CHAPTER 3

KLIMONTOVICH EQUATION, B.B.G.K.Y.–HIERARCHY AND VLASOV–MAXWELL EQUATIONS

In addition to the conservation laws of Chapter 2 an equation describing the conservation of particles in μ–space can be derived. This may serve as the starting point for the derivation of a hierarchy of equations for the multiple distribution functions introduced in Chapter 1. As a simple example the description of collisionless plasmas is based on this general theory.

3.1. DENSITIES IN μ–SPACE

A microscopic density may be defined by

t

$$f_\mu(\xi,r) = \sum_{i=1}^{N} \delta(\xi-\xi_i), \tag{3.1.1}$$

where the abbreviations

$$\xi = (\boldsymbol{r},\boldsymbol{v}), \quad \delta(\xi-\xi_i) = \delta(\boldsymbol{r}-\boldsymbol{r}_i)\delta(\boldsymbol{v}-\boldsymbol{v}_i)$$

have been used. Averaging we have:

$$f(\xi,t) = \langle f_\mu \rangle = \sum_{i=1}^{N} \int \delta(\xi-\xi_i)D(\Gamma,t)d\Gamma = N\int \delta(\xi-\xi_1)D(\Gamma,t)d\Gamma =$$

$$= n_0\int \delta(\xi-\xi_1)F_1(\xi_1,t)d\xi_1 = n_0F_1(\xi,t), \tag{3.1.2}$$

i.e. the average density in μ–space is the product of the total number of particles N and the probability density $V^{-1}F_1(\xi,t)$ associated with a single particle at position ξ. This is, of course, not amazing, since f and F_1 are defined with the same probability density $D(\Gamma,t)$ in Γ–space. The function $f(\xi,t)$ is called *molecular distribution function.*

Next we consider the statistical correlation between the deviations from the average at two points of μ–space:

$$Q(\xi,\xi',t) = \langle\{f_\mu(\xi,\Gamma)-f(\xi,t)\}\{f_\mu(\xi',\Gamma)-f(\xi',t)\}\rangle. \tag{3.1.3}$$

P. P. J. M. Schram (ed.), *Kinetic Theory of Gases and Plasmas*

Evaluation of the product leads to

$$Q(\xi,\xi',t) = < f_\mu(\xi,\Gamma)f_\mu(\xi',\Gamma)> - f(\xi,t)f(\xi',t). \tag{3.1.4}$$

From the definition (3.1.1) of f_μ it follows that

$$<f_\mu(\xi,\Gamma)f_\mu(\xi',\Gamma)> = \sum_{i=1}^{N}\sum_{j=1}^{N}\int \delta(\xi-\xi_i)\delta(\xi'-\xi_j)D(\Gamma,t)d\Gamma$$

$$= \sum_{i=1}^{N}\int \delta(\xi-\xi_i)\delta(\xi'-\xi_i)D(\Gamma,t) + \sum_{i=1}^{N}\sum_{j=1}^{N}{}' \int \delta(\xi-\xi_i)\delta(\xi'-\xi_j)D(\Gamma,t)d\Gamma.$$

In the right hand side the contributions of $i = j$ and $i \neq j$ have been seperated. From the usual procedure we now find:

$$<f_\mu(\xi,\Gamma)f_\mu(\xi',\Gamma)> = n_0\delta(\xi-\xi')F_0(\xi,t)+n_0^2F_2(\xi,\xi',t).$$

Substitution into (3.1.4) gives together with (3.1.2) and (2.2.29):

$$Q(\xi,\xi',t) = n_0\delta(\xi-\xi')F_1(\xi,t)+n_0^2g_2(\xi,\xi',t), \tag{3.1.5}$$

in which the pair correlation g_2 is present again. If the particles are statistically independent, the term with g_2 disappears but the first term of the right hand side remains. It indicates that the standard deviation of densities at ξ is infinitely large in an integrable manner. The meaning of this term becomes clearer, when we consider the number of particles in a finite cell of μ-space:

$$N_\Delta(\Gamma) = \int_\Delta f_\mu(\xi,\Gamma)d\xi. \tag{3.1.6}$$

Averaging we have:

$$<N_\Delta> = n_0\int_\Delta F_1(\xi,t)d\xi \tag{3.1.7}$$

and the deviation from this is given by

$$N_\Delta - <N_\Delta> = \int_\Delta \{f_\mu(\xi,\Gamma) - n_0F_1(\xi,t)\}d\xi. \tag{3.1.8}$$

The square of this expression leads to a double integral with respect to ξ and ξ'. It is easily seen that the average is given by

$$\sigma_\Delta^2 = <(N_\Delta - <N_\Delta>)^2> = \int_\Delta d\xi \int_\Delta d\xi' Q(\xi,\xi',t). \tag{3.1.9}$$

In the case that the particles are statistically independent ($g_2 = 0$) substitution of (3.1.5) shows that

$$\sigma_\Delta^2 = n_0 \int_\Delta d\xi \int_\Delta d\xi' \delta(\xi-\xi')F_1(\xi,t) = n_0 \int_\Delta d\xi F_1(\xi,t) = <N_\Delta>$$

For the relative standard deviation the well–known result

$$\sigma_\Delta / <N_\Delta> = <N_\Delta>^{-\frac{1}{2}} \tag{3.1.10}$$

is found, so that the relative deviation from the average becomes very small, if the number of particles in the cell is very large.

3.2. KLIMONTOVICH EQUATION.

In order to obtain an equation for f_μ we differentiate $\tilde{f_\mu}(\xi,\Gamma_0,t)$ with respect to time:

constant $\quad \xi = \langle \vec{r}, \vec{v} \rangle \quad \frac{d}{dt}\xi_i = \frac{d\xi_i}{dr_i}\frac{dr_i}{dt} + \frac{d\xi_i}{dv_i}\frac{dv_i}{dt} \Rightarrow \frac{d}{dt}\xi_i = \left(\dot{\vec{r}}_i \cdot \frac{\partial}{\partial \vec{r}_i} + \dot{\vec{v}}_i \cdot \frac{\partial}{\partial \vec{v}_i}\right)\xi_i$

$\dot{r} \quad \dot{v} \quad \vec{v}_i \quad \vec{a}_i$

$$\partial \tilde{f_\mu} / \partial t = \partial / \partial t \sum_{i=1}^{N} \delta\{\xi - \xi_i(\Gamma_0, t)\} = -\sum_{i=1}^{N} (\dot{\boldsymbol{r}}_i \cdot \nabla + \dot{\boldsymbol{v}}_i \cdot \nabla_v)\delta(\xi - \xi_i)$$

