EMTH 211 Linear Algebra Assignment

Danny Choo
student ID - 28842156
kxc11@uclive.ac.nz

Grace Dain Lee
student ID - 51455525
dil15@uclive.ac.nz

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Q1.a

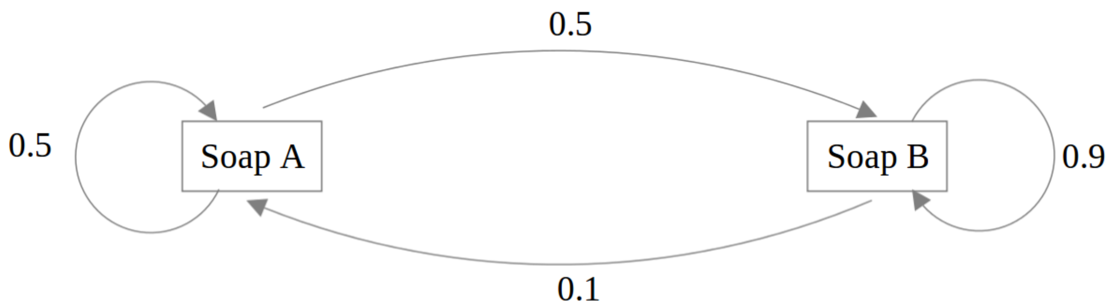

Figure 1: A directed graph of the senario.

Assume that, Soap A and B have 100 soaps respectively

$$A_{k'} = 0.5A_k + 0.1B_k$$
$$B_{k'} = 0.5A_k + 0.9B_k$$

where A_k = customers of soap A for this month and B_k = customers of soap B for this month, and $A_{k'}$ = customers of soap A for next month and $B_{k'}$ = customers of soap B for next month.

Q1.b

$$\begin{bmatrix} A_{k'} \\ B_{k'} \end{bmatrix} = \begin{bmatrix} 0.5 & 0.1 \\ 0.5 & 0.9 \end{bmatrix} \begin{bmatrix} A_k \\ B_k \end{bmatrix} \text{(from 1.1)}$$

Q1.c

In one month's time

$$\Rightarrow \begin{bmatrix} A_{k'} \\ B_{k'} \end{bmatrix} = \begin{bmatrix} 0.5 & 0.1 \\ 0.5 & 0.9 \end{bmatrix} \begin{bmatrix} 100 \\ 100 \end{bmatrix}$$
$$= \begin{bmatrix} 50+10 \\ 50+90 \end{bmatrix} = \begin{bmatrix} 60 \\ 140 \end{bmatrix}$$

Q1.d

In two month's time

$$
\begin{aligned}
\Rightarrow \begin{bmatrix} A_k\prime\prime \\ B_k\prime\prime \end{bmatrix} &= \begin{bmatrix} 0.5 & 0.1 \\ 0.5 & 0.9 \end{bmatrix} \begin{bmatrix} A_k\prime \\ B_k\prime \end{bmatrix} \\
&= \begin{bmatrix} 0.5 & 0.1 \\ 0.5 & 0.9 \end{bmatrix} \begin{bmatrix} 60 \\ 140 \end{bmatrix} \\
&= \begin{bmatrix} 30+14 \\ 30+126 \end{bmatrix} = \begin{bmatrix} 44 \\ 156 \end{bmatrix}
\end{aligned}
$$

Q1.e

In one year's time, the equation is:

$$\begin{bmatrix} A_k\prime\prime\prime \\ B_k\prime\prime\prime \end{bmatrix} = \begin{bmatrix} 0.5 & 0.1 \\ 0.5 & 0.9 \end{bmatrix}^{12} \begin{bmatrix} 100 \\ 100 \end{bmatrix} \text{(from 1.1)}$$

To find the eignevalues:

$$
\begin{aligned}
det(A - \lambda I) &= 0 \\
det \begin{bmatrix} 0.5-\lambda & 0.1 \\ 0.5 & 0.9-\lambda \end{bmatrix} &= 0 \\
\Rightarrow (0.5-\lambda)(0.4-\lambda) - 0.05 &= 0 \\
\Rightarrow \lambda^2 - 1.4\lambda = 0.4 &= 0 \\
\Rightarrow 5\lambda^2 - 7\lambda + 2 &= 0 \\
\Rightarrow (5\lambda - 2)(\lambda - 1) &= 0
\end{aligned}
$$

