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Supposedly explains the kappa0 definition
in Kappa Toolbox paper

# Beyond kappa distributions: Exploiting Tsallis statistical mechanics in space plasmas

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[1] Empirically derived kappa distributions are becoming increasingly widespread in space physics as the power law nature of various suprathermal tails is melded with more classical quasi-Maxwellian cores. Two different mathematical definitions of kappa distributions are commonly used and various authors characterize the power law nature of suprathermal tails in different ways. In this study we examine how kappa distributions arise naturally from Tsallis statistical mechanics, which provides a solid theoretical basis for describing and analyzing complex systems out of equilibrium. This analysis exposes the possible values of kappa, which are strictly limited to certain ranges. We also develop the concept of temperature out of equilibrium, which differs significantly from the classical equilibrium temperature. This analysis clarifies which of the kappa distributions has primacy and, using this distribution, the kinetic and physical temperatures become one, both in and out of equilibrium. Finally, we extract the general relation between both types of kappa distributions and the spectral indices commonly used to parameterize space plasmas. With this relation, it is straightforward to compare both spectral indices from various space physics observations, models, and theoretical studies that use kappa distributions on a consistent footing that minimizes the chances for misinterpretation and error. Now that the connection is complete between empirically derived kappa distributions and Tsallis statistical mechanics, the full strength and capability of Tsallis statistical tools are available to the space physics community for analyzing and understanding the kappa-like properties of the various particle and energy distributions observed in space.

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## 1. Introduction

[2] Kappa distributions have become increasingly important in space plasma physics. An empirical expression for these distributions was introduced into the field by *Vasyliũnas* [1968], and since then, kappa distributions have been utilized in numerous studies of the solar wind [e.g., *Gloeckler and Geiss*, 1998; *Chotoo et al.*, 2000; *Mann et al.*, 2002; *Marsch*, 2006] and planetary magnetospheres [e.g., *Christon*, 1987; *Mauk et al.*, 2004; *Schippers et al.*, 2008, *Dialynas et al.*, 2009]. Quite recent observations from the Voyager spacecraft [*Decker and Krimigis*, 2003; *Decker et al.*, 2005] indicate that ions in the outer heliosphere are well described by kappa distributions; theoretical analyses of the ions and energetic neutral atoms (ENAs) have already begun to rely heavily on these kappa distributions [e.g., *Prested et al.*, 2008; *Heerikhuisen et al.*, 2008].

[3] The use of kappa distributions has become increasingly widespread across space physics and astrophysics. In order to document this growth, we conducted a survey of the Astrophysics Data System (ADS) for papers related to kappa distributions. Figure 1 summarizes the results of this survey, where we identified the ~400 papers that mention kappa distributions in their title or abstract from 1980 through mid-February 2009. It is remarkable that over 15% of these papers were published during 2008 alone and that the number published in the first 6 weeks of 2009 is already roughly equal to the number per year from 1991 to 1997 and far more than in any year prior to that.

[4] Since their introduction, several modified versions of kappa distributions have been suggested [e.g., *Hawkins et al.*, 1998; *Mauk et al.*, 2004]. However, two definitions of kappa distributions currently dominate the field of space plasmas (referred to here as first and second kinds), with their primary difference being in their kappa indices and temperature-like parameters. More generally, various thermal parameters have been considered in the expressions of different types of kappa distributions. However, the exact interpretation of temperature is not something that can be simply chosen or roughly defined; rather the true definition of temperature must emerge from statistical mechanics.

[5] Boltzmann-Gibbs (BG) statistical mechanics has stood the test of time for describing classical equilibrium

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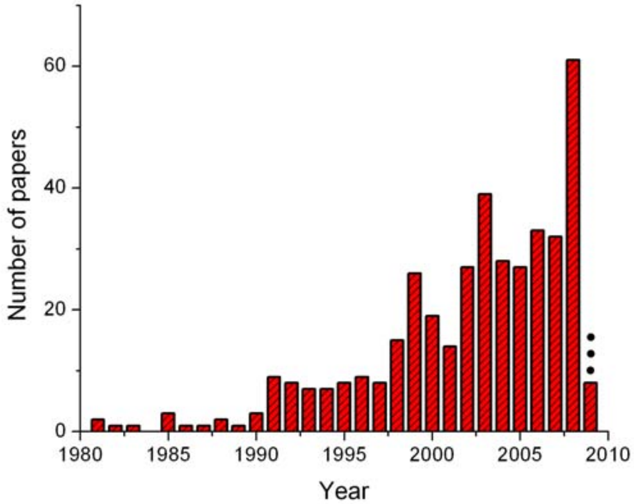


**Figure 1.** Distribution of the published papers in space physics and astrophysics since 1980 that are related to kappa distributions and mention these distributions in their title or abstract. The last bar represents just the first 6 weeks of 2009.

systems; however, this formalism cannot adequately describe most space plasmas, which are systems that are not in equilibrium. In contrast, Tsallis statistical mechanics, based on a nonextensive formulation of entropy [*Tsallis*, 1988], and a consistent generalization of the concept of expectation value [*Tsallis et al.*, 1998], has offered a theoretical basis for describing and analyzing complex systems out of equilibrium [e.g., see *Borges et al.*, 2002, and references therein]. In particular, Tsallis entropy, $S_q$, is expressed in terms of a $q$ index and recovers the classical Boltzmannian entropy in the limit of $q \to 1$. Moreover, the expectation value is expressed in terms of the so-called escort probability distribution, which characterizes a system after its relaxation into stationary states out of equilibrium [*Gell-Mann and Tsallis*, 2004]. This is constructed in terms of the ordinary probability distribution and the $q$ index.

[6] The Tsallis-like stationary probability distribution is derived from the "extremization" of entropy $S_q$, under the constraints of a Canonical Ensemble [*Tsallis*, 1999]. This is the so-called $q$-deformed exponential distribution [e.g., *Silva et al.*, 1998; *Yamano*, 2002], which was considered an anomalous distribution [*Abe*, 2002] from the point of view of the standard BG exponential distribution. However, $q$-deformed exponential distributions are observed quite frequently in nature, and it is now widely accepted that these distributions constitute a suitable generalization of the BG exponential distribution, rather than describing a kind of rare or anomalous behavior. Applications of the $q$-deformed exponential distribution can be found in a wide variety of topics, for example, in sociology-sociometry (e.g., the Internet [*Abe and Suzuki*, 2003]; citation networks of scientific papers [*Tsallis and de Albuquerque*, 2000]; urban agglomeration [*Malacarne et al.*, 2001]; linguistics [*Montemurro*, 2001]); in economics [*Borland*, 2002]; in biology [*Andricioaei and Straub*, 1996; *Tsallis et al.*, 1999]; in applied statistics [*Habeck et al.*, 2005]; in physics (e.g., nonlinear dynamics [*Robledo*, 1999; *Borges et al.*, 2002]; condensed-matter [*Hasegawa*, 2005]; earthquakes [*Sotolongo-Costa et al.*, 2000; *Sotolongo-Costa and Posadas*, 2004; *Silva et al.*, 2006]; turbulent fluids [*Beck et al.*, 2001]); and in astrophysics and space plasmas [*Tsallis et al.*, 2003; *Jiulin*, 2004; *Sakagami and Taruya*, 2004]. A more extended bibliography of $q$-deformed exponential distributions can be found in the work of *Swinney and Tsallis* [2004], *Gell-Mann and Tsallis* [2004], and *Tsallis* [2009a, 2009b] (for a complete bibliography on "nonextensive statistical mechanics and thermodynamics," see http://tsallis.cat.cbpf.br/TEMUCO.pdf).

[7] The origin of the kappa distribution in Tsallis statistical mechanics has already been examined by several authors [e.g., *Milovanov and Zelenyi*, 2000; *Leubner*, 2002, 2004a, 2004b; *Shizgal*, 2007; *Nieves-Chinchilla and Viñas*, 2008a, 2008b]. In the Tsallis framework, the phenomenologically introduced kappa distribution and the Tsallis-like Maxwellian distribution of velocities are accidentally of the same form, using the transformation of indices: $q = 1 + 1/\kappa$. As we shall see in this study, the first and second kind of kappa distributions, which are widely used in space physics, coincide with the ordinary and escort Tsallis-Maxwellian probability distributions, respectively.

[8] Once the exact characterization of the statistical mechanics that justifies the kappa distribution is specified, then the exact definition of temperature can also be determined. Having interpreted the kappa distribution as the Tsallis-Maxwellian probability distribution, the exact definition of temperature is given by the so-called physical temperature, $T_q$ [*Abe*, 1999; *Rama*, 2000].

[9] In classical BG statistical mechanics, temperature is primarily defined in one of three ways: (1) thermodynamics: the thermodynamic definition $T_S \equiv (\partial S/\partial U)^{-1}$ (with $S$ and $U$ stand for the classical BG entropy and internal energy, respectively) [e.g., see *Tsallis*, 1999; *Milovanov and Zelenyi*, 2000]; (2) kinetic theory: the kinetic temperature $T_K$, determined by the second statistical moment of the probability distribution of velocities; and (3) statistics: the Lagrangian temperature $T$, defined by the second Lagrangian multiplier that corresponds to the constraint of internal energy in the Canonical Ensemble. All of these three definitions coincide in equilibrium, $T_S = T_K = T$, but they are typically different when the system is out of equilibrium.

[10] In Tsallis statistical mechanics, the thermodynamic definition of temperature is generalized to the physical temperature $T_q$, and again, all the three definitions coincide in equilibrium $T_q = T_K = T$. In contrast to the BG formalism, the Tsallis approach maintains the equality of $T_q = T_K$, even when the system is relaxing into stationary states out of equilibrium. In this way, the kinetic temperature $T_K$, which is used in the majority of space plasmas analyses, even in the primary work of *Vasyliũnas* [1968], is now provided with a solid foundation given by the concept of physical temperature $T_q$ within the formalism of Tsallis statistical mechanics. In contrast to these more recent developments, the use of BG statistical mechanics in space physics is highly problematic, since it provides neither a reliable derivation of kappa distribution, nor a well-defined temperature out of equilibrium.

[11] The purpose of this paper is to clarify the precise connection of kappa distributions with Tsallis statistical mechanics and develop a robust definition of temperature;

these results have broad implications for use in space plasmas as well as other nonequilibrium systems. In section 2 we provide a brief mathematical motivation for utilizing the kappa distribution: the deformation of the Maxwell distribution. In section 3 we present a survey of the different kinds of kappa distributions that are most frequently considered in space plasmas, while their establishment within the framework of Tsallis statistical mechanics is thoroughly examined in section 4. The relation of the first and second kinds of kappa distributions with the ordinary and escort Tsallis-Maxwellian probability distributions, respectively, is also provided in this section, while the inconsistency of kappa distributions with the BG statistical mechanics is examined in detail. In section 5, we develop the concept of the kinetic temperature for systems relaxing into stationary states out of equilibrium. In particular, the physical temperature coincides with the kinetic temperature, highlighting its substantial difference from the classical Lagrangian temperature that coincides with the kinetic temperature only in equilibrium. In the last section of the paper, section 6, we extract a general expression between the kappa index (of both the kinds) and spectral or spectral-like indices commonly used to parameterize space plasma distributions. We also argue that various thermal quantities that have been considered previously need to be replaced by the physical temperature. Appendix A comprises a complete analysis for defining and studying the $q$-deformed Gamma function, which is a generalization of the classical Gamma function, covering both kinds of kappa distributions, and provides a compressed nomenclature for expressing the Tsallis mathematical formalism. Finally, Appendix B gives a compilation of the definitions, derivation and related calculations of Tsallis Canonical probability distributions needed for space physicists to be able to use the power of Tsallis statistics in their own work.

## 2. A Mathematical Motivation: Deformation of the Maxwell Distribution

[12] The Maxwell distribution is widely known as the basis of the kinetic theory of gases. It describes the velocities $\vec{u}$ of the gas particles and can be readily derived, by substituting the kinetic energy $\varepsilon = \frac{1}{2}\mu \cdot u^2$ (of gas particles with mass $\mu$) into the Boltzmannian distribution of energies

$$p(\varepsilon; T) \sim e^{-\frac{\varepsilon}{k_B T}}, \tag{1}$$

resulting in

$$p(u;\theta) \sim e^{-(u/\theta)^2},\ \frac{1}{2}\mu \cdot \theta^2 \equiv k_B T, \tag{2}$$

where $k_B$ is the Boltzmann's constant, $T$ is the temperature, and $\theta$ is the characteristic speed-scale parameter. Now, let us rewrite the Maxwell distribution as follows. One of the formal definitions of the exponential function is given by the following limit:

$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n, n \in \aleph, \tag{3}$$

or equivalently, by

$$e^x = (e^{-x})^{-1} = \left[\lim_{n\to\infty}\left(1-\frac{x}{n}\right)^n\right]^{-1} = \lim_{n\to\infty}\left(1-\frac{x}{n}\right)^{-n}.$$

Even though $n$ denotes a positive integer ($n \in \aleph$), the above limit can be approached also by a positive real number $\kappa$. Indeed, since any real number $\kappa$ is included between the two sequential integers, $n \equiv Int(\kappa) \leq \kappa < Int(\kappa) + 1 \equiv n + 1$, then

$$\lim_{n\to\infty} = \lim_{\kappa\to\infty}.$$

Hence (3) is rewritten as follows:

$$e^x = \lim_{\kappa\to\infty}\left(1-\frac{x}{\kappa}\right)^{-\kappa}, \kappa \in \Re^+, \tag{4}$$

and by substituting $x = -(u/\theta)^2$, we write the Maxwell distribution (2) as

$$p(u;\theta) \sim \lim_{\kappa\to\infty}\left[1+\frac{1}{\kappa}\cdot\left(\frac{u}{\theta}\right)^2\right]^{-\kappa}. \tag{5}$$

In the generic case, we consider that the speed-scale parameter $\theta$ depends also on $\kappa$ and thus is denoted by $\theta_\kappa$. However, the ordinary parameter $\theta$ has to be recovered in the limit

$$\theta = \lim_{\kappa\to\infty}\theta_\kappa.$$

Thus we have

$$p(u;\theta_\kappa;\kappa) \sim \left[1+\frac{1}{\kappa}\cdot\left(\frac{u}{\theta_\kappa}\right)^2\right]^{-\kappa},\ p(u;\theta) = \lim_{\kappa\to\infty} p(u;\theta_\kappa;\kappa), \tag{6}$$

where $p(u;\ \theta_\kappa;\kappa)$ gives the deformation of the Maxwell distribution in terms of the $\kappa$ index (in regards to the deformation of the exponential distribution [see, e.g., *Silva et al.*, 1998; *Yamano*, 2002]). Then, the following questions arise:

[13] 1. Why should $p(u;\theta_\kappa;\kappa)$ describe systems only for one single index, $\kappa \to +\infty$?

[14] 2. If $\kappa \to +\infty$ stands for systems in equilibrium, could finite values of $\kappa$ correspond to stationary states out of equilibrium?

[15] 3. The solar wind, as well as any other space plasmas, are driven nonlinear nonequilibrium systems, tending slowly to such stationary states out of equilibrium [e.g., *Burlaga and Viñas*, 2005]. Could these be described by $p(u;\theta_\kappa;\kappa)$?

[16] Indeed, in the light of (6), we claim that the deformed Maxwellian $p(u;\theta_\kappa;\kappa)$ describes systems not just for the specific value of $\kappa \to \infty$, which coincides with the classical Maxwellian, but for any other finite values of $\kappa$. As we shall see, finite values of $\kappa$ correspond to stationary states out of equilibrium, while $p(u;\theta_\kappa;\kappa)$ has its origin to the Tsallis statistical mechanics. The constructed, deformed Maxwellian,

**Table 1.** Examples of Observational Values of the Power Indices $\kappa = \kappa^* - 1 = \gamma = \gamma_E - \frac{1}{2} = \frac{1}{2}\gamma_V$, Used in Space Plasmas[a]

| Publication | $\gamma_E$ | $\gamma_v$ | $\gamma$ | $\kappa$ | $\kappa^*$ | Comments |
|---|---|---|---|---|---|---|
| *Decker et al.* [2005] | 2.13 | 3.26 | 1.63 | **1.63** | 2.63 | second kind kappa distribution |
| *Fisk and Gloeckler* [2006] | 2 | **3** | 1.5 | 1.5 | 2.5 | suprathermal power law tail |
| *Dialynas et al.* [2009] | >3 | >5 | >2.5 | >2.5 | **>3.5** | first kind kappa distribution |
| *Dayeh et al.* [2009] | <3 | <5 | **<2.5** | <2.5 | <3.5 | suprathermal power law tail |

[a]Bold values concern the quantities that were extracted directly by the authors. In particular, through the analysis of *Decker et al.* [2005], the value of spectral index $\gamma$ was extracted for hydrogen ions by utilizing the second kind of kappa distribution. *Fisk and Gloeckler* [2006] plotted the probability distribution $p(u)$, so that the contribution of the density states of velocities, $g_V(u)$, was excluded. They argued for a universal power law in the suprathermal region, $p_{H-E}(u) \sim u^{-5}$, and thus, $p_{H-E}(u)\, g_V(u) \sim u^{-3}$, or $\gamma_V \cong 3$. *Dialynas et al.* [2009] expressed their results directly in terms of the spectral index $\gamma$, while the bold $\kappa^*$ value means that they utilized the first kind of kappa distribution. *Dayeh et al.* [2009] estimated directly the spectral index in the spectra of the heavy ions CNO, Ne-S, and Fe.

