CHAPTER

SIX

ASYMPTOTIC EXPANSION OF INTEGRALS

In five minutes you will say that it is all so absurdly simple.

—Sherlock Holmes, *The Adventure of the Dancing Men*
Sir Arthur Conan Doyle

(E) 6.1 INTRODUCTION

The analysis of differential and difference equations in Chaps. 3 to 5 is pure local analysis; there we predict the behavior of solutions near one point, but we do not incorporate initial-value or boundary-value data at other points. As a result, our predictions of the local behavior usually contain unknown constants. However, when the differential or difference equation is soluble, we can use the boundary and initial data to make parameter-free predictions of local behavior.

Example 1 *Effect of initial data upon asymptotic behavior.* The solution to the initial-value problem $y'' = y$ $[y(0) = 1, y'(0) = 0]$ is $y(x) = \cosh x$. Local analysis of the differential equation $y'' = y$ at the irregular singular point at $x = \infty$ gives two kinds of possible behaviors for $y(x)$ as $x \to +\infty$: $y(x) \sim Ae^x$ $(x \to +\infty)$, where A is some nonzero constant, or, if $A = 0$, $y(x) \sim Be^{-x}$ $(x \to +\infty)$, where B is some constant. Since the equation is soluble in closed form, we know from the initial conditions that $A = \frac{1}{2}$. Similarly, if the initial conditions were $y(0) = 1$, $y'(0) = -1$, we would find that $A = 0$ and $B = 1$.

In very rare cases, it is possible to solve a differential equation in closed form. There, as we have seen in Example 1, it is easy to incorporate initial or boundary conditions. Among those equations that are not soluble in terms of elementary functions, it is sometimes possible (though still very rare) to find a representation for the solution of the equation as an integral in which the independent variable x appears as a parameter. Typically, this integral representation of the solution contains all of the initial-value or boundary-value data. In this chapter we will show how to perform a local analysis of integral representations containing x as a parameter. We will see that the local behavior of the solution $y(x)$ is completely determined by this analysis of the integral and contains no arbitrary constants.

Example 2 *Integral representation of a solution to an initial-value problem.* The solution to the initial-value problem $y' = xy + 1$ $[y(0) = 0]$ can be expressed as the integral $y(x) =$

$e^{x^2/2} \int_0^x e^{-t^2/2}\,dt$. This integral representation implies that $y(x) \sim \sqrt{\pi/2}\, e^{x^2/2}$ $(x \to +\infty)$ because $\lim_{x\to+\infty} \int_0^x e^{-t^2/2}\,dt = \int_0^\infty e^{-t^2/2}\,dt = \sqrt{\pi/2}$. Note that a direct local analysis of the differential equation $y' = xy + 1$ predicts only that $y(x) \sim Ae^{x^2/2}$ $(x \to +\infty)$ (where A is unknown) because it does not utilize the initial condition $y(0) = 0$.

Here is a more striking example of the usefulness of integral representations.

Example 3 *Integral representation of a solution to a boundary-value problem.* Consider the boundary-value problem

$$xy''' + 2y = 0, \qquad y(0) = 1, \qquad y(+\infty) = 0. \tag{6.1.1}$$

The problem is to find the behavior of $y(x)$ as $x \to +\infty$. Using the techniques of Chap. 3, local analysis gives three possible behaviors of $y(x)$ for large positive x: $y(x) \sim Ax^{1/3}e^{-3\omega(x/2)^{2/3}}$ $(x \to +\infty)$, where ω is one of the cube roots of unity: $1, (-1 \pm i\sqrt{3})/2$. The condition $y(+\infty) = 0$ implies that we must choose $\omega = 1$ to avoid the exponentially growing solutions. Therefore, the solution to (6.1.1) satisfies

$$y(x) \sim Ax^{1/3}e^{-3(x/2)^{2/3}}, \qquad x \to +\infty. \tag{6.1.2}$$

Here A is a constant that cannot be determined by the methods of Chap. 3.

However, this problem is rigged; there is a delightful integral representation for the solution of the boundary-value problem (6.1.1):

$$y(x) = \int_0^\infty \exp\left(-t - \frac{x}{\sqrt{t}}\right) dt. \tag{6.1.3}$$

Notice that this integral satisfies the boundary conditions $y(0) = 1$ (set $x = 0$ and evaluate the integral) and $y(+\infty) = 0$. (Why?) To prove that (6.1.3) satisfies the differential equation when $x > 0$, we differentiate three times under the integral sign and integrate the result by parts once (see Prob. 6.1).

The integral representation (6.1.3) can be used to evaluate the constant A in the asymptotic behavior (6.1.2) of the solution $y(x)$. Laplace's method (see Sec. 6.4 and Prob. 6.31) gives $A = \pi^{1/2}2^{2/3}3^{-1/2}$.

Example 4 *Integral representation for $n!$.* The factorial function $a_n = (n - 1)!$ satisfies the first-order difference equation $a_{n+1} = na_n$ $(a_1 = 1)$. The direct local analysis of this difference equation in Sec. 5.4 gives the large-n behavior of a_n (the Stirling formula) apart from an overall multiplicative constant: $a_n \sim Cn^n e^{-n} n^{-1/2}$ $(n \to \infty)$. The constant C is determined by the initial condition $a_1 = 1$ which cannot be used in the analysis of the difference equation at $n = \infty$.

The integral representation (2.2.2) for a_n, $a_n = \int_0^\infty t^{n-1}e^{-t}\,dt$, is equivalent to *both* the difference equation and the initial condition. When n is large, the asymptotic behavior of this integral may be found using Laplace's method (see Example 10 in Sec. 6.4). The result is $C = \sqrt{2\pi}$.

Example 5 *Integral representation for the solution of a difference equation.* The solution of the initial-value problem

$$(n + 1)a_{n+1} = 2na_n - na_{n-1}, \qquad a_0 = 1,\, a_1 = 0, \tag{6.1.4}$$

may be expressed as an integral which is equivalent to the difference equation together with the initial conditions:

$$a_n = \frac{1}{n!}\int_0^\infty e^{1-t}t^n J_0(2\sqrt{t})\,dt, \tag{6.1.5}$$

where J_0 is the Bessel function of order zero (see Prob. 6.2).

Local analysis of the difference equation gives two possible behaviors for large n, $a_n \sim A_{\pm} \exp(\pm 2i\sqrt{n})n^{-1/4}$ $(n \to +\infty)$, where the constants $A_{\pm}$ cannot be determined. However, an analysis of the integral representation (6.1.5) using Laplace's method (see Prob. 6.32) shows that

$$a_n \sim \sqrt{e/\pi}\, n^{-1/4} \cos(2\sqrt{n} - \tfrac{1}{4}\pi), \qquad n \to +\infty. \tag{6.1.6}$$

The asymptotic expansion of integral representations is an extremely important technique because all of the special (Bessel, Airy, gamma, parabolic cylinder, hypergeometric) functions commonly used in mathematical physics and applied mathematics have integral representations. The asymptotic properties of these special functions are derived from their integral representations. Many of these properties are used in Part IV to obtain the global behavior of general classes of differential equations whose solutions are not expressible as integrals.

Sometimes, an integral representation of the solution to a differential equation is derived by following the systematic procedure of taking an integral (Fourier, Laplace, Hankel) transform of the equation. However, many integral representations are the product of imaginative guesswork. Unfortunately, apart from a small number of equations, one cannot hope to find integral representations for the solutions. This chapter is concerned only with the local analysis of integrals and not with the construction of integral representations. After all, discovering an integral representation is the same as solving the equation in closed form in terms of known functions, and this book is primarily concerned with those equations which *cannot* be solved exactly. Thus, the construction and subsequent expansion of integral representations is not a general method of global analysis. That is why, although the examples in this chapter provide a first glimpse of how to obtain global information about the solutions to differential and difference equations, we include this chapter in Part II, Local Analysis, rather than in Part IV, Global Analysis. This chapter concerns the local analysis of integrals rather than the global analysis of differential and difference equations.

(E) 6.2 ELEMENTARY EXAMPLES

It is sometimes possible to determine the behavior of an integral without using any techniques beyond those introduced in our first exposure to asymptotic analysis in Chap. 3. Consider, for example, the integral

$$I(x) = \int_0^2 \cos\left[(xt^2 + x^2t)^{1/3}\right] dt.$$

It is hard to evaluate this integral in closed form when x is nonzero. However, to determine the leading behavior of $I(x)$ as $x \to 0$, we simply set $x = 0$ in the integral and do the trivial integration. The result is $I(x) \sim 2$ $(x \to 0)$.

More generally, suppose we are asked to find the leading behavior of the integral

$$I(x) = \int_a^b f(x, t)\, dt \qquad \text{as } x \to x_0.$$

If it is given that

$$f(t, x) \sim f_0(t), \qquad x \to x_0,$$

uniformly for $a \le t \le b$ [that is, $\lim_{x\to x_0} f(t, x)/f_0(t) = 1$ uniformly in t], then the leading behavior of $I(x)$ as $x \to x_0$ is just

$$I(x) = \int_a^b f(t, x)\, dt \sim \int_a^b f_0(t)\, dt, \qquad x \to x_0, \tag{6.2.1}$$

provided that the right side of this relation is finite and nonzero. (See Prob. 6.3 for the details of the argument.)

This simple idea may be easily extended to give the full asymptotic expansion of $I(x)$ as $x \to x_0$. If $f(t, x)$ possesses the asymptotic expansion

$$f(t, x) \sim \sum_{n=0}^{\infty} f_n(t)(x - x_0)^{\alpha n}, \qquad x \to x_0,$$

for some $\alpha > 0$, uniformly for $a \le t \le b$, then

$$\int_a^b f(t, x)\, dt \sim \sum_{n=0}^{\infty} (x - x_0)^{\alpha n} \int_a^b f_n(t)\, dt, \qquad x \to x_0, \tag{6.2.2}$$

provided that all the terms on the right are finite (see Prob. 6.3). In the following examples we illustrate the use of formulas (6.2.1) and (6.2.2) and introduce some new twists that extend the applicability of these elementary ideas.

Example 1 $\int_0^1 [(\sin tx)/t]\, dt$ as $x \to 0$. Since the Taylor expansion $(\sin tx)/t = x - x^3t^2/6 + x^5t^4/120 - \cdots$ converges uniformly for $0 \le t \le 1$, $|x| \le 1$, it follows that

$$\int_0^1 \frac{\sin tx}{t}\, dt \sim x - \frac{1}{18}x^3 + \frac{1}{600}x^5 - \cdots, \qquad x \to 0.$$

The series on the right converges for all x.

Example 2 $\int_0^x t^{-1/2}e^{-t}\, dt$ as $x \to 0+$. The expansion

$$t^{-1/2}e^{-t} = t^{-1/2} - t^{1/2} + \tfrac{1}{2}t^{3/2} - \tfrac{1}{6}t^{5/2} + \cdots$$

converges for all $t \ne 0$ but does not converge at $t = 0$. However, this series is asymptotic uniformly for $0 \le t \le x$ as $x \to 0+$. Thus, term-by-term integration gives

$$\int_0^x t^{-1/2}e^{-t}\, dt \sim 2x^{1/2} - \tfrac{2}{3}x^{3/2} + \tfrac{1}{5}x^{5/2} - \tfrac{1}{21}x^{7/2} + \cdots, \qquad x \to 0+. \tag{6.2.3}$$

This result can be rederived by substituting $s = t^{1/2}$ in the integral:

$$\int_0^x t^{-1/2}e^{-t}\, dt = 2\int_0^{\sqrt{x}} e^{-s^2}\, ds.$$

The Taylor expansion of e^{-s^2} does converge uniformly for $0 \le s \le 1$, so

$$\int_0^{\sqrt{x}} e^{-s^2}\, ds = \int_0^{\sqrt{x}} [1 - s^2 + \tfrac{1}{2}s^4 - \tfrac{1}{6}s^6 + \cdots]ds$$

$$= [s - \tfrac{1}{3}s^3 + \tfrac{1}{10}s^5 - \tfrac{1}{42}s^7 + \cdots]_0^{\sqrt{x}},$$

which reproduces (6.2.3).

Example 3 $\int_x^\infty e^{-t^4}\,dt$ as $x \to 0$. Term-by-term integration of the convergent Taylor series $e^{-t^4} = 1 - t^4 + \frac{1}{2}t^8 - \frac{1}{6}t^{12} + \cdots$ gives a divergent result. The proper way to apply (6.2.2) is to write

$$\int_x^\infty e^{-t^4}\,dt = \int_0^\infty e^{-t^4}\,dt - \int_0^x e^{-t^4}\,dt$$

and then to substitute the Taylor series only in the second term on the right. The result is

$$\int_x^\infty e^{-t^4}\,dt = \Gamma(\tfrac{5}{4}) - x + \tfrac{1}{5}x^5 - \tfrac{1}{18}x^9 + \tfrac{1}{78}x^{13} - \cdots, \tag{6.2.4}$$

where we have used the substitution $s = t^4$ to obtain

$$\int_0^\infty e^{-t^4}\,dt = \tfrac{1}{4}\int_0^\infty s^{-3/4}e^{-s}\,ds = \tfrac{1}{4}\Gamma(\tfrac{1}{4}) = \Gamma(\tfrac{5}{4}).$$

The series (6.2.4) converges for all x, although it is not very useful if $|x|$ is large (see Example 2 of Sec. 6.3).

Example 4 *Incomplete gamma function* $\Gamma(a, x)$ *as* $x \to 0+$. The incomplete gamma function $\Gamma(a, x)$ is defined by $\Gamma(a, x) = \int_x^\infty t^{a-1}e^{-t}\,dt$ $(x > 0)$. To discuss the behavior of $\Gamma(a, x)$ as $x \to 0+$, we distinguish three cases: (a) $a > 0$; (b) $a < 0$ but nonintegral; and (c) $a = 0, -1, -2, -3, \ldots$. Only cases (b) and (c) present new difficulties.

(a) $a > 0$. As in Example 3, we find that

$$\begin{aligned}\Gamma(a, x) &= \int_0^\infty t^{a-1}e^{-t}\,dt - \int_0^x t^{a-1}(1 - t + \tfrac{1}{2}t^2 - \tfrac{1}{6}t^3 + \cdots)\,dt \\ &= \Gamma(a) - \sum_{n=0}^\infty (-1)^n \frac{x^{a+n}}{n!\,(a+n)},\end{aligned} \tag{6.2.5}$$

where the series converges for all x.

(b) $a < 0$ but nonintegral. In this case we write

$$\begin{aligned}\Gamma(a, x) = \int_x^\infty t^{a-1}\left[1 - t + \cdots + (-1)^N \frac{t^N}{N!}\right]dt - \int_0^x t^{a-1}\left[e^{-t} - \sum_{n=0}^N \frac{(-t)^n}{n!}\right]dt \\ + \int_0^\infty t^{a-1}\left[e^{-t} - \sum_{n=0}^N \frac{(-t)^n}{n!}\right]dt,\end{aligned} \tag{6.2.6}$$

where N is the largest integer less than $-a$. Notice that all three integrals on the right side of (6.2.6) converge. The first two integrals may be performed term by term and the third may be done by repeated integration by parts (see Prob. 6.4):

$$\int_0^\infty t^{a-1}\left[e^{-t} - \sum_{n=0}^N \frac{(-t)^n}{n!}\right]dt = \Gamma(a). \tag{6.2.7}$$

We conclude that the series in (6.2.5) is still valid for this case.

There is a slightly simpler derivation of (6.2.5) for case (b) which uses the relation $\partial\Gamma(a, x)/\partial x = -x^{a-1}e^{-x}$. Integrating the series expansion of $x^{a-1}e^{-x}$ gives

$$\Gamma(a, x) \sim C - \sum_{n=0}^N (-1)^n \frac{x^{a-n}}{n!\,(a+n)}, \qquad x \to 0+,$$

for some constant C. The value $C = \Gamma(a)$ is determined by evaluating an integral like that in (6.2.7).

(*c*) $a = 0, -1, -2, \ldots$. For simplicity, we consider the case $a = 0$:

$$E_1(x) = \int_x^\infty \frac{e^{-t}}{t}\,dt,$$

which is the *exponential integral*. Since

$$dE_1(x)/dx = -e^{-x}/x = -1/x + 1 - \tfrac{1}{2}x + \tfrac{1}{6}x^2 - \cdots,$$

we have

$$E_1(x) \sim C - \ln x + x - \tfrac{1}{4}x^2 + \tfrac{1}{18}x^3 - \cdots, \qquad x \to 0+, \tag{6.2.8}$$

where C is a constant. Note the appearance of the function $\ln x$ in this series.

Next, we compute C. From (6.2.8) we see that $E_1(x) + \ln x \sim C$ $(x \to 0+)$, so

$$C = \lim_{x \to 0+} \left(\int_x^\infty \frac{e^{-t}}{t}\,dt + \ln x \right)$$

$$= -\gamma, \tag{6.2.9}$$

where $\gamma \doteqdot 0.5772$ is Euler's constant [see Prob. 6.5(*a*)].

The expansion of $E_1(x)$ may be obtained more directly by writing

$$E_1(x) = \int_x^\infty \frac{1}{t(t+1)}\,dt + \int_0^\infty \left(e^{-t} - \frac{1}{t+1}\right)\frac{1}{t}\,dt - \int_0^x \left(e^{-t} - \frac{1}{t+1}\right)\frac{1}{t}\,dt. \tag{6.2.10}$$

The first integral on the right equals $-\ln x + \ln(1 + x) = -\ln x + x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots$, the second integral equals $-\gamma$ [see Prob. 6.5(*b*)], and Taylor expanding the third integrand gives

$$\int_0^x \left(e^{-t} - \frac{1}{t+1}\right)\frac{1}{t}\,dt = -\frac{1}{4}x^2 + \frac{5}{18}x^3 - \cdots.$$

Combining these results recovers (6.2.8) with $C = -\gamma$. More generally, if $a = -N$ $(N = 0, 1, 2, \ldots)$, then a similar analysis (see Prob. 6.6) gives

$$\Gamma(-N, x) \sim C_N + \frac{(-1)^{N+1}}{N!}\ln x - \sum_{\substack{n=0 \\ n \neq N}}^{\infty} (-1)^n \frac{x^{n-N}}{n!\,(n-N)}, \qquad x \to 0+, \tag{6.2.11}$$

where

$$C_N = \frac{(-1)^{N+1}}{N!}\left(\gamma - \sum_{n=1}^{N} \frac{1}{n}\right), \qquad N > 0, \tag{6.2.12}$$

and $C_0 = -\gamma$. This series contains the function $\ln x$, a new feature not found in the series (6.2.5). The appearance of a logarithmic term when $a = 0, -1, -2, \ldots$ is reminiscent of the special cases of the Frobenius method for differential equations in which logarithms also appear.

(E) 6.3 INTEGRATION BY PARTS

Integration by parts is a particularly easy procedure for developing asymptotic approximations to many kinds of integrals. We explain the technique by applying it to a variety of problems.

Example 1 *Derivation of an asymptotic power series.* If $f(x)$ is differentiable near $x = 0$, then the local behavior of $f(x)$ near 0 may be studied using integration by parts. We merely represent $f(x)$

as the integral $f(x) = f(0) + \int_0^x f'(t)\, dt$. Integrating once by parts gives

$$f(x) = f(0) + (t - x)f'(t)\Big|_0^x + \int_0^x (x - t)f''(t)\, dt$$

$$= f(0) + xf'(0) + \int_0^x (x - t)f''(t)\, dt.$$

Repeating this process $(N - 1)$ times gives

$$f(x) = \sum_{n=0}^{N} \frac{x^n}{n!} f^{(n)}(0) + \frac{1}{N!}\int_0^x (x - t)^N f^{(N+1)}(t)\, dt.$$

If the remainder term (the integral on the right) exists for all N and sufficiently small positive x, then

$$f(x) \sim \sum_{n=0}^{\infty} \frac{x^n}{n!} f^{(n)}(0), \qquad x \to 0+.$$

Why? Moreover, if this series converges, then it is just the Taylor expansion of $f(x)$ about $x = 0$.

Example 2 *Behavior of* $\int_x^\infty e^{-t^4}\, dt$ *as* $x \to +\infty$. We have already found the behavior of the integral

$$I(x) = \int_x^\infty e^{-t^4}\, dt \tag{6.3.1}$$

for small values of the parameter x. In Example 3 of Sec. 6.2 we showed that

$$I(x) = \int_0^\infty e^{-t^4}\, dt - \int_0^x (1 - t^4 + \tfrac{1}{2}t^8 - \tfrac{1}{6}t^{12} + \cdots)\, dt$$

$$= \Gamma(\tfrac{5}{4}) - x + \tfrac{1}{5}x^5 - \tfrac{1}{18}x^9 + \tfrac{1}{78}x^{13} - \cdots. \tag{6.3.2}$$

Although the series (6.3.2) converges for all x, it is not very useful if x is large (see Fig. 6.1).

In order to study $I(x)$ for large x we must develop an asymptotic series for $I(x)$ in inverse powers of x; to wit, we rewrite (6.3.1) as

$$I(x) = -\frac{1}{4}\int_x^\infty \frac{1}{t^3}\frac{d}{dt}(e^{-t^4})\, dt$$

and integrate by parts:

$$I(x) = -\frac{1}{4t^3}e^{-t^4}\Big|_x^\infty - \frac{3}{4}\int_x^\infty \frac{1}{t^4}e^{-t^4}\, dt$$

$$= \frac{1}{4x^3}e^{-x^4} - \frac{3}{4}\int_x^\infty \frac{1}{t^4}e^{-t^4}\, dt. \tag{6.3.3}$$

But
$$\int_x^\infty \frac{1}{t^4}e^{-t^4}\, dt < \frac{1}{x^4}\int_x^\infty e^{-t^4}\, dt = \frac{1}{x^4}I(x) \ll I(x), \qquad x \to +\infty,$$

so the leading behavior of $I(x)$ is

$$I(x) \sim \frac{1}{4x^3}e^{-x^4}, \qquad x \to +\infty. \tag{6.3.4}$$

Repeated integration by parts gives the full asymptotic expansion of $I(x)$. To systematize the argument we define the integrals

$$I_n(x) = \int_x^\infty \frac{1}{t^{4n}}e^{-t^4}\, dt. \tag{6.3.5}$$

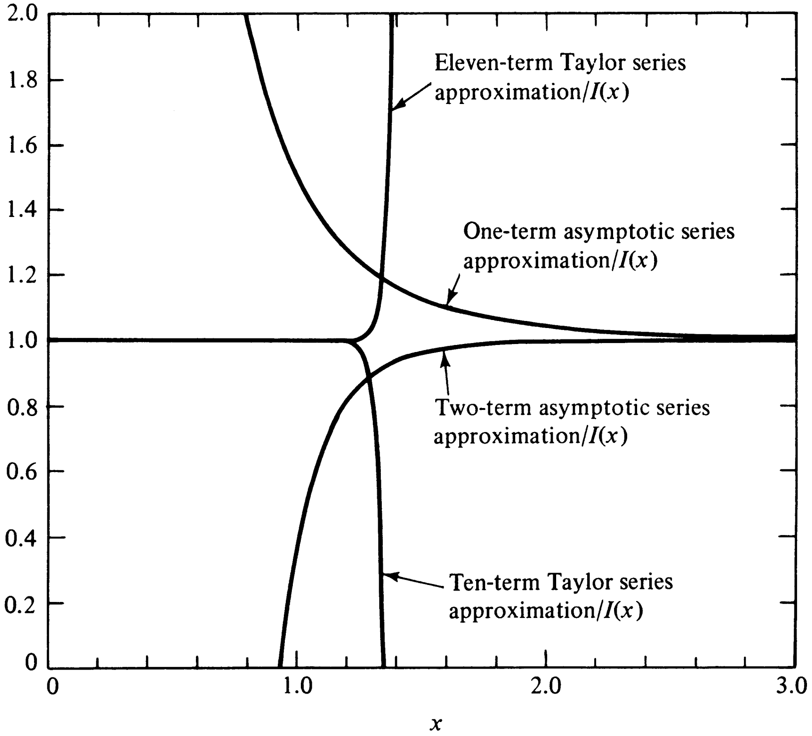

Figure 6.1 A comparison of small-x and large-x approximations to $I(x) = \int_x^\infty \exp(-t^4)\, dt$. The small-$x$ approximation to $I(x)$ is the Taylor series in (6.3.2) which converges for all x. The large-x approximation to $I(x)$ is the asymptotic series in (6.3.8) which is valid as $x \to +\infty$. All approximations to $I(x)$ are normalized by dividing by $I(x)$. It is clear that the truncated asymptotic series is useful in a much larger region than the truncated Taylor series.

Now, (6.3.3) becomes $I(x) = e^{-x^4}/4x^3 - \frac{3}{4}I_1(x)$. Integrating

$$I_n(x) = -\frac{1}{4}\int_x^\infty \frac{1}{t^{4n+3}} \frac{d}{dt} e^{-t^4}\, dt$$

by parts gives

$$\begin{aligned} I_n(x) &= -\frac{1}{t^{4n+3}} e^{-t^4}\Bigg|_x^\infty - \left(n + \frac{3}{4}\right)\int_x^\infty \frac{1}{t^{4n+4}} e^{-t^4}\, dt \\ &= \frac{1}{4x^{4n+3}} e^{-x^4} - \left(n + \frac{3}{4}\right) I_{n+1}(x). \end{aligned} \tag{6.3.6}$$

Therefore,

$$\begin{aligned} I(x) = \frac{1}{4x^3} e^{-x^4}&\left[1 - \frac{3}{4x^4} + \frac{(3)(7)}{(4x^4)^2} - \frac{(3)(7)(11)}{(4x^4)^3} + \cdots + (-1)^{n-1} \frac{(3)(7)(11)\cdots(4n-5)}{(4x^4)^{n-1}}\right] \\ &+ (-1)^n \frac{(3)(7)(11)\cdots(4n-1)}{4^n} I_n(x), \end{aligned} \tag{6.3.7}$$

which is an identity valid for all $x > 0$.

The integrand of (6.3.5) is positive, so $I_{n+1}(x) > 0$ for $x > 0$. Thus, (6.3.6) implies that

$$|I_n(x)| < \frac{1}{4x^{4n+3}} e^{-x^4} \ll \frac{1}{x^{4n-1}} e^{-x^4}, \qquad x \to +\infty,$$

so the term proportional to $I_n(x)$ in (6.3.7) is asymptotically much smaller than the last retained term in square brackets in (6.3.7). Therefore, the full asymptotic expansion of $I(x)$ is

$$I(x) \sim \frac{1}{4x^3} e^{-x^4} \left[1 + \sum_{n=1}^{\infty} (-1)^n \frac{(3)(7)(11)\cdots(4n-1)}{(4x^4)^n}\right], \qquad x \to +\infty. \tag{6.3.8}$$

The accuracy of this asymptotic approximation and its advantage over the convergent series in (6.3.2) when $x > 1$ is demonstrated in Fig. 6.1.

Example 3 *Behavior of* $\int_0^x t^{-1/2} e^{-t}\, dt$ *as* $x \to +\infty$. We may use integration by parts to find the behavior of $I(x) = \int_0^x t^{-1/2} e^{-t}\, dt$ for large x, but we must be careful. Immediate integration by parts gives an indeterminate expression which is the difference of two infinite terms:

$$I(x) = \int_0^x t^{-1/2} e^{-t}\, dt = -t^{-1/2} e^{-t}\Big|_0^x - \frac{1}{2}\int_0^x t^{-3/2} e^{-t}\, dt. \tag{6.3.9}$$

It is best to express $I(x)$ as the difference of two integrals:

$$\int_0^x t^{-1/2} e^{-t}\, dt = \int_0^\infty t^{-1/2} e^{-t}\, dt - \int_x^\infty t^{-1/2} e^{-t}\, dt.$$

The first integral on the right is finite and has the value $\Gamma(\frac{1}{2}) = \sqrt{\pi}$; the second may be integrated by parts successfully because the contribution from the endpoint at ∞ vanishes:

$$\begin{aligned}\int_0^x t^{-1/2} e^{-t}\, dt &= \sqrt{\pi} + \int_x^\infty t^{-1/2} \frac{d}{dt}(e^{-t})\, dt \\ &= \sqrt{\pi} - \frac{e^{-x}}{\sqrt{x}} + \frac{1}{2}\int_x^\infty t^{-3/2} e^{-t}\, dt.\end{aligned}$$

Repeated use of integration by parts gives the full asymptotic expansion of $I(x)$ (see Prob. 6.11):

$$\int_0^x t^{-1/2} e^{-t}\, dt \sim \sqrt{\pi} - \frac{e^{-x}}{\sqrt{x}} \left[1 + \sum_{n=1}^{\infty} (-1)^n \frac{(1)(3)(5)\cdots(2n-1)}{(2x)^n}\right], \qquad x \to +\infty. \tag{6.3.10}$$

The general rule to be learned from this example is that integration by parts will not work if the contribution from one of the limits of integration is much larger than the size of the integral. In this example the integral $I(x)$ is finite for all $x > 0$, but the endpoint $t = 0$ contributes a spurious infinity in (6.3.9).

Example 4 *Behavior of* $\int_0^x e^{t^2}\, dt$ *as* $x \to +\infty$. Here, it is wrong to write $\int_0^x e^{t^2}\, dt = \int_0^\infty e^{t^2}\, dt - \int_x^\infty e^{t^2}\, dt$ because the right side has the form $\infty - \infty$. But it is also wrong to integrate directly by parts:

$$\begin{aligned}\int_0^x e^{t^2}\, dt &= \frac{1}{2}\int_0^x \frac{1}{t}\frac{d}{dt}(e^{t^2})\, dt \\ &= \frac{1}{2t} e^{t^2}\Big|_0^x + \frac{1}{2}\int_0^x t^{-2} e^{t^2}\, dt,\end{aligned}$$

which also has the form $\infty - \infty$. To obtain a correct asymptotic expansion of this integral about $x = \infty$, we introduce a cutoff parameter a and write the integrals

$$\int_0^x e^{t^2}\, dt = \int_0^a e^{t^2}\, dt + \int_a^x e^{t^2}\, dt \tag{6.3.11}$$

for some fixed $0 < a < x$. We will see that for fixed a, the full asymptotic expansion of (6.3.11) is independent of the first integral on the right and is also independent of the cutoff a!

We begin our analysis by expanding the second integral on the right side of (6.3.11):

$$\int_a^x e^{t^2}\,dt = \frac{1}{2t}e^{t^2}\Big|_a^x + \frac{1}{2}\int_a^x \frac{1}{t^2}e^{t^2}\,dt$$

$$= \frac{1}{2x}e^{x^2} - \frac{1}{2a}e^{a^2} + \frac{1}{2}\int_a^x \frac{1}{t^2}e^{t^2}\,dt. \tag{6.3.12}$$

Note that $\frac{1}{2}\int_a^x e^{t^2}\,dt/t^2 \ll e^{x^2}/2x$ $(x \to +\infty)$ because by l'Hôpital's rule

$$\lim_{x\to+\infty} \frac{\frac{1}{2}\int_a^x t^{-2}e^{t^2}\,dt}{\frac{1}{2}x^{-1}e^{x^2}} = \lim_{x\to+\infty} \frac{x^{-2}e^{x^2}}{(2-x^{-2})e^{x^2}} = 0.$$

Also,
$$\int_0^a e^{t^2}\,dt \ll \int_0^x e^{t^2}\,dt, \qquad x \to +\infty,$$

$$\frac{1}{2a}e^{a^2} \ll \frac{1}{2x}e^{x^2}, \qquad x \to +\infty.$$

Hence, from (6.3.11) and (6.3.12) we have

$$\int_0^x e^{t^2}\,dt \sim \frac{1}{2x}e^{x^2}, \qquad x \to +\infty, \tag{6.3.13}$$

which is the leading asymptotic behavior of the integral. Observe that integration by parts works because the endpoint contribution in (6.3.12) from $t = a$ is negligible compared with that from $t = x$ when $0 < a < x$.

Repeated integration by parts in (6.3.12) establishes the full asymptotic expansion of $\int_0^x e^{t^2}\,dt$ as $x \to +\infty$ (see Prob. 6.11):

$$\int_0^x e^{t^2}\,dt \sim \frac{1}{2x}e^{x^2}\left[1 + \sum_{n=1}^{\infty} \frac{(1)(3)(5)\cdots(2n-1)}{(2x^2)^n}\right], \qquad x \to +\infty. \tag{6.3.14}$$

Notice that the arbitrary cutoff a does not appear in this asymptotic expansion. In fact, the endpoint contributions from $t = a$ are exponentially smaller than those from $t = x$ after any number of integrations by parts.

