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$$-x+2y=5$$
$$3x+5y=-3$$

$$\underset{2\times 2}{\begin{bmatrix}-1 & 2\\ 3 & 5\end{bmatrix}}\underset{2\times 1}{\begin{bmatrix}x\\ y\end{bmatrix}}=\begin{bmatrix}5\\ -3\end{bmatrix}$$

$$\underset{\text{Coeff. matrix}}{A}\;\underset{\text{Matrix of variables}}{X}=\underset{\text{Matrix of Constant}}{B}$$

If $A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$ and if $ad-bc\neq 0$, then the inverse of A, denoted by A^{-1}, is given as follows.

$$A^{-1}=\frac{1}{ad-bc}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}=\begin{bmatrix}\frac{d}{ad-bc} & \frac{-b}{ad-bc}\\ \frac{-c}{ad-bc} & \frac{a}{ad-bc}\end{bmatrix}$$

To verify $AA^{-1}=A^{-1}A=I_{2\times2}$

$$\underset{A}{\begin{bmatrix}a & b\\ c & d\end{bmatrix}}\underset{A^{-1}}{\begin{bmatrix}\frac{d}{ad-bc} & \frac{-b}{ad-bc}\\ \frac{-c}{ad-bc} & \frac{a}{ad-bc}\end{bmatrix}}=\begin{bmatrix}\frac{ad}{ad-bc}+\frac{-bc}{ad-bc} & \frac{-ab}{ad-bc}+\frac{ba}{ad-bc}\\ \frac{cd}{ad-bc}+\frac{-dc}{ad-bc} & \frac{-cb}{ad-bc}+\frac{da}{ad-bc}\end{bmatrix}$$

$$=I_{2\times 2}$$

$$=\begin{bmatrix}\frac{ad-bc}{ad-bc} & 0\\ 0 & \frac{ad-bc}{ad-bc}\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

Similarily $A^{-1}A=I_{2\times2}$.

Ex Find the inverses of the following matrices (if possible):

$$A=\begin{bmatrix}2 & 3\\ -4 & -6\end{bmatrix}\text{ and }B=\begin{bmatrix}5 & -6\\ 2 & 1\end{bmatrix}.$$

Matrix A: $ad-bc=2(-6)-3(-4)=-12+12=0\Rightarrow$ A does not have an inverse (A is not invertible).

non-invertible

singular.

Matrix B: $ad-bc = 5(1)-(-6)(2) = 5+12 = 17 \neq 0$

$\Rightarrow$ B does have an inverse.

$$B = \begin{bmatrix} 5 & -6 \\ 2 & 1 \end{bmatrix} \qquad B^{-1} = \frac{1}{17}\begin{bmatrix} 1 & -(-6) \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 1/17 & 6/17 \\ -2/17 & 5/17 \end{bmatrix}.$$

Check: $$BB^{-1} = \underbrace{\begin{bmatrix} 5 & -6 \\ 2 & 1 \end{bmatrix}}_{2\times 2} \underbrace{\begin{bmatrix} 1/17 & 6/17 \\ -2/17 & 5/17 \end{bmatrix}}_{2\times 2} = \begin{bmatrix} 17/17 & 0 \\ 0 & 17/17 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore, $B^{-1} = \begin{bmatrix} 1/17 & 6/17 \\ -2/17 & 5/17 \end{bmatrix}$

<u>Ex</u> Find the inverses of the following matrices (if possible):

$$A = \begin{bmatrix} -5 & 2 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -3 \\ 2 & -6 \end{bmatrix}.$$

$ad-bc = -17 \neq 0$ $\qquad$ $ad-bc = 0 \Rightarrow$ B is a singular matrix.

<u>Note</u>: In the previous example, A was not invertible and if we look at the 2 rows of A, we can see that they are scalar multiples of each other.