$p(u;\theta_\kappa;\kappa)$, is the so-called kappa distribution, which has already been used in space physics for more than four decades.

## 3. Describing the Space Plasmas: The Kappa Distribution

[17] The classical Maxwellian distribution does a good job of describing the velocities $\vec{u}$ of the ion populations of the solar wind (and other space plasmas), primarily in the low-energy (L-E) region [e.g., *Gruntman*, 1992; *Hammond et al.*, 1995], that is

$$p_{L-E}(\vec{u}) \sim e^{-(|\vec{u}-\vec{u}_b|/\theta)^2}, \tag{7}$$

where $\vec{u}$ and $\vec{u}_b$ stand for the ion and bulk flow velocities, measured with respect to the observing spacecraft's reference frame.

[18] On the other hand, the high-energy (H-E) (or suprathermal) region of ion distributions is non-Maxwellian, governed rather by power law tails [e.g., *Decker et al.*, 2005; *Fisk and Gloeckler*, 2006], that is

$$p_{H-E}(\vec{u}) \sim |\vec{u}-\vec{u}_b|^{-2(\gamma+1)}, \tag{8}$$

where the power parameter $\gamma$ is called the spectral index.

[19] An empirical functional form for describing the distribution of energy over the whole spectrum, both the low-energy Maxwellian core and the high-energy power law tail, was first proposed by *Vasyliũnas* [1968]. It is widely known as the kappa distribution, since it depends on an index symbolized with the Greek letter $\kappa$ ("kappa"), namely

$$p^{(1)}(\vec{u};\theta_\kappa;\kappa) \sim \left[1+\frac{1}{\kappa}\cdot\left(\frac{|\vec{u}-\vec{u}_b|}{\theta_\kappa}\right)^2\right]^{-\kappa}. \tag{9}$$

[20] The work of *Vasyliũnas* [1968] was related to a survey of low-energy electrons of the Earth's magnetosphere. Since then, this empirical distribution has been used for describing ions in various magnetospheres [e.g., *Dialynas et al.*, 2009]. This distribution has also been utilized by several solar wind studies [e.g., *Collier et al.*, 1996; *Chotoo et al.*, 2000; *Nieves-Chinchilla and Viñas*, 2008a, 2008b], where they succeeded in characterizing solar wind ions and magnetic clouds. However, these results do not claim a universal value of the index $\kappa$ and in some cases require different values of $\kappa$ to describe different energy ranges [*Collier et al.*, 1996]. Of particular interest are the results of *Dialynas et al.* [2009], where the values of $\kappa$ are calculated for a large number of samples, organized by the L shell of Saturn over 5–20 planet radii (see section 6 and Table 1).

[21] Speaking more precisely, the empirical distribution of *Vasyliũnas* [1968] was not referring to the formulation of (9) but to the following:

$$p^{(2)}(\vec{u};\theta_\kappa;\kappa) \sim \left[1+\frac{1}{\kappa}\cdot\left(\frac{|\vec{u}-\vec{u}_b|}{\theta_\kappa}\right)^2\right]^{-\kappa-1}. \tag{10}$$

The expressions of (9) and (10) constitute what we call the first and second kind of kappa distributions. It is apparent that there should be two different ways to denote for the $\kappa$-indices, $\kappa^{(1)}$, $\kappa^{(2)}$, and the speed-scale parameters, $\theta_\kappa^{(1)}$ and $\theta_\kappa^{(2)}$, characterizing the distributions $p^{(1)}$ and $p^{(2)}$, respectively. However, we adopt the simple symbolism of denoting with an asterisk the parameters of the first kind, i.e.,

$$p^{(1)}(\vec{u};\theta_\kappa^*;\kappa^*) \sim \left[1+\frac{1}{\kappa^*}\cdot\left(\frac{|\vec{u}-\vec{u}_b|}{\theta_\kappa^*}\right)^2\right]^{-\kappa^*},$$
$$p^{(2)}(\vec{u};\theta_\kappa;\kappa) \sim \left[1+\frac{1}{\kappa}\cdot\left(\frac{|\vec{u}-\vec{u}_b|}{\theta_\kappa}\right)^2\right]^{-\kappa-1}. \tag{11}$$

[22] The first kind of kappa distribution is less widely used than the second kind, which is adopted by the majority of the researchers in the field [e.g., see *Kivelson and Russell*, 1995; *Collier*, 1995; *Gloeckler and Geiss*, 1998; *Prested et al.*, 2008; *Heerikhuisen et al.*, 2008]. A possible reason for the dominance of the second kind is the coincidence of the spectral index $\gamma$ with the $\kappa$ index for three-dimensional systems (see section 6). Notice also that if the speed scales were related as $\sqrt{\kappa^*}\cdot\theta_\kappa{}^* = \sqrt{\kappa}\cdot\theta_\kappa$, then the two kinds would be equivalent under the transformation of $\kappa^* = \kappa + 1$ (see section 4.1). Under this transformation the two kinds of kappa distribution are identical. However, for a common index $\kappa^* = \kappa$, the distributions are different, even though they have similar shapes (especially for large values of kappa).

[23] Furthermore, we verify the high- and low-energy asymptotic limits of the kappa distribution. We show both

the asymptotic behaviors for $p^{(2)}$, while similar approximations can be found for $p^{(1)}$. Namely,

$$\ln p^{(2)}_{\text{L-E}}(\vec{u};\theta_\kappa;\kappa) \sim -(\kappa+1)\cdot\ln\left[1+\frac{1}{\kappa}\cdot\frac{|\vec{u}-\vec{u}_b|^2}{\theta_\kappa^2}\right] \cong -\frac{\kappa+1}{\kappa}\cdot\frac{|\vec{u}-\vec{u}_b|^2}{\theta_\kappa^2}, \Rightarrow$$
$$p^{(2)}_{\text{L-E}}(\vec{u};\theta_\kappa;\kappa) \sim e^{-\left(\vec{u}/\tilde{\theta}_\kappa\right)^2},\ \tilde{\theta}_\kappa^2 \equiv \frac{\theta_\kappa^2}{1+\frac{1}{\kappa}}, \tag{12}$$

while

$$p^{(2)}_{\text{H-E}}(\vec{u};\theta_\kappa;\kappa) \sim \left[1+\frac{1}{\kappa}\cdot\frac{|\vec{u}-\vec{u}_b|^2}{\theta_\kappa^2}\right]^{-\kappa-1} \cong \left[\frac{1}{\kappa}\cdot\frac{|\vec{u}-\vec{u}_b|^2}{\theta_\kappa^2}\right]^{-\kappa-1} \sim |\vec{u}-\vec{u}_b|^{-2(\kappa+1)}, \tag{13}$$

which prescribe a Maxwellian core as (7) and a power law tail as (8), respectively. Therefore we justify the role of kappa distribution in connecting in one single distribution, both the Maxwellian core observed in the low-energy region, and the power law tail observed in the high-energy region. Moreover, by comparing (8) and (13) we show that $\gamma = \kappa$ (see also section 6).

[24] Both the kappa distributions $p^{(1,2)}$ have been utilized for various positive values of the $\kappa$ index. However, it is remarkable that they can be defined also for negative values of $\kappa$. The restriction is that the quantity included in the outer brackets of the distributions has to be nonnegative. For example, for $p^{(2)}$ (and similarly for $p^{(1)}$),

$$1+\frac{1}{\kappa}\cdot\left(\frac{|\vec{u}-\vec{u}_b|}{\theta_\kappa}\right)^2 \geq 0, \tag{14}$$

which implies that for $\kappa < 0$, the restriction $|\vec{u}-\vec{u}_b| < \sqrt{|\kappa|}\,\theta_\kappa$ is required. In order to avoid any implications of this type in relevant computations, a cutoff condition is added through the operation,

$$[x]_+ \equiv \begin{cases} x, & \text{if } x \geq 0\,, \\ 0, & \text{if } x \leq 0\,, \quad x \in \Re\,. \end{cases} \tag{15}$$

This is widely known as the Tsallis cutoff condition, namely, for $\kappa < 0$, $|\vec{u}-\vec{u}_b| > \sqrt{|\kappa|}\,\theta_\kappa$, the distributions $p^{(1,2)}$ vanish. Thus the expressions in (11) are rewritten as

$$p^{(1)}(\vec{u};\theta^*_\kappa;\kappa^*) \sim \left[1+\frac{1}{\kappa^*}\cdot\left(\frac{|\vec{u}-\vec{u}_b|}{\theta^*_\kappa}\right)^2\right]_+^{-\kappa^*},$$
$$p^{(2)}(\vec{u};\theta_\kappa;\kappa) \sim \left[1+\frac{1}{\kappa}\cdot\left(\frac{|\vec{u}-\vec{u}_b|}{\theta_\kappa}\right)^2\right]_+^{-\kappa-1}. \tag{16}$$

[25] Yet another modified version of the first kind of kappa distribution was suggested by *Leubner and Vörös* [2005],

$$p^{(bk)}(\vec{u};\theta^{**}_\kappa;\kappa^{**}) \sim \left\{\left[1+\frac{1}{\kappa^{**}}\cdot\left(\frac{|\vec{u}-\vec{u}_b|}{\theta^{**}_\kappa}\right)^2\right]_+^{-\kappa^{**}} + \left[1-\frac{1}{\kappa^{**}}\cdot\left(\frac{|\vec{u}-\vec{u}_b|}{\theta^{**}_\kappa}\right)^2\right]_+^{\kappa^{**}}\right\}, \tag{17}$$

which combines the normalized sum of two kappa distributions of the first kind, having opposite indices, $\kappa^{**}$ and $-\kappa^{**}$, i.e., $p^{(bk)}(\vec{u};\theta_\kappa^{**};\kappa^{**}) \sim p^{(1)}(\vec{u};\theta_\kappa^{**};\kappa^{**}) + p^{(1)}(\vec{u};\theta_\kappa^{**};-\kappa^{**})$. For this reason it is called bi-kappa distribution (denoted by "$bk$"). For the values $-|\vec{u}-\vec{u}_b|^2/\theta_\kappa^{**2} < \kappa^{**} < 0$, the first term of bi-kappa distribution, i.e., $p^{(1)}(\vec{u};\theta_\kappa^{**};\kappa^{**})$, cannot be defined and vanishes through the Tsallis cutoff condition. But the second term, i.e., $p^{(1)}(\vec{u};\theta_\kappa^{**};-\kappa^{**})$, remains finite. Similarly, for the values $0 < \kappa^{**} < |\vec{u}-\vec{u}_b|^2/\theta_\kappa^{**2}$, the second term $p^{(1)}(\vec{u};\theta_\kappa^{**};-\kappa^{**})$ vanishes, while the first one, $p^{(1)}(\vec{u};\theta_\kappa^{**};\kappa^{**})$, persists. In such a way, the bi-kappa distribution suggests that for $|\kappa^{**}| > |\vec{u}-\vec{u}_b|^2/\theta_\kappa^{**2}$, both the terms $p^{(1)}(\vec{u};\theta_\kappa^{**};\kappa^{**})$ and $p^{(1)}(\vec{u};\theta_\kappa^{**};-\kappa^{**})$ persist and contribute to the whole distribution $p^{(bk)}(\vec{u};\theta_\kappa^{**};\kappa^{**})$; thus a duality of $\kappa^{**}$-indices characterizes the system. Namely, if $\kappa^{**} = \kappa_1^{**}$ is one observed $\kappa^{**}$ index, then $\kappa^{**} = \kappa_2^{**} = -\kappa_1^{**}$ is also a second $\kappa$ index that characterizes the system.

[26] In a similar way, other versions of kappa distributions have been modified in order to describe a power law of "multiscaling index," namely, a power law with its index being different for several scales, especially in the H-E region. However, they share this lack of theoretical grounding. For example, see the empirical expression of *Hawkins et al.* [1998], optimized for describing the anisotropic fluxes of energetic ions in the Jovian magnetosphere, modified even further by *Mauk et al.* [2004]. Even though these versions of kappa distributions are more flexible than the bi-kappa version (since the two involved $\kappa$ indices are not fixed to have opposite values), their expression is simply empirical with more free parameters available to fit the data. We argue that any modifications combining two kappa distributions [e.g., see *Leubner*, 2004a] should utilize a convolution of kappa distributions of the second kind, characterized by different indices $\kappa_1$, $\kappa_2$; this analysis is the topic of future work. Throughout this study, we will deal simply with the first and second kind of kappa distributions.

[27] First we discuss the permissible values of the kappa indices, $\kappa^*$ and $\kappa$. In the classical case, where the probability distribution decays exponentially, the relevant integrals of normalization and of mean energy (second statistical moment of velocity) converge for any power-like expression of the density of velocity states, $g_V(u)$ (see Appendix B (B16)),

$$\int_0^\infty p(u)\,g_V(u)du < +\infty,\quad \int_0^\infty u^2 p(u)\,g_V(u)du < +\infty.$$

[28] However, the convergence is not obvious for non-exponential decay, as in the case of kappa distributions where we have power law-like decay. As $u \to \infty$, $u^2\, p(u)\, g_V(u)$ is larger than $p(u)\, g_V(u)$, and thus if the second moment integral $\langle u^2 \rangle = \int_0^\infty u^2\, p(u)\, g_V(u) du$ converges, so does the normalization integral $\int_0^\infty p(u)\, g_V(u) du$. The integrals converge as soon as the integrant in the high-energy limit attains at least a power law decay of $1/u^r$, with $r > 1$ (see Appendix A (A8), (A9)). In the case of the first kind of kappa distribution, we have: $p(u) \sim u^{-2\kappa^*} \Rightarrow u^2\, p(u)\, g_V(u) \sim u^{-2\kappa^*+4}$, so that for $\langle u^2 \rangle < +\infty$, $2\kappa^* - 4 > 1$, or $\kappa^* > 5/2$. In the case of the second kind of kappa distribution, we have $p(u) \sim u^{-2\kappa-2} \Rightarrow u^2\, p(u)\, g_V(u) \sim u^{-2\kappa+2}$, so that for $\langle u^2 \rangle < +\infty$, $2\kappa - 2 > 1$, or $\kappa > 3/2$. In section 4.1 we will see that the two kinds of kappa distributions can be transformed to each other using $\kappa^* = \kappa + 1$, which is consistent with the relation between the lower limits of $\kappa^* > 5/2$ and $\kappa > 3/2$.

[29] Finally, we stress that the restrictions of $\kappa$-indices have been already considered [e.g., see *Leubner*, 2002; *Shizgal*, 2007] (see also the work of *Ferri et al.* [2005], which concerns the equivalent restriction on $q$ indices (see section 4.1)). Here, by considering the restriction of $\kappa$-indices for both the kinds of kappa distributions, we evaluate the consequences for spectral indices from space physics observations (section 6), as well as the influence on the physical temperature and its relation with the classical temperature in equilibrium (section 5).

# 4. Connection With Tsallis Statistical Mechanics

## 4.1. Consistency of Kappa Distributions With Tsallis Statistical Mechanics

[30] Consider the following transformation of the $\kappa$ index:

$$\kappa \equiv \frac{1}{q-1}, \text{ or, } q \equiv 1 + \frac{1}{\kappa}, \tag{18}$$

(similarly for the indices with asterisk (*)). Then, the two kinds of kappa distributions in (16) become

$$p^{(1)}(\vec{u};\theta_\kappa^*;\kappa^*) \sim \left[1 + \frac{1}{\kappa^*}\cdot\left(\frac{|\vec{u}-\vec{u}_b|}{\theta_\kappa^*}\right)^2\right]_+^{-\kappa^*} \Rightarrow$$
$$p^{(1)}\left(\vec{u};\theta_q^*;q^*\right) \sim \left[1 - \left(1-q^*\right)\cdot\left(\frac{|\vec{u}-\vec{u}_b|}{\theta_q^*}\right)^2\right]_+^{\frac{1}{1-q^*}}, \tag{19}$$

$$p^{(2)}(\vec{u};\theta_\kappa;\kappa) \sim \left[1 + \frac{1}{\kappa}\cdot\left(\frac{|\vec{u}-\vec{u}_b|}{\theta_\kappa}\right)^2\right]_+^{-\kappa-1} \Rightarrow$$
$$p^{(2)}\left(\vec{u};\theta_q;q\right) \sim \left[1 - (1-q)\cdot\left(\frac{|\vec{u}-\vec{u}_b|}{\theta_q}\right)^2\right]_+^{\frac{q}{1-q}}, \tag{20}$$

where we also set $\theta_q \equiv \theta_\kappa$, $\theta_q^* \equiv \theta_\kappa^*$. In the formalism of Tsallis statistical mechanics, there is a closed form for describing the function

$$f(x;q) = \left[1 + (1-q)\cdot x\right]_+^{\frac{1}{1-q}}, \tag{21}$$

that is the so-called $q$-deformed exponential, denoted by $\exp_q(x)$ [e.g., *Silva et al.*, 1998; *Yamano*, 2002]. Hence

$$p^{(1)}\left(\vec{u};\theta_q^*;q^*\right) \sim \exp_{q^*}\left[-\left(\frac{|\vec{u}-\vec{u}_b|}{\theta_q^*}\right)^2\right],$$
$$p^{(2)}\left(\vec{u};\theta_q;q\right) \sim \exp_q\left[-\left(\frac{|\vec{u}-\vec{u}_b|}{\theta_q}\right)^2\right]^q, \tag{22}$$

where we consider different indices $q^*$, $q$, and characteristic speed scales $\theta_q^*$, $\theta_q$, for each of the two kinds of distributions $p^{(1)}$, $p^{(2)}$, respectively.

