Improved Hardy Inequalities with Exact Remainder Terms

WEIJIA YIN

Supervisor: Nguyen LAM

University of British Columbia

weijia.ubc@gmail.com
nlam@math.ubc.ca

Abstract

We set up several identities that imply some versions of the Hardy type inequalities. These equalities give a straightforward understanding of several Hardy type inequalities as well as the nonexistence of nontrivial optimizers. These identities also provide the "virtual" extremizers for two-weight Hardy type inequalities.

1. Introduction

Let Ω be a domain containing the origin in $\mathbb{R}^N$, $N \geq 3$. The main subject in our research is the following Hardy inequality that plays extremely important roles in many areas such as analysis, probability and partial differential equations:

$$\int_{\Omega} |\nabla u|^2 \, dx \geq \left(\frac{N-2}{2}\right)^2 \int_{\Omega} \frac{|u|^2}{|x|^2} dx. \tag{1.1}$$

We also have the following version:

$$\int_{\Omega} \left|\frac{x}{|x|} \cdot \nabla u\right|^2 dx \geq \left(\frac{N-2}{2}\right)^2 \int_{\Omega} \frac{|u|^2}{|x|^2} dx. \tag{1.2}$$

(1.2) provides an improved version for the Hardy inequality since

$$|\nabla u| \geq \left|\frac{x}{|x|} \cdot \nabla u\right|.$$

The constant $\frac{N-2}{2}$ is optimal. However, the equalities in (1.1) and (1.2) are never achieved by nontrivial functions. Much research have been devoted to the problems of improving the Hardy inequalities by adding extra nonnegative terms to the RHS of (1.1).

On the whole space $\mathbb{R}^N$, Ghoussoub and Moradifam proved in [3] that there is no strictly positive $V \in C^1((0,\infty))$ such that the inequality

$$\int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \left(\frac{N-2}{2}\right)^2 \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} dx \geq \int_{\mathbb{R}^N} V(|x|) |u|^2 \, dx$$

holds.

However, it was showed that extra terms can be added to the Hardy inequality on bounded domains. For instance, let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 3$, with $0 \in \Omega$, then in order to investigate the stability of singular solutions of nonlinear elliptic equations, Brezis and Vázquez verified in [1] that for all $u \in W_0^{1,2}(\Omega)$:

$$\int_{\Omega} |\nabla u|^2 \, dx - \left(\frac{N-2}{2}\right)^2 \int_{\Omega} \frac{|u|^2}{|x|^2} dx \geq z_0^2 \omega_N^{\frac{2}{N}} |\Omega|^{-\frac{2}{N}} \int_{\Omega} |u|^2 \, dx \tag{1.3}$$

where ω_N is the volume of the unit ball and $z_0 = 2.4048...$ is the first zero of the Bessel function $J_0(z)$. The constant $z_0^2 \omega_N^{\frac{2}{N}} |\Omega|^{-\frac{2}{N}}$ is optimal when Ω is a ball.

It is interesting that $z_0^2 \omega_N^{\frac{2}{N}} |\Omega|^{-\frac{2}{N}}$ is again unattainable in $W_0^{1,2}(\Omega)$. Thus it is logical to conjecture that $z_0^2 \omega_N^{\frac{2}{N}} |\Omega|^{-\frac{2}{N}} \int_{\Omega} |u|^2 \, dx$ is just the first term of an infinite series of extra terms that can be added to the RHS of (1.3). This problem was investigated by many authors.

In an attempt to improve, extend and unify several results in this direction, Ghoussoub and Moradifam [3] introduced the Bessel pairs and studied their connections to the Hardy inequalities:

Theorem. *Let $0 < R \leq \infty$, V and W be positive C^1-functions on $(0,R)$ such that $\int_0^R \frac{1}{r^{N-1}V(r)} dr = \infty$ and $\int_0^R r^{N-1} V(r) \, dr < \infty$. Then the following are equivalent:*

(1) (V,W) is a $N-$dimensional Bessel pair on $(0,R)$.

(2) $\int_{B_R} V(|x|) |\nabla u|^2 \, dx \geq \int_{B_R} W(|x|) |u|^2 \, dx$ for all $u \in C_0^\infty(B_R)$.

2. Preliminaries

We say that a couple of C^1-functions (V,W) is a normalized $N-$dimensional Bessel pair on $(0,R)$ if the ordinary differential equation

$$y''(r) + \left(\frac{N-1}{r} + \frac{V_r(r)}{V(r)}\right) y'(r) + \frac{W(r)}{V(r)} y(r) = 0$$

has a positive solution $\varphi_{V,W;R}$ on the interval $(0,R)$.

