Improved Hardy Inequalities with Exact Remainder Terms

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- These identities also provide the "virtual" extremizers for two-weight Hardy type inequalities.

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- important in the study of partial differential equations
- allow one to trade regularity (in the sense of derivatives) for integrability

Hardy inequality

- *For all* $u \in C_0^\infty\left(\mathbb{R}^N\right)$:

$$\left(\frac{N-2}{2}\right)^2 \int\limits_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} dx \leq \int\limits_{\mathbb{R}^N} |\nabla u|^2 \, dx \tag{H}$$

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- No nontrivial optimizer

Why are Hardy inequalities interesting?

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$$\int_{\mathbb{R}^N} |u|^2 \, dx \leq \frac{2}{N-2} \left(\int_{\mathbb{R}^N} |x|^2 |u|^2 \, dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right)^{\frac{1}{2}}$$

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- Uncertainty Principle in the sense of quantum mechanics
- more precise than Heisenberg's original statement (the position and the velocity of an object cannot both be measured exactly, at the same time)

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- $\left(\frac{N-2}{2}\right)^2$ is the optimal constant but there is no nontrivial optimizer
- Much research have been devoted to the problems of improving the Hardy inequalities by adding extra nonnegative terms to the LHS of (H)

Improved Hardy inequalities

- On the whole space $\mathbb{R}^N$, Ghoussoub and Moradifam proved that there is no strictly positive $V \in C^1((0,\infty))$ such that the inequality

$$\int\limits_{\mathbb{R}^N} |\nabla u|^2 \, dx - \left(\frac{N-2}{2}\right)^2 \int\limits_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} dx \geq \int\limits_{\mathbb{R}^N} V(|x|) \, |u|^2 \, dx$$

holds

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  holds
- However, it was showed that extra terms can be added to the Hardy inequality on bounded domain.

- Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 3$, with $0 \in \Omega$: Brezis and Vázquez proved that for

$$\int_\Omega |\nabla u|^2 \, dx - \left(\frac{N-2}{2}\right)^2 \int_\Omega \frac{|u|^2}{|x|^2} dx \geq z_0^2 \omega_N^{\frac{2}{N}} |\Omega|^{-\frac{2}{N}} \int_\Omega |u|^2 \, dx$$

where ω_N is the volume of the unit ball and $z_0 = 2.4048...$ is the first zero of the Bessel function $J_0(z)$.

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- The constant $z_0^2 \omega_N^{\frac{2}{N}} |\Omega|^{-\frac{2}{N}}$ is optimal when Ω is a ball.

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- This problem was investigated by many authors.

Bessel pairs

- We say that a couple of C^1-functions (V, W) is a $N-$dimensional Bessel pair on $(0, R)$ if the ordinary differential equation

$$y''(r) + \left(\frac{N-1}{r} + \frac{V_r(r)}{V(r)}\right) y'(r) + \frac{W(r)}{V(r)} y(r) = 0$$

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- **Example 1.** $(V, W) = \left(r^{-\lambda}, \frac{(N-\lambda-2)^2}{4} r^{-\lambda-2}\right)$ is a N-dimensional Bessel pair on $(0, \infty)$ with $\varphi_{V,W;\infty}(r) = r^{-\frac{N-\lambda-2}{2}}$.

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- **Example 2.** For any $R > 0$,
  $(V, W) = \left(r^{-\lambda}, \frac{(N-\lambda-2)^2}{4} r^{-\lambda-2} + \frac{z_0^2}{R^2} r^{-\lambda}\right)$ is a N-dimensional Bessel pair on $(0, R)$ with $\varphi_{V,W;R}(r) = r^{-\frac{N-\lambda-2}{2}} J_0\left(\frac{rz_0}{R}\right) = r^{-\frac{N-\lambda-2}{2}} J_{0;R}(r)$.
  Here $z_0 = 2.4048...$ is the first zero of the Bessel function $J_0(z)$.

