FLIGHT DYNAMICS OF PROJECTILES

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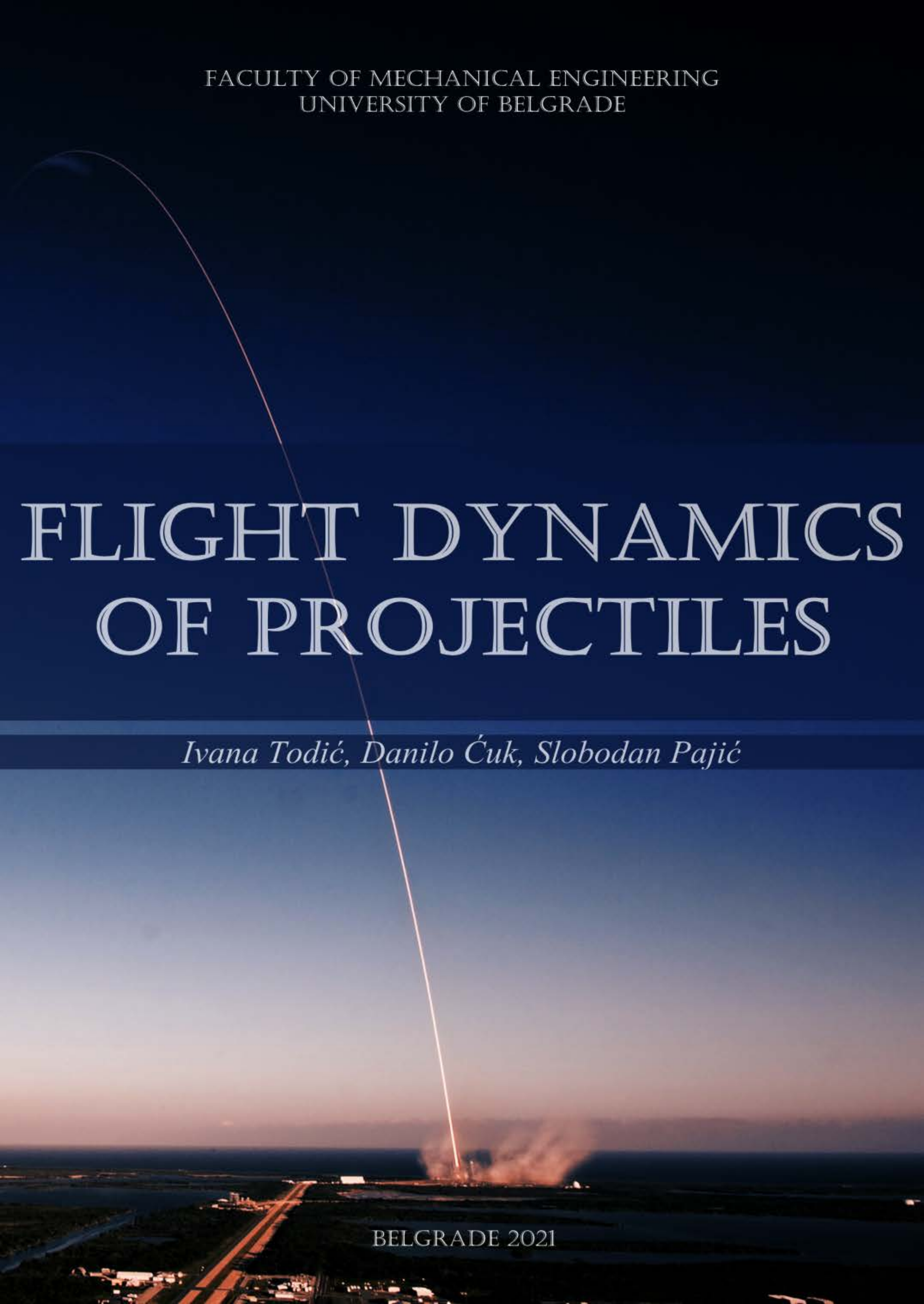

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Faculty of Mechanical Engineering
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“Scientists study the world as it is, engineers create the world that never has been”

- Theodore von Karman -

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PREFACE

For a long time, there has been a need for a book at the Faculty of Mechanical Engineering of the University of Belgrade that would give an overview of fundamental knowledge in the field of flight dynamics of both conventional and rocket projectiles. With selected materials, the book should provide an up-to-date guide to the methods for solving different problems in external ballistics.

The core of the material is based on the handouts of Ivana Todić, PhD, the present professor, and Danilo Ćuk, PhD, the previous professor, which were prepared for the course Flight Dynamics of Projectiles for international students at the faculty. The book will be of interest to students of the Weapons Systems Department who attend the courses in Flight Mechanics of the Projectile at the BSc and MSc levels. We hope that the book may also be useful to engineers for analysis and solving many practical problems in the design of projectiles.

The choice of materials and the setting of problems are in accordance with the other courses from the Weapons Systems Department. The book gives the basis of aerodynamics that is needed for an understanding of flight mechanics equations.

Within the material, valuable functions of the Matlab software package are given, which describe specific problems of projectile flight dynamics.

The book may be large for students in a single course. However, a number of sections were written in such a way that they may be omitted without affecting the other presented subject matters.

Authors

7 Equations of Motion

7.1 Equations of motion of an arbitrary system – rigid body

Newton's second law provides the equation of motion of the particle with mass δm:

$$\delta \boldsymbol{F} = \ddot{\boldsymbol{r}}\delta m = \dot{\boldsymbol{v}}\delta m \tag{7.1}$$

where

$\delta \boldsymbol{F}$ the resultant acting on the mass δm;
$\boldsymbol{r}$ the position vector of the mass δm ;
$\boldsymbol{v}$ the velocity of the mass δm.

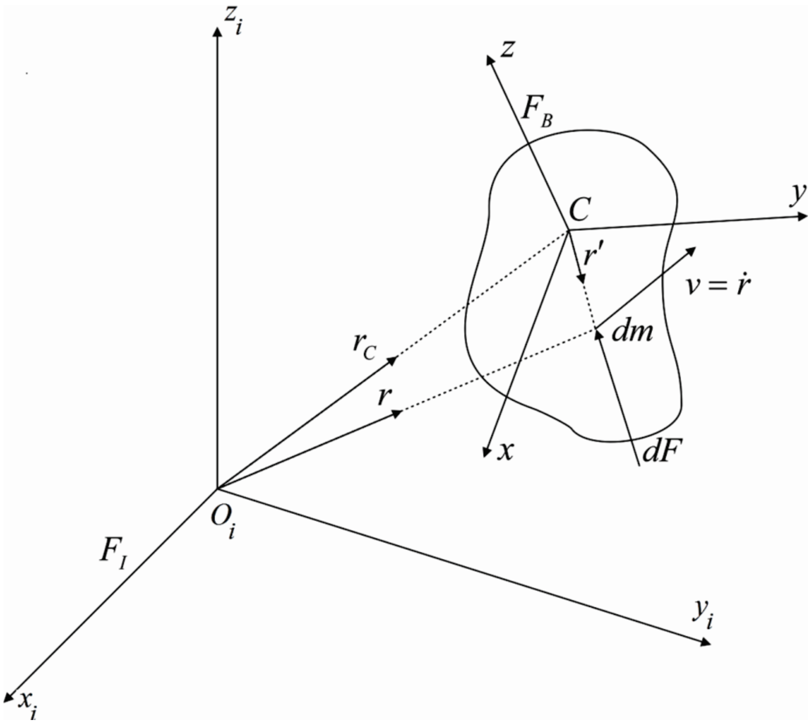

FIGURE 7-1 APPLICATION OF NEWTON'S LAW TO AN ELEMENT OF A BODY

The equation (7.1) is valid only in the inertial reference frame (Figure 7-1). Taking the cross product of Equation (7.1) with $\boldsymbol{r}$ gives the moment equation:

$$\boldsymbol{r} \times \delta \boldsymbol{F} = \boldsymbol{r} \times \dot{\boldsymbol{v}}\delta m \tag{7.2}$$

The angular momentum of δm with respect to $O_i \equiv O$ is:

$$\delta \boldsymbol{H}_O = \boldsymbol{r} \times \boldsymbol{v}\delta m \tag{7.3}$$

It follows that

$$\frac{d}{dt}(\delta \boldsymbol{H}_O) = \dot{\boldsymbol{r}} \times \boldsymbol{v}\delta m + \boldsymbol{r} \times \dot{\boldsymbol{v}}\delta m = \boldsymbol{v} \times \boldsymbol{v}\delta m + \boldsymbol{r} \times \dot{\boldsymbol{v}}\delta m = \boldsymbol{r} \times \dot{\boldsymbol{v}}\delta m$$
$$\frac{d}{dt}(\delta \boldsymbol{H}_O) = \boldsymbol{r} \times \dot{\boldsymbol{v}}\delta m \tag{7.4}$$

which is the right-hand side of Equation (7.2).