Relation ~~through Boltzmann equation??~~ Or other? chain rule

$$= -\sum_{i=1}^{N} (\boldsymbol{v}_i \cdot \nabla + 1/m \, f_i \cdot \nabla_v)\delta(\xi - \xi_i). \tag{3.2.1}$$

$f = ma; \quad a = \frac{1}{m}f$

Because $\boldsymbol{v}_i$ and ∇ commute and because the deltafunction in the i^{th} term differs from zero only for $\boldsymbol{v}_i = \boldsymbol{v}$ (and $\boldsymbol{r}_i = \boldsymbol{r}$), we may replace $\boldsymbol{v}_i$ by $\boldsymbol{v}$ in each term, so that

$$\sum_{i=1}^{N} \boldsymbol{v}_i \cdot \nabla \delta(\xi - \xi_i) = \boldsymbol{v} \cdot \nabla \sum_{i=1}^{N} \delta(\xi - \xi_i) = \boldsymbol{v} \cdot \nabla f_\mu. \tag{3.2.2}$$

In the other terms of the right hand side of (3.2.1) we observe the force f_i on particle i. A similar procedure is possible for these terms. We write:

$$f_i = F_\mu(\xi_i, \Gamma_0, t), \tag{3.2.3}$$

where $F_\mu(\xi,\Gamma_0,t)$ is the force field due to all particles in the system and possibly exterior sources, with the understanding that, if the position ξ_i of a particle in μ–space is substituted for ξ, no force of that particle on itself is allowed. Using the sieving property of the delta function we write:

$$\dot{f}_i \cdot \nabla_v \delta(\xi - \xi_i) = \nabla_v \cdot \{F_\mu(\xi_i, \Gamma_0, t)\}$$
$$= \nabla_v \cdot \{F_\mu(\xi, \Gamma_0, t)\delta(\xi - \xi_i)\}.$$

If

$$\nabla_v \cdot F_\mu(\xi, \Gamma_0, t) = 0, \tag{3.2.4}$$

Need whatever force we choose to satisfy this condition.

it follows that

$$\sum_{i=1}^{N} \dot{f}_i \cdot \nabla_v(\xi - \xi_i) = F_\mu(\xi, \Gamma_0, t) \cdot \nabla_v \sum_{i=1}^{N} \delta(\xi - \xi_i)$$
$$= F_\mu(\xi, \Gamma_0, t) \cdot \nabla_v f_\mu. \tag{3.2.5}$$

microscopic density

From (3.2.1,2,5) we conclude that

$$\partial \tilde{f}_\mu / \partial t + \boldsymbol{v} \cdot \nabla f_\mu + 1/m\, F_\mu \cdot \nabla_v f_\mu = 0 \tag{3.2.6}$$

and this is (KLI1967) called the *Klimontovich equation.*

In order to apply this formalism to a plasma we first generalize (3.2.6) for the case of several species:

$$\partial f_\mu^{(a)} / \partial t + \boldsymbol{v} \cdot \nabla f_\mu^{(a)} + 1/m_a\, F_\mu^{(a)} \cdot \nabla_v f_\mu^{(a)} = 0, \tag{3.2.7}$$

where we have omitted the wiggle (~) in the first term. Furthermore m_a is the mass of a particle of species a, with $a = 1,2,..,p$,

$$f_\mu^{(a)}(\xi, \Gamma_0, t) = \sum_{i=1}^{N_a} \delta\{\xi - \xi_i^{(a)}(\Gamma_0, t)\}, \tag{3.2.8}$$

N_a the number of particles of species a and $\xi_i^{(a)}(\Gamma_0, t)$ is the trajectory of particle i of species a, which depends on the initial conditions of *all* particles (Γ_0). In a plasma we have:

$$F_\mu^{(a)} = q_a(E_\mu + \boldsymbol{v} \times B_\mu), \tag{3.2.9}$$

where q_a is the electric charge of a particle of species a, whereas E_μ and B_μ are the electric and magnetic microfield respectively. It should be noted that the Lorentz force (3.2.9) depends on the velocity, but nevertheless satisfies conditon (3.2.4). The microfields E_μ and B_μ are connected with the fields D_μ and H_μ through the vacuum relations

$$D_\mu(\boldsymbol{r},\Gamma_0,t) = \epsilon_0\, E_\mu(\boldsymbol{r},\Gamma_0,t),$$
$$H_\mu(\boldsymbol{r},\Gamma_0,t) = 1/\mu_0\, B_\mu(\boldsymbol{r},\Gamma_0,t). \tag{3.2.10}$$

For the vacuum constants ϵ_0 and μ_0 we have:

$$\epsilon_0\mu_0 = 1/c^2 \ , \tag{3.2.11}$$

where c is the velocity of light in vacuo. The magnetic field is divergence free:

$$\nabla\cdot B_\mu = 0 \tag{3.2.12}$$

and the microscopic Poisson equation, $\nabla\cdot D_\mu = \lambda_\mu$, becomes with (3.2.10):

$$\nabla\cdot E_\mu = 1/\epsilon_0\, \lambda_\mu(\boldsymbol{r},\Gamma) \tag{3.2.13}$$

with the microscopic charge density

$$\lambda_\mu(\boldsymbol{r},\Gamma) = \sum_a q_a \sum_{i=1}^{N_a} \delta(\boldsymbol{r}-\boldsymbol{r}_i^{(a)}) = \sum_a q_a n_\mu^{(a)}. \tag{3.2.14}$$

The homogeneous Maxwell–Lorentz equation reads:

$$\nabla \times E_\mu = -\,\partial B_\mu/\partial t. \tag{3.2.15}$$

Ampère's law, $\nabla \times H_\mu = j_\mu + \partial D_\mu\,/\partial t$ is multiplied with μ_0 and becomes with (3.2.10,11):

$$\nabla \times B_\mu = \mu_0 j_\mu + c^{-2}\partial E_\mu/\partial t\ , \tag{3.2.16}$$

where the microscopic current density is given by

$$j_\mu(\boldsymbol{r},\Gamma) = \sum_a q_a \sum_{i=1}^{N_a} \boldsymbol{v}_i^{(a)}\delta(\boldsymbol{r}-\boldsymbol{r}_i^{(a)}) = \sum_a q_a\, J_\mu^{(a)}. \tag{3.2.17}$$

The equations (3.2.12,13,14) are not completely independent of the system (3.2.7,9,15,16,17). In the first place we derive from (3.2.15) that

$$\partial/\partial t(\nabla\cdot B_\mu) = 0, \tag{3.2.18}$$

so that for time dependent processes (3.2.12) is only needed as an initial conditon. A similar observation applies to (3.2.13). Taking the divergence of (3.2.16) and using (3.2.17) we write:

$$c^{-2}\partial/\partial t\,(\nabla\cdot E_\mu) = -\,\mu_0\nabla\cdot j_\mu = -\,\mu_a \sum_a q_a \nabla\cdot J_\mu^{(a)}.$$

From the continuity equations for all species, i.e. the direct generalization of (2.13):

$$\partial n_\mu^{(a)}/\partial t + \nabla \cdot J_\mu^{(a)} = 0, \tag{3.2.19}$$

and the definition (3.2.14) we then conclude that

$$\partial/\partial t(\nabla . E_\mu) = 1/\epsilon_0 \, \partial\lambda_\mu/\partial t, \tag{3.2.20}$$

so that also (3.2.13) is only needed as an initial condition in the case of instationary processes. Note that (3.2.19) can also be obtained by integration of the Klimontovich equation (3.2.7) over velocity space.

