EMTH 211 Linear Algebra
Assignment

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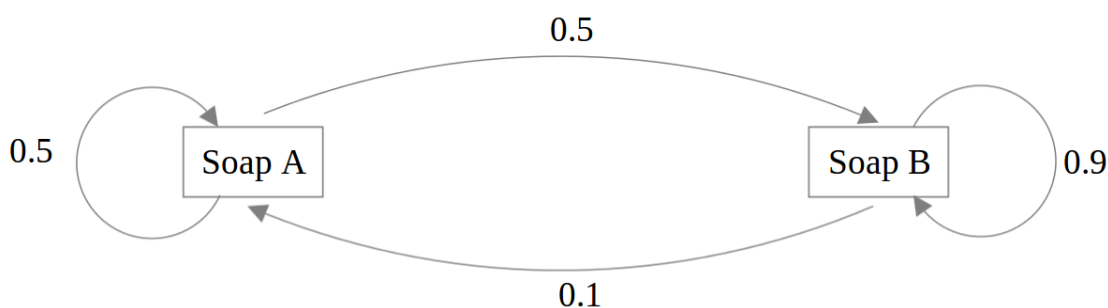
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Q1.a

$$0.5$$

$$0.5 \quad \boxed{\text{Soap A}} \qquad \boxed{\text{Soap B}} \quad 0.9$$

$$0.1$$

Figure 1: A directed graph of the senario.

Assume that, Soap A and B have 100 soaps respectively

$$A_k{\prime} = 0.5A_k + 0.1B_k$$
$$B_k{\prime} = 0.5A_k + 0.9B_k$$

where A_k = customers of soap A for this month and B_k = customers of soap B for this month, and $A_k{\prime}$ = customers of soap A for next month and $B_k{\prime}$ = customers of soap B for next month.

Q1.b

$$\begin{bmatrix} A_k{\prime} \\ B_k{\prime} \end{bmatrix} = \begin{bmatrix} 0.5 & 0.1 \\ 0.5 & 0.9 \end{bmatrix} \begin{bmatrix} A_k \\ B_k \end{bmatrix} \text{ (from 1.1)}$$

Q1.c

In one month's time

$$\Rightarrow \begin{bmatrix} A_k{\prime} \\ B_k{\prime} \end{bmatrix} = \begin{bmatrix} 0.5 & 0.1 \\ 0.5 & 0.9 \end{bmatrix} \begin{bmatrix} 100 \\ 100 \end{bmatrix}$$
$$= \begin{bmatrix} 50 + 10 \\ 50 + 90 \end{bmatrix} = \begin{bmatrix} 60 \\ 140 \end{bmatrix}$$

Q1.d

In two month's time

$$\Rightarrow \begin{bmatrix} A_k'' \\ B_k'' \end{bmatrix} = \begin{bmatrix} 0.5 & 0.1 \\ 0.5 & 0.9 \end{bmatrix} \begin{bmatrix} A_k' \\ B_k' \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 & 0.1 \\ 0.5 & 0.9 \end{bmatrix} \begin{bmatrix} 60 \\ 140 \end{bmatrix}$$

$$= \begin{bmatrix} 30 + 14 \\ 30 + 126 \end{bmatrix} = \begin{bmatrix} 44 \\ 156 \end{bmatrix}$$

Q1.e

In one year's time, the equation is:

$$\begin{bmatrix} A_k''' \\ B_k''' \end{bmatrix} = \begin{bmatrix} 0.5 & 0.1 \\ 0.5 & 0.9 \end{bmatrix}^{12} \begin{bmatrix} 100 \\ 100 \end{bmatrix} \text{(from 1.1)}$$