Since

$$\delta \boldsymbol{M}_O = \boldsymbol{r} \times \delta \boldsymbol{F}$$

is the moment of $\delta \boldsymbol{F}$ about $O_i \equiv O$, we have:

$$\delta \boldsymbol{M}_O = \frac{d}{dt}(\delta \boldsymbol{H}_O) \tag{7.5}$$

If we integrate Equations (7.1) and (7.5) for the system particles comprising a general deformable-body of mass *m*, we obtain:

$$\boldsymbol{F} = \int_V \dot{\boldsymbol{v}}\delta m \tag{7.6}$$

$$\boldsymbol{M}_O = \frac{d\boldsymbol{H}_O}{dt} \tag{7.7}$$

where

$$\boldsymbol{F} = \int_V \delta \boldsymbol{F} \tag{7.8}$$

$$\boldsymbol{M}_O = \int_V \boldsymbol{r} \times \delta \boldsymbol{F} \tag{7.9}$$

The vector $\boldsymbol{F}$ is the vector sum of all forces acting on all the elements. The internal forces vanish by Newton's third law of motion, so $\boldsymbol{F}$ is the resultant external force acting on the system of mass *m*.

The vector $\boldsymbol{M}_O$ is the resultant external moment about $O_i \equiv O$.

The position of the mass centre *C* is defined by $\boldsymbol{r}_C$:

$$m\boldsymbol{r}_C = \int_V \boldsymbol{r}\delta m \tag{7.10}$$

Differentiating once gives:

$$m\dot{\boldsymbol{r}}_C = \int_V \dot{\boldsymbol{r}}\delta m$$
$$m\boldsymbol{v}_C = \int_V \boldsymbol{v}\delta m \tag{7.11}$$

and second time

$$m\dot{\boldsymbol{v}}_C = \int_V \dot{\boldsymbol{v}}\delta m$$
$$m\boldsymbol{a}_C = \int_V \dot{\boldsymbol{v}}\delta m \tag{7.12}$$

The vectors $\boldsymbol{v}_C$ and $\boldsymbol{a}_C$ are respectively the velocity and acceleration of the mass centre relative to F_i.

From Equations (7.6) and (7.12) we obtain:

$$\boldsymbol{F} = m\boldsymbol{a}_C \tag{7.13}$$

The angular momentum with respect to O_i is:

$$\boldsymbol{H}_O = \int_V \boldsymbol{r}\times\boldsymbol{v}\delta m \tag{7.14}$$

Let

$$\boldsymbol{r} = \boldsymbol{r}_C + \boldsymbol{r}' \tag{7.15}$$

It may be shown from (7.10) that:

$$\int_V \boldsymbol{r}'\delta m = 0 \tag{7.16}$$

Expanding Equation (7.7) gives

$$\int_V (\boldsymbol{r}_C + \boldsymbol{r}')\times\delta\boldsymbol{F} = \frac{d}{dt}\int_V (\boldsymbol{r}_C + \boldsymbol{r}')\times\boldsymbol{v}\delta m$$

$$\boldsymbol{r}_C\times\boldsymbol{F} + \int_V \boldsymbol{r}'\times\delta\boldsymbol{F} = \boldsymbol{r}_C\times\int_V \dot{\boldsymbol{v}}\delta m + \frac{d}{dt}\int_V \boldsymbol{r}'\times\boldsymbol{v}\delta m = \boldsymbol{r}_C\times\boldsymbol{F} + \frac{d}{dt}\int_V \boldsymbol{r}'\times\boldsymbol{v}\delta m$$

$$\boldsymbol{M} = \frac{d\boldsymbol{H}}{dt} \tag{7.17}$$

where

$$\boldsymbol{M} = \boldsymbol{M}_C = \int_V \boldsymbol{r}' \times \delta \boldsymbol{F} \tag{7.18}$$

$$\boldsymbol{H} = \boldsymbol{H}_C = \int_V \boldsymbol{r}' \times \boldsymbol{v} \delta m \tag{7.19}$$

$\boldsymbol{M}$ the moment of the forces acting on a system about the mass centre, *C*.
$\boldsymbol{H}$ the angular momentum about the mass centre, *C*.

The equation (7.17) has a simple form, as in the case of a fixed point in the inertial space. The law about the angular momentum has the same form in the case of two points: fixed point in the inertial space and the mass centre of the system of particles.

Equations (7.13) and (7.17) are the two fundamental equations in the mechanics, and they are equivalent to six scalar differential equations.

THE ANGULAR MOMENTUM

The angular momentum with components in the inertial space F_i is (matrix notation):

$$\mathbf{H}^i = \int_V \mathbf{r}'^i \times \mathbf{v}^i \delta m = \int_V \mathbf{r}'^i \times \dot{\mathbf{r}}'^i \delta m \tag{7.20}$$

We want its components along the body axes (see Appendix A):

$$\dot{\mathbf{r}}'^i = \mathbf{C}_b^i \left(\dot{\mathbf{r}}'^b + \boldsymbol{\omega}_{ib}^b \times \mathbf{r}'^b \right) \tag{7.21}$$

$$\mathbf{H}^b = \mathbf{C}_i^b \mathbf{H}^i = \int_V \mathbf{C}_i^b \left(\mathbf{r}'^i \times \right) \mathbf{C}_b^i \left(\dot{\mathbf{r}}'^b + \boldsymbol{\omega}_{ib}^b \times \mathbf{r}'^b \right) \delta m$$

$$\mathbf{H}^b = \int_V \mathbf{r}'^b \times \left(\dot{\mathbf{r}}'^b + \boldsymbol{\omega}_{ib}^b \times \mathbf{r}'^b \right) \delta m = \int_V \mathbf{r}'^b \times \dot{\mathbf{r}}'^b \delta m + \int_V \left(\mathbf{r}'^b \times \right) \left(\boldsymbol{\omega}_{ib}^b \times \right) \mathbf{r}'^b \delta m \tag{7.22}$$

In the case of rigid body $\dot{\mathbf{r}}'^b = 0$ and the first term vanishes:

$$\mathbf{H}^b = \int_V \left(\mathbf{r}'^b \times \right) \left(\boldsymbol{\omega}_{ib}^b \times \right) \mathbf{r}'^b \delta m \tag{7.23}$$

Since

$$\boldsymbol{\omega} \times \boldsymbol{r}' = -\boldsymbol{r}' \times \boldsymbol{\omega}$$

it will be

$$\left(\boldsymbol{\omega}_{ib}^b \times \right) \mathbf{r}'^b = -\left(\mathbf{r}'^b \times \right) \boldsymbol{\omega}_{ib}^b$$

and

$$\mathbf{H}^b = -\int_V \left(\mathbf{r}'^b \times\right)\left(\mathbf{r}'^b \times\right)\boldsymbol{\omega}_{ib}^b \delta m \tag{7.24}$$

The matrix column $\boldsymbol{\omega}_{ib}^b$ is constant with respect to the integration, and we obtain:

$$\mathbf{H}^b = \mathbf{J}^b \boldsymbol{\omega}_{ib}^b \tag{7.25}$$

where

$$\mathbf{J}^b = -\int_V \left(\mathbf{r}'^b \times\right)\left(\mathbf{r}'^b \times\right)\delta m \tag{7.26}$$

$$\boldsymbol{\omega}_{ib}^b = \begin{bmatrix} p \\ q \\ r \end{bmatrix} \tag{7.27}$$

Having in mind the identity

$$\left(\mathbf{r}'^b \times\right)\left(\mathbf{r}'^b \times\right) = \mathbf{r}'^b \left(\mathbf{r}'^b\right)^T - \left(\mathbf{r}'^b\right)^T \mathbf{r}'^b \mathbf{I} \tag{7.28}$$

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{r}'^b = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

The matrix column $\mathbf{r}'^b$ represents the coordinates of the particle in the body reference frame. The matrix (7.26) may be written as:

$$\mathbf{J}^b = \begin{bmatrix} I_x & -I_{xy} & -I_{xz} \\ -I_{xy} & I_y & -I_{yz} \\ -I_{xz} & -I_{yz} & I_z \end{bmatrix} = \begin{bmatrix} A & -F & -E \\ -F & B & -D \\ -E & -D & C \end{bmatrix} \tag{7.29}$$

(Both types of notation are in current use in flight mechanics literature.) The notations in Equation (7.29) have the following meanings:

$$I_x = A = \int_V (y^2 + z^2)\delta m \quad I_y = B = \int_V (x^2 + z^2)\delta m \quad I_z = C = \int_V (x^2 + y^2)\delta m \tag{7.30}$$

$$I_{xy} = F = \int_V xy\delta m \quad \text{etc.} \tag{7.31}$$

I_x, I_y, I_z the moments of inertia;

I_{xy}, I_{xz}, I_{yz} the products of inertia.

