MS&E 346 Assignment 7

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1 Problem 1

Merton's Portfolio problem with $\log(\cdot)$ utility function

The infinitesimal change in wealth dW_t can be calculated as

$$dW_t = [(r + \pi_t(\mu - r))W_t - c_t] \cdot dt + \sigma\pi_t W_t \cdot dz_t,$$

and our goal is to determine optimal (π_t, c_t) at time t to maximize

$$\mathbb{E}_t \left[\int_t^T e^{-\rho(s-t)} \cdot \log(c_s)ds + e^{-\rho(T-t)} \cdot [B(T) + \log(W_T)] \right].$$

Therefore, the optimal value function at time t is

$$V^*(t, W_t) = \max_{\pi_t, c_t} \mathbb{E}_t \left[\int_t^T e^{-\rho(s-t)} \cdot \log(c_s)ds + e^{-\rho(T-t)} \cdot [B(T) + \log(W_T)] \right].$$

We have the relation

$$\max_{\pi_t, c_t} \mathbb{E}_t [dV^* + \log(c_t)dt] = \rho \cdot V^* \cdot dt,$$

which can be transformed to

$$\max_{\pi_t, c_t} \mathbb{E}_t \left[\frac{\partial V^*}{\partial t} dt + \frac{\partial V^*}{\partial W_t} dW_t + \frac{1}{2} \frac{\partial^2 V^*}{\partial W_t^2} (dW_t)^2 + \log(c_t)dt \right] = \rho \cdot V^* \cdot dt$$

using Itô Lemma on dV^*. This implies that

$$\max_{\pi_t, c_t} \left\{ \frac{\partial V^*}{\partial t} + \frac{\partial V^*}{\partial W_t} [(r + \pi_t(\mu - r))W_t - c_t] + \frac{1}{2} \frac{\partial^2 V^*}{\partial W_t^2} \sigma^2 \pi_t^2 W_t^2 + \log(c_t) \right\} = \rho V^* \tag{1}$$

subject to the terminal condition

$$V^*(T, W_T) = B(T) + \log(W_T). \tag{2}$$

Assume the left hand side on (1) as $\max_{\pi_t, c_t} \Phi(t, W_t; \pi_t, c_t)$ and take partial derivatives w.r.t. π_t and c_t, we get

$$\begin{aligned} \pi_t^* &= -\frac{\frac{\partial V^*}{\partial W_t} \cdot (\mu - r)}{\frac{\partial^2 V^*}{\partial W_t^2} \cdot \sigma^2 \cdot W_t}, \\ c_t^* &= \left(\frac{\partial V^*}{\partial W_t} \right)^{-1}. \end{aligned} \tag{3}$$

Plugging this into (1), we have

$$\frac{\partial V^*}{\partial t} - \frac{(\mu - r)^2}{2\sigma^2} \frac{(\frac{\partial V^*}{\partial W_t})^2}{\frac{\partial^2 V^*}{\partial W_t^2}} + rW_t \frac{\partial V^*}{\partial W_t} - 1 - \log(\frac{\partial V^*}{\partial W_t}) = \rho V^*. \tag{4}$$

We guess the solution to be

$$V^* = B(t) + \log(W_t).$$

When $\rho = 1$, we plug this into (4) and derive

$$B(t) = \frac{(\mu - r)^2}{2\sigma^2} + r - 1 - e^{-(T-t)},$$

Finally, we can plug this into the expression of V^*, π_t^* and c_t^*.

2 Problem 3

The state space is $S = \{(b, s)\}$ where b denotes the employment status, i.e. $b \in \{U, E\}$ in which U represents unemployed and E represents employed, and s denotes the skill level at the beginning of a day. The action space is $A = \{\alpha | \alpha \in [0, 1]\}$ where a is the fraction of time you take to work. Suppose T_0 is the total minutes available each day and define $G(s) = (1 + g(s)) \cdot s$. The transition probability is

$$\mathbb{P}((b', s')|(b, s), a) = \begin{cases} p, & \text{if } b' = U, b = E, s' = G^{T_0(1-\alpha)}(s) \\ 1 - p, & \text{if } b' = E, b = E, s' = G^{T_0(1-\alpha)}(s) \\ h(s), & \text{if } b' = E, b = U, s' = s \cdot e^{-\lambda} \\ 1 - h(s), & \text{if } b' = U, b = U, s' = s \cdot e^{-\lambda} \\ 0, & \text{otherwise} \end{cases}$$

The reward function is

$$R((b, s), a) = f(s) \cdot \alpha T_0, \quad \text{if } b = E$$

and $R((b, s), a) = 0$ if $b = U$.