It is now possible to write the Klimontovich and Maxwell–Lorentz equations formally as a closed system of equations for $f_\mu^{(a)}$, E_μ and B_μ as functions of t, r and $\mathbf{v}$. The emphasis is on the word "formally", because the parametric dependence on Γ_0 prevents actual solutions. This follows from the definition of f_μ in (3.1.1), which implies that an initial value problem for f_μ is equivalent with a complete dynamical problem in Γ–space. The formal system of equations is nevertheless useful as a starting point for statistical operations. The system is obtained by the interpretation of the sources λ_μ and j_μ in the Maxwell–Lorentz equations as integrals over the distribution functions, since

$$\int f_\mu^{(a)} d^3v = \sum_{i=1}^{N_a} \int \delta(r - r_i^{(a)}) \delta(v - v_i^{(a)}) d^3v = n_\mu^{(a)} \tag{3.2.21}$$

and

$$\int v f_\mu^{(a)} d^3v = \sum_{i=1}^{N_a} \int v\delta(r - r_i^{(a)}) \delta(v - v_i^{(a)}) d^3v = J_\mu^{(a)}. \tag{3.2.22}$$

Therefore:

$$\lambda_\mu = \sum_a q_a \underbrace{\int f_\mu^{(a)} d^3v}_{n_\mu^{(a)}}, \qquad \lambda_\mu = \sum q_a n_\mu^{(a)} \quad \text{Microscopic charge density} \tag{3.2.23}$$

$$\mathbf{j}_\mu = \sum_a q_a \underbrace{\int \mathbf{v} f_\mu^{(a)} d^3v}_{J_\mu^{(a)}}. \qquad \vec{J}_\mu = \sum q_a \vec{J}_\mu^{(a)} \quad \text{Microscopic current density} \tag{3.2.24}$$

The formal closed system of equations consists of (3.2.7,12,13,15,16,23,24) with the understanding, as discussed before, that (3.2.12,13) are *partially* superfluous.

3.3. VLASOV–MAXWELL EQUATIONS

The Maxwell–Lorentz equations lend themselves perfectly to averaging. In analogy to (3.1.2) we define:

$$f_a(\xi,t) = <f_\mu^{(a)}> = n_0^{(a)} F_1^{(a)}(\xi,t). \tag{3.3.1}$$

Similarly:

$$E(r,t) = <E_\mu>, \quad B(r,t) = <B_\mu>. \tag{3.3.2}$$

Averaging (3.2.12,13,15,16,23,24) we immediately obtain:

$$\nabla \cdot B = 0, \tag{3.3.3}$$

$$\nabla \cdot E = 1/\epsilon_0 \sum_a q_a \int f_a(\xi,t) d^3v, \tag{3.3.4}$$

$$\nabla \times E = - \partial B/\partial t \tag{3.3.5}$$

and

$$\nabla \times B - c^{-2} \partial E/\partial t = \mu_0 \sum_a q_a \int v \, f_a(\xi,t) d^3v. \tag{3.3.6}$$

Averaging the Klimontovich equation (3.2.7) and using (3.2.9) we write:

$$\partial f_a/\partial t + v \cdot \nabla f_a + q_a/m_a <(E_\mu + v \times B_\mu) \cdot \nabla_v f_\mu^{(a)}> = 0. \tag{3.3.7}$$

We now express E_μ, B_μ and $f_\mu^{(a)}$ as sums of their averaged values and the deviations from these:

$$E_\mu(r,t,\Gamma_0) = E(r,t) + \delta E_\mu(r,t,\Gamma_0),$$

$$B_\mu(r,t,\Gamma_0) = B(r,t) + \delta B_\mu(r,t,\Gamma_0),$$

$$f_\mu^{(a)}(\xi,t,\Gamma_0) = f_a(\xi,t) + \delta f_\mu^{(a)}(\xi,t,\Gamma_0). \tag{3.3.8}$$

Substitution into (3.3.7) and the obvious fact that $<\delta E_\mu> = <\delta B_\mu> = <\delta f_\mu^{(a)}> = 0$ lead to

$$\partial f_a/\partial t + v \cdot \nabla f_a + q_a/m_a \, (E_\mu + v \times B) \cdot \nabla_v f_a = - q_a/m_a <(\delta E_\mu + v \times \delta B_\mu) \cdot \nabla_v \delta f_\mu^{(a)}>. \tag{3.3.9}$$

If the right hand side of (3.3.9) is neglected, the Vlasov equation for species a results. The equations (3.3.3,4,5,6,9) then constitute a closed system of equations at the kinetic level. At the present stage the neglect of the right hand side of (3.3.9) is an ad hoc simplification, which boils down to the assumption that the statistical correlation between fields and particles is small. As far as the irrotational part of the electric field is concerned this means that the pair correlation g_2 should be small, as will be shown explicitly later on.

The *electrostatic approximation* is obtained by the neglect of the internal magnetic field. Only a time independent external magnetic field is allowed in this case. It follows from (3.3.5) that **E** can then be derived from an electrostatic potential:

$\nabla \times \vec{E} = 0$

$\Rightarrow \vec{E} = -\nabla \Phi$

$$E = -\nabla\Phi(\boldsymbol{r},t). \tag{3.3.10}$$

The Vlasov equations are transformed into

$$\partial f_a/\partial t + \boldsymbol{v}\cdot\nabla f_a + q_a/m_a\,(-\nabla\Phi + \boldsymbol{v}\times \boldsymbol{B}_0)\cdot\nabla_v f_a = 0, \tag{3.3.11}$$

where $\boldsymbol{B}_0$ is a vacuum field satisfying

$$\nabla\cdot\boldsymbol{B}_0 = 0, \qquad \nabla\times\boldsymbol{B}_0 = 0. \tag{3.3.12}$$