The angular momentum of the rigid body with components in the body reference frame (omitting subscripts and superscripts for the body reference frame) is;

$$\mathbf{H} = \mathbf{J}\boldsymbol{\omega} \tag{7.32}$$

where

$$\mathbf{H} = \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} \qquad \boldsymbol{\omega} = \begin{bmatrix} p \\ q \\ r \end{bmatrix} \qquad \mathbf{J} = \mathbf{J}^b$$

In the case of a cruciform projectile, there are two planes of symmetry: hence products of inertia are equal to zero, and the transversal moments of inertia are equal one to the other:

$$I_{xy} = I_{xz} = I_{yz} = 0 \tag{7.33}$$

$$I_y = I_z \tag{7.34}$$

Hence

$$\mathbf{H} = \mathbf{J}\boldsymbol{\omega} = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_y \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} = \begin{bmatrix} I_x p \\ I_y q \\ I_y r \end{bmatrix} \tag{7.35}$$

By using the rate of change of a vector on Equation (7.17) with relative derivative in the body reference frame (see Appendix A), we obtain:

$$\boldsymbol{M} = \left(\frac{d\boldsymbol{H}}{dt}\right)_i = \left(\frac{d\boldsymbol{H}}{dt}\right)_b + \boldsymbol{\omega}_{ib} \times \boldsymbol{H} \tag{7.36}$$

The equation (7.36) may be written by using matrix notations with elements in the body reference frame:

$$\begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_y \end{bmatrix} \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} + \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix} \begin{bmatrix} I_x p \\ I_y q \\ I_z r \end{bmatrix} = \begin{bmatrix} L + L^F \\ M + M^F \\ N + N^F \end{bmatrix} \tag{7.37}$$

where L, M, N are aerodynamic moments, and L^F, M^F, N^F are the moments of the thrust in the body frame for the centre of gravity.

The development of Equation (7.37) gives:

$$\begin{aligned} I_x\dot{p} &= L + L^F \\ I_y\dot{q} - (I_y - I_x)pr &= M + M^F \\ I_y\dot{r} - (I_x - I_y)pq &= N + N^F \end{aligned} \tag{7.38}$$

The angular momentum in the aeroballistic reference frame:

$$\mathbf{H}^{ab} = \mathbf{C}_b^{ab}\begin{bmatrix} I_x p \\ I_y q \\ I_y r \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}\begin{bmatrix} I_x p \\ I_y q \\ I_y r \end{bmatrix} = \begin{bmatrix} I_x p \\ I_y(q\cos\phi - r\sin\phi) \\ I_y(q\sin\phi + r\cos\phi) \end{bmatrix} = \begin{bmatrix} I_x p \\ I_y\tilde{q} \\ I_y\tilde{r} \end{bmatrix} \tag{7.39}$$

By using the rate of change of a vector on Equation (7.17) with relative derivative in the aeroballistic reference frame, we obtain:

$$\boldsymbol{M} = \left(\frac{d\boldsymbol{H}}{dt}\right)_i = \left(\frac{d\boldsymbol{H}}{dt}\right)_{ab} + \boldsymbol{\omega}_{i,ab} \times \boldsymbol{H} \tag{7.40}$$

The equation (7.40) may be written by using matrix notations with elements in the aeroballistic reference frame:

$$\begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_y \end{bmatrix}\begin{bmatrix} \dot{p} \\ \dot{\tilde{q}} \\ \dot{\tilde{r}} \end{bmatrix} + \begin{bmatrix} 0 & -\tilde{r} & \tilde{q} \\ \tilde{r} & 0 & -p_{ab} \\ -\tilde{q} & p_{ab} & 0 \end{bmatrix}\begin{bmatrix} I_x p \\ I_y\tilde{q} \\ I_z\tilde{r} \end{bmatrix} = \begin{bmatrix} L + L^F \\ \tilde{M} + \tilde{M}^F \\ \tilde{N} + \tilde{N}^F \end{bmatrix} \tag{7.41}$$

The development of Equation (7.37) gives:

$$\begin{aligned} I_x\dot{p} &= L + L^F \\ I_y\dot{\tilde{q}} - I_y p_{ab}\tilde{r} + I_x p\tilde{r} &= \tilde{M} + \tilde{M}^F \\ I_y\dot{\tilde{r}} + I_y p_{ab}\tilde{q} - I_x p\tilde{q} &= \tilde{N} + \tilde{N}^F \end{aligned} \tag{7.42}$$

or, since $p_{ab} \approx 0$

$$\begin{aligned} I_x\dot{p} &= L + L^F \\ I_y\dot{\tilde{q}} + I_x p\tilde{r} &= \tilde{M} + \tilde{M}^F \\ I_y\dot{\tilde{r}} - I_x p\tilde{q} &= \tilde{N} + \tilde{N}^F \end{aligned} \tag{7.43}$$

where $L, \tilde{M}, \tilde{N}$ are aerodynamic moments and $L^F, \tilde{M}^F, \tilde{N}^F$ are the moments of the thrust in the aeroballistic frame for the centre of gravity.

7.2 GENERAL EQUATIONS OF MISSILE MOTION

In the continuation of this chapter, the equations of motion of a projectile in different coordinate frames and under different assumptions will be defined. Figure 7-2 shows a block diagram of the generalised equations of motion of the rocket. This block diagram summarises the relationship between all the concepts defined so far that describe the motion of a rocket. From the block diagram itself, it can already be seen that the solution of the equations of motion does not have an analytical form and that it is necessary to solve them numerically.

The atmosphere model, gravity and the Earth model, in terms of the shape and rotation of the Earth, are defined in Chapter 2 and directly depend on the rocket's position in relation to the Earth.

The aerodynamic forces and moments, defined in Chapter 4, directly depend on the atmosphere and velocity in the body frame and thus also indirectly depend on the rocket's position.

The thrust and its moment are defined in Chapter 5. During the flight, they depend on the characteristics of the atmosphere, primarily atmospheric pressure, and thus again indirectly depend on the position of the rocket.

The postulate of equations of motion and their solution is closely related to the definition of coordinate frames and transformations, which are defined in Chapter 3.

The emphasis of this chapter is on the solution of Newton's second law and the equation of angular momentum, as well as on the calculation of the velocity and position of the rocket under different assumptions and thus in different coordinate frames.