Hence, $\lambda_1 = 0.4$ and $\lambda_2 = 1$

To find the eigenvectors at $\lambda_1 = 0.4$,

$$\left[\begin{array}{cc|c} 0.1 & 0.1 & 0 \\ 0.5 & 0.5 & 0 \end{array}\right] R_2 -> R_2 - 5R_1 \quad \sim \quad \left[\begin{array}{cc|c} 0.1 & 0.1 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad x_1 = -x_2$$

Thus, $x_1 = \begin{bmatrix} 44 \\ 156 \end{bmatrix}$

To find the eigenvectors At $\lambda_2 = 1$,

$$\left[\begin{array}{l|l} -0.5 & 0\ 0.1 \\ 0.5 & -0\ 0.1 \end{array}\right] R_2 -> R_2 + R_1 \quad \sim \quad \left[\begin{array}{cc|c} -0.5 0.1 & & 0 \\ 0 & 0 & 0 \end{array}\right] \quad x_2 = 5x_1$$

Thus, $x_2 = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

Therefore,

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 0.4 & 0 \\ 0 & 1 \end{bmatrix}, P = \begin{bmatrix} 1 & 1 \\ -1 & 5 \end{bmatrix}, P^{-1} = \begin{bmatrix} \frac{5}{6} & -\frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} \end{bmatrix}$$

Let

$$\begin{bmatrix} A_{k+2} \\ B_{k+2} \end{bmatrix} = \begin{bmatrix} 0.5 & 0.1 \\ 0.5 & 0.9 \end{bmatrix}^{12} \begin{bmatrix} 100 \\ 100 \end{bmatrix}$$

be $\vec{v}_1 = A^2\vec{v}_0$, where $A = PDP^{-1}$.
Thus,

$$\Rightarrow \vec{v}_1 = PDP^{-1}\vec{v}_0 = \begin{bmatrix} 1 & 1 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} 0.4^{12} & 0 \\ 0 & 1^{12} \end{bmatrix} \begin{bmatrix} \frac{5}{6} & -\frac{1}{6} \\ \frac{1}{6} & -\frac{1}{6} \end{bmatrix} \begin{bmatrix} 100 \\ 100 \end{bmatrix}$$

The calculation was implemted via matlab. Matlab shows

In the long run:

$$\begin{aligned} \vec{v}_k &= A^k\vec{v}_0 \\ &= PDP^{-1}\vec{v}_0 \\ &= \begin{bmatrix} 1 & 1 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} 0.4^{k} & 0 \\ 0 & 1^{k} \end{bmatrix} \begin{bmatrix} \frac{5}{6} & -\frac{1}{6} \\ \frac{1}{6} & -\frac{1}{6} \end{bmatrix} \begin{bmatrix} 100 \\ 100 \end{bmatrix} \end{aligned}$$

As $k \to \infty$,

$$D^{\infty} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{aligned} \vec{v}_k &= \begin{bmatrix} 1 & 1 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{5}{6} & -\frac{1}{6} \\ \frac{1}{6} & -\frac{1}{6} \end{bmatrix} \begin{bmatrix} 100 \\ 100 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} \frac{5}{6} & -\frac{1}{6} \\ \frac{1}{6} & -\frac{1}{6} \end{bmatrix} \begin{bmatrix} 100 \\ 100 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{6} & \frac{1}{6} \\ \frac{5}{6} & \frac{5}{6} \end{bmatrix} \begin{bmatrix} 100 \\ 100 \end{bmatrix} \\ &= \begin{bmatrix} \frac{200}{6} \\ \frac{1000}{6} \end{bmatrix} \end{aligned}$$

Since,

$$\Rightarrow \frac{200}{6} + \frac{1000}{6} = \frac{1200}{6} = 200$$

The solution is correct.

