Theory of Plasma Oscillations. A. Origin of Medium-Like Behavior

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A theory of electron oscillations of an unbounded plasma of uniform ion density is developed, taking into account the effects of random thermal motions, but neglecting collisions.

The first problem considered is that of finding the frequencies at which a plasma can undergo organized steady-state oscillations of small enough amplitude so that a linear approximation applies. It is found that long wave-length oscillations of plasmas with a Maxwell distribution of electron velocities are characterized by the steady-state dispersion relation $\omega^2=\omega_P{}^2+(3\kappa T/m)(2\pi/\lambda)^2$. Here ω_P is the plasma frequency, T the absolute temperature of the electron gas, λ the wave-length, and ω the angular frequency of oscillation. It is also shown that organized oscillations of wave-lengths smaller than the Debye length for the electron gas are not possible.

The theory is then extended to describe the processes by which oscillations are set up. It is found that, for a given wave-length, a plasma can oscillate with arbitrary frequency, but that those frequencies not given by the steady-state dispersion relation describe motions in which, after some time, there is no contribution to macroscopic averages. These additional frequencies lead asymptotically only to microscopic fluctuations of the charge density about the organized oscillation of the plasma. In this way, one can describe the manner in which the system develops organized behavior.

The treatment is then applied to large steady-state oscillations for which the equations are non-linear. One obtains solutions in which particles close to the wave velocity are trapped in the trough of the potential, oscillating back and forth about a mean velocity equal to that of the wave. One can also obtain non-linear traveling pulse solutions in which a group of particles, moving as a pulse, creates a reaction on the surrounding charge, which traps the particles and holds them together.

I. INTRODUCTION

A GAS containing a suitably high density of free positive and negative charges is known as a plasma. As a result of the electrical interactions between the charges, a plasma displays certain forms of ordered behavior which make a description of the system regarded as a whole more appropriate than one in which the individual particles are treated separately. The ordering processes characteristics of a plasma result in a tendency to remain approximately field-free and electrically neutral. If electric fields are introduced, either by an external disturbance or by incomplete space-charge neutralization, the highly mobile free charges automatically respond to the forces in such a way as to shield out the fields. One can therefore regard a plasma as a medium which tends to remain near a field-free and neutral equilibrium state, resisting efforts to produce deviations from this state, just as a liquid tends to remain near an equilibrium state of definite volume, resisting efforts to produce changes in this volume.

In order that the concept of a medium be generally applicable, it is necessary that the dynamic behavior as well as the static behavior show characteristic organized properties. Let us return to our example of the liquid. When a given volume contains an excess of molecules the resulting pressure gradient creates a net flux of particles out of the region, but after a uniform density is reached, the particles still have a net outwardly directed momentum. This eventually results in a deficiency of molecules in the volume so that the motion is reversed; systematic oscillations about the equilibrium state will occur, and as a result, sound waves will be transmitted through the liquid. Because the particles are locked together by interatomic forces so that the system responds more or less as a unit, one does not have to take into account the details of individual particle motions.

The behavior of a medium near its equilibrium state can be described with the aid of a dispersion relation defining the angular frequency, ω, as a function of the wave number,[a] $k=2\pi/\lambda$. Because of the linearity of the equations of motion for small displacements, one can Fourier-analyze the motion, and thus regard an arbitrary displacement as a superposition of waves. From the value of the displacement and of its rate of change at every point in space at a given time, combined with the dispersion relation, one can then calculate the subsequent behavior of the medium. For sound waves in a liquid, for example, the angular frequency is $\omega=\pm kv_s$, where v_s is the speed of sound, which is independent of wave-length. The group velocity, $v_g=\partial\omega/\partial k$, then yields the speed at which energy is transmitted through the system. For a liquid one obtains $v_g=\pm v_s$, a well-known result.

In a plasma a similar medium-like organization of the particle motions is made possible by the electrical forces. If, for example, a given region contains an excess of electrons, they repel each other and therefore begin to move out. By the time neutrality has been established the electrons have gained momentum so that they keep on going and create a deficiency of negative charge which at-

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[a] There may be one or more values of ω for each k; e.g., in a crystal, there may be one frequency corresponding to ordinary sound waves and a higher frequency corresponding to excitation of intra-molecular vibrations with the same k.

tracts the electrons back in. In time the motion is reversed, and a systematic oscillation of the charged region is set up. For the case in which the random thermal motions of the charges are slow enough to be neglected, Langmuir and Tonks[1] have studied these oscillations and have shown that their angular frequency is given by

$$\omega_P = ((4\pi n_0 \epsilon^2)/m)^{\frac{1}{2}}, \tag{1}$$

where n_0 is the density of charged particles, and m is their mass. For a typical density of 10^{12} electrons per cm^3, the plasma frequency is about 10^{10} c.p.s. This equation is modified somewhat by the effects of random thermal motions in a way which will be discussed further ahead.

These oscillations are irrotational and therefore do not radiate. Transverse plasma oscillations are, however, also possible, but we shall not study them here because a reasonably complete theory[2] already exists for them. Because the ions are so much heavier than the electrons, their motions will be so small that we can neglect them altogether and assume they remain at rest. The positive ions can also oscillate, but at a much lower frequency.[3] In this paper, however, we shall consider only electron oscillations. For wave-lengths much greater than interionic spacing, it is a good approximation to regard the charge of the positive ions as uniformly smeared over the region.

For a plasma we observe that ω is independent of k, so that the group velocity is zero. This means that in this approximation plasma oscillations are not transmitted through the system at all. Consider, for example, an arbitrary disturbance, which at $t=0$ is localized in a definite region of space. The displacement $\boldsymbol{\xi}(\mathbf{x}, t)$ can be expressed at $t=0$ as a Fourier integral, $\boldsymbol{\xi}(\mathbf{x}, 0)=\int \mathbf{f}(\mathbf{k}) \exp i\mathbf{k}\cdot\mathbf{x} d\mathbf{k}$. The behavior as a function of time can be found by inserting the proper frequency $\omega(\mathbf{k})$, and we obtain $\boldsymbol{\xi}(\mathbf{x}, t)=\int \mathbf{f}(\mathbf{k}) \exp i(\mathbf{k}\cdot\mathbf{x}-\omega t) d\mathbf{k}$. Since ω is independent of $\mathbf{k}$, $\exp -i\omega t$ can be taken out of the integral so that the disturbance remains localized, regardless of the size or shape of the region.[b] Such an oscillation is reminiscent of the behavior of a very thin jelly; hence, the name plasma oscillation.

The dispersion relation (1) does not take into account the effect of random thermal motions of the electrons. In Paper A we extend the theory to include random motions, showing when and how an ionized gas exhibits medium-like properties. This problem has a general interest because the plasma is the only system which is simple enough so that the origin of medium-like behavior can be traced out in detail with the aid of kinetic theory. In addition to giving a complete discussion of the case of small potentials, for which the equations are linear, we also obtain exact non-linear solutions for several cases where the linear approximation fails. In Paper B we discuss the conditions under which plasma oscillations can be excited and damped. In Paper C we discuss the effects of spatial boundaries. These are particularly important in plasmas occurring in discharge tubes where the extent is limited by the walls of the tube.

II. MICROSCOPIC PROCESSES LEADING TO MEDIUM-LIKE BEHAVIOR

In any medium each particle moves in a field of force which is the sum of the forces due to all the other particles, plus those arising from externally imposed fields, if any. In general, this problem is too complex to be solved, because as a given particle collides with other particles, it experiences a rapidly fluctuating force, which is very difficult to take into account. In order to simplify this problem, one must take some sort of average of the field, and one must therefore consider aggregates large enough to contain many particles at once. If a disturbance is to be discussed in terms of oscillations of a medium, it is therefore necessary, at the very least, that the wave-length be considerably larger than the inter-particle distance $n_0^{-\frac{1}{3}}$.

Further conditions must be satisfied, however, before the motion predicted by the average forces is a good approximation to the motion produced by the actual forces. In general, such a simplification is possible in either of two limiting conditions: (a) the forces have a short range and the density is so high that many collisions occur during the period of an oscillation, (b) the forces have a long range and the speed of the particles is so low that the mean distance between most of the interacting particles does not change appreciably during the period of an oscillation. The former occurs with sound waves in a gas or a liquid, the latter in a plasma.

If condition (a) is satisfied, each particle experiences so many small impulses in a short time that its mean motion is determined very accurately by a short-time average momentum transfer. It is in this way that organized medium-like behavior is produced. If, for example, the density in a given region is not uniform, so that in macroscopic terms there is a pressure gradient, each particle will be struck more often from the denser side than from the rarer, and will therefore tend to accelerate in such a way as to leave the denser region. This is the process responsible for the tendency of a liquid or a gas to maintain its static property of constant density.