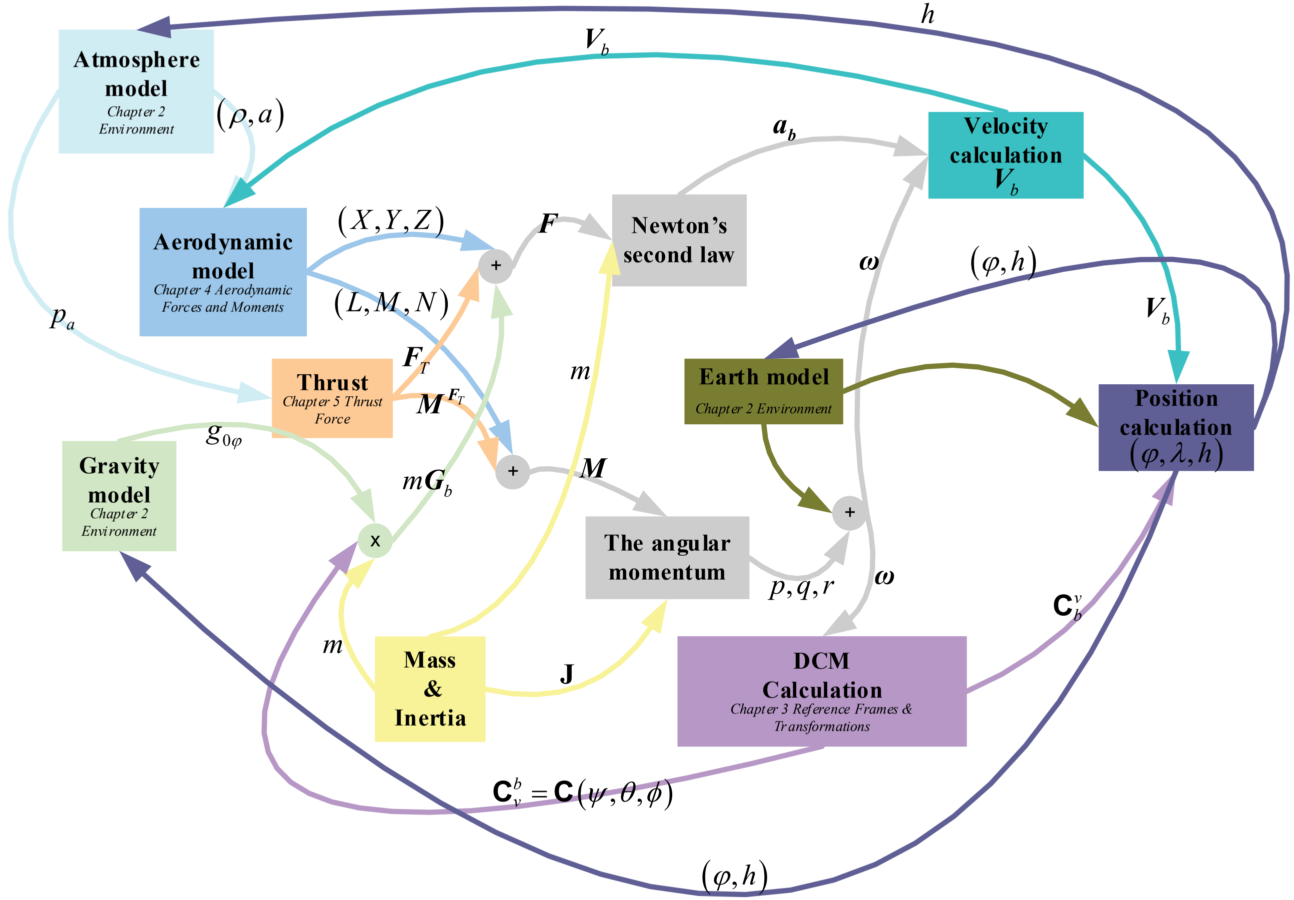

FIGURE 7-2 BLOCK DIAGRAM OF GENERAL EQUATIONS OF MISSILE MOTION

Consider the situation where it is required to compute the trajectory with respect to the inertial reference frame and non-accelerating and non-rotating set of axes. Let $\boldsymbol{r}$ represent the position vector of the point P with respect to the origin O (Figure 7-3).

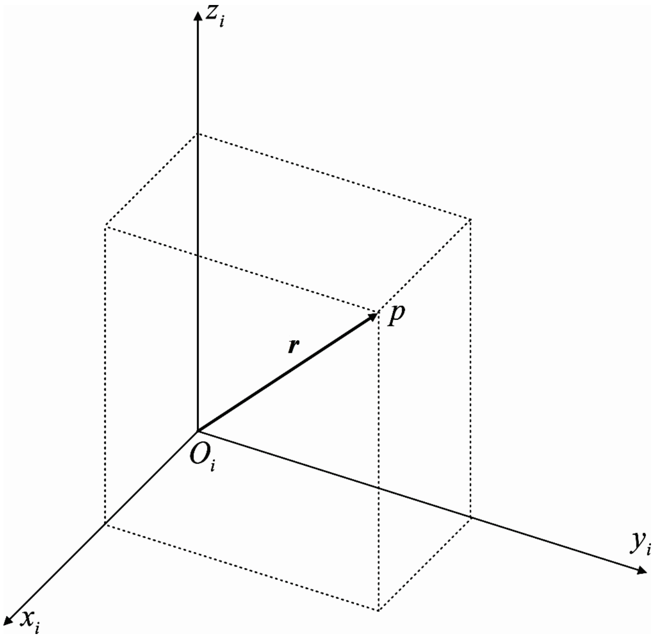

FIGURE 7-3 POSITION VECTOR IN THE INERTIAL REFERENCE FRAME

The acceleration of P with respect to the inertial frame is defined by

$$\boldsymbol{a}_i = \left.\frac{d^2\boldsymbol{r}}{dt^2}\right|_i \tag{7.44}$$

The difference between the total acceleration and gravitational acceleration is the specific force (non-gravitation acceleration) acting at the point P

$$\boldsymbol{f} = \left.\frac{d^2 r}{dt^2}\right|_i - \boldsymbol{g} \tag{7.45}$$

In which $\boldsymbol{g}$ is the mass attraction gravitation vector. The non-gravitational forces are aerodynamic force, thrust, etc.

The equation (7.45) can be rearranged as follows

$$\left.\frac{d^2\boldsymbol{r}}{dt^2}\right|_i = \boldsymbol{f} + \boldsymbol{g} \tag{7.46}$$

In practice, we often need to estimate a vehicle's velocity with respect to the Earth, i.e., the relative velocity – or according to the definition in flight mechanics, the kinematical velocity.

By using the Coriolis equation, the inertial velocity may be expressed in terms of kinematical velocity (see Appendix A)

$$\boldsymbol{v}_i = \left.\frac{d\boldsymbol{r}}{dt}\right|_i = \left.\frac{d\boldsymbol{r}}{dt}\right|_e + \boldsymbol{\omega}_{ie} \times \boldsymbol{r} \tag{7.47}$$

where

$$\left.\frac{d\boldsymbol{r}}{dt}\right|_e = \boldsymbol{v}_e = \boldsymbol{V}_k \tag{7.48}$$

($\boldsymbol{V}_k$ is the notation for a vehicle's velocity with respect to the Earth.)

The turn rate of the Earth $\boldsymbol{\omega}_{ie}$ is constant, and its derivative with respect to time is equal to zero.

$$\frac{d\boldsymbol{\omega}_{ie}}{dt} = 0 \tag{7.49}$$

Differentiating the expression (7.46) with the definition of kinematical velocity (ground velocity), we have

$$\left.\frac{d^2\boldsymbol{r}}{dt^2}\right|_i = \left.\frac{d\boldsymbol{v}_e}{dt}\right|_i + \left.\frac{d}{dt}\left[\boldsymbol{\omega}_{ie} \times \boldsymbol{r}\right]\right|_i \tag{7.50}$$

Substituting (7.47) and (7.49) into (7.50) one can obtain

$$\left.\frac{d^2\boldsymbol{r}}{dt^2}\right|_i = \left.\frac{d\boldsymbol{v}_e}{dt}\right|_i + \boldsymbol{\omega}_{ie} \times \boldsymbol{v}_e + \boldsymbol{\omega}_{ie} \times \left[\boldsymbol{\omega}_{ie} \times \boldsymbol{r}\right] \tag{7.51}$$

Keeping in mind Equation (7.46) the expression (7.51) can be rearranged

$$\left.\frac{d\boldsymbol{v}_e}{dt}\right|_i = \boldsymbol{f} - \boldsymbol{\omega}_{ie} \times \boldsymbol{v}_e - \boldsymbol{\omega}_{ie} \times \left[\boldsymbol{\omega}_{ie} \times \boldsymbol{r}\right] + \boldsymbol{g} \tag{7.52}$$

where

$\boldsymbol{f}$	represents the specific force to which the projectile is subjected
$\boldsymbol{\omega}_{ie} \times \boldsymbol{v}_e$	the Coriolis acceleration
$\boldsymbol{\omega}_{ie} \times \left[\boldsymbol{\omega}_{ie} \times \boldsymbol{r}\right]$	the centripetal acceleration due to the Earth's rotation

The centripetal acceleration is not separately distinguishable from the gravitational acceleration due to mass attraction $\boldsymbol{g}$. The sum of the accelerations caused by the mass attraction force and centripetal force is known as the local gravity vector (Figure 7-4).

$$\boldsymbol{g}_l = \boldsymbol{g} - \boldsymbol{\omega}_{ie} \times \left[\boldsymbol{\omega}_{ie} \times \boldsymbol{r}\right] \tag{7.53}$$

The local gravity vector is given by functions in terms of latitude and the height above the Earth's surface.