To find the eignevalues:

$$det(A - \lambda I) = 0$$

$$det \begin{bmatrix} 0.5 - \lambda & 0.1 \\ 0.5 & 0.9 - \lambda \end{bmatrix} = 0$$

$$\Rightarrow (0.5 - \lambda)(0.4 - \lambda) - 0.05 = 0$$

$$\Rightarrow \lambda^2 - 1.4\lambda = 0.4 = 0$$

$$\Rightarrow 5\lambda^2 - 7\lambda + 2 = 0$$

$$\Rightarrow (5\lambda - 2)(\lambda - 1) = 0$$

Hence, $\lambda_1 = 0.4$ and $\lambda_2 = 1$

To find the eigenvectors at $\lambda_1 = 0.4$,

$$\begin{bmatrix} 0.1 & 0.1 & | & 0 \\ 0.5 & 0.5 & | & 0 \end{bmatrix} \quad R_2 -> R_2 - 5R_1 \qquad \sim \quad \begin{bmatrix} 0.1 & 0.1 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}_{x_1 = -x_2}$$

Thus, $x_1 = \begin{bmatrix} 44 \\ 156 \end{bmatrix}$

To find the eigenvectors At $\lambda_2 = 1$,

$$\begin{bmatrix} -0.5 & 0.1 & | & 0 \\ 0.5 & -0.1 & | & 0 \end{bmatrix} \quad R_2 -> R_2 + R_1 \qquad \sim \quad \begin{bmatrix} -0.5 & 0.1 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}_{x_2 = 5x_1}$$

Thus, $x_2 = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

Therefore,

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 0.4 & 0 \\ 0 & 1 \end{bmatrix}, P = \begin{bmatrix} 1 & 1 \\ -1 & 5 \end{bmatrix}, P^{-1} = \begin{bmatrix} \dfrac{5}{6} & -\dfrac{1}{6} \\ \dfrac{1}{6} & \dfrac{1}{6} \end{bmatrix}$$

Let

$$\begin{bmatrix} A_{k+2} \\ B_{k+2} \end{bmatrix} = \begin{bmatrix} 0.5 & 0.1 \\ 0.5 & 0.9 \end{bmatrix}^{12} \begin{bmatrix} 100 \\ 100 \end{bmatrix}$$

2

be $\vec{v}_1 = A^2 \vec{v}_0$, where $A = PDP^{-1}$.

Thus,

$$\Rightarrow \vec{v}_1 = PDP^{-1}\vec{v}_0 = \begin{bmatrix} 1 & 1 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} 0.4^{12} & 0 \\ 0 & 1^{12} \end{bmatrix} \begin{bmatrix} \dfrac{5}{6} & -\dfrac{1}{6} \\ \dfrac{1}{6} & \dfrac{1}{6} \end{bmatrix} \begin{bmatrix} 100 \\ 100 \end{bmatrix}$$

The calculation was implemted via matlab. Matlab shows

```
>> Z = P*(D^12)*P_Inv*v

Z =

    33.3345
   166.6655

>> Z

Z =

    33.3345
   166.6655

>> Z(1) + Z(2)

ans =

   200
```

which is correct.

In the long run:

$$\vec{v}_k = A^k \vec{v}_0$$
$$= PDP^{-1}\vec{v}_0$$

$$= \begin{bmatrix} 1 & 1 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} 0.4^k & 0 \\ 0 & 1^k \end{bmatrix} \begin{bmatrix} \dfrac{5}{6} & -\dfrac{1}{6} \\ \dfrac{1}{6} & -\dfrac{1}{6} \end{bmatrix} \begin{bmatrix} 100 \\ 100 \end{bmatrix}$$

As $k \to \infty$,

$$D^{\infty} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

3

$$\vec{v}_k = \begin{bmatrix} 1 & 1 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dfrac{5}{6} & -\dfrac{1}{6} \\ \dfrac{1}{6} & -\dfrac{1}{6} \end{bmatrix} \begin{bmatrix} 100 \\ 100 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} \dfrac{5}{6} & -\dfrac{1}{6} \\ \dfrac{1}{6} & -\dfrac{1}{6} \end{bmatrix} \begin{bmatrix} 100 \\ 100 \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{1}{6} & \dfrac{1}{6} \\ \dfrac{5}{6} & \dfrac{5}{6} \end{bmatrix} \begin{bmatrix} 100 \\ 100 \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{200}{6} \\ \dfrac{1000}{6} \end{bmatrix}$$

Since,

$$\Rightarrow \frac{200}{6} + \frac{1000}{6} = \frac{1200}{6} = 200$$

The solution is correct.

