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INVARIANT KAPPA DISTRIBUTION IN SPACE PLASMAS OUT OF EQUILIBRIUM

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ABSTRACT

Recent advances in Space Physics theory have shown the connection between non-extensive Statistical Mechanics and space plasmas by providing a theoretical basis for the empirically derived kappa distributions commonly used to describe the phase-space distribution functions of these systems. The non-equilibrium temperature and the kappa index that govern these distributions are the two independent controlling parameters of non-equilibrium systems. The significance of the kappa index is primarily given by its role in identifying the non-equilibrium stationary states and measuring their "thermodynamic distance" from thermal equilibrium, while its physical meaning is connected to the correlation between the system's particles. The classical, single stationary state at equilibrium is generalized into a whole set of different non-equilibrium stationary states labeled by the kappa index. This paper addresses certain crucial issues about the physical meaning and role of the kappa index in identifying stationary states. The origin of the emerged inconsistencies is that the kappa index is not an invariant physical quantity, but instead depends on the degrees of freedom of the system's particles. This leads in several misleading conclusions, such as (1) only large kappa index, practically infinite, can characterize the many-particle kappa distribution, and (2) the correlation between particles depends on the total number of the system's particles. Here we show that a modified kappa index, invariant for any number of degrees of freedom, can be naturally defined. Then, we develop and examine the relevant corrected formulation of many-particle multidimensional kappa distribution, and discuss the physical meaning of the invariant kappa index.

Key words: interplanetary medium – ISM: kinematics and dynamics – methods: analytical – plasmas – solar wind – Sun: heliosphere

1. INTRODUCTION

Classical Statistical Physics studies weakly interacting, low correlation systems, whose distribution function has stabilized into a Maxwell–Boltzmann distribution, characteristic of thermal equilibrium. Their phase-space distributions are stationary, namely they do not depend explicitly on time, and thus all the macroscopic thermal observables have ceased to change with time. However, thermal equilibrium is not the only possible stationary state that many systems may attain.

Space plasmas from the solar wind to planetary magnetospheres and the outer heliosphere are largely collisionless systems of particles, with long-range interactions, residing in *non-equilibrium* stationary states. These states are not Maxwellian and instead are characterized by the empirical distribution of particle velocities introduced by Vasyliūnas (1968). This is the so-called kappa distribution, parameterized by the index κ ("kappa"). Since then, kappa distributions have been utilized in numerous studies of space plasmas (e.g., ions, electrons, photons, cosmic rays, etc.) in the inner heliosphere, such as the solar wind (e.g., Chotoo et al. 2000; Mann et al. 2002; Zouganelis et al. 2004; Leubner & Vörös 2005; Marsch 2006; Yoon et al. 2006; Pierrard & Lazar 2010) and the planetary magnetospheres (e.g., Christon 1987; Mauk et al. 2004; Schippers et al. 2008; Dialynas et al. 2009), in the outer heliosphere and the inner heliosheath (e.g., Decker & Krimigis 2003; Decker et al. 2005; Livadiotis et al. 2011), and in other various plasma-related analyses (e.g., Milovanov & Zelenyi 2000; Saito et al. 2000; Leubner 2004a, 2004b; Raadu & Shafiq 2007; Hellberg et al. 2009; Livadiotis & McComas 2009, 2010a, 2010b; le Roux et al. 2010).

Boltzmann–Gibbs (BG) Statistical Mechanics has stood the test of time for describing classical equilibrium systems. However, this formalism cannot adequately describe most space plasmas, which are systems residing in non-equilibrium stationary states and characterized by stationary probability distributions of velocities different than Maxwellian. In contrast, the generalized framework of non-extensive Statistical Mechanics, based on a non-extensive formulation of entropy S_q that is parameterized by the index q (Tsallis 1988; Tsallis et al. 1998), has offered a solid theoretical basis for describing systems out of equilibrium (e.g., see Borges et al. 2002 and references therein).

The non-equilibrium generalization of the Maxwellian distribution of velocities is deduced by maximizing the Tsallis entropy under the constraints of the Canonical ensemble. The derived distribution is called q-exponential or q-Maxwellian distribution (e.g., Silva et al. 1998; Yamano 2002), which had been considered an anomalous distribution (Abe 2002) from the point of view of the standard BG exponential distribution. However, the q-exponential distributions are observed quite frequently in nature and it is now widely accepted that these distributions constitute a suitable generalization of the BG exponential distribution, rather than describing a kind of rare or anomalous behavior. An extended bibliography of applications of the q-exponential distribution can be found in Tsallis (2009a, 2009b), Livadiotis & McComas (2009), and references therein.

The origin of the kappa distribution in Tsallis Statistical Mechanics has already been examined by several authors (e.g., Milovanov & Zelenyi 2000; Leubner 2002, 2004a, 2004b; Shizgal 2007; Nieves-Chinchilla & Viñas 2008a, 2008b; Livadiotis & McComas 2009). Livadiotis & McComas (2009) showed that the q-exponential distribution coincides precisely with the kappa distribution, while the entropic index q that characterizes the Tsallis entropy is equivalent to the kappa index κ of the kappa distribution through

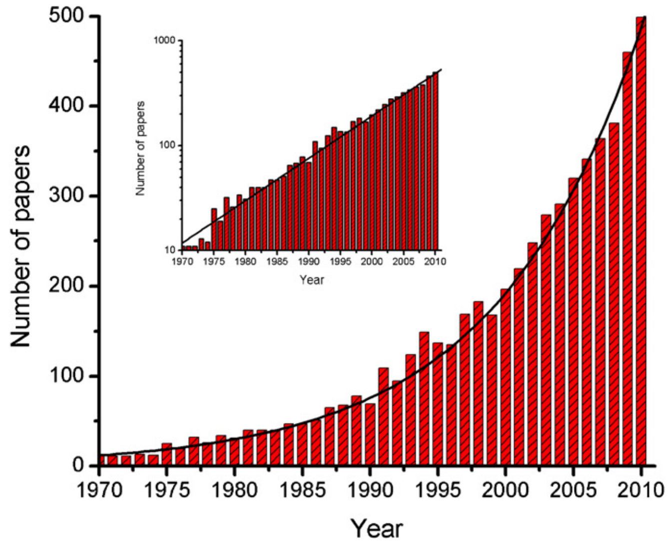

Figure 1. Distribution of the 5638 papers, cataloged in ADS in the last four decades (through 2010), that are related to kappa distributions and mention these in their title or abstract. The fit solid curve and the semi-log inset show the exponential growth of these papers.

the simple transformation

$$\kappa = \frac{1}{q-1} \Leftrightarrow q = 1 + \frac{1}{\kappa}. \tag{1}$$

Therefore, the q-Maxwellian distribution is generally expressed in terms of the q-index in the Statistical Physics community, while it is completely equivalent to the kappa distribution that is expressed in terms of the κ-index, which is more standard in the Space Physics community. The use of kappa distributions has become increasingly widespread across Space Physics and generally across the broader Physics community. In order to document this growth, we conducted a survey of ADS for papers in Physics related to kappa distributions from the last four decades, where we identified 5638 papers characterized by an exponential growth (Figure 1). This can be empirically written as $N(t) = N_0 \cdot \exp[\lambda \cdot (t - t_0)]$ with $t_0 = 1970$, $N_0 \cong 5.24 \pm 0.24$, and $\lambda \cong 0.095 \pm 0.002$, which means that the percent annual increase of publications is $(\Delta N/\Delta t)/N \sim 10\%$ (cf. Figure 1 in Livadiotis & McComas 2009 for which we had taken into account only the published papers in Astrophysics).

Prior to the connection of kappa distributions with non-extensive Statistical Mechanics, there was confusion in the Space Physics community about the physical meaning of the non-equilibrium temperature and κ-index. (Hereafter, this index that governs the kappa distribution will be referred to either as the kappa index or by its Greek letter notation "κ".) For instance, several modified versions of the kappa distribution have been suggested (e.g., Hawkins et al. 1998; Mauk et al. 2004), with all having different definitions of the kappa index. The associated temperature was involved in even more confused scenarios and there were numerous publications, defining different temperature-like parameters (cf. Table 2 in Livadiotis & McComas 2009). However, the exact definition of the non-equilibrium temperature is not something that can be simply chosen; rather, it must emerge from Statistical Mechanics. The non-extensive Statistical Mechanics not only offers a solid foundation of kappa distributions, but also provides a set of proven tools for understanding these distributions, including a consistent definition of temperature for non-equilibrium systems (e.g., see Abe 2001; Abe et al. 2001; Rama 2000; Livadiotis & McComas 2009, 2010a), and the physical meaning of the kappa index as a measure of how far a stationary state that the system resides in is from equilibrium (Livadiotis & McComas 2010a, 2010b).

In the pioneering work of Vasyliūnas (1968), the temperature was given by its kinetic definition that is through the variance of the distribution of velocities. This involves the kinetic definition of temperature given by the system's kinetic energy, as follows:

$$U = \langle \varepsilon \rangle \equiv \frac{1}{2} \cdot f \cdot k_B \cdot T, \tag{2}$$

where for $f = 3$ degrees of freedom (per particle), we have $T \equiv (2/3k_B) \cdot U$. Livadiotis & McComas (2009) showed the equality between the kinetic definition and the primary thermodynamic definition $T \equiv (\partial S_q/\partial U)^{-1} \cdot [1-(q-1) \cdot S_q/k_B]$ (also called "physical temperature", e.g., see Abe 2001; Abe et al. 2001; Rama 2000) within the framework of non-extensive Statistical Mechanics. This equivalence is sufficient to provide a consistent and unique definition of non-equilibrium temperature (see references in Livadiotis & McComas 2009). In addition, Equation (2) ensures that the internal energy is invariant under variations of the kappa index. The internal energy U is a characteristic of the system and cannot be different for different stationary states. All the stationary states must equivalently describe a system that has a particular internal energy U. Since the kappa index is a characteristic of stationary states (it differs for different states), the internal energy U must be independent of this index. Therefore, the temperature T, defined out of equilibrium, and the kappa index κ, are the *two independent and controlling parameters* of non-equilibrium systems such as space plasmas.

The significant role of the kappa index is that it identifies non-equilibrium stationary states and gives a measure of their "thermodynamic distance" from thermal equilibrium. The kappa distribution identifies a single stationary state, corresponding to a certain value of the kappa index. Different stationary states are indicated by different values of the kappa index. The temperature,

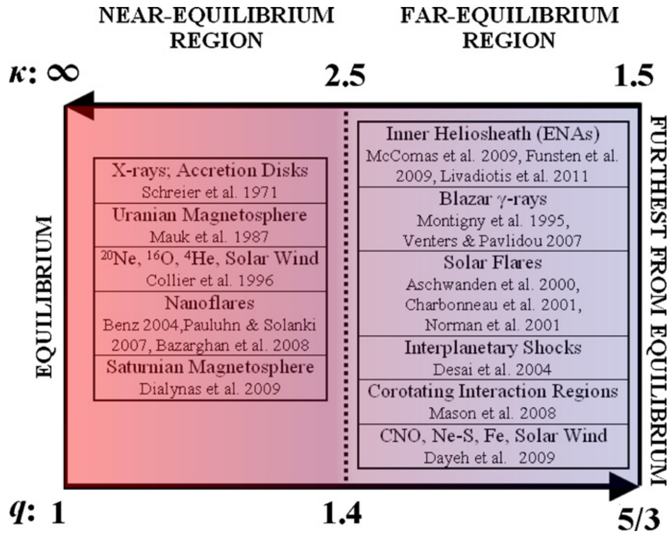

Figure 2. Spectrum of all the stationary states. In non-extensive Statistical Mechanics, the classical stationary state at equilibrium is generalized into a whole set of different non-equilibrium stationary states, labeled by the κ-index, or its equivalent, the entropic index $q = 1 + 1/\kappa$. The q-index is primarily used in non-extensive Statistical Mechanics, but the κ-index is better known in the Space Physics community.

on the other hand, is independent of the kappa index, and thus, the characterization of the stationary state of a system is irrelevant to its temperature in contrast to the kappa index. Therefore, stationary states can be arranged according to their κ-indices, so that the smaller the kappa index is, the further from equilibrium the system is. At equilibrium, the kappa index is $\kappa \to \infty$. In practice, for kappa indices larger than $\kappa \sim 10$, the kappa distribution is extremely close to a Maxwellian (quasi-equilibrium). On the other hand, for $\kappa \to 3/2$, the farthest possible stationary state from equilibrium is attained. This extreme stationary state opposite to equilibrium is called the "q-frozen" state, because a different sort of "freezing-like" procedure occurs when the kappa index decreases reaching $\kappa \to 3/2$ (Livadiotis & McComas 2010a). This extreme is not reached by the familiar method of decreasing temperature, but by the extraordinary process of decreasing the kappa index (or by increasing the q-index) that increases the "thermodynamic distance" of the system from thermal equilibrium. Therefore, the whole set of stationary states can be realized in a spectrum-like arrangement of the kappa index in the interval $\kappa \in (3/2, \infty)$.

Livadiotis & McComas (2010a) showed that the kappa indices characterizing space plasmas can be divided into two regions: the "near-equilibrium" region with indices $\kappa \in (2.5, \infty]$ and the "far-equilibrium" region with indices $\kappa \in (1.5, 2.5]$. While various space plasmas appear to have indices distributed over both regions (e.g., Dialynas et al. 2009; Livadiotis et al. 2011; for more information, see Figure 2), analytical derivations also support this near/far equilibrium separation (this issue is discussed in detail in Section 6).

Up to now we have dealt with three-dimensional velocity space, where the possible values of the kappa index are $3/2 < \kappa \leqslant \infty$ (or equivalently, of the q-index $1 \leqslant q < 5/3$), with the q-frozen state corresponding to the minimum kappa index $\kappa \to 3/2$ (or $q \to 5/3$). The dimensionality $f = 3$ of the velocity space corresponds to the 3 degrees of freedom of the velocity vector of each particle. When taking into account all the N particles in a system, we have $f = 3N$ microscopically independent kinetic degrees of freedom that are involved in the N-particle distribution function, or more generally, $f = D \cdot N$, when we have D degrees per particle. On the other hand, starting from the N-particle distribution we can easily arrive at the well-known picture of the one-particle distribution. Indeed, applying multiple integrations of the N-particle distribution function over $D \cdot (N-1)$ degrees of freedom (which correspond to $(N-1)$ particles), we end up with the description of one-particle D-dimensional distribution function. Nevertheless, the concept of a one-particle distribution function makes sense when the particles that constitute the system can be considered as weakly interacting and having low correlation. This is true for many physically interesting cases of systems residing at equilibrium. However, for non-equilibrium systems, a great deal of information is lost when getting from an N-particle to a one-particle description.

The correlation of the degrees of freedom is an issue interwoven with the form of the kappa distribution, and specifically, with the kappa index. When the degrees of freedom are uncorrelated, then the joint probability distribution $P(u_1, u_2, \ldots, u_f)$ is factorized into the marginal probabilities $P(u_i)$ of all the degrees of freedom $i = 1, \ldots, f$, i.e., $P(u_1, u_2, \ldots, u_f) = \prod_{i=1}^{f} P(u_i)$ (and vice versa). For the kappa distribution, the factorization holds only for $\kappa \to \infty$ (thermal equilibrium), that is, for the exponential form of Maxwellian distribution. The kappa distribution for $\kappa < \infty$ is not factorized and, therefore, correlation between the particles is naturally inferred by the non-extensitivity of Statistical Mechanics (see also Abe 1999).

The minimum value of the kappa index corresponds to a characteristic universal behavior of the probability distribution that is independent of the degrees of freedom, f, of the system. Namely, the distribution function of the velocity magnitude or the energy always has the same power law, i.e., $\sim u^{-3}$ or $\sim \varepsilon^{-2}$, respectively, which is independent of the dimensionality, f. Therefore, the minimum value of the kappa index refers always to the same state, the q-frozen state. The q-frozen state is characterized by the kappa index $\kappa \to 3/2$ in the simple case of the one-particle distribution and $f = 3$. For arbitrary degrees of freedom f, the farthest stationary state is given by $\kappa \to f/2$ (Livadiotis & McComas 2010a, 2010b). Therefore, the q-frozen state's kappa index is f-dependent, i.e., $\kappa_{\mathrm{Min}}(f) = f/2$.

Then, the following crucial issues are questioning the consistency and the physical meaning of the kappa index (or equivalently the q-index), in the form in which it was defined and utilized since now.

Table 1
Number of Solar Wind Particles for Various Characteristic Length Scales

Description	Length Scale	Number of Particles
Coulomb-free path	0.1–10 AU	10^{37}–10^{43}
Alfvén waves	(0.3–1) $\times\ 10^8$ m	10^{29}–10^{31}
Ion inertial length	10^5 m	10^{22}
Ion gyroradius	5×10^4 m	10^{20}
Debye length	10 m	10^{10}

Note. The number of particles refers to a typical solar wind density of $n \sim 4$ cm^{-3} (in the inner heliosphere).

1. How can each value of the kappa index consistently identify a specific stationary state, if the q-frozen state is not invariant, but dependent on the degrees of freedom? For example, when $f = 3$, the kappa index $\kappa = 1.5$ identifies the q-frozen state, but when the degrees of freedom are restricted to $f = 2$ (e.g., in the presence of an external magnetic field) the same kappa index $\kappa = 1.5$ identifies a certain state in the far-equilibrium region, different from the q-frozen state which is identified in this dimensionality by $\kappa = 1$.
2. The one-particle distribution constitutes the simplest way to study the phase-space distribution of particles, but we can also use the two-particle distribution, or any other number of particles up to the entire N-particle distribution. For the latter, the number of the microscopically independent degrees of freedom is $f = D \cdot N$, and thus the minimum value of the kappa index, which corresponds to the kappa index of the q-frozen state, is $\kappa \to DN/2$. Taking into account the huge number of correlated particles that can be present in space plasmas (e.g., see Table 1 for examples of N in solar wind), we have $DN/2 \gg 10$, and thus, the minimum value of the kappa index is practically equal to its maximum value, i.e., $\kappa \to \infty$. Remarkably, for a large number of particles N, the q-frozen state practically coincides with equilibrium, and the only possible stationary state should be equilibrium, precluding the commonly observed non-equilibrium states in space plasmas.

   Even worse, the minimum kappa index depends on the number of particles chosen in the study of a system. For example, for the n-particle distribution, with n be any subset of the system's particles $n = 1, \ldots, N$, and for $D = 3$, we have the minimum kappa index to be given by $\kappa \to 3n/2$. However, how can the kappa index that characterizes a specific state (e.g., the q-frozen state) be dependent on our mathematical choice of the number of particles n included in the distribution function?
3. As we will show in this paper, the misleading identification of the kappa index is not referred to the q-frozen state alone. On the contrary, the same identification problem arises for any other stationary state with a kappa index in the interval $f/2 < \kappa \leqslant \infty$. The origin of this is that the kappa index is not an invariant physical quantity, but it does depend on the degrees of freedom f. (Note that the kappa index should not consistently identify the stationary states only for a given dimensionality f, e.g., for $f = 3$. This would be meaningless, since the number of correlated degrees of freedom may vary between several space plasmas or it can even be variable within a certain space plasma.)

As we have shown, the kappa index is of great importance for understanding, classifying, and studying the non-equilibrium stationary states. However, the kappa index must be invariant, namely, independent of the degrees of freedom and the number of particles. The primary purpose of this paper is to resolve all the above inconsistencies and elucidate the physical meaning of the kappa index. In Section 2, we review the basic preliminary elements necessary for accomplishing this task for readers who may not be fully versed in non-extensive Statistical Mechanics. The dependence of the kappa index on the degrees of freedom is shown in Section 3. The exact relation between the kappa index and the degrees of freedom is developed in Section 4, where a modified kappa index that is invariant for any number of degrees of freedom is naturally defined. The f-dimensional velocity distribution is found either in terms of the invariant or of the old, f-dependent, kappa index. In Section 5, we find the distribution in terms of the energy of each particle, the total energy of all the particles, and the (average) energy per particle; the flux is also calculated. In Section 6, we discuss the physical meaning of the invariant kappa index, namely, (1) its role in identifying the non-equilibrium stationary states and constructing the whole arrangement of stationary states that is called the κ-spectrum; (2) its connection with the correlation between two different degrees of freedom; (3) its connection to a suitable measure of the thermodynamic distance of stationary states from thermal equilibrium; and (4) its role in characterizing a new kind of freezing (when the kappa index decreases) with the particles decelerating to near zero velocities. Finally, Section 7 summarizes the conclusions. Note that Appendix A calculates the necessary integrations in order to prove the dependence of the kappa index on the degrees of freedom f, and Appendix B calculates the statistical moments of the invariant N-particle D-dimensional kappa distribution that are associated with the correlation between two degrees of freedom or two particles. All the equations that involve the multidimensional kappa distribution and the N-particle D-dimensional kappa distribution are given in Table 2.

2. PRELIMINARIES ON NON-EXTENSIVE STATISTICAL MECHANICS

For the continuous phase space of a system with probability distribution $P(\vec{u})$, the entropy is typically written in terms of a velocity scale δu. This holds either for the classical BG entropy

$$S_1 = -\int_{-\infty}^{\infty} \ln[\delta u^3 P(\vec{u})] \cdot P(\vec{u})\, d^3\vec{u}, \tag{3a}$$

or the Tsallis entropy, which is parameterized by the q-index,

$$S_q = \frac{1 - \left\{\int_{-\infty}^{\infty} [\delta u^3 P(\vec{u})]^{1/q-1} \cdot P(\vec{u})\, d^3\vec{u}\right\}^{-q}}{q-1}, \tag{3b}$$

where we denoted the BG entropy by the subscript "1", meaning the classical limit of $q \to 1$ where the Tsallis entropy S_q recovers the BG entropy S_1.