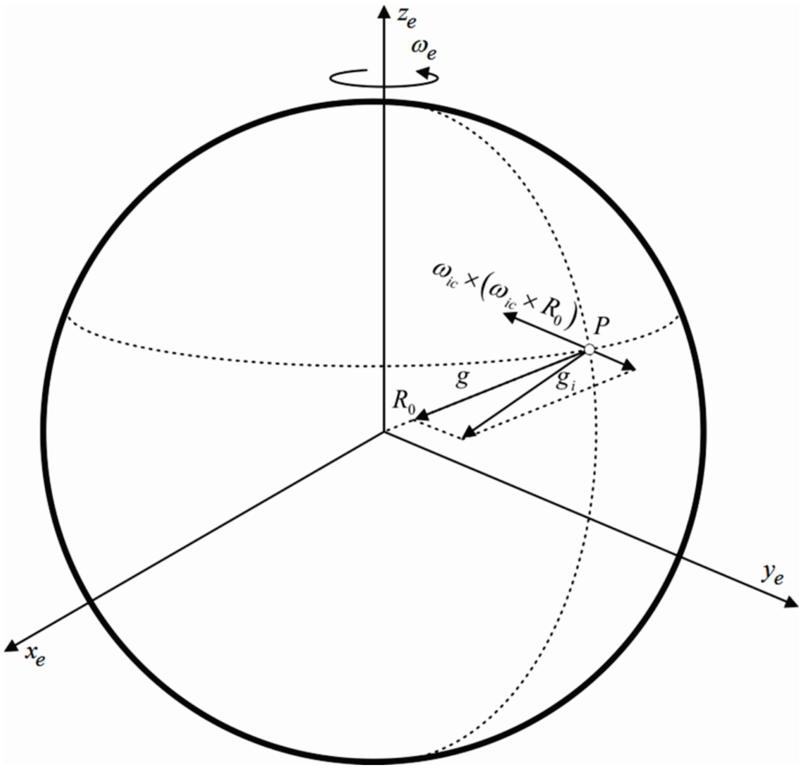

FIGURE 7-4 COMPONENTS OF THE GRAVITY VECTOR

By using the definition of the local gravity vector $\boldsymbol{g}_l$, the Equation (7.52) is transformed into

$$\left.\frac{d\boldsymbol{v}_e}{dt}\right|_i = \boldsymbol{f} - \boldsymbol{\omega}_{ie} \times \boldsymbol{v}_e + \boldsymbol{g}_l \tag{7.54}$$

We have not yet specified the reference frame in which we want to integrate the differential equations to obtain the missile's attitude, velocity, and position. This reference frame will be denoted as n -frame F_n . This reference frame may have an arbitrary angular rate. (In the case of an inertial navigation system, this frame is named the navigation reference frame because it is used to navigate a vehicle. We will use a terminology navigation frame for the reference frame in which the differential equations are integrated.)

Let us, for the present, leave the angular velocity of the navigation frame general and denote it by $\boldsymbol{\omega}_{in}$. Differentiation components will be given with respect to the navigation reference frame. The derivative of the kinematical velocity with respect to navigation frame is given by the following expression:

$$\left.\frac{d\boldsymbol{v}_e}{dt}\right|_i = \left.\frac{d\boldsymbol{v}_e}{dt}\right|_n + \boldsymbol{\omega}_{in} \times \boldsymbol{v}_e \tag{7.55}$$

Substituting Equation (7.55) into Equation (7.54) yields

$$\left.\frac{d\boldsymbol{v}_e}{dt}\right|_n = \boldsymbol{f} - (\boldsymbol{\omega}_{ie} + \boldsymbol{\omega}_{in}) \times \boldsymbol{v}_e + \boldsymbol{g}_l \tag{7.56}$$

If the navigation reference frame rotates with respect to the Earth, its angular velocity can be expressed as follows:

$$\boldsymbol{\omega}_{in} = \boldsymbol{\omega}_{ie} + \boldsymbol{\omega}_{en} \tag{7.57}$$

where:

$\boldsymbol{\omega}_{ie}$ the angular velocity of the Earth with respect to the inertial frame

$\boldsymbol{\omega}_{en}$ the angular velocity of the navigation frame with respect to the Earth.

Substituting (7.57) into (7.56) yields

$$\left.\frac{d\boldsymbol{v}_e}{dt}\right|_n = \boldsymbol{f} - (2\boldsymbol{\omega}_{ie} + \boldsymbol{\omega}_{en}) \times \boldsymbol{v}_e + \boldsymbol{g}_l \tag{7.58}$$

This equation may be expressed in navigation frame by using matrix notation:

$$\dot{\mathbf{v}}_e^n = \mathbf{f}^n - \left[2\boldsymbol{\omega}_{ie}^n + \boldsymbol{\omega}_{en}^n\right] \times \mathbf{v}_e^n + \mathbf{g}_l^n \tag{7.59}$$

Since the components of the specific force are calculated in the body reference frame, F_b, specific force in F_n can be expressed as:

$$\mathbf{f}^n = \mathbf{C}_b^n \mathbf{f}^b \tag{7.60}$$

Where $\mathbf{C}_b^n$ is a direction cosine matrix used to transform measured specific force vector into navigation frame. This matrix is computed by solving the matrix differential equation

$$\dot{\mathbf{C}}_b^n = \mathbf{C}_b^n \left(\boldsymbol{\omega}_{nb}^b \times\right) \tag{7.61}$$

Where $\left(\boldsymbol{\omega}_{nb}^{b}\times\right)$ is the skew-symmetric matrix of $\boldsymbol{\omega}_{nb}^{b}$, the body rate with respect to the navigation frame.

The body rate with respect to the navigation frame is derived by differencing the measured body rate $\boldsymbol{\omega}_{ib}^{b}$ and the estimates of the components of the navigation frame rate $\boldsymbol{\omega}_{in}$. Having in mind, (7.57) one can obtain.

$$\boldsymbol{\omega}_{nb}^{b} = \boldsymbol{\omega}_{ib}^{b} - \mathbf{C}_{n}^{b}\left[\boldsymbol{\omega}_{ie}^{n} + \boldsymbol{\omega}_{en}^{n}\right] \tag{7.62}$$

7.3 INTEGRATION OF DIFFERENTIAL EQUATIONS IN THE LOCAL GEOGRAPHIC REFERENCE FRAME $(F_n \equiv F_l)$

In order to determine the trajectory over large distances around the Earth, navigation information is most commonly required in the local geographic frame in terms of North, East and down velocity components

$$\mathbf{v}_{e}^{l} = \mathbf{v}_{e}^{n} = \begin{bmatrix} v_N & v_E & v_D \end{bmatrix} \tag{7.63}$$

and longitude, latitude and height above the Earth

$$\lambda,\quad \varphi,\quad h \tag{7.64}$$

Keeping in mind the properties of the local geographic reference frame as a navigation frame, we can write complete differential equations to determine the trajectory of a vehicle.

7.3.1 ATTITUDE RATE

$$\dot{\mathbf{C}}_{b}^{n} = \mathbf{C}_{b}^{n}\left(\boldsymbol{\omega}_{nb}^{b}\times\right) \tag{7.65}$$

where

$$\boldsymbol{\omega}_{nb}^{b} = \boldsymbol{\omega}_{ib}^{b} - \mathbf{C}_{n}^{b}\left[\boldsymbol{\omega}_{ie}^{n} + \boldsymbol{\omega}_{en}^{n}\right] \tag{7.66}$$

$$\boldsymbol{\omega}_{ie}^{n} = \mathbf{C}_{e}^{n}\boldsymbol{\omega}_{ie}^{e} = \begin{bmatrix} \Omega\cos\varphi & 0 & -\Omega\sin\varphi \end{bmatrix}^{T} \tag{7.67}$$

$$\boldsymbol{\omega}_{en}^{n} = \begin{bmatrix} \dfrac{v_E}{R_0 + h} & \dfrac{-v_N}{R_0 + h} & \dfrac{-v_E\tan\varphi}{R_0 + h} \end{bmatrix}^{T} \tag{7.68}$$

7.3.2 ACCELERATION TRANSFORMATION

$$\mathbf{f}^n = \mathbf{C}_b^n \mathbf{f}^b \tag{7.69}$$

7.3.3 VELOCITY RATE

$$\dot{\mathbf{v}}_e^n = \mathbf{f}^n - \left[2\boldsymbol{\omega}_{ie}^n + \boldsymbol{\omega}_{en}^n\right] \times \mathbf{v}_e^n + \mathbf{g}_l^n \tag{7.70}$$

where

$$\mathbf{g}_l^n = [\xi g \quad -\eta g \quad g]^T \tag{7.71}$$

g is the magnitude of the local gravity vector, which includes combined effects of the mass attraction of the Earth and the centripetal acceleration

ξ, η represent angular deflections in the direction of the local gravity vector with respect to the local vertical due to gravity anomalies (Figure 7-5)