The potential Φ satisfies the Poisson equation (3.3.4) with the field (3.3.10):

$$\nabla^2\Phi = -1/\epsilon_0 \sum_a q_a \int f_a d^3v. \tag{3.3.13}$$

The validity of the electrostatic approximation can be tested with (3.3.6). The divergence of this equation is the time deriative of (3.3.13). Taking the curl we see that the electrostatic approximation provides an exact solution to the complete Vlasov–Maxwell system if the auxiliary condtion

$$\nabla\times\sum_a q_a \int \boldsymbol{v} f_a d^3v = 0 \tag{3.3.14}$$

is satisfied. A trivial solution of this kind corresponds to a stationary homogeneous system and vanishing fields:

$$f_a(\xi,t) = f_a(\boldsymbol{v}), \qquad E(\boldsymbol{r},t) = 0, \qquad B(\boldsymbol{r},t) = 0. \tag{3.3.15}$$

Then, of course, the sources of the fields should be zero. According to (3.3.4,6) this implies that

$$\sum_a q_a \int f_a(\boldsymbol{v})d^3v = 0, \quad \sum_a q_a \int \boldsymbol{v} f_a(\boldsymbol{v})d^3v = 0.$$

These conditions pose only weak requirements with respect to the functional dependence of the distribution functions $f_a(\mathbf{v})$ on the velocity. Later on it will be shown that collisions induce relaxation towards thermal equilibrium where the molecular distribution functions must be Maxwellians. Some such relaxation does *not* follow from the Vlasov equation, which is, as may be checked easily, reversible. It would be wrong, however, to state that the Vlasov equation is devoid of interactions. The fields can be caused by charge densities and electric currents in the plasma and these fields exert forces on plasma particles. These interactions are, however, of a *collective* nature, as if the plasma were a continuous medium. Only collisions in the sense of interactions between discrete particles lead to an irreversible relaxation towards thermal equilibrium.

3.4. THE FIRST EQUATION OF THE B.B.G.K.Y.–HIERARCHY.

We now return to the Klimontovich equation (3.2.6) for a one–component system. It is easy to generalize to multiple systems, when the need arises.
We assume that the force field F_μ consists of an external field plus a contribution from molecular interaction potentials:

$$F_\mu(\xi,\Gamma_0,t) = F_e(\xi,t) - \sum_{j=1}^{N} \nabla\phi(r-r_j). \tag{3.4.1}$$

Substituting into (3.2.6) and averaging we obtain:

$$(\partial/\partial t + v\cdot\nabla + m^{-1} F_e \cdot \nabla_v) f(\xi,t) = m^{-1} \sum_{i,j} \int \nabla\phi(r-r_j) \cdot \nabla_v \{\delta(r-r_i)\delta(v-v_i)\} D(\Gamma,t) d\Gamma. \tag{3.4.2}$$

In the sum $\sum_{i=1}^{N} \sum_{j=1}^{N}{}'$ the terms with $i = j$ have been excluded, because the particles are not allowed to exert forces on themselves. Using the symmetry of $D(\Gamma,t)$ with respect to interchange of particles we rewrite the right hand side of (3.4.2) as

$$N(N-1)/(mV^2) \int \nabla\phi(r-r_2)\cdot\nabla_v\delta(v-v_1)\delta(r-r_1)F_2(\xi_1,\xi_2,t)d\xi_1 d\xi_2$$

$$= n_0^2/m \int \nabla\phi(r-r')\cdot\nabla_v F_2(\xi,\xi',t)d\xi'$$

With $f = nF_1$ (3.4.2) then becomes:

$$(\partial/\partial t + v\cdot\nabla + m^{-1} F_e \cdot \nabla_v)\, F_1(\xi,t) = n_0/m \int \nabla\phi(r-r')\cdot \nabla_v F_2(\xi,\xi',t)d\xi'. \tag{3.4.3}$$

This is the first equation of the B.B.G.K.Y.– hierarchy, named after Bogoliubov, Born, Green, Kirkwood and Yvon, who have derived these equations independently: [BOG1946], [BOR1946], [KIR1946], [YVO1935].
In the case of long range interaction (plasma) it is convenient to decompose F_2 according to (2.2.29). The result is[1]:

$$\{ \partial/\partial t + v\cdot\nabla + m^{-1}(F_e + F_c)\cdot\nabla_v\}F_1(\xi,t) = n_0/m \int \nabla\phi(r-r')\cdot\nabla_v g_2(\xi,\xi',t)d\xi' \tag{3.4.4}$$

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[1]The notation $\mathbf{F}_e$, $\mathbf{F}_c$ discriminates these force from the analogous force densities $\mathbf{F}_{ext}$, $\mathbf{F}_{col}$ in Chapter 2.

with the collective force:

$$F_c(r,t) = -n_0 \int \nabla\phi(r-r')F_1(r',v,t)d^3vd^3r'. \tag{3.4.5}$$

The approximation of the (electrostatic) Vlasov theory consists of the neglect of the right hand side of (3.4.4). This is justified, if the pair correlation is sufficiently small, a condition which was announced in section 3.3. Extension to multiple species is easy:

$$\{\partial/\partial t + v\cdot\nabla + m_a^{-1}(F_e^{(a)} + F_c^{(a)}) \cdot \nabla_v\}f_a(\xi,t) = 0 \tag{3.4.6}$$

and

$$F_c^{(a)} = -\sum_b \int \nabla\phi_{ab}(r-r')f_b(r',v,t)d^3vd^3r' \tag{3.4.7}$$

In the case of a plasma ϕ_{ab} is the Coulomb interaction potential:

$$\phi_{ab}(r-r') = q_a q_b/(4\pi\epsilon_0 |r-r'|\ ) \tag{3.4.8}$$

The collective force may then be written as

$$F_c^{(a)} = -q_a\nabla\Phi \tag{3.4.9}$$

with the potential

$$\Phi(r,t) = (4\pi\epsilon_0)^{-1}\sum_b q_b \int f_b(r',v,t)/(|r-r'|)d^3vd^3r'. \tag{3.4.10}$$

This potential is a solution of the Poisson equation, since $\nabla^2\, 1/r = -4\pi\delta(r)$ implies that

$$\begin{aligned}\nabla^2\Phi &= -\epsilon_0^{-1}\sum_b q_b \int \delta(r-r')f_b(r',v,t)d^3vd^3r' \\ &= -\epsilon_0^{-1}\sum_b q_b \int f_b(r,v,t)d^3v.\end{aligned} \tag{3.4.11}$$

The system of equations (3.4.6,9,11) is in the case that the external force is magnetic, i.e. $F_{ext}^{(a)} = q_a v \times B_0$, equivalent to (3.3.11,13). In the electrostatic approximation only the Coulomb interaction between the particles plays a role.