[31] On the other hand, within the framework of Tsallis statistical mechanics, the Canonical probability distribution in the continuous description of an energy spectrum is given by (see Appendix B (B15)),

$$p\left(\varepsilon;T_q;q\right) \sim \exp_q\left[-\frac{1}{1_q\left(\frac{3}{2}\right)}\cdot\frac{\varepsilon}{k_B T_q}\right], \tag{23}$$

which is expressed in terms of the physical temperature $T_q$. We use the notation $1_q(u) \equiv 1 + (1-q)\,u$, that is the $q$-deformed "unit function," defined in Appendix A (equation (A12)). On the other hand, one of the fundamental aspects of Tsallis statistical mechanics concerns the escort probability distribution $P$, which can be expressed in terms of the ordinary probability distribution $p$, and vice versa [*Beck and Schlogl*, 1993] (see also Appendix B (B8, B15)),

$$P\left(\varepsilon;T_q;q\right) \sim p\left(\varepsilon;T_q;q\right)^q \sim \exp_q\left[-\frac{1}{1_q\left(\frac{3}{2}\right)}\cdot\frac{\varepsilon}{k_B T_q}\right]^q. \tag{24}$$

The escort probability distribution has a fundamental role in contrast to the ordinary probability distribution, since the expectation values are expressed in terms of the escort probability (called escort expectation values or escort mean values) (*Tsallis et al.* [1998] is the pioneer work on this topic [see also *Tsallis*, 1999; *Gell-Mann and Tsallis*, 2004; *Tsallis*, 2009b]). Thus the physical meaning of the statistical moments is carried out only by the escort probability distribution (denoted by the symbol $\langle\ \rangle_q$) [e.g., see *Prato and Tsallis*, 1999]. Following Tsallis, the escort mean of a function of energy, $f(\varepsilon)$, is given by

$$\langle f(\varepsilon)\rangle_q = \frac{\int_0^\infty P\left(\varepsilon;T_q;q\right) f(\varepsilon)\, g_E(\varepsilon) d\varepsilon}{\int_0^\infty P\left(\varepsilon;T_q;q\right) g_E(\varepsilon) d\varepsilon} = \frac{\int_0^\infty p\left(\varepsilon;T_q;q\right)^q f(\varepsilon)\, g_E(\varepsilon) d\varepsilon}{\int_0^\infty p\left(\varepsilon;T_q;q\right)^q g_E(\varepsilon) d\varepsilon}, \tag{25}$$

where $g_E(\varepsilon)$ is the density of energy states. As a specific case, the internal energy $U_q$ is estimated as the escort expectation value of energy $\langle\varepsilon\rangle_q$, that is

$$U_q = \langle\varepsilon\rangle_q = \frac{\int_0^\infty P\left(\varepsilon;T_q;q\right) \varepsilon\, g_E(\varepsilon) d\varepsilon}{\int_0^\infty P\left(\varepsilon;T_q;q\right) g_E(\varepsilon) d\varepsilon} = \frac{\int_0^\infty p\left(\varepsilon;T_q;q\right)^q \varepsilon\, g_E(\varepsilon) d\varepsilon}{\int_0^\infty p\left(\varepsilon;T_q;q\right)^q g_E(\varepsilon) d\varepsilon}, \tag{26}$$

and by considering the (three-dimensional) density of energy states, that is $g_E(\varepsilon) \sim \varepsilon^{1/2}$ (B16), we find

$$U_q = \frac{3}{2} k_B T_q, \tag{27}$$

(see Appendix B (B19)). Therefore the kinetic temperature $T_K$, defined by

$$U_q \equiv \frac{3}{2} k_B T_K, \tag{28}$$

coincides with the physical temperature, $T_K = T_q$. This result is remarkable in that the system is characterized by the same internal energy (mean kinetic energy) or kinetic temperature, independently of the specific stationary state that is relaxing. This implies that the physical temperature $T_q$ constitutes the appropriate definition of temperature, since it is common for all the stationary states, independently of their $q$ index.

[32] Hence the ordinary and escort probability distributions are readily written as

$$p(\varepsilon; T_q; q) \sim \exp_q\left[-\frac{1}{1_q\left(\frac{3}{2}\right)} \cdot \frac{\varepsilon}{k_B T_q}\right],$$
$$P(\varepsilon; T_q; q) \sim \exp_q\left[-\frac{1}{1_q\left(\frac{3}{2}\right)} \cdot \frac{\varepsilon}{k_B T_q}\right]^q, \tag{29}$$

or, in terms of velocities

$$p(u; \theta_q; q) \sim \exp_q\left[-\left(\frac{u}{\theta_q}\right)^2\right], P(u; \theta_q; q) \sim \exp_q\left[-\left(\frac{u}{\theta_q}\right)^2\right]^q,$$
$$\text{with } \theta_q \equiv \sqrt{1_q\left(\frac{3}{2}\right) \cdot \frac{2k_B T_q}{\mu}}. \tag{30}$$

The coincidence of the escort probability distribution in (30) with the kappa distribution of the second kind in (22) is evident:

$$p^{(2)}(\vec{u}; \theta_q; q) = P(\vec{u}; \theta_q; q) \sim \exp_q\left[-\left(\frac{|\vec{u} - \vec{u}_b|}{\theta_q}\right)^2\right]^q \tag{31}$$

where we restored the bulk flow velocity, $\vec{u}_b$.

[33] However, if the statistical moments were carried out by the ordinary probability distribution, then (25) and (26) would be written as

$$\langle f(\varepsilon)\rangle = \frac{\int_0^\infty p\left(\varepsilon; T_{q^*}; q^*\right) f(\varepsilon)\, g_E(\varepsilon) d\varepsilon}{\int_0^\infty p\left(\varepsilon; T_{q^*}; q^*\right) g_E(\varepsilon) d\varepsilon}, \tag{32}$$

and

$$U_{q^*} = \langle\varepsilon\rangle = \frac{\int_0^\infty p\left(\varepsilon; T_{q^*}; q^*\right) \varepsilon\, g_E(\varepsilon) d\varepsilon}{\int_0^\infty p\left(\varepsilon; T_{q^*}; q^*\right) g_E(\varepsilon) d\varepsilon} = \frac{1}{1_{q^*}(1)} \cdot \frac{3}{2} k_B T_{q^*}, \tag{33}$$

(see Appendix B (B18), (B19)) where we continue to use asterisks to indicate parameters associated with ordinary probability distribution (e.g., $q^*$, $T_q^*$) as opposed to the escort distribution. Of course, the system has to be characterized by the same internal energy, independently of the probability distribution that is being considered, namely,

$$U_{q^*} = U_q \equiv \frac{3}{2} k_B T_K \Rightarrow T_K = \frac{1}{1_{q^*}(1)} \cdot T_{q^*}. \tag{34}$$

Therefore the kinetic temperature $T_K$ coincides with the physical temperature, $T_q$, only when the expectation values are estimated by means of the escort probability distribution. On the contrary, when the expectation values are estimated by means of the ordinary probability distribution, $T_q^*$ does not constitute a well-defined temperature, since it depends on the value of $q^*$ index, $T_{q^*}(q^*) \sim 1_{q^*}(1) = (2 - q^*)$ and does not coincide with $T_K$. Hence we express the (ordinary) probability distribution in terms of the kinetic temperature $T_K$, that is to say, in terms of the physical temperature $T_q$. Namely,

$$p(\varepsilon; T_q; q^*) \sim \exp_{q^*}\left[-\frac{1}{1_{q^*}\left(\frac{5}{2}\right)} \frac{\varepsilon}{k_B T_q}\right], \text{or}$$
$$p\left(u; \theta_q^*; q^*\right) \sim \exp_{q^*}\left[-\left(\frac{u}{\theta_q^*}\right)^2\right], \text{with}$$
$$\theta_q^* \equiv \sqrt{1_{q^*}\left(\frac{5}{2}\right) \frac{2k_B T_q}{\mu}}, \tag{35}$$

where we observe the coincidence of the ordinary probability distribution in (35) with the kappa distribution of the first kind in (22), namely,

$$p^{(1)}\left(\vec{u}; \theta_q^*; q^*\right) = p\left(\vec{u}; \theta_q^*; q^*\right) \sim \exp_{q^*}\left[-\left(\frac{|\vec{u} - \vec{u}_b|}{\theta_q^*}\right)^2\right]. \tag{36}$$

As we have seen in section 3, both the first and the second kind of kappa distributions have been utilized to describe space plasmas. In order to use both types of distributions, the equality $p(\varepsilon; T_q; q^*) = P(\varepsilon; T_q; q)$ is required. Hence we need to find a transformation between the $q$ and $q^*$ indices (or equivalently, between $\kappa$ and $\kappa^*$ indices) in order to ensure that this equality is valid. Indeed, by comparing (31) and (36) we find

$$q^* = 2 - \frac{1}{q}, \text{ or, } \kappa^* = \kappa + 1. \tag{37}$$

[34] The derivation of a kappa distribution through Tsallis statistical mechanics was referred to in the analysis of *Milovanov and Zelenyi* [2000] and *Leubner* [2002]. They showed that the kappa distribution constitutes the Canonical probability distribution by extremizing the Tsallis entropy under the constraints of Canonical Ensemble. However, in

regards to the second constraint, the one of internal energy, they did not consider the escort expectation value. In this case, after extremizing the Tsallis entropy $S_q$, instead of the already derived Canonical probability distribution (see Appendix B (B14)),

$$p\left(\varepsilon;T_q;q\right)\ \sim\ \left[1-(1-q)\cdot\frac{\varepsilon-U_q}{k_BT_q}\right]_+^{\frac{1}{1-q}}, \tag{38}$$

one finds

$$p(\varepsilon;T;q)\ \sim\ \left[1-(q-1)\cdot\frac{\varepsilon}{k_BT}\right]_+^{\frac{1}{q-1}}. \tag{39}$$

In fact, (39) was the first extracted distribution [*Tsallis*, 1988]. However, this result was highly problematic, mainly because it was not invariant for an arbitrary selection of the ground level of the energy. Subsequently, by considering the escort expectation values, *Tsallis et al.* [1998] succeeded in recovering this feature. *Milovanov and Zelenyi* [2000] and *Leubner* [2002, 2004a, 2004b] used (39) to find the first kind of kappa distribution using the transformation

$$\kappa\equiv\frac{1}{1-q},\text{or, } q\equiv 1-\frac{1}{\kappa}, \tag{40}$$

which has the opposite sign compared to (18). *Leubner* [2002] at least mentioned the second kind of kappa distribution but described it as a ‘‘reduced’’ form of the first kind of kappa distribution.

[35] Further analyses [e.g., *Shizgal*, 2007; *Nieves-Chinchilla and Viñas*, 2008a, 2008b] also focused on the first kind of kappa distribution. All of these above analyses were restricted to only $\kappa^* > 0$ or $q^* > 1$. The analysis of *Leubner and Vörös* [2005] was extended to the bi-kappa distribution (17), in order to be valid for $\kappa^* < 0$ or $q^* < 1$, which in terms of $q$ index, can be written as

$$\begin{aligned}p^{(bk)}\left(\vec{u};\theta_q^{**};q^{**}\right)\sim \exp_{q^{**}}&\left[-\left(\frac{|\vec{u}-\vec{u}_b|}{\theta_q^{**}}\right)^2\right]\\&+\exp_{2-q^{**}}\left[-\left(\frac{|\vec{u}-\vec{u}_b|}{\theta_q^{**}}\right)^2\right].\end{aligned} \tag{41}$$

We remark that within these prior analyses there was no reference to the physical temperature $T_q$, to its coincidence with the kinetic temperature $T_K$, or in general, to its relation to the kappa distribution.

[36] Finally, we provide the well known normalized escort Canonical probability distribution, but in a new form (where the Maxwellian is recovered simply as the normalization constant tends to $A(q) \to 1$ (see below)), expressed either in terms of the $q$ index,

$$P\left(\varepsilon;T_q;q\right)=\pi^{-\frac{3}{2}}\cdot\theta_{\text{eff}}^{-3}\cdot A(q)\cdot\left[1+\frac{2(q-1)}{5-3q}\cdot\frac{\varepsilon}{k_BT_q}\right]_+^{-\frac{q}{q-1}}, \tag{42}$$

$$P(u;\theta_{\text{eff}};q)\ =\pi^{-\frac{3}{2}}\cdot\theta_{\text{eff}}^{-3}\cdot A(q)\cdot\left[1+\frac{2(q-1)}{5-3q}\ \cdot\left(\frac{u}{\theta_{\text{eff}}}\right)^2\right]_+^{-\frac{q}{q-1}}, \tag{43}$$

or in terms of the $\kappa$ index,

$$P\left(\varepsilon;T_q;\kappa\right)\ =\pi^{-\frac{3}{2}}\cdot\theta_{\text{eff}}^{-3}\cdot A(\kappa)\cdot\left(1+\frac{1}{\kappa-\frac{3}{2}}\cdot\frac{\varepsilon}{k_BT_q}\right)_+^{-\kappa-1}, \tag{44}$$

$$P(u;\theta_{\text{eff}};\kappa)\ =\pi^{-\frac{3}{2}}\cdot\theta_{\text{eff}}^{-3}\cdot A(\kappa)\cdot\left[1+\frac{1}{\kappa-\frac{3}{2}}\ \cdot\left(\frac{u}{\theta_{\text{eff}}}\right)^2\right]_+^{-\kappa-1}, \tag{45}$$

where we set the effective speed-scale parameter $\theta_{\text{eff}} = \sqrt{2k_BT_q/\mu}$, and the normalization constants

$$\begin{aligned}A(q)&\equiv\sqrt{8}\cdot\left(\frac{q-1}{5-3q}\right)^{\frac{3}{2}}\frac{\Gamma\left(\frac{q}{q-1}\right)}{\Gamma\left(\frac{1}{q-1}-\frac{1}{2}\right)},\\ A(\kappa)&\equiv A\left(q=1+\frac{1}{\kappa}\right)=\frac{\left(\kappa-\frac{3}{2}\right)^{-\frac{3}{2}}\Gamma(\kappa+1)}{\Gamma\left(\kappa-\frac{1}{2}\right)}.\end{aligned} \tag{46}$$

All the above expressions are derived from the normalization relations (Appendix B)

$$\begin{aligned}1&=\int_0^\infty P\left(\varepsilon;T_q;q\right)\,g_{\text{E}}(\varepsilon)d\varepsilon,\text{or,}\\ 1&=\int_0^\infty P(u;\theta_{\text{eff}};q)\,g_{\text{V}}(u)du.\end{aligned} \tag{47}$$

### 4.2. Inconsistency of Kappa Distributions With the Boltzmann-Gibbs Statistical Mechanics

[37] In contrast to using Tsallis statistical mechanics, attempting to theoretically derive a kappa distribution from the standard BG statistical mechanics is highly problematic. However, such an approach was attempted by several authors [e.g., *Montroll and Shlesinger*, 1983; *Treumann et al.*, 1999, 2004; *Collier*, 2004]. The idea was quite simple; the Boltzmannian entropy was maximized with the constraint of energy $\langle\varepsilon\rangle$ replaced by the constraint of the logarithm of energy, $\langle\ln(\varepsilon)\rangle$. Indeed, this constraint $\langle\ln(\varepsilon)\rangle$ yields the Boltzmannian entropy to be maximized for a power law probability distribution, $p(\varepsilon) \sim \varepsilon^{-\kappa}$. Let us examine this topic further.

