - n\Omega_a} \,, \tag{249}$$

where $\varsigma = 1$ for $\Im\omega > 0$ and $\varsigma = -1$ for $\Im\omega < 0$. There is a striking resemblance between this expression and Eq. (246) for the conductivity tensor that can be exploited to establish a relation between the latter and the source fluctuations:

$$\langle \delta\boldsymbol{E}_0 \otimes \delta\boldsymbol{E}_0\rangle_{\boldsymbol{k},\omega} = \left(\varsigma k_{\mathrm{B}}T\big/\pi^2|\omega|^2\right)\hat{\boldsymbol{\sigma}}_{\boldsymbol{k},\omega} \,. \tag{250}$$

The conductivity tensor appearing here is directly related to the Maxwell tensor via (232). This relation can be used to rewrite (250) in terms of the Maxwell tensor. Inserting the resulting equation into (247) leads to the following expression for the total field fluctuations:

$$\langle \delta\boldsymbol{E} \otimes \delta\boldsymbol{E}\rangle_{\boldsymbol{k},\omega} = -\frac{\imath\varsigma k_{\mathrm{B}}T\omega}{4\pi^3|\omega|^2}\left(\hat{\boldsymbol{\Lambda}}^{-1\dagger}_{\boldsymbol{k},\omega} - \hat{\boldsymbol{\Lambda}}^{-1}_{\boldsymbol{k},\omega}\cdot\hat{\boldsymbol{\Lambda}}^{(0)}_{\boldsymbol{k},\omega}\cdot\hat{\boldsymbol{\Lambda}}^{-1\dagger}_{\boldsymbol{k},\omega}\right). \tag{251}$$

In the last section, the spectral correlations of the field fluctuations were derived by computing the product $\langle \delta\boldsymbol{E}_{\boldsymbol{k},\omega} \otimes \delta\boldsymbol{E}^*_{\boldsymbol{k}',\omega'}\rangle$. The adjoint tensor of this expression is given by

$$\langle \delta\boldsymbol{E}_{\boldsymbol{k},\omega} \otimes \delta\boldsymbol{E}^*_{\boldsymbol{k}',\omega'}\rangle^\dagger = \langle \delta\boldsymbol{E}^*_{\boldsymbol{k},\omega} \otimes \delta\boldsymbol{E}_{\boldsymbol{k}',\omega'}\rangle^{\mathrm{T}} = \langle \delta\boldsymbol{E}_{\boldsymbol{k}',\omega'} \otimes \delta\boldsymbol{E}^*_{\boldsymbol{k},\omega}\rangle \tag{252}$$

Thus, taking the adjoint in this case simply means interchanging the primed and unprimed spectral variables. In view of the resonance conditions $\boldsymbol{k}' = \boldsymbol{k}$ and $\omega' = \omega$ entering the formula for the spectral correlations one can conclude that the latter are not affected by taking the adjoint at all,

$$\langle \delta\boldsymbol{E} \otimes \delta\boldsymbol{E}\rangle^\dagger_{\boldsymbol{k},\omega} = \langle \delta\boldsymbol{E} \otimes \delta\boldsymbol{E}\rangle_{\boldsymbol{k},\omega} \,, \tag{253}$$

so the tensor describing the spectrum of the field correlations is hermitian. Since every hermitian matrix $\hat{\boldsymbol{H}}$ equals half the sum of itself and its adjoint, $\hat{\boldsymbol{H}} = \frac{1}{2}(\hat{\boldsymbol{H}} + \hat{\boldsymbol{H}}^\dagger)$, Eq. (251) implies

$$\begin{aligned}\langle \delta\boldsymbol{E} \otimes \delta\boldsymbol{E}\rangle_{\boldsymbol{k},\omega} = -\frac{\imath\varsigma k_{\mathrm{B}}T}{8\pi^3|\omega|^2}\Big[&\omega\hat{\boldsymbol{\Lambda}}^{-1\dagger}_{\boldsymbol{k},\omega} - \omega^*\hat{\boldsymbol{\Lambda}}^{-1}_{\boldsymbol{k},\omega}\\ &- \hat{\boldsymbol{\Lambda}}^{-1}_{\boldsymbol{k},\omega}\cdot\left(\omega\hat{\boldsymbol{\Lambda}}^{(0)}_{\boldsymbol{k},\omega} - \omega^*\hat{\boldsymbol{\Lambda}}^{(0)\dagger}_{\boldsymbol{k},\omega}\right)\cdot\hat{\boldsymbol{\Lambda}}^{-1\dagger}_{\boldsymbol{k},\omega}\Big]\,,\end{aligned} \tag{254}$$

where the product rule $(\hat{\boldsymbol{A}}\hat{\boldsymbol{B}}\hat{\boldsymbol{C}})^\dagger = \hat{\boldsymbol{C}}^\dagger\hat{\boldsymbol{B}}^\dagger\hat{\boldsymbol{A}}^\dagger$ was applied. In view of (233) the expression inside the parentheses may be rewritten as

$$\omega\hat{\boldsymbol{\Lambda}}^{(0)}_{\boldsymbol{k},\omega} - \omega^*\hat{\boldsymbol{\Lambda}}^{(0)\dagger}_{\boldsymbol{k},\omega} = (\omega-\omega^*)\left[\hat{\mathbf{1}} + |\eta|^2\left(\hat{\mathbf{1}} - \boldsymbol{k}\otimes\boldsymbol{k}/k^2\right)\right], \tag{255}$$

$$\eta = kc/\omega \,. \tag{256}$$

The new quantity η introduced here is the complex-valued refractive index. Inserting the second last equation into (254) yields

$$\begin{aligned}\langle \delta\boldsymbol{E} \otimes \delta\boldsymbol{E}\rangle_{\boldsymbol{k},\omega} = \frac{\imath\varsigma k_{\mathrm{B}}T}{8\pi^3|\omega|^2}\Big\{&\omega^*\hat{\boldsymbol{\Lambda}}^{-1}_{\boldsymbol{k},\omega} - \omega\hat{\boldsymbol{\Lambda}}^{-1\dagger}_{\boldsymbol{k},\omega}\\ &+(\omega-\omega^*)\hat{\boldsymbol{\Lambda}}^{-1}_{\boldsymbol{k},\omega}\cdot\left[\hat{\mathbf{1}} + |\eta|^2\left(\hat{\mathbf{1}} - \boldsymbol{k}\otimes\boldsymbol{k}/k^2\right)\right]\cdot\hat{\boldsymbol{\Lambda}}^{-1\dagger}_{\boldsymbol{k},\omega}\Big\}\,.\end{aligned} \tag{257}$$

This is the main result of this section, a generalization of the fluctuation–dissipation theorem for thermal equilibrium plasmas that is valid for arbitrary complex frequencies. It expresses the thermal field fluctuations in terms of the inverse Maxwell tensor alone. One can readily see that this expression for the correlation tensor still is hermitian as it should be.

5.3. Weakly damped and aperiodic fluctuations

Since this result for the fluctuation–dissipation theorem is a generalization to arbitrary complex frequencies $\omega = \omega_{\mathrm{R}} + \imath\gamma$, the question immediately arises if it is consistent with the formulations for real-valued frequencies found in the literature. To this end, the case $\gamma \to 0$ must be considered that corresponds to the limit of weak damping:

$$\lim_{\gamma\to 0}\langle \delta\boldsymbol{E} \otimes \delta\boldsymbol{E}\rangle_{\boldsymbol{k},\omega} = \frac{\imath k_{\mathrm{B}}T}{8\pi^3\omega_{\mathrm{R}}}\lim_{\gamma\to 0}\varsigma\left(\hat{\boldsymbol{\Lambda}}^{-1}_{\boldsymbol{k},\omega} - \hat{\boldsymbol{\Lambda}}^{-1\dagger}_{\boldsymbol{k},\omega}\right). \tag{258}$$

In this limit, the theorem reproduces the well-known feature that only the antihermitian part of the (inverse) Maxwell tensor is required to describe the dissipative properties of the plasma. The appearance of the parameter ς indicates that the result is sensitive to the (in-)stability of the fluctuations because $\varsigma = 1$ for $\Im\omega > 0$ and $\varsigma = -1$ for $\Im\omega < 0$. Formally, this corresponds to a discontinuity considering that the limits $\gamma \to +0$ and $\gamma \to -0$ differ by a minus sign due to the presence of the factor ς. Ignoring the prefactors stemming from the different conventions regarding the Fourier–Laplace transforms, the equation above states that the spectral correlations are proportional to $\imath k_{\mathrm{B}}T/\omega_{\mathrm{R}}$ times the antihermitian part of the Maxwell tensor. Hence, the result found here agrees with the classical limit of the fluctuation–dissipation theorem obtained by Sitenko (1967), viz. (235).

Apart from the oscillations with negligible damping, $\omega \in \mathbb{R}$, the second case of interest are aperiodic fluctuations that are characterized by purely imaginary frequencies, $\omega \in \imath\mathbb{R}$. Setting $\omega = \imath\gamma$ in Eq. (257) yields

$$\begin{aligned}\langle \delta\boldsymbol{E} \otimes \delta\boldsymbol{E} \rangle_{\boldsymbol{k},\imath\gamma} = &\frac{\varsigma k_{\mathrm{B}}T}{8\pi^3\gamma}\big\{\hat{\boldsymbol{\Lambda}}^{-1}_{\boldsymbol{k},\imath\gamma} + \hat{\boldsymbol{\Lambda}}^{-1\dagger}_{\boldsymbol{k},\imath\gamma}\\ &- 2\hat{\boldsymbol{\Lambda}}^{-1}_{\boldsymbol{k},\imath\gamma}\cdot\big[\hat{\mathbf{1}} + (k^2c^2/\gamma^2)\big(\hat{\mathbf{1}} - \boldsymbol{k}\otimes\boldsymbol{k}/k^2\big)\big]\cdot\hat{\boldsymbol{\Lambda}}^{-1\dagger}_{\boldsymbol{k},\imath\gamma}\big\}\,.\end{aligned} \tag{259}$$

It is evident that the correlations are considerably different from the previous case of the weakly damped limit. Firstly, it is no longer the antihermitian part of the Maxwell tensor but the hermitian part that appears on the right-hand side. As a consequence, the well-known correspondence between the dissipation of the plasma and the antihermitian part of the response tensor must be revised if the complex frequencies stemming from a Laplace transform are considered rather than the real frequencies of a Fourier transform. Furthermore, the entire term in the second line of Eq. (259) constitutes a new feature in comparison with the case of real frequencies. This can be seen more clearly by inspecting the universal equation (257). There, the term in question is preceded by the factor $(\omega - \omega^*)$ that is proportional to the imaginary part of the frequency, so it remains hidden as long as ω is a real number.

5.4. Unmagnetized plasmas

The fluctuation–dissipation theorem simplifies if no external magnetic field is present. The reason is that, apart from the wavevector, there is no longer a privileged direction in space because the thermal Maxwell–Jüttner distribution of the plasma particles is isotropic. The Maxwell tensor also reflects this symmetry as will be shown next. The resulting expression can then be inserted into the fluctuation–dissipation theorem.

According to Eq. (232) the main task is to compute the conductivity tensor for the case at hand, the corresponding Maxwell tensor follows immediately. It should be emphasized that the computation of the conductivity tensor presented here does not only hold for the Maxwell–Jüttner distribution but, more generally, for any isotropic distribution function. The coordinate system will be chosen such that its z-axis is parallel to the wavevector. In view of the axial symmetry it is reasonable to introduce cylindrical coordinates $p_\perp$, φ and $p_\parallel$ in momentum space. The Cartesian components of the conductivity tensor (168) can then be written as

$$\sigma^{ij}_{\boldsymbol{k},\omega} = -\sum_a \frac{\imath e_a^2 n_a}{\omega}\int \mathrm{d}^3p\; v_i\left(\frac{\partial f^a_{\boldsymbol{p}}}{\partial p_j} + \frac{kv_j}{\omega - kv_\parallel}\frac{\partial f^a_{\boldsymbol{p}}}{\partial p_\parallel}\right), \tag{260}$$

where the notations $p_\parallel = p_3 = p_z$ are used interchangeably. In order to make use of the underlying symmetry, the following relations are employed that hold for any gyrotropic function $g = g(p_\perp, p_\parallel)$, that is, for any function of momentum that is independent of the angular variable φ:

$$\frac{\partial g}{\partial p_i} = \frac{v_i}{v_\perp}\frac{\partial g}{\partial p_\perp} \qquad \text{for } i \in \{1,2\}, \tag{261}$$

$$\int \mathrm{d}p_1 \int \mathrm{d}p_2\; v_i\, g = 0 \qquad \text{for } i \in \{1,2\}, \tag{262}$$

$$\int \mathrm{d}p_1 \int \mathrm{d}p_2\; v_i v_j\, g = \frac{\delta_{ij}}{2}\int \mathrm{d}p_1 \int \mathrm{d}p_2\; v_\perp^2\, g \qquad \text{for } i,j \in \{1,2\}. \tag{263}$$

The proofs are straightforward computations in polar coordinates as will be shown in Appendix A.2. It should be noted that the distribution function $f^a_{\boldsymbol{p}}$ itself obeys these relations because it is isotropic. Applying the first and third relation to the conductivity tensor one obtains

$$\sigma^{ij}_{\boldsymbol{k},\omega} = \delta_{ij}\,\sigma^{\perp}_{\boldsymbol{k},\omega} \qquad \text{for } i,j \in \{1,2\} \tag{264}$$

with

$$\sigma^{\perp}_{\boldsymbol{k},\omega} \equiv -\sum_a \frac{\imath e_a^2 n_a}{2\omega}\int \mathrm{d}^3p\; v_\perp\left(\frac{\partial f^a_{\boldsymbol{p}}}{\partial p_\perp} + \frac{kv_\perp}{\omega - kv_\parallel}\frac{\partial f^a_{\boldsymbol{p}}}{\partial p_\parallel}\right). \tag{265}$$

The first relation implies further that the tensor elements σ_{13} and σ_{23} vanish, and in view of the first two relations the components σ_{31} and σ_{32} must be zero as well. Therefore, the only remaining entry of the tensor is $\sigma^{33}_{\boldsymbol{k},\omega} \equiv \sigma^{\parallel}_{\boldsymbol{k},\omega}$, and it simplifies to

$$\sigma^{\parallel}_{\boldsymbol{k},\omega} = -\sum_a \imath e_a^2 n_a \int \mathrm{d}^3p\, \frac{v_\parallel}{\omega - kv_\parallel} \frac{\partial f^a_{\boldsymbol{p}}}{\partial p_\parallel}\,. \tag{266}$$

In summary, the conductivity tensor has diagonal form in the current coordinate system with $\boldsymbol{e}_3 = \boldsymbol{k}/k$. In view of Eq. (232) one can infer that the Maxwell operator inherits this symmetry:

$$\Lambda^{ij}_{\boldsymbol{k},\omega} = \mathrm{diag}\big(\Lambda^{\perp}_{\boldsymbol{k},\omega}, \Lambda^{\perp}_{\boldsymbol{k},\omega}, \Lambda^{\parallel}_{\boldsymbol{k},\omega}\big)_{ij}\,, \tag{267}$$

$$\Lambda^{\perp}_{\boldsymbol{k},\omega} = 1 - \frac{k^2c^2}{\omega^2} + \sum_a \frac{2\pi e_a^2 n_a}{\omega^2} \int \mathrm{d}^3p\, v_\perp \left(\frac{\partial f^a_{\boldsymbol{p}}}{\partial p_\perp} + \frac{kv_\perp}{\omega - kv_\parallel} \frac{\partial f^a_{\boldsymbol{p}}}{\partial p_\parallel} \right), \tag{268}$$

$$\Lambda^{\parallel}_{\boldsymbol{k},\omega} = 1 + \sum_a \frac{4\pi e_a^2 n_a}{\omega} \int \mathrm{d}^3p\, \frac{v_\parallel}{\omega - kv_\parallel} \frac{\partial f^a_{\boldsymbol{p}}}{\partial p_\parallel}\,. \tag{269}$$

This representation of the Maxwell tensor describing an unmagnetized plasma agrees with the result of Schlickeiser (2010). However, it only holds for complex frequencies in the upper half plane. For frequencies with negative imaginary parts on the other hand, analytical continuations of these expressions must be computed in order to obtain representations that are valid in the entire complex plane. As mentioned earlier, this issue was addressed by Felten et al. (2013). The diagonal form of the tensor remains unchanged, of course, but the transverse and longitudinal dispersion functions become, respectively,

$$\begin{aligned}\Lambda^{\perp}_{\boldsymbol{k},\omega} &= 1 - \eta^2 + \sum_a \frac{\omega^2_{\mathrm{p},a}\eta}{4k^2c^2\Theta_a K_2(1/\Theta_a)} \int_1^\infty \mathrm{d}\Gamma_a \Bigg[(2/\eta)(1-\Gamma_a^{-2})^{1/2} \frac{\partial U_a}{\partial \Gamma_a} \\ &\quad + \ln \frac{1/\eta + (1-\Gamma_a^{-2})^{1/2}}{1/\eta - (1-\Gamma_a^{-2})^{1/2}} \frac{\partial}{\partial \Gamma_a} \Big((1 - 1/\eta^2) U_a - \exp(-\Gamma_a/\Theta_a) \Big) \Bigg]\end{aligned} \tag{270}$$

and

$$\begin{aligned}\Lambda^{\parallel}_{\boldsymbol{k},\omega} &= 1 + \sum_a \frac{\omega^2_{\mathrm{p},a}}{k^2c^2\Theta_a K_2(1/\Theta_a)} \int_1^\infty \mathrm{d}\Gamma_a \\ &\quad \frac{\partial U_a}{\partial \Gamma_a} \left((1/2\eta) \ln \frac{1/\eta + (1-\Gamma_a^{-2})^{1/2}}{1/\eta + (1+\Gamma_a^{-2})^{1/2}} - (1-\Gamma_a^{-2})^{1/2} \right)\end{aligned} \tag{271}$$

with

$$U_a(\Gamma_a) = \big[(\Gamma_a + \Theta_a)^2 + \Theta_a^2 \big] \exp(-\Gamma_a/\Theta_a)\,. \tag{272}$$

For the further developments, however, it suffices to note that the Maxwell tensor possesses the diagonal form (267) for arbitrary values of the complex frequency.

The fluctuation–dissipation theorem in the formulation presented here requires the knowledge of the inverse Maxwell tensor. Due to the diagonal form (267) it can readily be found by replacing each element with its reciprocal:

$$\Lambda^{-1,ij}_{\boldsymbol{k},\omega} = \mathrm{diag}\big(1/\Lambda^{\perp}_{\boldsymbol{k},\omega} \big| 1/\Lambda^{\perp}_{\boldsymbol{k},\omega} \big| 1/\Lambda^{\parallel}_{\boldsymbol{k},\omega}\big)_{ij}\,. \tag{273}$$

Thus, every term entering the right-hand side of (257) also has diagonal form in the current situation. Since the 3 × 3-diagonal matrices constitute a ring with respect to addition and multiplication and since these operations are performed elementwise, one can infer that the left-hand side of this equation can be rewritten as

$$\langle \delta E_i\, \delta E_j \rangle_{\boldsymbol{k},\omega} = \mathrm{diag}\big(\langle \delta E_1^2 \rangle_{\boldsymbol{k},\omega} \big| \langle \delta E_2^2 \rangle_{\boldsymbol{k},\omega} \big| \langle \delta E_3^2 \rangle_{\boldsymbol{k},\omega} \big)_{ij} \tag{274}$$

with

$$\begin{aligned}\langle \delta E_1^2 \rangle_{\boldsymbol{k},\omega} &= \langle \delta E_2^2 \rangle_{\boldsymbol{k},\omega} \\ &= \frac{\imath \varsigma k_{\mathrm{B}} T}{8\pi^3 |\omega|^2} \left(\frac{\omega^*}{\Lambda^{\perp}_{\boldsymbol{k},\omega}} - \frac{\omega}{\Lambda^{\perp *}_{\boldsymbol{k},\omega}} + (\omega - \omega^*) \frac{1 + |\eta|^2}{|\Lambda^{\perp}_{\boldsymbol{k},\omega}|^2} \right) \\ &= -\frac{\varsigma k_{\mathrm{B}} T}{4\pi^3} \left(\Im(1/\omega \Lambda^{\perp}_{\boldsymbol{k},\omega}) - \Im(1/\omega) \frac{1 + |\eta|^2}{|\Lambda^{\perp}_{\boldsymbol{k},\omega}|^2} \right)\end{aligned} \tag{275}$$

and

$$\begin{aligned}\langle \delta E_3^2\rangle_{\boldsymbol{k},\omega} &= \frac{\imath\varsigma k_{\mathrm{B}}T}{8\pi^3|\omega|^2}\left(\frac{\omega^*}{\Lambda^{\parallel}_{\boldsymbol{k},\omega}} - \frac{\omega}{\Lambda^{\parallel *}_{\boldsymbol{k},\omega}} + (\omega-\omega^*)\,\frac{1}{|\Lambda^{\parallel}_{\boldsymbol{k},\omega}|^2}\right)\\ &= -\frac{\varsigma k_{\mathrm{B}}T}{4\pi^3}\left(\Im(1/\omega\Lambda^{\parallel}_{\boldsymbol{k},\omega}) - \Im(1/\omega)\,\frac{1}{|\Lambda^{\parallel}_{\boldsymbol{k},\omega}|^2}\right).\end{aligned} \tag{276}$$

The last equation specifies the longitudinal field fluctuations. The transverse fluctuations are simply the sum of the respective contributions of the 1- and 2-components. The latter also determine the magnetic field fluctuations because from (160) one infers $|\delta\boldsymbol{B}_{\boldsymbol{k},\omega}| = |\eta|\,|\delta\boldsymbol{E}^{\perp}_{\boldsymbol{k},\omega}|$, which implies

$$\langle \delta B^2\rangle_{\boldsymbol{k},\omega} = |\eta|^2\langle \delta E_{\perp}^2\rangle_{\boldsymbol{k},\omega} = 2|\eta|^2\langle \delta E_1^2\rangle_{\boldsymbol{k},\omega}\,. \tag{277}$$

The preceding computations were simplified by the choosing a coordinate system such that the z-axis is parallel with respect to the wavevector. This ensured that all occurring tensors had diagonal form. The results obtained in this fashion may now be reformulated without relying on such restrictions. To begin with, the Maxwell tensor (267) can be expressed as

$$\hat{\boldsymbol{\Lambda}}_{\boldsymbol{k},\omega} = \Lambda^{\perp}_{\boldsymbol{k},\omega}\hat{\mathbf{1}} + \left(\Lambda^{\parallel}_{\boldsymbol{k},\omega} - \Lambda^{\perp}_{\boldsymbol{k},\omega}\right)\boldsymbol{k}\otimes\boldsymbol{k}/k^2\,. \tag{278}$$

In the same fashion, Eq. (273) for the inverse tensor can be rewritten in such a way that it becomes independent of any particular coordinate system as well,

$$\hat{\boldsymbol{\Lambda}}^{-1}_{\boldsymbol{k},\omega} = \left(1/\Lambda^{\perp}_{\boldsymbol{k},\omega}\right)\hat{\mathbf{1}} + \left(1/\Lambda^{\parallel}_{\boldsymbol{k},\omega} - 1/\Lambda^{\perp}_{\boldsymbol{k},\omega}\right)\boldsymbol{k}\otimes\boldsymbol{k}/k^2\,. \tag{279}$$

One can readily verify that this is the inverse of (278), indeed, by noticing that the matrix $\boldsymbol{k}\otimes\boldsymbol{k}/k^2$ is idempotent,

$$\frac{\boldsymbol{k}\otimes\boldsymbol{k}}{k^2}\cdot\frac{\boldsymbol{k}\otimes\boldsymbol{k}}{k^2} = \frac{\boldsymbol{k}\otimes\boldsymbol{k}}{k^2}\,, \tag{280}$$

because

$$\begin{aligned}[(\boldsymbol{k}\otimes\boldsymbol{k})\cdot(\boldsymbol{k}\otimes\boldsymbol{k})]_{ij} &= (\boldsymbol{k}\otimes\boldsymbol{k})_{il}(\boldsymbol{k}\otimes\boldsymbol{k})_{lj}\\ &= k_ik_lk_lk_j\\ &= k^2k_ik_j\\ &= k^2(\boldsymbol{k}\otimes\boldsymbol{k})_{ij}\,.\end{aligned} \tag{281}$$

Finally, for the field fluctuations themselves one infers from Eq. (274)

$$\langle \delta\boldsymbol{E}\otimes\delta\boldsymbol{E}\rangle_{\boldsymbol{k},\omega} = \tfrac{1}{2}\langle \delta E_{\perp}^2\rangle_{\boldsymbol{k},\omega}\hat{\mathbf{1}} + \left(\langle \delta E_{\parallel}^2\rangle_{\boldsymbol{k},\omega} - \tfrac{1}{2}\langle \delta E_{\perp}^2\rangle_{\boldsymbol{k},\omega}\right)\boldsymbol{k}\otimes\boldsymbol{k}/k^2\,. \tag{282}$$

Thus, all results obtained in this subsection are also available independent of any particular choice of coordinate system now.

In order to model the case of weak damping in unmagnetized plasmas, too, the limit of vanishing imaginary part of the frequency $\omega = \omega_{\mathrm{R}} + \imath\gamma$ will be considered next. As argued before, the appearance of ς makes this limit sensitive to whether it is approached from the lower or upper half plane, corresponding to stable or unstable fluctuations:

$$\lim_{\gamma\to 0}\langle \delta E_{\perp}^2\rangle_{\boldsymbol{k},\omega} = \frac{k_{\mathrm{B}}T}{2\pi^3\omega_{\mathrm{R}}}\lim_{\gamma\to 0}\frac{\varsigma\Im(\Lambda^{\perp}_{\boldsymbol{k},\omega})}{|\Lambda^{\perp}_{\boldsymbol{k},\omega}|^2}\,, \tag{283}$$

$$\lim_{\gamma\to 0}\langle \delta E_{\parallel}^2\rangle_{\boldsymbol{k},\omega} = \frac{k_{\mathrm{B}}T}{4\pi^3\omega_{\mathrm{R}}}\lim_{\gamma\to 0}\frac{\varsigma\Im(\Lambda^{\parallel}_{\boldsymbol{k},\omega})}{|\Lambda^{\parallel}_{\boldsymbol{k},\omega}|^2}\,. \tag{284}$$

Here, the relation $\Im(1/z) = -\Im z/|z|^2$ for an arbitrary complex number z was employed. Similarly, $\Im(1/\imath z) = -\Re z/|z|^2$ can be used in the case of aperiodic fluctuations. The latter are characterized by imaginary frequencies $\omega = \imath\gamma \in \imath\mathbb{R}$. Since ς reproduces the sign of γ, one obtains

$$\langle \delta E_{\perp}^2\rangle_{\boldsymbol{k},\imath\gamma} = \frac{k_{\mathrm{B}}T}{2\pi^3|\gamma|\,|\Lambda^{\perp}_{\boldsymbol{k},\imath\gamma}|^2}\left(\Re(\Lambda^{\perp}_{\boldsymbol{k},\imath\gamma}) - (1 + k^2c^2/\gamma^2)\right), \tag{285}$$

$$\langle \delta E_{\parallel}^2\rangle_{\boldsymbol{k},\imath\gamma} = \frac{k_{\mathrm{B}}T}{4\pi^3|\gamma|\,|\Lambda^{\parallel}_{\boldsymbol{k},\imath\gamma}|^2}\left(\Re(\Lambda^{\parallel}_{\boldsymbol{k},\imath\gamma}) - 1\right). \tag{286}$$

5.5. Ohmic dissipation in the spectral domain

In the well-known fluctuation–dissipation theorem for real-valued frequencies only the antihermitian part of the Maxwell tensor appears while the hermitian part is irrelevant for the description of dissipation (Sitenko, 1967). This feature is confirmed by a separate analysis of Ohmic losses which confirms that only the antihermitian part of the response tensor is associated with dissipation (Melrose and McPhedran, 1991). Since the generalized theorem derived in this section contains both the antihermitian and the hermitian part of the Maxwell tensor, it is instructive to recap the computation of Ohmic losses and to point out where and why it must be modified for complex frequencies. The whole purpose is to defend the earlier findings by invalidating possible counter arguments against them.