There are further processes, however, which cause the particle motions to interlock to a much higher

[1] I. Langmuir and L. Tonks, Phys. Rev. **33**, 195 (1929).
[2] H. R. Mimno, Rev. Mod. Phys. **9**, 1 (1937); H. Lassen, Ann d. Physik **1**, 415–28 (1947); H. Margenau, Phys. Rev. **73**, 297 (1948).
[3] R. Rompe and H. Steenbeck—Ergeb. d. exakt. Naturwiss. Bd **18**, 303 (1939).
[b] Actually one must take into account the fact that $\omega = \pm\omega_p$ and that both initial displacement and velocity must be specified. The above analysis can easily be carried through with this correction.

degree than that necessary simply to produce a tendency towards uniform density. If, for example, a particle moves faster than the mean velocity of the surrounding particles, it will be struck more often from the forward than from the backward directions, so that it tends to slow down to the mean velocity. If the particle is slower than the surrounding particles, it will in a similar way tend to speed up to the mean velocity. Thus, the collision processes interlock the time average velocity of each particle to the average flow velocity prevailing at each point. This high degree of interlocking makes the hydrodynamic equations, in terms of the velocity and density prevailing at each point, a good approximation to the actual motion.

Let us now contrast with the above, the type of interlocking process which occurs when condition (b) is satisfied. The forces then have such a long range that each particle is continually colliding with many particles at once, but with small momentum transfers for each collision. Here, it is necessary to trace the actual orbit of the particle resulting from the forces, rather than to regard the interaction as a sudden process, completed in a time too short for any significant average motion to occur. In fact, each particle moves almost freely, except that it experiences a gradual change of velocity caused by the cumulative and simultaneous forces produced by all of the other particles. Under these conditions one can simplify the problem by taking a space average of the potential, in the sense that one smooths out the fluctuations resulting from the point character of the charges. This method is more or less analogous to the Hartree self-consistent field method, used in quantum theory, and was first applied to the plasma by Vlasov.[4] He applied this method also to the short range forces, but this is not permissible because the smoothed-out average field, neglecting the large fluctuations which occur in collision, is then a poor approximation to the actual field.

The use of the smoothed-out average field neglects those few Coulomb collisions which occur at short range, and which involve large momentum transfers delivered during a time which is short compared with the period of a plasma oscillation. Collisions between electrons and gas molecules are of the same kind. At the low pressures typical of gaseous discharges, the mean time between such close collisions is much longer than the period of a plasma oscillation. When a collision does occur, however, its effect is to destroy the ordered component of the motion, since the time of collision is independent of the state of the average field. In Paper B this problem is treated in detail, and it is shown that in most plasmas collisions lead to a small damping of oscillations which can usually be neglected without much error. In this paper we shall therefore ignore collisions altogether.

[4] A. Vlasov—J. Phys. U.S.S.R. 9, 25, 130 (1945).

III. THE DISPERSION RELATION

We shall now solve for the organized motion of the particles in a plasma. It is sufficient to seek solutions in which the potential varies trigonometrically in space and time, i.e., $\varphi = Re\, \varphi_0 \exp i(\mathbf{k}\cdot\mathbf{x}-\omega t)$. In any small disturbance in which the plasma acts as a medium, the potential can be represented as the sum of waves of this kind, but ω must satisfy the dispersion relation which we shall now obtain.

One can simplify the problem considerably by going to the coordinate system in which the wave is at rest; such a coordinate system moves with the velocity $\mathbf{V}_W=\omega\mathbf{k}/k^2$, and in this system, the potential is equal to $\varphi=\varphi_0 \exp i(\mathbf{k}\cdot\mathbf{x})$. It consists simply of a static potential wave. In general, particles which are far from the wave velocity will move across this wave, suffering small periodic changes of velocity as they go from crest to trough and on to the next trough. Particles which are close to the wave velocity, however, may be unable to go over the crest of a wave, and can thus be trapped, so that they oscillate back and forth inside a single trough.

Since the potential is static in the wave system, one can integrate the equations of motions by means of conservation of energy. Let us take the x axis in the $\mathbf{k}$ direction. If $\mathbf{U}_{0x}$ is the x component of the velocity in the wave system[c] at the point where $\varphi=0$, one obtains for the velocity, $\mathbf{U}_x$, at any other point

$$U_x^2 = U_{0x}^2+(2\epsilon\varphi/m). \tag{2}$$

(Note that ϵ is positive by definition, and that we are dealing with electrons.)

The condition for trapping is then obtained by setting $U_x=0$ where $\varphi=\varphi_{\mathrm{min.}}$ The result is, for a trapped particle,

$$U_{0x}^2 < -(2\epsilon\varphi_{\mathrm{min.}}/m). \tag{3}$$

In order to obtain solutions which are static in the wave system, and therefore oscillations in the laboratory system which have reached a steady state, we shall have to wait long enough for the number of particles of any given velocity to become constant at each point in space. It will be necessary to take as given, the final distribution of velocities at some specified point. For the untrapped particles, it is convenient to choose this point at $\varphi=0$ (where $\mathbf{V}=\mathbf{V}_0$). We suppose that the velocity distribution

[c] In general, velocities in the laboratory system will be denoted by $\mathbf{V}$, those in the wave system by $\mathbf{U}$.

function for these particles is given by

$$dN_0 = n_0 f(\mathbf{V}_0)d\mathbf{V}_0 = n_0 f(\mathbf{U}_0+\mathbf{V}_W)d\mathbf{U}_0,$$

where f is normalized to unity. In most plasmas, $f(\mathbf{V}_0)$ will be close to a Maxwellian function. We shall see that the wave velocity, $\mathbf{V}_W$, is usually far above the mean thermal speeds. This means that for most particles, $\mathbf{U}_x=\mathbf{V}_x-\mathbf{V}_W$, will be very large, so that for moderate potentials, very few particles are trapped. These few trapped particles make only a small change in the final result; hence, it is permissible to neglect them. We shall therefore assume in this section that $f(\mathbf{V}_0)=0$ over a small region in the neighborhood of $\mathbf{V}_0=\mathbf{V}_W$. In the next section, however, we shall study in detail the effects of the trapped particles.

Because of the average potential, $\varphi(\mathbf{x})$, each particle undergoes a periodic change of velocity and a corresponding change of its contribution to the density. To obtain the particle density at any point $\mathbf{x}$, we note that if the particle distribution has reached a steady state in the wave system, the contribution of a given particle to the density is inversely proportional to its speed. Thus, one obtains

$$dN=\frac{n_0 f(\mathbf{V}_0)|\mathbf{U}_{0x}|d\mathbf{U}_0}{|\mathbf{U}_x|}=\frac{n_0 f(\mathbf{V}_0)d\mathbf{U}_0}{\left(1+\frac{2\epsilon\varphi(\mathbf{x})}{mU_{0x}^2}\right)^{\frac{1}{2}}}. \quad (4)$$

The total electron density is found by integrating over-all $\mathbf{U}_0$. The positive ion density does not change appreciably; hence, it remains equal to the mean density, n_0. The total charge density is then

$$\rho(\mathbf{x})=+n_0\epsilon-n_0\epsilon\int\frac{f(\mathbf{V}_0)d\mathbf{U}_0}{\left(1+\frac{2\epsilon\varphi(\mathbf{x})}{mU_{0x}^2}\right)^{\frac{1}{2}}}. \quad (5)$$

This charge density results from the action of the assumed average potential, $\varphi(\mathbf{x})$. In order that $\varphi(\mathbf{x})$ be a solution it is necessary that the potential generated by the charge $\rho(\mathbf{x})$ be equal to the potential causing the charge; or, according to Poisson's equation, that

$$-\nabla^2\varphi=+4\pi\rho=4\pi\epsilon n_0-4\pi\epsilon n_0\int\frac{f(\mathbf{V}_0)d\mathbf{V}_0}{\left(1+\frac{2\epsilon\varphi(\mathbf{x})}{mU_{0x}^2}\right)^{\frac{1}{2}}}. \quad (6)$$

The above is a non-linear integro-differential equation, defining $\varphi(\mathbf{x})$. In general, $\varphi(\mathbf{x})$ may not be trigonometric in form, but the assumption that φ is static in the wave system restricts us to solutions in which all quantities are functions only of $(\mathbf{x}-\mathbf{V}_W t)$; i.e., to travelling wave solutions.

For small values of $2\epsilon\varphi/mU_{0x}^2$, one can expand the square root, obtaining the linear approximation

$$-\nabla^2\varphi=\frac{4\pi n_0\epsilon^2}{m}\varphi\int\frac{f(\mathbf{V}_0)d\mathbf{V}_0}{U_{0x}^2}$$
$$=\frac{4\pi n_0\epsilon^2}{m}\varphi\int\frac{f(\mathbf{V}_0)d\mathbf{V}_0}{(\mathbf{V}_{0x}-\mathbf{V}_W)^2}. \quad (7)$$

In the above, it is essential that $f(\mathbf{V}_0)$ vanish in a range near $\mathbf{V}=\mathbf{V}_W$; otherwise, the expansion is not permissible. This is equivalent to neglecting the trapped particles.

