Global Sensitivity Analysis of Plasma Instabilities via Active Subspaces

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Colorado Nonlinear Day

*supported by NSF (DMS-1614586 and DMS-2107938)[1]

April 2022

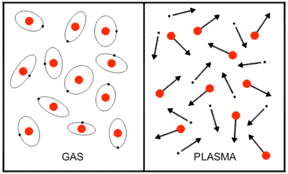
[1] Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the National Science Foundation (NSF).

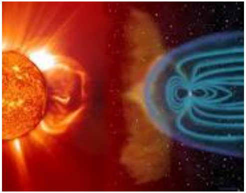
Outline

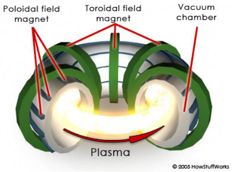
- Background & Models
- Plasma Instabilities
- Sensitivity Analysis & Active Subspaces
- Applications

What is a plasma?

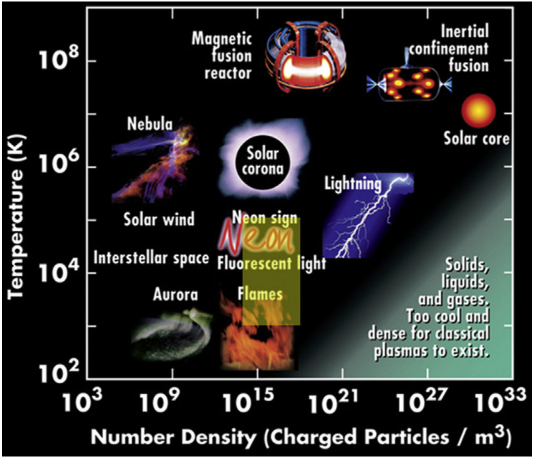
A plasma is an ionized gas:







Plasmas at Different Temperatures & Densities



Consider

- large, but non-relativistic velocities (i.e., temp $10^5 - 10^9$K)
- dilute (low density) conditions

Fundamental (1D) model is **Vlasov-Poisson** system

$$(VP) \qquad \begin{cases} \partial_t f + v\partial_x f - E\partial_v f = 0 \\ \partial_x E(t,x) = 1 - \int f(t,x,v)\, dv \end{cases}$$

$$f(t,x,v) \sim \text{distribution of electrons}$$

$$1 \sim \text{normalized ionic background density}$$

$$E(t,x) \sim \text{electric field}$$

- Bounded domain $0 \leq x \leq L$ (periodic BCs), $v \in \mathbb{R}$, $t \geq 0$
- Statistical mechanics model $\Longrightarrow f(t,x,v)$ depends on v
- VP is nonlocally nonlinear $\Longrightarrow E\partial_v f$
- Parameters (e.g., m, q, ρ_0, ϵ_0) removed via rescaling

Equlibria

Note: Any $f_{\text{eq}}(v)$ with
$$\int f_{\text{eq}}(v)\ dv = 1$$
satisfies (VP) $\Rightarrow$ spatially-homogeneous equilibria

Examples:

- Maxwellian (Gaussian)
$$f_M(v) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}|v-\mu|^2}$$
- Bi-Maxwellian (Multi-Gaussian)
$$f_{BM}(v) = \frac{\beta}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{1}{2\sigma_1^2}|v-\mu_1|^2} + \frac{1-\beta}{\sqrt{2\pi\sigma_2^2}} e^{-\frac{1}{2\sigma_2^2}|v-\mu_2|^2}$$
- Lorentzian (Cauchy dist.)
$$f_L(v) = \frac{1}{\pi} \frac{\sigma}{(v-\mu)^2 + \sigma^2}$$

Parameters (physical properties) influence stability/rate

Stability Properties

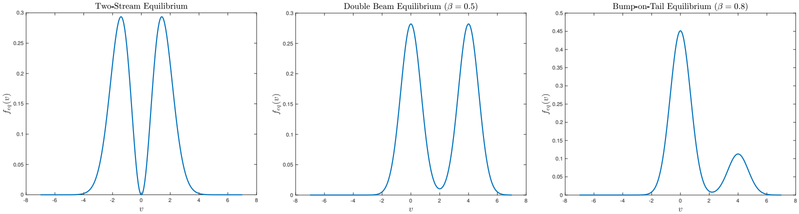


Figure: Two-Stream equilibrium (left) and Bi-Maxwellian equilibria for $\beta = 0.5$ (center) and $\beta = 0.8$ (right).