Q1.f

Transition matrix, let $Ax = \lambda x$, then,

$$\begin{aligned} &\Rightarrow Ax - \lambda x \\ &\Rightarrow (A - \lambda I) = 0 \end{aligned}$$

If, $\lambda = 1$,

$$\left[\begin{array}{c|c} -0.50.1 & 0 \\ 0.5 \quad -0.1 & 0 \end{array}\right] \; R_2 -> R_2 + R_1 \qquad \sim \qquad \left[\begin{array}{cc|c} -0.50.1 & & 0 \\ 0 & 0 & 0 \end{array}\right] \; x_1 = -x_2$$

As done in Q 1.e, the eigenvector of $\lambda = 1$ is $x = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

Q2.a

(i)

The function is written as below

```

function [final_eig_val, final_eig_vect] = power_method_infinity_norm (A, tol)
%
% This function performs the Power Method until two successive iterations
% have a difference of less than the tolerance
%
% Function declaration: [final_eig_val, final_eig_vect] = power_method_infinity_norm (A, tol)
%
% Inputs:
% A = n x n matrix
% tol = tolerance value
%
% Outputs:
% final_eig_val = eigen-value
% final_eig_vect = corresponding eigen-vector
%
% Authors: Danny, Grace


%% Obtain starting matrix (y)
row_col = size(A); % Returns 1 x 2 matrix with dimensions of A
n = row_col(1);
y = ones(n,1); % Create n x 1 starting matrix

%% Find 0th eigen-value
eig_val_now = norm(A*y, Inf); % Infinity norm of (A*y)
eig_vect_now = y;
% Hence, 0th eigen-value (eig_val) and eigen-vector (eig_vect) obtained

%% Initialise diff to 0 (to be used to compare with tol)
diff = abs(dot((A * eig_vect_now), eig_vect_now) / dot(eig_vect_now, eig_vect_now));

%% Iterate
while diff >= tol

    eig_val_before = eig_val_now;
    eig_vect_before = eig_vect_now;

    % Get latest eigen-vector and eigen-value
    eig_vect_now = (A * eig_vect_before) / norm((A * eig_vect_before), Inf);
    eig_val_now = norm((A * eig_vect_now), Inf);

    % Get Rayleigh Quotient
    rayleigh_before = dot((A * eig_vect_before), eig_vect_before) / dot(eig_vect_before, eig_vect_before);
    rayleigh_now = dot((A * eig_vect_now), eig_vect_now) / dot(eig_vect_now, eig_vect_now);

    % Find difference
    diff = abs(rayleigh_before - rayleigh_now);

end
```

```

final_eig_val = eig_val_now;
final_eig_vect = eig_vect_now;

end
```

The approximated eigenvalue and the corresponding eigenvalue was tested with the test file below:

```
clear all
clc

A = [-3.9 0.1 0.5 0.6;0.1 7.2 0.1 -0.5;0.5 0.1 1.1 0.3;0.6 -0.5 0.3 -10];
tol = 0.001;

[eigenvalue, eigenvector] = power_method_infinity_norm (A, tol)
```

Thus, the output is shown as:

```
eigenvalue =

   10.0755


eigenvector =

    0.0958
   -0.0223
    0.0229
   -1.0000
```

as expected

(ii)

Similarly, the Inverse power method function is written as below

```
function [eig_val_now,x, i]=inverse_method(A,tol)
%Power method for computing eigenvalues
%% Obtain starting matrix (y)
%row_col = size(A); % Returns 1 x 2 matrix with dimensions of A
%n = row_col(1);
%x = ones(n,1); % Create n x 1 starting matrix

%% Find 0th eigen-value
%eig_val_now = norm(A*y, Inf); % Infinity norm of (A*y)
%eig_vect_now = y;
% Hence, 0th eigen-value (eig_val) and eigen-vector (eig_vect) obtained

row_col = size(A);
eig_val_now = 0;
shift = 0;

N = row_col(1);
I = eye(N);
x = ones(N,1);
x(1) = 0;
x(N) = 0;
conv = 10000;
eig_val_before= 0;
i = 0;

 % compute shifted inverse matrix
B = inv(A - shift*I);

while (conv > tol)
   i = i + 1;
   y = B*x;
   x = y / norm(y,2);
   eig_val_now = x'*A*x;
   conv = abs(eig_val_now-eig_val_before) / abs(eig_val_now);
   eig_val_before = eig_val_now;
end
```