To obtain the dispersion relation we write $\mathbf{V}_W=\omega/k$, and $\nabla^2\varphi=-k^2\varphi$. The result is

$$k^2\varphi=\frac{4\pi n_0\epsilon^2}{m}k^2\varphi\int\frac{f(\mathbf{V}_0)d\mathbf{V}_0}{(\omega-\mathbf{k}\cdot\mathbf{V}_0)^2}.$$

There are two conditions yielding a solution

$$k^2\varphi=0 \quad (8)$$

or

$$1=\frac{4\pi n_0\epsilon^2}{m}\int\frac{f(\mathbf{V}_0)d\mathbf{V}_0}{(\omega-\mathbf{k}\cdot\mathbf{V}_0)^2}. \quad (9)$$

The first condition is equivalent to $\nabla^2\varphi=0$. In other words, any solution of Laplace's equation yields a solution of the plasma equations. To obtain a non-zero solution of this kind, however, one must have bounding electrodes on which the charge does not vanish. This type of solution will be discussed further in Paper C. For the present, however, we consider only condition (9), which is an integral equation defining ω as a function of $\mathbf{k}$; this constitutes the dispersion relation. In the linear approximation, this result reduces to a relation first obtained by Vlasov[4] by a different method.

Under the assumption that $\mathbf{V}_W$ is so large that very few particles are present for $|\mathbf{V}_{0x}|>|\mathbf{V}_W|$, the above result can be simplified by expansion in a series of powers of $\mathbf{k}\cdot\mathbf{V}_0/\omega$. Since it turns out that ω is usually somewhat larger than the plasma frequency without random motion (Eq. (1)) it is clear that this expansion is good for small k, i.e., long waves. One can then neglect the small contributions of those particles in the regions for which the expansion does not converge, and one obtains

$$1=\frac{4\pi n_0\epsilon^2}{m\omega^2}\int\left[1+2\frac{\mathbf{k}\cdot\mathbf{V}_0}{\omega}+3\frac{(\mathbf{k}\cdot\mathbf{V}_0)^2}{\omega^2}+\cdots\right]f(\mathbf{V}_0)d\mathbf{V}_0. \quad (10)$$

By definition, $\int f(\mathbf{V}_0)d\mathbf{V}_0=1$; $\int f(\mathbf{V}_0)(\mathbf{k}\cdot\mathbf{V}_0)d\mathbf{V}_0$

$=k\bar{V}_\kappa$ where $\bar{V}_k$ is the mean plasma velocity in the k direction, and $\int f(\mathbf{V}_0)(\mathbf{k}\cdot\mathbf{V}_0)^2 d\mathbf{V}_0=k^2\bar{V}_\kappa^2$, where $\bar{V}_k^2$ is the mean square velocity in the $\mathbf{k}$ direction. For an isotropic distribution one obtains $\bar{V}_k^2=\frac{1}{3}\bar{V}^2$, and $\bar{V}_k$ vanishes. Solving for ω up to second order in k, one obtains

$$\omega^2\cong(4\pi n_0\epsilon^2/m)+k^2\bar{V}^2=\omega_P^2+(3\kappa T/m)k^2. \quad (11)$$

As k approaches zero, the above becomes equivalent to Eq. (1), obtained with the neglect of thermal motions. We see that, for small k at least, plasma oscillations are possible for any distribution function which falls off rapidly enough to make the integrals converge. In this approximation the form of the dispersion relation is also independent of the precise form of $f(\mathbf{V}_0)$.

J. J. Thomson and G. P. Thomson[5] have obtained a similar relation using a macroscopic transport treatment, but their result is not exactly the same. Instead it is $\omega^2=\omega_P^2+(\kappa T/m)k^2$. The macroscopic theory, however, makes arbitrary assumptions about the distribution function which are not quite correct, and which lead to an error of a factor of 3 in the latter term.

IV. DESCRIPTION OF ORDERED MOTION

Let us now consider in detail the processes by which ordered plasma oscillations are maintained. We first observe that except for the particles near the wave velocity, which we are not now considering, each particle experiences only a small perturbation in its velocity and in its contribution to the density. Despite the random motion, however, the contributions of all particles are coherent because they are all in phase with the average force which produces them. Because of the long range of the Coulomb force, the small perturbations in density suffered by the individual particles can still produce a large cumulative contribution to the net potential. The ordering is unlike the ordering of motion in a liquid where the velocity in each particle is interlocked with the local average. In a plasma, the local average velocity is of no significance because it does not directly control any forces; the local average density is not even what is important. Hence, the motion of a plasma shows only long range organization, while locally it is almost indistinguishable from a perfect gas.

The oscillations obtained so far describe only the behavior after the distribution has become constant in the wave system; the process by which it approaches constancy will be discussed later.

It is instructive to view the motion of the particles in the laboratory system. From Eq. (2), one obtains to a first approximation

$$U_x=U_{0x}+\frac{\epsilon\varphi}{mU_{0x}};\quad V_x=V_{0x}+\frac{\epsilon\varphi_0\exp i(kx-\omega t)}{m(V_0-V_W)}. \quad (12)$$

Thus, each particle suffers a wave-like perturbation in its velocity, which is larger for particles moving in the direction of the wave than for those moving in the opposite direction. The reason is that particles moving in the direction of the wave stay in phase longer and, therefore, experience a larger change of momentum in any given direction before the electric field, which imparts the momentum, is reversed. Particles going with the speed of the wave would stay in phase with the wave indefinitely, and thus the change of momentum, $\Delta p=\epsilon\int Edt$, would grow indefinitely with time. This is the description in the laboratory system of the reason for the infinity in Eq. (12) at $\mathbf{V}_0=\mathbf{V}_W$ (in the linear approximation).

It is clear that random thermal actions make the localization of an oscillation impossible, because these motions carry the disturbance from one region to another. The average effect is not, however, isotropic, as one might first suppose. Because the particles moving in the direction of the wave experience a larger change of density, there will be a net tendency to carry the disturbance in the direction of the wave. This can be demonstrated by calculating the group velocity

$$\mathbf{V}_g=\partial\omega/\partial\mathbf{k}=(3\kappa T/m)(\mathbf{k}/\omega). \quad (13)$$

This equation shows that for long wave-lengths and low temperatures, the energy transport is small, so that almost complete localization is possible.

It is noteworthy that in plasma oscillations energy is transported by random thermal drift, which bodily carries the excitation from one region of space to another. This is in contrast to the process of transfer in a liquid or in a gas, where the energy transfer is mainly by direct impacts of molecules which are very frequent during a period of an oscillation. This means that in plasmas of low density such as we have been considering, the hydrodynamic description in terms of a fluid with a definite velocity at each point, in which the force on a particle is determined by the pressure gradient, is inappropriate. Thus, the treatment of Linder,[6] which is along these general lines, can be applied only to plasmas of very much higher density than are commonly met with in applications.[d]

[5] J. J. Thomson and G. P. Thomson, *Conduction of Electricity in Gases* (Cambridge University Press, London, 1933), third edition, Vol. 2, p. 353.

[6] E. G. Linder, Phys. Rev. **49**, 753 (1936).

[d] Linder also assumes that the oscillations are isothermal. This assumption requires more study, even at plasma densities so high that formulation in terms of a pressure gradient is permissible.

V. ORIGIN OF MEDIUM-LIKE BEHAVIOR

In this section we shall investigate the non-steady-state solutions for plasma motion, and show that the steady-state oscillations, obtained in the previous work, describe the limiting behavior of the plasma, which is approached after a suitable period of time. The corresponding non-steady-state solutions in a liquid would, for example, involve a description of the details of the collision processes responsible for interlocking the time average of the individual particle velocities to the local average velocity.

In order to obtain a steady state of plasma oscillation with a given **k**, it is necessary that the initial perturbations in velocity and density match those demanded by Eqs. (12) and (4). If the actual initial conditions are different we shall see that waves of a given **k** are not restricted to frequencies given by the dispersion relation, Eq. (9), but can exist with arbitrary frequency. In time, however, oscillations with frequencies not given by the dispersion relation tend to get out of phase with each other, and only the organized plasma oscillations, for which the frequency is given by the dispersion relation, continue to contribute to macroscopic averages such as the mean potential. The other frequencies then correspond to the excitation of random microscopic motions, which are essentially a form of heat energy. One can therefore regard the plasma oscillations as a dynamically stable limiting form of motion, about which small deviations corresponding to random or disorganized particle motions occur. Hence, for calculating macroscopic averages, one can ignore all frequencies of oscillation other than the plasma frequency; and regard the system as a medium constrained to oscillate only with the plasma frequency, and with the perturbations of the individual particle velocities apparently interlocked in such a way that Eq. (12) holds.