In addition, we can express the entropy in terms of the kappa index $\kappa \equiv 1/(q-1)$, which is more popular in Space Physics, where the BG entropy is recovered for $\kappa \to \infty$,

$$S_\kappa = \kappa - \kappa \cdot \left\{\int_{-\infty}^{\infty} [\delta u^3 P(\vec{u})]^{-1/(\kappa+1)} \cdot P(\vec{u})\, d^3\vec{u}\right\}^{-1-1/\kappa}. \tag{3c}$$

The entropy formula in Equation (3b) appears to be different from the original Tsallis entropy (e.g., see Tsallis 1988, 2009). The reason is that the probability distribution $P(\vec{u})$ that is involved in Equations (3a)–(3c) represents the so-called escort probability, while the original equations of entropy involve the ordinary probability $p(\vec{u})$. Escort and ordinary distributions are related to each other by the relations

$$P(\vec{u}) = p(\vec{u})^q \bigg/ \int_{-\infty}^{\infty} p(\vec{u})^q d^3\vec{u} \Leftrightarrow p(\vec{u}) = P(\vec{u})^{1/q} \bigg/ \int_{-\infty}^{\infty} P(\vec{u})^{1/q} d^3\vec{u}. \tag{4}$$

The famous Gibbs' path of entropy maximization involves maximizing the BG entropy under the Canonical ensemble's constraints and deriving the canonical probability distribution that is the Boltzmannian distribution of energy. Then, by substituting the kinetic energy, one finds the Maxwellian distribution of velocities that involves the familiar Gaussian exponential

$$P(\vec{u}) \sim \exp\left\{-\frac{\vec{u}^2}{\theta^2}\right\}. \tag{5a}$$

Similarly, maximizing the Tsallis generalized formulation of entropy as given by Equation (3c) and under the same Canonical ensemble constraints (Tsallis et al. 1998), the Tsallis-like Canonical probability distribution is derived (cf. Appendix B in Livadiotis & McComas 2009). This is the kappa distribution, expressed in terms of the kappa index κ,

$$P(\vec{u}) \sim \left[1 + \frac{1}{\kappa - \frac{3}{2}} \cdot \frac{\vec{u}^2}{\theta^2}\right]^{-\kappa-1}, \tag{5b}$$

(where we assumed zero mean velocity in both the distributions given in Equations (5a) and (5b)). When Equation (5b) is expressed in terms of the q-index (according to Equation (1)) that is better known in Statistical Physics community, it is transformed into the q-Maxwellian distribution, which is given by the so-called q-exponential distribution,

$$p(\varepsilon; T; q) \sim \exp_q\left(-\frac{\varepsilon - U}{k_{\mathrm{B}} T}\right), \tag{6}$$

and replaces the classical Boltzmann exponential distribution of energy. Here the q-exponential function is denoted by $\exp_q(x) \equiv [1 + (1-q) \cdot x]_+^{\frac{1}{1-q}}$, while the subscript "+" means the operation $[x]_+ = x$, if $x \geqslant 0$ and $[x]_+ = 0$, if $x \leqslant 0$, in accordance with the Tsallis cut-off condition. In our case, the quantity inside the q-exponential is given by $y \equiv 1 + (q-1) \cdot (\varepsilon - U)/(k_{\mathrm{B}} T)$. For f degrees of freedom, we have $U/(k_{\mathrm{B}} T) = f/2$; then, the minimum value of y is obtained for the smaller energy, which is $\varepsilon = 0$, and the maximum value of the q-index (for $q \geqslant 1$), which is $q = 1 + 2/f$ (Livadiotis & McComas 2010a); this minimum value is found to be $y = 0$. Therefore, y is always non-negative, and hereafter, we abandon the use of the subscript "+".

The formalism of Tsallis Statistical Mechanics is interwoven with the fundamental concept of escort probabilities (Tsallis 1999, 2009b; Gell-Mann & Tsallis 2004). The escort probability distribution $P(\varepsilon; T; q)$ is constructed from the ordinary probability distribution, $p(\varepsilon; T; q)$, as $P(\varepsilon; T; q) \sim p(\varepsilon; T; q)^q$ (Beck & Schlogl 1993). Hence,

$$P(\varepsilon; T; q) = \frac{p(\varepsilon; T; q)^q}{\int_0^\infty p(\varepsilon; T; q)^q g_{\mathrm{E}}(\varepsilon)\, d\varepsilon}, \tag{7}$$

where $g_{\mathrm{E}}(\varepsilon) \sim \varepsilon^{\frac{f}{2}-1}$ is the density of energy states.

In contrast to the ordinary probability distribution, the escort probability distribution has a fundamental role because the expectation values are expressed in terms of the escort probability (called escort-expectation values or escort-mean values; Tsallis et al. 1998; Tsallis 1999, 2009b; Gell-Mann & Tsallis 2004; Naudts 2011). Thus, the physical meaning of the statistical moments is obtained only by the escort probability distribution (denoted by the symbol $\langle\rangle_q$) (e.g., see Prato and Tsallis 1999). Following Tsallis, the escort-mean of a function of energy, $f(\varepsilon)$, is given by

$$\langle f(\varepsilon)\rangle_q = \frac{\int_0^\infty f(\varepsilon) P(\varepsilon; T; q) g_{\mathrm{E}}(\varepsilon) d\varepsilon}{\int_0^\infty P(\varepsilon; T; q) g_{\mathrm{E}}(\varepsilon) d\varepsilon} = \frac{\int_0^\infty f(\varepsilon) p(\varepsilon; T; q)^q g_{\mathrm{E}}(\varepsilon) d\varepsilon}{\int_0^\infty p(\varepsilon; T; q)^q g_{\mathrm{E}}(\varepsilon) d\varepsilon}. \tag{8}$$

Therefore, the escort probability distribution characterizes a system after its relaxation into a stationary state out of equilibrium (Gell-Mann & Tsallis 2004). Ignoring for now the relevant normalization, the escort probability distribution is written as

$$P(\varepsilon; T; q) \sim \left[\exp_q\left(-\frac{\varepsilon - U}{k_B T}\right)\right]^q, \tag{9}$$

where

$$\left[\exp_q\left(-\frac{\varepsilon - U}{k_B T}\right)\right]^q = \left[1 + (q-1)\cdot\frac{\varepsilon - U}{k_B T}\right]^{\frac{q}{1-q}} \sim \left(1 + \frac{1}{\frac{1}{q-1} - \frac{U}{k_B T}}\cdot\frac{\varepsilon}{k_B T}\right)^{-\frac{1}{q-1}-1}.$$

Given the equipartition of internal energy to all the f degrees of freedom, i.e., $U = (f/2)\cdot k_B\, T$, we have

$$P(\varepsilon; T; q) \sim \left(1 + \frac{1}{\frac{1}{q-1} - \frac{f}{2}}\cdot\frac{\varepsilon}{k_B T}\right)^{-\frac{1}{q-1}-1}, \tag{10}$$

or, after the transformation (1),

$$P(\varepsilon; T; \kappa) \sim \left(1 + \frac{1}{\kappa - \frac{f}{2}}\cdot\frac{\varepsilon}{k_B T}\right)^{-\kappa-1}. \tag{11}$$

When the energy is given by the kinetic energy in the three-dimensional velocity space, i.e., $\varepsilon = \frac{1}{2}\mu\vec{u}^2$, we obtain the so-called kappa distribution (Livadiotis & McComas 2009),

$$P(\vec{u}; \theta; \kappa) = P(u_x, u_y, u_z; \theta; \kappa) = \pi^{-3/2}\cdot \mathrm{A}(\kappa)\cdot\theta^{-3}\cdot\left(1 + \frac{1}{\kappa - \frac{3}{2}}\cdot\frac{\vec{u}^2}{\theta^2}\right)^{-\kappa-1}, \tag{12a}$$

$$\mathrm{A}(\kappa) \equiv \left(\kappa - \frac{3}{2}\right)^{-3/2}\cdot\frac{\Gamma(\kappa+1)}{\Gamma(\kappa - \frac{1}{2})}, \tag{12b}$$

with $\mathrm{A}(\kappa \to \infty) = 1$ (at equilibrium).

For an f-dimensional velocity space, we set $\vec{u}_f^2 \equiv \sum_{i=1}^{f} u_i^2$ and thus have

$$P(\vec{u}_f; \theta; \kappa; f) = \pi^{-\frac{f}{2}}\cdot\left(\kappa - \frac{f}{2}\right)^{-\frac{f}{2}}\cdot\frac{\Gamma(\kappa+1)}{\Gamma\left(\kappa - \frac{f}{2} + 1\right)}\cdot\theta^{-f}\cdot\left(1 + \frac{1}{\kappa - \frac{f}{2}}\cdot\frac{\vec{u}_f^2}{\theta^2}\right)^{-\kappa-1}. \tag{12c}$$

However, as we will see in the next section, this equation needs reconsideration and revision. The reason is that the involved kappa index κ is not independent of the degrees of freedom f. Therefore, κ does not constitute an independent quantity, capable of identifying non-equilibrium stationary states and measuring their thermodynamic distance from thermal equilibrium.

3. DEPENDENCE OF THE KAPPA INDEX ON THE DEGREES OF FREEDOM

Consider the three-dimensional probability distribution as shown in Equation (12a). Its integration over the three-dimensional velocity space gives unity (normalization)

$$1 = \int_{-\infty}^{\infty} P(u_x, u_y, u_z; \theta; \kappa)\, du_x du_y du_z, \tag{13a}$$

while a single and double integration would give the two- and one-dimensional distributions, respectively,

$$P(u_x, u_y; \theta; \kappa) = \int P(u_x, u_y, u_z; \theta; \kappa)\, du_z, \tag{13b}$$

$$P(u_x; \theta; \kappa) = \int P(u_x, u_y, u_z; \theta; \kappa)\, du_y du_z. \tag{13c}$$

The one-dimensional distribution of Equation (13c) is found to be

$$P_{3\to1}(u_x; \theta; \kappa) = \pi^{-\frac{1}{2}}\cdot\left(\kappa - \frac{3}{2}\right)^{-\frac{1}{2}}\cdot\frac{\Gamma(\kappa)}{\Gamma\left(\kappa - \frac{1}{2}\right)}\cdot\theta^{-1}\cdot\left(1 + \frac{1}{\kappa - \frac{3}{2}}\cdot\frac{u_x^2}{\theta^2}\right)^{-\kappa}, \tag{14}$$

(where the subscript $3 \to 1$ indicates that the one-dimensional distribution has been derived from the double integration of the three-dimensional distribution).

On the other hand, the one-dimensional kappa distribution can be derived directly, simply by substituting $f = 1$ in Equation (12c), that is,

$$P_{f=1}(u_x;\theta;\kappa) = \pi^{-\frac{1}{2}} \cdot \left(\kappa - \frac{1}{2}\right)^{-\frac{1}{2}} \cdot \frac{\Gamma(\kappa+1)}{\Gamma\left(\kappa-\frac{1}{2}\right)} \cdot \theta^{-1} \cdot \left(1 + \frac{1}{\kappa-\frac{1}{2}} \cdot \frac{u_x^2}{\theta^2}\right)^{-\kappa-1}, \tag{15}$$

(where the subscript $f = 1$ indicates that the distribution has been derived directly for the one-dimensional case). Then, we have two different representations for the one-dimensional distribution. Namely,

$$P_{3\to1}(u_x;\theta;\kappa) \sim \left(1 + \frac{1}{\kappa-\frac{3}{2}} \cdot \frac{u_x^2}{\theta^2}\right)^{-\kappa} \overset{?}{\longleftrightarrow} P_{f=1}(u_x;\theta;\kappa) \sim \left(1 + \frac{1}{\kappa-\frac{1}{2}} \cdot \frac{u_x^2}{\theta^2}\right)^{-\kappa-1} \tag{16a}$$

(ignoring the normalization constants for simplicity). The obvious question is why we have different results in Equations (14) and (15). Should not these two distributions be identical? How can this be possible?

It can easily be seen that the two distributions can be indeed identical when the kappa index is a function of the system's dimensionality. Namely, if κ_1, κ_3 are the kappa indices parameterizing the one- and three-dimensional kappa distributions, respectively, then relation (16a) can be written as

$$P_{3\to1}(u_x;\theta;\kappa_3) \sim \left(1 + \frac{1}{\kappa_3-\frac{3}{2}} \cdot \frac{u_x^2}{\theta^2}\right)^{-\kappa_3} \overset{?}{\longleftrightarrow} P_{f=1}(u_x;\theta;\kappa_1) \sim \left(1 + \frac{1}{\kappa_1-\frac{1}{2}} \cdot \frac{u_x^2}{\theta^2}\right)^{-\kappa_1-1}, \tag{16b}$$

where we see that the two distributions are identical if and only if $\kappa_3 = \kappa_1 + 1$.

Not surprisingly, the same problem arises when we derive the two-dimensional kappa distribution in both ways, i.e., directly from Equation (12c), which is $P_{f=2}(u_x, u_y;\theta;\kappa_2)$ with κ_2 indicating the two-dimensional kappa index, and by integrating the three-dimensional kappa distribution (13b), $P_{3\to2}(u_x, u_y;\theta;\kappa_3)$. Then, similar to Equation (16b), we have

$$P_{3\to2}(u_x,u_y;\theta;\kappa_3) \sim \left(1 + \frac{1}{\kappa_3-\frac{3}{2}} \cdot \frac{u_x^2+u_y^2}{\theta^2}\right)^{-\kappa_3-\frac{1}{2}} \overset{?}{\longleftrightarrow} P_{f=2}(u_x,u_y;\theta;\kappa_2) \sim \left(1 + \frac{1}{\kappa_2-1} \cdot \frac{u_x^2+u_y^2}{\theta^2}\right)^{-\kappa_2-1}, \tag{16c}$$

where the two distributions are identical if and only if $\kappa_3 = \kappa_2 + \frac{1}{2}$.

It is apparent that the kappa index depends on the degrees of freedom f. This specific dependence seems to be $\kappa_3 = \kappa_f + (3 - f)/2$; then for $f = 1$ and 2 we conclude in the previous relations of $\kappa_3 = \kappa_1 + \frac{2}{2}$ and $\kappa_3 = \kappa_2 + \frac{1}{2}$. Thus, the quantity $\kappa_3 - \frac{3}{2} = \kappa_f - \frac{f}{2}$ appears to be invariant; namely, in contrast to the kappa index κ_f itself, the difference $\kappa_f - \frac{f}{2}$ is independent of the degrees of freedom f. In the next section, we prove the generality of $\kappa_f - \frac{f}{2} =$ constant.

4. DERIVATION OF THE INVARIANT KAPPA INDEX

Having shown that the kappa index κ is generally dependent on the degrees of freedom f, and thus better indicated by κ_f, the f-dimensional kappa distribution (12c) of the corresponding velocity $\vec{u}_f \equiv (u_1, \ldots, u_f)$ is written as

$$P(\vec{u}_f;\theta;\kappa_f;f) = \pi^{-\frac{f}{2}} \cdot \left(\kappa_f - \frac{f}{2}\right)^{-\frac{f}{2}} \cdot \frac{\Gamma(\kappa_f+1)}{\Gamma(\kappa_f-\frac{f}{2}+1)} \cdot \theta^{-f} \cdot \left(1 + \frac{1}{\kappa_f-\frac{f}{2}} \cdot \frac{\vec{u}_f^2}{\theta^2}\right)^{-\kappa_f-1}, \tag{17}$$

where $\vec{u}_f^2 = \sum_{i=1}^{f} u_i^2$, and the subscript f shows the dimensionality of the velocity vector.

The one-dimensional integration of the f-dimensional distribution $P(\vec{u}_f;\theta;\kappa_f;f)$, e.g., over the component u_f, would give a distribution that must be identical to the $(f-1)$-dimensional distribution $P(\vec{u}_{f-1};\theta;\kappa_{f-1};f-1)$. In Appendix A, we show that this integration gives $P(\vec{u}_{f-1};\theta;\kappa_f - 1/2;f-1)$, i.e.,

$$P(\vec{u}_{f-1};\theta;\kappa_{f-1};f-1) \equiv \int P(\vec{u}_f;\theta;\kappa_f;f)\,du_f = P\left(\vec{u}_{f-1};\theta;\kappa_f - \frac{1}{2};f-1\right). \tag{18}$$

Hence, we have $\kappa_{f-1} = \kappa_f - \frac{1}{2}$. In the same way, the two, three, or $\tilde{f}$ multiple integrations of the f-dimensional distribution $P(\vec{u}_f;\theta;\kappa_f;f)$, e.g., over the $\tilde{f}$ components $u_f, u_{f-1}, u_{f-2}, \ldots, u_{f-\tilde{f}+1}$, give

$$P(\vec{u}_{f-2};\theta;\kappa_{f-2};f-2) \equiv \int P(\vec{u}_f;\theta;\kappa_f;f)\,du_{f-1}du_f = P(\vec{u}_{f-2};\theta;\kappa_f - 1;f-2), \tag{19a}$$

$$P(\vec{u}_{f-3};\theta;\kappa_{f-3};f-3) \equiv \int P(\vec{u}_f;\theta;\kappa_f;f)\,du_{f-2}du_{f-1}du_f = P\left(\vec{u}_{f-3};\theta;\kappa_f - \frac{3}{2};f-3\right), \tag{19b}$$

$$P(\vec{u}_{f-\tilde{f}};\theta;\kappa_{f-\tilde{f}};f-\tilde{f}) \equiv \int P(\vec{u}_f;\theta;\kappa_f;f)\,du_{f-\tilde{f}+1}\cdots du_{f-1}du_f = P\left(\vec{u}_{f-\tilde{f}};\theta;\kappa_f-\frac{\tilde{f}}{2};f-\tilde{f}\right). \tag{19c}$$

The corresponding kappa index relations are $\kappa_{f-2}=\kappa_f-\frac{2}{2}$, $\kappa_{f-3}=\kappa_f-\frac{3}{2}$, and generally, $\kappa_{f-\tilde{f}}=\kappa_f-\frac{\tilde{f}}{2}$. Then, by substituting $f\to f_1$ and $f-\tilde{f}\to f_2$, we obtain $\kappa_{f_2}=\kappa_{f_1}-(f_1-f_2)/2$, or

$$\kappa_{f_1}-\frac{f_1}{2}=\kappa_{f_2}-\frac{f_2}{2},\quad \forall f_1,\ f_2. \tag{20}$$

Apparently, the quantity $\kappa_f-\frac{f}{2}$ is invariant under variations of the degrees of freedom, f; in other words, $\kappa_f-\frac{f}{2}$ is independent of f. In fact, Equation (20) shows that any kappa index κ_f of an arbitrary dimensionality f can be related to the kappa index of a fixed dimensionality such as the usual kappa index κ_3 for $f=3$, or the kappa index κ_0 for $f=0$ that equals exactly the difference $\kappa_f-\frac{f}{2}$, i.e.,

$$\kappa_0\equiv\kappa_f-\frac{f}{2}. \tag{21}$$

Therefore, the f-dimensional kappa distribution can be written either in terms of the dependent κ_f index,

$$P(u_1,\ldots,u_f;\theta;\kappa_f;f)=\pi^{-\frac{f}{2}}\cdot\left(\kappa_f-\frac{f}{2}\right)^{-\frac{f}{2}}\cdot\frac{\Gamma(\kappa_f+1)}{\Gamma\left(\kappa_f+1-\frac{f}{2}\right)}\cdot\theta^{-f}\cdot\left(1+\frac{1}{\kappa_f-\frac{f}{2}}\cdot\frac{1}{\theta^2}\sum_{i=1}^{f}u_i^2\right)^{-\kappa_f-1}, \tag{22a}$$

or in an invariant form, i.e., in terms of the three-dimensional kappa index κ_3, i.e.,

$$P(u_1,\ldots,u_f;\theta;\kappa_3;f)=\pi^{-\frac{f}{2}}\cdot\left(\kappa_3-\frac{3}{2}\right)^{-\frac{f}{2}}\cdot\frac{\Gamma\left(\kappa_3-\frac{1}{2}+\frac{f}{2}\right)}{\Gamma\left(\kappa_3-\frac{1}{2}\right)}\cdot\theta^{-f}\cdot\left(1+\frac{1}{\kappa_3-\frac{3}{2}}\cdot\frac{1}{\theta^2}\sum_{i=1}^{f}u_i^2\right)^{-\kappa_3+\frac{1}{2}-\frac{f}{2}}, \tag{22b}$$

(this coincides with the kappa index κ used since today), or in terms of the invariant difference $\kappa_0\equiv\kappa_f-\frac{f}{2}$, i.e.,

$$P(u_1,\ldots,u_f;\theta;\kappa_0;f)=\pi^{-\frac{f}{2}}\cdot\kappa_0^{-\frac{f}{2}}\cdot\frac{\Gamma\left(\kappa_0+1+\frac{f}{2}\right)}{\Gamma(\kappa_0+1)}\cdot\theta^{-f}\cdot\left(1+\frac{1}{\kappa_0}\cdot\frac{1}{\theta^2}\sum_{i=1}^{f}u_i^2\right)^{-\kappa_0-1-\frac{f}{2}}. \tag{22c}$$

Note that the above invariant relations (Equations (22b), (22c)) are very useful when dealing with N-particle distributions. For example, for two interacting particles A and B with three degrees of freedom each, we have six degrees of freedom, i.e., $u_1=u_{x_A}, u_2=u_{y_A}, u_3=u_{z_A}$, and $u_4=u_{x_B}, u_5=u_{y_B}, u_6=u_{z_B}$. Therefore, the N-particle kappa distribution for $N=2$, which is $f=6$, can be used in its invariant forms which are given either in terms of κ_3, or κ_0, i.e.,

$$P(u_1,\ldots,u_6;\theta;\kappa_3)=\pi^{-3}\cdot\left(\kappa_3-\frac{3}{2}\right)^{-3}\left(\kappa_3^2-\frac{1}{4}\right)\left(\kappa_3+\frac{3}{2}\right)\cdot\theta^{-6}\cdot\left(1+\frac{1}{\kappa_3-\frac{3}{2}}\cdot\frac{1}{\theta^2}\sum_{i=1}^{6}u_i^2\right)^{-\kappa_3+\frac{5}{2}}, \tag{23a}$$

$$P(u_1,\ldots,u_6;\theta;\kappa_0)=\pi^{-3}\cdot\kappa_0^{-3}(\kappa_0+1)(\kappa_0+2)(\kappa_0+3)\cdot\theta^{-6}\cdot\left(1+\frac{1}{\kappa_0}\cdot\frac{1}{\theta^2}\sum_{i=1}^{6}u_i^2\right)^{-\kappa_0-4}. \tag{23b}$$

Throughout the remainder of this paper, we use exclusively the modified kappa index given by the invariant kappa $\kappa_0\equiv\kappa_f-\frac{f}{2}$. One can easily convert any relation in terms of the three-dimensional kappa index κ_3, using $\kappa_0=\kappa_3-\frac{3}{2}$.