The derivation starts from Poynting's theorem, one of the central results of classical electrodynamics. It states that energy is a conserved quantity. More precisely, it states that the change of the local energy density contained in the electromagnetic fields is balanced by the radiative energy transport to other regions and by the mechanical work done on the charge carriers (Landau and Lifshitz, 1971):

$$\frac{\partial}{\partial t}\,\frac{\delta \boldsymbol{E}_{\boldsymbol{r},t}^2+\delta \boldsymbol{B}_{\boldsymbol{r},t}^2}{8\pi}+\nabla\cdot\left(\frac{c}{4\pi}\,\delta\boldsymbol{E}_{\boldsymbol{r},t}\times\delta\boldsymbol{B}_{\boldsymbol{r},t}\right)=-\delta\boldsymbol{J}_{\boldsymbol{r},t}\cdot\delta\boldsymbol{E}_{\boldsymbol{r},t}\,. \tag{287}$$

(Considering that Maxwell's equations hold for the field fluctuations in their usual form without any modification, viz. (144)–(147), Poynting's theorem can directly be transferred as well.) This balance equation identifies the Ohmic losses on the right-hand side as the type of dissipation encountered in electrodynamics. These losses are attributed to the work done by the fields on the electric charges, i. e., to the thermal particle motion generated by the fields. Hence, only the *induced* current density is relevant in this regard (Melrose and McPhedran, 1991; Schlickeiser, 2002). Since the equation above has the dimension of energy per time and volume, the total energy losses are given by

$$\varepsilon_{\mathrm{tot}}=\int \mathrm{d}^3r\int \mathrm{d}t\;\delta\boldsymbol{J}^{\mathrm{ind}}_{\boldsymbol{r},t}\cdot\delta\boldsymbol{E}_{\boldsymbol{r},t}\,. \tag{288}$$

In the idealized case of stationary and homogeneous turbulence the correlations do not vanish as t and $|\boldsymbol{r}|$ approach infinity, so the integrals become infinite, then. Therefore the dissipative losses are best described in terms of the average over a volume V and a time interval T of infinite size (Melrose and McPhedran, 1991; Schlickeiser, 2002),

$$\varepsilon_{\mathrm{av}}=\lim_{V,T\to\infty}\frac{1}{VT}\int_V \mathrm{d}^3r\int_T \mathrm{d}t\;\delta\boldsymbol{J}^{\mathrm{ind}}_{\boldsymbol{r},t}\cdot\delta\boldsymbol{E}_{\boldsymbol{r},t}=\lim_{V,T\to\infty}\frac{\varepsilon_{\mathrm{tot}}}{VT}\,. \tag{289}$$

So even in this case $\varepsilon_{\mathrm{tot}}$ remains the quantity of interest. For the present goal of establishing a connection between the dissipation in the plasma and the Maxwell tensor, therefore, it suffices to investigate (288). This equation must be carried over to the spectral domain. Since the purpose here is to recap the theory of real-valued frequencies, a Fourier transform both in space and time will be applied. This can be achieved by means of the Plancherel or multiplication theorem. It states that any two functions $f_{\boldsymbol{r}}, g_{\boldsymbol{r}}$ as well as their respective Fourier transforms $f_{\boldsymbol{k}}$ and $g_{\boldsymbol{k}}$ obey the relation

$$\begin{aligned}\int \mathrm{d}^3k\, f^*_{\boldsymbol{k}}g_{\boldsymbol{k}}&=\int \mathrm{d}^3k\int\frac{\mathrm{d}^3r}{(2\pi)^3}\int\frac{\mathrm{d}^3r'}{(2\pi)^3}\,\mathrm{e}^{\imath\boldsymbol{k}\cdot\boldsymbol{r}}\mathrm{e}^{-\imath\boldsymbol{k}\cdot\boldsymbol{r}'}f^*_{\boldsymbol{r}}g_{\boldsymbol{r}'}\\&=\int\frac{\mathrm{d}^3r}{(2\pi)^3}\int\frac{\mathrm{d}^3r'}{(2\pi)^3}\,(2\pi)^3\,\delta(\boldsymbol{r}-\boldsymbol{r}')f^*_{\boldsymbol{r}}g_{\boldsymbol{r}'}\\&=\int\frac{\mathrm{d}^3r}{(2\pi)^3}\,f^*_{\boldsymbol{r}}g_{\boldsymbol{r}}\,.\end{aligned} \tag{290}$$

If both $f_{\boldsymbol{r}}$ and $g_{\boldsymbol{r}}$ are real-valued functions, complex conjugation has no effect on them. Therefore, adding this theorem to its complex conjugate and dividing by 2 yields

$$\int\frac{\mathrm{d}^3r}{(2\pi)^3}\,f_{\boldsymbol{r}}g_{\boldsymbol{r}}=\int \mathrm{d}^3k\,\tfrac{1}{2}(f^*_{\boldsymbol{k}}g_{\boldsymbol{k}}+f_{\boldsymbol{k}}g^*_{\boldsymbol{k}})\qquad \text{for } f_{\boldsymbol{r}},g_{\boldsymbol{r}}\in\mathbb{R}. \tag{291}$$

Evidently this result applies accordingly to a one-dimensional Fourier transform with respect to time. Thus, the total energy loss is given by

$$\varepsilon_{\mathrm{tot}}=(2\pi)^4\int \mathrm{d}^3k\int \mathrm{d}\omega\;\tfrac{1}{2}\left(\delta\boldsymbol{J}^{\mathrm{ind}\,*}_{\boldsymbol{k},\omega}\cdot\delta\boldsymbol{E}_{\boldsymbol{k},\omega}+\delta\boldsymbol{J}^{\mathrm{ind}}_{\boldsymbol{k},\omega}\cdot\delta\boldsymbol{E}^*_{\boldsymbol{k},\omega}\right). \tag{292}$$

The induced current density is determined by Ohm's law. In preparation for the corresponding substitution, the following auxiliary formula will prove useful. It holds for an arbitrary matrix $\hat{\boldsymbol{M}}$ and a vector $\boldsymbol{a}$:

$$\begin{aligned}(\hat{\boldsymbol{M}}\cdot\boldsymbol{a})\cdot\boldsymbol{a}^*+(\hat{\boldsymbol{M}}\cdot\boldsymbol{a})^*\cdot\boldsymbol{a}&=(\hat{\boldsymbol{M}}\cdot\boldsymbol{a})_i a^*_i+(\hat{\boldsymbol{M}}^*\cdot\boldsymbol{a}^*)_j a_j\\&=M_{ij}a_ja^*_i+M^*_{ji}a^*_ia_j\\&=(M_{ij}+M^*_{ji})a_ja^*_i\\&=2M^{\mathrm{H}}_{ij}(\boldsymbol{a}\otimes\boldsymbol{a}^*)_{ji}\\&=2\,\mathrm{tr}[\hat{\boldsymbol{M}}^{\mathrm{H}}\cdot(\boldsymbol{a}\otimes\boldsymbol{a}^*)]\,.\end{aligned} \tag{293}$$

Now the induced current density will be expressed in terms of the electric field as stated by Ohm's law (167), which takes the same form if the theory is developed for a Fourier–Fourier transform rather than a Fourier–Laplace transform. Applying the auxiliary formula leads to

$$\varepsilon_{\rm tot} = (2\pi)^4 \int \mathrm{d}^3k \int \mathrm{d}\omega \;\; \mathrm{tr}[\hat{\boldsymbol{\sigma}}^{\rm H}_{\boldsymbol{k},\omega} \cdot (\delta\boldsymbol{E}_{\boldsymbol{k},\omega} \otimes \delta\boldsymbol{E}^*_{\boldsymbol{k},\omega})]\,. \tag{294}$$

Thus, only the hermitian part of the conductivity tensor describes the dissipative properties of the plasma. In order to find a similar relation involving the Maxwell tensor, Eq. (232) may be used. Since $\hat{\boldsymbol{\Lambda}}^{(0)}_{\boldsymbol{k},\omega}$ is hermitian for real frequencies, one obtains

$$\hat{\boldsymbol{\sigma}}^{\rm H}_{\boldsymbol{k},\omega} = (\omega/4\pi\imath)\hat{\boldsymbol{\Lambda}}^{\rm A}_{\boldsymbol{k},\omega}\,. \tag{295}$$

This relation confirms the usual result that only the antihermitian part of the Maxwell tensor enters the dissipated energy. The situation changes drastically if the frequency is complex. In this case the hermitian part of the conductivity tensor becomes

$$\begin{aligned}\hat{\boldsymbol{\sigma}}^{\rm H}_{\boldsymbol{k},\omega} &= \frac{\omega\hat{\boldsymbol{\Lambda}}_{\boldsymbol{k},\omega} - \omega^*\hat{\boldsymbol{\Lambda}}^{\dagger}_{\boldsymbol{k},\omega}}{8\pi\imath} - \frac{(\omega-\omega^*)[\hat{\mathbf{1}} + |\eta|^2(\hat{\mathbf{1}} - \boldsymbol{k}\otimes\boldsymbol{k}/k^2)]}{8\pi\imath} \\ &= \frac{(\omega+\omega^*)\hat{\boldsymbol{\Lambda}}^{\rm A}_{\boldsymbol{k},\omega}}{8\pi\imath} - \frac{(\omega-\omega^*)[\hat{\mathbf{1}} + |\eta|^2(\hat{\mathbf{1}} - \boldsymbol{k}\otimes\boldsymbol{k}/k^2) - \hat{\boldsymbol{\Lambda}}^{\rm H}_{\boldsymbol{k},\omega}]}{8\pi\imath}\,,\end{aligned} \tag{296}$$

which reproduces the previous result (295) for real-valued frequencies. In general, however, it is evident that both the hermitian and the antihermitian part of the Maxwell tensor appear once complex frequencies are considered. This supports the corresponding result that was obtained earlier regarding the fluctuation–dissipation theorem.

6. Magnetic field fluctuations in the intergalactic medium

Outline. In this section the general theory developed in the first part will be applied to the unmagnetized, intergalactic plasma in the cosmic voids. Due to the newly discovered stable branch of the Weibel mode, the strength of the spontaneously emitted magnetic field fluctuations is high enough to provide the necessary seed level that is required for the operation of amplification processes such as magnetohydrodynamic dynamos, plasma instabilities, or flux-conserving compression (Yoon et al., 2014). Here, it is demonstrated that an even further enhancement of the seed fields can occur if a highly relativistic electron–positron pair beam traverses the IGM and supplies free energy by disturbing the plasma fluctuations. The impact of the beam is investigated by computing the spectral balance equation that accounts for spontaneous emission and absorption in the perturbed plasma. On that basis, the conditions for negative values of the absorption coefficient are analyzed because effectively these correspond to an amplification of the fluctuations and, thus, to a transformation of the stable mode into an instability. It will be shown that the resulting mathematical criterion possesses a simple geometrical interpretation.

Reference. Parts of the results presented in this section were published by Kolberg et al. (2016).

6.1. Outline of the model

The scenario investigated here is depicted in Fig. 3: Tev-radiation emitted by a sufficiently distant active galactic nucleus (AGN) interacts with the soft photons of the extragalactic background light (EBL). In the annihilation process, an electron–positron pair beam is created that propagates through the equilibrium plasma constituting the IGM. Correspondingly, there are two components that need to be modeled mathematically, namely the beam and the background plasma, and both will be considered here one after another. Regarding the former, Schlickeiser et al. (2012a) confirmed by proper analytical proofs what one might intuitively expect: (1) The created electron–positron pairs are highly relativistic. Typical Lorentz factors are of the order of $\Gamma_{\rm b} = 10^6$. (2) Their energy spectrum is sharply peaked. (3) The pairs are strongly collimated in the forward direction of the TeV-photons emitted by the AGN. These results justify the assumption that the beam is monoenergetic and monodirectional, i. e., that all beam particles have the same relativistic momentum $\boldsymbol{P}$. In plasma physics, this scenario is known as the *cold beam approximation*. The corresponding distribution function simply reads

$$f^{\rm beam}_a(\boldsymbol{p}) = \delta(\boldsymbol{p} - \boldsymbol{P})\,, \tag{297}$$

where the index a labels the particle species e^+ and e^-. It should be noted that the right-hand side is independent of the species because electrons and positrons are assumed to have the same momentum in this approximation. As a consequence, both currents neutralize each other so there is no net current density associated with the beam. In contrast to purely hadronic beams, therefore, the pair beam does not induce any return currents in the background plasma.

The description of the latter is a nontrivial task because the thermal state of the IGM after the reionization onset is not yet fully understood and still remains a vivid field of research (Kovetz et al., 2017; Rorai et al., 2017; Lidz, 2016; Becker et al., 2011; McQuinn, 2016). In order to obtain results that are commensurable with the preceding investigations by Schlickeiser (2012); Schlickeiser and Felten (2013), as well as Yoon et al. (2014), the current analysis will be based on the same model of the cosmic voids in the unmagnetized IGM: Electrons and protons are assumed to have the same temperature and the same number density with the respective values (Hui and Gnedin, 1997; Hui and Haiman, 2003)

$$T \equiv T_e = T_p = 10^4\,\mathrm{K}\,, \tag{298}$$

Table 2
Relevant parameters of the intergalactic medium, assuming a temperature of $T = 10^4$ K and a density of $n = 10^{-7}$ cm^{-3} for each species. Numerical values of parameters depending on the particle species a are given for electrons ($a = e$).

Parameter	Symbol	Definition	Value
Debye Length	λ_D	$(k_B T/4\pi n e^2)^{1/2}$	$2.2 \cdot 10^6$ cm
Plasma Frequency	$\omega_{p,a}$	$(4\pi n e^2/m_a)^{1/2}$	$18\ \mathrm{s}^{-1}$ ($a = e$)
Skin Depth	δ_a	$c/\omega_{p,a}$	$1.7 \cdot 10^9$ cm ($a = e$)
Plasma Parameter	ε_p	$[(4/3)\pi n\lambda_D^3]^{-1}$	$2.3 \cdot 10^{-13}$
Relativistic Temperature	Θ_a	$k_B T/m_a c^2$	$1.7 \cdot 10^{-6}$ ($a = e$)
Thermal Speed	β_a	$(2k_B T/m_a c^2)^{1/2}$	$1.8 \cdot 10^{-3}$ ($a = e$)
Closest Approach	r_c	$e^2/k_B T$	$1.7 \cdot 10^{-7}$ cm
Fermi Temperature	$T_{F,a}$	$(\hbar^2/2m_a k_B)(3\pi^2 n)^{2/3}$	$9.1 \cdot 10^{-16}$ K ($a = e$)

$$n \equiv n_e = n_p = 10^{-7}\ \mathrm{cm}^{-3}\,. \tag{299}$$

(It should be noted that, according to this definition, n does not denote the total particle density $n_e + n_p$ as one might expect.) These two state variables, temperature and density, already determine several other parameters. Among them are the relativistic temperature Θ_a and the thermal speed β_a,

$$\beta_a \equiv (2\Theta_a)^{1/2} \equiv (2k_B T/m_a c^2)^{1/2}\,. \tag{300}$$

Table 2 summarizes these and some further parameters along with their respective definitions and the corresponding numerical values. Considering that the plasma parameter is very small, the approximation of a collisionless plasma is certainly justified, here. Moreover, the smallness of the relativistic temperature and the dimensionless thermal speed indicate that the plasma is nonrelativistic. On the other hand, the temperature is high enough (that is, much higher than the Fermi temperature) to allow for a classical, non-quantum description. The remaining and so far unmentioned parameters in the table are listed there for later reference.

It is implicitly understood in the specification of a temperature that the particles are distributed thermally. Mathematically, this can be implemented in the form of the Maxwell–Jüttner distribution function, viz. (239)–(241),

$$f_a^{\mathrm{IGM}}(\boldsymbol{p}) = [4\pi m_a^3 c^3 \Theta_a K_2(1/\Theta_a)]^{-1} \exp[-\Gamma_a(\boldsymbol{p})/\Theta_a]\,, \tag{301}$$

where a labels electrons and protons in this case (Schlickeiser, 2002). Just as in Section 5.2, Γ_a denotes the Lorentz factor of particle species a, and K_2 is the modified Bessel function of the second kind and second order.

Recently it was shown that the field fluctuations in such a thermal, unmagnetized plasma are dominated by an aperiodic, damped, and transverse mode that can be interpreted as the stable branch of the Weibel mode:

$$\omega_M(\boldsymbol{k}) = \imath\gamma(k) \qquad (\gamma \in \mathbb{R}^-)\,. \tag{302}$$

Felten et al. (2013) discovered it by performing the proper analytical continuation of the form factors and the Maxwell tensor so as to generalize the fluctuation theory to arbitrary complex frequencies. In order to specify the dispersion relation of the mode, it is advantageous to introduce the dimensionless wavenumber κ and the dimensionless frequency y as follows:

$$\kappa = k/k_c\,, \tag{303}$$

$$y = -\gamma/ck = -\gamma/ck_c\kappa\,. \tag{304}$$

Since the mode is a damped one, the minus sign in the definition of y guarantees that this frequency is a positive number. The reference wavenumber k_c is given by the inverse electron skin depth $\delta_e = c/\omega_{pe}$ (see Table 2) multiplied by some further constants such that all prefactors appearing in the dispersion relation are absorbed in its definition:

$$k_c = \frac{(4\pi m_p/m_e)^{1/4}}{\beta_e^{3/2}}\,\frac{1}{\delta_e} = 9.32 \cdot 10^{-5}\ \mathrm{cm}^{-1}\,. \tag{305}$$

In terms of the dimensionless variables κ and y the dispersion relation of the new mode can be specified in the following simple form derived by Schlickeiser and Felten (2013):

$$\kappa^2 = y(y^2 + \zeta)/(y^2 + 1)\,, \tag{306}$$

where

$$\zeta = (\beta_e^2/2)(m_e/m_p)^{1/2} = 3.95 \cdot 10^{-8}\,. \tag{307}$$

In the same reference, a good approximation for the inverse of this dispersion relation was found that is valid for the entire wavenumber range. Later on in this section it will be convenient to consider the frequency y as the independent variable rather than the wavenumber κ, so the inverse relation will be of great benefit, then:

$$y \simeq \kappa^2(1 + \kappa^{4/3})/(\zeta + \kappa^{4/3})\,. \tag{308}$$

Up to this point, beam and background plasma were modeled separately. So far, however, there is still one parameter missing that relates both components, namely the density ratio. Evidently, the beam density strongly depends on the distance to the source (Broderick et al., 2012). Sufficiently far away from the AGN, a typical value is $10^{-22}\,\mathrm{cm}^{-3}$ (Schlickeiser et al., 2012b), implying a density ratio of

$$\varepsilon = n_e^{\mathrm{beam}}/n_e^{\mathrm{IGM}} = 10^{-15}\,. \tag{309}$$

In order to describe the scenario in closer proximity to the source, a ratio of $\varepsilon = 10^{-5}$ will be adopted. Above this threshold, the cold beam approximation becomes questionable (Cairns, 1989; Schlickeiser et al., 2013). In any case, the low density ratio underlying the present analysis defies a numerical simulation but provides an ideal basis for the perturbative treatment employed in the subsequent investigations.

In order to derive the spectral balance equation for the field fluctuations eventually, the first task is the computation of the Maxwell tensor because it constitutes an essential part of the wave equation. It contains the entire information about the particular plasma model under investigation, in this case a thermal background plasma with an additional beam. Since the conductivity tensor (168) is a linear functional of the particle distribution function, the Maxwell tensor (171) is an affine functional. Therefore, the beam and background contributions may be computed separately:

$$\hat{\boldsymbol{\Lambda}}_{\boldsymbol{k},\omega} = \hat{\boldsymbol{\Lambda}}^{\mathrm{IGM}}_{\boldsymbol{k},\omega} + \varepsilon\,\hat{\boldsymbol{\Lambda}}^{\mathrm{beam}}_{\boldsymbol{k},\omega}\,, \tag{310}$$

where

$$\begin{aligned}\hat{\boldsymbol{\Lambda}}^{\mathrm{IGM}}_{\boldsymbol{k},\omega} &= \hat{\mathbf{1}} - \frac{k^2c^2}{\omega^2}\left(\hat{\mathbf{1}} - \frac{\boldsymbol{k}\otimes\boldsymbol{k}}{k^2}\right) \\ &\quad + \sum_{a=e,p}\frac{4\pi e_a^2 n}{\omega^2}\int \mathrm{d}^3p\ \boldsymbol{v}\otimes\left(\nabla_p f_a^{\mathrm{IGM}}(\boldsymbol{p}) + \frac{\boldsymbol{k}\cdot\nabla_p f_a^{\mathrm{IGM}}(\boldsymbol{p})}{\omega - \boldsymbol{k}\cdot\boldsymbol{v}}\,\boldsymbol{v}\right)\end{aligned} \tag{311}$$

and

$$\hat{\boldsymbol{\Lambda}}^{\mathrm{beam}}_{\boldsymbol{k},\omega} = \sum_{a=e^\pm}\frac{4\pi e_a^2 n}{\omega^2}\int \mathrm{d}^3p\ \boldsymbol{v}\otimes\left(\nabla_p f_a^{\mathrm{beam}}(\boldsymbol{p}) + \frac{\boldsymbol{k}\cdot\nabla_p f_a^{\mathrm{beam}}(\boldsymbol{p})}{\omega - \boldsymbol{k}\cdot\boldsymbol{v}}\,\boldsymbol{v}\right). \tag{312}$$

Here, the beam density was expressed as εn in terms of the background density, hence the appearance of the density ratio in the first equation. The Maxwell tensor of a thermal, unmagnetized plasma has already been computed in Section 5.4, so it suffices to repeat the result (278) here and adjust the notation appropriately:

$$\hat{\boldsymbol{\Lambda}}^{\mathrm{IGM}}_{\boldsymbol{k},\omega} = \Lambda^{\perp}_{\boldsymbol{k},\omega}\hat{\mathbf{1}} + \left(\Lambda^{\parallel}_{\boldsymbol{k},\omega} - \Lambda^{\perp}_{\boldsymbol{k},\omega}\right)\boldsymbol{k}\otimes\boldsymbol{k}/k^2\,. \tag{313}$$

The scalar functions $\Lambda^{\perp}_{\boldsymbol{k},\omega}$ and $\Lambda^{\parallel}_{\boldsymbol{k},\omega}$ are given by Eqs. (268) and (269). In order to ease the notation, they were not equipped with the label "IGM", although they are associated with the unperturbed thermal component. Regarding the beam component, the integration is carried out in Appendix A.3, yielding

$$\hat{\boldsymbol{\Lambda}}^{\mathrm{beam}}_{\boldsymbol{k},\omega} = -\frac{2\omega_{\mathrm{p},e}^2}{\Gamma_{\mathrm{b}}\omega^2}\left(\hat{\mathbf{1}} + \frac{c(\boldsymbol{k}\otimes\boldsymbol{\beta} + \boldsymbol{\beta}\otimes\boldsymbol{k})}{\omega - c\boldsymbol{k}\cdot\boldsymbol{\beta}} - \frac{(\boldsymbol{\beta}\otimes\boldsymbol{\beta})(\omega^2 - k^2c^2)}{(\omega - c\boldsymbol{k}\cdot\boldsymbol{\beta})^2}\right), \tag{314}$$

where Γ_{b} and $\boldsymbol{\beta} = \boldsymbol{P}/\Gamma_{\mathrm{b}}m_e c$ denote the Lorentz factor and the dimensionless velocity of the beam, respectively. Moreover, the electron plasma frequency $\omega_{\mathrm{p},e} = (4\pi ne^2/m_e)^{1/2}$ of the background plasma was employed for brevity.

6.2. The aperiodic mode in the perturbed plasma

According to the linear theory discussed in Section 2 and 4, the field fluctuations occur in the form of eigenmodes with a fixed relation between wavenumber and frequency, $\omega = \omega_{\mathrm{M}}(\boldsymbol{k})$. These modes are the roots of the dispersion function, i. e., the determinant of the Maxwell tensor vanishes there, viz. (33). The determinant of the Maxwell tensor (313) describing the IGM component is given by

$$\det\hat{\boldsymbol{\Lambda}}^{\mathrm{IGM}}_{\boldsymbol{k},\omega} = \det\left[\mathrm{diag}\left(\Lambda^{\perp}_{\boldsymbol{k},\omega},\,\Lambda^{\perp}_{\boldsymbol{k},\omega},\,\Lambda^{\parallel}_{\boldsymbol{k},\omega}\right)\right] = \left(\Lambda^{\perp}_{\boldsymbol{k},\omega}\right)^2\Lambda^{\parallel}_{\boldsymbol{k},\omega}\,. \tag{315}$$

This can easily be verified in a coordinate system with one axis aligned to the wavevector since the tensor becomes diagonal, there. Such a procedure is allowed because the determinant is independent of the coordinate representation of its argument. The transverse, damped, aperiodic mode introduced in Section 6.1 obeys the dispersion relation

$$\Lambda^{\perp}_{\boldsymbol{k},\omega} = 0 \quad \text{for}\ \ \omega = \omega_{\mathrm{M}}(\boldsymbol{k})\,, \tag{316}$$

so the mode is a root of the dispersion function that corresponds to the unperturbed thermal background plasma.

