1D Kappa Distribution

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The 3D Kappa (or generalized Lorentzian) plasma distribution function is given from papers [1, 2, 3, 4, 5] and many others by:

$$f_\kappa(\boldsymbol{v}) = \left(\pi\sigma^2\right)^{-3/2} A_\kappa \left[1 + \frac{|\boldsymbol{v} - \boldsymbol{v}_0|^2}{(\kappa - 3/2)\sigma^2}\right]^{-\kappa-1} \tag{1a}$$

$$A_\kappa = \left(\kappa - \frac{3}{2}\right)^{-3/2} \frac{\Gamma(\kappa+1)}{\Gamma(\kappa - 1/2)} \tag{1b}$$

where $\boldsymbol{v}$ is the particle velocity, $\boldsymbol{v}_0$ is the mean/bulk velocity, $\sigma = \sqrt{2k_BT/m}$ is the thermal velocity, T is the temperature and second velocity moment, k_B is the Boltzmann constant, m is the particle mass, and $\kappa \in (3/2, \infty)$ is the index of the distribution.

Integrate out two of the dimensions (assume $\boldsymbol{v}_0 = \boldsymbol{0}$ for simplicity).

$$\begin{aligned}
\int_{-\infty}^{\infty} f_\kappa(v_1, v_2, v_3) dv_2 &= \left(\pi\sigma^2\right)^{-3/2} A_\kappa \int_{-\infty}^{\infty} \left[1 + \frac{v_1^2 + v_2^2 + v_3^2}{(\kappa - 3/2)\sigma^2}\right]^{-\kappa-1} dv_2 \\
&= \left(\pi\sigma^2\right)^{-3/2} A_\kappa \frac{\Gamma(\kappa+1/2)}{\Gamma(\kappa+1)} \frac{\sqrt{\pi(2\kappa-3)}\sigma}{\sqrt{2}} \left(1 + \frac{2(v_1^2 + v_3^2)}{(2\kappa-3)\sigma^2}\right)^{-1/2} \left[1 + \frac{2(v_1^2 + v_3^2)}{(2\kappa-3)\sigma^2}\right]^{-\kappa} \\
&= \left(\pi\sigma^2\right)^{-3/2} A_\kappa \frac{\Gamma(\kappa+1/2)}{\Gamma(\kappa+1)} \sqrt{\frac{\pi(2\kappa-3)}{2}}\, \sigma \left[1 + \frac{2(v_1^2 + v_3^2)}{(2\kappa-3)\sigma^2}\right]^{-\kappa-1/2} \\
f_\kappa(v_1, v_3) &= \frac{B_\kappa}{\pi\sigma^2} \left[1 + \frac{2(v_1^2 + v_3^2)}{(2\kappa-3)\sigma^2}\right]^{-\kappa-1/2}.
\end{aligned}$$

$$\begin{aligned}
\int_{-\infty}^{\infty} f_\kappa(v_1, v_3) dv_3 &= \frac{B_\kappa}{\pi\sigma^2} \int_{-\infty}^{\infty} \left[1 + \frac{2(v_1^2 + v_3^2)}{(2\kappa-3)\sigma^2}\right]^{-\kappa-1/2} dv_3 \\
&= \frac{B_\kappa}{\pi\sigma^2} \frac{\Gamma(\kappa)}{\Gamma(\kappa+1/2)} \frac{1}{\sqrt{2}} \sqrt{(2\kappa-3)\pi\sigma^2 + 2\pi v_1^2} \left[1 + \frac{2v_1^2}{(2\kappa-3)\sigma^2}\right]^{-\kappa-1/2} \\
&= \frac{B_\kappa}{\pi\sigma^2} \frac{\Gamma(\kappa)}{\Gamma(\kappa+1/2)} \sqrt{\frac{(2\kappa-3)\pi\sigma^2}{2}} \left(1 + \frac{2v_1^2}{(2\kappa-3)\sigma^2}\right)^{1/2} \left[1 + \frac{2v_1^2}{(2\kappa-3)\sigma^2}\right]^{-\kappa-1/2} \\
f_\kappa(v_1) &= \frac{C_\kappa}{\sqrt{\pi}\sigma} \left[1 + \frac{2v_1^2}{(2\kappa-3)\sigma^2}\right]^{-\kappa}.
\end{aligned}$$

where

$$\begin{aligned}
C_\kappa &= B_\kappa \frac{\Gamma(\kappa)}{\Gamma(\kappa+1/2)} \sqrt{\frac{2\kappa-3}{2}} \\
&= A_\kappa \frac{\Gamma(\kappa+1/2)}{\Gamma(\kappa+1)} \sqrt{\frac{2\kappa-3}{2}} \frac{\Gamma(\kappa)}{\Gamma(\kappa+1/2)} \sqrt{\frac{2\kappa-3}{2}} \\
&= \left(\frac{2}{2\kappa-3}\right)^{3/2} \frac{\Gamma(\kappa+1)}{\Gamma(\kappa-1/2)} \frac{\Gamma(\kappa+1/2)}{\Gamma(\kappa+1)} \sqrt{\frac{2\kappa-3}{2}} \frac{\Gamma(\kappa)}{\Gamma(\kappa+1/2)} \sqrt{\frac{2\kappa-3}{2}} = \sqrt{\frac{2}{2\kappa-3}} \frac{\Gamma(\kappa)}{\Gamma(\kappa-1/2)}.
\end{aligned}$$