Example 5 *Behavior of integrals of Airy functions.* The asymptotic expansion of the integral $\int_0^x \mathrm{Ai}(t)\,dt$ as $x \to +\infty$ is obtained by following the procedure of Example 3. First, we write $\int_0^x \mathrm{Ai}(t)\,dt = \frac{1}{3} - \int_x^\infty \mathrm{Ai}(t)\,dt$. [Here we have used the property of Ai (t) that $\int_0^\infty \mathrm{Ai}(t)\,dt = \frac{1}{3}$; see Prob. 10.20.] This decomposition avoids endpoint contributions (from $t = 0$) that are larger than the integral itself. Next, we integrate by parts using $\mathrm{Ai}''(t) = t\,\mathrm{Ai}(t)$, the differential equation satisfied by the Airy function:

$$\int_x^\infty \mathrm{Ai}(t)\,dt = \int_x^\infty \frac{1}{t}\mathrm{Ai}''(t)\,dt$$

$$= \frac{1}{t}\mathrm{Ai}'(t)\Big|_x^\infty + \int_x^\infty \frac{1}{t^2}\mathrm{Ai}'(t)\,dt.$$

But
$$\mathrm{Ai}'(x) \sim -\frac{1}{2\sqrt{\pi}}x^{1/4}e^{-2x^{3/2}/3}, \qquad x \to +\infty,$$

and $$\left|\int_x^\infty \frac{1}{t^2}\operatorname{Ai}'(t)\,dt\right| < \frac{1}{x^2}\left|\int_x^\infty \operatorname{Ai}'(t)\,dt\right| = \frac{1}{x^2}\operatorname{Ai}(x)$$

$$\sim \frac{1}{2\sqrt{\pi}}x^{-9/4}e^{-2x^{3/2}/3}, \qquad x \to +\infty.$$

Thus, $$\int_x^\infty \operatorname{Ai}(t)\,dt \sim \frac{1}{2\sqrt{\pi}}x^{-3/4}e^{-2x^{3/2}/3}, \qquad x \to +\infty. \tag{6.3.15}$$

The full asymptotic expansion of $\int_x^\infty \operatorname{Ai}(t)\,dt$ as $x \to +\infty$ is most easily obtained by using the differential equation $y''' = xy'$ that it satisfies to find suitable recursion relations (see Prob. 6.12).

Another Airy function integral is $\int_0^x \operatorname{Bi}(t)\,dt$. The asymptotic behavior of this integral as $x \to +\infty$ may be found by following the procedure in Example 4. First, we write $\int_0^x \operatorname{Bi}(t)\,dt = \int_0^a \operatorname{Bi}(t)\,dt + \int_a^x \operatorname{Bi}(t)\,dt$, where $0 < a < x$, in order to avoid infinite endpoint contributions. Then, integrating by parts in the second integral on the right using $\operatorname{Bi}''(t) = t\operatorname{Bi}(t)$ gives

$$\begin{aligned}\int_a^x \operatorname{Bi}(t)\,dt &= \int_a^x \frac{1}{t}\operatorname{Bi}''(t)\,dt\\ &= \frac{1}{x}\operatorname{Bi}'(x) - \frac{1}{a}\operatorname{Bi}'(a) + \int_a^x \frac{1}{t^2}\operatorname{Bi}'(t)\,dt.\end{aligned}$$

Next, we note a number of asymptotic relations:

$$\frac{1}{a}\operatorname{Bi}'(a) \ll \frac{1}{x}\operatorname{Bi}'(x), \qquad x \to +\infty,$$

$$\int_0^a \operatorname{Bi}(t)\,dt \ll \int_0^x \operatorname{Bi}(t)\,dt, \qquad x \to +\infty,$$

$$\int_a^x \frac{1}{t^2}\operatorname{Bi}'(t)\,dt \ll \frac{1}{x}\operatorname{Bi}'(x), \qquad x \to +\infty,$$

this last asymptotic relation following from l'Hôpital's rule, and

$$\operatorname{Bi}(x) \sim \frac{1}{\sqrt{\pi}}x^{-1/4}e^{2x^{3/2}/3}, \qquad x \to +\infty,$$

$$\operatorname{Bi}'(x) \sim \frac{1}{\sqrt{\pi}}x^{1/4}e^{2x^{3/2}/3}, \qquad x \to +\infty.$$

From this heap of asymptotic inequalities we deduce that

$$\int_0^x \operatorname{Bi}(t)\,dt \sim \frac{1}{\sqrt{\pi}}x^{-3/4}e^{2x^{3/2}/3}, \qquad x \to +\infty \tag{6.3.16}$$

(see Prob. 6.13).

Since integration of one-signed asymptotic relations is permissible (see Prob. 3.28), (6.3.16) could have been obtained by integrating the asymptotic behavior of Bi (x):

$$\int_0^x \operatorname{Bi}(t)\,dt \sim \frac{1}{\sqrt{\pi}}\int_0^x t^{-1/4}e^{2t^{3/2}/3}\,dt, \qquad x \to +\infty.$$

Integration by parts gives, for any $a > 0$,

$$\int_a^x t^{-1/4} e^{2t^{3/2}/3}\, dt = \int_a^x t^{-3/4} \frac{d}{dt} e^{2t^{3/2}/3}\, dt$$
$$= t^{-3/4} e^{2t^{3/2}/3} \Big|_a^x + \frac{3}{4} \int_a^x t^{-7/4} e^{2t^{3/2}/3}\, dt,$$

and (6.3.16) is easily recovered.

Integration by Parts for Laplace Integrals

Until now we have only considered integrals where the parameter x appears as a limit of integration. A *Laplace* integral has the form

$$I(x) = \int_a^b f(t) e^{x\phi(t)}\, dt \tag{6.3.17}$$

in which x appears as part of the integrand.

To obtain the asymptotic behavior of $I(x)$ as $x \to +\infty$, we try integrating by parts:

$$I(x) = \frac{1}{x} \int_a^b \frac{f(t)}{\phi'(t)} \frac{d}{dt} [e^{x\phi(t)}]\, dt$$
$$= \frac{1}{x} \frac{f(t)}{\phi'(t)} e^{x\phi(t)} \Big|_a^b - \frac{1}{x} \int_a^b \frac{d}{dt} \left[\frac{f(t)}{\phi'(t)}\right] e^{x\phi(t)}\, dt. \tag{6.3.18}$$

(We assume that the new integral on the right exists, of course.) The formula in (6.3.18) is useful if the integral on the right side is asymptotically smaller than the boundary terms as $x \to \infty$. If this is true, then the boundary terms in (6.3.18) are asymptotic to $I(x)$:

$$I(x) \sim \frac{1}{x} \frac{f(b)}{\phi'(b)} e^{x\phi(b)} - \frac{1}{x} \frac{f(a)}{\phi'(a)} e^{x\phi(a)}, \qquad x \to +\infty. \tag{6.3.19}$$

In general, (6.3.19) is a correct asymptotic relation if $\phi(t)$, $\phi'(t)$, and $f(t)$ are continuous (possibly complex) functions and one of the following three conditions is satisfied:

1. $\phi'(t) \neq 0$ $(a \le t \le b)$ and at least one of $f(a)$ and $f(b)$ are not zero. These conditions are sufficient to ensure that the remainder integral on the right side of (6.3.18) exists. Once we know that this integral exists we can prove (see Prob. 6.15) that it becomes negligible compared with the boundary term in (6.3.18) as $x \to \infty$ and therefore that (6.3.19) is valid.
2. Re $\phi(t) <$ Re $\phi(b)$ $(a \le t < b)$, Re $\phi'(b) \neq 0$, and $f(b) \neq 0$. These conditions are insufficient to imply that the integral on the right side of (6.3.18) exists. Nevertheless, they *are* strong enough to ensure that

$$I(x) \sim \frac{1}{x} \frac{f(b)}{\phi'(b)} e^{x\phi(b)}, \qquad x \to +\infty. \tag{6.3.20}$$

This result is explained using Laplace's method in Sec. 6.4 [see (6.4.19*b*)].

3. Re $\phi(t) <$ Re $\phi(a)$ $(a \le t \le b)$, Re $\phi'(a) \ne 0$, and $f(a) \ne 0$. As in condition 2, these conditions are again insufficient to imply that the integral on the right side of (6.3.18) exists, but they are strong enough to ensure that

$$I(x) \sim -\frac{1}{x}\frac{f(a)}{\phi'(a)}e^{x\phi(a)}, \qquad x \to +\infty \tag{6.3.21}$$

[see (6.4.19*a*)].

Example 6 *Leading behavior of simple Laplace integrals.* Using the formulas in (6.3.19) to (6.3.21) we have

$$(a)\ \int_1^2 e^{x \cosh t}\,dt \sim \frac{e^{x \cosh 2}}{x \sinh 2}, \qquad x \to +\infty;$$

$$(b)\ \int_{-1}^3 e^{x \cosh^2 t} \sim \frac{e^{x \cosh^2 3}}{2x \sinh 3 \cosh 3}, \qquad x \to +\infty.$$

If the integral on the right side of (6.3.18) meets one of the three conditions stated above, we may continue integrating by parts. Apparently, each integration by parts introduces a new factor of $1/x$; for example, if Re $\phi(b) >$ Re $\phi(a)$ the full asymptotic expansion of $I(x)$ in (6.3.17) has the form

$$I(x) \sim e^{x\phi(b)} \sum_{n=1}^{\infty} A_n x^{-n}, \qquad x \to +\infty. \tag{6.3.22}$$

Failure of Integration by Parts

The method of integration by parts is rather inflexible; it can only produce asymptotic series of the form in (6.3.22) which contain integral powers of $1/x$. However, Laplace integrals like $I(x)$ in (6.3.17) can have large-x asymptotic expansions which contain fractional powers of x. It is clear, therefore, that the method of integration by parts is inadequate to find the asymptotic expansion of all such integrals. If we have no prior knowledge of the correct expansion of an integral, how then do we know whether or not integration by parts will work? Generally, the symptoms that integration by parts is breaking down are easy to detect and interpret: when integration by parts produces an integral which does not exist it is not working. We know that integration by parts is about to fail when $\phi'(t)$ has a zero somewhere in $a \le t \le b$. Here are some examples.

Example 7 *Failure of integration by parts for $\int_0^\infty e^{-xt^2}\,dt$.* The integral $\int_0^\infty e^{-xt^2}\,dt$ has the exact value $\frac{1}{2}\sqrt{\pi/x}$. Since its asymptotic behavior as $x \to +\infty$ is not an asymptotic series of powers of $1/x$ like that in (6.3.22), we expect integration by parts to fail. Comparing this integral with the general form in (6.3.17) shows that $\phi(t) = -t^2$. Since $\phi'(t) = -2t$ vanishes at $t = 0$, integration by parts gives a nonexistent integral,

$$\int_0^\infty e^{-xt^2}\,dt = \int_0^\infty \left(\frac{1}{-2xt}\right)(-2xte^{-xt^2})\,dt = \frac{e^{-xt^2}}{-2xt}\bigg|_0^\infty - \int_0^\infty \frac{1}{2xt^2}e^{-xt^2}\,dt,$$

a sure sign that integration by parts is not applicable.

Example 8 *Failure of integration by parts for* $\int_0^\infty e^{-x\sinh^2 t}\,dt$. For this integral $\phi(t) = -\sinh^2 t$. Since $\phi'(t)$ vanishes at $t = 0$, we do not expect that integration by parts will be useful for finding the large-x behavior of this integral.

In Example 6(*b*) of Sec. 6.4 we will use Laplace's method to show that the leading behavior of this integral as $x \to +\infty$ is $\frac{1}{2}\sqrt{\pi/x}$.

Example 9 *Leading behavior of* $\int_0^\infty \ln(1+t)\, e^{-x\sinh^2 t}\,dt$. For some integrals, integration by parts yields several terms in the expansion of $I(x)$ and then breaks down. The integral $I(x) = \int_0^\infty \ln(1+t)\, e^{-x\sinh^2 t}\,dt$ has this property. For this integral $f(t) = \ln(1+t)$ and $\phi(t) = \sinh^2 t$. Thus, although $\phi'(t) = \sinh(2t)$ vanishes at $t = 0$, $f(t)$ also vanishes there. As a result, it is correct to integrate by parts once:

$$I(x) = -\frac{1}{x}\int_0^\infty \frac{\ln(1+t)}{\sinh(2t)}\frac{d}{dt}\left(e^{-x\sinh^2 t}\right)dt$$

$$= -\frac{1}{x}\frac{\ln(1+t)}{\sinh(2t)}e^{-x\sinh^2 t}\Big|_0^\infty + \frac{1}{x}\int_0^\infty \left[\frac{d}{dt}\frac{\ln(1+t)}{\sinh(2t)}\right]e^{-x\sinh^2 t}\,dt. \tag{6.3.23}$$

Using Laplace's method, one can show that the last integral on the right vanishes like $x^{-3/2}$ as $x \to +\infty$ (see Prob. 6.33). Hence, $I(x) \sim 1/2x$ $(x \to +\infty)$. Thus, integration by parts gives the leading behavior of $I(x)$ correctly. However, integration by parts cannot be used to find the next term in the asymptotic expansion of $I(x)$ for large x because the next term is proportional to $x^{-3/2}$, which is not an integer power of $1/x$.

Example 10 *Stieltjes integral.* Integration by parts is useful for finding the behavior of the Stieltjes integral (see Sec. 3.8):

$$I(x) = \int_0^\infty \frac{e^{-t}}{1+xt}\,dt$$

for small positive x but not for large x. (The Stieltjes integral is not a Laplace integral.)

To derive the small-x behavior of $I(x)$, we organize the integration by parts so that one new factor of x is introduced at each stage:

$$I(x) = -\int_0^\infty \frac{1}{1+xt}\frac{d}{dt}e^{-t}\,dt$$

$$= -\frac{1}{1+xt}e^{-t}\Big|_0^\infty - x\int_0^\infty \frac{1}{(1+xt)^2}e^{-t}\,dt$$

$$= 1 + x\frac{1}{(1+xt)^2}e^{-t}\Big|_0^\infty + 2x^2\int_0^\infty \frac{1}{(1+xt)^3}e^{-t}\,dt$$

$$\vdots$$

$$= 1 - x + 2!\,x^2 - \cdots + (-1)^{n-1}(n-1)!\,x^{n-1}$$

$$+ (-1)^n n!\int_0^\infty \frac{1}{(1+xt)^{n+1}}e^{-t}\,dt.$$

Since this procedure works for all n, we have successfully derived the full asymptotic behavior of $I(x)$ for small x: $I(x) \sim \sum_{n=0}^\infty (-1)^n n!\,x^n$ $(x \to 0+)$.

Now let us see how integration by parts fails to give the large-x behavior of $I(x)$. If integration by parts did work it would have to introduce an additional factor of $1/x$ at each stage. Thus, our best hope is to write $I(x) = (1/x)\int_0^\infty e^{-t}(d/dt)\ln(1+xt)\,dt$ and to use integration by parts to obtain $I(x) = (1/x) \int_0^\infty e^{-t} \ln(1 + xt)\, dt$. Integrating by parts once again gives

$$I(x) = \frac{1}{x^2} \int_0^\infty e^{-t} \frac{d}{dt} [(1 + xt) \ln(1 + xt) - (1 + xt)]$$

$$= \frac{1}{x^2} + \frac{1}{x^2} \int_0^\infty e^{-t}[(1 + xt) \ln(1 + xt) - (1 + xt)]\, dt.$$

Have we shown that the leading behavior of $I(x)$ is x^{-2} $(x \to +\infty)$? No, because the last term on the right is not small compared to x^{-2}. In fact

$$\lim_{x \to +\infty} \int_0^\infty e^{-t}[(1 + xt) \ln(1 + xt) - (1 + xt)]\, dt = \infty \tag{6.3.24}$$

(see Prob. 6.16).

To understand why integration by parts has failed we need only recall [see Example 3 of Sec. 3.8 and Prob. 3.39(i)] that $I(x) \sim (\ln x)/x$ $(x \to +\infty)$; the large-x behavior of $I(x)$ is not a power series in $1/x$.

(E) 6.4 LAPLACE'S METHOD AND WATSON'S LEMMA

Laplace's method is a very general technique for obtaining the asymptotic behavior as $x \to +\infty$ of integrals in which the large parameter x appears in an exponential:

$$I(x) = \int_a^b f(t) e^{x\phi(t)}\, dt. \tag{6.4.1}$$

Here, we assume that $f(t)$ and $\phi(t)$ are real continuous functions. Integrals of this form are called Laplace integrals and were introduced in Sec. 6.3.

Laplace's method rests on an important idea involved in many standard techniques of asymptotic analysis of integrals, such as the methods of stationary phase and steepest descents which are discussed in Secs. 6.5 and 6.6. The idea is this: if the real continuous function $\phi(t)$ has its maximum on the interval $a \le t \le b$ at $t = c$ and if $f(c) \ne 0$, then it is only the immediate neighborhood of $t = c$ that contributes to the full asymptotic expansion of $I(x)$ for large x. That is, we may approximate the integral $I(x)$ by $I(x; \varepsilon)$, where

$$I(x; \varepsilon) = \int_{c-\varepsilon}^{c+\varepsilon} f(t) e^{x\phi(t)}\, dt \tag{6.4.2a}$$

if $a < c < b$,

$$I(x; \varepsilon) = \int_a^{a+\varepsilon} f(t) e^{x\phi(t)}\, dt \tag{6.4.2b}$$

if the maximum of $\phi(t)$ is at $t = a$, and

$$I(x; \varepsilon) = \int_{b-\varepsilon}^{b} f(t) e^{x\phi(t)}\, dt \tag{6.4.2c}$$

if the maximum of $\phi(t)$ is at $t = b$. Here ε may be chosen to be an arbitrary positive number (such that the restricted integration range $c - \varepsilon \le t \le c + \varepsilon$ is a subinterval of $a \le t \le b$). It is crucial that the full asymptotic expansion of $I(x; \varepsilon)$ as $x \to +\infty$ (1) does not depend on ε and (2) is identical to the full asymptotic expansion of $I(x)$ as $x \to +\infty$. Both of these rather surprising results are true because (we assume here that $a < c < b$) $\left|\int_a^{c-\varepsilon} f(t)e^{x\phi(t)}\,dt\right| + \left|\int_{c+\varepsilon}^{b} f(t)e^{x\phi(t)}\,dt\right|$ is subdominant (exponentially small) with respect to $I(x)$ as $x \to +\infty$. This is so because for all t on the intervals $a \le t \le c - \varepsilon$ and $c + \varepsilon \le t \le b$, $e^{x\phi(t)}$ is exponentially smaller than $e^{x\phi(c)}$ as $x \to +\infty$. To show that $I(x) - I(x; \varepsilon)$ is subdominant as $x \to +\infty$, we use integration by parts (see Prob. 6.23). The result that $I(x) - I(x; \varepsilon)$ is subdominant as $x \to +\infty$ sometimes holds even if $f(c) = 0$; we discuss this point later.

It is helpful to approximate $I(x)$ by $I(x; \varepsilon)$ because $\varepsilon > 0$ may be chosen so small that it is valid to replace $f(t)$ and $\phi(t)$ by their Taylor or asymptotic series expansions about $t = c$.

Example 1 *Leading behavior of* $\int_0^{10} (1 + t)^{-1}e^{-xt}\,dt$ *as* $x \to +\infty$. We use Laplace's method to approximate this integral. Here $\phi(t) = -t$ has a maximum in the integration region $0 \le t \le 10$ at $t = 0$. Therefore, we may replace the integral by

$$I(x; \varepsilon) = \int_0^{\varepsilon} (1 + t)^{-1}e^{-xt}\,dt$$

for any $\varepsilon > 0$ at the cost of introducing errors which are exponentially small as $x \to +\infty$. Next we choose ε so small that we can replace $(1 + t)^{-1}$ by 1, the first term in its Taylor series about $t = 0$. This replacement makes the integral easy to evaluate. Thus,

$$\int_0^{10} (1 + t)^{-1}e^{-xt}\,dt \sim \int_0^{\varepsilon} e^{-xt}\,dt = (1 - e^{-\varepsilon x})/x, \qquad x \to +\infty.$$

Since $e^{-\varepsilon x} \ll 1$ as $x \to +\infty$ for any $\varepsilon > 0$, we obtain

$$\int_0^{10} (1 + t)^{-1}e^{-xt}\,dt \sim \frac{1}{x}, \qquad x \to +\infty. \tag{6.4.3}$$

Note that the final result in (6.4.3) does not depend on the arbitrary parameter ε; ε appears in a subdominant term only.

Example 1 (revisited) *Full asymptotic expansion of* $\int_0^{10} (1 + t)^{-1}e^{-xt}\,dt$ *as* $x \to +\infty$. Laplace's method gives the full asymptotic series expansion of this integral. As in Example 1, we can replace 10, the upper limit of integration, by $\varepsilon < 1$ and introduce only exponentially small errors as $x \to +\infty$. Instead of replacing $(1 + t)^{-1}$ by 1, as we did in Example 1, we use the full Taylor expansion $(1 + t)^{-1} = \sum (-t)^n$, which converges for $|t| < 1$:

$$I(x; \varepsilon) = \int_0^{\varepsilon} (1 + t)^{-1}e^{-xt}\,dt = \sum_{n=0}^{\infty} \int_0^{\varepsilon} (-t)^n e^{-xt}\,dt.$$

The easiest way to evaluate the integral $\int_0^{\varepsilon} (-t)^n e^{-xt}\,dt$ for any n is to replace it by $\int_0^{\infty} (-t)^n e^{-xt}\,dt$. It may seem surprising that it is valid to replace the small parameter ε by ∞! However, this replacement introduces only exponentially small errors as $x \to +\infty$ because the integral from ε to ∞ is subdominant with respect to $\int_0^{\infty} (-t)^n e^{-xt}\,dt = (-1)^n n!\, x^{-n-1}$ as

$x \to +\infty$. We verify this using integration by parts; integration by parts gives

$$\int_{\varepsilon}^{\infty} (-t)^n e^{-xt}\,dt \sim (-\varepsilon)^n e^{-\varepsilon x}/x, \qquad x \to +\infty,$$

which is indeed exponentially smaller than $\int_0^\infty (-t)^n e^{-xt}\,dt$ as $x \to +\infty$. Assembling these results, we obtain the full asymptotic expansion of $\int_0^{10} (1+t)^{-1} e^{-xt}\,dt$ as $x \to +\infty$:

$$\int_0^{10} (1+t)^{-1} e^{-xt}\,dt \sim \sum_{n=0}^{\infty} (-1)^n n!\, x^{-n-1}, \qquad x \to +\infty. \tag{6.4.4}$$

Let us pause a moment to review the procedure we have just used. There are three steps involved in Laplace's method applied to an integral $I(x)$. First, we approximate $I(x)$ by $I(x; \varepsilon)$ by restricting the original integration region to a narrow region surrounding the maximum of $\phi(t)$. Second, we expand the functions $f(t)$ and $\phi(t)$ in series which are valid near the location of the maximum of $\phi(t)$. This allows us to expand $I(x; \varepsilon)$ into a series of integrals. Finally, the most convenient way to evaluate the integrals in the series for $I(x; \varepsilon)$ is to extend the integration region in each integral to infinity. It is this third step that is hardest to grasp. It may seem foolish to first replace the finite number 10 in Example 1 (revisited) by ε and then to replace ε by ∞! However, we must choose ε to be small in order to expand the integrand of $I(x; \varepsilon)$ and thereby obtain a series. We then let the integration region become infinite in order to evaluate the terms in the series. Each time we change the limits of integration, we introduce only exponentially small errors. Note that had we not replaced the integration limit 10 by $\varepsilon < 1$, we could not have used the Taylor expansion for $(1+t)^{-1}$, which is only valid for $|t| < 1$.

Watson's Lemma

'Pon my word, Watson, you are coming along wonderfully.
You have really done very well indeed.

—Sherlock Holmes, *A Case of Identity*
Sir Arthur Conan Doyle

In Example 1 we obtained the asymptotic expansion valid as $x \to +\infty$ of an integral which belongs to a broad class of integrals of the form

$$I(x) = \int_0^b f(t) e^{-xt}\,dt, \qquad b > 0. \tag{6.4.5}$$

There is a general formula, usually referred to as Watson's lemma, which gives the full asymptotic expansion of any integral of this type provided that $f(t)$ is continuous on the interval $0 \le t \le b$ and that $f(t)$ has the asymptotic series expansion

$$f(t) \sim t^{\alpha} \sum_{n=0}^{\infty} a_n t^{\beta n}, \qquad t \to 0+. \tag{6.4.6}$$

Note that we must require that $\alpha > -1$ and $\beta > 0$ for the integral to converge at $t = 0$. Note also that if $b = +\infty$, it is necessary that $f(t) \ll e^{ct}$ $(t \to +\infty)$, for some positive constant c, for the integral (6.4.5) to converge.

Watson's lemma states that if the above conditions hold then

$$I(x) \sim \sum_{n=0}^{\infty} \frac{a_n \Gamma(\alpha + \beta n + 1)}{x^{\alpha + \beta n + 1}}, \qquad x \to +\infty. \tag{6.4.7}$$

We prove this result as follows. First, we replace $I(x)$ by $I(x;\varepsilon)$, where

$$I(x;\varepsilon) = \int_0^{\varepsilon} f(t) e^{-xt}\, dt. \tag{6.4.8}$$

This approximation introduces only exponentially small errors for any positive value of ε. In particular, we can choose ε so small that the first n terms in the asymptotic series for $f(t)$ are a good approximation to $f(t)$:

$$\left| f(t) - t^{\alpha} \sum_{n=0}^{N} a_n t^{\beta n} \right| \le K t^{\alpha + \beta(N+1)}, \qquad 0 \le t \le \varepsilon, \tag{6.4.9}$$

where K is a nonzero constant. Substituting the first N terms in the series for $f(t)$ into (6.4.8) and using (6.4.9) gives

$$\begin{aligned}\left| I(x;\varepsilon) - \sum_{n=0}^{N} a_n \int_0^{\varepsilon} t^{\alpha+\beta n} e^{-xt}\, dt \right| &\le K \int_0^{\varepsilon} t^{\alpha+\beta(N+1)} e^{-xt}\, dt \\ &\le K \int_0^{\infty} t^{\alpha+\beta(N+1)} e^{-xt}\, dt \\ &= K \frac{\Gamma(\alpha + \beta + \beta N + 1)}{x^{\alpha+\beta+\beta N+1}}.\end{aligned}$$

Finally, we replace ε by ∞ and use the identity $\int_0^{\infty} t^{\alpha+\beta n} e^{-xt}\, dt = [\Gamma(\alpha + \beta n + 1)]/x^{\alpha+\beta n+1}$ to obtain

$$I(x) - \sum_{n=0}^{N} a_n \Gamma(\alpha + \beta n + 1)/x^{\alpha+\beta n+1} \ll x^{-\alpha-\beta N-1}, \qquad x \to +\infty.$$

Since this asymptotic relation is valid for all N, we have established the validity of the asymptotic series in (6.4.7) and have proved Watson's lemma.

Example 2 *Application of Watson's lemma.* To expand the integral

$$I(x) = \int_0^5 \frac{e^{-xt}}{1+t^2}\, dt$$

for large x, we replace $(1 + t^2)^{-1}$ by its Taylor series about $t = 0$:

$$\frac{1}{1+t^2} = 1 - t^2 + t^4 - t^6 + \cdots. \tag{6.4.10}$$

Watson's lemma allows us to substitute (6.4.10) into the integral, interchange orders of integration and summation, and replace the upper limit of integration 5 by ∞. This gives $I(x) \sim 1/x - 2!/x^3 + 4!/x^5 - 6!/x^7 + \cdots (x \to +\infty)$.

Example 3 *Asymptotic expansion of $K_0(x)$ using Watson's lemma.* A standard integral representation of the modified Bessel function $K_0(x)$ is $K_0(x) = \int_1^\infty (s^2 - 1)^{-1/2} e^{-xs}\, ds$. In order to apply Watson's lemma, we substitute $s = t + 1$. This shifts the lower endpoint of integration to $t = 0$:

$$K_0(x) = e^{-x} \int_0^\infty (t^2 + 2t)^{-1/2} e^{-xt}\, dt. \tag{6.4.11}$$

When $|t| < 2$, the binomial theorem gives

$$\begin{aligned}(t^2 + 2t)^{-1/2} &= (2t)^{-1/2}(1 + t/2)^{-1/2} \\ &= (2t)^{-1/2} \sum_{n=0}^\infty (-t/2)^n \frac{\Gamma(n + \frac{1}{2})}{n!\, \Gamma(\frac{1}{2})}.\end{aligned}$$

Watson's lemma then gives

$$K_0(x) \sim e^{-x} \sum_{n=0}^\infty (-1)^n \frac{[\Gamma(n + \frac{1}{2})]^2}{2^{n+1/2} n!\, \Gamma(\frac{1}{2}) x^{n+1/2}}, \qquad x \to +\infty. \tag{6.4.12}$$

Example 4 *Asymptotic expansion of $D_\nu(x)$ using Watson's lemma.* An integral representation of the parabolic cylinder function $D_\nu(x)$ which is valid when Re $(\nu) < 0$ is

$$D_\nu(x) = \frac{e^{-x^2/4}}{\Gamma(-\nu)} \int_0^\infty t^{-\nu-1} e^{-t^2/2} e^{-xt}\, dt. \tag{6.4.13}$$

To obtain the behavior of $D_\nu(x)$ as $x \to +\infty$, we expand $e^{-t^2/2}$ as a power series in t: $e^{-t^2/2} = \sum_{n=0}^\infty (-1)^n t^{2n}/2^n n!$. Watson's lemma then gives

$$D_\nu(x) \sim x^\nu \frac{e^{-x^2/4}}{\Gamma(-\nu)} \sum_{n=0}^\infty (-1)^n \frac{\Gamma(2n - \nu)}{2^n n!\, x^{2n}}, \qquad x \to +\infty, \tag{6.4.14}$$

in agreement with (3.5.13) and (3.5.14). [The expansion in (6.4.14) is also valid when Re $\nu \ge 0$.]

Asymptotic Expansion of General Laplace Integrals

Watson's lemma only applies to Laplace integrals $I(x)$ of the form (6.4.1) in which $\phi(t) = -t$. For more general $\phi(t)$, there are two possible approaches. If $\phi(t)$ is sufficiently simple, it may be useful to make a change of variable by substituting

$$s = -\phi(t) \tag{6.4.15}$$

into (6.4.1) and to rewrite the integral in the form $\int_{-\phi(a)}^{-\phi(b)} F(s) e^{-xs}\, ds$, where $F(s) = -f(t)/\phi'(t)$. Watson's lemma applies to this transformed integral.

Example 5 *Indirect use of Watson's lemma.* Watson's lemma does not apply directly to the integral

$$I(x) = \int_0^{\pi/2} e^{-x \sin^2 t}\, dt \tag{6.4.16}$$

because $\phi(t) = -\sin^2 t$. However, if we let $s = \sin^2 t$ and rewrite the integral as $I(x) = \frac{1}{2} \int_0^1 [s(1 - s)]^{-1/2} e^{-xs}\, ds$, then Watson's lemma does apply. Since

$$[s(1 - s)]^{-1/2} = \sum_{n=0}^\infty \frac{\Gamma(n + \frac{1}{2}) s^{n-1/2}}{n!\, \Gamma(\frac{1}{2})}$$

for $|s| < 1$, Watson's lemma gives

$$I(x) \sim +\frac{1}{2} \sum_{n=0}^\infty \frac{[\Gamma(n + \frac{1}{2})]^2}{n! \Gamma(\frac{1}{2}) x^{n+1/2}}, \qquad x \to +\infty. \tag{6.4.17}$$

Sometimes the substitution (6.4.15) is unwieldy because the inverse function $t = \phi^{-1}(-s)$ is a complicated multivalued function. In this case, it may be simpler to use a more direct method than Watson's lemma for obtaining the first few terms in the asymptotic series for $I(x)$. We discuss the mechanical aspects of this calculation first and postpone a more theoretical discussion till Examples 7 and 8.