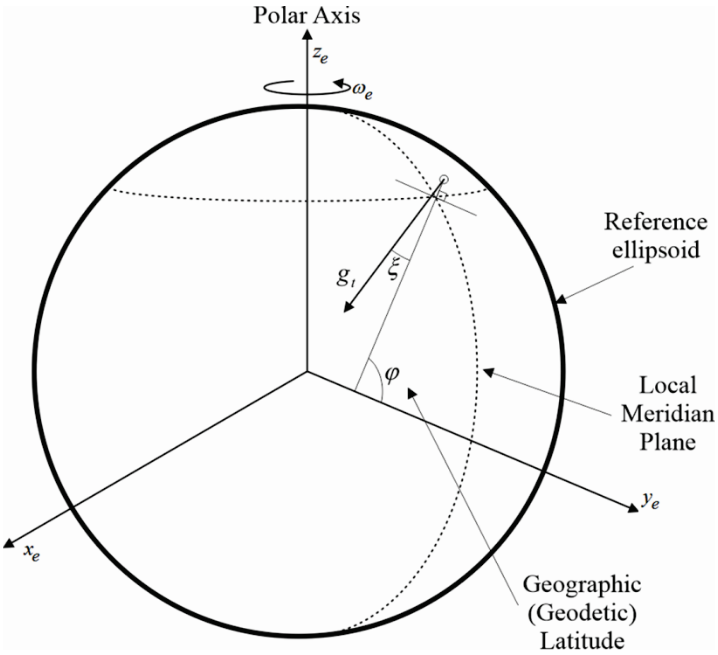

FIGURE 7-5 DEFLECTION OF LOCAL GRAVITY DUE TO ANOMALIES

7.3.4 POSITION RATE

$$\dot{\mathbf{C}}_n^e = \mathbf{C}_n^e \left(\boldsymbol{\omega}_{en}^n \times\right) \tag{7.72}$$

$$\dot{h} = -v_D \tag{7.73}$$

The initial value of the matrix $\mathbf{C}_b^n$ can be computed on the basis of the initial angular orientation of the body with respect to the local geographic reference frame as a navigation frame

$$\left(\mathbf{C}_b^n\right)_0 = \left(\mathbf{C}_b^l\right)_0 = \left(\mathbf{C}_l^b\right)_0^T = \mathbf{C}^T\left(\psi_0, \theta_0, \phi_0\right) \tag{7.74}$$

The initial value of the matrix $\left(\mathbf{C}_n^e\right)_0$ can be determined by using the latitude and longitude of the launch point

$$\left(\mathbf{C}_n^e\right)_0 = \left[\mathbf{C}_e^l\left(\lambda_0, \varphi_0\right)\right]^T \tag{7.75}$$

The initial height h_0 defines the initial height of a vehicle at the launch point.

The initial velocity with respect to the Earth is

$$\mathbf{v}_e^n = \begin{bmatrix} v_{N_0} & v_{E_0} & v_{D_0} \end{bmatrix}^T \tag{7.76}$$

Instead of Equation (7.65), with (7.66) one can apply the following expressions:

$$\dot{\mathbf{C}}_b^n = \mathbf{C}_b^n\left(\boldsymbol{\omega}_{ib}^b \times\right) - \left(\boldsymbol{\omega}_{in}^n \times\right)\mathbf{C}_b^n$$

$$\boldsymbol{\omega}_{in}^n = \boldsymbol{\omega}_{ie}^n + \boldsymbol{\omega}_{en}^n$$

The inputs for the integration of the differential equations are the body angular velocity and specific force components

$$\boldsymbol{\omega}_{ib}^b = \begin{bmatrix} p & q & r \end{bmatrix}^T \tag{7.77}$$

$$\mathbf{f}^b = \begin{bmatrix} f_x & f_y & f_z \end{bmatrix}^T \tag{7.78}$$

Instead of the matrix differential equation,(7.72) we can use two scalar differential equations:

$$\begin{aligned} \dot{\lambda} &= \frac{v_E}{\left(R_0 + h\right)\cos\varphi} \\ \dot{\varphi} &= \frac{v_N}{R_0 + h} \end{aligned} \tag{7.79}$$

7.6 EQUATIONS OF MOTION FOR SIX-DEGREES-OF-FREEDOM (6-DOF) TRAJECTORIES OF SPINNING PROJECTILE

The basis of the Six-Degrees-of-Freedom mathematical model is from Ref. *Modern Exterior Ballistics* by R. L. McCoy for the gyro-stabilised projectiles, where the differential equation of roll angle is turned off, was presented first. In comparison with previous 6DOF models, this one has an advantage in the choice of the smaller step of integration and better accuracy in the calculation of the trajectory of these types of projectiles. Then, the above approach is extended here to the gyro and fin-stabilised rocket projectiles with flat and wrap-around fins (WAF). Different perturbations such as small aerodynamic asymmetries and thrust misalignment are included to provide the estimation of dispersion.

We will adopt a right-handed, rectangular basic reference coordinate system $Ox_0y_0z_0$, with the origin of coordinates located at the gun muzzle. For the purpose of developing a mathematical model, the 6-DOF coordinate axis labels are changed to [1,2,3] instead of [x_0, y_0, z_0] labels we have used in the previous chapter on definitions of the reference frame. In our Earth's fixed coordinate system, the 1-3 plane is tangent to the Earth's surface at the launch point, the 1-axis points downrange, the 2-axis points vertically upward through the launch point, and the 3-axis points to the right when looking downrange. Figure 7-9 illustrates the trajectory and the 6-DOF coordinate system used.

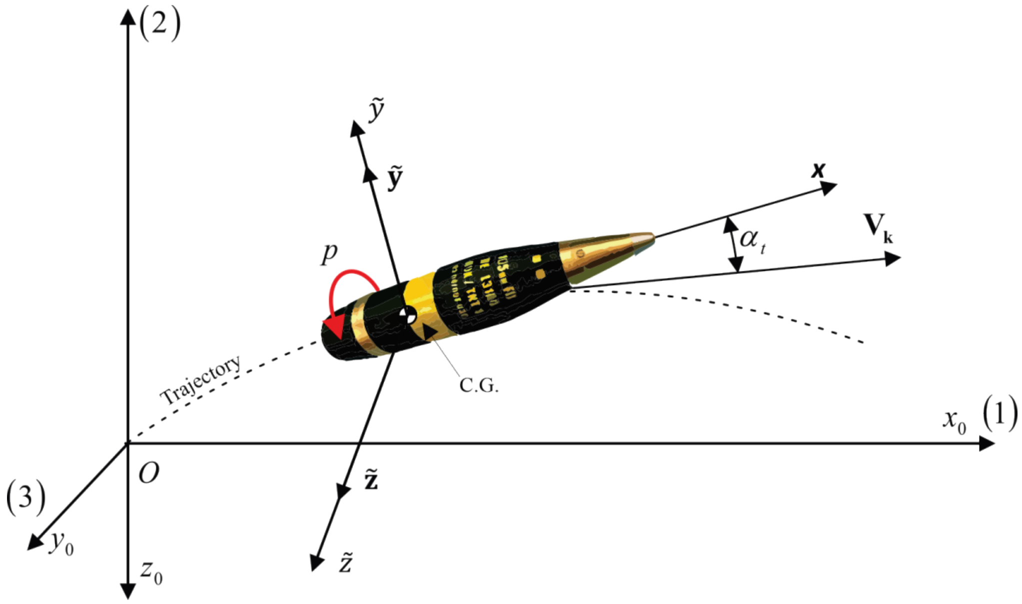

FIGURE 7-9 REFERENCE FRAME FOR SIX-DEGREES-OF-FREEDOM TRAJECTORIES

Newton's laws of motion state that the rate of change of linear momentum must be equal to the sum of all the externally applied forces. The rate of change of angular momentum must be equal to the sum of all the externally applied moments for the centre of gravity. Our 6-DOF equations of motion include additional forces and moments for the rocket projectile: thus, forces and moments due to rocket thrust and jet damping terms must be included as well. Newton's laws for the projectile are:

$$
\begin{aligned}
m\frac{d\boldsymbol{V}_k}{dt} &= \boldsymbol{R}^a + m\boldsymbol{g} + m\boldsymbol{a}_{cor} + \boldsymbol{F} \\
\frac{d\boldsymbol{H}}{dt} &= \boldsymbol{M} + \boldsymbol{M}^F
\end{aligned}
\tag{7.125}
$$

where

$\boldsymbol{V}_k$	vector velocity with respect to the ground-fixed coordinate axes
$\boldsymbol{R}^a$	the vector sum of all the aerodynamic forces
$\boldsymbol{g}$	acceleration due to gravity
$\boldsymbol{a}_{cor}$	Coriolis acceleration due to the Earth's rotation
$\boldsymbol{F}$	rocket thrust forces
$\boldsymbol{H}$	total vector angular momentum of the projectile
$\boldsymbol{M}$	the vector sum of all the aerodynamic moments referenced to the centre of mass
$\boldsymbol{M}^F$	rocket thrust moments referenced to the centre of mass
$\boldsymbol{L}^F$	rolling moment due to rocket spin-torque