We shall, for simplicity, consider a one-dimensional plasma. If there are N particles in this system, there are N degrees of freedom, which may be taken as the initial coordinates of each particle. We shall find it convenient to group particles of the same initial velocity together, and to regard each of these groups as the origin of a beam. The continuity of the range of velocities leads to formal difficulties which can be avoided, as is commonly done in other problems, by considering only a discrete set of velocities, V_i, which are separated by an interval, λ, so small that no important physical quantity changes in the step from one velocity to the next. We shall further assume that the system contains so many particles that the fluctuations in each beam due to the particle nature of the charges can be neglected.

One can then specify the state of this system by giving at each point in space the initial charge density in each beam, $\lambda n_i(x, 0)$. For our purpose it is more convenient to Fourier-analyze this function, writing

$$n_i(x, 0) = n_{0i} + \int \delta n_{ki} e^{ikx} dk, \tag{14}$$

where λn_{0i} is the mean density, and $\lambda \delta n_{ki}$ yields the variation about this mean. To avoid problems introduced by continuity, we also restrict k to a set of discrete, but closely spaced, values, separated by Δk. The total initial density can then be written

$$N(x, 0) = \lambda \sum_i (n_{0i} + \Delta k \sum_k \delta n_{ki} e^{ikx}), \tag{15}$$

and in the limit, as λ and Δk approach zero, the sums become integrals, so that we get

$$N(x, 0) = \int n_0(V_0) dV_0 + \int\int \delta n_k(V_0) e^{ikx} dV_0 dk.$$

By suitably specifying δn_{ki}, one can produce an arbitrary initial distribution of charge. To take into account the fact that each beam is made up of a finite number of particles, however, one should limit the maximum k so that the number of degrees of freedom is equal to the number of particles in the beam. We shall assume here that there are enough particles so that we can describe in this way waves of as short a length as interest us.[e]

Our program will be to treat first a perfect gas of non-interacting particles, and then to show how these results are modified by the introduction of Coulomb forces.

In the perfect gas, each particle moves at a constant velocity equal to its initial value, V_{0i}. Its position is therefore given by $x = x_0 + V_{0i}t$. To obtain the density at any other time, we simply replace x in Eq. (15) by $x - tV_{0i}$; this means that each group of particles carries its own perturbation in density bodily with its own velocity. We get

$$N(x, t) = \lambda \sum_i (n_{0i} + \Delta k \sum_k \delta n_{ki} e^{ikx} \cdot e^{-ikV_{0i}t}). \tag{16}$$

It is instructive to consider first a special case in which all of the δn_{ki} are zero except one, which we denote as δn_{kj}. We then get for the variable portion of the density,

$$\delta N_{kj}(x, t) = \lambda \delta n_{kj} \exp i(kx - kV_{0j}t). \tag{17}$$

This shows that if one starts out with a trigonometric spatial variation of density of particles of a given velocity, one obtains a wave with a definite angular frequency, $\omega = kV_{0j}$. This is because the wave is

[e] We are interested here in only giving a schematic description, which will apply rigorously to adequately high densities, but is intended more generally to give a qualitative picture of the processes by which medium-like behavior is set up.

being carried only by the motion of the particles. The frequency of this wave is determined by the Doppler shift.

One can now build up an arbitrary initial distribution from waves of this kind, simply by adding with suitable coefficients terms coming from different values of k and j. Each different wave can be regarded formally as a normal mode, from which one builds up arbitrary solutions. One obtains essentially a double Fourier series for $N(x, t)$. This is in contrast to the results obtained for steady-state plasma oscillations (Eq. (9)) as well as those of any other medium, where for each k there are, in general, at most a limited small number of allowable values of ω.

In order to obtain a definite frequency in a perfect gas, however, it is, as we have seen, necessary to vary the density of only one beam of particles. As λ approaches zero, there will be fewer and fewer particles in each beam, hence, only a negligible variation in charge density having a definite frequency can be obtained in this way.[f] To obtain appreciable variations, one must sum over particles of many velocities. In order to illustrate the effects of this, let us assume that all particles in the range $V_{0i}-(\delta/2)$ to $V_{0i}+(\delta/2)$ have been given the same trigonometric variations in density, $\delta n_k e^{ikx}$.

From Eq. (16), one obtains the total variation in density,

$$\delta N(x, t) = \lambda\Delta k \sum_{V_{0i}-(\delta/2)}^{V_{0i}+(\delta/2)} \delta n_k e^{i(kx-kV_{0j}t)}. \qquad (18)$$

For small λ, this may be approximated by the integral,

$$\delta N(x, t) = \delta n_k e^{ikx}\Delta k \int_{V_0=V_{0i}-(\delta/2)}^{V_0=V_{0i}+(\delta/2)} e^{-ikV_0t} dV_0$$
$$= -2\Delta k \frac{\delta n_k e^{ik(x-V_{0i}t)}\sin(k\delta t/2)}{kt}.$$

We see that as t gets large, $\delta N(x, t)$ approaches zero. This means that a wave-like disturbance involving a range, r, of particle velocities dies out in a time of order $t=2/kr$. The reason is that the waves, associated with particles of different velocity, get out of phase with each other. (One can obtain similar results with almost any distribution, which falls off with increasing velocity. For example, a Gaussian initial distribution,

$$\delta n_k(V_0) = (2/\sigma\sqrt{\pi}) \exp - V_0^2/\sigma^2,$$

leads to

$$\delta N_k(x, t) = \exp ikx \exp - k^2\sigma^2t^2/4\cdot(\Delta k).$$

The physical process responsible for the decay of these waves is essentially the random diffusion of particles of different velocities, which tends to carry particles away from regions where they are initially in excess and into regions in which they are initially deficient. Thus, we see that in a perfect gas, there can be no medium-like oscillations.[g]

VI. EFFECTS OF ELECTRICAL FORCES

The effect of electrical forces is, as we have seen in Section III, to couple the motions of particles of different velocity, and thus to make possible organized oscillations. In this section we shall trace out in detail how the normal modes of the perfect gas, consisting of waves carried along by the motion of each group of particles, go over into organized plasma oscillations.

Let us begin with the equation of motion for each particle,

$$m(dV/dt) = \epsilon\nabla\varphi. \qquad (19)$$

Suppose that in the absence of an oscillation, the velocity of a particle is V_{0i}, and that, more generally, the velocity is $V_i = V_{0i}+\delta V_i(x, t)$, where we note that the small perturbation, δV_i depends, in general, on the position and time. One can therefore write $dV/dt = (\partial V/\partial t) + V(\partial V/\partial x)$, and with the neglect of the second order term, $\delta V_i(\partial\delta V_i/\partial x)$, one obtains

$$(\partial\delta V_i/\partial t) + V_{0i}(\partial\delta V_i/\partial x) = (\epsilon/m)(\partial\varphi/\partial x). \qquad (20)$$

The charge density is obtained from the equation of continuity, applied to each beam,

$$(\partial n_i/\partial t) + (\partial/\partial x)(n_i V_i) = 0. \qquad (21)$$

For a small oscillation, the density takes the form

$$n_i = n_{0i} + \delta n_i(x, t), \qquad (22)$$

and in the linear approximation, Eq. (21) becomes

$$(\partial/\partial t)\delta n_i + V_{0i}(\partial\delta n_i/\partial x) = -n_{0i}(\partial\delta V_i/\partial x). \qquad (23)$$

For a trigonometric perturbation, where δV_i and δn_i are proportional to $\exp ikx$, one obtains

$$\begin{aligned}(\partial/\partial t)\delta V_i + ikV_{0i}\delta V_i &= (i\epsilon/m)k\varphi,\\ (\partial/\partial t)\delta n_i + ikV_{0i}\delta n_i &= -ikn_{0i}\delta V_i.\end{aligned} \qquad (24)$$

The most general initial conditions on the above equations are that, at $t=0$, $\delta V_i = \delta V_{0i}$ and $\delta n_i = \delta n_{0i}$.

[f] It can be seen that the possibility of obtaining waves of arbitrary frequency comes from splitting the distribution into beams of discrete velocities. This step is a way of taking into account some of the fluctuations resulting from the particle nature of the elementary charges.

[g] Sound waves in a real gas are made possible by collisions, which are caused by short range interactions of particles, as a result of which the gas ceases to be perfect.