5. ENERGY DISTRIBUTION AND FLUX

Equations (22a)–(22c) address the probability density $P(u_1,\ldots,u_f)$ and the infinitesimal probability $P(u_1,\ldots,u_f)du_1\cdots du_f$ with normalization $P(u_1,\ldots,u_f)\int du_1\cdots\int du_f=1$. These equations can be written in terms of the relevant energy $\varepsilon_i=\frac{1}{2}\mu\cdot u_i^2$ of each degree of freedom, $i=1,\ldots,f$, so that the infinitesimal probability of energies $P(\varepsilon_1,\ldots,\varepsilon_f)$ is defined by

$$P(\varepsilon_1,\ldots,\varepsilon_f)\,d\varepsilon_1\cdots d\varepsilon_f=P(u_1,\ldots,u_f)\frac{du_1}{d\varepsilon_1}\cdots\frac{du_f}{d\varepsilon_f}d\varepsilon_1\cdots d\varepsilon_f,$$

with normalization $P(\varepsilon_1,\ldots,\varepsilon_f)\int d\varepsilon_1\cdots\int d\varepsilon_f=1$. The relevant energy distributions are given in terms of the dependent κ_f index, and the invariant three- and zero- dimensional kappa indices, κ_3 and κ_0, respectively,

$$P(\varepsilon_1,\ldots,\varepsilon_f;T;\kappa_f;f)=(4\pi)^{-\frac{f}{2}}\cdot\left(\kappa_f-\frac{f}{2}\right)^{-\frac{f}{2}}\cdot\frac{\Gamma(\kappa_f+1)}{\Gamma\left(\kappa_f+1-\frac{f}{2}\right)}\cdot(k_BT)^{-\frac{f}{2}}\cdot\left(1+\frac{1}{\kappa_f-\frac{f}{2}}\cdot\frac{1}{k_BT}\sum_{i=1}^{f}\varepsilon_i\right)^{-\kappa_f-1}\cdot\prod_{i=1}^{f}\varepsilon_i^{-\frac{1}{2}}, \tag{24a}$$

$$P(\varepsilon_1, \ldots, \varepsilon_f; T; \kappa_3; f) = (4\pi)^{-\frac{f}{2}} \cdot \left(\kappa_3 - \frac{3}{2}\right)^{-\frac{f}{2}} \cdot \frac{\Gamma\left(\kappa_3 - \frac{1}{2} + \frac{f}{2}\right)}{\Gamma\left(\kappa_3 - \frac{1}{2}\right)} \cdot (k_B T)^{-\frac{f}{2}} \cdot \left(1 + \frac{1}{\kappa_3 - \frac{3}{2}} \cdot \frac{1}{k_B T} \sum_{i=1}^{f} \varepsilon_i\right)^{-\kappa_3 + \frac{1}{2} - \frac{f}{2}} \cdot \prod_{i=1}^{f} \varepsilon_i^{-\frac{1}{2}}, \quad \text{(24b)}$$

$$P(\varepsilon_1, \ldots, \varepsilon_f; T; \kappa_0; f) = (4\pi)^{-\frac{f}{2}} \cdot \kappa_0^{-\frac{f}{2}} \cdot \frac{\Gamma\left(\kappa_0 + 1 + \frac{f}{2}\right)}{\Gamma(\kappa_0 + 1)} \cdot (k_B T)^{-\frac{f}{2}} \cdot \left(1 + \frac{1}{\kappa_0} \cdot \frac{1}{k_B T} \sum_{i=1}^{f} \varepsilon_i\right)^{-\kappa_0 - 1 - \frac{f}{2}} \cdot \prod_{i=1}^{f} \varepsilon_i^{-\frac{1}{2}}. \quad \text{(24c)}$$

Furthermore, these equations can be written in terms of the velocity magnitude $u_{\{f\}} \equiv \left|\vec{u}_f\right| = \sqrt{\sum_{i=1}^{f} u_i^2}$ and the energy that is distributed to the f degrees and given by the kinetic square term $\varepsilon_{\{f\}} = \frac{1}{2}\mu \cdot u_{\{f\}}^2$. From now on we will avoid using the term $\{f\}$ in the velocity magnitude u and energy ε, for simplicity. The velocity and energy density of states, $g_{\rm V}(u; f)$ and $g_{\rm E}(\varepsilon; f)$, for an isotropic velocity space are defined by

$$du_1 \cdots du_f \equiv g_{\rm V}(u; f)\, du, du_1 \cdots du_f \equiv g_{\rm E}(\varepsilon; f)\, d\varepsilon, \quad \text{(25)}$$

and can be derived as follows. The f-dimensional infinitesimal volume in velocity space is

$$du_1 \cdots du_f = u^{f-1} du d\Omega_{\{f\}}, \quad \text{(26a)}$$

where $\Omega_{\{f\}}$ is the solid angle embedded in the f-dimensional space. For an isotropic velocity space, we integrate over all the solid angle $\Omega_{\{f\}}$, and obtain

$$du_1 \cdots du_f = B(f) \cdot u^{f-1} du, \quad \text{(26b)}$$

where the proportionality coefficient $B(f)$ represents the surface area of the f-dimensional sphere, i.e.,

$$B(f) \equiv 2\pi^{\frac{f}{2}} / \Gamma\left(\frac{f}{2}\right). \quad \text{(26c)}$$

In addition, from an energy perspective,

$$du_1 \cdots du_f = \frac{1}{2} B(f) \cdot (u^2)^{\frac{f}{2}-1} d(u^2), \text{ or, } du_1 \cdots du_f = \left(\frac{\mu}{2}\right)^{-\frac{f}{2}} \cdot \frac{1}{2} B(f) \cdot \varepsilon^{\frac{f}{2}-1} d\varepsilon. \quad \text{(27)}$$

Comparing Equations (25) and (27) we calculate the velocity and energy density of states, $g_{\rm V}(u)$ and $g_{\rm E}(\varepsilon)$,

$$g_{\rm V}(u; f) = B(f) \cdot u^{f-1}, \quad \text{and} \quad g_{\rm E}(\varepsilon; f) = \left(\frac{\mu}{2}\right)^{-\frac{f}{2}} \cdot \frac{1}{2} B(f) \cdot \varepsilon^{\frac{f}{2}-1}, \quad \text{or,}$$

$$g_{\rm V}(u; f) = \frac{2\pi^{\frac{f}{2}}}{\Gamma\left(\frac{f}{2}\right)} \cdot u^{f-1}, \quad \text{and} \quad g_{\rm E}(\varepsilon; f) = \frac{\left(\frac{2}{\mu}\pi\right)^{\frac{f}{2}}}{\Gamma\left(\frac{f}{2}\right)} \cdot \varepsilon^{\frac{f}{2}-1}. \quad \text{(28)}$$

Given the distribution of the velocity magnitude,

$$P(u; \theta; \kappa_0; f) = \pi^{-\frac{f}{2}} \cdot \kappa_0^{-\frac{f}{2}} \cdot \frac{\Gamma\left(\kappa_0 + 1 + \frac{f}{2}\right)}{\Gamma(\kappa_0 + 1)} \cdot \theta^{-f} \cdot \left(1 + \frac{1}{\kappa_0} \cdot \frac{u^2}{\theta^2}\right)^{-\kappa_0 - 1 - \frac{f}{2}}, \quad \text{(29a)}$$

or expressed in terms of the kinetic energy $\varepsilon = \frac{1}{2}\mu \cdot u^2$,

$$P(\varepsilon; T; \kappa_0; f) = \left(\frac{2}{\mu}\pi\right)^{-\frac{f}{2}} \cdot \kappa_0^{-\frac{f}{2}} \cdot \frac{\Gamma\left(\kappa_0 + 1 + \frac{f}{2}\right)}{\Gamma(\kappa_0 + 1)} \cdot (k_B T)^{-\frac{f}{2}} \cdot \left(1 + \frac{1}{\kappa_0} \cdot \frac{\varepsilon}{k_{\rm B} T}\right)^{-\kappa_0 - 1 - \frac{f}{2}}, \quad \text{(29b)}$$

then the entire probability distributions of velocity magnitude and energy, defined respectively by

$$P_{\rm V}(u; \theta; \kappa_0; f) \equiv P(u; \theta; \kappa_0; f) \cdot g_{\rm V}(u; f), \quad \text{(29c)}$$

and

$$P_{\rm E}(u; T; \kappa_0; f) \equiv P(\varepsilon; T; \kappa_0; f) \cdot g_{\rm E}(\varepsilon; f), \quad \text{(29d)}$$

are given by

$$P_{\rm V}(u; \theta; \kappa_0; f) = \frac{2\kappa_0^{-\frac{f}{2}}}{{\rm B}\left(\kappa_0 + 1, \frac{f}{2}\right)} \cdot \theta^{-f} \cdot \left(1 + \frac{1}{\kappa_0} \cdot \frac{u^2}{\theta^2}\right)^{-\kappa_0 - 1 - \frac{f}{2}} u^{f-1}, \quad \text{(29e)}$$

and

$$P_{\mathrm{E}}(\varepsilon; T; \kappa_0; f) = \frac{\kappa_0^{-\frac{f}{2}}}{\mathrm{B}\left(\kappa_0+1, \frac{f}{2}\right)} \cdot (k_{\mathrm{B}}T)^{-\frac{f}{2}} \cdot \left(1 + \frac{1}{\kappa_0} \cdot \frac{\varepsilon}{k_{\mathrm{B}}T}\right)^{-\kappa_0-1-\frac{f}{2}} \varepsilon^{\frac{f}{2}-1}. \tag{29f}$$

where we used the Beta function formalism, $\mathrm{B}(x, y) \equiv \Gamma(x)\Gamma(y)/\Gamma(x+y)$.

It is interesting that the above equations (29e) and (29f) can be expressed in terms of the F-distribution (Abramowitz & Stegun 1972). Indeed, Equation (29f) written as

$$P_{\mathrm{E}}(\varepsilon; T; \kappa_0; f)\, d\varepsilon = \frac{1}{\mathrm{B}\left(\kappa_0+1, \frac{f}{2}\right)} \cdot \left(1 + \frac{\varepsilon}{\kappa_0 \cdot k_{\mathrm{B}}T}\right)^{-\kappa_0-1-\frac{f}{2}} \cdot \left(\frac{\varepsilon}{\kappa_0 \cdot k_{\mathrm{B}}T}\right)^{\frac{f}{2}-1} d\left(\frac{\varepsilon}{\kappa_0 \cdot k_{\mathrm{B}}T}\right)$$

is expressed in terms of the F-distribution

$$F(x; m, n)\, dx = \frac{1}{B\left(\frac{m}{2}, \frac{n}{2}\right)} \cdot x^{\frac{n}{2}-1} \cdot (1+x)^{-\frac{m+n}{2}} dx, \tag{30}$$

for $n \equiv f$, $m \equiv 2(\kappa_0+1)$, $x \equiv \varepsilon/(\kappa_0 \cdot k_{\mathrm{B}}T)$, namely,

$$P_{\mathrm{E}}(\varepsilon; T; \kappa_0; f)\, d\varepsilon = F\left(\frac{\varepsilon}{\kappa_0 \cdot k_{\mathrm{B}}T}; 2\kappa_0+2; f\right) d\left(\frac{\varepsilon}{\kappa_0 \cdot k_{\mathrm{B}}T}\right). \tag{31}$$

Equation (29f) can be expressed in terms of the average energy $\bar{\varepsilon}$, defined by the total energy ε, either (1) per degree of freedom, $\bar{\varepsilon} = \varepsilon/f$, or (2) per particle, $\bar{\varepsilon} = \varepsilon/N = (D/f) \cdot \varepsilon$, where each particle has D degrees of freedom. These distributions are respectively

$$P_{\mathrm{E}}(\bar{\varepsilon}; T; \kappa_0; f) = \frac{f^{\frac{f}{2}} \cdot \kappa_0^{-\frac{f}{2}}}{\mathrm{B}\left(\kappa_0+1, \frac{f}{2}\right)} \cdot (k_{\mathrm{B}}T)^{-\frac{f}{2}} \cdot \left(1 + \frac{f}{\kappa_0} \cdot \frac{\bar{\varepsilon}}{k_{\mathrm{B}}T}\right)^{-\kappa_0-1-\frac{f}{2}} \bar{\varepsilon}^{\frac{f}{2}-1}, \tag{32a}$$

$$P_{\mathrm{E}}(\bar{\varepsilon}; T; \kappa_0; D \cdot N) = \frac{N^{\frac{D}{2}N} \cdot \kappa_0^{-\frac{D}{2}N}}{\mathrm{B}\left(\kappa_0+1, \frac{D}{2}N\right)} \cdot (k_{\mathrm{B}}T)^{-\frac{D}{2}N} \cdot \left(1 + \frac{N}{\kappa_0} \cdot \frac{\bar{\varepsilon}}{k_{\mathrm{B}}T}\right)^{-\kappa_0-1-\frac{D}{2}N} \bar{\varepsilon}^{\frac{D}{2}N-1}. \tag{32b}$$

In the thermodynamic limit, $N \to \infty$, Equation (32b) becomes

$$P_{\mathrm{E}}(\bar{\varepsilon}; T; \kappa_0; D \cdot N \to \infty) = \frac{[(D/2)\kappa_0]^{\kappa_0+1}}{\Gamma(\kappa_0+1)} \cdot (k_{\mathrm{B}}T)^{\kappa_0+1} \cdot \bar{\varepsilon}^{-\kappa_0-2} \exp\left\{-(D/2)\kappa_0 k_{\mathrm{B}}T \cdot \frac{1}{\bar{\varepsilon}}\right\} \tag{33}$$

(this is easily proved using Stirling's approximation formula for Gamma functions). This can be expressed in terms of the inverse Gamma distribution function defined by

$$f_{\mathrm{Inv}\Gamma}(x; a) \equiv \frac{1}{\Gamma(a)} \cdot x^{-a-1} \exp\left\{-\frac{1}{x}\right\}, \tag{34}$$

for $a \equiv \kappa_0 + 1$ and $x \equiv \bar{\varepsilon}/[(D/2) \cdot \kappa_0 \cdot k_{\mathrm{B}}T]$, namely,

$$P_{\mathrm{E}}(\bar{\varepsilon}; T; \kappa_0; D \cdot N \to \infty)\, d\bar{\varepsilon} = f_{\mathrm{Inv}\Gamma}\left[\frac{\bar{\varepsilon}}{(D/2)\kappa_0 k_{\mathrm{B}}T}; \kappa_0+1\right] d\left[\frac{\bar{\varepsilon}}{(D/2)\kappa_0 k_{\mathrm{B}}T}\right]. \tag{35}$$

Furthermore, the differential flux j of particles with number density η that are described by the f-dimensional kappa distribution (considering f degrees of freedom per particle) $P(u_1, \ldots, u_f; \theta; \kappa_0; f)$ is given by

$$j = \frac{\eta \cdot P(u_1, \ldots, u_f; \theta; \kappa_0; f)\, du_1 \cdots du_f}{d\Omega_{\{f\}} d\varepsilon_0} \cdot u. \tag{36}$$

(Note that in order to calculate the flux we use the one-particle distribution, and thus, ε and f indicate the energy and degrees of freedom per particle.) Utilizing $du_1 \cdots du_f = u^{f-1} du\, d\Omega_{\{f\}}$ (Equation (26a)) we obtain

$$j = \frac{\eta}{\mu} \cdot P(u_1, \ldots, u_f; \theta; \kappa_0; f) \cdot u^{f-1}, \quad \text{or} \quad j(\varepsilon; T; \kappa_0; f) = \frac{\eta}{\mu} \cdot \left(\frac{2}{\mu}\right)^{\frac{f-1}{2}} \cdot P(\varepsilon; T; \kappa_0; f) \cdot \varepsilon^{\frac{f-1}{2}},$$

that is,

$$j(\varepsilon; T; \kappa_0; f) = \frac{\eta}{\sqrt{2\mu}} \cdot \pi^{-\frac{f}{2}} \cdot \kappa_0^{-\frac{f}{2}} \cdot \frac{\Gamma\left(\kappa_0 + \frac{f}{2} + 1\right)}{\Gamma(\kappa_0+1)} \cdot (k_{\mathrm{B}}T)^{-\frac{f}{2}} \cdot \left(1 + \frac{1}{\kappa_0} \cdot \frac{\varepsilon}{k_{\mathrm{B}}T}\right)^{-\kappa_0-1-\frac{f}{2}} \cdot \varepsilon^{\frac{f-1}{2}}. \tag{37}$$

For $\varepsilon \gg \kappa_0 k_{\rm B} T$ we obtain the high-energy asymptotic behavior

$$j(\varepsilon; T; \kappa_0; f) \cong \frac{\eta}{\sqrt{2\mu}} \cdot \pi^{-\frac{f}{2}} \cdot \kappa_0^{\kappa_0} \cdot \frac{\Gamma\left(\kappa_0 + \frac{f}{2} + 1\right)}{\Gamma(\kappa_0)} \cdot (k_{\rm B} T)^{\kappa_0+1} \cdot \varepsilon^{-\kappa_0 - \frac{3}{2}},$$

that gives the power law that commonly characterizes suprathermal tails in space plasmas

$$j(\varepsilon) \cong j_0 \cdot \varepsilon^{-\gamma}. \tag{38a}$$

Note that the flux-related quantity j_0 depends on the energy units [E], i.e., $j_0 \cdot [{\rm E}]^{\gamma} = j_0' \cdot [{\rm E}]'^{\gamma}$ and is called [E]-flux, e.g., for [E] = 1 keV, j_0 is the 1 keV flux (see Livadiotis et al. 2011). The [E]-flux, j_0, and the spectral index γ, are given by

$$j_0 \equiv \frac{\eta}{\sqrt{2\mu}} \cdot \pi^{-\frac{f}{2}} \cdot \kappa_0^{\kappa_0} \cdot \frac{\Gamma\left(\kappa_0 + \frac{f}{2} + 1\right)}{\Gamma(\kappa_0)} \cdot (k_{\rm B} T)^{\kappa_0+1}, \quad \gamma \equiv \kappa_0 + \frac{3}{2} (= \kappa_3). \tag{38b}$$

Note that the spectral index γ does not depend on f. The stationary state farthest from thermal equilibrium, $\kappa_0 \sim 0$, corresponds to flux with spectral index $\gamma \sim 1.5$,

$$j(\varepsilon; T; \kappa_0 \to 0; f) \cong \frac{\eta}{\sqrt{2\mu}} \cdot \pi^{-\frac{f}{2}} \cdot \kappa_0 \cdot \Gamma\left(\frac{f}{2} + 1\right) \cdot (k_{\rm B} T) \cdot \varepsilon^{-\frac{3}{2}} \sim \varepsilon^{-\frac{3}{2}}. \tag{38c}$$

6. DISCUSSION: CHARACTERISTICS AND PHYSICAL MEANING OF THE KAPPA INDEX

We have shown that the kappa index is not an invariant physical quantity, but instead is dependent on the degrees of freedom f, and thus indicated by κ_f, where $\kappa_f = \text{constant} + \frac{f}{2}$. On the other hand, the difference $\kappa_0 \equiv \kappa_f - \frac{f}{2}$ is invariant, namely, it is independent of the degrees of freedom $f = D \cdot N$ of the system and the particles N (which have D degrees of freedom each). This generalization to N-particle distribution function and the modified kappa index presented in this study should provide a new paradigm for how kappa distributions are used and interpreted in space plasmas. In particular, we have developed an invariant formulation of the f-dimensional kappa distribution that can give either the one-particle or the entire N-particle description. The non-equilibrium stationary states should hereafter be characterized by the invariant kappa index κ_0 and not by the dependent index κ_f. Further, any physical meaning of the kappa index, which has been missing in most prior studies, can now be examined in light of this invariant kappa index κ_0.

In this section, we (1) clarify the role of the invariant kappa index in identifying the non-equilibrium stationary states and constructing the whole arrangement of the kappa indices, which we call the κ-spectrum, (2) show how the whole κ-spectrum is separated into the near- and far- equilibrium regions, (3) calculate the correlation between two different particles and degrees of freedom, and show how this leads to a suitable measure of the thermodynamic distance of stationary states from thermal equilibrium, and (4) show how the kappa index can act in a temperature-like fashion. While the kappa index is irrelevant to the mean energy (the mean energy is independent of the kappa index and defines the temperature (see Equation (2)), it turns out that by decreasing the kappa index a new kind of "freezing" occurs where particles can decelerate to near zero velocities. (Apparently then, when increasing the kappa index a new kind of "heating" occurs and particles can accelerate from near zero velocities.)

Table 2 summarizes the main equations that involve the multidimensional kappa distribution. Any relation can easily be convert to be expressed in terms of the traditional three-dimensional kappa index κ_3, using $\kappa_0 = \kappa_3 - \frac{3}{2}$, or generally, in terms of the f-dimensional index, using $\kappa_0 = \kappa_f - \frac{f}{2}$.

6.1. κ-Spectrum: Near- and Far-equilibrium Stationary States

We have mentioned that the spectrum of all the kappa indices $\kappa_0 \in (0, \infty]$ that characterize space plasmas can be divided into two regions; the "near-equilibrium" region with indices $\kappa_0 \in (1, \infty]$ and the "far-equilibrium" region with indices $\kappa_0 \in (0, 1]$ (Livadiotis & McComas 2010a). Observations of various space plasmas showed that the kappa indices selectively distributed over one of the two regions (e.g., Dialynas et al. 2009; Livadiotis et al. 2011; for more information, see Figure 2 in the introduction of this paper). In addition, analytical derivations support this near/far equilibrium separation.

One line of theoretical evidence that supports the separate thermodynamics that characterize the two regions of near- and far-equilibrium states is the statistical deviation of the mean energy.

