MS&E 346 Assignment 8

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1 Problem 1

The state space $S_t = \{c_t, r_t, l_t\}$ where c_t denotes the cash amount at the end of day t (before borrowing and investing), r_t denotes the amount of money invested in risky assets (before borrowing and investing) and l_t denotes the liability. The action space is $A_t = \{y_t, x_t\}$ where y_t is the cash amount to borrow and x_t is the new amount of money invested in risk assets (if x_t is negative, that means we take money out of the risky assets). So $r_{t+1} = r_t + x_t$ and $c_{t+1} = c_t - R \cdot y_t - x_t + (l_{t+1} - l_t) - R_t$, where $R_t = K \cdot \cot(\frac{\pi c}{2C}) \cdot I(c_t - x_t + y_t < C)$. Our goal is to maximize the expected utility of assets less liabilities at the end of a T-day horizon conditional on any current situation of assets and liabilities, i.e.

$$\max_{y_t, x_t} \mathbb{E}[U(c_T + r_T - I_T) | (c_t, r_t, I_t)],$$

where $U(\cdot)$ is the utility function. This is a finite time MDP, so I will use backward reduction method to solve it.

2 Problem 2

2.1

We need to find the optimal supply S which minimizes the expected cost

$$g(S) = p \cdot g_1(S) + h \cdot g_2(S) = p \cdot \int_S^\infty (x - S) f(x) dx + h \cdot \int_{-\infty}^S (S - x) f(x) dx.$$

Taking derivatives w.r.t. S and setting it equals to 0, we get

$$g'(S) = -p \cdot \int_S^\infty f(x) dx + h \cdot \int_{-\infty}^S f(x) dx = -p + (p + h) \cdot \int_{-\infty}^S f(x) dx = 0,$$

which implies that $S^* = F^{-1}(\frac{p}{p+h})$, where $F(\cdot)$ is the cumulative distribution function of x.

2.2

If we denote the strike price be S and the underlying stock price at expiration be the random variable x, then the payoff of a European call option at expiration is $\max(x - S, 0)$ and the payoff of a European put option at expiration is $\max(S - x, 0)$. Suppose we decide to sell p units of European call option as well as h units of European put option at the same strike price S. Additionally, we know the probability distribution function of x as f. Then, the optimal strike price S which minimizes our expected obligation at expiration is

$$S^* = \arg\min_S \{p \cdot \mathbb{E}[\max(x - S, 0)] + h \cdot \mathbb{E}[\max(S - x, 0)]\} = \arg\min_S g(S).$$

This is the same as our original optimization problem.