CHAPTER 5

Collisionless Kinetic Equations from Astrophysics – The Vlasov–Poisson System

Gerhard Rein

Department of Mathematics, University of Bayreuth, 95440 Bayreuth, Germany
E-mail: gerhard.rein@uni-bayreuth.de

Contents

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Introduction

Many important developments and concepts in mathematics originate with the N-body problem. It describes the motion of N mass points which move according to Newton's equations of motion under the influence of their mutual attraction governed by Newton's law of gravity. The N-body problem has many applications in astronomy and astrophysics, the most notable one being our solar system. Looking at larger astronomical scales further N-body systems come into view, for example globular clusters or galaxies. If the internal structure of the stars, interstellar material, processes leading to the birth or death of stars, and various other effects are neglected, then a galaxy can be described as an N-body system. For the solar system N is a fairly small number, and it is to an excellent degree of precision possible to predict the exact positions of all N bodies. For a galaxy however, N is of the order of 10^{10}–10^{12}, and keeping track of all these mass points is neither feasible nor even desired. Instead, the evolution of the in some sense averaged mass distribution of the galaxy is the issue. Such a statistical description of a large ensemble of gravitationally interacting mass points leads to a mathematical problem which is far more tractable in certain of its aspects than the N-body problem is even for very moderate N. In the present treatise results for certain nonlinear systems of partial differential equations are presented, which are used in the modeling of galaxies, globular clusters, and many other systems where a large ensemble of mass points interacts by a force field which the ensemble creates collectively.

In order to motivate the equations which describe such a particle ensemble let us continue to think of a galaxy. If $U = U(t, x)$ denotes its gravitational potential depending on time $t \in \mathbb{R}$ and position $x \in \mathbb{R}^3$, then an individual star of unit mass with position x and velocity $v \in \mathbb{R}^3$ obeys Newton's equations of motion

$$\dot{x} = v, \quad \dot{v} = -\partial_x U(t, x), \tag{1}$$

as long as it has no close encounters with other stars. Here $\partial_x U$ denotes the gradient of U with respect to x. To describe the galaxy as a whole we introduce its density $f = f(t, x, v) \geqslant 0$ on phase space $\mathbb{R}^3 \times \mathbb{R}^3$. The integral of f over any region of phase space gives the mass or number of particles (stars) which at that instant of time have phase space coordinates in that region. In a typical galaxy collisions among stars are sufficiently rare to be (in a first approximation) negligible. Hence f is constant along solutions of the equations of motion (1) and satisfies a first-order conservation law on phase space, the characteristic system of which are the equations of motion (1) of a single test particle,

$$\partial_t f + v \cdot \partial_x f - \partial_x U \cdot \partial_v f = 0. \tag{2}$$

Of course this equation can be derived in a more rigorous way like other conservation laws, using Gauss' theorem. The spatial mass density $\rho = \rho(t, x)$ induced by f determines the gravitational potential U according to Newton's law for gravity, subject to the usual boundary condition at spatial infinity,

$$\Delta U = 4\pi\gamma\rho, \qquad \lim_{|x|\to\infty} U(t,x) = 0, \tag{3}$$

$$\rho(t,x) = \int f(t,x,v)\,\mathrm{d}v; \tag{4}$$

for the moment $\gamma = 1$. Equations (2)–(4) form a closed, nonlinear system of partial differential equations which governs the time evolution of a self-gravitating collisionless ensemble of particles. No additional equations such as an equation of state are needed to close this system, as would be the case for fluid type models. A more detailed discussion of the derivation of such a system and its underlying physical assumptions can be found in [73].

At the beginning of the last century the astrophysicist Sir J. Jeans used this system to model stellar clusters and galaxies [65] and to study their stability properties. In this context it appears in many textbooks on astrophysics such as [13,26]. If we want to model an ensemble of mass points which interact by a repulsive electrostatic potential, we choose $\gamma = -1$. This form of the system is important in plasma physics where it was introduced by A.A. Vlasov around 1937 [109,110]. In the mathematics literature the system of equations (2)–(4) has become known as the Vlasov–Poisson system.

Besides being nonlinear the specific mathematical difficulty of this system lies in the fact that an equation on phase space is coupled to an equation on space. The Vlasov equation easily provides a priori bounds on L^p-norms of $f(t)$ for any $p \in [1,\infty]$, but upon integration with respect to v only an L^1-bound on $\rho(t)$ survives, which does not give good bounds for $\partial_x U$.

The Vlasov–Poisson system is just one example of a class of partial differential equations known as kinetic equations. Other such systems are the Vlasov–Maxwell system, a Lorentz invariant model for a dilute plasma where the particles interact by electrodynamic fields, and the Vlasov–Einstein system which describes a self-gravitating collisionless ensemble of mass points in the framework of General Relativity; for more details we refer to Section 1.1. In these systems the standing assumptions are that the particle ensemble is sufficiently large to justify a description by a (smooth) density function on phase space and that collisions are sufficiently rare to be neglected. If collisions are to be included a Boltzmann collision operator replaces the zero on the right-hand side of the Vlasov equation (2). One can then consider situations where collisions are the only interaction among the particles, the case of the classical Boltzmann equation of gas dynamics, or situations where both short and long range interactions are taken into account, like in the Vlasov–Poisson–Boltzmann or Vlasov–Maxwell–Boltzmann systems.

We refer to [18,22,28,43,44] for systems including collisions. The present treatise is concerned with the collisionless case. We essentially consider two topics: The existence of classical, smooth solutions to the initial value problem, and the nonlinear stability of stationary solutions. In dealing with these problems we focus on the Vlasov–Poisson system, for the stability problem we restrict ourselves even further and consider only the gravitational case. The motivation for this approach is as follows. For the Vlasov–Poisson system the mathematical understanding of the initial value problem is fairly complete, while on the other hand the techniques which were successful there can provide a guide to attacking open problems for related systems. For the stability problem in the gravitational case, which as noted above was one of the starting points of the whole field, a successful approach began to appear in the last few years, with techniques which hopefully will reach beyond kinetic theory. In spite of the restriction to the Vlasov–Poisson system we frequently comment on results and open problems for related systems so that this treatise can serve as a guide into the whole field of collisionless kinetic equations.

There are important questions concerning kinetic equations and even concerning the Vlasov–Poisson system which we do not discuss. An obvious one, which comes to mind in connection with our point of departure, the N-body problem, is the following: Consider a sequence of N-body problems where N increases to infinity and where the initial data, which can be interpreted as sums of Dirac δ distributions on phase space, converge in an appropriate sense to a smooth initial distribution function on phase space. Do the solutions of the N-body problems at later times then converge to the solution of the Vlasov–Poisson system launched by this initial distribution? A positive answer to this question could be considered as a rigorous derivation of the Vlasov–Poisson system from the N-body problem, but the question is open. Partial results, where the Newtonian interaction potential $1/|x|$ is replaced by less singular ones, are given in [54,85].

Notation and preliminaries

Our notation is mostly standard or self-explaining, but to avoid misunderstandings we fix some of it here. For $x, y \in \mathbb{R}^n$ the Euclidean scalar product and norm are denoted by

$$x \cdot y := \sum_{i=1}^{n} x_i y_i, \quad |x| := \sqrt{x \cdot x}.$$

The open ball of radius $R > 0$ with center $x \in \mathbb{R}^n$ is denoted by

$$B_R(x) := \{ y \in \mathbb{R}^n \mid |x - y| < R \}, \quad B_R := B_R(0).$$

For $\xi \in \mathbb{R}$ the positive part of this number is

$$\xi_+ := \max\{\xi, 0\}.$$

For a set $M \subset \mathbb{R}^n$, $\mathbb{1}_M$ denotes its indicator function,

$$\mathbb{1}_M(x) = 1 \text{ if } x \in M, \quad \mathbb{1}_M(x) = 0 \text{ if } x \notin M.$$

For a differentiable function $f = f(t, x, v)$, $t \in \mathbb{R}$, $x, v \in \mathbb{R}^3$,

$$\partial_t f, \quad \partial_x f, \quad \partial_v f$$

denote its partial derivatives with respect to the indicated variable; in the case of x or v these are actually gradients. If $U = U(x)$ we also write $\nabla U = \partial_x U$ for the gradient. For

$t \in \mathbb{R}$ we denote by $f(t)$ the function

$$f(t): \mathbb{R}^3 \times \mathbb{R}^3 \ni (x, v) \mapsto f(t, x, v).$$

By

$$C^k(\mathbb{R}^n), \quad C^k_c(\mathbb{R}^n)$$

we denote the space of k times continuously differentiable functions on $\mathbb{R}^n$, the subscript "c" indicates compactly supported functions. The Lebesgue measure of a measurable set $M \subset \mathbb{R}^n$ is denoted by $\mathrm{vol}(M)$. The norm on the usual Lebesgue spaces $L^p(\mathbb{R}^n)$, $1 \leqslant p \leqslant \infty$, is denoted by $\|\cdot\|_p$, where by default the corresponding integral extends over $\mathbb{R}^n$ with $n = 3$ or $n = 6$ as the case may be. We denote by $L^1_+(\mathbb{R}^n)$ the set of nonnegative integrable functions on $\mathbb{R}^n$. For $f \in L^1_+(\mathbb{R}^6)$ we define the induced spatial density $\rho_f \in L^1_+(\mathbb{R}^3)$ by

$$\rho_f(x) := \int f(x, v)\,\mathrm{d}v;$$

integrals without explicitly specified domain of integration always extend over $\mathbb{R}^3$. For $\rho: \mathbb{R}^3 \to \mathbb{R}$ measurable the induced potential is denoted by

$$U_\rho(x) := -\gamma \int \frac{\rho(y)}{|x - y|}\,\mathrm{d}y,$$

provided the latter convolution integral exists, also $U_f := U_{\rho_f}$, and if $f = f(t, x, v)$ or $\rho = \rho(t, x)$ also depend on time t we write $\rho_f(t, x) := \rho_{f(t)}(x)$, $U_\rho(t, x) := U_{\rho(t)}(x)$ etc. It will often not be necessary to write the subscripts at all.

For the convenience of the reader we collect some facts from potential theory.

LEMMA P1. *Let $\rho \in C^1_c(\mathbb{R}^3)$. Then the following holds*:

(a) *U_ρ is the unique solution of*

$$\Delta U = 4\pi\gamma\rho, \qquad \lim_{|x|\to\infty} U(x) = 0$$

in $C^2(\mathbb{R}^3)$. Moreover,

$$\nabla U_\rho(x) = \gamma \int \frac{x - y}{|x - y|^3}\rho(y)\,\mathrm{d}y,$$

$$U_\rho(x) = \mathrm{O}\big(|x|^{-1}\big), \quad \nabla U_\rho(x) = \mathrm{O}\big(|x|^{-2}\big) \quad \textit{for } |x| \to \infty.$$

(b) *For any $p \in [1, 3[$,*

$$\|\nabla U_\rho\|_\infty \leqslant c_p \|\rho\|_p^{p/3} \|\rho\|_\infty^{1-p/3},$$

where the constant $c_p > 0$ *depends only on* p, *in particular,* $c_1 = 3(2\pi)^{2/3}$. *Moreover, the second-order derivative satisfies, for any* $0 < d \leqslant R$,

$$\left\| D^2 U_\rho \right\|_\infty \leqslant c\left[R^{-3}\|\rho\|_1 + d\|\nabla\rho\|_\infty + \left(1 + \ln\frac{R}{d}\right)\|\rho\|_\infty \right],$$

with $c > 0$ *independent of* ρ, R, d, *and*

$$\left\| D^2 U_\rho \right\|_\infty \leqslant c\left[\left(1 + \|\rho\|_\infty\right)\left(1 + \ln_+ \|\nabla\rho\|_\infty\right) + \|\rho\|_1 \right].$$

PROOF. We only sketch the proof since most of this is well known. The formula for ∇U_ρ is obtained by shifting the x-variable into the argument of ρ first. After differentiating under the integral once the derivative can be moved from ρ to the kernel $1/|x-y|$ using Gauss' theorem; one has to exclude a small ball of radius ε about the singularity $y = x$ when doing so, but the corresponding boundary term vanishes as $\varepsilon \to 0$. This procedure is then applied to the formula for ∇U_ρ, except that now the boundary term survives in the limit $\varepsilon \to 0$, and the resulting singularity is no longer integrable at $y = x$. Hence for any $d > 0$ and $i, j = 1, 2, 3$,

$$\begin{aligned} \partial_{x_i}\partial_{x_j} U_\rho(x) = {} & -\gamma \int_{|x-y|\geqslant d} \left[3\frac{(x_i - y_i)(x_j - y_j)}{|x-y|^5} - \frac{\delta_{ij}}{|x-y|^3} \right]\rho(y)\,\mathrm{d}y \\ & - \gamma \int_{|x-y|\leqslant d} [\cdots]\big(\rho(y) - \rho(x)\big)\,\mathrm{d}y + \frac{4\pi}{3}\gamma\delta_{ij}\rho(x); \end{aligned}$$

the difference $\rho(y) - \rho(x)$ in the latter integral kills one power of the singularity so the integral exists. The uniqueness assertion is usually referred to as Liouville's theorem, and the asymptotic behavior is easy to deduce from the compact support of ρ. We consider (b) in more detail. For any $R > 0$, Hölder's inequality implies that

$$\begin{aligned} \left|\nabla U_\rho(x)\right| & \leqslant \int_{|x-y|<R} \frac{|\rho(y)|}{|x-y|^2}\,\mathrm{d}y + \int_{|x-y|\geqslant R} \frac{|\rho(y)|}{|x-y|^2}\,\mathrm{d}y \\ & \leqslant 4\pi R\|\rho\|_\infty + \left(\frac{4\pi}{2q-3}R^{3-2q}\right)^{1/q}\|\rho\|_p, \end{aligned}$$

where $1/p + 1/q = 1$ and hence $q > 3/2$. We optimize this estimate by choosing $R = (c\|\rho\|_p/\|\rho\|_\infty)^{p/3}$ with a suitable constant $c > 0$ and obtain the estimate for ∇U_ρ. With $0 < d \leqslant R$ the formula for $\partial_{x_i}\partial_{x_j} U_\rho$ implies that

$$\begin{aligned} \left|D^2 U_\rho(x)\right| & \leqslant \frac{4\pi}{3}\left|\rho(x)\right| + \|\nabla\rho\|_\infty \int_{|x-y|\leqslant d} \frac{4}{|x-y|^2}\,\mathrm{d}y \\ & \quad + \int_{d<|x-y|\leqslant R} \frac{4}{|x-y|^3}\left|\rho(y)\right|\mathrm{d}y + \int_{|x-y|>R} \frac{4}{|x-y|^3}\left|\rho(y)\right|\mathrm{d}y \\ & \leqslant c\left[\|\rho\|_\infty + d\|\nabla\rho\|_\infty + \|\rho\|_\infty \ln\frac{R}{d} + R^{-3}\|\rho\|_1 \right]. \end{aligned}$$

The second form of the estimate results by the choice $R = 1$, and $d = 1/\|\nabla\rho\|_\infty$ if $\|\nabla\rho\|_\infty \geqslant 1$, else $d = 1$. $\square$

We also need certain L^p-estimates for the potential, based on the weak Young's inequality.

LEMMA P2. (a) *Let $1 < p, q, r < \infty$ with $1/p + 1/q = 1 + 1/r$. Then for all functions $g \in L^p(\mathbb{R}^n)$, $h \in L^q_w(\mathbb{R}^n)$ the convolution $g * h := \int g(\cdot - y)h(y)\,\mathrm{d}y \in L^r(\mathbb{R}^n)$ satisfies*

$$\|g * h\|_r \leqslant c\|g\|_p\|h\|_{q,w}.$$

Here $c = c(p, q, n) > 0$, and by definition, $h \in L^q_w(\mathbb{R}^n)$ iff h is measurable and

$$\|h\|_{q,w} := \sup_{\tau>0} \tau\bigl(\mathrm{vol}\bigl\{x \in \mathbb{R}^n \mid \bigl|h(x)\bigr| > \tau\bigr\}\bigr)^{1/q} < \infty;$$

the latter expression does not define a norm.

(b) *If $\rho \in L^{6/5}(\mathbb{R}^3)$ then $U_\rho \in L^6(\mathbb{R}^3)$ with weak derivative $\nabla U_\rho := \gamma \cdot /|\cdot|^3 * \rho \in L^2(\mathbb{R}^3)$, and there exists a constant $c > 0$ such that*

$$\frac{1}{8\pi}\int |\nabla U_\rho|^2\,\mathrm{d}x = \frac{1}{2}\iint \frac{\rho(x)\rho(y)}{|x-y|}\,\mathrm{d}x\,\mathrm{d}y \leqslant c\|\rho\|^2_{6/5}.$$

PROOF. As to the weak Young's inequality recalled in (a), cf. [74], Section 4.3. The assertions for U_ρ and ∇U_ρ follow with $n = 3$, $p = 6/5$, and $h = |\cdot|^{-1}$, $q = 3$ or $h = |\cdot|^{-2}$, $q = 3/2$. The estimate in (b), a special case of the Hardy–Littlewood–Sobolev inequality, follows by Hölder's inequality. If $\rho \in C^1_c(\mathbb{R}^3)$ then $\gamma \cdot /|\cdot|^3 * \rho$ is the gradient of U_ρ, and integration by parts together with the Poisson equation yields the equality of the two integrals. The general case follows by a density argument. $\square$

1. Classical solutions to the initial value problem

1.1. *The initial value problem – an overview*

Before going into details we give an overview of this chapter and of the history and current state of the mathematical treatment of the initial value problem for collisionless kinetic systems. We also introduce some systems which are related to the Vlasov–Poisson system

$$\partial_t f + v \cdot \partial_x f - \partial_x U \cdot \partial_v f = 0, \tag{1.1}$$

$$\Delta U = 4\pi\gamma\rho, \qquad \lim_{|x|\to\infty} U(t, x) = 0, \tag{1.2}$$

$$\rho(t, x) = \int f(t, x, v)\,\mathrm{d}v; \tag{1.3}$$

as a rule, $t \in \mathbb{R}$, $x, v \in \mathbb{R}^3$. If the particles are allowed to move at relativistic speeds a first modification is to replace the Vlasov equation (1.1) by

$$\partial_t f + \frac{v}{\sqrt{1+|v|^2}} \cdot \partial_x f - \partial_x U \cdot \partial_v f = 0. \tag{1.4}$$

Here v should be viewed as momentum so that $v/\sqrt{1+|v|^2}$ is the corresponding relativistic velocity; like all other physical constants the speed of light is normalized to unity. The system (1.4), (1.2), (1.3) is called the relativistic Vlasov–Poisson system, and again one distinguishes the gravitational case $\gamma = 1$ and the plasma physics case $\gamma = -1$. In spite of its name this system is not fully relativistic, i.e., not Lorentz invariant. To obtain a Lorentz invariant system the field equation has to be modified accordingly. In the plasma physics case this yields the relativistic Vlasov–Maxwell system which we write for a plasma with two particle species of opposite charge; otherwise, all physical constants are again normalized to unity:

$$\begin{aligned}
&\partial_t f^\pm + \frac{v}{\sqrt{1+|v|^2}} \cdot \partial_x f^\pm \pm \left(E + \frac{v}{\sqrt{1+|v|^2}} \times B \right) \cdot \partial_v f^\pm = 0,\\
&\partial_t E - \operatorname{curl} B = -4\pi j, \qquad \partial_t B + \operatorname{curl} E = 0,\\
&\operatorname{div} E = 4\pi\rho, \qquad \operatorname{div} B = 0,\\
&\rho(t,x) = \int \big(f^+ - f^-\big)(t,x,v)\,\mathrm{d}v,\\
&j(t,x) = \int \frac{v}{\sqrt{1+|v|^2}} \big(f^+ - f^-\big)(t,x,v)\,\mathrm{d}v.
\end{aligned}$$

Here $f^\pm = f^\pm(t,x,v)$ are the densities of the positively or negatively charged particles on phase space respectively, $E = E(t,x)$ and $B = B(t,x)$ denote the electric and magnetic field, and the source terms in the Maxwell field equations are the charge and current density $\rho = \rho(t,x)$ and $j = j(t,x)$. According to [71], p. 124, it was this system (with v instead of $v/\sqrt{1+|v|^2}$) which Vlasov introduced into the plasma physics literature in 1937. The major assumption that collisions can be neglected is satisfied if the plasma is very hot and/or very dilute, a good example being the solar wind. If a fully relativistic description in the gravitational case is desired the corresponding Vlasov equation has to be coupled to Einstein's field equations

$$G^{\alpha\beta} = 8\pi T^{\alpha\beta}.$$

Here the Einstein tensor $G^{\alpha\beta}$ is a nonlinear second-order differential expression in terms of the Lorentz metric $g_{\alpha\beta}$ on the space–time manifold M, and $T^{\alpha\beta}$ is the energy momentum tensor. The equations of motion of a test particle are the geodesic equations in the metric $g_{\alpha\beta}$ so that the corresponding Vlasov equation is that first-order differential equation on the tangent bundle TM of the space–time manifold M which has the geodesic equations as its characteristic system. The corresponding density f on phase space TM determines the energy momentum tensor $T^{\alpha\beta}$. For the present treatise there is no need to make this more precise, and we refer to [3] for an introduction to the Vlasov–Einstein system.

We first consider classical solutions of the Vlasov–Poisson system (1.1)–(1.3), i.e., solutions where all relevant derivatives exist in the classical sense. A local existence and uniqueness result to the initial value problem was established by Kurth [69]. A first global existence result was proven by Batt [6] for a modified system where the spatial density ρ is regularized. The first global existence result for the original problem was again obtained by Batt [7] for spherically symmetric data. In the course of this proof an important continuation criterion was established: A local solution can be extended as long as its velocity support is under control. In Section 1.2 we prove the local existence result together with this continuation criterion, since it forms the basis for results toward global existence. The analogous result is valid for the relativistic Vlasov–Poisson and Vlasov–Maxwell systems [36], see also [15,66], and for the Vlasov–Einstein system in the spherically symmetric, asymptotically flat case [100]. In Section 1.3 spherical symmetry is shown to imply global existence for the Vlasov–Poisson system. It is also shown to be essential that the particle distribution is given by a regular function on phase space; if the particles are allowed to be δ-distributed in velocity space blow-up in finite time can occur.

The next major step was a global existence result for the Vlasov–Poisson system with sufficiently small data by Bardos and Degond [5]. The analogous result was achieved for the Vlasov–Maxwell system by Glassey and Strauss [37], and for the Vlasov–Einstein system in the spherically symmetric, asymptotically flat case by Rein and Rendall [100]. The corresponding techniques are discussed in Section 1.4 for the case of the Vlasov–Poisson system.

The development for the Vlasov–Poisson system culminated in 1989 when independently and almost simultaneously two different proofs for global existence of classical solutions for general data were given, one by Pfaffelmoser [89] and one by Lions and Perthame [80]. Since the two approaches are quite different from each other and both have their strengths we present them both in Section 1.6. In sharp contrast to the N-body problem global existence is obtained both for the repulsive and for the attractive case. In the former case the total energy is positive definite while in the latter it is indefinite, but as shown in Section 1.5, the same a priori bounds can be derived in both cases.

It may seem strange to discuss results for spherically symmetric and for small initial data when there is a result for general ones. The reason is that for the restricted data more information on the behavior of the solution is obtained and, more importantly, the techniques employed may be useful for similar problems where a general result is not yet available.

For the Vlasov–Maxwell system no analogous global existence result for general data has been proven yet. However, two points have to be emphasized here: Firstly, our discussion so far refers to the full three-dimensional problem, and much progress has been made for lower-dimensional versions of the Vlasov–Maxwell system [31–33]. Secondly, our discussion so far is restricted to classical, smooth solutions. In a celebrated paper R. DiPerna and P.-L. Lions proved global existence of appropriately defined weak solutions for the Vlasov–Maxwell system, cf. [21]. A somewhat simplified proof of this result under somewhat more restrictive assumptions can be found in [99]. A variety of tools for kinetic equations, which are used in these results and in many others, is discussed in a much broader context in [88].

Granted that global existence holds for both the attractive and the repulsive case of the Vlasov–Poisson system one certainly expects a different behavior of the two cases for large times. Results in this direction were obtained in [24,63,87], and they are discussed in Section 1.7.

1.2. *Local existence*

A local existence and uniqueness theorem is the necessary starting point for all further investigations. Since the basic approach to proving such a result is used for many related systems again and again, it is worthwhile to give a complete proof here. To begin with we make precise what we mean by a classical solution:

DEFINITION. *A function* $f : I \times \mathbb{R}^3 \times \mathbb{R}^3 \to [0, \infty[$ *is a classical solution of the Vlasov–Poisson system on the interval* $I \subset \mathbb{R}$ *if the following holds*:

(i) *The function* f *is continuously differentiable with respect to all its variables.*

(ii) *The induced spatial density* $\rho = \rho_f$ *and potential* $U = U_f$ *exist on* $I \times \mathbb{R}^3$. *They are continuously differentiable*, *and* U *is twice continuously differentiable with respect to* x.

(iii) *For every compact subinterval* $J \subset I$ *the field* $\partial_x U$ *is bounded on* $J \times \mathbb{R}^3$.

(iv) *The functions* f, ρ, U *satisfy the Vlasov–Poisson system* (1.1)–(1.3) *on* $I \times \mathbb{R}^3 \times \mathbb{R}^3$.

It is essential that the local existence result not only provides unique local solutions for a sufficiently large class of initial data, but also says in which way a solution can possibly stop to exist after a finite time.

THEOREM 1.1. *Every initial datum* $\mathring{f} \in C^1_c(\mathbb{R}^6), \mathring{f} \geqslant 0$, *launches a unique classical solution* f *on some time interval* $[0, T[$ *with* $f(0) = \mathring{f}$. *For all* $t \in [0, T[$ *the function* $f(t)$ *is compactly supported and nonnegative. If* $T > 0$ *is chosen maximal and if*

$$\sup\{|v| \mid (x, v) \in \operatorname{supp} f(t), 0 \leqslant t < T\} < \infty$$

or

$$\sup\{\rho(t, x) \mid 0 \leqslant t < T, x \in \mathbb{R}^3\} < \infty,$$

then the solution is global, *i.e.*, $T = \infty$.

A classical solution can be extended as long as its velocity support or its spatial density remain bounded. This rules out a breakdown of the solution by shock formation where typically the solution remains bounded but a derivative blows up; if the solution blows up, ρ must blow up due to a concentration effect.

Due to the requirement that the initial datum be compactly supported, the solutions obtained in the theorem enjoy stronger properties than what is required in the definition. This requirement can be replaced by suitable fall-off conditions at infinity [57]. Since such an extension of the result is mostly technical we adopt the simpler case. One can weaken the requirements on a solution and still retain uniqueness. Exactly how far one can weaken the solution concept without loosing uniqueness is not yet completely understood.

In order to prove the theorem we need to be able to solve the Vlasov equation for a given field $F = -\partial_x U$. First we consider the characteristic flow.

LEMMA 1.2. *Let $I \subset \mathbb{R}$ be an interval and let $F \in C(I \times \mathbb{R}^3; \mathbb{R}^3)$ be continuously differentiable with respect to x and bounded on $J \times \mathbb{R}^3$ for every compact subinterval $J \subset I$. Then for every $t \in I$ and $z = (x, v) \in \mathbb{R}^3 \times \mathbb{R}^3$ there exists a unique solution $I \ni s \mapsto (X, V)(s, t, x, v)$ of the characteristic system*

$$\dot{x} = v, \quad \dot{v} = F(s, x) \tag{1.5}$$

with $(X, V)(t, t, x, v) = (x, v)$. The characteristic flow $Z := (X, V)$ has the following properties:

(a) *$Z : I \times I \times \mathbb{R}^6 \to \mathbb{R}^6$ is continuously differentiable.*

(b) *For all $s, t \in I$ the mapping $Z(s, t, \cdot) : \mathbb{R}^6 \to \mathbb{R}^6$ is a C^1-diffeomorphism with inverse $Z(t, s, \cdot)$, and $Z(s, t, \cdot)$ is measure preserving, i.e.,*

$$\det \frac{\partial Z}{\partial z}(s, t, z) = 1, \quad s, t \in I, z \in \mathbb{R}^6.$$

PROOF. Most of this is standard theory for ordinary differential equations, in particular, $Z(r, t, Z(t, s, z)) = Z(r, s, z)$ by uniqueness. Hence $Z(s, t, Z(t, s, z)) = z$, i.e., $Z^{-1}(s, t, \cdot) = Z(t, s, \cdot)$. In order to see that the flow is measure preserving, we rewrite the characteristic system in the form

$$\dot{z} = G(s, z), \quad G(s, x, v) := \big(v, F(s, x)\big).$$

The assertion then follows from the fact that

$$\frac{\mathrm{d}}{\mathrm{d}s} \det \frac{\partial Z}{\partial z}(s, t, z) = \operatorname{div}_z G\big(s, Z(s, t, z)\big) \det \frac{\partial Z}{\partial z}(s, t, z) = 0;$$

divergence-free vector fields induce measure preserving flows, a fact also known as Liouville's theorem. □

The relation between the characteristic flow and the Vlasov equation is as follows.

LEMMA 1.3. *Under the assumptions of Lemma 1.2 the following holds*:

(a) *A function $f \in C^1(I \times \mathbb{R}^6)$ satisfies the Vlasov equation*

$$\partial_t f + v \cdot \partial_x f + F(t, x) \cdot \partial_v f = 0 \tag{1.6}$$

iff it is constant along every solution of the characteristic system (1.5).

(b) *For* $\mathring{f} \in C^1(\mathbb{R}^6)$ *the function*

$$f(t,z) := \mathring{f}\big(Z(0,t,z)\big), \quad t \in I, z \in \mathbb{R}^6$$

is the unique solution of (1.6) *in the space* $C^1(I \times \mathbb{R}^6)$ *with* $f(0) = \mathring{f}$. *If* $\mathring{f}$ *is non-negative then so is* f,

$$\operatorname{supp} f(t) = Z\big(t, 0, \operatorname{supp} \mathring{f}\big), \quad t \in I,$$

and for every $p \in [1, \infty]$,

$$\|f(t)\|_p = \|\mathring{f}\|_p, \quad t \in I.$$

PROOF. For a solution $z(s)$ of the characteristic system (1.5),

$$\frac{\mathrm{d}}{\mathrm{d}s} f\big(s, z(s)\big) = \big(\partial_t f + v \cdot \partial_x f + F \cdot \partial_v f\big)\big(s, z(s)\big).$$

This proves (a), since through each point (t, x, v) there passes a characteristic curve. The remaining assertions follow immediately with Lemma 1.2. □

Before giving a rigorous proof of Theorem 1.1 an "exploratory" computation is instructive, which the experts of the trade would usually accept as a proof in itself. For this computation let f be a solution on some time interval $[0, T[$. What are the crucial quantities that must be controlled in order to control the solution? Assuming that $f(t)$ has compact support as will indeed be the case for the solution constructed below, let

$$P(t) := \sup\big\{|v| \mid (x, v) \in \operatorname{supp} f(t)\big\}, \quad t \in [0, T[. \tag{1.7}$$

Since $\|f(t)\|_\infty$ and $\|f(t)\|_1$ are constant by Lemma 1.3,

$$\|\rho(t)\|_\infty \leqslant C P^3(t), \qquad \|\rho(t)\|_1 = C,$$

so that by Lemma P1,

$$\|\partial_x U(t)\|_\infty \leqslant C P^2(t).$$

By the characteristic system,

$$P(t) \leqslant P(0) + C \int_0^t P^2(s)\, \mathrm{d}s, \tag{1.8}$$

where C depends only on $\mathring{f}$ and changes its value from line to line. This estimate gives local-in-time control on P and the quantities which we estimated against P. In order to get a smooth solution we need to control derivatives, i.e., we need to go through another

Gronwall loop as above, but for the differentiated quantities. The x-derivative of ρ can be estimated against the x-derivative of f and hence of the characteristics; note that the support of the solution is now under control. If we differentiate the characteristic system with respect to initial data we get a Gronwall inequality for $\partial_x Z(s,t,x,v)$, involving $\partial_x^2 U$ so that $\partial_x Z(s,t,x,v)$ will be bounded by the exponential of the time integral of $\partial_x^2 U$. The crucial point is that in the estimate for the latter quantity in Lemma P1(b), $\partial_x \rho$ enters only logarithmically, and the whole chain of estimates leads to a linear Gronwall estimate for $\partial_x \rho$. Hence the derivatives are under control as long as the function P is.

If is useful to go through the arguments of the above exploratory computation in the form of a rigorous proof at least once in a mathematical lifetime, and here is your chance:

PROOF OF THEOREM 1.1. We fix an initial datum $\mathring{f} \in C_c^1(\mathbb{R}^6)$ with $\mathring{f} \geqslant 0$. For later use we also fix two constants $\mathring{R} > 0$ and $\mathring{P} > 0$ such that

$$\mathring{f}(x,v) = 0 \quad \text{for } |x| \geqslant \mathring{R} \text{ or } |v| \geqslant \mathring{P}.$$

We consider the following iterative scheme. The 0th iterate is defined by

$$f_0(t,z) := \mathring{f}(z), \quad t \geqslant 0, z \in \mathbb{R}^6.$$

If the nth iterate $f_n : [0,\infty[\, \times \mathbb{R}^6 \to [0,\infty[$ is already defined, we define

$$\rho_n := \rho_{f_n}, \quad U_n := U_{\rho_n}$$

on $[0,\infty[\times\mathbb{R}^3$, and we denote by

$$Z_n(s,t,z) = (X_n, V_n)(s,t,x,v)$$

the solution of the characteristic system

$$\dot{x} = v, \quad \dot{v} = -\partial_x U_n(s,x)$$

with $Z_n(t,t,z) = z$. Then

$$f_{n+1}(t,z) := \mathring{f}\big(Z_n(0,t,z)\big), \quad t \geqslant 0, z \in \mathbb{R}^6,$$

defines the next iterate. The idea of the proof is to show that these iterates converge on some time interval in a sufficiently strong sense and to identify the limit as the desired solution.