Example 1. $(V,W) = \left(r^{-\lambda}, \frac{(N-\lambda-2)^2}{4} r^{-\lambda-2}\right)$ is a N-dimensional Bessel pair on $(0,\infty)$ with $\varphi_{V,W;\infty}(r) = r^{-\frac{N-\lambda-2}{2}}$.

Example 2. For any $R > 0$, $(V,W) = \left(r^{-\lambda}, \frac{(N-\lambda-2)^2}{4} r^{-\lambda-2} + \frac{z_0^2}{R^2} r^{-\lambda}\right)$ is a N-dimensional Bessel pair on $(0,R)$ with $\varphi_{V,W;R}(r) = r^{-\frac{N-\lambda-2}{2}} J_0\left(\frac{r z_0}{R}\right) = r^{-\frac{N-\lambda-2}{2}} J_{0;R}(r)$. Here $z_0 = 2.4048...$ is the first zero of the Bessel function $J_0(z)$.

Example 3. $\left(1, \left(\frac{N-2}{2}\right)^2 \frac{1}{r^2\left(1-\left(\frac{R}{r}\right)^{2-N}\right)^2}\right)$ is a N-dimensional Bessel pair on $(0,R)$ with $\varphi_{V,W;R}(r) = R^{-\frac{N-2}{2}} \sqrt{\left(\frac{R}{r}\right)^{N-2} - 1}$.

3. Main results

Theorem. *Let $0 < R \leq \infty$, V and W be positive C^1-functions on $(0,R)$ such that $\int_0^R \frac{1}{r^{N-1}V(r)} dr = \infty$ and $\int_0^R r^{N-1} V(r) \, dr < \infty$. Assume that (V,W) is a $N-$dimensional Bessel pair on $(0,R)$.* Then for all $u \in C_0^\infty(B_R)$:

$$\begin{aligned}&\int_{B_R} V(|x|) \left|\frac{x}{|x|} \cdot \nabla u\right|^2 dx - \int_{B_R} W(|x|) |u|^2 \, dx \\ &= \int_{B_R} V(|x|) \left|\frac{x}{|x|} \cdot \nabla \left(\frac{u}{\varphi_{V,W;R}}\right)\right|^2 \varphi_{V,W;R}^2 dx\end{aligned}$$

and

$$\begin{aligned}&\int_{B_R} V(|x|) |\nabla u|^2 \, dx - \int_{B_R} W(|x|) |u|^2 \, dx \\ &= \int_{B_R} V(|x|) \left|\nabla \left(\frac{u}{\varphi_{V,W;R}}\right)\right|^2 \varphi_{V,W;R}^2 dx\end{aligned}$$

where $\varphi_{V,W;R}$ is the positive solution of

$$y''(r) + \left(\frac{N-1}{r} + \frac{V_r(r)}{V(r)}\right) y'(r) + \frac{W(r)}{V(r)} y(r) = 0$$

on the interval $(0,R)$.

Our result implies a version of the Heisenberg-Pauli-Weyl type Uncertainty principle:

$$\begin{aligned}&\left(\int_{\mathbb{R}^N} |u|^2 \, dx\right)^2 \\ &\leq \left(\int_{\mathbb{R}^N} W(|x|) |u|^2 \, dx\right) \left(\int_{\mathbb{G}} \frac{1}{W(|x|)} |u|^2 \, dx\right) \\ &\leq \left(\int_{\mathbb{R}^N} V(|x|) \left|\frac{x}{|x|} \cdot \nabla u\right|^2 dx\right) \left(\int_{\mathbb{R}^N} \frac{1}{W(|x|)} |u|^2 \, dx\right).\end{aligned}$$

This covers the classical Heisenberg-Pauli-Weyl uncertainty principle on $\mathbb{R}^N$:

$$\begin{aligned}\left(\int_{\mathbb{R}^N} |u|^2 \, dx\right)^2 &\leq \left(\int_{\mathbb{R}^N} \frac{1}{|x|^2} |u|^2 \, dx\right) \left(\int_{\mathbb{R}^N} |x|^2 |u|^2 \, dx\right) \\ &\leq \left(\frac{2}{N-2}\right)^2 \left(\int_{\mathbb{R}^N} \left|\frac{x}{|x|} \cdot \nabla u\right|^2 dx\right) \left(\int_{\mathbb{R}^N} |x|^2 |u|^2 \, dx\right) \\ &\leq \left(\frac{2}{N-2}\right)^2 \left(\int_{\mathbb{R}^N} |\nabla u|^2 \, dx\right) \left(\int_{\mathbb{R}^N} |x|^2 |u|^2 \, dx\right).\end{aligned}$$