Let each of the N particles be characterized by the same degrees of freedom D. For the nth particle, $n = 1, \ldots, N$, the velocity vector is $\vec{u}_{(n)} \equiv (u_{1,_{(1)}}, u_{2,_{(1)}}, \ldots, u_{D,_{(1)}})$ and the energy $\varepsilon_{(n)} \equiv \frac{\mu}{2} \cdot \vec{u}_{(n)}^2 = \frac{\mu}{2} \cdot \sum_{i=1}^{D} u_{i,_{(n)}}^2$. Then, the mean energy $\langle\varepsilon\rangle$, that is the internal energy U, defines the temperature as shown in Equation (2), i.e., $U = \langle\varepsilon\rangle = 1/2\mu \cdot \langle\vec{u}^2\rangle \equiv DN/2 \cdot k_B T$. The standard deviation of the mean energy is given by $\delta\varepsilon = \sqrt{\langle\varepsilon^2\rangle - \langle\varepsilon\rangle^2} = 1/2\mu \cdot \sqrt{\langle\vec{u}^4\rangle - \langle\vec{u}^2\rangle^2}$. (Note that the second statistical moment $\langle\vec{u}^2\rangle$ is necessary for deriving the mean energy, but we need also the fourth $\langle\vec{u}^4\rangle$ for the energy deviation.) One can easily calculate this energy deviation $\delta\varepsilon$ and show that it is finite only for the near-equilibrium region $\kappa_0 > 1$, while for the far-equilibrium region $\kappa_0 \leqslant 1$ this is indefinite. Indeed, the total energy ε is the sum of the N particles energy $\varepsilon = \varepsilon_{(1)} + \cdots + \varepsilon_{(N)}$. Hence, we have

$$\langle\varepsilon^2\rangle = \langle(\varepsilon_{(1)} + \cdots + \varepsilon_{(N)})^2\rangle = N \cdot \left\langle\varepsilon_{(n)}^2\right\rangle + N(N-1) \cdot \langle\varepsilon_{(n)} \cdot \varepsilon_{(m)}\rangle, \tag{39a}$$

where for the nth and mth particles we have

$$\left\langle\varepsilon_{(n)}^2\right\rangle = \left(\frac{1}{2}\mu\right)^2 \cdot \left\langle\vec{u}_{(n)}^4\right\rangle, \qquad \langle\varepsilon_{(n)} \cdot \varepsilon_{(m)}\rangle = \left(\frac{1}{2}\mu\right)^2 \cdot \left\langle\vec{u}_{(n)}^2 \cdot \vec{u}_{(m)}^2\right\rangle. \tag{39b}$$

In Appendix B, we show that $\langle \vec{u}_{(n)}^2 \cdot \vec{u}_{(m)}^2 \rangle$ and $\langle \vec{u}_{(n)}^4 \rangle$ do not depend on specific particles, namely, they do not depend on subscripts (n) and (m), and they are given by

$$\langle \vec{u}_{(n)}^2 \cdot \vec{u}_{(m)}^2 \rangle = \theta^4 \cdot \frac{\kappa_0}{\kappa_0 - 1} \times \begin{cases} \left(\frac{D}{2}\right)^2 & \text{if } n \neq m, \\ \left(\frac{D}{2}\right) \cdot \left(\frac{D}{2} + 1\right) & \text{if } n = m. \end{cases} \tag{40}$$

Therefore, we find

$$\langle \varepsilon^2 \rangle = \left[\frac{D}{2}\left(\frac{D}{2} + 1\right) \cdot N + \left(\frac{D}{2}\right)^2 \cdot N(N-1) \right] \cdot \frac{\kappa_0}{\kappa_0 - 1} \cdot (k_B T)^2, \tag{41a}$$

$$\langle \varepsilon \rangle = \frac{D}{2} N \cdot k_B T, \tag{41b}$$

which leads to the standard deviation

$$\delta\varepsilon = \sqrt{\langle \varepsilon^2 \rangle - \langle \varepsilon \rangle^2} = \frac{D}{2} N \cdot \sqrt{\frac{\kappa_0}{\kappa_0 - 1} \cdot \frac{1}{\frac{D}{2} N} + \frac{1}{\kappa_0 - 1}} \cdot k_B T, \tag{41c}$$

and the relative deviation

$$\frac{\delta\varepsilon}{\langle \varepsilon \rangle} = \sqrt{\frac{\kappa_0}{\kappa_0 - 1} \cdot \frac{1}{\frac{D}{2} N} + \frac{1}{\kappa_0 - 1}} \xrightarrow{N \to \infty} \frac{1}{\sqrt{\kappa_0 - 1}}. \tag{41d}$$

Note that for the thermodynamic limit, $N \to \infty$, Equation (41d) gives non-zero relative deviation, unless the system resides at equilibrium $\kappa_0 \to \infty$.

Moreover, we observe that the fourth statistical moment does not converge for $\kappa_0 \leqslant 1$. In general, in order for the integral of the ath statistical moment to converge, i.e.,

$$\int_0^\infty u^a \cdot P(u) g_{\mathrm{V}}(u)\, du < +\infty, \tag{42}$$

the integrand in the high-energy limit of $u \to \infty$ has to attain at least a power-law decay $1/u^r$, with $r > 1$. Then, given the degrees of freedom $f = D \cdot N$, the distribution's high-energy asymptotic behavior $P(u) \sim u^{-2\kappa_0 - 2 - D \cdot N}$, and the density of velocity states $g_{\mathrm{V}}(u) \sim u^{D \cdot N - 1}$, we have $u^a P(u) g_{\mathrm{V}}(u) \sim u^a u^{-2\kappa_0 - 2 - D \cdot N} u^{D \cdot N - 1} \sim u^{a - 2\kappa_0 - 3}$, so that $a - 2\kappa_0 - 3 < -1$ or $\kappa_0 > a/2 - 1$. For the energy deviation $\delta\varepsilon$, the fourth statistical moment $\langle u^4 \rangle$ has to converge ($a = 4$), and thus we obtain $\kappa_0 > 1$.

Another possible reason that makes the thermodynamics of the far-equilibrium region $\kappa_0 \in (0, 1]$ much more complicated than that of the near-equilibrium region $\kappa_0 \in (1, \infty]$ is the entropy of the system of particles in non-equilibrium stationary states. This entropy can be expressed in terms of the kappa index alone, revealing the possible paths by which space plasmas transit through stationary states toward, or away from, equilibrium (Livadiotis & McComas 2010a, 2010c). The entropy is monotonic in $\kappa_0 \in (1, \infty]$ and non-monotonic convex with a minimum in $\kappa_0 \in (0, 1]$ (Figure 3(a)). As the kappa index increases over $\kappa_0 \in (1, \infty]$, the entropy is monotonically increasing toward its maximum value at equilibrium ($\kappa_0 \to \infty$). However, within the other subinterval $\kappa_0 \in (0, 1]$ the entropy is non-monotonic and the dynamics of the transitions of the system between the states is much more complicated. This dynamical behavior separates the stationary states to those near equilibrium, with $\kappa_0 > 1$ (near-equilibrium region), and to those near the q-frozen state, with $1 \geqslant \kappa_0$ (far-equilibrium region), where their boundary is the specific state at $\kappa_0 = 1$. This is called the "escape state" since a transient system can escape from the far-equilibrium toward the near-equilibrium region only by passing through this state.

Spontaneous procedures that can increase entropy move the system gradually toward equilibrium ($\kappa_0 \to \infty$), the state with the maximum entropy (red arrow in Figure 4). On the other hand, external factors that may decrease entropy move the system back to states farther from equilibrium and closer to the q-frozen state ($\kappa_0 \to 0$; blue arrow in Figure 4). Newly formed pick-up ions (PUIs) can play this critical role in the case of solar wind and other space plasmas because of their highly ordered motion. Livadiotis & McComas (2010a, 2010c) proposed that the pick-up protons, which are ions with highly organized phase-space distributions, reduce the entropy of the combined system of solar wind and pick-up protons (incorporated solar wind), and thus push the values of the kappa index to be far from equilibrium and within the far-equilibrium region $\kappa_0 \in (0, 1]$. This result was shown analytically (Livadiotis & McComas 2011), and finally verified by the *IBEX* observations as shown in the analysis of Livadiotis et al. (2011; Figure 3(b)).

6.2. Correlation between Degrees of Freedom and Particles

Degrees of freedom are the variables that construct the phase space of a physical system. The degrees of freedom are dynamically independent when the Hamiltonian can be written in the form $H = \sum_{i=1}^{f} H_i$, where H_i is a function of the ith variable, e.g., as a function of the kinetic energy $H_i = \frac{1}{2\mu} \cdot p_i^2$ (in terms of momentum p_i), or of both the kinetic and potential energies $H_i = \frac{1}{2\mu} \cdot p_i^2 + \frac{k}{2} \cdot x_i^2$ with the latter expressed in terms of the square position $\frac{k}{2} \cdot x_i^2$. For all these cases, the Hamiltonian is completely determined (there are no mixed nonlinear terms leading to chaos, e.g., $H_i = \frac{1}{2\mu} \cdot p_i^2 + \frac{k}{2} \cdot x_i^2 + \varepsilon \cdot |p_i|^\alpha \cdot x_i^\beta$). In the absence of any sort of chaos, the system's

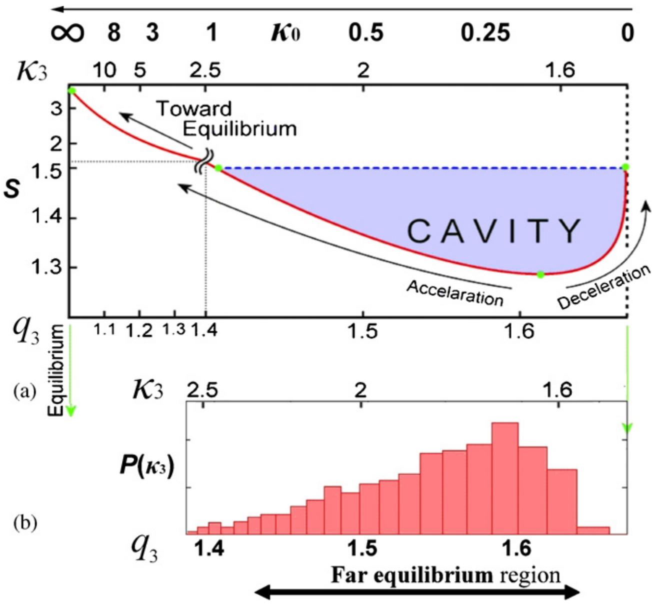

Figure 3. (a) Entropy is monotonic in the near-equilibrium region and non-monotonic convex with a minimum in the far-equilibrium region. This far region is characterized by more complicated thermodynamics that separates it from the near-equilibrium region (see Livadiotis & McComas 2010a). When the entropy increases in the near-equilibrium region, $\kappa_0 \in (1, \infty]$, the kappa index increases, the particles are accelerated, and the system moves toward equilibrium (see the text), while in the far-equilibrium region, $\kappa_0 \in (0, 1]$, the kappa index either increases or decreases, causing either phenomenological acceleration or deceleration, respectively. (b) The distribution of the kappa indices ($\kappa_3 = \kappa_0 - 3/2$) of the plasma in the inner heliosheath as estimated by the work of Livadiotis et al. (2011) covers the far-equilibrium region $\kappa_0 \in (0, 1]$.

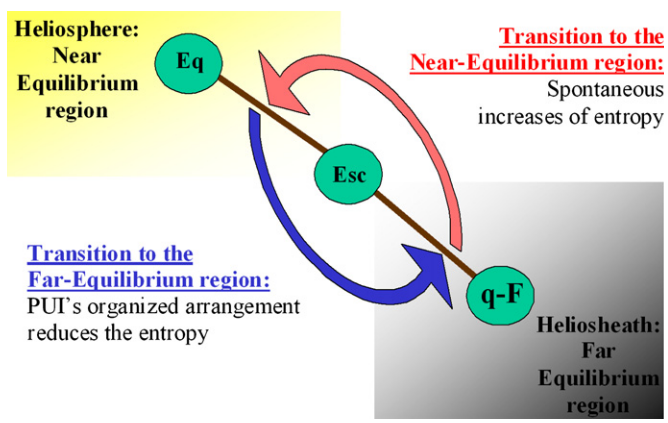

Figure 4. Scheme of non-equilibrium transitions of the heliosphere. While spontaneous increases of entropy move the system toward thermal equilibrium and the near-equilibrium region $\kappa_0 \in (1, \infty]$, the impact of highly organized pick-up ions (PUI's) turns it back away from equilibrium and into the far-equilibrium region $\kappa_0 \in (0, 1]$. As a result, the inner heliosphere, with comparatively fewer PUIs resides in near-equilibrium stationary states, while the much more distant inner heliosheath resides in far from equilibrium states. (We use Eq for Equilibrium, Esc for the Escape state, and q-F for the q-Frozen state.)

phase space is microscopically deterministic, but then, it is harder and more subtle to realize the fact of the system's macroscopic thermodynamical irreversibility that is observed macroscopically (second law of thermodynamics). Where has the information been lost? At which scale and for what physical reason does this happen?

Even though the microscopic f degrees of freedom are considered to be dynamically independent, they do have complicated correlations because of the presence of numerous other degrees of freedom that we cannot observe in practice. In fact, the evolution of a system transforms simple information into information that in practice is inaccessible, because it is associated with extremely complicated correlations between the positions and velocities of a macroscopic number of particles. The key to understanding a system's transitions is its correlations; this is one of the crucial limitations of the classical BG Statistical approach where correlations are not considered. By contrast, the non-extensitivity of Statistical Mechanics naturally captures the correlations between the particles and the degrees of freedom.

The physical correlation is mathematically described by the correlation inferred by the specific formulation of the kappa distribution that cannot be factorized (unless $\kappa_0 \to \infty$, corresponding to thermal equilibrium and the exponential form of the Maxwellian distribution). Indeed, for non-equilibrium states,

$$P(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_f) \neq \prod_{i=1}^{f} P(\varepsilon_i), \quad \forall \kappa_0 < \infty, \tag{43a}$$

while at thermal equilibrium,

$$P(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_f) = \prod_{i=1}^{f} P(\varepsilon_i), \quad \text{for} \quad \kappa_0 \to \infty, \tag{43b}$$

because

$$P(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_f) \sim \exp\left(-\frac{\sum_{i=1}^{f} \varepsilon_i}{k_B T}\right) \cdot \prod_{i=1}^{f} \varepsilon_i^{-\frac{1}{2}} = \prod_{i=1}^{f} \left[\exp\left(-\frac{\varepsilon_i}{k_B T}\right) \cdot \varepsilon_i^{-\frac{1}{2}} \cdot\right].$$

Therefore, the invariant kappa index κ_0 not only gives a measure of how far the stationary state that the system is residing in from thermal equilibrium, it also characterizes the correlation between the degrees of freedom. In fact, as we will see further below, this correlation itself provides a physically meaningful measure of this thermodynamic distance of a stationary state from equilibrium. For example, at equilibrium the kappa distribution recovers the Maxwellian exponential ($\kappa_0 \to \infty$), which are fully uncorrelated degrees of freedom. Namely, when the measure of the thermodynamic distance from equilibrium is zero, the correlation is also zero. Similarly, the absence of correlation leads the system to stabilize at the stationary state with zero thermodynamic distance, that is, thermal equilibrium.

Below we deal with the correlation $\rho(\kappa_0)$ between two particles (or degrees of freedom), inferred by a non-equilibrium kappa index κ_0. The stronger the correlation, the smaller the invariant kappa index κ_0, and thus, the farther from equilibrium the stationary state is stabilized. We expect the correlation to be maximized, $\rho(\kappa_0) = 1$, when κ_0 attains its minimum value, i.e., $\kappa_0 = 0$, at the q-frozen state. On the other hand, we do not expect the correlation to be dependent on the total number of the correlated particles or degrees of freedom that are involved in the kappa distribution. In other words, the correlation between two particles and the degrees of freedom should be the same, independently of the whole number N of the correlated particles or the whole number f of the correlated degrees of freedom. This is because each correlation value $\rho(\kappa_0)$ is connected to a specific stationary state, and thus, neither the degrees of freedom f nor the number of particles N should be involved within the correlation's formulation $\rho(\kappa_0)$; instead, the invariant kappa index κ_0 does.

Again, let us examine the case where each of the N particles is characterized by the same degrees of freedom D. The correlation $\rho(\kappa_0; D)$ between any two particles (1) and (2), having velocity vectors $\vec{u}_{(1)} = (u_{1,_{(1)}}, u_{2,_{(1)}}, \ldots, u_{D,_{(1)}})$ and $\vec{u}_{(2)} = (u_{1,_{(2)}}, u_{2,_{(2)}}, \ldots, u_{D,_{(2)}})$, respectively, and is given either in terms of the energies $\varepsilon_{(1)} = \frac{1}{2}\mu\vec{u}_{(1)}^2$, $\varepsilon_{(2)} = \frac{1}{2}\mu\vec{u}_{(2)}^2$, or of the squares of the velocity magnitudes $\vec{u}_{(1)}^2$, $\vec{u}_{(2)}^2$, by

$$\rho(\kappa_0; D) = \frac{\langle\varepsilon_{(1)} \cdot \varepsilon_{(2)}\rangle - \langle\varepsilon_{(1)}\rangle \cdot \langle\varepsilon_{(2)}\rangle}{\sqrt{\langle\varepsilon_{(1)}^2\rangle - \langle\varepsilon_{(1)}\rangle^2}\sqrt{\langle\varepsilon_{(2)}^2\rangle - \langle\varepsilon_{(2)}\rangle^2}} = \frac{\langle\vec{u}_{(1)}^2 \cdot \vec{u}_{(2)}^2\rangle - \langle\vec{u}_{(1)}^2\rangle \cdot \langle\vec{u}_{(2)}^2\rangle}{\sqrt{\langle\vec{u}_{(1)}^4\rangle - \langle\vec{u}_{(1)}^2\rangle^2}\sqrt{\langle\vec{u}_{(2)}^4\rangle - \langle\vec{u}_{(2)}^2\rangle^2}}, \tag{44}$$

where $\langle\vec{u}_{(n)}^2 \cdot \vec{u}_{(m)}^2\rangle$ and $\langle\vec{u}_{(n)}^4\rangle$ are given by Equation (40) so that the correlation becomes

$$\rho(\kappa_0; D) = \frac{\frac{D}{2}}{\kappa_0 + \frac{D}{2}}. \tag{45}$$

It is important to remember that D is the degrees of freedom per particle and not the whole number of degrees given by $f = D \cdot N$. As we expect, the correlation does not depend on the number of particles N.

In addition, it is worth mentioning that while neither $\langle\vec{u}_{(n)}^2 \cdot \vec{u}_{(m)}^2\rangle$ nor $\langle\vec{u}_{(n)}^4\rangle$ converge for $0 < \kappa_0 \leqslant 1$ (far-equilibrium region), their ratio involved in the correlation's formulation (44) does converge. The proof is given in Appendix B.

In terms of the f-dependent kappa index, $\kappa_f = \kappa_0 + f/2 = \kappa_0 + DN/2$, the correlation is written as

$$\rho\left(\kappa_0 = \kappa_f - \frac{D}{2}N; D\right) = \frac{\frac{D}{2}}{\kappa_f - \frac{D}{2}(N-1)}, \tag{46a}$$

or in terms of the q-index, $q_f = 1 + 1/\kappa_f$ (see Equation (1)), becomes

$$\rho\left(\kappa_0 = \frac{1}{q_f - 1} - \frac{D}{2}N; D\right) = \frac{(q_f - 1) \cdot D}{2 + (1 - q_f) \cdot D(N-1)}. \tag{46b}$$

Equation (46b) was exactly the result found by Abe (1999). However, that study suggested that the correlation depends on the number of particles N, so that when $N \to \infty$, the correlation vanishes, $\rho \to 0$. Nevertheless, this would be the case only if the q- or κ-indices have an absolute notion of being independent of the total degrees of freedom, $f = D \cdot N$, and the number of particles, N. Clearly, this is not true, because q_f and κ_f are indeed dependent on N, in a way that the dependence of correlation ρ on N cancels out. The actual dependence of the correlation ρ is on the invariant kappa index κ_0 that characterizes the stationary states, and therefore, the correlation $\rho(\kappa_0)$ itself can also be used to characterize the stationary states and their thermodynamic distance from thermal equilibrium.

Finally, we note that in the case where the particles have different numbers of degrees of freedom, that is $D_{(1)}$ and $D_{(2)}$, with velocity vectors $\vec{u}_{(1)} = (u_{1,_{(1)}}, u_{2,_{(1)}}, \ldots, u_{D_{(1)},_{(1)}})$ and $\vec{u}_{(2)} = (u_{1,_{(2)}}, u_{2,_{(2)}}, \ldots, u_{D_{(2)},_{(2)}})$, respectively, Equation (40) is now given by

$$\langle\vec{u}_{(n)}^2 \cdot \vec{u}_{(m)}^2\rangle = \theta^4 \cdot \frac{\kappa_0}{\kappa_0 - 1} \times \begin{cases} \left(\frac{D_{(n)}}{2}\right) \cdot \left(\frac{D_{(m)}}{2}\right) & \text{if } n \neq m, \\ \left(\frac{D_{(n)}}{2}\right) \cdot \left(\frac{D_{(n)}}{2} + 1\right) & \text{if } n = m, \end{cases} \tag{47}$$

and the correlation given in Equation (45) becomes

$$\rho_{(1,2)}(\kappa_0;\, D_{(1)}, D_{(2)}) = \sqrt{\rho(\kappa_0;\, D_{(1)}) \cdot \rho(\kappa_0;\, D_{(2)})} = \sqrt{\frac{\frac{D_{(1)}}{2}}{\kappa_0 + \frac{D_{(1)}}{2}} \cdot \frac{\frac{D_{(2)}}{2}}{\kappa_0 + \frac{D_{(2)}}{2}}}. \tag{48}$$

6.3. *Measure of the Thermodynamic Distance*

The invariant kappa index κ_0 gives a measure of how far the stationary state that the system is residing in from thermal equilibrium. In fact, the inverse, $1/\kappa_0$, can be naturally used as such a measure of the thermodynamic distance of a stationary state from equilibrium, which is denoted by $M(\kappa_0)$,

$$M(\kappa_0) = \frac{1}{\kappa_0}, \quad 0 \leqslant M(\kappa_0) < \infty. \tag{49}$$

Even though this does not constitute a bounded measure, the escape state equally separates the near- and far-equilibrium regions. Namely, $1/\kappa_0$ is 0 at equilibrium, 1 at the escape state that separates near/far-equilibrium regions, and ∞ at the q-frozen state, the farthest state from equilibrium.

A mathematical description of the thermodynamical measure was previously developed by Livadiotis & McComas (2010a, 2010b). The formulated measure, M_q, called q-metastability, is given by

$$M_q(\kappa_0;\, D) = \frac{\frac{D+1}{2}}{\kappa_0 + \frac{D+1}{2}} \tag{50}$$

(which is equal to $\rho(\kappa_0;\, D+1)$). Note that the measure given in Equation (50) is bounded in the interval $0 \leqslant M(\kappa_0;\, D) < 1$, and thus can be referred to as a percentage, i.e., $0\% \leqslant M(\kappa_0;\, D) \times 100\% < 100\%$. In general, any measure that can be written in the general form

$$M(\kappa_0;\, a) = \frac{a}{\kappa_0 + a} \tag{51}$$

is bounded, and can also be referred to as a percentage. Obviously, M_q (Equation (50)) can be given for $M_q(\kappa_0;\, D) = M(\kappa_0;\, a = \frac{D+1}{2})$. In addition, the correlation given in Equation (45) can naturally define a measure. This correlation measure can be written in terms of Equation (51), as $\rho(\kappa_0;\, D) = M(\kappa_0;\, a = D/2)$. For example, the correlation between two different degrees of freedom (of the same or different particle) is

$$\rho(\kappa_0;\, D = 1) = \frac{\frac{1}{2}}{\kappa_0 + \frac{1}{2}}, \tag{52a}$$

while for a pair of degrees of freedom the correlation is

$$\rho(\kappa_0;\, D = 2) = \frac{1}{\kappa_0 + 1}. \tag{52b}$$

It is more fundamental to ask about the correlation of two Hamiltonian pairs of degrees of freedom instead of any other number of degrees or particles (D degrees). This is because in the Hamiltonian description of phase space the degrees of freedom come in pairs (position/velocity). Then, this is the correlation specifically between two Hamiltonian pairs of degrees of freedom, which is called the Hamiltonian correlation measure $\rho_H(\kappa_0) \equiv \rho(\kappa_0;\, D = 2)$, and equally separates the near- and far-equilibrium regions.

The measures given by Equations (45), (52a), and (52b) have the advantages (1) of being bounded; (2) of being connected to the physically meaningful feature of correlation; especially for the one given by Equation (52b); we have also that (3) the correlation refers to a Hamiltonian pair of degrees of freedom, which is a more realistic situation; and (4) the near- and far-equilibrium regions are equally separated the whole κ-spectrum of stationary states. Finally, Figure 5 shows these measures and their values in the near- and far-equilibrium regions.

