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$\vec{w} = k_1\vec{v}_1 + k_2\vec{v}_2 + \cdots + k_n\vec{v}_n$ $\quad$ $\vec{w}$ is a linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ with coefficients $k_1, k_2, \ldots, k_n$ respectively.

$\vec{v}_1\,(1,0)$ and $(0,1) = \vec{v}_2$

Any vector in the xy-coordinate system can be written as a linear combination of $\vec{v}_1$ and $\vec{v}_2$.

$\vec{u} = (5,-2) = 5(1,0) + (-2)(0,1) = 5(1,0) - 2(0,1).$

$\vec{w} = (-2,15) = -2(1,0) + 15(0,1)$

$\{(1,0),(0,1)\}$ is the $\underset{\text{orthonormal}}{\text{}}$ basis of the xy-coordinate system.

Ex
Write the vector $(3,5)$ as a linear combination of $(1,0)$ and $(0,1)$.

$$(3,5) = 3(1,0) + 5(0,1).$$

The orthonormal basis of the xyz-coordinate system are $\vec{i} = (1,0,0)$, $\vec{j} = (0,1,0)$, $\vec{k} = (0,0,1)$

$$(-3,4,2) = -3(1,0,0) + 4(0,1,0) + 2(0,0,1).$$
$$= -3\vec{i} + 4\vec{j} + 2\vec{k}$$

A <u>unit vector</u> is a vector that has magnitude equal to 1.

$(1,0)$ and $(0,1)$ are unit vectors in xy-coordinate system ($\mathbb{R}^2$).
$\vec{i}, \vec{j}, \vec{k}$ 〃 〃 〃 〃 xyz- ($\mathbb{R}^3$) 〃 〃

$\left(\frac{3}{5}, -\frac{4}{5}\right)$ is a unit vector because $\left\|\left(\frac{3}{5}, -\frac{4}{5}\right)\right\| = \sqrt{\frac{9}{25} + \frac{16}{25}} = 1.$

Property of Magnitudes: If $\vec{v}$ is a vector in $\mathbb{R}^2$ or $\mathbb{R}^3$ and k is a scalar, then $\|k\vec{v}\| = |k|\,\|\vec{v}\|$.

Ex Let $\vec{v} = (1,5,-3)$. Find $-2\vec{v}$, $\|\vec{v}\|$ and $\|-2\vec{v}\|$.

$-2\vec{v} = (-2,-10,6)$.

$\|\vec{v}\| = \sqrt{1^2+(5)^2+(-3)^2} = \sqrt{35}$.

$\|-2\vec{v}\| = \sqrt{(-2)^2+(-10)^2+6^2} = \sqrt{4+100+36} = \sqrt{140} \stackrel{?}{=} |-2|\sqrt{35}$

$= \sqrt{4(35)}$

$= 2\sqrt{35}$

Ex Verify the above property for the vector $\vec{u} = (1,-3)$ and $\vec{v} = (-2,6)$. $(\vec{v} = -2\vec{u})$.

$\|\vec{u}\| = \sqrt{1^2+(-3)^2} = \sqrt{10}$.

$\|\vec{v}\| = \sqrt{(-2)^2+6^2} = \sqrt{40} = \sqrt{4(10)} = 2\sqrt{10}$

$\|\vec{v}\| = |-2|\,\|\vec{u}\|$.

The vector $\frac{1}{\|\vec{v}\|}\vec{v}$ is a unit vector that is parallel to $\vec{v}$.

Ex Let $\vec{v} = (5,0,-3)$. Find $\frac{1}{\|\vec{v}\|}\vec{v}$ and its magnitude.

$\|\vec{v}\| = \sqrt{5^2+0^2+(-3)^2} = \sqrt{34}$.

$\frac{1}{\|\vec{v}\|}\vec{v} = \frac{1}{\sqrt{34}}(5,0,-3) = \left(\frac{5}{\sqrt{34}}, 0, \frac{-3}{\sqrt{34}}\right)$ is a unit vector.