3.5. THE COMPLETE HIERARCHY.

In order to construct the B.B.G.K.Y.–hierarchy, of which (3.4.3) is the first equation, we now consider a microscopic density of particle pairs:

$$f_{2\mu}(\xi,\xi',\Gamma) = \sum_{i,j} \delta(\xi-\xi_i)\delta(\xi'-\xi_j). \tag{3.5.1}$$

We take the time derivative of $\tilde{f}_{2\mu}(\xi,\xi',\Gamma_0,t)$, omit the wiggle (~) and use the notation $\partial/\partial\boldsymbol{r}$, $\partial/\partial\boldsymbol{v}$ instead of ∇, ∇_v. It follows that

$$\partial f_{2\mu}/\partial t = -\sum_{i,j} (\boldsymbol{v}_i \cdot \partial/\partial\boldsymbol{r} + m^{-1} \boldsymbol{f}_i \cdot \partial/\partial\boldsymbol{v} + \boldsymbol{v}_j \cdot \partial/\partial\boldsymbol{r}'$$
$$+ m^{-1} \boldsymbol{f}_j \cdot \partial/\partial\boldsymbol{v}')\, \delta(\xi-\xi_i)\delta(\xi'-\xi_j).$$

In exactly the same way as we derived the Klimontovich equation (3.2.6) and under the same condition (3.2.4) we now arrive at a Klimontovich equation for particle pairs:

$$(\partial/\partial t + \boldsymbol{v} \cdot \partial/\partial\boldsymbol{r} + m^{-1} \boldsymbol{F}_\mu\, \partial/\partial\boldsymbol{v} + \boldsymbol{v}' \cdot \partial/\partial\boldsymbol{r}' + m^{-1}\boldsymbol{F}_\mu{}' \cdot$$
$$\partial/\partial\boldsymbol{v}')f_{2\mu}(\xi,\xi',\Gamma_0,t) = 0. \tag{3.5.2}$$

We restrict ourselves to a $\boldsymbol{F}_\mu(\xi,\Gamma_0,t)$ of the form (3.4.1) and $\boldsymbol{F}_\mu{}'$ is an abbreviation for $\boldsymbol{F}_\mu(\xi',\Gamma_0,t)$.
Substitution of (3.4.1) into (3.5.2) leads to

$$\Big[\partial/\partial t + \boldsymbol{v} \cdot \partial/\partial\boldsymbol{r} + m^{-1} \boldsymbol{F}_e \cdot \partial/\partial\boldsymbol{v} + m^{-1} \boldsymbol{F}_e{}' \cdot \partial/\partial\boldsymbol{v}' - m^{-1} \sum_{j=1}^{N}$$
$$\{\partial\phi(\boldsymbol{r}-\boldsymbol{r}_j)/\partial\boldsymbol{r} \cdot \partial/\partial\boldsymbol{v} + \partial\phi(\boldsymbol{r}'-\boldsymbol{r}_j)/\partial\boldsymbol{r}' \cdot \partial/\partial\boldsymbol{v}'\}\Big]\, f_{2\mu}(\xi,\xi',\Gamma_0 t) = 0, \tag{3.5.3}$$

where, of course, $\boldsymbol{F}_e{}' = \boldsymbol{F}_e(\xi',t)$. This equation is now averaged. In analogy to (3.1.2) we have:

$$\langle f_{2\mu}(\xi,\xi',t)\rangle = N(N-1)/V^2 F_2(\xi,\xi',t). \tag{3.5.4}$$

Only the sum in (3.5.3) presents some difficulty in the process of averaging. We consider:

$$\sum_j \partial\phi(\boldsymbol{r}-\boldsymbol{r}_j)/\partial\boldsymbol{r} \cdot \partial f_{2\mu}/\partial\boldsymbol{v} = \sum_j \sum_{i,k} \partial\phi(\boldsymbol{r}-\boldsymbol{r}_j)/\partial\boldsymbol{r} \cdot \partial/\partial\boldsymbol{v}$$
$$\delta(\xi-\xi_i)\delta(\xi'-\xi_k). \tag{3.5.5}$$

In the triple sum of the right hand side we may have $j = k$, but $j \neq i$, because self–forces are excluded. In the terms with $j = k$ the potential $\phi(\boldsymbol{r}-\boldsymbol{r}_j)$ can be replaced by $\phi(\boldsymbol{r}-\boldsymbol{r}')$ because of $\delta(\xi'-\xi_j)$. The contribution of these terms can then be seen to be:

$$\partial\phi(\boldsymbol{r}-\boldsymbol{r}')/\partial\boldsymbol{r} \cdot \partial f_{2\mu}/\partial\boldsymbol{v}\,. \tag{3.5.6}$$

The terms with $j \neq k$ may be written as

$$\sum_{i,j,k} \int \partial\phi(\boldsymbol{r}-\boldsymbol{r}'')/\partial\boldsymbol{r} \cdot \partial/\partial\boldsymbol{v}$$

$$\delta(\xi-\xi_i)\delta(\xi'-\xi_k)\delta(\xi''-\xi_j)d\xi'',$$

or, after we have interchanged sum and integral, as

$$\int \partial\phi(\boldsymbol{r}-\boldsymbol{r}'')/\partial\boldsymbol{r} \cdot \partial f_{3\mu}(\xi,\xi',\xi'')/\partial\boldsymbol{v}\, d\xi'' \tag{3.5.7}$$

with the density of particle triplets

$$f_{3\mu}(\xi,\xi',\xi'',\Gamma) = \sum_{i,j,k} \delta(\xi-\xi_i)\delta(\xi'-\xi_k)\delta(\xi''-\xi_j). \tag{3.5.8}$$

The other part of the sum in (3.5.3) can be dealt with quite analogously. This leads to contributions of the form (3.5.6,7) with an interchange of ξ and ξ'. With $\boldsymbol{F}_\alpha = \boldsymbol{F}_e(\xi_\alpha,t)$ we arrive at

$$\{\partial/\partial t + \sum_{\alpha=1}^{2} (\boldsymbol{v}_\alpha \cdot \partial/\partial\boldsymbol{r}_\alpha + m^{-1}\boldsymbol{F}_\alpha \cdot \partial/\partial\boldsymbol{v}_\alpha) - m^{-1}\sum_{\alpha=1}^{2}\sum_{\beta=1}^{2}{}'$$

$$\partial\phi(\boldsymbol{r}_\alpha-\boldsymbol{r}_\beta)/\partial\boldsymbol{r}_\alpha \cdot \partial/\partial\boldsymbol{v}_\alpha\} f_{2\mu}(\xi_1,\xi_2,\Gamma) = m^{-1}\sum_{\alpha=1}^{2}\int \partial\phi(\boldsymbol{r}_\alpha-\boldsymbol{r}_3)/\partial\boldsymbol{r}_\alpha \cdot$$

$$\partial f_{3\mu}(\xi_1,\xi_2,\xi_3,\Gamma)/\partial\boldsymbol{v}_\alpha\, \partial\xi_3. \tag{3.5.9}$$

The numbering 1,2,3 of the ξ–variables in (3.5.9) replaces the primes and should not be confused with the numbering of the Γ–coordinates.
Analogously to (3.5.4) we have

$$\langle f_{3\mu}\rangle = N(N-1)(N-2)/V^3\, F_3 \;,$$

so that averaging (3.5.9) we obtain an equation of exactly the same structure with $F_2\,(\xi_1,\xi_2,t)$ instead of $f_{2\mu}$, $F_3(\xi_1,\xi_2,\xi_3,t)$ instead of $f_{3\mu}$ and an extra factor $(N-2)/V \simeq n_0$ in the right hand side.