The issue that needs to be addressed now is how this situation is affected by the presence of the pair beam. To this end, the determinant of the perturbed Maxwell tensor (310) must be computed:

$$\Lambda(\boldsymbol{k},\omega) \equiv \det\hat{\boldsymbol{\Lambda}}_{\boldsymbol{k},\omega} = \det\left(\hat{\boldsymbol{\Lambda}}^{\mathrm{IGM}}_{\boldsymbol{k},\omega} + \varepsilon\,\hat{\boldsymbol{\Lambda}}^{\mathrm{beam}}_{\boldsymbol{k},\omega}\right). \tag{317}$$

Since the determinant is only columnwise (multi-) linear, but in general not a linear function of its entire argument, the best way to proceed is to express both matrices in coordinates. In order to benefit from the available diagonal form of the first tensor, the coordinate system is chosen such that its z-axis is oriented parallel with respect to the wavevector. Neglecting all terms of nonlinear order in ε, one obtains

$$\Lambda(\boldsymbol{k},\omega) \simeq \Lambda^{\perp}_{\boldsymbol{k},\omega}\Lambda^{\parallel}_{\boldsymbol{k},\omega}\left(\Lambda^{\perp}_{\boldsymbol{k},\omega} + \varepsilon L(\boldsymbol{k},\omega)\right), \tag{318}$$

where

$$L(\boldsymbol{k},\omega) \equiv \Lambda^{\mathrm{beam},11}_{\boldsymbol{k},\omega} + \Lambda^{\mathrm{beam},22}_{\boldsymbol{k},\omega} + \left(\Lambda^{\perp}_{\boldsymbol{k},\omega}/\Lambda^{\parallel}_{\boldsymbol{k},\omega}\right)\Lambda^{\mathrm{beam},33}_{\boldsymbol{k},\omega}. \tag{319}$$

As expected, the zeroth order term reproduces the dispersion function (315) of the unperturbed IGM. The function $L(\boldsymbol{k},\omega)$ and its prefactors constitute the first order correction that accounts for the disturbance by the pair beam. Since it remains finite at $\omega_{\mathrm{M}}(\boldsymbol{k})$, the dispersion relation (316) implies

$$\Lambda(\boldsymbol{k},\omega) \simeq 0 \quad \text{for} \;\; \omega = \omega_{\mathrm{M}}(\boldsymbol{k}). \tag{320}$$

Thus, the mode in question also solves the dispersion relation of the perturbed system in the first order approximation, indicating that it still dominates the field fluctuations in the plasma.

6.3. *The perturbed wave equation*

Now that the occurrence of the damped, aperiodic mode in the perturbed plasma is established, the next task is to derive the wave equation for the field fluctuations. The general theory discussed in Section 4 asserts that the field fluctuations can be expressed in the form (180), so one obtains

$$\delta\boldsymbol{E}_{\boldsymbol{k},\omega} = \hat{\boldsymbol{\Lambda}}^{-1}_{\boldsymbol{k},\omega}\cdot\delta\boldsymbol{E}^{0}_{\boldsymbol{k},\omega}, \tag{321}$$

where

$$\delta\boldsymbol{E}^{0}_{\boldsymbol{k},\omega} \equiv -(4\pi\imath/\omega)\,\delta\boldsymbol{J}^{0}_{\boldsymbol{k},\omega}. \tag{322}$$

An equation of the same type also holds for the thermal fluctuations in the unperturbed IGM,

$$\delta\boldsymbol{E}^{\mathrm{IGM}}_{\boldsymbol{k},\omega} = \hat{\boldsymbol{\Lambda}}^{\mathrm{IGM},-1}_{\boldsymbol{k},\omega}\cdot\delta\boldsymbol{E}^{0}_{\boldsymbol{k},\omega}, \tag{323}$$

and all quantities appearing here are known from previous investigations. In particular, it is known that the Maxwell tensor is diagonal in a coordinate system with $\boldsymbol{e}_z \parallel \boldsymbol{k}$ and that its inverse inherits this property:

$$\Lambda^{\mathrm{IGM},-1,ij}_{\boldsymbol{k},\omega} = \mathrm{diag}\left(1/\Lambda^{\perp}_{\boldsymbol{k},\omega}\middle|1/\Lambda^{\perp}_{\boldsymbol{k},\omega}\middle|1/\Lambda^{\parallel}_{\boldsymbol{k},\omega}\right)_{ij}. \tag{324}$$

As a consequence, longitudinal and transverse fluctuations are governed by two separate, decoupled equations. As far as the stable branch of the Weibel mode is concerned, only the transverse fluctuations are relevant:

$$\delta E^{\mathrm{IGM},i}_{\boldsymbol{k},\omega} = \delta E^{0,i}_{\boldsymbol{k},\omega}/\Lambda^{\perp}_{\boldsymbol{k},\omega} \qquad (i=1,2). \tag{325}$$

Due to the isotropy of the unperturbed plasma, there is no distinguished direction in space other than the one specified by the wavevector. Therefore, the transverse fluctuations do not depend on the angle that determines the orientation within the plane perpendicular to $\boldsymbol{k}$,

$$\delta E^{\mathrm{IGM}\perp}_{\boldsymbol{k},\omega} = \delta E^{0\perp}_{\boldsymbol{k},\omega}/\Lambda^{\perp}_{\boldsymbol{k},\omega}. \tag{326}$$

Since the damped, aperiodic, and transverse mode also dominates the fluctuations of the perturbed plasma, it is reasonable to ask if such a decoupled equation for the transverse fluctuations also exists in this case. Of course, one cannot expect an axisymmetric solution again because the beam distinguishes another preferred direction in addition to the wavevector and therefore breaks the symmetry of the unperturbed case. So the goal here is to derive a decoupled equation for the transverse fluctuations in the x-y-plane that still depends on the orientation within this plane, just like equation (325) above. To this end, a new matrix $\hat{\boldsymbol{Q}}_{\boldsymbol{k},\omega}$ is introduced as follows:

$$\hat{\boldsymbol{Q}}_{\boldsymbol{k},\omega} \equiv \hat{\boldsymbol{1}} + \varepsilon\,\hat{\boldsymbol{\Lambda}}^{\mathrm{IGM},-1}_{\boldsymbol{k},\omega}\cdot\hat{\boldsymbol{\Lambda}}^{\mathrm{beam}}_{\boldsymbol{k},\omega}. \tag{327}$$

By design, this matrix is the factor that relates the respective Maxwell tensors of the perturbed and unperturbed plasma, because from (310) one infers

$$\hat{\boldsymbol{\Lambda}}_{\boldsymbol{k},\omega} = \hat{\boldsymbol{\Lambda}}^{\mathrm{IGM}}_{\boldsymbol{k},\omega}\cdot\hat{\boldsymbol{Q}}_{\boldsymbol{k},\omega}. \tag{328}$$

Plugging the inverse of this expression into the wave equation (321) and making use of (323) yields

$$\delta\boldsymbol{E}_{\boldsymbol{k},\omega} = \hat{\boldsymbol{Q}}^{-1}_{\boldsymbol{k},\omega}\cdot\delta\boldsymbol{E}^{\mathrm{IGM}}_{\boldsymbol{k},\omega} = \hat{\boldsymbol{q}}_{\boldsymbol{k},\omega}\cdot\delta\boldsymbol{E}^{\mathrm{IGM}}_{\boldsymbol{k},\omega}/(\det\hat{\boldsymbol{Q}}_{\boldsymbol{k},\omega}), \tag{329}$$

where the inverse of $\hat{\boldsymbol{Q}}_{\boldsymbol{k},\omega}$ was expressed in terms of its adjugate $\hat{\boldsymbol{q}}_{\boldsymbol{k},\omega}$ and its determinant as discussed in Section 2.4. Thus, the new matrix does not only link the perturbed and unperturbed Maxwell tensors, but it also relates the corresponding field fluctuations. Now a coordinate system will be chosen such that its z-axis is parallel with respect to the wavevector. In coordinate form, the equation above becomes a sum with contributions from all three components $\delta E^{\mathrm{IGM},i}_{\boldsymbol{k},\omega}$. Since these fluctuations are dominated by the damped aperiodic mode, and since the latter is a transverse mode, the contribution from the longitudinal field component can be neglected. Applying the summation convention with an accordingly restricted index range yields

$$\delta E^{i}_{\boldsymbol{k},\omega} \simeq \frac{q^{ij}_{\boldsymbol{k},\omega}\,\delta E^{\mathrm{IGM},j}_{\boldsymbol{k},\omega}}{\det \hat{\boldsymbol{Q}}_{\boldsymbol{k},\omega}} \simeq \frac{q^{ij}_{\boldsymbol{k},\omega}\,\delta E^{0,j}_{\boldsymbol{k},\omega}}{\Lambda^{\perp}_{\boldsymbol{k},\omega}\det \hat{\boldsymbol{Q}}_{\boldsymbol{k},\omega}}\,, \qquad i,j \in \{1,2\}\,. \tag{330}$$

In order to compute the denominator, one can take the determinant of Eq. (328) and make use of (315) and (318) afterwards, yielding

$$\delta E^{i}_{\boldsymbol{k},\omega} = \frac{q^{ij}_{\boldsymbol{k},\omega}\,\delta E^{0,j}_{\boldsymbol{k},\omega}}{\Lambda^{\perp}_{\boldsymbol{k},\omega} + \varepsilon L(\boldsymbol{k},\omega)}\,, \qquad i,j \in \{1,2\}\,. \tag{331}$$

Evidently, the result of the last subsection, that the stable branch of the Weibel mode also occurs in the perturbed plasma, is reproduced here in the wave equation as expected. In order to simplify this equation for the subsequent computations, it is advantageous to dispose of the indices and to rewrite everything as a scalar product:

$$\delta \boldsymbol{E}^{\perp}_{\boldsymbol{k},\omega} = \frac{\hat{\boldsymbol{q}}^{\perp}_{\boldsymbol{k},\omega}\cdot\delta \boldsymbol{E}^{0}_{\boldsymbol{k},\omega}}{\Lambda^{\perp}_{\boldsymbol{k},\omega} + \varepsilon L(\boldsymbol{k},\omega)}\,, \tag{332}$$

The new matrix $\hat{\boldsymbol{q}}^{\perp}_{\boldsymbol{k},\omega}$ is identical to $\hat{\boldsymbol{q}}_{\boldsymbol{k},\omega}$ except that its third row and its third column solely contain zeros. This representation of the matrix, however, is only valid in the underlying coordinate system with $\boldsymbol{e}_z \parallel \boldsymbol{k}$. A coordinate-free formulation is given by

$$\hat{\boldsymbol{q}}^{\perp}_{\boldsymbol{k},\omega} = (\hat{\mathbf{1}} - \boldsymbol{k}\otimes\boldsymbol{k}/k^2)\cdot\hat{\boldsymbol{q}}_{\boldsymbol{k},\omega}\cdot(\hat{\mathbf{1}} - \boldsymbol{k}\otimes\boldsymbol{k}/k^2)\,. \tag{333}$$

According to Faraday's law of induction in the form (160), the magnetic field fluctuations are entirely determined by the transverse electric ones. So although the longitudinal field fluctuations were not addressed in this subsection, the result obtained here provides a sufficient amount of information to determine the entire fluctuations of the magnetic field because the latter is transverse by nature.

6.4. Spectral balance equation

It is already known from previous discussions, in particular from (181), that the solution of the wave equation in real space is composed of one or several eigenmodes, each of which is a superposition of plane waves with a fixed relation between wavenumber and frequency:

$$\delta \boldsymbol{E}_{\boldsymbol{r},t} = \sum_{M} \delta \boldsymbol{E}^{(M)}_{\boldsymbol{r},t} = \sum_{M} \int \mathrm{d}^3k\; \mathrm{e}^{\imath[\boldsymbol{k}\cdot\boldsymbol{r} - \omega_M(\boldsymbol{k})t]}\,\delta \boldsymbol{E}^{(M)}_{\boldsymbol{k}}\,. \tag{334}$$

Due to the randomness of the fluctuations, the physically meaningful quantities are the variances or correlations rather than the fluctuations themselves. Apart from a constant prefactor, the latter represent the average electric energy density contained in a given mode M (Krall and Trivelpiece, 1973; Melrose and McPhedran, 1991),

$$\langle|\delta \boldsymbol{E}^{(M)}_{\boldsymbol{r},t}|^2\rangle = \int \mathrm{d}^3k \int \mathrm{d}^3k'\; \mathrm{e}^{\imath(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{r}}\,\mathrm{e}^{-\imath[\omega_M(\boldsymbol{k}) - \omega^{*}_M(\boldsymbol{k}')]t}\,\langle\delta \boldsymbol{E}^{(M)}_{\boldsymbol{k}}\cdot\delta \boldsymbol{E}^{(M)*}_{\boldsymbol{k}'}\rangle\,. \tag{335}$$

According to Eq. (183), the mode amplitude $\delta \boldsymbol{E}^{(M)}_{\boldsymbol{k}}$ is a linear functional of the natural statistical current fluctuations, so it inherits their spatial homogeneity that was established in Section 4.6:

$$\langle\delta \boldsymbol{E}^{(M)}_{\boldsymbol{k}}\cdot\delta \boldsymbol{E}^{(M)*}_{\boldsymbol{k}'}\rangle = \langle\delta \boldsymbol{E}^{2}_{M}\rangle_{\boldsymbol{k}}\,\delta(\boldsymbol{k}-\boldsymbol{k}')\,. \tag{336}$$

In the preceding sections, correlations were expressed in terms of the tensor product of two vectors, which is defined by $(\boldsymbol{a}\otimes\boldsymbol{b})_{ij} = a_i b_j$. It contains more information than the scalar product $\boldsymbol{a}\cdot\boldsymbol{b} = a_i b_i = \mathrm{tr}(\boldsymbol{a}\otimes\boldsymbol{b})$ appearing here, and in this sense it is more general. Thus, all previous results regarding homogeneous turbulence can immediately be adopted to the current situation, so the equation above is well justified. Due to the resonance condition, the integration over the primed wavevector appearing in the energy density can readily be carried out, yielding

$$\langle|\delta \boldsymbol{E}^{(M)}_{\boldsymbol{r},t}|^2\rangle = \int \mathrm{d}^3k\; I^{(M)}_{\boldsymbol{k},t} \tag{337}$$

with

$$I^{(M)}_{\boldsymbol{k},t} = \exp[2\Im\omega_M(\boldsymbol{k})t]\,\langle\delta \boldsymbol{E}^{2}_{M}\rangle_{\boldsymbol{k}}\,. \tag{338}$$

Since the left-hand side of (337) is the integrated or total electric energy density, the quantity $I^{(M)}_{\boldsymbol{k},t}$ introduced here can be identified as the *spectral* energy density of the mode. According to (338) it exhibits an exponential time dependency, so in agreement with the assertion of Section 4.3 it reproduces the well-known result from quasi-linear theory (Vedenov et al., 1961, 1962; Drummond and Pines, 1962; Sitenko, 1982):

$$\partial I^{(M)}_{\boldsymbol{k},t}/\partial t = 2[\Im\omega_M(\boldsymbol{k})]I^{(M)}_{\boldsymbol{k},t}\,. \tag{339}$$

The energy density considered here involves the scalar product of the field fluctuations. In the same fashion the more general correlation tensor can be investigated which contains the tensor product instead. Keeping in mind that the Laplace transform is associated with the short fluctuation timescale introduced in Section 4.3, one has to formulate the time dependency in terms of t_1 rather than t in this context. Denoting the imaginary part of the mode frequency by $\gamma(\boldsymbol{k})$ again, one obtains

$$\hat{\boldsymbol{I}}^{(M)}_{\boldsymbol{k},t_1} = \exp[2\gamma(\boldsymbol{k})t_1]\,\langle\delta\boldsymbol{E}_M\otimes\delta\boldsymbol{E}_M\rangle_{\boldsymbol{k}}\,, \tag{340}$$

$$\partial\hat{\boldsymbol{I}}^{(M)}_{\boldsymbol{k},t_1}/\partial t_1 = 2\gamma(\boldsymbol{k})\hat{\boldsymbol{I}}^{(M)}_{\boldsymbol{k},t_1}\,. \tag{341}$$

The trace of this tensor reproduces the energy density considered before, so both results are consistent with each other. Since the increase or decrease with time is proportional to the energy density itself, the spectral balance equation (339) describes only the induced effects. The same applies to the generalized version (341) as well. Spontaneous emission, by contrast, manifests itself as an additional term that is independent of the energy density. In order to implement these spontaneous effects into the spectral balance equation, the derivative with respect to the adiabatic timescale t_2 must be recovered that was absorbed into the frequency according to (158), that is, one must substitute $\omega\to\omega+\imath\,\partial/\partial t_2$. For an arbitrary function of the frequency, $h(\omega)$, this can be achieved by the rule (159),

$$h(\omega)\,\delta a_\omega \to \left(h(\omega)+\imath\,\frac{\partial h}{\partial\omega}\,\frac{\partial}{\partial t_2}\right)\delta a_\omega\,. \tag{342}$$

Basically, it corresponds to a linearization in the correction term $\imath\,\partial/\partial t_2$. A subtlety that requires some care is the application of this rule to second order terms such as $\delta a_\omega\,\delta b_\omega$. The reason is that the time derivative obeys the product rule of calculus. In order to obtain consistent results, the product must be symmetrized:

$$\begin{aligned} h(\omega)(\delta a_\omega\,\delta b_\omega) &= \frac{1}{2}\Big[\delta a_\omega\,h(\omega)\,\delta b_\omega+\delta b_\omega\,h(\omega)\,\delta a_\omega\Big]\\ &\to \frac{1}{2}\Big[\delta a_\omega\Big(h(\omega)+\imath\,\frac{\partial h}{\partial\omega}\,\frac{\partial}{\partial t_2}\Big)\delta b_\omega+\delta b_\omega\Big(h(\omega)+\imath\,\frac{\partial h}{\partial\omega}\,\frac{\partial}{\partial t_2}\Big)\delta a_\omega\Big]\\ &= \frac{1}{2}\Big[2h(\omega)\,\delta a_\omega\,\delta b_\omega+\imath\,\frac{\partial h}{\partial\omega}\,\frac{\partial}{\partial t_2}(\delta a_\omega\,\delta b_\omega)\Big]\\ &= \Big[h(\omega)+\frac{\imath}{2}\,\frac{\partial h}{\partial\omega}\,\frac{\partial}{\partial t_2}\Big](\delta a_\omega\,\delta b_\omega)\,. \end{aligned} \tag{343}$$

Thus, a factor 1/2 emerges in comparison with the naive application of (342) to second order terms. This was also taken into account in earlier applications of the two-timescale method (Yoon, 2000, 2006; Yoon et al., 2014). In order to employ this formalism in the current situation, one needs a wave equation for the correlations, so this must be derived first. In view of the product rule (220), the wave equation (332) implies

$$\langle\delta\boldsymbol{E}^{\perp}_{\boldsymbol{k},\omega}\otimes\delta\boldsymbol{E}^{\perp *}_{\boldsymbol{k}',\omega'}\rangle = \frac{\hat{\boldsymbol{q}}^{\perp}_{\boldsymbol{k},\omega}\cdot\langle\delta\boldsymbol{E}^{0}_{\boldsymbol{k},\omega}\otimes\delta\boldsymbol{E}^{0*}_{\boldsymbol{k}',\omega'}\rangle\cdot\hat{\boldsymbol{q}}^{\perp\dagger}_{\boldsymbol{k},\omega}}{[\Lambda^{\perp}_{\boldsymbol{k},\omega}+\varepsilon L(\boldsymbol{k},\omega)][\Lambda^{\perp}_{\boldsymbol{k}',\omega'}+\varepsilon L(\boldsymbol{k}',\omega')]^*}\,. \tag{344}$$

According to their definition (322), the source fields are directly related to the spontaneous current density. Since the correlations of the latter are homogeneous and stationary, viz. (218), the field fluctuations possess this property as well:

$$\langle\delta\boldsymbol{E}^{\perp}_{\boldsymbol{k},\omega}\otimes\delta\boldsymbol{E}^{\perp *}_{\boldsymbol{k}',\omega'}\rangle = \langle\delta\boldsymbol{E}_\perp\otimes\delta\boldsymbol{E}_\perp\rangle_{\boldsymbol{k},\omega}\,\delta(\boldsymbol{k}-\boldsymbol{k}')\,\delta_{\mathbb{C}}(\omega-\omega')\,, \tag{345}$$

$$\langle\delta\boldsymbol{E}_\perp\otimes\delta\boldsymbol{E}_\perp\rangle_{\boldsymbol{k},\omega} = \frac{\hat{\boldsymbol{q}}^{\perp}_{\boldsymbol{k},\omega}\cdot\langle\delta\boldsymbol{E}_0\otimes\delta\boldsymbol{E}_0\rangle_{\boldsymbol{k},\omega}\cdot\hat{\boldsymbol{q}}^{\perp\dagger}_{\boldsymbol{k},\omega}}{|\Lambda^{\perp}_{\boldsymbol{k},\omega}+\varepsilon L(\boldsymbol{k},\omega)|^2}\,. \tag{346}$$

For the present purpose, it is convenient to rewrite the last equation in the following manner:

$$[\Lambda^{\perp}_{\boldsymbol{k},\omega}+\varepsilon L(\boldsymbol{k},\omega)]\langle\delta\boldsymbol{E}_\perp\otimes\delta\boldsymbol{E}_\perp\rangle_{\boldsymbol{k},\omega} = \frac{\hat{\boldsymbol{q}}^{\perp}_{\boldsymbol{k},\omega}\cdot\langle\delta\boldsymbol{E}_0\otimes\delta\boldsymbol{E}_0\rangle_{\boldsymbol{k},\omega}\cdot\hat{\boldsymbol{q}}^{\perp\dagger}_{\boldsymbol{k},\omega}}{[\Lambda^{\perp}_{\boldsymbol{k},\omega}+\varepsilon L(\boldsymbol{k},\omega)]^*}\,. \tag{347}$$

Now the adiabatic timescale is reintroduced by applying the substitution (343). Following earlier work, this procedure is only carried out for the left-hand side, but not for the denominator on the right (Yoon et al., 2014). Since it is merely a small correction, the substitution is not applied to the already small quantity $\varepsilon L(\boldsymbol{k},\omega)$ but only to the dispersion function $\Lambda^{\perp}_{\boldsymbol{k},\omega}$:

$$\left(\Lambda^{\perp}_{\boldsymbol{k},\omega}+\frac{\imath}{2}\,\frac{\partial\Lambda^{\perp}_{\boldsymbol{k},\omega}}{\partial\omega}\,\frac{\partial}{\partial t_2}+\varepsilon L(\boldsymbol{k},\omega)\right)\langle\delta\boldsymbol{E}_\perp\otimes\delta\boldsymbol{E}_\perp\rangle_{\boldsymbol{k},\omega} = \frac{\hat{\boldsymbol{q}}^{\perp}_{\boldsymbol{k},\omega}\cdot\langle\delta\boldsymbol{E}_0\otimes\delta\boldsymbol{E}_0\rangle_{\boldsymbol{k},\omega}\cdot\hat{\boldsymbol{q}}^{\perp\dagger}_{\boldsymbol{k},\omega}}{[\Lambda^{\perp}_{\boldsymbol{k},\omega}+\varepsilon L(\boldsymbol{k},\omega)]^*}\,. \tag{348}$$

In order to simplify this equation, the smallness of the density ratio ε is employed once more. In this spirit, $\varepsilon L(\boldsymbol{k}, \omega)$ can be omitted altogether in the denominator on the right-hand side. Furthermore, the imaginary part of this function can be neglected on the left-hand side because it is small compared to $\imath\, \partial \Lambda^{\perp}_{\boldsymbol{k},\omega}/\partial\omega$ due to the smallness parameter ε:

$$\left(\Lambda^{\perp}_{\boldsymbol{k},\omega} + \frac{\imath}{2}\, \frac{\partial \Lambda^{\perp}_{\boldsymbol{k},\omega}}{\partial\omega}\, \frac{\partial}{\partial t_2} + \varepsilon \Re L(\boldsymbol{k}, \omega)\right) \langle \delta\boldsymbol{E}_{\perp} \otimes \delta\boldsymbol{E}_{\perp}\rangle_{\boldsymbol{k},\omega} \simeq \frac{\hat{\boldsymbol{q}}^{\perp}_{\boldsymbol{k},\omega} \cdot \langle \delta\boldsymbol{E}_0 \otimes \delta\boldsymbol{E}_0\rangle_{\boldsymbol{k},\omega} \cdot \hat{\boldsymbol{q}}^{\perp\dagger}_{\boldsymbol{k},\omega}}{\Lambda^{\perp *}_{\boldsymbol{k},\omega}}\,. \tag{349}$$

It was already established previously that the stable branch of the Weibel mode also dominates the transverse fluctuations of the perturbed plasma. Hence, the frequency dependence is of a resonant nature,

$$\langle \delta\boldsymbol{E}_{\perp} \otimes \delta\boldsymbol{E}_{\perp}\rangle_{\boldsymbol{k},\omega} = -\hat{\boldsymbol{I}}^{\perp}_{\boldsymbol{k}}\, \delta_{\mathbb{C}}(\omega - \omega_M(\boldsymbol{k}))\,. \tag{350}$$

Since the tensor $\hat{\boldsymbol{I}}^{\perp}_{\boldsymbol{k}}$ is a spectral density by definition, it can be interpreted as the correlation of the transverse field fluctuations emitted into the mode *per spectral linewidth* of the resonance. As an integrand of (closed) line integrals in the complex plane, it is sensitive to the orientation of the integration path. Thus, it is a matter of convention whether the correlations are counted per positively or per negatively oriented differentials $\mathrm{d}\omega$. According to (198), the function $\delta_{\mathbb{C}}$ behaves like the complex generalization of Dirac's δ-function if the integration is performed along negatively oriented (counterclockwise) curves, so the additional minus sign in the equation above reverses that behavior. This convention was adopted here for the sake of consistency with the results that were already published beforehand (Kolberg et al., 2016) in which simply a real-valued δ-function was employed.