To obtain the leading behavior of $I(x)$ as $x \to +\infty$, we argue as follows. If $\phi(t)$ has a maximum at $t = c$, then we can approximate $I(x)$ by $I(x; \varepsilon)$, as in (6.4.2). In the narrow region $|t - c| \le \varepsilon$, we replace $\phi(t)$ by the first few terms in its Taylor series. There are several cases to consider. If c lies at one of the endpoints a or b and if $\phi'(c) \ne 0$, then we approximate $\phi(t)$ by

$$\phi(c) + (t - c)\phi'(c). \tag{6.4.18a}$$

If $\phi'(c) = 0$ (this must happen when c is interior to the interval $a \le t \le b$) but $\phi''(c) \ne 0$, then we approximate $\phi(t)$ by

$$\phi(c) + \tfrac{1}{2}(t - c)^2\phi''(c). \tag{6.4.18b}$$

More generally, if $\phi'(c) = \phi''(c) = \cdots = \phi^{(p-1)}(c) = 0$ and $\phi^{(p)}(c) \ne 0$, then we approximate $\phi(t)$ by

$$\phi(c) + \frac{1}{p!}(t - c)^p\phi^{(p)}(c). \tag{6.4.18c}$$

In each of these cases, we also expand $f(t)$ about $t = c$ and retain just the leading term. For simplicity, let us assume that $f(t)$ is continuous and that $f(c) \ne 0$ [that is, the leading behavior of $f(t)$ as $t \to c$ is $f(c)$]. We treat cases in which $f(c) = 0$ later in Example 6 and in Prob. 6.24.

Now we substitute these approximations in $I(x; \varepsilon)$ and evaluate the leading behavior of the resulting integral by extending the range of integration to infinity. When (6.4.18a) holds, c must be an endpoint, $c = a$ or $c = b$. If $c = a$, then $\phi'(a) < 0$ and

$$I(x; \varepsilon) \sim \int_a^{a+\varepsilon} f(a)e^{x[\phi(a) + (t-a)\phi'(a)]}\,dt, \qquad x \to +\infty,$$

$$\sim f(a)e^{x\phi(a)} \int_a^{\infty} e^{x(t-a)\phi'(a)}\,dt, \qquad x \to +\infty.$$

Thus,

$$I(x) \sim -\frac{f(a)e^{x\phi(a)}}{x\phi'(a)}, \qquad x \to +\infty. \tag{6.4.19a}$$

If $c = b$, then $\phi'(b) > 0$ and a similar computation gives

$$I(x) \sim \frac{f(b)e^{x\phi(b)}}{x\phi'(b)}, \qquad x \to +\infty. \tag{6.4.19b}$$

When (6.4.18b) holds and $a < c < b$, then $\phi''(c) < 0$ [because $\phi(t)$ has a maximum at $t = c$] and

$$I(x;\varepsilon) \sim \int_{c-\varepsilon}^{c+\varepsilon} f(c)e^{x[\phi(c)+(t-c)^2\phi''(c)/2]}\,dt, \qquad x \to +\infty,$$

$$\sim f(c)e^{x\phi(c)}\int_{-\infty}^{\infty} e^{x(t-c)^2\phi''(c)/2}\,dt, \qquad x \to +\infty,$$

$$= \frac{\sqrt{2}\,f(c)e^{x\phi(c)}}{\sqrt{-x\phi''(c)}}\int_{-\infty}^{\infty} e^{-s^2}\,ds,$$

where we have substituted $s^2 = -x(t-c)^2\phi''(c)$. Recall that $\int_{-\infty}^{\infty} e^{-s^2}\,ds = \int_0^\infty u^{-1/2}e^{-u}\,du = \Gamma(\frac{1}{2}) = \sqrt{\pi}$, so

$$I(x) \sim \frac{\sqrt{2\pi}\,f(c)e^{x\phi(c)}}{\sqrt{-x\phi''(c)}}, \qquad x \to +\infty. \tag{6.4.19c}$$

This result holds if c is interior to the interval (a, b); if $c = a$ or $c = b$, the result in (6.4.19c) must be multiplied by a factor $\frac{1}{2}$. (Why?)

When (6.4.18c) holds and $a < c < b$, then p must be even and $\phi^{(p)}(c) < 0$ [otherwise $\phi(c)$ would not be a maximum]. Then

$$I(x;\varepsilon) \sim \int_{c-\varepsilon}^{c+\varepsilon} f(c)e^{x[\phi(c)+(t-c)^p\phi^{(p)}(c)/p!]}\,dt, \qquad x \to +\infty,$$

$$\sim f(c)e^{x\phi(c)}\int_{-\infty}^{\infty} e^{x(t-c)^p\phi^{(p)}(c)/p!}\,dt, \qquad x \to +\infty,$$

$$= f(c)e^{x\phi(c)}[-x\phi^{(p)}(c)/p!]^{-1/p}\int_{-\infty}^{\infty} e^{-s^p}\,ds.$$

Now recall that $\int_{-\infty}^{\infty} e^{-s^p}\,ds = 2\Gamma(1/p)/p$, so

$$I(x) \sim \frac{2\Gamma(1/p)(p!)^{1/p}}{p[-x\phi^{(p)}(c)]^{1/p}}f(c)e^{x\phi(c)}, \qquad x \to +\infty. \tag{6.4.19d}$$

Example 6 *Use of Laplace's method to determine leading behavior.*

(a) $\int_0^{\pi/2} e^{-x\tan t}\,dt \sim 1/x$ $(x \to +\infty)$ because (6.4.19a) applies.
(b) $\int_0^\infty e^{-x\sinh^2 t}\,dt \sim \frac{1}{2}\sqrt{\pi/x}$ $(x \to +\infty)$. Here (6.4.19c) with the result multiplied by $\frac{1}{2}$ applies because $c = 0$ is an endpoint.
(c) $\int_{-1}^{1} e^{-x\sin^4 t}\,dt \sim [\Gamma(\frac{1}{4})]/2x^{1/4}$ $(x \to +\infty)$ because (6.4.19b) applies with $p = 4$.
(d) $\int_{-\pi/2}^{\pi/2} (t+2)e^{-x\cos t}\,dt \sim 4/x$ $(x \to +\infty)$, where we have added together contributions obtained using both (6.4.19a) and (6.4.19b) because there are maxima at both $t = -\pi/2$ and $t = \pi/2$.

(e) $$\int_0^1 \sin t e^{-x \sinh^4 t}\, dt \sim \int_0^{\varepsilon} t e^{-xt^4}\, dt, \qquad x \to +\infty,$$
$$\sim \int_0^{\infty} t e^{-xt^4}\, dt, \qquad x \to +\infty,$$
$$= \frac{1}{4x^{1/2}} \int_0^{\infty} s^{-1/2} e^{-s}\, ds = \frac{\Gamma(\frac{1}{2})}{4x^{1/2}} = \frac{1}{4}\sqrt{\frac{\pi}{x}}.$$

This example is interesting because the maximum of ϕ occurs at a point where $f(t)$ vanishes. Laplace's method works because the contribution to the integral from outside the interval $0 \le t \le \varepsilon$ is subdominant for any $\varepsilon > 0$. In general, if $f(t)$ vanishes algebraically at the maximum of ϕ, Laplace's method works as explained above.

(f) $$\int_0^{\infty} \frac{e^{-x \cosh t}}{\sqrt{\sinh t}}\, dt \sim \int_0^{\varepsilon} \frac{e^{-x(1+t^2/2)}}{\sqrt{t}}\, dt, \qquad x \to +\infty,$$
$$\sim e^{-x} \int_0^{\infty} \frac{e^{-xt^2/2}}{\sqrt{t}}\, dt, \qquad x \to +\infty,$$
$$= (8x)^{-1/4} e^{-x} \int_0^{\infty} s^{-3/4} e^{-s}\, ds$$
$$= \Gamma\left(\frac{1}{4}\right) (8x)^{-1/4} e^{-x}.$$

This example is interesting because $f(t)$ is infinite at the maximum of ϕ. Again, Laplace's method works because the contribution to the integral from outside the interval $0 \le t \le \varepsilon$ is subdominant.

(g) The modified Bessel function $K_\nu(x)$ of order ν has the integral representation

$$K_\nu(x) = \int_0^{\infty} e^{-x \cosh t} \cosh(\nu t)\, dt, \tag{6.4.20}$$

which is valid for $x > 0$ (see Prob. 6.36). Therefore, (6.4.19c) with an extra factor $\frac{1}{2}$ gives

$$K_\nu(x) \sim \sqrt{\frac{\pi}{2x}}\, e^{-x}, \qquad x \to +\infty, \tag{6.4.21}$$

in agreement with (3.8.11).

(h) The integral representation (6.4.20) can also be used to find the asymptotic behavior of $K_\nu(p\nu)$ as $\nu \to +\infty$ with p fixed: $K_\nu(p\nu) = \frac{1}{2}\int_0^\infty e^{-\nu(p \cosh t + t)}\, dt + \frac{1}{2}\int_0^\infty e^{-\nu(p\cosh t - t)}\, dt$, where we have substituted $\cosh(\nu t) = (e^{\nu t} + e^{-\nu t})/2$. Laplace's method applied to the first integral on the right gives its leading behavior as $e^{-p\nu}/2\nu$. Laplace's method also applies to the second integral on the right. Here $\phi(t) = t - p \cosh t$ has a maximum at $t = c$ where $p \sinh c = 1$. Therefore, (6.4.19c) gives

$$K_\nu(p\nu) \sim \sqrt{\frac{\pi}{2\nu}}\, \frac{e^{-\nu q}}{(1+p^2)^{1/4}}, \qquad \nu \to +\infty;\ p > 0, \tag{6.4.22}$$

where $q = p \cosh c - c = \sqrt{1+p^2} - \ln[(1 + \sqrt{1+p^2})/p]$. The derivation of (6.4.22) is facilitated by the formulas $p \cosh c = \sqrt{1+p^2}$ and $e^c = \cosh c + \sinh c = (\sqrt{1+p^2} + 1)/p$.

The procedures used to derive the results in (6.4.19) are correct but not fully justified. In the next two examples, we apply Laplace's method to an integral taking care to justify and explain our approximations more carefully.

Example 7 *Careful application of Laplace's method to* $\int_0^{\pi/2} e^{-x\sin^2 t}\,dt$. Here $\phi(t) = -\sin^2 t$ has a maximum at $t = 0$, so (6.4.2) shows that for any ε $(0 < \varepsilon < \pi/2)$, $I(x) = \int_0^{\pi/2} e^{-x\sin^2 t}\,dt \sim \int_0^{\varepsilon} e^{-x\sin^2 t}\,dt$ $(x \to +\infty)$ with only exponentially small errors. Recall that this step is justified in Prob. 6.23. If ε is small, $\sin t$ can be approximated by t for all t $(0 \le t \le \varepsilon)$. Thus, we expect that it is valid to approximate $I(x)$ by $I(x) \sim \int_0^{\varepsilon} e^{-xt^2}\,dt$ $(x \to +\infty)$. For finite ε, the Gaussian integral on the right cannot be evaluated in terms of elementary functions (it is an error function). However, the evaluation of the integral on the right is easy if we extend the integration region to ∞:

$$\int_0^{\infty} e^{-xt^2}\,dt = \frac{1}{2\sqrt{x}}\int_0^{\infty} s^{-1/2}e^{-s}\,ds = \frac{1}{2\sqrt{x}}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\frac{\pi}{x}},$$

where we have used the substitution $s = xt^2$. Thus,

$$I(x) \sim \tfrac{1}{2}\sqrt{\pi/x}, \qquad x \to +\infty. \tag{6.4.23}$$

Note that this result agrees with the first term in (6.4.17).

To justify the asymptotic relation (6.4.23) we must verify that

$$\int_0^{\varepsilon} e^{-x\sin^2 t}\,dt \sim \int_0^{\varepsilon} e^{-xt^2}\,dt, \qquad x \to +\infty, \tag{6.4.24}$$

when $0 < \varepsilon < 2\pi$.

The proof of (6.4.24) is delicate and illustrates one of the more subtle aspects of Laplace's method, so we go through it in some detail. The idea of the proof is that over the range of t that contributes substantially to either of the integrals in (6.4.24), the integrands of both integrals are nearly identical. Outside of this range the two integrands are quite different, but both are exponentially small.

The proof of (6.4.24) begins by breaking up the range of integration $0 \le t \le \varepsilon$ into the two ranges $0 \le t \le x^{-\alpha}$ and $x^{-\alpha} < t \le \varepsilon$, where $\frac{1}{4} < \alpha < \frac{1}{2}$. This restriction on α is made so that when $t = x^{-\alpha}$, $xt^2 \to +\infty$ but $xt^4 \to 0$ as $x \to +\infty$. When $t \le x^{-\alpha}$ with $\frac{1}{4} < \alpha < \frac{1}{2}$,

$$x\sin^2 t - xt^2 \ll 1, \qquad x \to +\infty. \tag{6.4.25}$$

To prove (6.4.25) we use the inequalities $t - t^3/6 \le \sin t \le t$ which hold for all $t > 0$ (see Prob. 6.40) to obtain

$$|x\sin^2 t - xt^2| = x|\sin t + t|\,|\sin t - t| < x(2t)(t^3/6) = xt^4/3.$$

But $xt^4/3 \to 0$ as $x \to +\infty$ when $t \le x^{-\alpha}$, with $\alpha > \frac{1}{4}$, which proves (6.4.25). Exponentiating (6.4.25) gives

$$e^{-x\sin^2 t} \sim e^{-xt^2}, \qquad t \le x^{-\alpha}, x \to +\infty. \tag{6.4.26}$$

Integrating this asymptotic relation gives

$$\int_0^{x^{-\alpha}} e^{-x\sin^2 t}\,dt \sim \int_0^{x^{-\alpha}} e^{-xt^2}\,dt, \qquad x \to +\infty.$$

To complete the proof of (6.4.24), we must estimate the contribution to each integral from the interval $x^{-\alpha} \le t \le \varepsilon$. Note that (6.4.26) is not true for all t in this range. In fact, when t is of order ε and $x \to +\infty$, $e^{-x\sin^2 t}$ is exponentially larger in magnitude than e^{-xt^2}. Here lies the subtlety in the proof of (6.4.24); (6.4.24) remains valid despite the discrepancy between the magnitudes of the two integrands $e^{-x\sin^2 t}$ and e^{-xt^2} when t is of the order ε. The point is simply that the contribution to each integral in (6.4.24) from the interval $x^{-\alpha} < t \le \varepsilon$ is subdominant as $x \to +\infty$ with respect to the contribution from $0 \le t \le x^{-\alpha}$ when $\alpha < \frac{1}{2}$. In fact, if $t > x^{-\alpha}$ with $\alpha < \frac{1}{2}$, then $e^{-x\sin^2 t}$ and e^{-xt^2} are both smaller than $e^{-x\sin^2(x^{-\alpha})} \sim e^{-x^{1-2\alpha}}$ $(x \to +\infty)$ which is exponentially small. In Fig. 6.2, we plot the integrands on the left and right sides of (6.4.24) for $x = 100$. Observe that the integrands are nearly the same for those values of t that contribute substantially to the integrals and that the integrands differ substantially only for those values of t that make a negligible contribution to the integrals.

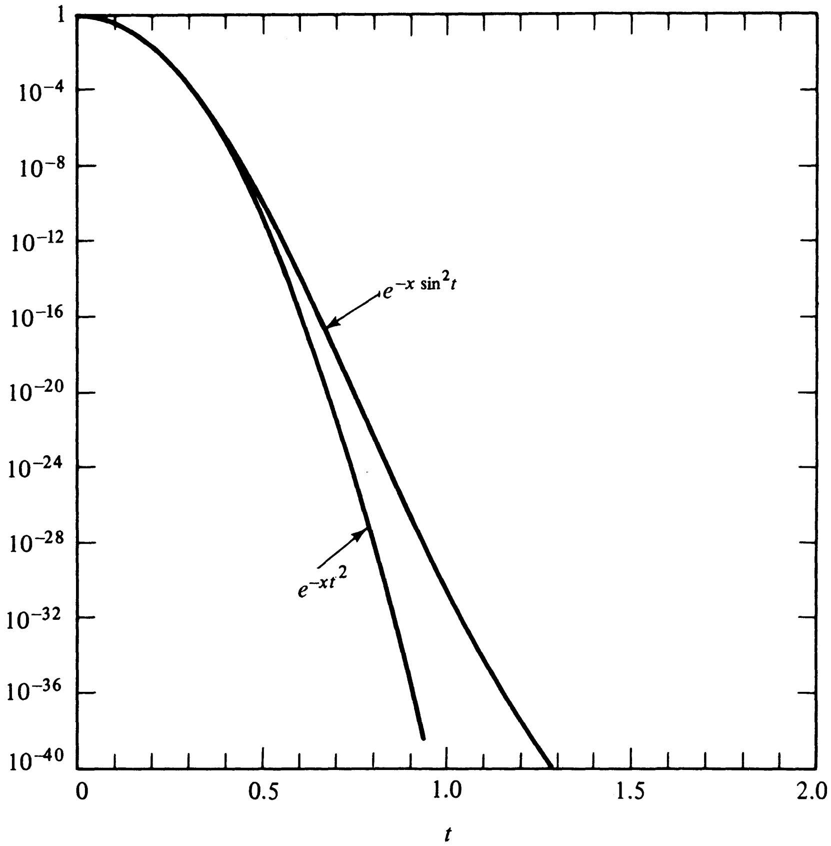

Figure 6.2 A graphical demonstration that $\int_0^\varepsilon \exp(-x\sin^2 t)\,dt \sim \int_0^\varepsilon \exp(-xt^2)\,dt$ $(x \to +\infty)$ for any positive $\varepsilon < 2\pi$ [see (6.4.24)]. The graph compares the two integrands for $x = 100$. Observe that over the range of t that contributes significantly to either of the integrands, both integrands are nearly equal. Outside of this range the integrands differ substantially but are negligibly small.

Example 8 *Leading behavior of $I_n(x)$ as $x \to +\infty$.* It may be shown (see Prob. 6.41) that

$$I_n(x) = \frac{1}{\pi}\int_0^\pi e^{x\cos t}\cos(nt)\,dt, \tag{6.4.27}$$

where $I_n(x)$ is the modified Bessel function of order n. Local analysis of the modified Bessel equation (see Example 2 of Sec. 3.5) establishes only that

$$I_n(x) \sim \frac{1}{\sqrt{x}}(c_1 e^x + c_2 e^{-x}), \qquad x \to +\infty, \tag{6.4.28}$$

for some constants c_1 and c_2, but it does not establish the values of c_1 and c_2. However, asymptotic analysis of the integral representation (6.4.27) shows that $c_1 = 1/\sqrt{2\pi}$; the value of c_2 is *not* determined by asymptotic analysis of (6.4.27) because e^{-x} is subdominant with respect to e^x as $x \to +\infty$. Observe that integration by parts cannot be used to derive (6.4.28) because the leading behavior of $I_n(x)$ involves $1/\sqrt{x}$ and not $1/x$.

To use Laplace's method we note that $\phi(t) = \cos t$ has a maximum at $t = 0$. Thus,

$$I_n(x) \sim \int_0^\varepsilon e^{x \cos t} \cos (nt)\, dt, \qquad x \to +\infty. \tag{6.4.29}$$

According to (6.4.18*b*), we must replace $\cos t$ by the first two terms in its Taylor expansion about 0. What happens if, instead, we approximate $\cos t$ by 1 and $\cos (nt)$ by 1 for $0 \le t \le \varepsilon$? The resulting approximation to $I_n(x)$, $\int_0^\varepsilon e^x(1)\, dt = \varepsilon e^x$, is *not* correct because the dependence on the arbitrary constant ε has not dropped out. The trouble here is that e^x is not a good approximation to $e^{x \cos t}$ over the subinterval of $0 \le t \le \varepsilon$ that gives the dominant contribution to the integral (see Fig. 6.3).

A correct application of Laplace's method to (6.4.29) is obtained by approximating $\cos t$ by the first *two* terms of its Taylor series $1 - t^2/2$ and by approximating $\cos (nt)$ by 1:

$$I_n(x) \sim \frac{1}{\pi} \int_0^\varepsilon e^{x(1-t^2/2)}\, dt, \qquad x \to +\infty, \tag{6.4.30}$$

$$\sim \frac{1}{\sqrt{2\pi x}} e^x, \qquad x \to +\infty. \tag{6.4.31}$$

The dependence on ε has disappeared, as it should.

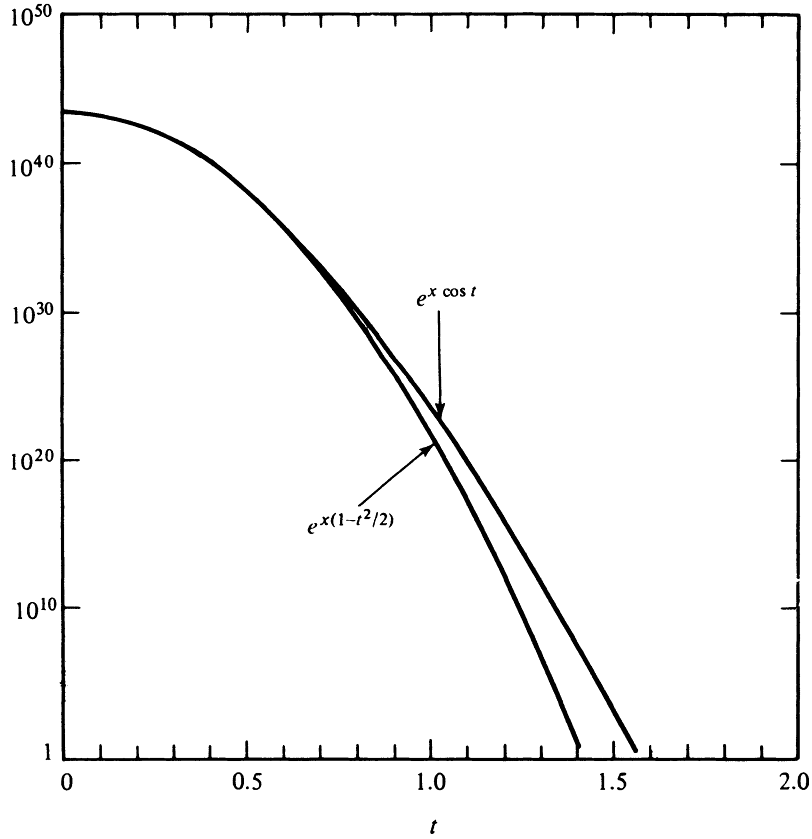

Figure 6.3 A graphical justification that it is valid to replace $\exp(x \cos t)$ with $\exp(x - \frac{1}{2}xt^2)$ in the integrand in (6.4.29) when $x \to \infty$. The graph compares these two expressions when $x = 100$. Note that the two curves are nearly identical over many orders of magnitude.

The leading behavior of $I_n(x)$ as $x \to +\infty$ does not depend on n; however, higher-order terms in the asymptotic expansion of $I_n(x)$ do depend on n. In fact, the complete asymptotic expansion of $I_n(x)$ as $x \to +\infty$ is given in (3.5.8) and (3.5.9) with $c_1 = 1/\sqrt{2\pi}$.

The only step that really needs justification is (6.4.30). The argument is nearly the same as that used to justify (6.4.24). The range of integration from $t = 0$ to $t = \varepsilon$ is broken up into the two ranges $0 \le t \le x^{-\alpha}$ and $x^{-\alpha} < t \le \varepsilon$, where $\frac{1}{4} < \alpha < \frac{1}{2}$. Now for fixed n, $\cos(nt)\, e^{x\cos t} \sim e^{x(1-t^2/2)}$ $(x \to +\infty)$ uniformly for all t satisfying $0 \le t \le x^{-\alpha}$ because $1 - t^2/2 \le \cos t \le 1 - t^2/2 + t^4/24$. Therefore, $\int_0^{x^{-\alpha}} \cos(nt)\, e^{x\cos t}\, dt \sim \int_0^{x^{-\alpha}} e^{x(1-t^2/2)}\, dt$ $(x \to +\infty)$. Also, when $x^{-\alpha} < t \le \varepsilon$, the integrands on both sides of (6.4.30) are subdominant with respect to e^x, so the contribution to (6.4.30) from the integration range $x^{-\alpha} < t \le \varepsilon$ is exponentially small compared to the contribution from the range $0 \le t \le x^{-\alpha}$.

Laplace's Method—Determination of Higher-Order Terms

The approach we have used to obtain the leading asymptotic behavior of integrals by Laplace's method can be extended to give the higher-order terms in the asymptotic expansion of the integral. To do this one would naturally expect to have to retain more terms in the expansions of $\phi(t)$ and $f(t)$ than those used to obtain (6.4.19). We illustrate the mechanics of this procedure for the case in which $\phi'(c) = 0$, $\phi''(c) < 0$, $f(c) \ne 0$, and $a < c < b$, where c is the location of the maximum of $\phi(t)$.

By (6.4.2), $I(x) \sim \int_{c-\varepsilon}^{c+\varepsilon} f(t)e^{x\phi(t)}\, dt$ $(x \to +\infty)$ with exponentially small errors. The leading behavior of $I(x)$ given by (6.4.19c) is obtained by replacing $f(t)$ by $f(c)$ and $\phi(t)$ by $\phi(c) + \frac{1}{2}(t-c)^2\phi''(c)$. To compute the first correction to (6.4.19c) we must approximate $f(t)$ and $\phi(t)$ by two *more* terms in their Taylor series:

$$I(x) \sim \int_{c-\varepsilon}^{c+\varepsilon} [f(c) + f'(c)(t-c) + \tfrac{1}{2}f''(c)(t-c)^2] \times \exp\{x[\phi(c) + \tfrac{1}{2}(t-c)^2\phi''(c) + \tfrac{1}{6}(t-c)^3\phi'''(c) + \tfrac{1}{24}(t-c)^4(d^4\phi/dt^4)(c)]\}\, dt, \qquad x \to +\infty. \tag{6.4.32}$$

It is somewhat surprising that *two* additional terms in the series for $\phi(t)$ and $f(t)$ are required to compute just the next term in (6.4.19c). We will see shortly why this is so.

Because ε may be chosen small, we Taylor expand the integrand in (6.4.32) as follows:

$$\exp\{x[\tfrac{1}{6}(t-c)^3\phi'''(c) + \tfrac{1}{24}(t-c)^4(d^4\phi/dt^4)(c)]\} = 1 + x[\tfrac{1}{6}(t-c)^3\phi'''(c) + \tfrac{1}{24}(t-c)^4(d^4\phi/dt^4)(c)] + \tfrac{1}{72}x^2(t-c)^6[\phi'''(c)]^2 + \cdots.$$

Substituting this expansion into (6.4.32) and collecting powers of $t - c$ gives

$$I(x) \sim \int_{c-\varepsilon}^{c+\varepsilon} e^{x\phi(c) + x(t-c)^2\phi''(c)/2} \times \left\{ f(c) + \frac{(t-c)^2}{2} f''(c) + (t-c)^4[\tfrac{1}{24}xf(c)(d^4\phi/dt^4)(c) + \tfrac{1}{6}xf'(c)\phi'''(c)] + \tfrac{1}{72}(t-c)^6x^2f(c)[\phi'''(c)]^2 + \cdots \right\} dt, \qquad x \to +\infty, \tag{6.4.33}$$

where we have excluded odd powers of $t - c$ because they vanish upon integration. Only the displayed terms in (6.4.33) contribute to the next term in (6.4.19c). Notice that we do *not* Taylor expand $\exp\left[\frac{1}{2}x(t-c)^2\phi''(c)\right]$; we return to this point shortly.

Next we extend the range of integration in (6.4.33) to $(-\infty, \infty)$ and substitute $s = \sqrt{x}(t-c)$:

$$I(x) \sim \frac{1}{\sqrt{x}} e^{x\phi(c)} \int_{-\infty}^{\infty} e^{s^2\phi''(c)/2}$$
$$\times \left\{ f(c) + \frac{1}{x}\left[\tfrac{1}{2}s^2 f''(c) + \tfrac{1}{24}s^4 f(c)(d^4\phi/dt^4)(c) + \tfrac{1}{6}s^4 f'(c)\phi'''(c)\right.\right.$$
$$\left.\left. + \tfrac{1}{72}s^6[\phi'''(c)]^2 f(c)\right]\right\} ds, \qquad x \to +\infty. \tag{6.4.34}$$

Observe that all the displayed terms in (6.4.33) contribute to the coefficient of $1/x$ in (6.4.34); the additional terms that we have neglected in going from (6.4.32) to (6.4.34) contribute to the coefficients of $1/x^2$, $1/x^3$, and so on.

To evaluate the integrals in (6.4.34) we use integration by parts to derive the general formula $\int_{-\infty}^{\infty} e^{-s^2/2} s^{2n}\, ds = \sqrt{2\pi}(2n-1)(2n-3)(2n-5)\cdots(5)(3)(1)$. Thus, we have

$$I(x) \sim \sqrt{\frac{2\pi}{-x\phi''(c)}}\, e^{x\phi(c)} \left\{ f(c) + \frac{1}{x}\left[-\frac{f''(c)}{2\phi''(c)} + \frac{f(c)(d^4\phi/dt^4)(c)}{8[\phi''(c)]^2} \right.\right.$$
$$\left.\left. + \frac{f'(c)\phi'''(c)}{2[\phi''(c)]^2} - \frac{5[\phi'''(c)]^2 f(c)}{24[\phi''(c)]^3} \right]\right\}, \qquad x \to +\infty. \tag{6.4.35}$$

One aspect of the derivation of (6.4.35) requires explanation. In proceeding from (6.4.32) to (6.4.34) we did not Taylor expand $\exp\left[\frac{1}{2}x(t-c)^2\phi''(c)\right]$, but we did Taylor expand the cubic and quartic terms in the exponential. If we had Taylor expanded $\exp\left[\frac{1}{2}x(t-c)^2\phi''(c)\right]$ and retained only a finite number of terms, the resulting approximation to $I(x)$ would depend on ε (see Example 8). If we had not expanded the cubic and quartic terms and if $(d^4\phi/dt^4)(c)$ were nonnegative, then extending the range of integration from $(c-\varepsilon, c+\varepsilon)$ to $(-\infty, \infty)$ would yield a divergent integral which would be a poor approximation to $I(x)$ indeed! If we had not expanded the cubic and quartic terms and if $(d^4\phi/dt^4)(c) < 0$, then extending the range of integration from $(c-\varepsilon, c+\varepsilon)$ to $(-\infty, \infty)$ would yield a convergent integral. However, this convergent integral might not be asymptotic to $I(x)$ because replacing $\phi(t)$ by the four-term Taylor series in (6.4.32) can introduce new relative maxima which lie outside $(c-\varepsilon, c+\varepsilon)$ which would dominate the integral on the right side of (6.4.32). In summary, there are three reasons why we must Taylor expand the cubic and quartic terms in the exponential before we extend the range of integration to $(-\infty, \infty)$:

1. The resulting integrals are always convergent and depend on ε only through subdominant terms.
2. It is easy to evaluate the resulting Gaussian integrals.
3. We avoid introducing any spurious maxima into the integrand.

To illustrate the above discussion we consider the integral

$$I(x) = \int_0^{\pi/2} e^{-x \sin^2 t}\, dt. \tag{6.4.36}$$

To obtain a higher-order approximation to this integral than that in (6.4.23), we Taylor expand $\sin^2 t$ through t^4:

$$I(x) \sim \int_0^{\varepsilon} e^{-x(t^2 - t^4/3)}\, dt, \qquad x \to +\infty. \tag{6.4.37}$$

Taylor expanding the quartic term, we obtain

$$\begin{aligned} I(x) &\sim \int_0^{\varepsilon} e^{-xt^2}\left(1 + \frac{1}{3}xt^4\right) dt, \qquad x \to +\infty, \\ &\sim \int_0^{\infty} e^{-xt^2}\left(1 + \frac{1}{3}xt^4\right) dt \\ &= \frac{1}{2}\sqrt{\frac{\pi}{x}}\left(1 + \frac{1}{4x}\right), \qquad x \to +\infty, \end{aligned} \tag{6.4.38}$$

in agreement with (6.4.17). In Fig. 6.4 we plot the integrands of (6.4.36) to (6.4.38) for $x = 100$. Observe that all three integrands are nearly identical for small t, but that the integrand in (6.4.37) blows up as $t \to +\infty$. The integrands of (6.4.36) and (6.4.38) do differ when t is large, but large t makes a negligible contribution to both (6.4.36) and (6.4.38).

Laplace's Method for Integrals with Movable Maxima

There are two kinds of problems where Laplace's method is useful but does not apply directly. First, we know what to do when $f(t)$ vanishes algebraically at $t = c$, the maximum of $\phi(t)$. But what if $f(t)$ vanishes exponentially fast at c? Second, it can happen that the Laplace integral (6.4.1) converges but that $\max \phi(t) = \infty$. What do we do then? We consider each of these cases in the following two examples.