Step 1. Using Lemmae 1.2, 1.3, and Lemma P1 it is a simple proof by induction to see that the iterates are well defined and enjoy the following properties:

$$f_n \in C^1\big([0,\infty[\times\mathbb{R}^6\big), \qquad \|f(t)\|_\infty = \|\mathring{f}\|_\infty, \qquad \|f(t)\|_1 = \|\mathring{f}\|_1, \quad t \geqslant 0,$$

$$f_n(t,x,v) = 0 \quad \text{for } |v| \geqslant P_n(t) \text{ or } |x| \geqslant \mathring{R} + \int_0^t P_n(s)\,\mathrm{d}s,$$

where

$$P_0(t) := \mathring{P}, \qquad P_n(t) := \sup\bigl\{\bigl|V_{n-1}(s,0,z)\bigr| \mid z \in \operatorname{supp} \mathring{f}, 0 \leqslant s \leqslant t\bigr\}, \quad n \in \mathbb{N},$$

$$\rho_n \in C^1\bigl([0,\infty[\times\mathbb{R}^3\bigr),$$

$$\bigl\|\rho(t)\bigr\|_1 = \|\mathring{f}\|_1, \qquad \bigl\|\rho(t)\bigr\|_\infty \leqslant \frac{4\pi}{3}\|\mathring{f}\|_\infty P_n^3(t), \quad t \geqslant 0,$$

$$\rho_n(t,x) = 0 \quad \text{for } |x| \geqslant \mathring{R} + \int_0^t P_n(s)\,\mathrm{d}s,$$

and finally

$$\partial_x U_n \in C^1\bigl([0,\infty[\times\mathbb{R}^3\bigr), \qquad \|\partial_x U_n(t)\|_\infty \leqslant C\bigl(\mathring{f}\bigr) P_n^2(t),$$

where by Lemma P1(b) with $p = 1$,

$$C\bigl(\mathring{f}\bigr) := 4 \cdot 3^{1/3}\pi^{4/3}\|\mathring{f}\|_1^{1/3}\|\mathring{f}\|_\infty^{2/3}. \tag{1.9}$$

Since this particular constant enters into the length of the interval on which the iterates converge, the information on which parameters it depends is important for the proof of the continuation criterion.

Step 2. Let $P : [0,\delta[\, \to\,]0,\infty[$ denote the maximal solution of the integral equation

$$P(t) = \mathring{P} + C\bigl(\mathring{f}\bigr)\int_0^t P^2(s)\,\mathrm{d}s,$$

i.e.,

$$P(t) = \mathring{P}\bigl(1 - \mathring{P}C\bigl(\mathring{f}\bigr)t\bigr)^{-1}, \quad 0 \leqslant t < \delta := \bigl(\mathring{P}C\bigl(\mathring{f}\bigr)\bigr)^{-1};$$

without loss of generality $\mathring{f} \neq 0$. We claim that for every $n \in \mathbb{N}_0$ and $t \in [0,\delta[$ the estimate

$$P_n(t) \leqslant P(t)$$

holds. The assertion is obvious for $n = 0$. Assume it holds for some $n \in \mathbb{N}_0$. Then by Step 1,

$$\begin{aligned}\bigl|V_n(s,0,z)\bigr| &\leqslant |v| + \int_0^s \bigl\|\partial_x U_n(\tau)\bigr\|_\infty \mathrm{d}\tau \leqslant \mathring{P} + C\bigl(\mathring{f}\bigr)\int_0^s P_n^2(\tau)\,\mathrm{d}\tau \\ &\leqslant \mathring{P} + C\bigl(\mathring{f}\bigr)\int_0^t P^2(\tau)\,\mathrm{d}\tau = P(t)\end{aligned}$$

for any $0 \leqslant s \leqslant t < \delta$ and $z \in \operatorname{supp} \mathring{f}$ so that the assertion follows by induction. On the interval $[0,\delta[$ the following estimates hold:

$$\bigl\|\rho_n(t)\bigr\|_\infty \leqslant \frac{4\pi}{3}\|\mathring{f}\|_\infty P^3(t), \qquad \bigl\|\partial_x U_n(t)\bigr\|_\infty \leqslant C\bigl(\mathring{f}\bigr)P^2(t), \quad n \in \mathbb{N}_0.$$

We aim to show that the iterative scheme converges uniformly on any compact subinterval of $[0, \delta[$. Hence we fix $0 < \delta_0 < \delta$. In order to estimate terms like

$$\partial_x U_n(t, X_n) - \partial_x U_n(t, X_{n+1}),$$

a bound on $\partial_x^2 U_n$ is needed, uniformly in n.

Step 3. There exists some constant $C > 0$ depending on the initial datum and on δ_0 such that

$$\left\|\partial_x \rho_n(t)\right\|_\infty + \left\|\partial_x^2 U_n(t)\right\|_\infty \leqslant C, \quad t \in [0, \delta_0], n \in \mathbb{N}_0.$$

In the following proof of this assertion the constant C may change its value from line to line; it is only important that it does not depend on $t \in [0, \delta_0]$ or on $n \in \mathbb{N}_0$. First we note that

$$\left|\partial_x \rho_{n+1}(t, x)\right| \leqslant \int_{|v| \leqslant P(t)} \left|\partial_x\left[\mathring{f}\left(Z_n(0, t, x, v)\right)\right]\right| \mathrm{d}v \leqslant C \left\|\partial_x Z_n(0, t, \cdot)\right\|_\infty.$$

We fix $x, v \in \mathbb{R}^3$ and $t \in [0, \delta_0]$ and write $(X_n, V_n)(s)$ instead of $(X_n, V_n)(s, t, x, v)$. If we differentiate the characteristic system defining Z_n with respect to x we obtain the estimates

$$\left|\partial_x \dot{X}_n(s)\right| \leqslant \left|\partial_x V(s)\right|, \qquad \left|\partial_x \dot{V}_n(s)\right| \leqslant \left\|\partial_x^2 U_n(s)\right\|_\infty \left|\partial_x X_n(s)\right|.$$

If we integrate these estimates, observe that $\partial_x X_n(t) = \mathrm{id}$, $\partial_x V_n(t) = 0$, and add the results we find that

$$\begin{aligned} &\left|\partial_x X_n(s)\right| + \left|\partial_x V_n(s)\right| \\ &\quad \leqslant 1 + \int_s^t \left(1 + \left\|\partial_x^2 U_n(\tau)\right\|_\infty\right)\left(\left|\partial_x X_n(\tau)\right| + \left|\partial_x V_n(\tau)\right|\right) \mathrm{d}\tau. \end{aligned}$$

By Gronwall's lemma,

$$\left|\partial_x X_n(s)\right| + \left|\partial_x V_n(s)\right| \leqslant \exp \int_0^t \left(1 + \left\|\partial_x^2 U_n(\tau)\right\|_\infty\right) \mathrm{d}\tau,$$

and hence

$$\left\|\partial_x \rho_{n+1}(t)\right\|_\infty \leqslant C \exp \int_0^t \left\|\partial_x^2 U_n(\tau)\right\|_\infty \mathrm{d}\tau, \quad 0 \leqslant s \leqslant t \leqslant \delta_0.$$

We insert the estimate on ρ_{n+1} from Step 2 and the above estimate on $\partial_x \rho_{n+1}$ into the second estimate for $\partial_x^2 U_{n+1}$ from Lemma P1(b) to find that

$$\left\|\partial_x^2 U_{n+1}(t)\right\|_\infty \leqslant C\left(1 + \int_0^t \left\|\partial_x^2 U_n(\tau)\right\|_\infty \mathrm{d}\tau\right).$$

By induction,

$$\left\|\partial_x^2 U_n(t)\right\|_\infty \leqslant C\mathrm{e}^{Ct}, \quad t \in [0, \delta_0], n \in \mathbb{N}_0,$$

if we increase C so that $\|\partial_x^2 U_0\|_\infty \leqslant C$, and the claim of Step 3 is established.

Step 4. We show that the sequence (f_n) converges to some function f, uniformly on $[0, \delta_0] \times \mathbb{R}^6$. Firstly, for $n \in \mathbb{N}$, and $t \in [0, \delta_0]$, $z \in \mathbb{R}^6$,

$$\left|f_{n+1}(t, z) - f_n(t, z)\right| \leqslant C\left|Z_n(0, t, z) - Z_{n-1}(0, t, z)\right|.$$

For $0 \leqslant s \leqslant t$ we have, suppressing the t and z arguments of the characteristics,

$$\begin{aligned}
\left|X_n(s) - X_{n-1}(s)\right| &\leqslant \int_s^t \left|V_n(\tau) - V_{n-1}(\tau)\right| \mathrm{d}\tau, \\
\left|V_n(s) - V_{n-1}(s)\right| &\leqslant \int_s^t \Big[\left|\partial_x U_n\big(\tau, X_n(\tau)\big) - \partial_x U_{n-1}\big(\tau, X_n(\tau)\big)\right| \\
&\qquad + \left|\partial_x U_{n-1}\big(\tau, X_n(\tau)\big) - \partial_x U_{n-1}\big(\tau, X_{n-1}(\tau)\big)\right|\Big] \mathrm{d}\tau \\
&\leqslant \int_s^t \Big[\left\|\partial_x U_n(\tau) - \partial_x U_{n-1}(\tau)\right\|_\infty \\
&\qquad + C\left|X_n(\tau) - X_{n-1}(\tau)\right|\Big] \mathrm{d}\tau.
\end{aligned}$$

If we add these estimates and apply Gronwall's lemma we obtain the estimate

$$\begin{aligned}
\left|Z_n(s) - Z_{n-1}(s)\right| &\leqslant C \int_0^t \left\|\partial_x U_n(\tau) - \partial_x U_{n-1}(\tau)\right\|_\infty \mathrm{d}\tau \\
&\leqslant C \int_0^t \left\|\rho_n(\tau) - \rho_{n-1}(\tau)\right\|_\infty^{2/3} \left\|\rho_n(\tau) - \rho_{n-1}(\tau)\right\|_1^{1/3} \mathrm{d}\tau \\
&\leqslant C \int_0^t \left\|\rho_n(\tau) - \rho_{n-1}(\tau)\right\|_\infty \mathrm{d}\tau \\
&\leqslant C \int_0^t \left\|f_n(\tau) - f_{n-1}(\tau)\right\|_\infty \mathrm{d}\tau;
\end{aligned}$$

note that the support of both $\rho_n(t)$ and $f_n(t)$ is bounded, uniformly in n and $t \in [0, \delta_0]$. Summing up we obtain

$$\left\|f_{n+1}(t) - f_n(t)\right\|_\infty \leqslant C_* \int_0^t \left\|f_n(\tau) - f_{n-1}(\tau)\right\|_\infty \mathrm{d}\tau,$$

and by induction,

$$\left\|f_{n+1}(t) - f_n(t)\right\|_\infty \leqslant C \frac{C_*^n t^n}{n!} \leqslant C \frac{C^n}{n!}, \quad n \in \mathbb{N}_0, 0 \leqslant t \leqslant \delta_0.$$

This implies that the sequence is uniformly Cauchy and converges uniformly on $[0,\delta_0]\times\mathbb{R}^6$ to some function $f\in C([0,\delta_0]\times\mathbb{R}^6)$. The limit has the following properties:

$$f(t,x,v)=0 \quad \text{for } |v|\geqslant P(t) \text{ or } |x|\geqslant \mathring{R}+\int_0^t P(s)\,\mathrm{d}s$$

and

$$\rho_n\to\rho:=\rho_f,\qquad U_n\to U:=U_f$$

as $n\to\infty$, uniformly on $[0,\delta_0]\times\mathbb{R}^3$.

Step 5. In this step we show that the limiting function f has the regularity required of a solution to the Vlasov–Poisson system. Since

$$\big\|\partial_x U_n(t)-\partial_x U_m(t)\big\|_\infty\leqslant C\big\|\rho_n(t)-\rho_m(t)\big\|_\infty^{2/3}\big\|\rho_n(t)-\rho_m(t)\big\|_1^{1/3}$$

and

$$\begin{aligned}&\big\|\partial_x^2 U_n(t)-\partial_x^2 U_m(t)\big\|_\infty\\&\quad\leqslant C\bigg[\bigg(1+\ln\frac{R}{d}\bigg)\big\|\rho_n(t)-\rho_m(t)\big\|_\infty\\&\qquad\qquad+d\big\|\partial_x\rho_n(t)-\partial_x\rho_m(t)\big\|_\infty+R^{-3}\big\|\rho_n(t)-\rho_m(t)\big\|_1\bigg]\end{aligned}$$

for any $0<d\leqslant R$ the sequences $(\partial_x U_n)$ and $(\partial_x^2 U_n)$ are uniformly Cauchy on $[0,\delta_0]\times\mathbb{R}^3$; notice that due to the compact support in x, uniformly in n, the L^1-difference of the ρ's can be estimated against the L^∞-difference which converges to zero by the previous step, and while the L^∞-difference of the derivatives of the ρ's can according to Step 3 only be estimated by a uniform and not necessarily small constant, it has the factor d in front which can be chosen smaller than any prescribed ε. Hence

$$U,\partial_x U,\partial_x^2 U\in C\big([0,\delta_0]\times\mathbb{R}^3\big).$$

This in turn implies that

$$Z:=\lim_{n\to\infty} Z_n\in C^1\big([0,\delta_0]\times[0,\delta_0]\times\mathbb{R}^6\big),$$

which is the characteristic flow induced by the limiting field $-\partial_x U$. Hence

$$f(t,z)=\lim_{n\to\infty}\mathring{f}\big(Z_n(0,t,z)\big)=\mathring{f}\big(Z(0,t,z)\big),$$

and $f\in C^1([0,\delta_0]\times\mathbb{R}^6)$ is a classical solution of the Vlasov–Poisson system. Since the arguments from Steps 3–5 hold on any compact subinterval of the interval $[0,\delta[$ this solution exists on the latter interval, and it is straight forward to verify the remaining properties from the above definition of a classical solution such as the differentiability of ρ and U with respect to t.

Step 6. In order to show uniqueness we take two solutions f and g according to the definition with $f(0) = g(0)$, which both exist on some interval $[0, \delta]$. By (iii) in the definition of solution and Lemma 1.3 both $f(t)$ and $g(t)$ are supported in a compact set in $\mathbb{R}^6$ which can be chosen independent of $t \in [0, \delta]$. The estimates for the difference of two iterates $f_n - f_{n-1}$ can now be repeated for the difference $f - g$ to obtain the estimate

$$\|f(t) - g(t)\|_\infty \leqslant C \int_0^t \|f(s) - g(s)\|_\infty \, \mathrm{d}s$$

on the interval $[0, \delta]$, and uniqueness follows.

Step 7. In order to prove the continuation criterion, let $f \in C^1([0, T[\times\mathbb{R}^6)$ be the maximally extended classical solution obtained above, and assume that

$$P^* := \sup\{|v| \mid (t, x, v) \in \operatorname{supp} f\} < \infty,$$

but $T < \infty$. By Lemma 1.3,

$$\|f(t)\|_\infty = \|\mathring{f}\|_\infty, \qquad \|f(t)\|_1 = \|\mathring{f}\|_1, \quad 0 \leqslant t < T.$$

The idea is to use the control of the length δ of the interval on which we constructed the solution in Steps 1–5 to show that, if we use the procedure above for the new initial value problem where we prescribe $f(\mathring{t})$ as initial datum at time $t = \mathring{t}$, we extend the solution beyond T if $\mathring{t}$ is chosen sufficiently close to T. This is then the desired contradiction. To carry this out we notice first that $C(f(\mathring{t})) = C(\mathring{f})$, cf. (1.9). The maximal solution of the equation

$$P(t) = P^* + C\big(f(\mathring{t})\big) \int_{\mathring{t}}^t P^2(s) \, \mathrm{d}s$$

exists on some interval $[\mathring{t}, \mathring{t} + \delta^*[$ the length δ^* of which is independent of $\mathring{t}$. But since $f(\mathring{t}, x, v) = 0$ for $|v| \geqslant P^*$ by definition of the latter quantity, the functions P_n will be bounded by P on this interval, and all the estimates from Steps 2–5 can be repeated on the interval $[\mathring{t}, \mathring{t} + \delta^*[$ so that our solution does exist there. If the a priori bound on ρ holds this gives an a priori bound on the field $-\partial_x U$ and hence on the quantity P^* as well, and the proof is complete. □

Concluding remarks. (a) The above proof is essentially given in [7] although the result is not stated there.

(b) The analogous result, in particular the analogous continuation criterion is valid for the relativistic Vlasov–Maxwell system [36] or the Vlasov–Einstein system in the case of spherical symmetry and asymptotic flatness [100]; for these systems the control of the velocity support has to be replaced by control of the momentum support. Due to the nonlinear nature of the Einstein equations the continuation criterion is not valid for the Vlasov–Einstein system in general.

(c) Uniqueness within weaker solution concepts is considered in [81,103,114].

(d) Uniqueness is violated within the framework of measure-valued, weak solutions to the Vlasov–Poisson system in one space dimension [82], cf. also [83].

1.3. *Spherically symmetric solutions*

The estimates used in the proof of the local existence result are not strong enough to yield global existence. Indeed, it is a priori not clear that solutions should exist globally in time. In the plasma physics case one might argue that the particles repulse each other and hence the spatial density should remain bounded. But in the gravitational case the particles attract each other, and a gravitational collapse seems conceivable. To make these doubts more substantial we consider spherically symmetric solutions which by definition are invariant under simultaneous rotations of both x and v,

$$f(t,x,v) = f(t,Ax,Av), \quad A \in \mathrm{SO}(3).$$

If f is a solution, this transformation produces another one which by uniqueness coincides with f if the initial datum is spherically symmetric. Hence spherical symmetry is preserved by the Vlasov–Poisson system. For a spherically symmetric solution

$$\rho(t,x) = \rho(t,r), \qquad U(t,x) = U(t,r), \quad r := |x|,$$

with some abuse of notation, and

$$\partial_r U(t,r) = \frac{4\pi\gamma}{r^2}\int_0^r \rho(t,s)s^2\,\mathrm{d}s, \qquad \partial_x U(t,x) = \partial_r U(t,r)\frac{x}{r}. \tag{1.10}$$

Example of a "dust" solution which blows up. Let $\gamma = 1$. A likely candidate for a gravitational collapse is a spherically symmetric ensemble of particles which all move radially inward in such a way that they all arrive at the center at the same time. Hence let us consider a distribution function of the form

$$f(t,x,v) = \rho(t,x)\delta\big(v - u(t,x)\big),$$

where δ is the Dirac distribution. In such a distribution there is at each point in space only one particle velocity given by the velocity field $u(t,x) \in \mathbb{R}^3$, in other words, there is no velocity scattering. Notice that such an f is forbidden in our definition of solution, more importantly, it is not a distribution *function* on phase space, but we consider it anyway. Formally, such an f satisfies the Vlasov–Poisson system provided the spatial density ρ and the velocity field u, which now are the dynamical variables, satisfy the following special case of the Euler–Poisson system

$$\partial_t \rho + \mathrm{div}(\rho u) = 0, \tag{1.11}$$

$$\partial_t u + (u\cdot\partial_x)u = -\partial_x U, \tag{1.12}$$

$$\Delta U = 4\pi\rho, \qquad \lim_{|x|\to\infty} U(t,x) = 0.$$

This system represents a self-gravitating, ideal, compressible fluid with the strange equation of state that the pressure is zero, a situation which in astrophysics is sometimes referred to as *dust*. More information on how to pass from the Vlasov–Poisson system to the pressure-less Euler–Poisson system can be found in [20]. As initial datum to this system we take a homogeneous ball about the origin with all the particles at rest

$$\rho(0) = \frac{3}{4\pi}\mathbb{1}_{B_1}, \qquad u(0) = 0.$$

For such data the following ansatz is reasonable:

$$\rho(t,x) = \frac{3}{4\pi}\frac{1}{R^3(t)}\mathbb{1}_{B_{R(t)}}(x), \qquad u(t,x) = \frac{\dot{R}(t)}{R(t)}x,$$

i.e., we assume that the system retains the shape of a homogeneous ball, but the ball may contract (or expand). The radius $R : [0, T[\to]0, \infty[$ of the ball has to be determined such that a solution of the system above is obtained. It is straight forward to see that the continuity equation (1.11) is satisfied for $|x| \neq R(t)$; it holds in a weak sense everywhere. Moreover,

$$\partial_x U(t,x) = R^{-3}(t)x, \qquad \partial_t u(t,x) = \left(\frac{\ddot{R}}{R} - \frac{\dot{R}^2}{R^2}\right)(t)x,$$
$$(u \cdot \partial_x)u = \frac{\dot{R}^2}{R^2}(t)x.$$

Hence Newton's law (1.12) is equivalent to the equation

$$\ddot{R} = -R^{-2} \tag{1.13}$$

which is nothing but the equation for the radial motion of a mass point in a central gravitational field. Our initial data translate into the condition

$$R(0) = 1, \qquad \dot{R}(0) = 0.$$

The corresponding solution of (1.13) becomes 0 in finite time which means that all the mass of the solution of the pressure-less Euler–Poisson system collapses to a point in finite time.

This blow-up result for a system which does not belong to kinetic theory but seems to be closely related to the Vlasov–Poisson system might motivate one to look for a corresponding blow-up example for the latter. But in 1977, J. Batt proved the following result, which was the first global existence result for the Vlasov–Poisson system in three space dimensions – the result holds for both $\gamma = 1$ and $\gamma = -1$.

THEOREM 1.4. *Let* $\mathring{f} \in C^1_c(\mathbb{R}^6), \mathring{f} \geqslant 0$ *be spherically symmetric. Then there exists a constant* $P_0 > 0$ *such that for the corresponding classical solution of the Vlasov–Poisson system*

$$f(t,x,v) = 0 \quad \textit{for } |v| \geqslant P_0, 0 \leqslant t < T, x \in \mathbb{R}^3,$$

in particular, the solution is global in time, $T = \infty$. *The constant* P_0 *depends only on* $\|\mathring{f}\|_1, \|\mathring{f}\|_\infty$, *and* $\mathring{P}$, *where* $\mathring{f}(x,v) = 0$ *for* $|v| \geqslant \mathring{P}$.

PROOF. With $M := \|\mathring{f}\|_1 = \|f(t)\|_1$, formula (1.10) implies that

$$\left|\partial_x U(t,x)\right| \leqslant \frac{M}{r^2}, \quad r = |x|, t \geqslant 0.$$

On the other hand, by Lemma P1(b) with $p = 1$,

$$\left|\partial_x U(t,x)\right| \leqslant C\left\|\rho(t)\right\|_\infty^{2/3} \leqslant C P^2(t);$$

for technical reasons P is redefined to be nondecreasing:

$$P(t) := \sup\left\{|v| \mid (x,v) \in \operatorname{supp} f(s), 0 \leqslant s \leqslant t\right\}.$$

Combining both estimates we find that

$$\left|\partial_x U(t,x)\right| \leqslant C \min\left\{\frac{1}{r^2}, P^2(t)\right\}.$$

Hence for any characteristic $(x(s), v(s))$ which starts in the support of $\mathring{f}$ we have for $i = 1, 2, 3$ and $0 \leqslant s \leqslant t < T$ the estimate

$$\begin{aligned}\left|\ddot{x}_i(s)\right| &\leqslant \left|\partial_{x_i} U\big(s, x(s)\big)\right| \\ &\leqslant C^* \min\left\{\frac{1}{|x_i(s)|^2}, P^2(t)\right\},\end{aligned}$$

where the constant C^* depends only on the L^1 and L^∞-norms of $\mathring{f}$. Let $\xi := x_i$. Then $\xi \in C^2([0,t])$ with

$$\left|\ddot{\xi}(s)\right| \leqslant g\big(\xi(s)\big), \quad 0 \leqslant s \leqslant t,$$

where

$$g(r) := C^* \min\left\{\frac{1}{r^2}, P^2(t)\right\} \geqslant 0, \quad r \in \mathbb{R}.$$

If $\dot{\xi}(s) \neq 0$ on $]0, t[$, i.e., $\dot{\xi}$ does not change sign, it follows that

$$\begin{aligned}
\left|\dot{\xi}(t) - \dot{\xi}(0)\right|^2 &\leqslant \left|\dot{\xi}(t) - \dot{\xi}(0)\right|\left|\dot{\xi}(t) + \dot{\xi}(0)\right| \\
&= \left|\dot{\xi}(t)^2 - \dot{\xi}^2(0)\right| = 2\left|\int_0^t \dot{\xi}(s)\ddot{\xi}(s)\,\mathrm{d}s\right| \\
&\leqslant 2\int_0^t \left|\dot{\xi}(s)\right| g\big(\xi(s)\big)\,\mathrm{d}s = 2\int_{\xi([0,t])} g(r)\,\mathrm{d}r \\
&\leqslant 2\int g(r)\,\mathrm{d}r = 8C^* P(t),
\end{aligned}$$

and hence

$$\left|\dot{\xi}(t) - \dot{\xi}(0)\right| \leqslant 2\sqrt{2C^*}P^{1/2}(t).$$

If $\dot{\xi}(s) = 0$ for some $s \in]0, t[$ we define

$$s_- := \inf\big\{s \in]0, t[\mid \dot{\xi}(s) = 0\big\}, \qquad s_+ := \sup\big\{s \in]0, t[\mid \dot{\xi}(s) = 0\big\}$$

so that $0 \leqslant s_- \leqslant s_+ \leqslant t$, $\dot{\xi}(s_-) = \dot{\xi}(s_+) = 0$, and the first case applies on the intervals $[0, s_-]$ and $[s_+, t]$. Hence

$$\left|\dot{\xi}(t) - \dot{\xi}(0)\right| \leqslant \left|\dot{\xi}(t) - \dot{\xi}(s_+)\right| + \left|\dot{\xi}(s_-) - \dot{\xi}(0)\right| \leqslant 4\sqrt{2C^*}P^{1/2}(t).$$

Since $\dot{\xi} = \dot{x}_i = v_i$, this implies that

$$P(t) \leqslant P(0) + 4\sqrt{6C^*}P^{1/2}(t), \quad t \in [0, T[\,.$$

The proof is complete. □

Given the blow-up example for the pressure-less Euler–Poisson system on the one hand and the global existence result for the spherically symmetric Vlasov–Poisson system on the other, the question arises whether there are similar semi-explicit solutions to the Vlasov–Poisson system and how they behave. A family of such examples has been constructed by Kurth [70].

Semiexplicit spherically symmetric solutions. It is easy to check that

$$f_0(x, v) := \frac{3}{4\pi^3}\begin{cases} (1 - |x|^2 - |v|^2 + |x \times v|^2)^{-1/2}, \\ \qquad \text{where } (\cdots) > 0 \text{ and } |x \times v| < 1, \\ 0 \quad \text{else} \end{cases}$$

defines a time independent solution with spatial density and potential

$$\rho_0(x) = \frac{3}{4\pi}\mathbb{1}_{B_1}(x), \qquad U_0(x) = \begin{cases} |x|^2/2 - 3/2, & |x| \leqslant 1, \\ -1/|x|, & |x| > 1. \end{cases}$$

Note that due to spherical symmetry the particle angular momentum $x \times v$ is preserved along characteristics, the particle energy $|v|^2/2 + U_0(x)$ is preserved because U_0 is time independent, and f_0 is a function of these invariants. The transformation

$$f(t,x,v) := f_0\left(\frac{x}{R(t)}, R(t)v - \dot{R}(t)x\right)$$

turns this steady state into a time dependent solution with spatial mass density

$$\rho(t) = \frac{3}{4\pi}\frac{1}{R^3(t)}\mathbb{1}_{B_{R(t)}},$$

provided the function $R = R(t)$ solves the differential equation

$$\ddot{R} - R^{-3} + R^{-2} = 0,$$

and $R(0) = 1$. Notice that in this example the spatial density is constant on a ball with a time dependent radius, like for the Euler–Poisson example stated above. Depending on $\alpha := \dot{R}(0)$ the solution behaves as follows:

- If $\alpha = 0$ then $R(t) = 1, t \in \mathbb{R}$, and we recover the steady state f_0.
- If $0 < |\alpha| < 1$ then

  $$R(t) = \frac{1 - \alpha\cos\phi(t)}{1-\alpha^2},$$

  where $\phi(t)$ is uniquely determined by

  $$\phi(t) - \alpha\sin\phi(t) = (1-\alpha^2)^{3/2}(t - t_0),$$
  $$t_0 := -(1-\alpha^2)^{-3/2}(\phi_0 - \alpha\sin\phi_0), \quad \phi_0 := \arccos\alpha.$$

  The solution is time periodic with period $2\pi(1-\alpha^2)^{-3/2}$.
- If $|\alpha| = 1$ then

  $$R(t) = \frac{1+\phi^2(t)}{2},$$

  where $\phi(t)$ is uniquely determined by

  $$\phi(t) + \frac{\phi^3(t)}{3} = 2\left(\alpha t + \frac{2}{3}\right).$$

The solution is global, but $R(t) \to \infty$ for $|t| \to \infty$, and R is strictly decreasing on $]-\infty, t_0]$ and strictly increasing on $[t_0, \infty[$ where $t_0 = -2/(3\alpha)$.

- If $|\alpha| > 1$ then

$$R(t) = \frac{|\alpha| \cosh \phi(t) - 1}{\alpha^2 - 1},$$

where $\phi(t)$ is uniquely determined by

$$\phi(t) - |\alpha| \sinh \phi(t) = -(\alpha^2 - 1)^{3/2}(t - t_0),$$
$$t_0 := (\alpha^2 - 1)^{-3/2}\big(\phi_0 - |\alpha| \sinh \phi_0\big), \quad \cosh \phi_0 = \alpha, \operatorname{sgn} \phi_0 = \operatorname{sgn} \alpha.$$

The solution is global with $R(t) \to \infty$ for $|t| \to \infty$, and R is strictly decreasing on $]-\infty, t_0]$ and strictly increasing on $[t_0, \infty[$.

Rigorously speaking, this example does not fit into our definition of solution, because f becomes singular at the boundary of its support, but the induced field allows for well defined characteristics, and f is constant along these. In no case does the solution blow up. To understand this difference to the dust example discussed above it should be observed that as opposed to the former there is velocity scattering in these solutions.

Concluding remarks. (a) The original proof of Theorem 1.4 given in [7] considered only the gravitational case $\gamma = 1$, which is the more difficult case anyway. It relied on a detailed analysis of the characteristic system, written in coordinates adapted to the symmetry: For a spherically symmetric solution, $f(t, x, v) = f(t, r, u, \alpha)$, where $r := |x|$, $u := |v|$, and α is the angle between x and v. It is easy to check that the modulus of angular momentum $ru \sin \alpha = |x \times v|$ is conserved along characteristics, and this fact was exploited in [7]. The above proof of Theorem 1.4 is due to Horst [58], where an analogous result is shown also for axially symmetric solutions which by definition are invariant under rotations about some fixed axis. To prove the latter result a priori bounds on the kinetic energy of the solution and on $\|\rho(t)\|_{5/3}$ were established first. These a priori bounds are discussed in Section 1.5, since they become essential for the global results in Section 1.6.

(b) Angular momentum invariants have proven useful in related situations. For example, in the plasma physics case global existence of classical solutions to the relativistic Vlasov–Poisson system has been shown for spherically symmetric and for axially symmetric initial data [29,34]. In the gravitational case blow-up occurs for that system, cf. Section 1.7.

(c) Under the assumption of spherical symmetry the relativistic Vlasov–Maxwell system reduces to the plasma physics case of the relativistic Vlasov–Poisson system, and global existence holds.

(d) The above Kurth solutions are to our knowledge the only time dependent solutions to the Vlasov–Poisson system, the behavior of which can be determined analytically. Notice that the boundary condition $\lim_{|x| \to \infty} U(t, x) = 0$ is part of our formulation of this system, i.e., we consider only isolated systems. Other semi-explicit solution families which do not satisfy this boundary condition are considered in [8]. For a cosmological interpretation of these solutions, which do not represent bounded particle ensembles, we refer to [25]. No Kurth type examples are known for the related systems.

1.4. *Small data solutions*

For a nonlinear evolution equation a natural question is whether sufficiently small initial data lead to solutions which decay and hence are global in time. This happens if the linear part of the equation has some dispersive property which is strong enough to dominate the nonlinearity as long as the solution is small. By no means all nonlinear PDEs have this property, but kinetic equations as a rule do. For the Vlasov–Poisson system this was established in [5]. In the following discussion of this result initial data are always taken from the set

$$\mathcal{D} := \big\{ \mathring{f} \in C_c^1(\mathbb{R}^6) \mid \mathring{f} \geqslant 0, \|\mathring{f}\|_\infty \leqslant 1, \|\partial_{(x,v)} \mathring{f}\|_\infty \leqslant 1,$$
$$\mathring{f}(x,v) = 0 \text{ for } |x| \geqslant \mathring{R} \text{ or } |v| \geqslant \mathring{P} \big\},$$

where $\mathring{R}, \mathring{P} > 0$ are arbitrary but fixed. Constants denoted by C may depend on these parameters and may change from line to line. The following theorem holds for both $\gamma = 1$ and $\gamma = -1$.

THEOREM 1.5. *There exists some $\delta > 0$ such that for any initial datum $\mathring{f} \in \mathcal{D}$ with $\|\mathring{f}\|_\infty < \delta$ the corresponding solution is global and satisfies the following decay estimates for $t > 0$:*

$$\|\rho(t)\|_\infty \leqslant Ct^{-3}, \qquad \|\partial_x U(t)\|_\infty \leqslant Ct^{-2}, \qquad \|\partial_x^2 U(t)\|_\infty \leqslant Ct^{-3}\ln(1+t).$$

The idea of the proof is as follows:

- If the field is zero, $\partial_x U = 0$, the free motion of the particles causes ρ to decay:

$$\rho(t,x) = \int \mathring{f}\big(X(0,t,x,v), V(0,t,x,v)\big)\,\mathrm{d}v = \int \mathring{f}(x - tv, v)\,\mathrm{d}v$$
$$= t^{-3} \int \mathring{f}\left(X, \frac{x - X}{t}\right)\mathrm{d}X \leqslant Ct^{-3}.$$

  If the field is not zero but decays sufficiently fast this argument remains valid, i.e., the determinant of the matrix $\partial_v X(0,t,x,v)$ which comes up in the change of variables above grows like t^3.
- By Lemma P1 a decay of ρ translates into a decay of the field.

If the decay of the field which is needed in the first step is asymptotically slower than the one resulting in the second step, then one can "bootstrap" this argument and obtain the decay estimates on the whole existence interval of the solution, which implies that the solution is global.