As a consequence of the resonant nature of the correlations, the first term on the left-hand side of (349) can be omitted. The reason is that the fields vanish everywhere except at the eigenfrequency ω_M, but there the dispersion function vanishes by definition of the mode. Furthermore, the resonant denominator on the right-hand side can be linearized around the eigenfrequency of the mode. Keeping in mind that there is no zeroth order contribution in this expansion because of the dispersion relation, one obtains

$$-\left(\frac{\imath}{2}\, \frac{\partial \Lambda^{\perp}_{\boldsymbol{k},\omega}}{\partial\omega}\, \frac{\partial}{\partial t_2} + \varepsilon \Re L(\boldsymbol{k}, \omega)\right) \hat{\boldsymbol{I}}^{\perp}_{\boldsymbol{k}}\, \delta_{\mathbb{C}}(\omega - \omega_M(\boldsymbol{k})) \simeq \frac{\hat{\boldsymbol{q}}^{\perp}_{\boldsymbol{k},\omega} \cdot \langle \delta\boldsymbol{E}_0 \otimes \delta\boldsymbol{E}_0\rangle_{\boldsymbol{k},\omega} \cdot \hat{\boldsymbol{q}}^{\perp\dagger}_{\boldsymbol{k},\omega}}{(\partial \Lambda^{\perp}_{\boldsymbol{k},\omega_M}/\partial\omega_M)^*\, (\omega - \omega_M)^*}\,. \tag{351}$$

In order to solve this equation for $\hat{\boldsymbol{I}}^{\perp}_{\boldsymbol{k}}$ eventually, it has to be integrated along a closed circle in the complex plane. The radius must be chosen sufficiently large such that all relevant frequency values lie in the interior of the enclosed area. To simplify matters, the radius will be considered infinite, here. In view of (198) and (199), the computation of the integrals is a trivial task. Rearranging terms in the resulting equation yields

$$\frac{\partial \hat{\boldsymbol{I}}^{\perp}_{\boldsymbol{k}}}{\partial t_2} = \left[\frac{2\imath\varepsilon \Re L(\boldsymbol{k}, \omega)}{\partial \Lambda^{\perp}_{\boldsymbol{k},\omega}/\partial\omega}\, \hat{\boldsymbol{I}}^{\perp}_{\boldsymbol{k}} + \frac{4\pi\, \hat{\boldsymbol{q}}^{\perp}_{\boldsymbol{k},\omega} \cdot \langle \delta\boldsymbol{E}_0 \otimes \delta\boldsymbol{E}_0\rangle_{\boldsymbol{k},\omega} \cdot \hat{\boldsymbol{q}}^{\perp\dagger}_{\boldsymbol{k},\omega}}{|\partial \Lambda^{\perp}_{\boldsymbol{k},\omega}/\partial\omega|^2}\right]_{\omega_M(\boldsymbol{k})}. \tag{352}$$

This spectral balance equation describes the evolution with respect to the adiabatic timescale t_2. According to the prescription of the two-timescale approximation given in (156), the entire time derivative consists of the contributions from both timescales, $\partial/\partial t = \partial/\partial t_1 + \partial/\partial t_2$. In view of Eq. (341) describing the variation with respect to t_1, therefore, one obtains

$$\partial \hat{\boldsymbol{I}}^{\perp}_{\boldsymbol{k}}/\partial t = \hat{\boldsymbol{\alpha}}^{\perp}_{\boldsymbol{k}} - \mu_{\boldsymbol{k}} \hat{\boldsymbol{I}}^{\perp}_{\boldsymbol{k}}\,. \tag{353}$$

Here, the respective coefficients for spontaneous emission and absorption were introduced as follows:

$$\hat{\boldsymbol{\alpha}}^{\perp}_{\boldsymbol{k}} \equiv \left.\frac{4\pi\, \hat{\boldsymbol{q}}^{\perp}_{\boldsymbol{k},\omega} \cdot \langle \delta\boldsymbol{E}_0 \otimes \delta\boldsymbol{E}_0\rangle_{\boldsymbol{k},\omega} \cdot \hat{\boldsymbol{q}}^{\perp\dagger}_{\boldsymbol{k},\omega}}{|\partial \Lambda^{\perp}_{\boldsymbol{k},\omega}/\partial\omega|^2}\right|_{\omega=\omega_M(\boldsymbol{k})}, \tag{354}$$

$$\mu_{\boldsymbol{k}} \equiv -2\gamma(\boldsymbol{k}) - \left.\frac{2\imath\varepsilon \Re L(\boldsymbol{k}, \omega)}{\partial \Lambda^{\perp}_{\boldsymbol{k},\omega}/\partial\omega}\right|_{\omega=\omega_M(\boldsymbol{k})}. \tag{355}$$

By consulting Faraday's law of induction, (160), the corresponding radiation law for the magnetic field fluctuations can readily be found as well. One possible procedure is to repeat all previous computations of this subsection accordingly. In view of (230), an alternative and easier approach is to apply the tensor product $(c^2/|\omega_M(\boldsymbol{k})|^2)\hat{\boldsymbol{k}}_{\times} \cdot (\ldots) \cdot \hat{\boldsymbol{k}}^{\mathrm{T}}_{\times}$ to the results for the electric field fluctuations. Then, the correlation tensor $\hat{\boldsymbol{I}}^{\perp}_{\boldsymbol{k}}$ transforms into the corresponding correlation tensor $\hat{\boldsymbol{I}}^{B}_{\boldsymbol{k}}$ of the magnetic field fluctuations. Hence, only the spontaneous emission coefficient is affected by this procedure at all while the absorption coefficient remains the same:

$$\partial \hat{\boldsymbol{I}}^{B}_{\boldsymbol{k}}/\partial t = \hat{\boldsymbol{\alpha}}^{B}_{\boldsymbol{k}} - \mu_{\boldsymbol{k}} \hat{\boldsymbol{I}}^{B}_{\boldsymbol{k}}\,, \tag{356}$$

$$\hat{\boldsymbol{\alpha}}^{B}_{\boldsymbol{k}} = (c^2/|\omega_M(\boldsymbol{k})|^2)\hat{\boldsymbol{k}}_{\times} \cdot \hat{\boldsymbol{\alpha}}^{\perp}_{\boldsymbol{k}} \cdot \hat{\boldsymbol{k}}^{\mathrm{T}}_{\times}\,. \tag{357}$$

As expected, in both Eqs. (353) and (356) there appears a term which is linear in the field correlations as well as a zeroth order term which is independent of the field strength. Thus, these results constitute the desired spectral balance equations that self-consistently account for the competing effects of spontaneous emission and absorption. In the unperturbed limit $\varepsilon \to 0$, the previous results obtained by Yoon et al. (2014) are reproduced. In the perturbed case $\varepsilon \neq 0$, the presence of the pair beam has no impact on the spontaneous emission, only the absorption coefficient $\mu_{\boldsymbol{k}}$ is affected. There, an additional correction term appears which is linear in the smallness parameter ε. This is consistent with the linear perturbation theory employed here.

6.5. Effective growth rate

Since the mode under investigation is a damped one, its growth rate $\gamma(\boldsymbol{k})$ is always negative. Hence, the absorption coefficient fulfills $\mu_{\boldsymbol{k}} > 0$ in the unperturbed case $\varepsilon = 0$. The additional term accounting for the presence of the pair beam can alter this property as will be shown below, i. e., it can lead to negative values of the absorption coefficient that correspond to an amplification of the fluctuations. Considering that the unperturbed mode is stable, $\gamma(\boldsymbol{k}) < 0$, this seemingly paradox situation must be interpreted as an effective transition of the mode itself which becomes an instability (at least in a certain wavenumber range as will be shown in the next subsection). Hence, it is reasonable to introduce an effective growth rate that, in contrast to the true growth rate $\gamma(\boldsymbol{k})$, reflects this transition by exhibiting positive values whenever amplification occurs. In view of the relation $\mu_{\boldsymbol{k}} = -2\gamma(\boldsymbol{k})$ describing the unperturbed case both in the context of quasilinear theory as well as in the two-timescale approximation, viz. (341) and (355), such an effective growth rate is suitably defined as

$$\gamma_{\text{eff}}(\boldsymbol{k}) \equiv -\frac{\mu_{\boldsymbol{k}}}{2} = \gamma(\boldsymbol{k}) + \varepsilon \left.\frac{\imath \Re L(\boldsymbol{k},\omega)}{\partial \Lambda^{\perp}_{\boldsymbol{k},\omega}/\partial\omega}\right|_{\omega=\omega_M(\boldsymbol{k})}. \tag{358}$$

Again, the impact of the pair beam manifests itself as a first order correction term proportional to the smallness parameter ε, implying that the true and the effective growth rate become identical in the unperturbed case. According to the definition above, the effective growth rate is positive whenever the absorption coefficient indicates amplification by attaining negative values. Vice versa, the case of absorption is characterized by either of the equivalent statements $\gamma_{\text{eff}} < 0$ and $\mu_{\boldsymbol{k}} > 0$.

If and under which conditions amplification can occur is an issue that will be addressed in the next subsection. As a prerequisite, the effective growth rate must first be formulated as an explicit function of the wavenumber. To this end, the quantities appearing in the right-hand side of Eq. (358) will be computed one at a time, starting with the function $L(\boldsymbol{k},\omega)$. According to the dispersion relation (316), the last term entering the definition (319) vanishes if $L(\boldsymbol{k},\omega)$ is evaluated at the mode frequency. The only remaining contributions stem from the diagonal elements of $\hat{\boldsymbol{\Lambda}}^{\text{beam}}_{\boldsymbol{k},\omega}$, which can be inferred from the coordinate representation (A.29):

$$L(\boldsymbol{k},\omega_M(\boldsymbol{k})) = -\frac{8\pi e^2 n}{\Gamma_{\text{b}} m_e \omega^2}\left(2 - \frac{(\beta_1^2+\beta_2^2)(\omega^2-k^2c^2)}{(\omega - kc\beta_\parallel)^2}\right)\Bigg|_{\omega=\omega_M(\boldsymbol{k})}. \tag{359}$$

The mode under investigation is aperiodic so its frequency is an imaginary number, viz. (302). Therefore, the real part of the equation above reads

$$\Re L(\boldsymbol{k},\omega_M(\boldsymbol{k})) = \frac{8\pi e^2 n}{\Gamma_{\text{b}} m_e \gamma^2}\left(2 - \frac{\beta_\perp^2(\gamma^2+k^2c^2)(\gamma^2-k^2c^2\beta_\parallel^2)}{(\gamma^2+k^2c^2\beta_\parallel^2)^2}\right)\Bigg|_{\gamma=\gamma(\boldsymbol{k})}, \tag{360}$$

where $\beta_1^2+\beta_2^2$ was substituted by $\beta_\perp^2$ on account of the underlying coordinate system whose 3-axis is parallel to the wavevector. In order to implement the dispersion relation $\gamma = \gamma(\boldsymbol{k})$, the frequency and the wavenumber must be replaced by their dimensionless counterparts κ and y defined in (303) and (304). Then, the dispersion relation (306) can be used to obtain a function of y alone. After some purely algebraic, albeit a trifle lengthy manipulations, one obtains

$$\Re L(\boldsymbol{k},\omega_M(\boldsymbol{k})) = \frac{s_0(y^4 - s_1 y^2 + s_2)(y^2+1)}{y^3(y^2+\beta_\parallel^2)^2(y^2+\zeta)}, \tag{361}$$

where

$$s_0 = \frac{8\pi e^2 n(2-\beta_\perp^2)}{\Gamma_{\text{b}} m_e k_c^2 c^2}, \tag{362}$$

$$s_1 = \frac{\beta_\perp^2 - 4\beta_\parallel^2 - \beta_\perp^2\beta_\parallel^2}{2-\beta_\perp^2}, \tag{363}$$

$$s_2 = \frac{\beta_\parallel^2(\beta_\perp^2 + 2\beta_\parallel^2)}{2-\beta_\perp^2}. \tag{364}$$

The next quantity to be computed is the transverse dispersion function of the unperturbed thermal plasma. According to equation (A.3) of Schlickeiser and Felten (2013) it is given by

$$\Lambda^{\perp}_{k,\imath\gamma} = 1 + \frac{k^2c^2}{\gamma^2} + \frac{k_c^2 c}{k\gamma}\left(\frac{\gamma^2}{k^2c^2} + \zeta\right). \tag{365}$$

Here, the notation used in the reference was adapted to the current work. In the same fashion as before, (303) and (304) are employed now to introduce the dimensionless variables κ and y that enable the application of the dispersion relation (306).

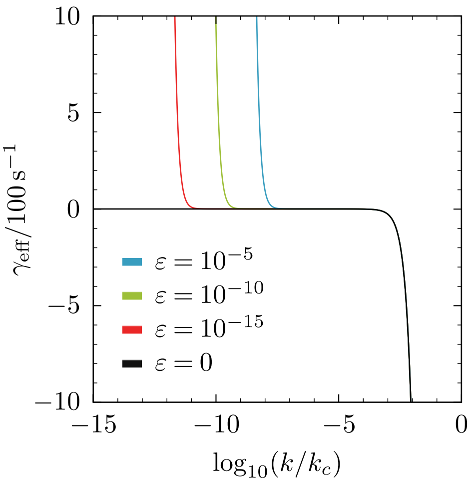

Fig. 8. Effective growth rate of the fluctuations with a wavevector perpendicular to the propagation direction of the relativistic ($\Gamma_{\rm b} = 10^6$) electron–positron pair beam, $\beta_\parallel = 0$, for several values of the density ratio ε.

After some straightforward algebra one obtains

$$\frac{1}{\imath}\frac{\partial \Lambda^{\perp}_{\boldsymbol{k},\omega}}{\partial\omega}\bigg|_{\omega=\omega_M(\boldsymbol{k})} = -\frac{(y^2+1)^{1/2}[y^4+(3-\zeta)y^2+\zeta]}{k_c c y^{7/2}(y^2+\zeta)^{3/2}} \,. \tag{366}$$

In view of the intermediate results (361) and (366), rewriting (358) becomes a simple task:

$$\gamma_{\rm eff} = -\frac{k_c c y^{3/2}(y^2+\zeta)^{1/2}}{(y^2+1)^{1/2}}[1+\varepsilon s_0 H(y)]\,, \tag{367}$$

$$H(y) = \frac{(y^4 - s_1 y^2 + s_2)(y^2+1)}{y[y^4+(3-\zeta)y^2+\zeta](y^2+\beta_\parallel^2)^2}\,. \tag{368}$$

This is the desired equation that describes the effective growth rate as a function of one variable alone without any remaining implicit dependencies hidden in the wavenumber. It still possesses the structure of a linear perturbation expansion in the density ratio ε as the term inside the squared brackets distinctly shows.

6.6. *The occurrence of amplification*

Now that the spectrum of the effective growth rate is known, the conditions for amplification can be investigated. In the unperturbed case $\varepsilon = 0$, the growth rate is negative in the entire spectrum. This corresponds to an absorption of the field fluctuations in compliance with the fact that the mode is a damped one. The correction term accounting for the disturbance by the beam, however, might change this situation. If $H(y) < -1/\varepsilon s_0$, then the effective growth rate becomes positive and the field fluctuations are amplified. The corresponding derivation of criteria for the appearance of amplification is carried out in Appendix A.4. The most important result is the certainty that the effective growth rate can really become positive. In essence, this confirms that the electron–positron pair beam modifies the formerly damped, aperiodic mode in the IGM in such a way that the field fluctuations are amplified. However, this effect is subject to certain conditions and restrictions (see Fig. 8 and Fig. 9): First of all, positive growth rates do not occur in the entire wavenumber spectrum, they are limited to a spectral region. Secondly, amplification does not occur for arbitrary values of the parallel component of the beam velocity but only for those satisfying the criterion

$$\frac{\Gamma_{\rm b}}{\varepsilon}\beta_\parallel^3(4\beta_\parallel^2+\zeta/3) < 1.64\cdot 10^{-12}\,. \tag{369}$$

The latter determines a critical velocity β_0 such that field fluctuations are only amplified if $\beta_\parallel < \beta_0$. For the standard configuration $\Gamma_{\rm b} = 10^6$ and $\varepsilon = 10^{-15}$ this critical value amounts to $\beta_0 = 4.99\cdot 10^{-9}$. For other configurations that still satisfy $4\beta_0^2 \ll \zeta/3$, the inequality (369) leads to

$$\beta_\parallel < \beta_0 \simeq \left(\frac{12\pi e^2 n}{25 m_e k_c^2 c^2 \zeta}\,\frac{\varepsilon}{\Gamma_{\rm b}}\right)^{1/3} = 5.00\cdot 10^{-2}\left(\frac{\varepsilon}{\Gamma_{\rm b}}\right)^{1/3}\,. \tag{370}$$

This criterion for the local occurrence of amplified field fluctuations is always satisfied for sufficiently low parallel velocities. Moreover, it favors large values of the density ratio ε, making the effect more pronounced in the vicinity of the AGN. A little

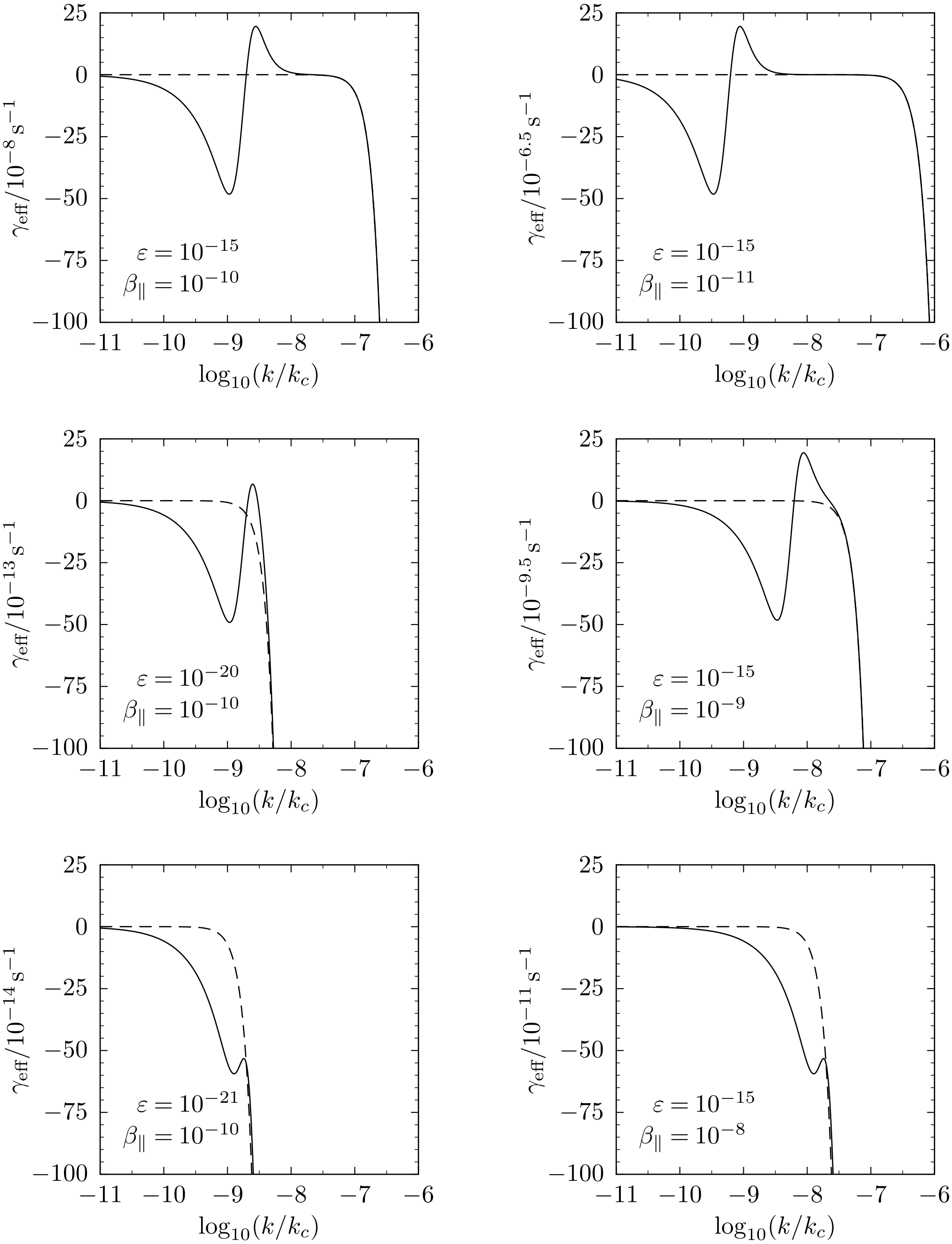

Fig. 9. Effective growth rates for $0 < \beta_\| \ll 1$ and $\Gamma_{\rm b} = 10^6$. The left panel displays the spectra for different values of the density ratio ε while $\beta_\| = 10^{-10}$ remains unchanged. Conversely, $\varepsilon = 10^{-15}$ is fixed in all diagrams exhibited in the right panel and the velocity component parallel to the wavevector is varied. The dashed curve always represents the unperturbed case $\varepsilon = 0$. It should be noted that the scaling of the ordinate is adjusted for each plot individually.

counterintuitive is the fact that positive growth rates develop more easily for small Lorentz factors of the beam than larger ones. Nevertheless, it should be noted that the criterion is valid only for relativistic pairs because $\Gamma_{\rm b} \gg 1$ was a crucial assumption entering the derivation.

While the parallel velocity component was a convenient variable during the analytical calculations, it is not the best choice when it comes to interpreting the results. In this regard, a geometrical approach is more suitable. Concerning the latter, it is instructive to note that $\beta_\|$ completely determines the angle between wavevector and beam velocity because the total speed β is fixed by the prescribed Lorentz factor. The criterion (370) can therefore be understood as a lower limit for

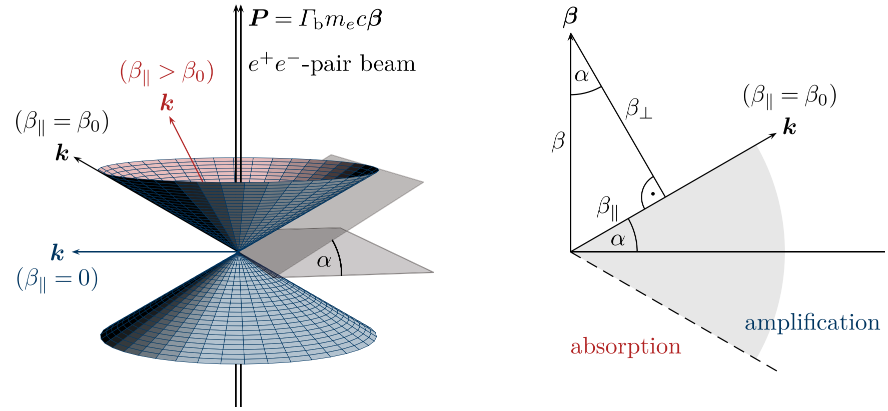

Fig. 10. Illustration of the double cone structure defined by the criterion $\beta_\parallel < \beta_0$ for the occurrence of amplified field fluctuations. Assuming that the Lorentz factor of the beam is given, the total speed β is fixed as well, so there is a one-to-one correspondence between the parallel component $\beta_\parallel$ and the half opening angle α.

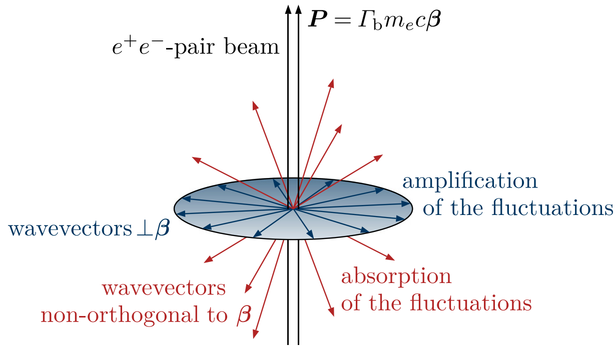

Fig. 11. Since the half opening angle of the double cone structure depicted in Fig. 10 is very small, the latter degenerates into a two-dimensional plane. Thus, only the fluctuations associated with wavevectors perpendicular to the beam velocity are amplified in certain spectral regions, while nonorthogonal wavevectors entail absorption in the entire spectrum.

this very angle between $\boldsymbol{\beta}$ and $\boldsymbol{k}$. Thus, all wavevectors accessible to locally positive growth rates are enclosed by the double cone shape (more precisely: its exterior) depicted in Fig. 10. The half opening angle α is determined by the critical velocity β_0,

$$\sin\alpha = \beta_0/\beta = \beta_0(1 - \Gamma_b^{-2})^{-1/2} \simeq \beta_0\,, \tag{371}$$

where the highly relativistic Lorentz factor of the beam was employed in the approximation at the end. The field fluctuations associated with wavevectors within the symmetry plane of the double cone, $\beta_\parallel = 0$, exhibit positive growth rates of unbounded magnitude. The effective growth rates of all other wavevectors enveloped by the two cones possess a finite maximum instead. All previous remarks about the preference of high values of ε/Γ_b equally apply in the context of the geometrical interpretation: The higher this ratio is, the larger is the half opening angle α, and the more wavevectors satisfy the condition for positive growth rates.

Considering the smallness of β_0 one can further conclude from (371) that $\alpha \simeq \beta_0$. The half opening angle, therefore, is extremely small itself. Measured in degrees, it amounts to as little as $\alpha = (2.86 \cdot 10^{-7})^\circ$ for the standard configuration. In the geometrical interpretation this means that the double cone structure is (almost) degenerated into the plane shown in Fig. 11. The fluctuations associated with wavevectors within this plane are amplified in certain spectral ranges. All other wavevectors that are not orthogonal to the beam direction exhibit negative growth rates in the entire spectrum.

7. Mode-driven velocity fluctuations in the intergalactic medium

Outline. This section is devoted to possible means and ways of gaining observational evidence for the newly discovered stable branch of the Weibel mode. To this end, it is shown that the mode is associated with incompressible turbulence, i. e., that it does not generate any density fluctuations. This excludes the applicability of dispersion measure and scintillation data techniques from the outset. The subsequent estimate for the fluctuations of the rotation measure yield so low a value that this method must be discarded as well. Therefore, the mode-driven proton velocity fluctuations in the IGM are computed.

It is found that these are significantly larger than the thermal velocity. Assuming that the neutral hydrogen atoms adopt the turbulent proton velocities, it is concluded that line broadening studies are a suitable technique to eventually detect the mode after all.

Reference. Parts of the results presented in this section were published by Kolberg et al. (2017).

7.1. Observational evidence for the mode

The previous section stands witness to the significance of the damped aperiodic mode regarding the field fluctuations in the IGM. To this date, however, observational evidence for the mode is still pending. Since a field-free plasma can hardly be maintained under laboratory conditions, it stands to reason that conclusive data can, if ever, only be collected in connection with astrophysical plasmas. Since all such plasmas in the vicinity of the Earth, like its radiation belts or the solar wind, are magnetized, the mode can only occur in very distant environments beyond the reach of *in situ* measurements by spacecraft instruments. Many of the modern techniques designed for the purpose of observing remote objects, such as scintillation data, rotation measure, and dispersion measure, crucially rely on the presence of density fluctuations.

This raises the question if the mode can be traced this way, i. e., whether it leaves a footprint on the turbulence spectrum of the density fluctuations or not. In principle, due to the fundamental coupling between particles and fields in a plasma, it actually does affect the turbulent motion of the charge carriers by means of the Lorentz force. In addition to the spontaneous thermal fluctuations of the density, therefore, there is also an induced contribution stemming from the field fluctuations. Unfortunately, the turbulence driven by a *transverse* mode in an unmagnetized and isotropic plasma is always incompressible, i. e., it does not sustain any density fluctuations as will be shown in the next subsection. This excludes measurements based on scintillation data and dispersion measure from the outset. The corresponding rotation measure is also too small for all practical purposes as the following computations will affirm.