Q1.f

Transition matrix, let $Ax = \lambda x$, then,

$$\Rightarrow Ax - \lambda x$$
$$\Rightarrow (A - \lambda I) = 0$$

If, $\lambda = 1$,

$$\begin{bmatrix} -0.5 & 0.1 & \big| & 0 \\ 0.5 & -0.1 & \big| & 0 \end{bmatrix} R_2 -> R_2 + R_1 \qquad \sim \qquad \begin{bmatrix} -0.5 & 0.1 & \big| & 0 \\ 0 & 0 & \big| & 0 \end{bmatrix}_{x_1 = -x_2}$$

As done in Q 1.e, the eigenvector of $\lambda = 1$ is $x = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

Q2.a

(i)

The function is written as below

```
1
2  function [final_eig_val, final_eig_vect] = power_method_infinity_norm (A, tol)
3  %
4  % This function performs the Power Method until two successive iterations
5  % have a difference of less than the tolerance
6  %
7  % Function declaration: [final_eig_val, final_eig_vect] = power_method_infinity_norm (A,
         tol)
8  %
9  % Inputs:
10 % A = n x n matrix
```

```
11    % tol = tolerance value
12    %
13    % Outputs:
14    % final_eig_val = eigen-value
15    % final_eig_vect = corresponding eigen-vector
16    %
17    % Authors: Danny, Grace
18
19
20    %% Obtain starting matrix (y)
21    row_col = size(A); % Returns 1 x 2 matrix with dimensions of A
22    n = row_col(1);
23    y = ones(n,1); % Create n x 1 starting matrix
24
25    %% Find 0th eigen-value
26    eig_val_now = norm(A*y, Inf); % Infinity norm of (A*y)
27    eig_vect_now = y;
28    % Hence, 0th eigen-value (eig_val) and eigen-vector (eig_vect) obtained
29
30    %% Initialise diff to 0 (to be used to compare with tol)
31    diff = abs(dot((A * eig_vect_now), eig_vect_now) / dot(eig_vect_now, eig_vect_now));
32
33    %% Iterate
34    while diff >= tol
35
36        eig_val_before = eig_val_now;
37        eig_vect_before = eig_vect_now;
38
39        % Get latest eigen-vector and eigen-value
40        eig_vect_now = (A * eig_vect_before) / norm((A * eig_vect_before), Inf);
41        eig_val_now = norm((A * eig_vect_now), Inf);
42
43        % Get Rayleigh Quotient
44        rayleigh_before = dot((A * eig_vect_before), eig_vect_before) / dot(eig_vect_before,
                eig_vect_before);
45        rayleigh_now = dot((A * eig_vect_now), eig_vect_now) / dot(eig_vect_now,
                eig_vect_now);
46
47        % Find difference
48        diff = abs(rayleigh_before - rayleigh_now);
49
50    end
51
52    final_eig_val = eig_val_now;
53    final_eig_vect = eig_vect_now;
54
55    end
```

The approximated eigenvalue and the corresponding eigenvalue was tested with the test file below:

```
1    clear all
2    clc
3
4    A = [-3.9 0.1 0.5 0.6; 0.1 7.2 0.1 -0.5; 0.5 0.1 1.1 0.3;0.6 -0.5 0.3 -10];%
5    %A = [3.5 1.5;1.5 -0.5];
6    noplot = 1;
7    tol = 0.001;
8
9    [final_eig_val, final_eig_vect] = power_method_infinity_norm (A, tol)
10   gershdisc(A)
11   [eig_val_now,x, i]=inverse_method(A,tol)
```