Improved Hardy inequalities

- Ghoussoub and Moradifam introduced the Bessel pairs and studied their connections to the Hardy inequalities:
  Theorem. *Let* $0 < R \leq \infty$, *V and W be positive C^1-functions on $(0, R)$ such that* $\int_0^R \frac{1}{r^{N-1}V(r)} dr = \infty$ *and* $\int_0^R r^{N-1} V(r)\, dr < \infty$. *Then the following are equivalent:*
  (1) (V, W) *is a $N-$dimensional Bessel pair on* $(0, R)$.
  (2) $\int_{B_R} V(|x|) |\nabla u|^2 dx \geq \int_{B_R} W(|x|) |u|^2 dx$ *for all* $u \in C_0^\infty(B_R)$.

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  Theorem. *Let* $0 < R \leq \infty$*,* V *and* W *be positive* C^1-*functions on* $(0, R)$ *such that* $\int_0^R \frac{1}{r^{N-1}V(r)} dr = \infty$ *and* $\int_0^R r^{N-1} V(r)\, dr < \infty$*. Then the following are equivalent:*
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- This result improves, extends and unifies several results in this direction.

Improved Hardy inequalities

- We also have the following improved version of Hardy Inequality:

$$\int_\Omega \left| \frac{x}{|x|} \cdot \nabla u \right|^2 dx \geq \left(\frac{N-2}{2} \right)^2 \int_\Omega \frac{|u|^2}{|x|^2} dx. \tag{0.1}$$

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- In the polar coordinate: $\frac{x}{|x|} \cdot \nabla = \partial_r$: radial derivative
- can be used to study Hardy type inequalities on homogeneous group where the full gradient is not homogeneous.

Theorem. *Let* $0 < R \leq \infty$, V *and* W *be positive* C^1-*functions on* $(0, R)$ *such that* $\int_0^R \frac{1}{r^{N-1}V(r)} dr = \infty$ *and* $\int_0^R r^{N-1} V(r)\, dr < \infty$. *Assume that* (V, W) *is a* $N-$*dimensional Bessel pair on* $(0, R)$. Then for all $u \in C_0^\infty(B_R)$:

$$\int_{B_R} V(|x|) \left| \frac{x}{|x|} \cdot \nabla u \right|^2 dx - \int_{B_R} W(|x|) |u|^2 dx$$
$$= \int_{B_R} V(|x|) \left| \frac{x}{|x|} \cdot \nabla \left(\frac{u}{\varphi_{V,W;R}} \right) \right|^2 \varphi_{V,W;R}^2 dx$$

Main result

and

$$\int_{B_R} V(|x|)|\nabla u|^2 dx - \int_{B_R} W(|x|)|u|^2 dx$$
$$= \int_{B_R} V(|x|)\left|\nabla\left(\frac{u}{\varphi_{V,W;R}}\right)\right|^2 \varphi_{V,W;R}^2 dx$$

where $\varphi_{V,W;R}$ is the positive solution of

$$y''(r) + \left(\frac{N-1}{r} + \frac{V_r(r)}{V(r)}\right) y'(r) + \frac{W(r)}{V(r)} y(r) = 0$$

on the interval $(0, R)$.

- $\varphi_{V,W;R}$ is the "virtual" optimizer for the Hardy inequality.

Heisenberg-Pauli-Weyl type Uncertainty principle

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$$\left(\int_{\mathbb{R}^{\mathbb{N}}} |u|^2\, dx\right)^2 \leq \left(\int_{\mathbb{R}^{\mathbb{N}}} W(|x|)\, |u|^2\, dx\right)\left(\int_{\mathbb{G}} \frac{1}{W(|x|)}\, |u|^2\, dx\right) \leq \left(\int_{\mathbb{R}^N} V(|x|) \left|\frac{x}{|x|}\cdot\nabla u\right|^2 dx\right)\left(\int_{\mathbb{R}^N} \frac{1}{W(|x|)}\, |u|^2\, dx\right).$$

Heisenberg-Pauli-Weyl uncertainty principle

- This covers the classical Heisenberg-Pauli-Weyl uncertainty principle on $\mathbb{R}^N$:

