Analytic Appx of Initial Guess

$$D(\omega,k) = 1 - \frac{1}{k^2}\int \frac{f_{eq}'(v)}{v - \omega/k}\,dv$$

$$= 1 + \frac{1}{k}\int \frac{f_{eq}'(v)}{\omega - kv}\,dv$$

Taylor expand $f(x) = \frac{1}{x}$ about a

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \frac{1}{6}f'''(a)(x-a)^3 + O((x-a)^4)$$

or

$$\frac{1}{x} = \frac{1}{a} - \frac{1}{a^2}(x-a) + \frac{1}{a^3}(x-a)^2 - \frac{1}{a^4}(x-a)^3 + O((x-a)^4)$$

Take $\begin{cases} x = \omega - kv \\ a = \omega \end{cases} \Longrightarrow \boxed{\frac{1}{\omega - kv} \approx \frac{1}{\omega} + \frac{1}{\omega^2}kv + \frac{1}{\omega^3}k^2v^2 + \frac{1}{\omega^4}k^3v^3}$

appx is valid for

$|x-a| = \underset{\text{small}}{kv}$

Using this in the dielectric function gives

$$D(\omega,k) = 1 + \frac{1}{k}\int \frac{f_{eq}'(v)}{\omega - kv}\,dv \approx 1 + \frac{1}{k}\left(\frac{1}{\omega}\int f_{eq}'(v)\,dv + \frac{k}{\omega^2}\int f_{eq}'(v)\,v\,dv + \frac{k^2}{\omega^3}\int f_{eq}'(v)\,v^2\,dv + \frac{k^3}{\omega^4}\int f_{eq}'(v)\,v^3\,dv\right)$$

Then, Ⓐ $\int f_{eq}'(v)\,dv = \lim_{b\to\infty} f_{eq}(b) - \lim_{a\to-\infty} f_{eq}(a) = 0$

Ⓑ $\int f_{eq}'(v)\,v\,dv = v f_{eq}(v)\Big|_{v\to-\infty}^{v\to\infty} - \int f_{eq}(v)\,dv = -1$

Ⓒ $\int f_{eq}'(v)\, v^2\, dv = \underbrace{v^2 f_{eq}(v)\Big|_{v\to-\infty}^{v\to\infty}}_{\text{need } f_{eq}(v)\sim |v|^{-2-\varepsilon}} - 2\int f_{eq}(v)\, v\, dv$

$$= -2\int v\, f_{eq}(v)\, dv$$

Ⓓ $\int f_{eq}'(v)\, v^3\, dv = v^3 f_{eq}(v)\Big|_{v\to-\infty}^{v\to\infty} - 3\int f_{eq}(v)\, v^2\, dv$

$$= -3\int v^2 f_{eq}(v)\, dv$$

Let $C_1 = -2\int v\, f_{eq}(v)\, dv = 0$ if f_{eq} even

and $C_2 = 3\int v^2 f_{eq}(v)\, dv \geq 0$

Then, D becomes

$$D(\omega, k) \approx 1 - \frac{1}{\omega^2} + \frac{C_1 k}{\omega^3} - \frac{C_2 k^2}{\omega^4} \overset{\text{set}}{=} 0$$

① If f_{eq} is an even function (e.g. Maxwellian, kappa) then this becomes

$$1 - \frac{1}{\omega^2}\left(1 + \frac{C_2 k^2}{\omega^2}\right) = 0$$

$\sqrt{1-x} \approx 1 - \frac{1}{2}x$

or $\quad \omega^4 - \omega^2 - C_2 k^2 = 0$

Take $y = \omega^2$:

$$y^2 - y - C_2 k^2 = 0$$

$$y = \frac{1 \pm \sqrt{1 + 4C_2 k^2}}{2}$$

<u>NOTE</u>: $C_2 \geq 0$

Then, $\sqrt{1+x} \approx 1 + \frac{1}{2}x$

So $\sqrt{1+4C_2k^2} \approx 1 + 2C_2k^2 > 1$

Hence, as $y \geq 0$ is needed we take $\oplus$ of $\pm$ and find

$$y \approx \frac{1 + (1+2C_2k^2)}{2} = 1 + C_2k^2$$

Finally, $\omega^2 \approx 1 + C_2k^2$

and

$$\boxed{\begin{gathered} \mathrm{Re}(\omega) \approx \sqrt{1+C_2k^2} \\ \text{where} \quad C_2 = 3\int v^2 f_{eq}(v)\,dv. \end{gathered}}$$

② More generally, for asymmetric distributions, we need to solve

$$1 - \frac{1}{\omega^2} + \frac{C_1k}{\omega^3} - \frac{C_2k^2}{\omega^4} = 0$$

or

$$\boxed{\begin{gathered} \omega^4 - \omega^2 + C_1k\omega - C_2k^2 = 0 \\ \text{where} \\ C_1 = -2\int v f_{eq}(v)\,dv \\ C_2 = 3\int v^2 f_{eq}(v)\,dv. \end{gathered}}$$