Thus, the output is shown as:

```
1    eigenvalue =
2
3        10.0755
4
5
6    eigenvector =
7
8        0.0958
9       -0.0223
10       0.0229
11      -1.0000
```

as expected

(ii)

Similarly, the Inverse power method function is written as below

```
1  function [eig_val_now,x, i]=inverse_method(A,tol)
2  %Power method for computing eigenvalues
3  %% Obtain starting matrix (y)
4  %row_col = size(A); % Returns 1 x 2 matrix with dimensions of A
5  %n = row_col(1);
6  %x = ones(n,1); % Create n x 1 starting matrix
7
8  %% Find 0th eigen-value
9  %eig_val_now = norm(A*y, Inf); % Infinity norm of (A*y)
10 %eig_vect_now = y;
11 % Hence, 0th eigen-value (eig_val) and eigen-vector (eig_vect) obtained
12
13 row_col = size(A);
14 eig_val_now = 0;
15 shift = 0;
16
17 N = row_col(1);
18 I = eye(N);
19 x = ones(N,1);
20 x(1) = 0;
21 x(N) = 0;
22 conv = 10000;
23 eig_val_before= 0;
24 i = 0;
25
26  % compute shifted inverse matrix
27 B = inv(A - shift*I);
28
29 while (conv > tol)
30     i = i + 1;
31     y = B*x;
32     x = y / norm(y,2);
33     eig_val_now = x'*A*x;
34     conv = abs(eig_val_now-eig_val_before) / abs(eig_val_now);
35     eig_val_before = eig_val_now;
36 end
```

The approximated eigenvalue and the corresponding eigenvalue was tested with the test file below:

```
1  clear all
2  clc
3
4  A = [-3.9 0.1 0.5 0.6;0.1 7.2 0.1 -0.5;0.5 0.1 1.1 0.3;0.6 -0.5 0.3 -10];
5  tol = 0.001;
6
7  [eigenvalue,eigenvector, iteration]=inverse_method(A,tol)
```

Thus, the output is shown as:

```
1  eig_val_now =
2
3      1.1596
4
5
6  x =
7
8      0.1010
9     -0.0147
10     0.9942
11     0.0328
12
13
14 i =
15
16      4
```

as expected

6

Q2.b

(i)

Choose starting vector, $y^{(0)} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. To find the eigenvalue:

$$\Rightarrow Ay^{(0)} = \begin{bmatrix} -3.9 & 0.1 & 0.5 & 0.6 \\ 0.1 & 7.2 & 0.1 & -0.5 \\ 0.5 & 0.1 & 1.1 & 0.3 \\ 0.6 & -0.5 & 0.3 & -10 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2.7 \\ 6.9 \\ 2 \\ -9.6 \end{bmatrix}$$

Thus, $\|Ay^{(0)}\|_\infty = 9.6$, which indicates $\lambda_0 = 9.6$

1st iteration:

$$\Rightarrow y^{(0)} = \frac{A_{y^{(0)}}}{\|Ay^{(0)}\|_\infty} = \begin{bmatrix} -2.7 \\ 6.9 \\ 2 \\ -9.6 \end{bmatrix} / 9.6 = \begin{bmatrix} -0.28128 \\ 0.71375 \\ 0.20833 \\ -1 \end{bmatrix}$$

To find the eigenvalue:

$$\Rightarrow Ay^{(1)} = \begin{bmatrix} -3.9 & 0.1 & 0.5 & 0.6 \\ 0.1 & 7.2 & 0.1 & -0.5 \\ 0.5 & 0.1 & 1.1 & 0.3 \\ 0.6 & -0.5 & 0.3 & -10 \end{bmatrix} \begin{bmatrix} -0.28128 \\ 0.71375 \\ 0.20833 \\ -1 \end{bmatrix} = \begin{bmatrix} 0.0729 \\ 5.6677 \\ -0.1346 \\ 9.5344 \end{bmatrix}$$