Although the previously mentioned observational techniques are not applicable for the detection of the transverse mode, there is still one alternative worth investigating. The very same arguments regarding the coupling between particles and fields imply that the velocity fluctuations also contain an induced contribution beyond the spontaneous thermal level. These are driven by the field fluctuations and, in this sense, by the mode itself. The latter might thus become accessible to observational evidence after all due to the line broadening it causes. If an atomic or molecular line is emitted by a distant source, its spectral line profile becomes widened during the passage through the turbulent IGM on account of the Doppler shift caused by the random particle motion. The relative width of a spectral line with the linear frequency ν is given by

$$\frac{\delta\nu}{\nu} = \left(\frac{2k_{\mathrm{B}}T}{mc^2} + \frac{\delta V^2}{c^2}\right)^{1/2}, \tag{372}$$

where the terms inside the parentheses are the respective squares of the thermal and the turbulent velocity, both in units of the speed of light (Rybicki and Lightman, 1979).

A subtle, yet crucial point that must be emphasized here is that only the neutrals in the IGM participate in line broadening. The reason is that Doppler broadening occurs by absorption and re-emission of photons. Neither protons nor electrons possess the discrete energy levels necessary for this process, only bound systems such as hydrogen atoms meet this requirement. Hence, Eq. (372) contains the thermal and turbulent velocity of electrically neutral particles. Although the degree of ionization is large, a sparse amount of neutral hydrogen exists in the IGM nonetheless and, indeed, it makes this environment capable of line broadening. It is not directly affected by the mode because field fluctuations only influence charged particles. However, the motions of charged and uncharged particles are not independent of each other, but coupled due to mutual charge exchange and elastic collisions. This is well-known from star-forming regions in which protons, retained by an ambient magnetic field on account of the frozen-in condition, collide with the neutrals and slow down their self-gravitational collapse until a steady drift called *ambipolar diffusion*[3] is reached (Carroll and Ostlie, 2007). Similarly, collisions and charge exchange processes can also cause protons and neutrals in the IGM to adopt the same velocities. Considering that the protons are perpetually driven by the field fluctuations, it is arguably their induced velocity that will eventually be the one shared by both species. At first glance this argumentation might seem to contradict the assumption of a collisionless plasma that is based on the low particle densities of $n = 10^{-7}\ \mathrm{cm}^{-3}$ for electrons and protons. But despite the low collision frequency, after a sufficiently long period of time the turbulent velocity spectrum of atomic hydrogen can assimilate after all.

Consequently, line broadening studies can actually serve as a tool to probe the field fluctuations in the IGM provided that the turbulent velocity is comparable to or even larger than the thermal velocity. This condition is crucial as one can easily infer from Eq. (372). Since the momentum associated with charged particles is predominantly carried by the protons rather than the electrons, the former dominate the impact of collisions on the motion of neutrals, so their velocity is relevant in this context. For this reason, the present section is devoted to the proton density and velocity fluctuations driven by

[3] In plasmas physics, a different connotation of ambipolar diffusion is very common that must be distinguished from the astrophysical one employed here: Due to the small mass of electrons, their thermal motion is usually much faster than the one of the ions. The resulting charge separation generates an electric field that slows down the electrons and increases the ion speed. Hence, the diffusion is enhanced in the case of the electrons and decreased for ions, and this effect is also termed ambipolar diffusion (Bittencourt, 2004; Krall and Trivelpiece, 1973).

the stable branch of the Weibel mode in the IGM. Both the wavenumber spectrum as well as the integrated total value will be computed. Since the spectral distribution of the velocity correlations is the most important quantity in stochastic turbulence theory (Yoshizawa et al., 2002), the approach followed here also leads to some valuable insights into the state of the microturbulence in the IGM.

The computations will be based on the same plasma model for the IGM that was already discussed in detail in the previous section. Most notably, this means that the thermal Maxwell–Jüttner distribution (301) is adopted with a common, non-relativistic temperature of electrons and protons. In order to keep track of the scaling, temperature and density will be denoted as follows:

$$T = 10^4 T_4\,\mathrm{K}\,, \qquad n = 10^{-7} n_{-7}\,\mathrm{cm}^{-3}\,. \tag{373}$$

Introducing the dimensionless parameters T_4 and n_{-7} as linear prefactors allows for an easy reconstruction of the scaling as intended. At the same time, setting both equal to unity readily displays the correct numerical value of the respective quantity at hand for the specific temperature and density adopted in the current model.

7.2. Density fluctuations

The preceding sections already established the fact that the fluctuations of the 6-dimensional phase space density consist of two contributions. According to (166), one of them is proportional to the field fluctuations while the other one is independent of them,

$$\delta N^a_{\boldsymbol{k},\boldsymbol{p},\omega} = \delta N^{a0}_{\boldsymbol{k},\boldsymbol{p},\omega} + \delta N^{a,\mathrm{ind}}_{\boldsymbol{k},\boldsymbol{p},\omega} = \delta N^{a0}_{\boldsymbol{k},\boldsymbol{p},\omega} + \boldsymbol{Q}^a_{\boldsymbol{k},\boldsymbol{p},\omega} \cdot \delta \boldsymbol{E}_{\boldsymbol{k},\omega}\,. \tag{374}$$

Thus, the respective first term corresponds to spontaneous fluctuations while the second one describes induced effects. The local particle density can be inferred from these quantities by collecting the contributions from the entire momentum space (Klimontovich, 1982):

$$\delta n^a_{\boldsymbol{k},\omega} = \int \mathrm{d}^3 p\; \delta N^a_{\boldsymbol{k},\boldsymbol{p},\omega}\,. \tag{375}$$

Due to the linearity of the integral, the density fluctuations inherit the decomposition into a spontaneous and an induced part. Evidently, the damped aperiodic mode investigated here only affects the latter:

$$\delta n^{a,\mathrm{ind}}_{\boldsymbol{k},\omega} = \int \mathrm{d}^3 p\; \delta N^{a,\mathrm{ind}}_{\boldsymbol{k},\boldsymbol{p},\omega} = \delta \boldsymbol{E}_{\boldsymbol{k},\omega} \cdot \int \mathrm{d}^3 p\; \boldsymbol{Q}^a_{\boldsymbol{k},\boldsymbol{p},\omega}\,. \tag{376}$$

In order to carry out the integration, the linear coefficient $\boldsymbol{Q}^a_{\boldsymbol{k},\boldsymbol{p},\omega}$ must be specified as a function of momentum. Consulting equation (166) once again, one obtains

$$\boldsymbol{Q}^a_{\boldsymbol{k},\boldsymbol{p},\omega} = -\frac{\imath e_a n_a}{\omega}\left(\nabla_p f^a_{\boldsymbol{p}} + \frac{\boldsymbol{k}\cdot\nabla_p f^a_{\boldsymbol{p}}}{\omega - \boldsymbol{k}\cdot\boldsymbol{v}}\,\boldsymbol{v}\right). \tag{377}$$

As before, it is expedient to choose a coordinate system whose 3-axis is parallel to the wavevector. Furthermore, cylindrical coordinates $p_\perp$, φ and $p_\parallel$ are introduced in momentum space. Then, the transverse components of the coefficient vector read

$$Q^{a,i}_{\boldsymbol{k},\boldsymbol{p},\omega} = -\frac{\imath e_a n_a}{\omega}\left(\frac{v_i}{v_\perp}\frac{\partial f^a_{\boldsymbol{p}}}{\partial p_\perp} + \frac{k v_i}{\omega - k v_\parallel}\frac{\partial f^a_{\boldsymbol{p}}}{\partial p_\parallel}\right) \qquad \text{for } i \in \{1, 2\}. \tag{378}$$

Here, the symmetry relation (A.17) could be employed because the distribution function is isotropic. Since the right-hand side is the product of the transverse velocity components v_i and a gyrotropic function of $p_\perp$ and $p_\parallel$ alone, the integral of this expression over the entire momentum space vanishes due to the further symmetry statement (A.22) derived in the appendix:

$$\int \mathrm{d}^3 p\; Q^{a,i}_{\boldsymbol{k},\boldsymbol{p},\omega} = 0 \qquad \text{for } i \in \{1, 2\}. \tag{379}$$

Consequently, in an unmagnetized and isotropically distributed plasma only longitudinal field fluctuations contribute to density fluctuations,

$$\delta n^{a,\mathrm{ind}}_{\boldsymbol{k},\omega} = \delta E^{\parallel}_{\boldsymbol{k},\omega} \int \mathrm{d}^3 p\; Q^{a,\parallel}_{\boldsymbol{k},\boldsymbol{p},\omega}\,. \tag{380}$$

Since the stable branch of the Weibel mode is only associated with transverse fields, it does not induce any density fluctuations at all. In other words, the turbulence driven by this mode is incompressible. This proves the previous assertion that the mode cannot be detected by techniques relying on density fluctuations such as dispersion measure and scintillation data.

7.3. Fluctuations of the rotation measure

The applicability of Faraday rotation diagnostics is a more subtle matter. In principle, the magnetic field fluctuations in the IGM really do cause a rotation of the polarization plane, but the effect is very small and difficult to detect. In order to prove and quantify this claim, the rotation measure (RM) will be estimated. The plasma is unmagnetized, $\boldsymbol{B} = \delta\boldsymbol{B}$, and the mode-driven turbulence is incompressible, $n_e = \langle n_e \rangle$, so the rotation measure is given by the line-of-sight integral (Ruzmaikin et al., 1988)

$$\mathrm{RM} = \alpha \int_0^L \mathrm{d}s\ \langle n_e \rangle\, \delta B_0(s)\,. \tag{381}$$

Here, L is the distance traveled by the polarized radiation through the turbulent fields, and δB_0 denotes the component of the magnetic field fluctuations along the line of sight. In cgs units the constant prefactor reads

$$\alpha = e^3/(2\pi m_e^2 c^4) = 2.631 \cdot 10^{-17}\ \mathrm{cm}^{1/2}\mathrm{g}^{-1/2}\mathrm{s}\,. \tag{382}$$

The average of a fluctuating quantity is always zero, viz. (52), so one immediately infers that the average of the rotation measure vanishes, $\langle \mathrm{RM} \rangle = 0$. Apart from this mathematical argumentation this is also plausible in view of the underlying physics: Evidently, the rotation measure is sensitive to whether the aligned component of the magnetic field is oriented parallel or anti-parallel to the line of sight. Since the field fluctuations are randomly distributed with equal probabilities for all directions, contributions to both orientations exist and cancel each other. Consequently, as the average provides no information, one has to resort to the fluctuations of the rotation measure and compute the variance instead:

$$\sigma_{\mathrm{RM}}^2 = \alpha^2 \langle n_e \rangle^2 \int_0^L \mathrm{d}s \int_0^L \mathrm{d}s'\ \langle \delta B_0(s)\, \delta B_0(s') \rangle\,. \tag{383}$$

Due to the spatial homogeneity of the magnetic turbulence, the integrand, that is, the correlation function, does not depend on both positions along the line of sight independently, but only on their difference, viz. (187):

$$\langle \delta B_0(s)\, \delta B_0(s') \rangle = C(s' - s)\,. \tag{384}$$

Obviously, the left-hand side of this equation is invariant under permutations of s and s', so the right-hand side must also have this property, i. e., the correlation function C must be an even function. Based on this symmetry, Jokipii and Lerche (1969) integrated by parts to obtain

$$\sigma_{\mathrm{RM}}^2 = 2\alpha^2 \langle n_e \rangle^2 \int_0^L \mathrm{d}s\ (L - s) C(s)\,. \tag{385}$$

As their paper does not contain an explicit calculation or any further details, the derivation of the equation above is carried out in Appendix A.5. In order to proceed with the computation of the rotation measure fluctuations, the correlation function must be specified. This specification need not even be exact, a suitable approximation will suffice for the present purpose of finding an order-of-magnitude estimate for σ_{RM}. In this spirit, a reasonable assumption is that the field fluctuations are significantly correlated only on short distances, i. e., below a characteristic correlation length of the turbulence l_c. On larger scales, however, the fluctuations are almost uncorrelated. Following Ruzmaikin et al. (1988), this scenario can be modeled by

$$C(s) = C_0 \exp(-s/l_c)\,. \tag{386}$$

The corresponding integral entering equation (385) for the rotation measure fluctuations can readily be evaluated,

$$\begin{aligned} \sigma_{\mathrm{RM}}^2 &= 2\alpha^2 \langle n_e \rangle^2 C_0 l_c^2 \left[L/l_c - 1 + \exp(-L/l_c) \right] \\ &\simeq 2\alpha^2 \langle n_e \rangle^2 C_0 l_c L\,. \end{aligned} \tag{387}$$

The approximation in the last step is based on the assumption that the correlation length of the turbulence is much shorter than the distance traveled by the polarized radiation through the turbulent fields along the line of sight. As an aside it is reassuring to note that, in this approximation, one obtains the same result by applying the model $C(s) = C_0 H(l_c - s)$ instead, where H denotes the Heaviside step function. Yoon et al. (2014) found that the spatial turbulence scale of the mode-driven magnetic field fluctuations amounts to $l_t = 2.4 \cdot 10^{15}$ cm, so this characteristic length can serve as an estimate for the correlation length of the microturbulence:

$$\sigma_{\mathrm{RM}} = 3.2 \cdot 10^{-22}\ \mathrm{rad\ cm}^{-2}\ \frac{\sqrt{C_0}}{10^{-18}\ \mathrm{G}} \sqrt{\frac{L}{1\ \mathrm{Mpc}}\ \frac{l_c}{l_t}}\,. \tag{388}$$

The parameter C_0 refers to the component of the magnetic field fluctuations along the line of sight, so it must be even smaller than their total strength of $6 \cdot 10^{-18}$ G derived by Yoon et al. (2014). Therefore, the rotation measure does not only possess a vanishing average on account of the incompressibility of the turbulence, but its fluctuations are also very small. For these reasons, it seems improbable, if not impossible, that observational evidence of the stable branch of the Weibel mode can be obtained by means of Faraday rotation. Considering that measurements based on the dispersion measure or scintillation data had to be excluded as well, investigating the feasibility of line broadening studies becomes all the more important.

7.4. The spectrum of the velocity fluctuations

The spectral fluctuations of the velocity can be defined in a similar manner as before in the case of the density. Following Klimontovich (1982) again, they are the first moment with respect to the phase space density:

$$\delta \boldsymbol{V}^{a}_{\boldsymbol{k},\omega} = \frac{1}{n_a} \int \mathrm{d}^3p \; \boldsymbol{v} \, \delta N^{a}_{\boldsymbol{k},\boldsymbol{p},\omega} \, . \tag{389}$$

The prefactor had to be added for dimensional reasons as one can infer from the normalization conditions (65) and (142). Again, the separation into spontaneous and induced parts is carried over from the phase space density to the velocity,

$$\delta \boldsymbol{V}^{a,\mathrm{ind}}_{\boldsymbol{k},\omega} = \frac{1}{n_a} \int \mathrm{d}^3p \; \boldsymbol{v} \, \delta N^{a,\mathrm{ind}}_{\boldsymbol{k},\boldsymbol{p},\omega} = \frac{1}{n_a} \int \mathrm{d}^3p \; \boldsymbol{v} \, (\boldsymbol{Q}^{a}_{\boldsymbol{k},\boldsymbol{p},\omega} \cdot \delta \boldsymbol{E}_{\boldsymbol{k},\omega}) \, . \tag{390}$$

Analogously to the procedure of the last subsection, the next step is to exploit the independence of the field fluctuations from the momentum variable and to pull them outside the integral. In contrast to the computation of the density fluctuations, however, a prerequisite from tensor algebra is required here because the triple product inside the integral yields a vector now. To this end, the relation

$$[\boldsymbol{a}(\boldsymbol{b} \cdot \boldsymbol{c})]_i = a_i b_j c_j = (\boldsymbol{a} \otimes \boldsymbol{b})_{ij} c_j = [(\boldsymbol{a} \otimes \boldsymbol{b}) \cdot \boldsymbol{c}]_i \tag{391}$$

is employed that holds for arbitrary vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$. Then, the equation above for the velocity fluctuations can be rewritten as

$$e_a n_a \, \delta \boldsymbol{V}^{a,\mathrm{ind}}_{\boldsymbol{k},\omega} = \hat{\boldsymbol{\sigma}}^{a}_{\boldsymbol{k},\omega} \cdot \delta \boldsymbol{E}_{\boldsymbol{k},\omega} \tag{392}$$

in terms of the newly defined tensor

$$\hat{\boldsymbol{\sigma}}^{a}_{\boldsymbol{k},\omega} \equiv -\frac{\imath e_a^2 n_a}{\omega} \int \mathrm{d}^3p \; \boldsymbol{v} \otimes \left(\nabla_{p} f^{a}_{\boldsymbol{p}} + \frac{\boldsymbol{k} \cdot \nabla_{p} f^{a}_{\boldsymbol{p}}}{\omega - \boldsymbol{k} \cdot \boldsymbol{v}} \, \boldsymbol{v} \right) . \tag{393}$$

From (168) one infers that this tensor is precisely the contribution to the conductivity tensor that stems from particle species a, hence the choice of notation. Similarly, the left-hand side of (392) is the induced part of the current density fluctuation (162) associated with particles of type a. Thus, summing Eq. (392) over a reproduces the induced part of Ohm's law (167), so the statement above is a species-resolved generalization thereof.

The conductivity tensor of an unmagnetized and isotropically distributed plasma has already been computed in Section 5.4. Revisiting the derivation carried out there immediately shows that the obtained results also hold accordingly for the contribution of every particle species individually. In a coordinate system whose 3-axis is aligned to the wavevector, the conductivity tensor becomes diagonal. Moreover, in terms of the cylindrical momentum coordinates $p_\perp$, φ and $p_\parallel$, the diagonal elements are given by

$$\hat{\boldsymbol{\sigma}}^{a}_{\boldsymbol{k},\omega} = \mathrm{diag}\big(\sigma^{a\perp}_{\boldsymbol{k},\omega} \big| \sigma^{a\perp}_{\boldsymbol{k},\omega} \big| \sigma^{a\parallel}_{\boldsymbol{k},\omega}\big) \, , \tag{394}$$

$$\sigma^{a\perp}_{\boldsymbol{k},\omega} = -\frac{\imath e_a^2 n_a}{2\omega} \int \mathrm{d}^3p \; v_\perp \left(\frac{\partial f^{a}_{\boldsymbol{p}}}{\partial p_\perp} + \frac{k v_\perp}{\omega - k v_\parallel} \, \frac{\partial f^{a}_{\boldsymbol{p}}}{\partial p_\parallel} \right) , \tag{395}$$

$$\sigma^{a\parallel}_{\boldsymbol{k},\omega} = -\imath e_a^2 n_a \int \mathrm{d}^3p \; \frac{v_\parallel}{\omega - k v_\parallel} \, \frac{\partial f^{a}_{\boldsymbol{p}}}{\partial p_\parallel} \, . \tag{396}$$

In view of this particular form of the conductivity tensor, equation (392) for the velocity fluctuations simplifies to

$$\delta \boldsymbol{V}^{a,\mathrm{ind}}_{\boldsymbol{k},\omega} = \big(\sigma^{a\perp}_{\boldsymbol{k},\omega} \, \delta \boldsymbol{E}^{\perp}_{\boldsymbol{k},\omega} + \sigma^{a\parallel}_{\boldsymbol{k},\omega} \, \delta \boldsymbol{E}^{\perp}_{\boldsymbol{k},\omega}\big) / e_a n_a \, . \tag{397}$$

In the statistical framework adopted here to model the plasma, the fluctuations are random variables. Therefore, only the correlations are physically meaningful:

$$\langle |\delta \boldsymbol{V}^{a,\mathrm{ind}}_{\boldsymbol{k},\omega}|^2 \rangle = \left| \frac{\sigma^{a\perp}_{\boldsymbol{k},\omega}}{e_a n_a} \right|^2 \langle |\delta \boldsymbol{E}^{\perp}_{\boldsymbol{k},\omega}|^2 \rangle + \left| \frac{\sigma^{a\parallel}_{\boldsymbol{k},\omega}}{e_a n_a} \right|^2 \langle |\delta \boldsymbol{E}^{\parallel}_{\boldsymbol{k},\omega}|^2 \rangle \, . \tag{398}$$

Apparently, both longitudinal and transverse field fluctuations are able to drive velocity fluctuations. According to the earlier discussion, they exist in the form of modes with a fixed relation between frequency and wavenumber, $\omega = \omega_M(\boldsymbol{k})$. The field fluctuations emitted into a particular mode are given by (Sitenko, 1982, see also (350))

$$\langle |\delta \boldsymbol{E}_{\boldsymbol{k},\omega}|^2 \rangle = I_{\boldsymbol{k}} \, \delta_{\mathbb{C}}(\omega - \omega_M(\boldsymbol{k})) \, . \tag{399}$$

Here, $\delta_{\mathbb{C}}$ was implemented instead of the conventional δ-function in order to account for the complex frequency considered in this work. In the same manner, the velocity fluctuations driven by the mode can be identified as

$$\langle |\delta \boldsymbol{V}^{a,\mathrm{ind}}_{\boldsymbol{k},\omega}|^2\rangle = \langle |\delta \boldsymbol{V}^{a}_{\boldsymbol{k}}|^2\rangle \, \delta_{\mathbb{C}}(\omega - \omega_M(\boldsymbol{k})) \,. \tag{400}$$

The damped aperiodic mode under investigation is a transverse one, i. e., it is only associated with transverse field fluctuations. Consequently, the longitudinal contribution on the right-hand side of (398) vanishes, here. Taking this into account, the preceding three equations imply

$$\langle |\delta \boldsymbol{V}^{a}_{\boldsymbol{k}}|^2\rangle = I^{\perp}_{\boldsymbol{k}} \left| \frac{\sigma^{a\perp}_{\boldsymbol{k},\omega}}{e_a n_a} \right|^2_{\omega=\omega_M(\boldsymbol{k})} . \tag{401}$$

The spectrum of the field fluctuations emitted into the stable branch of the Weibel mode has already been computed by Yoon et al. (2014). In terms of the dimensionless variables κ and y introduced in (303) and (304), the spectral intensity is given by

$$I^{\perp}_{\boldsymbol{k}} = \frac{m_e c^2 \beta_e^4}{4\pi^2} \left(\frac{m_e}{m_p}\right)^{1/2} \frac{y(\kappa)[1+y^2(\kappa)]}{m^2(\kappa)} \, D\!\left(\frac{y(\kappa)}{\beta_e}\right). \tag{402}$$

The variables are coupled by the dispersion relation (306), hence the notation $y(\kappa)$. The function D appearing in the previous equation is an exponentially modulated complementary error function,

$$D(x) = \exp(x^2)\,\mathrm{erfc}(x)\,. \tag{403}$$

Furthermore, $m(\kappa)$ is an abbreviation originally declared in the article of Yoon et al. (2014) that is rewritten here as a function of y alone by means of the dispersion relation (306):

$$m(\kappa) = \frac{y^4 + (3-\zeta)y^2 + \zeta}{\kappa(y^2+1)} = \frac{y^4 + (3-\zeta)y^2 + \zeta}{y^{1/2}(y^2+1)^{1/2}(y^2+\zeta)^{1/2}} \,. \tag{404}$$

As far as the spectrum of the velocity fluctuations is concerned, the only remaining task is to compute the transverse element $\sigma^{a\perp}_{\boldsymbol{k},\omega}$ of the conductivity tensor. Fortunately, the lengthy integration involved can be circumvented. The reason is that the integral in question can be reduced to the expression

$$J_a = \int \mathrm{d}^3p \, \frac{k^2 v_\perp^2 f^a_{\boldsymbol{p}}}{\gamma^2(\boldsymbol{k}) + k^2 v_\parallel^2} \,, \tag{405}$$

and that the latter has already been computed by Yoon et al. (2014) who derived the approximation

$$J_a \simeq \pi^{1/2} \beta_a \frac{1+y^2(\kappa)}{y(\kappa)} \, D\!\left(\frac{y(\kappa)}{\beta_a}\right). \tag{406}$$

In order to establish the asserted relation between J_a as defined above and the transverse component of the conductivity tensor, it is expedient to start from (395). Since the Maxwell–Jüttner distribution that underlies the present analysis is, in essence, an exponential function, partial derivation merely reproduces it with some additional factors, viz. (245):

$$\sigma^{a\perp}_{\boldsymbol{k},\omega} = \frac{\imath e_a^2 n_a}{2 m_a c^2 \Theta_a} \int \mathrm{d}^3p \, \frac{v_\perp^2 f^a_{\boldsymbol{p}}}{\omega - k v_\parallel} \,. \tag{407}$$

Eq. (401) demands that this expression be evaluated at the mode frequency which is an imaginary number for aperiodic modes, $\omega_M = \imath\gamma(\boldsymbol{k})$. Expanding the resulting fraction inside the integral such that the denominator becomes a real number one obtains

$$\sigma^{a\perp}_{\boldsymbol{k},\omega}\Big|_{\omega=\omega_M(\boldsymbol{k})} = \frac{\imath e_a^2 n_a}{2 m_a c^2 \Theta_a} \int \mathrm{d}^3p \, \frac{[-\imath\gamma(\boldsymbol{k}) - k v_\parallel] v_\perp^2 f^a_{\boldsymbol{p}}}{\gamma^2(\boldsymbol{k}) + k^2 v_\parallel^2} \,. \tag{408}$$

Due to the $p_\parallel$-integration implicitly contained in the volume integral, the second term inside the square brackets does not contribute because the integral of an odd function over symmetric integration limits always vanishes. The remaining integral is proportional to J_a,

$$\sigma^{a\perp}_{\boldsymbol{k},\omega}\Big|_{\omega=\omega_M(\boldsymbol{k})} = \frac{e_a^2 n_a \gamma(\boldsymbol{k})}{2 m_a c^2 \Theta_a} \int \mathrm{d}^3p \, \frac{v_\perp^2 f^a_{\boldsymbol{p}}}{\gamma^2(\boldsymbol{k}) + k^2 v_\parallel^2} = -\frac{e_a^2 n_a J_a y}{2 m_a c \, \Theta_a k_c \kappa} \,. \tag{409}$$

In the last step, k and $\gamma(\boldsymbol{k})$ were substituted with the dimensionless variables κ and y according to the definitions (303) and (304). This completes the computation of the transverse conductivity component because J_a is already known from the approximation stated earlier. Since the spectral intensity of the field fluctuations is also known, the remaining task is merely to collect all terms entering (401). This procedure results in the desired spectrum of the proton velocity

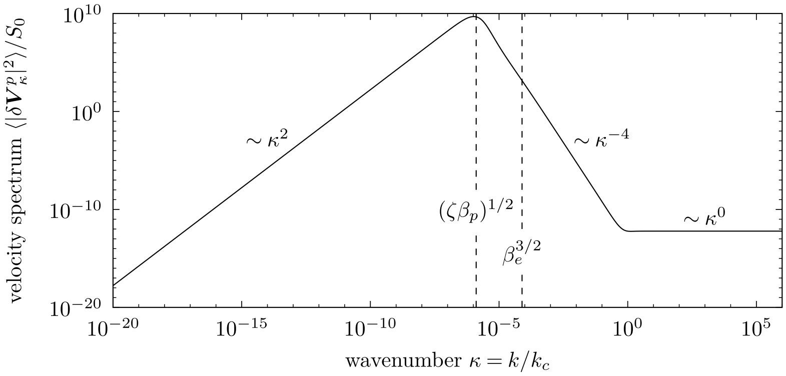

Fig. 12. Spectrum of the induced proton velocity fluctuations that are driven by the stable branch of the Weibel mode in the IGM for a temperature of $T = 10^4$ K and a particle density of $n_e = 10^{-7}$ cm^{-3} for electrons and protons.

fluctuations,

$$\langle|\delta \boldsymbol{V}^p_\kappa|^2\rangle = S_0\,\frac{y(\kappa)[1+y^2(\kappa)]^3}{\kappa^2 m^2(\kappa)}\,D^2\left(\frac{y(\kappa)}{\beta_p}\right)D\left(\frac{y(\kappa)}{\beta_e}\right). \tag{410}$$

All constant prefactors were collected in the new quantity S_0 introduced here. Taking the definitions given in Section 6.1 into account, the corresponding expression eventually simplifies to

$$S_0 \equiv \frac{c^2\zeta^2\beta_e}{8\pi^{5/2}n_e}\,\frac{m_e}{m_p} = 1.00\cdot 10^5\ \mathrm{cm^5 s^{-2}}\,T_4^{5/2}n_{-7}^{-1}\,. \tag{411}$$

The wavenumber spectrum of the mode-driven proton velocity fluctuations in the IGM is shown in Fig. 12. The computation of its asymptotic behavior will be carried out next.