4. Some examples

Corollary 1. We have

$$\begin{aligned}&\int_{\mathbb{R}^N} \frac{1}{|x|^\lambda} \left|\frac{x}{|x|} \cdot \nabla u\right|^2 dx - \frac{(N-\lambda-2)^2}{4} \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{\lambda+2}} dx \\ &= \int_{\mathbb{R}^N} \left|\frac{x}{|x|} \cdot \nabla \left(|x|^{\frac{N-\lambda-2}{2}} u\right)\right|^2 \left|\frac{1}{|x|^{\frac{N-\lambda-2}{2}}}\right|^2 dx\end{aligned}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^N} \frac{1}{|x|^\lambda} |\nabla u|^2 \, dx - \frac{(N-\lambda-2)^2}{4} \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{\lambda+2}} dx \\ &= \int_{\mathbb{R}^N} \left|\nabla \left(|x|^{\frac{N-\lambda-2}{2}} u\right)\right|^2 \left|\frac{1}{|x|^{\frac{N-\lambda-2}{2}}}\right|^2 dx.\end{aligned}$$

Corollary 2. For $u \in C_0^\infty(B_R)$, we have

$$\begin{aligned}&\int_{B_R} \frac{1}{|x|^\lambda} \left|\frac{x}{|x|} \cdot \nabla u\right|^2 dx - \frac{(N-\lambda-2)^2}{4} \int_{B_R} \frac{|u|^2}{|x|^{\lambda+2}} dx \\ &= \frac{z_0^2}{R^2} \int_{B_R} \frac{|u|^2}{|x|^\lambda} dx + \int_{B_R} \left|\frac{x}{|x|} \cdot \nabla \left(\frac{|x|^{\frac{N-\lambda-2}{2}}}{J_{0;R}(|x|)} u\right)\right|^2 \left|\frac{J_{0;R}(|x|)}{|x|^{\frac{N-\lambda-2}{2}}}\right|^2 dx\end{aligned}$$

and

$$\begin{aligned}&\int_{B_R} \frac{1}{|x|^\lambda} |\nabla u|^2 \, dx - \frac{(N-\lambda-2)^2}{4} \int_{B_R} \frac{|u|^2}{|x|^{\lambda+2}} dx \\ &= \frac{z_0^2}{R^2} \int_{B_R} \frac{|u|^2}{|x|^\lambda} dx + \int_{B_R} \left|\nabla \left(\frac{|x|^{\frac{N-\lambda-2}{2}}}{J_{0;R}(|x|)} u\right)\right|^2 \left|\frac{J_{0;R}(|x|)}{|x|^{\frac{N-\lambda-2}{2}}}\right|^2 dx.\end{aligned}$$

Corollary 3. We have

$$\begin{aligned}&\int_{B_R} \left|\frac{x}{|x|} \cdot \nabla u\right|^2 dx - \left(\frac{N-2}{2}\right)^2 \int_{B_R} \frac{|u|^2}{|x|^2 \left(1 - \left(\frac{R}{|x|}\right)^{2-N}\right)^2} dx \\ &= \int_{B_R} \left|\frac{x}{|x|} \cdot \nabla \left(\frac{u}{\sqrt{\left(\frac{R}{|x|}\right)^{N-2} - 1}}\right)\right|^2 \left[\left(\frac{R}{|x|}\right)^{N-2} - 1\right] dx\end{aligned}$$

and

$$\begin{aligned}&\int_{B_R} |\nabla u|^2 \, dx - \left(\frac{N-2}{2}\right)^2 \int_{B_R} \frac{|u|^2}{|x|^2 \left(1 - \left(\frac{R}{|x|}\right)^{2-N}\right)^2} dx \\ &= \int_{B_R} \left|\nabla \left(\frac{u}{\sqrt{\left(\frac{R}{|x|}\right)^{N-2} - 1}}\right)\right|^2 \left[\left(\frac{R}{|x|}\right)^{N-2} - 1\right] dx\end{aligned}$$

References

[1] Brezis, H.; Vázquez, J. L. *Blow-up solutions of some nonlinear elliptic problems*, Rev. Mat. Univ.Complut. Madrid,10, 1997, 443–469.

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