We will choose a unit vector $\boldsymbol{x}$ along the projectile's axis of rotational symmetry, directed positive from tail to nose shown in Figure 7-10. The projectile is assumed to be both rigid (non-flexible) and rotationally symmetric about its spin axis; therefore, every transverse axis passes through the centre of mass and is perpendicular to the axis of symmetry, the principal axis of inertia. The total projectile vector angular momentum may be expressed as the sum of two vectors:

- the angular momentum about $\boldsymbol{x}$,
- and the angular momentum about an axis perpendicular to $\boldsymbol{x}$ and passing through the projectile's centre of mass.

$$
\boldsymbol{H} = I_x\boldsymbol{p} + I_y\boldsymbol{q}_t = I_x p\boldsymbol{x} + I_y q_t\boldsymbol{z}
\tag{7.126}
$$

where $\boldsymbol{z}$ is the transverse unit vector and q_t is the transverse angular velocity.

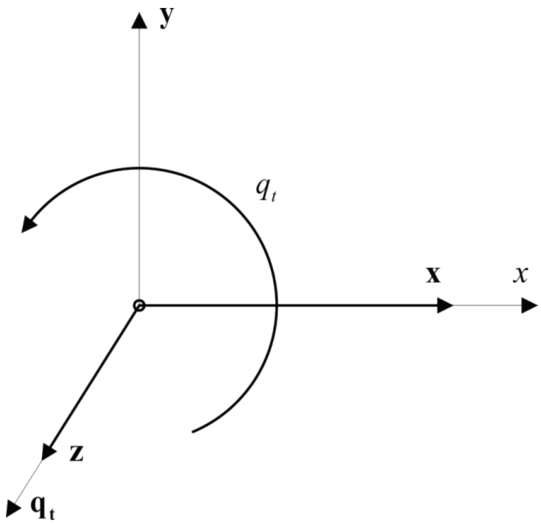

FIGURE 7-10 DERIVATIVE OF THE UNIT VECTOR *X*

Starting from Figure 7-10 we get

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{q}_t \times \boldsymbol{x} = q_t \boldsymbol{z} \times \boldsymbol{x} = q_t \boldsymbol{y} \tag{7.127}$$

$$\boldsymbol{x} \times \frac{d\boldsymbol{x}}{dt} = q_t \boldsymbol{x} \times \boldsymbol{y} = q_t \boldsymbol{z} \tag{7.128}$$

Substituting Equation (7.128) into (7.126) gives:

$$\boldsymbol{H} = I_x p \boldsymbol{x} + I_y \left(\boldsymbol{x} \times \frac{d\boldsymbol{x}}{dt} \right) \tag{7.129}$$

Now we set $\boldsymbol{h} = \frac{\boldsymbol{H}}{I_y}$, and divide both sides of Equation (7.129) by I_y:

$$\boldsymbol{h} = \frac{I_x}{I_y} p \boldsymbol{x} + \left(\boldsymbol{x} \times \frac{d\boldsymbol{x}}{dt} \right) \tag{7.130}$$

The rate of change of the vector angular momentum divided by I_y, is now given by:

$$\frac{d\boldsymbol{h}}{dt} = \frac{I_x \dot{p}}{I_y} \boldsymbol{x} + \frac{I_x p}{I_y} \frac{d\boldsymbol{x}}{dt} + \left(\boldsymbol{x} \times \frac{d^2 \boldsymbol{x}}{dt^2} \right) = \frac{I_x \dot{p}}{I_y} \boldsymbol{x} + \left(1 - \frac{I_x}{I_y} \right) p q_t \boldsymbol{y} + \dot{q}_t \boldsymbol{z} \tag{7.131}$$

We will need both the vector dot product and the cross product of $\boldsymbol{h}$ and $\boldsymbol{x}$. Performing the two vector operations on Equation (7.130):

$$\left(\boldsymbol{h}\bullet\boldsymbol{x}\right)=\frac{I_x p}{I_y} \tag{7.132}$$

$$\left(\boldsymbol{h}\times\boldsymbol{x}\right)=\left(\boldsymbol{x}\times\frac{d\boldsymbol{x}}{dt}\right)\times\boldsymbol{x}=\frac{d\boldsymbol{x}}{dt} \tag{7.133}$$

The six-degree-of-freedom vector differential equations of motion, for a rigid, rotationally symmetric projectile, acted on by all significant aerodynamic forces and moments, in addition to the wind, gravity and Coriolis forces, rocket thrust and spin torque, and jet damping forces and moments will be summarised in two vector differential equations.

DRAG

$$\begin{aligned}\boldsymbol{D}&=-\frac{1}{2}\rho V^2 SC_D\boldsymbol{i}_v=-\frac{1}{2}\rho VSC_D\boldsymbol{V}\\ \frac{\boldsymbol{D}}{m}&=-\frac{\rho VSC_D}{2m}\boldsymbol{V}\end{aligned} \tag{7.134}$$

C_D drag force coefficient

$\boldsymbol{V}$ vector velocity of the projectile with respect to the air ($\boldsymbol{V}=\boldsymbol{V}_k-\boldsymbol{W}$)

$\boldsymbol{W}$ vector wind velocity, relative to the Earth fixed frame

LIFT

$$\boldsymbol{L}=\frac{1}{2}\rho V^2 SC_{L_\alpha}\sin\alpha_t\boldsymbol{j}_v \tag{7.135}$$

C_{L_α} the gradient of lift force coefficient

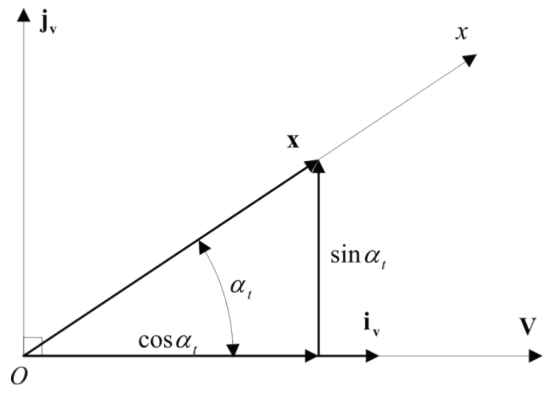

FIGURE 7-11 UNIT VECTOR OF THE LIFT, $\boldsymbol{j}_v$

From Figure 7-11, we obtain;

$$\begin{aligned} \cos\alpha_t \boldsymbol{i}_v + \sin\alpha_t \boldsymbol{j}_v &= \boldsymbol{x} \\ \boldsymbol{x} - \cos\alpha_t \boldsymbol{i}_v &= \sin\alpha_t \boldsymbol{j}_v \end{aligned} \tag{7.136}$$

Substitution of Equation (7.136) into (7.135) gives:

$$\boldsymbol{L} = \frac{1}{2}\rho V^2 S C_{L_\alpha}\left(\boldsymbol{x} - \cos\alpha_t \boldsymbol{i}_v\right) = \frac{1}{2}\rho S C_{L_\alpha}\left[V^2 \boldsymbol{x} - \left(\boldsymbol{V}\bullet\boldsymbol{x}\right)\boldsymbol{V}\right] \tag{7.137}$$

because of

$$V\cos\alpha_t = \left(\boldsymbol{V}\bullet\boldsymbol{x}\right) \tag{7.138}$$

The acceleration due to the lift:

$$\frac{\boldsymbol{L}}{m} = \frac{\rho S C_{L_\alpha}}{2m}\left[V^2 \boldsymbol{x} - \left(\boldsymbol{V}\bullet\boldsymbol{x}\right)\boldsymbol{V}\right] \tag{7.139}$$

MAGNUS FORCE

$$\boldsymbol{N}_{magnus} = \frac{1}{2}\rho V^2 S C_{N_{p\alpha}} \frac{dp}{V}\sin\alpha_t \boldsymbol{k}_v \tag{7.140}$$