The solutions satisfying these conditions are

$$\delta V_i = \frac{ik\epsilon}{m} e^{-ikV_{0i}t} \int_0^t e^{ikV_{0i}\tau} \varphi(\tau) d\tau + \delta V_{0i} e^{-ikV_{0i}t},$$

$$\begin{aligned} \delta n_i &= -ikn_{0i} e^{-ikV_{0i}t} \int_0^t e^{ikV_{0i}\tau} \delta V_i(\tau) d\tau + \delta n_{0i} e^{-ikV_{0i}t} \\ &= \frac{k^2 n_{0i}\epsilon}{m} e^{-ikV_{0i}t} \int_0^t e^{ikV_{0i}\tau} (t-\tau) \varphi(\tau) d\tau \\ &\quad + \delta n_{0i} e^{-ikV_{0i}t} - ikn_{0i} \delta V_{0i} t e^{-ikV_{0i}t}. \end{aligned} \tag{25}$$

The equation determing φ is obtained from Poisson's equation

$$\begin{aligned} \nabla^2 \varphi = -k^2 \varphi &= 4\pi\epsilon\lambda \textstyle\sum_i \delta n_i \\ &= 4\pi\epsilon\lambda \textstyle\sum_i \Big[k^2 n_{0i} (\epsilon/m) e^{-ikV_{0i}t} \\ &\quad \times \displaystyle\int_0^t e^{ikV_{0i}\tau} (t-\tau) \varphi(\tau) d\tau \\ &\quad + \delta n_{0i} e^{-ikV_{0i}t} - ikn_{0i} \delta V_{0i} t e^{-ikV_{0i}t} \Big]. \end{aligned} \tag{26}$$

In the limit as λ goes to zero, this becomes an integral equation defining φ as a function of the time. It can be solved with the aid of a Laplace transform, as was done first by Landau.[7] We shall adopt, however, the method of obtaining the normal coordinates, and subsequently expanding an arbitrary solution as a sum of normal modes, because in this way, the physical processes responsible for the origin of medium-like behavior can be made more evident.

We begin, therefore, by seeking solutions in which $\varphi = \varphi_0 \exp - i\omega t$. One then obtains from Eqs. (25)

$$\begin{aligned} \delta V_i &= \frac{k\epsilon}{m} \frac{\varphi_0 e^{-i\omega t}}{(kV_{0i}-\omega)} (1 - e^{i(\omega - kV_{0i})t}) + \delta V_{0i} e^{-ikV_{0i}t}, \\ \delta n_i &= \frac{i\epsilon k n_{0i}\varphi_0}{m} e^{-i\omega t} \frac{\partial}{\partial V_{0i}} \\ &\quad \times \left(\frac{1 - e^{i(\omega - kV_{0i})t}}{(kV_{0i}-\omega)} \right) - ikn_{0i} \delta V_{0i} t e^{-ikV_{0i}t} \\ &\quad + \delta n_{0i} e^{-ikV_{0i}t}. \end{aligned} \tag{27}$$

The above equations show that, in general, the response of a particle of velocity V_{0i} to the potential, $\varphi_0 \exp - i\omega t$, is not only to produce a component of charge density which has the "forcing frequency," ω, but also other components having the frequency, kV_{0i}, which result from the "free oscillation" terms. In order that δn_i vary only with the frequency, ω, one must choose the following boundary conditions:

$$\begin{aligned} \delta V_{0i} &= (k\epsilon/m)(\varphi_0 / kV_{0i} - \omega), \\ \delta n_{0i} &= (k^2\epsilon/m) n_{0i} [\varphi_0 / (kV_{0i} - \omega)^2] \\ &\quad + g_\omega \delta(\omega - kV_{0i}), \end{aligned} \tag{28}$$

where $\delta(\omega - kV_{0i}) = 0$ unless $V_{0i} = \omega/k$, in which case it is unity and gives an arbitrary constant.[h] The term $g_\omega \delta(\omega - kV_{0i})$ represents an initial trigonometric perturbation in the density of particles of a definite velocity, V_{0i}. As in the perfect gas, this leads to a wave of definite frequency, $\omega = kV_{0i}$. The remaining terms represent the response of the rest of the plasma to the total potential. Note that this response is the same as that given in Eq. (12), as it should be, since in both cases one is solving for the steady-state solution in the linear approximation.

We then obtain for the velocity and density perturbation of each particle

$$\begin{aligned} \delta V_i &= \frac{k\epsilon}{m} \frac{\varphi_0 e^{-i\omega t}}{(kV_{0i} - \omega)}, \\ \delta n_i &= \left[\frac{k^2 \epsilon n_{0i} \varphi_0}{m(kV_{0i} - \omega)^2} + g_\omega \delta(\omega - kV_{0i}) \right] e^{-i\omega t}. \end{aligned} \tag{29}$$

At this point we encounter the difficulty that, according to the above formula, the response of the particles to the potential becomes infinite at the wave velocity, $V_{0i} = \omega/k$. This is because, as shown in Sections (III) and (VII), the linear approximation breaks down when $|V_0 - \omega/k|^2 < (2\epsilon\varphi_0/m)$. We shall see in the next section that the actual response of these particles, in the exact non-linear treatment, is finite and not usually very important, except when there are many particles near the wave velocity. In order to keep the procedure of expanding the oscillations as a sum of normal modes, however, we shall find it desirable, if possible, to retain the linear approximation. From Eq. (27), one sees that over a finite time, t, the response of δV_i and δn_i to the potential always remains finite, even at the wave velocity. Since this response is proportional to φ_0, one concludes that over any finite time, t, however long, it will always be possible to choose a φ_0 so small that the linear approximation is good for all particles. The time, t, can still be chosen long enough so that plasma oscillations can go through many cycles. Although this procedure

[7] L. Landau, J. Phys. U.S.S.R. 10, 25 (1946).

[h] It will turn out that, for all of the permissible frequencies of oscillation, ω/k will be equal to the velocity of some one of the groups of particles.

may restrict us to rather small potentials, it should still give a qualitatively correct description of what happens with moderately large φ_0, since only a few particles near the wave velocity need then be treated in a non-linear way.

Let us separate the range of velocities into two regions. In Region I, $|V_0-\omega/k|>\alpha(2\epsilon\varphi_0/m)^{\frac{1}{2}}$, where α is a number of the order of 10 or more, while in Region II, $|V_0-\omega/k|<\alpha(2\epsilon\varphi_0/m)^{\frac{1}{2}}$. In Region I, where the linear approximation applies, we adopt the boundary conditions leading to a steady state, Eqs. (28). In Region II, we cannot adopt these boundary conditions, because they imply an infinite perturbation in the velocity and density. We shall see, however, that if we adopt instead the conditions, $\delta V_{0i}=0$, and $\delta n_{0i}=g_\omega\delta(\omega-kV_{0i})$, then over a finite period of time, however long, it is always possible to choose φ_0 so small that the response of these particles to the potential can be neglected altogether. To show this, we note first that the number of particles in Region II is proportional to the range of velocities for which the linear approximation fails, which is $\alpha(2\epsilon\varphi_0/m^{\frac{1}{2}})$. Since, for a finite time, the response of each group of particles is proportional to φ_0, the total charge density contributed by the response of particles in Region II is proportional to $\varphi_0^{\frac{3}{2}}$. It is always possible, by choosing φ_0 small enough, to make this negligible in comparison with the response of the particles in Region I, which is proportional to φ_0, since the number of particles in this region is practically independent of φ_0.

To complete the calculation, one must now satisfy Poisson's equation, which now takes the form,

$$k^2\left(1-4\pi\lambda\sum_{\mathrm{I}}\frac{\epsilon^2}{m}\frac{n_{0i}}{(\omega-kV_{0i})^2}\right)\varphi_0=4\pi\epsilon g_\omega, \quad (30)$$

where the summation is carried out only over Region I. As λ goes to zero, the sum may be replaced by an integral, and one obtains

$$k^2\left(1-\frac{4\pi\epsilon^2}{m}\int_{\mathrm{I}}\frac{n_0 f(V_0)dV_0}{(\omega-kV_0)^2}\right)\varphi_0=4\pi\epsilon g_\omega, \quad (31)$$

where $\omega=kV_{0j}$ for some j. Note that when the expression on the left hand side is zero, ω is equal to the "plasma frequency," from the dispersion relation (9).

From the above equation, one concludes that an oscillation of definite frequency, ω_j can be set up by starting the particles in Region I according to Eq. (28), with ω set equal to ω_j, while in Region II, one takes the initial conditions, $\delta V_{0i}=0$, $\delta n_{0i}=g_\omega\delta(\omega-kV_{0i})$. Each one of these oscillations is a normal mode, and since one obtains an oscillation only when ω_j/k is equal to the velocity of some beam, one concludes that there are as many oscillations of this kind as there are beams of particles. One obtains, therefore, just as many normal modes as were obtained for the perfect gas.