7.5. Power law spectral indices

The fact that the spectrum appears as a concatenation of straight lines in the log–log graph depicted in Fig. 12 indicates a distinct power law relation between the spectral velocity fluctuations and the wavenumber. Obviously, the corresponding spectral index is not the same throughout the entire spectrum but it changes every time the slope of the line changes. It is the purpose of the present subsection to comprehend this behavior by deriving the underlying power law relations from the universal representation (410) of the spectrum. To this end, $\kappa m(\kappa)$ is substituted in this equation by means of (404):

$$\frac{\langle|\delta \boldsymbol{V}^p_\kappa|^2\rangle}{S_0} = \frac{y(y^2+1)^5}{[y^4+(3-\zeta)y^2+\zeta]^2}\,D^2\left(\frac{y}{\beta_p}\right)D\left(\frac{y}{\beta_e}\right). \tag{412}$$

Now the spectrum is a function of y alone as the wavenumber κ no longer appears explicitly, here. In this form, it is much easier to analyze the asymptotics. Once this is accomplished, the results can be reformulated in terms of the wavenumber by applying the dispersion relation (306),

$$\kappa = \frac{y^{1/2}(y^2+\zeta)^{1/2}}{(y^2+1)^{1/2}} \simeq \begin{cases} \zeta^{1/2}y^{1/2} & \text{for } y \ll \zeta^{1/2} \\ y^{3/2} & \text{for } \zeta^{1/2} \ll y \ll 1 \\ y^{1/2} & \text{for } y \gg 1. \end{cases} \tag{413}$$

As a prerequisite for the arguments to come, the asymptotic behavior of the exponentially modulated error function (403) must be specified. Consulting the standard reference by Abramowitz and Stegun (1972) one obtains

$$D(x) \simeq \begin{cases} 1 & \text{for } x \ll 1 \\ (\pi^{1/2}x)^{-1} & \text{for } x \gg 1. \end{cases} \tag{414}$$

A closer inspection of the representation (412) reveals the critical values of the variable y that mark the positions in the spectrum with a possible break in the power law, namely $\zeta^{1/2}$, β_p, β_e, and 1. Their respective numerical values imply the order

$$0 < \beta_p < \zeta^{1/2} < \beta_e \ll 1\,. \tag{415}$$

The three intermediate numbers hardly cover two orders of magnitude in the spectrum, which is a very narrow range. For the present purpose it suffices, therefore, to investigate only the cases $y \ll \beta_p$, $\beta_e \ll y \ll 1$, and $y \gg 1$. As it will turn out, this excludes merely the range in between the dashed lines in Fig. 12, so this approach seems justified, here.

Low wavenumbers. Assuming $y \ll \beta_p$ first, the dispersion relation further implies that $\kappa \simeq (\zeta y)^{1/2}$. Hence, this case corresponds to wavenumbers with $\kappa \ll (\zeta\beta_p)^{1/2}$. Since both terms involving the function D are approximately unity in this case, (412) becomes

$$\langle|\delta \boldsymbol{V}_\kappa^p|^2\rangle/S_0 \simeq y/\zeta^2 \simeq \kappa^2/\zeta^3\,. \tag{416}$$

This result is in perfect agreement with the increase that the spectrum exhibits to the left of the dashed lines.

Intermediate wavenumbers. The values $\beta_e \ll y \ll 1$ considered now entail the approximation $\kappa \simeq y^{3/2}$, so they deal with wavenumbers $\beta_e^{3/2} \ll \kappa \ll 1$. This time, both arguments of the function D are much larger than one, so the appropriate approximation must be applied:

$$\frac{\langle|\delta \boldsymbol{V}_\kappa^p|^2\rangle}{S_0} \simeq \frac{\beta_p^2\beta_e}{(3-\zeta)^2\pi^{3/2}}\,\frac{1}{y^6} \simeq \frac{\beta_p^2\beta_e}{(3-\zeta)^2\pi^{3/2}}\,\frac{1}{\kappa^4}\,. \tag{417}$$

Again, this result is in accordance with Fig. 12. It correctly describes the comparatively steep decrease of the spectrum between the right dashed line and $\kappa = 10^0$.

High wavenumbers. The final case to be considered here is $y \gg 1$. Consulting the dispersion relation once more, one infers $\kappa \simeq y^{1/2}$, so this case addresses the spectral range defined by $\kappa \gg 1$. This time all powers of y cancel each other leaving merely a constant,

$$\langle|\delta \boldsymbol{V}_\kappa^p|^2\rangle/S_0 \simeq \beta_p^2\beta_e/\pi^{3/2}\,. \tag{418}$$

This result perfectly reproduces the horizontal line constituting the spectrum of the velocity fluctuations in the spectral region beyond $\kappa = 10^0$.

7.6. Integrated total velocity fluctuations

In order to find the total value of the turbulent velocities that determine the relative line broadening, the inverse Fourier transform of the spectrum (410) must be computed. According to inversion formula (192) for spatially homogeneous turbulence, this is achieved by collecting all spectral contributions, and the exponential Fourier kernel $\exp(\pm\imath \boldsymbol{k}\cdot\boldsymbol{r})$ is not involved anymore. However, the integration over the entire wavevector space poses a serious problem. As demonstrated in the preceding subsection, the spectrum asymptotically approaches a constant value for high wavenumbers, so the integral diverges. This clearly unphysical feature indicates that some assumptions are violated that implicitly enter the plasma model and limit its applicability. Since high wavenumbers correspond to small spatial scales, the divergence is probably due to the fact that the interaction energy between two distinct particles becomes very high at short distances. Once it becomes higher than their respective kinetic energy, the characteristic features of a plasma are no longer fulfilled that were a crucial ingredient in the derivation of the linear model. Such a situation is well known in plasma physics because the Landau collision integral exhibits a logarithmic divergence for high wavenumbers that can be attributed to the same causes[4] (Klimontovich, 1982; Alexandrov et al., 1984). The customary method to remedy this deficiency in the case of the Landau integral is to truncate the integral artificially at a maximum wavenumber, and this procedure will also be adopted here:

$$\langle\delta V_p^2\rangle = \int_{|\boldsymbol{k}|\le k_{\max}} \mathrm{d}^3k\,\langle|\delta \boldsymbol{V}_\kappa^p|^2\rangle\,. \tag{419}$$

Considering the origin of the divergence, it is appropriate to assign the artificial limit to the length scale defined by the condition that the average kinetic energy and the binary interaction energy are equal, i. e., to the *classical distance of closest approach*, $r_{\mathrm{c}} = e^2/k_{\mathrm{B}}T$. The associated wavenumber is an adequate upper limit for the divergent integral (Klimontovich, 1982),

$$k_{\max} = 2\pi/r_{\mathrm{c}} = 2\pi k_{\mathrm{B}}T/e^2 = 3.76\cdot 10^7\,\mathrm{cm}^{-1}\,T_4\,. \tag{420}$$

On account of the isotropy exhibited by the spectrum (410), the integration over the solid angle is trivial. Normalizing the remaining one-dimensional integration variable according to $\kappa = k/k_c$ yields

$$\langle\delta V_p^2\rangle = 4\pi k_c^3 \int_0^{\kappa_{\max}} \mathrm{d}\kappa\;\kappa^2\langle|\delta \boldsymbol{V}_\kappa^p|^2\rangle\,. \tag{421}$$

The dimensionless new upper integration limit is much higher than unity, a property that will be exploited in the following computations:

$$\kappa_{\max} = k_{\max}/k_c = 4.05\cdot 10^{11} T_4^{7/4} n_{-7}^{-1/2} \gg 1\,. \tag{422}$$

[4] Balescu (1975, 1997) mentions another reason for the divergence of the Landau collision integral: A proper quantum mechanical treatment reveals that repulsive interactions set in at short distances that appear as a hard core of the ions. They prevent the occurrence of infinite Coulomb energies. In this sense, the divergence at high wavenumbers can also be attributed to a violation of the conditions associated with a classical treatment.

It has already been pointed out at the beginning of Section 7.5 that analytical operations concerning the velocity spectrum are more conveniently carried out in terms of the variable y rather than κ. The corresponding change of variables in the integral at hand is governed by the relation

$$\frac{\mathrm{d}\kappa}{\mathrm{d}y} = \frac{y^4 + (3-\zeta)y^2 + \zeta}{2y^{1/2}(y^2+1)^{3/2}(y^2+\zeta)^{1/2}} \tag{423}$$

that can immediately be inferred from the dispersion relation (306). The inverse dispersion relation (308) determines the upper limit of the integral, which can be approximated due to the high numerical value of $\kappa_{\max}$,

$$Y \equiv y(\kappa_{\max}) = \frac{\kappa_{\max}^2(1+\kappa_{\max}^{4/3})}{\zeta + \kappa_{\max}^{4/3}} \simeq \kappa_{\max}^2 \,. \tag{424}$$

With these auxiliary results at disposal, the change of variables can be finalized. Making use of Eq. (410) or (412) for the spectrum one obtains

$$\langle \delta V_p^2 \rangle = 2\pi k_c^3 S_0 H = \left(5.06 \cdot 10^{-7}\,\mathrm{cm^2 s^{-2}}\, T_4^{1/4} n_{-7}^{1/2}\right) H \tag{425}$$

with

$$H = \int_0^Y \mathrm{d}y\, \frac{y^{3/2}(y^2+1)^{5/2}(y^2+\zeta)^{1/2}}{y^4+(3-\zeta)y^2+\zeta} D\left(\frac{y}{\beta_e}\right) D^2\left(\frac{y}{\beta_p}\right). \tag{426}$$

This integral can readily be computed numerically, yielding $H = 2.65 \cdot 10^{22}$. The disadvantage of this method, however, is that it provides no information about the scaling with temperature and density. In order to rectify this deficiency, an analytical expression must be derived. This task is simplified by the fact that the contribution stemming from the interval $[0, \sqrt{3}]$ is only $4.29 \cdot 10^{-8}$ as one can verify numerically, so it is many orders of magnitude below the total value. Since this range can be neglected, therefore, the highest power of y is the dominating term in every sum appearing in the integrand. Applying the approximation (414) again, one obtains

$$H \simeq \int_{\sqrt{3}}^Y \mathrm{d}y\, \frac{\beta_e \beta_p^2 y^{1/2}}{\pi^{3/2}} = \frac{2\beta_e\beta_p^2(Y^{3/2} - 3^{3/4})}{3\pi^{3/2}} \simeq \frac{2\beta_e\beta_p^2 Y^{3/2}}{3\pi^{3/2}} \,. \tag{427}$$

This analytical expression reproduces the numerical result of $H = 2.65 \cdot 10^{22}$ again, so the approximation is very accurate. Moreover, it contains the required scaling with temperature and density. Plugging this outcome into Eq. (425) and substituting the constants according to their respective definitions yields the final result for the proton velocity fluctuations driven by the stable branch of the Weibel mode in the IGM,

$$\delta V_p^{\mathrm{mode}} \equiv \sqrt{\langle \delta V_p^2 \rangle} = \sqrt{\frac{\beta_e^2 \beta_p^6 c^2 k_{\max}^3}{24\pi^3 n_e}} = 1.16 \cdot 10^8\,\mathrm{cm\,s^{-1}} T_4^{7/2} n_{-7}^{-1/2} \,. \tag{428}$$

This value corresponds to 0.39% of the speed of light, so the assumption that the plasma is non-relativistic is not contravened ex post. In comparison with the thermal velocity

$$V_p^{\mathrm{th}} = \beta_p c = 1.28 \cdot 10^6\,\mathrm{cm\,s^{-1}} T_4^{1/2} \tag{429}$$

the induced velocity fluctuations are very strong, in fact they are 90.2 times larger than the former. Consequently, the relative line broadening (372) is governed by the mode-driven turbulence rather than the thermal motion. In the case of hydrogen one obtains a combined total effect of $\delta\nu/\nu = 3.9 \cdot 10^{-3}$, so the purely thermal broadening is enhanced by a factor 90.2 on account of the mode. This result confirms that line broadening is indeed a suitable means to provide access to the first observational evidence for the stable branch of the Weibel mode.

8. Magnetization of the early universe by dark aperiodic fluctuations

As outlined in the preceding sections, one of the major scientific problems of modern physics and astrophysics is the understanding of the origin of cosmic magnetic fields. In order to advance their previous work addressing the onset of the recombination era, Schlickeiser et al. (2018) (hereafter referred to as SKY) proposed an electromagnetic origin based on the stimulated emission of aperiodic transverse fluctuations by charged electron–positron pairs in the early universe prior to the pair annihilation epoch at photon temperatures $k_{\mathrm{B}}T > 1$ MeV. We refer to this early cosmological phase as pamyprepanne, standing for past myon and prior electron–positron annihilation epoch.

During the pamyprepanne the charged pair density was a factor $\eta^{-1} \simeq 10^9$ higher than the density of the surviving electrons and protons from the baryon–antibaryon asymmetry, where η denotes the current matter–photon ratio. These pairs produce by spontaneous emission damped, incompressible, subluminal, aperiodic transverse electromagnetic fluctuations without density fluctuations (Kolberg et al., 2017) which are unobservable (dark), but have a substantial energy density though smaller than the superluminal photon energy density. The associated field decay with time by damping is compensated by their perpetually reexcitation due to the spontaneous emission by the pairs to maintain a lasting fluctuation level. This additional subluminal fluctuation field persists when after the pair annihilation most of the pairs annihilated into superluminal photons. Using classical fluctuation theory SKY demonstrated that during the pamyprepanne random electric and magnetic fields with very high tera-Gauss strengths $|\delta B|$ and $|\delta E|$ were generated by the relativistic electron–positron pairs on spatial scales greater than $L \simeq 10^{-10}$ cm $\cdot T_{\mathrm{MeV}}^{-1}$ with 100 percent volume filling factor ($T_{\mathrm{MeV}} = k_{\mathrm{B}}T/\mathrm{MeV}$).

The thermal evolution of the early universe is rather well understood (Kolb and Turner, 1990; Peebles, 1993; Weinberg, 2008; Husdal, 2016). A tiny fraction of a second after the Big Bang, the early universe material was close to thermal equilibrium with $k_{\mathrm{B}}T = 10^{12}$ eV and consisted of all the elementary particles P^+ and their antiparticles P^- of the standard model. The particle masses ranged from the heaviest known particles, the top quark down to the lightest particles, the neutrinos and the photon γ with superluminal phase speed. The frequent annihilation ($P^+ + P^- \rightarrow \gamma + \gamma$) and pair production ($\gamma + \gamma \rightarrow P^+ + P^-$) processes provided roughly the same abundance of particles and antiparticles (ignoring the negligibly small baryon–antibaryon asymmetry), with bosons and fermions obeying the ideal Bose–Einstein of temperature T_B and the Fermi–Dirac of temperature T_F distribution functions, respectively, in thermal equilibrium with the photons with the blackbody Planckian distribution function T, so that $T_B = T_F = T$.

In contrast to the annihilation interactions the pair production interactions require a threshold center-of-mass energy of $\simeq 2\ \mathrm{mc}^2$, where m denotes the mass of the bosonic and fermionic particles and antiparticles. Hence, due to the cooling of the expanding universe, the pair production rate of massive particles could not keep up with the annihilation rate, so that most of the massive particles vanished one by one, except for the small number of light massive particles (protons, neutrons and electrons) surviving due to baryon asymmetry with $n_m = \eta n_\gamma$ and $\eta \simeq 10^{-9}$, which at later cosmological epochs was enough to form all the stars and galaxies of our present-day universe (Boesgaard and Steigman, 1985; Whittet and Chiar, 1993).

During the pamyprepanne, i. e. after the myon annihilation epoch but before the electron–positron annihilation epoch, i. e. at temperatures $1\,\mathrm{MeV} < k_{\mathrm{B}}T < 210\,\mathrm{MeV}$, corresponding to times between $2 \cdot 10^{-5}\,\mathrm{s} < t < 1\,\mathrm{s}$ after the Big Bang and redshifts between $6 \cdot 10^9 \leq Z \leq 1.4 \cdot 10^{12}$, the universe consisted of a fully-ionized electron–positron–proton plasma with number densities $n_{e^+} \simeq n_\gamma$, $n_{e^-} = (1+\eta)n_\gamma$, $n_{p^+} = \eta n_\gamma$, coexisting with neutrons, neutrinos and superluminal photons of density n_γ in thermodynamic equilibrium. The overall charge-neutrality of the electron–positron–proton plasma requires a slight excess by the small factor η of electrons and protons compared to positrons. In the pamyprepanne the isotropic phase space density of electron and positrons are given by the Fermi–Dirac distribution function

$$f_F(p) = \frac{g_d}{(2\pi)^3\hbar^3}\frac{1}{e^{\mu E}+1}, \quad E = \sqrt{1+\frac{p^2}{m_e^2c^2}}, \quad \mu = \frac{m_ec^2}{k_BT} = \frac{0.511}{T_{\mathrm{MeV}}}, \quad T_{\mathrm{MeV}} = \frac{k_BT}{\mathrm{MeV}} \tag{430}$$

with the effective degeneracy parameter (Husdal, 2016) $g_d = 9.5$. In order to make use of the extensive earlier work (Felten et al., 2013; Felten and Schlickeiser, 2013c) we use the very accurate thermal approximation

$$f_F(p) \simeq f_a(p) = n_0F_a(p), \quad F_a(p) = \frac{\mu e^{-\mu E}}{4\pi(m_ec)^3K_2(\mu)} \tag{431}$$

in terms of the modified Bessel functions $K_\nu(\mu)$, with a representing positrons and electrons, yielding for the mean number densities

$$n_{e^+} = n_{e^-} = n_0 = \int d^3p\, f_F(p) = \frac{g_d}{\pi^2}\left(\frac{m_ec}{\hbar\mu}\right)^3 = \frac{g_d}{\pi^2}\left(\frac{k_BT}{\hbar c}\right)^3 = 1.25 \cdot 10^{32}T_{\mathrm{MeV}}^3\ \ \mathrm{cm}^{-3}\,. \tag{432}$$

The positron and electron total number densities n_0 are solely determined by the photon temperature T. At this and earlier phases the dynamics of the universe was radiation dominated so that T_{MeV} decreased with time as (Baumann, 2018)

$$T_{\mathrm{MeV}} = 1.5g_d^{-1/4}(t/t_0)^{-1/2} \simeq (t/t_0)^{-1/2} \tag{433}$$

with $t_0 = 1$ s. Additionally the electrons and protons surviving from the baryon asymmetry have the phase space distribution functions

$$f_{be}(p) = \eta n_0F_a(p), \quad f_{bp}(p) = \eta n_0F_{bp}(p), \quad F_{bp}(p) = \frac{\mu_pe^{-\mu_pE}}{4\pi(m_pc)^3K_2(\mu_p)}, \quad \mu_p = 1836\mu_e\,. \tag{434}$$

Their temperature is also determined by the photon temperature, as these electrons are tightly coupled to the photons via Compton scattering even after the pamyprepanne, and the protons strongly interact with the electrons via Coulomb scattering.

As all plasmas have fluctuations due to the random motion of charged particles for finite temperatures, their state variables such as energy density, pressure and electromagnetic fields fluctuate in position and time around their mean values, which for electric and magnetic fields vanish (unmagnetized plasma). Using the classical general theory of electromagnetic fluctuations SKY calculated quantitatively the level of aperiodic fluctuations in the pamyprepanne. Such quantitative estimates of the stochastic field strengths are also of high interest for future laser-plasma experiments that are able to generate laboratory conditions characteristic for the early universe plasma (Gregori et al., 2015; Huntington et al., 2017).

However, the classical fluctuation theory is applicable as long as the de Broglie wavelength of individual plasma particles $\lambda_B < n_0^{-1/3}$ is smaller than the average interparticle distance $n_0^{-1/3}$, i. e.

$$\lambda_B = \hbar/p \simeq \hbar c/k_B T = \frac{\hbar\mu}{m_e c} < n_0^{-1/3}, \tag{435}$$

so that the plasma particles can be considered as pointlike, ignoring overlapping wave functions and quantum interference effects. As $\eta \ll 1$ we ignore the electron–proton-plasma surviving from the baryon asymmetry, so that with the density (432) the criterion (435) becomes

$$\left(\frac{\hbar\mu}{m_e c}\right)^3 n_0 = \frac{g_d}{\pi^2} = 0.96 < 1, \tag{436}$$

which for $g_d = 9.5$ is only just fulfilled. At the same time the wavelength of the fluctuations $\lambda \geq \lambda_{\min} \gg \lambda_B$ should be large compared to the de Broglie wavelength of individual plasma particles. Adopting $\lambda_{\min} = 2\pi\lambda_B$ then provides for the maximum wavenumber of fluctuations

$$k_{\max} = \frac{2\pi}{\lambda_{\min}} = \frac{1}{\lambda_B} = \frac{m_e c}{\hbar\mu} = 5 \cdot 10^{10} T_{\mathrm{MeV}} \ \ \mathrm{cm}^{-1} \tag{437}$$

For nonrelativistic plasma temperatures fluidlike quantum descriptions of plasma fluctuations based on the Wigner–Schrödinger equations are available (Haas, 2008; Haas and Lazar, 2008). However, a kinetic quantum description for ultrarelativistic temperatures on the basis of the Dirac equation currently is not available.

Due to the small value of η the small contribution to the fluctuation spectra from the excess protons and electrons is negligibly small, so that one is dealing with an equal-density pure electron–positron plasma with $\sum_a \omega_{p.a}^2/m_a = 2\omega_{p,e}^2/m_e$.

8.1. Non-collective aperiodic fluctuations during the pamyprepanne

SKY calculated with the distribution function (431) the transverse aperiodic non-collective fluctuation spectra as

$$\langle(\delta B)^2\rangle(k, \Gamma) = \frac{4k_B T k^2 c^2}{(2\pi)^3 \Gamma^3} \frac{\Lambda_T(k, \Gamma) - 1 - \frac{k^2c^2}{\Gamma^2}}{|\Lambda_T(k, \Gamma)|^2}, \quad \langle(\delta E)^2\rangle(k, \Gamma) = \frac{\Gamma^2}{k^2c^2} \langle(\delta B)^2\rangle(k, \Gamma), \tag{438}$$

in terms of the thermal ultrarelativistic ($\mu \ll 1$) transverse dispersion function

$$\Lambda_T(k, \Gamma) \simeq 1 + \frac{1}{I^2} + \frac{\omega_{p,e}^2\mu}{k^2c^2}\left(\frac{1+I^2}{I}[\arctan\frac{1}{I} + \frac{\pi\sigma}{2}] - 1\right), \quad I = \frac{\Gamma}{kc}, \tag{439}$$

where $\sigma = 0$ for $\Gamma > 0$ and $\sigma = 2$ for $\Gamma < 0$ accounts for the analytical continuation. In Fig. 13 we show the resulting transverse aperiodic non-collective magnetic fluctuation spectrum as a function of Γ and k. Obviously, the spectrum is dominated by a damped mode with negative values of Γ.