It is not difficult anymore to guess the general structure of the hierarchy equations. The s–multiple distribution functions $F_s(\xi_1,\xi_2,...,\xi_s,t)$ satisfy:

$$\{\partial/\partial t + \sum_{\alpha=1}^{s} (\boldsymbol{v}_\alpha \cdot \partial/\partial \boldsymbol{r}_\alpha + m^{-1} \boldsymbol{F}_\alpha \cdot \partial/\partial \boldsymbol{v}_\alpha) - m^{-1} \sum_{\alpha=1}^{s} \sum_{\beta=1}^{s}{}'$$

$$\partial\phi(\boldsymbol{r}_\alpha - \boldsymbol{r}_\beta)/\partial \boldsymbol{r}_\alpha \cdot \partial/\partial \boldsymbol{v}_\alpha\} F_s(\xi_1,\xi_2,\ldots,\xi_s,t) = (mV)^{-1}(N-s) \sum_{\alpha=1}^{s} \int$$

$$\partial\phi(\boldsymbol{r}_\alpha - \boldsymbol{r}_{s+1})/\partial \boldsymbol{r}_\alpha \cdot \partial F_{s+1}(\xi_1,\xi_2,\ldots,\xi_s,\xi_{s+1},t)/\partial \boldsymbol{v}_a \, d\xi_{s+1}. \qquad (3.5.10)$$

If the right hand side were zero, this would be the Liouville equation for a group of s particles. The meaning would be that the probability density is conserved along the trajectory of the group in $3s$–dimensional phase space. This trajectory is, of course, influenced by the mutual interactions of the s particles and by external forces. The characteristics of the left hand side are the equations of motion of the s particles. The right hand side of (3.5.10) represents the interaction of the s particle group with the remainder of the system. It is due to the symmetry of $D(\Gamma,t)$ and the binary character of the interaction forces, that the influence of the entire remainder can be given credit by integration over one additional (six dimensional) μ–space.

The rigorous derivation of (3.5.10) is not essential for a good understanding of this book. It will nevertheless be given in the next section for amateurs.

3.6. DERIVATION OF THE B.B.G.K.Y.–HIERARCHY.

For the density of clusters of s particles we have in analogy to (3.5.3):

$$\partial f_{s\mu}/\partial t + \sum_{\alpha=1}^{s} (\boldsymbol{v}_\alpha \cdot \partial f_{s\mu}/\partial \boldsymbol{r}_\alpha + m^{-1} \boldsymbol{F}_a \cdot \partial f_{s\mu}/\partial \boldsymbol{v}_\alpha) - m^{-1} \sum_{\alpha=1}^{s} \sum_{j=1}^{N}$$

$$\partial\phi(\boldsymbol{r}_\alpha - \boldsymbol{r}_j)/\partial \boldsymbol{r}_\alpha \cdot \partial f_{s\mu}/\partial \boldsymbol{v}_\alpha = 0, \qquad (3.6.1)$$

where $f_{s\mu}$ is defined as

$$f_{s\mu}(\xi_1,\xi_2,\ldots,\xi_s,\Gamma) = \sum_{i_1=1}^{N}{}' \sum_{i_2=2}^{N}{}' \ldots \sum_{i_s=1}^{N}{}'$$

$$\delta(\xi_1 - \zeta_{i_1})\delta(\xi_2 - \zeta_{i_2})\ldots\delta(\xi_s - \zeta_{i_s}) \qquad (3.6.2)$$

with the new notation $\zeta_i(\Gamma_0,t)$ for the trajectories of the particles and with primed summation signs to indicate that the subscripts i_1, i_2,...,i_s should all be different from each other. The last term of (3.6.1) is written as

$$m^{-1} \sum_{\alpha=1}^{s} \sum_{i_1=1}^{N}{}' \sum_{i_2=1}^{N}{}' \cdots \sum_{i_s=1}^{N}{}' \sum_{j=1}^{N} \partial\phi(\boldsymbol{r}_\alpha-\boldsymbol{r}_j)/\partial\boldsymbol{r}_\alpha \cdot \partial/\partial\boldsymbol{v}_\alpha$$

$$\delta(\xi_1-\zeta_{i_1})\ldots\delta(\xi_s-\zeta_{i_s}) \qquad (3.6.3)$$

In the $(s+2)$–multiple sum of (3.6.3) $j = i_\beta$ $(\beta=1,2,\ldots,s)$ is allowed, but $\alpha \neq j$ because of the exclusion of self forces. The terms with j equal to one of the i_β give ($j = i_\beta$ implies $\boldsymbol{r}_j = \boldsymbol{r}_{i_\beta} = \boldsymbol{r}_\beta$):

$$m^{-1} \sum_{\alpha=1}^{s}{}' \sum_{\beta=1}^{s}{}' \sum_{i_1=1}^{N}{}' \cdots \sum_{i_s=1}^{N}{}' \partial\phi(\boldsymbol{r}_\alpha-\boldsymbol{r}_\beta)/\partial\boldsymbol{r}_\alpha \cdot \partial/\partial\boldsymbol{v}_\alpha$$

$$\delta(\xi_1-\zeta_{i_1})\ldots\delta(\xi_s-\zeta_{i_s}) = m^{-1} \sum_{\alpha=1}^{s}{}' \sum_{\beta=1}^{s}{}' \partial\phi(\boldsymbol{r}_\alpha-\boldsymbol{r}_\beta)/\partial\boldsymbol{r}_\alpha \cdot \partial f_{s\mu}/\partial\boldsymbol{v}_\alpha .$$