The approximated eigenvalue and the corresponding eigenvalue was tested with the test file below:

```
clear all
clc

A = [-3.9 0.1 0.5 0.6;0.1 7.2 0.1 -0.5;0.5 0.1 1.1 0.3;0.6 -0.5 0.3 -10];
tol = 0.001;

[eigenvalue,eigenvector, iteration]=inverse_method(A,tol)
```

Thus, the output is shown as:

```
eig_val_now =

    1.1596


x =

    0.1010
   -0.0147
    0.9942
    0.0328


i =

     4
```

as expected

Q2.b

(i)

Choose starting vector, $y^{(0)} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.To find the eigenvalue:

$$\Rightarrow Ay^{(0)} = \begin{bmatrix} -3.9 & 0.1 & 0.5 & 0.6 \\ 0.1 & 7.2 & 0.1 & -0.5 \\ 0.5 & 0.1 & 1.1 & 0.3 \\ 0.6 & -0.5 & 0.3 & -10 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2.7 \\ 6.9 \\ 2 \\ -9.6 \end{bmatrix}$$

Thus, $\|Ay^{(0)}\|_\infty = 9.6$, which indicates $\lambda_0 = 9.6$

1st iteration:

$$\Rightarrow y^{(0)} = \frac{A_{y^{(0)}}}{\|Ay^{(0)}\|_\infty} = \begin{bmatrix} -2.7 \\ 6.9 \\ 2 \\ -9.6 \end{bmatrix} / \, 9.6 = \begin{bmatrix} -0.28128 \\ 0.71375 \\ 0.20833 \\ -1 \end{bmatrix}$$

To find the eigenvalue:

$$\Rightarrow Ay^{(1)} = \begin{bmatrix} -3.9 & 0.1 & 0.5 & 0.6 \\ 0.1 & 7.2 & 0.1 & -0.5 \\ 0.5 & 0.1 & 1.1 & 0.3 \\ 0.6 & -0.5 & 0.3 & -10 \end{bmatrix} \begin{bmatrix} -0.28128 \\ 0.71375 \\ 0.20833 \\ -1 \end{bmatrix} = \begin{bmatrix} 0.0729 \\ 5.6677 \\ -0.1346 \\ 9.5344 \end{bmatrix}$$

Thus, $\|Ay^{(1)}\|_\infty = 9.5344$, which indicates $\lambda_0 = 9.5344$

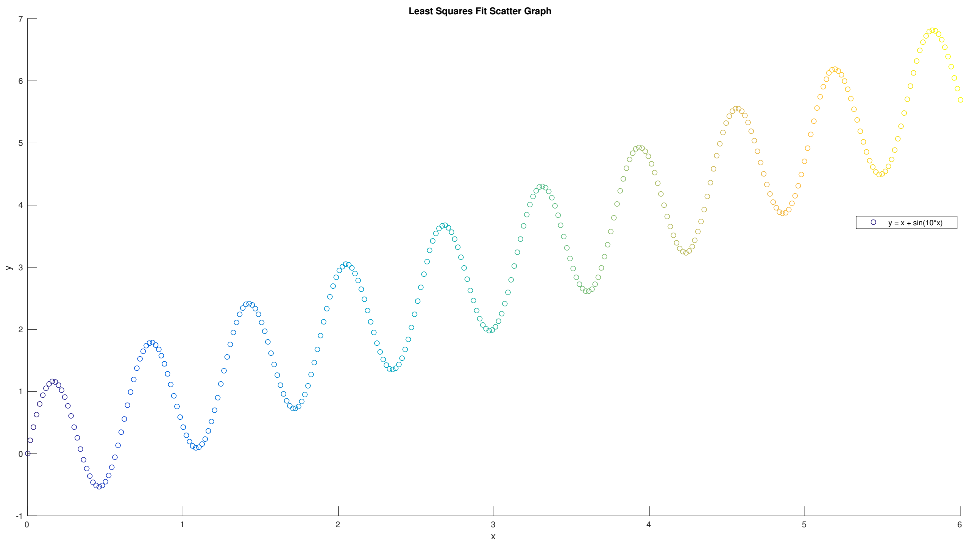

Figure 2: Scatter plot of the sin function.

(ii)

Q3

(a-d)

(e)