8.2. Collective aperiodic eigenmode during the pamyprepanne

A collective aperiodic plasma eigenmode with Γ_k has to fulfill the dispersion relation $\Lambda_T(k, \Gamma_k) = 0$, yielding for negative values[5] of $\Gamma < 0$ as solution the dispersion relation

$$I(k) = I_k \simeq -\frac{k^2}{k_c^2 T_{\mathrm{MeV}}^2}, \quad \Gamma_k = kcI_k = -\frac{k^3 c}{k_c^2 T_{\mathrm{MeV}}^2} \tag{440}$$

with the characteristic wavenumber $k_c = 2.32 \cdot 10^{10}$ cm^{-1}. $k_c T_{\mathrm{MeV}}$ equals within a factor 2.1 the maximum wavenumber (437), so that we adopt $k_c T_{\mathrm{MeV}}$ as the maximum wavenumber for which the classical fluctuation theory is applicable. Using the formalism of (Yoon et al., 2014) yields for the associated spontaneous emission coefficient

$$\alpha(k, t) = \frac{2k_B T(1 + I_k^2)}{\pi^2 I_k^4 |\Gamma_k| |\frac{\partial\Lambda_T(k,\Gamma)}{\partial\Gamma}|_{\Gamma_k}^2} \simeq 2\alpha_0 c k_c \kappa^3 (t/t_0)^{1/2}, \quad \alpha_0 = \frac{9\mathrm{MeV}}{(4\pi)^2}, \quad \kappa = k/k_c, \tag{441}$$

The magnetic eigenmode wavenumber spectrum then obeys the Kirchhoff law

$$\frac{\partial S(k, t)}{\partial t} = 2b(\kappa)[\alpha_0(t/t_0)^{1/2} - (t/t_0)S(k, t)], \quad b(\kappa) = ck_c\kappa^3 \tag{442}$$

With the initial condition that no fluctuations exist before the pamyprepanne, SKY found that at the end of the pamyprepanne, about $t_0 = 1$ s after the big bang, the magnetic and electric aperiodic eigenmode wavenumber spectra are given by

$$S(\kappa \leq \kappa_0, t_0) \simeq \alpha_0(\frac{\kappa}{\kappa_0})^3, \quad S(\kappa_0 < \kappa \leq 1, t_0) \simeq a_0, \quad S(\kappa > 1, t_0) = 0,$$

5 No eigenmode solution exists for positive values of Γ.

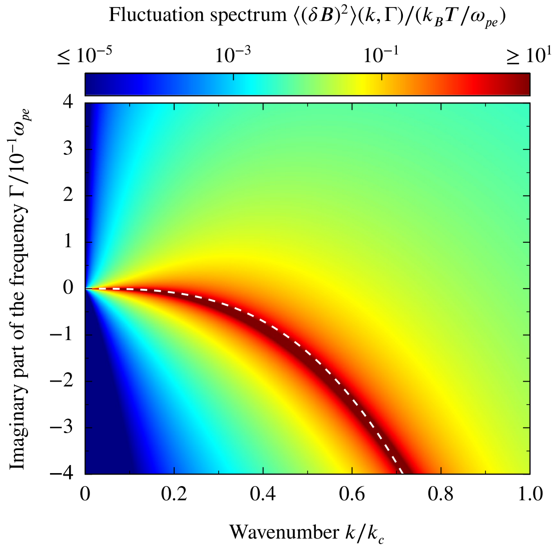

Fig. 13. Contour plot of the spontaneously emitted thermal aperiodic magnetic noise at the end of the pamyprepanne in units of $k_BT/(2\pi^3\omega_{p,e})$ for a temperature of $k_BT = 1$ MeV. The electron plasma frequency $\omega_{p,e} = 6.3 \cdot 10^{20} T_{\rm MeV}^{3/2}$ Hz is also determined by the temperature. The color scale is logarithmic in powers of 10. The dashed line shows the dispersion relation (440) of the damped eigenmode. From Schlickeiser et al. (2018).

$$\alpha_0 = \frac{9\,\rm MeV}{(4\pi)^2}, \quad \kappa_0 = 1.13 \cdot 10^{-7}, \quad S^E(k, t_0) = \kappa^4 S(k, t_0) \tag{443}$$

By integrating these wavenumber fluctuation spectra over all wavenumber values up to the maximum wavenumber $k_c T_{\rm MeV}$ the random magnetic and electric fields in the form of aperiodic fluctuations have tera-Gauss strengths

$$|\delta B(t_0)| = \sqrt{(\delta B)^2(t_0)} = 2.2 \cdot 10^{12}\,\rm G\,, \tag{444}$$

$$|\delta E(t_0)| = \sqrt{(\delta E)^2(t_0)} = 1.4 \cdot 10^{12}\,\rm statvolt\,cm^{-1}\,. \tag{445}$$

The electric plus magnetic energy density in aperiodic fluctuations

$$w_{B+E}(t) = \frac{(\delta B)^2 + (\delta E)^2}{8\pi} = \frac{5}{28\pi}(\delta B)^2(t) = 2.7 \cdot 10^{23} T_{\rm MeV}^4\,\rm erg\,cm^{-3} = 2.7 \cdot 10^{23}(\frac{t}{t_0})^{-2}\,\rm erg\,cm^{-3}, \tag{446}$$

which is about three orders of magnitude smaller than the photon energy density $w_p(t) = 1.4 \cdot 10^{26} T_{\rm MeV}^4$ erg cm^{-3}, so that the aperiodic fluctuations have a negligible influence on the radiation-driven cosmological evolution during the pamyprepanne.

9. Summary and conclusions

The description of electromagnetic fluctuations in fully ionized plasmas is commonly based on the system of Maxwell's equations for the fields accompanied by a suitable kinetic equation for the particles. With respect to collisionless plasmas, the latter is usually either the Vlasov equation or, if the spontaneously emitted noise is to be taken into account, the Klimontovich equation. In both cases, the analysis is expediently carried out in the spectral domain in terms of harmonics with the spatio-temporal profile $\propto \exp[\imath(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)]$. While the indefinite spatial boundary conditions corresponding to an infinitely extended medium demand a real-valued Fourier variable $\boldsymbol{k}$, the initial condition attributed to a moment of time in the definite past entails a complex Laplace variable $\omega = \Omega + \imath\gamma$. Thus, there are two characteristic features associated with modes, propagation and amplitude modulation, quantified by the real and imaginary part of the frequency, respectively. Ever since its advent in the middle of the last century, plasma fluctuation theory has predominantly focused on weakly damped or weakly growing modes with $|\Omega| \gg |\gamma|$. Regarding the opposite case, prominent examples such as the Buneman, firehose, mirror, filamentation, and Weibel modes attest to the fact that weakly propagating fluctuations obeying $|\Omega| \ll |\gamma|$ and non-propagating, aperiodic modes characterized by $\Omega = 0$ also exist and are discussed in the literature (Buneman, 1958; Fried, 1959; Weibel, 1959). However, the general framework, as it stands, still suffers major restrictions as far as the complex frequency is concerned. Many results, such as the Landau formula for the growth or damping rate of a mode, crucially rely on the assumption that the imaginary part of the frequency is much smaller than the real part, or they assume the frequency to be a real number altogether, thereby excluding weakly propagating and aperiodic modes from the outset. An even more serious constraint arises from the divergence of the Laplace integral for frequencies located in the half-plane below the abscissa of convergence (Doetsch, 1974). Efforts to overcome these limitations by performing proper analytical continuations of the form factors and dispersion functions to the entire complex plane are recent developments of the past few years, culminating in the discovery of a new damped, aperiodic, and transverse mode in unmagnetized thermal plasmas (Felten et al., 2013). This mode can be interpreted as the stable branch of the Weibel mode that operates in isotropic systems. It owes its significance to the fact that it is the only spontaneously excited mode in unmagnetized equilibrium plasmas that is associated with magnetic field fluctuations. By comparison with the textbook results unaware of its existence, it leads to a considerably enhanced noise level (Schlickeiser and Felten, 2013; Yoon et al., 2014). The present article advances this line of research by contributing new results both to the generic plasma theoretical foundation as well as astrophysical applications.

As a prerequisite for the strictly research-related sections, in the first part of this work the underlying concepts were motivated and the required equations were provided by deriving them from first principles. In order to reveal the respective electrodynamic or statistical origin of every method, both areas were addressed separately at first. Thus, in Section 2, Maxwell's equations were solved in the spectral domain by performing a linear response expansion of the current density. This automatically led to the notion of an eigenmode and to the necessity to perform an analytical continuation. Since the linear response coefficients are abstract parameters in this context that cannot be determined from within the theory, this procedure very naturally exposed the need to include a microscopic plasma model—quite in agreement with the viewpoint "that statistical mechanics is something like a 'transfer mechanics' whose role is to transmit information from the microscopic to the macroscopic level". (Balescu, 1975, page vi). Consequently, the kinetic theory of charged particles was outlined in Section 3, based on the Klimontovich equation for the 6-dimensional phase space density. There, it was argued that the chain of equations for the moments can be truncated if the plasma parameter is sufficiently small, leading to a linear (or rather: affine) relation for the fluctuations. It was also demonstrated that the spontaneously emitted noise is due to the self-correlations of discrete particles. After their individual analysis, the electrodynamic and statistical aspects were unified into the full plasma fluctuation theory in Section 4, emphasizing that both branches complement each other: Kinetic theory supplies a sound justification for the linear response relation (i. e., Ohm's law) along with a specification of the response tensor in terms of the distribution function. Vice versa, Maxwell's equations account for the full electrodynamic effects, generalizing the non-relativistic Coulomb interactions considered before for the sake of simplicity. Particular attention was paid to the Fourier–Laplace inversion of correlation functions describing homogeneous and stationary turbulence. Since Dirac's δ-distribution is no longer properly defined for the complex frequencies considered in this work, the expression $\delta(\omega + \omega')$ associated with stationarity had to be generalized appropriately. To this end, Balescu's solution to this problem was introduced (Balescu, 2005) that is relevant not only for the inversion formula but also for the spectral representation of the spontaneously emitted natural statistical fluctuations.

Having these concepts and equations at disposal, the way was paved for the second part of this article which is devoted to the new research results that were obtained in response to the objectives defined in the introductory section.

Result 1 (Schlickeiser and Kolberg, 2015). Although important efforts were made in recent years to generalize plasma fluctuation theory to the entire complex plane, a fundamental cornerstone of modern theoretical physics was still not considered yet, and this is the fluctuation–dissipation theorem (Callen and Welton, 1951; Kubo, 1957, 1966; Martin, 1968; Landau and Lifshitz, 1980). Among the various different formulations, the one relevant in the present context constitutes a relation between the Maxwell-tensor of a classical equilibrium plasma to the correlation tensor of the field fluctuations (Sitenko, 1967). Despite the use of complex arguments in linear response theory for mathematical convenience, leading to important results such as the Kramers–Kronig relations, the theorem itself is still not valid for arbitrary complex frequencies. The derivations of the theorem found in the literature crucially rely on the assumption that the frequency is a real number, or at least they consider the limit $\Im\omega \to 0$ corresponding to weakly damped or weakly amplified fluctuations (ibid.).

The generalization of the fluctuation–dissipation theorem derived in Section 5 disposes of all restrictions regarding the frequency and holds for classical, weakly coupled equilibrium plasmas, even allowing for a large-scale magnetization as well as relativistic particle energies. Basically, the derivation compares the general expression for the Maxwell tensor with the fluctuation spectrum obtained by Schlickeiser and Yoon (2015), which is valid for the entire complex frequency plane. The main result is equation (257). As expected, it reproduces Sitenko's previous formulation for real-valued frequencies. In the general case, however, it displays a new feature: Not only the anti-hermitian part of the inverse Maxwell tensor enters the theorem, but the hermitian part as well. This remains true in the limit $\Re\omega \to 0$ describing aperiodic fluctuations. At first glance, this seems to contradict the well known fact that dissipation caused by Ohmic losses is only associated with the anti-hermitian part of the dielectric or Maxwell tensor (Melrose and McPhedran, 1991). However, in this article it was shown that this is no longer the case if the spectrum of a complex Laplace variable is considered. Then, the Poynting theorem implies that both the hermitian and anti-hermitian part are relevant. Finally, the theorem was also computed for unmagnetized plasmas. Due to the additional isotropy, the formulation simplifies considerably because the Maxwell tensor becomes diagonal and only its purely transversal and longitudinal components enter the theorem.

Result 2 (Kolberg et al., 2016). After these abstract theoretical investigations, the astrophysical consequences arising from the newly discovered stable branch of the Weibel mode were addressed. Prior to the discovery of the latter, only three modes operating in unmagnetized and isotropic equilibrium plasmas were known in the literature: Langmuir waves, ion acoustic waves, and electromagnetic waves (Melrose, 1986). Since the first two are longitudinal modes, only the last one is associated with magnetic field fluctuations. Considering that spontaneous emission in unmagnetized plasmas only occurs for subluminal phase speeds, therefore, it was believed that no mode is available for spontaneously emitted magnetic field fluctuations, implying that the latter merely exist as an uncollective effect. The stable branch of the Weibel mode, however, changes the picture, entailing a higher thermal fluctuation level. Assuming a temperature of 10^4 K and a number density of 10^{-7} cm^{-3} for both electrons and protons, the corresponding level of the self-excited magnetic field fluctuations in the cosmic voids of the intergalactic medium (IGM) amounts to $6 \cdot 10^{-18}$ G (Yoon et al., 2014). Hence, it is even high enough to serve as the initial seed field required for the operation of magnetohydrodynamic dynamos, plasma instabilities, flux conserving compression, and other means of field amplification causing cosmic magnetization (Schlickeiser, 2012; Kulsrud and Zweibel, 2008; Rees, 1987).

In Section 6 it was shown that a highly relativistic electron–positron pair beam has a tremendous impact on these fluctuations and causes an even higher level. Such a beam with a typical Lorentz factor of $\Gamma_b = 10^6$ is produced very naturally when the TeV-radiation emitted by an active galactic nucleus (AGN) annihilates with the low-energy photons of the extragalactic background light. The number density of the beam is considerably lower than the one of the background plasma, the corresponding ratio $\varepsilon = n_e^{\mathrm{beam}}/n_e^{\mathrm{IGM}}$ is about 10^{-15} at sufficiently large distances (Schlickeiser et al., 2012b) or at most 10^{-5} in the vicinity of the AGN. By means of a first-order perturbation expansion in the small parameter ε, the spectral balance equations for the field fluctuations were computed that self-consistently account for the competing effects of spontaneous emission and absorption in the perturbed thermal plasma, viz. (353) and (356). These equations were derived from first principles of kinetic plasma fluctuation theory and are covariantly valid within the limits of special relativity. In contrast to their unperturbed counterparts, these Kirchhoff type radiation laws are tensor equations due to the symmetry breaking influence of the beam. It was found that the perturbation does not affect the spontaneous emission, but that the absorption coefficient contains an additional term that enables the possibility of negative absorption, i. e., amplification, feeding on the free energy provided by the beam. This effect is similar to the well known bump-in-tail instability (Treumann and Baumjohann, 2001), but, in contrast to the latter, the instability emerges from the stable branch of the Weibel mode and is therefore associated with aperiodic and transverse fluctuations. Nevertheless, the instability can still be interpreted as the equilibrating process initiated by the local bump in the distribution function at the energy level of the beam particles. In this sense, the situation is the classical counterpart of the quantum mechanical population inversion leading to laser action.

In order to elucidate the conditions for amplification in more detail, the circumstances were investigated under which the absorption coefficient becomes negative. Since the latter is closely related to the growth rate of the mode, an effective growth rate was introduced that accounts for the disturbance of the beam and that indicates amplification by taking positive values. It was found that the latter can occur only locally in a narrow band of the wavenumber spectrum (see Figs. 8 and 9). Moreover, only those fluctuations are amplified whose wavevector lies within the double cone depicted in Fig. 10 that is coaxially oriented with respect to the propagation direction of the beam. It was shown that the half opening angle is determined by the density ratio and the Lorentz factor of the beam, viz. (370)–(371). Approximately, it scales as $(\varepsilon/\Gamma_b)^{1/3}$, so the effect of amplification is more pronounced in the vicinity of the AGN where ε is larger. While this result is quite plausible, it appears somewhat counterintuitive that the criterion favors smaller beam energies. It should be noted, however, that a relativistic Lorentz factor $\Gamma_b \gg 1$ was assumed throughout the entire analysis. For all sensible values of density ratio and Lorentz factor, the half opening angle is negligibly small, so the double cone effectively degenerates into a plane (Fig. 11). Thus, amplification occurs only for fluctuations attributed to wavevectors that are perpendicular to the beam.

Having established the certainty that the magnetic field fluctuations are partially subject to amplification, it is evident that the unperturbed level of $6 \cdot 10^{-18}$ G must become enhanced due to the presence of the beam. The next logical step to be pursued by future research, therefore, is to quantify this statement in more detail by integrating the spectrum in order to determine the total strength of the magnetic noise. In the context of magnetogenesis, this provides valuable constraints on the relevant timescales and the required efficiency of the subsequent amplification processes. A further refinement of the current results may be achieved by weakening the precondition of a cold beam distribution, i. e., by allowing for a finite spread of the beam energy spectrum. Nevertheless, the assumption of a collimated and monoenergetic beam underlying the present analysis is well justified in view of the strongly peaked energy spectrum and angular distribution of the created pairs that was computed by Schlickeiser et al. (2012a).

Result 3 (Kolberg et al., 2017). In the light of these findings, it is evident that the stable branch of the Weibel mode is of ample relevance not only from a purely theory-immanent point of view but also with respect to astrophysical applications. It is all the more unsatisfactory that observational evidence for its existence is still pending. Even worse, considering that the mode operates in unmagnetized stable plasmas, there are hardly any prospects of gathering data by either laboratory experiments or *in situ* measurements in natural plasmas within the reach of spacecraft instruments. One must, therefore, inevitably resort to more distant environments such as the voids of the large-scale structure. The most suitable messengers providing information about such remote objects are photons traveling through the turbulent fields in the IGM during their passage to Earth. In this regard, the repertoire of standard techniques to study turbulence comprises rotation measure, dispersion measure, line broadening, and scintillation data. All of these tools rely on the presence of either density or velocity fluctuations in the IGM that affect the traversing radiation. Due to the coupling of particles and fields in a plasma, the field fluctuations governed by the new mode can, in general, induce such characteristic particle fluctuations by means of the Lorentz force. This opens the possibility to finally achieve the desired empirical confirmation for the existence of the mode after all.

In order to examine the feasibility from a theoretical point of view, the density fluctuations generated by the mode were computed in Section 7. It was demonstrated anew that transverse modes in an isotropic, unmagnetized plasma are always associated with incompressible turbulence bare of any density fluctuations. This already disqualifies dispersion and scintillation measurements. Although the average of the rotation measure also vanishes in this case, this does not necessarily hold for its standard deviation as well. Therefore, the corresponding fluctuations of the rotation measure were estimated next, yielding $\sigma_{\mathrm{RM}} \leq 3.2 \cdot 10^{-22}$ rad cm^{-2}. Resolving so low a value is truly demanding at the very least, so the applicability of this technique for the purpose at hand appears rather improbable, if not impossible.

That leaves line broadening analysis as the only remaining method, and consequently it becomes even more important to learn whether it can actually detect the mode. To this end, the spectrum of the mode-driven proton velocity fluctuations was computed, resulting in (410). Since it already is of considerable interest in its own right as it characterizes the state of the microturbulence, the corresponding power-law spectral indices were calculated as well, and they exactly reproduce the asymptotics exhibited in Fig. 12. In order to obtain the total value of the induced velocity fluctuations, the contributions of all wavenumbers were collected by an integration. In this fashion, a value of $1.16 \cdot 10^8$ cm s^{-1} $\propto T^{7/2} n^{-1/2}$ was found for the mode-driven proton velocity fluctuations which is 90 times higher than the thermal velocity. As only neutrals participate in line broadening, a crucial assumption in this regard is that mutual charge exchange and elastic collision events (ambipolar diffusion) are sufficiently frequent such that the hydrogen atoms adopt the velocity profile of the protons which are under the perpetual influence of the mode. Under this condition, a spectral line with the frequency ν that traverses the IGM fluctuations is subject to a relative broadening of $\delta\nu/\nu = 3.9 \cdot 10^{-3}$, which exceeds pure Doppler broadening by a factor of 90. In view of this high value, one can conclude that line broadening measurements are indeed a very promising tool to obtain observational evidence for the stable branch of the Weibel mode.

Considering that the ratio between the mode-driven velocity fluctuations and the thermal velocity scales with temperature as T^3, the effect is expected to be even more pronounced in hotter regions. It should be noted, however, that the results obtained here are not valid for arbitrarily high temperatures, mainly for two reasons: Firstly, the steep increase of the velocity fluctuations themselves ($\propto T^{7/2}$) quickly leads to values that are comparable to the speed of light, so this scaling is bound to break down eventually due to the restrictions arising from special relativity. Secondly, Yoon et al. (2014) have shown that the driving fluctuations of the magnetic field also increase with temperature, albeit much slower: $\propto T^{1/8}$. Hence, above a certain temperature threshold the fluctuations outgrow the linear regime and nonlinear effects become significant.

As a final remark it is worth pointing out that the spectrum of the induced velocity fluctuations approaches a constant value for large wavenumbers. For this reason, a maximum wavenumber was introduced artificially in order to truncate the integral entering the total velocity fluctuations and, in this vein, prevent its divergence. Consequently, the outcome is sensitive to the particular choice of the upper limit ($\propto k_{\max}^{3/2}$). Despite this fact, the value obtained here is by no means afflicted with arbitrariness because the declaration $k_{\max} = 2\pi/r_{\mathrm{c}}$ in terms of the classical distance of closest approach is specifically adapted to the physical source of the divergence. Both the problem and the cure are the very same as in the well-known case of the Coulomb logarithm that characterizes the divergence exhibited by the Landau collision integral. One might possibly remedy this issue altogether in future analyses by taking non-linear effects into account. These provide higher-order corrections for the spectrum that can rid the current theory of the divergence and, therefore, of the need for an artificial upper integration limit.

Acknowledgments

We gratefully acknowledge discussions and cooperations with P.H. Yoon, M. Lazar, and H. Fichtner. This work was financially supported by the Deutsche Forschungsgemeinschaft, Germany grants Schl 201/31-1 and Schl 201/32-1.

Appendix. Derivation of auxiliary formulas

A.1. Fourier and Laplace transforms

In order to fix prefactor conventions and notation concerning Fourier and Laplace transforms, a brief definition supplemented by some technical remarks will be given here. In this work, the Fourier transform of a field $f_{\boldsymbol{r}}$ and the Laplace transform of a time-dependent function g_t are defined as

$$f_{\boldsymbol{k}} = \int \frac{\mathrm{d}^3 r}{(2\pi)^3}\, \mathrm{e}^{-\imath \boldsymbol{k}\cdot\boldsymbol{r}} f_{\boldsymbol{r}} \qquad (\boldsymbol{k} \in \mathbb{R}^3)\,, \tag{A.1}$$

$$g_\omega = \int_0^\infty \frac{\mathrm{d}t}{2\pi}\, \mathrm{e}^{\imath\omega t} g_t \qquad (\omega \in \mathbb{C})\,. \tag{A.2}$$

This definition of the Laplace variable ω follows plasma physicists such as Nicholson (1983); Schlickeiser (2002), and Yoon et al. (2016), and it differs from the variable $s = -\imath\omega$ commonly used in the literature by a factor of $-\imath$, corresponding to a rotation of the complex plane by $\pi/2$. This must be taken into account when the references are consulted. The rationale behind this convention is that the resulting kernel of the combined transform, $\exp[-\imath(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)]$, has the same form as in the case of an ordinary normal mode analysis with a real-valued frequency.

Regarding the convergence of these integrals and the existence of the transformed functions, the interested reader is referred to the mathematical literature (e.g., Papoulis, 1962; Doetsch, 1974). However, one particular theorem is worth mentioning here explicitly for later reference: For a given function g_t there is a number $a \in \mathbb{R} \cup \{\pm\infty\}$, called *abscissa of convergence*, such that the Laplace integral converges in the upper half-plane $\Im\omega > a$ and diverges for all ω with an imaginary

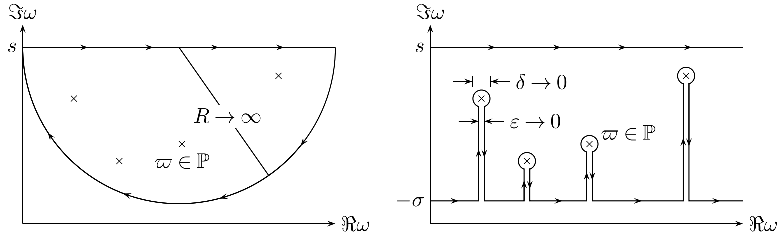

Fig. A.14. Possible contours for computing the Bromwich integral. Left: Closing the path of integration by a semi-circle of infinite radius. Right: Landau contour wrapping around the poles.

part less than a (Doetsch, 1974, theorem 3.5). For the line $\Im\omega = a$, a general statement is not possible as it can belong to the region of convergence either entirely, or in parts, or not at all. In the case of the Fourier transform such a restriction on the wavenumber domain is not necessary because the exponential kernel $\exp(\imath \boldsymbol{k} \cdot \boldsymbol{r})$ does not grow beyond all boundaries as its absolute value is always one.

The main reason to apply these transforms is that they convert any linear differential equation into an algebraic one because

$$(\nabla f)_{\boldsymbol{k}} = \imath \boldsymbol{k} f_{\boldsymbol{k}} \,, \tag{A.3}$$

$$(\nabla \cdot \boldsymbol{f})_{\boldsymbol{k}} = \imath \boldsymbol{k} \cdot \boldsymbol{f}_{\boldsymbol{k}} \,, \tag{A.4}$$

$$(\nabla \times \boldsymbol{f})_{\boldsymbol{k}} = \imath \boldsymbol{k} \times \boldsymbol{f}_{\boldsymbol{k}} \,, \tag{A.5}$$

$$(\dot{g})_\omega = -g_{t=0}/2\pi - \imath\omega g_\omega \,. \tag{A.6}$$

Once the resulting algebraic equations are solved in Fourier–Laplace space, the solutions must be transformed back into real space by means of the inverse transforms (Papoulis, 1962)

$$f_{\boldsymbol{r}} = \int \mathrm{d}^3k \; \mathrm{e}^{\imath \boldsymbol{k}\cdot\boldsymbol{r}} f_{\boldsymbol{k}} \,, \tag{A.7}$$

$$g_t = \int_{\mathcal{B}} \mathrm{d}\omega \; \mathrm{e}^{-\imath\omega t} g_\omega \qquad (t > 0), \tag{A.8}$$

where $\mathcal{B} = \mathbb{R} + \imath s$ denotes the Bromwich contour, a horizontal line above the abscissa of convergence. According to the inversion theorem for the Laplace transform, the Bromwich integral (A.8) does not depend on the choice of the number $s > a$, superseding the necessity to specify it any further (ibid.). Geometrically, this means the freedom to perform arbitrary parallel shifts of the contour of integration within the domain of g_ω—an operation that alters the start and end point of the contour. In addition, Cauchy's integral theorem allows a homotopic deformation of the contour in the interior of the half-plane of convergence because, there, the Laplace transform g_ω constitutes a holomorphic function (Doetsch, 1974, theorem 6.1).

In most cases, the Bromwich integral (A.8) is rather difficult to compute. In view of this, it proves useful to utilize the theorems of complex analysis again. In this respect, the only required assumption is that g_ω can be continued meromorphically. This means that there is a function $\tilde{g}_\omega$ such that (1) it is defined on the entire complex plane, (2) it equals g_ω in its region of convergence, and (3) it is holomorphic with exception of a finite number of isolated poles. Since g_ω is holomorphic in its domain, every pole of $\tilde{g}_\omega$ is located on or below the line $\Im\omega = a$. This meromorphic continuation $\tilde{g}_\omega$ extends the freedom to deform the Bromwich contour in the region below the abscissa of convergence.

One possible way to proceed is to close the path of integration by extending it by a semi-circle of infinite radius (see Fig. A.14), and to apply the residue theorem. Then, the Bromwich integral is given by the residues of its integrand minus the surplus integral along the semi-circle. This method is useful in those cases in which the integrand vanishes sufficiently fast as $|\omega| \to \infty$ such that the surplus integral becomes zero. In general, however, there is no a priori reason to assume that this is fulfilled by the transformed electromagnetic fields, and in order to avoid such restrictive assumptions whenever possible, an alternative method must be found.