[38] We consider the case where the constraint of the mean energy $U = \langle\varepsilon\rangle$ is replaced by the one of $\varphi$ mean, defined by $\varphi\ (U_\varphi) \equiv \langle\varphi\ (\varepsilon)\rangle$, with $\varphi$ being a strictly monotonic function. Then, the maximization of BG entropy and along the Gibb's path (see Appendix B), is derived from

$$\frac{\partial}{\partial p_j}G\left(\{p_k\}_{k=1}^W\right)=0,\ \forall\ j=1,\ldots,W, \tag{48}$$

with

$$G\left(\{p_k\}_{k=1}^{W}\right) = S\left(\{p_k\}_{k=1}^{W}\right) + \lambda_1 \sum_{k=1}^{W} p_k + \lambda_2 \sum_{k=1}^{W} p_k \varphi(\varepsilon_k),$$
$$S\left(\{p_k\}_{k=1}^{W}\right) = -\sum_{k=1}^{W} p_k \, \ln(p_k), \tag{49}$$

(for simplicity the Boltzmann constant is temporary ignored), where we return to the discrete description of states $k = 1, \ldots, W$. Then, we have

$$-\ln(p_j) - 1 + \lambda_1 + \lambda_2 \varphi(\varepsilon_j) = 0, \text{ or, } p_j = \frac{1}{Z_\varphi} \cdot e^{-\beta\varphi(\varepsilon_j)}, \tag{50}$$

where $Z_\varphi \equiv e^{1-\lambda_1} = \sum_{k=1}^{W} e^{-\beta\varphi(\varepsilon_k)}$ is the relevant partition function, while $\lambda_2$ is the second Lagrangian multiplier, related to the temperature as $\lambda_2 = -\beta = -(k_B T)^{-1}$. By choosing $\varphi$ to be the logarithmic function, $\varphi(\varepsilon) = \ln(\varepsilon)$, then from (50) we derive the power law distribution

$$p_j \sim e^{-\beta \ln(\varepsilon_j)} = \varepsilon_j^{-\beta}, \tag{51}$$

or, by recalling the continuous description of states,

$$p(\varepsilon) \sim \varepsilon^{-\beta}. \tag{52}$$

Furthermore, the kappa distribution can be attained by considering that the energy is the sum of the kinetic energy $\frac{1}{2}\mu \cdot u^2$ and a nonkinetic factor $\varepsilon_0$, that is, $\varepsilon = \varepsilon_0 + (1/2)\mu \cdot u^2$. Hence

$$p(u) \sim \left(\varepsilon_0 + \frac{1}{2}\mu \cdot u^2\right)^{-\beta}, \tag{53}$$

which provides the kappa distribution of the first kind,

$$p(u;\kappa) \sim \left[1 + \frac{1}{\kappa} \cdot \left(\frac{u}{\theta_\kappa}\right)^2\right]^{-\kappa}, \tag{54}$$

under the considerations

$$\kappa \equiv \beta, \theta_\kappa \equiv \sqrt{\frac{2\varepsilon_0}{\mu} \cdot \frac{1}{\kappa}}. \tag{55}$$

In order for the temperature to appear in (54), we calculate the logarithmic mean of energy, $U_{\ln}$,

$$\ln(U_{\ln}) \equiv \langle \ln(\varepsilon) \rangle = \ln(\varepsilon_0) + A_\kappa \times \int_0^\infty \left[1 + \frac{1}{\kappa} \cdot \left(\frac{u}{\theta_\kappa}\right)^2\right]^{-\kappa} \cdot \ln\left[1 + \frac{1}{\kappa} \cdot \left(\frac{u}{\theta_\kappa}\right)^2\right] \cdot 4\pi \left(\frac{u}{\theta_\kappa}\right)^2 d\left(\frac{u}{\theta_\kappa}\right), \tag{56}$$

where $A_\kappa \, \theta_\kappa^{-3}$ constitutes the normalization constant. By setting

$$c_\kappa \equiv \frac{2\kappa}{3} \cdot A_\kappa \cdot \int_0^\infty \left[1 + \frac{1}{\kappa} \cdot \left(\frac{u}{\theta_\kappa}\right)^2\right]^{-\kappa} \times \ln\left[1 + \frac{1}{\kappa} \cdot \left(\frac{u}{\theta_\kappa}\right)^2\right] \cdot 4\pi \left(\frac{u}{\theta_\kappa}\right)^2 d\left(\frac{u}{\theta_\kappa}\right), \tag{57}$$

so that

$$c_\kappa = c_\kappa(\kappa), \lim_{\kappa\to\infty} c_\kappa = 1,$$

then, from (56) we obtain

$$U_{\ln} = \varepsilon_0 \cdot e^{\frac{3}{2} c_\kappa/\kappa}, \tag{58}$$

and (54) can be rewritten as follows:

$$p(u;\kappa) \sim \left[1 + \frac{1}{\kappa/c_\kappa} \cdot \left(\frac{u}{\theta_{\rm eff}}\right)^2\right]^{-\kappa}, \tag{59}$$

where $\theta_{\rm eff} = \sqrt{2k_B T_{\rm K}/\mu}$ determines the effective speed-scale parameter that is independent of $\kappa$ index, while the kinetically defined, temperature $T_{\rm K}$, is now given by

$$U_{\ln} \equiv \frac{3}{2} k_B T_{\rm K} \cdot d_\kappa, \quad d_\kappa \equiv \frac{e^{\frac{3}{2} c_\kappa/\kappa}}{\frac{3}{2} c_\kappa/\kappa}, \tag{60}$$

so that the Maxwellian distribution to be recovered for $\kappa \to \infty$, as expected:

$$p(u) \sim e^{-(u/\theta_{\rm eff})^2}. \tag{61}$$

[39] This whole procedure seems to produce reasonable results. As we will show, however, there is a fundamental problem with this approach. This is caused by the assumption $\kappa \equiv \beta = -\lambda_2$, which clearly postulates that the kinetic temperature $T_{\rm K}$ is not related to the second Lagrangian multiplier $\lambda_2$. Hence if the temperature does not have its origin in $\lambda_2$, then we have to ask where it comes from. Since $k_B \, T_{\rm K} = \varepsilon_0 \cdot c_\kappa/\kappa$, from (58) and (60), then it is apparent that the origin of the temperature is encrypted in the expression of the nonkinetic energy factor $\varepsilon_0$, i.e., $\varepsilon_0 = \varepsilon_0\,(T_{\rm K};\kappa) = k_B T_{\rm K} \cdot \kappa/c_\kappa$, instead of being related to $\lambda_2$.

[40] Such a result is unacceptable, as the definition of temperature has to be developed from statistics and not simply given by an energy expression, such as $\varepsilon\,(T_{\rm K};\kappa) = \varepsilon_0\,(T_{\rm K};\kappa) + \frac{1}{2}\mu \cdot u^2$. In addition, the dependence of $U_{\ln} = U_{\ln}\,(T_{\rm K};\kappa)$ reads that under isothermal procedures, in which $\kappa$ index varies, the internal energy $U_{\ln}$ (or its logarithm) will not have fixed value. After these failures, it is not surprising that the Maxwellian distribution, recovered for $\kappa \to \infty$, requires the second Lagrangian multiplier $\lambda_2$ to be infinite ($\kappa = \beta = -\lambda_2$). In contrast, using Tsallis statistical mechanics, the kinetically defined temperature is given by the physical temperature $T_q$, with $T_q = T \cdot \phi_q\,(T_q;q)$ (see Appendix B (B13)), so that $\lambda_2\,(T_q;q) = -\phi_q\,(T_q;q)/(k_B T_q)$. Since the argument $\phi_q$ is always finite (see Appendix B (B51)), the

same holds for the Lagrangian multiplier $\lambda_2$, even for $q \to 1$, where the Maxwellian distribution is recovered. In fact, it is no surprise that it is not possible to develop a robust grounding kappa distribution within the framework of BG statistical mechanics, since BG statistics does not cover systems in stationary states out of equilibrium. In contrast, the Tsallis generalized framework of statistical mechanics provides a set of proven tools, including a grounded definition of temperature for systems in stationary states out of thermodynamic equilibrium. Moreover, the extracted values of the $q$ index, or of the $\kappa$ index, provide a robust measure of the departure of these systems (such as space plasmas) from equilibrium [e.g., *Burlaga and Viñas*, 2005].

## 5. Definition of Temperature out of Equilibrium and the Physical Temperature

[41] The definition of temperature is controversial whenever the classical weak interactions scenario of BG statistical mechanics is no longer valid. Over the last 2 decades, different concepts of "nonequilibrium temperatures" have been examined. For a classical gas in equilibrium, the definition of the kinetic temperature, $T_{\mathrm{K}}$, emerges from the equipartition of the internal energy

$$U \equiv \frac{f}{2} M\, k_B T_{\mathrm{K}}, \tag{62}$$

where $f$ is the degrees of freedom and $M$ is the number of the gas particles. This definition is often adopted for systems in nonequilibrium [e.g., *Chapman and Cowling*, 1990; *Fort et al.*, 1999].

[42] Alternatively, a completely different definition of nonequilibrium temperature is possible in terms of a nonequilibrium entropy, by analogy to an equilibrium expression [e.g., *Luzzi et al.*, 1997], namely,

$$T_S \equiv \left(\frac{\partial S}{\partial U}\right)^{-1}, \tag{63}$$

which constitutes the thermodynamic definition of temperature. However, *Hoover and Hoover* [2008] claim that away from equilibrium, the phase space probability distribution $p(\vec{x}, \vec{u})$ is typically fractal [*Hoover*, 2001; *Hoover et al.*, 2004]. Hence, the Boltzmannian entropy, that is the phase space average logarithm of $p(\vec{x}, \vec{u})$, diverges. Thus the existence of a nonequilibrium temperature, based on (63) and the BG entropic formulation, appears to be doubtful.

[43] In 1988, Tsallis introduced the generalized formulation of entropy $S_q$, given in Appendix B (equation (B11)). Eventually, it was shown that Tsallis entropy can successfully describe complex systems that are either out of equilibrium or characterized by the presence of long-range interactions [*Tsallis*, 1999]. This was achieved under specific values of the entropic index $q$ (different from $q$ = 1, which recovers the Boltzmannian entropy). Still, when the Tsallis generalized entropy is utilized in (63), it is dubious that a thermometer immersed in a complex system will measure the quantity $(\partial S_q/\partial U_q)^{-1}$. In contrast to this quantity, the definition of the temperature given in (63) is generalized to the physical temperature $T_q$ [*Abe*, 1999; *Rama*, 2000],

$$T_q = \left(\frac{\partial S_q}{\partial U_q}\right)^{-1} \left[1 + (1-q)\cdot S_q/k_B\right]. \tag{64}$$

In this way, the physical temperature $T_q$ generalizes the zeroth law of thermodynamics (that two bodies in thermal equilibrium with a third, are also in thermal equilibrium with each other). Note that *Baranyai* [2000a, 2000b] showed that the (Boltzmannian) kinetic temperature does not absolutely satisfy the zeroth law of thermodynamics. However, the physical temperature $T_q$ is obtained in accordance with the generalized zeroth law [*Abe et al.*, 2001; *Wang et al.*, 2002; *Toral*, 2003]. As mentioned above, the physical temperature $T_q$ serves the role of the kinetic definition of temperature within the framework of Tsallis statistical mechanics. Therefore all the advantages of a kinetically defined temperature, in contrast to other configurational definitions [*Hoover and Hoover*, 2008], can be ascribed to $T_q$. In addition, the inconsistencies concerning the kinetic definition in regards to the zeroth law of thermodynamics [*Baranyai*, 2000a, 2000b] are fully recovered, since the origin of $T_q$ establishes the generalized zeroth law.

[44] Equation (64) shows that the physical temperature $T_q$ is connected with the "Lagrangian temperature" $T$ (the one related to the second Lagrangian multiplier, i.e., $\lambda_2 = -\beta = -1/(k_B\, T)$), through the argument $\phi_q$, defined in Appendix B (equations (B13) and (B51)), namely,

$$T_q = T \cdot \phi_q. \tag{65}$$

In general, $T_q$ and $T$ differ from each other, except at equilibrium ($q \to 1$). In Appendix B we calculate the expression of $\phi_q$ that holds for stationary states out of equilibrium. This is given by

$$\phi_q(q; T_q; \sigma) = 1_q\left(\frac{3}{2}\right)^{1_q\left(\frac{1}{2}\right)} \cdot \left[2\pi\, \Gamma_q\left(\frac{3}{2}\right)\right]^{1-q} \cdot \left(\frac{1}{\sigma^2}\frac{2k_B T_q}{\mu}\right)^{\frac{3}{2}(1-q)}, \tag{66}$$

where $\sigma$ is a characteristic speed scale. Hence we deduce the following relation between $T_q$ and $T$,

$$T_q = C(q) \cdot \left(\frac{1}{\sigma^2}\frac{2k_B}{\mu}\right)^{\frac{\frac{3}{2}(1-q)}{1_q\left(-\frac{3}{2}\right)}} \cdot T^{\frac{1}{1_q\left(-\frac{3}{2}\right)}}, \tag{67}$$

with

$$C(q) \equiv \left\{ 1_q\left(\frac{3}{2}\right)^{1_q\left(\frac{1}{2}\right)} \cdot \left[2\pi\, \Gamma_q\left(\frac{3}{2}\right)\right]^{1-q} \right\}^{\frac{1}{1_q\left(-\frac{3}{2}\right)}}. \tag{68}$$

Notice that from (67), if $\sigma$ is independent of $q$, $T$, and $T_q$, we obtain $T_q \propto T^{1/1_q\left(-\frac{3}{2}\right)}$. If, on the other hand, $\sigma$ is dependent on $T$ or $T_q$ then we can equate the temperature-like (dimensions of temperature) quantity $\sigma^2\, \mu/(2k_B)$ that appears in (67), with one of these, which implies that $T_q \propto T$.

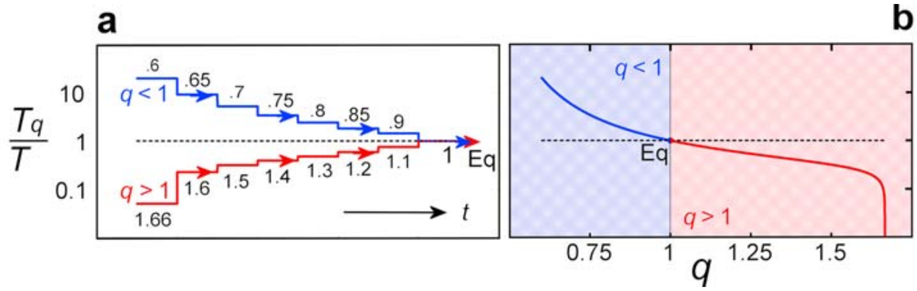


**Figure 2.** (a) Two hypothetical routes of transient (metastable) stationary states toward equilibrium. (b) The relation of physical temperature $T_q$ with the Lagrangian temperature $T$.

Here, we follow the same path as Gibbs, where the only temperature-like quantity emerging from statistics is the inverse of the second Lagrangian multiplier, that is to say $\sigma^2 \mu/(2k_B) = T$, or equivalently, $\sigma \equiv \theta = \sqrt{2k_BT/\mu}$. Then, we have

$$T_q = T \cdot \phi_q(q), \text{with } \phi_q(q) = C(q) = \left\{ 1_q\left(\frac{3}{2}\right)^{1_q\left(\frac{1}{2}\right)} \cdot \left[2\pi\, \Gamma_q\left(\frac{3}{2}\right)\right]^{1-q} \right\}^{\frac{1}{1_q\left(-\frac{3}{2}\right)}}. \tag{69}$$

[45] In such a case, the ratio $T_q/T$ (that is the argument $\phi_q$) depends only on the $q$ index that characterizes a particular stationary state. For the specific stationary state at equilibrium ($q \to 1$), this ratio equals $\phi_q = 1$. For all the other stationary states, this ratio can be either greater or lesser than $\phi_q = 1$, depending on the value of the $q$ index, namely, for $q < 1$ and $q > 1$, respectively, as shown in Figure 2. In Figure 2a we demonstrate two hypothetical routes, in which the system passing through various stationary states gradually approaches equilibrium. Each route shows a monotonic switching of the system between stationary states, characterized either by $q < 1$, where $q$ is gradually increasing, or by $q > 1$, where $q$ is gradually decreasing. In the former case, the values of the ratio $T_q/T$ lie above the horizontal line (which corresponds to equilibrium), while in the latter case, the values lie below the horizontal line. The dependence of the ratio $T_q/T = \phi_q(q)$ is depicted in Figure 2b. As shown, $\phi_q(q)$ constitutes a monotonically decreasing function of $q$, lying in the interval $3/5 = q_{\mathrm{Min}} < q < q_{\mathrm{Max}} = 5/3$. Its largest value is attained for $q = q_{\mathrm{Min}} = 3/5$, that is $\phi_{q,\mathrm{Max}} \cong 19.975$, while its smallest value is zero, attained for $q = q_{\mathrm{Max}} = 5/3$. Given the duality of ordinary escort probabilities and the extracted symmetry on $q$ indices, that is $p \xrightarrow{q} P \xrightarrow{1/q} p$ (equation (B8)), we set the $q$ index domain values so that for each value $q > 1$ corresponds a value $q < 1$. In other words, if $q < 5/3$ are the allowable values for $q > 1$, then $3/5 < q$ shall be the allowable values for $q < 1$, namely, $3/5 < q < 5/3$.