$\left\|\frac{1}{\|\vec{v}\|}\vec{v}\right\| = \sqrt{\left(\frac{5}{\sqrt{34}}\right)^2+0^2+\left(\frac{-3}{\sqrt{34}}\right)^2} = \sqrt{\frac{25}{34}+0^2+\frac{9}{34}} = 1$.

$$\|k\vec{v}\| = |k|\,\|\vec{v}\|.$$

Ex Find two unit vectors that are parallel to vector $\vec{u} = (5,6)$

$\frac{1}{\|\vec{u}\|}\vec{u} = \frac{1}{\sqrt{5^2+6^2}}(5,6) = \left(\frac{5}{\sqrt{61}}, \frac{6}{\sqrt{61}}\right)$. This is one unit vector. that has the same direction as $\vec{u}$.

The other unit vector that is parallel to $\vec{u}$ is the vector

$\left(\frac{-5}{\sqrt{61}}, \frac{-6}{\sqrt{61}}\right)$ and has the opposite direction of $\vec{u}$.

Ex (a) find a unit vector with the same direction as $\vec{u} = (-5,12)$

(b) " " " " " the opposite " of $\vec{u} = (-5,12)$.

(a) $\left(\frac{-5}{13}, \frac{12}{13}\right)$ is a unit vector with same direction as $\vec{u}$.

(b) $\left(\frac{5}{13}, -\frac{12}{13}\right)$ " " " " " opp. " " $\vec{u}$.

Dot Product.

$\vec{u} = (u_1, u_2)$ $\qquad \vec{v} = (v_1, v_2)$.

$\vec{u}\cdot\vec{v} = (u_1, u_2)\cdot(v_1, v_2) = u_1v_1 + u_2v_2$.

for example, $\vec{u} = (-1,4)$ and $\vec{v} = (5,-3)$

$\vec{u}\cdot\vec{v} = (-1,4)\cdot(5,-3) = (-1)(5)+(4)(-3) = -5-12 = -17$

Let $\vec{x} = (x_1, x_2, x_3)$ and $\vec{y} = (y_1, y_2, y_3)$.

$\vec{x}\cdot\vec{y} = x_1y_1 + x_2y_2 + x_3y_3$.

for example, $\vec{x} = (1,-5,7)$ and $\vec{y} = (-3,2,1)$

$\vec{x}\cdot\vec{y} = (1,-5,7)\cdot(-3,2,1) = 1(-3)+(-5)(2)+7(1)$

$= -6$.

Ex (a) Find the dot product between vectors $\vec{u} = (3, -5, -2)$ and $\vec{v} = (1, 4, -8)$.

(b) Find the dot product between vectors $\vec{a} = (1, -3)$ and $\vec{b} = (4, 7)$

(a) $\vec{u} \cdot \vec{v} = (3, -5, -2) \cdot (1, 4, -8) = 3 - 20 + 16 = -1$.

(b) $\vec{a} \cdot \vec{b} = (1, -3) \cdot (4, 7) = 4 - 21 = -17$

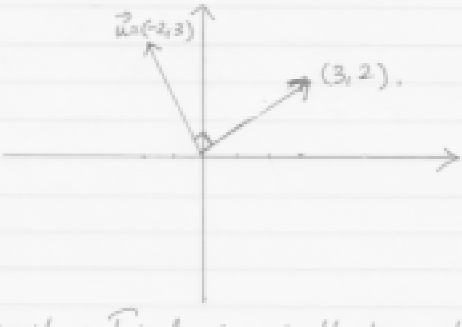
Rule: Let $\vec{u}$ and $\vec{v}$ be two non-zero vectors in $\mathbb{R}^2$ (or $\mathbb{R}^3$). Then $\vec{u}$ and $\vec{v}$ are perpendicular to each other (orthogonal) if and only if $\vec{u} \cdot \vec{v} = 0$.

For example, the vectors $(1, 0)$ and $(0, 1)$ are orthogonal vectors

$$(1, 0) \cdot (0, 1) = 0.$$

$$(-2, 3) \cdot (3, +2) = 0.$$

The vectors $(-2, 3)$ and $(3, 2)$ are orthogonal vectors.



Example: Find x so that vectors $\vec{u} = (4, x)$ and $\vec{v} = (-x, 2)$ are orthogonal to each other.

$$\vec{u} \cdot \vec{v} = (4, x) \cdot (-x, 2) = -4x + 2x = -2x = 0$$
$$x = 0$$

Example: Find y so that the vectors $\vec{u} = (1, 3, y)$ and $\vec{v} = (5, -3, 1)$ are orthogonal vectors.

$\vec{u} \cdot \vec{v} = 5 - 9 + y = 0 \Rightarrow -4 + y = 0 \Rightarrow y = 4.$