Example 9 *Leading behavior of* $\int_0^\infty e^{-xt-1/t}\, dt$. Here $f(t) = e^{-1/t}$ vanishes exponentially fast at $t = 0$, the maximum of $\phi(t) = -t$. If we apply Watson's lemma (6.4.7), we obtain the asymptotic series expansion $0 + 0/x + 0/x^2 + \cdots$ $(x \to +\infty)$ because the coefficients of the asymptotic power series of $e^{-1/t}$ as $t \to 0+$ are all zero. Watson's lemma does not determine the behavior of $I(x)$ because Watson's lemma can only produce a series of inverse powers of x. Here, $I(x)$ is smaller than any power of x; it decreases exponentially fast as $x \to +\infty$.

In order to determine the correct behavior of $I(x)$, let us determine the location of the true maximum of the full integrand $e^{-xt-1/t}$. This maximum occurs when $(d/dt)(-xt - 1/t) = 0$ or $t = 1/\sqrt{x}$. We call such a maximum a movable maximum because its location depends on x.

For this kind of movable maximum problem, Laplace's method can be applied if we first transform the movable maximum to a fixed maximum. This is done by making the change of variables $t = s/\sqrt{x}$:

$$I(x) = \frac{1}{\sqrt{x}} \int_0^\infty e^{-\sqrt{x}(s + 1/s)}\, ds.$$

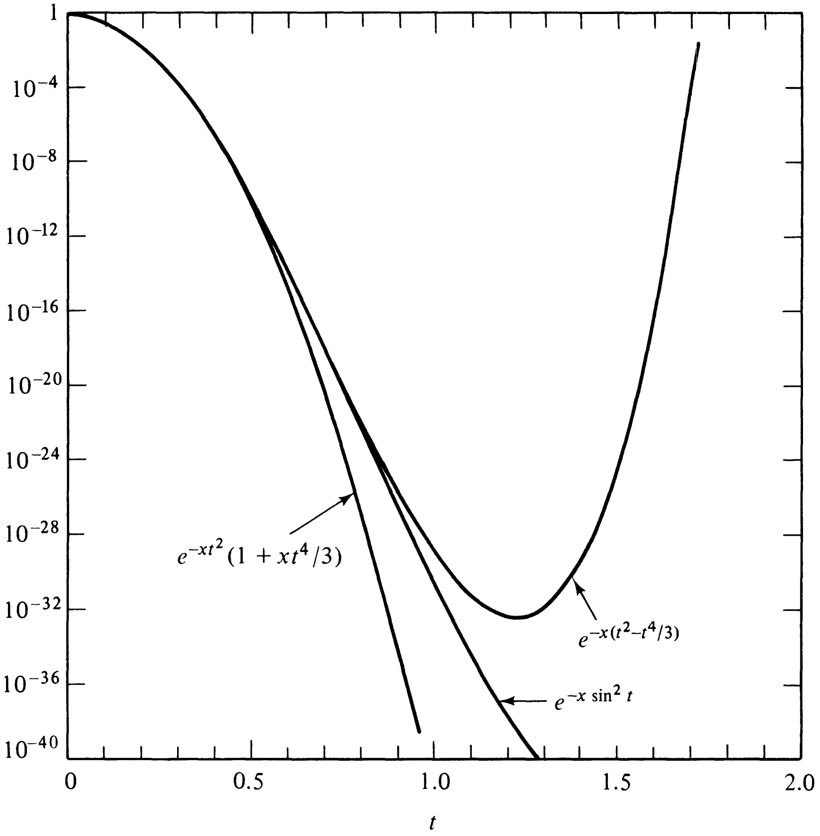

Figure 6.4 A comparison of the three integrands in (6.4.36) to (6.4.38) for $x = 100$. All three differ when t is large, but in the restricted range near $t = 0$ all three contribute equally to the integral as $x \to \infty$.

In this form, $f(s) = 1$ and $\phi(s) = s + 1/s$ and Laplace's method applies directly. The maximum of the new function $\phi(s)$ occurs at $s = 1$, so (6.4.19c) gives

$$I(x) \sim \sqrt{\pi}\, e^{-2\sqrt{x}}/x^{3/4}, \qquad x \to +\infty.$$

Example 10 *Derivation of Stirling's formula for* $\Gamma(x)$. A convergent integral representation for $\Gamma(x)$ is $\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}\,dt$ $(x > 0)$ (see Sec. 2.2). Here $f(t) = e^{-t}/t$ and $\phi(t) = \ln t$. Note that max $\phi(t) = \infty$ for $0 \le t < \infty$, so Laplace's method is not immediately applicable. This example is very similar to the previous example because the maximum of $\phi(t)$ occurs as $t \to \infty$ where $f(t)$ is exponentially small. As in Example 9, we find the location of the maximum of $e^{-t}t^x$, neglecting the factor $1/t$ which vanishes algebraically at ∞. This maximum occurs where $(d/dt)e^{-t}t^x = 0$ or $t = x$. Again, we encounter a movable maximum.

If we make the change of variables $t = sx$, we obtain

$$\Gamma(x) = x^x \int_0^\infty e^{-x(s - \ln s)}\, \frac{ds}{s}.$$

Now $f(s) = 1/s$ and $\phi(s) = -s + \ln s$. Laplace's method applies directly to this transformed integral. The maximum of $\phi(s)$ occurs at $s = 1$ so (6.4.19c) gives

$$\Gamma(x) \sim x^x e^{-x}\sqrt{2\pi/x}, \qquad x \to +\infty, \tag{6.4.39}$$

in agreement with (5.4.1). To obtain the next term in the Stirling series we note that $\phi(1) = -1$, $\phi'(1) = 0$, $\phi''(1) = -1$, $\phi'''(1) = 2$, $(d^4\phi/ds^4)(1) = -6$, $f(1) = 1$, $f'(1) = -1$, $f''(1) = 2$. Substituting these coefficients into the formula (6.4.35), we obtain

$$\Gamma(x) \sim x^x e^{-x}\sqrt{\frac{2\pi}{x}}\left(1 + \frac{1}{12x}\right), \qquad x \to +\infty, \tag{6.4.40}$$

in agreement with (5.4.1).

The distinction between ordinary and movable maxima is examined in Probs. 6.45 to 6.47.

(I) 6.5 METHOD OF STATIONARY PHASE

There is an immediate generalization of the Laplace integrals studied in Sec. 6.4 which we obtain by allowing the function $\phi(t)$ in (6.4.1) to be complex. Note that, if we wish, we may assume that $f(t)$ is real; if it were complex, $f(t)$ could be decomposed into a sum of its real and imaginary parts. However, allowing $\phi(t)$ to be complex poses new and nontrivial problems. In this section we consider the special case in which $\phi(t)$ is pure imaginary: $\phi(t) = i\psi(t)$, where $\psi(t)$ is real. The resulting integral

$$I(x) = \int_a^b f(t)e^{ix\psi(t)}\,dt \tag{6.5.1}$$

with $f(t)$, $\psi(t)$, a, b, x all real is called a generalized Fourier integral. When $\psi(t) = t$, $I(x)$ is an ordinary Fourier integral. The general case in which $\phi(t)$ is complex is considered in Sec. 6.6.

To study the behavior of $I(x)$ in (6.5.1) as $x \to +\infty$, we can use integration by parts to develop an asymptotic expansion in inverse powers of x so long as the boundary terms are finite and the resulting integrals exist.

Example 1 *Asymptotic expansion of a Fourier integral as $x \to +\infty$.* We use integration by parts to find an asymptotic approximation to the Fourier integral

$$I(x) = \int_0^1 \frac{e^{ixt}}{1+t}\,dt.$$

After one integration by parts we obtain

$$I(x) = -\frac{i}{2x}e^{ix} + \frac{i}{x} - \frac{i}{x}\int_0^1 \frac{e^{ixt}}{(1+t)^2}\,dt. \tag{6.5.2}$$

The integral on the right side of (6.5.2) is negligible compared with the boundary terms as $x \to +\infty$; in fact, it vanishes like $1/x^2$ as $x \to +\infty$. To see this, we integrate by parts again:

$$-\frac{i}{x}\int_0^1 \frac{e^{ixt}}{(1+t)^2}\,dt = -\frac{1}{4x^2}e^{ix} + \frac{1}{x^2} - \frac{2}{x^2}\int_0^1 \frac{e^{ixt}}{(1+t)^3}\,dt.$$

The integral on the right is bounded because

$$\left| \int_0^1 \frac{e^{ixt}}{(1+t)^3} dt \right| \le \int_0^1 (1+t)^{-3}\, dt = \frac{3}{8}.$$

Since the integral on the right in (6.5.2) does vanish like $1/x^2$ as $x \to +\infty$, $I(x)$ is asymptotic to the boundary terms: $I(x) \sim -(i/2x)e^{ix} + i/x$ $(x \to +\infty)$.

Repeated application of integration by parts gives the complete asymptotic expansion of $I(x)$ as $x \to +\infty$: $I(x) = e^{ix}u(x) + v(x)$ where

$$u(x) \sim -\frac{i}{2x} - \frac{1}{4x^2} + \cdots + \frac{(-i)^n(n-1)!}{(2x)^n} + \cdots, \qquad x \to +\infty,$$

$$v(x) \sim \frac{i}{x} + \frac{1}{x^2} + \cdots - \frac{(-i)^n(n-1)!}{x^n} + \cdots, \qquad x \to +\infty.$$

Example 2 *Integration by parts applied to* $\int_0^1 \sqrt{t}\, e^{ixt}\, dt$. Integration by parts can be used just once for the Fourier integral $I(x) = \int_0^1 \sqrt{t}\, e^{ixt}\, dt$. One integration by parts gives

$$I(x) = -\frac{i}{x} e^{ix} + \frac{i}{2x} \int_0^1 \frac{e^{ixt}}{\sqrt{t}} dt. \tag{6.5.3}$$

The integral on the right side of (6.5.3) vanishes more rapidly than the boundary term as $x \to +\infty$. We cannot use integration by parts to verify this because the resulting integral does not exist. (Why?) However, we can use the following simple scaling argument. We let $s = xt$ and obtain

$$\frac{i}{2x} \int_0^1 \frac{e^{ixt}}{\sqrt{t}} dt = \frac{i}{2x^{3/2}} \int_0^x \frac{e^{is}}{\sqrt{s}} ds \sim \frac{i}{2x^{3/2}} \int_0^\infty \frac{e^{is}}{\sqrt{s}} ds, \qquad x \to +\infty.$$

To evaluate the last integral we rotate the contour of integration from the real-s axis to the positive imaginary-s axis in the complex-s plane and obtain

$$\int_0^\infty \frac{e^{is}}{\sqrt{s}} ds = \sqrt{\pi}\, e^{i\pi/4}. \tag{6.5.4}$$

(See Prob. 6.49 for the details of this calculation.) Therefore,

$$I(x) + \frac{i}{x} e^{ix} \sim \frac{i}{2x^{3/2}} \sqrt{\pi}\, e^{i\pi/4}, \qquad x \to +\infty. \tag{6.5.5}$$

Clearly, this result cannot be found by direct integration by parts of the integral on the right side of (6.5.3) because a fractional power of x has appeared. However, it is possible to find the full asymptotic expansion of $I(x)$ as $x \to +\infty$ by an indirect application of integration by parts (see Prob. 6.50).

In Example 1 we used integration by parts to argue that the integral on the right side of (6.5.2) vanishes more rapidly than the boundary terms as $x \to +\infty$. In Example 2 we used a scaling argument to show that the integral on the right side of (6.5.3) vanishes more rapidly than the boundary terms as $x \to +\infty$. There is, in fact, a very general result called the Riemann-Lebesgue lemma that guarantees that

$$\int_a^b f(t)e^{ixt}\, dt \to 0, \qquad x \to +\infty; \tag{6.5 6}$$

provided that $\int_a^b |f(t)|\,dt$ exists. This result is valid even when $f(t)$ is not differentiable and integration by parts or scaling do not work. We will cite the Riemann-Lebesgue lemma repeatedly throughout this section; we could have used it to justify neglecting the integrals on the right sides of (6.5.2) and (6.5.3).

We reserve a proof of the Riemann-Lebesgue lemma for Prob. 6.51. Although the proof of (6.5.6) is messy, it is easy to understand the result heuristically. When x becomes large, the integrand $f(t)e^{ixt}$ oscillates rapidly and contributions from adjacent subintervals nearly cancel.

The Riemann-Lebesgue lemma can be extended to cover generalized Fourier integrals of the form (6.5.1). It states that $I(x) \to 0$ as $x \to +\infty$ so long as $|f(t)|$ is integrable, $\psi(t)$ is continuously differentiable for $a \le t \le b$, and $\psi(t)$ is not constant on any subinterval of $a \le t \le b$ (see Prob. 6.52). The lemma implies that $\int_0^{10} t^3 e^{ix\sin^2 t}\,dt \to 0$ $(x \to +\infty)$, but it does not apply to $\int_0^{10} t^3 e^{2ix}\,dt$.

Integration by parts gives the leading asymptotic behavior as $x \to +\infty$ of generalized Fourier integrals of the form (6.5.1), provided that $f(t)/\psi'(t)$ is smooth for $a \le t \le b$ and nonvanishing at one of the endpoints a or b. Explicitly,

$$I(x) = \frac{f(t)}{ix\psi'(t)} e^{ix\psi(t)}\Bigg|_{t=a}^{t=b} - \frac{1}{ix}\int_a^b \frac{d}{dt}\frac{f(t)}{\psi'(t)}\, e^{ix\psi(t)}\,dt.$$

The Riemann-Lebesgue lemma shows that the integral on the right vanishes more rapidly than $1/x$ as $x \to +\infty$. Therefore, $I(x)$ is asymptotic to the boundary terms (assuming that they do not vanish):

$$I(x) \sim \frac{f(t)}{ix\psi'(t)} e^{ix\psi(t)}\Bigg|_{t=a}^{t=b}, \qquad x \to +\infty. \tag{6.5.7}$$

Observe that when integration by parts applies, $I(x)$ vanishes like $1/x$ as $x \to +\infty$.

Integration by parts may not work if $\psi'(t) = 0$ for some t in the interval $a \le t \le b$. Such a point is called a *stationary* point of ψ. When there are stationary points in the interval $a \le t \le b$, $I(x)$ must still vanish as $x \to +\infty$ by the Riemann-Lebesgue lemma, but $I(x)$ usually vanishes less rapidly than $1/x$ because the integrand $f(t)e^{ix\psi(t)}$ oscillates less rapidly near a stationary point than it does near a point where $\psi'(t) \ne 0$. Consequently, there is less cancellation between adjacent subintervals near the stationary point.

The method of stationary phase gives the *leading* asymptotic behavior of generalized Fourier integrals having stationary points. This method is very similar to Laplace's method in that the leading contribution to $I(x)$ comes from a small interval of width ε surrounding the stationary points of $\psi(t)$. We will show that if c is a stationary point and if $f(c) \ne 0$, then $I(x)$ goes to zero like $x^{-1/2}$ as $x \to +\infty$ if $\psi''(c) \ne 0$, like $x^{-1/3}$ if $\psi''(c) = 0$ but $\psi'''(c) \ne 0$, and so on; as $\psi(t)$ becomes flatter at $t = c$, $I(x)$ vanishes less rapidly as $x \to +\infty$.

Since any generalized Fourier integral can be written as a sum of integrals in which $\psi'(t)$ vanishes only at an endpoint, we can explain the method of stationary phase for the special integral (6.5.1) in which $\psi'(a) = 0$ and $\psi'(t) \ne 0$ for $a < t \le b$.

We decompose $I(x)$ into two terms:

$$I(x) = \int_a^{a+\varepsilon} f(t)e^{ix\psi(t)}\,dt + \int_{a+\varepsilon}^{b} f(t)e^{ix\psi(t)}\,dt, \tag{6.5.8}$$

where ε is a small positive number to be chosen later. The second integral on the right side of (6.5.8) vanishes like $1/x$ as $x \to +\infty$ because there are no stationary points in the interval $a + \varepsilon \le t \le b$.

To obtain the leading behavior of the first integral on the right side of (6.5.8), we replace $f(t)$ by $f(a)$ and $\psi(t)$ by $\psi(a) + \psi^{(p)}(a)(t - a)^p/p!$ where $\psi^{(p)}(a) \ne 0$ but $\psi'(a) = \cdots = \psi^{(p-1)}(a) = 0$:

$$I(x) \sim \int_a^{a+\varepsilon} f(a) \exp\left\{ix\left[\psi(a) + \frac{1}{p!}\psi^{(p)}(a)(t-a)^p\right]\right\} dt, \qquad x \to +\infty. \tag{6.5.9}$$

Next, we replace ε by ∞, which introduces error terms that vanish like $1/x$ as $x \to +\infty$ and thus may be disregarded, and let $s = (t - a)$:

$$I(x) \sim f(a)e^{ix\psi(a)} \int_0^\infty \exp\left[\frac{ix}{p!}\psi^{(p)}(a)s^p\right] ds, \qquad x \to +\infty. \tag{6.5.10}$$

To evaluate the integral on the right, we rotate the contour of integration from the real-s axis by an angle $\pi/2p$ if $\psi^{(p)}(a) > 0$ and make the substitution

$$s = e^{i\pi/2p}\left[\frac{p!\,u}{x\psi^{(p)}(a)}\right]^{1/p} \tag{6.5.11a}$$

with u real or rotate the contour by an angle $-\pi/2p$ if $\psi^{(p)}(a) < 0$ and make the substitution

$$s = e^{-i\pi/2p}\left[\frac{p!\,u}{x\,|\psi^{(p)}(a)|}\right]^{1/p} \tag{6.5.11b}$$

Thus,

$$I(x) \sim f(a)e^{ix\psi(a) \pm i\pi/2p}\left[\frac{p!}{x\,|\psi^{(p)}(a)|}\right]^{1/p} \frac{\Gamma(1/p)}{p}, \qquad x \to +\infty, \tag{6.5.12}$$

where we use the factor $e^{i\pi/2p}$ if $\psi^{(p)}(a) > 0$ and the factor $e^{-i\pi/2p}$ if $\psi^{(p)}(a) < 0$.

The formula in (6.5.12) gives the leading behavior of $I(x)$ if $f(a) \ne 0$ but $\psi'(a) = 0$. If $f(a)$ vanishes, it is necessary to decide whether the contribution from the stationary point still dominates the leading behavior. When it does, the behavior is slightly more complicated than (6.5.12) (see Prob. 6.53).

Example 3 *Leading behavior of* $\int_0^{\pi/2} e^{ix\cos t}\,dt$ *as* $x \to +\infty$. The function $\psi(t) = \cos t$ has a stationary point at $t = 0$. Since $\psi''(0) = -1$, (6.5.12) with $p = 2$ gives $I(x) \sim \sqrt{\pi/2x}\, e^{i(x-\pi/4)}$ $(x \to +\infty)$.

Example 4 *Leading behavior of* $\int_0^\infty \cos(xt^2 - t)\, dt$ *as* $x \to +\infty$. To use the method of stationary phase, we write this integral as $\int_0^\infty \cos(xt^2 - t)\, dt = \text{Re} \int_0^\infty e^{i(xt^2 - t)}\, dt$. The function $\psi(t) = t^2$ has a stationary point at $t = 0$. Since $\psi''(0) = 2$, (6.5.12) with $p = 2$ gives $\int_0^\infty \cos(xt^2 - t)\, dt \sim \text{Re}\, \frac{1}{2}\sqrt{\pi/x}\, e^{i\pi/4} = \frac{1}{2}\sqrt{\pi/2x}$ $(x \to +\infty)$.

Example 5 *Leading behavior of* $J_n(n)$ *as* $n \to \infty$. When n is an integer, the Bessel function $J_n(x)$ has the integral representation

$$J_n(x) = \frac{1}{\pi} \int_0^\pi \cos(x \sin t - nt)\, dt \tag{6.5.13}$$

(see Prob. 6.54). Therefore, $J_n(n) = \text{Re} \int_0^\pi e^{in(\sin t - t)}\, dt/\pi$. The function $\psi(t) = \sin t - t$ has a stationary point at $t = 0$. Since $\psi''(0) = 0$, $\psi'''(0) = -1$, (6.5.12) with $p = 3$ gives

$$\begin{aligned} J_n(n) &\sim \frac{1}{\pi} \text{Re} \left[\frac{1}{3} e^{-i\pi/6} \left(\frac{6}{n}\right)^{1/3} \Gamma\left(\frac{1}{3}\right) \right], \qquad x \to +\infty, \\ &= \frac{1}{\pi} 2^{-2/3} 3^{-1/6} \Gamma\left(\frac{1}{3}\right) n^{-1/3}, \qquad n \to \infty. \end{aligned} \tag{6.5.14}$$

Observe that because $\psi''(0) = 0$, $J_n(n)$ vanishes less rapidly than $n^{-1/2}$ as $n \to \infty$.

If n is not an integer, (6.5.14) still holds (see Prob. 6.55).

In this section we have obtained only the leading behavior of generalized Fourier integrals. Higher-order approximations can be complicated because nonstationary points may also contribute to the large-x behavior of the integral. Specifically, the second integral on the right in (6.5.8) must be taken into account when computing higher-order terms because the error incurred in neglecting this integral is usually algebraically small. By contrast, recall that the approximation in (6.4.2) for Laplace's method is valid to all orders because the errors are exponentially, rather than algebraically, small. To obtain the higher-order corrections to (6.5.12), one can either use the method of asymptotic matching (see Sec. 7.4) or the method of steepest descents (see Sec. 6.6).

(I) 6.6 METHOD OF STEEPEST DESCENTS

The method of steepest descents is a technique for finding the asymptotic behavior of integrals of the form

$$I(x) = \int_C h(t) e^{x\rho(t)}\, dt \tag{6.6.1}$$

as $x \to +\infty$, where C is an integration contour in the complex-t plane and $h(t)$ and $\rho(t)$ are analytic functions of t. The idea of the method is to use the analyticity of the integrand to justify deforming the contour C to a new contour C' on which $\rho(t)$ has a constant imaginary part. Once this has been done, $I(x)$ may be evaluated asymptotically as $x \to +\infty$ using Laplace's method. To see why, observe that on the contour C' we may write $\rho(t) = \phi(t) + i\psi$, where ψ is a real constant and $\phi(t)$ is a real function. Thus, $I(x)$ in (6.6.1) takes the form

$$I(x) = e^{ix\psi} \int_{C'} h(t) e^{x\phi(t)}\, dt. \tag{6.6.2}$$

Although t is complex, (6.6.2) can be treated by Laplace's method as $x \to +\infty$ because $\phi(t)$ is real.

Our motivation for deforming C into a path C' on which $\operatorname{Im} \rho(t)$ is a constant is to eliminate rapid oscillations of the integrand when x is large. Of course, one could also deform C into a path on which $\operatorname{Re} \rho(t)$ is a constant and then apply the method of stationary phase. However, we have seen that Laplace's method is a much better approximation scheme than the method of stationary phase because the full asymptotic expansion of a generalized Laplace integral is determined by the integrand in an arbitrarily small neighborhood of the point where $\operatorname{Re} \rho(t)$ is a maximum on the contour. By contrast, the full asymptotic expansion of a generalized Fourier integral typically depends on the behavior of the integrand along the entire contour. As a consequence, it is usually easier to obtain the full asymptotic expansion of a generalized Laplace integral than of a generalized Fourier integral.

Before giving a formal exposition of the method of steepest descents, we consider three preliminary examples which illustrate how shifting complex contours can greatly simplify asymptotic analysis. In the first example we consider a Fourier integral whose asymptotic expansion is difficult to find by the methods used in Sec. 6.5. However, deforming the contour reduces the integral to a pair of integrals that are easy to evaluate by Laplace's method.

Example 1 *Conversion of a Fourier integral into a Laplace integral by deforming the contour.* The behavior of the integral

$$I(x) = \int_0^1 \ln t \, e^{ixt} \, dt \tag{6.6.3}$$

as $x \to +\infty$ cannot be found directly by the methods of Sec. 6.5 because there is no stationary point. Also, integration by parts is useless because $\ln 0 = -\infty$. Integration by parts is doomed to fail because, as we will see, the leading asymptotic behavior of $I(x)$ contains the factor $\ln x$ which is not a power of $1/x$.

To approximate $I(x)$ we deform the integration contour C, which runs from 0 to 1 along the real-t axis, to one which consists of three line segments: C_1, which runs up the imaginary-t axis from 0 to iT; C_2, which runs parallel to the real-t axis from iT to $1 + iT$; and C_3, which runs down from $1 + iT$ to 1 along a straight line parallel to the imaginary-t axis (see Fig. 6.5). By Cauchy's theorem, $I(x) = \int_{C_1+C_2+C_3} \ln t \, e^{ixt} \, dt$. Next we let $T \to +\infty$. In this limit the contribution from C_2 approaches 0. (Why?) In the integral along C_1 we set $t = is$, and in the integral along C_3 we set $t = 1 + is$, where s is real in both integrals. This gives

$$I(x) = i \int_0^\infty \ln (is) \, e^{-xs} \, ds - i \int_0^\infty \ln (1 + is) \, e^{ix(1+is)} \, ds. \tag{6.6.4}$$

The sign of the second integral on the right is negative because C_3 is traversed downward.

Observe that both integrals in (6.6.4) are Laplace integrals. The first integral can be done exactly. We substitute $u = xs$ and use $\ln (is) = \ln s + i\pi/2$ and the identity $\int_0^\infty e^{-u} \ln u \, du = -\gamma$, where $\gamma = 0.5772\ldots$ is Euler's constant, and obtain

$$i \int_0^\infty \ln (is) \, e^{-xs} \, ds = -i(\ln x)/x - (i\gamma + \pi/2)/x.$$

We apply Watson's lemma to the second integral on the right in (6.6.4) using the Taylor expansion $\ln (1 + is) = -\sum_{n=1}^\infty (-is)^n/n$, and obtain

$$-i \int_0^\infty \ln (1 + is) \, e^{ix(1+is)} \, ds \sim ie^{ix} \sum_{n=1}^\infty \frac{(-i)^n(n-1)!}{x^{n+1}}, \qquad x \to +\infty.$$

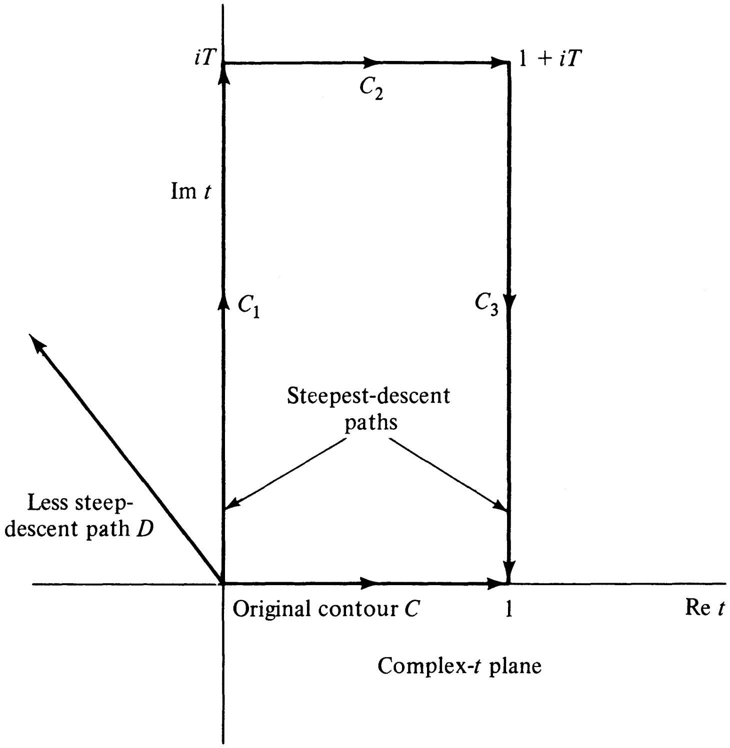

Figure 6.5 It is possible to convert the Fourier integral $I(x)$ in (6.6.3) into a Laplace integral merely by deforming the original contour C into $C_1 + C_2 + C_3$ as shown above and then allowing $T \to \infty$. C_1 and C_3 are called steepest-descent paths because $|\exp[x\rho(t)]|$ decreases most rapidly along these paths as t moves up from the real-t axis; $|\exp[x\rho(t)]|$ also decreases along D, but less rapidly per unit length than along C_1.

Combining the above two expansions gives the final result:

$$I(x) \sim -\frac{i \ln x}{x} - \frac{i\gamma + \pi/2}{x} + ie^{ix} \sum_{n=1}^{\infty} \frac{(-i)^n(n-1)!}{x^{n+1}}, \qquad x \to +\infty.$$

Let us review the calculation in the preceding example. For the integral (6.6.3), $\rho(t) = it$. For this function, paths of constant Im $\rho(t)$ are straight lines parallel to the imaginary-t axis. On the particular contours C_1 and C_3, Im $\rho(t) = 0$ and 1, respectively. Note that Im $\rho(t)$ is not the same constant on C_1 and C_3, but this does not matter; we have applied Laplace's method separately to each of the integrals on the right side of (6.6.4). Since Im $\rho(t = 0) \neq$ Im $\rho(t = 1)$, it is clear that there is no continuous contour joining $t = 0$ and $t = 1$ on which Im $\rho(t)$ is constant. This is why it is necessary to deform the original contour C into C_1 and C_3 which are joined at ∞ by C_2 along which the integrand vanishes. In general, we expect that if Im $\rho(t)$ is not the same at the endpoints of the original integration contour C, then we cannot deform C into a continuous contour on which Im $\rho(t)$ is constant; the best one can hope for is to be able to deform C into distinct constant-phase contours which are joined by a contour on which the integrand vanishes.

Now we can explain why the procedure used in Example 1 is called the method of steepest descents. The contours C_1 and C_3 are called contours of constant phase because the phase of the complex number $e^{x\rho(t)}$ is constant. At the same time, C_1 and C_3 are also called steepest-descent paths because $|e^{x\rho(t)}|$ decreases most rapidly along these paths as t ranges from the endpoints 0 and 1 toward ∞. Any path originating at the endpoints 0 and 1 and moving upward in the complex-t plane is a path on which $|e^{x\rho(t)}|$ decreases (see Fig. 6.5). However, after traversing any given length of arc, $|e^{x\rho(t)}|$ decreases more along the vertical paths C_1 and C_3 than along any other path leaving the endpoints 0 and 1, respectively. We will explain this feature of steepest-descent paths later in this section.

Example 2 *Full asymptotic behavior of* $\int_0^1 e^{ixt^2}\,dt$ *as* $x \to +\infty$. The method of stationary phase can be used to find the leading behavior of the integral $I(x) = \int_0^1 e^{ixt^2}\,dt$. Here $\psi(t) = t^2$, so the stationary point lies at $t = 0$ and, using (6.5.12), $I(x) \sim \frac{1}{2}\sqrt{\pi/x}\, e^{i\pi/4}$ $(x \to +\infty)$. The method of steepest descents gives an easy way to determine the full asymptotic behavior of $I(x)$. [The method of integration by parts also works (see Prob. 6.57).]

As in Example 1, we try to deform the contour C: $0 \le t \le 1$ into contours along which Im $\rho(t)$ is constant, where $\rho(t) = it^2$. We begin by finding a contour which passes through $t = 0$ and on which Im $\rho(t)$ is constant. Writing $t = u + iv$ with u and v real, we obtain Im $\rho(t) = u^2 - v^2$. At $t = 0$, Im $\rho = 0$. Therefore, constant-phase contours passing through $t = 0$ must satisfy $u = v$ or $u = -v$ everywhere along the contour (see Fig. 6.6). On the contour $u = -v$, Re $\rho(t) = 2v^2$, so $|e^{x\rho(t)}| = e^{2xv^2}$ increases as $t = (i - 1)v \to \infty$. This is called a steepest-ascent contour; since there is no maximum of $|e^{x\rho(t)}|$ on this contour, Laplace's method cannot be applied. On the other hand, the contour $u = v$ is a steepest-descent contour because Re $\rho(t) = -2v^2$, so $|e^{x\rho(t)}| = e^{-2xv^2}$ decreases as $t = (1 + i)v \to \infty$. The contour C_1: $t = (1 + i)v$ $(0 \le v < \infty)$ is comparable to the contour C_1 employed in Example 1.