[46] The similarity of the relations $\theta_{\mathrm{eff}} \equiv \sqrt{2k_BT_q/\mu}$ and $\theta \equiv \sqrt{2k_BT/\mu}$ might give the wrong impression that the physical temperature $T_q$ coincides with the Lagrangian temperature $T$. For example, *Heerikhuisen et al.* [2008] referred to $\theta_{\mathrm{eff}}$ as the relevant Maxwellian thermal speed $\theta$, signifying that the difference between $\theta_q \equiv \sqrt{1_q\left(\frac{3}{2}\right) \cdot 2k_BT_q/\mu}$ and $\theta$ is realized only in the presence of the factor $1_q(3/2)$, that is $(\kappa - (3/2))/\kappa$. But this is not correct, because even though in equilibrium we have $T_{\mathrm{K}} = T_q = T$, in stationary states out of equilibrium we have $T_{\mathrm{K}} = T_q \neq T$.

[47] Furthermore, a question that might arise given their difference, is which of the two definitions, $T_q$ or $T$, serves the role of the actual, effective temperature that correctly describes the stationary states out of equilibrium? If $T_q$ is the temperature, then $T$ is a dependent parameter, given by $T = T(T_q;q)$, implying also that $\lambda_2 = \lambda_2(T_q;q)$. On the other hand, if $T$ is the temperature, then $T_q$ is a dependent parameter, given by $T_q = T_q(T;q)$. We address this temperature-like duality as follows: the physical temperature $T_q$ is connected with the escort mean of kinetic energy, $U_q = \langle\varepsilon\rangle_q$, in a similar way that $T$ is connected with $U = \langle\varepsilon\rangle$ at equilibrium ($q \to 1$). Indeed, in Appendix B we show that for a power law density of states $g_{\mathrm{E}}(\varepsilon) \propto \varepsilon^{a-1}$, the internal energy $U_q$ (mean kinetic energy) is given by $U_q = a\, k_BT_q$ (B17). Then, for a system of $M$ particles, with $f$ degrees of freedom each, i.e., of $f \cdot M$ total degrees of freedom, the density of states is given by $g_{\mathrm{E}}(\varepsilon) \propto \varepsilon^{(f\,M/2)-1}$ (that is, $a = fM/2$), and the internal energy is

$$U_q = \frac{f}{2} M\, k_BT_q \equiv \frac{f}{2} M\, k_BT_{\mathrm{K}}. \tag{70}$$

This expression formulates the generalization of the classical expression (62) and shows the equipartition of kinetic energy. Therefore within the framework of Tsallis statistical mechanics (and for the continued description of energy states) the kinetic definition of temperature $T_{\mathrm{K}}$ coincides with the physical temperature $T_q$. If the Lagrangian temperature $T$ were the temperature, then it would be independent of the $q$ index that characterizes the stationary states. Therefore the switching of the system over the stationary states by an isothermal procedure would be characterized by an invariant form for $T$, that is $T = T_q(q)/\phi_q(q)$: constant. This fact has the consequence that the internal energy $U_q$, for each stationary state, would not be invariant, since $U_q = U_q(q) = \frac{3}{2}k_B\, T \cdot \phi_q(q) \propto \phi_q(q)$. In this case, the stationary states cannot be considered as being equivalent, since they describe different internal energies for the same system. In other words, the kinetic temperature is dependent on the value of $q$ index $T_{\mathrm{K}}(q) = T \cdot \phi_q(q) \propto \phi_q(q)$. This inconsistency is recovered if and only if the physical temperature $T_q$ is the temperature. Then, both $T_q = T_{\mathrm{K}}$ and $U_q$ remain invariant, independently of the $q$

index of a stationary state. All the above considerations support our conclusion that the physical temperature $T_q$ is the actual, effective temperature describing the stationary states of a system out of equilibrium.

## 6. Discussion and Conclusions

[48] The tools developed in this paper make it straightforward to compare observations of various space plasma distributions both with each other and with the diverse theories that seek to explain them. For the high-energy suprathermal tails, the asymptotic behavior of the first and second kinds of kappa distributions, $p(\vec{u};\theta_{\text{eff}};\kappa^*)$ and $P(\vec{u};\theta_{\text{eff}};\kappa)$, respectively, are given by

$$p_{\text{H-E}}(\vec{u};\theta_{\text{eff}};\kappa^*) \sim \left[1+\frac{1}{\kappa^*-\frac{5}{2}}\cdot\left(\frac{|\vec{u}-\vec{u}_b|}{\theta_{\text{eff}}}\right)^2\right]_+^{-\kappa^*} \cong \left[\frac{1}{\kappa^*-\frac{5}{2}}\cdot\left(\frac{|\vec{u}-\vec{u}_b|}{\theta_\kappa^*}\right)^2\right]^{-\kappa^*} \sim |\vec{u}-\vec{u}_b|^{-2\kappa^*}, \quad (71)$$

$$P_{\text{H-E}}(\vec{u};\theta_{\text{eff}};\kappa) \sim \left[1+\frac{1}{\kappa-\frac{3}{2}}\cdot\left(\frac{|\vec{u}-\vec{u}_b|}{\theta_{\text{eff}}}\right)^2\right]_+^{-\kappa-1} \cong \left[\frac{1}{\kappa-\frac{3}{2}}\cdot\frac{|\vec{u}-\vec{u}_b|^2}{\theta_\kappa^2}\right]^{-\kappa-1} \sim |\vec{u}-\vec{u}_b|^{-2(\kappa+1)}, \quad (72)$$

where we derive the spherical symmetry, $p_{\text{H-E}}(\vec{u}) \cong p_{\text{H-E}}(u)$, because of the approximation $u \gg u_b$, with $u \equiv |\vec{u}|$, $u_b \equiv |\vec{u}_b|$. This holds because $|\vec{u}-\vec{u}_b|^2 = \vec{u}^2+\vec{u}_b^2-2\vec{u}\cdot\vec{u}_b = u^2 + u_b^2 - 2u\cdot u_b\cdot\cos\hat{w}$ (where $\hat{w}$ is the angle between $\vec{u}$ and $\vec{u}_b$), so that $|\vec{u}-\vec{u}_b| = u\cdot[1+(\frac{u_b}{u})^2 - 2\cos\hat{w}\cdot(\frac{u_b}{u})]^{\frac{1}{2}} \cong u$. Hence

$$p_{\text{H-E}}(u;\kappa^*) \sim u^{-2\kappa^*}, P_{\text{H-E}}(u;\kappa) \sim u^{-2(\kappa+1)}, \quad (73)$$

and by also taking into account the (three-dimensional) density of velocity states, that is $g_{\text{V}}(u) \sim u^2$ (B16), we obtain

$$p_{\text{H-E}}(u;\kappa^*)\,g_{\text{V}}(u) \sim u^{-2\kappa^*}u^2 = u^{-2(\kappa^*-1)} \equiv u^{-\gamma_{\text{V}}}, P_{\text{H-E}}(u;\kappa)g_{\text{V}}(u) \sim u^{-2(\kappa+1)}u^2 = u^{-2\kappa} \equiv u^{-\gamma_{\text{V}}}. \quad (74)$$

The velocity distribution yields a power law with index $\gamma_{\text{V}} = 2(\kappa^* - 1) = 2\kappa$. Similarly, the probability distributions (73) can also be expressed in terms of the (kinetic) energy, given that $u \sim \varepsilon^{1/2}$,

$$p_{\text{H-E}}(\varepsilon;\kappa^*) \cong \varepsilon^{-\kappa^*}, \; P_{\text{H-E}}(\varepsilon;\kappa) \cong \varepsilon^{-(\kappa+1)}, \quad (75)$$

and by considering the (three-dimensional) density of energy states, that is $g_{\text{E}}(\varepsilon) \sim \varepsilon^{1/2}$ (B16), we obtain

$$p_{\text{H-E}}(\varepsilon;\kappa^*)g_{\text{E}}(\varepsilon) \cong \varepsilon^{-\kappa^*}\varepsilon^{\frac{1}{2}} \sim \varepsilon^{-(\kappa^*-\frac{1}{2})} \equiv \varepsilon^{-\gamma_{\text{E}}}, P_{\text{H-E}}(\varepsilon;\kappa)\,g_{\text{E}}(\varepsilon) \cong \varepsilon^{-(\kappa+1)}\varepsilon^{\frac{1}{2}} \sim \varepsilon^{-(\kappa+\frac{1}{2})} \equiv \varepsilon^{-\gamma_{\text{E}}}. \quad (76)$$

Namely, the energy distribution yields a power law with index $\gamma_{\text{E}} = \kappa^* - (1/2) = \kappa + (1/2)$. We also recall the relation of the particle flux, that is $j(u) \sim u \cdot p(u)g_{\text{V}}(u)$, that is,

$$j(\varepsilon) \sim \varepsilon^{\frac{1}{2}}\cdot p(\varepsilon)g_{\text{E}}(\varepsilon) \sim \varepsilon^{\frac{1}{2}}\cdot\varepsilon^{-\gamma_{\text{E}}} = \varepsilon^{-(\gamma_{\text{E}}-\frac{1}{2})} \equiv \varepsilon^{-\gamma}, \quad (77)$$

hence the flux yields a power law with spectral index given by $\gamma = \gamma_{\text{E}} - (1/2)$. Therefore given the value of one of the power indices $\gamma$, $\gamma_{\text{E}}$, $\gamma_{\text{V}}$, we derive the value of $\kappa$ and $\kappa^*$ indices, namely,

$$\kappa = \kappa^* - 1 = \gamma = \gamma_{\text{E}} - \frac{1}{2} = \frac{1}{2}\gamma_{\text{V}}. \quad (78)$$

Table 1 compares the results of *Decker et al.* [2005], *Fisk and Gloeckler* [2006], *Dialynas et al.* [2009], and *Dayeh et al.* [2009]. Each of these analyses estimates a different primary index, which we easily convert it to all of the others using (78). In particular, *Decker et al.* [2005] estimated the value of $\kappa$ index by utilizing the second kind of kappa distribution, while *Fisk and Gloeckler* [2006] estimated the value of $\gamma_{\text{V}}$. *Dialynas et al.* [2009] expressed their results directly in terms of the spectral index $\gamma$, but they dealt with the $\kappa^*$ index, since the first kind of kappa distribution was used. Finally, *Dayeh et al.* [2009] estimated the spectral index $\gamma$ directly.

[49] With respect to the relation between the $\kappa$ index and the power indices $\gamma$, $\gamma_{\text{E}}$, $\gamma_{\text{V}}$, it is important to avoid two common errors. First, if one does not take into account the density of states, then they find that $p_{\text{H-E}}(\varepsilon;\kappa^*) \cong \varepsilon^{-\kappa^*}$, and thus $\kappa^* = \gamma_{\text{E}}$. Owing to this unfortunate coincidence, it is easy to confuse the index $\gamma_{\text{E}}$ with the $\kappa^*$ index of the first kind of kappa distribution. Another error arises when the transformation of velocity to energy, and vice versa, is derived by substituting the energy $\varepsilon = (1/2)\mu \cdot u^2$ into the density of states, $g_{\text{E}}(\varepsilon)$ or $g_{\text{V}}(u)$, instead of to the number of states, $g_{\text{E}}(\varepsilon)d\varepsilon$ or $g_{\text{V}}(u)du$. Indeed, the following relations

$$p_{\text{H-E}}(u;\kappa^*)\,g_{\text{V}}(u) \sim u^{-2(\kappa^*-1)} \sim \varepsilon^{-(\kappa^*-1)}, \quad P_{\text{H-E}}(u;\kappa)g_{\text{V}}(u) \sim u^{-2\kappa} \sim \varepsilon^{-\kappa}, \quad (79)$$

are obviously different from (76), respectively, and incorrect, in contrast to the relations

$$p_{\text{H-E}}(u;\kappa^*)\,g_{\text{V}}(u)\frac{du}{d\varepsilon} \sim \varepsilon^{-(\kappa^*-1)}\varepsilon^{-\frac{1}{2}} \sim \varepsilon^{-(\kappa^*-\frac{1}{2})}, \quad P_{\text{H-E}}(u;\kappa)g_{\text{V}}(u)\frac{du}{d\varepsilon} \sim \varepsilon^{-\kappa}\varepsilon^{-\frac{1}{2}} \sim \varepsilon^{-(\kappa+\frac{1}{2})}, \quad (80)$$

which are exactly the same as (76) and correct. In particular, if one uses the incorrect equation (79), they might again come to the wrong conclusion that $\kappa = \gamma_{\text{E}}$, and confuse the index $\gamma_{\text{E}}$ with the $\kappa$-index of the second kind of kappa distribution. The correct relation is that the $\kappa$ index coincides only with the spectral index, i.e., $\kappa = \gamma$.

[50] In the generic case of a $f$-dimensional system, the densities of states are given by $g_{\text{E}}(\varepsilon) \sim \varepsilon^{(f/2)-1}$ and

**Table 2.** Examples of Thermal Arguments That Have Been Utilized in Various Published Papers[a]

| Publication | Thermal Arguments | Relation With $T_q$ and $\theta_{\rm eff}$ | Comments |
|---|---|---|---|
| *Chotoo et al.* [2000]; | $v_{th}$; | $v_{th} \equiv \theta_\kappa{}^* = \sqrt{(\kappa^* - 3/2)/\kappa^*} \cdot \theta_{\rm eff}$ | first kind, one-dimensional |
| *Collier et al.* [1996] | $\omega_0$ ($\equiv v_{th}$) | | |
| *Kallenrode* [2001], *Decker and Krimigis* [2003]; | $E_T$; | $E_T \equiv k_B\, T'_\kappa \equiv [(\kappa - 3/2)/\kappa] \cdot k_B\, T_q$ | second kind, three-dimensional |
| *Ermakova and Antonova* [2006] | $\varepsilon_0(\equiv E_T)$ | | |
| *Gloeckler and Geiss* [1998] | $\theta$, $v_{th}$ | $\theta \equiv \theta_\kappa = \sqrt{(\kappa - 3/2)/\kappa} \cdot \theta_{\rm eff}$, $v_{th} \equiv \theta_{\rm eff}$ | second kind, three-dimensional |
| *Summers and Thorne* [1991]; | $\theta$, $T$; | $\theta \equiv \theta_\kappa = \sqrt{(\kappa - 3/2)/\kappa} \cdot \theta_{\rm eff}$, $T \equiv T_q$ | second kind, three-dimensional |
| *Marsch* [2006], *Prested et al.* [2008] | $\upsilon_\kappa(\equiv\theta)$, $T_\kappa(\equiv T)$ | | |
| *Heerikhuisen et al.* [2008] | $\Theta_p$, $v_{th}$, $T$ | $\Theta_p \equiv \theta_\kappa = \sqrt{(\kappa - 3/2)/\kappa} \cdot \theta_{\rm eff}$, $v_{th} \equiv \theta_{\rm eff}$, $T \equiv T_q$ | second kind, three-dimensional |
| *Mann et al.* [2002] | $E_\kappa$, $T$ | $E_\kappa \equiv k_B\, T'_\kappa \equiv [(\kappa - 3/2)/\kappa] \cdot k_B\, T_q$, $T \equiv T_q$ | second kind, three-dimensional |
| *Nieves-Chinchilla and Viñas* [2008a, 2008b] | $w_h$, $T_h$ | $w_h \equiv \theta_\kappa{}^* = \sqrt{(\kappa^* - 3/2)/\kappa^*} \cdot \theta_{\rm eff}$, $T_h \equiv T'^*_\kappa \equiv [(\kappa^* - 3/2)/\kappa^*] \cdot T_q$ | first kind, three-dimensional |
| *Vasyliũnas* [1968], *Schippers et al.* [2008]; | $w_0$; | $w_0 \equiv \theta_\kappa = \sqrt{(\kappa - 3/2)/\kappa} \cdot \theta_{\rm eff}$ | second kind, three-dimensional |
| *Christon* [1987] | $\upsilon_0(\equiv w_0)$ | | |
| *Dialynas et al.* [2009] | $T$ | $T \equiv T'^*_\kappa \equiv [(\kappa^* - 5/2)/\kappa^*] \cdot T_q$ | first kind, three-dimensional |
| *Mori et al.* [2004] | $E_0$, $E_C$ | $E_0 \equiv k_B\, T'_\kappa \equiv [(\kappa - 3/2)/\kappa] \cdot k_B\, T_q$, $E_C \equiv (3/4) \cdot k_B\, T_q$, $E_C \equiv (1/2) \cdot \langle\varepsilon\rangle_q = [\kappa/(\kappa - 3/2)] \cdot (3/4) \cdot E_0$ | second kind, three-dimensional. Instead of the referred relations, the paper presents the following $E_C \equiv (1/2) \cdot \langle\varepsilon\rangle_q = [\kappa/(\kappa - 2)] \cdot E_0$. This is true only in the four-dimensional case. |
| *Zaharia et al.* [2000] | $\varepsilon$ | $\varepsilon \equiv k_B\, T_q$ | second kind, three-dimensional |
| *Saito et al.* [2000] | $\upsilon_{th,\kappa}$, $T$ | $\upsilon_{th,\kappa} \equiv \theta_\kappa{}^* = \sqrt{(\kappa^* - 3/2)/\kappa^*} \cdot \theta_{\rm eff}$, $T \equiv T_q$ | first kind, one-dimensional |

[a]Their relations with the physical temperature $T_q$ (that coincides with the kinetic temperature $T_{\rm K}$) or the effective characteristic speed scale $\theta_{\rm eff} \equiv \sqrt{2k_BT_q/\mu}$, and with various auxiliary quantities, such as $T'^*_\kappa$, $T'_\kappa$, $\theta^*_\kappa$, $\theta_\kappa$, are also shown. The relationships between these auxiliary thermal quantities and the primary ones, $T_q$, $\theta_{\rm eff}$, are given in equations (B33) and (B34) of Appendix B for the general case of the $f$-dimensional system.