We approach the result through a series of lemmas, the first one being a local perturbation result about the trivial solution. It provides some finite time interval which can be made as long as desired and on which the solution exists and the field is sufficiently small to start the bootstrap argument outlined above.

LEMMA 1.6. *For any $\varepsilon > 0$ and $T > 0$ there exists some $\delta > 0$ such that every solution with initial datum $\mathring{f} \in \mathcal{D}$ satisfying $\|\mathring{f}\|_\infty < \delta$ exists on the interval $[0, T]$ and satisfies the estimate*

$$\left\|\partial_x U(t)\right\|_\infty + \left\|\partial_x^2 U(t)\right\|_\infty \leqslant \varepsilon, \quad t \in [0, T].$$

PROOF. By Step 2 of the proof of Theorem 1.1 the solution for any initial datum $\mathring{f} \in \mathcal{D}$ exists on the interval $[0, (C(\mathring{f})\mathring{P})^{-1}[$, where by (1.9),

$$C(\mathring{f}) = 4 \cdot 3^{1/3}\pi^{4/3}\|\mathring{f}\|_1^{1/3}\|\mathring{f}\|_\infty^{2/3} \leqslant C_0\|\mathring{f}\|_\infty,$$

with C_0 depending only on the parameters $\mathring{R}$ and $\mathring{P}$. Hence if $\delta := (2C_0\mathring{P}T)^{-1}$, the solution exists on the prescribed time interval $[0, T]$, provided $\|\mathring{f}\|_\infty < \delta$. Let $\widetilde{P} : [0, 2T[\to]0, \infty[$ denote the maximal solution of

$$\widetilde{P}(t) = \mathring{P} + \frac{1}{2T\mathring{P}}\int_0^t \widetilde{P}^2(s)\,\mathrm{d}s,$$

a function which depends only on $\mathring{R}, \mathring{P}$ and T. If $\|\mathring{f}\|_\infty < \delta$ then $C(\mathring{f}) \leqslant (2T\mathring{P})^{-1}$, and hence $f(t, x, v) = 0$ for $|v| \geqslant \widetilde{P}(t)$ and $t \in [0, T]$. This implies that

$$\left\|\rho(t)\right\|_\infty \leqslant \frac{4\pi}{3}\widetilde{P}^3(T)\|\mathring{f}\|_\infty,$$

and we also have

$$\left\|\rho(t)\right\|_1 \leqslant \left(\frac{4\pi}{3}\right)^2 \mathring{R}^3\mathring{P}^3\|\mathring{f}\|_\infty$$

for all $t \in [0, T]$. After making δ smaller if necessary, Lemma P1(b) implies the desired estimates for $\partial_x U$ and $\partial_x^2 U$. For the latter quantity we have to go through the estimates in Step 3 of the proof of Theorem 1.1 applied to the solution instead of the iterates to find that $\|\partial_x\rho(t)\|_\infty < C$ where the constant depends only on $\mathring{R}$ and $\mathring{P}$. Then by Lemma P1(b),

$$\left\|\partial_x^2 U(t)\right\|_\infty \leqslant C\bigl(\delta + d + (1 - \ln d)\delta\bigr)$$

for any $0 < d < 1$, $t \in [0, T]$ and $\mathring{f} \in \mathcal{D}$ which $\|\mathring{f}\|_\infty < \delta$. The right-hand side can be made less than ε by first choosing d sufficiently small and then again making δ smaller if necessary. □

The following decay condition on the field is the substitute for $\partial_x U$ to vanish identically in the first step of the bootstrap argument. Let $a > 0$ and $\alpha > 0$. A solution satisfies the *free streaming condition with parameter* α on the interval $[0, a]$ if the solution exists on $[0, a]$ and satisfies the estimates

$$\begin{cases} \left\|\partial_x U(t)\right\|_\infty \leqslant \alpha(1 + t)^{-3/2}, \\ \left\|\partial_x^2 U(t)\right\|_\infty \leqslant \alpha(1 + t)^{-5/2} \end{cases} \tag{FSα}$$

there. The next lemma justifies this terminology: Under the assumption the long time asymptotics of certain quantities are like in the case where the field vanishes identically, provided the parameter α is chosen sufficiently small.

LEMMA 1.7. *If $\alpha > 0$ is small enough then any solution f with initial datum $\mathring{f} \in \mathcal{D}$, which satisfies the free streaming condition (FSα) on some interval $[0, a]$, has the following properties for all $t \in [0, a]$:*
(a) $f(t, x, v) = 0$ *for* $|v| \geqslant \mathring{P} + 1$ *and* $x \in \mathbb{R}^3$;
(b) $|\det \partial_v X(0, t, x, v)| \geqslant \frac{1}{2} t^3$ *for* $(x, v) \in \mathbb{R}^6$;
(c) *for* $t > 0, x \in \mathbb{R}^3$ *the mapping* $X(0, t, x, \cdot) : \mathbb{R}^3 \to \mathbb{R}^3$ *is a* C^1*-diffeomorphism*;
(d) $\|\partial_x \rho(t)\|_\infty \leqslant 4\pi(\mathring{P} + 1)^3$.

PROOF. Let $s \mapsto (x(s), v(s))$ be a characteristic with $|v(0)| \leqslant \mathring{P}$. Then for any $t \in [0, a]$ by (FSα),

$$\left|v(t)\right| \leqslant \mathring{P} + \int_0^t \left\|\partial_x U(s)\right\|_\infty \mathrm{d}s \leqslant \mathring{P} + \alpha \int_0^t (1+s)^{-3/2}\,\mathrm{d}s \leqslant \mathring{P} + 2\alpha,$$

which implies (a) if $\alpha \leqslant 1/2$. As to (b), we define for $0 \leqslant s \leqslant t \leqslant a$ and $(x, v) \in \mathbb{R}^6$ the function

$$\xi(s) := \partial_v X(s, t, x, v) - (s - t)\mathrm{id}.$$

Clearly,

$$\ddot{\xi}(s) = -\partial_x^2 U\big(s, X(s, t, x, v)\big) \cdot \partial_v X(s, t, x, v), \qquad \xi(t) = \dot{\xi}(t) = 0,$$

and by (FSα),

$$\left|\ddot{\xi}(s)\right| \leqslant \alpha(1+s)^{-5/2}\big(\left|\xi(s)\right| + (t-s)\big).$$

Upon integrating this inequality twice and switching the order of integration we obtain the estimate

$$\begin{aligned}\left|\xi(s)\right| &\leqslant \int_s^t \int_\tau^t \left|\ddot{\xi}(\sigma)\right| \mathrm{d}\sigma\,\mathrm{d}\tau = \int_s^t \int_s^\sigma \left|\ddot{\xi}(\sigma)\right| \mathrm{d}\tau\,\mathrm{d}\sigma \\ &\leqslant \alpha \int_s^t (1+\sigma)^{-3/2}\big(\left|\xi(\sigma)\right| + (t-\sigma)\big)\,\mathrm{d}\sigma \\ &\leqslant 2\alpha(t-s) + \alpha \int_s^t (1+\sigma)^{-3/2}\left|\xi(\sigma)\right| \mathrm{d}\sigma.\end{aligned}$$

By Gronwall's lemma,

$$\left|\xi(s)\right| \leqslant 2\alpha(t-s)\exp\left(\alpha \int_s^t (1+\sigma)^{-3/2}\,\mathrm{d}\sigma\right) \leqslant 2\alpha \mathrm{e}^{2\alpha}(t-s).$$

If we take $s = 0$, recall the definition of ξ, and divide by t we can rewrite this as

$$\left|\frac{1}{t}\partial_v X(0,t,x,v) + \mathrm{id}\right| \leqslant 2\alpha \mathrm{e}^{2\alpha}. \tag{1.14}$$

The assertion in (b) follows if $\alpha > 0$ is sufficiently small. In addition, by (1.14) the mapping considered in (c) is one-to-one

$$\begin{aligned}
&\left|X(0,t,x,v) - X(0,t,x,\bar{v})\right| \\
&\quad = \left|\int_0^1 \partial_v X\bigl(0,t,x,\tau v + (1-\tau)\bar{v}\bigr)(v-\bar{v})\,\mathrm{d}\tau\right| \\
&\quad = \left|\int_0^1 \bigl[-t\mathrm{id} + t\mathrm{id} + \partial_v X\bigl(0,t,x,\tau v + (1-\tau)\bar{v}\bigr)\bigr](v-\bar{v})\,\mathrm{d}\tau\right| \\
&\quad \geqslant t|v-\bar{v}| - 2\alpha\mathrm{e}^{2\alpha} t|v-\bar{v}| \geqslant \frac{1}{2} t|v-\bar{v}|
\end{aligned}$$

for $v, \bar{v} \in \mathbb{R}^3$, $x \in \mathbb{R}^3$, and $t \in]0,a]$; for the last estimate α is again chosen smaller if necessary. Hence the mapping $X(0,t,x,\cdot)$ is a C^1-diffeomorphism onto its range which is an open set. Assume that it were not onto $\mathbb{R}^3$. Then the range $X(0,t,x,\mathbb{R}^3)$ has a boundary point x_0 which is not an image point. Choose a sequence $(v_n) \subset \mathbb{R}^3$ such that $X(0,t,x,v_n) \to x_0$. By the previous estimate, $v_n \to v_0$ converges, and by continuity, $x_0 = X(0,t,x,v_0)$ is an image point. This is a contradiction, and the assertion in (c) is established. As to (d), clearly

$$\left\|\partial_x \rho(t)\right\|_\infty \leqslant \frac{4\pi}{3}\bigl(\mathring{P}+1\bigr)^3 \left\|\partial_x f(t)\right\|_\infty$$

and

$$\left|\partial_x f(t,z)\right| \leqslant \left\|\partial_z \mathring{f}\right\|_\infty \bigl(\left|\partial_x X(0,t,z)\right| + \left|\partial_x V(0,t,z)\right|\bigr).$$

By definition of the initial data set $\mathcal{D}$, $\|\partial_z \mathring{f}\|_\infty \leqslant 1$, so it remains to estimate the derivatives of the characteristics. Proceeding as above we define

$$\xi(s) := \partial_x X(s,t,x,v) - \mathrm{id}$$

so that

$$\left|\ddot{\xi}(s)\right| \leqslant \alpha(1+s)^{-5/2}\bigl(\left|\xi(s)\right| + 1\bigr), \qquad \xi(t) = \dot{\xi}(t) = 0.$$

The resulting Gronwall estimate yields, for α sufficiently small,

$$\left|\xi(s)\right| \leqslant 2\alpha\mathrm{e}^{2\alpha} \leqslant 1,$$

and

$$\left|\dot{\xi}(s)\right| \leqslant \int_s^t \left|\ddot{\xi}(\tau)\right| \mathrm{d}\tau \leqslant \alpha \int_s^t (1+\tau)^{-5/2}\left(\left|\xi(\tau)\right|+1\right) \mathrm{d}\tau \leqslant 1.$$

Since $\partial_x V(0,t,x,v) = \dot{\xi}(0)$, we have shown that

$$\left|\partial_x X(0,t,z)\right| + \left|\partial_x V(0,t,z)\right| \leqslant 3,$$

and the proof is complete. □

After these preparations we are ready to prove Theorem 1.5.

PROOF OF THEOREM 1.5. We start by fixing some $\alpha > 0$ sufficiently small for all the assertions of Lemma 1.7 to hold, and we consider some interval $[0,a]$ with $a > 1$ on which (FSα) holds for some solution f with initial datum $\mathring{f} \in \mathcal{D}$. For $t \in]0,a]$ and $x \in \mathbb{R}^3$ the change of variables $v \mapsto X = X(0,t,x,v)$ and Lemma 1.7(b),(c) imply that

$$\begin{aligned}
\rho(t,x) &= \int \mathring{f}\big(X(0,t,x,v), V(0,t,x,v)\big)\,\mathrm{d}v \\
&= \int \mathring{f}\big(X, V\big(0,t,x,v(X)\big)\big)\left|\det \partial_v X^{-1}\big(0,t,x,v(X)\big)\right| \mathrm{d}X \\
&\leqslant \frac{8\pi}{3}\mathring{R}^3 \|\mathring{f}\|_\infty t^{-3};
\end{aligned}$$

$v(X)$ denotes the inverse of the change of variables. Hence by Lemma 1.7(d),

$$\left\|\rho(t)\right\|_\infty \leqslant C_1 t^{-3}, \qquad \left\|\partial_x \rho(t)\right\|_\infty \leqslant C_1, \quad t \in [0,a],$$

where the constant C_1 depends only on $\mathring{R}$ and $\mathring{P}$. By Lemma P1,

$$\left\|\partial_x U(t)\right\|_\infty \leqslant 3(2\pi)^{2/3} \|\mathring{f}\|_1^{1/3} C_1^{2/3} t^{-2} \leqslant C_2 t^{-2},$$

and for $t \in [1,a]$ with $R = t$ and $d = t^{-3} \leqslant R$,

$$\left\|\partial_x^2 U(t)\right\|_\infty \leqslant C\left[t^{-3} + t^{-3} + t^{-3}\ln t^4\right] \leqslant C_2(1+\ln t)t^{-3},$$

where again the constant C_2 depends only on $\mathring{R}$ and $\mathring{P}$. We fix some time $T_0 > 1$ such that for all $t \geqslant T_0$,

$$C_2 t^{-2} \leqslant \frac{\alpha}{2}(1+t)^{-3/2}, \qquad C_2(1+\ln t)t^{-3} \leqslant \frac{\alpha}{2}(1+t)^{-5/2}$$

which means that the decay obtained as output in the above estimates is stronger than the one in the free streaming condition (FSα). Lemma 1.6 provides $\delta > 0$ such that any solution launched by an initial datum $\mathring{f} \in \mathcal{D}$ with $\|\mathring{f}\|_\infty < \delta$ exists on the maximal existence interval $[0, T[$ with $T > T_0$, and

$$\left\|\partial_x U(t)\right\|_\infty + \left\|\partial_x^2 U(t)\right\|_\infty < \frac{\alpha}{2}(1+T_0)^{-5/2}, \quad t \in [0, T_0].$$

By continuity the free streaming condition (FSα) holds on some interval $[0, T^*[$ with $T^* \in]T_0, T]$, and we choose T^* maximal with this property. On $[T_0, T^*[$,

$$\left\|\partial_x U(t)\right\|_\infty \leqslant C_2 t^{-2} \leqslant \frac{\alpha}{2}(1+t)^{-3/2},$$

$$\left\|\partial_x^2 U(t)\right\|_\infty \leqslant C_2(1+\ln t)t^{-3} \leqslant \frac{\alpha}{2}(1+t)^{-5/2},$$

which implies that $T^* = T$, and by Lemma 1.7(a) and the continuation criterion from Theorem 1.1, $T = \infty$. □

Concluding remarks. (a) Lemma 1.6 is a special case of the fact that solutions depend continuously on initial data, cf. [90], Theorem 1.

(b) Global existence for small initial data was established for the relativistic Vlasov–Maxwell system in [37], and these techniques lead to analogous results for nearly neutral and nearly spherically symmetric data [30,90]. Similar techniques have been employed for the spherically symmetric, asymptotically flat Vlasov–Einstein system [100].

1.5. *Conservation laws and a priori bounds*

Conservation laws represent physically relevant properties of the system, and they lead to a priori bounds on the solutions used for the global existence result. As a matter of fact we have already stated and used one such conservation law, namely conservation of phase space volume: The characteristic flow of the Vlasov equation is measure preserving, cf. Lemma 1.2, and this leads to the a priori bounds

$$\left\|f(t)\right\|_p = \left\|\mathring{f}\right\|_p, \quad p \in [1, \infty],$$

as long as the solution exists. For $p = 1$ this is conservation of mass

$$\iint f(t, x, v)\,\mathrm{d}v\,\mathrm{d}x = \int \rho(t, x)\,\mathrm{d}x = M,$$

which can also be viewed as a consequence of the local mass conservation law

$$\partial_t \rho + \operatorname{div} j = 0, \tag{1.15}$$

where the mass current j is defined as

$$j(t,x) := \int v f(t,x,v)\,\mathrm{d}v.$$

Equation (1.15) follows by integrating the Vlasov equation with respect to v and observing that the total v-divergence $\partial_x U \cdot \partial_v f = \mathrm{div}_v(f\partial_x U)$ vanishes upon integration. Conservation of phase space volume follows from the Vlasov equation alone, regardless of the field equation to which it is coupled. The resulting a priori bounds are much too weak to gain global existence, since in particular only a bound on the L^1-norm of ρ results.

However, the system is also conservative. There is no dissipative mechanism in the system, and hence energy is conserved. It is a straightforward computation to see that for a classical solution the total energy

$$\frac{1}{2}\iint |v|^2 f(t,x,v)\,\mathrm{d}v\,\mathrm{d}x - \frac{\gamma}{8\pi}\int \left|\partial_x U(t,x)\right|^2 \mathrm{d}x \tag{1.16}$$

is constant as long as the solution exists. There is however an immediate problem: In the gravitational case $\gamma = 1$ the energy does not have a definite sign, and hence it is conceivable that the individual terms in (1.16), kinetic and potential energy, become unbounded in finite time while the sum remains constant. This does indeed happen for solutions of the N-body problem when two bodies collide, and it also happens for the counterexample to global existence for the pressure-less Euler–Poisson system in Section 1.3. If we consider the plasma physics case both kinetic and potential energy are obviously bounded. But as we will see shortly the same is true also in the gravitational case, which may come as a surprise.

Since conservation of energy plays a vital role in the stability analysis in the second part of this treatise, our presentation in the rest of the present section is a bit more general than necessary for the existence problem. The kinetic and the potential energy of a state $f \in L^1_+(\mathbb{R}^6)$ are defined as

$$E_{\mathrm{kin}}(f) := \frac{1}{2}\iint |v|^2 f(x,v)\,\mathrm{d}v\,\mathrm{d}x,$$

$$E_{\mathrm{pot}}(f) := -\frac{\gamma}{8\pi}\int \left|\nabla U_f(x)\right|^2 \mathrm{d}x = \frac{1}{2}\int U_f(x)\rho_f(x)\,\mathrm{d}x.$$

The spatial density ρ_f is bounded in an appropriate norm by the kinetic energy $E_{\mathrm{kin}}(f)$. The reason is that the kinetic energy is a second-order moment in velocity of f, while ρ_f is a zeroth order moment. For later purposes we prove a more general result than needed right now.

LEMMA 1.8. *For $k \geqslant 0$ we denote the kth order moment density and the kth order moment in velocity of a nonnegative, measurable function $f : \mathbb{R}^6 \to [0,\infty[$ by*

$$m_k(f)(x) := \int |v|^k f(x,v)\,\mathrm{d}v$$

and

$$M_k(f) := \int m_k(f)(x)\,\mathrm{d}x = \iint |v|^k f(x,v)\,\mathrm{d}v\,\mathrm{d}x.$$

Let $1 \leqslant p, q \leqslant \infty$ *with* $1/p + 1/q = 1$, $0 \leqslant k' \leqslant k < \infty$, *and*

$$r := \frac{k + 3/q}{k' + 3/q + (k-k')/p}.$$

If $f \in L^p_+(\mathbb{R}^6)$ *with* $M_k(f) < \infty$ *then* $m_{k'}(f) \in L^r(\mathbb{R}^3)$ *and*

$$\left\| m_{k'}(f) \right\|_r \leqslant c \| f \|_p^{(k-k')/(k+3/q)} M_k(f)^{(k'+3/q)/(k+3/q)}$$

where $c = c(k, k', p) > 0$.

PROOF. We split the v-integral defining $m_{k'}(f)$ into small and large v's and optimize with respect to the splitting parameter, more precisely, for any $R > 0$,

$$\begin{aligned} m_{k'}(f)(x) &\leqslant \int_{|v|\leqslant R} |v|^{k'} f(x,v)\,\mathrm{d}v + \int_{|v|>R} |v|^{k'} f(x,v)\,\mathrm{d}v \\ &\leqslant \left\| f(x,\cdot) \right\|_p \left(\int_{|v|\leqslant R} |v|^{k'q}\,\mathrm{d}v \right)^{1/q} + R^{k'-k} \int |v|^k f(x,v)\,\mathrm{d}v \\ &\leqslant c \left\| f(x,\cdot) \right\|_p R^{k'+3/q} + R^{k'-k} m_k(f)(x), \end{aligned}$$

where we used Hölder's inequality. Let

$$R := \left[\frac{m_k(f)(x)}{\| f(x,\cdot) \|_p} \right]^{1/(k+3/q)},$$

which up to a constant is the choice which minimizes the right-hand side as a function of $R > 0$. Then

$$m_{k'}(f)(x) \leqslant c \left(\left\| f(x,\cdot) \right\|_p \right)^{(k-k')/(k+3/q)} \left(m_k(f)(x) \right)^{(k'+3/q)/(k+3/q)}.$$

If we take this estimate to the power r and integrate in x we can by the definition of r again apply Hölder's inequality, and the assertion follows. □

Together with Lemma P2 the potential energy can be estimated in terms of the kinetic energy in such a way that by conservation of energy both terms individually remain bounded along classical solutions also in the case $\gamma = 1$. This was first observed by Horst [57].

PROPOSITION 1.9. *Let f be a classical solution of the Vlasov–Poisson system on the time interval $[0, T[$ with induced spatial density ρ. Then for all $t \in [0, T[$,*

$$E_{\mathrm{kin}}\big(f(t)\big), \qquad \big|E_{\mathrm{pot}}\big(f(t)\big)\big|, \qquad \big\|\rho(t)\big\|_{5/3} \leqslant C,$$

where the constant depends only on the initial datum $f(0) = \mathring{f}$, more precisely on its L^1 and L^∞-norms and its kinetic energy.

PROOF. If $\gamma = -1$ the kinetic and potential energy are both nonnegative and hence bounded by conservation of energy. The bound on ρ follows by Lemma 1.8 with $k = 2$, $k' = 0$, $p = \infty$, $q = 1$, $r = 5/3$:

$$\big\|\rho(t)\big\|_{5/3} \leqslant c\big\|f(t)\big\|_\infty^{2/5} E_{\mathrm{kin}}\big(f(t)\big)^{3/5}.$$

If $\gamma = 1$ we use Lemma P2(b) and Lemma 1.8 with $k = 2, k' = 0, p = 9/7, r = 6/5$ to obtain

$$\big|E_{\mathrm{pot}}\big(f(t)\big)\big| \leqslant c\big\|\rho(t)\big\|_{6/5}^2 \leqslant c\big\|f(t)\big\|_{9/7}^{3/2} E_{\mathrm{kin}}\big(f(t)\big)^{1/2} = C E_{\mathrm{kin}}\big(f(t)\big)^{1/2},$$

where the constant C has the claimed dependence. By conservation of energy

$$E_{\mathrm{kin}}\big(f(t)\big) - C E_{\mathrm{kin}}\big(f(t)\big)^{1/2} \leqslant E_{\mathrm{kin}}\big(f(t)\big) + E_{\mathrm{pot}}\big(f(t)\big) \leqslant E_{\mathrm{kin}}\big(\mathring{f}\big),$$

which implies the bound on the kinetic energy also for the case $\gamma = 1$. □

With these additional bounds at hand one may hope to improve the estimates in the local existence result in such a way that global existence follows. Indeed, by Lemma P1(b) with $p = 5/3$ and Proposition 1.9,

$$\big\|\partial_x U(t)\big\|_\infty \leqslant C\big\|\rho(t)\big\|_\infty^{4/9} \leqslant C P^{4/3}(t) \tag{1.17}$$

with P as defined in (1.7). Hence

$$P(t) \leqslant P(0) + \int_0^t \big\|\partial_x U(s)\big\|_\infty \, \mathrm{d}s \leqslant P(0) + C\int_0^t P^{4/3}(s)\, \mathrm{d}s. \tag{1.18}$$

This certainly is an improvement compared to our first attempt at bounding P, cf. equation (1.8), but the improvement is not sufficient to yield a global bound.

One way to improve this argument is to observe that an a priori bound on a higher-order L^p-norm of $\rho(t)$ allows for a smaller power of the L^∞-norm of $\rho(t)$ in the estimate (1.17) and thus for a smaller power of $P(s)$ in the Gronwall inequality (1.18). In the estimate (1.17) we would need an exponent less or equal to $1/3$ on $\|\rho(t)\|_\infty$ to obtain a Gronwall estimate on P leading to a global bound. If we compare this to Lemma P1(b) and use Lemma 1.8 with $p = \infty, k = 3, k' = 0$ we obtain a less demanding continuation criterion which we note for later use.

PROPOSITION 1.10. *If for a local solution f on its maximal existence interval* $[0, T[$ *the quantity* $\|\rho(t)\|_2$ *or* $M_3(t)$ *is bounded, then the solution is global.*

Concluding remarks. (a) The a priori bounds in Proposition 1.9 together with compactness properties of the solution operator to the Poisson equation can be used to prove the existence of global weak solutions for the Vlasov–Poisson system [4,61]. These solutions are not known to be unique nor are they known to satisfy the above conservation laws.

(b) For the relativistic Vlasov–Poisson system the kinetic energy

$$\iint \sqrt{1+|v|^2}\, f(t,x,v)\,\mathrm{d}v\,\mathrm{d}x$$

is of lower order in v than in the nonrelativistic case, and the potential energy turns out to be of the same order in the sense of the above estimates. Indeed, in the gravitational case the a priori bounds from Proposition 1.9 do not hold and solutions can blow up, cf. Theorem 1.17. For the plasma physics case the bound on the kinetic energy yields only a bound on $\|\rho(t)\|_{4/3}$, and these a priori bounds are then too weak for the proofs of global existence in the next section to extend to the relativistic case.

1.6. *Global existence for general data*

The aim of this section is to prove the following theorem.

THEOREM 1.11. *Any nonnegative initial datum* $\mathring{f} \in C^1_c(\mathbb{R}^6)$ *launches a global classical solution of the Vlasov–Poisson system.*

Let $[0, T[$ be the right maximal existence interval of the local solution provided by Theorem 1.1; all the arguments apply also when going backward in time. For technical reasons we redefine the quantity $P(t)$ and make it nondecreasing

$$P(t) := \max\bigl\{|v| \mid (x,v) \in \operatorname{supp} f(s),\, 0 \leqslant s \leqslant t\bigr\}.$$

We need to show that this function is bounded on bounded time intervals. By Proposition 1.10 it also suffices to bound a sufficiently high-order moment in v. This is the approach followed by Lions and Perthame [80]. The approach followed by Pfaffelmoser [89] is to fix a characteristic $(X, V)(t)$ along which the increase in velocity

$$\bigl|V(t) - V(t-\Delta)\bigr| \leqslant \int_{t-\Delta}^{t} \iint \frac{f(s,y,w)}{|y-X(s)|^2}\,\mathrm{d}w\,\mathrm{d}y\,\mathrm{d}s \tag{1.19}$$

during the time interval $[t-\Delta, t]$ is estimated. In the Gronwall argument leading to (1.18) we first split x-space to obtain the estimate (1.17) and then split v-space to obtain the estimate for ρ in Proposition 1.9. Pfaffelmoser's idea is that instead of doing one after the other one should split (x, v)-space in (1.19) into suitably chosen sets. Since this approach is more elementary and gives better estimates on the possible growth of the solution, we discuss it first, following a greatly simplified version due to Schaeffer [105,106]. The Lions–Perthame approach, which has the greater potential to generalize to related situations, is presented second.

1.6.1. *The Pfaffelmoser–Schaeffer proof.* Let us single out one particle in our distribution, the increase in velocity of which we want to control over a certain time interval. Mathematically speaking, we fix a characteristic $(X, V)(t)$ with $(X, V)(0) \in \operatorname{supp} \mathring{f}$, and we take $0 \leqslant \Delta \leqslant t < T$. After the change of variables

$$y = X(s, t, x, v), \qquad w = V(s, t, x, v), \tag{1.20}$$

equation (1.19) takes the form

$$\left|V(t) - V(t-\Delta)\right| \leqslant \int_{t-\Delta}^{t} \iint \frac{f(t, x, v)}{|X(s, t, x, v) - X(s)|^2} \,\mathrm{d}v \,\mathrm{d}x \,\mathrm{d}s, \tag{1.21}$$

because f is constant along the volume preserving characteristic flow. For parameters $0 < p \leqslant P(t)$ and $r > 0$, which will be specified later, we split the domain of integration in (1.21) into the following sets:

$$\begin{aligned}
M_{\mathrm{g}} &:= \bigl\{(s, x, v) \in [t-\Delta, t] \times \mathbb{R}^6 \bigm| |v| \leqslant p \vee \left|v - V(t)\right| \leqslant p\bigr\},\\
M_{\mathrm{b}} &:= \bigl\{(s, x, v) \in [t-\Delta, t] \times \mathbb{R}^6 \bigm| |v| > p \wedge \left|v - V(t)\right| > p\\
&\qquad \wedge \bigl[\left|X(s, t, x, v) - X(s)\right| \leqslant r|v|^{-3}\\
&\qquad \vee \left|X(s, t, x, v) - X(s)\right| \leqslant r\left|v - V(t)\right|^{-3}\bigr]\bigr\},\\
M_{\mathrm{u}} &:= \bigl\{(s, x, v) \in [t-\Delta, t] \times \mathbb{R}^6 \bigm| |v| > p \wedge \left|v - V(t)\right| > p\\
&\qquad \wedge \left|X(s, t, x, v) - X(s)\right| > r|v|^{-3}\\
&\qquad \wedge \left|X(s, t, x, v) - X(s)\right| > r\left|v - V(t)\right|^{-3}\bigr\}.
\end{aligned}$$

The logic behind the names of these sets is as follows. In the set M_{g} velocities are bounded, either with respect to our frame of reference or with respect to the one particle which we singled out. Hence M_{g} is the *good* set – we know how to proceed if the velocities are bounded. The set M_{b} is the *bad* set, since here velocities are large, and in addition the particle whose contribution to the integral in (1.21) we are computing is close in space to the singled out particle, i.e., the singularity of the Newton force is strong. Notice however that the latter type of badness is coupled with the former via the condition $|X(s, t, x, v) - X(s)| \leqslant r|v|^{-3}$. Both M_{g} and M_{b} are going to be estimated in a straight forward manner, while on the set M_{u} the time integral in (1.21) will be exploited in a crucial way to bound its contribution in terms of the kinetic energy. It is the *ugly* set although the ideas involved in its estimate are beautiful.