Next, we must find a steepest-descent contour passing through $t = 1$ along which Im $\rho(t)$ is constant. At $t = 1$, the value of Im $\rho(t)$ is 1. Therefore, the constant-phase contour passing through $u = 1$, $v = 0$ is given by $u = \sqrt{v^2 + 1}$. Since Re $\rho(t) = -2uv$ decreases as $t = u + iv \to \infty$ along the portion of this constant-phase contour with $0 \le v < \infty$, the steepest-descent contour passing through $t = 1$ is given by C_3: $t = \sqrt{v^2 + 1} + iv$, $0 \le v < \infty$. Note that C_1 and C_3 become tangent as $v \to +\infty$ (see Fig. 6.6).

The next step is to deform the original contour C: $0 \le t \le 1$ into $C_1 + C_3$, in which C_3 is traversed from $t = \infty$ to $t = 1$. Along C_1, Im $\rho(t) = 0$, while along C_3, Im $\rho(t) = 1$. Since the value of Im $\rho(t)$ is different on C_1 and C_3, it is clear that the original contour cannot be continuously deformed into $C_1 + C_3$. Rather, we must include a third contour C_2 which bridges the gap between C_1 and C_3. We take C_2 to be the straight line connecting the points $(1 + i)V$ on C_1 and $\sqrt{V^2 + 1} + iV$ on C_3 (see Fig. 6.6). C can be continuously deformed into C_2 together with the portions of C_1 and C_3 satisfying $0 \le v \le V$. Now, as $V \to \infty$, the contribution from the contour C_2 vanishes. (Why?) Thus,

$$I(x) = \int_{C_1} e^{ixt^2}\,dt - \int_{C_3} e^{ixt^2}\,dt. \tag{6.6.5}$$

The integral along C_1 can be evaluated exactly. Setting $t = (1 + i)v$, we obtain

$$\int_{C_1} e^{ixt^2}\,dt = (1 + i)\int_0^\infty e^{-2xv^2}\,dv = \frac{1}{2}\sqrt{\frac{\pi}{x}}\,e^{i\pi/4}. \tag{6.6.6}$$

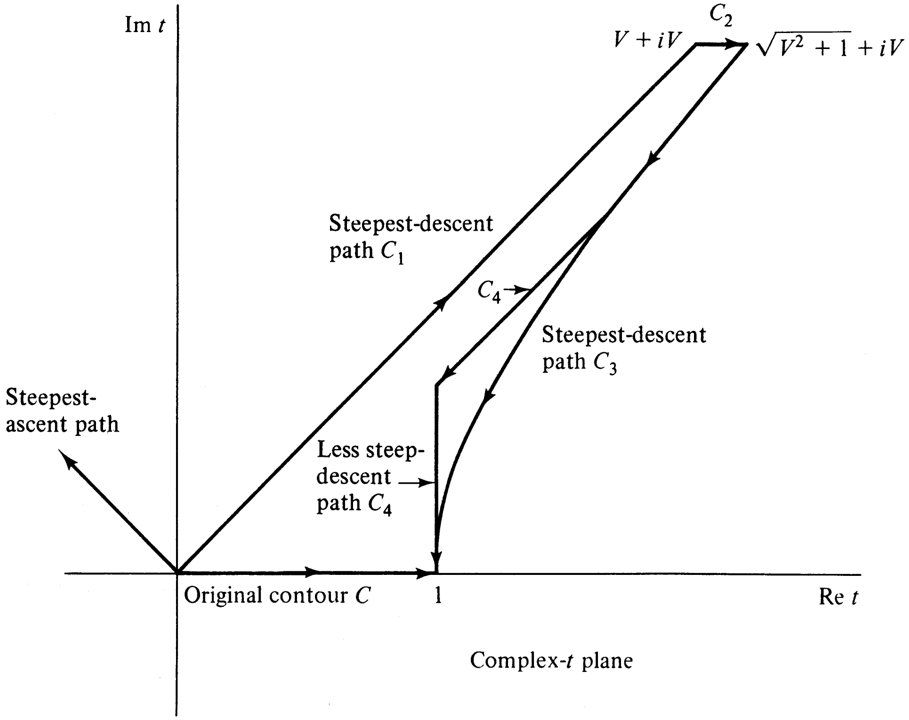

Figure 6.6 The Fourier integral $I(x)$ in Example 2 becomes a pair of Laplace integrals if the original contour C is distorted into $C_1 + C_2 + C_3$ and V is allowed to approach ∞. To simplify the evaluation of the integral along C_3, we can replace the lower part of the contour C_3 by C_4.

This contribution is precisely the leading behavior of $I(x)$ as $x \to +\infty$ that we found using the method of stationary phase.

Now we evaluate the contribution to $I(x)$ from the integral on C_3. Note that if we substitute $t = \sqrt{v^2+1} + iv$, $0 \le v < \infty$, then $\rho(t) = it^2 = i - 2v\sqrt{v^2+1}$. This verifies that C_3 is a curve of constant phase; it is also a curve of steepest descent. An easy way to obtain the full asymptotic expansion of the integral over C_3 is to use Watson's lemma. To do this, the integral must be expressed in the form $\int_0^\infty f(s)e^{-xs}\,ds$. This motivates the change of variables from t to s where s is defined by

$$\rho(t) = it^2 = i - s; \tag{6.6.7}$$

observe that $s = 2v\sqrt{v^2+1}$ is real and satisfies $0 \le s < \infty$ along C_3. Since $t = (1+is)^{1/2}$, $dt/ds = \frac{1}{2}i(1+is)^{-1/2}$, so

$$\int_{C_3} e^{ixt^2}\,dt = \tfrac{1}{2}ie^{ix}\int_0^\infty \frac{e^{-xs}}{\sqrt{1+is}}\,ds.$$

To apply Watson's lemma, we use the Taylor expansion

$$(1+is)^{-1/2} = \sum_{n=0}^{\infty} (-is)^n \Gamma(n+\tfrac{1}{2})/n!\,\Gamma(\tfrac{1}{2}).$$

We obtain

$$\int_{C_3} e^{ixt^2}\,dt \sim \frac{1}{2} i e^{ix} \sum_{n=0}^{\infty} (-i)^n \frac{\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})x^{n+1}}, \qquad x \to +\infty. \tag{6.6.8}$$

Combining this result with that in (6.6.6) gives the full asymptotic expansion of $I(x)$ as $x \to +\infty$:

$$I(x) \sim \frac{1}{2}\sqrt{\frac{\pi}{x}}\, e^{i\pi/4} - \frac{1}{2} i e^{ix} \sum_{n=0}^{\infty} (-i)^n \frac{\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})x^{n+1}}, \qquad x \to +\infty. \tag{6.6.9}$$

Finally, we mention an alternative way to obtain the result in (6.6.8) for the integral on C_3. The substitution in (6.6.7) is an exact parametrization of the curve C_3 in terms of the real parameter s. However, as we know from our discussion of Laplace's method in Sec. 6.4, it is only the immediate neighborhood of the maximum at $t = 1$ that contributes to the full asymptotic expansion of the integral on C_3. Therefore, it is not necessary to follow the curve C_3 exactly. It is correct to shift the integration path C_3 to one which still passes through the maximum at $t = 1$ and which is a descent contour in the sense that $|e^{x\rho(t)}|$ decreases and rejoins C_3 for large $|t|$. Any deformation of C_3 of this kind does not change the value of the integral because the integrand is analytic. For the present example, a convenient alternative to C_3 is a contour C_4 which originates at $t = 1$, goes vertically upward parallel to the imaginary-t axis, and then rejoins the contour C_3 at any point in the upper half plane (see Fig. 6.6). Only the vertical straight-line portion of C_4 in the immediate vicinity of $t = 1$ contributes to the full asymptotic expansion of the integral. We can parametrize the straight-line portion of C_4 near $t = 1$ by $t = 1 + iv$, where v is real and $0 \le v \le \varepsilon$ with ε small. Thus,

$$\int_{C_3} e^{ixt^2}\,dt = \int_{C_4} e^{ixt^2}\,dt \sim i \int_0^{\varepsilon} e^{ix(1+iv)^2}\,dv$$

$$= ie^{ix} \int_0^{\varepsilon} e^{-2xv} e^{-ixv^2}\,dv, \qquad x \to +\infty.$$

Using Laplace's method

$$\int_0^{\varepsilon} e^{-2xv} e^{-ixv^2}\,dv \sim \int_0^{\varepsilon} e^{-2xv} \sum_{n=0}^{\infty} \frac{(-ix)^n v^{2n}}{n!}\,dv$$

$$\sim \sum_{n=0}^{\infty} \frac{(-i)^n (2n)!}{2^{2n+1} n!\, x^{n+1}}, \qquad x \to +\infty.$$

Since $(2n)!/(2^{2n} n!) = \Gamma(n+\frac{1}{2})/\Gamma(\frac{1}{2})$, we have reproduced (6.6.8) exactly.

This alternative calculation, in which we have replaced the curved path C_3 by a path C_4 which begins as a straight line, is an important computational device that is frequently helpful in the method of steepest descents. Note that C_4 is neither a curve of constant phase nor a curve of steepest descent, although it is a curve of descent of $|e^{x\rho(t)}|$. Other descent curves could be used instead of C_4 (see Prob. 6.58).

Example 3 *Sophisticated example of the method of steepest descents.* What is the leading behavior of the generalized Fourier integral

$$I(x) = \int_0^1 \exp\,(ixe^{-1/s})\,ds \tag{6.6.10}$$

as $x \to +\infty$? This is a sophisticated example because $s = 0$ is an infinite-order stationary point; i.e., all derivatives of $e^{-1/s}$ vanish as $s \to 0+$. We know from our discussion of the method of stationary phase that if the first nonvanishing derivative of ψ in (6.5.1) at a stationary point is $\psi^{(p)}$, then $I(x)$ must vanish like $x^{-1/p}$ as $x \to +\infty$. Therefore, we expect that if the integrand has an infinite-order stationary point, $I(x)$ vanishes less rapidly than any power of $1/x$ as $x \to +\infty$. However, the Riemann-Lebesgue lemma guarantees that $I(x)$ does indeed vanish as $x \to +\infty$.

How fast does $I(x)$ in (6.6.10) vanish? It is hard to apply the method of stationary phase to $I(x)$ directly. (Try it!) However, the method of steepest descents provides a relatively easy approach. We begin by making the substitution $t = e^{-1/s}$:

$$I(x) = \int_0^{1/e} \frac{e^{ixt}}{t(\ln t)^2}\, dt.$$

The form of this integral is similar to that of the integral (6.6.3) considered in Example 1. Therefore, as in Example 1, we shift the contour $C: 0 \le t \le 1/e$ to two vertical lines parallel to the imaginary-t axis:

$$I(x) = \int_{C_1} \frac{e^{ixt}}{t(\ln t)^2}\, dt - \int_{C_3} \frac{e^{ixt}}{t(\ln t)^2}\, dt, \tag{6.6.11}$$

where C_1 is the path $t = iv$ $(0 \le v < \infty)$ and C_3 is the path $t = 1/e + iv$ $(0 \le v < \infty)$. Now we find the leading behavior of each of the integrals on the right in (6.6.11).

The integral on the path C_3 requires only a straightforward application of Laplace's method. We substitute $t = 1/e + iv$ $(0 \le v < \infty)$ and obtain

$$\int_{C_3} \frac{e^{ixt}}{t(\ln t)^2}\, dt = ie^{ix/e} \int_0^{\infty} \frac{e^{-xv}}{(1/e + iv)[\ln (1/e + iv)]^2}\, dv \sim ie^{ix/e} e/x, \qquad x \to +\infty. \tag{6.6.12}$$

The integral on C_1 is more difficult. We simplify the integral by substituting $t = iv$ $(0 \le v < \infty)$ and perform one integration by parts:

$$I_1(x) = \int_{C_1} \frac{e^{ixt}}{t(\ln t)^2}\, dt = \int_0^{\infty} \frac{e^{-xv}}{v[\ln (iv)]^2}\, dv = -x \int_0^{\infty} \frac{e^{-xv}}{\ln (iv)}\, dv. \tag{6.6.13}$$

The integral on the right side of (6.6.13) is a Laplace integral; we can restrict the range of integration to the vicinity of $v = 0$ without altering its asymptotic expansion as $x \to +\infty$. Thus,

$$I_1(x) \sim -x \int_0^{\varepsilon} \frac{e^{-xv}}{\ln (iv)}\, dv, \qquad x \to +\infty.$$

This integral does not yield to a straightforward application of Laplace's method because the integrand vanishes at $v = 0$. Moreover, the conventional treatment of a moving maximum [see the derivation of the Stirling series for $\Gamma(x)$ given in Example 10 of Sec. 6.4] does not work because the moving maximum of the integrand is too broad (see Prob. 6.47). A good way to proceed is to substitute $r = xv$ and thus obtain

$$I_1(x) \sim -\int_0^{\varepsilon x} \frac{e^{-r}}{\ln r - \ln x + i\pi/2}\, dr, \qquad x \to +\infty,$$

where we have used the relation $\ln (iv) = \ln v + i\pi/2$. Next, we argue that the immediate vicinity of the origin, say $0 \le r \le 1/x^{1/2}$, does not contribute to the asymptotic expansion of the integral as $x \to +\infty$. To prove this we bound the contribution to $I_1(x)$ from $0 \le r \le 1/x^{1/2}$:

$$\left| \int_0^{1/x^{1/2}} \frac{e^{-r}}{\ln r - \ln x + i\pi/2}\, dr \right| \le \frac{2}{\pi x^{1/2}},$$

because $|\ln r - \ln x + i\pi/2| \ge \pi/2$ and $|e^{-r}| \le 1$. This contribution to $I_1(x)$ is negligible because, as we shall see, the full asymptotic expansion of $I_1(x)$ is a series in inverse powers of $\ln x$. Thus,

$$I_1(x) \sim -\int_{1/x^{1/2}}^{\varepsilon x} \frac{e^{-r}}{\ln r - \ln x + i\pi/2}\, dr, \qquad x \to +\infty. \tag{6.6.14}$$

To expand the integral in (6.6.14), we Taylor expand the integrand in powers of $1/\ln x$:

$$\frac{1}{\ln r - \ln x + i\pi/2} = -\frac{1}{\ln x} \sum_{n=0}^{\infty} \left(\frac{i\pi/2 + \ln r}{\ln x} \right)^n, \qquad x^{-1/2} \le r \le \varepsilon x,\ x \to +\infty.$$

Thus, $$I_1(x) \sim \frac{1}{\ln x} \sum_{n=0}^{\infty} \int_{1/x^{1/2}}^{\varepsilon x} e^{-r} \left(\frac{i\pi/2 + \ln r}{\ln x} \right)^n dr, \qquad x \to +\infty. \tag{6.6.15}$$

The range of each of the integrals in (6.6.15) can be extended to $0 \le r < \infty$ with an error smaller than any inverse power of ln x. (Why?) Evaluating the first two integrals, we obtain

$$I_1(x) \sim \frac{1}{\ln x} + \frac{i\pi/2 - \gamma}{(\ln x)^2} + \cdots, \tag{6.6.16}$$

where we have used $\int_0^\infty \ln re^{-r}\, dr = -\gamma$. The coefficient of the general term in (6.6.16) may be expressed in terms of derivatives of $\Gamma(t)$ at $t = 1$ (see Prob. 6.59).

Combining the results (6.6.12) and (6.6.16) with (6.6.11), we obtain the final result

$$I(x) \sim \frac{1}{\ln x} + \frac{i\pi/2 - \gamma}{(\ln x)^2} + \cdots; \qquad x \to +\infty. \tag{6.6.17}$$

One could not have guessed this result from a cursory inspection of the original integral in (6.6.10)! Does this asymptotic series diverge? (See Prob. 6.60.) In Table 6.1 we compare numerical values of $I(x)$ with the asymptotic results for $I(x)$ given in (6.6.17).

Formal Discussion of Steepest-Descent Paths in the Complex Plane

In the previous three introductory examples, we have shown that deforming contours of integration in the complex-t plane can facilitate the asymptotic evaluation of integrals. It is now appropriate to give a more general discussion of steepest-descent (constant-phase) contours.

We begin by recalling the role of the gradient in elementary calculus. If $f(u, v)$ is a differentiable function of two variables, then the gradient of f is the vector $\nabla f = (\partial f/\partial u, \partial f/\partial v)$. This vector points in the direction of the most rapid change of f at the point (u, v). In terms of the gradient, the directional derivative df/ds in the direction of the unit vector $\mathbf{n}$ is $df/ds = \mathbf{n} \cdot \nabla f$. This directional derivative is the rate of change of f in the direction $\mathbf{n}$. Thus, the largest directional derivative is in the direction $\mathbf{n} = \nabla f / |\nabla f|$ and has magnitude $|\nabla f|$. On a two-dimensional contour plot of $f(u, v)$, the vector ∇f is perpendicular to the contours of constant f

Table 6.1 Comparison between the exact value of the integral $I(x)$ in (6.6.10) and one-term and two-term asymptotic approximations to $I(x)$ in (6.6.17) obtained using the method of steepest descents

ln x	Exact value of $I(x)$	One-term asymptotic approximation	Two-term asymptotic approximation
0	$0.9814 + 0.1467i$	∞	∞
2	$0.3077 + 0.5419i$	0.5000	$0.3557 + 0.3927i$
4	$0.2499 + 0.0643i$	0.2500	$0.2139 + 0.0982i$
6	$0.1428 + 0.0423i$	0.1667	$0.1506 + 0.0436i$
8	$0.1146 + 0.0227i$	0.1250	$0.1160 + 0.0245i$
10	$0.0935 + 0.0143i$	0.1000	$0.0942 + 0.0157i$
12	$0.0790 + 0.0100i$	0.0833	$0.0793 + 0.0109i$

(level curves). Note that the directional derivative in the direction of the tangents to a level curve is 0.

We will now give a formal proof that constant-phase contours are also steepest contours. Let $\rho(t) = \phi(t) + i\psi(t)$ be an analytic function of the complex variable $t = u + iv$. Also, for the moment, we restrict ourselves to regions of the complex-t plane in which $\rho'(t) \neq 0$.

We define a constant-phase contour of $e^{x\rho(t)}$ where $x > 0$ as a contour on which $\psi(t)$ is constant. A steepest contour is defined as a contour whose tangent is parallel to $\nabla |e^{x\rho(t)}| = \nabla e^{x\phi(t)}$, which is parallel to $\nabla\phi$. That is, a steepest contour is one on which the magnitude of $e^{x\rho(t)}$ is changing most rapidly with t.

Now we will show that if $\rho(t)$ is analytic, then constant-phase contours are steepest contours. If $\rho(t)$ is analytic, then it satisfies the Cauchy-Riemann equations

$$\partial\phi/\partial u = \partial\psi/\partial v, \qquad \partial\phi/\partial v = -\partial\psi/\partial u.$$

Therefore,

$$(\partial\phi/\partial u)(\partial\psi/\partial u) + (\partial\phi/\partial v)(\partial\psi/\partial v) = 0.$$

However, this equation can be written in vector form as $\nabla\phi \cdot \nabla\psi = 0$, so $\nabla\phi$ is perpendicular to $\nabla\psi$ and the directional derivative in the direction of $\nabla\phi$ satisfies $d\psi/ds = 0$. Thus, ψ is constant on contours whose tangents are parallel to $\nabla\phi$, showing that constant-phase contours are also steepest contours.

There is a slightly more sophisticated way to establish that constant-phase contours are steepest contours. It is well known that an analytic function $\rho(t)$ is a conformal (angle-preserving) mapping from the complex-t plane (u, v) to the complex-ρ plane (ϕ, ψ) if $\rho'(t) \neq 0$. Therefore, since lines of constant u are perpendicular to lines of constant v, lines of constant ϕ are perpendicular to lines of constant ψ. But lines of constant ϕ are also perpendicular to steepest curves of ϕ. This reestablishes the identity of steepest and constant-phase contours.

In the above two arguments, it was necessary to assume that $\rho'(t) \neq 0$. In the second argument, this condition was necessary because a map is not conformal at a point where $\rho'(t) = 0$. Where was this condition used in the first argument?

Saddle Points

When the contour of integration in (6.6.1) is deformed into constant-phase contours, the asymptotic behavior of the integral is determined by the behavior of the integrand near the local maxima of $\phi(t)$ along the contour. These local maxima of $\phi(t)$ may occur at endpoints of constant-phase contours (see Examples 1 to 3) or at an interior point of a constant-phase contour. If $\phi(t)$ has an interior maximum then the directional derivative along the constant-phase contour $d\phi/ds = |\nabla\phi|$ vanishes. The Cauchy-Riemann equations imply that $\nabla\phi = \nabla\psi = 0$ so $\rho'(t) = 0$ at an interior maximum of ϕ on a constant-phase contour.

A point at which $\rho'(t) = 0$ is called a *saddle point*. Saddle points are special because it is only at such a point that two distinct steepest curves can intersect. When $\rho'(t_0) \neq 0$, there is only one steepest curve passing through t and its tangent is parallel to $\nabla\phi$. In the direction of $\nabla\phi$, $|e^{x\rho}|$ is increasing so this portion of the curve is a steepest-*ascent* curve; in the direction of $-\nabla\phi$, $|e^{x\rho}|$ is decreasing so this portion of the curve is a steepest-*descent* curve. On the other hand, when $\rho'(t_0) = 0$ there are two or more steepest-ascent curves and two or more steepest-descent curves emerging from the point t_0.

To study the nature of the steepest curves emerging from a saddle point, let us study the region of the complex-t plane near t_0.

Example 4 *Steepest curves of e^{xt^2} near the saddle point $t = 0$.* Here $\rho(t) = t^2$. Observe that $\rho'(t) = 2t$ vanishes at $t = 0$, which verifies that 0 is a saddle point. We substitute $t = u + iv$ and identify the real and imaginary parts of $\rho(t)$:

$$\rho(t) = u^2 - v^2 + 2iuv, \qquad \phi(t) = u^2 - v^2, \qquad \psi(t) = 2uv.$$

Since $\rho(0) = 0$, the constant-phase contours that pass through $t = 0$ must satisfy $\psi(t) = 0$ everywhere. The constant-phase contours $u = 0$ (the imaginary axis) and $v = 0$ (the real axis) cross at the saddle point $t = 0$.

All four curves that emerge from $t = 0$, (a) $u = 0$ with v positive, (b) $u = 0$ with v negative, (c) $v = 0$ with u positive, and (d) $v = 0$ with u negative, are steepest curves because $\rho'(t) \neq 0$ except at $t = 0$. Which of these four curves are steepest-ascent curves and which are steepest-descent curves? On curves (a) and (b), $\phi(t) = -v^2$, so ϕ is decreasing away from $t = 0$; these curves are steepest-descent curves. On curves (c) and (d), $\phi(t) = u^2$, so ϕ is increasing away from $t = 0$; these curves are steepest-ascent curves. A plot showing these steepest-ascent and -descent curves as well as the level curves of ϕ away from $t = 0$ is given in Fig. 6.7.

Example 5 *Steepest curves of $e^{ix \cosh t}$ near the saddle point $t = 0$.* Here $\rho(t) = i \cosh t$, so $\rho'(t) = i \sinh t$ vanishes at $t = 0$. If we substitute $t = u + iv$ and use the identity

$$\cosh(u + iv) = \cosh u \cos v + i \sinh u \sin v,$$

we obtain the real and imaginary parts of $\rho(t)$:

$$\phi(t) = -\sinh u \sin v, \qquad \psi(t) = \cosh u \cos v.$$

Since $\rho(0) = i$, the constant-phase contours passing through $t = 0$ must satisfy $\psi(t) = \text{Im}\, \rho(t) = 1$. Thus, the constant-phase contours through $t = 0$ are given by

$$\cosh u \cos v = 1.$$

Other constant-phase contours (steepest-descent and -ascent curves) are given by $\cosh u \cos v = c$, where c is a constant. On Fig. 6.8 we plot the constant-phase contours for various values of c. Observe that steepest curves never cross except at saddle points.

Example 6 *Steepest curves of $e^{x(\sinh t - t)}$ near the saddle point at $t = 0$.* Here $\rho(t) = \sinh t - t$, so $\rho'(t) = \cosh t - 1$ vanishes at $t = 0$. Note that $\rho''(t) = \sinh t$ also vanishes at 0 and that the lowest nonvanishing derivative of ρ at $t = 0$ is $\rho'''(t)$. We call such a saddle point a third-order saddle point. At $t = 0$ six constant-phase contours meet. To find these contours we substitute $t = u + iv$ and identify the real and imaginary parts of ρ:

$$\rho = \phi + i\psi = (\sinh u \cos v - u) + i(\cosh u \sin v - v).$$

But $\rho(0) = 0$. Thus, constant-phase contours passing through $t = 0$ satisfy $\cosh u \sin v - v = 0$. Solutions to this equation are $v = 0$ (the u axis) and $u = \text{arc cosh}\,(v/\sin v)$.

In Prob. 6.61 you are asked to verify that (a) a total of six steepest paths emerge from $t = 0$; (b) paths emerge at 60° angles from adjacent paths; (c) as t moves away from 0, the paths alternate between steepest-ascent and steepest-descent paths; (d) the paths approach $\pm\infty$, $\pm\infty + i\pi$, $\pm\infty - i\pi$. All these results are shown on Fig. 6.9.

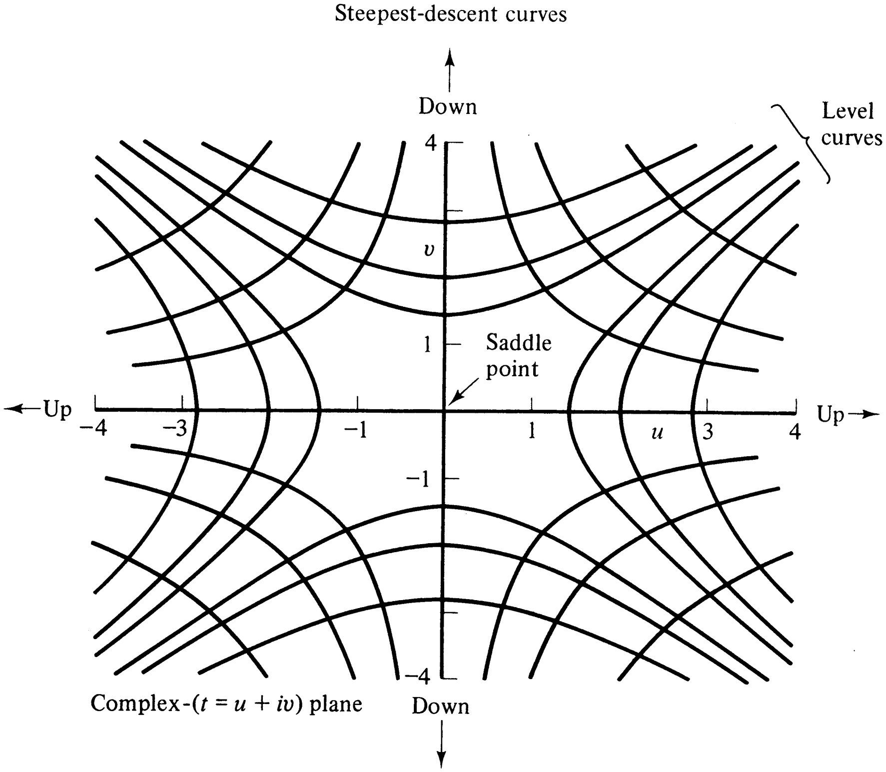

Figure 6.7 Steepest curves of e^{xt^2} near the saddle point at $t = 0$ in the complex-t plane. The steepest curves satisfy $uv =$ constant. The level curves of ϕ satisfy $u^2 - v^2 =$ constant and are orthogonal to the steepest curves.

Example 7 *Steepest curves of* $e^{x(\cosh t - t^2/2)}$ *near the saddle point at* $t = 0$. Here $\rho(t) = \cosh t - t^2/2$. Note that $\rho'(t)$, $\rho''(t)$, and $\rho'''(t)$ all vanish at $t = 0$. The first nonvanishing derivative of $\rho(t)$ at $t = 0$ is $d^4\rho/dt^4$, so we call $t = 0$ a fourth-order saddle point. Eight constant-phase curves meet at $t = 0$. Note that

$$\rho(t) = \cosh u \cos v + (v^2 - u^2)/2 + i(\sinh u \sin v - uv).$$

Thus, constant-phase contours emerging from $t = 0$ satisfy $\psi = \sinh u \sin v - uv = 0$. Solutions to this equation are $u = 0$ (the imaginary axis), $v = 0$ (the real axis), and $(\sinh u)/u = v/\sin v$.

In Prob. 6.62 you are asked to verify the results on Fig. 6.10. Namely, that (*a*) eight steepest paths emerge from $t = 0$, all equally spaced at 45° from each other; (*b*) as t moves away from 0, the paths alternate between steepest-ascent and steepest-descent paths; (*c*) the four steepest-ascent paths lie on the u and v axes; (*d*) the four steepest-descent paths approach $\pm\infty + i\pi$, $\pm\infty - i\pi$.

Steepest-Descent Approximation to Integrals with Saddle Points

We have seen that by shifting the integration contour so that it follows a path of constant phase we can treat an integral of the form in (6.6.1) by Laplace's method.

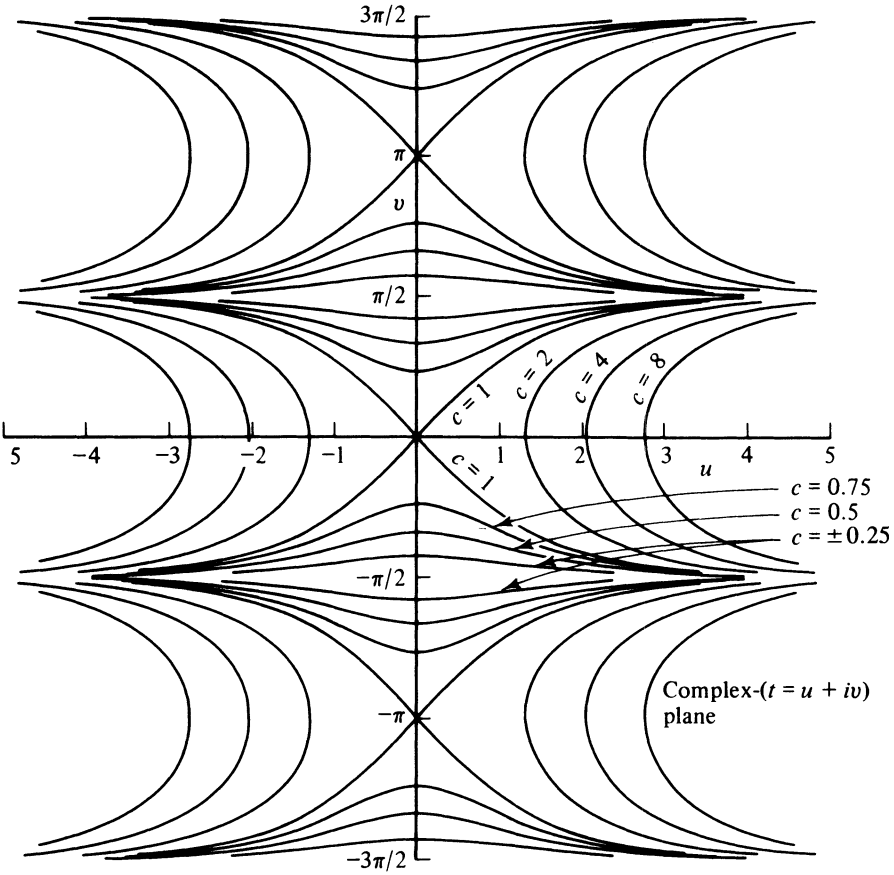

Figure 6.8 Constant-phase (steepest) contours of $\exp(ix \cosh t)$ in the complex-$(t = u + iv)$ plane. Constant-phase contours satisfy $(\cosh u)(\cos v) = c$, where c is a constant. Saddle points lying at $t = 0$ and $t = \pm i\pi$ are shown.

What happens when the constant-phase contour passes through a saddle point? In the following examples we encounter this situation.

Example 8 *Asymptotic expansion of $J_0(x)$ as $x \to +\infty$.* A standard integral representation for $J_0(x)$ [see (6.5.13)] is $J_0(x) = \int_{-\pi/2}^{\pi/2} \cos(x \cos\theta)\, d\theta/\pi$, which can be transformed into

$$J_0(x) = \operatorname{Re} \frac{1}{i\pi} \int_{-i\pi/2}^{i\pi/2} dt\, e^{ix \cosh t} \tag{6.6.18a}$$

by substituting $t = i\theta$.