Let us now investigate the general character of the oscillations. The second term on the left hand side of Eq. (31) represents the response of all particles to the total potential. In general, this response modifies the potential resulting from the $g_\omega\delta(\omega-kV_{0j})$ term, which latter represents the effects of particles at the wave velocity. When ω is far from the "plasma frequency," however, the coefficient of φ_0 remains large, so that the potential is of the same order of magnitude as $\varphi_0=(4\pi\epsilon g_\omega)/k^2$. Since there are very few particles near the wave velocity, the maximum potential that can be obtained from such an oscillation is very small. The general character of the wave is not very different from that of the waves in a perfect gas, since an oscillation is possible only if it is supported by inhomogeneities in density of particles at the wave velocity. Although the potential is somewhat modified by the response of the other particles, there is no real organization of the motion.

As ω approaches the plasma frequency, however, there is a qualitative change in the nature of the motion, resulting from the fact that the potential associated with a given value of g_ω becomes larger and larger. This means, for one thing, that the maximum potential attainable with waves of a definite frequency increases. When ω is equal to the plasma frequency, one obtains waves with g_ω set equal to zero. This means that the oscillations are no longer supported by periodic pulses of particles at the wave velocity, but that, instead, the charge density is made up of the cumulative and coherent contributions of all particles to the total potential. The motion therefore shows a considerable degree of organization. The amplitude of an oscillation at the plasma frequency is not limited by the number of particles at the wave velocity, since the latter play no role at all in maintaining the motion. Thus, even in the linear approximation, large potentials may be built up at the plasma frequency made possible by the fact that all particles act in unison with long range forces.

In order to obtain a single frequency, it is necessary, as we have seen, that all particles be started out with exactly the right phase relations. With most mechanisms of excitation, however, it will be very unlikely that exactly these initial conditions will be produced. More generally, one can expect that a whole range of frequencies will be excited. As in the perfect gas, it will be impossible to excite to any high degree these normal modes involving mostly a few particles near the wave velocity, and if these modes have any appreciable energy, there will be a corresponding range of frequencies, which eventually get out of phase with each other and produce no macroscopically observable results. The plasma frequency, however, can be highly excited, and its oscillations produce a potential which persists indefinitely. Thus, in the long run, only the plasma oscillations will be observable. This means that a system starting in an arbitrary way will eventually seem to have all of the particles interlocked with the velocity perturbations (Eq. (12)) characteristic of organized plasma oscillation. Thus, the complexities introduced by the degree of freedom in which oscillations are carried mainly by particles at the wave velocity can be ignored, because they produce no macroscopically observable effects. In this way, the system takes on the behavior of a medium.[i]

The above conclusions apply only to plasmas in which there are no large number of particles with a common, sharply defined velocity. If the latter are present, their macroscopically observable charge density does not cancel out after a long time. In other words, pulses of charge in a beam of well-defined velocity can persist for a long time. Hence, such beams continue to contribute to the total number of degrees of freedom, and are best described as separate plasmas, interpenetrating the original plasma, and interacting strongly with it. We shall have occasion to consider the effects of such beams in Paper B.

VII. MINIMUM WAVE-LENGTH FOR PLASMA OSCILLATIONS

Thus far we have studied the dispersion relation in detail only for the case of small k. To extend the investigation to arbitrary k, we rewrite Eq. (9) as follows: (we take the x axis in $\mathbf{k}$ direction and integrate over v_y, v_z.)

$$k^2=\omega_P{}^2\int\frac{g(V_x)dV_x}{(V_x-\omega/k)^2}=F(\omega/k);\qquad g(V_x)=\int\int f(\mathbf{V})dV_ydV_z, \tag{32}$$

where a small range of velocities near $V_x=\omega/k$ is excluded. $f(\mathbf{V})$ may be taken as the Maxwellian distribution with mean speed $\bar{V}$.

For large ω/k, F is approximately equal to $\omega_P{}^2/(\omega/k)^2$. Because $f(\omega/k)$ is so small, F has a value practically independent of the range near the wave velocity which has been cut out. As ω/k approaches zero, F reaches a maximum, the value of which depends fairly strongly on the range which has been cut out. It is clear, therefore, that there is a maximum k, for which ordered plasma oscillations are possible. (There is, of course, no maximum if we seek oscillations which are, as shown in Eq. (31), supported mainly by the few particles near the wave velocity.)

One can easily show that for oscillations in which ω/k is of the order of $\bar{V}$ or less, most of the response to the potential will come from particles as near the wave velocity as one can get before reaching the cut out region. (See, for example, Eq. (31), noting that if $f(\omega/k)$ is large, the main contribution to the integral comes from near $V_x=\omega/k$.) Hence, the degree of organization is very rudimentary in this region, and the motion is difficult to distinguish from disorganized free particle motion. A rough dividing velocity, above which the contribution of particles near the wave velocity becomes small, occurs where $\omega/k=\bar{V}$. Beyond this point, the frequency also depends only weakly on how large a range of velocities is cut out.

With $\bar{V}\approx(8\kappa T/\pi m)^{\frac{1}{2}}$, one obtains for the critical wave-length $\lambda_c=2\pi/k=2\pi(\bar{V}/\omega)$. From Eq. (11), one can show that when

$$\omega/k=\bar{V},\quad \omega^2\approx 2\omega_P{}^2=(8\pi n_0\epsilon^2/m)$$

and

$$\lambda_c=2\pi(\kappa T/\pi^2 n_0\epsilon^2)^{\frac{1}{2}}.$$

A similar limiting wave-length was obtained by Debye,[8] who showed that static disturbances could not be shielded out in a distance less than $(\kappa T/4\pi n_0\epsilon^2)^{\frac{1}{2}}$. Since shielding is the characteristic static property of a plasma, considered as a medium, one concludes that both statically and dynamically, aggregates smaller than a Debye length cease to act like an organized medium.[j]

Langmuir[9] has given a simple qualitative picture of why the Debye length should be the minimum wave-length for a plasma oscillation. A particle, moving much slower than the wave, experiences almost the same force as a particle at rest; hence, its contribution to the charge density is nearly the same as with the neglect of thermal motion. A particle moving much faster than the wave covers many wave-lengths during the period of a plasma oscillation, so that the average force on it tends to cancel. It therefore does not take part very strongly in the organized motion. When the wave velocity is so low that most particles are as fast as the wave or faster, organized oscillation becomes impossible.

[i] Landau (see reference 7), using a Laplace transform, has obtained essentially the same result as ours: i.e., no matter what are the initial conditions, the system oscillates asymptotically with the plasma frequency. Our method differs primarily in that it uses a discrete but closely spaced set of velocities, with the aid of which the particle nature of the charge is taken into account schematically in a more convenient way than can be done with a continuous distribution of velocities.

[8] P. Debye and E. Hückel, Physik. Zeits. 24, 185 (1923).

[j] This limitation has also been discussed by Vlasov (see reference 4), and Landau (see reference 7).

[9] I. Langmuir, Proc. Nat. Acad. Sci. 14, 627 (1928).

VIII. EXACT NON-LINEAR TRAVELING WAVE SOLUTIONS

In a steady state of oscillation,[k] the condition for breakdown of the linear approximation is that, in the wave system of coordinates, there exist many particles for which the kinetic energy is comparable with the potential, or for which

$$\epsilon\varphi \cong (m/2)U_0^2 = m/2(V_0 - V_W)^2. \qquad (33)$$

In the neighborhood of the wave velocity the expansion (10) breaks down, and the equation for φ is no longer linear. We shall indicate here the general lines on which an exact traveling wave solution, $\varphi = \varphi(x - V_W t)$, can be obtained, and shall also give the solutions for a few cases. For convenience, we shall restrict ourselves to the one-dimensional case. Although the restriction to traveling waves considerably decreases the generality of the treatment, one can still solve a wide variety of problems in this way. For example, if one has a standing wave resulting from reflections off boundaries, one can express the potential as the sum of two waves running in opposite directions, $\varphi = F(x - V_W t) + F(x + V_W t)$. Since the non-linearity is usually important only for particles close to the wave velocity, it is a good approximation to solve exactly for each running wave separately, and then to add the two solutions. This is because the wave running in the negative direction has only a slight influence on particles trapped in the wave running in the positive direction, and vice versa.

Since only a small fraction of the untrapped particles are usually near the wave velocity, and since, as we shall see, the non-linear effects introduced by their coming near the wave velocity are not qualitatively new, we shall assume that the distribution of untrapped particles is such that their density is given adequately by the linear approximation, where it is understood that $f(V_0)$ is to be taken zero in a region surrounding $V = V_W$, and broad enough to exclude the velocities for which the linear approximation fails.