$g_{\rm V}\,(u) \sim u^{f-1}$, while the two kinds of kappa distributions are given by

$$p(\vec{u};\theta_{\rm eff};\kappa^*) \sim \left[1+\frac{1}{\kappa^*-\frac{f+2}{2}}\cdot\left(\frac{|\vec{u}-\vec{u}_b|}{\theta_{\rm eff}}\right)^2\right]_+^{-\kappa^*},$$
$$P(\vec{u};\theta_{\rm eff};\kappa) \sim \left[1+\frac{1}{\kappa-\frac{f}{2}}\cdot\left(\frac{|\vec{u}-\vec{u}_b|}{\theta_{\rm eff}}\right)^2\right]_+^{-\kappa-1}. \quad (81)$$

Then, using the same steps given above ((71)–(77)), the expression (78) can be easily generalized to

$$\kappa = \kappa^* - 1 = \gamma + \frac{f-3}{2} = \gamma_{\rm E} - \frac{4-f}{2} = \frac{1}{2}(\gamma_{\rm V} + f - 3). \quad (82)$$

For example, in the case of one-dimensional systems, (82) gives

$$\kappa^* = \kappa + 1 = \gamma = \gamma_{\rm E} - \frac{1}{2} = \frac{1}{2}\gamma_{\rm V}, \quad (83)$$

where the index $\kappa^*$ of the first kind of kappa distribution is in this case the one that coincides with the spectral index, i.e., $\kappa^* = \gamma$. As a consequence, the first kind of kappa distribution was preferred when dealing with one-dimensional systems (see Table 2). Even worse, owing to the coincidence of $\kappa = \gamma$ for $f = 3$, and of $\kappa^* = \gamma$ for $f = 1$, the first kind of kappa distributions was sometimes called one-dimensional, while the second kind was called three-dimensional. Obviously, this is not true, since both kinds can be used for any $f$-dimensional system, as it is clarified by (81).

[51] Furthermore, having interpreted the first and the second kinds of kappa distribution as the ordinary and escort Tsallis-Maxwellian probability distributions, respectively, we deduced the exact well-defined temperature that should always be used for describing space plasmas in stationary states, that is, the physical temperature, $T_q$. The classical temperature, $T$, defined by the second Lagrangian multiplier, $\lambda_2 = -\beta = -1/(k_B\, T)$, coincides with the kinetic temperature $T_{\rm K}$ only in equilibrium. As soon as a system is relaxing at stationary states out of equilibrium, $T$ differs from $T_{\rm K}$. In these nonequilibrium states, however, $T_q$ coincides with $T_{\rm K}$, yielding a well-defined temperature that should be used with kappa distributions. Thus the physical temperature $T_q$, and the relevant, effective speed scale $\theta_{\rm eff} \equiv \sqrt{2k_BT_q/\mu}$, should be used instead of any other auxiliary thermal arguments that might be considered. Table 2 presents a variety of thermal arguments that have been utilized in various published papers. Their relations with $T_q$ and $\theta_{\rm eff}$ are also given. Hereafter, the kinetic temperature $T_{\rm K}$, is provided with a solid foundation given by the concept of physical temperature $T_q$ and the formalism of Tsallis statistical mechanics.

[52] The requirement of the convergence of the integrals that include the kappa distributions (second statistical moment), imply specific restrictions on the values of the relevant kappa indices. This is mainly specified by the integral of second statistical moment that provides the kinetic temperature. In the case of the second kind of kappa distribution, we obtain $\kappa > 3/2$. This restriction clearly precludes any theoretical consideration of values $\kappa \leq 3/2$.

For instance, when a distribution of kappa values is constructed, a cutoff has to be imposed for $\kappa \leq 3/2$. It is interesting to consider whether the common occurrence of $\kappa \sim 1.5$ [*Fisk and Gloeckler*, 2006] may originate because of this limit and the accumulation of transient stationary states approaching this limit; this topic is taken up in a subsequent study.

[53] With the results provided in this study, it is straightforward to compare both spectral indices from various space physics observations, and models and theoretical work that use kappa distributions on a consistent footing that minimizes the chances for misinterpretation and error. In addition, it is now clear how to compare the numerous different thermal and thermal-like quantities posed by various authors for parameterizing the nonequilibrium energy states in space plasmas; we evaluate these various parameters with the physical temperature, which evolves naturally from Tsallis statistical mechanics. Now that the connection is complete between empirically derived kappa distributions and Tsallis statistical mechanics, the full strength and capability of Tsallis statistical tools are available for the space physics community to analyze and understand the kappa-like properties of the various particle and energy distributions observed in space.

## Appendix A: The *q*-Deformed Gamma Function

[54] Tsallis statistical mechanics is related to the concept of $q$-deformation of functions [e.g., *Silva et al.*, 1998; *Yamano*, 2002]. For example the $q$-deformed exponential, defined by

$$\exp_q(x) = [1 + (1-q)\cdot x]_+^{\frac{1}{1-q}}, \tag{A1}$$

appears in the expression of the Tsallis Canonical probability distribution (e.g., see (22)), while its inverse function, the $q$-deformed logarithm, defined by

$$\ln_q(x) = \frac{1 - x^{1-q}}{q-1}, \tag{A2}$$

appears in the expression of the Tsallis entropy (B11) (e.g., see the interpretation of Tsallis entropy as the mean of Tsallis-Shannon information measure, that is given by *Gell-Mann and Tsallis* [2004]). The operation $[x]_+$ concerns the Tsallis cutoff condition, as indicated in (15). In addition, the gamma function, defined by the integral

$$\Gamma(a) \equiv \int_0^\infty \exp(-y)\, y^{a-1}\, dy, \tag{A3}$$

can be generalized to the $q$-deformed gamma function [*Duarte Queirós*, 2005], denoted by $\tilde{\Gamma}_q(a)$, with $a > 0$, defined by

$$\begin{aligned}\tilde{\Gamma}_q(a) &\equiv \int_0^\infty \exp_q(-y)\, y^{a-1}\, dy \\ &= \int_0^\infty [1-(1-q)y]_+^{\frac{1}{1-q}}\, y^{a-1}\, dy.\end{aligned} \tag{A4}$$

However, another adaptation of the $q$-deformed gamma function, can be given by

$$\begin{aligned}\Gamma_q(a) &\equiv \int_0^\infty \exp_q(-y)^q\, y^{a-1}\, dy \\ &= \int_0^\infty [1-(1-q)y]_+^{\frac{q}{1-q}}\, y^{a-1}\, dy,\end{aligned} \tag{A5}$$

and the functions $\tilde{\Gamma}_q(a)$ and $\Gamma_q(a)$ are characterized as the $q$-deformed gamma function of the first and the second kind, respectively, and both recover the ordinary gamma function $\Gamma(a)$ for $q \to 1$, i.e.,

$$\lim_{q\to 1}\Gamma_q(a) = \lim_{q\to 1}\tilde{\Gamma}_q(a) = \int_0^\infty \exp(-y)\, y^{a-1}\, dy = \Gamma(a). \tag{A6}$$

The two definitions (A4) and (A5) can be related as follows:

$$\begin{aligned}\tilde{\Gamma}_{q^*}(a) &= \int_0^\infty \exp_{q^*}(-y)\, y^{a-1}\, dy \\ &= \int_0^\infty [1-(1-q^*)y]_+^{\frac{1}{1-q^*}}\, y^{a-1}\, dy \\ &= \int_0^\infty \left[1-(1-q)\frac{y}{q}\right]_+^{\frac{q}{1-q}} y^{a-1}\, dy \\ &= q^a \int_0^\infty [1-(1-q)z]_+^{\frac{q}{1-q}}\, z^{a-1}\, dz = q^a \cdot \Gamma_q(a) \\ &= (2-q^*)^{-a}\cdot \Gamma_{(2-q^*)^{-1}}(a),\end{aligned} \tag{A7}$$

where we set $z \equiv y/q$ and $1/(1-q^*) \equiv q/(1-q)$, that is the transformation also mentioned in (37). In terms of the $\kappa$ index, i.e., $\kappa \equiv 1/(q-1)$ (equation (18)), this transformation is written as $\kappa^* \equiv \kappa + 1$, mentioned also in (37).

[55] The values of the $q$ index have to be specified in order for the integrals in (A4) and (A5) to converge. As $y \to \infty$ the relevant integrants should have the asymptotic behavior $y^{-r}$, with $r > 1$ [e.g., *Ferri et al.*, 2005]. Thus for the integrant of $\Gamma_q(a)$ (A4), i.e., $\exp_q(-y)^q\, y^{a-1} \sim y^{-r}$, we have

$$\begin{aligned}\exp_q(-y)^q\, y^{a-1} &= [1-(1-q)y]_+^{\frac{q}{1-q}}\, y^{a-1} \\ &= \begin{cases} [1-(1-q)y]_+^{\frac{q}{1-q}}\, y^{a-1} & \text{for } q<1, \\ \frac{y^{a-1}}{[1+(q-1)y]^{\frac{q}{q-1}}} & \text{for } q>1, \end{cases} \\ &\overset{y\to\infty}{\longrightarrow} \begin{cases} 0 & \text{for } q<1, \\ y^{a-1-\frac{q}{q-1}} & \text{for } q>1. \end{cases}\end{aligned}$$

The case $q < 1$ becomes zero because of the cutoff condition (which is activated, since the asymptotic behavior $y \to \infty$ implies that $1/(1-q) < y$). However, for $q > 1$ we have

$$\begin{aligned}&r \equiv \frac{q}{q-1} - a + 1 > 1 \;\Rightarrow\; q < \frac{a}{a-1} = 1 + \frac{1}{a-1}, \\ &\text{or } 1 < q < 1 + \frac{1}{a-1}.\end{aligned} \tag{A8}$$

Similarly, for the integral of $\tilde{\Gamma}_q(a)$ (A5), i.e., $\exp_q(-y)^q\, y^{a-1} \sim y^{-r}$, we have

$$\exp_q(-y)\, y^{a-1} = [1-(1-q)y]_+^{\frac{1}{1-q}}\, y^{a-1} \xrightarrow{y\to\infty} \begin{cases} 0 & \text{for} \quad q<1\,, \\ y^{a-1-\frac{1}{q-1}} & \text{for} \quad q>1\,, \end{cases}$$

which for $q > 1$ leads to

$$r \equiv \frac{1}{q-1} - a + 1 > 1 \quad \Rightarrow \quad q < 1 + \frac{1}{a}, \text{or } 1 < q < 1 + \frac{1}{a}, \tag{A9}$$

respectively. Furthermore, from the integration by parts and for $a > 1$, we have

$$\begin{aligned} &\int_0^\infty [1-(1-q)y]_+^{\frac{q}{1-q}}\, y^{a-1}\, dy = \\ &\quad (a-1)\cdot \int_x^\infty [1-(1-q)y]_+^{\frac{1}{1-q}}\, y^{a-2}\, dy \Rightarrow \\ &\quad \Gamma_q(a) = (a-1)\cdot \tilde{\Gamma}_q(a-1) \\ &\int_0^\infty [1-(1-q)y]_+^{\frac{1}{1-q}}\, y^{a-1}\, dy = \\ &\quad \int_0^\infty [1-(1-q)y]_+^{\frac{q}{1-q}}\, [1-(1-q)y]\, y^{a-1}\, dy \\ &\quad \Rightarrow \tilde{\Gamma}_q(a) = \Gamma_q(a) - (1-q)\cdot\Gamma_q(a+1) \\ &\quad \Rightarrow \tilde{\Gamma}_q(a-1) = \Gamma_q(a-1) - (1-q)\cdot\Gamma_q(a), \end{aligned}$$

from which, we derive the recurrent relations

$$\begin{aligned} &\Gamma_q(a) = \frac{a-1}{1+(1-q)(a-1)}\cdot\Gamma_q(a-1), \\ &\tilde{\Gamma}_q(a) = \frac{a-1}{1+(1-q)\,a}\cdot\tilde{\Gamma}_q(a-1), \tilde{\Gamma}_q(a) = \frac{\Gamma_q(a)}{1+(1-q)a}. \end{aligned} \tag{A10}$$

[56] If, in addition, $a = n \in \aleph$, then, the following simple close forms are derived from (A10), namely,

$$\begin{aligned} &\Gamma_q(n) = \frac{(n-1)!}{\prod_{k=0}^{n-1}[1+(1-q)\,k]}, \Gamma_q(1) = 1; \\ &\tilde{\Gamma}_q(n) = \frac{(n-1)!}{\prod_{k=0}^{n}[1+(1-q)\,k]}, \tilde{\Gamma}_q(1) = \frac{1}{2-q}. \end{aligned} \tag{A11}$$

Furthermore, by defining the $q$-deformed "unit function" $1_q(u)$ and "unit factorial" $1_q(n)!$ as

$$1_q(u) = 1 + (1-q)\,u, 1_q(n)! = \prod_{k=0}^{n} 1_q(k) = \prod_{k=0}^{n}[1+(1-q)\,k], \tag{A12}$$

then (A10) becomes

$$\begin{aligned} &\Gamma_q(a) = \frac{a-1}{1_q(a-1)}\cdot\Gamma_q(a-1), \tilde{\Gamma}_q(a) = \frac{a-1}{1_q(a)}\cdot\tilde{\Gamma}_q(a-1), \\ &\tilde{\Gamma}_q(a) = \frac{1}{1_q(a)}\cdot\Gamma_q(a), \end{aligned} \tag{A13}$$

while (A11) becomes

$$\begin{aligned} &\Gamma_q(n) = \frac{(n-1)!}{1_q(n-1)\,!}, \Gamma_q(1) = 1; \\ &\tilde{\Gamma}_q(n) = \frac{(n-1)!}{1_q(n)\,!}, \tilde{\Gamma}_q(1) = \frac{1}{1_q(1)}. \end{aligned} \tag{A14}$$

Finally, the $q$-deformed Gamma function can be expressed analytically in terms of the Beta function (this was also mentioned in the work of *Shizgal* [2007]). Basically, the Beta function $\mathrm{B}(\mu, \nu)$ is defined by the integral

$$\int_0^1 y^{\mu-1}(1-y)^{\nu-1}\, dy \equiv \mathrm{B}(\mu,\nu), \tag{A15}$$

while, by applying the transformation $y \to \tilde{y} = y/(1-y)$, and setting $m \equiv \mu$, $n \equiv \mu + \nu$, we obtain the integral

$$\int_0^\infty \tilde{y}^{m-1}(1+\tilde{y})^{-n}\, d\tilde{y} = \mathrm{B}(m, n-m). \tag{A16}$$

Now, for the first integral (A15), we set $\mu \equiv a$, $\nu - 1 \equiv q/(1-q)$ (or $\nu \equiv 1/(1-q)$), and $y \equiv (1-q)x$. Then, we have

$$\begin{aligned} &\mathrm{B}\left(a, \frac{1}{1-q}\right) = (1-q)^a \cdot \int_0^{\frac{1}{1-q}} x^{a-1}\exp_q(x)^q\, dx = \\ &\qquad \Gamma_{q<1}(a)\cdot(1-q)^a \Rightarrow \\ &\Gamma_{q<1}(a) = (1-q)^{-a}\cdot\mathrm{B}\left(a, \frac{1}{1-q}\right). \end{aligned}$$

For the other integral (A16), we set $m \equiv a$, $n \equiv q/(q-1)$ (or $n \equiv 1/(q-1)+1$), and $\tilde{y} \equiv (q-1)x$. Then, we have

$$\Gamma_{q>1}(a) = (q-1)^{-a}\cdot\mathrm{B}\left(a, \frac{1}{q-1} + 1 - a\right).$$

[57] Overall, then

$$\begin{aligned} &\Gamma_q(a) = \begin{cases} (1-q)^{-a}\cdot\mathrm{B}\left(a, \frac{1}{1-q}\right) & , \quad q<1 \\ \Gamma(a) & , \quad q=1 \\ (q-1)^{-a}\cdot\mathrm{B}\left(a, \frac{1}{q-1}+1-a\right) & , \quad q>1 \end{cases}, \\ &\tilde{\Gamma}_q(a) = \begin{cases} (1-q)^{-a}\cdot\mathrm{B}\left(a, \frac{1}{1-q}+1\right) & , \quad q<1 \\ \Gamma(a) & , \quad q=1 \\ (q-1)^{-a}\cdot\mathrm{B}\left(a, \frac{1}{q-1}-a\right) & , \quad q>1 \end{cases} \end{aligned} \tag{A17}$$