$C_{N_{p\alpha}}$ gradient of Magnus force coefficient

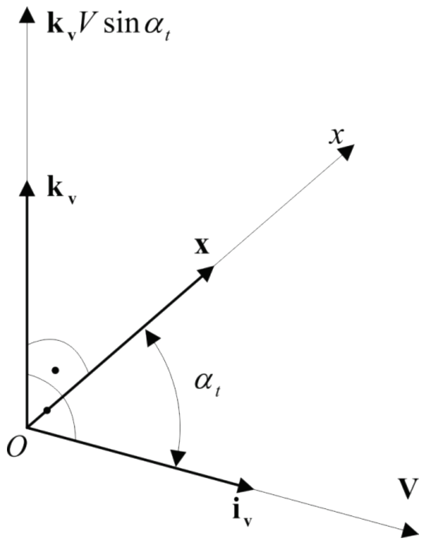

FIGURE 7-12 UNIT VECTOR OF THE MAGNUS FORCE, $\boldsymbol{k}_v$

Since

$$p = \frac{I_y}{I_x}(\boldsymbol{h}\bullet\boldsymbol{x}) \tag{7.141}$$

$$\boldsymbol{x}\times\boldsymbol{V} = -\boldsymbol{V}\times\boldsymbol{x} = -V\boldsymbol{i}_v\times\boldsymbol{x} = -V\sin\alpha_t\boldsymbol{k}_v \tag{7.142}$$

we have

$$\boldsymbol{N}_{magnus} = -\frac{1}{2}\rho SdC_{N_{p\alpha}}\frac{I_y}{I_x}(\boldsymbol{h}\bullet\boldsymbol{x})(\boldsymbol{x}\times\boldsymbol{V}) \tag{7.143}$$

The acceleration due to Magnus acceleration:

$$\frac{\boldsymbol{N}_{magnus}}{m} = -\frac{\rho SdC_{N_{p\alpha}}}{2m}\frac{I_y}{I_x}(\boldsymbol{h}\bullet\boldsymbol{x})(\boldsymbol{x}\times\boldsymbol{V}) \tag{7.144}$$

<u>DAMPING FORCE</u>

$$\boldsymbol{Y}_{Damping} = \frac{1}{2}\rho V^2 S\left(C_{N_q} + C_{N_{\dot\alpha}}\right)\frac{q_t d}{V}\boldsymbol{y} \tag{7.145}$$

$\left(C_{N_q} + C_{N_{\dot\alpha}}\right)$ pitch damping force coefficient

Having in mind Equations (7.128) and (7.133)

$$(\boldsymbol{h}\times\boldsymbol{x}) = \frac{d\boldsymbol{x}}{dt} = q_t\boldsymbol{y} \tag{7.146}$$

we obtain

$$\boldsymbol{Y}_{Damping} = \frac{1}{2}\rho VSd\left(C_{N_q} + C_{N_{\dot\alpha}}\right)(\boldsymbol{h}\times\boldsymbol{x}) \tag{7.147}$$

The acceleration due to damping force:

$$\frac{\boldsymbol{Y}_{Damping}}{m} = \frac{\rho VSd\left(C_{N_q} + C_{N_{\dot\alpha}}\right)}{2m}(\boldsymbol{h}\times\boldsymbol{x}) \tag{7.148}$$

ROCKET THRUST FORCES

The longitudinal component:

$$\boldsymbol{F}_x = F_x \boldsymbol{x} \tag{7.149}$$

The acceleration due to thrust:

$$\frac{\boldsymbol{F}_x}{m} = \frac{F_x}{m} \boldsymbol{x} \tag{7.150}$$

The components due to thrust misalignment are obtained from Figure 7-13:

$$F_n = F \varepsilon_F \tag{7.151}$$

$$F_{\tilde{y}} = -F_n \sin(\phi + \phi_\varepsilon) \tag{7.152}$$

$$F_{\tilde{z}} = F_n \cos(\phi + \phi_\varepsilon) \tag{7.153}$$

where ϕ is the roll angle with respect to the horizontal plane, and ϕ_ε is the angle of the plane of the thrust misalignment.

By using the unit vectors of the aeroballistics reference frame shown in Figure 7-9 we can write:

$$\begin{aligned} \boldsymbol{F}_{\tilde{y}} &= F_{\tilde{y}} \tilde{\boldsymbol{y}} \\ \boldsymbol{F}_{\tilde{z}} &= F_{\tilde{z}} \tilde{\boldsymbol{z}} \end{aligned} \tag{7.154}$$

or, in terms of accelerations

$$\begin{aligned} \frac{\boldsymbol{F}_{\tilde{y}}}{m} &= \frac{-F_n \sin(\phi + \phi_\varepsilon)}{m} \tilde{\boldsymbol{y}} \\ \frac{\boldsymbol{F}_{\tilde{z}}}{m} &= \frac{F_n \cos(\phi + \phi_\varepsilon)}{m} \tilde{\boldsymbol{z}} \end{aligned} \tag{7.155}$$

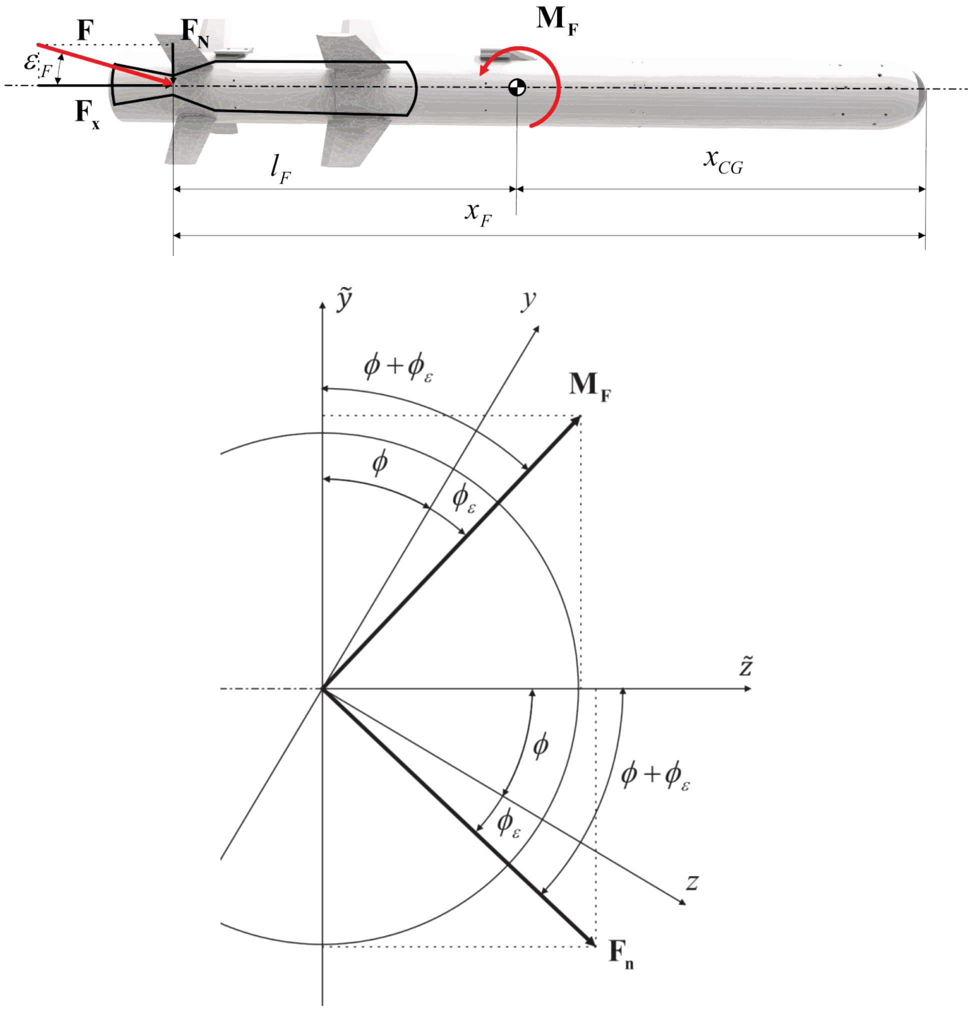

FIGURE 7-13 COMPONENTS OF THE THRUST AND ITS MOMENT DUE TO THRUST MISALIGNMENT

JET DAMPING FORCE

The velocity of the jet and the transverse angular rate generate the Coriolis force in the case of the rocket projectiles:

$$\frac{F_{jet\ damping}}{m} = \left(\frac{\dot{I}_y}{mr_t} - \frac{\dot{m}r_e}{m}\right) q_t \boldsymbol{y} = \left(\frac{\dot{I}_y}{mr_t} - \frac{\dot{m}r_e}{m}\right) (\boldsymbol{h} \times \boldsymbol{x}) \tag{7.156}$$

r_t distance from the projectile centre of mass to the throat of the rocket nozzle (positive if the throat is *aft* of the centre of mass)

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