$$(3.6.4)$$

The terms with $j \neq i_\beta$ $(\beta=1,2,\ldots,s)$ can be written with the notation i_{s+1} instead of j, an additional deltafunction $\delta(\xi_{s+1}-\zeta_{i_{s+1}})$ and an integration of all terms with respect to ξ_{s+1}. The summation over $i_1,i_2,\ldots,i_{s+1}$ then defines $f_{s+1,\mu}$ so that (3.6.1) transforms into

$$\{ \partial/\partial t + \sum_{\alpha=1}^{s} (\boldsymbol{v}_\alpha \cdot \partial/\partial\boldsymbol{r}_\alpha + m^{-1} \boldsymbol{F}_\alpha \cdot \partial/\partial\boldsymbol{v}_\alpha) - m^{-1} \sum_{\alpha=1}^{s}{}' \sum_{\beta=1}^{s}{}'$$

$$\partial\phi(\boldsymbol{r}_\alpha-\boldsymbol{r}_\beta)/\partial\boldsymbol{r}_\alpha \cdot \partial/\partial\boldsymbol{v}_\alpha\} f_{s\mu}(\xi_1,\xi_2,\ldots,\xi_s,\Gamma_0,t) = m^{-1} \sum_{\alpha=1}^{s} \int$$

$$\partial\phi(\boldsymbol{r}_\alpha-\boldsymbol{r}_{s+1})/\partial\boldsymbol{r}_\alpha \cdot \partial F_{s+1,\mu}(\xi_1,\xi_2,\ldots,\xi_{s+1},\Gamma_0,t)/\partial\boldsymbol{v}_a \, d\xi_{s+1}.$$

$$(3.6.5)$$

Averaging (3.6.5) and using

$$\langle f_{s\mu}(\xi_1,\xi_2,\ldots,\xi_s,\Gamma)\rangle = V^{-s}N(N-1)\ldots(N-s+1)\, F_s(\xi_1,\ldots,\xi_s,t)$$

$$(3.6.6)$$

we immediately derive the hierarchy (3.5.10). A mathematical observation about the hierarchy should be made.

Formally we can view (3.5.10) as a system of N equations with N unknowns, $s = 1,2,...,N$. Because of the factor $(N-s)$ the right hand side disappears for $s=N$. The equation for $s=N$ is simply the Liouville equation. (Because of the assumption that the intermolecular forces are derivable from potentials, the system possesses a Hamiltonian, of course. If this assumption is dropped the equation for F_N is more general than the Liouville equation.) If it were possible to solve this equation, then all the preceding equations $(s<N)$ would be superfluous, since al $F_s(s<N)$ are defined as integrals over $D(\Gamma,t)$ and $F_N = V^N D(\Gamma,t)$.
Of course it is desirable to restrict ourselves to a small number of equations, $s << N$. Formally one uses to take the so–called thermodynamic limit:

$$N \to \infty,\ V \to \infty.\ \ N/V = n_0(\text{finite}). \tag{3.6.7}$$

In this limit the factor $(N-s)/V$ in (3.5.10) is exactly, for finite s, transformed into n_0. In the thermodynamic limit the hierarchy is an *open* system of equations, i.e. the number of equations, no matter to which value of s one is willing to go, is always one less than the number of unknown functions. This poses the problem to find a method for *closing* the system of equations. We will attack that problem in the next chapter.

3.7. EXERCISES.

1. Consider a system of N statistically independent particles in a volume V. The particles are uniformly distributed. Calculate the probability of finding M particles in a volume xV, if $x << 1$ and $M << N$. Rederive (3.1.10).

<u>Solution</u>
The probability of finding a fixed group of M particles in xV is given by $x^M(1-x)^{N-M}$. The number of such groups is $N!(M!)^{-1}\{(N-M)!\}^{-1}$.
Therefore: $P(M) = N!(M!)^{-1}\{(N-M)!\}^{-1}x^M(1-x)^{N-M}$, or, approximately for $x << 1$ and $M << N$, the Poisson distribution $P(M) = (M!)^{-1}(xN)^M \exp(-xN)$.
We now calculate:

$$<M> = \sum_{M=0}^{N} MP(M) \simeq xN \sum_{M=1}^{\infty}$$

$$\{(M-1)!\}^{-1}(xN)^{M-1}\exp(-xN)=xN,$$

$$<M^2> = \sum_{M=0}^{N} M^2P(M) \simeq xN\exp(-xN)\ d/dx\,[x \sum_{M=1}^{\infty}$$

$$\{(M-1)!\}^{-1}(xN)^{M-1}]$$

$$= xN \exp(-xN)\ d/dx\{\exp(xN)\} = (Nx)^2 + Nx,$$

and find:

$$\sigma_M^2 = <M^2> - <M>^2 = (Nx)^2, \text{ so that}$$

$$<\sigma_M/<M> = (Nx)^{-1/2} = <M>^{-1/2}, \text{ i.e. equation (3.1.10).}$$

2. Derive the hierarchy for the configurational distribution functions $\mu_s(\boldsymbol{r}_1,\boldsymbol{r}_2,\ldots,\boldsymbol{r}_s)$ in thermal equilibrium. Determine μ_2 of a dilute gas in the case that no external forces are present.

Solution
In thermal equilibrium the distribution functions $F_s(\xi_1,\xi_2,\ldots,\xi_s)$ are of the form: $F_s = \prod_{i=1}^{s} F_M(\boldsymbol{v}_i)\ \mu_s(\boldsymbol{r}_1.\boldsymbol{r}_2,\ldots,\boldsymbol{r}_s)$ with the Maxwellian $F_M(\boldsymbol{v}) = (\alpha/\pi)^{3/2}$ $\exp(-\alpha v^2)$ and $\alpha = m/(2k_B T)$, where T is the absolute temperature and k_B Boltzmann's constant. Substituting this F_s into (3.5.10) and taking $\partial/\partial t = 0$ we obtain an equation of the form:

$$\prod_{i=1}^{s} F_M(\boldsymbol{v}_i) \sum_{\alpha=1}^{s} \boldsymbol{v}_\alpha \cdot \boldsymbol{Q}_\alpha(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_s) = 0.$$

This equation must be satisfied identically in all $\boldsymbol{v}_\alpha$. Therefore $\boldsymbol{Q}_\alpha = 0$, i.e.

$$k_B T\ \partial\mu_s/\partial\boldsymbol{r}_\alpha - \boldsymbol{F}_\alpha\,\mu_s + \sum_{\beta=1}^{s}{}'\ \partial\phi(\boldsymbol{r}_\alpha-\boldsymbol{r}_\beta)/\partial\boldsymbol{r}_\alpha\,\mu_s$$

$$= -\ N-s/V\int \partial\phi(\boldsymbol{r}_\alpha-\boldsymbol{r}_{s+1})/\partial\boldsymbol{r}_\alpha\,\mu_{s+1}(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_{s+1})d^3r_{s+1} \qquad (3.7.1)$$

with $\alpha = 1,2,\ldots,s$ and the notation $\sum'$ to indicate that $\beta \neq \alpha$.
In the case of a dilute gas the right hand side (proportional to n_0) can be neglected. With $\boldsymbol{F}_\alpha = 0$ and $\boldsymbol{r}_1 - \boldsymbol{r}_2 = \boldsymbol{s}$ we then find for μ_2 the Boltzmann factor: $\mu_2(s) = \exp[-\phi(s)/(k_B T)]$.