In terms of the $\kappa$ index, i.e., $\kappa \equiv 1/(q-1)$ (18), and for $\kappa > 0$, $q > 1$, gives

$$\begin{aligned} \Gamma_q(a) = &\Gamma(a)\cdot(q-1)^{1-a}\cdot\frac{\Gamma\left(\frac{q}{q-1}-a\right)}{\Gamma\left(\frac{1}{q-1}\right)} = \Gamma(a)\cdot\kappa^{a-1} \\ &\cdot\frac{\Gamma(\kappa+1-a)}{\Gamma(\kappa)}, \end{aligned} \tag{A18}$$

$$\tilde{\Gamma}_q(a) = \Gamma(a)\cdot(q-1)^{-a}\cdot\frac{\Gamma\left(\frac{1}{q-1}-a\right)}{\Gamma\left(\frac{1}{q-1}\right)} = \Gamma(a)\cdot\kappa^a\cdot\frac{\Gamma(\kappa-a)}{\Gamma(\kappa)}, \tag{A19}$$

e.g., for $a$ = 3/2 and $a$ = 5/2, we obtain

$$\Gamma_q\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}\cdot\kappa^{\frac{1}{2}}\cdot\frac{\Gamma\left(\kappa-\frac{1}{2}\right)}{\Gamma(\kappa)}, \tilde{\Gamma}_q\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}\cdot\kappa^{\frac{3}{2}}\cdot\frac{\Gamma\left(\kappa-\frac{3}{2}\right)}{\Gamma(\kappa)}, \tag{A20}$$

$$\Gamma_q\left(\frac{5}{2}\right) = \frac{3\sqrt{\pi}}{4}\cdot\kappa^{\frac{3}{2}}\cdot\frac{\Gamma\left(\kappa-\frac{3}{2}\right)}{\Gamma(\kappa)}, \tilde{\Gamma}_q\left(\frac{5}{2}\right) = \frac{3\sqrt{\pi}}{4}\cdot\kappa^{\frac{5}{2}}\cdot\frac{\Gamma\left(\kappa-\frac{5}{2}\right)}{\Gamma(\kappa)}. \tag{A21}$$

## Appendix B: Canonical Probability Distribution in Tsallis Statistical Mechanics

[58] In order to obtain the stationary probability distribution $\{p_k\}_{k=1}^W$, associated with a conservative physical system of energy spectrum $\{\varepsilon_k\}_{k=1}^W$, we follow along the famous Gibbs' path, where the entropy $S = S(\{p_k\}_{k=1}^W)$ is extremized (under constraints). The extremum is derived from $\vec{\nabla}_p\, S(\{p_k\}_{k=1}^W) = 0$, where

$$\vec{\nabla}_p \equiv \left(\frac{\partial}{\partial p_1}, \frac{\partial}{\partial p_2}, \cdots, \frac{\partial}{\partial p_W}\right), \tag{B1}$$

is the gradient in the $W$-dimensional probability space,

$$\vec{p} \equiv (p_1, p_2, \ldots, p_W) \in \{p_1 \in [0,1]\} \otimes \{p_2 \in [0,1]\} \otimes \cdots \{p_W \in [0,1]\} \subseteq \Re^W.$$

Hence, we have

$$\frac{\partial}{\partial p_j} S\left(\{p_k\}_{k=1}^W\right) = 0, \forall j = 1, \ldots, W. \tag{B2}$$

[59] On the other hand, the "variables" $\{p_k\}_{k=1}^W$ are not independent because of the two constraints: (1) normalization of the probability distribution,

$$\sum_{k=1}^W p_k = 1. \tag{B3}$$

(2) known internal energy,

$$\sum_{k=1}^W p_k\, \varepsilon_k = U. \tag{B4}$$

In such a case, the Lagrange method involves extremizing the functional form

$$G\left(\{p_k\}_{k=1}^W\right) = S\left(\{p_k\}_{k=1}^W\right) + \sum_{m=1}^M \lambda_m\, B_m\left(\{p_k\}_{k=1}^W\right), \tag{B5}$$

instead of directly extremizing the entropy $S(\{p_k\}_{k=1}^W)$, when we have constraints, such as $B_m(\{p_k\}_{k=1}^W) = b_m$, $\forall\, m = 1, \ldots, M$. The unknown Lagrange multipliers $\{\lambda_m\}_{m=1}^M$ are linearly expressed in terms of the constraints known values $\{b_m\}_{m=1}^M$. Thus we have

$$G\left(\{p_k\}_{k=1}^W\right) = S\left(\{p_k\}_{k=1}^W\right) + \lambda_1 \sum_{k=1}^W p_k + \lambda_2 \sum_{k=1}^W p_k\, \varepsilon_k, \tag{B6}$$

where the extremization follows by

$$\frac{\partial}{\partial p_j} G\left(\{p_k\}_{k=1}^W\right) = 0, \forall j = 1, \ldots, W. \tag{B7}$$

[60] The formalism of Tsallis statistical mechanics is interwoven with the concept of escort probabilities [*Tsallis*, 1999, 2009b; *Gell-Mann and Tsallis*, 2004]. The escort probability distribution $\{P_k\}_{k=1}^W$ is constructed from the ordinary probability distribution, $\{p_k\}_{k=1}^W$, as $P_k \sim p_k^q$, $\forall\, k = 1, \ldots, W$, coinciding thus with $\{p_k\}_{k=1}^W$ for $q \to 1$ [*Beck and Schlogl*, 1993]. In fact, there is a duality between the ordinary $\{p_k\}_{k=1}^W$ and escort probabilities $\{P_k\}_{k=1}^W$, such as $P_k = P_k(\{p_{k'}\}_{k'=1}^W; q)$ and $p_k = p_k(\{P_{k'}\}_{k'=1}^W; \frac{1}{q})$, $\forall\, k = 1, \ldots, W$, expressed by

$$P_k = \frac{p_k^q}{\sum_{k'=1}^W p_{k'}^q} \Leftrightarrow p_k = \frac{P_k^{1/q}}{\sum_{k'=1}^W P_{k'}^{1/q}}. \tag{B8}$$

Within the framework of Tsallis statistics, the interpretation for the internal energy $U_q$ is given by the escort expectation value of energy $\langle\varepsilon\rangle_q$, that is

$$U_q = \langle\varepsilon\rangle_q = \sum_{k=1}^W P_k\, \varepsilon_k, \tag{B9}$$

where the symbol $\langle\ \rangle_q$ denotes the escort expectation value.

[61] Therefore in the case of Tsallis statistics we maximized the functional form

$$G_q\left(\{p_k\}_{k=1}^W; q\right) = S_q\left(\{p_k\}_{k=1}^W; q\right) + \lambda_1 \sum_{k=1}^W p_k + \lambda_2 \sum_{k=1}^W P_k\left(\{p_{k'}\}_{k'=1}^W; q\right) \varepsilon_k, \tag{B10}$$

with the entropy $S_q$ given by

$$S_q\left(\{p_k\}_{k=1}^W; q\right) = \sum_{k=1}^W p_k\, \ln_q\left(\frac{1}{p_k}\right) = \frac{1 - \sum_{k=1}^W p_k^q}{q-1}, \tag{B11}$$

(that recovers the BG entropy $S(\{p_k\}_{k=1}^W) = -\sum_{k=1}^W p_k \ln(p_k)$ for $q \to 1$), while the second constraint is referred to the internal energy $U_q$ as interpreted by the escort expectation value of the energy spectrum $\{\varepsilon_k\}_{k=1}^W$ (B9). The Boltzmann constant $k_B$ is ignored during the entropy extremization for the sake of simplicity, that is choosing suitable units so that $k_B = 1$. However, it is restored directly after. The extremization of the functional $G_q(\{p_k\}_{k=1}^W; q)$ leads to

$$p_j = \frac{1}{\tilde{Z}_q}\left[1 - (1-q)\beta_q\left(\varepsilon_j - U_q\right)\right]^{\frac{1}{1-q}}, \tag{B12}$$

where we set

$$\tilde{Z}_q \equiv \left(\lambda_1 \frac{q-1}{q}\right)^{-\frac{1}{1-q}}$$

and $\beta \equiv -\lambda_2$. The quantity $\beta_q$, i.e.,

$$\beta_q \equiv \beta/\phi_q \Leftrightarrow T_q \equiv T \cdot \phi_q, \text{with } \phi_q \equiv \sum_{k=1}^{W} p_k^q, \tag{B13}$$

defines the inverse of the physical temperature $T_q$, $\beta_q \equiv 1/(k_B T_q)$, recovering the inverse of the ordinary temperature $\beta \equiv 1/(k_B T)$ for $q \to 1$. In addition, (B12) can be written as

$$p_j = \frac{1}{\tilde{Z}_q} \exp_q\left[-\beta_q\left(\varepsilon_j - U_q\right)\right], \tag{B14}$$

where $\exp_q(x) \equiv [1 + (1 - q)x]_+^{1/(1-q)}$ denotes the $q$-deformed exponential function, while the subscript "+" denotes the operation $[x]_+ = x$, if $x \geq 0$ and $[x]_+ = 0$, if $x \leq 0$, in accordance with the Tsallis cut-off condition. We remark that the Tsallis partition function has to be settled as $Z_q = \tilde{Z}_q \exp_q(-\beta_q U_q)$ in order for the relations that connect the statistical mechanics with thermodynamics to be valid (for details, see *Tsallis* [1999] and *Gell-Mann and Tsallis* [2004]). Finally, the ordinary and escort probability distributions can be rewritten as

$$p(\varepsilon_k; q) = \frac{\exp_q\left[-\frac{1}{1_q\left(\beta_q U_q\right)}\beta_q\, \varepsilon_k\right]}{\sum_{k'=1}^{W} \exp_q\left[-\frac{1}{1_q\left(\beta_q U_q\right)}\beta_q\, \varepsilon_{k'}\right]},$$

$$P(\varepsilon_k; q) = \frac{\exp_q\left[-\frac{1}{1_q\left(\beta_q U_q\right)}\beta_q\, \varepsilon_k\right]^q}{\sum_{k'=1}^{W} \exp_q\left[-\frac{1}{1_q\left(\beta_q U_q\right)}\beta_q\, \varepsilon_{k'}\right]^q},$$

where we use the notation of the $q$-deformed "unit function," $1_q(u) \equiv 1 + (1 - q)\, u$, as defined in (A12).

[62] In the case of the continuous energy spectrum, the probability distributions are written as

$$p(\varepsilon; q) = \frac{\exp_q\left[-\frac{1}{1_q\left(\beta_q U_q\right)}\beta_q\, \varepsilon\right]}{\int_0^\infty \exp_q\left[-\frac{1}{1_q\left(\beta_q U_q\right)}\beta_q\, \varepsilon\right] d\varepsilon},$$

$$P(\varepsilon; q) = \frac{\exp_q\left[-\frac{1}{1_q\left(\beta_q U_q\right)}\beta_q\, \varepsilon\right]^q}{\int_0^\infty \exp_q\left[-\frac{1}{1_q\left(\beta_q U_q\right)}\beta_q\, \varepsilon\right]^q d\varepsilon},$$

while, also considering the density of states $g_E(\varepsilon)$, the distributions become

$$p(\varepsilon; q) = \frac{\exp_q\left[-\frac{1}{1_q\left(\beta_q U_q\right)}\beta_q\, \varepsilon\right]}{\int_0^\infty \exp_q\left[-\frac{1}{1_q\left(\beta_q U_q\right)}\beta_q\, \varepsilon\right] g_E(\varepsilon)\, d\varepsilon},$$

$$P(\varepsilon; q) = \frac{\exp_q\left[-\frac{1}{1_q\left(\beta_q U_q\right)}\beta_q\, \varepsilon\right]^q}{\int_0^\infty \exp_q\left[-\frac{1}{1_q\left(\beta_q U_q\right)}\beta_q\, \varepsilon\right]^q g_E(\varepsilon)\, d\varepsilon}. \tag{B15}$$

Here we utilize the classical case of a power law density of energy states, $g_E(\varepsilon) \sim \varepsilon^{a-1}$, with $a = f/2$, where $f$ denotes the degrees of freedom for each of the particles. For three-dimensional monatomic particles, $f = 3$, $a = 3/2$, and thus $g_E(\varepsilon) \sim \varepsilon^{1/2}$.

[63] The exact expression of the density of states can be found as follows. We assume that we have a spherical symmetry, and thus $du_x\, du_y\, du_z = 4\pi\, u^2\, du = 2\pi\, (\frac{2}{\mu})^{3/2}\, \varepsilon^{1/2}\, d\varepsilon$ (after the substitution of the kinetic energy $\varepsilon = \frac{1}{2}\mu\, u^2$). Hence the density of speed and energy states, $g_V(u)$ and $g_E(\varepsilon)$, respectively, are given by

$$g_V(u) = 4\pi u^2, g_E(\varepsilon) = 2\pi\left(\frac{2}{\mu}\right)^{\frac{3}{2}} \varepsilon^{\frac{1}{2}}, \text{with } g_V(u)du = g_E(\varepsilon)d\varepsilon. \tag{B16}$$

Then, the internal energy in terms of the escort probability distribution is found as follows:

$$U_q = \frac{\int_0^\infty P(\varepsilon; a; q)\, g_E(\varepsilon)\, \varepsilon\, d\varepsilon}{\int_0^\infty P(\varepsilon; a; q)\, g_E(\varepsilon)\, d\varepsilon}$$

$$= \frac{\int_0^\infty \exp_q\left[-\frac{1}{1_q\left(\beta_q U_q\right)}\beta_q\, \varepsilon\right]^q \varepsilon^a\, d\varepsilon}{\int_0^\infty \exp_q\left[-\frac{1}{1_q\left(\beta_q U_q\right)}\beta_q\, \varepsilon\right]^q \varepsilon^{a-1}\, d\varepsilon} = \frac{1_q\left(\beta_q U_q\right)}{\beta_q}$$

$$\cdot \frac{\int_0^\infty \exp_q(-x)^q x^a\, dx}{\int_0^\infty \exp_q(-x)^q x^{a-1}\, dx} = \frac{1_q\left(\beta_q U_q\right)}{\beta_q} \cdot \frac{\Gamma_q(a+1)}{\Gamma_q(a)}$$

$$= \frac{1_q\left(\beta_q U_q\right)}{\beta_q} \cdot \frac{a}{1_q(a)}$$

where we utilized the $q$-deformed Gamma function of the second kind $\Gamma_q(a)$, as described in Appendix A. Therefore,

$$\frac{1_q(a)}{a} = \frac{1_q\left(\beta_q U_q\right)}{\beta_q U_q} \Rightarrow \beta_q U_q = a \Rightarrow U_q = a\, k_B T_q. \tag{B17}$$

The internal energy in terms of the ordinary probability distribution (of a $q^*$ index and a $\beta_{q^*}$ temperature-like parameter) is found as follows:

$$U_{q^*} = \frac{\int_0^\infty p(\varepsilon; a; q^*)\, g_E(\varepsilon)\, \varepsilon\, d\varepsilon}{\int_0^\infty p(\varepsilon; a; q^*)\, g_E(\varepsilon)\, d\varepsilon}$$

$$= \frac{\int_0^\infty \exp_{q^*}\left[-\frac{1}{1_q\left(\beta_{q^*} U_{q^*}\right)}\beta_{q^*}\, \varepsilon\right] \varepsilon^a\, d\varepsilon}{\int_0^\infty \exp_{q^*}\left[-\frac{1}{1_{q^*}\left(\beta_{q^*} U_{q^*}\right)}\beta_{q^*}\, \varepsilon\right] \varepsilon^{a-1}\, d\varepsilon}$$

$$= \frac{1_{q^*}\left(\beta_{q^*} U_{q^*}\right)}{\beta_{q^*}} \cdot \frac{\int_0^\infty \exp_{q^*}(-x) x^a\, dx}{\int_0^\infty \exp_{q^*}(-x) x^{a-1}\, dx} = \frac{1_{q^*}\left(\beta_{q^*} U_{q^*}\right)}{\beta_{q^*}}$$

$$\cdot \frac{\tilde{\Gamma}_{q^*}(a+1)}{\tilde{\Gamma}_{q^*}(a)} = \frac{1_{q^*}\left(\beta_{q^*} U_{q^*}\right)}{\beta_{q^*}} \cdot \frac{a}{1_{q^*}(a+1)},$$

where we utilized the $q$-deformed Gamma function of the first kind $\tilde{\Gamma}_q(a)$ (Appendix A). Hence

$$\frac{1_{q^*}(a+1)}{a} = \frac{1_{q^*}\left(\beta_{q^*}U_{q^*}\right)}{\beta_{q^*}U_{q^*}} \Rightarrow \beta_{q^*}U_{q^*} = \frac{a}{1_{q^*}(1)} \Rightarrow U_{q^*} = \frac{1}{1_{q^*}(1)} \cdot a\,k_B T_{q^*}. \tag{B18}$$

Then, for $a = 3/2$, we have for the ordinary and escort probability distribution,

$$U_{q^*} = \frac{1}{1_{q^*}(1)} \cdot \frac{3}{2} k_B T_{q^*},\, U_q = \frac{3}{2} k_B T_q, \tag{B19}$$

respectively.