The trapped particles, however, we shall treat rigorously, because they lead to qualitatively new effects. It is convenient to specify the velocity distribution of trapped particles in terms of the velocity U_1, with which they pass through the bottom of the potential trough. We denote the potential at this point by $\varphi = \varphi_1$; note that it is a *maximum* here, because we are dealing with electrons. The distribution of trapped particles at this point we denote by $dN_1 = g(U_1)dU_1$. To compute the distribution of trapped particles at any other point, x, we note that, in the steady state, the density is inversely proportional to the velocity. We obtain

$$dN_1 = \frac{g(U_1)\,|U_1|\,dU_1}{(U_1^2 + (2\epsilon/m)(\varphi(x) - \varphi_1))^{\frac{1}{2}}}. \qquad (34)$$

We must take into account the fact, however, that particles with velocities less than $U_1^2 = (2\epsilon/m) \times(\varphi_1 - \varphi(x))$ will never reach the point, x. If φ_2 is the potential at the top of the trough, we note also that particles with velocities greater than $U_1^2 = (2\epsilon/m)(\varphi_1 - \varphi_2)$ cannot be trapped. The upper limit on U_1 is therefore always $U_1 = ((2\epsilon/m)(\varphi_1 - \varphi_2))^{\frac{1}{2}}$. The total density of trapped particles at the point, x, is then

$$N_1 = \int_{((2\epsilon/m)(\varphi_1 - \varphi(x)))^{\frac{1}{2}}}^{((2\epsilon/m)(\varphi_1 - \varphi_2))^{\frac{1}{2}}} \frac{g(U_1)dU_1\,|U_1|}{(U_1^2 + (2\epsilon/m)(\varphi(x) - \varphi_1))^{\frac{1}{2}}}. \qquad (35)$$

This can be given a more convenient form with the substitution

$$\xi^2 = U_1^2 - (2\epsilon/m)(\varphi_1 - \varphi(x)); \quad \xi d\xi = U_1 dU_1. \qquad (36)$$

We get

$$N_1 = \int_0^{((2\epsilon/m)(\varphi(x) - \varphi_2))} \times g((\xi^2 - (2\epsilon/m)(\varphi(x) - \varphi_1))^{\frac{1}{2}})d\xi. \qquad (37)$$

The equation defining φ can be obtained from Poisson's equation, using the linear approximation (7), for the contribution of the untrapped particles to the charge density,

$$\nabla^2\varphi = 4\pi\epsilon\left[n_0 - n_+ - n_0\frac{\epsilon\varphi(x)}{m}\int\frac{f(V_0)dV_0}{(V_0 - V_W)^2} + \int_0^{((2\epsilon/m)(\varphi - \varphi_2))^{\frac{1}{2}}} g((\xi^2 - (2\epsilon/m)(\varphi - \varphi_1))^{\frac{1}{2}})d\xi\right], \qquad (38)$$

where n_0 is the mean density of untrapped particles, and n_+ is the density of positive ions.

To solve this equation for φ, one must first know $g(U_1)$. This function depends on the processes which cause particles to be trapped in the wave. In actual plasmas, the two most important processes of this kind are (a) collisions, (b) processes by which electrons enter the plasma, such as ionization, or injection from a hot cathode.

In the present work, we have assumed that collisions are so infrequent that their effects on particle motions can be neglected. Yet, they will still be important in determining $f(V_0)$ and $g(U_1)$. For example, they will tend to throw particles into the range of trapped velocities, with a more or less uniform distribution, and they will also tend to throw them back out. The net distribution function $g(U_1)$ is the result of the balance of the rate at which

[k] In this section, we restrict ourselves to steady-state oscillations, and do not solve the initial value problem considered in the section on the origin of medium-like behavior.

particles enter the trapped region and that at which they leave. The precise form of $g(U_1)$ is, however, very hard to predict, but one can easily see what are its main qualitative features. In general, one expects to find more particles with small velocities, U_1, than with values of U_1 so large that the particle is barely trapped. This is because particles enter the trapped region with a more or less uniform distribution in velocity, but leave more easily if barely trapped, since a comparatively small collision is then sufficient to throw them out of the trapped region. We shall take here for a typical function of this type

$$g(U_1)=a((2\epsilon/m)(\varphi_1-\varphi_2)-U_1^2)^{\frac{1}{2}}. \tag{39}$$

This function has $g(U_1)=0$, for the critical value of U_1 which leads to escape (see Eq. (33)), and $g(U_1)$ is a maximum for the most thoroughly trapped particles ($U_1=0$). This distribution in taken primarily because it is a plausible one leading to very simple mathematical results. We have, however, carried out calculations with other functions, and have obtained similar results which are considerably more complicated in mathematical form. The contribution of the trapped particles to the density is then

$$N_1(x)=a\int_0^{((2\epsilon/m)(\varphi(x)-\varphi_2))^{\frac{1}{2}}} \times((2\epsilon/m)(\varphi(x)-\varphi_2)-\xi^2)^{\frac{1}{2}}d\xi. \tag{40}$$

With the substitution, $\xi=((2\epsilon/m)(\varphi(x)-\varphi_2))^{\frac{1}{2}}\zeta$, the above becomes

$$N_1(x)=\frac{2a\epsilon}{m}(\varphi(x)-\varphi_2)\int_0^1(1-\zeta^2)^{\frac{1}{2}}d\zeta =\frac{\pi a\epsilon}{2m}(\varphi(x)-\varphi_2). \tag{41}$$

The number "a" can be evaluated in terms of n_1, the total number of trapped particles.

$$n_1=\int_0^{((2\epsilon/m)(\varphi_1-\varphi_2))^{\frac{1}{2}}} g(U_1)dU_1 =a\int_0^{((2\epsilon/m)(\varphi_1-\varphi_2))^{\frac{1}{2}}}((2\epsilon/m)(\varphi_1-\varphi_2)-U_1^2)^{\frac{1}{2}}dU_1,$$

or

$$a=\frac{2n_1m}{\pi\epsilon(\varphi_1-\varphi_2)}$$

and

$$N_1(x)=n_1\frac{(\varphi(x)-\varphi_2)}{(\varphi_1-\varphi_2)}. \tag{42}$$

This expression is especially simple in that it is linear in $\varphi(x)$, (but not in the amplitude φ_2). The simplicity results from the special choice of $g(U_1)$; other choices lead to functions, $N_1(x)$, which are not linear in $\varphi(x)$, but which nevertheless yield qualitatively similar results for the potential. Poisson's equation becomes

$$\nabla^2\varphi=\left[\frac{-4\pi n_0\epsilon^2}{m}\times\int\frac{f(V_0)dV_0}{(V_0-V_W)^2}+\frac{4\pi n_1\epsilon}{\varphi_1-\varphi_2}\right]\varphi +\left(n_0-n_+-\frac{n_1\varphi_2}{\varphi_1-\varphi_2}\right)4\pi\epsilon. \tag{43}$$

In order that the average field vanish over very long distances, it is necessary that

$$n_+=n_0-(n_1\varphi_2/\varphi_1-\varphi_2).$$

This relation will be brought about automatically as a result of the processes which insure over-all static neutralization.

The solutions of the remaining equation are of the form

$$\varphi=A\cos(kx+\alpha)=(\varphi_1-\varphi_2/2)\cos(kx+\alpha), \tag{44}$$

where

$$k^2-\omega_P^2\int\frac{f(V_0)dV_0}{(V_0-V_W)^2}=-\frac{4\pi\epsilon n_1}{\varphi_1-\varphi_2}, \tag{45}$$

and we have replaced $\varphi_1-\varphi_2$ by $2A$. For the case that $f(V_0)$ is negligible for $V_0>V_W$, one can expand the denominator of the integrand in the above equation, obtaining (for $\bar{V}=0$)

$$\omega_P^2/V_W^2(1+3\bar{V}^2/V_W^2+\cdots) =k^2+(4\pi\epsilon n_1/\varphi_1-\varphi_2). \tag{46}$$

This is an equation defining V_W in terms of k and the wave amplitude, $\varphi_1-\varphi_2$. The effect of increasing the number of trapped particles n_1 is to reduce the wave velocity. To obtain the frequency, we write $V_W=\omega/k$. For the special case of no thermal motion ($\bar{V}^2=0$), we get

$$\omega^2=\frac{\omega_P^2k^2}{k^2+(4\pi\epsilon n_1/\varphi_1-\varphi_2)}. \tag{47}$$

By comparison with (1), we see that the effect of increasing n_1 is always to lower the frequency of oscillation. This is exactly the opposite of what happens when one increases the density of untrapped particles, for this raises the frequency of plasma oscillation. The physical reason why trapped electrons lower the frequency is that they tend to concentrate in regions of positive potential, hence shield out the forces which are causing the space charge to oscillate. The untrapped electrons, on the other hand, move faster through regions of positive potential, hence tend to make the region more positive still, and increase the forces tending to cause oscillation.