We can certainly use the method of stationary phase to find the leading behavior of this integral as $x \to +\infty$ (see Prob. 6.54). However, it is better to use the method of steepest descents to find the higher-order corrections to the leading behavior. (Why?)

To apply the method of steepest descents we extend the contour to infinity. Note that the integrals

$$\frac{1}{i\pi} \int_{-\infty - i\pi/2}^{-i\pi/2} dt\, e^{ix \cosh t}, \tag{6.6.18b}$$

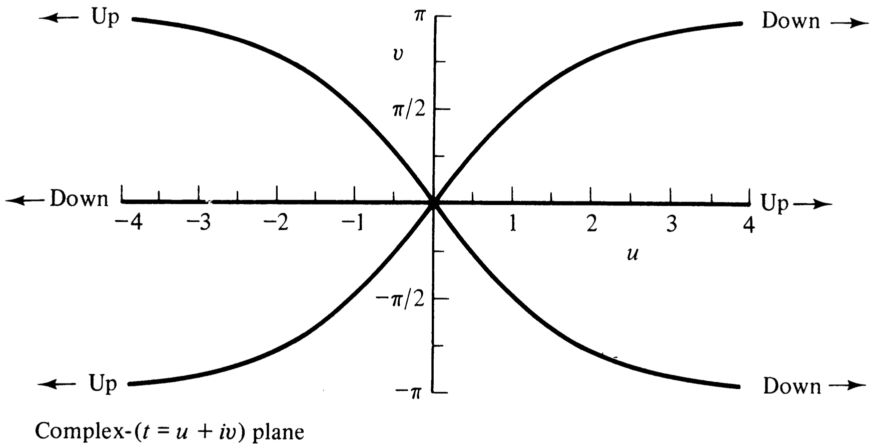

Figure 6.9 Steepest curves of $\exp[x(-t + \sinh t)]$ near the third-order saddle point at $t = 0$. The plot indicates that three steepest-descent curves and three steepest-ascent curves meet at $t = 0$.

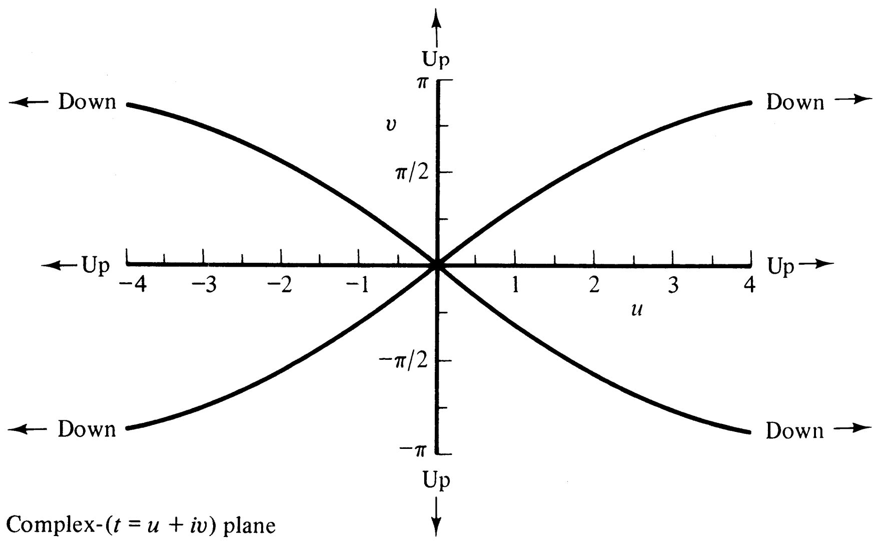

Figure 6.10 Steepest curves of $\exp[x(-\frac{1}{2}t^2 + \cosh t)]$ near the fourth-order saddle point at $t = 0$. The graph shows that four steepest-descent curves and four steepest-ascent curves meet at $t = 0$. In Example 12 the structure of the saddle point is the same as the one in this graph shifted by $i\pi$; the steepest-descent curve used in Example 12 consists of the curves in the third and fourth quadrants of this figure.

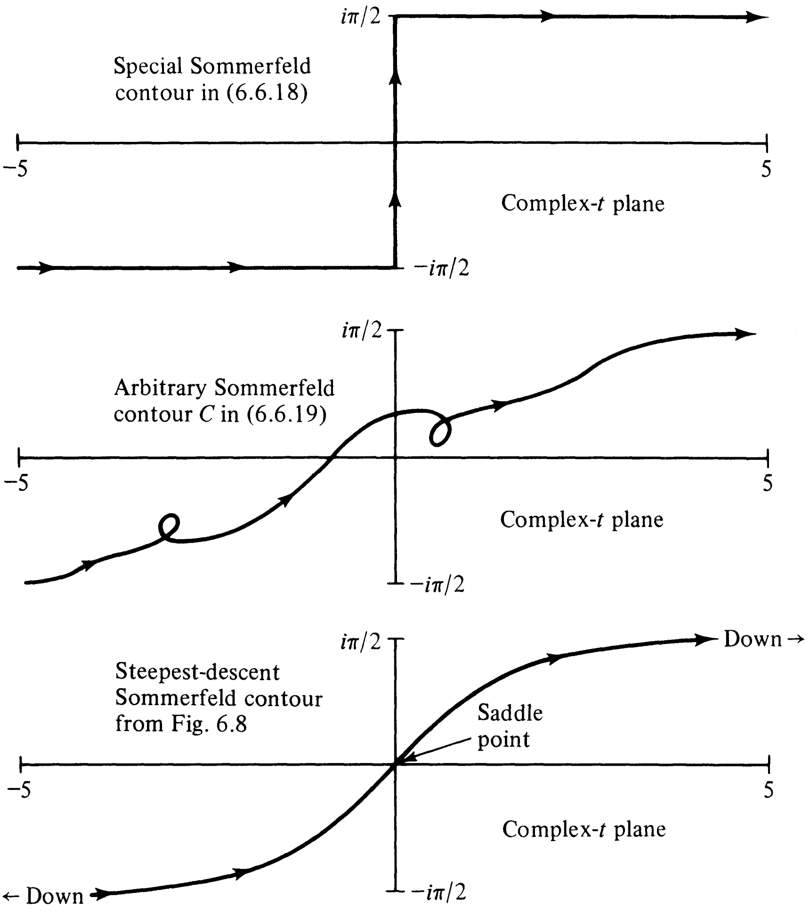

Figure 6.11 To find the asymptotic behavior of $J_0(x)$ as $x \to +\infty$ we first represent $J_0(x)$ as an integral along the special contour in (6.6.18*a*, *b*, *c*). Second, we observe that any Sommerfeld contour C from $-\infty - i\pi/2$ to $+\infty + i\pi/2$ is equally good. Third, to approximate the integral in (6.6.19) we choose that Sommerfeld contour which is also a path of steepest descent through the saddle point at $t = 0$.

where the contour extends along a line parallel to and below the real axis, and

$$\frac{1}{i\pi}\int_{i\pi/2}^{\infty+i\pi/2} dt\, e^{ix\cosh t}, \tag{6.6.18c}$$

where the contour extends along a line parallel to and above the real axis, are convergent and pure imaginary (see Prob. 6.63). Thus, we have constructed the rather fancy representation

$$J_0(x) = \text{Re}\,\frac{1}{i\pi}\int_C dt\, e^{ix\cosh t}, \tag{6.6.19}$$

where C is *any* contour which ranges from $-\infty - i\pi/2$ to $+\infty + i\pi/2$ (see Fig. 6.11). Such a contour is called a Sommerfeld contour.

From here on the steepest-descent analysis is easy because there is a curve of constant phase which ranges from $-\infty - i\pi/2$ to $+\infty + i\pi/2$ (see Fig. 6.8)! We have seen in Example 5 that this curve passes through a saddle point at $t = 0$. The equation for this curve is $\cosh u \cos v = 1$. Note that $|e^{ix \cosh t}|$ *attains its maximum value on the contour at the saddle point at* $t = 0$. Thus, we know from our study of Laplace's method that as $x \to +\infty$ the entire asymptotic expansion is determined by a small neighborhood about $t = 0$.

To find the leading behavior of $J_0(x)$ as $x \to \infty$, we approximate the steepest-descent path in a small neighborhood of $t = 0$ by the straight line $t = (1 + i)s$ (s real) and approximate $\cosh t$ near $s = 0$ by $\cosh t \sim 1 + is^2$ $(s \to 0)$. Thus,

$$J_0(x) \sim \text{Re}\,[(1 + i)/i\pi] \int_{s=-\varepsilon}^{\varepsilon} e^{ix - xs^2}\, ds, \qquad x \to +\infty.$$

Extending the limits of integration to ∞ and evaluating the integral gives

$$J_0(x) \sim \text{Re}\,[(1 + i)/i\pi] e^{ix} \sqrt{\pi/x} = \sqrt{2/\pi x} \cos(x - \pi/4), \qquad x \to +\infty.$$

To find the full asymptotic expansion of $J_0(x)$ as $x \to +\infty$, we use Watson's lemma. It is simplest to parametrize the integration path in terms of $\phi = \text{Re}\,\rho(t)$. We know that along the steepest-descent contour $\rho(t) = i + \phi(t)$, where $\phi(t)$ is real and ranges from $\phi = 0$ at $t = 0$ to $\phi = -\infty$ at $t = \pm(\infty \pm i\pi/2)$. Also, we have $\phi(t) = i \cosh t - i$, so $d\phi = i \sinh t\, dt$. Thus, $dt = d\phi/i\sqrt{-\phi^2 - 2i\phi}$. Substituting this result into (6.6.19) and replacing ϕ by $-\phi$ gives

$$J_0(x) = \text{Re}\, \frac{e^{ix - i\pi/4}}{\pi} \sqrt{2} \int_0^\infty \frac{d\phi}{\sqrt{\phi}} e^{-\phi x} \left(1 + \frac{i\phi}{2}\right)^{-1/2}$$

To apply Watson's lemma, we expand the square root:

$$\left(1 + \frac{i\phi}{2}\right)^{-1/2} = \sum_{n=0}^{\infty} \frac{(-i\phi/2)^n \Gamma(n + \frac{1}{2})}{n!\, \Gamma(\frac{1}{2})}$$

and integrate term by term:

$$J_0(x) \sim \text{Re}\, \frac{e^{ix - i\pi/4}}{\pi^{3/2}} \sqrt{2} \sum_{n=0}^{\infty} \frac{[\Gamma(n + \frac{1}{2})]^2}{n!\, \sqrt{x}} \left(\frac{-i}{2x}\right)^n, \qquad x \to +\infty.$$

Thus, the full asymptotic expansion of $J_0(x)$ is given by

$$J_0(x) = \sqrt{\frac{2}{x\pi}}\, [\alpha(x) \cos(x - \pi/4) + \beta(x) \sin(x - \pi/4)], \qquad (6.6.20)$$

where
$$\alpha(x) \sim \sum_{k=0}^{\infty} \frac{[\Gamma(2k + \frac{1}{2})]^2(-1)^k}{\pi(2k)!\,(2x)^{2k}}, \qquad x \to +\infty,$$

and
$$\beta(x) \sim \sum_{k=0}^{\infty} \frac{[\Gamma(2k + \frac{3}{2})]^2(-1)^k}{\pi(2k + 1)!\,(2x)^{2k+1}}, \qquad x \to +\infty.$$

The trick of adding the contour integrals (6.6.18*b*,*c*) to (6.6.18*a*) to derive (6.6.19) could have been avoided by deforming the contour from $-i\pi/2$ to $i\pi/2$ into three constant-phase contours: C_1: $t = -i\pi/2 + u$ $(-\infty < u \le 0)$; C_2: $\cosh u \cos v = 1$; and C_3: $t = i\pi/2 + u$ $(0 \le u < \infty)$. The contributions from C_1 and C_3 cancel exactly in this problem.

Example 9 *Asymptotic expansion of* $\Gamma(x)$ *as* $x \to +\infty$. In Example 10 of Sec. 6.4 we used Laplace's method to show that

$$\Gamma(x) \sim x^x e^{-x} \sqrt{2\pi/x} \qquad (6.6.21)$$

[see (6.4.39)]. In this example we use the method of steepest descents to rederive this result from a complex-contour integral representation of $\Gamma(x)$ [see Prob. 2.6(f)]:

$$\frac{1}{\Gamma(x)} = \frac{1}{2\pi i}\int_C e^t t^{-x}\, dt, \tag{6.6.22}$$

where C is a contour that begins at $t = -\infty - ia$ ($a > 0$), encircles the branch cut that lies along the negative real axis, and ends up at $-\infty + ib$ ($b > 0$) (see Fig. 6.12). The branch cut is present when x is nonintegral because t^{-x} is a multivalued function. The advantage of (6.6.22) over the integral representation used in Example 10 is that it converges for *all* complex values of x and not just those x for which $\text{Re}\, x > 0$. Nevertheless, in this example we will only investigate the behavior of $\Gamma(x)$ in the limit $x \to +\infty$.

We begin our analysis by making the same substitution that was made in Example 10 of Sec. 6.4; namely, $t = xs$. This substitution converts the integrand from one that has a movable saddle point to one that has a fixed saddle point. (Why?) The resulting integral representation is

$$\frac{1}{\Gamma(x)} = \frac{1}{2\pi i x^{x-1}}\int_C ds\, e^{x(s - \ln s)}. \tag{6.6.23}$$

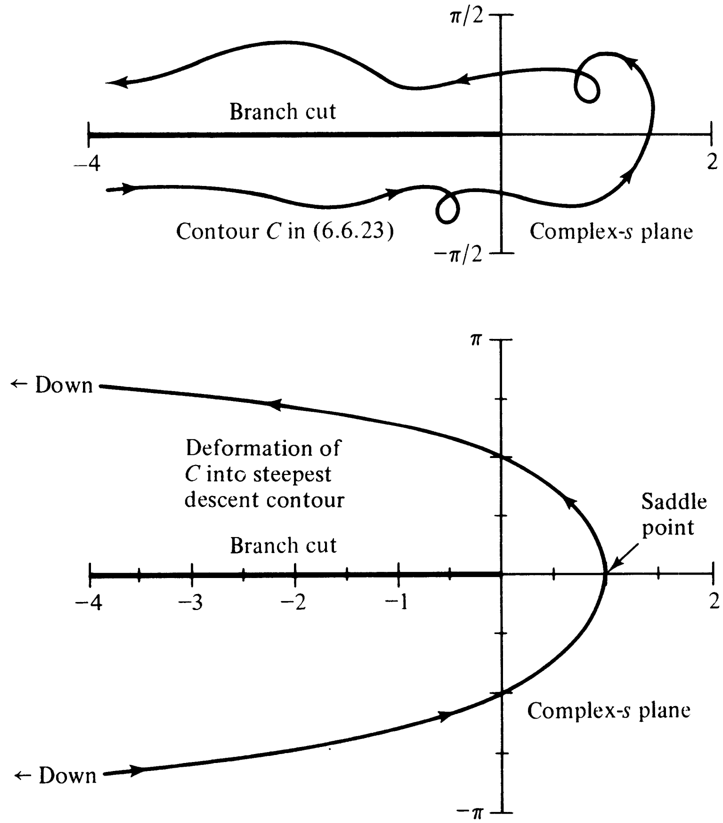

Figure 6.12 To find the asymptotic behavior of $\Gamma(x)$ as $x \to +\infty$, we represent $\Gamma(x)$ as the integral in (6.6.23) along a contour C in the complex-s plane which goes around the branch cut on the negative real axis. Then we distort C into a steepest-descent contour which passes through the saddle point at $s = 1$.

For this integral $\rho(s) = s - \ln s$. Thus, $\rho'(s) = 1 - 1/s$ and $\rho' = 0$ when $s = 1$. So there is a simple (second-order) saddle point at $s = 1$.

To ascertain the structure of the saddle point we let $s = u + iv$ and identify the real and imaginary parts of ρ: $\rho(s) = u - \ln \sqrt{u^2 + v^2} + i(v - \arctan v/u)$. At $s = 1$, $\rho = 1$. Therefore, paths of constant phase (steepest curves) emerging from $s = 1$ must satisfy

$$v - \arctan v/u = 0.$$

There are two solutions to this equation: $v = 0$ and $u = v \cot v$. These two curves are shown on Fig. 6.12. In Prob. 6.64 you are asked to verify that (*a*) the steepest-descent curves are correctly shown on Fig. 6.12; (*b*) as s moves away from $s = 1$, steepest-descent curves emerge from $s = 1$ initially parallel to the Im $s = v$ axis; (*c*) the steepest-descent curves cross the v axis at $\pm i\pi/2$ and approach $s = -\infty \pm i\pi$.

To use the method of steepest descents, we simply shift the contour C so that it is just the steepest-descent contour on Fig. 6.12 which passes through the saddle point at $s = 1$. Let us review why we choose such a contour. In general, we always choose a steepest-descent contour because on such a contour we can apply the techniques of Laplace's method directly to complex integrals. If the steepest-descent contour is finite and does not pass through a saddle point, then the maximum value of $|e^{x\rho}|$ must occur at an endpoint of the contour and we need only perform a local analysis of the integral at this endpoint. However, in the present example the contour has no endpoint and is infinitely long. It is crucial that it pass through a saddle point because $|e^{x\rho}|$ reaches its maximum at the saddle point and decays exponentially as $s \to \infty$ along *both* of the steepest-descent curves. If there were no saddle point, then, although $|e^{x\rho}|$ would decrease in one direction along the contour, it would increase in the other direction and the integral would not even converge!

Now we proceed with the asymptotic expansion of the integral in (6.6.23). We can approximate the steepest-descent contour in the neighborhood of $s = 1$ by the straight line $s = 1 + iv$. This gives the Laplace integral

$$\frac{1}{\Gamma(x)} \sim \frac{1}{2\pi x^{x-1}} \int_{-\varepsilon}^{\varepsilon} dv\; e^{x(1-v^2/2)}, \qquad x \to +\infty,$$

which we evaluate by letting $\varepsilon \to \infty$:

$$\frac{1}{\Gamma(x)} \sim \frac{1}{2\pi x^{x-1}} \frac{e^x}{\sqrt{x}} \sqrt{2\pi}, \qquad x \to +\infty.$$

We thereby recover the result in (6.6.21).

Example 10 *Steepest-descents approximation of a real integral where Laplace's method fails.* In this example we consider the real integral

$$I(x) = \int_0^1 dt\, e^{-4xt^2} \cos(5xt - xt^3) \tag{6.6.24}$$

in the limit $x \to +\infty$. This integral is *not* a Laplace integral because the argument of the cosine contains x. Nonetheless, one might think that one could use the ideas of Laplace's method to approximate the integral. To wit, one would argue that as $x \to +\infty$, the contribution to the integral is localized about $x = 0$. Thus, a very naive approach is simply to replace the argument of the cosine by 0. If this reasoning were correct, then we would conclude that

$$I(x) \sim \int_0^1 dt\, e^{-4xt^2} \sim \sqrt{\frac{\pi}{16x}}, \qquad x \to +\infty. \qquad \text{(WRONG)}$$

This result is clearly incorrect because e^{-xt^2} does not become exponentially small until t is larger than $1/\sqrt{x}$. Thus, when $t \sim 1/\sqrt{x}$ $(x \to +\infty)$, the argument of the cosine is *not* small. In particular, the term $5xt$ is large and the cosine oscillates rapidly. This suggests that there is destructive interference and that $I(x)$ decays much more rapidly than $\sqrt{\pi/16x}$ as $x \to +\infty$.

Can we correct this approach by including the $5xt$ term but neglecting the xt^3 term? After all, when t lies in the range from 0 to $1/\sqrt{x}$, the term $xt^3 \to 0$ as $x \to +\infty$. Thus, xt^3 does not even shift the phase of the cosine more than a fraction of a cycle. If we were to include just the $5xt$ term, we would obtain

$$\begin{aligned}
I(x) &\sim \int_0^1 dt\, e^{-4xt^2} \cos(5xt), \qquad x \to +\infty,\\
&\sim \int_0^\infty dt\, e^{-4xt^2} \cos(5xt), \qquad x \to +\infty,\\
&= \tfrac{1}{2} \int_{-\infty}^{\infty} dt\, e^{-4xt^2 + 5ixt}\\
&= \tfrac{1}{2} \int_{-\infty}^{\infty} dt\, e^{-x(2t - 5i/4)^2 - 25x/16}\\
&= \tfrac{1}{4}\sqrt{\pi/x}\, e^{-25x/16}, \qquad x \to +\infty. \qquad \text{(WRONG)}
\end{aligned}$$

Although this result is exponentially smaller than the previous wrong result, it is also wrong! It is incorrect to neglect the xt^3 term (see Prob. 6.65).

But if we cannot neglect even the xt^3 term, then how can we make any approximation at all? It should not be necessary to do the integral exactly to find its asymptotic behavior!

The correct approach is to use the method of steepest descents to approximate the integral at a saddle point in the complex plane. To prepare for this analysis we rewrite the integral in the following convenient form:

$$\begin{aligned}
I(x) &= \tfrac{1}{2} \int_{-1}^{1} dt\, e^{-4xt^2 + 5ixt - ixt^3}\\
&= \tfrac{1}{2} e^{-2x} \int_{-1}^{1} dt\, e^{x\rho(t)}, \qquad (6.6.25)
\end{aligned}$$

where
$$\rho(t) = -(t-i)^2 - i(t-i)^3. \qquad (6.6.26)$$

Our objective now is to find steepest-descent (constant-phase) contours that emerge from $t = 1$ and $t = -1$, to distort the original contour of integration t: $-1 \to 1$ into these contours, and then to use Laplace's method. To find these contours we substitute $t = u + iv$ and identify the real and imaginary parts of ρ:

$$\begin{aligned}
\rho(t) &= \phi + i\psi\\
&= -v^3 + 4v^2 - 5v + 3u^2 v - 4u^2 + 2 + i(3uv^2 - 8uv + 5u - u^3). \qquad (6.6.27)
\end{aligned}$$

Note that the phase of $\psi = \operatorname{Im} \rho$ at $t = 1$ and at $t = -1$ is different: $\operatorname{Im} \rho(-1) = -4$, $\operatorname{Im} \rho(1) = 4$. Thus, there is no single constant-phase contour which connects $t = -1$ to $t = 1$.

Our method is similar to that used in Examples 1 and 2. We follow steepest-descent contours C_1 and C_2 from $t = -1$ and from $t = 1$ out to ∞. Next, we join these two contours at ∞ by a third contour C_3 which is also a path of constant phase. C_3 must pass through a saddle point because its endpoints lie at ∞; otherwise, the integral along C_3 will not converge (see the discussion in Example 9).

There are two saddle points in the complex plane because $\rho'(t) = -2(t - i) - 3i(t - i)^2 = 0$ has two roots, $t = i$ and $t = 5i/3$. The contour C_3 happens to pass through the saddle point at i. On Fig. 6.13 we plot the three constant-phase contours C_1, C_2, and C_3. It is clear that the original contour C can be deformed into $C_1 + C_2 + C_3$. (In Prob. 6.66 you are to verify the results on Fig. 6.13.)

The asymptotic behavior of $I(x)$ as $x \to +\infty$ is determined by just three points on the contour $C_1 + C_2 + C_3$: the endpoints of C_1 and C_2 at $t = -1$ and at $t = +1$ and the saddle point at i. However, the contributions to $I(x)$ at $t = \pm 1$ are exponentially small compared with

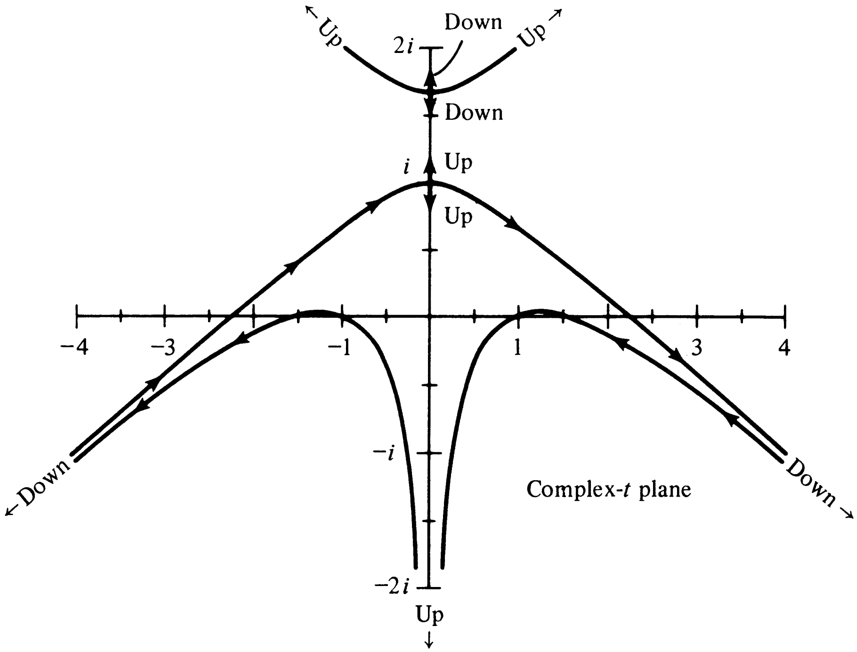

Figure 6.13 To approximate the integral in (6.6.25) by the method of steepest descents we deform the original contour connecting the points $t = -1$ to $t = 1$ along the real axis into the three distinct steepest-descent contours above, one of which passes through a saddle point at $t = i$. Steepest-ascent and -descent curves near a second saddle point at $t = 5i/3$ and steepest-ascent curves going from 1 and -1 to $-i\infty$ are also shown, but these curves play no role in the calculation.

that at $t = i$ (see Prob. 6.67). Near $t = i$ we can approximate the contour C_3 by the straight line $t = i + u$ and $\rho(t)$ by $\rho(t) \sim -u^2$ $(u \to 0)$. Thus,

$$I(x) \sim \tfrac{1}{2}e^{-2x} \int_{-\varepsilon}^{\varepsilon} e^{-xu^2}\, du, \qquad x \to +\infty,$$
$$\sim \tfrac{1}{2}e^{-2x}\sqrt{\pi/x}, \qquad x \to +\infty. \tag{6.6.28}$$

This, finally, is the correct asymptotic behavior of $I(x)$! This splendid example certainly shows the subtlety of asymptotic analysis and the power of the method of steepest descents.

Example 11 *Steepest-descents analysis with a third-order saddle point.* In Example 5 of Sec. 6.5 and Prob. 6.55 we showed that

$$J_x(x) \sim \frac{1}{\pi} 2^{-2/3} 3^{-1/6} \Gamma\left(\frac{1}{3}\right) x^{-1/3}, \qquad x \to +\infty. \tag{6.6.29}$$

Here we rederive (6.6.29) using the method of steepest descents.

We begin with the complex-contour integral representation for $J_\nu(x)$:

$$J_\nu(x) = \frac{1}{2\pi i} \int_C dt\, e^{x \sinh t - \nu t}, \tag{6.6.30}$$

where C is a Sommerfeld contour that begins at $+\infty - i\pi$ and ends at $+\infty + i\pi$. Setting $\nu = x$ gives

$$J_x(x) = \frac{1}{2\pi i} \int_C dt\, e^{x(\sinh t - t)}. \tag{6.6.31}$$

For this integral $\rho(t) = \sinh t - t$ has a third-order saddle point at $t = 0$.

We have already analyzed the steepest curves of this $\rho(t)$ in Example 6 (see Fig. 6.9). Note that we can deform the contour C so that it follows steepest-descent paths to and from the saddle point at $t = 0$.

The contribution to $J_x(x)$ as $x \to +\infty$ comes entirely from the neighborhood of the saddle point. In the vicinity of the saddle point we can approximate the contours approaching and leaving $t = 0$ by the straight lines $t = re^{-i\pi/3}$ and $t = re^{i\pi/3}$. Substituting into (6.6.30) gives

$$J_x(x) \sim \frac{1}{2\pi i}\int_{r=\varepsilon}^{0} dr\, e^{-i\pi/3} e^{-xr^3/6} + \frac{1}{2\pi i}\int_{r=0}^{\varepsilon} dr\, e^{i\pi/3} e^{-xr^3/6}, \qquad x \to +\infty.$$

To evaluate these integrals, we replace ε by ∞:

$$J_x(x) \sim \frac{e^{i\pi/3} - e^{-i\pi/3}}{2\pi i}\int_{r=0}^{\infty} dr\, e^{-xr^3/6}, \qquad x \to +\infty,$$

$$= \frac{\sin(\pi/3)}{\pi}(6/x)^{1/3}\int_0^{\infty} e^{-r^3}\, dr, \qquad x \to +\infty.$$

But $\int_0^\infty e^{-r^3}\, dr = \frac{1}{3}\int_0^\infty e^{-s} s^{-2/3}\, ds = \frac{1}{3}\Gamma(\frac{1}{3})$. Thus,

$$J_x(x) \sim \frac{\sin(\pi/3)}{3\pi}\left(\frac{6}{x}\right)^{1/3}\Gamma\left(\frac{1}{3}\right),$$

which reproduces the result in (6.6.29).

Example 12 *Steepest-descents analysis with a fourth-order saddle point.* What is the leading asymptotic behavior of the real integral

$$I(x) = \int_0^\infty dt\, \cos(x\pi t)\, e^{-x(\cosh t + t^2/2)} \tag{6.6.32}$$

as $x \to +\infty$? To analyze $I(x)$ we first rewrite the integral as

$$I(x) = \tfrac{1}{2}\int_{-\infty}^{\infty} e^{x(i\pi t - \cosh t - t^2/2)}$$

$$= \tfrac{1}{2}e^{-x\pi^2/2}\int_{-\infty}^{\infty} dt\, e^{x[\cosh(t - i\pi) - (t - i\pi)^2/2]}. \tag{6.6.33}$$

For this integral $\rho(t) = \cosh(t - i\pi) - (t - i\pi)^2/2$ has a fourth-order saddle point at $t = i\pi$ (see Example 7). The steepest-descent contours from this saddle point are drawn in Fig. 6.10 (the saddle point in Fig. 6.10 is shifted downward by π).

To approximate $I(x)$ we shift the original integration path t: $-\infty \to +\infty$ from the real axis into the complex plane so that it follows a steepest-descent curve passing through the saddle point. The asymptotic behavior of $I(x)$ is completely determined by the contribution from the saddle point. In the neighborhood of the saddle point at $i\pi$, we can approximate the steepest-descent contour by the straight lines $t = i\pi + re^{i\pi/4}$ to the left of the saddle point and $t = i\pi + re^{-i\pi/4}$ to the right of the saddle point. In terms of r, (6.6.33) becomes

$$I(x) \sim \tfrac{1}{2}e^{-x\pi^2/2}\left[\int_{-\varepsilon}^{0} e^{i\pi/4}\, dr\, e^{x(1 - r^4/24)} + \int_0^{\varepsilon} e^{-i\pi/4}\, dr\, e^{x(1 - r^4/24)}\right], \qquad x \to +\infty,$$

$$= e^{-x\pi^2/2 + x}\cos(\pi/4)\int_0^\infty dr\, e^{-xr^4/24}, \qquad x \to +\infty.$$

But $\int_0^\infty dr\, e^{-r^4} = \frac{1}{4}\int_0^\infty dr\, r^{-3/4}e^{-r} = \frac{1}{4}\Gamma(\frac{1}{4})$. Thus, we obtain the final result that

$$I(x) \sim \tfrac{1}{4}e^{x(1 - \pi^2/2)}(6/x)^{1/4}\Gamma(\tfrac{1}{4}). \tag{6.6.34}$$

This result could not have been obtained by performing a Laplace-like analysis of the real integral in (6.6.32). Suppose, for example, we argue that as $x \to +\infty$ the contribution to (6.6.32) comes entirely from the neighborhood of the origin $t = 0$. Then it would seem valid to replace $\cosh t$ by $1 + t^2/2$, the first two terms in its Taylor series. If we do this, we obtain an integral which we can evaluate exactly:

$$\begin{aligned}\int_0^\infty dt \cos(x\pi t)\, e^{-x(1+t^2)} &= \tfrac{1}{2}\int_{-\infty}^\infty dt\, e^{ix\pi t - x(1+t^2)}\\ &= \tfrac{1}{2}\int_{-\infty}^\infty dt\, e^{-x(t-i\pi/2)^2} e^{-x(1+\pi^2/4)}\\ &= \tfrac{1}{2}\sqrt{\pi/x}\, e^{-x(1+\pi^2/4)}.\end{aligned}$$

But this does not agree with (6.6.34) and is therefore *not* the asymptotic behavior of $I(x)$ as $x \to +\infty$! What is wrong with this argument? (See Prob. 6.68.)