Once more, the solution is due to the pioneering work of Landau (1946). According to complex analysis, a deformation of the Bromwich contour does not change the value of the integral as long as the deformed contour remains above all poles. Therefore, it is possible to choose the so-called *Landau contour* shown in Fig. A.14 instead of the line $\Im\omega = s$. In total, the vertical line segments do not contribute to the integral because the path approaching a pole cancels with the one departing from it in the limit of vanishing distance between them. Since the contributions from the circles enclosing the poles are, up to a prefactor, the residues of the integrand, one obtains

$$g_t = -2\pi\imath \sum_{\varpi\in\mathbb{P}} \mathrm{Res}_\varpi \{\mathrm{e}^{-\imath\omega t}\,\tilde{g}_\omega\} + \int_{-\infty-\imath\sigma}^{\infty-\imath\sigma} \mathrm{d}\omega \; \mathrm{e}^{-\imath\omega t}\,\tilde{g}_\omega \,, \tag{A.9}$$

where $\mathbb{P}$ denotes the set of all poles of $\tilde{g}_\omega$. The minus sign is due to the clockwise orientation of the circles enclosing the poles. Since the exponential is holomorphic, the summands can be rewritten as $\exp(-\imath\varpi t)\,\mathrm{Res}_\varpi\{\tilde{g}_\omega\}$. Therefore, they describe harmonic oscillations with exponentially modulated amplitudes. The contribution of the integral, on the other hand, describes a transient behavior which is exponentially damped with the damping rate σ:

$$\int_{-\infty-\imath\sigma}^{\infty-\imath\sigma} \mathrm{d}\omega\ \mathrm{e}^{-\imath\omega t}\ \tilde{g}_\omega = \mathrm{e}^{-\sigma t} \int_{-\infty}^{\infty} \mathrm{d}\kappa\ \mathrm{e}^{-\imath\kappa t}\ \tilde{g}_{\kappa+\imath\sigma}\ . \tag{A.10}$$

In the asymptotic long-time limit $t \to \infty$ this integral vanishes. But due to the freedom to choose a convenient value for the yet unspecified parameter σ, it is negligible even for finite times because for every time t there is a sufficiently large $\sigma \gg 1/t$ such that the integral becomes arbitrarily small (Michno, 2014). In particular, one may chose σ to be much less than the damping rates $\Im\varpi$ of all of the damped modes, which confirms the interpretation of the integral as mere transient, negligible contribution in comparison to the harmonic oscillations. Thus, one obtains the inversion formula

$$g_t \simeq -2\pi\imath \sum_{\varpi\in\mathbb{P}} \mathrm{e}^{-\imath\varpi t}\ \mathrm{Res}_\varpi\{\tilde{g}_\omega\}\ . \tag{A.11}$$

As outlined above, the method of computing the Bromwich integral by closing the contour with an infinitely large semi-circle leads to the same result (see also Doetsch, 1974). However, in contrast to the asymptotic derivation introduced by Landau, it relies on the crucial assumption that the meromorphic continuation $\tilde{g}_\omega$ vanishes fast enough as $|\omega| \to \infty$.

A.2. Some properties of axisymmetric functions

The goal of this subsection is to prove some useful properties of axisymmetric functions that were applied in the main text. In order to formulate and prove these identities and even define what is meant by axisymmetry (or gyrotropy), it is reasonable to introduce cylindrical coordinates $p_\perp$, φ and $p_\parallel$ in momentum space:

$$\left.\begin{aligned} p_\perp &= (p_1^2+p_2^2)^{1/2} \\ \varphi &= \arctan(p_2/p_1) \\ p_\parallel &= p_3 \end{aligned}\right\} \quad\longleftrightarrow\quad \left\{\begin{aligned} p_1 &= p_\perp\cos\varphi \\ p_2 &= p_\perp\sin\varphi \\ p_3 &= p_\parallel \end{aligned}\right. \tag{A.12}$$

These definitions of the coordinates determine the transformation of partial derivatives:

$$\frac{\partial}{\partial p_1} = \frac{p_1}{p_\perp}\frac{\partial}{\partial p_\perp} - \frac{p_2}{p_\perp^2}\frac{\partial}{\partial\varphi} = \cos\varphi\,\frac{\partial}{\partial p_\perp} - \frac{\sin\varphi}{p_\perp}\frac{\partial}{\partial\varphi} \tag{A.13}$$

$$\frac{\partial}{\partial p_2} = \frac{p_2}{p_\perp}\frac{\partial}{\partial p_\perp} + \frac{p_1}{p_\perp^2}\frac{\partial}{\partial\varphi} = \sin\varphi\,\frac{\partial}{\partial p_\perp} + \frac{\cos\varphi}{p_\perp}\frac{\partial}{\partial\varphi} \tag{A.14}$$

Moreover, the surface integral over the plane perpendicular to the symmetry axis transforms according to

$$\int_{-\infty}^{\infty}\mathrm{d}p_1\int_{-\infty}^{\infty}\mathrm{d}p_2 = \int_0^{\infty}\mathrm{d}p_\perp\ p_\perp\int_0^{2\pi}\mathrm{d}\varphi\ . \tag{A.15}$$

Since the theory developed in this work is supposed to be valid within the framework of special relativity, the velocity–momentum relation $\boldsymbol{p} = \Gamma m\boldsymbol{v}$ must be employed, where Γ denotes the Lorentz factor

$$\Gamma = \sqrt{1+\boldsymbol{p}^2/m^2c^2} = \sqrt{1+(p_\perp^2+p_\parallel^2)/m^2c^2}\ . \tag{A.16}$$

The Lorentz factor is a first example of a function that possesses axial symmetry. In general, an axisymmetric function $g = g(p_\perp, p_\parallel)$ is independent of the angular variable φ by definition. An immediate consequence is that the partial derivative with respect to φ vanishes so that

$$\frac{\partial g}{\partial p_i} = \frac{p_i}{p_\perp}\frac{\partial g}{\partial p_\perp} = \frac{v_i}{v_\perp}\frac{\partial g}{\partial p_\perp} \qquad \text{for } i\in\{1,2\}. \tag{A.17}$$

It should be noted that this relation remains valid even in the relativistic case $\boldsymbol{p} = \Gamma m\boldsymbol{v}$. This will also be true for the integral properties to be shown next. From (A.15) one obtains in view of the axial symmetry of both g and Γ

$$\int_{-\infty}^{\infty}\mathrm{d}p_1\int_{-\infty}^{\infty}\mathrm{d}p_2\begin{pmatrix}v_1\\v_2\end{pmatrix}g = \int_0^{\infty}\mathrm{d}p_\perp\ \frac{p_\perp^2 g}{\Gamma m}\int_0^{2\pi}\mathrm{d}\varphi\begin{pmatrix}\cos\varphi\\\sin\varphi\end{pmatrix} = 0\,, \tag{A.18}$$

because both integrals of the trigonometric functions vanish. In the same fashion one infers

$$\int_{-\infty}^{\infty}\mathrm{d}p_1\int_{-\infty}^{\infty}\mathrm{d}p_2\ v_1v_2\,g = \int_0^{\infty}\mathrm{d}p_\perp\ \frac{p_\perp^3 g}{\Gamma^2m^2}\int_0^{2\pi}\mathrm{d}\varphi\ \cos\varphi\sin\varphi = 0\,. \tag{A.19}$$

In the next computation one makes use of the fact that the integral from 0 to 2π is the same for $\cos^2\varphi$ and $\sin^2\varphi$:

$$\begin{aligned}\int_{-\infty}^{\infty} \mathrm{d}p_1 \int_{-\infty}^{\infty} \mathrm{d}p_2\; v_1^2\, g &= \int_0^{\infty} \mathrm{d}p_\perp\; \frac{p_\perp^3 g}{\Gamma^2 m^2} \int_0^{2\pi} \mathrm{d}\varphi\; \cos^2\varphi \\ &= \int_0^{\infty} \mathrm{d}p_\perp\; \frac{p_\perp^3 g}{\Gamma^2 m^2} \int_0^{2\pi} \mathrm{d}\varphi\; \sin^2\varphi \\ &= \int_{-\infty}^{\infty} \mathrm{d}p_1 \int_{-\infty}^{\infty} \mathrm{d}p_2\; v_2^2\, g\,. \end{aligned} \tag{A.20}$$

Adding the left-hand side of this equation to both sides, dividing by 2, and applying the definition of $v_\perp$ yields

$$\int_{-\infty}^{\infty} \mathrm{d}p_1 \int_{-\infty}^{\infty} \mathrm{d}p_2\; v_1^2\, g = \frac{1}{2} \int_{-\infty}^{\infty} \mathrm{d}p_1 \int_{-\infty}^{\infty} \mathrm{d}p_2\; v_\perp^2\, g\,. \tag{A.21}$$

The integral relations (A.18)–(A.21) obtained here can be summarized in the following more convenient form:

$$\int \mathrm{d}p_1 \int \mathrm{d}p_2\; v_i\, g = 0 \qquad \text{for } i \in \{1,2\}, \tag{A.22}$$

$$\int \mathrm{d}p_1 \int \mathrm{d}p_2\; v_i v_j\, g = \frac{\delta_{ij}}{2} \int \mathrm{d}p_1 \int \mathrm{d}p_2\; v_\perp^2\, g \qquad \text{for } i,j \in \{1,2\}. \tag{A.23}$$

A.3. The contribution of a pair beam to the Maxwell tensor

In order to determine the contribution of a beam to the Maxwell tensor one must compute the integral (312). To this end, it proves useful to choose a coordinate system whose z-axis is aligned with the wavevector, $\boldsymbol{k} = k\boldsymbol{e}_z$, and to integrate by parts. Using $p_z \equiv p_3 \equiv p_\parallel$ synonymously again, equation (312) can then be rewritten in the simplified coordinate form

$$\Lambda_{\boldsymbol{k},\omega}^{\mathrm{beam},ij} = \sum_{a=e^\pm} \frac{4\pi e_a^2 n}{\omega^2} \int \mathrm{d}^3p\; v_i \left(\frac{\partial f}{\partial p_j} + \frac{k v_j}{\omega - k v_\parallel} \frac{\partial f_a}{\partial p_\parallel} \right), \tag{A.24}$$

where the beam distribution function is temporarily abbreviated with f_a. Since it equals a δ-function, it is reasonable to integrate by parts in order to make f_a itself appear in the integral rather than its partial derivatives. This must be carried out in accordance with the relativistic relation $\boldsymbol{p} = \Gamma m_a \boldsymbol{v}$ between momentum and velocity, which implies

$$\partial v_i/\partial p_j = (\delta_{ij} - v_i v_j/c^2)/\Gamma m_a\,. \tag{A.25}$$

Thus, integrating the first term by parts with respect to p_j leads to

$$\int \mathrm{d}^3p\; v_i \frac{\partial f_a}{\partial p_j} = -\int \mathrm{d}^3p\; \frac{\delta_{ij} - v_i v_j/c^2}{\Gamma m_a} f_a\,, \tag{A.26}$$

because for physical reasons the distribution function vanishes as $|\boldsymbol{p}|$ approaches infinity. The second term can be treated in the same fashion, only this time the integration by parts must be performed with respect to $p_\parallel$,

$$\begin{aligned}\int \mathrm{d}^3p\; \frac{k v_i v_j}{\omega - k v_\parallel} \frac{\partial f_a}{\partial p_\parallel} = -\int \mathrm{d}^3p\; f_a \Bigg(& \frac{\delta_{i3} k v_j + \delta_{j3} k v_i}{\Gamma m_a (\omega - k v_\parallel)} \\ & - \frac{2 k v_i v_j v_\parallel}{\Gamma m_a c^2 (\omega - k v_\parallel)} + \frac{k^2 v_i v_j (c^2 - v_\parallel^2)}{\Gamma m_a c^2 (\omega - k v_\parallel)^2} \Bigg)\,. \end{aligned} \tag{A.27}$$

If the last two equations are added as required by (A.24), some terms can be rearranged and factorized differently by purely algebraic means, yielding

$$\Lambda_{\boldsymbol{k},\omega}^{\mathrm{beam},ij} = -\sum_{a=e^\pm} \frac{4\pi e_a^2 n}{\omega^2} \int \mathrm{d}^3p\; \frac{f_a}{\Gamma m_a} \left(\delta_{ij} + \frac{\delta_{i3} k v_j + \delta_{j3} k v_i}{\omega - k v_\parallel} - \frac{v_i v_j}{c^2} \frac{\omega^2 - k^2 c^2}{(\omega - k v_\parallel)^2} \right). \tag{A.28}$$

In this form, the evaluation of the integral is an easy task considering that the beam distribution (297) is a three-dimensional δ-function. In the process one needs to distinguish between the momentum variable $\boldsymbol{p}$ and the fixed beam momentum $\boldsymbol{P}$. To this end, the notations Γ_b and $\boldsymbol{\beta} = \boldsymbol{P}/\Gamma_\mathrm{b} m_e c$ will be used to denote the Lorentz factor and the dimensionless velocity of the pair beam, respectively. Since the summation over the electron and positron contributions merely provides a factor of 2, one obtains

$$\Lambda_{\boldsymbol{k},\omega}^{\mathrm{beam},ij} = -\frac{8\pi e^2 n}{\Gamma_\mathrm{b} m_e \omega^2} \left(\delta_{ij} + \frac{kc(\delta_{i3}\beta_j + \delta_{j3}\beta_i)}{\omega - kc\beta_\parallel} - \frac{\beta_i \beta_j (\omega^2 - k^2 c^2)}{(\omega - kc\beta_\parallel)^2} \right). \tag{A.29}$$

This equation can be rewritten in a coordinate-free form. Then, it becomes valid in arbitrary coordinate systems regardless of their orientation with respect to the wavevector:

$$\hat{\Lambda}^{\text{beam}}_{\boldsymbol{k},\omega} = -\frac{2\omega_{\text{p,e}}^2}{\Gamma_{\text{b}}\omega^2}\left(\hat{\mathbf{1}} + \frac{c(\boldsymbol{k}\otimes\boldsymbol{\beta}+\boldsymbol{\beta}\otimes\boldsymbol{k})}{\omega - c\boldsymbol{k}\cdot\boldsymbol{\beta}} - \frac{(\boldsymbol{\beta}\otimes\boldsymbol{\beta})(\omega^2-k^2c^2)}{(\omega - c\boldsymbol{k}\cdot\boldsymbol{\beta})^2}\right). \quad \text{(A.30)}$$

The electron plasma frequency $\omega_{\text{p},e} = (4\pi ne^2/m_e)^{1/2}$ was used here to shorten the notation.

A.4. Criteria for the occurrence of amplified fluctuations in the IGM

In this subsection a detailed analysis is carried out investigating the question if and under which conditions the effective growth rate (367) can become positive, that is, if and when $H(y) < -1/\varepsilon s_0$ is satisfied. Remembering that the variable y is always nonnegative and that $s_0 > 0$ and $s_2 \geq 0$ by definition, a closer inspection of Eq. (367) reveals that $H(y) > 0$ for sufficiently large arguments. As a consequence, the condition for amplification can never be met in the entire spectrum. All one can look for are spectral ranges in which the effective growth rate becomes positive. Moreover, it turns out that this effect depends on the parallel velocity component $\beta_\parallel$ and that one must distinguish the cases $\beta_\parallel = 0$, $0 < \beta_\parallel \ll 1$, and $\beta_\parallel \lesssim 1$. Negative values of the parallel velocity component require no further individual consideration beyond these ranges because only the squared term $\beta_\parallel^2$ enters the effective growth rate.

Case 1. The first case, characterized by $\beta_\parallel = 0$, corresponds to wavevectors that are perpendicular to the propagation direction of the pair beam. In view of its definition, the coefficient s_2 vanishes in this case, implying

$$\left.\begin{aligned} H(y) &= \frac{(y^2 - s_1)(y^2+1)}{y^3[y^4 + (3-\zeta)y^2 + \zeta]} \\ s_1 &= \frac{\beta_\perp^2}{2-\beta_\perp^2} = \frac{\Gamma_{\text{b}}^2 - 1}{\Gamma_{\text{b}}^2 + 1} \end{aligned}\right\} \quad \text{for } \beta_\parallel = 0. \quad \text{(A.31)}$$

As the beam particles are not resting but highly relativistic, the coefficient s_1 is greater than zero. Hence, the function $H(y)$ is positive for $y > s_1^{1/2}$. For arguments below this critical value, it is negative with a pole at $y = 0$. Consequently, the criterion for a positive growth rate, $H(y) < -1/\varepsilon s_0$, can always be fulfilled for sufficiently low values of the variable y, so the occurrence of amplification in a certain spectral range is guaranteed in this case. This conclusion is supported by Fig. 8 which shows the spectrum of the effective growth rate for different values of the density ratio ε. By means of the inverse dispersion relation (308), the frequency y was substituted by the wavenumber $\kappa = k/k_c$ in the plots in order to display the more natural representation of the spectrum. But due to the monotonic relation between y and κ, the important properties of the effective growth rate such as its sign and the occurrence of its pole remain the same. The plots clearly confirm that the field fluctuations are amplified in this case, irrespective of the underlying density ratio. It also indicates that the spectral region of positive growth rates becomes broader as ε increases. Thus, the effect is more pronounced in the vicinity of the AGN where the density ratio is higher than in the further distance.

Case 2. In the case considered now, the parallel velocity component is no longer vanishing, but still very small: $0 < \beta_\parallel \ll 1$. This corresponds to an almost perpendicular orientation of the wavevector with respect to the beam direction. In view of the relativistic Lorentz factor of the beam, its perpendicular velocity component must be near unity, $\beta_\perp \simeq 1$. Under these conditions, one infers the estimates $s_1 \simeq 1$ and $s_2 \simeq \beta_\parallel^2$ for the coefficients, implying

$$H(y) \simeq \frac{[(y^2 - \frac{1}{2})^2 - (\frac{1}{4} - \beta_\parallel^2)](y^2+1)}{y[y^4 + (3-\zeta)y^2 + \zeta](y^2 + \beta_\parallel^2)^2} \qquad \text{for } 0 < \beta_\parallel \ll 1. \quad \text{(A.32)}$$

In contrast to the previous case, the pole at the origin is positive now, $H(0) = +\infty$, so the minimum value of H is finite in the current case. In view of the high order of the polynomials in both the numerator and the denominator, an exact analytical computation of this minimum value is rather involved. Since an order-of-magnitude estimate suffices for the present purpose, a different approach will be taken, here. A closer inspection of the equation above reveals that the sign of $H(y)$ is entirely determined by the term in the square brackets in the numerator because all other factors are always positive. Thus, the function becomes negative if

$$|y^2 - \tfrac{1}{2}| < \sqrt{\tfrac{1}{4} - \beta_\parallel^2} \simeq \tfrac{1}{2} - \beta_\parallel^2\,, \quad \text{(A.33)}$$

where the square root was linearized on account of the assumption $\beta_\parallel \ll 1$. Therefore, the global minimum of H must be localized inside this interval, that is, $\beta_\parallel^2 < y^2 < 1 - \beta_\parallel^2$. Apart from the square bracket in the numerator, the remaining terms in (A.32) are monotonically decreasing, so $H(y)$ reaches its minimum shortly after crossing the abscissa at $y = \beta_\parallel$. Guided by this insight, the value at $y = 2\beta_\parallel$ is treated as a reference value in order to obtain the required estimate:

$$|H(y = 2\beta_\parallel)| \simeq [50\beta_\parallel^3(4\beta_\parallel^2 + \zeta/3)]^{-1} \equiv H_0\,. \quad \text{(A.34)}$$

Fig. A.15 displays the graph of $H(y)$ in units of H_0 for different values of $\beta_\parallel$. It supports the analytic result that H enters the lower half plane at $y = \beta_\parallel$. More importantly, the plots also confirm that $-H_0$ is a suitable order-of-magnitude estimate

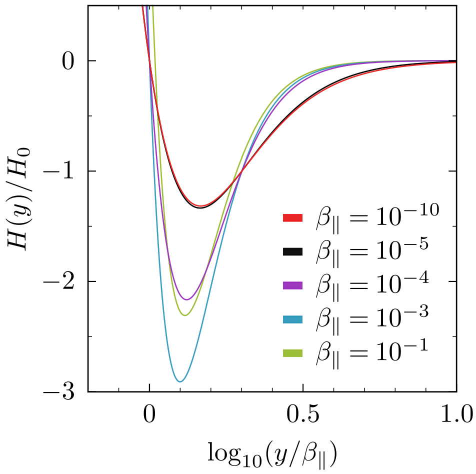

Fig. A.15. Plot of the auxiliary function $H(y)$ for different values of the beam velocity component parallel to the wavevector.

for the minimum value, indeed. If the criterion for positive growth rates, $H(y) < -1/\varepsilon s_0$, is met at all, then certainly at the minimum. Therefore, an amplification of the field fluctuations is guaranteed if $H_0 > 1/\varepsilon s_0$, that is, if

$$\frac{\Gamma_{\rm b}}{\varepsilon}\,\beta_\|^3(4\beta_\|^2+\zeta/3) < \frac{4\pi e^2 n}{25 m_e k_c^2 c^2} = 1.64\cdot 10^{-12}\,. \tag{A.35}$$

The previous results provide valuable guidelines, in which spectral region and under what conditions to look for positive values of the effective growth rate (A.32) in the corresponding plots. Since the Lorentz factor and the smallness parameter enter the criterion (369) only in terms of their quotient, it suffices to treat $\Gamma_{\rm b}$ as a fixed constant and regard ε and $\beta_\|$ as the only parameters. Consequently, the spectra exhibited in Fig. 9 are all based on a Lorentz factor of $\Gamma_{\rm b} = 10^6$, while different values of ε (left panel) or $\beta_\|$ (right panel) are considered. Most importantly, the diagrams confirm that, indeed, amplification can occur for $\beta_\| \neq 0$. In contrast to the previous case, there is no longer a pole involved as the graphs peak in a finite maximum instead. Moreover, the plots also support the above criterion: Amplification appears in all instances that fulfill the inequality. Vice versa, whenever the growth rate does not exhibit positive values (as in the two graphs displayed in the bottom row), the underlying parameters do not satisfy (369).

Case 3. The final case is characterized by $\beta_\| \lesssim 1$. An immediate consequence of this premise is the constraint $s_2 > 0$ which will be employed throughout the following argumentation. From (368) one infers that the sign of $H(y)$ is determined by the function

$$\begin{aligned} h(y) &= y^4 - s_1 y^2 + s_2 \\ &= (y^2 - s_1/2)^2 + s_2 - s_1^2/4 \end{aligned} \tag{A.36}$$

because all other factors are positive. The relevant question, therefore, is whether $h(y)$ locally becomes negative in certain regions of its domain or not. The simplest way to approach this matter is to examine the sign of the minimum value h_0. The two representations above imply

$$h_0 = \begin{cases} s_2 & \text{if } s_1 \le 0 \\ s_2 - s_1^2/4 & \text{if } s_1 \ge 0 \end{cases} = \begin{cases} +|s_2| & \text{if } s_1 \le 0 \\ +|s_2 - s_1^2/4| & \text{if } 0 < s_1 \le 2s_2^{1/2} \\ -|s_2 - s_1^2/4| & \text{if } s_1 > 2s_2^{1/2}. \end{cases} \tag{A.37}$$

All relevant cases are depicted in the left panel of Fig. A.16. According to (363)–(364), the parameters s_1 and s_2 are completely determined by the components of the dimensionless velocity, so the condition $s_1 > 2s_2^{1/2}$ can be translated into an equivalent statement in terms of $\beta_\|$ and $\beta_\perp$:

$$\beta_\perp > \beta_{\rm cr} \equiv 4\beta_\|(1-\beta_\|^2)^{1/2}/(1+\beta_\|^2)\,. \tag{A.38}$$

If this inequality is not satisfied, $H(y)$ cannot become negative, let alone obey $H(y) < -1/\varepsilon s_0$ as required for the occurrence of amplified field fluctuations. Additionally, special relativity demands that the beam velocity does not exceed the speed of light in vacuum, $\beta_\perp^2 + \beta_\|^2 \le 1$. Both constraints combined restrict the parallel velocity component to

$$\beta_\| < 2 - \sqrt{3} \approx 0.268 \tag{A.39}$$

as illustrated in the right panel of Fig. A.16. Only the shaded region within the $\beta_\|$–$\beta_\perp$-plane is compatible with $H(y) < 0$. In conclusion, an amplification of the field fluctuations is not possible in the case $\beta_\| \lesssim 1$ considered, here.

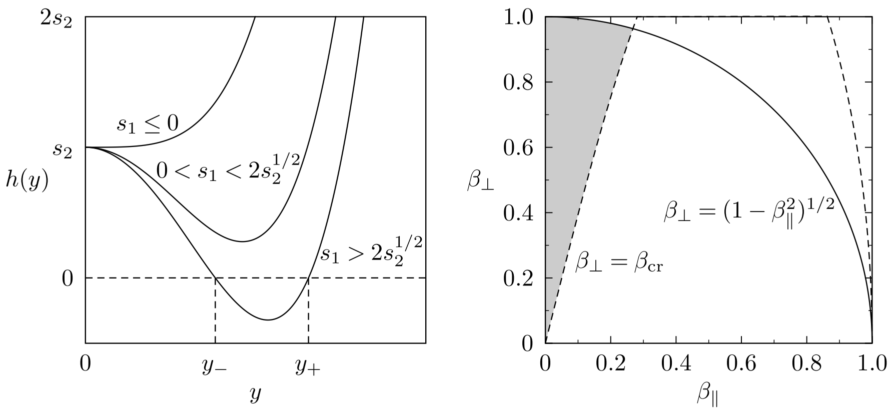

Fig. A.16. The left panel displays the function (A.36) for different values of the parameter s_1. The right panel visualizes constraints on the beam velocity components in the $\beta_\parallel$–$\beta_\perp$-plane. At most the shaded area is compatible with subluminal particle velocities and positive values of the effective growth rate.

A.5. Computation of the integral (383)

This subsection addresses the integral (383) with the purpose of deriving the simplified representation (385). More precisely, the goal here is to prove the integral relation

$$I \equiv \int_0^L \mathrm{d}s \int_0^L \mathrm{d}s'\; C(s'-s) = 2\int_0^L \mathrm{d}s\; (L-s)C(s)\,. \tag{A.40}$$

As stated in the main text, this result was already obtained by Jokipii and Lerche (1969), but they do not demonstrate the details of the computation explicitly. The only hints they give is to make use the fact that the correlation function C is an even function and that one must integrate by parts. The obvious first step is a change of variables from s' to $\xi = s - s'$. Afterwards, one can employ the symmetry of C as advised,

$$I = \int_0^L \mathrm{d}s \int_{s-L}^{s} \mathrm{d}\xi\; C(\xi) = \int_0^L \mathrm{d}s\; [H(s) - H(s-L)]\,. \tag{A.41}$$

The newly introduced function H denotes an anti-derivative of the correlation function. On account of the symmetry of its integrand it is an odd function itself,

$$H(x) = \int_0^x \mathrm{d}\xi\; C(\xi) = -H(-x)\,. \tag{A.42}$$

In the next step, the variable s is substituted by $\zeta = L - s$ in the second integrand of (A.41) and the symmetry of H is exploited:

$$I = \int_0^L \mathrm{d}s\; H(s) - \int_0^L \mathrm{d}\zeta\; H(-\zeta) = 2\int_0^L \mathrm{d}s\; H(s)\,. \tag{A.43}$$

Considering that the integrand is an anti-derivative, $H'(s) = C(s)$, it is reasonable to integrate by parts at this point:

$$I = 2sH(s)\Big|_{s=0}^{s=L} - 2\int_0^L \mathrm{d}s\; s\,H'(s) = 2L\,H(L) - 2\int_0^L \mathrm{d}s\; sC(s)\,. \tag{A.44}$$

Lastly, the auxiliary quantity $H(L)$ is expressed in terms of the defining integral again. This leads to the right-hand side of (A.40) and thus completes the proof.

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