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Bound on the Energy Available from a Plasma

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The motion of an ideal collisionless plasma subject to periodic boundary conditions, with velocity distribution given initially, is studied. The minimum value which the kinetic energy can subsequently attain is determined, on the assumption that the system is subject only to the constraint that phase-space volume is conserved.

I. INTRODUCTION

We consider an ideal collisionless plasma governed by the Vlasov equations. We assume that the velocity distribution function of one species of particle is a periodic function of the space coordinates. Let the initial value of the distribution function be given. In general, as time goes on, some of the particle kinetic energy will be converted into energy of the electromagnetic field, or energy of another species of particle. In particular, this can be expected to occur if the initial distribution is near an unstable equilibrium. There is an upper bound, however, on the energy thus available, which follows from a constraint on possible motions implied by the Vlasov equations; namely that phase-space volume must be conserved.

Newcomb[1] recognized the existence of this upper bound, and applied it to give a proof of the stability of a Maxwellian velocity distribution against small disturbances, employing the properties of entropy. An extension of Newcomb's proof to large disturbances has been given by Fowler.[2] Earlier, Rosenbluth[3] proved, using an appropriately modified definition of entropy, a theorem on the absolute stability of any equilibrium distribution which is a monotone decreasing function of energy. Related ideas have been discussed by Buneman.[4]

In this paper, we determine the best bound on the available energy if the plasma is subject *only* to the constraint of conservation of phase-space volume. The bound we give is small for any distribution function which is nearly a monotone decreasing function of energy.

Drummond and Pines[5] have studied the nonlinear growth of disturbances of a slightly instable one-dimensional plasma. These authors conclude that the system tends to a turbulent steady state, in which a certain small finite portion of the plasma kinetic energy has been converted into field energy. If the initial distribution function is an even function of velocity, our absolute upper bound on available energy is of the order of four times the quantity which, according to Drummond and Pines, will actually be yielded to the electromagnetic field. This discrepancy, it is felt, may be because in bounding the available energy we use only the constraint that phase-space volume is conserved; and by use of other, less obvious constraints to which the motion is no doubt subject, a more precise bound could be obtained.

II. STATEMENT OF THE RESULT

Let $f(t, \mathbf{x}, \mathbf{v})$ be the distribution function of one species of particle in the plasma. We assume that f satisfies the Vlasov equation

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla_x f + \frac{\mathbf{F}}{m}\cdot\nabla_v f = 0, \tag{1}$$

where the force $\mathbf{F}(t, \mathbf{x}, \mathbf{v})$ satisfies

$$\nabla_v\cdot\mathbf{F} = 0. \tag{2}$$

The motion of one particle is governed by the equations

$$d\mathbf{x}/dt = \mathbf{v}, \tag{3}$$

$$m(d\mathbf{v}/dt) = \mathbf{F}. \tag{4}$$

If one looks on Eq. (3) and Eq. (4) as defining a *flow* in the $\mathbf{x}$–$\mathbf{v}$ phase space, or a continuous transformation of phase space onto itself, then as is well

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[1] Quoted by I. B. Bernstein, Phys. Rev. **109**, 10 (1958) [see Appendix I, pp. 20–21].

[2] T. K. Fowler, "Lyapunoff's Stability Criteria for Plasmas," (appendix) (to be published).

[3] M. N. Rosenbluth, "Microinstabilities," article in Riso Report No. 18, Danish Atomic Energy Commission Research Establishment Riso, International Summer Course in Plasma Physics (1960).

[4] O. Buneman, in *Radiation and Waves in Plasmas*, edited by M. Mitchner (Stanford University Press, Stanford California, 1961).

[5] W. E. Drummond and D. Pines, Nucl. Fusion Suppl., Part 3 (to be published).

known, Eqs. (2), (3), and (4) show that the flow is *incompressible*. Furthermore, Eq. (1) says that the value of f, at the phase-space position of any fixed particle, does not change as the particle moves about in phase space. Interpreting f as *density*, we see that the phase-space motion is an incompressible flow of a fluid of variable density.

Let the initial value of f be given by

$$f(0, x, v) = f_0(x, v). \tag{5}$$

The kinetic energy of the plasma is

$$W(t) = \tfrac{1}{2}m \iint v^2 f(t, \mathbf{x}, \mathbf{v})\, d\mathbf{x}\, d\mathbf{v}. \tag{6}$$

Assuming f to be periodic in $\mathbf{x}$, we need integrate in $\mathbf{x}$ only over a finite box, the sides of which are the periods in the various coordinates.

Our object is to give a lower bound for $W(t)$, given f_0. We state that such a lower bound is W_1, given by

$$W_1 = \tfrac{1}{2}m \iint v^2 f_1(\mathbf{v})\, d\mathbf{x}\, d\mathbf{v}, \tag{7}$$

where $f_1(\mathbf{v})$ is (1) a monotone decreasing function of v^2; (2) for any number $\alpha > 0$, the phase-space volume of the region in $\mathbf{x}$–$\mathbf{v}$ space where $f_1(\mathbf{v}) > \alpha$ is equal to the phase-space volume of the region in $\mathbf{x}$–$\mathbf{v}$ space where $f_0(\mathbf{x}, \mathbf{v}) > \alpha$.

Our condition (2) is true of any possible $f(t, \mathbf{x}, \mathbf{v})$ arising from the motion since the phase-space flow is incompressible—i.e., phase-space volumes are conserved, and the value of f at a particle is conserved during any possible motion.

The question now is, subject only to condition (2), what is the minimum value possible for $W(t)$. By Eq. (6) we see that $W(t)$ is the potential energy the fluid would have if it were acted on by a body force of magnitude mvf directed toward $\mathbf{v} = 0$. Of course the lowest-energy state is the one in which the more massive particles (f large) are the closer to $\mathbf{v} = 0$. That is, if two particles have masses f, f', and velocities v, v', then in the lowest-energy state, if $f > f'$, then $v < v'$. This state corresponds to the distribution function $f_1(\mathbf{v})$ defined above. Thus the amount of kinetic energy originally in $f_0(\mathbf{x}, \mathbf{v})$ which can in principle be yielded up is not larger than

$$W(0) - W_1.$$

It may be noted that the same conclusion is valid if instead of periodicity we impose on f the condition of confinement in a finite box with perfectly reflecting walls.

Also, if the particles are acted on by a conservative external force field with a potential $\varphi(\mathbf{x})$ a quite similar conclusion is valid. Namely, if we define

$$W(t) = \iint [\tfrac{1}{2}mv^2 + \varphi(\mathbf{x})] f(t, \mathbf{x}, v)\, d\mathbf{x},\, d\mathbf{v}$$

then it follows that

$$W(t) \geq W_1 = \iint [\tfrac{1}{2}mv^2 + \varphi(\mathbf{x})] f_1(\tfrac{1}{2}mv^2 + \varphi(\mathbf{x}))\, d\mathbf{x},\, d\mathbf{v},$$

where (1) f_1 is a monotone decreasing function of its argument; (2) for any number $\alpha > 0$, the phase-space volume of the region in $\mathbf{x}$–$\mathbf{v}$ space where

$$f_1(\tfrac{1}{2}mv^2 + \varphi(\mathbf{x})) > \alpha$$

is equal to the phase-space volume of the region where

$$f_0(\mathbf{x}, \mathbf{v}) > \alpha.$$

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