[64] Moreover, we calculate the normalized probabilities

$$p\left(\varepsilon;\beta_{q^*};q^*\right)\, g_{\mathrm{E}}(\varepsilon) = \beta_{q^*}^a \left[\frac{1_{q^*}(1)}{1_{q^*}(a+1)}\right]^a \frac{1}{\tilde{\Gamma}_{q^*}(a)} \cdot \exp_{q^*}\left[-\frac{1_{q^*}(1)}{1_{q^*}(a+1)}\beta_{q^*}\,\varepsilon\right]\varepsilon^{a-1}, \tag{B20}$$

$$P\left(\varepsilon;\beta_q;q\right)\, g_{\mathrm{E}}(\varepsilon) = \beta_q^a \frac{1}{1_q(a)^a\,\Gamma_q(a)} \cdot \exp_q\left[-\frac{1}{1_q(a)}\beta_q\,\varepsilon\right]^q \varepsilon^{a-1}. \tag{B21}$$

The relevant Maxwellian-like distributions are extracted after the substitution of the kinetic energy $\varepsilon = \frac{1}{2}\,\mu\,u^2$. However, we also take into account that $g_{\mathrm{V}}(u) = g_{\mathrm{E}}(\varepsilon)(d\varepsilon/du)$ (equation (B16)), i.e.,

$$p\left(u;\beta_{q^*};q^*\right)\, g_{\mathrm{V}}(u) = 2\left(\frac{\mu}{2}\right)^{\alpha} \beta_{q^*}^a \left[\frac{1_{q^*}(1)}{1_{q^*}(a+1)}\right]^a \frac{1}{\tilde{\Gamma}_{q^*}(a)} \cdot \exp_{q^*}\left[-\frac{1_{q^*}(1)}{1_{q^*}(a+1)}\frac{\mu}{2}\beta_{q^*}\,u^2\right]u^{2a-1}, \tag{B22}$$

$$P\left(u;\beta_q;q\right)\, g_{\mathrm{V}}(u) = 2\left(\frac{\mu}{2}\right)^{\alpha} \beta_q^a \frac{1}{1_q(a)^a\,\Gamma_q(a)} \cdot \exp_q\left[-\frac{1}{1_q(a)}\frac{\mu}{2}\beta_q\,u^2\right]^q u^{2a-1}, \tag{B23}$$

or

$$p\left(u;\theta_q{}^*;q^*\right)\, g_{\mathrm{V}}(u) = \theta_q^{*\,-2a}\frac{2}{\tilde{\Gamma}_{q^*}(a)} \cdot \exp_{q^*}\left[-\left(\frac{u}{\theta_q{}^*}\right)^2\right]u^{2a-1},$$
$$\theta_q^* \equiv \sqrt{\frac{1_{q^*}(a+1)}{1_{q^*}(1)} \cdot \frac{2k_B T_{q^*}}{\mu}}, \tag{B24}$$

$$P\left(u;\theta_q;q\right)\, g_{\mathrm{V}}(u) = \theta_q^{-2a}\frac{2}{\Gamma_q(a)} \cdot \exp_q\left[-\left(\frac{u}{\theta_q}\right)^2\right]^q u^{2a-1},$$
$$\theta_q \equiv \sqrt{1_q(a) \cdot \frac{2k_B T_q}{\mu}}. \tag{B25}$$

The kinetic temperature $T_{\mathrm{K}}$ is defined by the mean kinetic energy (internal energy), i.e.,

$$U_q \equiv a\,k_B T_{\mathrm{K}}. \tag{B26}$$

Given (B17), (B18), and (B26), we conclude that for the ordinary probability distribution, $T_q{}^*$ depends on $q^*$ index and does not coincide with $T_{\mathrm{K}}$, i.e.,

$$T_{\mathrm{K}} = \frac{1}{1_{q^*}(1)} \cdot T_{q^*}, \tag{B27}$$

while for the escort probability distribution, $T_q$ is independent of $q$ index, coinciding with $T_{\mathrm{K}}$,

$$T_{\mathrm{K}} = T_q. \tag{B28}$$

Then, we express the probability distributions in terms of the kinetic temperature $T_{\mathrm{K}} = T_q$ and the effective speed-scale parameter $\theta_{\mathrm{eff}} = \sqrt{2k_B T_q/\mu}$, namely,

$$p\left(\varepsilon;T_q;q^*\right)\, g_{\mathrm{E}}(\varepsilon) = \left(k_B T_q\right)^{-a} \cdot \frac{1}{1_{q^*}(a+1)^a\,\tilde{\Gamma}_{q^*}(a)} \cdot \exp_{q^*}\left[-\frac{1}{1_{q^*}(a+1)} \cdot \frac{\varepsilon}{k_B T_q}\right]\varepsilon^{a-1}, \tag{B29}$$

$$P\left(\varepsilon;T_q;q\right)\, g_{\mathrm{E}}(\varepsilon) = \left(k_B T_q\right)^{-a} \cdot \frac{1}{1_q(a)^a\,\Gamma_q(a)} \cdot \exp_q\left[-\frac{1}{1_q(a)} \cdot \frac{\varepsilon}{k_B T_q}\right]^q \varepsilon^{a-1}. \tag{B30}$$

and

$$p(u;\theta_{\mathrm{eff}};q^*)\, g_{\mathrm{V}}(u) = \theta_{\mathrm{eff}}^{-2a} \cdot \frac{2}{1_{q^*}(a+1)^a\,\tilde{\Gamma}_{q^*}(a)} \cdot \exp_{q^*}\left[-\frac{1}{1_{q^*}(a+1)} \cdot \left(\frac{u}{\theta_{\mathrm{eff}}}\right)^2\right]u^{2a-1}, \tag{B31}$$

$$P(u;\theta_{\mathrm{eff}};q)\, g_{\mathrm{V}}(u) = \theta_{\mathrm{eff}}^{-2a} \cdot \frac{2}{1_q(a)^a\,\Gamma_q(a)} \cdot \exp_q\left[-\frac{1}{1_q(a)} \cdot \left(\frac{u}{\theta_{\mathrm{eff}}}\right)^2\right]^q u^{2a-1}. \tag{B32}$$

Notice that the following auxiliary temperatures and speed scales $\theta_q{}^*$, $\theta_q$, that were utilized in various analyses (see Table 2), can be expressed in terms of the kinetic temperature $T_{\mathrm{K}} = T_q$ and the effective speed scale $\theta_{\mathrm{eff}}$, namely

$$T_q'^* \equiv 1_{q^*}(a+1) \cdot T_q,\, T_q' \equiv 1_q(a) \cdot T_q,$$
$$\theta_q{}^* \equiv \sqrt{1_{q^*}(a+1)} \cdot \theta_{\mathrm{eff}},\, \theta_q \equiv \sqrt{1_q(a)} \cdot \theta_{\mathrm{eff}}, \tag{B33}$$

or in terms of the $\kappa$ index,

$$T'^{*}_{\kappa} \equiv \frac{\kappa^* - (a+1)}{\kappa^*} \cdot T_q, T'_\kappa \equiv \frac{\kappa - a}{\kappa} \cdot T_q,$$
$$\theta_{\kappa}{}^* \equiv \sqrt{\frac{\kappa^* - (a+1)}{\kappa^*}} \cdot \theta_{\text{eff}}, \theta_\kappa \equiv \sqrt{\frac{\kappa - a}{\kappa}} \cdot \theta_{\text{eff}}, \tag{B34}$$

(with $a \equiv f/2$). In the three-dimensional case ($f = 3$), the distributions (B29)–(B32) are

$$p\left(\varepsilon; T_q; q^*\right) = \theta_{\text{eff}}^{-3} \cdot \frac{1}{2\pi} \cdot \frac{1}{1_{q^*}\left(\frac{5}{2}\right)^{\frac{3}{2}} \tilde{\Gamma}_{q^*}\left(\frac{3}{2}\right)} \cdot \exp_{q^*}\left[-\frac{1}{1_{q^*}\left(\frac{5}{2}\right)} \cdot \frac{\varepsilon}{k_B T_q}\right], \tag{B35}$$

$$P\left(\varepsilon; T_q; q\right) = \theta_{\text{eff}}^{*\,-3} \cdot \frac{1}{2\pi} \frac{1}{1_q\left(\frac{3}{2}\right)^{\frac{3}{2}} \Gamma_q\left(\frac{3}{2}\right)} \cdot \exp_q\left[-\frac{1}{1_q\left(\frac{3}{2}\right)} \cdot \frac{\varepsilon}{k_B T_q}\right]^q, \tag{B36}$$

and

$$p(u; \theta_{\text{eff}}; q^*) = \theta_{\text{eff}}^{-3} \cdot \frac{1}{2\pi} \cdot \frac{1}{1_{q^*}\left(\frac{5}{2}\right)^{\frac{3}{2}} \tilde{\Gamma}_{q^*}\left(\frac{3}{2}\right)} \cdot \exp_{q^*}\left[-\frac{1}{1_{q^*}\left(\frac{5}{2}\right)} \cdot \left(\frac{u}{\theta_{\text{eff}}}\right)^2\right], \tag{B37}$$

$$P(u; \theta_{\text{eff}}; q) = \theta_{\text{eff}}^{-3} \cdot \frac{1}{2\pi} \cdot \frac{1}{1_q\left(\frac{3}{2}\right)^{\frac{3}{2}} \Gamma_q\left(\frac{3}{2}\right)} \cdot \exp_q\left[-\frac{1}{1_q\left(\frac{3}{2}\right)} \cdot \left(\frac{u}{\theta_{\text{eff}}}\right)^2\right]^q. \tag{B38}$$

It is apparent that the ordinary and escort probability distribution are transformed to each other

$$p\left(\varepsilon; T_q; q^*\right) = P\left(\varepsilon; T_q; q\right), p(u; \theta_{\text{eff}}; q^*) = P(u; \theta_{\text{eff}}; q), \tag{B39}$$

under the $q^*$, $q$, or $\kappa^*$, $\kappa$, indices relation of (37),

$$q^* = 2 - \frac{1}{q}, \text{or}, q = \frac{1}{2 - q^*} (\Rightarrow \kappa^* = \kappa + 1), \tag{B40}$$

which satisfies

$$\frac{1}{1 - q^*} = \frac{q}{1 - q}, \frac{1 - q^*}{1_{q^*}\left(\frac{5}{2}\right)} = \frac{1 - q}{1_q\left(\frac{3}{2}\right)}. \tag{B41}$$

However, in Tsallis statistical mechanics the ordinary probability distribution is not utilized for determining the statistical moments, e.g., the mean energy $U_q$. Then, the ordinary probability distribution is constructed through the ordinary escort duality relations $P(\varepsilon;T_{\text{eff}};q) \propto p(\varepsilon;T_{\text{eff}};q)^q$ (equations (B8) and (B15)), that is,

$$p\left(\varepsilon; T_q; q\right) = \theta_{\text{eff}}^{-3} \cdot \frac{1}{2\pi} \frac{1}{1_q\left(\frac{3}{2}\right)^{\frac{3}{2}} \tilde{\Gamma}_q\left(\frac{3}{2}\right)} \cdot \exp_q\left[-\frac{1}{1_q\left(\frac{3}{2}\right)} \cdot \frac{\varepsilon}{k_B T_q}\right], \tag{B42}$$

and

$$p(u; \theta_{\text{eff}}; q) = \theta_{\text{eff}}^{-3} \cdot \frac{1}{2\pi} \cdot \frac{1}{1_q\left(\frac{3}{2}\right)^{\frac{3}{2}} \tilde{\Gamma}_q\left(\frac{3}{2}\right)} \cdot \exp_q\left[-\frac{1}{1_q\left(\frac{3}{2}\right)} \cdot \left(\frac{u}{\theta_{\text{eff}}}\right)^2\right]. \tag{B43}$$

The statistical moments are exclusively determined by the escort probability distribution, which is finally written as

$$P\left(\varepsilon; T_q; q\right) = \pi^{-\frac{3}{2}} \cdot \theta_{\text{eff}}^{-3} \cdot A(q) \cdot \left[1 + \frac{2(q-1)}{5-3q} \cdot \frac{\varepsilon}{k_B T_q}\right]_+^{-\frac{q}{q-1}}, \tag{B44}$$

$$P(u; \theta_{\text{eff}}; q) = \pi^{-\frac{3}{2}} \cdot \theta_{\text{eff}}^{-3} \cdot A(q) \cdot \left[1 + \frac{2(q-1)}{5-3q} \cdot \left(\frac{u}{\theta_{\text{eff}}}\right)^2\right]_+^{-\frac{q}{q-1}}, \tag{B45}$$

or, in terms of the $\kappa$ index, i.e., $\kappa \equiv 1/(q-1)$ (18),

$$P\left(\varepsilon; T_q; \kappa\right) = \pi^{-\frac{3}{2}} \cdot \theta_{\text{eff}}^{-3} \cdot A(\kappa) \cdot \left(1 + \frac{1}{\kappa - \frac{3}{2}} \cdot \frac{\varepsilon}{k_B T_q}\right)_+^{-\kappa-1}, \tag{B46}$$

$$P(u; \theta_{\text{eff}}; \kappa) = \pi^{-\frac{3}{2}} \cdot \theta_{\text{eff}}^{-3} \cdot A(\kappa) \cdot \left[1 + \frac{1}{\kappa - \frac{3}{2}} \cdot \left(\frac{u}{\theta_{\text{eff}}}\right)^2\right]_+^{-\kappa-1}, \tag{B47}$$

where the expressions of

$$A(q) \equiv \sqrt{8} \cdot \left(\frac{q-1}{5-3q}\right)^{\frac{3}{2}} \frac{\Gamma\left(\frac{q}{q-1}\right)}{\Gamma\left(\frac{1}{q-1} - \frac{1}{2}\right)}, A(\kappa) \equiv \frac{\left(\kappa - \frac{3}{2}\right)^{-\frac{3}{2}} \Gamma(\kappa + 1)}{\Gamma\left(\kappa - \frac{1}{2}\right)}. \tag{B48}$$

are determined by utilizing the relations (A20) and (A21) of the $q$-deformed gamma function.

[65] Finally, we calculate the expression of the argument $\phi_q$, that connects the physical temperature $T_q$ with the classical temperature $T$, which is related to the second Lagrangian multiplier $\lambda_2$. This is given in terms of a scale parameter $\sigma$ as follows:

$$\phi_q \equiv \int_0^\infty \left[p(u; \theta_{\text{eff}}; q) \cdot \sigma^f\right]^q \cdot \frac{g_V(u)\, du}{\sigma^f}, \tag{B49}$$

while for $f = 3$,

$$\phi_q = \sigma^{3(q-1)} \cdot \int_0^\infty p(u; \theta_{\text{eff}}; q)^q\, g_V(u)\, du. \tag{B50}$$

Then, from (B43) we obtain,

$$\phi_q = \sigma^{3(q-1)} \cdot \left[\theta_{\text{eff}}^3 \cdot 2\pi \cdot 1_q\left(\frac{3}{2}\right)^{\frac{3}{2}} \tilde{\Gamma}_q\left(\frac{3}{2}\right)\right]^{-q} \cdot 4\pi \cdot \int_0^\infty \exp_q\left[-\frac{1}{1_q\left(\frac{3}{2}\right)} \cdot \left(\frac{u}{\theta_{\text{eff}}}\right)^2\right]_+^q u^2\, du \Rightarrow$$

$$\phi_q = \sigma^{3(q-1)} \cdot \left[\theta_{\text{eff}}^3 \cdot 2\pi \cdot 1_q\left(\frac{3}{2}\right)^{\frac{3}{2}}\right]^{1-q} \cdot \frac{\Gamma_q\left(\frac{3}{2}\right)}{\tilde{\Gamma}_q\left(\frac{3}{2}\right)^q}$$

$$= 1_q\left(\frac{3}{2}\right)^{\frac{3}{2}(1-q)+q} \cdot \left[2\pi\, \Gamma_q\left(\frac{3}{2}\right)\right]^{1-q} \cdot \left(\frac{\theta_{\text{eff}}}{\sigma}\right)^{3(q-1)} \Rightarrow$$

$$\phi_q = 1_q\left(\frac{3}{2}\right)^{1_q\left(\frac{1}{2}\right)} \cdot \left[2\pi\, \Gamma_q\left(\frac{3}{2}\right)\right]^{1-q} \cdot \left(\frac{1}{\sigma^2}\frac{2k_B T_q}{\mu}\right)^{\frac{3}{2}(1-q)}. \tag{B51}$$

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