6.4. *q-Freezing/Heating processes*

As the κ-index increases, the stationary state becomes closer to thermal equilibrium, and the probability distribution includes larger values of velocity. If the system attains a state of larger κ-index, its particles are characterized by larger values of velocity, producing a phenomenological acceleration. Similarly, for the reverse procedure of decreasing the kappa index and approaching the extreme, q-frozen state at $\kappa_0 \to 0$, we have a phenomenological deceleration, with the particles accumulated at near zero velocity.

We have underlined that the kappa index and the temperature are two independent parameters, and thus, any change of the kappa index should have not affected the temperature. Then how can a phenomenological acceleration or deceleration be possible by varying the kappa index, since the mean energy that is the temperature remains constant? Let us examine the case with increasing kappa index. What really happens is that particles with close to zero velocities indeed increase their velocities (acceleration), but at the same time, particles with very high velocities decrease their velocities (deceleration). Acceleration and deceleration take place at the same time, preserving the balance in energy; however, a larger fraction of particles are accelerated rather than decelerated, and thus, acceleration is dominant.

	Equilibrium	Escape State	q-Frozen State
κ_0	∞	1	0
q_0	1	2	∞
κ_3	∞	2.5	1.5
q_3	1	1.4	5/3
κ_f	∞	$1+f/2$	$f/2$
q_f	1	$(4+f)/(2+f)$	$1+1/f$
$1/\kappa_0$	0	1	∞
$M_q(\kappa_0,D)$	0	$(D+1)/(D+3)$	1
$\rho(\kappa_0,D)$	0	$D/(D+2)$	1
$\rho_H(\kappa_0)$	0	1/2	1

Near-Equilibrium Region — Far-Equilibrium Region

Figure 5. Characterization of the whole spectrum of stationary states via various measures. The basic states are shown, i.e., the boundary states of equilibrium and q-frozen state, and the escape state that separates the spectrum in the near- and far-equilibrium regions. First used is the invariant zero-dimensional κ_0, which is the most fundamental index to characterize the stationary states, and, through its inverse, $1/\kappa_0$, to give a measure of their thermodynamic distance from thermal equilibrium. We also give the f-dimensional kappa index, κ_f, which is not invariant but dependent on the degrees of freedom f, and the usual three-dimensional κ_3, which has been used in all the past publications (without the subscript "3"). All the corresponding q-indices are also given. The last three rows show different quantities that may be used as bounded measures: the measure $M_q(\kappa_0; D)$ (Equation (50)) developed by Livadiotis & McComas (2010a, 2010b), the correlation $\rho(\kappa_0; D)$ (Equation (45); where D is the degrees of freedom per particle), and the correlation for a pair of degrees of freedom $M(\kappa_0; a = 1) = \rho(\kappa_0; 2)$ (Equation (52b)). The first, $M_q(\kappa_0; D)$, obeys to certain mathematical rules developed in Livadiotis & McComas (2010a, 2010b). The second, $\rho(\kappa_0; D)$, has the unique physical meaning of correlation between particles, while the third measure is specifically the correlation between two Hamiltonian pairs of degrees of freedom, which is called the Hamiltonian measure, and equally separate the near- and far-equilibrium regions.

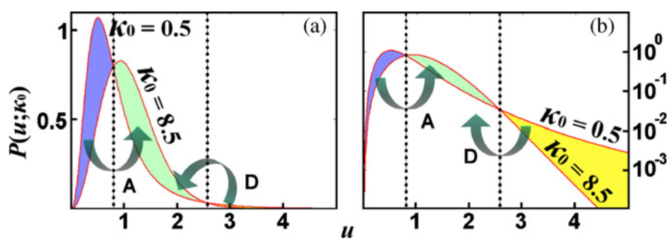

Figure 6. Illustration of the phenomenological acceleration and deceleration taking place at the same time and thus preserving the balance in energy (and temperature). When the kappa index increases, particles with low velocities and belonging to the region $u < u_A$ (blue) increase their velocities and depart to the region $u_A < u < u_D$ (green), that is, a phenomenological acceleration (A). On the other hand, particles with high velocities and within the region $u_D < u$ (yellow) decrease their velocities, and likewise, depart to the region $u_A < u < u_D$, appearing thus as a phenomenological deceleration (D). In particular, panels (a) and (b) (semi-log scale) show how the kappa distribution change when the kappa index increases from $\kappa_0 = 0.5$ to $\kappa_0 = 8.5$ (the region $u_A < u < u_D$ is clearer in panel (b)). In this case, the boundaries are $u_A/\theta \cong 0.79$, $u_D/\theta \cong 2.56$. Note that the distributions are depicted for $\theta = 1$.

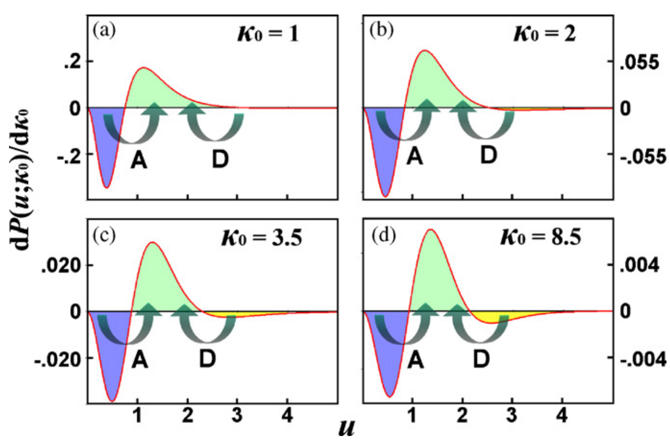

Figure 7. Phenomenological acceleration and deceleration illustrated by the variation of the distribution. The derivative $dP(u; \kappa_0)/d\kappa_0$ is depicted for $\kappa_0 = 1, 2, 3.5, 8.5$. We observe that this is negative for velocities $u < u_A$ and $u_D < u$, and positive for the velocities between $u_A < u < u_D$ (see the text). Therefore, particles with close to zero velocities in the region $u < u_A$, and extremely high velocities in the region $u_D < u$, depart to the mid velocities region $u_A < u < u_D$ in such a way that the balance in energy is preserved.

Figures 6 and 7 illustrate both acceleration and deceleration. In Figure 6, we depict the distribution $P(u; \kappa)$ for $\kappa_0 = 0.5$ and $\kappa_0 = 8.5$ (and $\theta = 1$, $f = 3$). Assuming that the kappa index increases from $\kappa_0 = 0.5$ to $\kappa_0 = 8.5$, we observe that the probability distribution decreases for velocities $u < u_A$ and $u_D < u$, while it increases for the velocities between, $u_A < u < u_D$. The boundaries u_A, u_D are the velocity values at the first and second intersection points of the two curves. In Figure 7, we depict the derivative

$dP(u;\kappa_0)/d\kappa_0$ for various kappa indices (and $\theta = 1$, $f = 3$). We observe that this is negative for velocities $u < u_{\mathrm{A}}$ and $u_{\mathrm{D}} < u$, while is positive for velocities between $u_{\mathrm{A}} < u < u_{\mathrm{D}}$. At the boundaries u_{A} and u_{D} the derivative $dP(u;\kappa_0)/d\kappa_0$ equals zero, with u_{A} corresponding to $d^2P(u;\kappa_0)/d\kappa_0^2 > 0$ and u_{D} to $d^2P(u;\kappa_0)/d\kappa_0^2 < 0$.

We show the dominance of the acceleration of the low-velocity particles on the deceleration of the high-velocity particles with the following numerical example. We examine the case of increasing kappa index from $\kappa_0 = 1.00$ to $\kappa_0 = 1.01$ ($\Delta\kappa_0 = 0.01$) in a system of $N = 10^6$ particles. We have $\Delta N_{\mathrm{A}} = \int_0^{u_{\mathrm{A}}} \Delta P(u)\, du$ accelerating and $\Delta N_{\mathrm{D}} = \int_{u_{\mathrm{D}}}^{\infty} \Delta P(u)\, du$ decelerating particles, while $\Delta N_{\mathrm{acc}} = \int_{u_{\mathrm{A}}}^{u_{\mathrm{D}}} \Delta P(u)\, du$ particles are accumulated in the mid-velocities; we find $u_{\mathrm{A}}/\theta \cong 0.76$, $u_{\mathrm{D}}/\theta \cong 3.07$. The probability distribution difference is given by $\Delta P(u) \cong (dP(u;\kappa_0)/d\kappa_0)\cdot\Delta\kappa_0$. Then, we estimate that $\Delta N_{\mathrm{A}} \cong 1427$ particles will accelerate, while only $\Delta N_{\mathrm{D}} \cong 65$ particles will decelerate to the mid velocities, that is, $\Delta N_{\mathrm{D}}/\Delta N_{\mathrm{A}} \cong 0.046$. The ratio $\Delta N_{\mathrm{D}}/\Delta N_{\mathrm{A}}$ increases with the kappa index up to ~ 0.39 for $\kappa_0 \to \infty$.

7. CONCLUSIONS

This paper deals with the invariant nature of the entire, N-particle multidimensional kappa distribution. This approach resolves several crucial problems that previously prevented construction of the entire kappa distribution.

1. For example, consider the farthest stationary state from thermal equilibrium, which is the one with the smallest possible kappa index, called the q-frozen state. While this boundary state is characterized by a universal probability distribution, i.e., it is independent of the degrees of freedom, f, of the system, it is characterized by a dependent value of the kappa index that behaves like $\sim f/2$.
2. Even worse, the minimum kappa index depends on the number of particles we choose to study the system. For example, for the n-particle distribution, with $n = 1, \ldots, N$, and for $D = 3$ degrees of freedom per particle, we have the minimum kappa index to be given by $\kappa \to 3n/2$. However, how can the kappa index that characterizes a specific state, here the q-frozen state, be dependent on our mathematical choice regarding the number of particles described by the distribution function?
3. This misleading identification of the kappa index does not refer to the q-frozen state alone. On the contrary, the same identification problem arises for any stationary state, because, as we showed in this paper, the kappa index is not an invariant quantity but depends on the degrees of freedom f. Therefore, how can each value of the kappa index consistently identify a specific stationary state?
4. Another inconsistency emerges when dealing with a large number of degrees of freedom, $f \to \infty$; having $f \to \infty$ (e.g., that is when $N \to \infty$) implies that the minimum kappa index, the index of the q-frozen state, is $\kappa \to \infty$. Hence, only one stationary state seems to be possible: thermal equilibrium. According to this, non-equilibrium stationary states should not be observed—but they are.

The inconsistency of the kappa index in characterizing the stationary states comes from the fact that it is dependent on the degrees of freedom and for this reason is denoted by κ_f. We derived the exact dependence of the kappa index on the degrees of freedom, which led naturally to the invariant definition of the kappa index, which is given by $\kappa_0 \equiv \kappa_f - f/2$. This index is independent of the degrees of freedom and represents the zero degrees of freedom kappa index. We then used κ_0 to construct invariant formulations of the kappa distribution function in terms of the velocities and energies of a complete N-particle system.

Prior to this analysis, the kappa distribution referred exclusively to the one-particle statistical description. The one-particle distribution constitutes the simplest way to study the phase-space distribution of particles, but this is just an approximation of the entire N-particle distribution function that has the whole statistical information. In fact, the concept of a one-particle distribution function only makes sense when the particles that constitute the system can be considered as weakly interacting and uncorrelated. This is true for many physically interesting cases of systems residing at or near thermal equilibrium. Even for systems residing in non-equilibrium states, it might be an acceptable approximation to go to one particle from the more complicated N-particle statistical description, but obviously, a great deal of information is lost.

There is another inconsistency related to the kappa index, which is of a physical nature instead of being simply mathematical. When using the one-particle kappa distribution rather than the entire N-particle distribution we assume that the correlation of the particles is almost zero and can be ignored. However, this is clearly not true. This paper showed that the correlation is interwoven with the formulation of the kappa distribution, and specifically, with the kappa index. Indeed, we calculated the correlation between two particles of 2 degrees of freedom and showed that is dependent only on the invariant kappa index κ_0, while it is independent of the total degrees of freedom, $f = D \cdot N$, i.e., of the total number of particles N. Previously, the correlation was mistakenly thought to be dependent on N, because the dependent kappa index κ_f was used, instead of the invariant kappa index κ_0.

The kappa index is of great importance for understanding, classifying, and studying the non-equilibrium stationary states. However, the relevant kappa index must be invariant, that is, independent of the degrees of freedom and the number of particles. The primary purpose of this paper is to solve these various inconsistencies and illuminate the physical meaning of the kappa index. Therefore, this analysis is a watershed in the applications of non-extensive Statistical Mechanics to the non-equilibrium space plasmas, primarily because it presents for the first time the invariant formulations of kappa distribution function of velocity and energy, and shows that these formulations encompass the entire N-particle phase-space distribution.

Having discovered the invariant kappa index, we finally were able to explain the physical meaning of the kappa index, namely, (1) its role in identifying the non-equilibrium stationary states and constructing the whole arrangement of stationary states that is called the κ-spectrum; (2) its connection with the correlation between different degrees of freedom; (3) its connection to a suitable measure of the thermodynamic distance of stationary states from thermal equilibrium; and (4) in characterizing a new kind of freezing (when the kappa index decreases) where particles decelerate to near zero velocities.

Table 2
Main Relations that Involve the Multidimensional Kappa Distribution in Terms of Velocity, Energy, and Flux

Description	Characteristics	Equation
Kappa distribution of velocities	f-dim distribution; κ_f-index (f-dependent)	$P(u_1,\ldots,u_f;\theta;\kappa_f;f)\,du_1\cdots du_f = \pi^{-\frac{f}{2}}\cdot\left(\kappa_f-\frac{f}{2}\right)^{-\frac{f}{2}}\cdot\frac{\Gamma(\kappa_f+1)}{\Gamma(\kappa_f+1-\frac{f}{2})}\cdot\theta^{-f}\cdot\left(1+\frac{1}{\kappa_f-\frac{f}{2}}\cdot\frac{1}{\theta^2}\sum_{i=1}^{f}u_i^2\right)^{-\kappa_f-1}du_1\cdots du_f$
Kappa distribution of velocities	f-dim distribution; κ_3-index (invariant)	$P(u_1,\ldots,u_f;\theta;\kappa_3;f)\,du_1\cdots du_f = \pi^{-\frac{f}{2}}\cdot(\kappa_3-\frac{3}{2})^{-\frac{f}{2}}\cdot\frac{\Gamma(\kappa_3-\frac{1}{2}+\frac{f}{2})}{\Gamma(\kappa_3-\frac{1}{2})}\cdot\theta^{-f}\cdot\left(1+\frac{1}{\kappa_3-\frac{3}{2}}\cdot\frac{1}{\theta^2}\sum_{i=1}^{f}u_i^2\right)^{-\kappa_3+\frac{1}{2}-\frac{f}{2}}du_1\cdots du_f$
Kappa distribution of velocities	f-dim distribution; κ_0-index (invariant)	$P(u_1,\ldots,u_f;\theta;\kappa_0;f)\,du_1\cdots du_f = \pi^{-\frac{f}{2}}\cdot\kappa_0^{-\frac{f}{2}}\cdot\frac{\Gamma(\kappa_0+1+\frac{f}{2})}{\Gamma(\kappa_0+1)}\cdot\theta^{-f}\cdot\left(1+\frac{1}{\kappa_0}\cdot\frac{1}{\theta^2}\sum_{i=1}^{f}u_i^2\right)^{-\kappa_0-1-\frac{f}{2}}du_1\cdots du_f$
Kappa distribution of energies	f-dim distribution; κ_f-index (f-dependent)	$P(\varepsilon_1,\ldots,\varepsilon_f;T;\kappa_f;f)\,d\varepsilon_1\cdots d\varepsilon_f = (4\pi)^{-\frac{f}{2}}\cdot(\kappa_f-\frac{f}{2})^{-\frac{f}{2}}\cdot\frac{\Gamma(\kappa_f+1)}{\Gamma(\kappa_f+1-\frac{f}{2})}\cdot(k_BT)^{-\frac{f}{2}}\cdot\left(1+\frac{1}{\kappa_f-\frac{f}{2}}\cdot\frac{1}{k_BT}\sum_{i=1}^{f}\varepsilon_i\right)^{-\kappa_f-1}\cdot\prod_{i=1}^{f}\varepsilon_i^{-\frac{1}{2}}d\varepsilon_1\cdots d\varepsilon_f$
Kappa distribution of energies	f-dim distribution; κ_3-index (invariant)	$P(\varepsilon_1,\ldots,\varepsilon_f;T;\kappa_3;f)\,d\varepsilon_1\cdots d\varepsilon_f = (4\pi)^{-\frac{f}{2}}\cdot(\kappa_3-\frac{3}{2})^{-\frac{f}{2}}\cdot\frac{\Gamma(\kappa_3-\frac{1}{2}+\frac{f}{2})}{\Gamma(\kappa_3-\frac{1}{2})}\cdot(k_BT)^{-\frac{f}{2}}\cdot\left(1+\frac{1}{\kappa_3-\frac{3}{2}}\cdot\frac{1}{k_BT}\sum_{i=1}^{f}\varepsilon_i\right)^{-\kappa_3+\frac{1}{2}-\frac{f}{2}}\cdot\prod_{i=1}^{f}\varepsilon_i^{-\frac{1}{2}}d\varepsilon_1\cdots d\varepsilon_f$
Kappa distribution of energies	f-dim distribution; κ_0-index (invariant)	$P(\varepsilon_1,\ldots,\varepsilon_f;T;\kappa_0;f)\,d\varepsilon_1\cdots d\varepsilon_f = (4\pi)^{-\frac{f}{2}}\cdot\kappa_0^{-\frac{f}{2}}\cdot\frac{\Gamma(\kappa_0+1+\frac{f}{2})}{\Gamma(\kappa_0+1)}\cdot(k_BT)^{-\frac{f}{2}}\cdot\left(1+\frac{1}{\kappa_0}\cdot\frac{1}{k_BT}\sum_{i=1}^{f}\varepsilon_i\right)^{-\kappa_0-1-\frac{f}{2}}\cdot\prod_{i=1}^{f}\varepsilon_i^{-\frac{1}{2}}d\varepsilon_1\cdots d\varepsilon_f$
Kappa distribution of velocities of N particles, each with D degrees of freedom	$D\cdot N$-dimensional distribution of the velocities of all the particles $\vec{u}_{(1)},\ldots,\vec{u}_{(N)}$; $D\cdot N$ degrees of freedom; κ_0-index	$P(\vec{u}_{(1)},\ldots,\vec{u}_{(N)};\theta;\kappa_0;D\cdot N) = \pi^{-\frac{D}{2}N}\cdot\kappa_0^{-\frac{D}{2}N}\cdot\frac{\Gamma(\kappa_0+1+\frac{D}{2}N)}{\Gamma(\kappa_0+1)}\cdot\theta^{-D\cdot N}\cdot\left(1+\frac{1}{\kappa_0}\cdot\frac{1}{\theta^2}\sum_{n=1}^{N}\vec{u}_{(n)}^2\right)^{-\kappa_0-1-\frac{D}{2}N}$
Kappa distribution of velocity magnitudes of N particles, each with D degrees of freedom	N-dim distribution of the velocity magnitudes of all the particles $u_{(1)},\ldots,u_{(N)}$; $D\cdot N$-degrees of freedom; κ_0-index	$P(u_{(1)},\ldots,u_{(N)};\theta;\kappa_0;D\cdot N) = \kappa_0^{-\frac{D}{2}N}\cdot\frac{\Gamma(\kappa_0+1+\frac{D}{2}N)\cdot 2^N}{\Gamma(\kappa_0+1)\cdot\Gamma(\frac{D}{2})^N}\cdot\theta^{-D\cdot N}\cdot\left(1+\frac{1}{\kappa_0}\cdot\frac{1}{\theta}\sum_{n=1}^{N}u_{(n)}^2\right)^{-\kappa_0-1-\frac{D}{2}N}\cdot\prod_{n=1}^{N}u_{(n)}^{D-1}$
Kappa distribution of energies of N particles, each with D degrees of freedom	N-dim distribution of the energies of all the particles $\varepsilon_{(1)},\ldots,\varepsilon_{(N)}$; $D\cdot N$-degrees of freedom; κ_0-index	$P(\varepsilon_{(1)},\ldots,\varepsilon_{(N)};T;\kappa_0;D\cdot N) = \kappa_0^{-\frac{D}{2}N}\cdot\frac{\Gamma(\kappa_0+1+\frac{D}{2}N)}{\Gamma(\kappa_0+1)\cdot\Gamma(\frac{D}{2})^N}\cdot(k_BT)^{-\frac{D}{2}\cdot N}\cdot\left(1+\frac{1}{\kappa_0}\cdot\frac{1}{k_BT}\sum_{n=1}^{N}\varepsilon_{(n)}\right)^{-\kappa_0-1-\frac{D}{2}N}\cdot\prod_{n=1}^{N}\varepsilon_{(n)}^{\frac{D}{2}-1}$
Density of states	f-degrees of freedom	Velocity magnitude: $g_{\mathrm{V}}(u;f)=[2\pi^{\frac{f}{2}}/\Gamma(\frac{f}{2})]\cdot u^{f-1}$; Energy: $g_{\mathrm{E}}(\varepsilon;f)=[(\frac{2\pi}{\mu})^{\frac{f}{2}}/\Gamma(\frac{f}{2})]\cdot\varepsilon^{\frac{f}{2}-1}$

Table 2
(Continued)