Both velocities - mean μ and thermal σ - are parameter inputs within the system

Dispersion Relation

Take perturbative solution

$$f(t,x,v) = f_{\mathrm{eq}}(v)\left(1 + \alpha e^{i[kx-\omega t]}\right)$$

$$\alpha \sim \text{amplitude}, \quad k \sim \text{wavenumber}, \quad \omega \sim \text{frequency} \in \mathbb{C}$$

and compute **linear** ($\Rightarrow \alpha \ll 1$) dispersion relation

$$\mathcal{D}(k,\omega) = 1 - \frac{1}{k^2}\int \frac{f'_{\mathrm{eq}}(v)}{v - \omega/k}\, dv.$$

Roots of $\mathcal{D}$ indicate solutions. Fix k and consider $\omega(k)$ such that

$$\mathcal{D}(k,\omega(k)) = 0$$

Separate into *Re* & *Im*

$$\omega(k) = \omega_R(k) + i\gamma(k)$$

so that solutions grow/decay exponentially with rate $\gamma(k)$

$$f(t,x,v) \sim f_{\mathrm{eq}}(v) + \mathcal{O}\left(e^{\gamma(k)t}\right)$$

Parameters Influence Stability

Each $f_{\mathrm{eq}}(v)$ depends upon physical parameters (e.g., μ, σ), that influence the rate of exponential growth/decay

For Lorentz equilibrium

$$f_L(v) = \frac{1}{\pi} \frac{\sigma}{(v-\mu)^2 + \sigma^2}$$

Residue Theorem $\Rightarrow$ Roots of $\mathcal{D}$ satisfy

$$\omega(k, \mu, \sigma) = k\mu \pm 1 - \sigma k i$$

so that

$$\gamma(k, \mu, \sigma) = -\sigma k.$$

As $\sigma > 0$, $f_L(v)$ is stable with

$$f(t, x, v) \sim f_{\mathrm{eq}}(v) + e^{-\sigma k t}$$

Problem: Typically, we cannot analytically compute $\gamma(k)$ or $\mathcal{D}$!

Maxwellian Stability

For Maxwellian equilibrium

$$f_M(v) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}|v-\mu|^2}$$

$\mathcal{D}$ involves the **plasma Z-function** $Z(z)$, namely

$$\mathcal{D}(k,\omega) = 1 + \frac{1}{\sigma^2 k^2}\left[1 + A\left(\frac{\omega}{k}\right) Z\left(A\left(\frac{\omega}{k}\right)\right)\right],$$

where

$$A(u) = \frac{1}{\sqrt{2\sigma^2}}(u-\mu)$$

$$Z(z) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} \frac{e^{-t^2}}{t-z}\, dt$$

Main Goal: Computationally approximate $\gamma(k,\mu,\sigma)$ & determine what parameters or range of values give rise to largest changes in γ

Issue: Expensive computations & range of parameters can be large

Solution: Use Active Subspaces to perform sensitivity analysis

Active Subspaces (I)

Consider $g(p)$ defined for a set of m (normalized) input parameters $p \in P \subset \mathbb{R}^m$, so that the average derivative matrix

$$C = \int_P \nabla g(p) \nabla g(p)^T dp \in \mathbb{R}^{m \times m}$$

captures directions of greatest change in $g(p)$.

Eigenvalue λ_ℓ measures average change in $g(p)$ subject to perturbations in p along eigenvector w_ℓ via

$$\lambda_\ell = \int_P |\nabla g(p) \cdot w_\ell|^2 dp \geq 0$$

Active Subspaces (II)

C is symmetric ($C = C^T$) $\Rightarrow$ spectral decomposition

$$C = W\Lambda W^T, \quad \Lambda = \mathrm{diag}(\lambda_1, ..., \lambda_m), \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$$

$$\Lambda = \begin{bmatrix} \Lambda_1 & \\ & \Lambda_2 \end{bmatrix}, \qquad W = \begin{bmatrix} W_1 & W_2 \end{bmatrix}$$

If spectral gap exists, i.e., $\min(\Lambda_1) \gg \max(\Lambda_2)$, this splits the domain into

1. Active variables
$$y = W_1^T p \in \mathbb{R}^n$$
2. Inactive variables
$$z = W_2^T p \in \mathbb{R}^{m-n}$$

Active Subspaces (III)

- In practice, Λ & W are appx via Monte Carlo (MC) & SVD where ∇g is appx via finite differences
- Once spectral gap is identified, perform nonlinear function fit of MC points & use as reduced surrogate model for $g(p)$.
- Often, $\lambda_1 \gg \lambda_2$ and $w_1 \cdot p$ is linear combination of parameters in which $g(p)$ changes most
- In the present context, we want to construct $\gamma = g(p)$ where $p =$ equilibrium parameters (e.g. μ, σ)