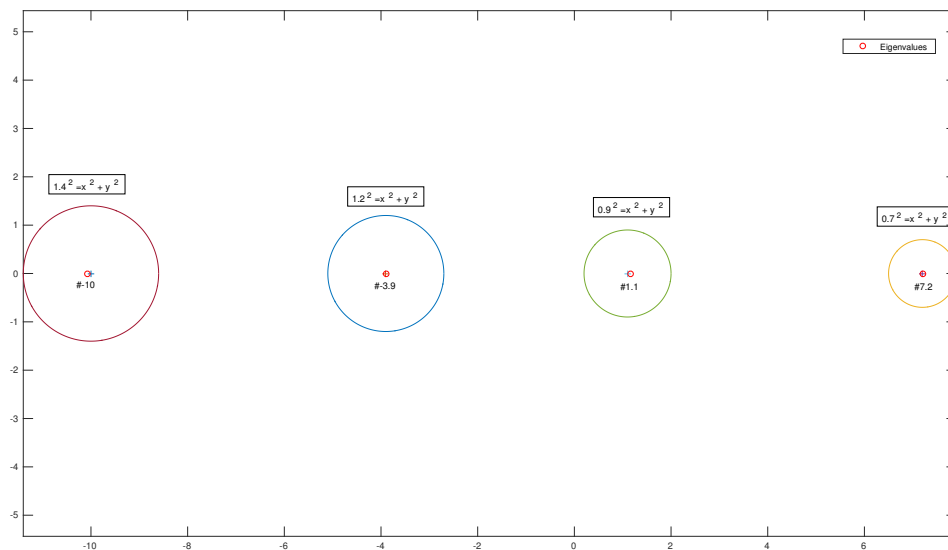Thus, $\|Ay^{(1)}\|_\infty = 9.5344$, which indicates $\lambda_0 = 9.5344$

(ii)

Figure 2: Gerschgorin Disc Circles.
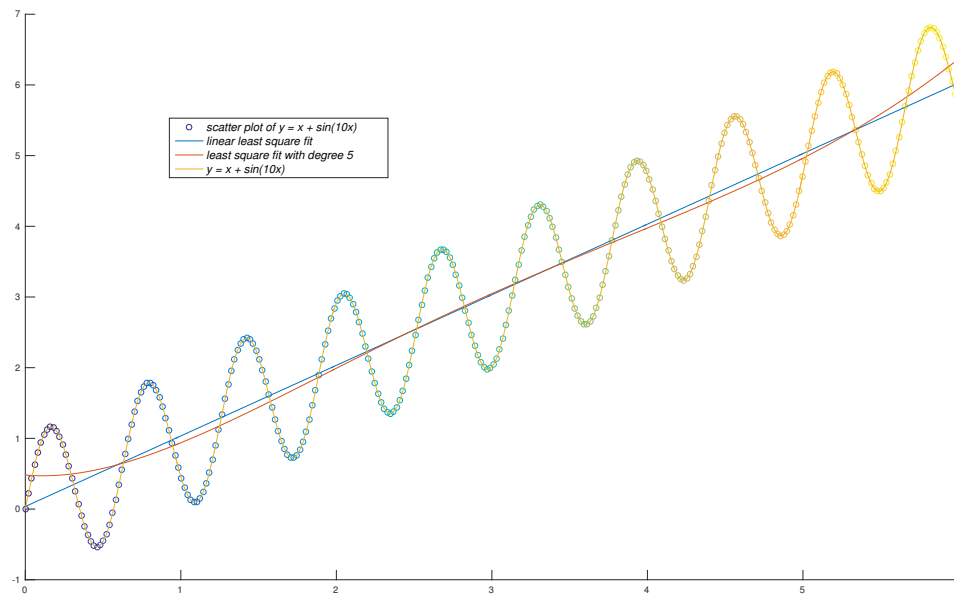
Q3

(a-d)

Figure 3: Scatter plot of the sin function.

(e)

The higher degree fits become less reliable than ones of lower degree because