Description	Characteristics	Equation
Distribution of velocity magnitude	1-dim distribution of the velocity magnitude (u); f degrees of freedom; κ_0-index	$P_{\mathrm{V}}(u;\theta;\kappa_0;f)\,du = \frac{2\kappa_0^{-\frac{f}{2}}}{\mathrm{B}(\kappa_0+1,\frac{f}{2})}\cdot\theta^{-f}\cdot\left(1+\frac{1}{\kappa_0}\cdot\frac{u^2}{\theta^2}\right)^{-\kappa_0-1-\frac{f}{2}}u^{f-1}du$
Distribution of energy	1-dim distribution of the total energy (ε); f degrees of freedom; κ_0-index	$P_{\mathrm{E}}(\varepsilon;T;\kappa_0;f)\,d\varepsilon = \frac{\kappa_0^{-\frac{f}{2}}}{\mathrm{B}(\kappa_0+1,\frac{f}{2})}\cdot(k_{\mathrm{B}}T)^{-\frac{f}{2}}\cdot\left(1+\frac{1}{\kappa_0}\cdot\frac{\varepsilon}{k_{\mathrm{B}}T}\right)^{-\kappa_0-1-\frac{f}{2}}\varepsilon^{\frac{f}{2}-1}d\varepsilon =$ $F\left(\frac{\varepsilon}{\kappa_0\cdot k_{\mathrm{B}}T};2\kappa_0+2;f\right)d\left(\frac{\varepsilon}{\kappa_0\cdot k_{\mathrm{B}}T}\right)$
Distribution of average energy (energy per particle)	1-dim distribution: energy per particle ($\bar{\varepsilon}$) of D degrees of freedom; κ_0-index, N particles	$P_{\mathrm{E}}(\bar{\varepsilon};T;\kappa_0;D\cdot N)d\bar{\varepsilon} = \frac{N^{\frac{D}{2}N}\cdot\kappa_0^{-\frac{D}{2}N}}{\mathrm{B}(\kappa_0+1,\frac{D}{2}N)}\cdot(k_{\mathrm{B}}T)^{-\frac{D}{2}N}\cdot\left(1+\frac{N}{\kappa_0}\cdot\frac{\bar{\varepsilon}}{k_{\mathrm{B}}T}\right)^{-\kappa_0-1-\frac{D}{2}N}\bar{\varepsilon}^{\frac{D}{2}N-1}d\bar{\varepsilon}$
Distribution of average energy (energy per particle)	1-dim distribution: average energy ($\bar{\varepsilon}$) per particle of D degrees of freedom; κ_0-index, $N\to\infty$ particles	$P_{\mathrm{E}}(\bar{\varepsilon};T;\kappa_0;D\cdot N\to\infty)\,d\bar{\varepsilon} = \frac{[(D/2)\kappa_0]^{\kappa_0+1}}{\Gamma(\kappa_0+1)}\cdot(k_{\mathrm{B}}T)^{\kappa_0+1}\cdot\bar{\varepsilon}^{-\kappa_0-2}$ $\cdot\exp\left\{-(D/2)\kappa_0 k_{\mathrm{B}}T\cdot\frac{1}{\bar{\varepsilon}}\right\}\,d\bar{\varepsilon} = f_{\mathrm{Inv}\Gamma}\left\{\frac{\bar{\varepsilon}}{(D/2)\kappa_0 k_{\mathrm{B}}T};\kappa_0+1\right\}d\left\{\frac{\bar{\varepsilon}}{(D/2)\kappa_0 k_{\mathrm{B}}T}\right\}$
Flux vs. energy	f-dim distribution; κ_0-index	$j(\varepsilon;T;\kappa_0;f) = \frac{\eta}{\sqrt{2\mu}}\cdot\pi^{-\frac{f}{2}}\cdot\kappa_0^{-\frac{f}{2}}\cdot\frac{\Gamma(\kappa_0+1+\frac{f}{2})}{\Gamma(\kappa_0+1)}\cdot(k_{\mathrm{B}}T)^{-\frac{f}{2}}\cdot\left(1+\frac{1}{\kappa_0}\cdot\frac{\varepsilon}{k_{\mathrm{B}}T}\right)^{-\kappa_0-1-\frac{f}{2}}\cdot\varepsilon^{\frac{f-1}{2}}$
Flux vs. energy (Power law)	f-dim distribution; $\varepsilon >> \kappa_0 k_{\mathrm{B}}T$; κ_0-index	$j(\varepsilon)\cong j_0\cdot\varepsilon^{-\gamma}$, $j_0\equiv\frac{\eta}{\sqrt{2\mu}}\cdot\pi^{-\frac{f}{2}}\cdot\kappa_0^{\kappa_0}\cdot\frac{\Gamma(\kappa_0+\frac{f}{2}+1)}{\Gamma(\kappa_0)}\cdot(k_{\mathrm{B}}T)^{\kappa_0+1}$, $\gamma\equiv\kappa_0+\frac{3}{2}$

Notes. We use three kappa indices: the f-dimensional kappa index, κ_f, which is not invariant, but dependent on the degrees of freedom f; the invariant zero-dimensional κ_0, and the usual three-dimensional κ_3, which has been used in all the past publications (without the subscript "3"). F and $f_{\mathrm{Inv}\Gamma}$ stand for the "F" and "Inverse-Gamma" distributions. Each of the N particles is characterized by the same degrees of freedom D, so that $f = D\cdot N$. For the nth particle, $n = 1,\ldots,N$, the velocity vector is $\vec{u}_{(n)} \equiv (u_{1,(1)}, u_{2,(1)},\ldots,u_{D,(1)})$ and the energy $\varepsilon_{(n)} \equiv \frac{\mu}{2}\cdot\vec{u}_{(n)}^2 = \frac{\mu}{2}\cdot\sum_{i=1}^{D}u_{i,(n)}^2$. Whole distribution notations: $P_{\mathrm{V}}(u;\theta;\kappa_0;f) = P(u;\theta;\kappa_0;f)\cdot g_{\mathrm{V}}(u;f)$, $P_{\mathrm{E}}(\varepsilon;T;\kappa_0;f) = P(\varepsilon;T;\kappa_0;f)\cdot g_{\mathrm{E}}(\varepsilon;f)$.

APPENDIX A

VARIANT AND INVARIANT KAPPA INDEX

In an f-dimensional velocity space, the kappa distribution in terms of the dependent κ_f index is

$$P(u_1, \ldots, u_f; \theta; \kappa_f; f) = \pi^{-\frac{f}{2}} \cdot \left(\kappa_f - \frac{f}{2}\right)^{-\frac{f}{2}} \cdot \frac{\Gamma(\kappa_f + 1)}{\Gamma\left(\kappa_f + 1 - \frac{f}{2}\right)} \cdot \theta^{-f} \cdot \left(1 + \frac{1}{\kappa_f - \frac{f}{2}} \cdot \frac{1}{\theta^2} \sum_{i=1}^{f} u_i^2\right)^{-\kappa_f - 1}, \tag{A1}$$

We set $\vec{u}_f^2 \equiv \sum_{i=1}^{f} u_i^2$ and Equation (A1) is rewritten as

$$P(\vec{u}_f; \theta; \kappa_f; f) = \pi^{-\frac{f}{2}} \cdot \left(\kappa_f - \frac{f}{2}\right)^{-\frac{f}{2}} \cdot \frac{\Gamma(\kappa_f + 1)}{\Gamma\left(\kappa_f + 1 - \frac{f}{2}\right)} \cdot \theta^{-f} \cdot \left(1 + \frac{1}{\kappa_f - \frac{f}{2}} \cdot \frac{\vec{u}_f^2}{\theta^2}\right)^{-\kappa_f - 1}. \tag{A2}$$

The (f-1)-dimensional distribution is given by

$$P(\vec{u}_{f-1}; \theta; \kappa_{f-1}; f-1) = \pi^{-\frac{f-1}{2}} \cdot \left(\kappa_{f-1} - \frac{f-1}{2}\right)^{-\frac{f-1}{2}} \cdot \frac{\Gamma(\kappa_{f-1} + 1)}{\Gamma\left(\kappa_{f-1} + 1 - \frac{f-1}{2}\right)} \cdot \theta^{-(f-1)} \cdot \left(1 + \frac{1}{\kappa_{f-1} - \frac{f-1}{2}} \cdot \frac{\vec{u}_{f-1}^2}{\theta^2}\right)^{-\kappa_{f-1} - 1} \tag{A3}$$

and can be found by one integration, i.e., of the f th degree u_f,

$$P(\vec{u}_{f-1}; \theta; \kappa_{f-1}; f-1) \equiv \int P(\vec{u}_f; \theta; \kappa_f; f)\, du_f, \tag{A4}$$

where $\vec{u}_f^2 = \vec{u}_{f-1}^2 + u_f^2$. Then, we have

$$\begin{aligned}
P(\vec{u}_f; \theta; \kappa_f; f) &\sim \left[1 + \frac{1}{\kappa_f - \frac{f}{2}} \cdot \frac{\vec{u}_f^2}{\theta^2}\right]^{-\kappa_f - 1} \sim \left[1 + \frac{1}{\kappa_f - \frac{f}{2}} \cdot \frac{\vec{u}_{f-1}^2}{\theta^2}\right]^{-\kappa_f - 1} \cdot \left[1 + \frac{1}{K} \cdot \frac{u_f^2}{\theta^2}\right]^{-\kappa_f - 1} \\
&\Rightarrow \int P(\vec{u}_f; \theta; \kappa_f; f)\, du_f = \pi^{-\frac{f}{2}} \cdot \left(\kappa_f - \frac{f}{2}\right)^{-\frac{f}{2}} \cdot \frac{\Gamma(\kappa_f + 1)}{\Gamma\left(\kappa_f + 1 - \frac{f}{2}\right)} \cdot \theta^{-f} \cdot \left(1 + \frac{1}{\kappa_f - \frac{f}{2}} \cdot \frac{\vec{u}_{f-1}^2}{\theta^2}\right)^{-\kappa_f - 1} \\
&\times \int \left(1 + \frac{1}{K} \cdot \frac{u_f^2}{\theta^2}\right)^{-\kappa_f - 1} du_f,
\end{aligned} \tag{A5}$$

where $K \equiv (\kappa_f - \frac{f}{2}) + \vec{u}_{f-1}^2/\theta^2$, and

$$\begin{aligned}
\int_{-\infty}^{\infty} \left(1 + \frac{1}{K} \cdot \frac{u_f^2}{\theta^2}\right)^{-\kappa_f - 1} du_f &= \int_0^{\infty} \left(1 + \frac{1}{K} \cdot \frac{u_f^2}{\theta^2}\right)^{-\kappa_f - 1} 2du_f \\
&= \left(\frac{\kappa_f}{K}\right)^{-\frac{1}{2}} \cdot \theta \cdot \int_0^{\infty} \left(1 + \frac{1}{\kappa_f} \cdot z\right)^{-\kappa_f - 1} z^{\frac{1}{2} - 1} dz = \left(\frac{\kappa_f}{K}\right)^{-\frac{1}{2}} \cdot \Gamma_{\kappa_f}\left(\frac{1}{2}\right) \cdot \theta \\
&= \pi^{\frac{1}{2}} \cdot K^{\frac{1}{2}} \cdot \frac{\Gamma\left(\kappa_f + \frac{1}{2}\right)}{\Gamma(\kappa_f + 1)} \cdot \theta = \pi^{\frac{1}{2}} \cdot \left(\kappa_f - \frac{f}{2}\right)^{\frac{1}{2}} \cdot \frac{\Gamma\left(\kappa_f - \frac{1}{2}\right)}{\Gamma(\kappa_f + 1)} \cdot \theta \cdot \left(1 + \frac{1}{\kappa_f - \frac{f}{2}} \cdot \frac{\vec{u}_{f-1}^2}{\theta^2}\right)^{\frac{1}{2}},
\end{aligned}$$

where $z \equiv (\kappa_f/K) \cdot (u_f^2/\theta^2)$. The q-Gamma function, $\Gamma_q(a)$, was defined and investigated in Livadiotis & McComas (2009, their Appendix A), and is given in terms of the q- or κ-index by

$$\Gamma_q(a) = \Gamma(a) \cdot (q-1)^{1-a} \cdot \frac{\Gamma\left(\frac{q}{q-1} - a\right)}{\Gamma\left(\frac{1}{q-1}\right)}, \tag{A6a}$$

or in terms of the kappa index

$$\Gamma_\kappa(a) = \Gamma(a) \cdot \kappa^{a-1} \cdot \frac{\Gamma(\kappa + 1 - a)}{\Gamma(\kappa)}. \tag{A6b}$$

Therefore, the q-Gamma function for the q-index q_f, $\Gamma_{q_f}(1/2)$, or equivalently, for the kappa index κ_f ($q_f = 1 + 1/\kappa_f$), $\Gamma_{\kappa_f}(1/2)$, equals

$$\Gamma_{\kappa_f}\left(\frac{1}{2}\right) = \Gamma\left(\frac{1}{2}\right) \cdot \kappa_f^{\frac{1}{2}} \cdot \frac{\Gamma\left(\kappa_f - \frac{1}{2} + 1\right)}{\Gamma(\kappa_f + 1)}.$$

Hence, we find

$$\int P(\vec{u}_f;\theta;\kappa_f;f)\,du_f = \pi^{-\frac{f-1}{2}}\cdot\left(\kappa_f-\frac{f}{2}\right)^{-\frac{f-1}{2}}\cdot\frac{\Gamma\left(\kappa_f+\frac{1}{2}\right)}{\Gamma\left(\kappa_f+1-\frac{f}{2}\right)}\cdot\theta^{-(f-1)}\cdot\left(1+\frac{1}{\kappa_f-\frac{f}{2}}\cdot\frac{\vec{u}_{f-1}^2}{\theta^2}\right)^{-\kappa_f-1+\frac{1}{2}}$$
$$= \pi^{-\frac{f-1}{2}}\cdot\left(\kappa_f-\frac{1}{2}-\frac{f-1}{2}\right)^{-\frac{f-1}{2}}\cdot\frac{\Gamma\left(\kappa_f-\frac{1}{2}+1\right)}{\Gamma\left(\kappa_f-\frac{1}{2}+1-\frac{f-1}{2}\right)}\cdot\theta^{-(f-1)}$$
$$\cdot\left(1+\frac{1}{\kappa_f-\frac{1}{2}-\frac{f-1}{2}}\cdot\frac{\vec{u}_{f-1}^2}{\theta^2}\right)^{-(\kappa_f-\frac{1}{2})-1}$$
$$= P\left(\vec{u}_{f-1};\theta;\kappa_f-\frac{1}{2};f-1\right).$$

Then the integration of the fth degree u_f is written in summary as

$$P(\vec{u}_{f-1};\theta;\kappa_{f-1};f-1)\equiv\int P(\vec{u}_f;\theta;\kappa_f;f)\,du_f = P\left(\vec{u}_{f-1};\theta;\kappa_f-\frac{1}{2};f-1\right) \tag{A7}$$

or

$$\kappa_{f-1}=\kappa_f-\frac{1}{2}. \tag{A8}$$

Now we proceed with the triple integration of the fth, $(f-1)$th, and $(f-2)$th degrees of freedom, i.e., the variables u_f, u_{f-1}, and u_{f-2}, namely,

$$P(\vec{u}_{f-3};\theta;\kappa_{f-3};f-3)\equiv\int P(\vec{u}_f;\theta;\kappa_f;f)\,du_{f-2}du_{f-1}du_f. \tag{A9}$$

We denote $\vec{u}_f^2=\vec{u}_{f-3}^2+\vec{w}^2$, $\vec{w}^2\equiv u_{f-2}^2+u_{f-1}^2+u_f^2$. Then, we have

$$P(\vec{u}_f;\theta;\kappa_f;f)\sim\left[1+\frac{1}{\kappa_f-\frac{f}{2}}\cdot\frac{\vec{u}_f^2}{\theta^2}\right]^{-\kappa_f-1}\sim\left[1+\frac{1}{\kappa_f-\frac{f}{2}}\cdot\frac{\vec{u}_{f-3}^2}{\theta^2}\right]^{-\kappa_f-1}\cdot\left[1+\frac{1}{K}\cdot\frac{\vec{w}^2}{\theta^2}\right]^{-\kappa_f-1}$$
$$\Rightarrow\int P(\vec{u}_f;\theta;\kappa_f;f)\,du_{f-2}\,du_{f-1}\,du_f=\pi^{-\frac{f}{2}}\cdot\left(\kappa_f-\frac{f}{2}\right)^{-\frac{f}{2}}\cdot\frac{\Gamma(\kappa_f+1)}{\Gamma\left(\kappa_f+1-\frac{f}{2}\right)}\cdot\theta^{-f}$$
$$\cdot\left(1+\frac{1}{\kappa_f-\frac{f}{2}}\cdot\frac{\vec{u}_{f-3}^2}{\theta^2}\right)^{-\kappa_f-1}\int\left(1+\frac{1}{K}\cdot\frac{\vec{w}^2}{\theta^2}\right)^{-\kappa_f-1}du_{f-2}du_{f-1}du_f, \tag{A10}$$

where $K\equiv(\kappa_f-\frac{f}{2})+\vec{u}_{f-3}^2/\theta^2$. We have

$$\int_{-\infty}^{\infty}\left(1+\frac{1}{K}\cdot\frac{\vec{w}^2}{\theta^2}\right)^{-\kappa_f-1}du_{f-2}du_{f-1}du_f=\int_0^{\infty}\left(1+\frac{1}{K}\cdot\frac{w^2}{\theta^2}\right)^{-\kappa_f-1}4\pi w^2dw$$
$$=2\pi\cdot\left(\frac{\kappa_f}{K}\right)^{-\frac{3}{2}}\cdot\theta^3\cdot\int_0^{\infty}\left(1+\frac{1}{\kappa_f}\cdot z\right)^{-\kappa_f-1}z^{\frac{3}{2}-1}dz=2\pi\cdot\left(\frac{\kappa_f}{K}\right)^{-\frac{3}{2}}\cdot\Gamma_{\kappa_f}\left(\frac{3}{2}\right)\cdot\theta^3$$
$$=\pi^{\frac{3}{2}}\cdot K^{\frac{3}{2}}\cdot\frac{\Gamma\left(\kappa_f-\frac{1}{2}\right)}{\Gamma(\kappa_f+1)}\cdot\theta^3=\pi^{\frac{3}{2}}\cdot\left(\kappa_f-\frac{f}{2}\right)^{\frac{3}{2}}\cdot\frac{\Gamma\left(\kappa_f-\frac{1}{2}\right)}{\Gamma(\kappa_f+1)}\cdot\theta^3$$
$$\cdot\left(1+\frac{1}{\kappa_f-\frac{f}{2}}\cdot\frac{\vec{u}_{f-3}^2}{\theta^2}\right)^{\frac{3}{2}},$$

where $z\equiv(\kappa_f/K)\cdot(w^2/\theta^2)$, $\Gamma_{\kappa_f}(3/2)=\Gamma(3/2)\cdot\kappa_f^{\frac{1}{2}}\cdot\Gamma(\kappa_f-\frac{1}{2})/\Gamma(\kappa_f)=\frac{1}{2}\pi^{\frac{1}{2}}\cdot\kappa_f^{\frac{3}{2}}\cdot\Gamma(\kappa_f-\frac{1}{2})/\Gamma(\kappa_f+1)$. Hence,

$$\int P(\vec{u}_f;\theta;\kappa_f;f)\,du_{f-2}du_{f-1}du_f=\pi^{-\frac{f-3}{2}}\cdot\left(\kappa_f-\frac{f}{2}\right)^{-\frac{f-3}{2}}\cdot\frac{\Gamma\left(\kappa_f-\frac{1}{2}\right)}{\Gamma\left(\kappa_f-\frac{f}{2}+1\right)}\cdot\theta^{-(f-3)}\cdot\left(1+\frac{1}{\kappa_f-\frac{f}{2}}\cdot\frac{\vec{u}_{f-3}^2}{\theta^2}\right)^{-\kappa_f-1+\frac{3}{2}}$$
$$=\pi^{-\frac{f-3}{2}}\cdot\left(\kappa_f-\frac{3}{2}-\frac{f-3}{2}\right)^{-\frac{f-3}{2}}\cdot\frac{\Gamma\left(\kappa_f-\frac{3}{2}+1\right)}{\Gamma\left(\kappa_f-\frac{3}{2}+1-\frac{f-3}{2}\right)}\cdot\theta^{-(f-3)}$$

$$\cdot\left(1+\frac{1}{\kappa_f-\frac{3}{2}-\frac{f-3}{2}}\cdot\frac{\vec{u}_{f-3}^2}{\theta^2}\right)^{-(\kappa_f-\frac{3}{2})-1}$$
$$=P\left(\vec{u}_{f-3};\theta;\kappa_f-\frac{3}{2};f-3\right).$$

Then the triple integration of the degrees of freedom u_f, u_{f-1}, and u_{f-2}, is written as

$$P(\vec{u}_{f-3};\theta;\kappa_{f-3};f-3)\equiv\int P(\vec{u}_f;\theta;\kappa_f;f)\,du_{f-2}du_{f-1}du_f=P\left(\vec{u}_{f-3};\theta;\kappa_f-\frac{3}{2};f-3\right)\tag{A11}$$

or

$$\kappa_{f-3}=\kappa_f-\frac{3}{2}.\tag{A12}$$

We continue with a more general case, where we integrate $\tilde{f}$ degrees of freedom, that is, $u_f, u_{f-1}, u_{f-2}, \ldots, u_{f-\tilde{f}+1}$. This is given by

$$P(\vec{u}_{f-\tilde{f}};\theta;\kappa_{f-\tilde{f}};f-\tilde{f})\equiv\int P(\vec{u}_f;\theta;\kappa_f;f)\,du_{f-\tilde{f}+1}\cdots du_{f-1}du_f.\tag{A13}$$

