data Nov 2, 2020

- $ax+by+c=0$ general form of a straight line in 3-space
  Ex: $-2y+3x-4=0$
  is equivalent to $3x-2y-4=0$ $\vec{n}_1=(3,-2)$ is a perpendicular vector to the line called the normal vector.

In general, the line $ax+by+c=0$ has the normal vector $\vec{n}=(a,b)$.

Ex: $y=-\frac{1}{2}x+4$ is equivalent to $\frac{1}{2}x+y=4$
which implies that the normal vector is $\vec{n}_2=(\frac{1}{2},1)$

- The distance D between the line $\ell \ldots ax+by=c$ and the point $P(x_0,y_0)$ is given by

$$D=\frac{|ax_0+by_0-c|}{\sqrt{a^2+b^2}}$$

Ex Find the distance D between the point $P(-3,4)$ and the line $y=-5x+4$.
We first have to write the equation of the line to be $5x+y=4 \Rightarrow a=5,\ b=1$ and $c=4$
and $(x_0,y_0)=(-3,4)$

$$D=\frac{|5(-3)+4-4|}{\sqrt{5^2+1^2}}=\frac{15}{\sqrt{26}}$$

Straight Lines in 3-space

$$(x,y,z)=t\underbrace{(a,b,c)}_{\text{direction vector}}+\underbrace{(x_0,y_0,z_0)}_{\text{point on the line}}$$

$\vec{d}=(a,b,c)$ is called a direction vector.

For example, the line $(x,y,z) = t(-3,4,1) + (-2,1,4)$ ✓
is a line through the point $(-2,1,4)$ and the direction vector $\vec{d} = (-3,4,1)$

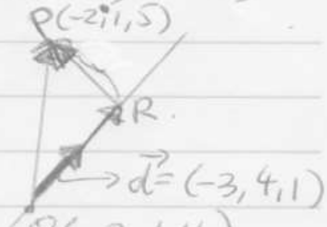
The equation of the line can also be written in this form:

$$(x,y,z) = (-3t, 4t, t) + (-2,1,4) \checkmark$$

or $L(t) = (x,y,z) = (-3t-2, 4t+1, t+4)$ ✓

(*) The three ticked equations are all equivalent equations of the same line

Ex Find the shortest distance D between the point $P(-2,1,5)$ and the line $(x,y,z) = (-3t-2, 4t+1, t+4)$ ✓



Reminder: $\text{proj}_{\vec{v}} \vec{u} = \dfrac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v}$

The distance D is the magnitude of $\vec{RP}$

$D = \|\vec{RP}\|$ and vector $\vec{QR} = \text{proj}_{\vec{d}} \vec{QP}$

$$\vec{QR} + \vec{RP} = \vec{QP}$$

$$\vec{QP} = (-2,1,5) - (-2,1,4) = (0,0,1)$$

$$\vec{QR} = \text{proj}_{\vec{d}} \vec{QP} = \frac{\vec{QP}\cdot\vec{d}}{\|\vec{d}\|^2}\vec{d} = \frac{(0,0,1)\cdot(-3,4,1)}{\left(\sqrt{(-3)^2+4^2+1^2}\right)^2}(-3,4,1)$$

$$= \frac{1}{26}(-3,4,1) = \left(\frac{-3}{26}, \frac{4}{26}, \frac{1}{26}\right)$$

$$\vec{QR} + \vec{RP} = \vec{QP} \Rightarrow \vec{RP} = \vec{QP} - \vec{QR}$$

$$= (0,0,1) - \left(\frac{-3}{26}, \frac{4}{26}, \frac{1}{26}\right)$$

$$= \left(\frac{3}{26}, \frac{-4}{26}, 1 - \frac{1}{26}\right) = \left(\frac{3}{26}, \frac{-4}{26}, \frac{25}{26}\right)$$

$$D = \|\vec{RP}\| = \sqrt{\left(\frac{3}{26}\right)^2 + \left(\frac{-4}{26}\right)^2 + \left(\frac{25}{26}\right)^2} = \sqrt{\frac{9}{(26)^2} + \frac{16}{(26)^2} + \frac{625}{(26)^2}} = \frac{\sqrt{650}}{26} = \frac{5\sqrt{26}}{26}$$

Therefore, the distance between $P(-2,1,5)$ and the line is $\frac{5\sqrt{26}}{26}$ units

In 2-space

EX Find the point $R(x,y)$ on the line $3x+4y-8=0$ which is closest to the point $P(-2,5)$.

Method 1 Using Slopes

$$3x+4y-8=0 \Rightarrow 4y=-3x+8$$
$$y=\frac{-3}{4}x+2 \qquad m_1=\frac{-3}{4} \text{ is the slope}$$

Any perpendicular line to the given line will have slope equal to $m_2=\frac{4}{3}$

$$P(-2,5) \quad m_2=\frac{4}{3} \Rightarrow y=\frac{4}{3}x+b$$
$$5=\frac{4}{3}(-2)+b$$
$$5+\frac{8}{3}=b$$
$$\frac{23}{3}=b$$

$\Rightarrow$ The perpendicular line (to the given line) that passes through $P(-2,5)$ is $y=\frac{4}{3}x+\frac{23}{3}$

The intersection point between the given line and the line that we just found will be the closest point to P.

Given line: $y=-\frac{3}{4}x+2$ }
Perpendicular line: $y=\frac{4}{3}x+\frac{23}{3}$ } Solve by comparison.

$$-\frac{3}{4}x+2=\frac{4}{3}x+\frac{23}{3}$$
$$-\frac{3}{4}x-\frac{4}{3}x=\frac{23}{3}-2$$
$$\left(-\frac{3}{4}-\frac{4}{3}\right)x=\frac{17}{3}$$
$$\frac{-25}{12}x=\frac{17}{3} \Rightarrow x=-\frac{12^{4}}{25}\cdot\frac{17}{\cancel{3}}=\frac{-68}{25}$$
$$y=\frac{4}{3}\left(\frac{-68}{25}\right)+\frac{23}{3}=\frac{4(-68)}{75}+\frac{23(25)}{3(25)}=\frac{-272+575}{75}=\frac{303}{75}=\frac{101}{25}$$

Therefore $R\left(\frac{-68}{25},\frac{101}{25}\right)$ is the point on the given line that is closest to the point $P(-2,5)$.