To estimate the contribution of each of these sets to the integral in (1.21) the length of the time interval $[t-\Delta, t]$ is chosen in such a way that velocities do not change very much on that interval. Recall that by (1.17),

$$\left\|\partial_x U(t)\right\|_\infty \leqslant C^* P(t)^{4/3}, \quad t \in [0, T[,$$

for some $C^* > 0$ so if

$$\Delta := \min\left\{t, \frac{p}{4C^* P(t)^{4/3}}\right\} \tag{1.22}$$

then

$$\left|V(s,t,x,v) - v\right| \leqslant \Delta C^* P(t)^{4/3} \leqslant \frac{1}{4}p, \quad s \in [t-\Delta, t], x, v \in \mathbb{R}^3. \tag{1.23}$$

The contribution of the good set M_g. For $(s,x,v) \in M_\mathrm{g}$ by (1.20) and (1.23),

$$|w| < 2p \vee \left|w - V(s)\right| < 2p.$$

Hence the change of variables (1.20) implies the estimate

$$\int_{M_\mathrm{g}} \frac{f(t,x,v)}{|X(s,t,x,v) - X(s)|^2}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s \leqslant \int_{t-\Delta}^t \int \frac{\tilde\rho(s,y)}{|y - X(s)|^2}\,\mathrm{d}y\,\mathrm{d}s,$$

where

$$\tilde\rho(s,y) := \int_{|w|<2p \vee |w-V(s)|<2p} f(s,y,w)\,\mathrm{d}w \leqslant Cp^3,$$

and by Proposition 1.9,

$$\left\|\tilde\rho(s)\right\|_{5/3} \leqslant \left\|\rho(s)\right\|_{5/3} \leqslant C.$$

Therefore, by the estimate (1.17),

$$\int_{M_\mathrm{g}} \frac{f(t,x,v)}{|X(s,t,x,v) - X(s)|^2}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s \leqslant Cp^{4/3}\Delta. \tag{1.24}$$

The contribution of the bad set M_b. For $(s,x,v) \in M_\mathrm{b}$ by (1.20) and (1.23),

$$\frac{1}{2}p < |w| < 2|v| \wedge \frac{1}{2}p < \left|w - V(s)\right| < 2\left|v - V(t)\right|$$
$$\wedge \left[\left|y - X(s)\right| < 8r|w|^{-3} \vee \left|y - X(s)\right| < 8r\left|w - V(s)\right|^{-3}\right].$$

On the other hand, $|w| \leqslant P(t)$ and $|w - V(s)| \leqslant 2P(t)$ for $w \in \operatorname{supp} f(s, y, \cdot)$, $0 \leqslant s \leqslant t$. Thus by (1.20) and since $\| f(s)\|_\infty = \|\mathring{f}\|_\infty$,

$$\begin{aligned}
&\int_{M_\mathrm{b}} \frac{f(t,x,v)}{|X(s,t,x,v) - X(s)|^2}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s \\
&\quad\leqslant \int_{t-\Delta}^{t} \int_{\frac{1}{2}p<|w|\leqslant P(t)} \int_{|y-X(s)|<8r|w|^{-3}} \frac{f(s,y,w)}{|y - X(s)|^2}\,\mathrm{d}y\,\mathrm{d}w\,\mathrm{d}s \\
&\qquad + \int_{t-\Delta}^{t} \int_{\frac{1}{2}p<|w-V(s)|\leqslant 2P(t)} \int_{|y-X(s)|<8r|w-V(s)|^{-3}} \frac{f(s,y,w)}{|y - X(s)|^2}\,\mathrm{d}y\,\mathrm{d}w\,\mathrm{d}s \\
&\quad\leqslant Cr \ln \frac{4P(t)}{p}\Delta. \qquad (1.25)
\end{aligned}$$

The contribution of the ugly set M_u. The main idea in estimating the contribution of the set M_u is to integrate with respect to time first, using the fact that on M_u the distance of $X(s,t,x,v)$ from $X(s)$ can be bounded from below linearly in time. Let $(x, v) \in \mathbb{R}^6$ with $|v - V(t)| > p$ and define

$$d(s) := X(s,t,x,v) - X(s), \quad s \in [t-\Delta, t].$$

We Taylor-expand this difference to first order around a point $s_0 \in [t-\Delta, t]$ where the difference is minimal

$$\big|d(s_0)\big| = \min\big\{\big|d(s)\big| \mid t - \Delta \leqslant s \leqslant t\big\}.$$

To this end, we define

$$\bar{d}(s) := d(s_0) + (s - s_0)\dot{d}(s_0), \quad s \in [t-\Delta, t].$$

Then

$$d(s_0) = \bar{d}(s_0), \qquad \dot{d}(s_0) = \dot{\bar{d}}(s_0),$$

and

$$\big|\ddot{d}(s) - \ddot{\bar{d}}(s)\big| = \big|\dot{V}(s,t,x,v) - \dot{V}(s)\big| \leqslant 2\big\|\partial_x U(s)\big\|_\infty \leqslant 2C^* P(t)^{4/3}.$$

Hence

$$\begin{aligned}
\big|d(s) - \bar{d}(s)\big| &\leqslant C^* P(t)^{4/3}(s - s_0)^2 \leqslant C^* P(t)^{4/3}\Delta|s - s_0| \\
&\leqslant \frac{1}{4}p|s - s_0| < \frac{1}{4}\big|v - V(t)\big||s - s_0|. \qquad (1.26)
\end{aligned}$$

On the other hand, by (1.23),

$$\left|\dot{d}(s_0)\right| = \left|V(s_0,t,x,v) - V(s_0)\right| \geqslant \left|v - V(t)\right| - \frac{1}{2}p > \frac{1}{2}\left|v - V(t)\right|,$$

and by the definition of s_0, distinguishing the cases $s_0 = t - \Delta$, $s_0 \in \,]t - \Delta, t[$, and $s_0 = t$,

$$(s - s_0)\,\mathrm{d}(s_0) \cdot \dot{d}(s_0) \geqslant 0.$$

Hence for all $s \in [t - \Delta, t]$ the estimate

$$\left|\bar{d}(s)\right|^2 \geqslant \frac{1}{4}\left|v - V(t)\right|^2 |s - s_0|^2$$

holds. Combining this with (1.26) finally implies that the estimate

$$\left|d(s)\right| \geqslant \frac{1}{4}\left|v - V(t)\right||s - s_0| \tag{1.27}$$

holds for all $s \in [t - \Delta, t]$ and $(x, v) \in \mathbb{R}^6$ with $|v - V(t)| > p$. To exploit this we define auxiliary functions

$$\sigma_1(\xi) := \begin{cases} \xi^{-2}, & \xi > r|v|^{-3}, \\ \left(r|v|^{-3}\right)^{-2}, & \xi \leqslant r|v|^{-3}, \end{cases}$$

and

$$\sigma_2(\xi) := \begin{cases} \xi^{-2}, & \xi > r\left|v - V(t)\right|^{-3}, \\ \left(r|v - V(t)|^{-3}\right)^{-2}, & \xi \leqslant r\left|v - V(t)\right|^{-3}. \end{cases}$$

The definition of M_{u}, the fact that the functions σ_i are nonincreasing and the estimate (1.27) imply that

$$\left|d(s)\right|^{-2} \mathbb{1}_{M_{\mathrm{u}}}(s,x,v) \leqslant \sigma_i\left(\left|d(s)\right|\right) \leqslant \sigma_i\left(\frac{1}{4}\left|v - V(t)\right||s - s_0|\right)$$

for $i = 1, 2$ and $s \in [t - \Delta, t]$. Hence we can estimate the time integral in the contribution of M_{u} in the following way:

$$\begin{aligned} \int_{t-\Delta}^{t} \left|d(s)\right|^{-2} \mathbb{1}_{M_{\mathrm{u}}}(s,x,v)\,\mathrm{d}s &\leqslant 8\left|v - V(t)\right|^{-1} \int_0^\infty \sigma_i(\xi)\,\mathrm{d}\xi \\ &= 16\left|v - V(t)\right|^{-1} \begin{cases} r^{-1}|v|^3, & i = 1, \\ r^{-1}\left|v - V(t)\right|^3, & i = 2, \end{cases} \end{aligned}$$

and since this estimate holds for both $i = 1$ and $i = 2$,

$$\int_{t-\Delta}^{t} \bigl|d(s)\bigr|^{-2} \mathbb{1}_{M_{\mathrm{u}}}(s, x, v)\,\mathrm{d}s \\ \leqslant 16 r^{-1} \bigl|v - V(t)\bigr|^{-1} \min\bigl\{|v|^3, \bigl|v - V(t)\bigr|^3\bigr\} \leqslant 16 r^{-1} |v|^2.$$

Therefore,

$$\begin{aligned} &\int_{M_{\mathrm{u}}} \frac{f(t, x, v)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s}{|X(s, t, x, v) - X(s)|^2} \\ &\quad\leqslant \iint f(t, x, v) \int_{t-\Delta}^{t} \bigl|d(s)\bigr|^{-2} \mathbb{1}_{M_{\mathrm{u}}}(s, x, v)\,\mathrm{d}s\,\mathrm{d}v\,\mathrm{d}x \\ &\quad\leqslant C r^{-1} \iint |v|^2 f(t, x, v)\,\mathrm{d}v\,\mathrm{d}x \\ &\quad\leqslant C r^{-1}, \end{aligned} \tag{1.28}$$

since according to Proposition 1.9 the kinetic energy is bounded.

Adding up the estimates (1.24), (1.25), (1.28) we arrive at the following control on the increase in velocity along the characteristic which we singled out

$$\begin{aligned} \bigl|V(t) - V(t-\Delta)\bigr| &\leqslant C\left(p^{4/3} + r \ln \frac{4P(t)}{p} + r^{-1}\Delta^{-1}\right)\Delta \\ &= C\left(p^{4/3} + r \ln \frac{4P(t)}{p} + r^{-1} \max\left\{\frac{1}{t}, \frac{4C^* P(t)^{4/3}}{p}\right\}\right)\Delta; \end{aligned}$$

recall the definition of $\Delta = \Delta(t)$ in (1.22). We choose the parameters p and r in such a way that the terms in the sum on the right-hand side of this estimate are of the same order in $P(t)$,

$$p = P(t)^{4/11}, \qquad r = P(t)^{16/33};$$

without loss of generality, $P(t) \geqslant 1$ so that $p \leqslant P(t)$, otherwise we replace $P(t)$ by $P(t) + 1$. Since P is nondecreasing and by Theorem 1.1, $\lim_{t\to T} P(t) = \infty$ if $T < \infty$, there exists a unique $T^* \in]0, T[$ such that $1/t \leqslant 4C^* P(t)^{4/3}/p = 4C^* P(t)^{32/33}$ for $t \geqslant T^*$. Hence for $t \geqslant T^*$,

$$\bigl|V(t) - V(t-\Delta)\bigr| \leqslant C P(t)^{16/33} \ln P(t) \Delta.$$

Thus, for any $\varepsilon > 0$ there exists a constant $C > 0$ such that

$$\bigl|V(t) - V(t-\Delta)\bigr| \leqslant C P(t)^{16/33+\varepsilon} \Delta, \quad t \geqslant T^*. \tag{1.29}$$

Let $t > T^*$ and define $t_0 := t$ and $t_{i+1} := t_i - \Delta(t_i)$ as long as $t_i \geqslant T^*$. Since

$$t_i - t_{i+1} = \Delta(t_i) \geqslant \Delta(t_0)$$

there exists $k \in \mathbb{N}$ such that

$$t_k < T^* \leqslant t_{k-1} < \cdots < t_0 = t.$$

Repeated application of (1.29) yields

$$\begin{aligned}
\left|V(t) - V(t_k)\right| &\leqslant \sum_{i=1}^{k} \left|V(t_{i-1}) - V(t_i)\right| \\
&\leqslant C P(t)^{16/33+\varepsilon} \sum_{i=1}^{k} (t_{i-1} - t_i) \\
&\leqslant C P(t)^{16/33+\varepsilon} t.
\end{aligned}$$

By the definition of P,

$$P(t) \leqslant P(t_k) + C P(t)^{16/33+\varepsilon} t$$

so that for any $\delta > 0$ there exists a constant $C > 0$ such that

$$P(t) \leqslant C(1+t)^{33/17+\delta}, \quad t \in [0, T[\,,$$

and by Theorem 1.1 the proof is complete.

1.6.2. *The Lions–Perthame proof.* We present the ideas developed in [80] within the framework of classical solutions and use them to verify the continuation criterion in Proposition 1.10. Let

$$m_k(t,x) := m_k\big(f(t)\big)(x), \qquad M_k(t) := M_k\big(f(t)\big), \quad t \in [0, T[\,, x \in \mathbb{R}^3;$$

the right-hand terms were defined in Lemma 1.8. The field induced by the potential U is denoted by

$$F(t,x) := -\partial_x U(t,x).$$

The proof is split into a number of steps; constants denoted by C may depend on the initial datum, and their value may change from line to line. The order k of the moment to bounded is specified below, but in any case, $k \geqslant 3$.

Step 1: *A differential inequality for* M_k. Using the Vlasov equation, integration by parts, Hölder's inequality, and Lemma 1.8 with $p = \infty, q = 1, k' = k - 1$, and hence

$r = (k+3)/(k+2)$ we obtain the following differential inequality

$$
\begin{aligned}
\left|\frac{\mathrm{d}}{\mathrm{d}t} M_k(t)\right| &= \left|\iint |v|^k (-v \cdot \partial_x f - F \cdot \partial_v f)\, \mathrm{d}v\, \mathrm{d}x\right| \\
&= \left|k \iint |v|^{k-2} v \cdot F f\, \mathrm{d}v\, \mathrm{d}x\right| \leqslant k \iint |v|^{k-1} f\, \mathrm{d}v\, |F|\, \mathrm{d}x \\
&\leqslant k \big\| F(t) \big\|_{k+3} \big\| m_{k-1}(t) \big\|_{(k+3)/(k+2)} \\
&\leqslant C \big\| F(t) \big\|_{k+3} M_k(t)^{(k+2)/(k+3)}. \qquad (1.30)
\end{aligned}
$$

Step 2: *Straight forward estimates for the field.* By Lemma P2(a) and Proposition 1.9 the estimate

$$\big\| F(t) \big\|_p \leqslant C, \quad t \in [0, T[, \qquad (1.31)$$

holds for any $p \in]3/2, 15/4]$, and the constant C can be chosen to be independent of p. Hence the estimate

$$\tau^p \operatorname{vol}\big\{x \in \mathbb{R}^3 \mid \big|F(t,x)\big| > \tau\big\} \leqslant C^p, \quad \tau > 0,$$

holds for all $3/2 < p \leqslant 15/4$, with C independent of p, so that the estimate also holds in the limiting case $p = 3/2$, which implies that

$$\big\| F(t) \big\|_{3/2,w} \leqslant C, \quad t \in [0, T[. \qquad (1.32)$$

This estimate also follows from the fact that the mapping $L^1(\mathbb{R}^3) \ni \rho \mapsto \rho * 1/|\cdot|^2$ is of weak-type $(1, 3/2)$, cf. [108], Section V.1.2, Theorem 1.

Step 3: *A representation formula for ρ and further estimates for the field.* In order to proceed with the differential inequality (1.30) we need a suitable estimate for $\|F(t)\|_{k+3}$, which is not provided by Step 2. To this end we first derive a representation formula for the spatial density ρ. The Vlasov equation can be rewritten as follows:

$$\partial_t f + v \cdot \partial_x f = -\operatorname{div}_v (f F).$$

We treat the right-hand side as an inhomogeneity and integrate this equation along the free streaming characteristics to obtain the following formula:

$$
\begin{aligned}
f(t,x,v) &= \mathring{f}(x - tv, v) - \int_0^t \operatorname{div}_v (f F)\big(s, x + (s-t)v, v\big)\, \mathrm{d}s \\
&= \mathring{f}(x - tv, v) - \operatorname{div}_v \int_0^t (f F)\big(s, x + (s-t)v, v\big)\, \mathrm{d}s \\
&\quad + \operatorname{div}_x \int_0^t (s-t)(f F)\big(s, x + (s-t)v, v\big)\, \mathrm{d}s.
\end{aligned}
$$

Integration with respect to v yields

$$\begin{aligned}\rho(t,x) &= \int \mathring{f}(x-tv,v)\,\mathrm{d}v + \mathrm{div}_x \int_0^t (s-t)\int (fF)\big(s, x+(s-t)v, v\big)\,\mathrm{d}v\,\mathrm{d}s\\ &=: \rho_0(t,x) + \mathrm{div}_x\,\sigma(t,x).\end{aligned}$$

We split the field accordingly,

$$F = F_0 + F_1 := -\partial_x U_{\rho_0} - \partial_x U_{\mathrm{div}\,\sigma}.$$

The first term is easy to control. Because of the estimate

$$\rho_0(t,x) = \int \mathring{f}(x-tv,v)\,\mathrm{d}v = t^{-3}\int \mathring{f}\left(X, \frac{x-X}{t}\right)\mathrm{d}X \leqslant Ct^{-3},$$

the density contribution $\rho_0(t)$ is bounded on $[0,T[$ in any L^p-norm. Hence Lemma P2(a) implies that $\|F_0(t)\|_r$ is bounded on $[0,T[$ for any $r > 3/2$. To proceed with F_1 we need an auxiliary result, a consequence of the Calderon–Zygmund inequality.

LEMMA 1.12. *For any $p \in]1,\infty[$ there is a constant $c > 0$ such that for all $\sigma \in C^1_{\mathrm{c}}(\mathbb{R}^3;\mathbb{R}^3)$,*

$$\big\|\big(\cdot/|\cdot|^3\big) * (\mathrm{div}\,\sigma)\big\|_p \leqslant c\|\sigma\|_p.$$

PROOF. Let $E := (\cdot/|\cdot|^3) * (\mathrm{div}\,\sigma)$. Integration by parts shows that for $i = 1,2,3$,

$$E_i(x) = \lim_{\varepsilon\to 0}\sum_{j=1}^{3}\big(I^{ij}_{1,\varepsilon}(x) - I^{ij}_{2,\varepsilon}(x)\big),$$

where

$$\begin{aligned}I^{ij}_{1,\varepsilon}(x) &= \int_{|x-y|=\varepsilon}\frac{x_i-y_i}{|x-y|^3}\sigma_j(y)\frac{x_j-y_j}{|x-y|}\,\mathrm{d}\Sigma(y),\\ I^{ij}_{2,\varepsilon}(x) &= \int_{|x-y|>\varepsilon}\partial_{y_j}\frac{x_i-y_i}{|x-y|^3}\sigma_j(y)\,\mathrm{d}y.\end{aligned}$$

For $i \neq j$ the surface integral of the kernel in $I^{ij}_{1,\varepsilon}$ vanishes, and since $\sigma \in C^1_{\mathrm{c}}(\mathbb{R}^3)$, $I^{ij}_{1,\varepsilon} \to 0$ for $\varepsilon \to 0$. For $i = j$ the integral of this kernel equals $4\pi/3$ so that $\sum_j I^{ij}_{1,\varepsilon} \to 4\pi\sigma/3$ for $\varepsilon \to 0$, uniformly on $\mathbb{R}^3$, and by the compact support assumption this convergence holds in L^p. The limit of $I^{ij}_{2,\varepsilon}$ can be estimated in the desired way by [108], Section II.4.2, Theorem 3, and the proof is complete. □

Using this lemma for the term F_1 in the above splitting and the established bound on F_0 we arrive at the estimate

$$\big\| F(t) \big\|_{k+3} \leqslant C\big(1 + \|\sigma(t)\|_{k+3}\big), \quad t \in [0, T[. \tag{1.33}$$

In order to proceed we need another auxiliary result.

LEMMA 1.13. *For all functions $g \in L^1 \cap L^\infty(\mathbb{R}^3)$ and $h \in L_w^{3/2}(\mathbb{R}^3)$,*

$$\int |gh| \, \mathrm{d}x \leqslant 3\left(\frac{3}{2}\right)^{2/3} \|g\|_1^{1/3} \|g\|_\infty^{2/3} \|h\|_{3/2,w}.$$

PROOF. For any $\tau > 0$ the "layer cake representation" [74], Section 1.13, implies that

$$\int_{|h|>\tau} |h| \, \mathrm{d}x \leqslant 3\|h\|_{3/2,w}^{3/2} \tau^{-1/2},$$

and hence

$$\begin{aligned} \int |gh| \, \mathrm{d}x &= \int_{|h|\leqslant\tau} |gh| \, \mathrm{d}x + \int_{|h|>\tau} |gh| \, \mathrm{d}x \\ &\leqslant \tau \|g\|_1 + 3\|h\|_{3/2,w}^{3/2} \tau^{-1/2} \|g\|_\infty. \end{aligned}$$

If we choose $\tau := \|h\|_{3/2,w} (3/2)^{2/3} (\|g\|_\infty / \|g\|_1)^{2/3}$ the assertion follows. □

Step 4: *Gronwall estimate for M_k*. In order to derive a Gronwall inequality for the moment M_k we need to estimate $\|\sigma(t)\|_{k+3}$ in terms of a moment, cf. Steps 1 and 3. We fix some time $t_0 \in]0, T[$, to be chosen in a suitable way later on. Then for any $t \in]t_0, T[$ we have by the definition of the quantity σ in Step 2,

$$\begin{aligned} \big\|\sigma(t)\big\|_{k+3} &= \left\| \int_0^t (s-t) \int (fF)\big(s, \cdot + (s-t)v, v\big) \, \mathrm{d}v \, \mathrm{d}s \right\|_{k+3} \\ &\leqslant \left\| \int_0^{t_0} \cdots \right\|_{k+3} + \left\| \int_{t_0}^t \cdots \right\|_{k+3} =: I_1 + I_2. \end{aligned}$$

By Lemma 1.13, equation (1.32), a change of variables, and the boundedness of f,

$$\int \big(|F| f\big)\big(s, x + (s-t)v, v\big) \, \mathrm{d}v \leqslant C(t-s)^{-2} \left(\int f\big(s, x + (s-t)v, v\big) \, \mathrm{d}v \right)^{1/3}.$$

We use this estimate and Lemma 1.8 for the term I_1,

$$
\begin{aligned}
I_1 &\leqslant C \left\| \int_0^{t_0} (t-s)^{-1} \left(\int f\big(s, \cdot + (s-t)v, v\big)\,\mathrm{d}v \right)^{1/3} \mathrm{d}s \right\|_{k+3} \\
&\leqslant C \int_0^{t_0} (t-s)^{-1} \left\| \int f\big(s, \cdot + (s-t)v, v\big)\,\mathrm{d}v \right\|_{(k+3)/3}^{1/3} \mathrm{d}s \\
&\leqslant C \int_0^{t_0} (t-s)^{-1} M_k(s)^{1/(k+3)}\,\mathrm{d}s.
\end{aligned}
$$

This estimate is good as long as we stay away from the singularity of the integrand, but for s close to t, i.e., on the interval $[t_0, t]$ we have to argue differently. We fix some parameter $3/2 < r \leqslant 15/4$, to be specified later, and its dual exponent defined by $1/r + 1/r' = 1$. Then by Hölder's inequality and (1.31),

$$
\begin{aligned}
I_2 \leqslant{}& \left\| \int_{t_0}^{t} (t-s) \left(\int \big| F\big(s, \cdot + (s-t)v\big)\big|^r \,\mathrm{d}v \right)^{1/r} \right. \\
&\left. \times \|\mathring{f}\|_\infty^{(r'-1)/r'} \left(\int f\big(s, \cdot + (s-t)v, v\big)\,\mathrm{d}v \right)^{1/r'} \mathrm{d}s \right\|_{k+3} \\
\leqslant{}& C \int_{t_0}^{t} (t-s)^{1-3/r} \left\| \int f\big(s, \cdot + (s-t)v, v\big)\,\mathrm{d}v \right\|_{(k+3)/r'}^{1/r'} \mathrm{d}s.
\end{aligned}
$$

Let $l > 0$ be such that $(l+3)/3 = (k+3)/r'$; such a choice is possible since

$$
\frac{k+3}{r'} \geqslant \frac{6}{r'} > 1.
$$

Applying Lemma 1.8 to the v-integral in the last estimate we conclude that

$$
I_2 \leqslant C \int_{t_0}^{t} (t-s)^{1-3/r} M_l(s)^{1/(k+3)}\,\mathrm{d}s.
$$

In order to continue it is convenient to have M_k nondecreasing in t so we replace $M_k(t)$ by $\sup_{0 \leqslant s \leqslant t} M_k(s)$. Collecting the estimates for I_1 and I_2 we obtain by (1.33) the following estimate for the field:

$$
\begin{aligned}
&\big\| F(t) \big\|_{k+3} \\
&\quad \leqslant C\left(1 + M_k(t)^{1/(k+3)} \ln \frac{t}{t-t_0} + M_l(t)^{1/(k+3)} (t-t_0)^{2-3/r} \right). \qquad (1.34)
\end{aligned}
$$

If on the other hand, we integrate the differential inequality from Step 1 we obtain the estimate

$$M_l(s) \leqslant M_l(0) + C \sup_{0 \leqslant s \leqslant t} \|F(s)\|_{l+3} \int_0^s M_l(\tau)^{(l+2)/(l+3)}\,\mathrm{d}\tau,$$

which implies that for $0 \leqslant s \leqslant t$,

$$M_l(s) \leqslant C\Big(1 + t^{l+3} \sup_{0 \leqslant s \leqslant t} \|F(s)\|_{l+3}^{l+3}\Big). \tag{1.35}$$

In order to close the Gronwall loop for the quantity M_k we must estimate the L^{l+3}-norm of the field $F(s)$ in terms of $M_k(s)$, which by Lemma 1.8 means that we must estimate it in terms of the $L^{(k+3)/3}$-norm of the spatial density. The way to do the latter is to again use Lemma P2(a), so we now must adjust the exponents k, l, r properly. We need that

$$\frac{3}{k+3} + \frac{2}{3} = 1 + \frac{1}{l+3}, \quad \text{i.e., } l+3 = \left(\frac{3}{k+3} - \frac{1}{3}\right)^{-1} = \frac{3(k+3)}{6-k}.$$

Since l must be positive, k must satisfy the restriction

$$3 \leqslant k < 6.$$

On the other hand, we have to observe the relation between k, l, r' used above

$$\frac{k+3}{6-k} = \frac{l+3}{3} = \frac{k+3}{r'}$$

which implies that

$$r' = 6-k, \quad \text{i.e., } \frac{1}{r} = 1 - \frac{1}{r'} = \frac{5-k}{6-k}.$$

Now we recall that for the estimates above the restriction

$$\frac{3}{2} < r \leqslant \frac{15}{4}, \quad \text{i.e., } \frac{3}{2} < \frac{6-k}{5-k} \leqslant \frac{15}{4}$$

was required. We end up with the result that the exponents k, l, r can be chosen such that all the relations introduced so far do indeed hold iff

$$3 < k \leqslant \frac{51}{11}.$$

This is the range of exponents for which we now establish a bound on M_k. By Lemma P2(a) and Lemma 1.8,

$$\|F(s)\|_{l+3} \leqslant C\|\rho(s)\|_{(k+3)/3} \leqslant C M_k(t)^{3/(k+3)}, \quad 0 \leqslant s \leqslant t < T,$$

and by (1.34) and (1.35),

$$\begin{aligned}&\|F(t)\|_{k+3}\\&\quad\leqslant C\left(1+M_k(t)^{1/(k+3)}\ln\frac{t}{t-t_0}+M_k(t)^{3(l+3)/(k+3)^2}t^{l+3}(t-t_0)^{2-3/r}\right).\end{aligned}$$

We have to examine the various exponents. Clearly,

$$2-\frac{3}{r}>0.$$

Since $1/r'>1/3$ we have $l+3>k+3$ and hence

$$\frac{1}{k+3}-\frac{3(l+3)}{(k+3)^2}=\frac{1}{k+3}\left(1-\frac{3(l+3)}{k+3}\right)<0.$$

By monotonicity there exists a unique time $t^*\in]0,T[$ such that

$$M_k(t)^{1/(k+3)-3(l+3)/(k+3)^2}<t^{2-3/r},\quad t\geqslant t^*;$$

without loss of generality $M_k(0)>0$. For $t\geqslant t^*$ we choose $t_0\in]0,t[$ such that

$$(t-t_0)^{2-3/r}=M_k(t)^{1/(k+3)-3(l+3)/(k+3)^2},$$

and hence

$$\|F(t)\|_{k+3}\leqslant Ct^{l+3}\ln t\,M_k(t)^{1/(k+3)}\ln M_k(t).$$

If we insert this estimate into the integrated differential inequality from Step 1 we finally arrive at the estimate

$$M_k(t)\leqslant C+C\int_{t^*}^t s^{l+3}\ln s\,M_k(s)\ln M_k(s)\,\mathrm{d}s,\quad t\in[t^*,T[.$$

Hence M_k is bounded on bounded time intervals, and by Proposition 1.10 the proof is complete.

Concluding remarks. (a) In addition to being more elementary the Pfaffelmoser and Schaeffer proof yields better bounds on $P(t)$ and $\|\rho(t)\|_\infty$. Using a somewhat different refinement of the original Pfaffelmoser proof Horst [60] showed that for any $\delta>0$,

$$P(t)\leqslant C(1+t)^{1+\delta},\qquad \|\rho(t)\|\leqslant C(1+t)^{3+\delta},\quad t\geqslant 0.$$

(b) The Pfaffelmoser and Schaeffer proof has been employed for the Vlasov–Poisson system in a spatially periodic, plasma physics setting [12] and in a cosmological setting [101].

(c) The Lions and Perthame ideas have the greater potential to generalize to related systems. In [14] global existence was established for the Vlasov–Fokker–Planck–Poisson system. In this system collisional effects are included by a linear approximation of the Boltzmann collision operator, and instead of the Vlasov equation (1.1) the so-called Vlasov–Fokker–Planck equation

$$\partial_t f + v \cdot \partial_x f - (\partial_x U + \beta v) \cdot \partial_v f = 3\beta f + \sigma \Delta_v f$$

is coupled to (1.2), (1.3), where $\beta, \sigma > 0$. Notice that the method of characteristics does not apply to this equation. The Lions and Perthame techniques were also successful for a version of the Vlasov–Poisson system which includes a damping term modeling the fact that charges in motion radiate energy [68].

(d) A proof based on the ideas of Lions and Perthame but using moments with respect to x and v is given in [27].

(e) The above global existence results extend easily to the plasma physics case with several particle species, with a fixed ion background, or with a fixed exterior field. If, on the other hand, the system is considered on a spatial domain with boundary, where a variety of boundary conditions for the particles can be posed like specular reflexion, absorption, or an inflow boundary condition, then the situation changes drastically, and in general not even a local existence and uniqueness result is known. We refer to [38–40,62,111] for results on Vlasov-type systems with boundary conditions.

(f) For the plasma physics case of the relativistic Vlasov–Poisson system and for the relativistic Vlasov–Maxwell system no global classical existence result for general data has been proven yet, cf. also the concluding remarks of Section 1.5. In addition to the papers mentioned in Section 1.1 we mention [86] where Pallard introduced significant new ideas for the Vlasov–Maxwell system.

(g) The ideas of Pallard have very recently been exploited by Calogero [17] to prove global existence of classical solutions for general data to the Vlasov–Nordström system

$$\begin{aligned}
&\partial_t f + \frac{v}{\sqrt{1+|v|^2}} \cdot \partial_x f - \left[\left(\partial_t \phi + \frac{v}{\sqrt{1+|v|^2}} \cdot \partial_x \phi\right) v + \frac{1}{\sqrt{1+|v|^2}} \partial_x \phi\right] \cdot \partial_v f \\
&\quad = 0, \\
&\partial_t^2 \phi - \Delta \phi = -\mathrm{e}^{4\phi} \int f \frac{\mathrm{d}v}{\sqrt{1+|v|^2}}.
\end{aligned}$$

In view of the notoriously difficult Vlasov–Einstein system the Vlasov–Nordström system can serve as a toy model to gain experience with relativistic, gravitationally interacting particle ensembles. It is much simpler than the former, physically correct system, but the Vlasov equation is relativistic in the sense that its characteristic system are the geodesic equations in the metric $\mathrm{e}^{2\phi} \operatorname{diag}(-1, 1, 1, 1)$, and the field equation for the function $\phi = \phi(t, x)$ is hyperbolic. Notice that compared to the Vlasov–Maxwell system the source term in the field equation here is of lower order in v, which is important for the success of the proof.

1.7. *Asymptotic behavior*

Due to Proposition 1.9 the global existence proofs go through and give the same bounds for both the plasma physics and the gravitational case. The latter fact is not satisfactory, since in the plasma physics case the particles repulse each other – in the case of several particle species with charges of different sign this is true at least on the average – and thus the spatial density should decay as $t \to \infty$. In the present subsection we present results of this type, which also form a link to the next chapter where we study the stability of steady states. The main tool are certain identities satisfied by a solution; the second one was introduced and exploited in [63,87] and put into a larger context in [23].

THEOREM 1.14. *Let f be a classical solution of the Vlasov–Poisson system with nonnegative initial datum $\mathring{f} \in C_c^1(\mathbb{R}^6)$. Then the following identities hold for all times*:

$$\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\iint |x|^2 f(t,x,v)\,\mathrm{d}v\,\mathrm{d}x = 2E_{\mathrm{kin}}\big(f(t)\big) + E_{\mathrm{pot}}\big(f(t)\big), \tag{1.36}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\iint |x-tv|^2 f(t,x,v)\,\mathrm{d}v\,\mathrm{d}x + t^2 E_{\mathrm{pot}}\big(f(t)\big)\right] = tE_{\mathrm{pot}}\big(f(t)\big). \tag{1.37}$$

PROOF. We start by proving yet another identity which is sometimes referred to as the dilation identity, cf. [35]. By the Vlasov equation,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\iint x\cdot v f(t,x,v)\,\mathrm{d}v\,\mathrm{d}x &= \iint x\cdot v(-v\cdot\partial_x f + \partial_x U\cdot\partial_v f)\,\mathrm{d}v\,\mathrm{d}x\\ &= \iint |v|^2 f\,\mathrm{d}v\,\mathrm{d}x - \int x\cdot\partial_x U\rho\,\mathrm{d}x.\end{aligned}$$

Now the formulas for the potential and its gradient imply that

$$\begin{aligned}\int x\cdot\partial_x U\rho\,\mathrm{d}x &= \gamma\iint x\frac{x-y}{|x-y|^3}\rho(t,y)\rho(t,x)\,\mathrm{d}y\,\mathrm{d}x\\ &= \frac{1}{2}\gamma\iint (x-y)\frac{x-y}{|x-y|^3}\rho(t,y)\rho(t,x)\,\mathrm{d}y\,\mathrm{d}x\\ &= -\frac{1}{2}\int U\rho\,\mathrm{d}x = -E_{\mathrm{pot}}\big(f(t)\big).\end{aligned}$$

Hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint x\cdot v f(t,x,v)\,\mathrm{d}v\,\mathrm{d}x = 2E_{\mathrm{kin}}\big(f(t)\big) + E_{\mathrm{pot}}\big(f(t)\big). \tag{1.38}$$

Together with

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\iint |x|^2 f \,\mathrm{d}v\,\mathrm{d}x = \frac{1}{2}\iint |x|^2(-v\cdot\partial_x f + \partial_x U\cdot\partial_v f)\,\mathrm{d}v\,\mathrm{d}x$$
$$= \iint x\cdot vf\,\mathrm{d}v\,\mathrm{d}x,$$

this implies the identity (1.36). Since the total energy $\mathcal{H} := E_{\mathrm{kin}} + E_{\mathrm{pot}}$ is conserved,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\iint |x-tv|^2 f(t,x,v)\,\mathrm{d}v\,\mathrm{d}x + t^2 E_{\mathrm{pot}}\big(f(t)\big)\right]\\ &\quad=\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\iint |x|^2 f\,\mathrm{d}v\,\mathrm{d}x + t^2\mathcal{H}\big(f(t)\big) - \frac{1}{2}t\frac{\mathrm{d}}{\mathrm{d}t}\iint |x|^2 f\,\mathrm{d}v\,\mathrm{d}x\right]\\ &\quad= 2t\mathcal{H}\big(f(t)\big) - \frac{1}{2}t\frac{\mathrm{d}^2}{\mathrm{d}t^2}\iint |x|^2 f\,\mathrm{d}v\,\mathrm{d}x = tE_{\mathrm{pot}}\big(f(t)\big),\end{aligned}$$

and the proof is complete. □

In the plasma physics case the identity (1.37) implies that solutions decay.

COROLLARY 1.15. *In the plasma physical case $\gamma = -1$ there exists for every solution of the Vlasov–Poisson system with initial datum as above a constant $C > 0$ such that the following estimates hold for all $t \geqslant 0$:*

$$\big\|\partial_x U(t)\big\|_2 \leqslant C(1+t)^{-1/2}, \tag{1.39}$$

$$\iint |v - x/t|^2 f(t,x,v)\,\mathrm{d}v\,\mathrm{d}x \leqslant C(1+t)^{-1}, \tag{1.40}$$

$$\big\|\rho(t)\big\|_{5/3} \leqslant C(1+t)^{-3/5}. \tag{1.41}$$

PROOF. Since $\gamma = -1$, the quantity

$$g(t) := t^2 E_{\mathrm{pot}}\big(f(t)\big) \geqslant 0$$

is nonnegative. The identity (1.37) takes the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\iint |x-tv|^2 f(t,x,v)\,\mathrm{d}v\,\mathrm{d}x + g(t)\right] = \frac{g(t)}{t}, \quad t > 0.$$

Integration of this identity from 1 to $t \geqslant 1$ yields

$$\frac{1}{2}\iint |x-tv|^2 f(t,x,v)\,\mathrm{d}v\,\mathrm{d}x + g(t) = C + \int_1^t \frac{g(s)}{s}\,\mathrm{d}s \tag{1.42}$$

for some constant $C > 0$ which depends on $f(1)$. We drop the double integral and apply Gronwall's lemma to the resulting inequality to obtain the estimate $g(t) \leqslant Ct$ for $t \geqslant 1$, and this proves (1.39). Insertion of the estimate for g into (1.42) proves (1.40). To obtain the estimate (1.41) we repeat the argument from the proof of Lemma 1.8, but instead of splitting the ρ integral according to $|v| < / > R$ we split according to $|v - x/t| < / > R$, and instead of the kinetic energy density $\int |v|^2 f \,\mathrm{d}v$ we use the quantity $\int |v - x/t|^2 f \,\mathrm{d}v$. $\square$

It should be emphasized that this type of decay remains true if the plasma consists of several species of particles with charges of different sign. In particular, this shows that in the plasma physics case the Vlasov–Poisson system as stated above does not have stationary solutions.