It is interesting to note that for very long wavelengths ($k \rightarrow 0$), the frequency approaches

$$\omega^2 = \frac{\omega_P{}^2 k^2}{4\pi n_1 \epsilon}(\varphi_1 - \varphi_2) = \frac{n_0}{n_1}\frac{\epsilon}{m}(\varphi_1 - \varphi_2)k^2. \tag{48}$$

The dispersion relation here resembles that of sound waves in a gas except that the speed of the waves, $V_W = \omega/k$, is proportional to the square root of the amplitude. It should be noted that the type of plasma oscillation represented by these waves is very different from those described by Eq. (11). The low frequency waves appear only in the non-linear approximation, and are made possible by the contributions of particles near the wave velocity. These waves are, in fact, of exactly the same type as the oscillations supplied mainly by particles near the wave velocity, obtained in Section VI except that here, of course, the treatment is non-linear.

IX. TRAVELING PULSE SOLUTIONS

One can use the results of the last section to obtain an interesting new type of solution, in which a group of particles is trapped in a potential pulse traveling through the rest of the plasma at constant velocity. In order to see qualitatively how the particles are trapped, let us go to the coordinate system in which the pulse is at rest, while the plasma electrons move past at high speed. In the region where the potential is positive, the plasma electrons speed up, thus contributing less to the density, and tending to create an excess of positive charge. This excess of positive charge in turn produces the positive potential, which we assumed to begin with, and also traps a certain amount of negative charge, overcoming the tendency of the latter to blow up by mutual repulsion of its parts.

On either side of the pulse, which we take to be symmetric about $x=0$, and to be within the limits, $x=\pm\beta$, the plasma electrons are present with the same density as that of the positive ions (n_+). It is consistent then, to assume no field in this region. Since the positive ions have no time to respond as the pulse moves past, their density inside the pulse is the same as outside.

In order to show that pulses of this kind can exist, one must now seek solutions of Eq. (38) for the intermediate region which satisfy the following conditions, which permit one to fit the pulse solution continuously to the constant solutions outside the pulse.[1]

1. The density of untrapped electrons at the edge of the pulse must remain continuous. Thus, the density must be $N(\pm\beta) = n_+$.
2. The electric field is zero at $x = \pm\beta$.

For simplicity, we shall neglect thermal motions of the untrapped electrons, which do not alter the qualitative results in any important way. To do this, write $f(V_0) = \delta(V_0)$. Using the linear approximation (7), we obtain for the density of untrapped electrons.

$$N(x) = n_0\{1 + (\epsilon/m)[\varphi(x)/V_W{}^2]\}.$$

The first condition becomes

$$n_+ = n_0(1 + \epsilon\varphi(\pm\beta)/mV_W{}^2).$$

The equation determining $\varphi(x)$ inside the pulse is

$$\nabla^2\varphi = -\left(\omega_P{}^2/V_W{}^2 - \frac{4\pi n_1\epsilon}{\varphi_1 - \varphi_2}\right)\varphi - 4\pi\epsilon\left(n_+ - n_0 + \frac{n_1\varphi_1}{\varphi_1 - \varphi_2}\right). \tag{49}$$

The most general solution, symmetric about $x=0$, is

$$\varphi = A\cos kx - 2\pi\epsilon\left(n_+ - n_0 + \frac{n_1\varphi_1}{\varphi_1 - \varphi_2}\right)x^2, \tag{50}$$

where k is given by Eq. (47). We must satisfy the definitions that at $x=0$, $\varphi=\varphi_1$ and at $x=\pm\beta$, $\varphi=\varphi_2$ (i.e., $x=\pm\beta$ is the last point at which particles can be trapped). These conditions yield

$$\begin{aligned}&\varphi_1 = A,\\ &\varphi_2 = A\cos ka - 2\pi\epsilon\beta^2(n_+ - n_0 + n_1\varphi_1/\varphi_1 - \varphi_2).\end{aligned} \tag{51}$$

The condition that $\partial\varphi/\partial x$ vanishes at $x=\pm\beta$ is

$$0 = -kA\sin k\beta - 4\pi\epsilon\beta(n_+ - n_0 + n_1\varphi_1/\varphi_1 - \varphi_2). \tag{52}$$

The condition (1) on $N(x)$ yields

$$n_+ = n_0(1 + \epsilon\varphi_2/mV_W{}^2). \tag{53}$$

These conditions reduce to

$$\begin{aligned}\varphi_2 &= \varphi_1\cos k\beta - 2\pi\epsilon\beta^2\left(\frac{n_0\epsilon\varphi_2}{mV_W{}^2} + \frac{n_1\varphi_1}{\varphi_1 - \varphi_2}\right),\\ 0 &= -k\varphi_1\sin k\beta - 4\pi\epsilon\beta\left(\frac{n_0\epsilon\varphi_2}{mV_W{}^2} + \frac{n_1\varphi_1}{\varphi_1 - \varphi_2}\right).\end{aligned} \tag{54}$$

[1] In this work we do not attempt to follow the processes by which such a pulse solution can be set up. We merely wish to show that in the steady state, such solutions can exist.

Division of the first of these equations by the second yields

$$\varphi_2=\varphi_1\cos k\beta+\frac{\beta k}{2}\varphi_1\sin ka$$
$$=\varphi_1\left(\cos k\beta+\frac{k\beta}{2}\sin k\beta\right)\cdot \quad (55)$$

Insertion into Eq. (52) yields

$$0=-k\varphi_1\sin k\beta$$
$$-4\pi\epsilon\beta\left(\frac{n_0\epsilon}{mV_W{}^2}\left(\cos k\beta+\frac{k\beta}{2}\sin k\beta\right)\right)\varphi_1$$
$$-4\pi\epsilon\beta n_1\bigg/1-\cos k\beta-\frac{k\beta}{2}\sin k\beta, \quad (56)$$

or

$$\varphi_1\left(k\sin k\beta+\frac{4\pi\epsilon^2}{m}\frac{\beta n_0}{V_W{}^2}\times\left(\cos k\beta+\frac{k\beta}{2}\sin k\beta\right)\right)=\frac{-4\pi\epsilon\beta n_1}{1-\cos k\beta-\frac{k\beta}{2}\sin k\beta}.$$

We can solve for φ_1 from the above, then for φ_2 from (55), and finally for n_+-n_0 from (53). This provides a pulse solution, in which V_W, the wave velocity, and n_1, the number of trapped particles, may be specified arbitrarily.

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Theory of Plasma Oscillations. B. Excitation and Damping of Oscillations

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The theory of electron oscillations of an unbounded plasma is extended to take into account the effects of collisions and special groups of particles having well-defined ranges of velocities. It is found that as a result of collisions a wave tends to be damped in a time of the order of the mean time between collisions. If beams of sharply defined velocity or groups of particles far above mean thermal speeds are present, however, they introduce a tendency toward instability so that small oscillations grow until limited by effects not taken into account in the linear approximation. An estimate is made of the steady-state amplitude for plasma oscillations in which excitation occurs because of a peak at high velocities in the electron velocity distribution, and in which the main damping arises from collisions. It is also found that in variable density plasmas, waves moving in the direction of decreasing plasma density show even stronger instability.

In absence of plasma oscillations, any beam of well-defined velocity is scattered by the individual plasma electrons acting at random, but, when all particles act in unison in the form of a plasma oscillation, the scattering can become much greater. Because of the instability of the plasma when special beams are present, the beams are scattered by the oscillations which they produce. It is suggested that this type of instability can explain the results of Langmuir, which show that beams of electrons traversing a plasma are scattered much more rapidly than can be accounted for by random collisions alone. It is also suggested that this type of instability may be responsible for radio noises received from the sun's atmosphere and from interstellar space.

I. INTRODUCTION

In the preceding paper (referred to as A), we gave a theory of oscillations of an unbounded plasma, neglecting collisions, and treating in detail only ion gases with a continuous distribution of velocities, which decreases monotonically with increasing velocity. In this paper, we extend the theory to include effects of collisions and more general velocity distributions, showing how these can bring about excitation and damping of plasma oscillations.

II. EFFECTS OF COLLISIONS

A collision may be said to occur whenever two particles come so close together that a sudden transfer of momentum takes place, which is so rapid that for macroscopic phenomena, such as wave motion, it may be regarded as instantaneous. These momentum transfers occur at random relative to the phase of organized wave motion; hence, their general effect is to disrupt it and to cause damping. Because of persistence of velocity, not all of the organized motion will be lost, but in a close collision of an electron with a heavy object, such as a neutral atom or an ion, the persistence of velocity is not very important, and one can, in a rough quantitative treatment such as this, neglect it altogether. We therefore take a simplified model of these collision processes, and assume that particles emerge from a collision with no relation to their previous velocity, but with a velocity distri-

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