Active Subspaces for Bi-Maxwellian

Recall Bi-Maxwellian equilibrium

$$f_{BM}(v) = \frac{\beta}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{1}{2\sigma_1^2}|v-\mu_1|^2} + \frac{1-\beta}{\sqrt{2\pi\sigma_2^2}} e^{-\frac{1}{2\sigma_2^2}|v-\mu_2|^2}$$

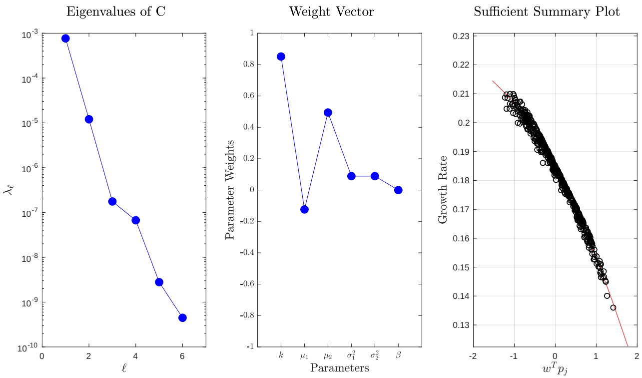
so that

$$\gamma = \gamma(k, \beta, \mu_1, \sigma_1, \mu_2, \sigma_2)$$

Questions:

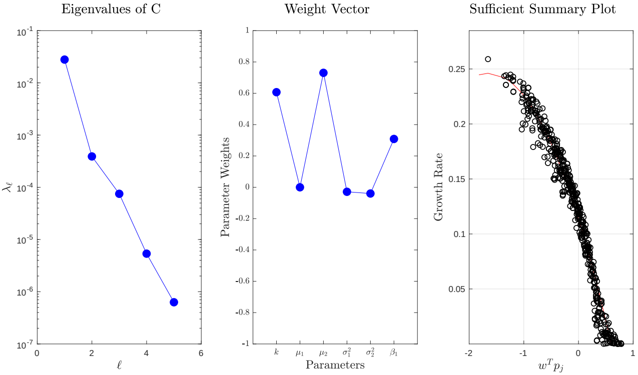
- Is there a low-dimensional active parameter subspace $\subset \mathbb{R}^6$?
- Of these 6 parameters, which has greatest influence on γ?
- Can we well-appx γ along its direction(s) of greatest change?
- If so, can we quantify uncertainty in this appx?

Result - Double Beam ($\beta \approx 0.5$)



- Eigenvalues (left), parameter weights (center), γ appx (right) of 1D active subspace captures $\eta = 98.42\%$ of variation.
- $k, \beta, \sigma_1^2, \sigma_2^2 \in [0.45, 0.55], \mu_1 \in [-0.1, 0.1], \mu_2 \in [3.6, 4.4]$

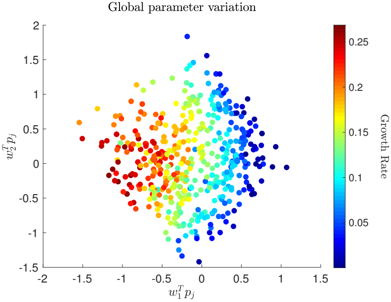
Result - Bump on Tail ($\beta \approx 0.8$)

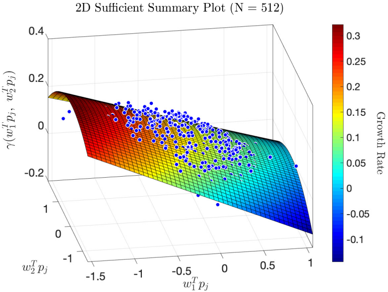


- Eigenvalues (left), parameter weights (center), γ appx (right) of 1D active subspace captures $\eta = 98.34\%$ of variation.
- Recover functional form of γ via linear/nonlinear curve fitting

Breakdown of 1D Active subspace

Large p intervals (e.g., $> \pm 50\%$ base) $\Rightarrow$ May need 2D projection





1D: $\eta_1 = 75.09\%$ of variation $\Rightarrow$ 2D: $\eta_2 = 93.58\%$ of variation

Summary & Future Directions

1. Physical characteristics of velocity distribution are crucial to rate of stability & instability in collisionless plasmas
2. Active subspaces can allow appx of unknown/expensive high-dimensional scalar functions - could be useful in your applications too!
3. Limitations - Active subspaces may fail to be low-dimensional, MC samples may be too expensive!
4. Additional plasma instabilities (Weibel, Dicotron) with greater complexity (3D, B field, collisions) - need HPC