3. Derive the B.B.G.K.Y.–hierarchy from the Liouville equation.

Solution

The Hamiltonian of the system is $H = \sum_{i=1}^{N} p_i^2/(2m) + \sum_{i,j=1}^{N}{}' \phi_{ij}$, where $p_i = |\boldsymbol{p}_i|$ is the magnitude of the momentum vector $\boldsymbol{p}_i$ and $\phi_{ij} = \phi(|\boldsymbol{r}_i - \boldsymbol{r}_j|)$ the intermolecular potential. The Liouville equation (1.4.8) is then written as

$$\partial D/\partial t + \sum_{i=1}^{N} \boldsymbol{v}_i \cdot \partial D/\partial \boldsymbol{r}_i - m^{-1} \sum_{i,j=1}^{N}{}' \partial \phi_{ij}/\partial \boldsymbol{r}_i \cdot \partial D/\partial \boldsymbol{v}_i = 0. \tag{3.7.2}$$

Multiplying (3.7.2) with V^s, integrating over $\xi_{s+1}, \xi_{s+2}, \ldots, \xi_N$ and using (1.4.11) we obtain:

$$\partial F_s/\partial t + \sum_{i=1}^{s} \boldsymbol{v}_i \cdot \partial F_s/\partial \boldsymbol{r}_i - m^{-1} \sum_{i,j=1}^{s}{}' \partial \phi_{ij}/\partial \boldsymbol{r}_i \cdot \partial F_s/\partial \boldsymbol{v}_i - V^s/m$$
$$\sum_{i=1}^{s} \sum_{j=s+1}^{N} \int \partial \phi_{ij}/\partial \boldsymbol{r}_i \cdot \partial D/\partial \boldsymbol{v}_i \, d\xi_{s+1} \ldots d\xi_N = 0. \tag{3.7.3}$$

Note that the sum $\sum_{i=s+1}^{N} \sum_{j=1}^{N} \int \partial \phi_{ij}/\partial \boldsymbol{r}_i \cdot \partial D/\partial \boldsymbol{v}_i \, d\xi_{s+1} \ldots d\xi_N = 0$ because of

the theorem of Gauss in $\boldsymbol{v}_i$–spaces. In (3.7.3) we have also omitted

$$V^s \sum_{i=s+1}^{N} \int \boldsymbol{v}_i \cdot \partial D/\partial \boldsymbol{r}_i \, d\xi_{s+1} \ldots d\xi_N = (N-s)/V \int \boldsymbol{v}_{s+1} \cdot \partial F_{s+1}/\partial \boldsymbol{r}_{s+1} d\xi_{s+1}$$

The last equality follows from the symmetry (1.4.10). The integral of the right hand side is also transformed by means of the theorem of Gauss:

$$\int \boldsymbol{v}_s \cdot \partial F_s/\partial \boldsymbol{r}_s \, d\xi_s = \int \boldsymbol{n} \cdot \boldsymbol{v}_s \, F_s \, d^3 v_s d^2 S = 0.$$

The surface integral ($\boldsymbol{n}$ is the unit normal vector at the surface of the system) vanishes, because N is constant and therefore no particle flux leaving the system is allowed. Using the symmetry of D again we transform the last term of (3.7.3) into

$$(N-s)/(mV) \sum_{i=1}^{s} \int \partial \phi_{i,s+1}/\partial \boldsymbol{r}_i \, \partial F_{s+1}/\partial \boldsymbol{v}_i \;\; d\xi_{s+1}$$

and obtain the hierarchy (3.5.10) with $F_\alpha = 0$.

4. Calculate the electrostatic potential due to a fixed electron at the origin surrounded by an infinite electron plasma in thermal equilibrium.

Solution
We use the Vlasov–Poisson equations in the form:

$$\boldsymbol{v} \cdot \nabla f + e/m \, \nabla \Phi \cdot \nabla_v f = 0, \tag{3.7.4}$$
$$\nabla^2 \Phi = e/\epsilon_0 \left[\int f d^3v - n_0\right], \tag{3.7.5}$$

substitute

$$f = n_0\{F_M(v) + \delta F\} + \delta(\boldsymbol{r})\delta(\boldsymbol{v}),$$

where $F_M(v)$ is the Maxwellian distribution and $n_0 \delta \mathrm{F} + \delta(\boldsymbol{r})\delta(\boldsymbol{v})$ is considered to be a small perturbation, and linearize (3.7.4). The result is:

$$\boldsymbol{v} \cdot \nabla \delta F - eF_M/(\mathrm{k_B} T) \cdot \nabla \Phi = 0, \tag{3.7.6}$$
$$\nabla^2 \Phi = e/\epsilon_0 \left[n_0 \int \delta F \, d^3v + \delta(\boldsymbol{r})\right]. \tag{3.7.7}$$

The solution of (3.7.6) satisfying the condition $\delta F(r \to \infty) = \Phi(r \to \infty) = 0$ is given by $\delta F = \mathrm{e} F_M/(\mathrm{k_B} T) \, \Phi$. Substituting this into (3.7.7) we obtain:

$$\nabla^2 \Phi - 1/\lambda_D{}^2 \, \Phi = e/\epsilon_0 \, \delta(\boldsymbol{r}), \tag{3.7.8}$$

where λ_D is the Debye length given in (2.2.30) or $\lambda_D{}^2 = \epsilon_0 \mathrm{k_B} T/(n_0 \mathrm{e}^2)$. At small distances the Coulomb potential $-e/(4\pi\epsilon_0 r)$ satisfies (3.7.8), because $\nabla^2(1/r) = -4\pi\delta(\boldsymbol{r})$. We therefore write $\Phi = -e/(4\pi\epsilon_0 r) g(r)$ and impose the boundary conditions $g(0) = 1$, $g(r \to \infty) = 0$. Substitution into (3.7.8) leads, for $r \neq 0$, to $d^2g/(dr^2) - g/(\lambda_D{}^2) = 1$. Therefore $g(r) = \exp(-r/\lambda_D)$. In this way we obtain the screened potential

$$\Phi = -e/(4\pi\epsilon_0 r) \exp(-r/\lambda_D). \tag{3.7.9}$$