Steepest Descents for Complex x and the Stokes Phenomenon

Until now, x in (6.6.1) has been treated as a large *real* parameter. However, the method of steepest descents can be used to treat problems where x is complex. As we have already seen in Secs. 3.7 and 3.8, an asymptotic relation is valid as $x \to \infty$ in a wedge-shaped region of the complex-x plane. At the edge of the wedge, the asymptotic relation ceases to be valid and must be replaced by another asymptotic relation. This change from one asymptotic relation to another is called the Stokes phenomenon.

The Stokes phenomenon usually surfaces in the method of steepest descents in a relatively simple way. For example, as x rotates in the complex plane, the structure of steepest-descent paths can change abruptly. When this happens, the asymptotic behavior of the integral changes accordingly. The integral representation of Ai (x) behaves in this manner (see Prob. 6.75). The Stokes phenomenon can also appear when the contribution from an endpoint of the contour suddenly becomes subdominant relative to the contribution from a saddle point (or vice versa). We consider this case in the next example.

Example 13 *Reexamination of Example 10 for complex x.* In this example we explain how the Stokes phenomenon arises in the integral (6.6.25). It is essential that the reader master Example 10 before reading further.

The integral $I(x)$ in (6.6.25) exhibits the Stokes phenomenon at $\arg x = \pm \arctan \frac{1}{2} \doteq 26.57°$ and at $\pm\pi$. When $|\arg x| < \arctan \frac{1}{2}$, the contribution to $I(x)$ from the saddle point at $t = i$ dominates the endpoint contributions. As in (6.6.28), this gives

$$I(x) \sim \tfrac{1}{2}e^{-2x}\sqrt{\pi/x}, \qquad x \to \infty,\ |\arg x| < \arctan \tfrac{1}{2}. \tag{6.6.35}$$

When $\arctan \frac{1}{2} < \arg x < \pi$, the endpoint contribution from $t = -1$ dominates. We obtain (see Prob. 6.69)

$$I(x) \sim \frac{i-4}{68x} e^{-4(i+1)x}, \qquad x \to \infty,\ \arctan \tfrac{1}{2} < \arg x < \pi. \tag{6.6.36}$$

When $-\pi < \arg x < -\arctan \frac{1}{2}$, the endpoint contribution from $t = 1$ dominates, giving

$$I(x) \sim -\frac{i+4}{68x} e^{4(i-1)x}, \qquad x \to \infty,\ -\pi < \arg x < -\arctan \tfrac{1}{2}. \tag{6.6.37}$$

It is interesting to see what happens to the steepest-descent contours as x is rotated into the complex-x plane. We have plotted the steepest-descent contours for $I(x)$ for $\arg x = 0°, 30°, 75°$, and $135°$ in Figs. 6.13 to 6.16. Observe that as $\arg x$ increases from $0°$ to $75°$, the contours through the endpoints at $t = \pm 1$ and the saddle point at $t = i$ tilt and distort slightly. Note that the asymptotes of these contours at ∞ rotate by $-(\arg x)/3$ as $\arg x$ increases. This is so because

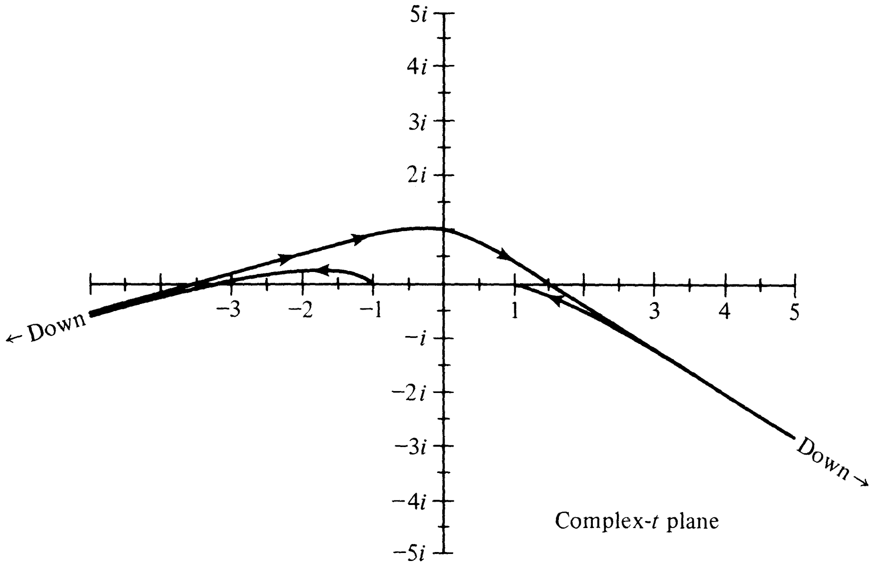

Figure 6.14 Steepest-descent path for $I(x)$ in (6.6.25) when $\arg x = 30°$. (See Fig. 6.13.)

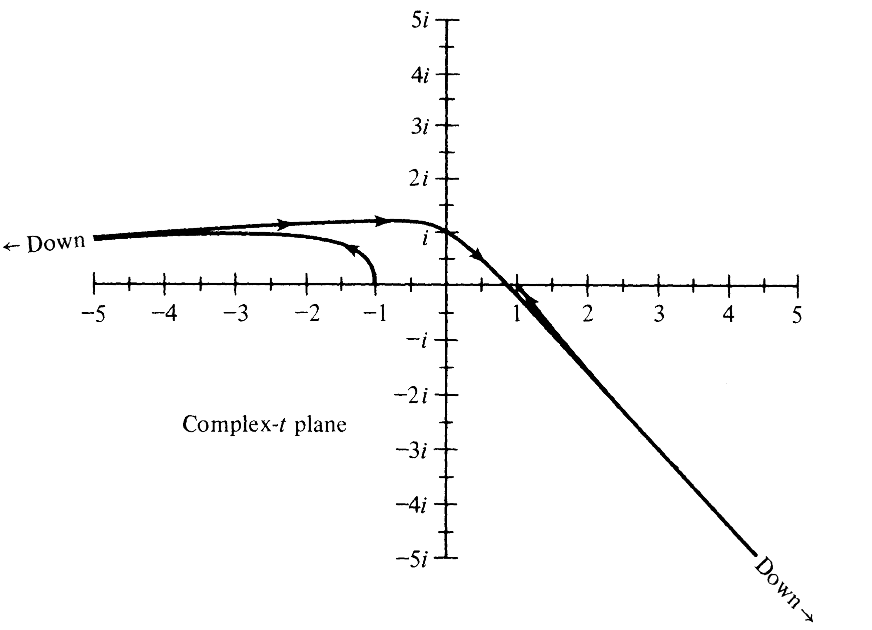

Figure 6.15 Steepest-descent path for $I(x)$ in (6.6.25) when $\arg x = 75°$.

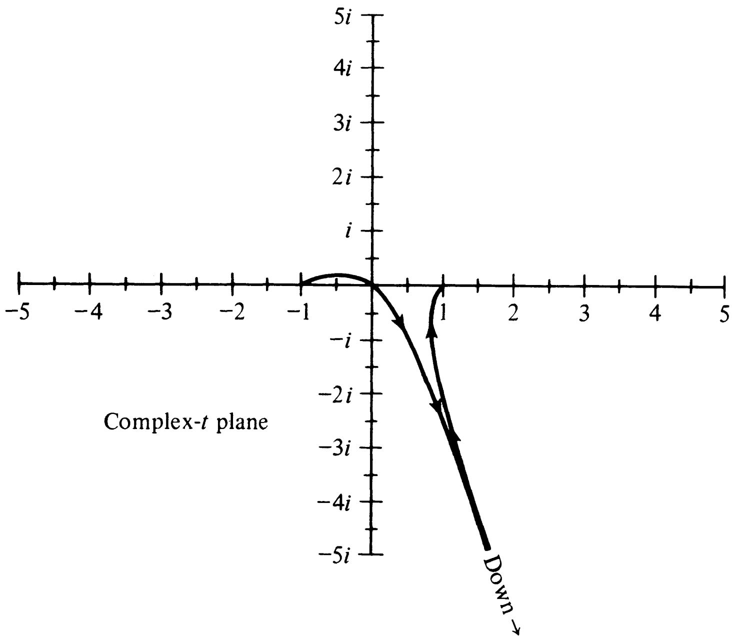

Figure 6.16 Steepest-descent path for $I(x)$ in (6.6.25) when $\arg x = 135°$. Note that the steepest-descent path no longer passes through the saddle point at $t = i$ as it does in Figs. 6.13 to 6.15.

Im $[x\rho(t)]$ must be constant at $t = \infty$. The constancy of Im $[x\rho(t)]$ on the steepest-descent contours implies that the endpoint contours passing through $t = \pm 1$ rotate by $-\arg x$ near $t = \pm 1$ and that the contour through $t = i$ rotates by $-(\arg x)/2$ near $t = i$. There is *no* abrupt or discontinuous change in the configuration of the steepest-descent contours as $\arg x$ increases past arc tan $\frac{1}{2}$. In this example the Stokes phenomenon is not associated with any discontinuity in the structure of the steepest-descent path. It occurs because the contribution from the saddle point becomes subdominant with respect to the contribution from the endpoint as $\arg x$ increases past arc tan $\frac{1}{2}$.

When $\arg x$ reaches $\pi - \text{arc tan } 2 \doteqdot 116.57°$, there is a discontinuous change in the steepest-descent path for $I(x)$ (see Prob. 6.69). As illustrated in Fig. 6.16, when $\arg x = 135°$, the steepest-descent contour no longer passes through the saddle point at $t = i$. When $\arg x > 116.57°$, the steepest-descent contours from $t = \pm 1$ meet at ∞, so it is no longer necessary to join them by a constant-phase contour passing through the saddle point at i. The abrupt disappearance of the saddle-point contour from the steepest-descent path when $\arg x$ increases beyond $116.57°$ does not affect the asymptotic behavior of $I(x)$ because the saddle-point contribution from $t = i$ is subdominant when arc tan $\frac{1}{2} < |\arg x| < \pi$.

(I) 6.7 ASYMPTOTIC EVALUATION OF SUMS

In this section we discuss methods for finding the asymptotic behavior of sums which depend on a large parameter x. We consider four methods in all: truncating the sum after a finite number of terms, approximating the sum by a Riemann integral, Laplace's method for sums, and the Euler-Maclaurin sum formula. The first method is very elementary and rarely applicable, but for completeness we illustrate it in the following brief example:

Example 1 *Behavior of* $\sum_{n=0}^{\infty} e^{-n^2x^2}$ *as* $x \to +\infty$. The behavior of this convergent series is easy to find because the sum of the terms with $n = 1, 2, 3, \ldots$ is clearly subdominant with respect to the first term as $x \to +\infty$. Thus, $1 + e^{-x^2} + e^{-4x^2} + \cdots \sim 1$ $(x \to +\infty)$.

Approximation of Sums by Riemann Integrals

The Riemann integral $I = \int_a^b f(t)\, dt$ is defined as the limit of the Riemann sum,

$$I = \lim_{N \to \infty} \sum_{n=0}^{N-1} f(\bar{t}_n)(t_{n+1} - t_n),$$

where $f(t)$ is continuous, $\bar{t}_n$ is any point in the interval $t_n \le \bar{t}_n \le t_{n+1}$, and $t_n = a + n(b - a)/N$. In the next two examples we show how to use this formula to find the leading behavior of sums.

Example 2 *Behavior of* $S(x) = \sum_{n=0}^{\infty} 1/(n^2 + x^2)$ *as* $x \to +\infty$. Note that each term in this series decays to 0 like x^{-2} as $x \to +\infty$ and that this series converges. One might therefore be tempted to conclude that $S(x)$ decays to 0 like x^{-2}. However, the correct behavior is

$$S(x) \sim \frac{\pi}{2x}, \qquad x \to +\infty. \tag{6.7.1}$$

The origin of this surprising result is that as x increases, more terms contribute significantly to the leading behavior of $S(x)$. Roughly speaking, x terms, each of size x^{-2}, contribute to $S(x)$ causing $S(x)$ to decay like x^{-1} as $x \to +\infty$.

We can establish (6.7.1) by converting the series to a Riemann sum. We simply multiply by x, rewrite the series as

$$xS(x) = \sum_{n=0}^{\infty} \frac{1}{1 + (n/x)^2} \frac{1}{x},$$

and observe that as $x \to +\infty$ the series becomes a Riemann sum for the convergent integral

$$\int_0^{\infty} \frac{1}{1 + t^2}\, dt = \frac{\pi}{2}. \tag{6.7.2}$$

Specifically, a Riemann sum for this integral is obtained by choosing the discrete points $t_n = \bar{t}_n = n/x$ $(n = 0, 1, 2, \ldots)$. This gives

$$\sum_{n=0}^{\infty} \frac{1}{1 + t_n^2} (t_{n+1} - t_n).$$

This Riemann sum converges to the integral in (6.7.2) as the interval $t_{n+1} - t_n = x^{-1} \to 0$. Thus, we obtain $\lim_{x \to \infty} xS(x) = \pi/2$, which is just (6.7.1). For a higher-order approximation to $S(x)$ see Prob. 6.91.

Example 3 *Behavior of* $\sum_{1 \le n < x} n^\alpha$ *as* $x \to +\infty$. We will show that the leading behavior of

$$S(x) = \sum_{1 \le n < x} n^\alpha \tag{6.7.3}$$

for large x is

$$S(x) \sim \begin{cases} \dfrac{x^{\alpha+1}}{\alpha+1}, & x \to +\infty;\ \alpha > -1, \qquad (6.7.4a) \\ \ln x, & x \to +\infty;\ \alpha = -1, \qquad (6.7.4b) \\ \zeta(-\alpha), & x \to +\infty;\ \alpha < -1, \qquad (6.7.4c) \end{cases}$$

where $\zeta(p)$ is the Riemann zeta function defined by the convergent series

$$\zeta(p) = \sum_{n=1}^{\infty} n^{-p}, \qquad \mathrm{Re}\ p > 1. \tag{6.7.5}$$

To verify (6.7.4a) we reason as in Example 2. We observe that as $x \to \infty$, $x^{-\alpha-1}S(x) = \sum_{1 \le n < x} (n/x)^{\alpha}/x$ is a Riemann sum for the integral $\int_0^1 t^{\alpha}\, dt = 1/(\alpha+1)$, where we have approximated the integral as a Riemann sum on the discrete points $t_n = n/x$ for $n = 1, 2, \ldots, [x]$ ($[x]$ stands for the largest integer less than x). This justifies (6.7.4a). The proof of (6.7.4b) was given in Prob. 5.8. Finally, when $\alpha < -1$, the sum $S(x)$ converges as $x \to +\infty$ to $S(+\infty) = \zeta(-\alpha)$. This proves (6.7.4$c$).

Laplace's Method for Sums

In the next example we show how to obtain the leading behavior of a sum by following the philosophy of Laplace's method which was introduced in Sec. 6.4.

Example 4 *Leading behavior of* $\sum_{n=0}^{\infty} x^n (n!)^{-k}$ *as* $x \to +\infty$. We have already encountered this problem in Example 1 of Sec. 3.5 and Prob. 3.42. There we solved the problem by finding a differential equation satisfied by the sum and then determining the asymptotic behavior of its solutions. However, you will recall that because the differential equation was linear, local analysis could not determine the overall multiplicative constant in the asymptotic behavior of the sum. In this example we find the leading behavior of the sum directly and encounter no such ambiguities.

To find the leading behavior of this sum by Laplace's method, we must identify the largest term in the series. Let us examine the ratio of the $(n-1)$th term to the nth term in the series:

$$x^{n-1}[(n-1)!]^{-k}/x^n(n!)^{-k} = n^k/x.$$

Note that this ratio is less than 1 if $n < x^{1/k}$ and greater than 1 if $n > x^{1/k}$. Therefore, the terms in the series increase as n increases until n reaches $[x^{1/k}]$. This is the largest term in the series. The remaining terms decrease with increasing n.

As $x \to +\infty$, the terms in the series peak sharply near the $n = [x^{1/k}]$ term. Therefore, using the principles of Laplace's method, we expect that for any $\varepsilon > 0$

$$S(x) = \sum_{n=0}^{\infty} x^n (n!)^{-k} \sim \sum_{n=[x^{1/k}(1-\varepsilon)]}^{[x^{1/k}(1+\varepsilon)]} \frac{x^n}{(n!)^k}, \tag{6.7.6}$$

with errors that are subdominant with respect to every power of $1/x$ as $x \to +\infty$.

Next we use the Stirling formula (see Sec. 5.4) to approximate each of the terms retained in (6.7.6). If $n = x^{1/k} + t$, where t is small compared with $x^{1/k}$, then by Stirling's formula we have

$$\begin{aligned} n! &\sim (n/e)^n \sqrt{2\pi n}, && n \to +\infty, \\ &= (x^{1/k} + t)^n e^{-n} \sqrt{2\pi n} \\ &= x^{n/k} e^{n \ln(1 + tx^{-1/k})} e^{-n} \sqrt{2\pi n} \\ &\sim x^{n/k} e^{-x^{1/k}} e^{t^2 x^{-1/k}/2} \sqrt{2\pi}\, x^{1/2k}, && x \to +\infty;\ t^3 \ll x^{2/k}, \end{aligned}$$

where we have truncated the Taylor expansion of $\ln(1 + tx^{-1/k})$ after quadratic terms in t. Therefore,

$$x^n(n!)^{-k} \sim e^{kx^{1/k}} e^{-t^2kx^{-1/k}/2}(2\pi)^{-k/2}x^{-1/2}, \qquad x \to +\infty; n = x^{1/k} + t, t^3 \ll x^{2/k}.$$

Finally, we substitute this result into the sum on the right side of (6.7.6) and extend the region of summation to ∞, just as we do with integrals. We obtain

$$S(x) \sim \sum_{t=-\infty}^{\infty} e^{kx^{1/k}} e^{-t^2kx^{-1/k}/2}(2\pi)^{-k/2}x^{-1/2}, \qquad x \to +\infty. \tag{6.7.7}$$

The leading behavior of the sum on the right side of (6.7.7) can be obtained as in Examples 2 and 3 by observing that the sum is a Riemann sum for the integral $\int_{-\infty}^{\infty} e^{kx^{1/k}} e^{-t^2kx^{-1/k}/2}(2\pi)^{-k/2}x^{-1/2}\,dt$. Hence, we evaluate this integral and find that the leading behavior of $S(x)$ as $x \to +\infty$ is

$$S(x) \sim (2\pi)^{(1-k)/2}k^{-1/2}x^{(1-k)/2k}e^{kx^{1/k}}, \qquad x \to +\infty. \tag{6.7.8}$$

Can you find the next-higher-order correction to this result? (See Prob. 6.89.)

Euler-Maclaurin Sum Formula

The Euler-Maclaurin sum formula is an elegant and general expression for the asymptotic expansion of sums of the form

$$F(n) = \sum_{k=0}^{n} f(k) \tag{6.7.9}$$

as $n \to \infty$. For example, this formula can be used to find the full asymptotic expansion of sums like $\sum_{k=1}^{n} k^\alpha$, $\sum_{k=1}^{n} \ln k$ as $n \to \infty$ and even sums like $\sum_{k=0}^{\infty} 1/(k^2 + x^2)$, $\sum_{k=0}^{\infty} (k + x)^{-\alpha}$ as $x \to +\infty$.

The Euler-Maclaurin sum formula involves Bernoulli polynomials $B_n(x)$ which are defined as the nth derivative of $te^{xt}/(e^t - 1)$ evaluated at $t = 0$. The first few Bernoulli polynomials are $B_0(x) = 1$, $B_1(x) = x - \frac{1}{2}$, $B_2(x) = x^2 - x + \frac{1}{6}$. $B_n(x)$ is a polynomial of degree n. Some of the properties of these polynomials are studied in Prob. 6.88. The Bernoulli numbers B_n are defined in terms of Bernoulli polynomials as $B_n = B_n(0)$. The first few Bernoulli numbers are $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_3 = 0$, $B_4 = -\frac{1}{30}$. Some of the properties of Bernoulli numbers are examined in Prob. 5.18. There it is shown that $B_{2n+1} = 0$ for $n \geq 1$.

In terms of B_n and $B_n(x)$, the full asymptotic expansion of $F(n)$ in (6.7.9) is

$$F(n) \sim \frac{1}{2}f(n) + \int_0^n f(t)\,dt + C + \sum_{j=1}^{\infty} (-1)^{j+1}\frac{B_{j+1}}{(j+1)!}f^{(j)}(n), \qquad n \to \infty, \tag{6.7.10}$$

where C is a constant given by the rather messy formula

$$C = \lim_{m\to\infty}\left[\sum_{j=1}^{m}\frac{(-1)^j B_{j+1}}{(j+1)!}f^{(j)}(0) + \frac{1}{2}f(0) + \frac{(-1)^m}{(m+1)!}\int_0^\infty B_{m+1}(t - [t])f^{(m+1)}(t)\,dt\right] \tag{6.7.11}$$

Table 6.2 Comparison between the exact sum of the first n reciprocal integers and the one-, two-, and three-term asymptotic approximations to this sum given by the Euler-Maclaurin sum formula in (6.7.12)

n	$\sum_{k=1}^{n} \frac{1}{k}$	$\ln n$	$\ln n + \gamma$	$\ln n + \gamma + \frac{1}{2n}$
10	2.928 968 25	2.302 585 09	2.879 800 76	2.929 800 76
50	4.499 205 34	3.912 023 01	4.489 238 67	4.499 238 67
100	5.187 377 52	4.605 170 19	5.182 385 85	5.187 385 85
500	6.792 823 43	6.214 608 10	6.791 823 76	6.792 823 76
1,000	7.485 470 86	6.907 755 28	7.484 970 94	7.485 470 94
5,000	9.094 508 85	8.517 193 19	9.094 408 86	9.094 508 86

and $[t]$ is the largest integer less than t. The proof of these formulas is left to Prob. 6.88.

Example 5 *Full asymptotic behavior of* $\sum_{k=1}^{n} 1/k$ *as* $n \to \infty$. By (6.7.10) with $f(t) = 1/(t+1)$ and n replaced by $n-1$,

$$\sum_{k=1}^{n} \frac{1}{k} \sim \ln n + C + \frac{1}{2n} - \frac{B_2}{2n^2} - \frac{B_4}{4n^4} - \frac{B_6}{6n^6} - \cdots, \qquad n \to \infty. \tag{6.7.12}$$

The constant C, as given in (6.7.11), is evaluated in Prob. 6.90; the result is $C = \gamma \doteqdot 0.5772$. In Table 6.2 we test the accuracy of the expansion in (6.7.12).

Example 6 *Full asymptotic behavior of* $\ln (n!)$ *as* $n \to \infty$. Using $f(t) = \ln (1+t)$ and n replaced by $n-1$, (6.7.10) gives

$$\ln (n!) \sim \left(n + \frac{1}{2}\right) \ln n - n + C + \frac{B_2}{1 \cdot 2n} + \frac{B_4}{3 \cdot 4n^3} + \frac{B_6}{5 \cdot 6n^5} + \cdots, \qquad n \to \infty. \tag{6.7.13}$$

Here $C = \lim_{n \to \infty} [\ln (n!) - (n + \frac{1}{2}) \ln n + n] = \frac{1}{2} \ln 2\pi$, by Stirling's formula.
For more examples, see the problems for Sec. 6.7.

PROBLEMS FOR CHAPTER 6

Sections 6.1 and 6.2

(E) **6.1** Show that the integral in (6.1.3) satisfies the differential equation (6.1.1).
Clue: Differentiate three times under the integral sign and integrate by parts once.

(E) **6.2** Show that a_n in (6.1.5) satisfies the difference equation (6.1.4) and the initial conditions $a_0 = 1$, $a_1 = 0$.

(TE) **6.3** (*a*) Prove (6.2.1).
Clue: Assuming that the asymptotic relation $f(t, x) \sim f_0(t)$ $(x \to x_0)$ is uniform in t, show that for any $\varepsilon > 0$, $|\int_a^b f(t, x)\, dt - \int_a^b f_0(t)\, dt| < \varepsilon |\int_a^b f_0(t)\, dt|$ for x sufficiently close to x_0.
(*b*) Prove (6.2.2).

(I) **6.4** Verify (6.2.7). (Note that N is the largest integer less than $-a$ and $a < 0$.)

(D) **6.5** Euler's constant γ is defined by $\gamma \equiv \lim_{n\to\infty} (1 + \frac{1}{2} + \frac{1}{3} + \cdots + 1/n - \ln n) \doteqdot 0.5772$. Show that γ can also be represented as

$$(a)\ \gamma = -\lim_{x\to 0+}\left(\int_x^\infty t^{-1}e^{-t}\,dt + \ln x\right);$$

$$(b)\ \gamma = \int_0^\infty (dt/t)[1/(1+t) - e^{-t}].$$

(I) **6.6** Verify (6.2.11) and (6.2.12).

(E) **6.7** Find the leading behavior as $x \to 0+$ of the following integrals:

$$(a)\ \int_x^1 \cos(xt)\,dt;$$

$$(b)\ \int_0^1 \sqrt{\sinh(xt)}\,dt;$$

$$(c)\ \int_0^1 e^{-x/t}\,dt;$$

$$(d)\ \int_x^1 e^{-1/t}\,dt;$$

$$(e)\ \int_x^1 \sin(xt)\,dt;$$

$$(f)\ \int_0^1 [dt/(1-t)](e^x - e^{xt});$$

$$(g)\ \int_0^{1/x} e^{-t^2}\,dt;$$

$$(h)\ \int_0^1 [e^{-xt}/(1+t^2)]\,dt;$$

$$(i)\ \int_0^\infty J_0(xt)t^{-1/2}\,dt.$$

(E) **6.8** Find the full asymptotic behavior as $x \to 0+$ of the following integrals:

$$(a)\ \int_0^1 [e^{-t}/(1+x^2t^3)]\,dt;$$

$$(b)\ \int_0^1 [dt/(1-t)](e^x - e^{xt});$$

$$(c)\ \int_0^x [e^{-t}/(t+a)]\,dt.$$

(I) **6.9** Show that $\int_0^1 [dt/(1-t)](e^x - e^{xt}) \sim e^x \ln x + e^x\gamma + \cdots\ (x \to +\infty)$.

(D) **6.10** Let $I(x) = \int_0^\infty [e^{-t}/(1 + xe^{t^2})]\,dt$.
(*a*) Show that $I(x) - 1 \sim -\exp(\sqrt{-\ln x})\ (x \to 0+)$.
(*b*) Find the full asymptotic expansion of $I(x)$ as $x \to 0+$.

Section 6.3

(I) **6.11** (*a*) Verify (6.3.10).
(*b*) Verify (6.3.14).

(I) **6.12** Find the full asymptotic expansion of $\int_0^x \mathrm{Ai}\,(t)\,dt$ as $x \to +\infty$ and compare three methods for obtaining this result:

(*a*) Use direct integration by parts.

(*b*) Use the asymptotic expansion of Ai (x) valid for large x.

(*c*) Show that $\int_0^x \mathrm{Ai}\,(t)\,dt$ satisfies the differential equation $y''' = xy'$ and use this equation to generate the series.

Which method is easiest?

(I) **6.13** Verify (6.3.16).

(I) **6.14** Find the full asymptotic expansion of $\int_0^x \mathrm{Bi}\,(t)\,dt$ as $x \to +\infty$.

(TD) **6.15** Show that the integral term on the right side of (6.3.18) becomes negligible compared with the boundary term as $x \to +\infty$, assuming that the integral exists and that $\phi'(t) \neq 0$ for $a \le t \le b$.

Clue: Divide the integration region into small subintervals and bound the integral on each subinterval.

(E) **6.16** Verify (6.3.24).

Clue: $\int_0^\infty e^{-t}(1+xt)\ln(1+xt)\,dt > x\int_0^\infty te^{-t}\ln(xt)\,dt$.

(I) **6.17** (*a*) The Fresnel integrals are $\int_x^\infty \cos(t^2)\,dt$, $\int_x^\infty \sin(t^2)\,dt$. Find the full asymptotic expansions as $x \to 0$ and as $x \to +\infty$.

(*b*) The generalized Fresnel integral is defined by $F(x, a) = \int_x^\infty t^{-a}e^{it}\,dt$ for $a > 0$. Find the full asymptotic expansion of $F(x, a)$ as $x \to +\infty$.

(E) **6.18** Find the leading behaviors of:

(*a*) $\displaystyle\int_x^\infty e^{-at^b}\,dt$ as $x \to +\infty$, where $a > 0$ and $b > 0$;

(*b*) $\displaystyle\int_1^\infty \cos(xt)t^{-1}\,dt$ as $x \to 0+$.

(I) **6.19** (*a*) How many terms of the asymptotic expansion of $\int_0^{\pi/4} \cos(xt^2)\tan^2 t\,dt$ as $x \to +\infty$ can be computed using integration by parts? Compute them.

(*b*) Do the same for $\int_0^\pi \cos(xt)\sin(t^2)\,dt$ as $x \to +\infty$.

(I) **6.20** Find the leading behavior as $x \to +\infty$ of the following integrals:

(*a*) $\int_x^\infty K_0(t)\,dt$, where K_0 is a modified Bessel function of order 0;

(*b*) $\int_0^x I_0(t)\,dt$, where I_0 is a modified Bessel function of order 0;

(*c*) $\int_x^\infty D_\nu(t)\,dt$, where D_ν is a parabolic cylinder function of order ν.

(I) **6.21** (*a*) Show that if f is infinitely differentiable, then $A_n = \int_0^\pi f(\cos\theta)\cos(n\theta)\,d\theta$ defined for integer n vanishes more rapidly than any finite power of $1/n$ as $n \to \infty$. (This proves that the Fourier expansion of any even, 2π-periodic, infinitely differentiable function is uniformly convergent and can be differentiated termwise an arbitrary number of times.)

(*b*) What is the leading behavior of A_n as $n \to \infty$ through nonintegral values?

(I) **6.22** Find the leading behavior as $n \to \infty$ (through integer values) of the integral $A_n = \int_0^1 t\sin(\frac{1}{2}\pi t)J_0(\lambda_n t)\,dt$, where J_0 is the Bessel function of order 0 and λ_n is its nth zero.

Clue: Use the asymptotic expansion of $J_0(x)$ as $x \to +\infty$ to show that $\lambda_n \sim (n - \frac{1}{4})\pi$ $(n \to \infty)$. Also, note that $J_0(\lambda_n t)$ satisfies the differential equation $(ty')' + t\lambda_n^2 y = 0$. Note that the Fourier-Bessel expansion of $\sin(\frac{1}{2}\pi x)$ is $2\sum_{n=1}^\infty A_n J_0(\lambda_n x)/J_0'(\lambda_n)^2$.

Section 6.4

(TI) **6.23** Use integration by parts to show that the difference between $I(x)$ in (6.4.1) and $I(x; \varepsilon)$ in (6.4.2) is subdominant with respect to $I(x)$.

(TI) **6.24** (*a*) Find the leading behavior as $x \to +\infty$ of Laplace integrals of the form $\int_a^b (t-a)^\alpha g(t)e^{x\phi(t)}\,dt$, where $\phi(t)$ has a maximum at $t = a$ and $g(a) = 1$. Here $\alpha > -1$ and $\phi'(a) < 0$.

(*b*) Repeat the analysis of part (*a*) when $\alpha > -1$ and $\phi'(a) = \phi''(a) = \cdots = \phi^{(p-1)}(a) = 0$ and $\phi^{(p)}(a) < 0$.

(I) **6.25** (*a*) Use the integral representation

$$J_\nu(x) = \frac{(x/2)^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\int_0^\pi \cos(x\cos\theta)\sin^{2\nu}\theta\,d\theta,$$

which is valid for $\nu > -\frac{1}{2}$, to show that the Bessel function $J_\nu(x)$ satisfies $J_\nu(x) \sim (x/2)^\nu/\Gamma(\nu+1)$ $(\nu \to \infty)$.

(*b*) Use the integral representation

$$Y_\nu(x) = \frac{1}{\pi}\int_0^\pi \sin(x\sin\theta - \nu\theta)\,d\theta - \frac{1}{\pi}\int_0^\infty [e^{\nu t} + e^{-\nu t}\cos(\nu\pi)]e^{-x\sinh t}\,dt$$

to show that the Bessel function $Y_\nu(x)$ satisfies $Y_\nu(x) \sim -\Gamma(\nu)(2/x)^\nu/\pi/(\nu \to \infty)$.