In the stellar dynamics case, Theorem 1.14 yields a dispersion result for solutions with positive energy. This was first observed in [24].

COROLLARY 1.16. *Consider a solution f of the Vlasov–Poisson system in the stellar dynamics case $\gamma = 1$ with positive energy*: $\mathcal{H}(\mathring{f}) = E_{\mathrm{kin}}(\mathring{f}) + E_{\mathrm{pot}}(\mathring{f}) > 0$. *Then there exist constants $C_1, C_2 > 0$ which depend on $\mathcal{H}(\mathring{f})$, $\|\mathring{f}\|_1$, $\|\mathring{f}\|_\infty$ such that for all sufficiently large times*,

$$C_1 t^2 \leqslant \iint |x|^2 f(t,x,v)\,\mathrm{d}v\,\mathrm{d}x \leqslant C_2 t^2.$$

In particular,

$$\sup\bigl\{|x| \mid (x,v) \in \operatorname{supp} f(t)\bigr\} \geqslant \left(\frac{C_1}{\|\mathring{f}\|_1}\right)^{1/2} t.$$

PROOF. By Proposition 1.9 and since the potential energy is negative,

$$0 < 2\mathcal{H}(\mathring{f}) \leqslant 2E_{\mathrm{kin}}\bigl(f(t)\bigr) + E_{\mathrm{pot}}\bigl(f(t)\bigr) \leqslant C_2,$$

and the assertion follows from the identity (1.36). $\square$

The corollary implies in particular that any stationary solution in the stellar dynamics case must have negative energy.

As pointed out for example in the remarks following Proposition 1.10, global existence does not hold for the relativistic Vlasov–Poisson system (1.4), (1.2), (1.3) in the gravitational case [29]. This result can be seen using an identity similar to the above.

THEOREM 1.17. *Let f be a spherically symmetric, classical solution to the gravitational, relativistic Vlasov–Poisson system* (1.4), (1.2), (1.3) *with nonnegative initial datum $\mathring{f} \in C_c^1(\mathbb{R}^6)$ and with negative energy*

$$\mathcal{H}(t) := \frac{1}{2}\iint \sqrt{1+|v|^2}\, f(t,x,v)\,\mathrm{d}v\,\mathrm{d}x - \frac{1}{8\pi}\int \bigl|\partial_x U(t,x)\bigr|^2\,\mathrm{d}x = \mathcal{H}(0) < 0.$$

Then this solution blows up in finite time.

PROOF. A computation analogous to the one leading to equation (1.38) yields the following relativistic dilation identity, which holds for any classical solution as long as it exists,

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint x\cdot v f(t,x,v)\,\mathrm{d}v\,\mathrm{d}x = \mathcal{H}(t) - \iint \frac{1}{\sqrt{1+|v|^2}} f(t,x,v)\,\mathrm{d}v\,\mathrm{d}x.$$

This implies that, for $t \geqslant 0$,

$$\iint x\cdot v f(t,x,v)\,\mathrm{d}v\,\mathrm{d}x \leqslant C + t\mathcal{H}(0).$$

Moreover, with

$$j(t,x) := \int \frac{v}{\sqrt{1+|v|^2}} f(t,x,v)\,\mathrm{d}v$$

and the previous estimate, we find that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\iint |x|^2\sqrt{1+|v|^2} f(t)\,\mathrm{d}v\,\mathrm{d}x \\
&\quad = 2\iint x\cdot v f(t)\,\mathrm{d}v\,\mathrm{d}x - \int |x|^2(\partial_x U\cdot j)(t,x)\,\mathrm{d}x \\
&\quad \leqslant C + 2\mathcal{H}(0)t - \int |x|^2(\partial_x U\cdot j)(t,x)\,\mathrm{d}x.
\end{aligned}$$

Due to spherical symmetry, using (1.10),

$$\left|\int |x|^2(\partial_x U\cdot j)(t,x)\,\mathrm{d}x\right| \leqslant \|f(t)\|_1 \int \bigl|j(t,x)\bigr|\,\mathrm{d}x \leqslant \|f(t)\|_1^2 = C$$

and hence

$$0 \leqslant \iint |x|^2\sqrt{1+|v|^2} f(t)\,\mathrm{d}v\,\mathrm{d}x \leqslant C(1+t) + \mathcal{H}(0)t^2.$$

But this estimate cannot hold for all $t > 0$, since by assumption, $\mathcal{H}(0) < 0$. □

Concluding remarks. (a) In [93] the decay estimates from Corollary 1.15 for the plasma physics case are used as input in the Pfaffelmoser and Schaeffer proof to obtain the improved estimates

$$P(t) \leqslant C(1+t)^{2/3}, \qquad \|\rho(t)\|_\infty \leqslant C(1+t)^2, \quad t \geqslant 0.$$

(b) In the plasma physics case, at least for the case of only one particle species, one may *conjecture* that the decay estimate

$$\|\rho(t)\|_\infty \leqslant Ct^{-3}, \quad t \geqslant 0,$$

holds. If true, this decay rate would be sharp,

$$t^3\|\rho(t)\|_\infty \not\to 0, \quad t \to \infty,$$

since with such a decay the velocities remain bounded, the diameter of the spatial support grows at most linearly in t, but the mass is conserved. For small data the above decay rate does hold. Using the techniques from Section 1.4 one can show the following: If a solution satisfies the decay estimate $\|\rho(t)\|_\infty \leqslant Ct^{-\alpha}$ with some $\alpha > 2$ then any solution starting in a small neighborhood satisfies the decay estimate with $\alpha = 3$. For spherically symmetric solutions the estimate $\|\rho(t)\|_\infty \leqslant Ct^{-3}\ln(1+t)$ was established in [59]. In space dimension one, $\|\rho(t)\|_\infty \leqslant Ct^{-1}$, cf. [10].

(c) Since the analogue of Theorem 1.1 holds for the relativistic Vlasov–Poisson system, blow-up means that ρ blows up in the L^∞-norm. Moreover, it is easy to see that in the spherically symmetric situation this blow-up has to occur at the origin. It is an interesting open problem to show that this blow-up behavior persists without the symmetry assumption. It is maybe not of physical but of mathematical interest that such blow-up results hold for the (nonrelativistic) Vlasov–Poisson system in space dimensions greater than or equal to 4, cf. [58,72].

2. Stability

2.1. *Introduction – steady states, stability and energy-Casimir functionals*

The question of which steady states of the Vlasov–Poisson system are stable in the gravitational case has over decades received a lot of attention in the astrophysics literature, and it still is an active field of research in astrophysics. The stability problem is of course also of considerable importance in plasma physics, and we will make occasional remarks on this case, but except for such remarks we consider only the gravitational case $\gamma = 1$ in this section. The corresponding results in the plasma physics case are in comparison easy to obtain. The results of this chapter originate in the collaboration of Y. Guo and the author [41,42,46–49,94,96–98]. They are presented here in a unified way.

Before entering into a discussion of the stability question there arises a presumably simpler question: Does the system *have* steady states?

2.1.1. *A strategy to construct steady states.* If $U_0 = U_0(x)$ is a time independent potential then the local or particle energy

$$E = E(x, v) := \frac{1}{2}|v|^2 + U_0(x) \tag{2.1}$$

is constant along solutions of the characteristic system

$$\dot{x} = v, \quad \dot{v} = -\nabla U_0(x).$$

Hence E as well as any function of E solves the Vlasov equation for the potential U_0. This leads to the following ansatz for a stationary solution:

$$f_0(x, v) = \phi(E) = \phi\big(E(x, v)\big),$$

where ϕ is a suitably chosen function. By this ansatz the Vlasov equation is satisfied. The spatial density ρ_0 becomes a functional of the potential U_0, and in order to obtain a self-consistent stationary solution of the Vlasov–Poisson system the remaining, semilinear Poisson equation must be solved, cf. (2.2). If a solution exists then the above ansatz *defines* a steady state with induced potential U_0. However, not just any solution obtained in this manner is acceptable. The resulting phase space density f_0 needs to have finite mass and possibly finite support in space; it should be noticed that the semilinear Poisson equation has to be solved on the whole space $\mathbb{R}^3$, since it is a priori not known where the support of the steady state will end and whether it will be bounded in the first place. It can be shown that these properties can hold only if the distribution vanishes for large values of the local energy E, cf. [102], Theorem 2.1. It turns out to be convenient to slightly reformulate the problem.

Steady state existence problem. Specify conditions on a measurable function $\phi : \mathbb{R} \to [0, \infty[$ with $\phi(\eta) = 0$ for $\eta < 0$ such that there exists a cut-off energy $E_0 \in \mathbb{R}$ and a solution U_0 of the semilinear elliptic problem

$$\Delta U_0 = 4\pi \int \phi\left(E_0 - \frac{1}{2}|v|^2 - U_0\right) \mathrm{d}v, \quad \lim_{|x| \to \infty} U_0(x) = 0, \tag{2.2}$$

with

$$U_0(x) \geqslant E_0 \quad \text{for } |x| \text{ sufficiently large.}$$

If U_0 is a solution of this problem then up to regularity issues

$$f_0(x, v) := \phi\left(E_0 - \frac{1}{2}|v|^2 - U_0(x)\right)$$

defines a steady state which is compactly supported and hence, if for example f_0 is bounded, has finite mass.

We do not enter more deeply into the matter of the existence of steady states for two reasons: Firstly, in our stability analysis we actually prove the existence of stable steady states. Secondly, steady states of the form discussed above must a posteriori be spherically symmetric so that the semilinear Poisson equation (2.2) becomes an ordinary differential equation with respect to the radial variable $r = |x|$, and its analysis does not really fit into the present treatise. In order to demonstrate the wide variety of possible steady states we present some of the known results without proofs. If U_0 is spherically symmetric the square of the modulus of angular momentum

$$L := |x \times v|^2 \tag{2.3}$$

is conserved along characteristics, and the distribution function can be taken to depend on E and L. The so-called polytropic ansatz

$$f_0(x, v) = (E_0 - E)_+^k L^l$$

with $k > -1, l > -1, k + l + 1/2 > 0, k < 3l + 7/2$, leads to steady states with finite mass and compact support, cf. [9]. In the limiting case $k = 3l + 7/2$ the mass is still finite but the support is the whole space, and for $k > 3l + 7/2$ the resulting steady state has infinite mass.

The dependence on L entails additional problems in the stability analysis which we will comment on later. In the sequel we restrict ourselves to steady states which depend only on the particle energy, so-called isotropic states. In [102], Theorem 3.1, it is shown that the approach above leads to a steady state with finite mass and compact support, provided $\phi \in L^\infty_{\mathrm{loc}}(\mathbb{R})$ and

$$\phi(E) = c(E_0 - E)^k + \mathrm{O}\big((E_0 - E)^{k+\delta}\big) \quad \text{as } E \to E_0-$$

with parameters $0 < k < 3/2$ and $\delta, c > 0$. This result covers the isotropic polytropes with $0 < k < 3/2$. More recently, a similar generalization of the isotropic polytropes with $0 < k < 7/2$ has been found, cf. [55], but then a condition on the global behavior of ϕ is required. We will repeatedly encounter the threshold $k = 3/2$ in what follows.

The steady states mentioned so far are spherically symmetric, and much less is known about the existence of steady states with less symmetry. In addition to the results in [49,94], which will be discussed in the stability context, we mention that axially symmetric steady states can be obtained as perturbations of spherically symmetric ones via the implicit function theorem [95].

For the plasma physics case the system as stated above has no steady states, cf. Section 1, Corollary 1.15. In order to have steady states in the plasma physics case one needs to include an exterior field or a fixed ion background or to consider the system on a bounded domain with appropriate boundary conditions. In these situations steady states are fairly easy to obtain, cf. [11,91]. The problem becomes more challenging if one is interested in steady states with a nontrivial magnetic field [45].

2.1.2. *Stability via linearization?* We do not wish to enter a general discussion of possible stability concepts, for which we refer to [56]. An often successful strategy to analyze the stability properties of some steady state of a dynamical system is linearization. For the Vlasov–Poisson system this approach is often followed in the astrophysics literature. We briefly review some of the arguments which can for example be found in the monographs

[13,26], where the interested reader will find many further references. Assume that f_0 is a steady state with induced spatial density ρ_0 and potential U_0, and let

$$f = f_0 + g, \qquad \rho = \rho_0 + \sigma, \qquad U = U_0 + W,$$

denote a solution of the time dependent problem which starts close to the steady state. We obtain the linearized system (linearized about f_0) if we substitute the above into the Vlasov–Poisson system, use the fact that (f_0, ρ_0, U_0) satisfies the system, and drop the quadratic term in the Vlasov equation

$$\begin{aligned}
&\partial_t g + v \cdot \partial_x g - \nabla U_0 \cdot \partial_v g = \partial_v f_0 \cdot \partial_x W,\\
&\Delta W = 4\pi\sigma, \qquad \lim_{|x|\to\infty} W(t,x) = 0,\\
&\sigma(t,x) = \int g(t,x,v)\,\mathrm{d}v.
\end{aligned}$$

Now we assume that the steady state is of the form discussed above, i.e., $f_0(x,v) = \phi(E)$. If we use the abbreviation

$$D := v \cdot \partial_x - \nabla U_0 \cdot \partial_v,$$

observe that $DE = 0$, and substitute the formula for the Newtonian potential we obtain the equivalent equation

$$\partial_t g + D\left[g + \phi'(E) \int\!\!\int \frac{g(t,y,w)}{|x-y|}\,\mathrm{d}w\,\mathrm{d}y\right] = 0.$$

What we have in mind here is only an exploratory calculation so we assume that everything is as regular as necessary for our manipulations. With some abuse of notation we make the ansatz

$$g(t,x,v) = \mathrm{e}^{\lambda t} g(x,v)$$

so that the linearized problem takes the form

$$\lambda g + D\left[g + \phi'(E) \int\!\!\int \frac{g(y,w)}{|x-y|}\,\mathrm{d}w\,\mathrm{d}y\right] = 0. \tag{2.4}$$

If $\mathrm{Re}\,\lambda < 0$ for all solutions (λ, g) then the steady state f_0 is expected to be stable, if $\mathrm{Re}\,\lambda > 0$ for one solution then it should be unstable. For finite-dimensional dynamical systems (ordinary differential equations) this expectation is of course justified by rigorous theorems. However, for infinite-dimensional dynamical systems such as the Vlasov–Poisson system no general such results exist.

There is however a more specific problem with an attempt to prove stability via the above spectral analysis of the linearized system. We split g into its even and odd parts in v, i.e.,

$$g = g_+ + g_-, \quad \text{where } g_\pm(x, v) := \frac{1}{2}\big[g(x, v) \pm g(x, -v)\big].$$

If we substitute this into the eigenvalue equation (2.4) and group together the even and odd parts we see that we have to solve the system of equations

$$\begin{aligned}&\lambda g_+ + D g_- = 0,\\ &\lambda g_- + D g_+ + D\left[\phi'(E)\iint \frac{g_+(y, w)}{|x-y|}\,\mathrm{d}w\,\mathrm{d}y\right] = 0.\end{aligned}$$

We eliminate g_+, and it remains to investigate the equation

$$\lambda^2 g_- = D^2 g_- + D\left[\phi'(E)\iint \frac{D g_-(y, w)}{|x-y|}\,\mathrm{d}w\,\mathrm{d}y\right] = 0,$$

where only solutions $g_- = g_-(x, v)$ are relevant which are odd in v. But if the pair (λ, g_-) solves this equation then so does the pair $(-\lambda, g_-)$. Hence as far as stability is concerned the best we may hope for is that all the eigenvalues λ are purely imaginary. Since in this situation one can in general draw no conclusion about the nonlinear stability of the steady state, not even for finite-dimensional dynamical systems, we do not pursue linearization any further.

For the plasma physics case a linearized analysis based on conserved quantities instead of spectral properties is carried out in [11].

2.1.3. *Energy-Casimir functionals.* As noted in Section 1.5, the Vlasov–Poisson system conserves energy: The functional

$$\mathcal{H}(f) := E_{\mathrm{kin}}(f) + E_{\mathrm{pot}}(f) = \frac{1}{2}\int |v|^2 f(x, v)\,\mathrm{d}v\,\mathrm{d}x - \frac{1}{8\pi}\int \big|\nabla U_f(x)\big|^2\,\mathrm{d}x$$

is constant along solutions. A natural approach to the stability question for a conservative system is to use the energy as a Lyapunov function. This idea meets an immediate obstacle: For the Lyapunov approach to work the steady state must first of all be a critical point of the energy, but in the present case the energy does not have critical points, i.e., the linear part in an expansion about any state f_0 with potential U_0 does not vanish,

$$\begin{aligned}\mathcal{H}(f) = \mathcal{H}(f_0) &+ \iint \left(\frac{1}{2}|v|^2 + U_0\right)(f - f_0)\,\mathrm{d}v\,\mathrm{d}x\\ &- \frac{1}{8\pi}\int |\nabla U_f - \nabla U_0|^2\,\mathrm{d}x.\end{aligned}$$

However, the characteristic flow corresponding to the Vlasov equation preserves phase space volume, cf. Section 1, Lemma 1.2(b), and hence for any reasonable function Φ the so-called *Casimir functional*

$$\mathcal{C}(f) := \iint \Phi\big(f(x,v)\big)\,\mathrm{d}v\,\mathrm{d}x$$

is conserved as well. If the energy-Casimir functional

$$\mathcal{H}_C := \mathcal{H} + \mathcal{C}$$

is expanded about an isotropic steady state

$$f_0(x,v) = \phi(E)$$

with the particle energy E defined as in (2.1), then

$$\begin{aligned}\mathcal{H}_C(f) = \mathcal{H}_C(f_0) + \iint \big(E + \Phi'(f_0)\big)(f - f_0)\,\mathrm{d}v\,\mathrm{d}x \\ - \frac{1}{8\pi}\int |\nabla U_f - \nabla U_0|^2\,\mathrm{d}x + \frac{1}{2}\iint \Phi''(f_0)(f-f_0)^2\,\mathrm{d}v\,\mathrm{d}x + \cdots.\end{aligned} \tag{2.5}$$

At least formally, we can choose Φ such that f_0 is a critical point of $\mathcal{H}_C$, namely $\Phi' = -\phi^{-1}$, provided ϕ is invertible. In more abstract terms we can say that the Hamiltonian $\mathcal{H}$ does not have critical points when we take as state space the space of all phase space densities f, but given a Casimir functional defined as above the corresponding steady state is a critical point of the Hamiltonian restricted to the manifold which is defined by the constraint $\mathcal{C}(f) = \mathcal{C}(f_0)$. A mostly formal discussion of this energy-Casimir approach in the context of so-called degenerate Hamiltonian or Lie–Poisson systems can be found in [56]. We make no use of this abstract background of our problem.

The question now is whether the quadratic term in the expansion (2.5) is positive (or negative) definite. As was noted in Section 2.1.1, in order for the steady state to have finite total mass the function ϕ must vanish above a certain cut-off energy. For ϕ^{-1} to exist ϕ should thus be decreasing, at least on its support. But then Φ'' is positive and the quadratic part in the expansion indefinite. Since one would like to use this quadratic part for defining the concept of distance or neighborhood, the method seems to fail.

If the issue is the stability of a plasma, the sign in front of the potential energy difference in the expansion (2.5) is reversed, and up to some technicalities stability follows, cf. [92]. The technical difficulties are among other things due to the fact that ϕ can at best be invertible on its support which is bounded from above by the cut-off energy.

2.1.4. *A variational problem and stability.* As noted above, certain steady states of the Vlasov–Poisson system are critical points of an energy-Casimir functional. At the same time the quadratic term in the Taylor expansion of this functional about such a steady state looks at first glance indefinite, which bodes ill for a stability analysis.

In this section we reverse our strategy in the following sense: We do not start with a given steady state whose stability we want to investigate, but instead we start with an energy-Casimir functional, i.e., with a function Φ defining the Casimir part, and we ask whether this functional attains its minimum on a suitable set of states f. Such a minimizer, if it exists, is a critical point of the energy-Casimir functional and hence should be a steady state, and its minimizing property can hopefully lead to a stability assertion.

Hence let $\Phi : [0,\infty[\, \to [0,\infty[$ be given – the necessary assumptions on this function are stated below. We investigate two closely related, but different variational problems, both of which have their merits. The difference between the two problems lies in the role of the Casimir functional – in the first formulation it is part of the functional to be minimized, in the second one it is part of the following constraint.

VARIATIONAL PROBLEM – VERSION 1. *Minimize the energy-Casimir functional $\mathcal{H}_C = \mathcal{H} + \mathcal{C}$ under a mass constraint, i.e., prove that the functional $\mathcal{H}_C$ has a minimizer $f_0 \in \mathcal{F}_M$,*

$$\mathcal{H}_C(f) \geqslant \mathcal{H}_C(f_0) \quad \textit{for all } f \in \mathcal{F}_M,$$

where the constraint set is defined as

$$\mathcal{F}_M := \left\{ f \in L^1_+(\mathbb{R}^6) \,\Big|\, \iint f \,\mathrm{d}v\,\mathrm{d}x = M,\, E_{\mathrm{kin}}(f) + \mathcal{C}(f) < \infty \right\}.$$

VARIATIONAL PROBLEM – VERSION 2. *Minimize the energy functional $\mathcal{H}$ under a mass-Casimir constraint, i.e., prove that the functional $\mathcal{H}$ has a minimizer $f_0 \in \mathcal{F}_{MC}$,*

$$\mathcal{H}(f) \geqslant \mathcal{H}(f_0) \quad \textit{for all } f \in \mathcal{F}_{MC},$$

where the constraint set is defined as

$$\mathcal{F}_{MC} := \left\{ f \in L^1_+(\mathbb{R}^6) \,\Big|\, \iint f \,\mathrm{d}v\,\mathrm{d}x + \mathcal{C}(f) = M,\, E_{\mathrm{kin}}(f) < \infty \right\}.$$

In both cases the parameter $M > 0$ is a prescribed positive number. In order to obtain solutions to these problems we make the following assumptions on Φ:

ASSUMPTIONS ON Φ. Let $\Phi \in C^1([0,\infty[)$ with $\Phi(0) = 0 = \Phi'(0)$, and

(Φ1) Φ is strictly convex,

(Φ2) $\Phi(f) \geqslant C f^{1+1/k}$ for $f \geqslant 0$ large,
where $0 < k < 3/2$ for Version 1,
$0 < k < 7/2$ for Version 2.

(Φ3) In addition for Version 1
$\Phi(f) \leqslant C f^{1+1/k'}$ for $f \geqslant 0$ small, where $0 < k' < 3/2$.

A typical function Φ is

$$\Phi(f) = \frac{k}{k+1} f^{1+1/k}, \quad f \geqslant 0. \tag{2.6}$$

The first version covers the parameter range $0 < k < 3/2$, the second one covers $0 < k < 7/2$, indeed, with some technical extra effort also the limiting case $k = 7/2$ can be covered, cf. [48]. The main advantage of the first approach, which covers a smaller range of the polytropic steady states, is that it can be attacked via a reduction procedure, and this reduction procedure brings out a relation between the stability problem for the Vlasov–Poisson system, i.e., for a self-gravitating collisionless gas, and the one for a self-gravitating perfect fluid as described by the Euler–Poisson system.

As will be seen below, the potential energy is finite for states in the constraint sets. The major step in the stability analysis is to prove the following theorem.

THEOREM 2.1. *Consider Version* 1 *of the variational problem under the above assumptions on Φ. Then the energy-Casimir functional $\mathcal{H}_C$ is bounded from below on $\mathcal{F}_M$ with $h_M := \inf_{\mathcal{F}_M} \mathcal{H}_C < 0$. Let $(f_j) \subset \mathcal{F}_M$ be a minimizing sequence of $\mathcal{H}_C$, i.e., $\mathcal{H}_C(f_j) \to h_M$. Then there exists a function $f_0 \in \mathcal{F}_M$, a subsequence, again denoted by (f_j) and a sequence $(a_j) \subset \mathbb{R}^3$ of shift vectors such that*

$$\begin{aligned} T^{a_j} f_j &:= f_j(\cdot + a_j, \cdot) \rightharpoonup f_0 && \text{weakly in } L^{1+1/k}(\mathbb{R}^6),\ j \to \infty, \\ T^{a_j} \nabla U_{f_j} &= \nabla U_{f_j}(\cdot + a_j) \to \nabla U_{f_0} && \text{strongly in } L^2(\mathbb{R}^3),\ j \to \infty. \end{aligned}$$

The state f_0 minimizes the energy-Casimir functional: $\mathcal{H}_C(f_0) = h_M$.

The analogous assertions hold for Version 2 *of the variational problem, with $\mathcal{H}_C$ replaced by $\mathcal{H}$ and $\mathcal{F}_M$ by $\mathcal{F}_{MC}$.*

Since the functionals under consideration are invariant under spatial translations a trivial minimizing sequence is obtained by shifting a given minimizer in space. If for example it is shifted off to infinity no subsequence can tend weakly to a minimizer, unless one moves with the sequence. Hence the spatial shifts in the theorem arise from the physical properties of the problem.

In Section 2.6 stability of the state f_0 will follow quite easily from the theorem. The point is that in the Taylor expansion (2.5) the negative definite part, i.e., the L^2 difference of the gravitational fields, converges to zero along minimizing sequences. Hence it is essential that the latter is part of Theorem 2.1 – the mere fact that f_0 be a minimizer is by itself not sufficient for stability.

The main difficulty of the proof of Theorem 2.1 is seen from the following sketch. To obtain a lower bound for the functional on the constraint set is easy, and by Assumption (Φ2) minimizing sequences can be seen to be bounded in $L^{1+1/k}$. Hence such a sequence has a weakly convergent subsequence, cf. [74], Section 2.18. The weak limit f_0 is the candidate for the minimizer, and one has to pass the limit into the various functionals. This is easy for the kinetic energy, the latter being linear. The Casimir functional is convex due to Assumption (Φ1), and so one can use Mazur's lemma for the same purpose, cf. [74], Section 2.13. The difficult part is the potential energy, for which one has to prove that the induced gravitational fields converge strongly in L^2. Since the latter do not depend directly on the phase space density f but only on the induced spatial density ρ_f the state space $\mathcal{F}_M$ seems inappropriate for the latter problem. This is the mathematical motivation for passing to a reduced functional which is defined on a suitable set of spatial densities. The reduction procedure is explained in the next section. Then we turn to the proof of Theorem 2.1 for the case of Version 1. For Version 2 reduction does not work and the necessary additional arguments are discussed in Section 2.4. The pay-off of reduction in terms of stability results for the Euler–Poisson system is discussed in Section 2.7.

2.2. *Reduction*

In this section we consider Version 1 of the variational problem with Φ satisfying the assumptions above; the results below are based on [97]. The aim is to factor out the v-dependence and obtain a reduced variational problem in terms of spatial densities. For $r \geqslant 0$ let

$$\mathcal{G}_r := \left\{ g \in L^1_+\big(\mathbb{R}^3\big) \;\Big|\; \int \left(\frac{1}{2}|v|^2 g(v) + \Phi\big(g(v)\big)\right) \mathrm{d}v < \infty, \int g(v)\,\mathrm{d}v = r \right\} \tag{2.7}$$

and

$$\Psi(r) := \inf_{g \in \mathcal{G}_r} \int \left(\frac{1}{2}|v|^2 g(v) + \Phi\big(g(v)\big)\right) \mathrm{d}v. \tag{2.8}$$

In addition to the variational problem of minimizing $\mathcal{H}_C$ over the set $\mathcal{F}_M$ we consider the problem of minimizing the functional

$$\mathcal{H}_r(\rho) := \int \Psi\big(\rho(x)\big)\,\mathrm{d}x + E_{\mathrm{pot}}(\rho) \tag{2.9}$$

over the set

$$\mathcal{R}_M := \left\{ \rho \in L^1_+\big(\mathbb{R}^3\big) \;\Big|\; \int \Psi\big(\rho(x)\big)\,\mathrm{d}x < \infty, \int \rho(x)\,\mathrm{d}x = M \right\}; \tag{2.10}$$

it will be seen further that the potential energy $E_{\mathrm{pot}}(\rho)$ which is defined in the obvious way is finite for states in this constraint set. The topic of the present section is the relation between the minimizers of $\mathcal{H}_C$ and $\mathcal{H}_r$. The following remark should convince the reader that the construction above is indeed a very natural one.

REMARK. Consider the intermediate functional

$$\mathcal{P}(\rho) := \inf_{f \in \mathcal{F}_\rho} \iint \left(\frac{1}{2}|v|^2 f(x,v) + \Phi\big(f(x,v)\big)\right) \mathrm{d}v\,\mathrm{d}x,$$

where for $\rho \in \mathcal{R}_M$,

$$\mathcal{F}_\rho := \{f \in \mathcal{F}_M \mid \rho_f = \rho\}.$$

Clearly, for $\rho = \rho_f$ with $f \in \mathcal{F}_M$,

$$\begin{aligned} \mathcal{C}(f) + E_{\mathrm{kin}}(f) &\geqslant \inf_{\tilde{f} \in \mathcal{F}_\rho} \big(\mathcal{C}(\tilde{f}) + E_{\mathrm{kin}}(\tilde{f})\big) \\ &\geqslant \inf_{\tilde{f} \in \mathcal{F}_\rho} \int \left[\inf_{g \in \mathcal{G}_{\rho(x)}} \int \left(\frac{1}{2}|v|^2 g(v) + \Phi\big(g(v)\big) \right) \mathrm{d}v \right] \mathrm{d}x \\ &= \int \left[\inf_{g \in \mathcal{G}_{\rho(x)}} \int \left(\frac{1}{2}|v|^2 g(v) + \Phi\big(g(v)\big) \right) \mathrm{d}v \right] \mathrm{d}x \\ &= \int \Psi\big(\rho(x)\big)\,\mathrm{d}x. \end{aligned} \tag{2.11}$$

This shows that

$$\mathcal{H}_C(f) \geqslant \mathcal{P}(\rho_f) + E_{\mathrm{pot}}(\rho_f) \geqslant \int \Psi\big(\rho_f(x)\big)\,\mathrm{d}x + E_{\mathrm{pot}}(\rho_f) = \mathcal{H}_r(\rho_f),$$

and it will be seen below that equality holds for minimizers. The functional $\mathcal{P}(\rho)$ is obtained by minimizing the positive contribution to $\mathcal{H}_C$, which also happens to be the part depending on phase space densities f directly, over all f's which generate a given spatial density ρ. Then in a second step one minimizes for each point x over all functions $g = g(v)$ the integral of which has the value $\rho(x)$.

These constructions owe much to [112] where they appear for the special case $\Phi(f) = f^{1+1/k}$ in a spherically symmetric situation. The main result of the present section is the following theorem.

THEOREM 2.2. (a) *For every function* $f \in \mathcal{F}_M$,

$$\mathcal{H}_C(f) \geqslant \mathcal{H}_r(\rho_f),$$

and if $f = f_0$ *is a minimizer of* $\mathcal{H}_C$ *over* $\mathcal{F}_M$ *then equality holds.*

(b) *Let* $\rho_0 \in \mathcal{R}_M$ *be a minimizer of* $\mathcal{H}_r$ *with induced potential* U_0. *Then there exists a Lagrange multiplier* $E_0 \in \mathbb{R}$ *such that a.e.*,

$$\rho_0 = \begin{cases} \big(\Psi'\big)^{-1}(E_0 - U_0), & U_0 < E_0, \\ 0, & U_0 \geqslant E_0. \end{cases} \tag{2.12}$$

With the particle energy E *defined as in* (2.1) *the function*

$$f_0 := \begin{cases} \big(\Phi'\big)^{-1}(E_0 - E), & E < E_0, \\ 0, & E \geqslant E_0, \end{cases}$$

is a minimizer of $\mathcal{H}_C$ *in* $\mathcal{F}_M$.

(c) *Assume that* $\mathcal{H}_r$ *has a minimizer in* $\mathcal{R}_M$. *If* $f_0 \in \mathcal{F}_M$ *is a minimizer of* $\mathcal{H}_C$ *then* $\rho_0 := \rho_{f_0} \in \mathcal{R}_M$ *is a minimizer of* $\mathcal{H}_r$, *this map is one-to-one and onto between the sets of minimizers of* $\mathcal{H}_C$ *in* $\mathcal{F}_M$ *and of* $\mathcal{H}_r$ *in* $\mathcal{R}_M$ *respectively, and is the inverse of the map* $\rho_0 \mapsto f_0$ *described in* (b).

In the next section we show that the reduced functional $\mathcal{H}_r$ does have a minimizer, and then the theorem guarantees that we recover all minimizers of $\mathcal{H}_C$ in $\mathcal{F}_M$ by "lifting" the ones of $\mathcal{H}_r$ as described in (b).

The above relation between Φ and Ψ arises in a natural way, but it can be made more explicit. Denote the Legendre transform of a function $h : \mathbb{R} \to]-\infty, \infty]$ by

$$\bar{h}(\lambda) := \sup_{r\in\mathbb{R}} \bigl(\lambda r - h(r)\bigr).$$

LEMMA 2.3. *Let* Ψ *be defined by* (2.7), (2.8), *and extend both* Φ *and* Ψ *by* $+\infty$ *to the interval* $]-\infty, 0[$.

(a) *For* $\lambda \in \mathbb{R}$,

$$\overline{\Psi}(\lambda) = \int \overline{\Phi}\left(\lambda - \frac{1}{2}|v|^2\right) \mathrm{d}v,$$

and in particular, $\overline{\Phi}(\lambda) = 0 = \overline{\Psi}(\lambda)$ *for* $\lambda < 0$.

(b) $\Psi \in C^1([0,\infty[)$ *is strictly convex, and* $\Psi(0) = \Psi'(0) = 0$.

(c) *With positive constants* C *which depend on* Φ *and* M,
$\Psi(\rho) \geqslant C\rho^{1+1/n}$ *for* $\rho \geqslant 0$ *large, where* $n := k + 3/2$, *and*
$\Psi(\rho) \leqslant C\rho^{1+1/n'}$ *for* $\rho \geqslant 0$ *small, where* $n' := k' + 3/2$.

