MS&E 346 Assignment 6

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1.1

Since $U(x) = x - \frac{\alpha}{2}x^2$ with $x \sim \mathcal{N}(\mu, \sigma^2)$, we have

$$\mathbb{E}[U(x)] = \mathbb{E}[x - \frac{\alpha}{2}x^2] = \mu - \frac{\alpha}{2}(\sigma^2 + \mu^2).$$

1.2

According to definition, the certainty-equivalent value x_{CE} satisfies

$$x_{CE} - \frac{\alpha}{2}x_{CE}^2 = \mu - \frac{\alpha}{2}(\sigma^2 + \mu^2).$$

(1) When $\alpha > 0$, then if $\mu - \frac{\alpha}{2}(\sigma^2 + \mu^2) \geq 0$,

$$x_{CE} = \frac{1 - \sqrt{1 - 2\alpha(\mu - \frac{\alpha}{2}(\sigma^2 + \mu^2))}}{\alpha};$$

if $\mu - \frac{\alpha}{2}(\sigma^2 + \mu^2) < 0$,

$$x_{CE} = \frac{1 + \sqrt{1 - 2\alpha(\mu - \frac{\alpha}{2}(\sigma^2 + \mu^2))}}{\alpha}.$$

(2) When $\alpha < 0$, then

$$x_{CE} = \frac{1 + \sqrt{1 - 2\alpha(\mu - \frac{\alpha}{2}(\sigma^2 + \mu^2))}}{\alpha}$$

since $x \geq 0$. (3) When $\alpha = 0$, then $x_{CE} = \mu$.

1.3

Absolute risk-premium can be calculated by

$$\pi_A = \mathbb{E}[x] - x_{CE} = \mu - x_{CE}.$$

Plugging x_{CE} calculated in Section 1.2, we derive our result.

1.4

Our expected utility after one year is

$$\begin{aligned}\mathbb{E}[U(z)] &= \mathbb{E}[U(zx + (10^6 - z)r + 10^6)] \\ &= z\mu + (10^6 - z)r + 10^6 - \frac{\alpha}{2}\left[z^2(\sigma^2 + \mu^2) + ((10^6 - z)r + 10^6)^2 + 2((10^6 - z)r + 10^6)z\mu\right],\end{aligned}$$

which implies that

$$z^* = \arg\max_z \mathbb{E}[U(z)] = \frac{\left[1 - \alpha(10^6 r + 10^6)\right](\mu - r)}{\alpha(\sigma^2 + \mu^2 + r^2 - 2r\mu)}.$$

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2.1

My initial wealth is W_0 and suppose my bet is $f \cdot W_0$, then the wealth $W = (1-f)W_0 + fW_0(1+\alpha)$ with probability p and $W = (1-f)W_0 + fW_0(1-\beta)$ with probability $1-p$.

2.2

The two outcomes for utility is respectively: $U(W) = \log[(1-f)W_0 + fW_0(1+\alpha)]$ with probability p and $U(W) = \log[(1-f)W_0 + fW_0(1-\beta)]$ with probability $1-p$.

2.3

The expected utility can be calculated as

$$\mathbb{E}[\log(W)] = p \cdot \log[(1-f)W_0 + fW_0(1+\alpha)] + (1-p) \cdot \log[(1-f)W_0 + fW_0(1-\beta)].$$

2.4

Take derivative of $\mathbb{E}[\log(W)]$ with respect to f, we get

$$\begin{aligned}
\frac{\partial \mathbb{E}[\log(W)]}{\partial f} &= p \cdot \frac{\alpha W_0}{(1-f)W_0 + fW_0(1+\alpha)} + (1-p) \cdot \frac{-\beta W_0}{(1-f)W_0 + fW_0(1-\beta)} \\
&= p \cdot \frac{\alpha}{1+\alpha f} + (1-p) \cdot \frac{-\beta}{1-\beta f}.
\end{aligned}$$

2.5

We set $\partial \mathbb{E}[\log(W)]/\partial f = 0$ and obtain

$$f^* = \frac{\alpha p - \beta(1-p)}{\alpha\beta}.$$

2.6

This result makes intuitive sense, because when p and α increase, we want to increase the fraction f^* in betting; when β increases, the fraction f^* will be smaller.