(E) **6.26** (*a*) Obtain three terms of the asymptotic expansion of $\int_0^{\pi/2} e^{-x\tan^2\theta}\,d\theta$ as $x \to \infty$.

(*b*) Find the leading behavior of $\int_0^{2\pi}(1+t^2)e^{x\cos t}\,dt$ as $x \to +\infty$. Note that two maxima contribute to this leading behavior.

(I) **6.27** Show that $\int_0^\infty \mathrm{Ai}(xt)/(1+t^2)\,dt \sim 1/(3x)$ $(x \to +\infty)$, where $\mathrm{Ai}(s)$ is the Airy function. Can you find the full asymptotic behavior of this integral as $x \to +\infty$?

(I) **6.28** Find the leading behaviors of

(*a*) $\displaystyle\int_0^{\pi/2} \sqrt{\sin t}\, e^{-x\sin^4 t}\,dt$ as $x \to \infty$;

(*b*) $\displaystyle\int_0^1 \sqrt{t(1-t)}\,(t+a)^{-x}\,dt$ as $x \to +\infty$ with $a > 0$;

(*c*) $\displaystyle\int_0^{\pi/4} \sqrt{\tan t}\, e^{-xt^2}\,dt$ as $x \to +\infty$;

(*d*) $\displaystyle\int_0^{\pi^2/2} ds \int_0^{\pi^2/2} dt\, e^{x\cos\sqrt{s+t}}$ as $x \to +\infty$.

(I) **6.29** Define $P_n(z) = (1/\pi)\int_0^\pi [z + (z^2-1)^{1/2}\cos\theta]^n\,d\theta$ $(z > 1)$ where the positive square root is taken. $P_n(z)$ is the nth-degree Legendre polynomial. Show that for large n,

$$P_n(z) \sim \frac{1}{\sqrt{2\pi n}}\frac{[z+(z^2-1)^{1/2}]^{n+1/2}}{(z^2-1)^{1/4}}.$$

(I) **6.30** (*a*) The digamma function $\psi(z) = \Gamma'(z)/\Gamma(z)$ has the integral representation

$$\psi(z) = \ln z - 1/2z - \int_0^\infty [(e^t-1)^{-1} - t^{-1} + \tfrac{1}{2}]e^{-tz}\,dt.$$

Use this integral representation to generate the first three terms of the asymptotic expansion of $\psi(z) - \ln z + 1/2z$ as $z \to \infty$.

(*b*) Show that

$$\psi(z) + \gamma = \int_0^\infty \frac{e^{-t} - e^{-zt}}{1 - e^{-t}}\,dt.$$

(*c*) Show that $\psi(z) + 1/z + \gamma = \sum_{k=2}^\infty (-1)^k\zeta(k)z^{k-1}$ $(|z| < 1)$, where the Riemann ζ function is defined by $\zeta(k) = \sum_{n=1}^\infty n^{-k}$.

(*d*) Use the series in part (*c*) to derive a recursion relation for the coefficients C_j in the Taylor series (5.4.4) for $1/\Gamma(z)$.

(I) **6.31** Use Laplace's method for a moving maximum on (6.1.3) to show that A in (6.1.2) is given by $\pi^{1/2}2^{2/3}3^{-1/2}$.

(I) **6.32** Use Laplace's method for a moving maximum on (6.1.5) to verify (6.1.6).

(I) **6.33** Show that the last integral on the right side of (6.3.23) vanishes like $x^{-3/2}$ as $x \to +\infty$.

(D) **6.34** Calculate two terms in the asymptotic expansion of $\int_0^\infty e^{-t-x/t^2}\,dt$ and $\int_0^\infty e^{-xt}(1+t)^{-1/2}\,dt$ as $x \to 0+$ and as $x \to +\infty$.

(I) **6.35** Find the leading behavior of $\int_0^\infty e^{-t-x/t^\alpha}\,dt$ for $\alpha > 0$ as $x \to 0+$ and as $x \to +\infty$.

(I) **6.36** (*a*) Verify that the integral representation in (6.4.20) satisfies the differential equation for $K_\nu(x)$.
(*b*) Use the integral representation (6.4.20) to show that $K_\nu(x) \sim \sqrt{\pi/2\nu}\,(2\nu/ex)^\nu$ $(\nu \to +\infty)$.

(I) **6.37** (*a*) Show that

$$\int_0^\infty \frac{t^{x-1}e^{-t}}{t+x}\,dt \sim \frac{1}{2x}\Gamma(x), \qquad x \to +\infty.$$

(*b*) Find the leading behavior as $x \to +\infty$ of

$$\frac{\displaystyle\int_0^\infty [t^{x-1}/(x+t)]\,e^{-t^\alpha}\,dt}{\displaystyle\int_0^\infty t^{x-1}e^{-t^\alpha}\,dt}.$$

(I) **6.38** Solve Prob. 3.77 using Watson's lemma.

(I) **6.39** The logarithmic integral function li (x) is defined as li $(x) = P\int_0^x dt/\ln t$, where P indicates that the Cauchy principal part of the integral is taken when $x > 1$. Show that li $(e^a) \sim e^a \sum_{n=0}^\infty n!/a^{n+1}$ $(a \to +\infty)$.

(I) **6.40** Prove that

$$\sum_{n=0}^{2N+1} (-1)^n \frac{t^{2n+1}}{(2n+1)!} < \sin t < \sum_{n=0}^{2N} (-1)^n \frac{t^{2n+1}}{(2n+1)!}$$

for all $t > 0$ and all integers N.

Clue: Prove that $\sin t < t$ by integrating $\cos t < 1$. In the same way, use repeated integration to establish the general result.

(I) **6.41** Show that (6.4.27) is an integral representation of the modified Bessel function $I_n(x)$. In other words, show that the integral satisfies the differential equation $x^2y'' + xy' - (x^2 + \nu^2)y = 0$ and the relation $I_n(x) \sim (x/2)^n/n!$ $(x \to 0+)$.

(I) **6.42** Use Laplace's method for a movable maximum to find the next correction to (6.4.40). In particular, show that

$$\Gamma(x) \sim x^{x-1/2}e^{-x}\sqrt{2\pi}\left(1 + \frac{1}{12x} + \frac{1}{288x^2} + \cdots\right), \qquad x \to +\infty.$$

(I) **6.43** (*a*) Show that Laplace's method for expanding integrals consists of approximating the integrand by a δ function. In particular, show how the representation $\delta(t) = \lim_{x\to\infty} \sqrt{x/\pi}\,e^{-xt^2}$ reproduces the leading behavior of a Laplace integral for which $\phi'(c) = 0$ but $\phi''(c) < 0$. [See (6.4.19*c*) and (1.5.10*c*).]

(*b*) What is the appropriate δ-function representation for the case in which $\phi(t) < \phi(a)$ for $a < t < b$ and $\phi'(a) < 0$? [See (6.4.19*a*).]

(*c*) What is the appropriate δ-function representation for the case in which $\phi'(c) = \phi''(c) = \cdots = \phi^{(p-1)}(c) = 0$, $\phi^{(p)}(c) < 0$ with p even? [See (6.4.19*d*).]

(D) (*d*) Extend the δ-function analysis of parts (*a*) to (*c*) to give the higher-order corrections to the leading behavior.

Clue: The answer is given in (6.4.35).

(D) **6.44** Find the leading behavior of the double integral $\int_0^\infty ds \int_0^\pi dt\, e^{-x[\nu s + (\cos t)(\sin s)]}$ as $x \to +\infty$ for $0 < \nu < 1$, $\nu = 1$, and $\nu > 1$. Sketch the function for large x.

Clue: Show that when $0 < \nu < 1$, the exponent has four stationary points. As $\nu \to 1-$, these stationary points merge into two. When $\nu > 1$, there are no stationary points.

(I) **6.45** What happens if we try to treat an ordinary Laplace integral $\int_a^b f(t)e^{x\phi(t)}\,dt$ using the methods appropriate for a moving maximum? Suppose we rewrite the integral as $\int_a^b e^{x\phi(t)+\ln f(t)}\,dt$ and expand about the maximum of the integrand. Show that now an interior maximum is shifted slightly from $t = c$ where $\phi'(c) = 0$, but that this does not affect the result given by Laplace's method in (6.4.19*c* to *d*).

(I) **6.46** Show that naive application of Laplace's method for a moving maximum to the integral $I(x, \alpha) = \int_0^\infty t^\alpha e^{-xt}\,dt = x^{-\alpha-1}\Gamma(\alpha + 1)$ gives the wrong answer! Show that the maximum of the integrand occurs at $t = \alpha/x$ and retaining only quadratic terms gives

$$I(x, \alpha) \sim e^{-\alpha}\alpha^{\alpha+1/2}x^{-\alpha-1}\sqrt{2}\int_{-\sqrt{\alpha/2}}^{\infty} e^{-u^2}\,du, \qquad x \to +\infty.$$

Explain why we have obtained the wrong answer.

(I) **6.47** (*a*) Show that

$$\int_0^{1/e} \frac{e^{-xt}}{\ln t}\,dt \sim -\frac{1}{x \ln x}, \qquad x \to +\infty.$$

Clue: See Example 3 of Sec. 6.6.

(*b*) Show that

$$\int_0^{1/e} \frac{e^{-xt}}{\ln t}\,dt \sim -\frac{1}{x \ln x}\sum_{n=0}^{\infty} (\ln x)^{-n}\int_0^\infty (\ln s)^n e^{-s}\,ds, \qquad x \to +\infty.$$

(*c*) Explain why naive use of Laplace's method for a moving maximum fails to give the results (*a*) and (*b*) above.

(D) **6.48** Find the leading behaviors of

(*a*) $\int_0^\infty e^{-xt}e^{-a(\ln t)^2}t^{-1}\,dt$ as $x \to \infty$;

(*b*) $(d^n/dx^n)\Gamma(x)|_{x=1}$ as $n \to \infty$.

Section 6.5

(I) **6.49** Show that $\int_0^\infty e^{is}s^{\alpha-1}\,ds = e^{i\pi\alpha/2}\Gamma(\alpha)$ for $0 < \text{Re}\,\alpha < 1$.

Clue: Substitute $s = it$ and rotate the contour of integration from the negative imaginary-t axis to the positive real-t axis.

(I) **6.50** Use integration by parts to show that the full asymptotic expansion $I(x)$ in (6.5.3) is

$$I(x) \sim \frac{i\sqrt{\pi}}{2x^{3/2}}e^{i\pi/4} - \frac{i}{x\sqrt{\pi}}e^{ix}\left[1 + \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{-i}{x}\right)^{n+1}\Gamma\left(n + \frac{1}{2}\right)\right], \qquad x \to +\infty.$$

Clue: Write

$$\int_0^1 \frac{e^{ixt}}{\sqrt{t}}\,dt = \int_0^\infty \frac{e^{ixt}}{\sqrt{t}}\,dt - \int_1^\infty \frac{e^{ixt}}{\sqrt{t}}\,dt$$

and use integration by parts on the second integral on the right.

(TD) **6.51** Prove the Riemann-Lebesgue lemma by showing that $\int_a^b f(t)e^{ixt}\,dt \to 0$ $(x \to +\infty)$ provided that $|f(t)|$ is integrable.

Clue: Break up the region of integration into small subintervals and bound the integral on each subinterval.

(TI) **6.52** Show that $\int_a^b f(t)e^{ix\psi(t)}\,dt \to 0$ $(x \to +\infty)$ provided that $|f(t)|$ is integrable, $\psi(t)$ is continuously differentiable, and $\psi(t)$ is not constant on any subinterval of $a \le t \le b$.

Clue: Use the Riemann-Lebesgue lemma.

(D) **6.53** Find the leading behavior of $\int_a^b f(t)e^{ix\psi(t)}\,dt$ under the following assumptions: $\psi'(a) = \cdots = \psi^{(p-1)}(a) = 0$; $\psi^{(p)}(a) \ne 0$; $f(t) \sim A(t - a)^\alpha$ $(t \to a+)$ with $\alpha > -1$.

(*a*) What is the leading contribution to the behavior of $I(x)$ from the neighborhood of the stationary point at $t = a$? This result is a generalization of the formula in (6.5.12).

(*b*) For which values of α and p does the contribution found in (*a*) equal the leading behavior of $I(x)$ as $x \to \infty$. Assume that $\psi'(t) \neq 0$ for $a < t \le b$ and that $f(t)$ is continuous and nonvanishing for $a < t \le b$.

(I) **6.54** (*a*) Show that (6.5.13) is an integral representation of $J_n(x)$.

Clue: Show that the integral satisfies the Bessel equation $x^2y'' + xy' + (x^2 - n^2)y = 0$ and behaves like $(x/2)^n/n!$ as $x \to 0$.

(*b*) Use the integral representation (6.5.13) to find the leading behavior of $J_n(x)$ as $x \to +\infty$.

(I) **6.55** When ν is not an integer, (6.5.13) generalizes for positive x to

$$J_\nu(x) = \frac{1}{\pi}\int_0^\pi \cos(x \sin t - \nu t)\, dt - \frac{\sin(\nu\pi)}{\pi}\int_0^\infty e^{-x\sinh t - \nu t}\, dt.$$

(*a*) Verify the validity of this integral representation of $J_\nu(x)$.

(*b*) Use this integral representation to find the leading behavior of $J_\nu(\nu)$ as $\nu \to +\infty$. The result shows that (6.5.14) remains valid when n is not an integer.

(E) **6.56** Use the method of stationary phase to find the leading behavior of the following integrals as $x \to +\infty$:

(*a*) $\displaystyle\int_0^1 e^{ixt^2} \cosh t^2\, dt$;

(*b*) $\displaystyle\int_0^1 \cos(xt^4) \tan t\, dt$;

(*c*) $\displaystyle\int_0^1 e^{ix(t - \sin t)}\, dt$;

(*d*) $\displaystyle\int_0^1 \sin[x(t + \tfrac{1}{6}t^3 - \sinh t)] \cos t\, dt$;

(*e*) $\displaystyle\int_{-1}^1 \sin[x(t - \sin t)] \sinh t\, dt$.

Section 6.6

(I) **6.57** Evaluate the full asymptotic behavior of $\int_0^1 e^{ixt^2}\, dt$ using integration by parts (see Example 2 of Sec. 6.6).

(I) **6.58** Explain why the contour C_3 can be replaced by the contour C_4 in Fig. 6.6. Show that this replacement introduces errors that are smaller than any term in the asymptotic expansion (6.6.8).

(E) **6.59** Show that the coefficient of the general term in (6.6.16) can be expressed in terms of derivatives of $\Gamma(t)$ at $t = 1$.

(I) **6.60** Find the radius of convergence of the asymptotic series (6.6.17).

Clue: See Probs. 6.48(*b*) and (6.59).

(E) **6.61** Verify the features of Fig. 6.9. In particular, show that six steepest paths emerge from $t = 0$ with 60° angular separations, that the six paths are alternately ascent and descent curves, and that the paths approach $\pm\infty$, $\pm\infty + i\pi$, $\pm\infty - i\pi$.

(E) **6.62** Verify the features of Fig. 6.10. In particular, show that eight steepest paths emerge from $t = 0$ with 45° angular separations, that the eight paths are alternately ascent and descent curves, and that the paths approach $\pm\infty$, $\pm i\infty$, $\pm\infty + i\pi$, $\pm\infty - i\pi$.

(E) **6.63** Verify (6.6.19) by showing that the integrals (6.6.18) are pure imaginary.

(E) **6.64** Verify that the steepest-descent curve through the saddle point of (6.6.23) is correctly drawn in Fig. 6.12. Show that this curve is vertical at $s = 1$, crosses the imaginary-s axis at $s = \pm i\pi/2$, and approaches $-\infty \pm i\pi$.

(I) **6.65** Formulate a convincing explanation that if we neglect the term xt^3 in (6.6.24), we will obtain the wrong asymptotic behavior of the integral as $x \to +\infty$.

Clue: On the assumption that the term is small, expand the cosine and estimate the error terms. Show that the error is not smaller than the exponentially small integral obtained by neglecting xt^3.

(E) **6.66** Verify the results in Fig. 6.13. In particular, show that C_1, C_2, and C_3 are steepest-descent contours.

(E) **6.67** Show that the contributions to $I(x)$ in (6.6.25) from C_1 and C_2 in Fig. 6.13 are exponentially small compared with the contribution from the saddle point on C_3.

(I) **6.68** Formulate a convincing explanation that if we replace $\cosh t$ by $1 + t^2/2$ in (6.6.32), we will obtain the wrong asymptotic behavior of (6.6.32) as $x \to +\infty$.

Clue: See Prob. 6.65.

(D) **6.69** (*a*) Verify (6.6.35) to (6.6.37).

(*b*) Show that as $\arg x$ in (6.6.25) increases past $\pi - \arctan 2$ the steepest descent path through the endpoints at $t = \pm 1$ no longer passes through the saddle point at $t = i$. (See Figs. 6.13 to 6.16.)

(I) **6.70** The integrand in (6.6.19) has saddle points at $t = in\pi$ ($n = 0, \pm 1, \pm 2, \ldots$). Why don't we distort the contour C to pass through all or some of these points instead of just through $t = 0$ as in Example 8 of Sec. 6.6?

(D) **6.71** An integral representation of the modified Bessel function $K_\nu(x)$ is $K_\nu(x) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-x\cosh t + \nu t}\, dt$. Show that $K_{ip}(x) = \sqrt{2\pi}\,(p^2 - x^2)^{-1/4} e^{-p\pi/2} \sin \phi(x)$, where $\phi(x) - p\cosh^{-1}(p/x) + \sqrt{p^2 - x^2} \sim \pi/4$ ($x \to +\infty$, $p/x \to +\infty$).

Clue: The contribution comes from the neighborhood of two saddle points satisfying $\sinh t = ip/x$. Explain why it is that although there are an infinite number of saddle points, only two contribute to the leading behavior.

(D) **6.72** Investigate the Stokes phenomenon for the integral $I(x) = \int_0^1 e^{xt^3}\, dt$ as $x \to \infty$. Specifically, evaluate the asymptotic behavior of $I(x)$ as $x \to \infty$ with $\arg x$ fixed for several values of $\arg x$ and show that there are Stokes lines at $|\arg x| = \pi/2$.

(I) **6.73** Find the first two terms in the asymptotic behavior of $\int_0^{\pi/4} \cos(xt^2) \tan^2 t\, dt$ as $x \to +\infty$.

(I) **6.74** Find three terms in the asymptotic behavior of $\int_0^1 \ln(1 + t)\, e^{ix\sin^2 t}\, dt$ as $x \to +\infty$.

(D) **6.75** (*a*) Show that an integral representation of the Airy function Ai (x) is given by

$$\mathrm{Ai}\,(x) = \frac{1}{2\pi i}\int_C e^{xt - t^3/3}\, dt,$$

where C is a contour which originates at $\infty e^{-2\pi i/3}$ and terminates at $\infty e^{2\pi i/3}$.

(*b*) Use this integral representation to show that the Taylor series expansion of Ai (x) about $x = 0$ is as given in (3.2.1).

(*c*) Using the method of steepest descents, find the asymptotic behavior of Ai (x) as $x \to +\infty$.

(*d*) Extend the steepest-descent argument used in part (*c*) to show that the same asymptotic behavior is valid for $x \to \infty$ with $|\arg x| < \pi$ and that there is no Stokes phenomenon at $|\arg x| = \pi/3$.

(*e*) Show that there is no Stokes phenomenon at $|\arg x| = \pi/3$ in a different way. Transform the integral in (*a*) to

$$\mathrm{Ai}\,(x) = \frac{1}{\pi} e^{-2x^{3/2}/3} \int_0^\infty e^{-x^{1/2}t^2} e^{it^3/3}\, dt$$

and then use Laplace's method.

Clue: For real positive x, deform C in (*a*) into the straight-line contour connecting $-x - i\infty$ to $-x + i\infty$ and then allow x to be complex with $|\arg x^{1/2}| < \pi/2$.

(*f*) Find the leading behavior of Ai (x) as $x \to -\infty$.

Clue: Show that the steepest-descent contour connecting $\infty e^{-2\pi i/3}$ to $\infty e^{2\pi i/3}$ consists of two pieces, one passing through the saddle point at $t = -i\sqrt{-x}$ and one passing through the saddle point at $t = +i\sqrt{-x}$.

(D) **6.76** (*a*) Show that an integral representation of the Airy function Bi (x) is given by

$$\mathrm{Bi}\,(x) = \frac{1}{2\pi}\int_{C_+} e^{xt - t^3/3}\,dt + \frac{1}{2\pi}\int_{C_-} e^{xt - t^3/3}\,dt,$$

where $C_\pm$ is a contour which originates at $\infty e^{\pm 2\pi i/3}$ and terminates at $+\infty$.

(*b*) Use this integral representation to show that the Taylor series expansion of Bi (x) about $x = 0$ is given in (3.2.2).

(*e*) Using the method of steepest descents, find the asymptotic behavior of Bi (x) as $x \to +\infty$.

(*d*) Extend the steepest-descent argument used in (*c*) to show that this leading asymptotic behavior of Bi (x) is still valid for $x \to \infty$ with $|\arg x| < \pi/3$.

(*e*) Use the integral representation of (*a*) to find the asymptotic behavior of Bi (x) as $x \to \infty$ with $\pi/3 < |\arg x| < \pi$. Show that as $|\arg x|$ increases beyond $\pi/3$, the contribution from the saddle point at $t = -\sqrt{x}$ overwhelms the contribution from the saddle point at $t = +\sqrt{x}$.

(D) **6.77** Consider the integral $I(n) = \int_0^\pi \cos\,(nt)\, e^{ia\cos t}\,dt$ where n is an integer.

(*a*) Use integration by parts to show that $I(n)$ for n integral decays to 0 faster than any power of $1/n$ as $n \to \infty$. What if n is nonintegral?

(*b*) Use the method of steepest descents in the complex-t plane to show that $I(n) \sim \sqrt{\pi/2n}\,(\tfrac{1}{2}iea/n)^n$ $(n \to \infty)$.

(*c*) By setting $z = \cos t$ show that

$$I(n) = \frac{1}{\pi}\int_C \frac{e^{iaz}}{(z + \sqrt{z^2 - 1})^n}\,\frac{dz}{\sqrt{1 - z^2}}$$

where the branch cut of $\sqrt{z^2 - 1}$ is chosen to lie along the real axis from $z = -1$ to $z = +1$, the branch is chosen so that $\sqrt{z^2 - 1} \to +z$ as $z \to \infty$, and C is any contour that loops around the branch cut just once in the counterclockwise sense.

(*d*) Use the integral representation in (*c*) and the method of steepest descents to reproduce the results in (*b*). Notice that there are no endpoint contours required with this alternative derivation.

(D) **6.78** Let $f(z)$ be an entire (everywhere analytic) function of z whose Taylor series about $z = 0$ is $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Cauchy's theorem gives a contour integral representation for a_n:

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(t)}{t^{n+1}}\,dt,$$

where C loops the origin once in the counterclockwise sense. Using the method of steepest descents, find the leading asymptotic behavior of a_n as $n \to \infty$ for the following functions:

(*a*) $f(z) = e^z$ (the result is a rederivation of the Stirling formula for $n!$);

(*b*) $f(z) = e^{z^m}$;

(*c*) $f(z) = \exp\,(e^z)$ (see Prob. 5.71).

(I) **6.79** Using the approach of Prob. 6.78, verify the result stated in Prob. 5.20 for the asymptotic behavior of the Bernoulli numbers $B_n = n!\,f^{(n)}(0)$ where $f(z) = z/(e^z - 1)$.

(D) **6.80** Find two terms in the asymptotic expansion of $\int_0^\pi e^{ix\cos t}\,dt$ as $x \to +\infty$. Where are the Stokes lines in the complex plane?

(D) **6.81** Use the method of steepest descents to find the full asymptotic behaviors of

(*a*) $\displaystyle\int_0^1 e^{ixt^3}\,dt$ $(x \to +\infty)$;

(*b*) $\displaystyle\int_0^1 t^\alpha e^{ixt^3}\,dt$ $(x \to +\infty)$.

(D) **6.82** Find the leading behavior of the integral $\int_{-\infty+ia}^{\infty+ia} t^{-1/2}e^{ix(t^3/3-t)}\,dt$ as $x \to +\infty$, where $a > 0$ and the contour of integration is an infinite straight line parallel to the real-t axis lying in the upper half of the complex-t plane. What is the effect of the branch point at $t = 0$?

(D) **6.83** (*a*) Show that $y(x) = \int_C e^{xt - t^5/5}\,dt$, where C is any contour connecting the points ∞, $\infty e^{\pm 2\pi i/5}$, $\infty e^{\pm 4\pi i/5}$ at which the integrand vanishes, satisfies the hyperairy equation (3.5.22).

(*b*) Use the integral representation given in (*a*) to find the possible leading behaviors of solutions to the hyperairy equation as $x \to +\infty$.

(*c*) Investigate fully the Stokes phenomenon for that solution of the hyperairy equation obtained by choosing C to be a contour connecting $\infty e^{-4\pi i/5}$ to $\infty e^{-2\pi i/5}$.

(D) **6.84** (*a*) Show that $D_\nu(x) = (e^{x^2/4}/i\sqrt{2\pi}) \int_C t^\nu e^{-xt + t^2/2}\,dt$, where C is a contour connecting $-i\infty$ to $+i\infty$ on which $\text{Re}\, t > 0$, satisfies the parabolic cylinder equation (3.5.11) and the initial conditions $D_\nu(0) = \pi^{1/2}2^{\nu/2}/\Gamma(\frac{1}{2} - \nu/2)$, $D_\nu'(0) = -\pi^{1/2}2^{(\nu+1)/2}\Gamma(-\nu/2)$.

(*b*) Use the method of steepest descents to show that $D_\nu(x) \sim x^\nu e^{-x^2/4}$ $(x \to +\infty)$.

(*c*) Extend the steepest descent argument of (*b*) to show that $D_\nu(x) \sim x^\nu e^{-x^2/4}$ is still valid if $x \to \infty$ with $|\arg x| < \pi/2$.

Clue: Show that the branch point of t^ν at $t = 0$ does not affect the steepest descent calculation if $|\arg x| < \pi/2$.

(*d*) Show that the asymptotic behavior found in (*b*) and (*c*) is valid beyond $|\arg x| = \pi/2$ and only breaks down at $|\arg x| = 3\pi/4$ when ν is not a nonnegative integer.

Clue: Show that the contribution from the branch point at $t = 0$ found in part (*c*) becomes significant as $|\arg x|$ increases beyond $3\pi/4$.

(*e*) Find the asymptotic behavior of $D_\nu(x)$ as $x \to \infty$ with $3\pi/4 < |\arg x| < 5\pi/4$.

(D) **6.85** (*a*) Using the integral representation of Prob. 6.84(*a*), show that if $\text{Re}\,(\nu) > -1$, $D_\nu(x) = \sqrt{2/\pi}\, e^{x^2/4} \int_0^\infty e^{-t^2/2}t^\nu \cos(xt - \nu\pi/2)\,dt$.

(*b*) Show that $D_\nu(x) = \sqrt{2}\,(\nu/e)^{\nu/2} \cos\theta$, where $\theta \sim x\nu^{1/2} - \nu\pi/2$ $(\nu \to \infty;\ x \text{ fixed})$.

(D) **6.86** Find the leading behavior as $x \to +\infty$ of the integral

$$\int_C \frac{e^{ix(t^3/3 - t)}}{t - a}\,dt,$$

where C is a contour connecting $-\infty$ to $+\infty$ in the upper-half t plane and a is a real constant. Investigate separately the cases $|a| < 1$, $|a| > 1$, $a = \pm 1$.

(D) **6.87** (*a*) Find the leading behavior of $I_n(x) = \int_0^\infty e^{xt - t^n/n}\,dt$ as $x \to +\infty$. (See Prob. 6.83.)

(*b*) Investigate the Stokes phenomenon for $I_n(x)$ as $x \to \infty$ in the complex-x plane.

Section 6.7

(I) **6.88** Derive the Euler-Maclaurin sum formula (6.7.10) as follows:

(*a*) Verify that

$$\tfrac{1}{2}[f(k) + f(k+1)] - \int_k^{k+1} f(t)\,dt = \int_k^{k+1} (t - k - \tfrac{1}{2})f'(t)\,dt.$$

(*b*) Sum the identity in (*a*) to show that $F(n)$ in (4.7.9) satisfies $F(n) = \frac{1}{2}[f(0) + f(n)] + \int_0^n f(t)\,dt + \int_0^n B_1(t - [t])f'(t)\,dt$, where $B_1(s)$ is the Bernoulli polynomial of degree 1, $B_1(s) = s - \frac{1}{2}$.

(*c*) Show that the Bernoulli polynomials $B_n(x)$, which are defined by the formula $te^{xt}/(e^t - 1) = \sum_{n=0}^\infty B_n(x)t^n/n!$, satisfy $B_n'(x) = nB_{n-1}(x)$ and $B_n(0) = B_n(1)$ for $n \geq 2$.

(*d*) Integrate the result of (*b*) repeatedly by parts to show that

$$F(n) = \tfrac{1}{2}[f(0) + f(n)] + \int_0^n f(t)\,dt + \sum_{j=1}^{m} (-1)^{j+1}\frac{B_{j+1}}{(j+1)!}$$

$$\times\,[f^{(j)}(n) - f^{(j)}(0)] + \frac{(-1)^m}{(m+1)!}\int_0^n B_{m+1}(t)f^{(m+1)}(t)\,dt.$$

(*e*) Let $m \to \infty$ and thus derive (6.7.10) and (6.7.11).

(D) **6.89** Find the first correction to the result in (6.7.8).

(D) **6.90** Euler's constant γ is defined as $\gamma = \lim_{n\to\infty} (\sum_{k=1}^{n} 1/k - \ln n)$. Use the formula (6.7.11) to show that the constant C in (6.7.12) is Euler's constant γ.

Clue: Reverse the process of integration by parts used in Prob. 6.88.

(D) **6.91** (*a*) Show that $\sum_{k=0}^{\infty} 1/(k^2 + x^2) - \pi/2x \sim 1/2x^2$ $(x \to +\infty)$.

Clue: Follow the derivation given in Prob. 6.88 to derive an analog of the Euler-Maclaurin sum formula for $x \to +\infty$.

(*b*) Evaluate $\sum_{k=0}^{\infty} (k^2 + x^2)^{-1}$ exactly by representing the sum as the contour integral

$$\frac{1}{2\pi i} \int_C \frac{\cot t}{t^2 + x^2}\, dt$$

over an appropriate contour.

(*c*) By comparing the results of (*a*) and (*b*), show that the error in (*a*) is exponentially small as $x \to +\infty$.

(D) **6.92** Find two terms in the asymptotic behavior as $x \to +\infty$ of the following sums:

(*a*) $\sum_{k=0}^{\infty} (k + x)^{-\alpha}$ $(\alpha > 1)$;

(*b*) $\sum_{k=0}^{\infty} (k^2 + x^2)^{-2}$.

(D) **6.93** Find three terms in the asymptotic behavior as $n \to \infty$ of the following sums:

(*a*) $\sum_{k=1}^{n} (-1)^k/k$;

(*b*) $\sum_{k=1}^{n} B_{2k}$;

(*c*) $\sum_{k=1}^{n} \sin k/k$.

(D) **6.94** Show that

$$\sum_{k=1}^{\infty} \frac{1}{k(k^2 + x^2)} \sim \frac{\ln x}{x^2} + \frac{\gamma}{x^2} - \sum_{n=1}^{\infty} \frac{(-1)^n B_{2n}}{2nx^{2n+2}}, \qquad x \to +\infty.$$

(I) **6.95** Show that

(*a*) $\sum_{1 \le n \le x} 2^{-(1+1/2+1/3+\cdots+1/n)} \sim x^{1-\ln 2}/(1 - \ln 2)$ $(x \to +\infty)$;

(*b*) $\sum_{1 \le n \le x} n^{1/2} e^{n^{1/2}} \sim 2xe^{x^{1/2}}$ $(x \to +\infty)$;

(*c*) $\sum_{n=0}^{\infty} (n + 1)^3/[(n + \frac{1}{2})^2 + x^2]^{5/2} \sim 2/3x$ $(x \to +\infty)$;

(*d*) $\sum_{j=0}^{n} j^k(n - j)^k \sim (n/2)^{2k+1}\sqrt{\pi/k}$ $(n \to \infty)$.