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- $$\begin{aligned}\left(\int_{\mathbb{R}^N} |u|^2\,dx\right)^2 &\le \left(\int_{\mathbb{R}^N} \frac{1}{|x|^2}|u|^2\,dx\right)\left(\int_{\mathbb{R}^N} |x|^2|u|^2\,dx\right)\\ &\le \left(\frac{2}{N-2}\right)^2\left(\int_{\mathbb{R}^N} \left|\frac{x}{|x|}\cdot\nabla u\right|^2 dx\right)\left(\int_{\mathbb{R}^N} |x|^2|u|^2\,dx\right)\\ &\le \left(\frac{2}{N-2}\right)^2\left(\int_{\mathbb{R}^N} |\nabla u|^2\,dx\right)\left(\int_{\mathbb{R}^N} |x|^2|u|^2\,dx\right).\end{aligned}$$

Some examples

Corollary 1. We have

$$\int_{\mathbb{R}^N} \frac{1}{|x|^\lambda} \left| \frac{x}{|x|} \cdot \nabla u \right|^2 dx - \frac{(N-\lambda-2)^2}{4} \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{\lambda+2}} dx$$

$$= \int_{\mathbb{R}^N} \left| \frac{x}{|x|} \cdot \nabla \left(|x|^{\frac{N-\lambda-2}{2}} u \right) \right|^2 \left| \frac{1}{|x|^{\frac{N-\lambda-2}{2}}} \right|^2 dx$$

and

$$\int_{\mathbb{R}^N} \frac{1}{|x|^\lambda} |\nabla u|^2 dx - \frac{(N-\lambda-2)^2}{4} \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{\lambda+2}} dx$$

$$= \int_{\mathbb{R}^N} \left| \nabla \left(|x|^{\frac{N-\lambda-2}{2}} u \right) \right|^2 \left| \frac{1}{|x|^{\frac{N-\lambda-2}{2}}} \right|^2 dx.$$

Corollary 2. For $u \in C_0^\infty(B_R)$, we have

$$\int_{B_R} \frac{1}{|x|^\lambda}\left|\frac{x}{|x|}\cdot\nabla u\right|^2 dx - \frac{(N-\lambda-2)^2}{4}\int_{B_R}\frac{|u|^2}{|x|^{\lambda+2}}dx$$

$$= \frac{z_0^2}{R^2}\int_{B_R}\frac{|u|^2}{|x|^\lambda}dx + \int_{B_R}\left|\frac{x}{|x|}\cdot\nabla\left(\frac{|x|^{\frac{N-\lambda-2}{2}}}{J_{0;R}(|x|)}u\right)\right|^2\left|\frac{J_{0;R}(|x|)}{|x|^{\frac{N-\lambda-2}{2}}}\right|^2 dx$$

and

$$\int_{B_R}\frac{1}{|x|^\lambda}|\nabla u|^2 dx - \frac{(N-\lambda-2)^2}{4}\int_{B_R}\frac{|u|^2}{|x|^{\lambda+2}}dx$$

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Some examples

Corollary 3. We have

$$\int_{B_R} \left| \frac{x}{|x|} \cdot \nabla u \right|^2 dx - \left(\frac{N-2}{2} \right)^2 \int_{B_R} \frac{|u|^2}{|x|^2 \left(1 - \left(\frac{R}{|x|} \right)^{2-N} \right)^2} dx$$

$$= \int_{B_R} \left| \frac{x}{|x|} \cdot \nabla \left(\frac{u}{\sqrt{\left(\frac{R}{|x|} \right)^{N-2} - 1}} \right) \right|^2 \left[\left(\frac{R}{|x|} \right)^{N-2} - 1 \right] dx$$

and

$$\int_{B_R} |\nabla u|^2 dx - \left(\frac{N-2}{2} \right)^2 \int_{B_R} \frac{|u|^2}{|x|^2 \left(1 - \left(\frac{R}{|x|} \right)^{2-N} \right)^2} dx$$

$$= \int \left| \nabla \left(\frac{u}{\sqrt{\left(\quad \right)^{N-2}}} \right) \right|^2 \left[\left(\frac{R}{|x|} \right)^{N-2} - 1 \right] dx$$

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- Thank you very much for your attention!