We denote $\vec{u}_f^2=\vec{u}_{f-\tilde{f}}^2+\vec{w}^2$, $\vec{w}^2\equiv u_{f-\tilde{f}+1}^2+\cdots+u_{f-2}^2+u_{f-1}^2+u_f^2$. Then, we have

$$P(\vec{u}_f;\theta;\kappa_f;f)\sim\left[1+\frac{1}{\kappa_f-\frac{f}{2}}\cdot\frac{\vec{u}_f^2}{\theta^2}\right]^{-\kappa_f-1}\sim\left[1+\frac{1}{\kappa_f-\frac{f}{2}}\cdot\frac{\vec{u}_{f-\tilde{f}}^2}{\theta^2}\right]^{-\kappa_f-1}\cdot\left[1+\frac{1}{K}\cdot\frac{\vec{w}^2}{\theta^2}\right]^{-\kappa_f-1},$$
$$\Rightarrow\int P(\vec{u}_f;\theta;\kappa_f;f)\,du_{f-\tilde{f}+1}\cdots du_{f-1}du_f=\pi^{-\frac{f}{2}}\cdot\left(\kappa_f-\frac{f}{2}\right)^{-\frac{f}{2}}\cdot\frac{\Gamma(\kappa_f+1)}{\Gamma\left(\kappa_f-\frac{f}{2}+1\right)}\cdot\theta^{-f}$$
$$\cdot\left(1+\frac{1}{\kappa_f-\frac{f}{2}}\cdot\frac{\vec{u}_{f-\tilde{f}}^2}{\theta^2}\right)^{-\kappa_f-1}\int\left(1+\frac{1}{K}\cdot\frac{\vec{w}^2}{\theta^2}\right)^{-\kappa_f-1}du_{f-\tilde{f}+1}\cdots du_{f-1}du_f,\tag{A14}$$

where we defined $K\equiv(\kappa_f-\frac{f}{2})+\vec{u}_{f-\tilde{f}}^2/\theta^2$. Then,

$$\int\left(1+\frac{1}{K}\cdot\frac{\vec{w}^2}{\theta^2}\right)^{-\kappa_f-1}du_{f-\tilde{f}+1}\cdots du_{f-1}du_f=\int_0^\infty\left(1+\frac{1}{K}\cdot\frac{w^2}{\theta^2}\right)^{-\kappa_f-1}B(\tilde{f})w^{\tilde{f}-1}dw$$
$$=\frac{1}{2}B(\tilde{f})\cdot\left(\frac{\kappa_f}{K}\right)^{-\frac{\tilde{f}}{2}}\cdot\theta^{\tilde{f}}\cdot\int_0^\infty\left(1+\frac{1}{\kappa_f}\cdot z\right)^{-\kappa_f-1}z^{\frac{\tilde{f}}{2}-1}dz$$
$$=\frac{1}{2}B(\tilde{f})\cdot\left(\frac{\kappa_f}{K}\right)^{-\frac{\tilde{f}}{2}}\cdot\Gamma_{\kappa_f}\left(\frac{\tilde{f}}{2}\right)\cdot\theta^{\tilde{f}}$$
$$=\pi^{\frac{\tilde{f}}{2}}\cdot K^{\frac{\tilde{f}}{2}}\cdot\frac{\Gamma\left(\kappa_f-\frac{\tilde{f}}{2}+1\right)}{\Gamma(\kappa_f+1)}\cdot\theta^{\tilde{f}}=\pi^{\frac{\tilde{f}}{2}}\cdot\left(\kappa_f-\frac{f}{2}\right)^{\frac{\tilde{f}}{2}}\cdot\frac{\Gamma\left(\kappa_f-\frac{\tilde{f}}{2}+1\right)}{\Gamma(\kappa_f+1)}\cdot\theta^3$$
$$\cdot\left(1+\frac{1}{\kappa_f-\frac{f}{2}}\cdot\frac{\vec{u}_{f-\tilde{f}}^2}{\theta^2}\right)^{\frac{\tilde{f}}{2}},$$

where $B(\tilde{f})=2\pi^{\frac{\tilde{f}}{2}}/\Gamma(\frac{\tilde{f}}{2})$ (see Equation (26c)); the auxiliary argument z is defined by $z\equiv(\kappa_f/K)\cdot(w^2/\theta^2)$. We used the q-Gamma function $\Gamma_{\kappa_f}(\frac{\tilde{f}}{2})=\Gamma(\frac{\tilde{f}}{2})\cdot\kappa_f^{\frac{\tilde{f}}{2}}\cdot\Gamma(\kappa_f-\frac{\tilde{f}}{2}+1)/\Gamma(\kappa_f+1)$. Hence,

$$\int P(\vec{u}_f;\theta;\kappa_f;f)\,du_{f-\tilde{f}+1}\cdots du_{f-1}du_f=\pi^{-\frac{f-\tilde{f}}{2}}\cdot\left(\kappa_f-\frac{f}{2}\right)^{-\frac{f-\tilde{f}}{2}}\cdot\frac{\Gamma\left(\kappa_f-\frac{\tilde{f}}{2}+1\right)}{\Gamma\left(\kappa_f-\frac{f}{2}+1\right)}\cdot\theta^{-(f-\tilde{f})}\cdot\left(1+\frac{1}{\kappa_f-\frac{f}{2}}\cdot\frac{\vec{u}_{f-\tilde{f}}^2}{\theta^2}\right)^{-\kappa_f-1+\frac{\tilde{f}}{2}}$$
$$=\pi^{-\frac{f-\tilde{f}}{2}}\cdot\left(\kappa_f-\frac{\tilde{f}}{2}-\frac{f-\tilde{f}}{2}\right)^{-\frac{f-\tilde{f}}{2}}\cdot\frac{\Gamma\left(\kappa_f-\frac{\tilde{f}}{2}+1\right)}{\Gamma\left(\kappa_f-\frac{\tilde{f}}{2}-\frac{f-\tilde{f}}{2}+1\right)}\cdot\theta^{-(f-\tilde{f})}$$
$$\cdot\left(1+\frac{1}{\kappa_f-\frac{\tilde{f}}{2}-\frac{f-\tilde{f}}{2}}\cdot\frac{\vec{u}_{f-\tilde{f}}^2}{\theta^2}\right)^{-(\kappa_f-\frac{\tilde{f}}{2})-1}=P\left(\vec{u}_{f-\tilde{f}};\theta;\kappa_f-\frac{\tilde{f}}{2};f-\tilde{f}\right).$$

Then the multiple integration of the degrees of freedom $u_f, u_{f-1}, u_{f-2}, \ldots, u_{f-\tilde{f}+1}$, is written as

$$P(\vec{u}_{f-\tilde{f}};\theta;\kappa_{f-\tilde{f}};f-\tilde{f}) \equiv \int P(\vec{u}_f;\theta;\kappa_f;f)\,du_{f-\tilde{f}+1}\cdots du_{f-1}du_f = P\left(\vec{u}_{f-\tilde{f}};\theta;\kappa_f-\frac{\tilde{f}}{2};f-\tilde{f}\right), \tag{A15}$$

$$\kappa_{f-\tilde{f}} = \kappa_f - \frac{\tilde{f}}{2}. \tag{A16}$$

Then, setting $f \to f_1$, $f-\tilde{f} \to f_2$, we finally conclude in $\kappa_{f_2} = \kappa_{f_1} - \frac{f_1-f_2}{2}$ or

$$\kappa_{f_1} - \frac{f_1}{2} = \kappa_{f_2} - \frac{f_2}{2}, \tag{A17}$$

and thus, an invariant kappa index, independent of the degrees of freedom f, is naturally defined by

$$\kappa_0 \equiv \kappa_f - \frac{f}{2}. \tag{A18}$$

APPENDIX B
CORRELATION

In this appendix, we calculate the statistical moments that are used for deriving the correlation between 2 degrees of freedom or two particles (of a certain degrees of freedom each). Note that, in a similar way, one can calculate any other statistical moment.

We begin by considering that each of the N particles is characterized by the same degrees of freedom D, so that $f = D\cdot N$. For the nth particle, $n = 1,\ldots,N$, the velocity vector is $\vec{u}_{(n)} \equiv (u_{1,(1)}, u_{2,(1)},\ldots,u_{D,(1)})$, and the energy $\varepsilon_{(n)} \equiv 1\mu/2\cdot\vec{u}^2_{(n)} = 1\mu/2\cdot\sum_{i=1}^{D} u^2_{i,(n)}$. The N-particle distribution is given by

$$P(\vec{u}_{(1)},\ldots,\vec{u}_{(N)};\theta;\kappa_0;D\cdot N) = \pi^{-\frac{D}{2}N}\cdot\kappa_0^{-\frac{D}{2}N}\cdot\frac{\Gamma\left(\kappa_0+1+\frac{D}{2}N\right)}{\Gamma(\kappa_0+1)}\cdot\theta^{-D\cdot N}\cdot\left(1+\frac{1}{\kappa_0}\cdot\frac{1}{\theta^2}\sum_{n=1}^{N}\vec{u}^2_{(n)}\right)^{-\kappa_0-1-\frac{D}{2}N}, \tag{B1}$$

which is normalized by $D\cdot N$ integrations all over the N particles velocity vectors,

$$1 = \int P(\vec{u}_{(1)},\ldots,\vec{u}_{(N)};\theta;\kappa_0;D\cdot N)\,d\vec{u}_{(1)}\ldots d\vec{u}_{(N)}, \tag{B2}$$

where $d\vec{u}_{(n)} \equiv du_{1,(n)}\cdots du_{i,(n)}\cdots du_{D,(n)}$.

The mean velocity of each particle is zero for the form of distribution given by Equation (B1), i.e.,

$$\langle\vec{u}_{(n)}\rangle = \int_{-\infty}^{\infty} P(\vec{u}_{(1)},\ldots,\vec{u}_{(N)};\theta;\kappa_0;D\cdot N)\cdot\vec{u}_{(n)}d\vec{u}_{(1)}\cdots d\vec{u}_{(N)} = 0. \tag{B3}$$

A non-zero mean distribution is given by the modified formulation

$$P(\vec{u}_{(1)},\ldots,\vec{u}_{(N)};\theta;\kappa_0;D\cdot N) = \pi^{-\frac{D}{2}N}\cdot\kappa_0^{-\frac{D}{2}N}\cdot\frac{\Gamma\left(\kappa_0+1+\frac{D}{2}N\right)}{\Gamma(\kappa_0+1)}\cdot\theta^{-D\cdot N}\cdot\left(1+\frac{1}{\kappa_0}\cdot\frac{1}{\theta^2}\sum_{n=1}^{N}(\vec{u}_{(n)}-\vec{u}_{b(n)})^2\right)^{-\kappa_0-1-\frac{D}{2}N}, \tag{B4}$$

where the mean velocity of each particle is given by

$$\vec{u}_{b(n)} \equiv \langle\vec{u}_{(n)}\rangle = \int_{-\infty}^{\infty} P(\vec{u}_{(1)},\ldots,\vec{u}_{(N)};\theta;\kappa_0;D\cdot N)\,\vec{u}_{(n)}d\vec{u}_{(1)}\cdots d\vec{u}_{(N)}. \tag{B5}$$

The bulk velocity of the space plasma is given by the average of this velocity vector, i.e.,

$$\vec{u}_b \equiv (1/N)\cdot\sum_{n=1}^{N}\vec{u}_{b(n)}. \tag{B6}$$

Here we consider zero mean velocity. Further, the second statistical moment is given by

$$\left\langle\vec{u}^2_{(n)}\right\rangle = \int_{-\infty}^{\infty} P(\vec{u}_{(1)},\ldots,\vec{u}_{(N)};\theta;\kappa_0;D\cdot N)\cdot\vec{u}^2_{(n)}d\vec{u}_{(1)}\cdots d\vec{u}_{(N)}. \tag{B7}$$

We integrate the degrees of freedom of all the other particles (other than the nth particle), that is,

$$\left\langle \vec{u}_{(n)}^2 \right\rangle = \int_{-\infty}^{\infty} P(\vec{u}_{(n)}; \theta; \kappa_0; D) \vec{u}_{(n)}^2 d\vec{u}_{(n)}, \tag{B8}$$

where

$$P(\vec{u}_{(n)}; \theta; \kappa_0; D) = \int_{-\infty}^{\infty} P(\vec{u}_{(1)}, \ldots, \vec{u}_{(N)}; \theta; \kappa_0; D \cdot N)\, d\vec{u}_{(1)} \cdots d\vec{u}_{(n-1)} d\vec{u}_{(n+1)} \cdots d\vec{u}_{(N)},$$

which is the one-particle distribution given by

$$P(\vec{u}_{(n)}; \theta; \kappa_0; D) = \pi^{-\frac{D}{2}} \cdot \kappa_0^{-\frac{D}{2}} \cdot \frac{\Gamma\left(\kappa_0 + 1 + \frac{D}{2}\right)}{\Gamma(\kappa_0 + 1)} \cdot \theta^{-D} \cdot \left(1 + \frac{1}{\kappa_0} \cdot \frac{1}{\theta^2} \vec{u}_{(n)}^2\right)^{-\kappa_0 - 1 - \frac{D}{2}}. \tag{B9}$$

Then, we have

$$\left\langle \vec{u}_{(n)}^2 \right\rangle = \sum_{i=1}^{D} \left\langle u_{i,(n)}^2 \right\rangle = \sum_{i=1}^{D} \int_{-\infty}^{\infty} P\left(u_{i,(n)}; \theta; \kappa_0; D = 1\right) u_{i,(n)}^2 du_{i,(n)}, \tag{B10}$$

where $\vec{u}_{(n)}^2 = \sum_{i=1}^{D} u_{i,(n)}^2$ and the distribution of one-degree of freedom is given by integrating over all the rest of the degrees, i.e.,

$$P\left(u_{i,(n)}; \theta; \kappa_0; D = 1\right) = \int_{-\infty}^{\infty} P(\vec{u}_{(n)}; \theta; \kappa_0; D)\, du_{1,(n)} \cdots du_{i-1,(n)} du_{i+1,(n)} \cdots du_{D,(n)},$$

that is,

$$P\left(u_{i,(n)}; \theta; \kappa_0; D = 1\right) = \pi^{-\frac{1}{2}} \cdot \kappa_0^{-\frac{1}{2}} \cdot \frac{\Gamma\left(\kappa_0 + \frac{3}{2}\right)}{\Gamma(\kappa_0 + 1)} \cdot \theta^{-1} \cdot \left(1 + \frac{1}{\kappa_0} \cdot \frac{1}{\theta^2} u_{i,(n)}^2\right)^{-\kappa_0 - \frac{3}{2}}. \tag{B11}$$

Hence,

$$\begin{aligned}
\left\langle u_{i,(n)}^2 \right\rangle &= \int_{-\infty}^{\infty} P\left(u_{i,(n)}; \theta; \kappa_0; D = 1\right) \cdot u_{i,(n)}^2 du_{i,(n)} \\
&= \theta^2 \cdot \pi^{-\frac{1}{2}} \cdot \kappa_0^{-\frac{1}{2}} \cdot \frac{\Gamma\left(\kappa_0 + \frac{3}{2}\right)}{\Gamma(\kappa_0 + 1)} \cdot \left(\frac{\kappa_0 + \frac{1}{2}}{\kappa_0}\right)^{-\frac{3}{2}} \cdot \int_0^{\infty} \left(1 + \frac{1}{\kappa_0 + \frac{1}{2}} \cdot z\right)^{-(\kappa_0 + \frac{1}{2}) - 1} z^{\frac{3}{2} - 1} dz \\
&= \theta^2 \cdot \pi^{-\frac{1}{2}} \cdot \kappa_0^{-\frac{1}{2}} \cdot \frac{\Gamma\left(\kappa_0 + \frac{3}{2}\right)}{\Gamma(\kappa_0 + 1)} \cdot \left(\frac{\kappa_0 + \frac{1}{2}}{\kappa_0}\right)^{-\frac{3}{2}} \cdot \Gamma_{\kappa_0 + \frac{1}{2}}\left(\frac{3}{2}\right),
\end{aligned}$$

where the q-Gamma function is defined by $\Gamma_{\kappa_0+\frac{1}{2}}(\frac{3}{2}) = \int_0^{\infty} (1 + \frac{1}{\kappa_0 + \frac{1}{2}} \cdot z)^{-(\kappa_0 + \frac{1}{2}) - 1} z^{\frac{3}{2} - 1} dz$, with $z \equiv [(\kappa_0 + \frac{1}{2})/\kappa_0] \cdot (u_{i,(n)}/\theta)^2$. According to Equation (A6b), the q-Gamma function is $\Gamma_{\kappa_0+\frac{1}{2}}(\frac{3}{2}) = \frac{1}{2}\pi^{\frac{1}{2}} \cdot (\kappa_0 + \frac{1}{2})^{\frac{1}{2}} \cdot \Gamma(\kappa_0)/\Gamma(\kappa_0 + \frac{1}{2})$. Then, we have

$$\left\langle u_{i,(n)}^2 \right\rangle = \frac{1}{2} \cdot \theta^2. \tag{B12}$$

Since $\left\langle u_{i,(n)}^2 \right\rangle$ does not depend on the specific ith degree of freedom, Equation (B10) gives

$$\left\langle \vec{u}_{(n)}^2 \right\rangle = \frac{D}{2} \cdot \theta^2 \tag{B13a}$$

or in terms of energy

$$\langle \varepsilon_{(n)} \rangle = \frac{D}{2} \cdot k_{\mathrm{B}} T. \tag{B13b}$$

We continue to the calculation of the fourth statistical moment. In similar to Equation (B8), we take into account only the nth particle's D degrees of freedom, i.e.,

$$\left\langle \vec{u}_{(n)}^4 \right\rangle = \int_{-\infty}^{\infty} P(\vec{u}_{(n)}; \theta; \kappa_0; D) \vec{u}_{(n)}^4 d\vec{u}_{(n)}. \tag{B14}$$

Hence,

$$\left\langle \vec{u}_{(n)}^4 \right\rangle = \pi^{-\frac{D}{2}} \cdot \kappa_0^{-\frac{D}{2}} \cdot \frac{\Gamma\left(\kappa_0 + 1 + \frac{D}{2}\right)}{\Gamma(\kappa_0 + 1)} \cdot \theta^{-D} \cdot \int_{-\infty}^{\infty} \left(1 + \frac{1}{\kappa_0} \cdot \frac{1}{\theta^2} u_{(n)}^2\right)^{-\kappa_0 - 1 - \frac{D}{2}} \cdot B(D) \cdot u_{(n)}^{D+3} du_{(n)}, \tag{B15}$$

where $u_{(n)} \equiv \vec{u}_{(n)}$ and $B(D)$ is given by Equation (26c).

$$\left\langle \vec{u}_{(n)}^4 \right\rangle = \theta^4 \cdot \pi^{-\frac{D}{2}} \cdot \frac{1}{2} B(D) \cdot \kappa_0^{-\frac{D}{2}} \cdot \frac{\Gamma\left(\kappa_0 + 1 + \frac{D}{2}\right)}{\Gamma(\kappa_0 + 1)} \cdot \left(\frac{\kappa_0 + \frac{D}{2}}{\kappa_0}\right)^{-\frac{D}{2}-2} \cdot \Gamma_{\kappa_0 + \frac{D}{2}}\left(\frac{D}{2} + 2\right), \tag{B16}$$

where $\Gamma_{\kappa_0+\frac{D}{2}}(\frac{D}{2}+2) = \int_{-\infty}^{\infty} (1 + \frac{1}{\kappa_0+\frac{D}{2}} \cdot z)^{-(\kappa_0+\frac{D}{2})-1} z^{\frac{D}{2}+2-1} dz$, $z \equiv [(\kappa_0 + \frac{D}{2})/\kappa_0] \cdot (u_{(n)}/\theta)^2$. Equation (A6b) gives $\Gamma_{\kappa_0+\frac{D}{2}}(\frac{D}{2}+2) = \Gamma(\frac{D}{2}+2) \cdot (\kappa_0 + \frac{D}{2})^{\frac{D}{2}+1} \cdot \Gamma(\kappa_0 - 1)/\Gamma(\kappa_0 + \frac{D}{2})$. Hence, Equation (B16) becomes

$$\left\langle \vec{u}_{(n)}^4 \right\rangle = \theta^4 \cdot \frac{D}{2} \cdot \left(\frac{D}{2} + 1\right) \cdot \frac{\kappa_0}{\kappa_0 - 1}. \tag{B17}$$

Moreover, we calculate the second statistical moment of two particles together, namely,

$$\begin{aligned} \left\langle \vec{u}_{(n)}^2 \cdot \vec{u}_{(m)}^2 \right\rangle &= \sum_{i=1}^{D} \sum_{j=1}^{D} \left\langle u_{i,(n)}^2 \cdot u_{j,(m)}^2 \right\rangle \\ &= \sum_{i=1}^{D} \sum_{j=1}^{D} \int_{-\infty}^{\infty} P\left(u_{i,(n)}, u_{j,(m)}; \theta; \kappa_0; D = 2\right) \cdot u_{i,(n)}^2 \cdot u_{j,(m)}^2 du_{i,(n)} du_{j,(m)}, \end{aligned} \tag{B18}$$

where the kappa distribution of $D = 2$ degrees of freedom (1 degree for each particle) becomes

$$P\left(u_{i,(n)}, u_{j,(m)}; \theta; \kappa_0; D = 2\right) = \frac{\kappa_0 + 1}{\pi \cdot \kappa_0} \cdot \theta^{-2} \cdot \left[1 + \frac{1}{\kappa_0} \cdot \frac{1}{\theta^2}\left(u_{i,(n)}^2 + u_{j,(m)}^2\right)\right]^{-\kappa_0 - 2}. \tag{B19}$$

Then,

$$\begin{aligned} \left\langle u_{i,(n)}^2 \cdot u_{j,(m)}^2 \right\rangle &= \int_{-\infty}^{\infty} P\left(u_{i,(n)}, u_{j,(m)}; \theta; \kappa_0; D = 2\right) \cdot u_{i,(n)}^2 \cdot u_{j,(m)}^2 du_{i,(n)} du_{j,(m)} \\ &= \frac{\kappa_0 + 1}{\pi \cdot \kappa_0} \cdot \theta^{-2} \cdot \int_{-\infty}^{\infty} \left(1 + \frac{1}{\kappa_0} \cdot \frac{1}{\theta^2} u_{j,(m)}^2\right)^{-\kappa_0 - 2} \cdot u_{j,(m)}^2 \cdot I_{i,(n)} du_{j,(m)} \end{aligned}$$

with the involved integral I given by

$$\begin{aligned} I_{i,(n)} &\equiv \int_{-\infty}^{\infty} \left(1 + \frac{1}{K} \cdot \frac{1}{\theta^2} u_{i,(n)}^2\right)^{-\kappa_0 - 2} \cdot u_{i,(n)}^2 du_{i,(n)} \\ &= \theta^3 \cdot \left(\frac{\kappa_0 + 1}{K}\right)^{-\frac{3}{2}} \cdot \int_0^{\infty} \left(1 + \frac{1}{\kappa_0 + 1} \cdot z\right)^{-(\kappa_0 + 1) - 1} \cdot z^{\frac{3}{2} - 1} dz = \theta^3 \cdot \left(\frac{\kappa_0 + 1}{K}\right)^{-\frac{3}{2}} \cdot \Gamma_{\kappa_0 + 1}\left(\frac{3}{2}\right), \end{aligned}$$

where $K \equiv \kappa_0 + u_{j,(m)}^2/\theta^2$, $z \equiv [(\kappa_0 + 1)/K] \cdot (u_{i,(n)}^2/\theta^2)$. Hence,

$$\begin{aligned} \left\langle u_{i,(n)}^2 \cdot u_{j,(m)}^2 \right\rangle &= \frac{\kappa_0 + 1}{\pi \cdot \kappa_0} \cdot \theta \cdot \Gamma_{\kappa_0 + 1}\left(\frac{3}{2}\right) \cdot \int_{-\infty}^{\infty} \left(1 + \frac{1}{\kappa_0} \cdot \frac{1}{\theta^2} u_{j,(m)}^2\right)^{-\kappa_0 - 2} \cdot \left(\frac{\kappa_0 + 1}{K}\right)^{-\frac{3}{2}} \cdot u_{j,(m)}^2 du_{j,(m)} \\ &= \theta \cdot \frac{1}{\pi} \cdot \left(\frac{\kappa_0 + 1}{\kappa_0}\right)^{-\frac{1}{2}} \cdot \Gamma_{\kappa_0 + 1}\left(\frac{3}{2}\right) \cdot \int_{-\infty}^{\infty} \left(1 + \frac{1}{\kappa_0} \cdot \frac{1}{\theta^2} u_{j,(m)}^2\right)^{-\kappa_0 - \frac{1}{2}} \cdot u_{j,(m)}^2 du_{j,(m)} \\ &= \theta^4 \cdot \frac{1}{\pi} \cdot \left(\frac{\kappa_0 + 1}{\kappa_0}\right)^{-\frac{1}{2}} \cdot \Gamma_{\kappa_0 + 1}\left(\frac{3}{2}\right) \cdot \left(\frac{\kappa_0 - \frac{1}{2}}{\kappa_0}\right)^{-\frac{3}{2}} \cdot \int_0^{\infty} \left(1 + \frac{1}{\kappa_0 - \frac{1}{2}} \cdot z\right)^{-(\kappa_0 - \frac{1}{2}) - 1} \cdot z^{\frac{3}{2} - 1} dz \\ &= \theta^4 \cdot \frac{1}{\pi} \cdot \left(\frac{\kappa_0 + 1}{\kappa_0}\right)^{-\frac{1}{2}} \cdot \Gamma_{\kappa_0 + 1}\left(\frac{3}{2}\right) \cdot \left(\frac{\kappa_0 - \frac{1}{2}}{\kappa_0}\right)^{-\frac{3}{2}} \cdot \Gamma_{\kappa_0 - \frac{1}{2}}\left(\frac{3}{2}\right), \end{aligned}$$

with $\Gamma_{\kappa_0+1}(\frac{3}{2}) = (\pi^{\frac{1}{2}}/2) \cdot (\kappa_0 + 1)^{\frac{1}{2}} \cdot \Gamma(\kappa_0 + \frac{1}{2})/\Gamma(\kappa_0 + 1)$, $\Gamma_{\kappa_0-\frac{1}{2}}(\frac{3}{2}) = (\pi^{\frac{1}{2}}/2) \cdot (\kappa_0 - \frac{1}{2})^{\frac{1}{2}} \cdot \Gamma(\kappa_0 - 1)/\Gamma(\kappa_0 - \frac{1}{2})$, which gives

$$\left\langle u_{i,(n)}^2 \cdot u_{j,(m)}^2 \right\rangle = \frac{1}{4} \theta^4 \cdot \frac{\kappa_0}{\kappa_0 - 1}. \tag{B20}$$