Simplifying, this is

$$\boxed{f_\kappa(v_1) = \left(\pi(\kappa-3/2)\sigma^2\right)^{-\frac{1}{2}} \frac{\Gamma(\kappa)}{\Gamma(\kappa-1/2)} \left[1 + \frac{v_1^2}{(\kappa-3/2)\sigma^2}\right]^{-\kappa}} \tag{2}$$

$$f_\kappa(v_1) = \frac{C_\kappa}{\sqrt{\pi\sigma^2}} \left[1 + \frac{v_1^2}{(\kappa-3/2)\sigma^2}\right]^{-\kappa}, \; C_\kappa = \frac{1}{\sqrt{\kappa-3/2}} \frac{\Gamma(\kappa)}{\Gamma(\kappa-1/2)}. \tag{3}$$

Compare this result to the 1D distributions from other papers.

1. **Sarkar et al. (2015)**[1] Equation 8:

$$f_{e0}(v_z) \propto \left(\pi\kappa\theta^2\right)^{-\frac{1}{2}} \frac{\Gamma(\kappa)}{\Gamma(\kappa-1/2)} \left[1 + \frac{v_z^2}{\kappa\theta^2}\right]^{-\kappa},$$

where a κ-dependent thermal velocity is used, $\theta^2 = \frac{2\kappa-3}{\kappa}\frac{\sigma^2}{2} = \frac{\kappa-3/2}{\kappa}\sigma^2$.

2. **Summers and Thorne (1991)**[2] 1D distribution from Table 1:

$$\begin{aligned} f_\kappa(v) &\propto \frac{1}{\sqrt{\pi}} \frac{1}{\theta\kappa^{3/2}} \frac{\Gamma(\kappa+1)}{\Gamma(\kappa-1/2)} \left[1 + \frac{v^2}{\kappa\theta^2}\right]^{-\kappa} \\ &= \frac{1}{\sqrt{\pi\theta^2\kappa^3}} \frac{\kappa\Gamma(\kappa)}{\Gamma(\kappa-1/2)} \left[1 + \frac{v^2}{\kappa\theta^2}\right]^{-\kappa} \\ &= \frac{1}{\sqrt{\pi\theta^2\kappa}} \frac{\Gamma(\kappa)}{\Gamma(\kappa-1/2)} \left[1 + \frac{v^2}{\kappa\theta^2}\right]^{-\kappa}, \end{aligned}$$

where $\theta^2 = \frac{\kappa-3/2}{\kappa}\sigma^2$ again.

3. **Livadiotis and McComas (2013)**[3] 1-particle, 1-dimensional distribution with $\kappa = \kappa - 1$:

$$\begin{aligned} P(u;\theta,\kappa) &= \left(\pi(\kappa-1/2)\theta^2\right)^{-\frac{1}{2}} \frac{\Gamma(\kappa+1)}{\Gamma(\kappa+1/2)} \left[1 + \frac{v^2}{(\kappa-1/2)\theta^2}\right]^{-(\kappa+1)} \\ P(u;\theta,\kappa-1) &= \left(\pi(\kappa-3/2)\theta^2\right)^{-\frac{1}{2}} \frac{\Gamma(\kappa)}{\Gamma(\kappa-1/2)} \left[1 + \frac{v^2}{(\kappa-3/2)\theta^2}\right]^{-\kappa}, \end{aligned}$$

where $\theta = \sigma = \sqrt{2k_BT/m}$.

References

[1] Susmita Sarkar, Samit Paul, and Raicharan Denra. Bump-on-tail instability in space plasmas. *Physics of Plasmas*, 22(10):102109, 10 2015.

[2] Danny Summers and Richard M. Thorne. The modified plasma dispersion function. *Physics of Fluids B: Plasma Physics*, 3(8):1835–1847, 08 1991.

[3] George Livadiotis and David J. McComas. Understanding kappa distributions: A toolbox for space science and astrophysics. *Space Science Reviews*, 175(1-4):183–214, jun 2013.

[4] Georgios Nicolaou, George Livadiotis, Christopher J. Owen, Daniel Verscharen, and Robert T. Wicks. Determining the kappa distributions of space plasmas from observations in a limited energy range. *The Astrophysical Journal*, 864(1):3, aug 2018.

[5] Viviane Pierrard and Marian Lazar. Kappa distributions: Theory and applications in space plasmas. *Solar Physics*, 267, 03 2010.