PROOF. By definition,

$$\begin{aligned}
\overline{\Psi}(\lambda) &= \sup_{r\geqslant 0}\left[\lambda r - \inf_{g\in\mathcal{G}_r} \int \left(\frac{1}{2}|v|^2 g(v) + \Phi\bigl(g(v)\bigr)\right) \mathrm{d}v\right] \\
&= \sup_{r\geqslant 0}\sup_{g\in\mathcal{G}_r} \int \left[\left(\lambda - \frac{1}{2}|v|^2\right) g(v) - \Phi\bigl(g(v)\bigr)\right] \mathrm{d}v \\
&= \sup_{g\in L^1_+(\mathbb{R}^3)} \int \left[\left(\lambda - \frac{1}{2}|v|^2\right) g(v) - \Phi\bigl(g(v)\bigr)\right] \mathrm{d}v \\
&= \int \sup_{y\geqslant 0}\left[\left(\lambda - \frac{1}{2}|v|^2\right) y - \Phi(y)\right] \mathrm{d}v = \int \overline{\Phi}\left(\lambda - \frac{1}{2}|v|^2\right) \mathrm{d}v.
\end{aligned}$$

As to the last-but-one equality, observe that both sides are obviously zero for $\lambda \leqslant 0$. If $\lambda > 0$ then for any $g \in L^1_+(\mathbb{R}^3)$,

$$\int \left[\left(\lambda - \frac{1}{2}|v|^2\right) g(v) - \Phi\bigl(g(v)\bigr)\right] \mathrm{d}v \leqslant \int \sup_{y\geqslant 0}\left[\left(\lambda - \frac{1}{2}|v|^2\right) y - \Phi(y)\right] \mathrm{d}v.$$

If $|v| \geqslant \sqrt{2\lambda}$ then $\sup_{y\geqslant 0}[\cdots] = 0$, and for $|v| < \sqrt{2\lambda}$ the supremum of the term in brackets is attained at $y = y_v := (\Phi')^{-1}(\lambda - \frac{1}{2}|v|^2)$. Thus with

$$g_0(v) := \begin{cases} y_v, & |v| < \sqrt{2\lambda}, \\ 0, & |v| \geqslant \sqrt{2\lambda}, \end{cases}$$

we have

$$\begin{aligned} &\int \sup_{y\geqslant 0}\left[\left(\lambda - \frac{1}{2}|v|^2\right)y - \Phi(y)\right]\mathrm{d}v \\ &\quad = \int\left[\left(\lambda - \frac{1}{2}|v|^2\right)g_0(v) - \Phi\big(g_0(v)\big)\right]\mathrm{d}v \\ &\quad \leqslant \sup_{g\in L^1_+(\mathbb{R}^3)} \int\left[\left(\lambda - \frac{1}{2}|v|^2\right)g(v) - \Phi\big(g(v)\big)\right]\mathrm{d}v, \end{aligned}$$

and part (a) is established.

Since Φ is strictly convex and lower semicontinuous as a function on $\mathbb{R}$ with $\lim_{|f|\to\infty} \Phi(f)/|f| \to \infty$, $\overline{\Phi} \in C^1(\mathbb{R})$, cf. [84, Prop. 2.4]. Obviously, $\overline{\Phi}(\lambda) = 0$ for $\lambda \leqslant 0$, in particular, $(\overline{\Phi})'(0) = 0$. Also, $(\overline{\Phi})'$ is strictly increasing on $[0, \infty[$ since Φ' is strictly increasing on $[0, \infty[$ with range $[0, \infty[$. Since for $|\lambda| < \lambda_0$ with $\lambda_0 > 0$ fixed the integral in the formula for $\overline{\Psi}$ extends over a compact set we may differentiate under the integral sign to conclude that $\overline{\Psi} \in C^1(\mathbb{R})$ with derivative strictly increasing on $[0, \infty[$. This in turn implies the assertion of part (b).

Part (c) follows with (a) and the definition of the Legendre transform. □

We now prove Theorem 2.1.

PROOF OF THEOREM 2.2. We start by proving

The Euler–Lagrange equation for the reduced problem. Let $\rho_0 \in \mathcal{R}_M$ be a minimizer with induced potential U_0. For $\varepsilon > 0$ define

$$S_\varepsilon := \left\{x \in \mathbb{R}^3 \,\middle|\, \varepsilon \leqslant \rho_0(x) \leqslant \frac{1}{\varepsilon}\right\};$$

think of ρ_0 as a pointwise defined representative of the minimizer. For a test function $w \in L^\infty(\mathbb{R}^3)$ which has compact support and is nonnegative on $\mathbb{R}^3 \setminus S_\varepsilon$ define for $\tau \geqslant 0$ small,

$$\rho_\tau := \rho_0 + \tau w - \tau \frac{\int w\,\mathrm{d}y}{\operatorname{vol} S_\varepsilon}\mathbb{1}_{S_\varepsilon}.$$

Then $\rho_\tau \geqslant 0$ and $\int \rho_\tau = M$ so that $\rho_\tau \in \mathcal{R}_M$ for $\tau \geqslant 0$ small. Since ρ_0 is a minimizer of $\mathcal{H}_r$,

$$0 \leqslant \mathcal{H}_r(\rho_\tau) - \mathcal{H}_r(\rho_0) = \tau \int \big(\Psi'(\rho_0) + U_0\big)\left(w - \frac{\int w\,\mathrm{d}y}{\operatorname{vol} S_\varepsilon}\mathbb{1}_{S_\varepsilon}\right)\mathrm{d}x + \mathrm{o}(\tau).$$

Hence the coefficient of τ in this estimate must be nonnegative, which we can rewrite in the form

$$\int \left[\Psi'(\rho_0) + U_0 - \frac{1}{\operatorname{vol} S_\varepsilon}\left(\int_{S_\varepsilon} \big(\Psi'(\rho_0) + U_0\big)\,\mathrm{d}y\right)\right] w\,\mathrm{d}x \geqslant 0.$$

This holds for all test functions w as specified above, and hence $\Psi'(\rho_0) + U_0 = E_\varepsilon$ on S_ε and $\Phi'(\rho_0) + U_0 \geqslant E_\varepsilon$ on $\mathbb{R}^3 \setminus S_\varepsilon$ for all $\varepsilon > 0$ small enough. Here E_ε is a constant which by the first relation must be independent of ε, and taking $\varepsilon \to 0$ proves the relation between ρ_0 and U_0 in part (b).

The inequality in part (a) was established as part of the remark before Theorem 2.2.

An intermediate assertion. We claim that if $f \in \mathcal{F}_M$ is such that up to sets of measure zero,

$$\begin{cases} \Phi'(f) = E_0 - E > 0, & \text{where } f > 0, \\ E_0 - E \leqslant 0, & \text{where } f = 0, \end{cases} \tag{2.13}$$

with $E := \frac{1}{2}|v|^2 + U_f(x)$ and E_0 a constant, then equality holds in (a). To prove this, observe that since Φ is convex, we have for a.e. $x \in \mathbb{R}^3$ and every $g \in \mathcal{G}_{\rho_f(x)}$,

$$\begin{aligned} \frac{1}{2}|v|^2 g(v) + \Phi\big(g(v)\big) \geqslant{}& \frac{1}{2}|v|^2 f(x,v) + \Phi\big(f(x,v)\big) \\ &+ \left(\frac{1}{2}|v|^2 + \Phi'\big(f(x,v)\big)\right)\big(g(v) - f(x,v)\big) \quad \text{a.e.} \end{aligned}$$

Now by (2.13),

$$\begin{aligned} &\int \left(\frac{1}{2}|v|^2 + \Phi'(f)\right)(g - f)\,\mathrm{d}v \\ &\quad = \int_{\{f>0\}} \cdots + \int_{\{f=0\}} \cdots \\ &\quad = \big(E_0 - U_f(x)\big)\int_{\{f>0\}} (g - f)\,\mathrm{d}v + \int_{\{f=0\}} \frac{1}{2}|v|^2 g\,\mathrm{d}v \\ &\quad = -\big(E_0 - U_f(x)\big)\int_{\{f=0\}} (g - f)\,\mathrm{d}v + \int_{\{f=0\}} \frac{1}{2}|v|^2 g\,\mathrm{d}v \\ &\quad = \int_{\{f=0\}} (E - E_0) g\,\mathrm{d}v \geqslant 0; \end{aligned}$$

observe that $g \geqslant 0$ and $\int (g - f)\,\mathrm{d}v = 0$. Hence

$$\begin{aligned}\Psi\big(\rho_f(x)\big) &\geqslant \int \left(\frac{1}{2}|v|^2 f + \Phi(f)\right)\mathrm{d}v\\ &\geqslant \inf_{g \in \mathcal{G}_{\rho_f(x)}} \int \left(\frac{1}{2}|v|^2 g + \Phi(g)\right)\mathrm{d}v = \Psi\big(\rho_f(x)\big) \quad \text{a.e.,}\end{aligned}$$

and the proof of the intermediate assertion is complete.

Proof of the equality assertion in (a). If $f_0 \in \mathcal{F}_M$ is a minimizer of $\mathcal{H}_C$ then the Euler–Lagrange equation of the minimization problem implies that (2.13) holds for some Lagrange multiplier E_0; the proof is essentially the same as for the reduced problem above, cf. also Theorem 2.6. Thus equality holds in (a) by the intermediate assertion, and the proof of part (a) is complete.

Proof of the remaining part of (b). Let f_0 be defined as in (b). Then up to sets of measure zero,

$$\begin{aligned}\int f_0(x,v)\,\mathrm{d}v &= \int_{|v| \leqslant \sqrt{2(E_0 - U_0(x))}} \big(\Phi'\big)^{-1}\left(E_0 - U_0(x) - \frac{1}{2}|v|^2\right)\mathrm{d}v\\ &= (\overline{\Psi})'\big(E_0 - U_0(x)\big) = (\Psi')^{-1}\big(E_0 - U_0(x)\big) = \rho_0(x),\end{aligned}$$

where $U_0(x) < E_0$, and both sides are zero where $U_0(x) \geqslant E_0$. Thus $\rho_0 = \rho_{f_0}$, in particular, $f_0 \in \mathcal{F}_M$. By definition, f_0 satisfies the relation (2.13) and thus by our intermediate assertion $\mathcal{H}_C(f_0) = \mathcal{H}_r(\rho_0)$. Therefore again by part (a),

$$\mathcal{H}_C(f) \geqslant \mathcal{H}_r(\rho_f) \geqslant \mathcal{H}_r(\rho_0) = \mathcal{H}_C(f_0), \quad f \in \mathcal{F}_M,$$

so that f_0 is a minimizer of $\mathcal{H}_C$, and the proof of part (b) is complete.

Proof of part (c). Assume that $\mathcal{H}_r$ has a minimizer $\rho_0 \in \mathcal{R}_M$ and define f_0 as above. Then part (a), the fact that each $\rho \in \mathcal{R}_M$ can be written as $\rho = \rho_f$ for some $f \in \mathcal{F}_M$, and our intermediate assertion imply that

$$\begin{aligned}\inf_{f \in \mathcal{F}_M} \mathcal{H}_C(f) &\geqslant \inf_{f \in \mathcal{F}_M} \mathcal{H}_r(\rho_f) = \inf_{\rho \in \mathcal{R}_M} \mathcal{H}_r(\rho)\\ &= \mathcal{H}_r(\rho_0) = \mathcal{H}_C(f_0) \geqslant \inf_{f \in \mathcal{F}_M} \mathcal{H}_C(f).\end{aligned} \tag{2.14}$$

Now take any minimizer $g_0 \in \mathcal{F}_M$ of $\mathcal{H}_C$. Then by (2.14) and part (a),

$$\inf_{\rho \in \mathcal{R}_M} \mathcal{H}_r(\rho) = \inf_{f \in \mathcal{F}_M} \mathcal{H}_C(f) = \mathcal{H}_C(g_0) = \mathcal{H}_r(\rho_{g_0}),$$

that is, $\rho_{g_0} \in \mathcal{R}_M$ minimizes $\mathcal{H}_r$, and the proof of part (c) is complete. □

2.3. *Existence of minimizers via the reduced problem*

First the reduced variational problem is studied in its own right under the following assumptions on the function Ψ.

ASSUMPTIONS ON Ψ. Let $\Psi \in C^1([0,\infty[)$ with $\Psi(0) = 0 = \Psi'(0)$, and

(Ψ1) Ψ is strictly convex,
(Ψ2) $\Psi(\rho) \geqslant C\rho^{1+1/n}$ for $\rho \geqslant 0$ large, with $0 < n < 3$,
(Ψ3) $\Psi(\rho) \leqslant C\rho^{1+1/n'}$ for $\rho \geqslant 0$ small, with $0 < n' < 3$.

We shall prove the following central result:

THEOREM 2.4. *The functional $\mathcal{H}_r$ is bounded from below on $\mathcal{R}_M$. Let $(\rho_j) \subset \mathcal{R}_M$ be a minimizing sequence of $\mathcal{H}_r$. Then there exists a sequence of shift vectors $(a_j) \subset \mathbb{R}^3$ and a subsequence, again denoted by (ρ_j), such that*

$$T^{a_j}\rho_j := \rho_j(\cdot + a_j) \rightharpoonup \rho_0 \quad \text{weakly in } L^{1+1/n}\big(\mathbb{R}^3\big),\ j \to \infty,$$
$$T^{a_j}\nabla U_{\rho_j} \to \nabla U_{\rho_0} \quad \text{strongly in } L^2\big(\mathbb{R}^3\big),\ j \to \infty,$$

and $\rho_0 \in \mathcal{R}_M$ is a minimizer of $\mathcal{H}_r$.

The main difficulty is to prove that the fields induced by a minimizing sequence converge strongly in L^2. Such a compactness property holds if the sequence (ρ_j) remains concentrated. In view of the next section the corresponding result stated below is slightly more general than what is needed in the present section.

LEMMA 2.5. *Let $0 < n < 5$. Let $(\rho_j) \subset L_+^{1+1/n}(\mathbb{R}^3)$ be such that*

$$\rho_j \rightharpoonup \rho_0 \quad \text{weakly in } L^{1+1/n}\big(\mathbb{R}^3\big),$$
$$\forall \varepsilon > 0\ \exists R > 0: \quad \limsup_{j\to\infty} \int_{|x|\geqslant R} \rho_j(x)\,\mathrm{d}x < \varepsilon. \tag{2.15}$$

Then $\nabla U_{\rho_j} \to \nabla U_{\rho_0}$ strongly in L^2.

PROOF. Clearly, there exists a constant $m > 0$ such that for all sufficiently large $j \in \mathbb{N}$, $\int \rho_j \leqslant m$, and by weak convergence the limit ρ_0 is integrable as well. The sequence $\sigma_j := \rho_j - \rho_0$ converges weakly to 0 in $L^{1+1/n}$, $\int |\sigma_j| \leqslant 2m$, and (2.15) holds for $|\sigma_j|$ as well. We need to show that $\nabla U_{\sigma_j} \to 0$ strongly in L^2 which is equivalent to

$$I_j := \iint \frac{\sigma_j(x)\sigma_j(y)}{|x-y|}\,\mathrm{d}y\,\mathrm{d}x \to 0.$$

For $\delta > 0$ and $R > 0$ we split the integral above as follows:

$$I_j = I_{j,1} + I_{j,2} + I_{j,3},$$

where

$$|x-y|<\delta \quad \text{for } I_{j,1}, \qquad |x-y|\geqslant \delta \wedge \big(|x|\geqslant R \vee |y|\geqslant R\big) \quad \text{for } I_{j,2},$$
$$|x-y|\geqslant \delta \wedge |x|<R \wedge |y|<R \quad \text{for } I_{j,3}.$$

Since $2n/(n+1)+2/(n+1)=2$, Young's inequality [74], Section 4.2, implies that

$$|I_{j,1}| \leqslant C\|\sigma_j\|_{1+1/n}^2 \big\|\mathbb{1}_{B_\delta}|\cdot|^{-1}\big\|_{(n+1)/2} \leqslant C\delta^{(5-n)/(n+1)}.$$

Hence we can make $I_{j,1}$ as small as we wish, uniformly in j and independently of R, by choosing δ small. For $\delta>0$ now fixed,

$$|I_{j,2}| \leqslant \frac{4m}{\delta}\int_{|x|>R} \big|\sigma_j(x)\big|\,\mathrm{d}x,$$

which becomes small by (2.15), if we choose $R>0$ accordingly. Finally by Hölder's inequality,

$$|I_{j,3}| = \left|\int \sigma_j(x)h_j(x)\,\mathrm{d}x\right| \leqslant \|\sigma_j\|_{1+1/n}\|h_j\|_{1+n} \leqslant C\|h_j\|_{1+n},$$

where in a pointwise sense,

$$h_j(x) := \mathbb{1}_{B_R}(x)\int_{|x-y|\geqslant\delta} \mathbb{1}_{B_R}(y)\frac{1}{|x-y|}\sigma_j(y)\,\mathrm{d}y \to 0$$

due to the weak convergence of σ_j and the fact that the test function against which σ_j is integrated here is in L^{1+n}. Since $|h_j| \leqslant \frac{2m}{\delta}\mathbb{1}_{B_R}$ uniformly in j Lebesgue's dominated convergence theorem implies that $h_j \to 0$ in L^{1+n}, and the proof is complete. □

PROOF OF THEOREM 2.4. Constants denoted by C may only depend on M and Ψ and may change their value from line to line. The proof is split into a number of steps.

Step 1: *Lower bound for $\mathcal{H}_r$ and weak convergence of minimizing sequences.* By Lemma P2(b), interpolation, and (Ψ2),

$$\begin{aligned}-E_{\mathrm{pot}}(\rho) &\leqslant C\|\rho\|_{6/5}^2 \leqslant C\|\rho\|_1^{(5-n)/3}\|\rho\|_{1+1/n}^{(n+1)/3}\\ &\leqslant C + C\left(\int \Psi(\rho)\,\mathrm{d}x\right)^{n/3}, \quad \rho\in\mathcal{R}_M;\end{aligned}$$

note that $1<6/5<1+1/n$. Hence on $\mathcal{R}_M$

$$\mathcal{H}_r(\rho) \geqslant \int \Psi(\rho)\,\mathrm{d}x - C - C\left(\int \Psi(\rho)\,\mathrm{d}x\right)^{n/3}. \tag{2.16}$$

Since $n < 3$ this implies that $\mathcal{H}_r$ is bounded from below on $\mathcal{R}_M$,

$$h_M := \inf_{\mathcal{R}_M} \mathcal{H}_r > -\infty.$$

Let $(\rho_j) \subset \mathcal{R}_M$ be a minimizing sequence. By (2.16), $\int \Psi(\rho_j)$ is bounded, and by $(\Psi 2)$ and the fact that $\int \rho_j = M$, the minimizing sequence is bounded in $L^{1+1/n}(\mathbb{R}^3)$. Hence we can – after extracting a subsequence – assume that it converges weakly to some function $\rho_0 \in L^{1+1/n}(\mathbb{R}^3)$. By weak convergence, $\rho_0 \geqslant 0$ almost everywhere – if ρ_0 were strictly negative on some set S of positive, finite measure the test function $\sigma = \mathbb{1}_S$ would yield a contradiction.

The next two steps show that minimizing sequences remain concentrated and do not split into far apart pieces or spread out uniformly in space.

Step 2: *Behavior under rescaling.* For $\rho \in \mathcal{R}_M$ and $a, b > 0$ we define $\bar{\rho}(x) := a\rho(bx)$. Then

$$\int \bar{\rho}\,\mathrm{d}x = ab^{-3} \int \rho\,\mathrm{d}x, \qquad E_{\mathrm{pot}}(\bar{\rho}) = a^2 b^{-5} E_{\mathrm{pot}}(\rho),$$
$$\int \Psi(\bar{\rho}) = b^{-3} \int \Psi(a\rho)\,\mathrm{d}x.$$

First we fix a bounded and compactly supported function $\rho \in \mathcal{R}_M$ and choose $a = b^3$ so that $\bar{\rho} \in \mathcal{R}_M$ as well. By $(\Psi 3)$ and since $3/n' > 1$,

$$\mathcal{H}_r(\bar{\rho}) = b^{-3} \int \Psi(b^3\rho)\,\mathrm{d}x + bE_{\mathrm{pot}}(\rho) \leqslant Cb^{3/n'} + bE_{\mathrm{pot}}(\rho) < 0$$

for b sufficiently small, and hence for $M > 0$,

$$h_M < 0. \tag{2.17}$$

Next we fix two masses $0 < \overline{M} \leqslant M$. If we take $a = 1$ and $b = (M/\overline{M})^{1/3} \geqslant 1$ then for $\rho \in \mathcal{R}_M$ and $\bar{\rho} \in \mathcal{R}_{\overline{M}}$ rescaled with these parameters,

$$\begin{aligned}\mathcal{H}_r(\bar{\rho}) &= b^{-3} \int \Psi(\rho)\,\mathrm{d}x + b^{-5} E_{\mathrm{pot}}(\rho) \\ &\geqslant b^{-5}\left(\int \Psi(\rho)\,\mathrm{d}x + E_{\mathrm{pot}}(\rho)\right) = \left(\frac{\overline{M}}{M}\right)^{5/3} \mathcal{H}_r(\rho).\end{aligned}$$

Since for the present choice of a and b the map $\rho \mapsto \bar{\rho}$ is one-to-one and onto between $\mathcal{R}_M$ and $\mathcal{R}_{\overline{M}}$ this estimate gives the following relation between the infima of our functional for different mass constraints:

$$h_{\overline{M}} \geqslant \left(\frac{\overline{M}}{M}\right)^{5/3} h_M, 0 < \overline{M} \leqslant M. \tag{2.18}$$

Step 3: *Spherically symmetric minimizing sequences remain concentrated.* In this step we prove the concentration property needed to apply Lemma 2.5, but to make things easier we consider for a moment spherically symmetric functions $\rho \in \mathcal{R}_M$, i.e., $\rho(x) = \rho(|x|)$. For any radius $R > 0$ we split ρ into the piece supported in the ball B_R and the rest, i.e.,

$$\rho = \rho_1 + \rho_2, \quad \rho_1(x) = 0 \text{ for } |x| > R, \rho_2(x) = 0 \text{ for } |x| \leqslant R.$$

Clearly,

$$\mathcal{H}_r(\rho) = \mathcal{H}_r(\rho_1) + \mathcal{H}_r(\rho_2) - \int \frac{\rho_1(x)\rho_2(y)}{|x-y|} \,\mathrm{d}x\,\mathrm{d}y.$$

Due to spherical symmetry the potential energy of the interaction between the two pieces can be estimated as

$$\int \frac{\rho_1(x)\rho_2(y)}{|x-y|} \,\mathrm{d}x\,\mathrm{d}y = -\int U_{\rho_1}\rho_2 \,\mathrm{d}x \leqslant \frac{(M-m)m}{R},$$

where $m = \int \rho_2$ is the mass outside the radius R which we want to make small along the minimizing sequence. We define

$$R_0 := -\frac{3}{5}\frac{M^2}{h_M} > 0$$

and use the scaling estimate (2.18) together with (2.17) and the fact that $\xi^{5/3} + (1-\xi)^{5/3} \leqslant 1 - \frac{5}{3}\xi(1-\xi)$ for $0 \leqslant \xi \leqslant 1$ to conclude that

$$\begin{aligned} \mathcal{H}_r(\rho) &\geqslant h_{M-m} + h_m - \frac{(M-m)m}{R} \\ &\geqslant \left[\left(1 - \frac{m}{M}\right)^{5/3} + \left(\frac{m}{M}\right)^{5/3}\right] h_M - \frac{(M-m)m}{R} \\ &\geqslant h_M + \left[\frac{1}{R_0} - \frac{1}{R}\right](M-m)m. \end{aligned} \tag{2.19}$$

We claim that, if $R > R_0$, then for any spherically symmetric minimizing sequence $(\rho_j) \subset \mathcal{R}_M$ of $\mathcal{H}_r$, the following assertion holds, which is even a bit stronger than what is needed to apply lemma 2.5,

$$\lim_{j\to\infty} \int_{|x| \geqslant R} \rho_j(x)\,\mathrm{d}x = 0. \tag{2.20}$$

Assume this assertion were false so that up to a subsequence,

$$\lim_{j\to\infty} \int_{|x| \geqslant R} \rho_j = m > 0.$$

Choose $R_j > R$ such that

$$m_j := \int_{|x| \geqslant R_j} \rho_j = \frac{1}{2} \int_{|x| \geqslant R} \rho_j.$$

By (2.19),

$$\mathcal{H}_r(\rho_j) \geqslant h_M + \left[\frac{1}{R_0} - \frac{1}{R_j} \right] (M - m_j) m_j \geqslant h_M + \left[\frac{1}{R_0} - \frac{1}{R} \right] (M - m_j) m_j,$$

and letting $j \to \infty$ leads to a contradiction, and equation (2.20) is proven.

For the weak limit ρ_0 of the minimizing sequence clearly

$$\operatorname{supp} \rho_0 \subset B_{R_0}, \qquad \int \rho_0 = M.$$

Step 4: *Proof of Theorem* 2.4 *under the assumption of spherical symmetry.* Given a minimizing sequence (ρ_j) we already know that up to a subsequence it converges weakly in $L^{1+1/n}$ to a nonnegative limit ρ_0 of mass M. The functional $\rho \mapsto \int \Psi(\rho)\,\mathrm{d}x$ is convex by Assumption $(\Psi 1)$, so by Mazur's lemma [74], Section 2.13 and Fatou's lemma [74], Section 1.7

$$\int \Psi(\rho_0)\,\mathrm{d}x \leqslant \limsup_{j \to \infty} \int \Psi(\rho_j)\,\mathrm{d}x,$$

in particular, $\rho_0 \in \mathcal{R}_M$. If we assume in addition that the minimizing sequence is spherically symmetric then by Step 3 and Lemma 2.5, $E_{\mathrm{pot}}(\rho_j) \to E_{\mathrm{pot}}(\rho_0)$, and hence

$$\mathcal{H}_r(\rho_0) \leqslant \limsup_{j \to \infty} \mathcal{H}_r(\rho_j) = h_M$$

so that ρ_0 is a minimizer of $\mathcal{H}_r$ over the subset of spherically symmetric functions in $\mathcal{R}_M$.

The restriction to spherical symmetry would mean that stability would only hold against spherically symmetric perturbations. Fortunately, this restriction can be removed using a general result due to Burchard and Guo.

Step 5: *Removing the symmetry assumption.* To explain the result by Burchard and Guo we define for a given function $\rho \in L^1_+(\mathbb{R}^3)$ its spherically symmetric decreasing rearrangement ρ^* as the unique spherically symmetric, radially decreasing function with the property that for every $\tau \geqslant 0$ the sup-level-sets $\{x \in \mathbb{R}^3 \mid \rho(x) > \tau\}$ and $\{x \in \mathbb{R}^3 \mid \rho^*(x) > \tau\}$ have the same volume; the latter set is of course a ball about the origin whose radius is determined by the volume of the former. The integral $\int \Psi(\rho)\,\mathrm{d}x$ does not change under such a rearrangement, while the potential energy can only decrease, and it does not decrease iff ρ is already spherically symmetric (with respect to some center of symmetry) and decreasing. These facts can be found in [74], Chapter 3. In particular, a minimizer must a posteriori be spherically symmetric.

Now let $(\rho_j) \subset \mathcal{R}_M$ be a not necessarily spherically symmetric minimizing sequence. Obviously, the sequence of spherically symmetric decreasing rearrangements (ρ_j^*) is again minimizing. Hence by the previous steps, up to a subsequence (ρ_j^*) converges weakly to a minimizer $\rho_0 = \rho_0^*$ and

$$\nabla U_{\rho_j^*} \to \nabla U_0 \quad \text{in } L^2, \qquad \text{hence } \int \Psi\big(\rho_j^*\big) \to \int \Psi(\rho_0).$$

Moreover,

$$\begin{aligned} E_{\mathrm{pot}}(\rho_j) &= \mathcal{H}_r(\rho_j) - \int \Psi(\rho_j) = \mathcal{H}_r(\rho_j) - \int \Psi\big(\rho_j^*\big) \\ &\quad \to \mathcal{H}_r(\rho_0) - \int \Psi(\rho_0) = E_{\mathrm{pot}}(\rho_0). \end{aligned}$$

In this situation the result of Burchard and Guo [16], Theorem 1, says that there exists a sequence $(a_j) \subset \mathbb{R}^3$ of shift vectors such that

$$T^{a_j} \nabla U_{\rho_j} = \nabla U_{\rho_j}(\cdot + a_j) \to \nabla U_0 \quad \text{in } L^2.$$

Hence we can repeat the arguments of Step 4 for the sequence $(T^{a_j} \rho_j)$, which is again minimizing, and the proof of Theorem 2.4 is complete. □

The proof of the result by Burchard and Guo is by no means easy, and it is possible to obtain stability against general perturbations without resorting to it, cf. [48,96,97]. Since this general result may be useful for other problems of this nature we wanted to mention and exploit it here. On the other hand, Version 2 of the variational problem does not lend itself to a reduction mechanism like Version 1. Hence the result by Burchard and Guo does not apply, and we will show in the next section how to handle the concentration problem directly in the nonsymmetric situation. We also refer to [53] for an account of the result by Burchard and Guo and its relation to stability problems.

Theorem 2.4 implies the result that we were originally interested in.

PROOF OF THEOREM 2.1 FOR VERSION 1. By Lemma 2.3 we see that if Φ satisfies the assumptions (Φ1), (Φ2), (Φ3) then the function Ψ defined by (2.8) satisfies the assumptions (Ψ1), (Ψ2), (Ψ3), where the parameters k and n are related by $n = k + 3/2$, with the same relation holding for the primed parameters. Theorem 2.2 connects the original and the reduced variational problem in the appropriate way to derive Theorem 2.1 from Theorem 2.4: Firstly, $\mathcal{H}_C$ is bounded from below on $\mathcal{F}_M$ since this is true for $\mathcal{H}_r$ on $\mathcal{R}_M$. Let $(f_j) \subset \mathcal{F}_M$ be a minimizing sequence for $\mathcal{H}_C$. By Theorem 2.2, $(\rho_{f_j}) \subset \mathcal{R}_M$ is a minimizing sequence for $\mathcal{H}_r$. Again by Theorem 2.2 we can lift the minimizer ρ_0 of $\mathcal{H}_r$ obtained in Theorem 2.4 to a minimizer f_0 of $\mathcal{H}_C$. The properly shifted fields converge strongly in L^2 to ∇U_{f_0}. Hence after extracting a subsequence the Casimir functional as well as the kinetic energy converge along $(T^{a_j} f_j)$, and this sequence converges weakly in $L^{1+1/k}$ to the minimizer f_0. □

Notice that the weak convergence of a subsequence of $(T^{a_j} f_j)$ to the minimizer f_0, which was derived after the existence of the minimizer was established, will play a role in the stability analysis in Section 2.6.

2.4. *Existence of minimizers – the direct approach*

In this section we prove Theorem 2.1 for Version 2 of our variational problem, i.e., we minimize the energy functional $\mathcal{H}$ under the mass-Casimir constraint implemented in the constraint set $\mathcal{F}_{MC}$. In the reduction procedure employed above the kinetic energy and the Casimir functional were reduced into a new functional acting on spatial densities ρ. But in Version 2 of the variational problem the former two functionals appear in different places, namely as part of the functional to be minimized and in the constraint respectively. Hence reduction in the above sense does not apply, and a direct argument is given. This necessarily also shows how the use of the nontrivial result by Burchard and Guo for removing the symmetry assumption can be avoided.

PROOF OF THEOREM 2.1 FOR VERSION 2. Constants denoted by C may only depend on M and Φ and may change their value from line to line. The growth parameter k in the assumptions on Φ satisfies

$$0 < k < \frac{7}{2}, \qquad \text{hence } \frac{3}{2} < n := k + \frac{3}{2} < 5 \quad \text{and} \quad 1 + \frac{1}{n} > \frac{6}{5}.$$

The proof is again split into a number of steps, similar to Version 1.

Step 1: *Lower bound for $\mathcal{H}$ and bounds on minimizing sequences.* By the assumptions on Φ, Lemma P2, Lemma 1.8 of Chapter 1, and interpolation the following estimates hold for any $f \in \mathcal{F}_{MC}$:

$$\begin{aligned}
&\|f\|_1 + \|f\|_{1+1/k} \leqslant C,\\
&\|\rho_f\|_{1+1/n} \leqslant C \|f\|_{1+1/k}^{(k+1)/(n+1)} E_{\mathrm{kin}}(f)^{3/(2k+5)} \leqslant C E_{\mathrm{kin}}(f)^{3/(2(n+1))},\\
&-E_{\mathrm{pot}}(f) \leqslant C \|\rho_f\|_{6/5}^2 \leqslant C \|\rho_f\|_1^{(5-n)/3} \|\rho_f\|_{1+1/n}^{(n+1)/3} \leqslant C E_{\mathrm{kin}}(f)^{1/2}.
\end{aligned}$$

Hence the total energy $\mathcal{H}$ is bounded from below on $\mathcal{F}_{MC}$,

$$\mathcal{H}(f) \geqslant E_{\mathrm{kin}}(f) - C E_{\mathrm{kin}}(f)^{1/2} \quad \text{for } f \in \mathcal{F}_{MC}, \qquad h_M := \inf_{\mathcal{F}_{MC}} \mathcal{H} > -\infty,$$

and E_{kin} together with the quantities estimated above are bounded along minimizing sequences of $\mathcal{H}$ in $\mathcal{F}_{MC}$.

The observation that concentration implies compactness made in Lemma 2.5 is going to be used again in the present situation, and we turn to the investigation of the concentration properties of the energy functional under the mass-Casimir constraint.