In this example, B is not invertible. We can see again that the 2 rows of B are scalar multiple of each other.

$$A^{-1} = -\frac{1}{17}\begin{bmatrix} 3 & -2 \\ -1 & -5 \end{bmatrix} = \begin{bmatrix} -3/17 & 2/17 \\ 1/17 & 5/17 \end{bmatrix}$$

If we verify our answers, $AA^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

<u>Theorem</u> If matrix A is invertible, then the system of linear equations $AX=B$ has a unique solution, given by $A^{-1}B$.

In other words, if A is invertible, then $X=A^{-1}B$.

<u>Proof</u>. Given: A is invertible $\Rightarrow AA^{-1}=A^{-1}A=I$

$AX=B$ Multiply both sides by A^{-1} on the left

$$A^{-1}(AX)=A^{-1}B.$$
$$(A^{-1}A)X=A^{-1}B.$$
$$IX=A^{-1}B.$$
$$\boxed{X=A^{-1}B}$$

<u>Ex</u> Solve the following system using the inverse of the coefficient matrix.

$$\begin{aligned} 2x-5y&=2\\ -3x+4y&=15 \end{aligned} \qquad \begin{bmatrix}2 & -5\\ -3 & 4\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}2\\ 15\end{bmatrix}$$

$$A=\begin{bmatrix}2 & -5\\ -3 & 4\end{bmatrix}$$

$ad-bc=2(4)-(-5)(-3)=-7$

$$A^{-1}=\frac{1}{-7}\begin{bmatrix}4 & 5\\ 3 & 2\end{bmatrix}=\begin{bmatrix}-4/7 & -5/7\\ -3/7 & -2/7\end{bmatrix} \checkmark$$

Verifying: $x=-83/7$

$y=-\frac{36}{7}$

$2\left(\frac{-83}{7}\right)-5\left(\frac{-36}{7}\right)\overset{?}{=}2 \checkmark$

$-3\left(\frac{-83}{7}\right)+4\left(\frac{-36}{7}\right)\overset{?}{=}15 \checkmark$

$X=A^{-1}B$

$$X=\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}-4/7 & -5/7\\ -3/7 & -2/7\end{bmatrix}\begin{bmatrix}2\\ 15\end{bmatrix}=\begin{bmatrix}-\frac{4}{7}(2)+\left(\frac{-5}{7}\right)\cdot(15)\\ -\frac{3}{7}(2)+\left(-\frac{2}{7}\right)(15)\end{bmatrix}$$

$$2\times2 \underbrace{\quad=\quad}_{} 2\times1$$

$$=\begin{bmatrix}-\frac{8}{7}-\frac{75}{7}\\ -\frac{6}{7}-\frac{30}{7}\end{bmatrix}=\begin{bmatrix}-\frac{83}{7}\\ -\frac{36}{7}\end{bmatrix}$$

Ex Solve the following system using the inverse of the coefficient matrix

$$3x + 2y = -12$$
$$-3x + 4y = -6.$$

The matrix representation of the system is $\begin{bmatrix} 3 & 2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -12 \\ -6 \end{bmatrix}$

$$A = \begin{bmatrix} 3 & 2 \\ -3 & 4 \end{bmatrix}$$

$$A^{-1} = \frac{1}{12+6}\begin{bmatrix} 4 & -2 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} \frac{4}{18} & \frac{-2}{18} \\ \frac{3}{18} & \frac{3}{18} \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{4}{18} & -\frac{2}{18} \\ \frac{3}{18} & \frac{3}{18} \end{bmatrix}\begin{bmatrix} -12 \\ -6 \end{bmatrix} = \begin{bmatrix} \frac{4(-12)}{18} + \frac{(-2)(-6)}{18} \\ \frac{3(-12)}{18} + \frac{3(-6)}{18} \end{bmatrix} = \begin{bmatrix} \frac{-48+12}{18} \\ \frac{-36-18}{18} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{36}{18} \\ -\frac{54}{18} \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \end{bmatrix} \Rightarrow \begin{matrix} x = -2 \\ y = -3 \end{matrix}$$ is the solution.

Verifying the answer in the system:

$$3(-2) + 2(-3) = -12 \checkmark$$
$$-3(-2) + 4(-3) = -6 \checkmark$$