Then, Equation (B18) becomes

$$\left\langle \vec{u}_{(n)}^{2} \cdot \vec{u}_{(m)}^{2} \right\rangle = \theta^4 \cdot \frac{\kappa_0}{\kappa_0 - 1} \cdot \left(\frac{D}{2}\right)^2 \tag{B21a}$$

or in terms of energies,

$$\langle \varepsilon_{(n)} \cdot \varepsilon_{(m)} \rangle = (k_{\mathrm{B}}T)^2 \cdot \frac{\kappa_0}{\kappa_0 - 1} \cdot \left(\frac{D}{2}\right)^2 . \tag{B21b}$$

Both Equations (B17) and (B21) can be written as

$$\left\langle \vec{u}_{(n)}^{2} \cdot \vec{u}_{(m)}^{2} \right\rangle = \theta^4 \cdot \frac{\kappa_0}{\kappa_0 - 1} \times \begin{cases} \left(\frac{D}{2}\right)^2 & \text{if } n \neq m, \\ \left(\frac{D}{2}\right) \cdot \left(\frac{D}{2} + 1\right) & \text{if } n = m, \end{cases} \tag{B22a}$$

or in terms of energy,

$$\langle \varepsilon_{(n)} \cdot \varepsilon_{(m)} \rangle = (k_{\mathrm{B}}T)^2 \cdot \frac{\kappa_0}{\kappa_0 - 1} \times \begin{cases} \left(\frac{D}{2}\right)^2 & \text{if } n \neq m, \\ \left(\frac{D}{2}\right) \cdot \left(\frac{D}{2} + 1\right) & \text{if } n = m. \end{cases} \tag{B22b}$$

Then, the correlation is given by

$$\rho(\kappa_0; D) = \frac{\langle \varepsilon_{(n)} \cdot \varepsilon_{(m)} \rangle - \langle \varepsilon_{(n)} \rangle \cdot \langle \varepsilon_{(m)} \rangle}{\left\langle \varepsilon_{(n)}^2 \right\rangle - \langle \varepsilon_{(n)} \rangle^2} = \frac{\frac{D}{2}}{\kappa_0 + \frac{D}{2}} . \tag{B23}$$

Note that while neither $\langle \varepsilon_{(n)} \cdot \varepsilon_{(m)} \rangle$ nor $\langle \varepsilon_{(n)}^2 \rangle$ converge in the interval $0 < \kappa_0 \leqslant 1$ (far-equilibrium region), their ratio, as involved in the correlation's formulation (B23), does converge. We prove this as follows.

Equation (B23) is written as

$$-(1-\rho)\langle \varepsilon_{(n)} \rangle^2 = \rho \cdot \left\langle \varepsilon_{(n)}^2 \right\rangle - \langle \varepsilon_{(n)} \cdot \varepsilon_{(m)} \rangle . \tag{B24}$$

If $\varepsilon_{i,_{(n)}}$ denotes the energy of the ith degree of freedom of the nth particle, then the values of $\langle \varepsilon_{i,_{(n)}} \rangle$, $\langle \varepsilon_{i,_{(n)}}^2 \rangle$, $\langle \varepsilon_{i,_{(n)}} \cdot \varepsilon_{j,_{(m)}} \rangle$, and $\langle \varepsilon_{i,_{(n)}} \cdot \varepsilon_{j,_{(n)}} \rangle$ do not depend on the particular particle, so that $\langle \varepsilon_{i,_{(n)}} \rangle \equiv \langle \varepsilon_i \rangle$, $\langle \varepsilon_{i,_{(n)}}^2 \rangle \equiv \langle \varepsilon_i^2 \rangle$, and $\langle \varepsilon_{i,_{(n)}} \cdot \varepsilon_{j,_{(m)}} \rangle = \langle \varepsilon_{i,_{(n)}} \cdot \varepsilon_{j,_{(n)}} \rangle \equiv \langle \varepsilon_i \cdot \varepsilon_j \rangle$. Hence, we have

$$\left\langle \varepsilon_{(n)}^2 \right\rangle = \sum_{i=1}^{D} \sum_{j=1}^{D} \left\langle \varepsilon_{i,_{(n)}} \cdot \varepsilon_{j,_{(n)}} \right\rangle = D \cdot \left\langle \varepsilon_i^2 \right\rangle + D(D-1) \cdot \langle \varepsilon_i \cdot \varepsilon_j \rangle , \tag{B25a}$$

$$\langle \varepsilon_{(n)} \cdot \varepsilon_{(m)} \rangle = \sum_{i=1}^{D} \sum_{j=1}^{D} \left\langle \varepsilon_{i,_{(n)}} \cdot \varepsilon_{j,_{(m)}} \right\rangle = D^2 \cdot \langle \varepsilon_i \cdot \varepsilon_j \rangle , \tag{B25b}$$

$$\langle \varepsilon_{(n)} \rangle = D \cdot \langle \varepsilon_i \rangle , \tag{B25c}$$

leading to

$$-(1-\rho) D \cdot \langle \varepsilon_i \rangle^2 = \rho \cdot \left\langle \varepsilon_i^2 \right\rangle + [\rho \cdot (D-1) - D] \cdot \langle \varepsilon_i \cdot \varepsilon_j \rangle , \tag{B26}$$

where $\langle \varepsilon_i^2 \rangle$ and $\langle \varepsilon_i \cdot \varepsilon_j \rangle$ diverge for $\kappa_0 \leqslant 1$. Indeed, $\left\langle \varepsilon_i^2 \right\rangle$ equals

$$\left\langle \varepsilon_i^2 \right\rangle = \pi^{-\frac{1}{2}} \cdot \kappa_0^{-\frac{1}{2}} \cdot \frac{\Gamma\left(\kappa_0 + \frac{3}{2}\right)}{\Gamma(\kappa_0 + 1)} \cdot (k_{\mathrm{B}}T)^{-\frac{1}{2}} \cdot \lim_{E \to \infty} \int_0^E \left(1 + \frac{1}{\kappa_0} \cdot \frac{\varepsilon_i}{k_{\mathrm{B}}T}\right)^{-\kappa_0 - \frac{3}{2}} \cdot \varepsilon_i^{\frac{3}{2}} d\varepsilon_i \tag{B27a}$$

which, in the high energy limit, is written as

$$\left\langle \varepsilon_i^2 \right\rangle \cong \pi^{-\frac{1}{2}} \cdot \kappa_0^{\kappa_0 + 1} \cdot \frac{\Gamma\left(\kappa_0 + \frac{3}{2}\right)}{\Gamma(\kappa_0 + 1)} \cdot (k_{\mathrm{B}}T)^{\kappa_0 + 1} \cdot \int_0^\infty \varepsilon_i^{-\kappa_0} d\varepsilon_i \to \infty, \quad \text{for} \quad \kappa_0 \leqslant 1.$$

Similarly, $\langle \varepsilon_i \cdot \varepsilon_j \rangle$ equals

$$\begin{aligned} \langle \varepsilon_i \cdot \varepsilon_j \rangle &= \frac{\kappa_0 + 1}{\pi \cdot \kappa_0} \cdot (k_{\mathrm{B}}T)^{-1} \cdot \lim_{E \to \infty} \int_0^E \left(1 + \frac{1}{\kappa_0} \cdot \frac{\varepsilon_i + \varepsilon_j}{k_{\mathrm{B}}T}\right)^{-\kappa_0 - 2} \cdot \varepsilon_i^{\frac{1}{2}} \varepsilon_j^{\frac{1}{2}} \, d\varepsilon_i \, d\varepsilon_j \\ &= \frac{1}{2}\pi^{-\frac{1}{2}} \cdot \kappa_0^{\frac{1}{2}} \cdot \frac{\Gamma\left(\kappa_0 + \frac{1}{2}\right)}{\Gamma(\kappa_0 + 1)} \cdot (k_{\mathrm{B}}T)^{\frac{1}{2}} \cdot \lim_{E \to \infty} \int_0^E \left(1 + \frac{1}{\kappa_0} \cdot \frac{\varepsilon_i}{k_{\mathrm{B}}T}\right)^{-\kappa_0 - \frac{1}{2}} \cdot \varepsilon_i^{\frac{1}{2}} d\varepsilon_i \end{aligned}$$

$$= \frac{1}{2}\pi^{-\frac{1}{2}} \cdot \kappa_0^{\frac{3}{2}} \cdot \frac{\Gamma\left(\kappa_0+\frac{1}{2}\right)}{\Gamma(\kappa_0+1)} \cdot (k_B T)^{\frac{3}{2}} \cdot \lim_{E\to\infty} \left\{ \int_0^E \left(1+\frac{1}{\kappa_0}\cdot\frac{\varepsilon_i}{k_B T}\right)^{-\kappa_0+\frac{1}{2}} \cdot \varepsilon_i^{-\frac{1}{2}} d\varepsilon_i - \int_0^E \left(1+\frac{1}{\kappa_0}\cdot\frac{\varepsilon_i}{k_B T}\right)^{-\kappa_0-\frac{1}{2}} \cdot \varepsilon_i^{-\frac{1}{2}} d\varepsilon_i \right\}$$

$$= \frac{1}{2}\pi^{-\frac{1}{2}} \cdot \kappa_0^{\frac{3}{2}} \cdot \frac{\Gamma\left(\kappa_0+\frac{1}{2}\right)}{\Gamma(\kappa_0+1)} \cdot (k_B T)^{\frac{3}{2}} \cdot \lim_{E\to\infty} \int_0^E \left(1+\frac{1}{\kappa_0}\cdot\frac{\varepsilon_i}{k_B T}\right)^{-\kappa_0+\frac{1}{2}} \cdot \varepsilon_i^{-\frac{1}{2}} d\varepsilon_i - \frac{1}{2}\kappa_0 \cdot (k_B T)^2$$

$$\Rightarrow \langle \varepsilon_i \cdot \varepsilon_j \rangle = \frac{1}{2}\frac{\kappa_0^2}{\kappa_0+\frac{1}{2}} (k_B T)^2 \left\langle \left(1+\frac{1}{\kappa_0}\cdot\frac{\varepsilon_i}{k_B T}\right)^2 \right\rangle - \frac{1}{2}\kappa_0 \cdot (k_B T)^2. \tag{B27b}$$

Hence, we find that the divergence of $\langle \varepsilon_i \cdot \varepsilon_j \rangle$ is related to the divergence of $\langle \varepsilon_i^2 \rangle$, i.e.,

$$\langle \varepsilon_i \cdot \varepsilon_j \rangle = \frac{\frac{1}{2}}{\kappa_0+\frac{1}{2}} \langle \varepsilon_i^2 \rangle + \frac{1}{4}\frac{\kappa_0}{\kappa_0+\frac{1}{2}} (k_B T)^2. \tag{B27c}$$

Then, Equation (B26) becomes

$$-(1-\rho)D \cdot \frac{1}{4}(k_B T)^2 = \rho \cdot \langle \varepsilon_i^2 \rangle + [\rho\cdot(D-1)-D] \cdot \left[\frac{\frac{1}{2}}{\kappa_0+\frac{1}{2}} \langle \varepsilon_i^2 \rangle + \frac{1}{4}\frac{\kappa_0}{\kappa_0+\frac{1}{2}} (k_B T)^2 \right]$$

$$\Rightarrow \left\{ -(1-\rho)D - [\rho\cdot(D-1)-D]\cdot\frac{\kappa_0}{\kappa_0+\frac{1}{2}} \right\} \cdot \frac{1}{4}(k_B T)^2 = \left\{ \rho + [\rho\cdot(D-1)-D]\cdot\frac{\frac{1}{2}}{\kappa_0+\frac{1}{2}} \right\} \cdot \langle \varepsilon_i^2 \rangle$$

$$\Rightarrow \frac{\kappa_0+\frac{D}{2}}{\kappa_0+\frac{1}{2}} \cdot \left(\rho - \frac{\frac{D}{2}}{\kappa_0+\frac{D}{2}} \right) \cdot \left[\langle \varepsilon_i^2 \rangle - \frac{1}{4}(k_B T)^2 \right] = 0, \tag{B28}$$

which leads to Equation (B23) independently of the convergence of $\langle \varepsilon_i^2 \rangle$.

Note that in the case where each particle has different degrees of freedom, we can easily show that

$$\langle \vec{u}_{(n)}^2 \cdot \vec{u}_{(m)}^2 \rangle = \theta^4 \cdot \frac{\kappa_0}{\kappa_0-1} \times \begin{cases} \left(\frac{D_{(n)}}{2}\right)\cdot\left(\frac{D_{(m)}}{2}\right) & \text{if } n \neq m, \\ \left(\frac{D_{(n)}}{2}\right)\cdot\left(\frac{D_{(n)}}{2}+1\right) & \text{if } n = m, \end{cases} \tag{B29a}$$

or in terms of energy,

$$\langle \varepsilon_{(n)} \cdot \varepsilon_{(m)} \rangle = (k_B T)^2 \cdot \frac{\kappa_0}{\kappa_0-1} \times \begin{cases} \left(\frac{D_{(n)}}{2}\right)\cdot\left(\frac{D_{(m)}}{2}\right) & \text{if } n \neq m, \\ \left(\frac{D_{(n)}}{2}\right)\cdot\left(\frac{D_{(n)}}{2}+1\right) & \text{if } n = m. \end{cases} \tag{B29b}$$

Then, the correlation is

$$\rho(\kappa_0; D_{(n)}, D_{(m)}) = \frac{\langle \varepsilon_{(n)} \cdot \varepsilon_{(m)} \rangle - \langle \varepsilon_{(n)} \rangle \cdot \langle \varepsilon_{(m)} \rangle}{\sqrt{\langle \varepsilon_{(n)}^2 \rangle - \langle \varepsilon_{(n)} \rangle^2} \cdot \sqrt{\langle \varepsilon_{(m)}^2 \rangle - \langle \varepsilon_{(m)} \rangle^2}} = \sqrt{\frac{\frac{D_{(n)}}{2}}{\kappa_0+\frac{D_{(n)}}{2}}} \cdot \sqrt{\frac{\frac{D_{(m)}}{2}}{\kappa_0+\frac{D_{(m)}}{2}}} = \sqrt{\rho(\kappa_0; D_{(n)}) \cdot \rho(\kappa_0; D_{(m)})}. \tag{B30}$$

REFERENCES

Abe, S. 1999, Physica A, 269, 403

Abe, S. 2001, Physica A, 300, 417

Abe, S. 2002, Phys. Rev. E, 66, 046134

Abe, S., Martínez, S., Pennini, F., & Plastino, A. 2001, Phys. Lett. A, 281, 126

Abramowitz, M., & Stegun, I. A. (ed) 1972, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables (9th printing; New York: Dover), 946

Aschwanden, M. J., Tarbell, T. D., Nightingale, R. W., et al. 2000, ApJ, 535, 1047

Bazarghan, M., Safari, H., Innes, D. E., Karami, E., & Solanki, S. K. 2008, A&A, 492, L13

Beck, C., & Schlogl, F. 1993, Thermodynamics of Chaotic Systems (Cambridge: Cambridge Univ. Press)

Benz, A. O. 2004, in IAU Symp. 219, Stars as Suns: Activity, Evolution, and Planets, ed. A. K. Dupree & A. O. Benz (Cambridge: Cambridge Univ. Press), 461

Borges, E. P., Tsallis, C., Anãnõs, G. F. J., & de Oliveira, P. M. C. 2002, Phys. Rev. Lett., 89, 254103

Charbonneau, P., McIntosh, S. W., Liu, H. L., & Bogdan, T. J. 2001, Sol. Phys., 203, 321

Chotoo, K., Schwadron, N. A., Mason, G. M., et al. 2000, J. Geophys. Res A, 105, 23107

Christon, S. P. 1987, Icarus, 71, 448

Collier, M. R., Hamilton, D. C., Gloeckler, G., Bochsler, P., & Sheldon, R. B. 1996, J. Geophys. Res A, 23, 1191

Dayeh, M. A., Desai, M. I., Dwyer, J. R., et al. 2009, ApJ, 693, 1588

Decker, R. B., & Krimigis, S. M. 2003, Adv. Space Res., 32, 597

Decker, R. B., Krimigis, S. M., Roelof, E. C., et al. 2005, Science, 309, 2020

Desai, M. I., Mason, G. M., Wiedenbeck, M. E., et al. 2004, ApJ, 611, 1156
Dialynas, K., Krimigis, S. M., Mitchell, D. G., et al. 2009, J. Geophys. Res. A, 114, A01212
Funsten, H. O., Allegrini, F., Crew, G.B., et al. 2009, Science, 326, 964
Gell-Mann, M., & Tsallis, C. 2004, Nonextensive Entropy: Interdisciplinary Applications (New York: Oxford Univ. Press)
Hawkins, S. E., III, Cheng, A. F., & Lanzerotti, L. J. 1998, J. Geophys. Res A, 103, 20031
Hellberg, M. A., Mace, R. L., Baluku, T. K., Kourakis, I., & Saini, N. S. 2009, Phys. Plasmas, 16, 094701
le Roux, J. A., Webb, G. M., Shalchi, A., & Zank, G. P. 2010, ApJ, 716, 671
Leubner, M. P., & Vörös, Z. 2005, ApJ, 618, 547
Leubner, M. P. 2002, Ap&SS, 282, 573
Leubner, M. P. 2004a, ApJ, 604, 469
Leubner, M. P. 2004b, Phys. Plasmas, 11, 1308
Livadiotis, G., & McComas, D. J. 2009, J. Geophys. Res. A, 114, 11105
Livadiotis, G., & McComas, D. J. 2010a, ApJ, 714, 971
Livadiotis, G., & McComas, D. J. 2010b, Phys. Scr., 82, 035003
Livadiotis, G., & McComas, D. J. 2010c, in AIP Conf. Proc. 1302, Pickup Ions Throughout the Heliosphere and Beyond, ed. J. LeRoux et al. (Melville, NY: AIP), 70
Livadiotis, G., & McComas, D. J. 2011, ApJ, 738, 64
Livadiotis, G., McComas, D. J., Dayeh, M. A., Funsten, H. O., & Schwadron, N. A. 2011, ApJ, 734, 1
Mann, G., Classen, H. T., Keppler, E., & Roelof, E. C. 2002, A&A, 391, 749
Marsch, E. 2006, Living Rev. Solar Phys., 3, 1
Mason, G. M., Leske, R. A., Desai, M. I., et al. 2008, ApJ, 678, 1458
Mauk, B. H., Krimigis, S. M., Keath, E. P., et al. 1987, J. Geophys. Res. A, 92, 15283
Mauk, B. H., Mitchell, D. G., McEntire, R. W., et al. 2004, J. Geophys. Res. A, 109, A09S12
McComas, D. J., Allegrini, F., Bochsler, P., et al. 2009, Science, 326, 959
Milovanov, A. V., & Zelenyi, L. M. 2000, Nonlinear Process. Geophys., 7, 211
Naudts, J. 2011, Generalised Thermostatistics (New York: Springer)
Nieves-Chinchilla, T., & Viñas, A. F. 2008a, J. Geophys. Res. A, 113, A02105
Nieves-Chinchilla, T., & Viñas, A. F. 2008b, Geofis. Int., 47, 245
Norman, J. P., Charbonneau, P., McIntosh, S. W., & Liu, H. L. 2001, ApJ, 557, 891
Pauluhn, A., & Solanki, S. K. 2007, A&A, 462, 311
Pierrard, V., & Lazar, M. 2010, Sol. Phys., 267, 153
Prato, D., & Tsallis, C. 1999, Phys. Rev. E, 60, 2398
Raadu, M. A., & Shafiq, M. 2007, Phys. Plasmas, 14, 012105
Rama, S. K. 2000, Phys. Lett. A, 276, 103
Saito, S., Forme, F. R. E., Buchert, S. C., Nozawa, S., & Fujii, R. 2000, Ann. Geophys., 18, 1216
Schippers, P., Blanc, M., André, N., et al. 2008, J. Geophys. Res. A, 113, A07208
Schreier, E., Gursky, H., Kellogg, E., Tananbaum, H., & Giacconi, R. 1971, ApJ, 170, L21
Shizgal, B. D. 2007, Ap&SS, 312, 227
Silva, R., Plastino, A. R., & Lima, J. A. S. 1998, Phys. Lett. A, 249, 401
Tsallis, C. 1988, J. Stat. Phys., 52, 479
Tsallis, C. 1999, Braz. J. Phys., 29, 1
Tsallis, C. 2009a, J. Comput. Appl. Math., 227, 51
Tsallis, C. 2009b, Introduction to Nonextensive Statistical Mechanics (New York: Springer)
Tsallis, C., Mendes, R. S., & Plastino, A. R. 1998, Physica A, 261, 534
Vasyliūnas, V. M. 1968, J. Geophys. Res. A, 73, 2839
Venters, T. M., & Pavlidou, V. 2007, ApJ, 666, 128
Von Montigny, C., Bertsch, D. L., Chiang, J., et al. 1995, ApJ, 440, 525
Yamano, T. 2002, Physica A, 305, 486
Yoon, P. H., Rhee, T., & Ryu, C.-M. 2006, J. Geophys. Res. A, 111, A09106
Zouganelis, I., Maksimovic, M., Meyer-Vernet, N., Lamy, H., & Issautier, K. 2004, ApJ, 606, 542