Step 2: *Behavior under rescaling.* Given any function f, we define a rescaled function $\bar{f}(x,v) = f(ax,bv)$, where $a,b>0$; as opposed to Version 1 we do not scale the dependent variable but only its arguments. Then

$$\iint \big(\bar{f} + \Phi(\bar{f})\big)\,\mathrm{d}v\,\mathrm{d}x = (ab)^{-3}\iint \big(f + \Phi(f)\big)\,\mathrm{d}v\,\mathrm{d}x \tag{2.21}$$

i.e., $f \in \mathcal{F}_{MC}$ iff $\bar{f} \in \mathcal{F}_{\overline{M}C}$ where $\overline{M} := (ab)^{-3}M$. The kinetic and potential energy scale as follows:

$$E_{\mathrm{kin}}(\bar{f}) = a^{-3}b^{-5}E_{\mathrm{kin}}(f), \qquad E_{\mathrm{pot}}(\bar{f}) = a^{-5}b^{-6}E_{\mathrm{pot}}(f).$$

If $f \in \mathcal{F}_{MC}$ and $b = a^{-1}$ then $\bar{f} \in \mathcal{F}_{MC}$ and

$$\mathcal{H}(\bar{f}) = a^2 E_{\mathrm{kin}}(f) + aE_{\mathrm{pot}}(f) < 0$$

for $a>0$ sufficiently small, since $E_{\mathrm{pot}}(f)<0$. Hence for all $M>0$,

$$h_M < 0. \tag{2.22}$$

Next we choose a and b such that $a^{-3}b^{-5} = a^{-5}b^{-6}$, i.e., $b = a^{-2}$. Then

$$\mathcal{H}(\bar{f}) = a^7\mathcal{H}(f), \tag{2.23}$$

and since $a = (\overline{M}/M)^{1/3}$ and the mapping $\mathcal{F}_{MC} \to \mathcal{F}_{\overline{M}C}$, $f \mapsto \bar{f}$ is one-to-one and onto this shows that for all $M, \overline{M} > 0$,

$$h_{\overline{M}} = \left(\frac{\overline{M}}{M}\right)^{7/3} h_M. \tag{2.24}$$

Step 3: *Minimizing sequences do not vanish.* In the nonsymmetric case we cannot establish a result like equation (2.15) as easily as in the spherically symmetric situation. As a first step we show that along any minimizing sequence some minimal mass must remain in a sufficiently large ball. This is precisely the point where we have to allow spatial shifts: We cannot expect this nonvanishing property to hold unless we move with the sequence. Our assertion is that for any minimizing sequence $(f_j) \subset \mathcal{F}_{MC}$ of $\mathcal{H}$ there exist a sequence $(a_j) \subset \mathbb{R}^3$ and $m_0>0$, $R_0>0$ such that

$$\int_{a_j + B_{R_0}} \rho_j\,\mathrm{d}x \geqslant m_0 \tag{2.25}$$

for all sufficiently large $j \in \mathbb{N}$, where $\rho_j := \rho_{f_j}$. To see this we split for $R>1$,

$$-E_{\mathrm{pot}}(f_j) = \frac{1}{2}\iint \frac{\rho_j(x)\rho_j(y)}{|x-y|}\,\mathrm{d}y\,\mathrm{d}x = I_1 + I_2 + I_3,$$

where

$$|x-y| < \frac{1}{R} \quad \text{for } I_1, \qquad \frac{1}{R} \leqslant |x-y| \leqslant R \quad \text{for } I_2, \qquad |x-y| > R \quad \text{for } I_3.$$

Since (ρ_j) is bounded in $L^1(\mathbb{R}^3)$ and in $L^{1+1/n}(\mathbb{R}^3)$ by Step 1,

$$\begin{aligned}
I_1 &\leqslant \|\rho_j\|^2_{1+1/n} \big\| \mathbb{1}_{B_{1/R}} 1/|\cdot| \big\|_{(n+1)/2} \leqslant C R^{-(5-n)/(n+1)}, \\
I_2 &\leqslant R \iint_{|x-y|<R} \rho_j(x)\, \rho_j(y) \,\mathrm{d}x\, \mathrm{d}y \leqslant RC \sup_{y \in \mathbb{R}^3} \int_{y+B_R} \rho_j(x) \,\mathrm{d}x, \\
I_3 &\leqslant \frac{1}{R} \iint \rho_j(x) \rho_j(y) \,\mathrm{d}x\, \mathrm{d}y \leqslant C R^{-1};
\end{aligned}$$

for the first estimate we used Young's inequality [74], Section 4.2. Since (f_j) is minimizing and $h_M < 0$ we have, for any $R > 1$,

$$\frac{h_M}{2} > \mathcal{H}(f_j) \geqslant -I_1 - I_2 - I_3,$$

provided j is sufficiently large. Therefore,

$$\sup_{y \in \mathbb{R}^3} \int_{y+B_R} \rho_j \,\mathrm{d}x \geqslant R^{-1} \left[-\frac{h_M}{2C} - R^{-1} - R^{-(5-n)/(n+1)} \right].$$

Since $h_M < 0$ the right-hand side of this estimate is positive for R sufficiently large, and the proof of equation (2.25) is complete.

Step 4: *Nonvanishing, weakly convergent minimizing sequences remain concentrated.* In this step we show that a minimizing sequence $(f_j) \subset \mathcal{F}_{MC}$ for $\mathcal{H}$ remains concentrated in the sense that equation (2.15) holds, provided that

$$\int_{B_{R_0}} \rho_j \,\mathrm{d}x \geqslant m_0 \quad \text{and} \quad \rho_j \rightharpoonup \rho_0 \quad \text{weakly in } L^{1+1/n}\big(\mathbb{R}^3\big)$$

for some $R_0 > 0$ and $m_0 > 0$, where $\rho_j := \rho_{f_j}$. Notice that a minimizing sequence, if properly shifted in space, does not vanish by Step 3. Since the shifted minimizing sequence is again minimizing, the induced spatial densities do by Step 1 converge weakly as required after extracting a subsequence.

For $R > R_0$ we split f_j as follows:

$$f_j = f_j^1 + f_j^2 + f_j^3,$$

where

$$f_j^1(x,v)=0 \quad \text{for } |x|\geqslant R_0,$$
$$f_j^2(x,v)=0 \quad \text{for } |x|< R_0 \vee |x|>R,$$
$$f_j^3(x,v)=0 \quad \text{for } |x|\leqslant R.$$

Then

$$\begin{aligned}\mathcal{H}(f_j) &= \mathcal{H}\big(f_j^1\big)+\mathcal{H}\big(f_j^2\big)+\mathcal{H}\big(f_j^3\big)\\ &\quad -\iint \frac{\rho_j^2(x)(\rho_j^1+\rho_j^3)(y)}{|x-y|}\,\mathrm{d}x\,\mathrm{d}y-\iint \frac{\rho_j^1(x)\rho_j^3(y)}{|x-y|}\,\mathrm{d}x\,\mathrm{d}y\\ &=: \mathcal{H}\big(f_j^1\big)+\mathcal{H}\big(f_j^2\big)+\mathcal{H}\big(f_j^3\big)-I_1-I_2,\end{aligned} \tag{2.26}$$

with obvious definitions for $\rho_j^1,\rho_j^2,\rho_j^3$. Since $\|\nabla U_{\rho_j^1+\rho_j^3}\|_2$ is bounded by Step 1,

$$I_1\leqslant C\|\nabla U_{\rho_j^2}\|_2\leqslant C\big(\|\nabla U_{\rho_0^2}\|_2+\|\nabla U_{\rho_j^2}-\nabla U_{\rho_0^2}\|_2\big).$$

For $R>2R_0$ and $|x|\leqslant R_0$, $|y|>R$ we have $|x-y|\geqslant R/2$, and hence

$$I_2\leqslant 2M^2R^{-1}.$$

It is easy to show that $\xi^{7/3}+(1-\xi)^{7/3}\leqslant 1-\frac{7}{3}\xi(1-\xi)$ for $\xi\in[0,1]$. With equation (2.24) and obvious definitions of M_j^1, M_j^2, M_j^3 this implies that

$$\begin{aligned}\mathcal{H}\big(f_j^1\big)+\mathcal{H}\big(f_j^2\big)+\mathcal{H}\big(f_j^3\big) &\geqslant h_{M_j^1}+h_{M_j^2}+h_{M_j^3}\\ &=\left[\left(\frac{M_j^1}{M}\right)^{7/3}+\left(\frac{M_j^2}{M}\right)^{7/3}+\left(\frac{M_j^3}{M}\right)^{7/3}\right]h_M\\ &\geqslant\left[\left(\frac{M_j^1+M_j^2}{M}\right)^{7/3}+\left(\frac{M_j^3}{M}\right)^{7/3}\right]h_M\\ &\geqslant\left[1-\frac{7}{3}\frac{M_j^1+M_j^2}{M}\frac{M_j^3}{M}\right]h_M\\ &\geqslant\left[1-\frac{7}{3}\frac{m_0}{M^2}M_j^3\right]h_M;\end{aligned}$$

in the last estimate we used the nonvanishing property. With (2.26) and the estimates for I_1 and I_2 this implies that

$$C_1m_0M_j^3\leqslant\mathcal{H}(f_j)-h_M+C_2\big[\|\nabla U_{\rho_0^2}\|_2+\|\nabla U_{\rho_j^2}-\nabla U_{\rho_0^2}\|_2+R^{-1}\big].$$

Here $R > 2R_0$ is so far arbitrary, and the constants C_1, C_2 are independent of R and R_0. The first difference on the right-hand side converges to zero since the sequence (f_j) is minimizing. The first term in the bracket can be made as small as we wish by increasing R_0; notice that this does not affect the nonvanishing property. Choosing $R > 2R_0$ large makes the third term in the bracket small. For fixed $R_0 < R$ the middle term converges to zero by Lemma 2.5, since $\rho_j^2 \rightharpoonup \rho_0^2$ weakly in $L^{1+1/n}$ and these functions are supported in B_R. This shows that the sequence (f_j) satisfies the concentration property (2.15) as claimed.

Step 5: *Proof of Theorem* 2.1 *for Version* 2. Let (f_j) be a minimizing sequence and choose $(a_j) \subset \mathbb{R}^3$ according to (2.25). Since $\mathcal{H}$ is translation invariant $(T^{a_j} f_j)$ is again a minimizing sequence which by abuse of notation we denote by (f_j). By Step 1, (f_j) is bounded in $L^{1+1/k}(\mathbb{R}^6)$. Thus there exists a weakly convergent subsequence, again denoted by (f_j): $f_j \rightharpoonup f_0$. Clearly, $f_0 \geqslant 0$ a.e. Again by Step 1, $(E_{\mathrm{kin}}(f_j))$ is bounded, and by weak convergence

$$E_{\mathrm{kin}}(f_0) \leqslant \limsup_{j\to\infty} E_{\mathrm{kin}}(f_j) < \infty.$$

By Step 1, $(\rho_j) = (\rho_{f_j})$ is bounded in $L^{1+1/n}(\mathbb{R}^3)$. After extracting a further subsequence

$$\rho_j \rightharpoonup \rho_0 = \rho_{f_0} \quad \text{weakly in } L^{1+1/n}\big(\mathbb{R}^3\big);$$

it is easy to see that the weak limit of the spatial densities induced by (f_j) is indeed the spatial density induced by the weak limit of (f_j). By Step 4 and Lemma 2.5,

$$\nabla U_{\rho_j} \to \nabla U_0 \quad \text{strongly in } L^2\big(\mathbb{R}^3\big).$$

Hence $\mathcal{H}(f_0) \leqslant \lim_{j\to\infty} \mathcal{H}(f_j)$, and it remains to show that $\int f_0 + \mathcal{C}(f_0) = M$. By ($\Phi$2), Mazur's lemma, and Fatou's lemma

$$M_0 := \iint \big(f_0 + \Phi(f_0)\big)\,\mathrm{d}v\,\mathrm{d}x \leqslant \limsup_{j\to\infty} \iint \big(f_j + \Phi(f_j)\big)\,\mathrm{d}v\,\mathrm{d}x = M,$$

and $M_0 > 0$ since otherwise $f_0 = 0$ in contradiction to $\mathcal{H}(f_0) < 0$. Let

$$b := \left(\frac{M_0}{M}\right)^{2/3}, \qquad a := b^{-1/2},$$

so that by (2.21), $\bar{f}_0 \in \mathcal{F}_{MC}$. Then by (2.23),

$$h_M \leqslant \mathcal{H}\big(\bar{f}_0\big) = a^7 \mathcal{H}(f_0) = \left(\frac{M}{M_0}\right)^{7/3} h_M,$$

which implies that $M_0 \geqslant M$. □

REMARK. Instead of the explicit arguments above one can also employ the concentration–compactness principle due to Lions [79], cf. [96].

2.5. *Minimizers are steady states*

Via the corresponding Euler–Lagrange identity the minimizers obtained by Theorem 2.1 are shown to be steady states of the Vlasov–Poisson system. Once the minimizers are identified as steady states some further properties are investigated. The minimizers obtained for the reduced variational problem in Theorem 2.4 turn out to be steady states of the Euler–Poisson system. This fact and the relation between steady states of the Vlasov–Poisson and of the Euler–Poisson system are postponed to Section 2.7.

THEOREM 2.6. *Let $f_0 \in \mathcal{F}_M$ be a minimizer of $\mathcal{H}_C$ with potential U_0, and define the particle energy as in* (2.1). *Then*

$$f_0(x,v) = \begin{cases} (\Phi')^{-1}(E_0 - E), & E < E_0, \\ 0, & E \geqslant E_0, \end{cases} \quad a.e.$$

with Lagrange multiplier

$$E_0 := \frac{1}{M} \iint \bigl(E + \Phi'(f_0)\bigr) f_0 \, \mathrm{d}v \, \mathrm{d}x.$$

If $f_0 \in \mathcal{F}_{MC}$ is a minimizer of $\mathcal{H}$ then

$$f_0(x,v) = \begin{cases} (\Phi')^{-1}\bigl(\frac{E}{E_0} - 1\bigr), & E < E_0, \\ 0, & E \geqslant E_0, \end{cases} \quad a.e.$$

with Lagrange multiplier

$$E_0 := \frac{\iint E f_0 \, \mathrm{d}v \, \mathrm{d}x}{\iint (1 + \Phi'(f_0)) f_0 \, \mathrm{d}v \, \mathrm{d}x} < 0.$$

In particular, f_0 is in both cases a steady state of the Vlasov–Poisson system.

The Lagrange multiplier E_0 is negative also in case of Version 1, but the proof is different and postponed to Proposition 2.7. The choice (2.6) leads to the polytropic steady state

$$f_0(x,v) = (E_0 - E)_+^k$$

in the case of Version 1, and to a similar formula for Version 2.

PROOF OF THEOREM 2.6. We give the proof for Version 2, since due to the nonlinear nature of the constraint this case is slightly less trivial. Let f_0 and U_0 be a pointwise defined representative of a minimizer of $\mathcal{H}$ in $\mathcal{F}_{MC}$ and of its induced potential respectively. The following abbreviation will be useful:

$$Q(f) := f + \Phi(f), \quad f \geqslant 0.$$

For $\varepsilon > 0$ small,

$$S_\varepsilon := \left\{ (x, v) \in \mathbb{R}^6 \,\middle|\, \varepsilon \leqslant f_0(x, v) \leqslant \frac{1}{\varepsilon} \right\}$$

defines a set of positive, finite measure. Let $w \in L^\infty(\mathbb{R}^6)$ be compactly supported in $S_\varepsilon \cup f_0^{-1}(0)$ and nonnegative outside S_ε, and define

$$G(\sigma, \tau) := \iint Q(f_0 + \sigma \mathbb{1}_{S_\varepsilon} + \tau w)\, \mathrm{d}v\, \mathrm{d}x;$$

for τ and σ close to zero, $\tau \geqslant 0$, the function $f_0 + \sigma \mathbb{1}_{S_\varepsilon} + \tau w$ is bounded on S_ε, and nonnegative. Therefore, G is continuously differentiable for such τ and σ, and $G(0, 0) = M$. Since

$$\partial_\sigma G(0, 0) = \iint_{S_\varepsilon} Q'(f_0)\, \mathrm{d}v\, \mathrm{d}x \neq 0,$$

there exists by the implicit function theorem a continuously differentiable function $\tau \mapsto \sigma(\tau)$ with $\sigma(0) = 0$, defined for $\tau \geqslant 0$ small, such that $G(\sigma(\tau), \tau) = M$. Hence $f_0 + \sigma(\tau)\mathbb{1}_{S_\varepsilon} + \tau w \in \mathcal{F}_{MC}$. Furthermore,

$$\sigma'(0) = -\frac{\partial_\tau G(0, 0)}{\partial_\sigma G(0, 0)} = -\frac{\iint Q'(f_0) w}{\iint_{S_\varepsilon} Q'(f_0)}. \tag{2.27}$$

Since $\mathcal{H}(f_0 + \sigma(\tau)\mathbb{1}_{S_\varepsilon} + \tau w)$ attains its minimum at $\tau = 0$,

$$0 \leqslant \mathcal{H}\big(f_0 + \sigma(\tau)\mathbb{1}_{S_\varepsilon} + \tau w\big) - \mathcal{H}(f_0) = \tau \iint E\big[\sigma'(0)\mathbb{1}_{S_\varepsilon} + w\big]\, \mathrm{d}v\, \mathrm{d}x + \mathrm{o}(\tau)$$

for $\tau \geqslant 0$ small. With (2.27) we get

$$\iint \big[-E_\varepsilon Q'(f_0) + E\big] w\, \mathrm{d}v\, \mathrm{d}x \geqslant 0, \quad E_\varepsilon := \frac{\iint_{S_\varepsilon} E}{\iint_{S_\varepsilon} Q'(f_0)}.$$

By the choice for w this implies that $E = E_\varepsilon Q'(f_0)$ a.e. on S_ε and $E \geqslant E_\varepsilon Q'(f_0)$ a.e. on $f_0^{-1}(0)$. This shows that $E_\varepsilon = E_0$ does in fact not depend on ε. With $\varepsilon \to 0$,

$$E = E_0 Q'(f_0) \qquad \text{a.e. on } f_0^{-1}\big(]0, \infty[\big),$$

$$E \geqslant E_0 Q'(0) = E_0 \quad \text{a.e. on } f_0^{-1}(0).$$

Multiplication of the former by f_0 and integration yields the formula for E_0, and since

$$\iint E f_0 \,\mathrm{d}v\,\mathrm{d}x = E_{\mathrm{kin}}(f_0) + 2E_{\mathrm{pot}}(f_0) < \mathcal{H}(f_0) < 0$$

this Lagrange multiplier is negative; such a direct argument does not seem to work for Version 1 of the variational problem. □

The minimizers are steady states of the Vlasov–Poisson system in the following sense: By definition, U_0 is the gravitational potential induced by f_0. On the other hand, for a time independent potential the particle energy, and hence any function of the particle energy, is constant along characteristics and in this sense satisfies the Vlasov equation. The problem is that U_0 should be sufficiently smooth for the characteristic equations to have well-defined solutions.

PROPOSITION 2.7. *Let f_0 be a minimizer of $\mathcal{H}$ or $\mathcal{H}_C$ as obtained in Theorem* 2.1 *with induced spatial density ρ_0. Alternatively, let ρ_0 be a minimizer of $\mathcal{H}_r$ as obtained in Theorem* 2.4. *Let U_0 be the induced potential. Then the following holds*:

(a) *the functions ρ_0 and U_0 are spherically symmetric with respect to some point in $\mathbb{R}^3$, and ρ_0 is decreasing as a function of the radial variable*;

(b) $\rho_0 \in C_c(\mathbb{R}^3)$, $U_0 \in C^2(\mathbb{R}^3)$ *with* $\lim_{|x|\to\infty} U_0(x) = 0$, *and* $E_0 < 0$. *If ρ_0 comes from a minimizer of $\mathcal{H}_C$ or $\mathcal{H}$ then $\rho_0 \in C_c^1(\mathbb{R}^3)$. Minimizers f_0 are compactly supported also with respect to v.*

PROOF. In order to prove part (a) we consider first the case that $\rho_0 \in \mathcal{R}_M$ is a minimizer of the reduced functional $\mathcal{H}_r$ as obtained in Theorem 2.4. As observed in Step 5 of the proof of that theorem ρ_0 is spherically symmetric with respect to some point in $\mathbb{R}^3$, and decreasing as a function of the radial variable. Let f_0 be a minimizer of the energy-Casimir functional $\mathcal{H}_C$. Then the assertions for ρ_0 and U_0 remain true, since f_0 arises from a minimizer of the reduced functional by the lifting process in Theorem 2.2(b).

To prove the spherical symmetry of a minimizer f_0 of the energy $\mathcal{H}$ we denote by f_0^* its spherically symmetric rearrangement with respect to x. Arguing as above, $f_0(x,v) = f_0^*(x + a_v, v)$ for some possibly v-dependent shift vector a_v. Since both f_0 and f_0^* are minimizers they are both of the form stated in Theorem 2.6, so $E_0 \Phi'(f_0(x,v)) = \frac{1}{2}|v|^2 + U_{f_0}(x) - E_0$ and $E_0^* \Phi'(f_0^*(x,v)) = \frac{1}{2}|v|^2 + U_{f_0^*}(x) - E_0^*$. The explicit form of E_0 implies that $E_0 = E_0^*$, hence $U_{f_0}(x) = U_{f_0^*}(x + a_v)$, and a_v is independent of v. Hence the minimizer f_0 is a spatial translation of f_0^*, which proves the symmetry assertion.

As to part (b), we note first that by Theorem 2.6 a minimizer f_0 obtained in Theorem 2.1 satisfies a relation of the form

$$f_0(x,v) = \phi\big(E(x,v)\big)$$

with ϕ determined by the function Φ and E_0. This in turn implies a relation between ρ_0 and U_0,

$$\rho_0(x) = h\big(U_0(x)\big) := 4\pi\sqrt{2}\int_{U_0(x)}^{\infty} \phi(E)\sqrt{E - U_0(x)}\,\mathrm{d}E, \tag{2.28}$$

where h is continuously differentiable. For a minimizer ρ_0 of $\mathcal{H}_r$ such a relation holds by (2.12); in this case h is determined by Ψ and need only be continuous.

Let $\rho_0 \in L^p_+(\mathbb{R}^3)$ for some $p > 1$, and as usual $1/p + 1/q = 1$. For any $R > 1$ we split the convolution integral defining U_0 according to $|x - y| < 1/R$, $1/R \leqslant |x - y| < R$, and $|x - y| \geqslant R$ to obtain

$$-U_0(x) \leqslant C\|\rho_0\|_p \left(\int_0^{1/R} r^{2-q}\,\mathrm{d}r\right)^{1/q} + R\int_{|y|\geqslant|x|-R} \rho_0(y)\,\mathrm{d}y + \frac{M}{R}.$$

This implies that $U_0 \in L^\infty(\mathbb{R}^3)$ with $U_0(x) \to 0$, $|x| \to \infty$, provided $q < 3$, i.e., $p > 3/2$. Assume for the moment that this is true. Then by (2.28), $\rho_0 \in L^1 \cap L^\infty(\mathbb{R}^3)$. By spherical symmetry,

$$U_0(r) = -\frac{4\pi}{r}\int_0^r s^2\rho_0(s)\,\mathrm{d}s - 4\pi\int_r^\infty s\rho_0(s)\,\mathrm{d}s, \quad U_0'(r) = \frac{4\pi}{r^2}\int_0^r s^2\rho_0(s)\,\mathrm{d}s,$$

where $r = |x|$, in particular U_0 is continuous. Again by (2.28), ρ_0 is continuous as well, and the formulas above imply the asserted regularity of U_0.

If $0 < n < 2$ then $p := 1 + 1/n > 3/2$, and the regularity assumptions are established. If $2 \leqslant n < 5$ a little more work is required. By the assumptions (Φ1) and (Φ2) and the mean value theorem,

$$\Phi'(f) \geqslant \Phi'(\tau) = \frac{\Phi(f) - \Phi(0)}{f - 0} \geqslant Cf^{1/k}$$

for all f large, with some intermediate value $0 \leqslant \tau \leqslant f$. Similarly, for ρ large,

$$\Psi'(\rho) \geqslant C\rho^{1/n}.$$

In both cases the relation (2.28) together with these estimates imply that

$$\rho_0(x) \leqslant C\big(1 + \big(E_0 - U_0(x)\big)_+^n\big).$$

If we use this estimate on the set of finite measure where ρ_0 is large and the integrability of ρ_0 on the complement we find that

$$\int \rho_0(x)^p\,\mathrm{d}x \leqslant C + C\int \big(-U_0(x)\big)^{np}\,\mathrm{d}x. \tag{2.29}$$

Starting with $p_0 = 1 + 1/n$ we apply Lemma P2(a) to find that U_0 lies in L^q with $q = (1/p_0 - 2/3)^{-1} > 1$, and substituting this into (2.29) we conclude that $\rho_0 \in L^{p_1}$ with $p_1 = q/n$; note that by assumption $p_0 < 3/2$. If $p_1 > 3/2$ we are done. If $p_1 = 3/2$ we decrease p_1 slightly – note that $\rho_0 \in L^1$ – so that in the next step we find p_2 as large as we wish. If $p_1 < 3/2$ we repeat the process. By induction,

$$p_k = \frac{3(1+1/n)(n-1)}{n^k(n-5)+2n+2} > 1$$

as long as $p_{k-1} < 3/2$. But since $2 \leqslant n < 5$ the denominator would eventually become negative so that the process must stop after finitely many steps, and again $\rho_0 \in L^p(\mathbb{R}^3)$ for some $p > 3/2$.

The minimizer of $\mathcal{H}_C$ or $\mathcal{H}_r$ obtained in Theorem 2.1 or Theorem 2.4 has compact support by Step 3 of the proof of the latter theorem. The limiting behavior of U_0 together with Theorem 2.6 implies that $E_0 < 0$.

For Version 2 of the variational problem $E_0 < 0$ by Theorem 2.6. Hence $\lim_{|x|\to\infty} U_0(x) = 0$ implies that for $|x|$ sufficiently large, $E(x,v) > E_0$, and by Theorem 2.6, f_0 and ρ_0 have compact support also in this case. □

A question which is of interest in itself and which is also relevant for the stability discussion in the next section is the possible uniqueness or nonuniqueness of the minimizer. So far, only preliminary results in this direction exist.

REMARK. (a) Consider the polytropic case $\Phi(f) = f^{1+1/k}$ or $\Psi(\rho) = \rho^{1+1/n}$, respectively. If $0 < k < 3/2$ or $0 < n < 3$ then up to spatial translations the functional $\mathcal{H}_C$ or $\mathcal{H}_r$ has exactly one minimizer with prescribed mass $M > 0$. If $0 < k < 7/2$ then up to spatial translations the energy $\mathcal{H}$ has at most two minimizers in the constraint set $\mathcal{F}_{MC}$.

We show the uniqueness assertion under the mass constraint; the proof under the mass-Casimir constraint is more technical, cf. [42], Theorem 3. Up to some shift U_0 as a function of the radial variable $r := |x|$ solves the equation

$$\frac{1}{r^2}\bigl(r^2 U_0'\bigr)' = c(E_0 - U_0)_+^n, \quad r > 0, \tag{2.30}$$

with some appropriately defined constant $c > 0$. The function $E_0 - U_0$ is a solution of the singular ordinary differential equation

$$\frac{1}{r^2}\bigl(r^2 z'\bigr)' = -c z_+^n, \quad r > 0. \tag{2.31}$$

Solutions $z \in C([0,\infty[) \cap C^2(]0,\infty[)$ of (2.31) with z' bounded near $r = 0$ are uniquely determined by $z(0)$. If z is such a solution then so is

$$z_\alpha(r) := \alpha z\bigl(\alpha^\gamma r\bigr), \quad r \geqslant 0,$$

for any $\alpha > 0$ where $\gamma := (n-1)/2$, and $z_\alpha(0) = \alpha z(0)$. Now assume there exists another minimizer with mass M, i.e., up to a shift another solution U_1 of (2.30) with cut-off energy E_1. Uniqueness for (2.31) yields some $\alpha > 0$ such that

$$E_1 - U_1(r) = \alpha E_0 - \alpha U_0\big(\alpha^\gamma r\big), \quad r \geqslant 0.$$

However, both steady states have the same mass M, so that

$$\begin{aligned} M &= c\int_0^\infty r^2\big(E_1 - U_1(r)\big)_+^n \,\mathrm{d}r \\ &= \alpha^{n-3\gamma} c\int_0^\infty r^2\big(E_0 - U_0(r)\big)_+^n \,\mathrm{d}r = \alpha^{n-3\gamma} M. \end{aligned}$$

For $0 < n < 3$ the exponent of α is not zero, hence $\alpha = 1$, and considering limits at spatial infinity we conclude that $E_0 = E_1$ and $U_0 = U_1$.

(b) Let Ψ be such that $\Psi(0) = 0$ and

$$\Psi'(\rho) = \begin{cases} \rho, & 0 \leqslant \rho \leqslant 1, \\ \rho^{1/10}, & 1 < \rho < 10, \\ 10^{-9/10}\rho, & 10 \leqslant \rho. \end{cases}$$

This function satisfies (Ψ1), (Ψ2), (Ψ3); note however that the exponent used for $1 < \rho < 10$ corresponds to $n = 10$ which is well outside of the required range $0 < n < 3$. If the resulting equation for $z = E_0 - U_0$ is solved numerically then the choices $z(0) = 0.522, 1.641, 2.364$ give three different steady states with the same mass $M = 0.462$. The minimizers of $\mathcal{H}_r$ for this value of M must be among these three states, and it turns out that the values of $\mathcal{H}_r$ resulting from $z(0) = 0.522, 2.364$ are equal and smaller than the one resulting from $z(0) = 1.641$. Hence for this example there are two distinct minimizers. Clearly, this also provides a counterexample to uniqueness of the minimizer for the energy-Casimir functional $\mathcal{H}_C$.

A similar example of nonuniqueness of the minimizer of $\mathcal{H}$ is reported in [107], Section 5. We have found no numerical indication that under our general assumptions there might be infinitely many minimizers. In particular, minimizers always seem to be isolated.

2.6. *Dynamical stability*

We now come to the stability assertion for the steady states which are obtained as minimizers above. To this end we first rewrite the Taylor expansion of the energy or energy-Casimir functional, respectively.

REMARK. (a) In case of Version 1,

$$\mathcal{H}_C(f) - \mathcal{H}_C(f_0) = d(f, f_0) - \frac{1}{8\pi}\int |\nabla U_f - \nabla U_0|^2 \,\mathrm{d}x, \tag{2.32}$$

where for $f \in \mathcal{F}_M$,

$$
\begin{aligned}
d(f, f_0) &:= \iint \big[\Phi(f) - \Phi(f_0) + E(f - f_0)\big] \,\mathrm{d}v\,\mathrm{d}x \\
&= \iint \big[\Phi(f) - \Phi(f_0) + (E - E_0)(f - f_0)\big] \,\mathrm{d}v\,\mathrm{d}x \\
&\geqslant \iint \big[\Phi'(f_0) + (E - E_0)\big](f - f_0) \,\mathrm{d}v\,\mathrm{d}x \geqslant 0
\end{aligned}
$$

with $d(f, f_0) = 0$ iff $f = f_0$.

(b) In case of Version 2,

$$
\mathcal{H}(f) - \mathcal{H}(f_0) = d(f, f_0) - \frac{1}{8\pi} \int \big|\nabla U_f - \nabla U_0\big|^2 \,\mathrm{d}x, \tag{2.33}
$$

where for $f \in \mathcal{F}_{MC}$,

$$
\begin{aligned}
d(f, f_0) &:= \iint E(f - f_0) \,\mathrm{d}v\,\mathrm{d}x \\
&= \iint \big[(-E_0)\big(\Phi(f) - \Phi(f_0)\big) + (E - E_0)(f - f_0)\big] \,\mathrm{d}v\,\mathrm{d}x \\
&\geqslant \iint \big[(-E_0)\Phi'(f_0) + (E - E_0)\big](f - f_0) \,\mathrm{d}v\,\mathrm{d}x \geqslant 0
\end{aligned}
$$

with $d(f, f_0) = 0$ iff $f = f_0$.

This is due to the strict convexity of Φ, and the fact that on the support of f_0 the bracket vanishes by Theorem 2.6; note also that in the second equality we added a zero due to the respective constraint.

THEOREM 2.8. *Let f_0 be a minimizer as obtained in Theorem* 2.1, *in case of Version* 1 *assume that the minimizer is unique or at least isolated up to shifts in x. Then the following nonlinear stability assertion holds*:

For any $\varepsilon > 0$ there exists a $\delta > 0$ such that for any classical solution $t \mapsto f(t)$ of the Vlasov–Poisson system with $f(0) \in C_c^1(\mathbb{R}^6) \cap \mathcal{F}_M$ or $f(0) \in C_c^1(\mathbb{R}^6) \cap \mathcal{F}_{MC}$, respectively, the initial estimate

$$
d\big(f(0), f_0\big) + \frac{1}{8\pi} \int |\nabla U_{f(0)} - \nabla U_0|^2 \,\mathrm{d}x < \delta
$$

implies that for any $t \geqslant 0$ there is a shift vector $a \in \mathbb{R}^3$ such that

$$
d\big(T^a f(t), f_0\big) + \frac{1}{8\pi} \int \big|T^a \nabla U_{f(t)} - \nabla U_0\big|^2 \,\mathrm{d}x < \varepsilon, \quad t \geqslant 0.
$$

As above, $T^a f(x, v) := f(x + a, v)$.

PROOF. Let us first assume that the minimizer is unique up to spatial translations, and let us consider Version 1 first. Assume the assertion is false. Then there exist $\varepsilon > 0$, $t_j > 0$, $f_j(0) \in C^1_c(\mathbb{R}^6) \cap \mathcal{F}_M$ such that for $j \in \mathbb{N}$,

$$d\big(f_j(0), f_0\big) + \frac{1}{8\pi} \int |\nabla U_{f_j(0)} - \nabla U_0|^2 \, dx < \frac{1}{j},$$

but for any shift vector $a \in \mathbb{R}^3$,

$$d\big(T^a f_j(t_j), f_0\big) + \frac{1}{8\pi} \int \big|T^a \nabla U_{f_j(t_j)} - \nabla U_0\big|^2 \, dx \geqslant \varepsilon.$$

Since $\mathcal{H}_C$ is conserved, (2.32) and the assumption on the initial data imply that $\mathcal{H}_C(f_j(t_j)) = \mathcal{H}_C(f_j(0)) \to \mathcal{H}_C(f_0)$, i.e., $(f_j(t_j)) \subset \mathcal{F}_M$ is a minimizing sequence. Hence by Theorem 2.1, $\int |\nabla U_{f_j(t_j)} - \nabla U_0|^2 \to 0$ up to subsequences and shifts in x, provided that there is no other minimizer to which this sequence can converge. By (2.32), $d(f_j(t_j), f_0) \to 0$ as well, which is the desired contradiction. If the minimizer is unique up to shifts, the proof for Version 2 is completely analogous.

By definition, we call the minimizer isolated up to spatial translations if

$$\inf\big\{\|\nabla U_{f_0} - \nabla U_{\tilde{f}_0}\|_2 \;\big|\; \tilde{f}_0 \in \mathcal{M}_M \setminus \big\{T^a f_0 \mid a \in \mathbb{R}^3\big\}\big\} > 0,$$

where $\mathcal{M}_M$ denotes the set of all minimizers of the given functional under the given constraint. The argument above then has to be combined with a continuity argument to show that the assertion of the theorem still holds true, cf. [98], p. 124. For Version 2 a much less trivial argument due to Schaeffer [107] shows that the theorem remains true even if the minimizer is not isolated. □

The spatial shifts appearing in the stability statement are again due to the spatial invariance of the system. If f_0 is perturbed by giving all the particles an additional, fixed velocity, then in space the corresponding solution travels off from f_0 at a linear rate in t, no matter how small the perturbation. Hence without the spatial shifts the assertion of the theorem is false. A stability result of this type is sometimes referred to as *orbital stability*, cf. [72,104].

A weak point of the present approach is the fact that the proof is not constructive – given ε it is not known how small the corresponding δ must be. A nice feature of the result is that the same quantity is used to measure the deviation initially and at later times t. In infinite-dimensional dynamical systems initial control in a strong norm can be necessary to gain control in a weaker norm at later times. On the other hand, it certainly is desirable to achieve the stability estimate also in some norm for f. In [72,104] results in this direction are obtained by changing the variational approach. However, such improvements are easily obtained within the framework presented here. To see this we have to think for a moment about what perturbations are admissible from a physics point of view.

Remark on dynamically accessible perturbations. A galaxy in equilibrium represented by a steady state f_0 is typically perturbed by the gravitational pull of some (distant) outside object like a neighboring galaxy. This means that an external force acts on the particles in addition to the self-consistent one. The resulting perturbation simply consists in a reshuffling of the particles in phase space. Hence a physically natural class of perturbations are all states f which are equimeasurable to f_0, $f \sim f_0$, which by definition means that

$$\forall \tau \geqslant 0: \quad \mathrm{vol}\big(\big\{(x,v) \in \mathbb{R}^6 \,\big|\, f(x,v) > \tau\big\}\big) = \mathrm{vol}\big(\big\{(x,v) \in \mathbb{R}^6 \,\big|\, f_0(x,v) > \tau\big\}\big).$$

Notice that this class is invariant under the Vlasov–Poisson system. Clearly, if $f \sim f_0$ then $\|f\|_p = \|f_0\|_p$ for any $p \in [1, \infty]$.

With this remark in mind we arrive at the following stronger stability result; notice that we need not even exploit the full strength of the restriction on the perturbations introduced above.

COROLLARY 2.9. *If in Theorem* 2.8 *the assumption* $\|f(0)\|_{1+1/k} = \|f_0\|_{1+1/k}$ *is added then for any* $\varepsilon > 0$ *the parameter* $\delta > 0$ *can be chosen such that the additional stability estimate*

$$\big\|T^a f(t) - f_0\big\|_{1+1/k} < \varepsilon, \quad t \geqslant 0,$$

holds. If $K > 0$ *is such that* $\mathrm{vol}(\mathrm{supp}\, f_0) < K$, *then this stability estimate holds with any* $p \in [1, 1 + 1/k]$ *instead of* $1 + 1/k$, *provided the perturbations satisfy the additional restriction* $\mathrm{vol}(\mathrm{supp}\, f(0)) < K$. *If* $K > 0$ *is such that* $\|f_0\|_\infty < K$ *and the perturbations satisfy the restriction* $\|f(0)\|_\infty < K$ *then the same is true for any* $p \in [1 + 1/k, \infty[$.

PROOF. We repeat the proof of Theorem 2.8 except that in the contradiction assumption we have

$$\big\|T^a f_j(t_j) - f_0\big\|_{1+1/k} + d\big(T^a f_j(t_j), f_0\big) + \frac{1}{8\pi} \int \big|T^a \nabla U_{f_j(t_j)} - \nabla U_0\big|^2 \,\mathrm{d}x \geqslant \varepsilon.$$

Now we observe that from the minimizing sequence $(f_j(t_j))$ obtained in that proof we can extract a subsequence which converges weakly in $L^{1+1/k}$ to f_0 by Theorem 2.1. But due to our additional restriction on the perturbations

$$\big\|f_j(t_j)\big\|_{1+1/k} = \|f_0\|_{1+1/k}, \quad j \in \mathbb{N}.$$

By the Radon–Riesz–Theorem [74], Theorem 2.11, this implies that $f_j(t_j) \to f_0$ strongly in $L^{1+1/k}$. Together with the rest of the proof of Theorem 2.8 this proves the first assertion. Under the additional restriction on the perturbations, $\mathrm{vol}(\mathrm{supp}\, f(t)) < K$ or $\|f(t)\|_\infty < K$ for all times, and the additional assertions follow by Hölder's inequality and interpolation. □

Concluding remarks. (a) The conditions for stability are formulated in terms of the Casimir function Φ, but they can be translated into conditions on the steady state $f_0 = \phi(E)$. In particular, the crucial assumption that Φ be strictly convex means that ϕ is strictly decreasing on its support.

(b) In [47] spherically symmetric steady states depending also on the modulus of angular momentum L, defined in (2.3), are dealt with. In [49] axially symmetric minimizers depending on the particle angular momentum corresponding to the axis of symmetry are considered. The method yields stability against perturbations which respect the symmetry, but not against general perturbations. This is due to the fact that the function Φ in the Casimir functional $\mathcal{C}$ must in these cases depend on the additional particle invariant, and hence the Casimir functional is preserved only along solutions with the proper symmetry. Stability of nonisotropic steady states which for example depend also on L against nonsymmetric perturbations is an interesting open problem, in particular, since in view of the above discussion of dynamically accessible perturbations symmetry restrictions are unphysical and at best are mathematical stepping stones toward more satisfactory results.

(c) A similar problem arises with flat steady states where all the particles are restricted to a plane. They are used as models for extremely flattened, disk-like galaxies. Their stability was investigated by variational techniques in [94], but the perturbations were restricted to live in the plane.

(d) So far, no rigorous instability results are known in the stellar dynamics case for steady states which violate the stability conditions. Such results do exist in the plasma physics case, cf. [51,52,75–78].

(e) By similar techniques a preliminary result toward stability was established for the Vlasov–Einstein system [113].

(f) The above stability result brings up a question concerning the initial value problem for the Vlasov–Poisson system: Can one extend the class of admissible initial data in such a way that it contains the steady states considered above? Notice that f_0 need not be continuously differentiable. Going further one might wish to admit all dynamically accessible perturbations originating from these steady states as initial data. It is not hard to establish stability results within the context of weak solutions in the sense that the stability estimates then hold for such weak solutions which are obtained as limits of solutions to certain regularized systems, cf. [67,72]. However, due to the inherent nonuniqueness such a formulation is unsatisfactory. It is therefore desirable to have a global existence and uniqueness result which covers these states as initial data and provides solutions which preserve all the conserved quantities, cf. [114].

(g) An ansatz of the form

$$f_0(x, v) = \left(\mathrm{e}^{E_0 - E} - 1\right)_+$$

also leads to a steady state with compact support and finite mass, cf. [102]. This so-called King model is important in astrophysics, but since the corresponding Casimir function

$$\Phi(f) = (1 + f)\ln(1 + f) - f$$

does not satisfy the growth condition ($\Phi2$), it cannot be dealt with by the above variational approach. A nonvariational approach which covers the King model has recently been developed in [50].

2.7. *The reduced variational problem and the Euler–Poisson system*

So far the reduced variational problem played the role of a mathematical device. In the present section we demonstrate that the reduction procedure is much more than that: It points to a deep connection between the Vlasov–Poisson and the Euler–Poisson systems on the level of their steady states and their stability.

If $\rho_0 \in \mathcal{R}_M$ minimizes the reduced functional $\mathcal{H}_r$, then ρ_0 supplemented with the velocity field $u_0 = 0$ is a steady state of the Euler–Poisson system

$$\partial_t \rho + \operatorname{div}(\rho u) = 0,$$
$$\rho \partial_t u + \rho (u \cdot \partial_x) u = -\partial_x p - \rho \partial_x U,$$
$$\Delta U = 4\pi \rho, \qquad \lim_{|x| \to \infty} U(t, x) = 0,$$

with equation of state

$$p = P(\rho) := \rho \Psi'(\rho) - \Psi(\rho).$$

This follows from the Euler–Lagrange identity (2.12). Here u and p denote the velocity field and the pressure of an ideal, compressible fluid with mass density ρ, and the fluid self-interacts via its induced gravitational potential U. This system is sometimes used as a simple model for a gaseous, barotropic star. The beautiful thing now is that the state $(\rho_0, u_0 = 0)$ obviously minimizes the energy

$$\mathcal{H}(\rho, u) := \frac{1}{2} \int |u|^2 \rho \, \mathrm{d}x + \int \Psi(\rho) \, \mathrm{d}x + E_{\mathrm{pot}}(\rho)$$

of the system, which is a conserved quantity. Expanding as before we find that

$$\mathcal{H}(\rho, u) - \mathcal{H}(\rho_0, 0) = \frac{1}{2} \int |u|^2 \rho \, \mathrm{d}x + d(\rho, \rho_0) - \frac{1}{8\pi} \int |\nabla U_\rho - \nabla U_0|^2 \, \mathrm{d}x,$$

where for $\rho \in \mathcal{R}_M$,

$$d(\rho, \rho_0) := \int \big[\Psi(\rho) - \Psi(\rho_0) + (U_0 - E_0)(\rho - \rho_0)\big] \, \mathrm{d}x \geqslant 0,$$

with equality iff $\rho = \rho_0$. The same proof as for the Vlasov–Poisson system implies a stability result for the Euler–Poisson system – the term with the unfavorable sign in the expansion again tends to zero along minimizing sequences, cf. Theorem 2.4. However, there is an important caveat: While for the Vlasov–Poisson system we have global-in-time solutions for sufficiently nice data, and these solutions really preserve all the conserved quantities, no such result is available for the Euler–Poisson system, and we only obtain a

CONDITIONAL STABILITY RESULT. For every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every solution $t \mapsto (\rho(t), u(t))$ with $\rho(0) \in \mathcal{R}_M$ which preserves energy and mass the initial estimate

$$\frac{1}{2}\int \left|u(0)\right|^2 \rho(0)\,\mathrm{d}x + d\big(\rho(0), \rho_0\big) + \frac{1}{8\pi}\int |\nabla U_{\rho(0)} - \nabla U_0|^2\,\mathrm{d}x < \delta$$

implies that as long as the solution exists,

$$\frac{1}{2}\int \left|u(t)\right|^2 \rho(t)\,\mathrm{d}x + d\big(\rho(t), \rho_0\big) + \frac{1}{8\pi}\int |\nabla U_{\rho(t)} - \nabla U_0|^2\,\mathrm{d}x < \varepsilon$$

up to shifts in x and provided the minimizer is unique up to such shifts.

The same comments as on Theorem 2.8 apply. Because of the above caveat we prefer not to call this a theorem, although as far as the stability analysis itself is concerned it is perfectly rigorous. The open problem is whether a suitable concept of solution to the initial value problem exists.

The relation between the fluid and the kinetic steady states. Now that minimizers of the reduced functional are identified as stable steady states of the Euler–Poisson system it is instructive to reconsider the reduction procedure leading from the kinetic to the fluid dynamics picture. First we recall that for the Legendre transform $\bar{h}$ of a function h the following holds:

$$h'(\xi) = \eta \quad\Longleftrightarrow\quad h(\xi) + \bar{h}(\eta) = \xi\eta \quad\Longleftrightarrow\quad \big(\bar{h}\big)'(\eta) = \xi.$$

If f_0 is a minimizer of $\mathcal{H}_C$,

$$f_0 = \big(\Phi'\big)^{-1}(E_0 - E) = \big(\overline{\Phi}\big)'(E_0 - E),$$
$$\rho_0 = \int f_0\,\mathrm{d}v = \int \big(\overline{\Phi}\big)'\left(E_0 - U_0 - \frac{1}{2}|v|^2\right)\mathrm{d}v,$$

and

$$p_0 = \frac{1}{3}\int |v|^2 f_0\,\mathrm{d}v = \int \overline{\Phi}\left(E_0 - U_0 - \frac{1}{2}|v|^2\right)\mathrm{d}v$$

is the induced, isotropic pressure. On the other hand, if ρ_0 is a minimizer of the reduced functional $\mathcal{H}_r$,

$$\rho_0 = \big(\Psi'\big)^{-1}(E_0 - U_0) = \big(\overline{\Psi}\big)'(E_0 - U_0),$$
$$p_0 = P(\rho_0) = \rho_0\Psi'(\rho_0) - \Psi(\rho_0) = \overline{\Psi}\big(\Psi'(\rho_0)\big) = \overline{\Psi}(E_0 - U_0).$$

In both the kinetic and the fluid picture the spatial density and the pressure are functionals of the potential, and these functional relations on the kinetic and on the fluid level fit provided

$$\overline{\Psi}(\lambda) = \int \overline{\Phi}\left(\lambda - \frac{1}{2}|v|^2\right) \mathrm{d}v,$$

which is exactly the relation between Φ and Ψ obtained by the reduction mechanism.

The threshold $k = 3/2, n = 3$. It is worthwhile to review the role of the threshold $k = 3/2$, $n = 3$ in the context of the relation between the Vlasov–Poisson and the Euler–Poisson system. For the Vlasov–Poisson system the Casimir functional $\mathcal{C}$ is preserved, and hence it is possible to incorporate $\mathcal{C}$ into the functional to be minimized or into the constraint. We have seen that the former approach, which allows for reduction, works only for $0 < k < 3/2$ while the latter works for $0 < k < 7/2$. That this is not just a mathematical technicality can be seen from the following observation.

By Theorem 2.2 the energy-Casimir functional in the kinetic picture equals the energy functional in the fluid picture in the case of a minimizer. For polytropes, $\operatorname{sign} \mathcal{H}_C(f_0) = \operatorname{sign}(n-3)$, i.e., the energy-Casimir functional in the kinetic and the energy in the fluid picture changes sign at $n = 3$. The energy $\mathcal{H}$ in the kinetic picture however remains negative for $0 < k < 7/2$. Secondly, if the perturbation of a steady state has positive energy then this perturbation is unstable in the sense of Corollary 1.16 from Chapter 1. An analogous result holds for the Euler–Poisson system, cf. [19]. Hence stability is lost for the Euler–Poisson system at $n = 3$ and so reduction in the sense we used it cannot work for $k \geqslant 3/2$.

A nonlinear instability result for the Euler–Poisson system with equation of state $p = A\rho^{6/5}$ was recently established in [64]. Notice that this equation of state corresponds to $n = 5$, which is well outside the range of stability which was established above for the Euler–Poisson case. On the other hand, the corresponding Vlasov–Poisson steady state $f_0(x, v) = c(-E)_+^{7/2}$, the so-called Plummer sphere, is stable, cf. [48], Section 6. That this state is the minimizer of the energy under an appropriate constraint had been observed earlier in [1], cf. also [2].

References

[1] J.J. Aly, *On the lowest energy state of a collisionless selfgravitating system under phase space volume constraints*, Mon. Not. R. Astron. Soc. **241** (1989), 15–27.

[2] J.J. Aly, *Existence of a minimum energy state for a constrained collisionless gravitational system*, Preprint (2005).

[3] H. Andréasson, *The Einstein–Vlasov system/kinetic theory*, Living Reviews in Relativity, Max Planck Institute for Gravitational Physics (2005); available at *http://www.livingreviews.org/lrr-2005-2*.

[4] A.A. Arsen'ev, *Global existence of a weak solution of Vlasov's system of equations*. Comput. Math. Math. Phys. **15** (1975), 131–141.

[5] C. Bardos and P. Degond, *Global existence for the Vlasov–Poisson system in 3 space variables with small initial data*, Ann. Inst. H. Poincaré Anal. Non Linéaire **2** (1985), 101–118.

[6] J. Batt, *Ein Existenzbeweis für die Vlasov–Gleichung der Stellardynamik bei gemittelter Dichte*, Arch. Ration. Mech. Anal. **13** (1963), 296–308.

[7] J. Batt, *Global symmetric solutions of the initial value problem in stellar dynamics*, J. Differential Equations **25** (1977), 342–364.

[8] J. Batt, H. Berestycki, P. Degond and B. Perthame, *Some families of solutions of the Vlasov–Poisson system*, Arch. Ration. Mech. Anal. **104** (1988), 79–103.

[9] J. Batt, W. Faltenbacher and E. Horst, *Stationary spherically symmetric models in stellar dynamics*, Arch. Ration. Mech. Anal. **93** (1986), 159–183.

[10] J. Batt, M. Kunze and G. Rein, *On the asymptotic behaviour of a one-dimensional, monocharged plasma*, Adv. Differential Equations **3** (1998), 271–292.

[11] J. Batt, P. Morrison and G. Rein, *Linear stability of stationary solutions of the Vlasov–Poisson system in three dimensions*, Arch. Ration. Mech. Anal. **130** (1995), 163–182.

[12] J. Batt and G. Rein, *Global classical solutions of the periodic Vlasov–Poisson system in three dimensions*, C. R. Acad. Sci. Paris **313** (1991), 411–416.

[13] J. Binney and S. Tremaine, *Galactic Dynamics*, Princeton Univ. Press, Princeton, NJ (1987).

[14] F. Bouchut, *Existence and uniqueness of a global smooth solution for the Vlasov–Poisson–Fokker–Planck system in three dimensions*, J. Funct. Anal. **111** (1993), 239–258.

[15] F. Bouchut, F. Golse and C. Pallard, *Classical solutions and the Glassey–Strauss theorem for the 3d Vlasov–Maxwell system*, Arch. Ration. Mech. Anal. **170** (2003), 1–15.

[16] A. Burchard and Y. Guo, *Compactness via symmetrization*, J. Funct. Anal. **214** (2004), 40–73.

[17] S. Calogero, *Global classical solutions to the 3D Nordström–Vlasov system*, Preprint (2005); available at *arXiv:math-ph/0507030*.

[18] C. Cercignani, R. Illner and M. Pulvirenti, *The Mathematical Theory of Dilute Gases*, Springer-Verlag, New York (1994).

[19] Y. Deng, T.-P. Liu, T. Yang and Z.-A. Yao, *Solutions of Euler–Poisson equations for gaseous stars*, Arch. Ration. Mech. Anal. **164** (2002), 261–285.

[20] C. Dietz and V. Sandor, *The hydrodynamical limit of the Vlasov–Poisson system*, Transport Theory Statist. Phys. **28** (1999), 499–520.

[21] R.J. DiPerna and P.-L. Lions, Global weak solutions of Vlasov–Maxwell systems. *Comm. Pure Appl. Math.* **XLII** (1989), 729–757.

[22] R.J. DiPerna and P.-L. Lions, *On the Cauchy problem for Boltzmann equations*: *Global existence and weak stability*, Ann. of Math. **130** (1989), 321–366.

[23] J. Dolbeault and G. Rein, *Time-dependent rescalings and Lyapunov functionals for the Vlasov–Poisson and Euler–Poisson systems*, *and for related models of kinetic equations*, *fluid dynamics and quantum physics*, Math. Models Methods Appl. Sci. **11** (2001), 407–432.

[24] J. Dolbeault, O. Sanchez and J. Soler, *Asymptotic behaviour for the Vlasov–Poisson system in the stellar-dynamics case*, Arch. Ration. Mech. Anal. **171** (2004), 301–327.

[25] J. Ehlers and W. Rienstra, *The locally isotropic solutions of the Liouville and Poisson equations*, *Astrophys. J.* **155** (1969), 105–116.

[26] A.M. Fridman and V.L. Polyachenko, *Physics of Gravitating Systems I*, Springer-Verlag, New York (1984).

[27] I. Gasser, P.-E. Jabin and B. Perthame, *Regularity and propagation of moments in some nonlinear Vlasov systems*, Proc. Roy. Soc. Edinburgh Sect. A **130** (2000), 1259–1273.

[28] R.T. Glassey, *The Cauchy Problem in Kinetic Theory*, SIAM, Philadelphia, PA (1996).

[29] R.T. Glassey and J. Schaeffer, *On symmetric solutions of the relativistic Vlasov–Poisson system*, Comm. Math. Phys. **101** (1985), 459–473.

[30] R.T. Glassey and J. Schaeffer, *Global existence of the relativistic Vlasov–Maxwell system with nearly neutral initial data*, Comm. Math. Phys. **119** (1988), 353–384.

[31] R.T. Glassey and J. Schaeffer, *On the* "*one and one-half dimensional*" *relativistic Vlasov–Maxwell system*, Math. Methods Appl. Sci. **13** (1990), 169–179.

[32] R.T. Glassey and J. Schaeffer, *The* "*two and one-half dimensional*" *relativistic Vlasov–Maxwell system*, Comm. Math. Phys. **185** (1997), 257–284.

[33] R.T. Glassey and J. Schaeffer, *The relativistic Vlasov–Maxwell system in two space dimensions*, *I and II*, Arch. Ration. Mech. Anal. **141** (1998), 331–354; 355–374.

[34] R.T. Glassey and J. Schaeffer, *On global symmetric solutions of the relativistic Vlasov–Poisson equation in three space dimensions*, Math. Methods Appl. Sci. **24** (2001), 143–157.

[35] R.T. Glassey and W.A. Strauss, *Remarks on collisionless plasmas*, Contemp. Math. **28** (1984), 269–279.

[36] R.T. Glassey and W.A. Strauss, *Singularity formation in a collisionless plasma could occur only at high velocities*, Arch. Ration. Mech. Anal. **92** (1986), 59–90.

[37] R.T. Glassey and W.A. Strauss, *Absence of shocks in an initially dilute collisionless plasma*, Comm. Math. Phys. **113** (1987), 191–208.

[38] Y. Guo, *Global weak solutions to the Vlasov–Maxwell system with boundary conditions*, Comm. Math. Phys. **154** (1993), 245–263.

[39] Y. Guo, *Regularity for the Vlasov equations in a half space*, Indiana Univ. Math. J. **43** (1994), 255–320.

[40] Y. Guo, *Singular solutions of the Vlasov–Maxwell system on a half line*, Arch. Ration. Mech. Anal. **131** (1995), 241–304.

[41] Y. Guo, *Variational method in polytropic galaxies*, Arch. Ration. Mech. Anal. **150** (1999), 209–224.

[42] Y. Guo, *On the generalized Antonov's stability criterion*, Contemp. Math. **263** (2000), 85–107.

[43] Y. Guo, *The Vlasov–Poisson–Boltzmann system near vacuum*, Comm. Math. Phys. **218** (2001), 293–313.

[44] Y. Guo, *The Vlasov–Maxwell–Boltzmann system near Maxwellians*, Invent. Math. **153** (2003), 593–630.

[45] Y. Guo and C. Grotta Ragazzo, *On steady states in a collisionless plasma*, Comm. Pure Appl. Math. **XVIX** (1996), 1145–1174.

[46] Y. Guo and G. Rein, *Stable steady states in stellar dynamics*, Arch. Ration. Mech. Anal. **147** (1999), 225–243.

[47] Y. Guo and G. Rein, *Existence and stability of Camm type steady states in galactic dynamics*, Indiana Univ. Math. J. **48** (1999), 1237–1255.

[48] Y. Guo and G. Rein, *Isotropic steady states in galactic dynamics*, Comm. Math. Phys. **219** (2001), 607–629.

[49] Y. Guo and G. Rein, *Stable models of elliptical galaxies*, Mon. Not. R. Astron. Soc. **344** (2003), 1396–1406.

[50] Y. Guo and G. Rein, *A non-variational approach to nonlinear stability in stellar dynamics applied to the King model*, Preprint (2006); available at *arXiv:math-ph/0602058*.

[51] Y. Guo and W.A. Strauss, *Nonlinear instability of double-humped equilibria*, Ann. Inst. H. Poincaré Anal. Non Linéaire **12** (1995), 339–352.

[52] Y. Guo and W.A. Strauss, *Unstable BGK solitary waves and collisionless shocks*, Comm. Math. Phys. **195** (1998), 267–293.

[53] M. Hadžić, *Compactness and stability of some systems of nonlinear PDE-s in galactic dynamics*, Diploma thesis, University of Vienna (2005).

[54] M. Hauray and P.-E. Jabin, *N-particle approximation of the Vlasov equations with singular potential*, Arch. Ration. Mech. Anal., to appear.

[55] M. Heinzle, A.D. Rendall and C. Uggla, *Theory of Newtonian self-gravitating stationary spherically symmetric systems*, Preprint (2004); available at *arXiv: math-ph/0408045*.

[56] D.D. Holm, J.E. Marsden, T. Ratiu and A. Weinstein, *Nonlinear stability of fluid and plasma equilibria*, Phys. Reports **123** (1/2) (1985), 1–116.

[57] E. Horst, *On the classical solutions of the initial value problem for the unmodified non-linear Vlasov equation I*, Math. Methods Appl. Sci. **3** (1981), 229–248.

[58] E. Horst, *On the classical solutions of the initial value problem for the unmodified non-linear Vlasov equation II*, Math. Methods Appl. Sci. **4** (1982) 19–32.

[59] E. Horst, *Symmetric plasmas and their decay*, Comm. Math. Phys. **126** (1990) 613–633.

[60] E. Horst, *On the asymptotic growth of the solutions of the Vlasov–Poisson system*, Math. Methods Appl. Sci. **16** (1993), 75–85.

[61] E. Horst and R. Hunze, *Weak solutions of the initial value problem for the unmodified non-linear Vlasov equation*, Math. Methods Appl. Sci. **6** (1984), 262–279.

[62] H.J. Hwang, *Regularity for the Vlasov–Poisson system in a convex domain*, SIAM J. Math. Anal. **36** (2004), 121–171.

[63] R. Illner and G. Rein, *Time decay of the solutions of the Vlasov–Poisson system in the plasma physical case*, Math. Methods Appl. Sci. **19** (1996), 1409–1413.

[64] J. Jang, *Nonlinear instability in gravitational Euler–Poisson system for $\gamma = 6/5$*, Preprint (2005).

[65] J. Jeans, *On the theory of star-streaming and the structure of the universe*, Mon. Not. R. Astron. Soc. **76** (1915), 70–84; **76** (1916), 552–567.

[66] S. Klainerman and G. Staffilani, *A new approach to study the Vlasov–Maxwell system*. Commun. Pure Appl. Anal. **1** (2002), 103–125.
[67] K.-O. Kruse and G. Rein, *A stability result for the relativistic Vlasov–Maxwell system*, Arch. Ration. Mech. Anal. **121** (1992), 187–203.
[68] M. Kunze and A.D. Rendall, *The Vlasov–Poisson system with radiation damping*, Ann. Inst. H. Poincaré Anal. Non Linéaire **2** (2001), 857–886.
[69] R. Kurth, *Das Anfangswertproblem der Stellardynamik*, Z. Astrophys. **30** (1952), 213–229.
[70] R. Kurth, *A global particular solution to the initial-value problem of stellar dynamics*, Quart. Appl. Math., **36** (1978), 325–329.
[71] L.D. Landau and E.M. Lifschitz, *Lehrbuch der Theoretischen Physik, Bd. X. Physikalische Kinetik*, Akademie-Verlag, Berlin (1983).
[72] M. Lemou, F. Méhats and P. Raphael, *On the orbital stability of the ground states and the singularity formation for the gravitational Vlasov–Poisson system*, Preprint (2005).
[73] R.L. Liboff, *Kinetic Theory*: *Classical, Quantum, and Relativistic Descriptions*, Springer-Verlag, New York (2003).
[74] E.H. Lieb and M. Loss, *Analysis*, Amer. Math. Soc., Providence, RI (1996).
[75] Z. Lin, *Instability of periodic BGK waves*, Math. Res. Lett. **8** (2001), 521–534.
[76] Z. Lin, *Nonlinear instability of periodic BGK waves for Vlasov–Poisson system*, Comm. Pure Appl. Math. **LVIII** (2005), 505–528.
[77] Z. Lin and W.A. Strauss, *Linear stability and instability of relativistic Vlasov–Maxwell systems*, Comm. Pure Appl. Math., to appear.
[78] Z. Lin and W.A. Strauss, *Nonlinear stability and instability of relativistic Vlasov–Maxwell systems*, Comm. Pure Appl. Math., to appear.
[79] P.-L. Lions, *The concentration-compactness principle in the calculus of variations. The locally compact case*. Part 1, Ann. Inst. H. Poincaré Anal. Non Linéaire **1** (1984), 109–145.
[80] P.-L. Lions and B. Perthame, *Propagation of moments and regularity for the 3-dimensional Vlasov–Poisson system*, Invent. Math. **105** (1991), 415–430.
[81] G. Loeper, *Uniqueness of the solution to the Vlasov–Poisson system with bounded density*, Preprint (2005); available at *arXiv:math.AP/0504140*.
[82] A. Majda, G. Majda and Y. Zheng, *Concentrations in the one-dimensional Vlasov–Poisson equations*, *I*: *Temporal development and non-unique weak solutions in the single component case*, Phys. D **74** (1994), 268–300.
[83] A. Majda and Y. Zheng, *Existence of global weak solutions to one-component Vlasov–Poisson and Fokker–Planck–Poisson systems in one space dimension with measures as initial data*, Comm. Pure Appl. Math. **XLVII** (1994), 1365–1401.
[84] J. Mawhin and M. Willem, *Critical Point Theory and Hamiltonian Systems*, Springer-Verlag, New York (1989).
[85] H. Neunzert, *An Introduction to the Nonlinear Boltzmann–Vlasov Equation*, Kinetic Theories and the Boltzmann Equation, Lecture Notes in Math., Vol. 1048, Springer-Verlag, Berlin (1984).
[86] C. Pallard, *On the boundedness of the momentum support of solutions to the relativistic Vlasov–Maxwell system*, Indiana Univ. Math. J., **54** (2005), 1395–1410.
[87] B. Perthame, *Time decay*, *propagation of low moments and dispersive effects for kinetic equations*, Comm. Partial Differential Equations **21** (1996) 659–686.
[88] B. Perthame, *Mathematical tools for kinetic equations*, Bull. Amer. Math. Soc. **41** (2004), 205–244.
[89] K. Pfaffelmoser, *Global classical solutions of the Vlasov–Poisson system in three dimensions for general initial data*, J. Differential Equations **95** (1992), 281–303.
[90] G. Rein, *Generic global solutions of the relativistic Vlasov–Maxwell system of plasma physics*, Comm. Math. Phys. **135** (1990), 41–78.
[91] G. Rein, *Existence of stationary*, *collisionless plasmas in bounded domains*, Math. Methods Appl. Sci. **15** (1992), 365–374.
[92] G. Rein, *Nonlinear stability for the Vlasov–Poisson system – the energy-Casimir method*, Math. Methods Appl. Sci. **17** (1994), 1129–1140.
[93] G. Rein, *Growth estimates for the solutions of the Vlasov–Poisson system in the plasma physics case*, Math. Nachr. **191** (1998), 269–278.

[94] G. Rein, *Flat steady states in stellar dynamics – existence and stability*, Comm. Math. Phys. **205** (1999), 229–247.

[95] G. Rein, *Stationary and static stellar dynamic models with axial symmetry*, Nonlinear Anal. **41** (2000), 313–344.

[96] G. Rein, *Stability of spherically symmetric steady states in galactic dynamics against general perturbations*, Arch. Ration. Mech. Anal. **161** (2002), 27–42.

[97] G. Rein, *Reduction and a concentration-compactness principle for energy-Casimir functionals*, SIAM J. Math. Anal. **33** (2002), 896–912.

[98] G. Rein, *Nonlinear stability of gaseous stars*, Arch. Ration. Mech. Anal. **168** (2003), 115–130.

[99] G. Rein, *Global weak solutions to the relativistic Vlasov–Maxwell system revisited*, Commun. Math. Sci. **2** (2004), 145–158.

[100] G. Rein and A.D. Rendall, *Global existence of solutions of the spherically symmetric Vlasov–Einstein system with small initial data*, Comm. Math. Phys. **150** (1992), 561–583.

[101] G. Rein and A.D. Rendall, *Global existence of classical solutions to the Vlasov–Poisson system in a three-dimensional*, *cosmological setting*, Arch. Ration. Mech. Anal. **126** (1994), 183–201.

[102] G. Rein and A.D. Rendall, *Compact support of spherically symmetric equilibria in non-relativistic and relativistic galactic dynamics*, Math. Proc. Cambridge Philos. Soc. **128** (2000), 363–380.

[103] R. Robert, *Unicité de la solution faible á support compact de l'équation de Vlasov–Poisson*, C. R. Acad. Sci. Paris **324** (1997), 873–877.

[104] O. Sánchez and J. Soler, *Orbital stability for polytropic galaxies*, Ann. Inst. H. Poincaré Anal. Non Linéaire, to appear.

[105] J. Schaeffer, *The good*, *the bad, and the ugly*, Private communication (1991).

[106] J. Schaeffer, *Global existence of smooth solutions to the Vlasov–Poisson system in three dimensions*, Comm. Partial Differential Equations **16** (1991), 1313–1335.

[107] J. Schaeffer, *Steady states in galactic dynamics*, Arch. Ration. Mech. Anal. **172** (2004), 1–19.

[108] E. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, Princeton, NJ (1970).

[109] A.A. Vlasov, Zh. Eksper. Teor. Fiz. **8** (1938), 291.

[110] A.A. Vlasov, *Vlasov equation and plasma dispersion relation*, J. Phys. (U.S.S.R.) **9** (1945), 25.

[111] J. Weckler, *On the initial-boundary-value problem for the Vlasov–Poisson system*: *Existence of weak solutions and stability*, Arch. Ration. Mech. Anal. **130** (1995), 145–161.

[112] G. Wolansky, *On nonlinear stability of polytropic galaxies*, Ann. Inst. H. Poincaré Anal. Non Linéaire **16** (1999), 15–48.

[113] G. Wolansky, *Static solutions of the Vlasov–Einstein system*, Arch. Ration. Mech. Anal. **156** (2001), 205–230.

[114] P. Zhidkov, *On global solutions for the Vlasov–Poisson system*, Electron. J. Differ. Equ. **58** (2004), 1–11.